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Chapter 13

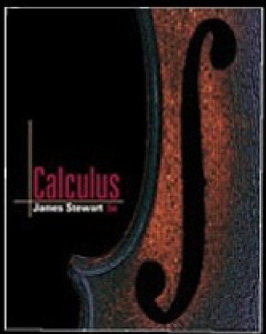
Adapted from the

Complete Solutions Manual

for

James Stewart's

Calculus – 5th Edition



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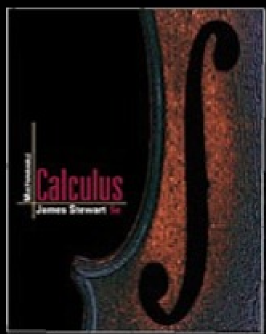

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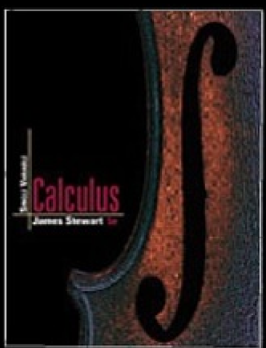

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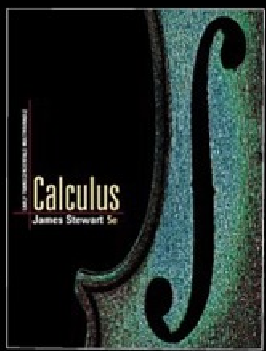

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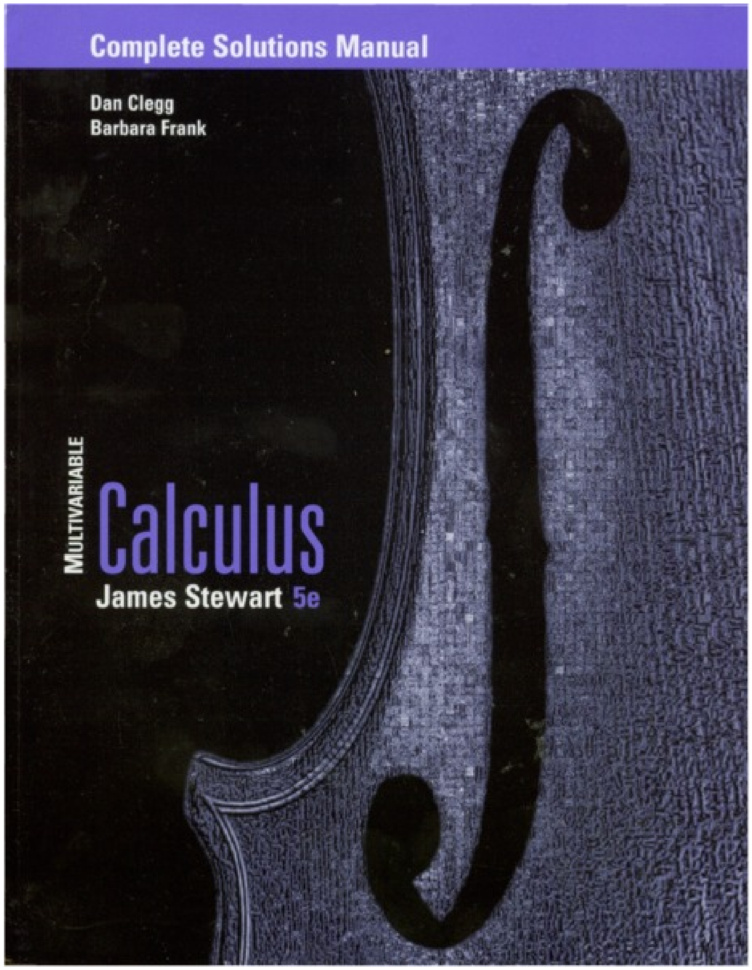
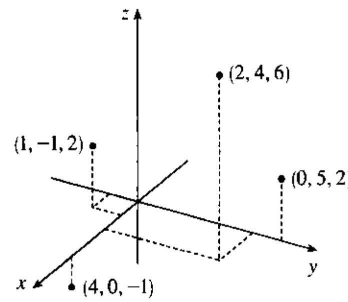


13 ☐ VECTORS AND THE GEOMETRY OF SPACE ☐ ET 12

13.1 Three-Dimensional Coordinate Systems ET 12.1

1. We start at the origin, which has coordinates $(0, 0, 0)$. First we move 4 units along the positive x-axis, affecting only the x-coordinate, bringing us to the point $(4, 0, 0)$. We then move 3 units straight downward, in the negative z-direction. Thus only the z-coordinate is affected, and we arrive at $(4, 0, -3)$.

2.

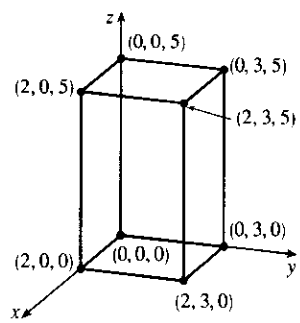


3. The distance from a point to the xz-plane is the absolute value of the y-coordinate of the point. $Q(-5, -1, 4)$ has the y-coordinate with the smallest absolute value, so Q is the point closest to the xz-plane. $R(0, 3, 8)$ must lie in the yz-plane since the distance from R to the yz-plane, given by the x-coordinate of R, is 0.

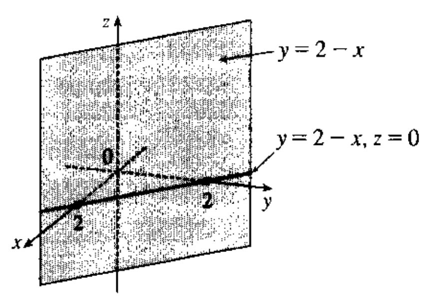
4. The projection of $(2, 3, 5)$ on the xy-plane is $(2, 3, 0)$; on the yz-plane, $(0, 3, 5)$; on the xz-plane, $(2, 0, 5)$.

The length of the diagonal of the box is the distance between the origin and $(2, 3, 5)$, given by

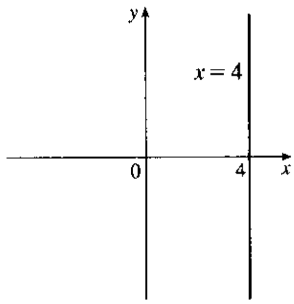
$$\sqrt{(2-0)^2 + (3-0)^2 + (5-0)^2} = \sqrt{38} \approx 6.16$$

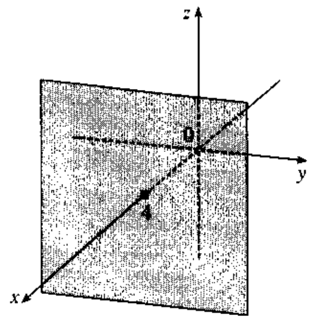


5. The equation $x + y = 2$ represents the set of all points in $\mathbb{R}^3$ whose x- and y-coordinates have a sum of 2, or equivalently where $y = 2 - x$. This is the set $\{(x, 2 - x, z) \mid x \in \mathbb{R}, z \in \mathbb{R}\}$ which is a vertical plane that intersects the xy-plane in the line $y = 2 - x$, $z = 0$.

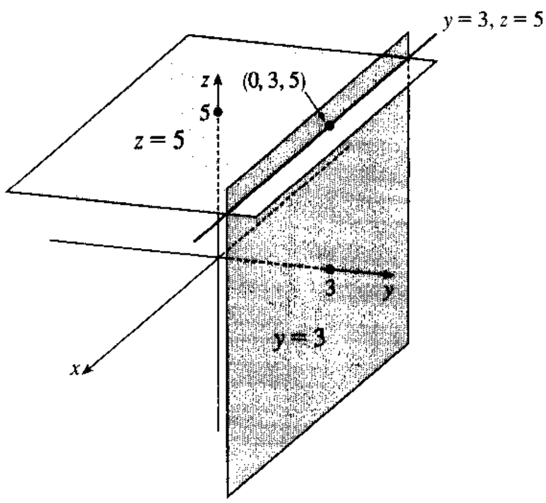


6. (a) In $\mathbb{R}^2$, the equation $x = 4$ represents a line parallel to the y-axis. In $\mathbb{R}^3$, the equation $x = 4$ represents the set $\{(x, y, z) \mid x = 4\}$, the set of all points whose x-coordinate is 4. This is the vertical plane that is parallel to the yz-plane and 4 units in front of it.





(b) In $\mathbb{R}^3$, the equation $y = 3$ represents a vertical plane that is parallel to the xz-plane and 3 units to the right of it. The equation $z = 5$ represents a horizontal plane parallel to the xy-plane and 5 units above it. The pair of equations $y = 3$, $z = 5$ represents the set of points that are simultaneously on both planes, or in other words, the line of intersection of the planes $y = 3$, $z = 5$. This line can also be described as the set $\{(x, 3, 5) \mid x \in \mathbb{R}\}$, which is the set of all points in $\mathbb{R}^3$ whose x-coordinate may vary but whose y- and z-coordinates are fixed at 3 and 5, respectively. Thus the line is parallel to the x-axis and intersects the yz-plane in the point $(0, 3, 5)$.



7. We first find the lengths of the sides of the triangle by using the distance formula between pairs of vertices:

$$|PQ| = \sqrt{[1 - (-2)]^2 + (2 - 4)^2 + (-1 - 0)^2} = \sqrt{9 + 4 + 1} = \sqrt{14}$$

$$|QR| = \sqrt{(-1 - 1)^2 + (1 - 2)^2 + [2 - (-1)]^2} = \sqrt{4 + 1 + 9} = \sqrt{14}$$

$$|PR| = \sqrt{[-1 - (-2)]^2 + (1 - 4)^2 + (2 - 0)^2} = \sqrt{1 + 9 + 4} = \sqrt{14}$$

Since all three sides have the same length, PQR is an equilateral triangle.

8. We can find the lengths of the sides of the triangle by using the distance formula between pairs of vertices:

$$|AB| = \sqrt{(3-1)^2 + (4-2)^2 + [-2-(-3)]^2} = \sqrt{4+4+1} = 3$$

$$|BC| = \sqrt{(3-3)^2 + (-2-4)^2 + [1-(-2)]^2} = \sqrt{0+36+9} = \sqrt{45} = 3\sqrt{5}$$

$$|AC| = \sqrt{(3-1)^2 + (-2-2)^2 + [1-(-3)]^2} = \sqrt{4+16+16} = 6$$

Since the Pythagorean Theorem is satisfied by $|AB|^2 + |AC|^2 = |BC|^2$, ABC is a right triangle. ABC is not ososceles, as no two sides have the same length.

9. (a) First we find the distances between points:

$$|AB| = \sqrt{(7-5)^2 + (9-1)^2 + (-1-3)^2} = \sqrt{84} = 2\sqrt{21}$$

$$|BC| = \sqrt{(1-7)^2 + (-15-9)^2 + [11-(-1)]^2} = \sqrt{756} = 6\sqrt{21}$$

$$|AC| = \sqrt{(1-5)^2 + (-15-1)^2 + (11-3)^2} = \sqrt{336} = 4\sqrt{21}$$

In order for the points to lie on a straight line, the sum of the two shortest distances must be equal to the longest distance. Since $|AB| + |AC| = |BC|$, the three points lie on a straight line.

(b) The distances between points are

$$|KL| = \sqrt{(1-0)^2 + (2-3)^2 + [-2-(-4)]^2} = \sqrt{6}$$

$$|LM| = \sqrt{(3-1)^2 + (0-2)^2 + [1-(-2)]^2} = \sqrt{17}$$

$$|KM| = \sqrt{(3-0)^2 + (0-3)^2 + [1-(-4)]^2} = \sqrt{43}$$

Since $\sqrt{6} + \sqrt{17} \neq \sqrt{43}$, the three points do not lie on a straight line.

10. (a) The distance from a point to the xy-plane is the absolute value of the z-coordinate of the point. Thus, the distance is $|-5| = 5$.

(b) Similarly, the distance is the absolute value of the x-coordinate of the point: $|3| = 3$.

(c) The distance is the absolute value of the y-coordinate of the point: $|7| = 7$.

(d) The point on the x-axis closest to $(3, 7, -5)$ is the point $(3, 0, 0)$. (Approach the x-axis perpendicularly.) The distance from $(3, 7, -5)$ to the x-axis is the distance between these two points:

$\sqrt{(3-3)^2 + (7-0)^2 + (-5-0)^2} = \sqrt{74} \approx 8.60.$

(e) The point on the y-axis closest to $(3, 7, -5)$ is $(0, 7, 0)$. The distance between these points is

$\sqrt{(3-0)^2 + (7-7)^2 + (-5-0)^2} = \sqrt{34} \approx 5.83.$

(f) The point on the z-axis closest to $(3, 7, -5)$ is $(0, 0, -5)$. The distance between these points is

$\sqrt{(3-0)^2 + (7-0)^2 + [-5-(-5)]^2} = \sqrt{58} \approx 7.62.$

11. An equation of the sphere with center $(1, -4, 3)$ and radius 5 is $(x-1)^2 + [y-(-4)]^2 + (z-3)^2 = 5^2$ or $(x-1)^2 + (y+4)^2 + (z-3)^2 = 25$. The intersection of this sphere with the xz-plane is the set of points on the sphere whose y-coordinate is 0. Putting $y = 0$ into the equation, we have $(x-1)^2 + 4^2 + (z-3)^2 = 25$, $y = 0$ or $(x-1)^2 + (z-3)^2 = 9$, $y = 0$, which represents a circle in the xz-plane with center $(1, 0, 3)$ and radius 3.

12. An equation of the sphere with center $(6, 5, -2)$ and radius $\sqrt{7}$ is $(x-6)^2 + (y-5)^2 + [z-(-2)]^2 = (\sqrt{7})^2$ or $(x-6)^2 + (y-5) + (z+2)^2 = 7$. The intersection of this sphere with the xy-plane is the set of points on the sphere whose z-coordinate is 0. Putting $z = 0$ into the equation, we have $(x-6)^2 + (y-5)^2 = 3$, $z = 0$ which represents a circle in the xy-plane with center $(6, 5, 0)$ and radius $\sqrt{3}$. To find the intersection with the xz-plane, we set $y = 0$: $(x-6)^2 + (z+2)^2 = -18$. Since no points satisfy this equation, the sphere does not intersect the xz-plane. (Also note that the distance from the center of the sphere to the xz-plane is greater than the radius of the sphere.) Similarly, the sphere does not intersect the yz-plane since substituting $x = 0$ into the equation gives $(y-5)^2 + (z+2)^2 = -29$.

13. The radius of the sphere is the distance between $(4, 3, -1)$ and $(3, 8, 1)$:
$r = \sqrt{(3-4)^2 + (8-3)^2 + [1-(-1)]^2} = \sqrt{30}$. Thus, an equation of the sphere is
$(x-3)^2 + (y-8)^2 + (z-1)^2 = 30$.

14. If the sphere passes through the origin, the radius of the sphere must be the distance from the origin to the point $(1, 2, 3)$: $r = \sqrt{(1-0)^2 + (2-0)^2 + (3-0)^2} = \sqrt{14}$. Then an equation of the sphere is
$(x-1)^2 + (y-2)^2 + (z-3)^2 = 14$.

15. Completing squares in the equation $x^2 + y^2 + z^2 - 6x + 4y - 2z = 11$ gives
$(x^2 - 6x + 9) + (y^2 + 4y + 4) + (z^2 - 2z + 1) = 11 + 9 + 4 + 1 \quad \Rightarrow$
$(x-3)^2 + (y+2)^2 + (z-1)^2 = 25$ which we recognize as an equation of a sphere with center $(3, -2, 1)$ and radius 5.

16. Completing squares in the equation gives
$(x^2 - 4x + 4) + (y^2 + 2y + 1) + z^2 = 0 + 4 + 1 \quad \Rightarrow \quad (x-2)^2 + (y+1)^2 + z^2 = 5$ which we recognize as an equation of a sphere with center $(2, -1, 0)$ and radius $\sqrt{5}$.

17. Completing squares in the equation gives $\left(x^2 - x + \frac{1}{4}\right) + \left(y^2 - y + \frac{1}{4}\right) + \left(z^2 - z + \frac{1}{4}\right) = \frac{1}{4} + \frac{1}{4} + \frac{1}{4} \quad \Rightarrow$
$\left(x - \frac{1}{2}\right)^2 + \left(y - \frac{1}{2}\right)^2 + \left(z - \frac{1}{2}\right)^2 = \frac{3}{4}$ which we recognize as an equation of a sphere with center $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$ and radius $\sqrt{\frac{3}{4}} = \frac{\sqrt{3}}{2}$.

18. Completing squares in the equation gives $4(x^2 - 2x + 1) + 4(y^2 + 4y + 4) + 4z^2 = 1 + 4 + 16 \quad \Rightarrow$
$4(x-1)^2 + 4(y+2)^2 + 4z^2 = 21 \quad \Rightarrow \quad (x-1)^2 + (y+2)^2 + z^2 = \frac{21}{4}$, which we recognize as an equation of a sphere with center $(1, -2, 0)$ and radius $\sqrt{\frac{21}{4}} = \frac{\sqrt{21}}{2}$.

19. (a) If the midpoint of the line segment from $P_1(x_1, y_1, z_1)$ to $P_2(x_2, y_2, z_2)$ is $Q = \left(\frac{x_1 + x_2}{2}, \frac{y_1 + y_2}{2}, \frac{z_1 + z_2}{2}\right)$, then the distances $|P_1Q|$ and $|QP_2|$ are equal, and each is half of $|P_1P_2|$. We verify that this is the case:

$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

$$\begin{aligned} |P_1Q| &= \sqrt{\left[\tfrac{1}{2}(x_1 + x_2) - x_1\right]^2 + \left[\tfrac{1}{2}(y_1 + y_2) - y_1\right]^2 + \left[\tfrac{1}{2}(z_1 + z_2) - z_1\right]^2} \\ &= \sqrt{\left(\tfrac{1}{2}x_2 - \tfrac{1}{2}x_1\right)^2 + \left(\tfrac{1}{2}y_2 - \tfrac{1}{2}y_1\right)^2 + \left(\tfrac{1}{2}z_2 - \tfrac{1}{2}z_1\right)^2} \\ &= \sqrt{\left(\tfrac{1}{2}\right)^2\left[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2\right]} \\ &= \tfrac{1}{2}\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \\ &= \tfrac{1}{2}\,|P_1P_2| \end{aligned}$$

$$\begin{aligned} |QP_2| &= \sqrt{\left[x_2 - \tfrac{1}{2}(x_1 + x_2)\right]^2 + \left[y_2 - \tfrac{1}{2}(y_1 + y_2)\right]^2 + \left[z_2 - \tfrac{1}{2}(z_1 + z_2)\right]^2} \\ &= \sqrt{\left(\tfrac{1}{2}x_2 - \tfrac{1}{2}x_1\right)^2 + \left(\tfrac{1}{2}y_2 - \tfrac{1}{2}y_1\right)^2 + \left(\tfrac{1}{2}z_2 - \tfrac{1}{2}z_1\right)^2} \\ &= \sqrt{\left(\tfrac{1}{2}\right)^2\left[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2\right]} \\ &= \tfrac{1}{2}\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2} \\ &= \tfrac{1}{2}\,|P_1P_2| \end{aligned}$$

So Q is indeed the midpoint of P_1P_2.

(b) By part (a), the midpoints of sides AB, BC and CA are $P_1\left(-\frac{1}{2}, 1, 4\right)$, $P_2\left(1, \frac{1}{2}, 5\right)$ and $P_3\left(\frac{5}{2}, \frac{3}{2}, 4\right)$. (Recall that a median of a triangle is a line segment from a vertex to the midpoint of the opposite side.) Then the lengths of the medians are:

$$|AP_2| = \sqrt{0^2 + \left(\tfrac{1}{2} - 2\right)^2 + (5 - 3)^2} = \sqrt{\tfrac{9}{4} + 4} = \sqrt{\tfrac{25}{4}} = \tfrac{5}{2}$$

$$|BP_3| = \sqrt{\left(\tfrac{5}{2} + 2\right)^2 + \left(\tfrac{3}{2}\right)^2 + (4 - 5)^2} = \sqrt{\tfrac{81}{4} + \tfrac{9}{4} + 1} = \sqrt{\tfrac{94}{4}} = \tfrac{1}{2}\sqrt{94}$$

$$|CP_1| = \sqrt{\left(-\tfrac{1}{2} - 4\right)^2 + (1 - 1)^2 + (4 - 5)^2} = \sqrt{\tfrac{81}{4} + 1} = \tfrac{1}{2}\sqrt{85}$$

20. By Exercise 19(a), the midpoint of the diameter (and thus the center of the sphere) is $C(3, 2, 7)$. The radius is half the diameter, so $r = \frac{1}{2}\sqrt{(4 - 2)^2 + (3 - 1)^2 + (10 - 4)^2} = \frac{1}{2}\sqrt{44} = \sqrt{11}$. Therefore an equation of the sphere is $(x - 3)^2 + (y - 2)^2 + (z - 7)^2 = 11$.

21. (a) Since the sphere touches the xy-plane, its radius is the distance from its center, $(2, -3, 6)$, to the xy-plane, namely 6. Therefore $r = 6$ and an equation of the sphere is $(x - 2)^2 + (y + 3)^2 + (z - 6)^2 = 6^2 = 36$.

(b) The radius of this sphere is the distance from its center $(2, -3, 6)$ to the yz-plane, which is 2. Therefore, an equation is $(x - 2)^2 + (y + 3)^2 + (z - 6)^2 = 4$.

(c) Here the radius is the distance from the center $(2, -3, 6)$ to the xz-plane, which is 3. Therefore, an equation is $(x - 2)^2 + (y + 3)^2 + (z - 6)^2 = 9$.

22. The largest sphere contained in the first octant must have a radius equal to the minimum distance from the center $(5, 4, 9)$ to any of the three coordinate planes. The shortest such distance is to the xz-plane, a distance of 4. Thus an equation of the sphere is $(x - 5)^2 + (y - 4)^2 + (z - 9)^2 = 16$.

23. The equation $y = -4$ represents a plane parallel to the xz-plane and 4 units to the left of it.

24. The equation $x = 10$ represents a plane parallel to the yz-plane and 10 units in front of it.

25. The inequality $x > 3$ represents a half-space consisting of all points in front of the plane $x = 3$.

26. The inequality $y \geq 0$ represents a half-space consisting of all points on or to the right of the xz-plane.

27. The inequality $0 \leq z \leq 6$ represents all points on or between the horizontal planes $z = 0$ (the xy-plane) and $z = 6$.

28. The equation $y = z$ represents a plane perpendicular to the yz-plane and intersecting the yz-plane in the line $y = z$, $x = 0$.

29. The inequality $x^2 + y^2 + z^2 > 1$ is equivalent to $\sqrt{x^2 + y^2 + z^2} > 1$, so the region consists of those points whose distance from the origin is greater than 1. This is the set of all points outside the sphere with radius 1 and center $(0, 0, 0)$.

30. The inequality $1 \leq x^2 + y^2 + z^2 \leq 25$ is equivalent to $1 \leq \sqrt{x^2 + y^2 + z^2} \leq 5$, so the region consists of those points whose distance from the origin is at least 1 and at most 5. This is the set of all points on or between the concentric spheres with radii 1 and 5 and center $(0, 0, 0)$.

31. Completing the square in z gives $x^2 + y^2 + (z^2 - 2z + 1) < 3 + 1$ or $x^2 + y^2 + (z - 1)^2 < 4$, which is equivalent to $\sqrt{x^2 + y^2 + (z - 1)^2} < 2$. Thus the region consists of those points whose distance from the point $(0, 0, 1)$ is less than 2. This is the set of all points inside the sphere with radius 2 and center $(0, 0, 1)$.

32. The equation $x^2 + y^2 = 1$ represents the set of all points in $\mathbb{R}^3$ where $x^2 + y^2 = 1$, a surface that intersects the xy-plane in the circle $x^2 + y^2 = 1$, $z = 0$. Since z can vary, the surface is a circular cylinder of radius 1. Thus, the equation represents the region consisting of all points on a circular cylinder of radius 1 with axis the z-axis.

33. Here $x^2 + z^2 \leq 9$ or equivalently $\sqrt{x^2 + z^2} \leq 3$ which describes the set of all points in $\mathbb{R}^3$ whose distance from the y-axis is at most 3. Thus, the inequality represents the region consisting of all points on or inside a circular cylinder of radius 3 with axis the y-axis.

34. The equation $xyz = 0$ is satisfied when any of x, y, or z is 0. Thus, the equation represents the region consisting of all points on the three coordinate planes $x = 0$, $y = 0$, and $z = 0$.

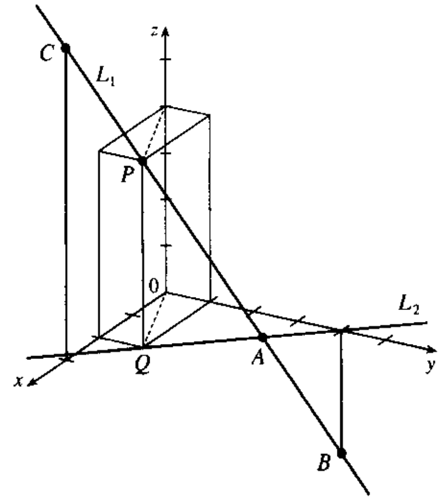
35. This describes all points with negative y-coordinates, that is, $y < 0$.

36. Because the box lies in the first quadrant, each point must comprise only nonnegative coordinates. So inequalities describing the region are $0 \leq x \leq 1$, $0 \leq y \leq 2$, $0 \leq z \leq 3$.

37. This describes a region all of whose points have a distance to the origin which is greater than r, but smaller than R. So inequalities describing the region are $r < \sqrt{x^2 + y^2 + z^2} < R$, or $r^2 < x^2 + y^2 + z^2 < R^2$.

38. The solid sphere itself is represented by $\sqrt{x^2 + y^2 + z^2} \leq 2$. Since we want only the upper hemisphere, we restrict the z-coordinate to nonnegative values. Then inequalities describing the region are $\sqrt{x^2 + y^2 + z^2} \leq 2$, $z \geq 0$, or $x^2 + y^2 + z^2 \leq 4$, $z \geq 0$.

39. (a) To find the x- and y-coordinates of the point P, we project it onto L_2 and project the resulting point Q onto the x- and y-axes. To find the z-coordinate, we project P onto either the xz-plane or the yz-plane (using our knowledge of its x- or y-coordinate) and then project the resulting point onto the z-axis. (Or, we could draw a line parallel to QO from P to the z-axis.) The coordinates of P are $(2, 1, 4)$.

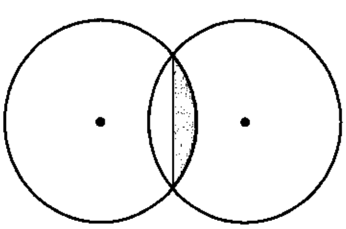


(b) A is the intersection of L_1 and L_2, B is directly below the y-intercept of L_2, and C is directly above the x-intercept of L_2.

40. Let $P = (x, y, z)$. Then $2\,|PB| = |PA| \;\Leftrightarrow\; 4\,|PB|^2 = |PA|^2 \;\Leftrightarrow$
$4\big((x-6)^2 + (y-2)^2 + (z+2)^2\big) = (x+1)^2 + (y-5)^2 + (z-3)^2 \;\Leftrightarrow$
$4(x^2 - 12x + 36) - x^2 - 2x + 4(y^2 - 4y + 4) - y^2 + 10y + 4(z^2 + 4z + 4) - z^2 + 6z = 35 \;\Leftrightarrow$
$3x^2 - 50x + 3y^2 - 6y + 3z^2 + 22z = 35 - 144 - 16 - 16 \;\Leftrightarrow\; x^2 - \frac{50}{3}x + y^2 - 2y + z^2 + \frac{22}{3}z = -\frac{141}{3}$.
By completing the square three times we get $\left(x - \frac{25}{3}\right)^2 + (y-1)^2 + \left(z + \frac{11}{3}\right)^2 = \frac{332}{9}$, which is an equation of a sphere with center $\left(\frac{25}{3}, 1, -\frac{11}{3}\right)$ and radius $\frac{\sqrt{332}}{3}$.

41. We need to find a set of points $\{P(x,y,z) \mid |AP| = |BP|\}$.
$\sqrt{(x+1)^2 + (y-5)^2 + (z-3)^2} = \sqrt{(x-6)^2 + (y-2)^2 + (z+2)^2} \;\Rightarrow$
$(x+1)^2 + (y-5) + (z-3)^2 = (x-6)^2 + (y-2)^2 + (z+2)^2 \;\Rightarrow$
$x^2 + 2x + 1 + y^2 - 10y + 25 + z^2 - 6z + 9 = x^2 - 12x + 36 + y^2 - 4y + 4 + z^2 + 4z + 4 \;\Rightarrow$
$14x - 6y - 10z = 9$. Thus the set of points is a plane perpendicular to the line segment joining A and B (since this plane must contain the perpendicular bisector of the line segment AB).

42. Completing the square three times in the first equation gives $(x+2)^2 + (y-1)^2 + (z+2)^2 = 2^2$, a sphere with center $(-2, 1, 2)$ and radius 2. The second equation is that of a sphere with center $(0, 0, 0)$ and radius 2. The distance between the centers of the spheres is $\sqrt{(-2-0)^2 + (1-0)^2 + (-2-0)^2} = \sqrt{4+1+4} = 3$. Since the spheres have the same radius, the volume inside both spheres is symmetrical about the plane containing the circle of intersection of the spheres. The distance from this plane to the center of the circles is $\frac{3}{2}$. So the region inside both spheres consists of two caps of spheres of height $h = 2 - \frac{3}{2} = \frac{1}{2}$. From Exercise 6.2.49 [ET 6.2.49], the volume of a cap of a sphere is $V = \frac{1}{3}\pi h^2(3r - h) = \frac{1}{3}\pi\left(\frac{1}{2}\right)^2\left(3 \cdot 2 - \frac{1}{2}\right) = \frac{11\pi}{24}$. So the total volume is $2 \cdot \frac{11\pi}{24} = \frac{11\pi}{12}$.



13.2 Vectors ET 12.2

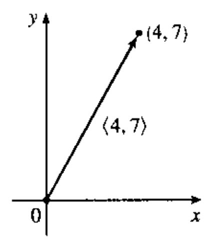
1. (a) The cost of a theater ticket is a scalar, because it has only magnitude.

   (b) The current in a river is a vector, because it has both magnitude (the speed of the current) and direction at any given location.

   (c) If we assume that the initial path is linear, the initial flight path from Houston to Dallas is a vector, because it has both magnitude (distance) and direction.

   (d) The population of the world is a scalar, because it has only magnitude.

2. If the initial point of the vector $\langle 4, 7\rangle$ is placed at the origin, then $\langle 4, 7\rangle$ is the position vector of the point $(4, 7)$.



3. Vectors are equal when they share the same length and direction (but not necessarily location). Using the symmetry of the parallelogram as a guide, we see that $\overrightarrow{AB} = \overrightarrow{DC}$, $\overrightarrow{DA} = \overrightarrow{CB}$, $\overrightarrow{DE} = \overrightarrow{EB}$, and $\overrightarrow{EA} = \overrightarrow{CE}$.

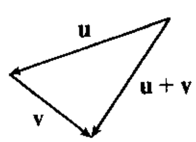
4. (a) The initial point of $\overrightarrow{QR}$ is positioned at the terminal point of $\overrightarrow{PQ}$, so by the Triangle Law the sum $\overrightarrow{PQ} + \overrightarrow{QR}$ is the vector with initial point P and terminal point R, namely $\overrightarrow{PR}$.

   (b) By the Triangle Law, $\overrightarrow{RP} + \overrightarrow{PS}$ is the vector with initial point R and terminal point S, namely $\overrightarrow{RS}$.

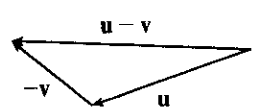
   (c) First we consider $\overrightarrow{QS} - \overrightarrow{PS}$ as $\overrightarrow{QS} + \left(-\overrightarrow{PS}\right)$. Then since $-\overrightarrow{PS}$ has the same length as $\overrightarrow{PS}$ but points in the opposite direction, we have $-\overrightarrow{PS} = \overrightarrow{SP}$ and so $\overrightarrow{QS} - \overrightarrow{PS} = \overrightarrow{QS} + \overrightarrow{SP} = \overrightarrow{QP}$.

   (d) We use the Triangle Law twice: $\overrightarrow{RS} + \overrightarrow{SP} + \overrightarrow{PQ} = \left(\overrightarrow{RS} + \overrightarrow{SP}\right) + \overrightarrow{PQ} = \overrightarrow{RP} + \overrightarrow{PQ} = \overrightarrow{RQ}$

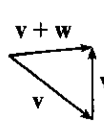
5. (a)



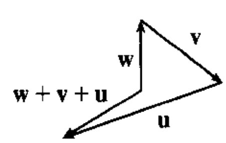
(b)



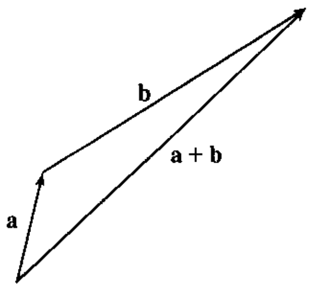
(c)



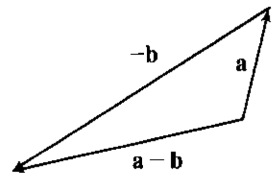
(d)



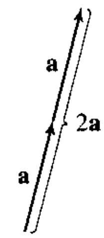
6. (a)



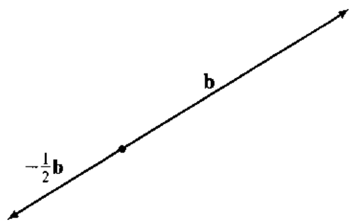
(b)



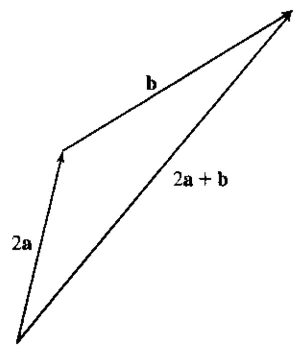
(c)



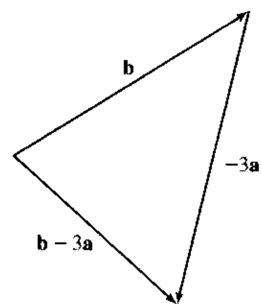
(d)



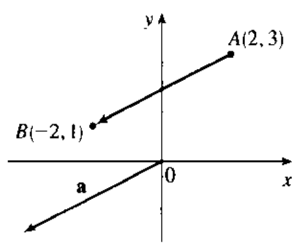
(e)



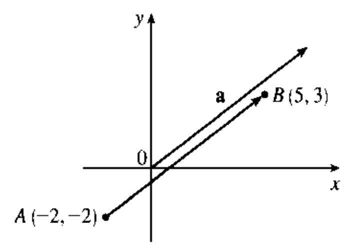
(f)



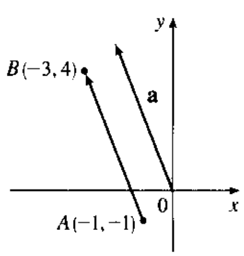
7. $\mathbf{a} = \langle -2-2, 1-3 \rangle = \langle -4, -2 \rangle$



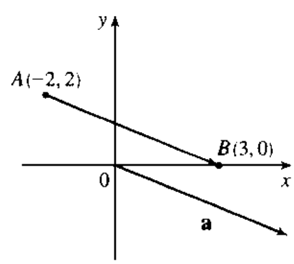
8. $\mathbf{a} = \langle 5-(-2), 3-(-2) \rangle = \langle 7, 5 \rangle$



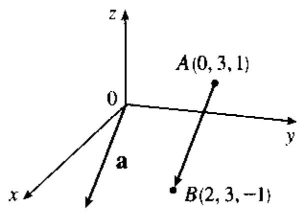
9. $\mathbf{a} = \langle -3-(-1), 4-(-1) \rangle = \langle -2, 5 \rangle$



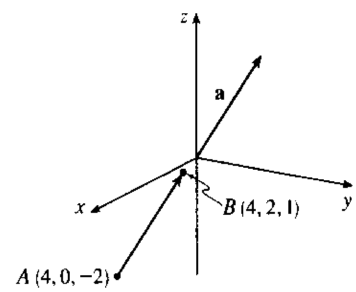
10. $\mathbf{a} = \langle 3-(-2), 0-2 \rangle = \langle 5, -2 \rangle$



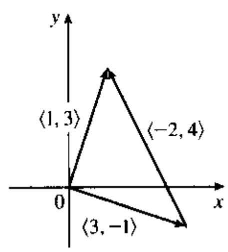
11. $\mathbf{a} = \langle 2-0, 3-3, -1-1\rangle = \langle 2, 0, -2\rangle$



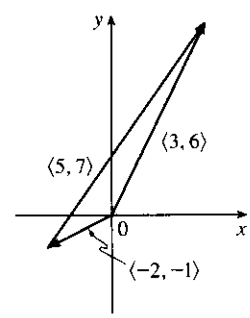
12. $\mathbf{a} = \langle 4-4, 2-0, 1-(-2)\rangle = \langle 0, 2, 3\rangle$



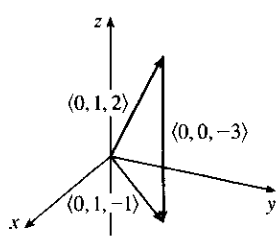
13. $\langle 3, -1\rangle + \langle -2, 4\rangle = \langle 3+(-2), -1+4\rangle$
$= \langle 1, 3\rangle$



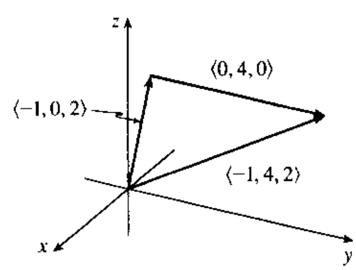
14. $\langle -2, -1\rangle + \langle 5, 7\rangle = \langle -2+5, -1+7\rangle$
$= \langle 3, 6\rangle$



15. $\langle 0, 1, 2\rangle + \langle 0, 0, -3\rangle = \langle 0+0, 1+0, 2+(-3)\rangle$
$= \langle 0, 1, -1\rangle$



16. $\langle -1, 0, 2\rangle + \langle 0, 4, 0\rangle = \langle -1+0, 0+4, 2+0\rangle$
$= \langle -1, 4, 2\rangle$



17. $|\mathbf{a}| = \sqrt{(-4)^2 + 3^2} = \sqrt{25} = 5$

$\mathbf{a}+\mathbf{b} = \langle -4+6, 3+2\rangle = \langle 2, 5\rangle$

$\mathbf{a}-\mathbf{b} = \langle -4-6, 3-2\rangle = \langle -10, 1\rangle$

$2\mathbf{a} = \langle 2(-4), 2(3)\rangle = \langle -8, 6\rangle$

$3\mathbf{a}+4\mathbf{b} = \langle -12, 9\rangle + \langle 24, 8\rangle = \langle 12, 17\rangle$

18. $|\mathbf{a}| = \sqrt{2^2 + (-3)^2} = \sqrt{13}$

$\mathbf{a}+\mathbf{b} = (2\,\mathbf{i} - 3\,\mathbf{j}) + (\mathbf{i} + 5\,\mathbf{j}) = 3\,\mathbf{i} + 2\,\mathbf{j}$

$\mathbf{a}-\mathbf{b} = (2\,\mathbf{i} - 3\,\mathbf{j}) - (\mathbf{i} + 5\,\mathbf{j}) = \mathbf{i} - 8\,\mathbf{j}$

$2\mathbf{a} = 2(2\,\mathbf{i} - 3\,\mathbf{j}) = 4\,\mathbf{i} - 6\,\mathbf{j}$

$3\mathbf{a}+4\mathbf{b} = 3(2\,\mathbf{i} - 3\,\mathbf{j}) + 4(\mathbf{i} + 5\,\mathbf{j})$
$= 6\,\mathbf{i} - 9\,\mathbf{j} + 4\,\mathbf{i} + 20\,\mathbf{j} = 10\,\mathbf{i} + 11\,\mathbf{j}$

19. $|\mathbf{a}| = \sqrt{6^2 + 2^2 + 3^2} = \sqrt{49} = 7$

$\mathbf{a} + \mathbf{b} = \langle 6 + (-1), 2 + 5, 3 + (-2) \rangle = \langle 5, 7, 1 \rangle$

$\mathbf{a} - \mathbf{b} = \langle 6 - (-1), 2 - 5, 3 - (-2) \rangle$

$= \langle 7, -3, 5 \rangle$

$2\mathbf{a} = \langle 2(6), 2(2), 2(3) \rangle = \langle 12, 4, 6 \rangle$

$3\mathbf{a} + 4\mathbf{b} = \langle 18, 6, 9 \rangle + \langle -4, 20, -8 \rangle$

$= \langle 14, 26, 1 \rangle$

20. $|\mathbf{a}| = \sqrt{(-3)^2 + (-4)^2 + (-1)^2} = \sqrt{26}$

$\mathbf{a} + \mathbf{b} = \langle -3 + 6, -4 + 2, -1 + (-3) \rangle$

$= \langle 3, -2, -4 \rangle$

$\mathbf{a} - \mathbf{b} = \langle -3 - 6, -4 - 2, -1 - (-3) \rangle$

$= \langle -9, -6, 2 \rangle$

$2\mathbf{a} = \langle 2(-3), 2(-4), 2(-1) \rangle = \langle -6, -8, -2 \rangle$

$3\mathbf{a} + 4\mathbf{b} = \langle -9, -12, -3 \rangle + \langle 24, 8, -12 \rangle$

$= \langle 15, -4, -15 \rangle$

21. $|\mathbf{a}| = \sqrt{1^2 + (-2)^2 + 1^2} = \sqrt{6}$

$\mathbf{a} + \mathbf{b} = (\mathbf{i} - 2\,\mathbf{j} + \mathbf{k}) + (\mathbf{j} + 2\,\mathbf{k}) = \mathbf{i} - \mathbf{j} + 3\,\mathbf{k}$

$\mathbf{a} - \mathbf{b} = (\mathbf{i} - 2\,\mathbf{j} + \mathbf{k}) - (\mathbf{j} + 2\,\mathbf{k}) = \mathbf{i} - 3\,\mathbf{j} - \mathbf{k}$

$2\mathbf{a} = 2(\mathbf{i} - 2\,\mathbf{j} + \mathbf{k}) = 2\,\mathbf{i} - 4\,\mathbf{j} + 2\,\mathbf{k}$

$3\mathbf{a} + 4\mathbf{b} = 3(\mathbf{i} - 2\,\mathbf{j} + \mathbf{k}) + 4(\mathbf{j} + 2\,\mathbf{k})$

$= 3\,\mathbf{i} - 6\,\mathbf{j} + 3\,\mathbf{k} + 4\,\mathbf{j} + 8\,\mathbf{k}$

$= 3\,\mathbf{i} - 2\,\mathbf{j} + 11\,\mathbf{k}$

22. $|\mathbf{a}| = \sqrt{3^2 + 0^2 + (-2)^2} = \sqrt{13}$

$\mathbf{a} + \mathbf{b} = (3\,\mathbf{i} - 2\,\mathbf{k}) + (\mathbf{i} - \mathbf{j} + \mathbf{k}) = 4\,\mathbf{i} - \mathbf{j} - \mathbf{k}$

$\mathbf{a} - \mathbf{b} = (3\,\mathbf{i} - 2\,\mathbf{k}) - (\mathbf{i} - \mathbf{j} + \mathbf{k}) = 2\,\mathbf{i} + \mathbf{j} - 3\,\mathbf{k}$

$2\mathbf{a} = 2(3\,\mathbf{i} - 2\,\mathbf{k}) = 6\,\mathbf{i} - 4\,\mathbf{k}$

$3\mathbf{a} + 4\mathbf{b} = 3(3\,\mathbf{i} - 2\,\mathbf{k}) + 4(\mathbf{i} - \mathbf{j} + \mathbf{k})$

$= 9\,\mathbf{i} - 6\,\mathbf{k} + 4\,\mathbf{i} - 4\,\mathbf{j} + 4\,\mathbf{k}$

$= 13\,\mathbf{i} - 4\,\mathbf{j} - 2\,\mathbf{k}$

23. $|\langle 9, -5 \rangle| = \sqrt{9^2 + (-5)^2} = \sqrt{106}$, so $\mathbf{u} = \dfrac{1}{\sqrt{106}} \langle 9, -5 \rangle = \left\langle \dfrac{9}{\sqrt{106}}, \dfrac{-5}{\sqrt{106}} \right\rangle$.

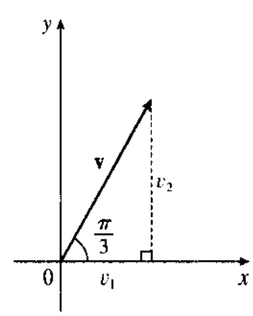
24. $|12\,\mathbf{i} - 5\,\mathbf{j}| = \sqrt{12^2 + (-5)^2} = \sqrt{169} = 13$, so $\mathbf{u} = \frac{1}{13}\,(12\,\mathbf{i} - 5\,\mathbf{j}) = \frac{12}{13}\,\mathbf{i} - \frac{5}{13}\,\mathbf{j}$.

25. The vector $8\,\mathbf{i} - \mathbf{j} + 4\,\mathbf{k}$ has length $|8\,\mathbf{i} - \mathbf{j} + 4\,\mathbf{k}| = \sqrt{8^2 + (-1)^2 + 4^2} = \sqrt{81} = 9$, so by Equation 4 the unit vector with the same direction is $\frac{1}{9}(8\,\mathbf{i} - \mathbf{j} + 4\,\mathbf{k}) = \frac{8}{9}\,\mathbf{i} - \frac{1}{9}\,\mathbf{j} + \frac{4}{9}\,\mathbf{k}$.

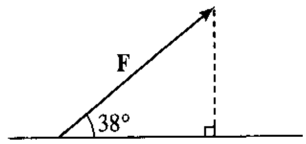
26. $|\langle -2, 4, 2 \rangle| = \sqrt{(-2)^2 + 4^2 + 2^2} = \sqrt{24} = 2\sqrt{6}$, so a unit vector in the direction of $\langle -2, 4, 2 \rangle$ is $\mathbf{u} = \dfrac{1}{2\sqrt{6}} \langle -2, 4, 2 \rangle$. A vector in the same direction but with length 6 is

$6\mathbf{u} = 6 \cdot \dfrac{1}{2\sqrt{6}} \langle -2, 4, 2 \rangle = \left\langle -\dfrac{6}{\sqrt{6}}, \dfrac{12}{\sqrt{6}}, \dfrac{6}{\sqrt{6}} \right\rangle$ or $\langle -\sqrt{6}, 2\sqrt{6}, \sqrt{6} \rangle$.

27. From the figure, we see that the x-component of $\mathbf{v}$ is $v_1 = |\mathbf{v}| \cos(\pi/3) = 4 \cdot \frac{1}{2} = 2$ and the y-component is $v_2 = |\mathbf{v}| \sin(\pi/3) = 4 \cdot \frac{\sqrt{3}}{2} = 2\sqrt{3}$. Thus $\mathbf{v} = \langle v_1, v_2 \rangle = \langle 2, 2\sqrt{3} \rangle$.



28. From the figure, we see that the horizontal component of the force $\mathbf{F}$ is $|\mathbf{F}|\cos 38^\circ = 50\cos 38^\circ \approx 39.4$ N, and the vertical component is $|\mathbf{F}|\sin 38^\circ = 50\sin 38^\circ \approx 30.8$ N.



29. $|\mathbf{F}_1| = 10$ lb and $|\mathbf{F}_2| = 12$ lb.

$$\mathbf{F}_1 = -|\mathbf{F}_1|\cos 45^\circ\,\mathbf{i} + |\mathbf{F}_1|\sin 45^\circ\,\mathbf{j} = -10\cos 45^\circ\,\mathbf{i} + 10\sin 45^\circ\,\mathbf{j}$$
$$= -5\sqrt{2}\,\mathbf{i} + 5\sqrt{2}\,\mathbf{j}$$

$$\mathbf{F}_2 = |\mathbf{F}_2|\cos 30^\circ\,\mathbf{i} + |\mathbf{F}_2|\sin 30^\circ\,\mathbf{j} = 12\cos 30^\circ\,\mathbf{i} + 12\sin 30^\circ\,\mathbf{j} = 6\sqrt{3}\,\mathbf{i} + 6\,\mathbf{j}$$

$$\mathbf{F} = \mathbf{F}_1 + \mathbf{F}_2 = \left(6\sqrt{3} - 5\sqrt{2}\right)\mathbf{i} + \left(6 + 5\sqrt{2}\right)\mathbf{j} \approx 3.32\,\mathbf{i} + 13.07\,\mathbf{j}$$

$$|\mathbf{F}| \approx \sqrt{(3.32)^2 + (13.07)^2} \approx 13.5 \text{ lb.} \quad \tan\theta = \frac{6 + 5\sqrt{2}}{6\sqrt{3} - 5\sqrt{2}} \quad \Rightarrow \quad \theta = \tan^{-1}\frac{6 + 5\sqrt{2}}{6\sqrt{3} - 5\sqrt{2}} \approx 76^\circ.$$

30. Set up the coordinate axes so that north is the positive y-direction, and east is the positive x-direction. The wind is blowing at 50 km/h from the direction N45° W, so that its velocity vector is 50 km/h S45° E, which can be written as $\mathbf{v}_{\text{wind}} = 50(\cos 45^\circ\,\mathbf{i} - \sin 45^\circ\,\mathbf{j})$. With respect to the still air, the velocity vector of the plane is 250 km/h N 60° E, or equivalently $\mathbf{v}_{\text{plane}} = 250(\cos 30^\circ\,\mathbf{i} + \sin 30^\circ\,\mathbf{j})$. The velocity of the plane relative to the ground is

$$\mathbf{v} = \mathbf{v}_{\text{wind}} + \mathbf{v}_{\text{plane}} = (50\cos 45^\circ + 250\cos 30^\circ)\,\mathbf{i} + (-50\sin 45^\circ + 250\sin 30^\circ)\,\mathbf{j}$$
$$= \left(25\sqrt{2} + 125\sqrt{3}\right)\mathbf{i} + \left(125 - 25\sqrt{2}\right)\mathbf{j} \approx 251.9\,\mathbf{i} + 89.6\,\mathbf{j}$$

The ground speed is $|\mathbf{v}| \approx \sqrt{(251.9)^2 + (89.6)^2} \approx 267$ km/h. The angle the velocity vector makes with the x-axis is $\theta \approx \tan^{-1}\left(\frac{89.6}{251.9}\right) \approx 20^\circ$. Therefore, the true course of the plane is about N$(90 - 20)^\circ$ E $=$ N 70° E.

31. With respect to the water's surface, the woman's velocity is the vector sum of the velocity of the ship with respect to the water, and the woman's velocity with respect to the ship. If we let north be the positive y-direction, then $\mathbf{v} = \langle 0, 22\rangle + \langle -3, 0\rangle = \langle -3, 22\rangle$. The woman's speed is $|\mathbf{v}| = \sqrt{9 + 484} \approx 22.2$ mi/h. The vector $\mathbf{v}$ makes an angle θ with the east, where $\theta = \tan^{-1}\left(\frac{22}{-3}\right) \approx 98^\circ$. Therefore, the woman's direction is about N$(98 - 90)^\circ$ W $=$ N8° W.

32. Call the two tensile forces $\mathbf{T}_3$ and $\mathbf{T}_5$, corresponding to the ropes of length 3 m and 5 m. In terms of vertical and horizontal components,

$$\mathbf{T}_3 = -|\mathbf{T}_3|\cos 52^\circ\,\mathbf{i} + |\mathbf{T}_3|\sin 52^\circ\,\mathbf{j} \quad (1) \qquad \text{and} \qquad \mathbf{T}_5 = |\mathbf{T}_5|\cos 40^\circ\,\mathbf{i} + |\mathbf{T}_5|\sin 40^\circ\,\mathbf{j} \quad (2)$$

The resultant of these forces, $\mathbf{T}_3 + \mathbf{T}_5$, counterbalances the force of gravity acting on the decoration [which is $-5g\,\mathbf{j} \approx -5(9.8)\,\mathbf{j} = -49\,\mathbf{j}$]. So $\mathbf{T}_3 + \mathbf{T}_5 = 49\,\mathbf{j}$. Hence

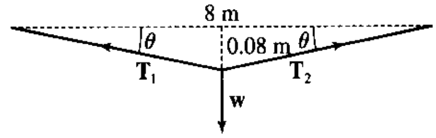
$\mathbf{T}_3 + \mathbf{T}_5 = (-|\mathbf{T}_3|\cos 52^\circ + |\mathbf{T}_5|\cos 40^\circ)\,\mathbf{i} + (|\mathbf{T}_3|\sin 52^\circ + |\mathbf{T}_5|\sin 40^\circ)\,\mathbf{j} = 49\,\mathbf{j}$. Thus

$-|\mathbf{T}_3|\cos 52^\circ + |\mathbf{T}_5|\cos 40^\circ = 0$ and $|\mathbf{T}_3|\sin 52^\circ + |\mathbf{T}_5|\sin 40^\circ = 49$.

From the first of these two equations $|\mathbf{T}_3| = |\mathbf{T}_5| \dfrac{\cos 40^\circ}{\cos 52^\circ}$. Substituting this into the second equation gives

$|\mathbf{T}_5| = \dfrac{49}{\cos 40^\circ \tan 52^\circ + \sin 40^\circ} \approx 30$ N. Therefore, $|\mathbf{T}_3| = |\mathbf{T}_5| \dfrac{\cos 40^\circ}{\cos 52^\circ} \approx 38$ N. Finally, from (1) and (2), $\mathbf{T}_3 \approx -23\,\mathbf{i} + 30\,\mathbf{j}$, and $\mathbf{T}_5 \approx 23\,\mathbf{i} + 19\,\mathbf{j}$.

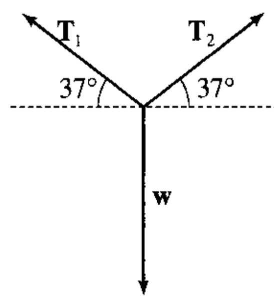
33. Let $\mathbf{T}_1$ and $\mathbf{T}_2$ represent the tension vectors in each side of the clothesline as shown in the figure. $\mathbf{T}_1$ and $\mathbf{T}_2$ have equal vertical components and opposite horizontal components, so we can write



$\mathbf{T}_1 = -a\,\mathbf{i} + b\,\mathbf{j}$ and $\mathbf{T}_2 = a\,\mathbf{i} + b\,\mathbf{j}$ $(a, b > 0)$. By similar triangles, $\dfrac{b}{a} = \dfrac{0.08}{4} \Rightarrow a = 50b$. The force due to gravity acting on the shirt has magnitude $0.8g \approx (0.8)(9.8) = 7.84$ N, hence we have $\mathbf{w} = -7.84\,\mathbf{j}$. The resultant $\mathbf{T}_1 + \mathbf{T}_2$ of the tensile forces counterbalances $\mathbf{w}$, so $\mathbf{T}_1 + \mathbf{T}_2 = -\mathbf{w} \Rightarrow (-a\,\mathbf{i} + b\,\mathbf{j}) + (a\,\mathbf{i} + b\,\mathbf{j}) = 7.84\,\mathbf{j}$ $\Rightarrow (-50b\,\mathbf{i} + b\,\mathbf{j}) + (50b\,\mathbf{i} + b\,\mathbf{j}) = 2b\,\mathbf{j} = 7.84\,\mathbf{j} \Rightarrow b = \frac{7.84}{2} = 3.92$ and $a = 50b = 196$. Thus the tensions are $\mathbf{T}_1 = -a\,\mathbf{i} + b\,\mathbf{j} = -196\,\mathbf{i} + 3.92\,\mathbf{j}$ and $\mathbf{T}_2 = a\,\mathbf{i} + b\,\mathbf{j} = 196\,\mathbf{i} + 3.92\,\mathbf{j}$.

Alternatively, we can find the value of θ and proceed as in Example 7.

34. We can consider the weight of the chain to be concentrated at its midpoint. The forces acting on the chain then are the tension vectors $\mathbf{T}_1$, $\mathbf{T}_2$ in each end of the chain and the weight $\mathbf{w}$, as shown in the figure. We know $|\mathbf{T}_1| = |\mathbf{T}_2| = 25$ N so, in terms of vertical and horizontal components, we have



$$\mathbf{T}_1 = -25\cos 37^\circ\,\mathbf{i} + 25\sin 37^\circ\,\mathbf{j}$$

$$\mathbf{T}_2 = 25\cos 37^\circ\,\mathbf{i} + 25\sin 37^\circ\,\mathbf{j}$$

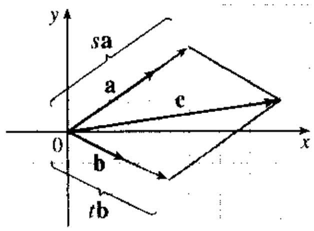
The resultant vector $\mathbf{T}_1 + \mathbf{T}_2$ of the tensions counterbalances the weight $\mathbf{w}$, giving $\mathbf{T}_1 + \mathbf{T}_2 = -\mathbf{w}$. Since $\mathbf{w} = -\,|\mathbf{w}|\,\mathbf{j}$, we have $(-25\cos 37^\circ\,\mathbf{i} + 25\sin 37^\circ\,\mathbf{j}) + (25\cos 37^\circ\,\mathbf{i} + 25\sin 37^\circ\,\mathbf{j}) = |\mathbf{w}|\,\mathbf{j} \Rightarrow$ $50\sin 37^\circ\,\mathbf{j} = |\mathbf{w}|\,\mathbf{j} \Rightarrow |\mathbf{w}| = 50\sin 37^\circ \approx 30.1$. So the weight is 30.1 N, and since $w = mg$, the mass is $\frac{30.1}{9.8} \approx 3.07$ kg.

35. By the Triangle Law, $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$. Then $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = \overrightarrow{AC} + \overrightarrow{CA}$, but $\overrightarrow{AC} + \overrightarrow{CA} = \overrightarrow{AC} + \left(-\overrightarrow{AC}\right) = \mathbf{0}$. So $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = \mathbf{0}$.

36. $\overrightarrow{AC} = \frac{1}{3}\overrightarrow{AB}$ and $\overrightarrow{BC} = \frac{2}{3}\overrightarrow{BA}$. $\mathbf{c} = \overrightarrow{OA} + \overrightarrow{AC} = \mathbf{a} + \frac{1}{3}\overrightarrow{AB} \Rightarrow \overrightarrow{AB} = 3\,\mathbf{c} - 3\,\mathbf{a}$.

$\mathbf{c} = \overrightarrow{OB} + \overrightarrow{BC} = \overrightarrow{OA} + \frac{2}{3}\overrightarrow{BA} \Rightarrow \overrightarrow{BA} = \frac{3}{2}\,\mathbf{c} - \frac{3}{2}\,\mathbf{b}$. $\overrightarrow{BA} = -\overrightarrow{AB}$, so $\frac{3}{2}\,\mathbf{c} - \frac{3}{2}\,\mathbf{b} = 3\,\mathbf{a} - 3\,\mathbf{c} \Leftrightarrow$ $\mathbf{c} + 2\,\mathbf{c} = 2\,\mathbf{a} + \mathbf{b} \Leftrightarrow \mathbf{c} = \frac{2}{3}\,\mathbf{a} + \frac{1}{3}\,\mathbf{b}$.

37. (a), (b)

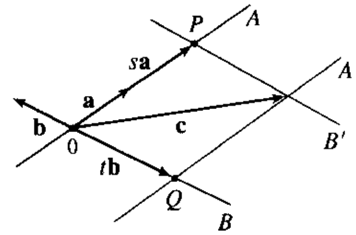


(c) From the sketch, we estimate that $s \approx 1.3$ and $t \approx 1.6$.

(d) $\mathbf{c} = s\,\mathbf{a} + t\,\mathbf{b} \;\Leftrightarrow\; 7 = 3s + 2t$ and $1 = 2s - t$.

Solving these equations gives $s = \frac{9}{7}$ and $t = \frac{11}{7}$.

38. Draw **a**, **b**, and **c** emanating from the origin. Extend **a** and **b** to form lines A and B, and draw lines A' and B' parallel to these two lines through the terminal point of **c**.



Since **a** and **b** are not parallel, A and B' must meet (at P), and A' and B must also meet (at Q). Now we see that $\overrightarrow{OP} + \overrightarrow{OQ} = \mathbf{c}$, so if $s = \dfrac{|\overrightarrow{OP}|}{|\mathbf{a}|}$ $\left(\text{or its negative, if } \mathbf{a} \text{ points in the direction opposite } \overrightarrow{OP}\right)$ and $t = \dfrac{|\overrightarrow{OQ}|}{|\mathbf{b}|}$ (or its negative, as in the diagram), then $c = s\mathbf{a} + t\mathbf{b}$, as required.

Argument using components: Since **a**, **b**, and **c** all lie in the same plane, we can consider them to be vectors in two dimensions. Let $\mathbf{a} = \langle a_1, a_2 \rangle$, $\mathbf{b} = \langle b_1, b_2 \rangle$, and $\mathbf{c} = \langle c_1, c_2 \rangle$. We need $sa_1 + tb_1 = c_1$ and $sa_2 + tb_2 = c_2$. Multiplying the first equation by a_2 and the second by a_1 and subtracting, we get $t = \dfrac{c_2 a_1 - c_1 a_2}{b_2 a_1 - b_1 a_2}$. Similarly $s = \dfrac{b_2 c_1 - b_1 c_2}{b_2 a_1 - b_1 a_2}$. Since $\mathbf{a} \neq \mathbf{0}$ and $\mathbf{b} \neq \mathbf{0}$ and **a** is not a scalar multiple of **b**, the denominator is not zero.

39. $|\mathbf{r} - \mathbf{r}_0|$ is the distance between the points (x, y, z) and (x_0, y_0, z_0), so the set of points is a sphere with radius 1 and center (x_0, y_0, z_0).

Alternate method: $|\mathbf{r} - \mathbf{r}_0| = 1 \;\Leftrightarrow\; \sqrt{(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2} = 1 \;\Leftrightarrow$

$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = 1$, which is the equation of a sphere with radius 1 and center (x_0, y_0, z_0).

40. Let P_1 and P_2 be the points with position vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ respectively. Then $|\mathbf{r} - \mathbf{r}_1| + |\mathbf{r} - \mathbf{r}_2|$ is the sum of the distances from (x, y) to P_1 and P_2. Since this sum is constant, the set of points (x, y) represents an ellipse with foci P_1 and P_2. The condition $k > |\mathbf{r}_1 - \mathbf{r}_2|$ assures us that the ellipse is not degenerate.

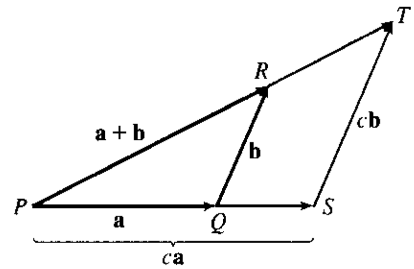
41.
$$\begin{aligned}
\mathbf{a} + (\mathbf{b} + \mathbf{c}) &= \langle a_1, a_2 \rangle + (\langle b_1, b_2 \rangle + \langle c_1, c_2 \rangle) = \langle a_1, a_2 \rangle + \langle b_1 + c_1, b_2 + c_2 \rangle \\
&= \langle a_1 + b_1 + c_1, a_2 + b_2 + c_2 \rangle = \langle (a_1 + b_1) + c_1, (a_2 + b_2) + c_2 \rangle \\
&= \langle a_1 + b_1, a_2 + b_2 \rangle + \langle c_1, c_2 \rangle = (\langle a_1, a_2 \rangle + \langle b_1, b_2 \rangle) + \langle c_1, c_2 \rangle \\
&= (\mathbf{a} + \mathbf{b}) + \mathbf{c}
\end{aligned}$$

42. *Algebraically:*

$$\begin{aligned} c(\mathbf{a}+\mathbf{b}) &= c(\langle a_1,a_2,a_3\rangle + \langle b_1,b_2,b_3\rangle) = c\langle a_1+b_1, a_2+b_2, a_3+b_3\rangle \\ &= \langle c(a_1+b_1), c(a_2+b_2), c(a_3+b_3)\rangle = \langle ca_1+cb_1, ca_2+cb_2, ca_3+cb_3\rangle \\ &= \langle ca_1, ca_2, ca_3\rangle + \langle cb_1, cb_2, cb_3\rangle = c\,\mathbf{a} + c\,\mathbf{b} \end{aligned}$$

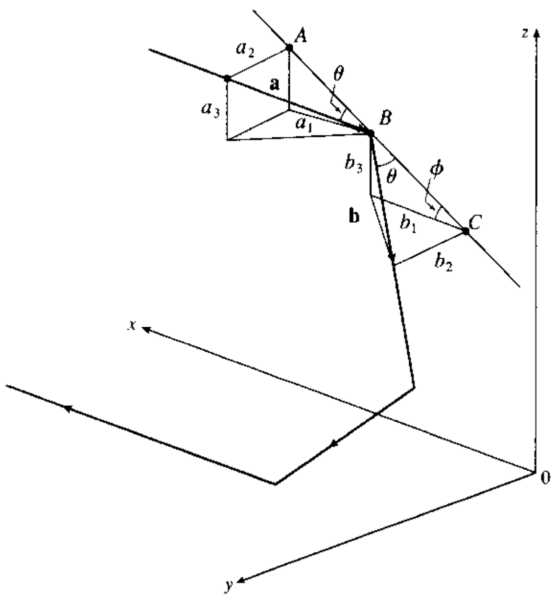
Geometrically:

According to the Triangle Law, if $\mathbf{a} = \overrightarrow{PQ}$ and $\mathbf{b} = \overrightarrow{QR}$, then $\mathbf{a}+\mathbf{b} = \overrightarrow{PR}$. Construct triangle PST as shown so that $\overrightarrow{PS} = c\,\mathbf{a}$ and $\overrightarrow{ST} = c\,\mathbf{b}$. (We have drawn the case where $c > 1$.) By the Triangle Law, $\overrightarrow{PT} = c\,\mathbf{a} + c\,\mathbf{b}$. But triangle PQR and triangle PST are similar triangles because $c\,\mathbf{b}$ is parallel to $\mathbf{b}$. Therefore, $\overrightarrow{PR}$ and $\overrightarrow{PT}$ are parallel and, in fact, $\overrightarrow{PT} = c\overrightarrow{PR}$. Thus, $c\,\mathbf{a} + c\,\mathbf{b} = c(\mathbf{a}+\mathbf{b})$.



43. Consider triangle ABC, where D and E are the midpoints of AB and BC. We know that $\overrightarrow{AB} + \overrightarrow{BC} = \overrightarrow{AC}$ (1) and $\overrightarrow{DB} + \overrightarrow{BE} = \overrightarrow{DE}$ (2). However, $\overrightarrow{DB} = \frac{1}{2}\overrightarrow{AB}$, and $\overrightarrow{BE} = \frac{1}{2}\overrightarrow{BC}$. Substituting these expressions for $\overrightarrow{DB}$ and $\overrightarrow{BE}$ into (2) gives $\frac{1}{2}\overrightarrow{AB} + \frac{1}{2}\overrightarrow{BC} = \overrightarrow{DE}$. Comparing this with (1) gives $\overrightarrow{DE} = \frac{1}{2}\overrightarrow{AC}$. Therefore $\overrightarrow{AC}$ and $\overrightarrow{DE}$ are parallel and $\left|\overrightarrow{DE}\right| = \frac{1}{2}\left|\overrightarrow{AC}\right|$.

44.



The question states that the light ray strikes all three mirrors, so it is not parallel to any of them and $a_1 \neq 0$, $a_2 \neq 0$ and $a_3 \neq 0$. Let $\mathbf{b} = \langle b_1, b_2, b_3\rangle$, as in the diagram. We can let $|\mathbf{b}| = |\mathbf{a}|$, since only its direction is important. Then $\dfrac{|b_2|}{|\mathbf{b}|} = \sin\theta = \dfrac{|a_2|}{|\mathbf{a}|} \quad\Rightarrow\quad |b_2| = |a_2|$. From the diagram $b_2\,\mathbf{j}$ and $a_2\,\mathbf{j}$ point in opposite directions, so $b_2 = -a_2$. $|AB| = |BC|$, so $|b_3| = \sin\phi\,|BC| = \sin\phi\,|AB| = |a_3|$, and $|b_1| = \cos\phi\,|BC| = \cos\phi\,|AB| = |a_1|$. $b_3\,\mathbf{k}$ and $a_3\,\mathbf{k}$ have the same direction, as do $b_1\,\mathbf{i}$ and $a_1\,\mathbf{i}$, so $\mathbf{b} = \langle a_1, -a_2, a_3\rangle$. When the ray hits the other mirrors, similar arguments show that these reflections will reverse the signs of the other two coordinates, so the final reflected ray will be $\langle -a_1, -a_2, -a_3\rangle = -\mathbf{a}$, which is parallel to $\mathbf{a}$.

13.3 The Dot Product

ET 12.3

1. (a) $\mathbf{a}\cdot\mathbf{b}$ is a scalar, and the dot product is defined only for vectors, so $(\mathbf{a}\cdot\mathbf{b})\cdot\mathbf{c}$ has no meaning.

(b) $(\mathbf{a}\cdot\mathbf{b})\,\mathbf{c}$ is a scalar multiple of a vector, so it does have meaning.

(c) Both $|\mathbf{a}|$ and $\mathbf{b}\cdot\mathbf{c}$ are scalars, so $|\mathbf{a}|\,(\mathbf{b}\cdot\mathbf{c})$ is an ordinary product of real numbers, and has meaning.

(d) Both $\mathbf{a}$ and $\mathbf{b}+\mathbf{c}$ are vectors, so the dot product $\mathbf{a}\cdot(\mathbf{b}+\mathbf{c})$ has meaning.

(e) $\mathbf{a}\cdot\mathbf{b}$ is a scalar, but $\mathbf{c}$ is a vector, and so the two quantities cannot be added and this expression has no meaning.

(f) $|\mathbf{a}|$ is a scalar, and the dot product is defined only for vectors, so $|\mathbf{a}|\cdot(\mathbf{b}+\mathbf{c})$ has no meaning.

2. Let the vectors be $\mathbf{a}$ and $\mathbf{b}$. Then by Theorem 3, $\mathbf{a}\cdot\mathbf{b}=|\mathbf{a}|\,|\mathbf{b}|\cos\theta=(6)\left(\frac{1}{3}\right)\cos\frac{\pi}{4}=\frac{6}{3\sqrt{2}}=\sqrt{2}$.

3. $\mathbf{a}\cdot\mathbf{b}=\langle 4,-1\rangle\cdot\langle 3,6\rangle=(4)\,(3)+(-1)\,(6)=6$

4. $\mathbf{a}\cdot\mathbf{b}=\left\langle \frac{1}{2},4\right\rangle\cdot\langle -8,-3\rangle=\left(\frac{1}{2}\right)(-8)+(4)(-3)=-16$

5. $\mathbf{a}\cdot\mathbf{b}=\langle 5,0,-2\rangle\cdot\langle 3,-1,10\rangle=(5)(3)+(0)(-1)+(-2)(10)=-5$

6. $\mathbf{a}\cdot\mathbf{b}=\langle s,2s,3s\rangle\cdot\langle t,-t,5t\rangle=(s)(t)+(2s)(-t)+(3s)(5t)=st-2st+15st=14st$

7. $\mathbf{a}\cdot\mathbf{b}=(\mathbf{i}-2\,\mathbf{j}+3\,\mathbf{k})\cdot(5\,\mathbf{i}+9\,\mathbf{k})=(1)(5)+(-2)(0)+(3)(9)=32$

8. $\mathbf{a}\cdot\mathbf{b}=(4\,\mathbf{j}-3\,\mathbf{k})\cdot(2\,\mathbf{i}+4\,\mathbf{j}+6\,\mathbf{k})=(0)(2)+(4)(4)+(-3)(6)=-2$

9. Use Theorem 3: $\mathbf{a}\cdot\mathbf{b}=|\mathbf{a}|\,|\mathbf{b}|\cos\theta=(12)(15)\cos\frac{\pi}{6}=180\cdot\frac{\sqrt{3}}{2}=90\sqrt{3}\approx 155.9$

10. Use Theorem 3: $\mathbf{a}\cdot\mathbf{b}=|\mathbf{a}|\,|\mathbf{b}|\cos\theta=(4)(10)\cos 120^\circ=40\left(-\frac{1}{2}\right)=-20$

11. $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$ are all unit vectors, so the triangle is an equilateral triangle. Thus the angle between $\mathbf{u}$ and $\mathbf{v}$ is 60° and $\mathbf{u}\cdot\mathbf{v}=|\mathbf{u}|\,|\mathbf{v}|\cos 60^\circ=(1)(1)\left(\frac{1}{2}\right)=\frac{1}{2}$. If $\mathbf{w}$ is moved so it has the same initial point as $\mathbf{u}$, we can see that the angle between them is 120° and we have $\mathbf{u}\cdot\mathbf{w}=|\mathbf{u}|\,|\mathbf{w}|\cos 120^\circ=(1)(1)\left(-\frac{1}{2}\right)=-\frac{1}{2}$.

12. $\mathbf{u}$ is a unit vector, so $\mathbf{w}$ is also a unit vector, and $|\mathbf{v}|$ can be determined by examining the right triangle formed by $\mathbf{u}$ and $\mathbf{v}$. Since the angle between $\mathbf{u}$ and $\mathbf{v}$ is 45°, we have $|\mathbf{v}|=|\mathbf{u}|\cos 45^\circ=\frac{\sqrt{2}}{2}$. Then $\mathbf{u}\cdot\mathbf{v}=|\mathbf{u}|\,|\mathbf{v}|\cos 45^\circ=(1)\left(\frac{\sqrt{2}}{2}\right)\frac{\sqrt{2}}{2}=\frac{1}{2}$. Since $\mathbf{u}$ and $\mathbf{w}$ are orthogonal, $\mathbf{u}\cdot\mathbf{w}=0$.

13. (a) $\mathbf{i}\cdot\mathbf{j}=\langle 1,0,0\rangle\cdot\langle 0,1,0\rangle=(1)(0)+(0)(1)+(0)(0)=0$. Similarly $\mathbf{j}\cdot\mathbf{k}=(0)(0)+(1)(0)+(0)(1)=0$ and $\mathbf{k}\cdot\mathbf{i}=(0)(1)+(0)(0)+(1)(0)=0$.

Another method: Because $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are mutually perpendicular, the cosine factor in each dot product (see Theorem 3) is $\cos\frac{\pi}{2}=0$.

(b) By Property 1 of the dot product, $\mathbf{i}\cdot\mathbf{i}=|\mathbf{i}|^2=1^2=1$ since $\mathbf{i}$ is a unit vector. Similarly, $\mathbf{j}\cdot\mathbf{j}=|\mathbf{j}|^2=1$ and $\mathbf{k}\cdot\mathbf{k}=|\mathbf{k}|^2=1$.

14. The dot product $\mathbf{A}\cdot\mathbf{P}$ is

$$\begin{aligned}\langle a,b,c\rangle\cdot\langle 2,1.5,1\rangle &= a(2)+b(1.5)+c(1)\\ &= (\text{number of hamburgers sold})(\text{price per hamburger})\\ &\quad+(\text{number of hot dogs sold})(\text{price per hot dog})\\ &\quad+(\text{number of soft drinks sold})(\text{price per soft drink})\end{aligned}$$

so it is equal to the vendor's total revenue for that day.

15. $|\mathbf{a}| = \sqrt{3^2+4^2} = 5$, $|\mathbf{b}| = \sqrt{5^2+12^2} = 13$, and $\mathbf{a}\cdot\mathbf{b} = (3)(5)+(4)(12) = 63$. Using Corollary 6, we have $\cos\theta = \dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|} = \dfrac{63}{5\cdot 13} = \dfrac{63}{65}$. So the angle between $\mathbf{a}$ and $\mathbf{b}$ is $\theta = \cos^{-1}\left(\frac{63}{65}\right) \approx 14^\circ$.

16. $|\mathbf{a}| = \sqrt{\left(\sqrt{3}\right)^2 + 1^2} = 2$, $|\mathbf{b}| = \sqrt{0+25} = 5$, and $\mathbf{a}\cdot\mathbf{b} = \left(\sqrt{3}\right)(0)+(1)(5) = 5$. Using Corollary 6, we have $\cos\theta = \dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|} = \dfrac{5}{2\cdot 5} = \dfrac{1}{2}$ and the angle between $\mathbf{a}$ and $\mathbf{b}$ is $\cos^{-1}\left(\frac{1}{2}\right) = 60^\circ$.

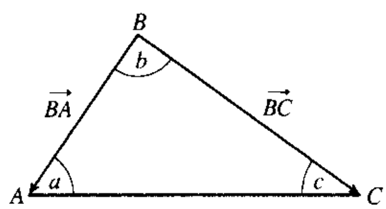
17. $|\mathbf{a}| = \sqrt{1^2+2^2+3^2} = \sqrt{14}$, $|\mathbf{b}| = \sqrt{4^2+0^2+(-1)^2} = \sqrt{17}$, and $\mathbf{a}\cdot\mathbf{b} = (1)(4)+(2)(0)+(3)(-1) = 1$. Then $\cos\theta = \dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|} = \dfrac{1}{\sqrt{14}\cdot\sqrt{17}} = \dfrac{1}{\sqrt{238}}$ and the angle between $\mathbf{a}$ and $\mathbf{b}$ is $\theta = \cos^{-1}\left(\frac{1}{\sqrt{238}}\right) \approx 86^\circ$.

18. $|\mathbf{a}| = \sqrt{6^2+(-3)^2+2^2} = 7$, $|\mathbf{b}| = \sqrt{2^2+1^2+(-2)^2} = 3$, and $\mathbf{a}\cdot\mathbf{b} = (6)(2)+(-3)(1)+(2)(-2) = 5$. Then $\cos\theta = \dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|} = \dfrac{5}{7\cdot 3} = \dfrac{5}{21}$ and $\theta = \cos^{-1}\left(\frac{5}{21}\right) \approx 76^\circ$.

19. $|\mathbf{a}| = \sqrt{0^2+1^2+1^2} = \sqrt{2}$, $|\mathbf{b}| = \sqrt{1^2+2^2+(-3)^2} = \sqrt{14}$, and $\mathbf{a}\cdot\mathbf{b} = (0)(1)+(1)(2)+(1)(-3) = -1$. Then $\cos\theta = \dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|} = \dfrac{-1}{\sqrt{2}\cdot\sqrt{14}} = \dfrac{-1}{2\sqrt{7}}$ and $\theta = \cos^{-1}\left(-\dfrac{1}{2\sqrt{7}}\right) \approx 101^\circ$.

20. $|\mathbf{a}| = \sqrt{2^2+(-1)^2+1^2} = \sqrt{6}$, $|\mathbf{b}| = \sqrt{3^2+2^2+(-1)^2} = \sqrt{14}$, and $\mathbf{a}\cdot\mathbf{b} = (2)(3)+(-1)(2)+(1)(-1) = 3$. Then $\cos\theta = \dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|} = \dfrac{3}{\sqrt{6}\cdot\sqrt{14}} = \dfrac{3}{2\sqrt{21}}$ and $\theta = \cos^{-1}\left(\dfrac{3}{2\sqrt{21}}\right) \approx 71^\circ$.

21. Let a, b, and c be the angles at vertices A, B, and C respectively. Then a is the angle between vectors $\overrightarrow{AB}$ and $\overrightarrow{AC}$, b is the angle between vectors $\overrightarrow{BA}$ and $\overrightarrow{BC}$, and c is the angle between vectors $\overrightarrow{CA}$ and $\overrightarrow{CB}$.



Thus $\cos a = \dfrac{\overrightarrow{AB}\cdot\overrightarrow{AC}}{\left|\overrightarrow{AB}\right|\,\left|\overrightarrow{AC}\right|} = \dfrac{\langle 2,6\rangle\cdot\langle -2,4\rangle}{\sqrt{2^2+6^2}\,\sqrt{(-2)^2+4^2}} = \dfrac{1}{\sqrt{40}\,\sqrt{20}}(-4+24) = \dfrac{20}{\sqrt{800}} = \dfrac{\sqrt{2}}{2}$

and $a = \cos^{-1}\left(\dfrac{\sqrt{2}}{2}\right) = 45^\circ$. Similarly,

$\cos b = \dfrac{\overrightarrow{BA}\cdot\overrightarrow{BC}}{\left|\overrightarrow{BA}\right|\,\left|\overrightarrow{BC}\right|} = \dfrac{\langle -2,-6\rangle\cdot\langle -4,-2\rangle}{\sqrt{4+36}\,\sqrt{16+4}} = \dfrac{1}{\sqrt{40}\,\sqrt{20}}(8+12) = \dfrac{20}{\sqrt{800}} = \dfrac{\sqrt{2}}{2}$ so

$b = \cos^{-1}\left(\dfrac{\sqrt{2}}{2}\right) = 45^\circ$ and $c = 180^\circ - (45^\circ + 45^\circ) = 90^\circ$.

Alternate solution: Apply the Law of Cosines three times as follows: $\cos a = \dfrac{\left|\overrightarrow{BC}\right|^2 - \left|\overrightarrow{AB}\right|^2 - \left|\overrightarrow{AC}\right|^2}{2\left|\overrightarrow{AB}\right|\,\left|\overrightarrow{AC}\right|}$,

$\cos b = \dfrac{\left|\overrightarrow{AC}\right|^2 - \left|\overrightarrow{AB}\right|^2 - \left|\overrightarrow{BC}\right|^2}{2\left|\overrightarrow{AB}\right|\,\left|\overrightarrow{BC}\right|}$, and $\cos c = \dfrac{\left|\overrightarrow{AB}\right|^2 - \left|\overrightarrow{AC}\right|^2 - \left|\overrightarrow{BC}\right|^2}{2\left|\overrightarrow{AC}\right|\,\left|\overrightarrow{BC}\right|}$.

22. As in Exercise 21, let d, e, and f be the angles at vertices D, E, and F. Then d is the angle between vectors $\overrightarrow{DE}$ and $\overrightarrow{DF}$, e is the angle between vectors $\overrightarrow{ED}$ and $\overrightarrow{EF}$, and f is the angle between vectors $\overrightarrow{FD}$ and $\overrightarrow{FE}$. Thus

$$\cos d = \frac{\overrightarrow{DE}\cdot\overrightarrow{DF}}{\left|\overrightarrow{DE}\right|\left|\overrightarrow{DF}\right|} = \frac{\langle -2,3,2\rangle\cdot\langle 1,1,-2\rangle}{\sqrt{(-2)^2+3^2+2^2}\sqrt{1^2+1^2+(-2)^2}} = \frac{1}{\sqrt{17}\sqrt{6}}(-2+3-4) = -\frac{3}{\sqrt{102}}$$

and $d = \cos^{-1}\left(-\frac{3}{\sqrt{102}}\right) \approx 107^\circ$. Similarly,

$$\cos e = \frac{\overrightarrow{ED}\cdot\overrightarrow{EF}}{\left|\overrightarrow{ED}\right|\left|\overrightarrow{EF}\right|} = \frac{\langle 2,-3,-2\rangle\cdot\langle 3,-2,-4\rangle}{\sqrt{4+9+4}\sqrt{9+4+16}} = \frac{1}{\sqrt{17}\sqrt{29}}(6+6+8) = \frac{20}{\sqrt{493}} \text{ so}$$

$e = \cos^{-1}\left(\frac{20}{\sqrt{493}}\right) \approx 26^\circ$ and $f \approx 180^\circ - (107^\circ + 26^\circ) = 47^\circ$.

Alternate solution: Apply the Law of Cosines three times as follows: $\cos d = \dfrac{\left|\overrightarrow{EF}\right|^2 - \left|\overrightarrow{DE}\right|^2 - \left|\overrightarrow{DF}\right|^2}{2\left|\overrightarrow{DE}\right|\left|\overrightarrow{DF}\right|}$,

$$\cos e = \frac{\left|\overrightarrow{DF}\right|^2 - \left|\overrightarrow{DE}\right|^2 - \left|\overrightarrow{EF}\right|^2}{2\left|\overrightarrow{DE}\right|\left|\overrightarrow{EF}\right|}, \text{ and } \cos f = \frac{\left|\overrightarrow{DE}\right|^2 - \left|\overrightarrow{DF}\right|^2 - \left|\overrightarrow{EF}\right|^2}{2\left|\overrightarrow{DF}\right|\left|\overrightarrow{EF}\right|}.$$

23. (a) $\mathbf{a}\cdot\mathbf{b} = (-5)(6) + (3)(-8) + (7)(2) = -40 \neq 0$, so $\mathbf{a}$ and $\mathbf{b}$ are not orthogonal. Also, since $\mathbf{a}$ is not a scalar multiple of $\mathbf{b}$, $\mathbf{a}$ and $\mathbf{b}$ are not parallel.

(b) $\mathbf{a}\cdot\mathbf{b} = (4)(-3) + (6)(2) = 0$, so $\mathbf{a}$ and $\mathbf{b}$ are orthogonal (and not parallel).

(c) $\mathbf{a}\cdot\mathbf{b} = (-1)(3) + (2)(4) + (5)(-1) = 0$, so $\mathbf{a}$ and $\mathbf{b}$ are orthogonal (and not parallel).

(d) Because $\mathbf{a} = -\frac{2}{3}\,\mathbf{b}$, $\mathbf{a}$ and $\mathbf{b}$ are parallel.

24. (a) Because $\mathbf{u} = -\frac{3}{4}\,\mathbf{v}$, $\mathbf{u}$ and $\mathbf{v}$ are parallel vectors (and thus not orthogonal).

(b) $\mathbf{u}\cdot\mathbf{v} = (1)(2) + (-1)(-1) + (2)(1) = 5 \neq 0$, so $\mathbf{u}$ and $\mathbf{v}$ are not orthogonal. Also, $\mathbf{u}$ is not a scalar multiple of $\mathbf{v}$, so $\mathbf{u}$ and $\mathbf{v}$ are not parallel.

(c) $\mathbf{u}\cdot\mathbf{v} = (a)(-b) + (b)(a) + (c)(0) = -ab + ab + 0 = 0$, so $\mathbf{u}$ and $\mathbf{v}$ are orthogonal (and not parallel).

25. $\overrightarrow{QP} = \langle -1,-3,2\rangle$, $\overrightarrow{QR} = \langle 4,-2,-1\rangle$, and $\overrightarrow{QP}\cdot\overrightarrow{QR} = -4+6-2 = 0$. Thus $\overrightarrow{QP}$ and $\overrightarrow{QR}$ are orthogonal, so the angle of the triangle at vertex Q is a right angle.

26. $\langle -6,b,2\rangle$ and $\langle b,b^2,b\rangle$ are orthogonal when $\langle -6,b,2\rangle\cdot\langle b,b^2,b\rangle = 0 \quad\Leftrightarrow\quad (-6)(b) + (b)(b^2) + (2)(b) = 0$ $\quad\Leftrightarrow\quad b^3 - 4b = 0 \quad\Leftrightarrow\quad b(b+2)(b-2) = 0 \quad\Leftrightarrow\quad b = 0$ or $b = \pm 2$.

27. Let $\mathbf{a} = a_1\,\mathbf{i} + a_2\,\mathbf{j} + a_3\,\mathbf{k}$ be a vector orthogonal to both $\mathbf{i}+\mathbf{j}$ and $\mathbf{i}+\mathbf{k}$. Then $\mathbf{a}\cdot(\mathbf{i}+\mathbf{j}) = 0 \quad\Leftrightarrow\quad a_1 + a_2 = 0$ and $\mathbf{a}\cdot(\mathbf{i}+\mathbf{k}) = 0 \quad\Leftrightarrow\quad a_1 + a_3 = 0$, so $a_1 = -a_2 = -a_3$. Furthermore $\mathbf{a}$ is to be a unit vector, so $1 = a_1^2 + a_2^2 + a_3^2 = 3a_1^2$ implies $a_1 = \pm\frac{1}{\sqrt{3}}$. Thus $\mathbf{a} = \frac{1}{\sqrt{3}}\,\mathbf{i} - \frac{1}{\sqrt{3}}\,\mathbf{j} - \frac{1}{\sqrt{3}}\,\mathbf{k}$ and $\mathbf{a} = -\frac{1}{\sqrt{3}}\,\mathbf{i} + \frac{1}{\sqrt{3}}\,\mathbf{j} + \frac{1}{\sqrt{3}}\,\mathbf{k}$ are two such unit vectors.

28. Let $\mathbf{u} = \langle a, b\rangle$ be a unit vector. By Theorem 3 we need $\mathbf{u}\cdot\mathbf{v} = |\mathbf{u}|\,|\mathbf{v}|\cos 60^\circ \;\Leftrightarrow\; 3a + 4b = (1)(5)\frac{1}{2} \;\Leftrightarrow$ $b = \frac{5}{8} - \frac{3}{4}a$. Since $\mathbf{u}$ is a unit vector, $|\mathbf{u}| = \sqrt{a^2+b^2} = 1 \;\Leftrightarrow\; a^2+b^2 = 1 \;\Leftrightarrow\; a^2 + \left(\frac{5}{8} - \frac{3}{4}a\right)^2 = 1 \;\Leftrightarrow$ $\frac{25}{16}a^2 - \frac{15}{16}a + \frac{25}{64} = 1 \;\Leftrightarrow\; 100a^2 - 60a - 39 = 0$. By the quadratic formula,

$$a = \frac{-(-60) \pm \sqrt{(-60)^2 - 4(100)(-39)}}{2(100)} = \frac{60 \pm \sqrt{19{,}200}}{200} = \frac{3 \pm 4\sqrt{3}}{10}.$$ If $a = \frac{3+4\sqrt{3}}{10}$ then

$$b = \frac{5}{8} - \frac{3}{4}\left(\frac{3+4\sqrt{3}}{10}\right) = \frac{4-3\sqrt{3}}{10},$$ and if $a = \frac{3-4\sqrt{3}}{10}$ then $b = \frac{5}{8} - \frac{3}{4}\left(\frac{3-4\sqrt{3}}{10}\right) = \frac{4+3\sqrt{3}}{10}$.

Thus the two unit vectors are $\left\langle \frac{3+4\sqrt{3}}{10}, \frac{4-3\sqrt{3}}{10}\right\rangle \approx \langle 0.9928, -0.1196\rangle$ and $\left\langle \frac{3-4\sqrt{3}}{10}, \frac{4+3\sqrt{3}}{10}\right\rangle \approx \langle -0.3928, 0.9196\rangle$.

29. Since $|\langle 3,4,5\rangle| = \sqrt{9+16+25} = \sqrt{50} = 5\sqrt{2}$, using Equations 8 and 9 we have $\cos\alpha = \frac{3}{5\sqrt{2}}$, $\cos\beta = \frac{4}{5\sqrt{2}}$, and $\cos\gamma = \frac{5}{5\sqrt{2}} = \frac{1}{\sqrt{2}}$. The direction angles are given by $\alpha = \cos^{-1}\left(\frac{3}{5\sqrt{2}}\right) \approx 65^\circ$, $\beta = \cos^{-1}\left(\frac{4}{5\sqrt{2}}\right) \approx 56^\circ$, and $\gamma = \cos^{-1}\left(\frac{1}{\sqrt{2}}\right) = 45^\circ$.

30. Since $|\langle 1,-2,-1\rangle| = \sqrt{1+4+1} = \sqrt{6}$, using Equations 8 and 9 we have $\cos\alpha = \frac{1}{\sqrt{6}}$, $\cos\beta = \frac{-2}{\sqrt{6}}$, and $\cos\gamma = \frac{-1}{\sqrt{6}}$. The direction angles are given by $\alpha = \cos^{-1}\left(\frac{1}{\sqrt{6}}\right) \approx 66^\circ$, $\beta = \cos^{-1}\left(-\frac{2}{\sqrt{6}}\right) \approx 145^\circ$, and $\gamma = \cos^{-1}\left(-\frac{1}{\sqrt{6}}\right) \approx 114^\circ$.

31. Since $|2\,\mathbf{i} + 3\,\mathbf{j} - 6\,\mathbf{k}| = \sqrt{4+9+36} = \sqrt{49} = 7$, Equations 8 and 9 give $\cos\alpha = \frac{2}{7}$, $\cos\beta = \frac{3}{7}$, and $\cos\gamma = \frac{-6}{7}$, while $\alpha = \cos^{-1}\left(\frac{2}{7}\right) \approx 73^\circ$, $\beta = \cos^{-1}\left(\frac{3}{7}\right) \approx 65^\circ$, and $\gamma = \cos^{-1}\left(-\frac{6}{7}\right) \approx 149^\circ$.

32. Since $|2\,\mathbf{i} - \mathbf{j} + 2\,\mathbf{k}| = \sqrt{4+1+4} = \sqrt{9} = 3$, Equations 8 and 9 give $\cos\alpha = \frac{2}{3}$, $\cos\beta = \frac{-1}{3}$, and $\cos\gamma = \frac{2}{3}$, while $\alpha = \gamma = \cos^{-1}\left(\frac{2}{3}\right) \approx 48^\circ$ and $\beta = \cos^{-1}\left(-\frac{1}{3}\right) \approx 109^\circ$.

33. $|\langle c,c,c\rangle| = \sqrt{c^2+c^2+c^2} = \sqrt{3}c$ (since $c > 0$), so $\cos\alpha = \cos\beta = \cos\gamma = \dfrac{c}{\sqrt{3}c} = \dfrac{1}{\sqrt{3}}$ and $\alpha = \beta = \gamma = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right) \approx 55^\circ$.

34. Since $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$, $\cos^2\gamma = 1 - \cos^2\alpha - \cos^2\beta = 1 - \cos^2\left(\frac{\pi}{4}\right) - \cos^2\left(\frac{\pi}{3}\right) = 1 - \left(\frac{1}{\sqrt{2}}\right)^2 - \left(\frac{1}{2}\right)^2 = \frac{1}{4}$. Thus $\cos\gamma = \pm\frac{1}{2}$ and $\gamma = \frac{\pi}{3}$ or $\gamma = \frac{2\pi}{3}$.

35. $|\mathbf{a}| = \sqrt{3^2 + (-4)^2} = 5$. The scalar projection of $\mathbf{b}$ onto $\mathbf{a}$ is $\operatorname{comp}_{\mathbf{a}}\mathbf{b} = \dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|} = \dfrac{3\cdot 5 + (-4)\cdot 0}{5} = 3$ and the vector projection of $\mathbf{b}$ onto $\mathbf{a}$ is $\operatorname{proj}_{\mathbf{a}}\mathbf{b} = \left(\dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|}\right)\dfrac{\mathbf{a}}{|\mathbf{a}|} = 3\cdot\frac{1}{5}\langle 3,-4\rangle = \left\langle \frac{9}{5}, -\frac{12}{5}\right\rangle$.

36. $|\mathbf{a}| = \sqrt{1^2+2^2} = \sqrt{5}$, so the scalar projection of $\mathbf{b}$ onto $\mathbf{a}$ is $\operatorname{comp}_{\mathbf{a}}\mathbf{b} = \dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|} = \dfrac{1(-4) + 2\cdot 1}{\sqrt{5}} = -\dfrac{2}{\sqrt{5}}$ and the vector projection of $\mathbf{b}$ onto $\mathbf{a}$ is $\operatorname{proj}_{\mathbf{a}}\mathbf{b} = \left(\dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|}\right)\dfrac{\mathbf{a}}{|\mathbf{a}|} = -\frac{2}{\sqrt{5}}\cdot\frac{1}{\sqrt{5}}\langle 1,2\rangle = \left\langle -\frac{2}{5}, -\frac{4}{5}\right\rangle$.

37. $|\mathbf{a}| = \sqrt{16+4+0} = 2\sqrt{5}$ so the scalar projection of $\mathbf{b}$ onto $\mathbf{a}$ is $\operatorname{comp}_\mathbf{a}\mathbf{b} = \dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|} = \dfrac{1}{2\sqrt{5}}(4+2+0) = \dfrac{3}{\sqrt{5}}$.

The vector projection of $\mathbf{b}$ onto $\mathbf{a}$ is $\operatorname{proj}_\mathbf{a}\mathbf{b} = \dfrac{3}{\sqrt{5}}\dfrac{\mathbf{a}}{|\mathbf{a}|} = \dfrac{3}{\sqrt{5}}\cdot\dfrac{1}{2\sqrt{5}}\langle 4,2,0\rangle = \frac{1}{5}\langle 6,3,0\rangle = \langle\frac{6}{5},\frac{3}{5},0\rangle$.

38. $|\mathbf{a}| = \sqrt{1+4+4} = 3$ so the scalar projection of $\mathbf{b}$ onto $\mathbf{a}$ is $\operatorname{comp}_\mathbf{a}\mathbf{b} = \dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|} = \dfrac{-3+(-6)+8}{3} = -\dfrac{1}{3}$, while

the vector projection is $\operatorname{proj}_\mathbf{a}\mathbf{b} = -\dfrac{1}{3}\dfrac{\mathbf{a}}{|\mathbf{a}|} = -\dfrac{1}{3}\cdot\dfrac{\langle -1,-2,2\rangle}{3} = \langle\frac{1}{9},\frac{2}{9},-\frac{2}{9}\rangle$.

39. $|\mathbf{a}| = \sqrt{1+0+1} = \sqrt{2}$ so the scalar projection of $\mathbf{b}$ onto $\mathbf{a}$ is $\operatorname{comp}_\mathbf{a}\mathbf{b} = \dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|} = \frac{1}{\sqrt{2}}(1+0+0) = \frac{1}{\sqrt{2}}$ while

the vector projection of $\mathbf{b}$ onto $\mathbf{a}$ is $\operatorname{proj}_\mathbf{a}\mathbf{b} = \dfrac{1}{\sqrt{2}}\dfrac{\mathbf{a}}{|\mathbf{a}|} = \frac{1}{\sqrt{2}}\cdot\frac{1}{\sqrt{2}}(\mathbf{i}+\mathbf{k}) = \frac{1}{2}(\mathbf{i}+\mathbf{k})$.

40. $|\mathbf{a}| = \sqrt{4+9+1} = \sqrt{14}$, so the scalar projection of $\mathbf{b}$ onto $\mathbf{a}$ is

$\operatorname{comp}_\mathbf{a}\mathbf{b} = \dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|} = \dfrac{2-18-2}{\sqrt{14}} = -\dfrac{18}{\sqrt{14}}$ while the vector projection of $\mathbf{b}$ onto $\mathbf{a}$ is

$\operatorname{proj}_\mathbf{a}\mathbf{b} = -\dfrac{18}{\sqrt{14}}\dfrac{\mathbf{a}}{|\mathbf{a}|} = -\dfrac{18}{\sqrt{14}}\cdot\dfrac{2\mathbf{i}-3\mathbf{j}+\mathbf{k}}{\sqrt{14}} = -\frac{9}{7}(2\mathbf{i}-3\mathbf{j}+\mathbf{k})$.

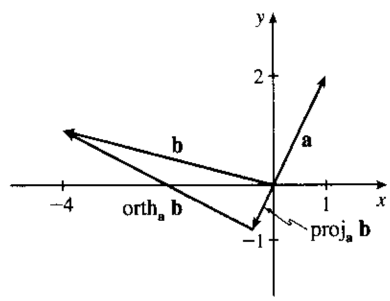
41. $$(\operatorname{orth}_\mathbf{a}\mathbf{b})\cdot\mathbf{a} = (\mathbf{b}-\operatorname{proj}_\mathbf{a}\mathbf{b})\cdot\mathbf{a} = \mathbf{b}\cdot\mathbf{a} - (\operatorname{proj}_\mathbf{a}\mathbf{b})\cdot\mathbf{a} = \mathbf{b}\cdot\mathbf{a} - \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|^2}\mathbf{a}\cdot\mathbf{a}$$
$$= \mathbf{b}\cdot\mathbf{a} - \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|^2}|\mathbf{a}|^2 = \mathbf{b}\cdot\mathbf{a} - \mathbf{a}\cdot\mathbf{b} = 0$$

So they are orthogonal by (7).

42. Using the formula in Exercise 41 and the result of Exercise 36,

we have

$$\operatorname{orth}_\mathbf{a}\mathbf{b} = \mathbf{b} - \operatorname{proj}_\mathbf{a}\mathbf{b} = \langle -4,1\rangle - \langle -\tfrac{2}{5},-\tfrac{4}{5}\rangle = \langle -\tfrac{18}{5},\tfrac{9}{5}\rangle$$



43. $\operatorname{comp}_\mathbf{a}\mathbf{b} = \dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|} = 2 \;\Leftrightarrow\; \mathbf{a}\cdot\mathbf{b} = 2|\mathbf{a}| = 2\sqrt{10}$. If $\mathbf{b} = \langle b_1,b_2,b_3\rangle$, then we need $3b_1 + 0b_2 - 1b_3 = 2\sqrt{10}$.

One possible solution is obtained by taking $b_1 = 0$, $b_2 = 0$, $b_3 = -2\sqrt{10}$.

In general, $\mathbf{b} = \langle s,t,3s-2\sqrt{10}\rangle$, $s,t\in\mathbb{R}$.

44. (a) $\operatorname{comp}_\mathbf{a}\mathbf{b} = \operatorname{comp}_\mathbf{b}\mathbf{a} \;\Leftrightarrow\; \dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|} = \dfrac{\mathbf{b}\cdot\mathbf{a}}{|\mathbf{b}|} \;\Leftrightarrow\; \dfrac{1}{|\mathbf{a}|} = \dfrac{1}{|\mathbf{b}|}$ or $\mathbf{a}\cdot\mathbf{b} = 0 \;\Leftrightarrow\; |\mathbf{b}| = |\mathbf{a}|$ or $\mathbf{a}\cdot\mathbf{b} = 0$.

That is, if $\mathbf{a}$ and $\mathbf{b}$ are orthogonal or if they have the same length.

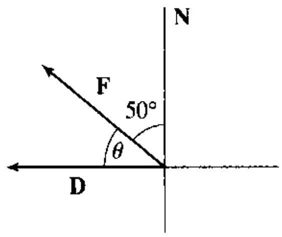
(b) $\operatorname{proj}_\mathbf{a}\mathbf{b} = \operatorname{proj}_\mathbf{b}\mathbf{a} \;\Leftrightarrow\; \dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|^2}\mathbf{a} = \dfrac{\mathbf{b}\cdot\mathbf{a}}{|\mathbf{b}|^2}\mathbf{b} \;\Leftrightarrow\; \mathbf{a}\cdot\mathbf{b} = 0$ or $\dfrac{\mathbf{a}}{|\mathbf{a}|^2} = \dfrac{\mathbf{b}}{|\mathbf{b}|^2}$. But $\dfrac{\mathbf{a}}{|\mathbf{a}|^2} = \dfrac{\mathbf{b}}{|\mathbf{b}|^2} \;\Rightarrow$

$\dfrac{|\mathbf{a}|}{|\mathbf{a}|^2} = \dfrac{|\mathbf{b}|}{|\mathbf{b}|^2} \;\Rightarrow\; |\mathbf{a}| = |\mathbf{b}|$. Substituting this into the previous equation gives $\mathbf{a} = \mathbf{b}$.

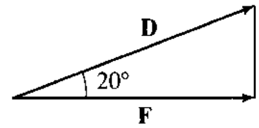
So $\operatorname{proj}_\mathbf{a}\mathbf{b} = \operatorname{proj}_\mathbf{b}\mathbf{a} \;\Leftrightarrow\;$ $\mathbf{a}$ and $\mathbf{b}$ are orthogonal, or they are equal.

45. Here $\mathbf{D} = (4-2)\,\mathbf{i} + (9-3)\,\mathbf{j} + (15-0)\,\mathbf{k} = 2\,\mathbf{i} + 6\,\mathbf{j} + 15\,\mathbf{k}$ so by Equation 12 we have $W = \mathbf{F}\cdot\mathbf{D} = 20 + 108 - 90 = 38$ joules.

46. $W = |\mathbf{F}|\,|\mathbf{D}|\cos\theta = (20)(4)\cos 40^\circ \approx 61$ ft-lb



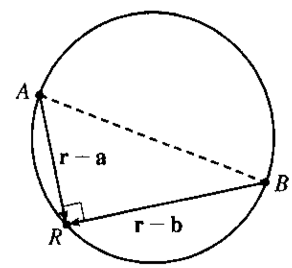
47. $W = |\mathbf{F}|\,|\mathbf{D}|\cos\theta = (25)(10)\cos 20^\circ \approx 235$ ft-lb



48. Here $|\mathbf{D}| = 100$ m, $|\mathbf{F}| = 50$ N, and $\theta = 30^\circ$. Thus $W = |\mathbf{F}|\,|\mathbf{D}|\cos\theta = (50)(100)\left(\frac{\sqrt{3}}{2}\right) = 2500\sqrt{3}$ joules.

49. First note that $\mathbf{n} = \langle a, b\rangle$ is perpendicular to the line, because if $Q_1 = (a_1, b_1)$ and $Q_2 = (a_2, b_2)$ lie on the line, then $\mathbf{n}\cdot\overrightarrow{Q_1Q_2} = aa_2 - aa_1 + bb_2 - bb_1 = 0$, since $aa_2 + bb_2 = -c = aa_1 + bb_1$ from the equation of the line. Let $P_2 = (x_2, y_2)$ lie on the line. Then the distance from P_1 to the line is the absolute value of the scalar projection of $\overrightarrow{P_1P_2}$ onto $\mathbf{n}$. $\operatorname{comp}_{\mathbf{n}}\left(\overrightarrow{P_1P_2}\right) = \dfrac{|\mathbf{n}\cdot\langle x_2 - x_1, y_2 - y_1\rangle|}{|\mathbf{n}|} = \dfrac{|ax_2 - ax_1 + by_2 - by_1|}{\sqrt{a^2+b^2}} = \dfrac{|ax_1 + by_1 + c|}{\sqrt{a^2+b^2}}$ since $ax_2 + by_2 = -c$. The required distance is $\dfrac{|3\cdot -2 + -4\cdot 3 + 5|}{\sqrt{3^2+4^2}} = \dfrac{13}{5}$.

50. $(\mathbf{r}-\mathbf{a})\cdot(\mathbf{r}-\mathbf{b}) = 0$ implies that the vectors $\mathbf{r}-\mathbf{a}$ and $\mathbf{r}-\mathbf{b}$ are orthogonal. From the diagram (in which A, B and R are the terminal points of the vectors), we see that this implies that R lies on a sphere whose diameter is the line from A to B. The center of this circle is the midpoint of AB, that is,



$\frac{1}{2}(\mathbf{a}+\mathbf{b}) = \left\langle \frac{1}{2}(a_1+b_1), \frac{1}{2}(a_2+b_2), \frac{1}{2}(a_3+b_3)\right\rangle$, and its radius is

$\frac{1}{2}|\mathbf{a}-\mathbf{b}| = \frac{1}{2}\sqrt{(a_1-b_1)^2 + (a_2-b_2)^2 + (a_3-b_3)^2}$.

Or: Expand the given equation, substitute $\mathbf{r}\cdot\mathbf{r} = x^2 + y^2 + z^2$ and complete the squares.

51. For convenience, consider the unit cube positioned so that its back left corner is at the origin, and its edges lie along the coordinate axes. The diagonal of the cube that begins at the origin and ends at $(1, 1, 1)$ has vector representation $\langle 1, 1, 1\rangle$. The angle θ between this vector and the vector of the edge which also begins at the origin and runs along the x-axis [that is, $\langle 1, 0, 0\rangle$] is given by $\cos\theta = \dfrac{\langle 1,1,1\rangle\cdot\langle 1,0,0\rangle}{|\langle 1,1,1\rangle|\,|\langle 1,0,0\rangle|} = \dfrac{1}{\sqrt{3}} \Rightarrow \theta = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right) \approx 55^\circ$.

52. Consider a cube with sides of unit length, wholly within the first octant and with edges along each of the three coordinate axes. $\mathbf{i}+\mathbf{j}+\mathbf{k}$ and $\mathbf{i}+\mathbf{j}$ are vector representations of a diagonal of the cube and a diagonal of one of its faces. If θ is the angle between these diagonals, then $\cos\theta = \dfrac{(\mathbf{i}+\mathbf{j}+\mathbf{k})\cdot(\mathbf{i}+\mathbf{j})}{|\mathbf{i}+\mathbf{j}+\mathbf{k}|\,|\mathbf{i}+\mathbf{j}|} = \dfrac{1+1}{\sqrt{3}\sqrt{2}} = \sqrt{\dfrac{2}{3}} \Rightarrow$

$\theta = \cos^{-1}\sqrt{\frac{2}{3}} \approx 35^\circ$.

53. Consider the H-C-H combination consisting of the sole carbon atom and the two hydrogen atoms that are at $(1,0,0)$ and $(0,1,0)$ (or any H-C-H combination, for that matter). Vector representations of the line segments emanating from the carbon atom and extending to these two hydrogen atoms are $\langle 1-\frac{1}{2}, 0-\frac{1}{2}, 0-\frac{1}{2}\rangle = \langle \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\rangle$ and $\langle 0-\frac{1}{2}, 1-\frac{1}{2}, 0-\frac{1}{2}\rangle = \langle -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\rangle$. The bond angle, θ, is therefore given by

$$\cos\theta = \frac{\langle \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\rangle \cdot \langle -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\rangle}{\left|\langle \frac{1}{2}, -\frac{1}{2}, -\frac{1}{2}\rangle\right| \left|\langle -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\rangle\right|} = \frac{-\frac{1}{4} - \frac{1}{4} + \frac{1}{4}}{\sqrt{\frac{3}{4}}\sqrt{\frac{3}{4}}} = -\frac{1}{3} \quad\Rightarrow\quad \theta = \cos^{-1}\left(-\tfrac{1}{3}\right) \approx 109.5^\circ.$$

54. Let α be the angle between $\mathbf{a}$ and $\mathbf{c}$ and β be the angle between $\mathbf{c}$ and $\mathbf{b}$. We need to show that $\alpha = \beta$. Now

$\cos\alpha = \dfrac{\mathbf{a}\cdot\mathbf{c}}{|\mathbf{a}|\,|\mathbf{c}|} = \dfrac{\mathbf{a}\cdot|\mathbf{a}|\,\mathbf{b} + \mathbf{a}\cdot|\mathbf{b}|\,\mathbf{a}}{|\mathbf{a}|\,|\mathbf{c}|} = \dfrac{|\mathbf{a}|\,\mathbf{a}\cdot\mathbf{b} + |\mathbf{a}|^2\,|\mathbf{b}|}{|\mathbf{a}|\,|\mathbf{c}|} = \dfrac{\mathbf{a}\cdot\mathbf{b} + |\mathbf{a}|\,|\mathbf{b}|}{|\mathbf{c}|}$. Similarly,

$\cos\beta = \dfrac{\mathbf{b}\cdot\mathbf{c}}{|\mathbf{b}|\,|\mathbf{c}|} = \dfrac{|\mathbf{a}|\,|\mathbf{b}| + \mathbf{b}\cdot\mathbf{a}}{|\mathbf{c}|}$. Thus $\cos\alpha = \cos\beta$. However $0^\circ \le \alpha \le 180^\circ$ and $0^\circ \le \beta \le 180^\circ$, so $\alpha = \beta$ and $\mathbf{c}$ bisects the angle between $\mathbf{a}$ and $\mathbf{b}$.

55. Let $\mathbf{a} = \langle a_1, a_2, a_3\rangle$ and $= \langle b_1, b_2, b_3\rangle$.

Property 2: $\mathbf{a}\cdot\mathbf{b} = \langle a_1, a_2, a_3\rangle \cdot \langle b_1, b_2, b_3\rangle = a_1b_1 + a_2b_2 + a_3b_3$
$= b_1a_1 + b_2a_2 + b_3a_3 = \langle b_1, b_2, b_3\rangle \cdot \langle a_1, a_2, a_3\rangle = \mathbf{b}\cdot\mathbf{a}$

Property 4: $(c\,\mathbf{a})\cdot\mathbf{b} = \langle ca_1, ca_2, ca_3\rangle \cdot \langle b_1, b_2, b_3\rangle = (ca_1)b_1 + (ca_2)b_2 + (ca_3)b_3$
$= c\,(a_1b_1 + a_2b_2 + a_3b_3) = c\,(\mathbf{a}\cdot\mathbf{b}) = a_1(cb_1) + a_2(cb_2) + a_3(cb_3)$
$= \langle a_1, a_2, a_3\rangle \cdot \langle cb_1, cb_2, cb_3\rangle = \mathbf{a}\cdot(c\,\mathbf{b})$

Property 5: $\mathbf{0}\cdot\mathbf{a} = \langle 0,0,0\rangle \cdot \langle a_1, a_2, a_3\rangle = (0)(a_1) + (0)(a_2) + (0)(a_3) = 0$

56. Let the figure be called quadrilateral $ABCD$. The diagonals can be represented by $\overrightarrow{AC}$ and $\overrightarrow{BD}$. $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$ and $\overrightarrow{BD} = \overrightarrow{BC} + \overrightarrow{CD} = \overrightarrow{BC} - \overrightarrow{DC} = \overrightarrow{BC} - \overrightarrow{AB}$ (Since opposite sides of the object are of the same length and parallel, $\overrightarrow{AB} = \overrightarrow{DC}$.) Thus

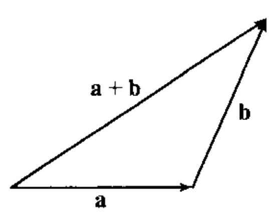
$$\overrightarrow{AC}\cdot\overrightarrow{BD} = \left(\overrightarrow{AB} + \overrightarrow{BC}\right)\cdot\left(\overrightarrow{BC} - \overrightarrow{AB}\right) = \overrightarrow{AB}\cdot\left(\overrightarrow{BC} - \overrightarrow{AB}\right) + \overrightarrow{BC}\cdot\left(\overrightarrow{BC} - \overrightarrow{AB}\right)$$
$$= \overrightarrow{AB}\cdot\overrightarrow{BC} - \left|\overrightarrow{AB}\right|^2 + \left|\overrightarrow{BC}\right|^2 - \overrightarrow{AB}\cdot\overrightarrow{BC} = \left|\overrightarrow{BC}\right|^2 - \left|\overrightarrow{AB}\right|^2$$

But $\left|\overrightarrow{AB}\right|^2 = \left|\overrightarrow{BC}\right|^2$ because all sides of the quadrilateral are equal in length. Therefore $\overrightarrow{AC}\cdot\overrightarrow{BD} = 0$, and since both of these vectors are nonzero this tells us that the diagonals of the quadrilateral are perpendicular.

57. $|\mathbf{a}\cdot\mathbf{b}| = \big|\,|\mathbf{a}|\,|\mathbf{b}|\cos\theta\big| = |\mathbf{a}|\,|\mathbf{b}|\,|\cos\theta|$. Since $|\cos\theta| \le 1$, $|\mathbf{a}\cdot\mathbf{b}| = |\mathbf{a}|\,|\mathbf{b}|\,|\cos\theta| \le |\mathbf{a}|\,|\mathbf{b}|$.

Note: We have equality in the case of $\cos\theta = \pm 1$, so $\theta = 0$ or $\theta = \pi$, thus equality when $\mathbf{a}$ and $\mathbf{b}$ are parallel.

58. (a)



The Triangle Inequality states that the length of the longest side of a triangle is less than or equal to the sum of the lengths of the two shortest sides.

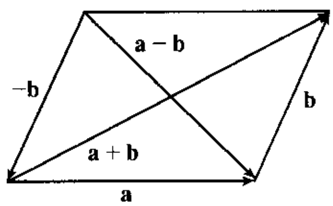
(b) $|\mathbf{a}+\mathbf{b}|^2 = (\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}+\mathbf{b}) = (\mathbf{a}\cdot\mathbf{a}) + 2(\mathbf{a}\cdot\mathbf{b}) + (\mathbf{b}\cdot\mathbf{b}) = |\mathbf{a}|^2 + 2(\mathbf{a}\cdot\mathbf{b}) + |\mathbf{b}|^2$

$\leq |\mathbf{a}|^2 + 2\,|\mathbf{a}|\,|\mathbf{b}| + |\mathbf{b}|^2$ [by the Cauchy-Schwartz Inequality]

$= (|\mathbf{a}| + |\mathbf{b}|)^2$

Thus, taking the square root of both sides, $|\mathbf{a}+\mathbf{b}| \leq |\mathbf{a}| + |\mathbf{b}|$.

59. (a)



The Parallelogram Law states that the sum of the squares of the lengths of the diagonals of a parallelogram equals the sum of the squares of its (four) sides.

(b) $|\mathbf{a}+\mathbf{b}|^2 = (\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}+\mathbf{b}) = |\mathbf{a}|^2 + 2(\mathbf{a}\cdot\mathbf{b}) + |\mathbf{b}|^2$ and

$|\mathbf{a}-\mathbf{b}|^2 = (\mathbf{a}-\mathbf{b})\cdot(\mathbf{a}-\mathbf{b}) = |\mathbf{a}|^2 - 2(\mathbf{a}\cdot\mathbf{b}) + |\mathbf{b}|^2$.

Adding these two equations gives $|\mathbf{a}+\mathbf{b}|^2 + |\mathbf{a}-\mathbf{b}|^2 = 2\,|\mathbf{a}|^2 + 2\,|\mathbf{b}|^2$.

13.4 The Cross Product ET 12.4

1. $\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & 0 \\ 0 & 3 & 1 \end{vmatrix} = \begin{vmatrix} 2 & 0 \\ 3 & 1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & 2 \\ 0 & 3 \end{vmatrix}\mathbf{k} = (2-0)\,\mathbf{i} - (1-0)\,\mathbf{j} + (3-0)\,\mathbf{k} = 2\,\mathbf{i} - \mathbf{j} + 3\,\mathbf{k}$

Now $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{a} = \langle 2,-1,3\rangle\cdot\langle 1,2,0\rangle = 2-2+0 = 0$ and $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{b} = \langle 2,-1,3\rangle\cdot\langle 0,3,1\rangle = 0-3+3 = 0$, so $\mathbf{a}\times\mathbf{b}$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$.

2. $\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 5 & 1 & 4 \\ -1 & 0 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 4 \\ 0 & 2 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 5 & 4 \\ -1 & 2 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 5 & 1 \\ -1 & 0 \end{vmatrix}\mathbf{k}$

$= (2-0)\,\mathbf{i} - [10-(-4)]\,\mathbf{j} + [0-(-1)]\,\mathbf{k} = 2\,\mathbf{i} - 14\,\mathbf{j} + \mathbf{k}$

Now $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{a} = \langle 2,-14,1\rangle\cdot\langle 5,1,4\rangle = 10-14+4 = 0$ and

$(\mathbf{a}\times\mathbf{b})\cdot\mathbf{b} = \langle 2,-14,1\rangle\cdot\langle -1,0,2\rangle = -2+0+2 = 0$, so $\mathbf{a}\times\mathbf{b}$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$.

3. $\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 1 & -1 \\ 0 & 1 & 2 \end{vmatrix} = \begin{vmatrix} 1 & -1 \\ 1 & 2 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 2 & -1 \\ 0 & 2 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 2 & 1 \\ 0 & 1 \end{vmatrix}\mathbf{k}$

$= [2-(-1)]\,\mathbf{i} - (4-0)\,\mathbf{j} + (2-0)\,\mathbf{k} = 3\,\mathbf{i} - 4\,\mathbf{j} + 2\,\mathbf{k}$

Now $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{a} = (3\,\mathbf{i} - 4\,\mathbf{j} + 2\,\mathbf{k})\cdot(2\,\mathbf{i} + \mathbf{j} - \mathbf{k}) = 6-4-2 = 0$ and

$(\mathbf{a}\times\mathbf{b})\cdot\mathbf{b} = (3\,\mathbf{i} - 4\,\mathbf{j} + 2\,\mathbf{k})\cdot(\mathbf{j} + 2\,\mathbf{k}) = 0-4+4 = 0$, so $\mathbf{a}\times\mathbf{b}$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$.

4. $\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -1 & 1 \\ 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & -1 \\ 1 & 1 \end{vmatrix}\mathbf{k}$

$= (-1-1)\,\mathbf{i} - (1-1)\,\mathbf{j} + [1-(-1)]\,\mathbf{k} = -2\,\mathbf{i} + 2\,\mathbf{k}$

Now $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{a} = (-2\,\mathbf{i} + 2\,\mathbf{k})\cdot(\mathbf{i} - \mathbf{j} + \mathbf{k}) = -2+0+2 = 0$ and

$(\mathbf{a}\times\mathbf{b})\cdot\mathbf{b} = (-2\,\mathbf{i} + 2\,\mathbf{k})\cdot(\mathbf{i} + \mathbf{j} + \mathbf{k}) = -2+0+2 = 0$, so $\mathbf{a}\times\mathbf{b}$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$.

5. $\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 2 & 4 \\ 1 & -2 & -3 \end{vmatrix} = \begin{vmatrix} 2 & 4 \\ -2 & -3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 3 & 4 \\ 1 & -3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 3 & 2 \\ 1 & -2 \end{vmatrix}\mathbf{k}$

$= [-6-(-8)]\,\mathbf{i} - (-9-4)\,\mathbf{j} + (-6-2)\,\mathbf{k} = 2\,\mathbf{i} + 13\,\mathbf{j} - 8\,\mathbf{k}$

Since $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{a} = (2\,\mathbf{i}+13\,\mathbf{j}-8\,\mathbf{k})\cdot(3\,\mathbf{i}+2\,\mathbf{j}+4\,\mathbf{k}) = 6+26-32 = 0$, $\mathbf{a}\times\mathbf{b}$ is orthogonal to $\mathbf{a}$.

Since $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{b} = (2\,\mathbf{i}+13\,\mathbf{j}-8\,\mathbf{k})\cdot(\mathbf{i}-2\,\mathbf{j}-3\,\mathbf{k}) = 2-26+24 = 0$, $\mathbf{a}\times\mathbf{b}$ is orthogonal to $\mathbf{b}$.

6. $\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & e^t & e^{-t} \\ 2 & e^t & -e^{-t} \end{vmatrix} = \begin{vmatrix} e^t & e^{-t} \\ e^t & -e^{-t} \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & e^{-t} \\ 2 & -e^{-t} \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & e^t \\ 2 & e^t \end{vmatrix}\mathbf{k}$

$= (-1-1)\,\mathbf{i} - (-e^{-t}-2e^{-t})\,\mathbf{j} + (e^t-2e^t)\,\mathbf{k} = -2\,\mathbf{i} + 3e^{-t}\,\mathbf{j} - e^t\,\mathbf{k}$

Since $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{a} = (-2\,\mathbf{i}+3e^{-t}\,\mathbf{j}-e^t\,\mathbf{k})\cdot(\mathbf{i}+e^t\,\mathbf{j}+e^{-t}\,\mathbf{k}) = -2+3-1 = 0$, $\mathbf{a}\times\mathbf{b}$ is orthogonal to $\mathbf{a}$.

Since $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{b} = (-2\,\mathbf{i}+3e^{-t}\,\mathbf{j}-e^t\,\mathbf{k})\cdot(2\,\mathbf{i}+e^t\,\mathbf{j}-e^{-t}\,\mathbf{k}) = -4+3+1 = 0$, $\mathbf{a}\times\mathbf{b}$ is orthogonal to $\mathbf{b}$.

7. $\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ t & t^2 & t^3 \\ 1 & 2t & 3t^2 \end{vmatrix} = \begin{vmatrix} t^2 & t^3 \\ 2t & 3t^2 \end{vmatrix}\mathbf{i} - \begin{vmatrix} t & t^3 \\ 1 & 3t^2 \end{vmatrix}\mathbf{j} + \begin{vmatrix} t & t^2 \\ 1 & 2t \end{vmatrix}\mathbf{k}$

$= (3t^4-2t^4)\,\mathbf{i} - (3t^3-t^3)\,\mathbf{j} + (2t^2-t^2)\,\mathbf{k} = t^4\,\mathbf{i} - 2t^3\,\mathbf{j} + t^2\,\mathbf{k}$

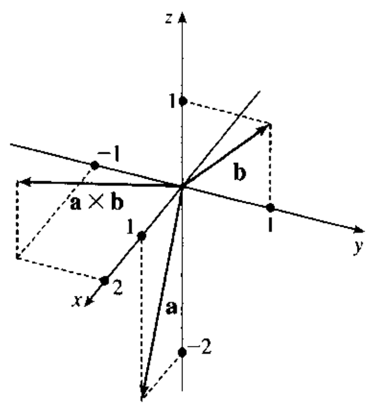
Since $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{a} = \langle t^4, -2t^3, t^2\rangle\cdot\langle t, t^2, t^3\rangle = t^5 - 2t^5 + t^5 = 0$, $\mathbf{a}\times\mathbf{b}$ is orthogonal to $\mathbf{a}$.

Since $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{b} = \langle t^4, -2t^3, t^2\rangle\cdot\langle 1, 2t, 3t^2\rangle = t^4 - 4t^4 + 3t^4 = 0$, $\mathbf{a}\times\mathbf{b}$ is orthogonal to $\mathbf{b}$.

8. $\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & -2 \\ 0 & 1 & 1 \end{vmatrix}$

$= \begin{vmatrix} 0 & -2 \\ 1 & 1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & -2 \\ 0 & 1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}\mathbf{k}$

$= 2\,\mathbf{i} - \mathbf{j} + \mathbf{k}$



9. (a) Since $\mathbf{b}\times\mathbf{c}$ is a vector, the dot product $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})$ is meaningful and is a scalar.

(b) $\mathbf{b}\cdot\mathbf{c}$ is a scalar, so $\mathbf{a}\times(\mathbf{b}\cdot\mathbf{c})$ is meaningless, as the cross product is defined only for two *vectors*.

(c) Since $\mathbf{b}\times\mathbf{c}$ is a vector, the cross product $\mathbf{a}\times(\mathbf{b}\times\mathbf{c})$ is meaningful and results in another vector.

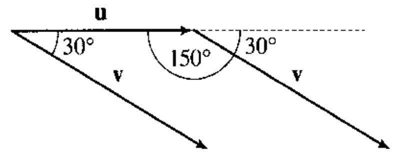
(d) $\mathbf{a}\cdot\mathbf{b}$ is a scalar, so the cross product $(\mathbf{a}\cdot\mathbf{b})\times\mathbf{c}$ is meaningless.

(e) Since $(\mathbf{a}\cdot\mathbf{b})$ and $(\mathbf{c}\cdot\mathbf{d})$ are both scalars, the cross product $(\mathbf{a}\cdot\mathbf{b})\times(\mathbf{c}\cdot\mathbf{d})$ is meaningless.

(f) $\mathbf{a}\times\mathbf{b}$ and $\mathbf{c}\times\mathbf{d}$ are both vectors, so the dot product $(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d})$ is meaningful and is a scalar.

10. Using Theorem 6, we have $|\mathbf{u}\times\mathbf{v}| = |\mathbf{u}|\,|\mathbf{v}|\sin\theta = (5)(10)\sin 60^\circ = 25\sqrt{3}$. By the right-hand rule, $\mathbf{u}\times\mathbf{v}$ is directed into the page.

11. If we sketch **u** and **v** starting from the same initial point, we see that the angle between them is 30°. Using Theorem 6, we have

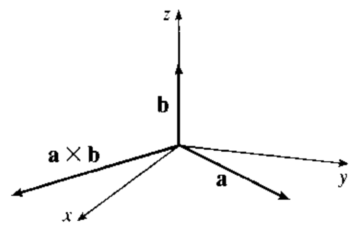


$$|\mathbf{u}\times\mathbf{v}| = |\mathbf{u}|\,|\mathbf{v}|\sin 30^\circ = (6)(8)\left(\tfrac{1}{2}\right) = 24$$

By the right-hand rule, $\mathbf{u}\times\mathbf{v}$ is directed into the page.

12. (a) $|\mathbf{a}\times\mathbf{b}| = |\mathbf{a}|\,|\mathbf{b}|\sin\theta = 3\cdot 2\cdot \sin\frac{\pi}{2} = 6$

(b) $\mathbf{a}\times\mathbf{b}$ is orthogonal to $\mathbf{k}$, so it lies in the xy-plane, and its z-coordinate is 0. By the right-hand rule, its y-component is negative and its x-component is positive.



13. $$\mathbf{a}\times\mathbf{b} = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & 2 & 1\\ 0 & 1 & 3\end{vmatrix} = \begin{vmatrix}2 & 1\\ 1 & 3\end{vmatrix}\mathbf{i} - \begin{vmatrix}1 & 1\\ 0 & 3\end{vmatrix}\mathbf{j} + \begin{vmatrix}1 & 2\\ 0 & 1\end{vmatrix}\mathbf{k} = (6-1)\,\mathbf{i} - (3-0)\,\mathbf{j} + (1-0)\,\mathbf{k} = 5\,\mathbf{i} - 3\,\mathbf{j} + \mathbf{k}$$

$$\mathbf{b}\times\mathbf{a} = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 0 & 1 & 3\\ 1 & 2 & 1\end{vmatrix} = \begin{vmatrix}1 & 3\\ 2 & 1\end{vmatrix}\mathbf{i} - \begin{vmatrix}0 & 3\\ 1 & 1\end{vmatrix}\mathbf{j} + \begin{vmatrix}0 & 1\\ 1 & 2\end{vmatrix}\mathbf{k} = (1-6)\,\mathbf{i} - (0-3)\,\mathbf{j} + (0-1)\,\mathbf{k} = -5\,\mathbf{i} + 3\,\mathbf{j} - \mathbf{k}$$

Notice $\mathbf{a}\times\mathbf{b} = -\mathbf{b}\times\mathbf{a}$ here, as we know is always true by Theorem 8.

14. $$\mathbf{b}\times\mathbf{c} = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ -1 & 1 & 0\\ 0 & 0 & -4\end{vmatrix} = \begin{vmatrix}1 & 0\\ 0 & -4\end{vmatrix}\mathbf{i} - \begin{vmatrix}-1 & 0\\ 0 & -4\end{vmatrix}\mathbf{j} + \begin{vmatrix}-1 & 1\\ 0 & 0\end{vmatrix}\mathbf{k} = -\,4\,\mathbf{i} - 4\,\mathbf{j} \text{ so}$$

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 3 & 1 & 2\\ -4 & -4 & 0\end{vmatrix} = \begin{vmatrix}1 & 2\\ -4 & 0\end{vmatrix}\mathbf{i} - \begin{vmatrix}3 & 2\\ -4 & 0\end{vmatrix}\mathbf{j} + \begin{vmatrix}3 & 1\\ -4 & -4\end{vmatrix}\mathbf{k} = 8\,\mathbf{i} - 8\,\mathbf{j} - 8\,\mathbf{k}.$$

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 3 & 1 & 2\\ -1 & 1 & 0\end{vmatrix} = \begin{vmatrix}1 & 2\\ 1 & 0\end{vmatrix}\mathbf{i} - \begin{vmatrix}3 & 2\\ -1 & 0\end{vmatrix}\mathbf{j} + \begin{vmatrix}3 & 1\\ -1 & 1\end{vmatrix}\mathbf{k} = -\,2\,\mathbf{i} - 2\,\mathbf{j} + 4\,\mathbf{k} \text{ so}$$

$$(\mathbf{a}\times\mathbf{b})\times\mathbf{c} = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ -2 & -2 & 4\\ 0 & 0 & -4\end{vmatrix} = \begin{vmatrix}-2 & 4\\ 0 & -4\end{vmatrix}\mathbf{i} - \begin{vmatrix}-2 & 4\\ 0 & -4\end{vmatrix}\mathbf{j} + \begin{vmatrix}-2 & -2\\ 0 & 0\end{vmatrix}\mathbf{k} = 8\,\mathbf{i} - 8\,\mathbf{j}.$$

Thus $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) \neq (\mathbf{a}\times\mathbf{b})\times\mathbf{c}$.

15. We know that the cross product of two vectors is orthogonal to both. So we calculate

$$\langle 1,-1,1\rangle\times\langle 0,4,4\rangle = \begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 1 & -1 & 1\\ 0 & 4 & 4\end{vmatrix} = \begin{vmatrix}-1 & 1\\ 4 & 4\end{vmatrix}\mathbf{i} - \begin{vmatrix}1 & 1\\ 0 & 4\end{vmatrix}\mathbf{j} + \begin{vmatrix}1 & -1\\ 0 & 4\end{vmatrix}\mathbf{k} = -8\,\mathbf{i} - 4\,\mathbf{j} + 4\,\mathbf{k}.$$

So two unit vectors orthogonal to both are $\pm\dfrac{\langle -8,-4,4\rangle}{\sqrt{64+16+16}} = \pm\dfrac{\langle -8,-4,4\rangle}{4\sqrt{6}}$, that is, $\left\langle -\frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right\rangle$ and $\left\langle \frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}\right\rangle$.

16. We know that the cross product of two vectors is orthogonal to both. So we calculate

$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 2 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & 1 \\ 2 & 0 \end{vmatrix}\mathbf{k} = \mathbf{i} + \mathbf{j} - 2\,\mathbf{k}$. Thus, two unit vectors orthogonal to both are $\pm\frac{1}{\sqrt{6}}\langle 1, 1, -2\rangle$, that is, $\left\langle \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}} \right\rangle$ and $\left\langle -\frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}} \right\rangle$.

17. Let $\mathbf{a} = \langle a_1, a_2, a_3\rangle$. Then

$$\mathbf{0} \times \mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 0 & 0 \\ a_1 & a_2 & a_3 \end{vmatrix} = \begin{vmatrix} 0 & 0 \\ a_2 & a_3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 0 & 0 \\ a_1 & a_3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 0 & 0 \\ a_1 & a_2 \end{vmatrix}\mathbf{k} = \mathbf{0},$$

$$\mathbf{a} \times \mathbf{0} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ 0 & 0 & 0 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ 0 & 0 \end{vmatrix}\mathbf{i} - \begin{vmatrix} a_1 & a_3 \\ 0 & 0 \end{vmatrix}\mathbf{j} + \begin{vmatrix} a_1 & a_2 \\ 0 & 0 \end{vmatrix}\mathbf{k} = \mathbf{0}.$$

18. Let $\mathbf{a} = \langle a_1, a_2, a_3\rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3\rangle$.

$$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{b} = \left\langle \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix}, \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix}, \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \right\rangle \cdot \langle b_1, b_2, b_3\rangle = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} b_1 - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} b_2 + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} b_3$$

$$= (a_2b_3b_1 - a_3b_2b_1) - (a_1b_3b_2 - a_3b_1b_2) + (a_1b_2b_3 - a_2b_1b_3) = 0$$

19. $\mathbf{a} \times \mathbf{b} = \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1\rangle$

$$= \langle (-1)(b_2a_3 - b_3a_2), (-1)(b_3a_1 - b_1a_3), (-1)(b_1a_2 - b_2a_1)\rangle$$

$$= -\langle b_2a_3 - b_3a_2, b_3a_1 - b_1a_3, b_1a_2 - b_2a_1\rangle = -\mathbf{b} \times \mathbf{a}$$

20. $c\mathbf{a} = \langle ca_1, ca_2, ca_3\rangle$, so

$$(c\,\mathbf{a}) \times \mathbf{b} = \langle ca_2b_3 - ca_3b_2, ca_3b_1 - ca_1b_3, ca_1b_2 - ca_2b_1\rangle$$

$$= \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1\rangle = c(\mathbf{a} \times \mathbf{b})$$

$$= \langle ca_2b_3 - ca_3b_2, ca_3b_1 - ca_1b_3, ca_1b_2 - ca_2b_1\rangle$$

$$= \langle a_2(cb_3) - a_3(cb_2), a_3(cb_1) - a_1(cb_3), a_1(cb_2) - a_2(cb_1)\rangle$$

$$= \mathbf{a} \times c\,\mathbf{b}$$

21. $\mathbf{a} \times (\mathbf{b} + \mathbf{c}) = \mathbf{a} \times \langle b_1 + c_1, b_2 + c_2, b_3 + c_3\rangle$

$$= \langle a_2(b_3 + c_3) - a_3(b_2 + c_2), a_3(b_1 + c_1) - a_1(b_3 + c_3), a_1(b_2 + c_2) - a_2(b_1 + c_1)\rangle$$

$$= \langle a_2b_3 + a_2c_3 - a_3b_2 - a_3c_2, a_3b_1 + a_3c_1 - a_1b_3 - a_1c_3, a_1b_2 + a_1c_2 - a_2b_1 - a_2c_1\rangle$$

$$= \langle (a_2b_3 - a_3b_2) + (a_2c_3 - a_3c_2), (a_3b_1 - a_1b_3) + (a_3c_1 - a_1c_3), (a_1b_2 - a_2b_1) + (a_1c_2 - a_2c_1)\rangle$$

$$= \langle a_2b_3 - a_3b_2, a_3b_1 - a_1b_3, a_1b_2 - a_2b_1\rangle + \langle a_2c_3 - a_3c_2, a_3c_1 - a_1c_3, a_1c_2 - a_2c_1\rangle$$

$$= (\mathbf{a} \times \mathbf{b}) + (\mathbf{a} \times \mathbf{c})$$

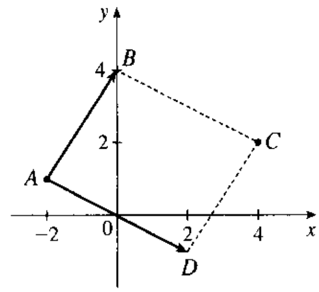
22. $(\mathbf{a}+\mathbf{b})\times\mathbf{c} = -\mathbf{c}\times(\mathbf{a}+\mathbf{b})$ by Property 1 of Theorem 8

$= -(\mathbf{c}\times\mathbf{a}+\mathbf{c}\times\mathbf{b})$ by Property 3 of Theorem 8

$= -(-\mathbf{a}\times\mathbf{c}+(-\mathbf{b}\times\mathbf{c}))$ by Property 1 of Theorem 8

$= \mathbf{a}\times\mathbf{c}+\mathbf{b}\times\mathbf{c}$ by Property 2 of Theorem 8

23. By plotting the vertices, we can see that the parallelogram is determined by the vectors $\overrightarrow{AB} = \langle 2,3\rangle$ and $\overrightarrow{AD} = \langle 4,-2\rangle$. We know that the area of the parallelogram determined by two vectors is equal to the length of the cross product of these vectors. In order to compute the cross product, we consider the vector $\overrightarrow{AB}$ as the three-dimensional vector $\langle 2,3,0\rangle$ (and similarly for $\overrightarrow{AD}$), and then the area of parallelogram $ABCD$ is



$$\left|\overrightarrow{AB}\times\overrightarrow{AD}\right| = \left\|\begin{matrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 2 & 3 & 0\\ 4 & -2 & 0\end{matrix}\right\| = |(0)\,\mathbf{i}-(0)\,\mathbf{j}+(-4-12)\,\mathbf{k}| = |-16\,\mathbf{k}| = 16$$

24. The parallelogram is determined by the vectors $\overrightarrow{KL} = \langle 0,1,3\rangle$ and $\overrightarrow{KN} = \langle 2,5,0\rangle$, so the area of parallelogram $KLMN$ is

$$\left|\overrightarrow{KL}\times\overrightarrow{KN}\right| = \left\|\begin{matrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ 0 & 1 & 3\\ 2 & 5 & 0\end{matrix}\right\| = |(-15)\,\mathbf{i}-(-6)\,\mathbf{j}+(-2)\,\mathbf{k}| = |-15\,\mathbf{i}+6\,\mathbf{j}-2\,\mathbf{k}| = \sqrt{265}\approx 16.28$$

25. (a) Because the plane through P, Q, and R contains the vectors $\overrightarrow{PQ}$ and $\overrightarrow{PR}$, a vector orthogonal to both of these vectors (such as their cross product) is also orthogonal to the plane. Here $\overrightarrow{PQ} = \langle -1,2,0\rangle$ and $\overrightarrow{PR} = \langle -1,0,3\rangle$, so

$$\overrightarrow{PQ}\times\overrightarrow{PR} = \langle (2)(3)-(0)(0),(0)(-1)-(-1)(3),(-1)(0)-(2)(-1)\rangle = \langle 6,3,2\rangle$$

Therefore, $\langle 6,3,2\rangle$ (or any scalar multiple thereof) is orthogonal to the plane through P, Q, and R.

(b) Note that the area of the triangle determined by P, Q, and R is equal to half of the area of the parallelogram determined by the three points. From part (a), the area of the parallelogram is $\left|\overrightarrow{PQ}\times\overrightarrow{PR}\right| = |\langle 6,3,2\rangle| = \sqrt{36+9+4} = 7$, so the area of the triangle is $\frac{1}{2}(7) = \frac{7}{2}$.

26. (a) $\overrightarrow{PQ} = \langle -3,2,-1\rangle$ and $\overrightarrow{PR} = \langle 1,-1,1\rangle$, so a vector orthogonal to the plane through P, Q, and R is $\overrightarrow{PQ}\times\overrightarrow{PR} = \langle (2)(1)-(-1)(-1),(-1)(1)-(-3)(1),(-3)(-1)-(2)(1)\rangle = \langle 1,2,1\rangle$ (or any scalar mutiple thereof).

(b) The area of the parallelogram determined by $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ is $\left|\overrightarrow{PQ}\times\overrightarrow{PR}\right| = |\langle 1,2,1\rangle| = \sqrt{1^2+2^2+1^2} = \sqrt{6}$, so the area of triangle PQR is $\frac{1}{2}\sqrt{6}$.

27. (a) $\overrightarrow{PQ} = \langle 4, 3, -2 \rangle$ and $\overrightarrow{PR} = \langle 5, 5, 1 \rangle$, so a vector orthogonal to the plane through P, Q, and R is $\overrightarrow{PQ} \times \overrightarrow{PR} = \langle (3)(1) - (-2)(5), (-2)(5) - (4)(1), (4)(5) - (3)(5) \rangle = \langle 13, -14, 5 \rangle$ (or any scalar mutiple thereof).

(b) The area of the parallelogram determined by $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ is $\left|\overrightarrow{PQ} \times \overrightarrow{PR}\right| = |\langle 13, -14, 5 \rangle| = \sqrt{13^2 + (-14)^2 + 5^2} = \sqrt{390}$, so the area of triangle PQR is $\frac{1}{2}\sqrt{390}$.

28. (a) $\overrightarrow{PQ} = \langle 1, 1, 3 \rangle$ and $\overrightarrow{PR} = \langle 3, 2, 5 \rangle$, so a vector orthogonal to the plane through P, Q, and R is $\overrightarrow{PQ} \times \overrightarrow{PR} = \langle (1)(5) - (3)(2), (3)(3) - (1)(5), (1)(2) - (1)(3) \rangle = \langle -1, 4, -1 \rangle$ (or any scalar multiple thereof).

(b) The area of the parallelogram determined by $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ is $\left|\overrightarrow{PQ} \times \overrightarrow{PR}\right| = |\langle -1, 4, -1 \rangle| = \sqrt{1 + 16 + 1} = \sqrt{18} = 3\sqrt{2}$, so the area of triangle PQR is $\frac{1}{2} \cdot 3\sqrt{2} = \frac{3}{2}\sqrt{2}$.

29. We know that the volume of the parallelepiped determined by **a**, **b**, and **c** is the magnitude of their scalar triple product, which is

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} 6 & 3 & -1 \\ 0 & 1 & 2 \\ 4 & -2 & 5 \end{vmatrix} = 6\begin{vmatrix} 1 & 2 \\ -2 & 5 \end{vmatrix} - 3\begin{vmatrix} 0 & 2 \\ 4 & 5 \end{vmatrix} + (-1)\begin{vmatrix} 0 & 1 \\ 4 & -2 \end{vmatrix}$$

$$= 6(5 + 4) - 3(0 - 8) - (0 - 4) = 82$$

Thus the volume of the parallelepiped is 82 cubic units.

30. $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 1 \end{vmatrix} = 1\begin{vmatrix} -1 & 1 \\ 1 & 1 \end{vmatrix} - 1\begin{vmatrix} 1 & 1 \\ -1 & 1 \end{vmatrix} + (-1)\begin{vmatrix} 1 & -1 \\ -1 & 1 \end{vmatrix} = -2 - 2 + 0 = -4.$

So the volume of the parallelepiped determined by **a**, **b**, and **c** is $|-4| = 4$ cubic units.

31. $\mathbf{a} = \overrightarrow{PQ} = \langle 2, 1, 1 \rangle$, $\mathbf{b} = \overrightarrow{PR} = \langle 1, -1, 2 \rangle$, and $\mathbf{c} = \overrightarrow{PS} = \langle 0, -2, 3 \rangle$.

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} 2 & 1 & 1 \\ 1 & -1 & 2 \\ 0 & -2 & 3 \end{vmatrix} = 2\begin{vmatrix} -1 & 2 \\ -2 & 3 \end{vmatrix} - 1\begin{vmatrix} 1 & 2 \\ 0 & 3 \end{vmatrix} + 1\begin{vmatrix} 1 & -1 \\ 0 & -2 \end{vmatrix} = 2 - 3 - 2 = -3,$$

so the volume of the parallelepiped is 3 cubic units.

32. $\mathbf{a} = \overrightarrow{PQ} = \langle 2, 3, 3 \rangle$, $\mathbf{b} = \overrightarrow{PR} = \langle -1, -1, -1 \rangle$ and $\mathbf{c} = \overrightarrow{PS} = \langle 6, -2, 2 \rangle$.

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} 2 & 3 & 3 \\ -1 & -1 & -1 \\ 6 & -2 & 2 \end{vmatrix} = 2\begin{vmatrix} -1 & -1 \\ -2 & 2 \end{vmatrix} - 3\begin{vmatrix} -1 & -1 \\ 6 & 2 \end{vmatrix} + 3\begin{vmatrix} -1 & -1 \\ 6 & -2 \end{vmatrix} = -8 - 12 + 24 = 4,$$

so the volume of the parallelepiped is 4 cubic units.

33. $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \begin{vmatrix} 2 & 3 & 1 \\ 1 & -1 & 0 \\ 7 & 3 & 2 \end{vmatrix} = 2\begin{vmatrix} -1 & 0 \\ 3 & 2 \end{vmatrix} - 3\begin{vmatrix} 1 & 0 \\ 7 & 2 \end{vmatrix} + 1\begin{vmatrix} 1 & -1 \\ 7 & 3 \end{vmatrix} = -4 - 6 + 10 = 0$, which says that the volume of the parallelepiped determined by **a**, **b** and **c** is 0, and thus these three vectors are coplanar.

34. $\mathbf{a} = \overrightarrow{PQ} = \langle 1,4,5 \rangle$, $\mathbf{b} = \overrightarrow{PR} = \langle 2,-1,1 \rangle$ and $\mathbf{c} = \overrightarrow{PS} = \langle 5,2,7 \rangle$.

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \begin{vmatrix} 1 & 4 & 5 \\ 2 & -1 & 1 \\ 5 & 2 & 7 \end{vmatrix} = 1\begin{vmatrix} -1 & 1 \\ 2 & 7 \end{vmatrix} - 4\begin{vmatrix} 2 & 1 \\ 5 & 7 \end{vmatrix} + 5\begin{vmatrix} 2 & -1 \\ 5 & 2 \end{vmatrix} = -9 - 36 + 45 = 0,$$

so the volume of the parallelepiped determined by **a**, **b** and **c** is 0, which says that these vectors lie in the same plane. Therefore, their initial and terminal points P, Q, R and S also lie in the same plane.

35. The magnitude of the torque is

$|\boldsymbol{\tau}| = |\mathbf{r}\times\mathbf{F}| = |\mathbf{r}|\,|\mathbf{F}|\sin\theta = (0.18\text{ m})(60\text{ N})\sin(70+10)^\circ = 10.8\sin 80^\circ \approx 10.6\text{ J}.$

36. $|\mathbf{r}| = \sqrt{4^2+4^2} = 4\sqrt{2}$ ft. A line drawn from the point P to the point of application of the force makes an angle of $180^\circ - (45+30)^\circ = 105^\circ$ with the force vector. Therefore,

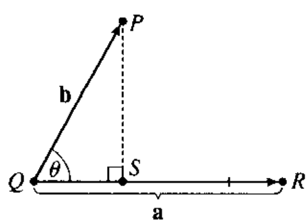
$|\boldsymbol{\tau}| = |\mathbf{r}\times\mathbf{F}| = |\mathbf{r}|\,|\mathbf{F}|\sin\theta = \left(4\sqrt{2}\right)(36)\sin 105^\circ \approx 197$ ft-lb.

37. Using the notation of the text, $\mathbf{r} = \langle 0, 0.3, 0\rangle$ and $\mathbf{F}$ has direction $\langle 0,3,-4\rangle$. The angle θ between them can be determined by $\cos\theta = \dfrac{\langle 0,0.3,0\rangle\cdot\langle 0,3,-4\rangle}{|\langle 0,0.3,0\rangle|\,|\langle 0,3,-4\rangle|} \;\Rightarrow\; \cos\theta = \dfrac{0.9}{(0.3)(5)} \;\Rightarrow\; \cos\theta = 0.6 \;\Rightarrow\; \theta \approx 53.1^\circ$.

Then $|\boldsymbol{\tau}| = |\mathbf{r}|\,|\mathbf{F}|\sin\theta \;\Rightarrow\; 100 = 0.3\,|\mathbf{F}|\sin 53.1^\circ \;\Rightarrow\; |\mathbf{F}| \approx 417$ N.

38. Since $|\mathbf{u}\times\mathbf{v}| = |\mathbf{u}|\,|\mathbf{v}|\sin\theta$, $0\le\theta\le\pi$, $|\mathbf{u}\times\mathbf{v}|$ achieves its maximum value for $\sin\theta = 1 \;\Rightarrow\; \theta = \frac{\pi}{2}$, in which case $|\mathbf{u}\times\mathbf{v}| = |\mathbf{u}|\,|\mathbf{v}| = 15$. The minimum value is zero, which occurs when $\sin\theta = 0 \;\Rightarrow\; \theta = 0$ or π, so when **u**, **v** are parallel. Thus, when **u** points in the same direction as **v**, so $\mathbf{u} = 3\,\mathbf{j}$, $|\mathbf{u}\times\mathbf{v}| = 0$. As **u** rotates counterclockwise, $\mathbf{u}\times\mathbf{v}$ is directed in the negative z-direction (by the right-hand rule) and the length increases until $\theta = \frac{\pi}{2}$, in which case $\mathbf{u} = -3\,\mathbf{i}$ and $|\mathbf{u}\times\mathbf{v}| = 15$. As **u** rotates to the negative y-axis, $\mathbf{u}\times\mathbf{v}$ remains pointed in the negative z-direction and the length of $\mathbf{u}\times\mathbf{v}$ decreases to 0, after which the direction of $\mathbf{u}\times\mathbf{v}$ reverses to point in the positive z-direction and $|\mathbf{u}\times\mathbf{v}|$ increases. When $\mathbf{u} = 3\,\mathbf{i}$ (so $\theta = \frac{\pi}{2}$), $|\mathbf{u}\times\mathbf{v}|$ again reaches its maximum of 15, after which $|\mathbf{u}\times\mathbf{v}|$ decreases to 0 as **u** rotates to the positive y-axis.

39. (a)

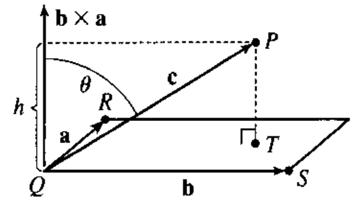


The distance between a point and a line is the length of the perpendicular from the point to the line, here $\left|\overrightarrow{PS}\right| = d$. But referring to triangle PQS, $d = \left|\overrightarrow{PS}\right| = \left|\overrightarrow{QP}\right|\sin\theta = |\mathbf{b}|\sin\theta$. But θ is the angle between $\overrightarrow{QP} = \mathbf{b}$ and $\overrightarrow{QR} = \mathbf{a}$. Thus by Theorem 6, $\sin\theta = \dfrac{|\mathbf{a}\times\mathbf{b}|}{|\mathbf{a}|\,|\mathbf{b}|}$ and so $d = |\mathbf{b}|\sin\theta = \dfrac{|\mathbf{b}|\,|\mathbf{a}\times\mathbf{b}|}{|\mathbf{a}|\,|\mathbf{b}|} = \dfrac{|\mathbf{a}\times\mathbf{b}|}{|\mathbf{a}|}$.

(b) $\mathbf{a} = \overrightarrow{QR} = \langle -1,-2,-1\rangle$ and $\mathbf{b} = \overrightarrow{QP} = \langle 1,-5,-7\rangle$. Then

$\mathbf{a}\times\mathbf{b} = \langle(-2)(-7)-(-1)(-5), (-1)(1)-(-1)(-7), (-1)(-5)-(-2)(1)\rangle = \langle 9,-8,7\rangle$. Thus the distance is $d = \dfrac{|\mathbf{a}\times\mathbf{b}|}{|\mathbf{a}|} = \frac{1}{\sqrt{6}}\sqrt{81+64+49} = \sqrt{\frac{194}{6}} = \sqrt{\frac{97}{3}}$.

40. (a) The distance between a point and a plane is the length of the perpendicular from the point to the plane, here $\left|\overrightarrow{TP}\right| = d$. But $\overrightarrow{TP}$ is parallel to $\mathbf{b} \times \mathbf{a}$ (because $\mathbf{b} \times \mathbf{a}$ is perpendicular to $\mathbf{b}$ and $\mathbf{a}$) and $d = \left|\overrightarrow{TP}\right| =$ the absolute value of the scalar projection of $\mathbf{c}$ along



$\mathbf{b} \times \mathbf{a}$, which is $|\mathbf{c}|\,|\cos\theta|$. (Notice that this is the same setup as the development of the volume of a parallelepiped with $h = |\mathbf{c}|\,|\cos\theta|$). Thus $d = |\mathbf{c}|\,|\cos\theta| = h = V/A$ where $A = |\mathbf{a} \times \mathbf{b}|$, the area of the base.

So finally $d = \dfrac{V}{A} = \dfrac{|\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|}{|\mathbf{a} \times \mathbf{b}|} = \dfrac{|(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}|}{|\mathbf{a} \times \mathbf{b}|}$ by Theorem 8 #5.

(b) $\mathbf{a} = \overrightarrow{QR} = \langle -1, 2, 0 \rangle$, $\mathbf{b} = \overrightarrow{QS} = \langle -1, 0, 3 \rangle$ and $\mathbf{c} = \overrightarrow{QP} = \langle 1, 1, 4 \rangle$. Then

$$(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = \begin{vmatrix} -1 & 2 & 0 \\ -1 & 0 & 3 \\ 1 & 1 & 4 \end{vmatrix} = (-1)\begin{vmatrix} 0 & 3 \\ 1 & 4 \end{vmatrix} - 2\begin{vmatrix} -1 & 3 \\ 1 & 4 \end{vmatrix} + 0 = 17$$

and

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -1 & 2 & 0 \\ -1 & 0 & 3 \end{vmatrix} = \begin{vmatrix} 2 & 0 \\ 0 & 3 \end{vmatrix}\mathbf{i} - \begin{vmatrix} -1 & 0 \\ -1 & 3 \end{vmatrix}\mathbf{j} + \begin{vmatrix} -1 & 2 \\ -1 & 0 \end{vmatrix}\mathbf{k} = 6\,\mathbf{i} + 3\,\mathbf{j} + 2\,\mathbf{k}$$

Thus $d = \dfrac{|(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}|}{|\mathbf{a} \times \mathbf{b}|} = \dfrac{17}{\sqrt{36 + 9 + 4}} = \dfrac{17}{7}$.

41.

$$\begin{aligned}
(\mathbf{a} - \mathbf{b}) \times (\mathbf{a} + \mathbf{b}) &= (\mathbf{a} - \mathbf{b}) \times \mathbf{a} + (\mathbf{a} - \mathbf{b}) \times \mathbf{b} && \text{by Theorem 8 \#3} \\
&= \mathbf{a} \times \mathbf{a} + (-\mathbf{b}) \times \mathbf{a} + \mathbf{a} \times \mathbf{b} + (-\mathbf{b}) \times \mathbf{b} && \text{by Theorem 8 \#4} \\
&= (\mathbf{a} \times \mathbf{a}) - (\mathbf{b} \times \mathbf{a}) + (\mathbf{a} \times \mathbf{b}) - (\mathbf{b} \times \mathbf{b}) && \text{by Theorem 8 \#2 (with } c = -1) \\
&= \mathbf{0} - (\mathbf{b} \times \mathbf{a}) + (\mathbf{a} \times \mathbf{b}) - \mathbf{0} && \text{by Example 2} \\
&= (\mathbf{a} \times \mathbf{b}) + (\mathbf{a} \times \mathbf{b}) && \text{by Theorem 8 \#1} \\
&= 2(\mathbf{a} \times \mathbf{b})
\end{aligned}$$

42. Let $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$, $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$ and $\mathbf{c} = \langle c_1, c_2, c_3 \rangle$, so $\mathbf{b} \times \mathbf{c} = \langle b_2c_3 - b_3c_2, b_3c_1 - b_1c_3, b_1c_2 - b_2c_1 \rangle$ and

$$\begin{aligned}
\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) &= \langle a_2(b_1c_2 - b_2c_1) - a_3(b_3c_1 - b_1c_3), a_3(b_2c_3 - b_3c_2) - a_1(b_1c_2 - b_2c_1), \\
&\qquad a_1(b_3c_1 - b_1c_3) - a_2(b_2c_3 - b_3c_2) \rangle \\
&= \langle a_2b_1c_2 - a_2b_2c_1 - a_3b_3c_1 + a_3b_1c_3, a_3b_2c_3 - a_3b_3c_2 - a_1b_1c_2 + a_1b_2c_1, \\
&\qquad a_1b_3c_1 - a_1b_1c_3 - a_2b_2c_3 + a_2b_3c_2 \rangle \\
&= \langle (a_2c_2 + a_3c_3)b_1 - (a_2b_2 + a_3b_3)c_1, (a_1c_1 + a_3c_3)b_2 - (a_1b_1 + a_3b_3)c_2, \\
&\qquad (a_1c_1 + a_2c_2)b_3 - (a_1b_1 + a_2b_2)c_3 \rangle
\end{aligned}$$

$$\begin{aligned}(\star)\quad &= \langle (a_2c_2 + a_3c_3)b_1 - (a_2b_2 + a_3b_3)c_1 + a_1b_1c_1 - a_1b_1c_1,\\ &\qquad (a_1c_1 + a_3c_3)b_2 - (a_1b_1 + a_3b_3)c_2 + a_2b_2c_2 - a_2b_2c_2,\\ &\qquad\qquad (a_1c_1 + a_2c_2)b_3 - (a_1b_1 + a_2b_2)c_3 + a_3b_3c_3 - a_3b_3c_3\rangle\\ &= \langle (a_1c_1 + a_2c_2 + a_3c_3)b_1 - (a_1b_1 + a_2b_2 + a_3b_3)c_1,\\ &\qquad (a_1c_1 + a_2c_2 + a_3c_3)b_2 - (a_1b_1 + a_2b_2 + a_3b_3)c_2,\\ &\qquad\qquad (a_1c_1 + a_2c_2 + a_3c_3)b_3 - (a_1b_1 + a_2b_2 + a_3b_3)c_3\rangle\\ &= (a_1c_1 + a_2c_2 + a_3c_3)\,\langle b_1, b_2, b_3\rangle - (a_1b_1 + a_2b_2 + a_3b_3)\,\langle c_1, c_2, c_3\rangle\\ &= (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}\end{aligned}$$

($\star$) Here we look ahead to see what terms are still needed to arrive at the desired equation. By adding and subtracting the same terms, we don't change the value of the component.

43. $\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) + \mathbf{b}\times(\mathbf{c}\times\mathbf{a}) + \mathbf{c}\times(\mathbf{a}\times\mathbf{b})$

$$= [(\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}] + [(\mathbf{b}\cdot\mathbf{a})\mathbf{c} - (\mathbf{b}\cdot\mathbf{c})\mathbf{a}] + [(\mathbf{c}\cdot\mathbf{b})\mathbf{a} - (\mathbf{c}\cdot\mathbf{a})\mathbf{b}] \qquad \text{by Exercise 42}$$

$$= (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c} + (\mathbf{a}\cdot\mathbf{b})\mathbf{c} - (\mathbf{b}\cdot\mathbf{c})\mathbf{a} + (\mathbf{b}\cdot\mathbf{c})\mathbf{a} - (\mathbf{a}\cdot\mathbf{c})\mathbf{b} = \mathbf{0}$$

44. Let $\mathbf{c}\times\mathbf{d} = \mathbf{v}$. Then

$$\begin{aligned}(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) &= (\mathbf{a}\times\mathbf{b})\cdot\mathbf{v} = \mathbf{a}\cdot(\mathbf{b}\times\mathbf{v}) && \text{by Theorem 8 \#5}\\ &= \mathbf{a}\cdot[\mathbf{b}\times(\mathbf{c}\times\mathbf{d})]\\ &= \mathbf{a}\cdot[(\mathbf{b}\cdot\mathbf{d})\mathbf{c} - (\mathbf{b}\cdot\mathbf{c})\mathbf{d}] && \text{by Exercise 42}\\ &= (\mathbf{b}\cdot\mathbf{d})(\mathbf{a}\cdot\mathbf{c}) - (\mathbf{b}\cdot\mathbf{c})(\mathbf{a}\cdot\mathbf{d}) && \text{by Properties 3 and 4 of the dot product}\\ &= \begin{vmatrix}\mathbf{a}\cdot\mathbf{c} & \mathbf{b}\cdot\mathbf{c}\\ \mathbf{a}\cdot\mathbf{d} & \mathbf{b}\cdot\mathbf{d}\end{vmatrix}\end{aligned}$$

45. (a) No. If $\mathbf{a}\cdot\mathbf{b} = \mathbf{a}\cdot\mathbf{c}$, then $\mathbf{a}\cdot(\mathbf{b}-\mathbf{c}) = 0$, so $\mathbf{a}$ is perpendicular to $\mathbf{b}-\mathbf{c}$, which can happen if $\mathbf{b}\neq\mathbf{c}$. For example, let $\mathbf{a} = \langle 1,1,1\rangle$, $\mathbf{b} = \langle 1,0,0\rangle$ and $\mathbf{c} = \langle 0,1,0\rangle$.

(b) No. If $\mathbf{a}\times\mathbf{b} = \mathbf{a}\times\mathbf{c}$ then $\mathbf{a}\times(\mathbf{b}-\mathbf{c}) = \mathbf{0}$, which implies that $\mathbf{a}$ is parallel to $\mathbf{b}-\mathbf{c}$, which of course can happen if $\mathbf{b}\neq\mathbf{c}$.

(c) Yes. Since $\mathbf{a}\cdot\mathbf{c} = \mathbf{a}\cdot\mathbf{b}$, $\mathbf{a}$ is perpendicular to $\mathbf{b}-\mathbf{c}$, by part (a). From part (b), $\mathbf{a}$ is also parallel to $\mathbf{b}-\mathbf{c}$. Thus since $\mathbf{a}\neq\mathbf{0}$ but is both parallel and perpendicular to $\mathbf{b}-\mathbf{c}$, we have $\mathbf{b}-\mathbf{c} = \mathbf{0}$, so $\mathbf{b} = \mathbf{c}$.

46. (a) $\mathbf{k}_i$ is perpendicular to $\mathbf{v}_j$ if $i\neq j$ by the definition of $\mathbf{k}_i$ and Theorem 5.

(b) $$\mathbf{k}_1\cdot\mathbf{v}_1 = \frac{\mathbf{v}_2\times\mathbf{v}_3}{\mathbf{v}_1\cdot(\mathbf{v}_2\times\mathbf{v}_3)}\cdot\mathbf{v}_1 = \frac{\mathbf{v}_1\cdot(\mathbf{v}_2\times\mathbf{v}_3)}{\mathbf{v}_1\cdot(\mathbf{v}_2\times\mathbf{v}_3)} = 1$$

$$\mathbf{k}_2\cdot\mathbf{v}_2 = \frac{\mathbf{v}_3\times\mathbf{v}_1}{\mathbf{v}_1\cdot(\mathbf{v}_2\times\mathbf{v}_3)}\cdot\mathbf{v}_2 = \frac{\mathbf{v}_2\cdot(\mathbf{v}_3\times\mathbf{v}_1)}{\mathbf{v}_1\cdot(\mathbf{v}_2\times\mathbf{v}_3)} = \frac{(\mathbf{v}_2\times\mathbf{v}_3)\cdot\mathbf{v}_1}{\mathbf{v}_1\cdot(\mathbf{v}_2\times\mathbf{v}_3)} = 1 \qquad \text{[by Theorem 8 \#5]}$$

$$\mathbf{k}_3\cdot\mathbf{v}_3 = \frac{(\mathbf{v}_1\times\mathbf{v}_2)\cdot\mathbf{v}_3}{\mathbf{v}_1\cdot(\mathbf{v}_2\times\mathbf{v}_3)} = \frac{\mathbf{v}_1\cdot(\mathbf{v}_2\times\mathbf{v}_3)}{\mathbf{v}_1\cdot(\mathbf{v}_2\times\mathbf{v}_3)} = 1 \qquad \text{[by Theorem 8 \#5]}$$

(c) $\mathbf{k}_1 \cdot (\mathbf{k}_2 \times \mathbf{k}_3) = \mathbf{k}_1 \cdot \left(\dfrac{\mathbf{v}_3 \times \mathbf{v}_1}{\mathbf{v}_1 \cdot (\mathbf{v}_2 \times \mathbf{v}_3)} \times \dfrac{\mathbf{v}_1 \times \mathbf{v}_2}{\mathbf{v}_1 \cdot (\mathbf{v}_2 \times \mathbf{v}_3)} \right) = \dfrac{\mathbf{k}_1}{[\mathbf{v}_1 \cdot (\mathbf{v}_2 \times \mathbf{v}_3)]^2} \cdot [(\mathbf{v}_3 \times \mathbf{v}_1) \times (\mathbf{v}_1 \times \mathbf{v}_2)]$

$$= \frac{\mathbf{k}_1}{[\mathbf{v}_1 \cdot (\mathbf{v}_2 \times \mathbf{v}_3)]^2} \cdot ([(\mathbf{v}_3 \times \mathbf{v}_1) \cdot \mathbf{v}_2]\, \mathbf{v}_1 - [(\mathbf{v}_3 \times \mathbf{v}_1) \cdot \mathbf{v}_1]\, \mathbf{v}_2) \qquad \text{[by Exercise 42]}$$

But $(\mathbf{v}_3 \times \mathbf{v}_1) \cdot \mathbf{v}_1 = 0$ since $\mathbf{v}_3 \times \mathbf{v}_1$ is orthogonal to $\mathbf{v}_1$, and $(\mathbf{v}_3 \times \mathbf{v}_1) \cdot \mathbf{v}_2 = \mathbf{v}_2 \cdot (\mathbf{v}_3 \times \mathbf{v}_1) = (\mathbf{v}_2 \times \mathbf{v}_3) \cdot \mathbf{v}_1 = \mathbf{v}_1 \cdot (\mathbf{v}_2 \times \mathbf{v}_3)$. Thus

$$\mathbf{k}_1 \cdot (\mathbf{k}_2 \times \mathbf{k}_3) = \frac{\mathbf{k}_1}{[\mathbf{v}_1 \cdot (\mathbf{v}_2 \times \mathbf{v}_3)]^2} \cdot [\mathbf{v}_1 \cdot (\mathbf{v}_2 \times \mathbf{v}_3)]\, \mathbf{v}_1 = \frac{\mathbf{k}_1 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot (\mathbf{v}_2 \times \mathbf{v}_3)}$$

$$= \frac{1}{\mathbf{v}_1 \cdot (\mathbf{v}_2 \times \mathbf{v}_3)} \qquad \text{[by part (b)]}$$

DISCOVERY PROJECT The Geometry of a Tetrahedron

1. Set up a coordinate system so that vertex S is at the origin, $R = (0, y_1, 0)$, $Q = (x_2, y_2, 0)$, $P = (x_3, y_3, z_3)$. Then $\overrightarrow{SR} = \langle 0, y_1, 0 \rangle$, $\overrightarrow{SQ} = \langle x_2, y_2, 0 \rangle$, $\overrightarrow{SP} = \langle x_3, y_3, z_3 \rangle$, $\overrightarrow{QR} = \langle -x_2, y_1 - y_2, 0 \rangle$, and $\overrightarrow{QP} = \langle x_3 - x_2, y_3 - y_2, z_3 \rangle$. Let

$$\begin{aligned} \mathbf{v}_S &= \overrightarrow{QR} \times \overrightarrow{QP} \\ &= (y_1 z_3 - y_2 z_3)\,\mathbf{i} + x_2 z_3\,\mathbf{j} + (-x_2 y_3 - x_3 y_1 + x_3 y_2 + x_2 y_1)\,\mathbf{k} \end{aligned}$$

Then $\mathbf{v}_S$ is an outward normal to the face opposite vertex S. Similarly, $\mathbf{v}_R = \overrightarrow{SQ} \times \overrightarrow{SP} = y_2 z_3\,\mathbf{i} - x_2 z_3\,\mathbf{j} + (x_2 y_3 - x_3 y_2)\,\mathbf{k}$, $\mathbf{v}_Q = \overrightarrow{SP} \times \overrightarrow{SR} = -y_1 z_3\,\mathbf{i} + x_3 y_1\,\mathbf{k}$, and $\mathbf{v}_P = \overrightarrow{SR} \times \overrightarrow{SQ} = -x_2 y_1\,\mathbf{k} \quad \Rightarrow \quad \mathbf{v}_S + \mathbf{v}_R + \mathbf{v}_Q + \mathbf{v}_P = \mathbf{0}$. Now

$$\begin{aligned} |\mathbf{v}_S| &= \text{area of the parallelogram determined by } \overrightarrow{QR} \text{ and } \overrightarrow{QP} \\ &= 2\,(\text{area of triangle } RQP) \\ &= 2|\mathbf{v}_1| \end{aligned}$$

So $\mathbf{v}_S = 2\mathbf{v}_1$, and similarly $\mathbf{v}_R = 2\mathbf{v}_2$, $\mathbf{v}_Q = 2\mathbf{v}_3$, $\mathbf{v}_P = 2\mathbf{v}_4$. Thus $\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 + \mathbf{v}_4 = \mathbf{0}$.

2. (a) Let $S = (x_0, y_0, z_0)$, $R = (x_1, y_1, z_1)$, $Q = (x_2, y_2, z_2)$, $P = (x_3, y_3, z_3)$ be the four vertices. Then

$$\begin{aligned} \text{Volume} &= \tfrac{1}{3}(\text{distance from } S \text{ to plane } RQP) \times (\text{area of triangle } RQP) \\ &= \frac{1}{3} \frac{\left|\mathbf{N} \cdot \overrightarrow{SR}\right|}{|\mathbf{N}|} \cdot \frac{1}{2} \left|\overrightarrow{RQ} \times \overrightarrow{RP}\right| \end{aligned}$$

where $\mathbf{N}$ is a vector which is normal to the face RQP. Thus $\mathbf{N} = \overrightarrow{RQ} \times \overrightarrow{RP}$. Therefore

$$V = \left| \tfrac{1}{6} \left(\overrightarrow{RQ} \times \overrightarrow{RP} \right) \cdot \overrightarrow{SR} \right| = \frac{1}{6} \left\| \begin{matrix} x_0 - x & y_0 - y_1 & z_0 - z_1 \\ x_2 - x_1 & y_2 - y_1 & z_2 - z_1 \\ x_3 - x_1 & y_3 - y_1 & z_3 - z_1 \end{matrix} \right\|.$$

(b) Using the formula from part (a), $V = \dfrac{1}{6} \left\| \begin{matrix} 1-1 & 1-2 & 1-3 \\ 1-1 & 1-2 & 2-3 \\ 3-1 & -1-2 & 2-3 \end{matrix} \right\| = \dfrac{1}{6} \left| 2(1-2) \right| = \dfrac{1}{3}$.

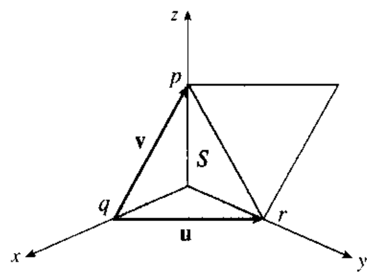
3. We define a vector $\mathbf{v}_1$ to have length equal to the area of the face opposite vertex P, so we can say $|\mathbf{v}_1| = A$, and direction perpendicular to the face and pointing outward, as in Problem 1. Similarly, we define $\mathbf{v}_2$, $\mathbf{v}_3$, and $\mathbf{v}_4$ so that $|\mathbf{v}_2| = B$, $|\mathbf{v}_3| = C$, and $|\mathbf{v}_4| = D$ and with the analogous directions. From Problem 1, we know $\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3 + \mathbf{v}_4 = \mathbf{0} \;\Rightarrow\; \mathbf{v}_4 = -(\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3) \;\Rightarrow\; |\mathbf{v}_4| = |-(\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3)| = |\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3|$ $\Rightarrow\; |\mathbf{v}_4|^2 = |\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3|^2 \;\Rightarrow$

$$\begin{aligned}\mathbf{v}_4 \cdot \mathbf{v}_4 &= (\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3) \cdot (\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3)\\ &= \mathbf{v}_1 \cdot \mathbf{v}_1 + \mathbf{v}_1 \cdot \mathbf{v}_2 + \mathbf{v}_1 \cdot \mathbf{v}_3 + \mathbf{v}_2 \cdot \mathbf{v}_1 + \mathbf{v}_2 \cdot \mathbf{v}_2 + \mathbf{v}_2 \cdot \mathbf{v}_3 + \mathbf{v}_3 \cdot \mathbf{v}_1 + \mathbf{v}_3 \cdot \mathbf{v}_2 + \mathbf{v}_3 \cdot \mathbf{v}_3\end{aligned}$$

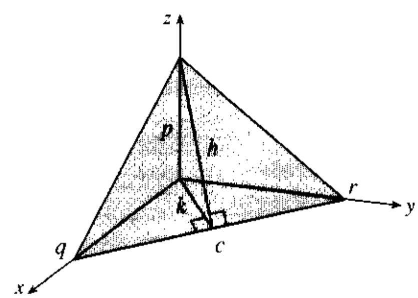
Since the vertex S is trirectangular, we know the three faces meeting at S are mutually perpendicular, so the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ are also mutually perpendicular. Therefore, $\mathbf{v}_i \cdot \mathbf{v}_j = 0$ for $i \neq j$ and $i, j \in \{1, 2, 3\}$. Thus we have $\mathbf{v}_4 \cdot \mathbf{v}_4 = \mathbf{v}_1 \cdot \mathbf{v}_1 + \mathbf{v}_2 \cdot \mathbf{v}_2 + \mathbf{v}_3 \cdot \mathbf{v}_3 \;\Rightarrow\; |\mathbf{v}_4|^2 = |\mathbf{v}_1|^2 + |\mathbf{v}_2|^2 + |\mathbf{v}_3|^2 \;\Rightarrow\; D^2 = A^2 + B^2 + C^2$.

Another method: We introduce a coordinate system, as shown. Recall that the area of the parallelogram spanned by two vectors is equal to the length of their cross product, so since $\mathbf{u} \times \mathbf{v} = \langle -q, r, 0\rangle \times \langle -q, 0, p\rangle = \langle pr, pq, qr\rangle$, we have $|\mathbf{u} \times \mathbf{v}| = \sqrt{(pr)^2 + (pq)^2 + (qr)^2}$, and therefore

$$\begin{aligned}D^2 &= \left(\tfrac{1}{2}|\mathbf{u} \times \mathbf{v}|\right)^2 = \tfrac{1}{4}\left[(pr)^2 + (pq)^2 + (qr)^2\right]\\ &= \left(\tfrac{1}{2}pr\right)^2 + \left(\tfrac{1}{2}pq\right)^2 + \left(\tfrac{1}{2}qr\right)^2 = A^2 + B^2 + C^2.\end{aligned}$$



A third method: We draw a line from S perpendicular to QR, as shown. Now $D = \frac{1}{2}ch$, so $D^2 = \frac{1}{4}c^2h^2$. Substituting $h^2 = p^2 + k^2$, we get $D^2 = \frac{1}{4}c^2(p^2 + k^2) = \frac{1}{4}c^2p^2 + \frac{1}{4}c^2k^2$. But $C = \frac{1}{2}ck$, so $D^2 = \frac{1}{4}c^2p^2 + C^2$. Now substituting $c^2 = q^2 + r^2$ gives $D^2 = \frac{1}{4}p^2q^2 + \frac{1}{4}q^2r^2 + C^2 = A^2 + B^2 + C^2$.



13.5 Equations of Lines and Planes — ET 12.5

1. (a) True; each of the first two lines has a direction vector parallel to the direction vector of the third line, so these vectors are each scalar multiples of the third direction vector. Then the first two direction vectors are also scalar multiples of each other, so these vectors, and hence the two lines, are parallel.

(b) False; for example, the x- and y-axes are both perpendicular to the z-axis, yet the x- and y-axes are not parallel.

(c) True; each of the first two planes has a normal vector parallel to the normal vector of the third plane, so these two normal vectors are parallel to each other and the planes are parallel.

(d) False; for example, the xy- and yz-planes are not parallel, yet they are both perpendicular to the xz-plane.

(e) False; the x- and y-axes are not parallel, yet they are both parallel to the plane $z = 1$.

(f) True; if each line is perpendicular to a plane, then the lines' direction vectors are both parallel to a normal vector for the plane. Thus, the direction vectors are parallel to each other and the lines are parallel.

(g) False; the planes $y = 1$ and $z = 1$ are not parallel, yet they are both parallel to the x-axis.

(h) True; if each plane is perpendicular to a line, then any normal vector for each plane is parallel to a direction vector for the line. Thus, the normal vectors are parallel to each other and the planes are parallel.

(i) True; see Figure 9 and the accompanying discussion.

(j) False; they can be skew, as in Example 3.

(k) True. Consider any normal vector for the plane and any direction vector for the line. If the normal vector is perpendicular to the direction vector, the line and plane are parallel. Otherwise, the vectors meet at an angle θ, $0^\circ \le \theta < 90^\circ$, and the line will intersect the plane at an angle $90^\circ - \theta$.

2. For this line, we have $\mathbf{r}_0 = \mathbf{i} - 3\,\mathbf{k}$ and $\mathbf{v} = 2\,\mathbf{i} - 4\,\mathbf{j} + 5\,\mathbf{k}$, so a vector equation is $\mathbf{r} = \mathbf{r}_0 + t\,\mathbf{v} = (\mathbf{i} - 3\,\mathbf{k}) + t(2\,\mathbf{i} - 4\,\mathbf{j} + 5\,\mathbf{k}) = (1 + 2t)\,\mathbf{i} - 4t\,\mathbf{j} + (-3 + 5t)\,\mathbf{k}$ and parametric equations are $x = 1 + 2t$, $y = -4t$, $z = -3 + 5t$.

3. For this line, we have $\mathbf{r}_0 = -2\,\mathbf{i} + 4\,\mathbf{j} + 10\,\mathbf{k}$ and $\mathbf{v} = 3\,\mathbf{i} + \mathbf{j} - 8\,\mathbf{k}$, so a vector equation is $\mathbf{r} = \mathbf{r}_0 + t\,\mathbf{v} = (-2\,\mathbf{i} + 4\,\mathbf{j} + 10\,\mathbf{k}) + t(3\,\mathbf{i} + \mathbf{j} - 8\,\mathbf{k}) = (-2 + 3t)\,\mathbf{i} + (4 + t)\,\mathbf{j} + (10 - 8t)\,\mathbf{k}$ and parametric equations are $x = -2 + 3t$, $y = 4 + t$, $z = 10 - 8t$.

4. This line has the same direction as the given line, $\mathbf{v} = 2\,\mathbf{i} - \mathbf{j} + 3\,\mathbf{k}$. Here $\mathbf{r}_0 = 0\,\mathbf{i} + 0\,\mathbf{j} + 0\,\mathbf{k}$, so a vector equation is $\mathbf{r} = (0\,\mathbf{i} + 0\,\mathbf{j} + 0\,\mathbf{k}) + t(2\,\mathbf{i} - \mathbf{j} + 3\,\mathbf{k}) = 2t\,\mathbf{i} - t\,\mathbf{j} + 3t\,\mathbf{k}$ and parametric equations are $x = 2t$, $y = -t$, $z = 3t$.

5. A line perpendicular to the given plane has the same direction as a normal vector to the plane, such as $\mathbf{n} = \langle 1, 3, 1\rangle$. So $\mathbf{r}_0 = \mathbf{i} + 6\,\mathbf{k}$, and we can take $\mathbf{v} = \mathbf{i} + 3\,\mathbf{j} + \mathbf{k}$. Then a vector equation is $\mathbf{r} = (\mathbf{i} + 6\,\mathbf{k}) + t(\mathbf{i} + 3\,\mathbf{j} + \mathbf{k}) = (1 + t)\,\mathbf{i} + 3t\,\mathbf{j} + (6 + t)\,\mathbf{k}$, and parametric equations are $x = 1 + t$, $y = 3t$, $z = 6 + t$.

6. The vector $\mathbf{v} = \langle 1 - 0, 2 - 0, 3 - 0\rangle = \langle 1, 2, 3\rangle$ is parallel to the line. Letting $P_0 = (0, 0, 0)$, parametric equations are $x = 0 + 1 \cdot t = t$, $y = 0 + 2 \cdot t = 2t$, $z = 0 + 3 \cdot t = 3t$, while symmetric equations are $x = \dfrac{y}{2} = \dfrac{z}{3}$.

7. The vector $\mathbf{v} = \langle -4 - 1, 3 - 3, 0 - 2\rangle = \langle -5, 0, -2\rangle$ is parallel to the line. Letting $P_0 = (1, 3, 2)$, parametric equations are $x = 1 - 5t$, $y = 3 + 0t = 3$, $z = 2 - 2t$, while symmetric equations are $\dfrac{x - 1}{-5} = \dfrac{z - 2}{-2}$, $y = 3$. Notice here that the direction number $b = 0$, so rather than writing $\dfrac{y - 3}{0}$ in the symmetric equation we must write the equation $y = 3$ separately.

8. $\mathbf{v} = \langle 2 - 6, 4 - 1, 5 - (-3)\rangle = \langle -4, 3, 8\rangle$, and letting $P_0 = (6, 1, -3)$, parametric equations are $x = 6 - 4t$, $y = 1 + 3t$, $z = -3 + 8t$, while symmetric equations are $\dfrac{x - 6}{-4} = \dfrac{y - 1}{3} = \dfrac{z + 3}{8}$.

9. $\mathbf{v} = \left\langle 2 - 0, 1 - \frac{1}{2}, -3 - 1\right\rangle = \left\langle 2, \frac{1}{2}, -4\right\rangle$, and letting $P_0 = (2, 1, -3)$, parametric equations are $x = 2 + 2t$, $y = 1 + \frac{1}{2}t$, $z = -3 - 4t$, while symmetric equations are $\dfrac{x - 2}{2} = \dfrac{y - 1}{1/2} = \dfrac{z + 3}{-4}$ or $\dfrac{x - 2}{2} = 2y - 2 = \dfrac{z + 3}{-4}$.

10. $\mathbf{v} = (\mathbf{i} + \mathbf{j}) \times (\mathbf{j} + \mathbf{k}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix} = \mathbf{i} - \mathbf{j} + \mathbf{k}$ is the direction of the line perpendicular to both $\mathbf{i} + \mathbf{j}$ and $\mathbf{j} + \mathbf{k}$.

With $P_0 = (2, 1, 0)$, parametric equations are $x = 2 + t$, $y = 1 - t$, $z = t$ and symmetric equations are $x - 2 = \dfrac{y - 1}{-1} = z$ or $x - 2 = 1 - y = z$.

11. The line has direction $\mathbf{v} = \langle 1, 2, 1 \rangle$. Letting $P_0 = (1, -1, 1)$, parametric equations are $x = 1 + t$, $y = -1 + 2t$, $z = 1 + t$ and symmetric equations are $x - 1 = \dfrac{y+1}{2} = z - 1$.

12. Setting $x = 0$, we see that $(0, 1, 0)$ satisfies the equations of both planes, so they do in fact have a line of intersection. $\mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2 = \langle 1, 1, 1 \rangle \times \langle 1, 0, 1 \rangle = \langle 1, 0, -1 \rangle$ is the direction of this line. Taking the point $(0, 1, 0)$ as P_0, parametric equations are $x = t$, $y = 1$, $z = -t$, and symmetric equations are $x = -z$, $y = 1$.

13. Direction vectors of the lines are $\mathbf{v}_1 = \langle -2 - (-4), 0 - (-6), -3 - 1 \rangle = \langle 2, 6, -4 \rangle$ and $\mathbf{v}_2 = \langle 5 - 10, 3 - 18, 14 - 4 \rangle = \langle -5, -15, 10 \rangle$, and since $\mathbf{v}_2 = -\frac{5}{2}\mathbf{v}_1$, the direction vectors and thus the lines are parallel.

14. Direction vectors of the lines are $\mathbf{v}_1 = \langle -2, 4, 4 \rangle$ and $\mathbf{v}_2 = \langle 8, -1, 4 \rangle$. Since $\mathbf{v}_1 \cdot \mathbf{v}_2 = -16 - 4 + 16 \neq 0$, the vectors and thus the lines are not perpendicular.

15. (a) A direction vector of the line with parametric equations $x = 1 + 2t$, $y = 3t$, $z = 5 - 7t$ is $\mathbf{v} = \langle 2, 3, -7 \rangle$ and the desired parallel line must also have $\mathbf{v}$ as a direction vector. Here $P_0 = (0, 2, -1)$, so symmetric equations for the line are $\dfrac{x}{2} = \dfrac{y-2}{3} = \dfrac{z+1}{-7}$.

(b) The line intersects the xy-plane when $z = 0$, so we need $\dfrac{x}{2} = \dfrac{y-2}{3} = \dfrac{1}{-7}$ or $x = -\frac{2}{7}$, $y = \frac{11}{7}$. Thus the point of intersection with the xy-plane is $\left(-\frac{2}{7}, \frac{11}{7}, 0\right)$. Similarly for the yz-plane, we need $x = 0 \quad \Leftrightarrow \quad 0 = \dfrac{y-2}{3} = \dfrac{z+1}{-7} \quad \Leftrightarrow \quad y = 2$, $z = -1$. Thus the line intersects the yz-plane at $(0, 2, -1)$. For the xz-plane, we need $y = 0 \quad \Leftrightarrow \quad \dfrac{x}{2} = -\dfrac{2}{3} = \dfrac{z+1}{-7} \quad \Leftrightarrow \quad x = -\frac{4}{3}$, $z = \frac{11}{3}$. So the line intersects the xz-plane at $\left(-\frac{4}{3}, 0, \frac{11}{3}\right)$.

16. (a) A vector normal to the plane $2x - y + z = 1$ is $\mathbf{n} = \langle 2, -1, 1 \rangle$, and since the line is to be perpendicular to the plane, $\mathbf{n}$ is also a direction vector for the line. Thus parametric equations of the line are $x = 5 + 2t$, $y = 1 - t$, $z = t$.

(b) On the xy-plane, $z = 0$. So $z = t = 0$ in the parametric equations of the line, and therefore $x = 5$ and $y = 1$, giving the point of intersection $(5, 1, 0)$. For the yz-plane, $x = 0$ which implies $t = -\frac{5}{2}$, so $y = \frac{7}{2}$ and $z = -\frac{5}{2}$ and the point is $\left(0, \frac{7}{2}, -\frac{5}{2}\right)$. For the xz-plane, $y = 0$ which implies $t = 1$, so $x = 7$ and $z = 1$ and the point of intersection is $(7, 0, 1)$.

17. From Equation 4, the line segment from $\mathbf{r}_0 = 2\,\mathbf{i} - \mathbf{j} + 4\,\mathbf{k}$ to $\mathbf{r}_1 = 4\,\mathbf{i} + 6\,\mathbf{j} + \mathbf{k}$ is
$\mathbf{r}(t) = (1 - t)\,\mathbf{r}_0 + t\,\mathbf{r}_1 = (1 - t)(2\,\mathbf{i} - \mathbf{j} + 4\,\mathbf{k}) + t(4\,\mathbf{i} + 6\,\mathbf{j} + \mathbf{k}) = (2\,\mathbf{i} - \mathbf{j} + 4\,\mathbf{k}) + t(2\,\mathbf{i} + 7\,\mathbf{j} - 3\,\mathbf{k})$, $0 \leq t \leq 1$.

18. From Equation 4, the line segment from $\mathbf{r}_0 = 10\,\mathbf{i} + 3\,\mathbf{j} + \mathbf{k}$ to $\mathbf{r}_1 = 5\,\mathbf{i} + 6\,\mathbf{j} - 3\,\mathbf{k}$ is

$$\begin{aligned}\mathbf{r}(t) &= (1 - t)\,\mathbf{r}_0 + t\,\mathbf{r}_1 = (1 - t)(10\,\mathbf{i} + 3\,\mathbf{j} + \mathbf{k}) + t(5\,\mathbf{i} + 6\,\mathbf{j} - 3\,\mathbf{k}) \\ &= (10\,\mathbf{i} + 3\,\mathbf{j} + \mathbf{k}) + t(-5\,\mathbf{i} + 3\,\mathbf{j} - 4\,\mathbf{k}), \quad 0 \leq t \leq 1\end{aligned}$$

The corresponding parametric equations are $x = 10 - 5t$, $y = 3 + 3t$, $z = 1 - 4t$, $0 \leq t \leq 1$.

19. Since the direction vectors are $\mathbf{v}_1 = \langle -6, 9, -3 \rangle$ and $\mathbf{v}_2 = \langle 2, -3, 1 \rangle$, we have $\mathbf{v}_1 = -3\mathbf{v}_2$ so the lines are parallel.

20. The lines aren't parallel since the direction vectors $\langle 2, 3, -1 \rangle$ and $\langle 1, 1, 3 \rangle$ aren't parallel. For the lines to intersect we must be able to find one value of t and one value of s that produce the same point from the respective parametric equations. Thus we need to satisfy the following three equations: $1 + 2t = -1 + s$, $3t = 4 + s$, $2 - t = 1 + 3s$. Solving the first two equations we get $t = 6$, $s = 14$ and checking, we see that these values don't satisfy the third equation. Thus L_1 and L_2 aren't parallel and don't intersect, so they must be skew lines.

21. Since the direction vectors $\langle 1, 2, 3\rangle$ and $\langle -4, -3, 2\rangle$ are not scalar multiples of each other, the lines are not parallel, so we check to see if the lines intersect. The parametric equations of the lines are L_1: $x = t$, $y = 1 + 2t$, $z = 2 + 3t$ and L_2: $x = 3 - 4s$, $y = 2 - 3s$, $z = 1 + 2s$. For the lines to intersect, we must be able to find one value of t and one value of s that produce the same point from the respective parametric equations. Thus we need to satisfy the following three equations: $t = 3 - 4s$, $1 + 2t = 2 - 3s$, $2 + 3t = 1 + 2s$. Solving the first two equations we get $t = -1$, $s = 1$ and checking, we see that these values don't satisfy the third equation. Thus the lines aren't parallel and don't intersect, so they must be skew lines.

22. Since the direction vectors $\langle 2, 2, -1\rangle$ and $\langle 1, -1, 3\rangle$ aren't parallel, the lines aren't parallel. Here the parametric equations are L_1: $x = 1 + 2t$, $y = 3 + 2t$, $z = 2 - t$ and L_2: $x = 2 + s$, $y = 6 - s$, $z = -2 + 3s$. Thus, for the lines to intersect, the three equations $1 + 2t = 2 + s$, $3 + 2t = 6 - s$, and $2 - t = -2 + 3s$ must be satisfied simultaneously. Solving the first two equations gives $t = 1$, $s = 1$ and, checking, we see that these values do satisfy the third equation, so the lines intersect when $t = 1$ and $s = 1$, that is, at the point $(3, 5, 1)$.

23. Since the plane is perpendicular to the vector $\langle -2, 1, 5\rangle$, we can take $\langle -2, 1, 5\rangle$ as a normal vector to the plane. $(6, 3, 2)$ is a point on the plane, so setting $a = -2$, $b = 1$, $c = 5$ and $x_0 = 6$, $y_0 = 3$, $z_0 = 2$ in Equation 7 gives $-2(x - 6) + 1(y - 3) + 5(z - 2) = 0$ or $-2x + y + 5z = 1$ to be an equation of the plane.

24. $\mathbf{j} + 2\,\mathbf{k} = \langle 0, 1, 2\rangle$ is a normal vector to the plane and $(4, 0, -3)$ is a point on the plane, so setting $a = 0$, $b = 1$, $c = 2$, $x_0 = 4$, $y_0 = 0$, $z_0 = -3$ in Equation 7 gives $0(x - 4) + 1(y - 0) + 2[z - (-3)] = 0$ or $y + 2z = -6$ to be an equation of the plane.

25. $\mathbf{i} + \mathbf{j} - \mathbf{k} = \langle 1, 1, -1\rangle$ is a normal vector to the plane and $(1, -1, 1)$ is a point on the plane, so setting $a = 1$, $b = 1$, $c = -1$, $x_0 = 1$, $y_0 = -1$, $z_0 = 1$ in Equation 7 gives $1\,(x - 1) + 1[y - (-1)] - 1(z - 1) = 0$ or $x + y - z = -1$ to be an equation of the plane.

26. Since the line is perpendicular to the plane, its direction vector $\langle 1, 2, -3\rangle$ is a normal vector to the plane. An equation of the plane, then, is $1[x - (-2)] + 2(y - 8) - 3(z - 10) = 0$ or $x + 2y - 3z = -16$.

27. Since the two planes are parallel, they will have the same normal vectors. So we can take $\mathbf{n} = \langle 2, -1, 3\rangle$, and an equation of the plane is $2(x - 0) - 1(y - 0) + 3(z - 0) = 0$ or $2x - y + 3z = 0$.

28. Since the two planes are parallel, they will have the same normal vectors. So we can take $\mathbf{n} = \langle 1, 1, 1\rangle$, and an equation of the plane is $1[x - (-1)] + 1(y - 6) + 1[z - (-5)] = 0$ or $x + y + z = 0$.

29. Since the two planes are parallel, they will have the same normal vectors. So we can take $\mathbf{n} = \langle 3, 0, -7\rangle$, and an equation of the plane is $3(x - 4) + 0[y - (-2)] - 7(z - 3) = 0$ or $3x - 7z = -9$.

30. First, a normal vector for the plane $2x + 4y + 8z = 17$ is $\mathbf{n} = \langle 2, 4, 8\rangle$. A direction vector for the line is $\mathbf{v} = \langle 2, 1, -1\rangle$, and since $\mathbf{n} \cdot \mathbf{v} = 0$ we know the line is perpendicular to $\mathbf{n}$ and hence parallel to the plane. Thus, there is a parallel plane which contains the line. By putting $t = 0$, we know the point $(3, 0, 8)$ is on the line and hence the new plane. We can use the same normal vector $\mathbf{n} = \langle 2, 4, 8\rangle$, so an equation of the plane is $2(x - 3) + 4(y - 0) + 8(z - 8) = 0$ or $x + 2y + 4z = 35$.

31. Here the vectors $\mathbf{a} = \langle 1 - 0, 0 - 1, 1 - 1\rangle = \langle 1, -1, 0\rangle$ and $\mathbf{b} = \langle 1 - 0, 1 - 1, 0 - 1\rangle = \langle 1, 0, -1\rangle$ lie in the plane, so $\mathbf{a} \times \mathbf{b}$ is a normal vector to the plane. Thus, we can take $\mathbf{n} = \mathbf{a} \times \mathbf{b} = \langle 1 - 0, 0 + 1, 0 + 1\rangle = \langle 1, 1, 1\rangle$. If P_0 is the point $(0, 1, 1)$, an equation of the plane is $1(x - 0) + 1(y - 1) + 1(z - 1) = 0$ or $x + y + z = 2$.

32. Here the vectors $\mathbf{a} = \langle 2, -4, 6\rangle$ and $\mathbf{b} = \langle 5, 1, 3\rangle$ lie in the plane, so $\mathbf{n} = \mathbf{a} \times \mathbf{b} = \langle -12 - 6, 30 - 6, 2 + 20\rangle = \langle -18, 24, 22\rangle$ is a normal vector to the plane and an equation of the plane is $-18(x - 0) + 24(y - 0) + 22(z - 0) = 0$ or $-18x + 24y + 22z = 0$.

33. Here the vectors $\mathbf{a} = \langle 8 - 3, 2 - (-1), 4 - 2\rangle = \langle 5, 3, 2\rangle$ and $\mathbf{b} = \langle -1 - 3, -2 - (-1), -3 - 2\rangle = \langle -4, -1, -5\rangle$ lie in the plane, so a normal vector to the plane is $\mathbf{n} = \mathbf{a} \times \mathbf{b} = \langle -15 + 2, -8 + 25, -5 + 12\rangle = \langle -13, 17, 7\rangle$ and an equation of the plane is $-13(x - 3) + 17[y - (-1)] + 7(z - 2) = 0$ or $-13x + 17y + 7z = -42$.

34. If we first find two nonparallel vectors in the plane, their cross product will be a normal vector to the plane. Since the given line lies in the plane, its direction vector $\mathbf{a} = \langle 3, 1, -1\rangle$ is one vector in the plane. We can verify that the given point $(1, 2, 3)$ does not lie on this line, so to find another nonparallel vector $\mathbf{b}$ which lies in the plane, we can pick any point on the line and find a vector connecting the points. If we put $t = 0$, we see that $(0, 1, 2)$ is on the line, so $\mathbf{b} = \langle 1 - 0, 2 - 1, 3 - 2\rangle = \langle 1, 1, 1\rangle$ and $\mathbf{n} = \mathbf{a} \times \mathbf{b} = \langle 1 + 1, -1 - 3, 3 - 1\rangle = \langle 2, -4, 2\rangle$. Thus, an equation of the plane is $2(x - 1) - 4(y - 2) + 2(z - 3) = 0$ or $2x - 4y + 2z = 0$. (Equivalently, we can write $x - 2y + z = 0$.)

35. If we first find two nonparallel vectors in the plane, their cross product will be a normal vector to the plane. Since the given line lies in the plane, its direction vector $\mathbf{a} = \langle -2, 5, 4\rangle$ is one vector in the plane. We can verify that the given point $(6, 0, -2)$ does not lie on this line, so to find another nonparallel vector $\mathbf{b}$ which lies in the plane, we can pick any point on the line and find a vector connecting the points. If we put $t = 0$, we see that $(4, 3, 7)$ is on the line, so $\mathbf{b} = \langle 6 - 4, 0 - 3, -2 - 7\rangle = \langle 2, -3, -9\rangle$ and $\mathbf{n} = \mathbf{a} \times \mathbf{b} = \langle -45 + 12, 8 - 18, 6 - 10\rangle = \langle -33, -10, -4\rangle$. Thus, an equation of the plane is $-33(x - 6) - 10(y - 0) - 4[z - (-2)] = 0$ or $33x + 10y + 4z = 190$.

36. Since the line $x = 2y = 3z$, or $x = \dfrac{y}{1/2} = \dfrac{z}{1/3}$, lies in the plane, its direction vector $\mathbf{a} = \left\langle 1, \frac{1}{2}, \frac{1}{3}\right\rangle$ is parallel to the plane. The point $(0, 0, 0)$ is on the line (put $t = 0$), and we can verify that the given point $(1, -1, 1)$ in the plane is not on the line. The vector connecting these two points, $\mathbf{b} = \langle 1, -1, 1\rangle$, is therefore parallel to the plane, but not parallel to $\langle 1, 2, 3\rangle$. Then $\mathbf{a} \times \mathbf{b} = \left\langle \frac{1}{2} + \frac{1}{3}, \frac{1}{3} - 1, -1 - \frac{1}{2}\right\rangle = \left\langle \frac{5}{6}, -\frac{2}{3}, -\frac{3}{2}\right\rangle$ is a normal vector to the plane, and an equation of the plane is $\frac{5}{6}(x - 0) - \frac{2}{3}(y - 0) - \frac{3}{2}(z - 0) = 0$ or $5x - 4y - 9z = 0$.

37. A direction vector for the line of intersection is $\mathbf{a} = \mathbf{n}_1 \times \mathbf{n}_2 = \langle 1, 1, -1\rangle \times \langle 2, -1, 3\rangle = \langle 2, -5, -3\rangle$, and $\mathbf{a}$ is parallel to the desired plane. Another vector parallel to the plane is the vector connecting any point on the line of intersection to the given point $(-1, 2, 1)$ in the plane. Setting $x = 0$, the equations of the planes reduce to $y - z = 2$ and $-y + 3z = 1$ with simultaneous solution $y = \frac{7}{2}$ and $z = \frac{3}{2}$. So a point on the line is $\left(0, \frac{7}{2}, \frac{3}{2}\right)$ and another vector parallel to the plane is $\left\langle -1, -\frac{3}{2}, -\frac{1}{2}\right\rangle$. Then a normal vector to the plane is $\mathbf{n} = \langle 2, -5, -3\rangle \times \left\langle -1, -\frac{3}{2}, -\frac{1}{2}\right\rangle = \langle -2, 4, -8\rangle$ and an equation of the plane is $-2(x + 1) + 4(y - 2) - 8(z - 1) = 0$ or $x - 2y + 4z = -1$.

38. $\mathbf{n}_1 = \langle 1, 0, -1\rangle$ and $\mathbf{n}_2 = \langle 0, 1, 2\rangle$. Setting $z = 0$, it is easy to see that $(1, 3, 0)$ is a point on the line of intersection of $x - z = 1$ and $y + 2z = 3$. The direction of this line is $\mathbf{v}_1 = \mathbf{n}_1 \times \mathbf{n}_2 = \langle 1, -2, 1\rangle$. A second vector parallel to the desired plane is $\mathbf{v}_2 = \langle 1, 1, -2\rangle$, since it is perpendicular to $x + y - 2z = 1$. Therefore, a normal of the plane in question is $\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2 = \langle 4 - 1, 1 + 2, 1 + 2\rangle = \langle 3, 3, 3\rangle$, or we can use $\langle 1, 1, 1\rangle$. Taking $(x_0, y_0, z_0) = (1, 3, 0)$, the equation we are looking for is $(x - 1) + (y - 3) + z = 0 \quad \Leftrightarrow \quad x + y + z = 4$.

39. Substitute the parametric equations of the line into the equation of the plane: $(3 - t) - (2 + t) + 2(5t) = 9 \quad \Rightarrow \quad 8t = 8 \quad \Rightarrow \quad t = 1$. Therefore, the point of intersection of the line and the plane is given by $x = 3 - 1 = 2$, $y = 2 + 1 = 3$, and $z = 5(1) = 5$, that is, the point $(2, 3, 5)$.

40. Substitute the parametric equations of the line into the equation of the plane: $(1+2t)+2(4t)-(2-3t)+1=0$ $\Rightarrow$ $13t=0$ $\Rightarrow$ $t=0$. Therefore, the point of intersection of the line and the plane is given by $x=1+2(0)=1$, $y=4(0)=0$, and $z=2-3(0)=2$, that is, the point $(1,0,2)$.

41. Parametric equations for the line are $x=t$, $y=1+t$, $z=\frac{1}{2}t$ and substituting into the equation of the plane gives $4(t)-(1+t)+3\left(\frac{1}{2}t\right)=8$ $\Rightarrow$ $\frac{9}{2}t=9$ $\Rightarrow$ $t=2$. Thus $x=2$, $y=1+2=3$, $z=\frac{1}{2}(2)=1$ and the point of intersection is $(2,3,1)$.

42. A direction vector for the line through $(1,0,1)$ and $(4,-2,2)$ is $\mathbf{v}=\langle 3,-2,1\rangle$ and, taking $P_0=(1,0,1)$, parametric equations for the line are $x=1+3t$, $y=-2t$, $z=1+t$. Substitution of the parametric equations into the equation of the plane gives $1+3t-2t+1+t=6$ $\Rightarrow$ $t=2$. Then $x=1+3(2)=7$, $y=-2(2)=-4$, and $z=1+2=3$ so the point of intersection is $(7,-4,3)$.

43. Setting $x=0$, we see that $(0,1,0)$ satisfies the equations of both planes, so that they do in fact have a line of intersection. $\mathbf{v}=\mathbf{n}_1\times\mathbf{n}_2=\langle 1,1,1\rangle\times\langle 1,0,1\rangle=\langle 1,0,-1\rangle$ is the direction of this line. Therefore, direction numbers of the intersecting line are $1, 0, -1$.

44. The angle between the two planes is the same as the angle between their normal vectors. The normal vectors of the two planes are $\langle 1,1,1\rangle$ and $\langle 1,2,3\rangle$. The cosine of the angle θ between these two planes is

$$\cos\theta=\frac{\langle 1,1,1\rangle\cdot\langle 1,2,3\rangle}{|\langle 1,1,1\rangle|\,|\langle 1,2,3\rangle|}=\frac{1+2+3}{\sqrt{1+1+1}\,\sqrt{1+4+9}}=\frac{6}{\sqrt{42}}=\sqrt{\frac{6}{7}}.$$

45. Normal vectors for the planes are $\mathbf{n}_1=\langle 1,4,-3\rangle$ and $\mathbf{n}_2=\langle -3,6,7\rangle$, so the normals (and thus the planes) aren't parallel. But $\mathbf{n}_1\cdot\mathbf{n}_2=-3+24-21=0$, so the normals (and thus the planes) are perpendicular.

46. Normal vectors for the planes are $\mathbf{n}_1=\langle -1,4,-2\rangle$ and $\mathbf{n}_2=\langle 3,-12,6\rangle$. Since $\mathbf{n}_2=-3\mathbf{n}_1$, the normals (and thus the planes) are parallel.

47. Normal vectors for the planes are $\mathbf{n}_1=\langle 1,1,1\rangle$ and $\mathbf{n}_2=\langle 1,-1,1\rangle$. The normals are not parallel, so neither are the planes. Furthermore, $\mathbf{n}_1\cdot\mathbf{n}_2=1-1+1=1\neq 0$, so the planes aren't perpendicular. The angle between them is given by $\cos\theta=\dfrac{\mathbf{n}_1\cdot\mathbf{n}_2}{|\mathbf{n}_1|\,|\mathbf{n}_2|}=\dfrac{1}{\sqrt{3}\,\sqrt{3}}=\dfrac{1}{3}$ $\Rightarrow$ $\theta=\cos^{-1}\left(\frac{1}{3}\right)\approx 70.5^\circ$.

48. The normals are $\mathbf{n}_1=\langle 2,-3,4\rangle$ and $\mathbf{n}_2=\langle 1,6,4\rangle$ so the planes aren't parallel. Since $\mathbf{n}_1\cdot\mathbf{n}_2=2-18+16=0$, the normals (and thus the planes) are perpendicular.

49. The normals are $\mathbf{n}_1=\langle 1,-4,2\rangle$ and $\mathbf{n}_2=\langle 2,-8,4\rangle$. Since $\mathbf{n}_2=2\mathbf{n}_1$, the normals (and thus the planes) are parallel.

50. The normal vectors are $\mathbf{n}_1=\langle 1,2,2\rangle$ and $\mathbf{n}_2=\langle 2,-1,2\rangle$. The normals are not parallel, so neither are the planes. Furthermore, $\mathbf{n}_1\cdot\mathbf{n}_2=2-2+4=4\neq 0$, so the planes aren't perpendicular. The angle between them is given by

$$\cos\theta=\frac{\mathbf{n}_1\cdot\mathbf{n}_2}{|\mathbf{n}_1|\,|\mathbf{n}_2|}=\frac{4}{\sqrt{9}\,\sqrt{9}}=\frac{4}{9}\quad\Rightarrow\quad\theta=\cos^{-1}\left(\tfrac{4}{9}\right)\approx 63.6^\circ.$$

51. (a) To find a point on the line of intersection, set one of the variables equal to a constant, say $z=0$. (This will only work if the line of intersection crosses the xy-plane; otherwise, try setting x or y equal to 0.) Then the equations of the planes reduce to $x+y=2$ and $3x-4y=6$. Solving these two equations gives $x=2$, $y=0$. So a point on the line of intersection is $(2,0,0)$. The direction of the line is $\mathbf{v}=\mathbf{n}_1\times\mathbf{n}_2=\langle 5-4,-3-5,-4-3\rangle=\langle 1,-8,-7\rangle$, and symmetric equations for the line are $x-2=\dfrac{y}{-8}=\dfrac{z}{-7}$.

(b) The angle between the planes satisfies $\cos\theta = \dfrac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1|\,|\mathbf{n}_2|} = \dfrac{3-4-5}{\sqrt{3}\,\sqrt{50}} = -\dfrac{\sqrt{6}}{5}$. Therefore $\theta = \cos^{-1}\left(-\frac{\sqrt{6}}{5}\right) \approx 119^\circ$ (or 61°).

52. (a) $x - 2y + z = 1 \;\Rightarrow\; \mathbf{n}_1 = \langle 1, -2, 1\rangle$ and $2x + y + z = 1 \;\Rightarrow\; \mathbf{n}_2 = \langle 2, 1, 1\rangle$. The vector that gives the direction of the line of intersection of these two planes is $\mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2 = \langle -2-1, 2-1, 1+4\rangle = \langle -3, 1, 5\rangle$. Setting $x = y = 0$, we see that both planes contain $(0, 0, 1)$ so that this point must lie on their line of intersection. Then symmetric equations for this line are $\dfrac{x}{-3} = y = \dfrac{z-1}{5}$.

(b) $\cos\theta = \dfrac{\mathbf{n}_1 \cdot \mathbf{n}_2}{|\mathbf{n}_1|\,|\mathbf{n}_2|} = \dfrac{2-2+1}{\sqrt{1+4+1}\sqrt{4+1+1}} = \dfrac{1}{6} \;\Rightarrow\; \theta = \cos^{-1}\left(\frac{1}{6}\right) \approx 80^\circ$.

53. Setting $x = 0$, the equations of the two planes become $z = y$ and $5y + z = -1$, which intersect at $y = -\frac{1}{6}$ and $z = -\frac{1}{6}$. Thus we can choose $(x_0, y_0, z_0) = \left(0, -\frac{1}{6}, -\frac{1}{6}\right)$. The vector giving the direction of this intersecting line, $\mathbf{v}$, is perpendicular to the normal vectors of both planes. So $\mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2 = \langle 2, -5, -1\rangle \times \langle 1, 1, -1\rangle = \langle 6, 1, 7\rangle$. Therefore, by Equations 2, parametric equations for this line are $x = 6t$, $y = -\frac{1}{6} + t$, $z = -\frac{1}{6} + 7t$.

54. Setting $y = 0$, the equations of the two planes become $2x + 5z = -3$ and $x + z = -2$, which intersect at $x = -\frac{7}{3}$ and $z = \frac{1}{3}$. Thus we can choose $(x_0, y_0, z_0) = \left(-\frac{7}{3}, 0, \frac{1}{3}\right)$. The vector giving the direction of this intersecting line, $\mathbf{v}$, is perpendicular to the normal vectors of both planes. So $\mathbf{v} = \mathbf{n}_1 \times \mathbf{n}_2 = \langle 2, 0, 5\rangle \times \langle 1, -3, 1\rangle = \langle 15, 5-2, -6\rangle = 3\langle 5, 1, -2\rangle$. Therefore, by Equations 2, parametric equations of the line of intersection of the two planes are $x = -\frac{7}{3} + 5t$, $y = t$, $z = \frac{1}{3} - 2t$.

55. The plane contains all perpendicular bisectors of the line segment joining $(1, 1, 0)$ and $(0, 1, 1)$. All of these bisectors pass through the midpoint of this segment $\left(\frac{1}{2}, \frac{1+1}{2}, \frac{1}{2}\right) = \left(\frac{1}{2}, 1, \frac{1}{2}\right)$. The direction of this line segment $\langle 1-0, 1-1, 0-1\rangle = \langle 1, 0, -1\rangle$ is perpendicular to the plane so that we can choose this to be $\mathbf{n}$. Therefore the equation of the plane is $1\left(x - \frac{1}{2}\right) + 0(y-1) - 1\left(z - \frac{1}{2}\right) = 0 \;\Leftrightarrow\; x = z$.

56. The plane will contain all perpendicular bisectors of the line segment joining the two points. Thus, a point in the plane is $P_0 = (-1, -1, 2)$, the midpoint of the line segment joining the two given points, and a normal to the plane is $\mathbf{n} = \langle 6, -6, 2\rangle$, the vector connecting the two points. So an equation of the plane is $6(x+1) - 6(y+1) + 2(z-2) = 0$ or $3x - 3y + z = 2$.

57. The plane contains the points $(a, 0, 0)$, $(0, b, 0)$ and $(0, 0, c)$. Thus the vectors $\mathbf{a} = \langle -a, b, 0\rangle$ and $\mathbf{b} = \langle -a, 0, c\rangle$ lie in the plane, and $\mathbf{n} = \mathbf{a} \times \mathbf{b} = \langle bc - 0, 0 + ac, 0 + ab\rangle = \langle bc, ac, ab\rangle$ is a normal vector to the plane. The equation of the plane is therefore $bcx + acy + abz = abc + 0 + 0$ or $bcx + acy + abz = abc$. Notice that if $a \neq 0$, $b \neq 0$ and $c \neq 0$ then we can rewrite the equation as $\dfrac{x}{a} + \dfrac{y}{b} + \dfrac{z}{c} = 1$. This is a good equation to remember!

58. (a) For the lines to intersect, we must be able to find one value of t and one value of s satisfying the three equations $1 + t = 2 - s$, $1 - t = s$ and $2t = 2$. From the third we get $t = 1$, and putting this in the second gives $s = 0$. These values of s and t do satisfy the first equation, so the lines intersect at the point $P_0 = (1+1, 1-1, 2(1)) = (2, 0, 2)$.

(b) The direction vectors of the lines are $\langle 1, -1, 2\rangle$ and $\langle -1, 1, 0\rangle$, so a normal vector for the plane is $\langle -1, 1, 0\rangle \times \langle 1, -1, 2\rangle = \langle 2, 2, 0\rangle$ and it contains the point $(2, 0, 2)$. Then the equation of the plane is $2(x-2) + 2(y-0) + 0(z-2) = 0 \;\Leftrightarrow\; x + y = 2$.

59. Two vectors which are perpendicular to the required line are the normal of the given plane, $\langle 1,1,1\rangle$, and a direction vector for the given line, $\langle 1,-1,2\rangle$. So a direction vector for the required line is $\langle 1,1,1\rangle \times \langle 1,-1,2\rangle = \langle 3,-1,-2\rangle$. Thus L is given by $\langle x,y,z\rangle = \langle 0,1,2\rangle + t\langle 3,-1,-2\rangle$, or in parametric form, $x = 3t$, $y = 1-t$, $z = 2-2t$.

60. Let L be the given line. Then $(1,1,0)$ is the point on L corresponding to $t=0$. L is in the direction of $\mathbf{a} = \langle 1,-1,2\rangle$ and $\mathbf{b} = \langle -1,0,2\rangle$ is the vector joining $(1,1,0)$ and $(0,1,2)$. Then

$\mathbf{b} - \operatorname{proj}_{\mathbf{a}} \mathbf{b} = \langle -1,0,2\rangle - \dfrac{\langle 1,-1,2\rangle \cdot \langle -1,0,2\rangle}{1^2+(-1)^2+2^2}\langle 1,-1,2\rangle = \langle -1,0,2\rangle - \frac{1}{2}\langle 1,-1,2\rangle = \left\langle -\frac{3}{2},\frac{1}{2},1\right\rangle$ is a direction vector for the required line. Thus $2\left\langle -\frac{3}{2},\frac{1}{2},1\right\rangle = \langle -3,1,2\rangle$ is also a direction vector, and the line has parametric equations $x = -3t$, $y = 1+t$, $z = 2+2t$. (Notice that this is the same line as in Exercise 59.)

61. Let P_i have normal vector $\mathbf{n}_i$. Then $\mathbf{n}_1 = \langle 4,-2,6\rangle$, $\mathbf{n}_2 = \langle 4,-2,-2\rangle$, $\mathbf{n}_3 = \langle -6,3,-9\rangle$, $\mathbf{n}_4 = \langle 2,-1,-1\rangle$. Now $\mathbf{n}_1 = -\frac{2}{3}\mathbf{n}_3$, so $\mathbf{n}_1$ and $\mathbf{n}_3$ are parallel, and hence P_1 and P_3 are parallel; similarly P_2 and P_4 are parallel because $\mathbf{n}_2 = 2\mathbf{n}_4$. However, $\mathbf{n}_1$ and $\mathbf{n}_2$ are not parallel. $\left(0,0,\frac{1}{2}\right)$ lies on P_1, but not on P_3, so they are not the same plane, but both P_2 and P_4 contain the point $(0,0,-3)$, so these two planes are identical.

62. Let L_i have direction vector $\mathbf{v}_i$. Then $\mathbf{v}_1 = \langle 1,1,-5\rangle$, $\mathbf{v}_2 = \langle 1,1,-1\rangle$, $\mathbf{v}_3 = \langle 1,1,-1\rangle$, $\mathbf{v}_4 = \langle 2,2,-10\rangle$. $\mathbf{v}_2$ and $\mathbf{v}_3$ are equal so they're parallel. $\mathbf{v}_4 = 2\mathbf{v}_1$, so L_4 and L_1 are parallel. L_3 contains the point $(1,4,1)$, but this point does not lie on L_2, so they're not equal. $(2,1,-3)$ lies on L_4, and on L_1, with $t=1$. So L_1 and L_4 are identical.

63. Let $Q = (2,2,0)$ and $R = (3,-1,5)$, points on the line corresponding to $t=0$ and $t=1$. Let $P = (1,2,3)$. Then $\mathbf{a} = \overrightarrow{QR} = \langle 1,-3,5\rangle$, $\mathbf{b} = \overrightarrow{QP} = \langle -1,0,3\rangle$. The distance is

$$d = \frac{|\mathbf{a}\times\mathbf{b}|}{|\mathbf{a}|} = \frac{|\langle 1,-3,5\rangle \times \langle -1,0,3\rangle|}{|\langle 1,-3,5\rangle|} = \frac{|\langle -9,-8,-3\rangle|}{|\langle 1,-3,5\rangle|} = \frac{\sqrt{9^2+8^2+3^2}}{\sqrt{1^2+3^2+5^2}} = \frac{\sqrt{154}}{\sqrt{35}} = \sqrt{\frac{22}{5}}.$$

64. Let $Q = (5,0,1)$ and $R = (4,3,3)$, points on the line corresponding to $t=0$ and $t=1$. Let $P = (1,0,-1)$. Then $\mathbf{a} = \overrightarrow{QR} = \langle -1,3,2\rangle$ and $\mathbf{b} = \overrightarrow{QP} = \langle -4,0,-2\rangle$. The distance is

$$d = \frac{|\mathbf{a}\times\mathbf{b}|}{|\mathbf{a}|} = \frac{|\langle -1,3,2\rangle \times \langle -4,0,-2\rangle|}{|\langle -1,3,2\rangle|} = \frac{|\langle -6,-10,12\rangle|}{|\langle -1,3,2\rangle|} = \frac{2\sqrt{3^2+5^2+6^2}}{\sqrt{1^2+3^2+2^2}} = \frac{2\sqrt{70}}{\sqrt{14}} = 2\sqrt{5}.$$

65. By Equation 9, the distance is $D = \dfrac{1}{\sqrt{1+4+4}}\left[(1)(2)+(-2)(8)+(-2)(5)-1\right] = \dfrac{25}{3}$.

66. By Equation 9, the distance is $D = \dfrac{1}{\sqrt{16+36+1}}\left[4(3)+(-6)(-2)+1(7)-5\right] = \dfrac{26}{\sqrt{53}}$.

67. Put $y=z=0$ in the equation of the first plane to get the point $(-1,0,0)$ on the plane. Because the planes are parallel, the distance D between them is the distance from $(-1,0,0)$ to the second plane. By Equation 9,

$$D = \frac{|3(-1)+6(0)-3(0)-4|}{\sqrt{3^2+6^2+(-3)^2}} = \frac{7}{3\sqrt{6}} \text{ or } \frac{7\sqrt{6}}{18}.$$

68. Put $y=z=0$ in the equation of the first plane to get the point $\left(\frac{4}{3},0,0\right)$ on the plane. Because the planes are parallel the distance D between them is the distance from $\left(\frac{4}{3},0,0\right)$ to the second plane. By Equation 9,

$$D = \frac{\left|1\left(\frac{4}{3}\right)+2(0)-3(0)-1\right|}{\sqrt{1^2+2^2+(-3)^2}} = \frac{1}{3\sqrt{14}}.$$

69. The distance between two parallel planes is the same as the distance between a point on one of the planes and the other plane. Let $P_0 = (x_0, y_0, z_0)$ be a point on the plane given by $ax + by + cz + d_1 = 0$. Then $ax_0 + by_0 + cz_0 + d_1 = 0$ and the distance between P_0 and the plane given by $ax + by + cz + d_2 = 0$ is, from Equation 9, $D = \dfrac{|ax_0 + by_0 + cz_0 + d_2|}{\sqrt{a^2 + b^2 + c^2}} = \dfrac{|-d_1 + d_2|}{\sqrt{a^2 + b^2 + c^2}} = \dfrac{|d_1 - d_2|}{\sqrt{a^2 + b^2 + c^2}}$.

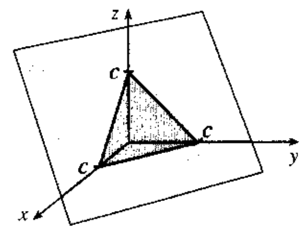
70. The planes must have parallel normal vectors, so if $ax + by + cz + d = 0$ is such a plane, then for some $t \neq 0$, $\langle a, b, c\rangle = t\langle 1, 2, -2\rangle = \langle t, 2t, -2t\rangle$. So this plane is given by the equation $x + 2y - 2z + e = 0$, where $e = d/t$. By Exercise 69, the distance between the planes is $2 = \dfrac{|1 - e|}{\sqrt{1^2 + 2^2 + (-2)^2}} \quad\Leftrightarrow\quad 6 = |1 - e| \quad\Leftrightarrow\quad e = 7$ or -5.
So the desired planes have equations $x + 2y - 2z = 7$ and $x + 2y - 2z = -5$.

71. L_1: $x = y = z \quad\Rightarrow\quad x = y$ (1). L_2: $x + 1 = y/2 = z/3 \quad\Rightarrow\quad x + 1 = y/2$ (2). The solution of (1) and (2) is $x = y = -2$. However, when $x = -2$, $x = z \quad\Rightarrow\quad z = -2$, but $x + 1 = z/3 \quad\Rightarrow\quad z = -3$, a contradiction. Hence the lines do not intersect. For L_1, $\mathbf{v}_1 = \langle 1, 1, 1\rangle$, and for L_2, $\mathbf{v}_2 = \langle 1, 2, 3\rangle$, so the lines are not parallel. Thus the lines are skew lines. If two lines are skew, they can be viewed as lying in two parallel planes and so the distance between the skew lines would be the same as the distance between these parallel planes. The common normal vector to the planes must be perpendicular to both $\langle 1, 1, 1\rangle$ and $\langle 1, 2, 3\rangle$, the direction vectors of the two lines. So set $\mathbf{n} = \langle 1, 1, 1\rangle \times \langle 1, 2, 3\rangle = \langle 3 - 2, -3 + 1, 2 - 1\rangle = \langle 1, -2, 1\rangle$. From above, we know that $(-2, -2, -2)$ and $(-2, -2, -3)$ are points of L_1 and L_2 respectively. So in the notation of Equation 8, $1(-2) - 2(-2) + 1(-2) + d_1 = 0 \quad\Rightarrow\quad d_1 = 0$ and $1(-2) - 2(-2) + 1(-3) + d_2 = 0 \quad\Rightarrow\quad d_2 = 1$.
By Exercise 69, the distance between these two skew lines is $D = \dfrac{|0 - 1|}{\sqrt{1 + 4 + 1}} = \dfrac{1}{\sqrt{6}}$.
Alternate solution (without reference to planes): A vector which is perpendicular to both of the lines is $\mathbf{n} = \langle 1, 1, 1\rangle \times \langle 1, 2, 3\rangle = \langle 1, -2, 1\rangle$. Pick any point on each of the lines, say $(-2, -2, -2)$ and $(-2, -2, -3)$, and form the vector $\mathbf{b} = \langle 0, 0, 1\rangle$ connecting the two points. The distance between the two skew lines is the absolute value of the scalar projection of $\mathbf{b}$ along $\mathbf{n}$, that is, $D = \dfrac{|\mathbf{n} \cdot \mathbf{b}|}{|\mathbf{n}|} = \dfrac{|1 \cdot 0 - 2 \cdot 0 + 1 \cdot 1|}{\sqrt{1 + 4 + 1}} = \dfrac{1}{\sqrt{6}}$.

72. First notice that if two lines are skew, they can be viewed as lying in two parallel planes and so the distance between the skew lines would be the same as the distance between these parallel planes. The common normal vector to the planes must be perpendicular to both $\mathbf{v}_1 = \langle 1, 6, 2\rangle$ and $\mathbf{v}_2 = \langle 2, 15, 6\rangle$, the direction vectors of the two lines respectively. Thus set $\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2 = \langle 36 - 30, 4 - 6, 15 - 12\rangle = \langle 6, -2, 3\rangle$. Setting $t = 0$ and $s = 0$ gives the points $(1, 1, 0)$ and $(1, 5, -2)$. So in the notation of Equation 8, $6 - 2 + 0 + d_1 = 0 \quad\Rightarrow\quad d_1 = -4$ and $6 - 10 - 6 + d_2 = 0 \quad\Rightarrow\quad d_2 = 10$. Then by Exercise 69, the distance between the two skew lines is given by
$$D = \frac{|-4 - 10|}{\sqrt{36 + 4 + 9}} = \frac{14}{7} = 2.$$
Alternate solution (without reference to planes): We already know that the direction vectors of the two lines are $\mathbf{v}_1 = \langle 1, 6, 2\rangle$ and $\mathbf{v}_2 = \langle 2, 15, 6\rangle$. Then $\mathbf{n} = \mathbf{v}_1 \times \mathbf{v}_2 = \langle 6, -2, 3\rangle$ is perpendicular to both lines. Pick any point on each of the lines, say $(1, 1, 0)$ and $(1, 5, -2)$, and form the vector $\mathbf{b} = \langle 0, 4, -2\rangle$ connecting the two points. Then the distance between the two skew lines is the absolute value of the scalar projection of $\mathbf{b}$ along $\mathbf{n}$, that is,
$$D = \frac{|\mathbf{n} \cdot \mathbf{b}|}{|\mathbf{n}|} = \frac{1}{\sqrt{36 + 4 + 9}}\,|0 - 8 - 6| = \frac{14}{7} = 2.$$

73. If $a \neq 0$, then $ax + by + cz + d = 0 \quad \Rightarrow \quad a(x + d/a) + b(y - 0) + c(z - 0) = 0$ which by (7) is the scalar equation of the plane through the point $(-d/a, 0, 0)$ with normal vector $\langle a, b, c \rangle$. Similarly, if $b \neq 0$ (or if $c \neq 0$) the equation of the plane can be rewritten as $a(x - 0) + b(y + d/b) + c(z - 0) = 0$ [or as $a(x - 0) + b(y - 0) + c(z + d/c) = 0$] which by (7) is the scalar equation of a plane through the point $(0, -d/b, 0)$ [or the point $(0, 0, -d/c)$] with normal vector $\langle a, b, c \rangle$.

74. (a) The planes $x + y + z = c$ have normal vector $\langle 1, 1, 1 \rangle$, so they are all parallel. Their x-, y-, and z-intercepts are all c. When $c > 0$ their intersection with the first octant is an equilateral triangle and when $c < 0$ their intersection with the octant diagonally opposite the first is an equilateral triangle.



(b) The planes $x + y + cz = 1$ have x-intercept 1, y-intercept 1, and z-intercept $1/c$. The plane with $c = 0$ is parallel to the z-axis. As c gets larger, the planes get closer to the xy-plane.

(c) The planes $y \cos \theta + z \cos \theta = 1$ have normal vectors $\langle 0, \cos \theta, \sin \theta \rangle$, which are perpendicular to the x-axis, and so the planes are parallel to the x-axis. We look at their intersection with the yz-plane. These are lines that are perpendicular to $\langle \cos \theta, \sin \theta \rangle$ and pass through $(\cos \theta, \sin \theta)$, since $\cos^2 \theta + \sin^2 \theta = 1$. So these are the tangent lines to the unit circle. Thus the family consists of all planes tangent to the circular cylinder with radius 1 and axis the x-axis.

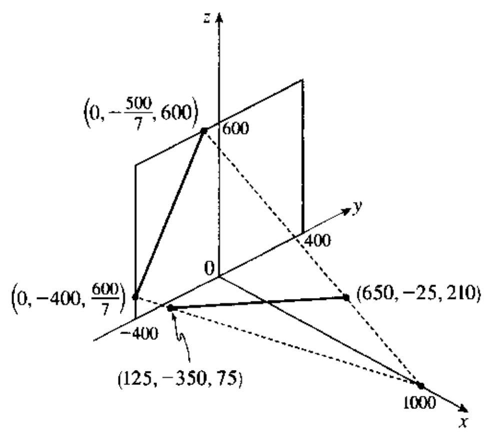
LABORATORY PROJECT Putting 3D in Perspective

1. If we view the screen from the camera's location, the vertical clipping plane on the left passes through the points $(1000, 0, 0)$, $(0, -400, 0)$, and $(0, -400, 600)$. A vector from the first point to the second is $\mathbf{v}_1 = \langle -1000, -400, 0 \rangle$ and a vector from the first point to the third is $\mathbf{v}_2 = \langle -1000, -400, 600 \rangle$. A normal vector for the clipping plane is $\mathbf{v}_1 \times \mathbf{v}_2 = -240{,}000\,\mathbf{i} + 600{,}000\,\mathbf{j}$ or $-2\,\mathbf{i} + 5\,\mathbf{j}$, and an equation for the plane is $-2(x - 1000) + 5(y - 0) + 0(z - 0) = 0 \quad \Rightarrow \quad 2x - 5y = 2000$. By symmetry, the vertical clipping plane on the right is given by $2x + 5y = 2000$. The lower clipping plane is $z = 0$. The upper clipping plane passes through the points $(1000, 0, 0)$, $(0, -400, 600)$, and $(0, 400, 600)$. Vectors from the first point to the second and third points are $\mathbf{v}_1 = \langle -1000, -400, 600 \rangle$ and $\mathbf{v}_2 = \langle -1000, 400, 600 \rangle$, and a normal vector for the plane is $\mathbf{v}_1 \times \mathbf{v}_2 = -480{,}000\,\mathbf{i} - 800{,}000\,\mathbf{k}$ or $3\,\mathbf{i} + 5\,\mathbf{k}$. An equation for the plane is $3(x - 1000) + 0(y - 0) + 5(z - 0) = 0 \quad \Rightarrow \quad 3x + 5z = 3000$.

A direction vector for the line L is $\mathbf{v} = \langle 630, 390, 162 \rangle$ and taking $P_0 = (230, -285, 102)$, parametric equations are $x = 230 + 630t$, $y = -285 + 390t$, $z = 102 + 162t$. L intersects the left clipping plane when $2(230 + 630t) - 5(-285 + 390t) = 2000 \quad \Rightarrow \quad t = -\frac{1}{6}$. The corresponding point is $(125, -350, 75)$. L intersects the right clipping plane when $2(230 + 630t) + 5(-285 + 390t) = 2000 \quad \Rightarrow \quad t = \frac{593}{642}$. The corresponding point is approximately $(811.9, 75.2, 251.6)$, but this point is not contained within the viewing volume. L intersects the upper clipping plane when $3(230 + 630t) + 5(102 + 162t) = 3000 \quad \Rightarrow \quad t = \frac{2}{3}$, corresponding to the point $(650, -25, 210)$, and L intersects the lower clipping plane when $z = 0 \quad \Rightarrow$ $102 + 162t = 0 \quad \Rightarrow \quad t = -\frac{17}{27}$. The corresponding point is approximately $(-166.7, -530.6, 0)$, which is not contained within the viewing volume. Thus L should be clipped at the points $(125, -350, 75)$ and $(650, -25, 210)$.

2. A sight line from the camera at $(1000, 0, 0)$ to the left endpoint $(125, -350, 75)$ of the clipped line has direction $\mathbf{v} = \langle -875, -350, 75 \rangle$. Parametric equations are $x = 1000 - 875t$, $y = -350t$, $z = 75t$. This line intersects the screen when $x = 0 \Rightarrow 1000 - 875t = 0 \Rightarrow t = \frac{8}{7}$, corresponding to the point $\left(0, -400, \frac{600}{7}\right)$. Similarly, a sight line from the camera to the right endpoint $(650, -25, 210)$ of the clipped line has direction $\langle -350, -25, 210 \rangle$ and parametric equations are $x = 1000 - 350t$, $y = -25t$, $z = 210t$. $x = 0 \Rightarrow 1000 - 350t = 0 \Rightarrow t = \frac{20}{7}$, corresponding to the point $\left(0, -\frac{500}{7}, 600\right)$. Thus the projection of the clipped line is the line segment between the points $\left(0, -400, \frac{600}{7}\right)$ and $\left(0, -\frac{500}{7}, 600\right)$.

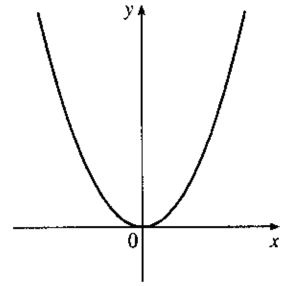
3. From Equation 13.5.4 [ET 12.5.4], equations for the four sides of the screen are $\mathbf{r}_1(t) = (1-t)\langle 0, -400, 0 \rangle + t\,\langle 0, -400, 600 \rangle$, $\mathbf{r}_2(t) = (1-t)\langle 0, -400, 600 \rangle + t\,\langle 0, 400, 600 \rangle$, $\mathbf{r}_3(t) = (1-t)\langle 0, 400, 0 \rangle + t\,\langle 0, 400, 600 \rangle$, and $\mathbf{r}_4(t) = (1-t)\langle 0, -400, 0 \rangle + t\,\langle 0, 400, 0 \rangle$. The clipped line segment connects the points $(125, -350, 75)$ and $(650, -25, 210)$, so an equation for the segment is $\mathbf{r}_5(t) = (1-t)\langle 125, -350, 75 \rangle + t\,\langle 650, -25, 210 \rangle$. The projection of the clipped segment connects the points $\left(0, -400, \frac{600}{7}\right)$ and $\left(0, -\frac{500}{7}, 600\right)$, so an equation is $\mathbf{r}_6(t) = (1-t)\left\langle 0, -400, \frac{600}{7} \right\rangle + t\left\langle 0, -\frac{500}{7}, 600 \right\rangle$. The sight line on the left connects the points $(1000, 0, 0)$ and $\left(0, -400, \frac{600}{7}\right)$, so an equation is $\mathbf{r}_7(t) = (1-t)\langle 1000, 0, 0 \rangle + t\left\langle 0, -400, \frac{600}{7} \right\rangle$. The other sight line connects $(1000, 0, 0)$ to $\left(0, -\frac{500}{7}, 600\right)$, so an equation is $\mathbf{r}_8(t) = (1-t)\langle 1000, 0, 0 \rangle + t\left\langle 0, -\frac{500}{7}, 600 \right\rangle$.



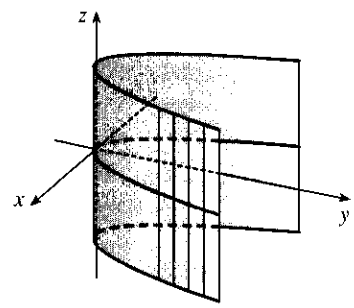
4. The vector from $(621, -147, 206)$ to $(563, 31, 242)$, $\mathbf{v}_1 = \langle -58, 178, 36 \rangle$, lies in the plane of the rectangle, as does the vector from $(621, -147, 206)$ to $(657, -111, 86)$, $\mathbf{v}_2 = \langle 36, 36, -120 \rangle$. A normal vector for the plane is $\mathbf{v}_1 \times \mathbf{v}_2 = \langle -1888, -142, -708 \rangle$ or $\langle 8, 2, 3 \rangle$, and an equation of the plane is $8x + 2y + 3z = 5292$. The line L intersects this plane when $8(230 + 630t) + 2(-285 + 390t) + 3(102 + 162t) = 5292 \Rightarrow t = \frac{1858}{3153} \approx 0.589$. The corresponding point is approximately $(601.25, -55.18, 197.46)$. Starting at this point, a portion of the line is hidden behind the rectangle. The line becomes visible again at the left edge of the rectangle, specifically the edge between the points $(621, -147, 206)$ and $(657, -111, 86)$. (This is most easily determined by graphing the rectangle and the line.) A plane through these two points and the camera's location, $(1000, 0, 0)$, will clip the line at the point it becomes visible. Two vectors in this plane are $\mathbf{v}_1 = \langle -379, -147, 206 \rangle$ and $\mathbf{v}_2 = \langle -343, -111, 86 \rangle$. A normal vector for the plane is $\mathbf{v}_1 \times \mathbf{v}_2 = \langle 10224, -38064, -8352 \rangle$ and an equation of the plane is $213x - 793y - 174z = 213{,}000$. L intersects this plane when $213(230 + 630t) - 793(-285 + 390t) - 174(102 + 162t) = 213{,}000 \Rightarrow t = \frac{44{,}247}{203{,}268} \approx 0.2177$. The corresponding point is approximately $(367.14, -200.11, 137.26)$. Thus the portion of L that should be removed is the segment between the points $(601.25, -55.18, 197.46)$ and $(367.14, -200.11, 137.26)$.

13.6 Cylinders and Quadric Surfaces ET 12.6

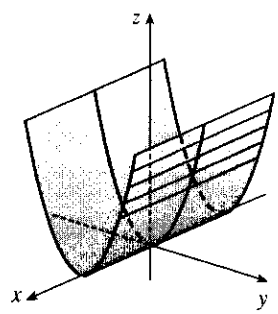
1. (a) In $\mathbb{R}^2$, the equation $y = x^2$ represents a parabola.



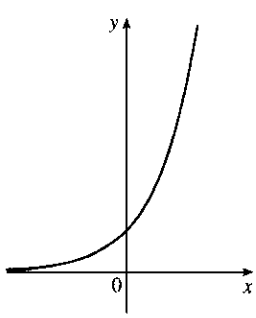
(b) In $\mathbb{R}^3$, the equation $y = x^2$ doesn't involve z, so any horizontal plane with equation $z = k$ intersects the graph in a curve with equation $y = x^2$. Thus, the surface is a parabolic cylinder, made up of infinitely many shifted copies of the same parabola. The rulings are parallel to the z-axis.



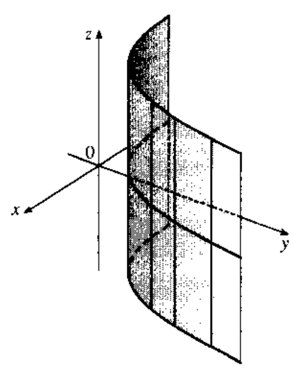
(c) In $\mathbb{R}^3$, the equation $z = y^2$ also represents a parabolic cylinder. Since x doesn't appear, the graph is formed by moving the parabola $z = y^2$ in the direction of the x-axis. Thus, the rulings of the cylinder are parallel to the x-axis.



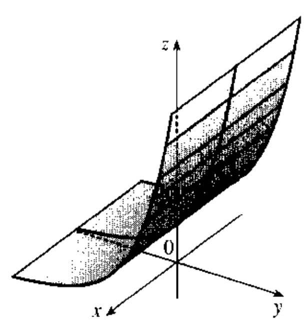
2. (a)



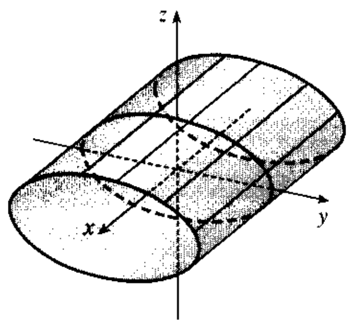
(b) Since the equation $y = e^x$ doesn't involve z, horizontal traces are copies of the curve $y = e^x$. The rulings are parallel to the z-axis.



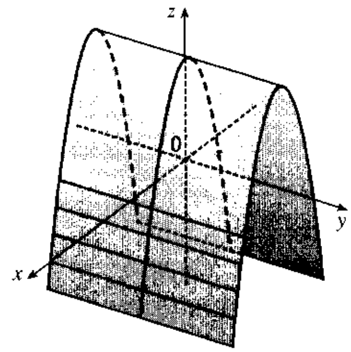
(c) The equation $z = e^y$ doesn't involve x, so vertical traces in $x = k$ (parallel to the yz-plane) are copies of the curve $z = e^y$. The rulings are parallel to the x-axis.



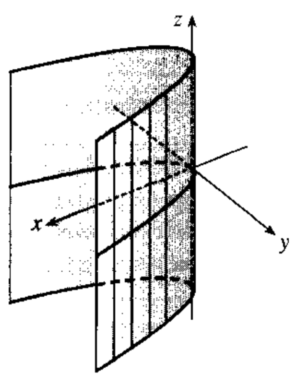
3. Since x is missing from the equation, the vertical traces $y^2 + 4z^2 = 4$, $x = k$, are copies of the same ellipse in the plane $x = k$. Thus, the surface $y^2 + 4z^2 = 4$ is an elliptic cylinder with rulings parallel to the x-axis.



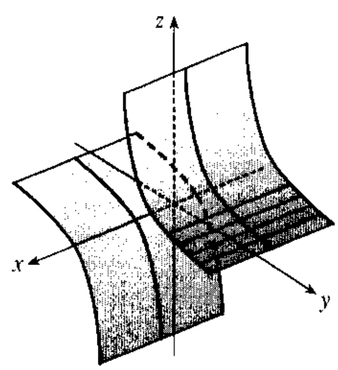
4. Since y is missing from the equation, each vertical trace $z = 4 - x^2$, $y = k$, is a copy of the same parabola in the plane $y = k$. Thus, the surface $z = 4 - x^2$ is a parabolic cylinder with rulings parallel to the y-axis.



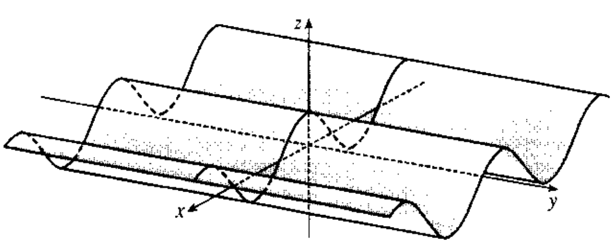
5. Since z is missing, each horizontal trace $x = y^2$, $z = k$, is a copy of the same parabola in the plane $z = k$. Thus, the surface $x - y^2 = 0$ is a parabolic cylinder with rulings parallel to the z-axis.



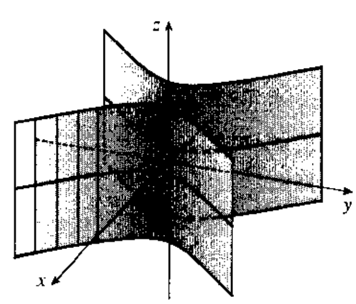
6. Since x is missing, each vertical trace $yz = 4$, $x = k$ is a copy of the same hyperbola in the plane $x = k$. Thus, the surface $yz = 4$ is a hyperbolic cylinder with rulings parallel to the x-axis.



7. Since y is missing, each vertical trace $z = \cos x$, $y = k$ is a copy of a cosine curve in the plane $y = k$. Thus, the surface $z = \cos x$ is a cylindrical surface with rulings parallel to the y-axis.

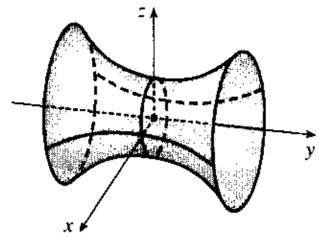


8. Since z is missing, each horizontal trace $x^2 - y^2 = 1$, $z = k$ is a copy of the same hyperbola in the plane $z = k$. Thus, the surface $x^2 - y^2 = 1$ is a hyperbolic cylinder with rulings parallel to the z-axis.

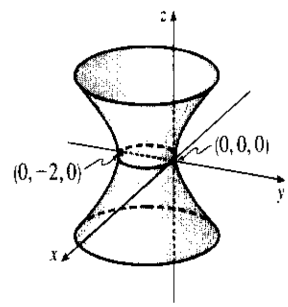


9. (a) The traces of $x^2 + y^2 - z^2 = 1$ in $x = k$ are $y^2 - z^2 = 1 - k^2$, a family of hyperbolas. (Note that the hyperbolas are oriented differently for $-1 < k < 1$ than for $k < -1$ or $k > 1$.) The traces in $y = k$ are $x^2 - z^2 = 1 - k^2$, a similar family of hyperbolas. The traces in $z = k$ are $x^2 + y^2 = 1 + k^2$, a family of circles. For $k = 0$, the trace in the xy-plane, the circle is of radius 1. As $|k|$ increases, so does the radius of the circle. This behavior, combined with the hyperbolic vertical traces, gives the graph of the hyperboloid of one sheet in Table 1.

(b) The shape of the surface is unchanged, but the hyperboloid is rotated so that its axis is the y-axis. Traces in $y = k$ are circles, while traces in $x = k$ and $z = k$ are hyperbolas.

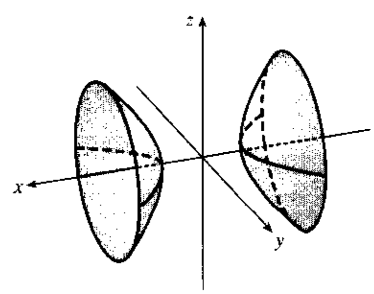


(c) Completing the square in y gives $x^2 + (y + 1)^2 - z^2 = 1$. The surface is a hyperboloid identical to the one in part (a) but shifted one unit in the negative y-direction.

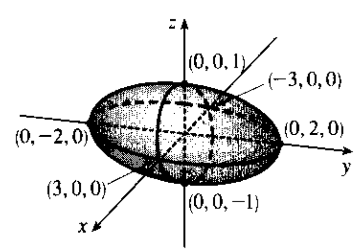


10. (a) The traces of $-x^2 - y^2 + z^2 = 1$ in $x = k$ are $-y^2 + z^2 = 1 + k^2$, a family of hyperbolas, as are the traces in $y = k$, $-x^2 + z^2 = 1 + k^2$. The traces in $z = k$ are $x^2 + y^2 = k^2 - 1$, a family of circles for $|k| > 1$. As $|k|$ increases, the radii of the circles increase; the traces are empty for $|k| < 1$. This behavior, combined with the vertical traces, gives the graph of the hyperboloid of two sheets in Table 1.

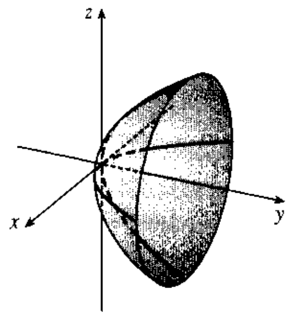
(b) The graph has the same shape as the hyperboloid in part (a) but is rotated so that its axis is the x-axis. Traces in $x = k$, $|k| > 1$, are circles, while traces in $y = k$ and $z = k$ are hyperbolas.



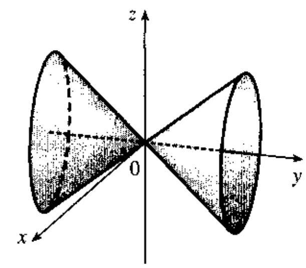
11. Traces: $x = k$, $9y^2 + 36z^2 = 36 - 4k^2$, an ellipse for $|k| < 3$; $y = k$, $4x^2 + 36z^2 = 36 - 9k^2$, an ellipse for $|k| < 2$; $z = k$, $4x^2 + 9y^2 = 36(1 - k^2)$, an ellipse for $|k| < 1$. Thus the surface is an ellipsoid with center at the origin and axes along the x-, y- and z-axes.



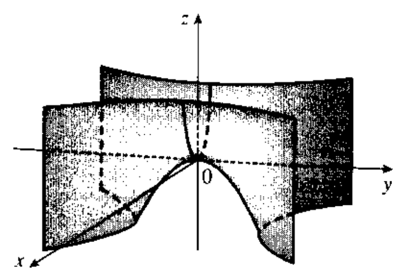
12. Traces: $x = k$, $4y = k^2 + z^2$, a parabola; $y = k$, $4k = x^2 + z^2$, a circle for $k > 0$; $z = k$, $4y = x^2 + k^2$ a parabola. Thus the surface is a circular paraboloid with axis the y-axis and vertex at $(0, 0, 0)$.



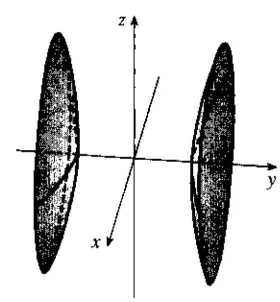
13. Traces: $x = k$, $y^2 = k^2 + z^2$ or $y^2 - z^2 = k^2$, a hyperbola for $k \neq 0$ and two intersecting lines for $k = 0$; $y = k$, $x^2 + z^2 = k^2$, a circle for $k \neq 0$; $z = k$, $y^2 = x^2 + k^2$ or $y^2 - x^2 = k^2$, a hyperbola for $k \neq 0$ and two intersecting lines for $k = 0$. Thus the surface is a cone (right circular) with axis the y-axis and vertex the origin.



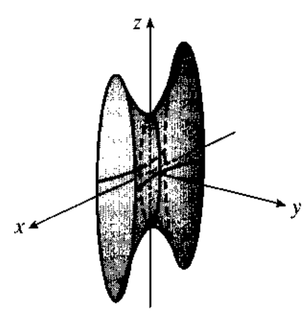
14. Traces: $x = k$, $z - k^2 = -y^2$, a parabola; $y = k$, $z + k^2 = x^2$, a parabola; $z = k$, $x^2 - y^2 = k$, a hyperbola. Thus the surface is a hyperbolic paraboloid with saddle point $(0, 0, 0)$ (and since $c > 0$, the saddle is upside down).



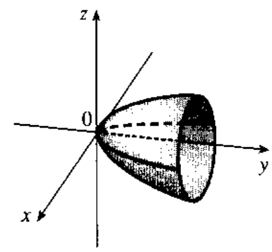
15. Traces: $x = k$, $4y^2 - z^2 = 4 + k^2$, a hyperbola; $y = k$, $x^2 + z^2 = 4k^2 - 4$, a circle for $|k| > 1$; $z = k$, $4y^2 - x^2 = 4 + k^2$, a hyperbola. Thus the surface is a hyperboloid of two sheets with axis the y-axis.



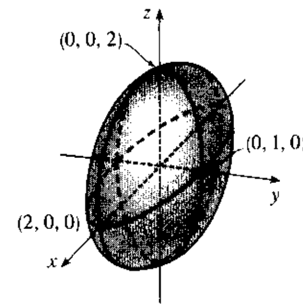
16. Traces: $x = k$, $25y^2 + z^2 = 100 + 4k^2$, an ellipse; $y = k$, $25k^2 + z^2 = 100 + 4x^2$ or $z^2 - 4x^2 = 100 - 25k^2$, a hyperbola for $|k| < 2$; $z = k$, $25y^2 + k^2 = 100 + 4x^2$ or $25y^2 - 4x^2 = 100 - k^2$, a hyperbola for $|k| < 10$. Thus the surface is a hyperboloid of one sheet with axis the x-axis.



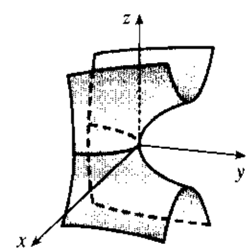
17. Traces: $x = k$, $k^2 + 4z^2 - y = 0$ or $y - k^2 = 4z^2$, a parabola; $y = k$, $x^2 + 4z^2 = k$, an ellipse for $k > 0$; $z = k$, $x^2 + 4k^2 - y = 0$ or $y - 4k^2 = x^2$, a parabola. Thus the surface is an elliptic paraboloid with axis the y-axis and vertex the origin.



18. Traces: $x = k$, $|k| \le 2 \;\Rightarrow\; y^2 + \dfrac{z^2}{4} = 1 - \dfrac{k^2}{4}$, ellipses; $y = k$, $|k| \le 1 \;\Rightarrow\; x^2 + z^2 = 4\left(1 - k^2\right)$, circles; $z = k$, $|k| \le 2 \;\Rightarrow\; \dfrac{x^2}{4} + y^2 = 1 - \dfrac{k^2}{4}$, ellipses. $x^2 + 4y^2 + z^2 = 4 \;\Leftrightarrow\; \dfrac{x^2}{2^2} + \dfrac{y^2}{1^2} + \dfrac{z^2}{2^2} = 1$, which is the equation of an ellipsoid.

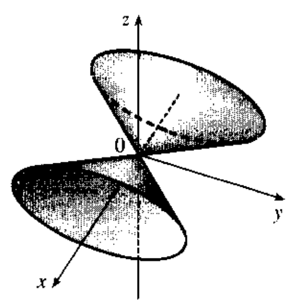


19. $y = z^2 - x^2$. The traces in $x = k$ are the parabolas $y = z^2 - k^2$; the traces in $y = k$ are $k = z^2 - x^2$, which are hyperbolas (note the hyperbolas are oriented differently for $k > 0$ than for $k < 0$); and the traces in $z = k$ are the parabolas $y = k^2 - x^2$. Thus, $\dfrac{y}{1} = \dfrac{z^2}{1^2} - \dfrac{x^2}{1^2}$ is a hyperbolic paraboloid.



20. Traces: $x = k \;\Rightarrow\; y^2 + 4z^2 = 16k^2$, ellipses; $y = k \;\Rightarrow\; 16x^2 - 4z^2 = k^2$, hyperbolas if $k \ne 0$ and two intersecting lines if $k = 0$; $z = k \;\Rightarrow\; 16x^2 - y^2 = 4k^2$, hyperbolas if $k \ne 0$ and two intersecting lines if $k = 0$.

$16x^2 = y^2 + 4z^2 \;\Leftrightarrow\; x^2 = \dfrac{y^2}{4^2} + \dfrac{z^2}{2^2}$ is an elliptic cone with axis the x-axis and vertex the origin.



21. This is the equation of an ellipsoid: $x^2 + 4y^2 + 9z^2 = x^2 + \dfrac{y^2}{(1/2)^2} + \dfrac{z^2}{(1/3)^2} = 1$, with x-intercepts ± 1, y-intercepts $\pm\frac{1}{2}$ and z-intercepts $\pm\frac{1}{3}$. So the major axis is the x-axis and the only possible graph is VII.

22. This is the equation of an ellipsoid: $9x^2 + 4y^2 + z^2 = \dfrac{x^2}{(1/3)^2} + \dfrac{y^2}{(1/2)^2} + z^2 = 1$, with x-intercepts $\pm\frac{1}{3}$, y-intercepts $\pm\frac{1}{2}$ and z-intercepts ± 1. So the major axis is the z-axis and the only possible graph is IV.

23. This is the equation of a hyperboloid of one sheet, with $a = b = c = 1$. Since the coefficient of y^2 is negative, the axis of the hyperboloid is the y-axis, hence the correct graph is II.

24. This is a hyperboloid of two sheets, with $a = b = c = 1$. This surface does not intersect the xz-plane at all, so the axis of the hyperboloid is the y-axis and the graph is III.

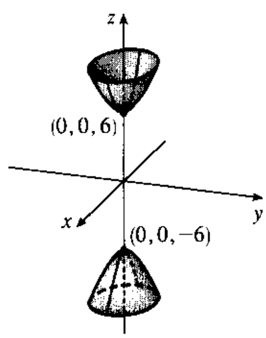
25. There are no real values of x and z that satisfy this equation for $y < 0$, so this surface does not extend to the left of the xz-plane. The surface intersects the plane $y = k > 0$ in an ellipse. Notice that y occurs to the first power whereas x and z occur to the second power. So the surface is an elliptic paraboloid with axis the y-axis. Its graph is VI.

26. This is the equation of a cone with axis the y-axis, so the graph is I.

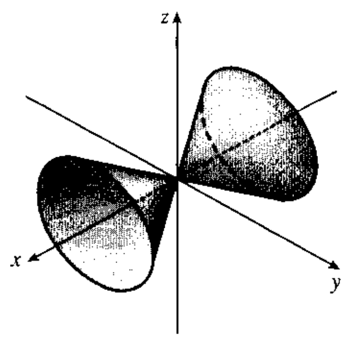
27. This surface is a cylinder because the variable y is missing from the equation. The intersection of the surface and the xz-plane is an ellipse. So the graph is VIII.

28. This is the equation of a hyperbolic paraboloid. The trace in the xy-plane is the parabola $y = x^2$. So the correct graph is V.

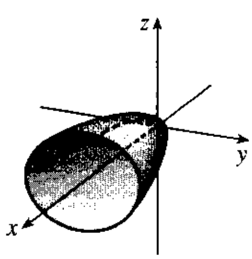
29. $z^2 = 4x^2 + 9y^2 + 36$ or $-4x^2 - 9y^2 + z^2 = 36$ or $-\frac{x^2}{9} - \frac{y^2}{4} + \frac{z^2}{36} = 1$ represents a hyperboloid of two sheets with axis the z-axis.



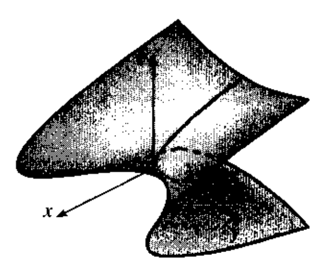
30. $x^2 = 2y^2 + 3z^2$ or $x^2 = \frac{y^2}{1/2} + \frac{z^2}{1/3}$ or $\frac{x^2}{6} = \frac{y^2}{3} + \frac{z^2}{2}$ represents an elliptic cone with vertex $(0, 0, 0)$ and axis the x-axis.



31. $x = 2y^2 + 3z^2$ or $x = \frac{y^2}{1/2} + \frac{z^2}{1/3}$ or $\frac{x}{6} = \frac{y^2}{3} + \frac{z^2}{2}$ represents an elliptic paraboloid with vertex $(0, 0, 0)$ and axis the x-axis.



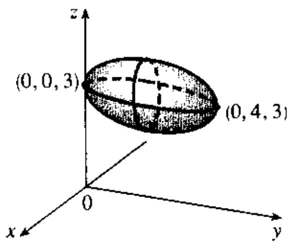
32. $4x - y^2 + 4z^2 = 0$ or $4x = y^2 - 4z^2$ or $x = \frac{y^2}{4} - z^2$ represents a hyperbolic paraboloid with center $(0, 0, 0)$.



33. Completing squares in y and z gives

$4x^2 + (y-2)^2 + 4(z-3)^2 = 4$ or

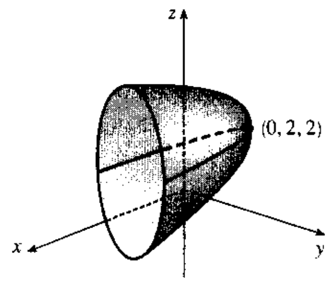
$x^2 + \dfrac{(y-2)^2}{4} + (z-3)^2 = 1$, an ellipsoid with center $(0, 2, 3)$.



34. Completing squares in y and z gives

$4(y-2)^2 + (z-2)^2 - x = 0$ or

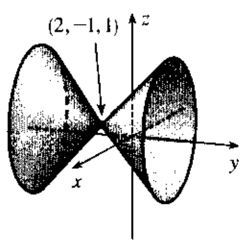
$\dfrac{x}{4} = (y-2)^2 + \dfrac{(z-2)^2}{4}$, an elliptic paraboloid with vertex $(0, 2, 2)$ and axis the horizontal line $y = 2$, $z = 2$.



35. Completing squares in all three variables gives

$(x-2)^2 - (y+1)^2 + (z-1)^2 = 0$ or

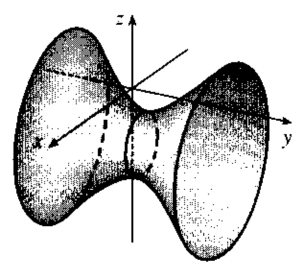
$(y+1)^2 = (x-2)^2 + (z-1)^2$, a circular cone with center $(2, -1, 1)$ and axis the horizontal line $x = 2$, $z = 1$.



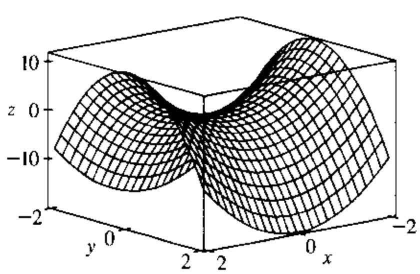
36. Completing squares in all three variables gives

$(x-1)^2 - (y-1)^2 + (z+2)^2 = 2$ or

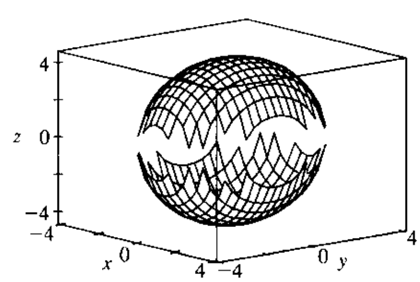
$\dfrac{(x-1)^2}{2} - \dfrac{(y-1)^2}{2} + \dfrac{(z+2)^2}{2} = 1$, a hyperboloid of one sheet with center $(1, 1, -2)$ and axis the horizontal line $x = 1$, $z = -2$.



37.

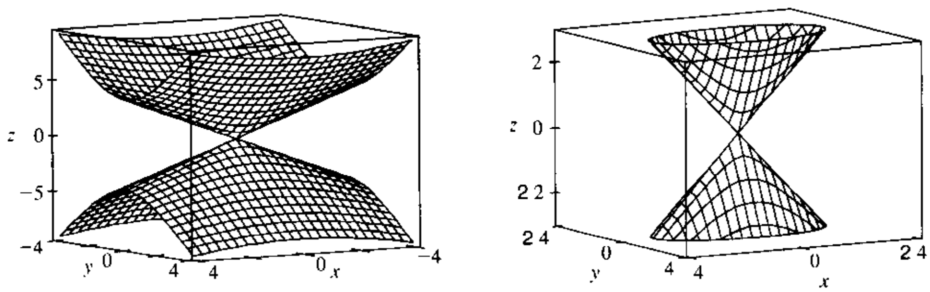


38.



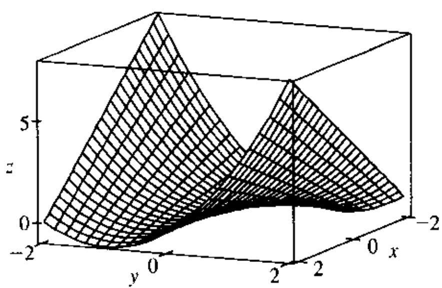
In Section 17.6 [ET 16.6], we will be able to graph ellipsoids without gaps; see Exercise 17.6.53 [ET 16.6.53].

39.

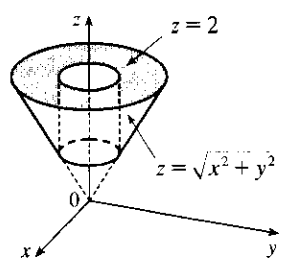


To restrict the z-range as in the second graph, we can use the option `view = -2..2` in Maple's `plot3d` command, or `PlotRange -> {-2,2}` in Mathematica's `Plot3D` command.

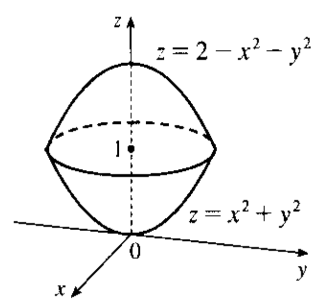
40.



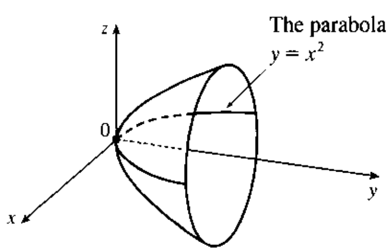
41.



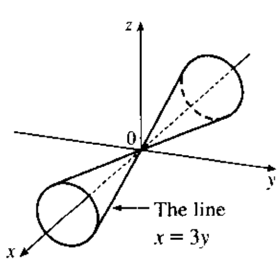
42.



43. The surface is a paraboloid of revolution (circular paraboloid) with vertex at the origin, axis the y-axis and opens to the right. Thus the trace in the yz-plane is also a parabola: $y = z^2$, $x = 0$. The equation is $y = x^2 + z^2$.



44. The surface is a right circular cone with vertex at $(0, 0, 0)$ and axis the x-axis. For $x = k \neq 0$, the trace is a circle with center $(k, 0, 0)$ and radius $r = y = \frac{x}{3} = \frac{k}{3}$. Thus the equation is $\frac{1}{3}x^2 = y^2 + z^2$.



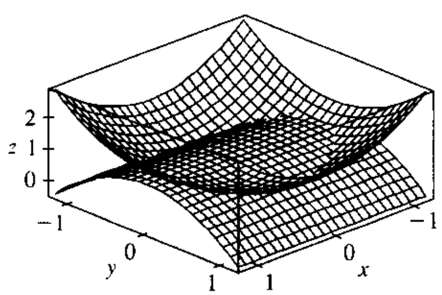
45. Let $P = (x, y, z)$ be an arbitrary point equidistant from $(-1, 0, 0)$ and the plane $x = 1$. Then the distance from P to $(-1, 0, 0)$ is $\sqrt{(x+1)^2 + y^2 + z^2}$ and the distance from P to the plane $x = 1$ is $|x - 1| / \sqrt{1^2} = |x - 1|$ (by Equation 13.5.9 [ET 12.5.9]). So $|x - 1| = \sqrt{(x+1)^2 + y^2 + z^2} \quad \Leftrightarrow \quad (x-1)^2 = (x+1)^2 + y^2 + z^2 \quad \Leftrightarrow$ $x^2 - 2x + 1 = x^2 + 2x + 1 + y^2 + z^2 \quad \Leftrightarrow \quad -4x = y^2 + z^2$. Thus the collection of all such points P is a circular paraboloid with vertex at the origin, axis the x-axis, which opens in the negative direction.

46. Let $P = (x, y, z)$ be an arbitrary point whose distance from the x-axis is twice its distance from the yz-plane. The distance from P to the x-axis is $\sqrt{(x-x)^2 + y^2 + z^2} = \sqrt{y^2 + z^2}$ and the distance from P to the yz-plane $(x = 0)$ is $|x| / 1 = |x|$. Thus $\sqrt{y^2 + z^2} = 2|x| \quad \Leftrightarrow \quad y^2 + z^2 = 4x^2 \quad \Leftrightarrow \quad x^2 = (y^2/2^2) + (z^2/2^2)$. So the surface is a right circular cone with vertex the origin and axis the x-axis.

47. If (a, b, c) satisfies $z = y^2 - x^2$, then $c = b^2 - a^2$. L_1: $x = a + t$, $y = b + t$, $z = c + 2(b - a)t$, L_2: $x = a + t$, $y = b - t$, $z = c - 2(b + a)t$. Substitute the parametric equations of L_1 into the equation of the hyperbolic paraboloid in order to find the points of intersection: $z = y^2 - x^2 \quad \Rightarrow$ $c + 2(b - a)t = (b + t)^2 - (a + t)^2 = b^2 - a^2 + 2(b - a)t \quad \Rightarrow \quad c = b^2 - a^2$. As this is true for all values of t, L_1 lies on $z = y^2 - x^2$. Performing similar operations with L_2 gives: $z = y^2 - x^2 \quad \Rightarrow$ $c - 2(b + a)t = (b - t)^2 - (a + t)^2 = b^2 - a^2 - 2(b + a)t \quad \Rightarrow \quad c = b^2 - a^2$. This tells us that all of L_2 also lies on $z = y^2 - x^2$.

48. Any point on the curve of intersection must satisfy both $2x^2 + 4y^2 - 2z^2 + 6x = 2$ and $2x^2 + 4y^2 - 2z^2 - 5y = 0$. Subtracting, we get $6x + 5y = 2$, which is linear and therefore the equation of a plane. Thus the curve of intersection lies in this plane.

49.



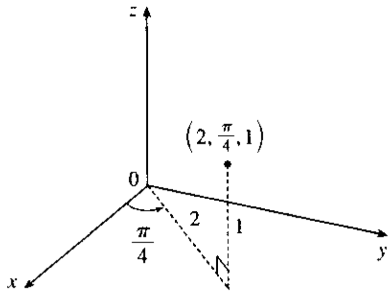
The curve of intersection looks like a bent ellipse. The projection of this curve onto the xy-plane is the set of points $(x, y, 0)$ which satisfy $x^2 + y^2 = 1 - y^2 \quad \Leftrightarrow$ $x^2 + 2y^2 = 1 \quad \Leftrightarrow \quad x^2 + \dfrac{y^2}{\left(1/\sqrt{2}\right)^2} = 1$. This is an equation of an ellipse.

13.7 Cylindrical and Spherical Coordinates ET 12.7

1. See Figure 1 and the accompanying discussion; see the paragraph accompanying Figure 3.

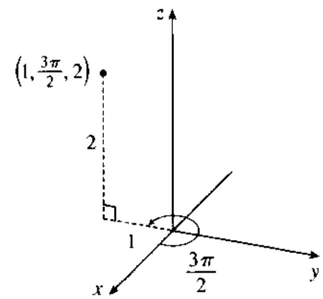
2. See Figure 5 and the accompanying discussion.

3.



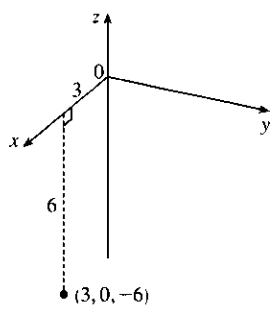
$x = 2\cos\frac{\pi}{4} = \sqrt{2}$, $y = 2\sin\frac{\pi}{4} = \sqrt{2}$, $z = 1$, so the point is $\left(\sqrt{2}, \sqrt{2}, 1\right)$ in rectangular coordinates.

4.



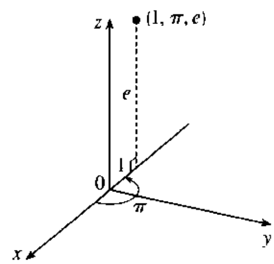
$x = 1\cos\frac{3\pi}{2} = 0$, $y = 1\sin\frac{3\pi}{2} = -1$, $z = 2$, so the point is $(0, -1, 2)$ in rectangular coordinates.

5.



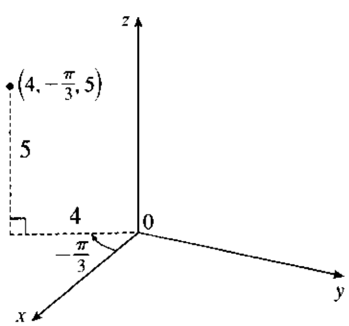
$x = 3\cos 0 = 3$, $y = 3\sin 0 = 0$, and $z = -6$, so the point is $(3, 0, -6)$ in rectangular coordinates.

6.



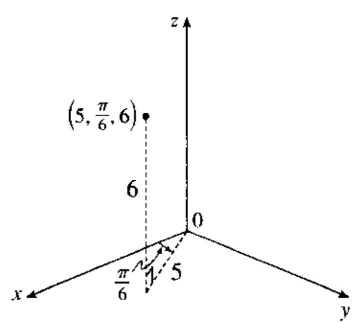
$x = 1\cos\pi = -1$, $y = 1\sin\pi = 0$, and $z = e$, so the point is $(-1, 0, e)$ in rectangular coordinates.

7.



$x = 4\cos\left(-\frac{\pi}{3}\right) = 2$, $y = 4\sin\left(-\frac{\pi}{3}\right) = -2\sqrt{3}$, and $z = 5$, so the point is $\left(2, -2\sqrt{3}, 5\right)$ in rectangular coordinates.

8.



$x = 5\cos\left(\frac{\pi}{6}\right) = \frac{5\sqrt{3}}{2}$, $y = 5\sin\left(\frac{\pi}{6}\right) = \frac{5}{2}$, and $z = 6$, so the point is $\left(\frac{5\sqrt{3}}{2}, \frac{5}{2}, 6\right)$ in rectangular coordinates.

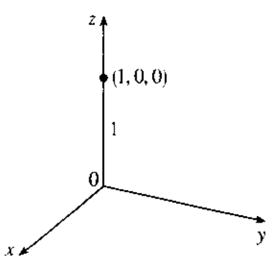
9. $r^2 = x^2 + y^2 = 1^2 + (-1)^2 = 2$ so $r = \sqrt{2}$; $\tan\theta = \dfrac{y}{x} = \dfrac{-1}{1} = -1$ and the point $(1, -1)$ is in the fourth quadrant of the xy-plane, so $\theta = \frac{7\pi}{4} + 2n\pi$; $z = 4$. Thus, one set of cylindrical coordinates is $\left(\sqrt{2}, \frac{7\pi}{4}, 4\right)$.

10. $r^2 = x^2 + y^2 = 3^2 + 3^2 = 18$ so $r = \sqrt{18} = 3\sqrt{2}$; $\tan\theta = \dfrac{y}{x} = \dfrac{3}{3} = 1$ and the point $(3, 3)$ is in the first quadrant of the xy-plane, so $\theta = \frac{\pi}{4} + 2n\pi$; $z = -2$. Thus, one set of cylindrical coordinates is $\left(3\sqrt{2}, \frac{\pi}{4}, -2\right)$.

11. $r^2 = (-1)^2 + \left(-\sqrt{3}\right)^2 = 4$ so $r = 2$; $\tan\theta = \frac{-\sqrt{3}}{-1} = \sqrt{3}$ and the point $\left(-1, -\sqrt{3}\right)$ is in the third quadrant of the xy-plane, so $\theta = \frac{4\pi}{3} + 2n\pi$; $z = 2$. Thus, one set of cylindrical coordinates is $\left(2, \frac{4\pi}{3}, 2\right)$.

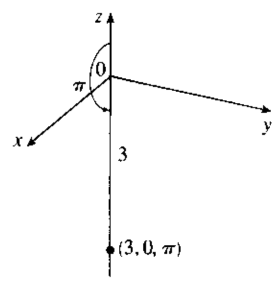
12. $r^2 = 3^2 + 4^2 = 25$ so $r = 5$; $\tan\theta = \frac{4}{3}$ and the point $(3, 4)$ is in the first quadrant of the xy-plane, so $\theta = \tan^{-1}\left(\frac{4}{3}\right) + 2n\pi \approx 0.93 + 2n\pi$; $z = 5$. Thus, one set of cylindrical coordinates is $\left(5, \tan^{-1}\left(\frac{4}{3}\right), 5\right) \approx (5, 0.93, 5)$.

13.



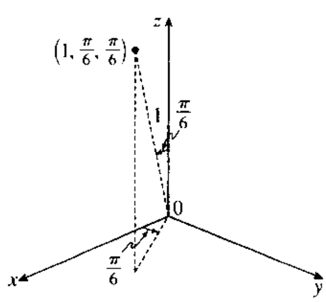
$x = \rho\sin\phi\cos\theta = (1)\sin 0\cos 0 = 0$,
$y = \rho\sin\phi\sin\theta = (1)\sin 0\sin 0 = 0$, and
$z = \rho\cos\phi = (1)\cos 0 = 1$ so the point is
$(0, 0, 1)$ in rectangular coordinates.

14.



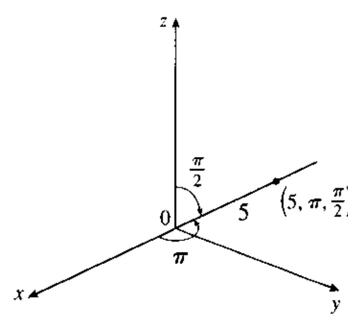
$x = 3\sin\pi\cos 0 = 0$, $y = 3\sin\pi\sin 0 = 0$,
$z = 3\cos\pi = -3$ and in rectangular
coordinates the point is $(0, 0, -3)$.

15.



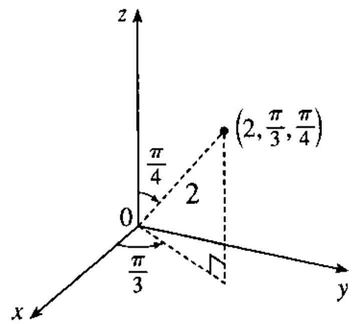
$x = \sin\frac{\pi}{6}\cos\frac{\pi}{6} = \frac{\sqrt{3}}{4}$, $y = \sin\frac{\pi}{6}\sin\frac{\pi}{6} = \frac{1}{4}$, and
$z = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$, so the point is $\left(\frac{\sqrt{3}}{4}, \frac{1}{4}, \frac{\sqrt{3}}{2}\right)$ in
rectangular coordinates.

16.

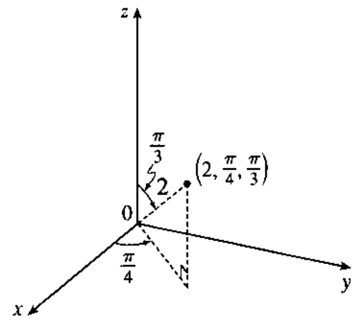


$x = 5\sin\frac{\pi}{2}\cos\pi = -5$, $y = 5\sin\frac{\pi}{2}\sin\pi = 0$,
$z = 5\cos\frac{\pi}{2} = 0$ so the point is $(-5, 0, 0)$ in
rectangular coordinates.

17. $x = 2\sin\frac{\pi}{4}\cos\frac{\pi}{3} = \frac{\sqrt{2}}{2}$, $y = 2\sin\frac{\pi}{4}\sin\frac{\pi}{3} = \frac{\sqrt{6}}{2}$, $z = 2\cos\frac{\pi}{4} = \sqrt{2}$ so the point is $\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{6}}{2}, \sqrt{2}\right)$ in rectangular coordinates.



18. $x = 2\sin\frac{\pi}{3}\cos\frac{\pi}{4} = \frac{\sqrt{6}}{2}$, $y = 2\sin\frac{\pi}{3}\sin\frac{\pi}{4} = \frac{\sqrt{6}}{2}$, $z = 2\cos\frac{\pi}{3} = 1$ so the point is $\left(\frac{\sqrt{6}}{2}, \frac{\sqrt{6}}{2}, 1\right)$ in rectangular coordinates.



19. $\rho = \sqrt{x^2+y^2+z^2} = \sqrt{1+3+12} = 4$, $\cos\phi = \dfrac{z}{\rho} = \dfrac{2\sqrt{3}}{4} = \dfrac{\sqrt{3}}{2} \Rightarrow \phi = \dfrac{\pi}{6}$, and $\cos\theta = \dfrac{x}{\rho\sin\phi} = \dfrac{1}{4\sin(\pi/6)} = \dfrac{1}{2} \Rightarrow \theta = \dfrac{\pi}{3}$ (since $y > 0$). Thus spherical coordinates are $\left(4, \dfrac{\pi}{3}, \dfrac{\pi}{6}\right)$.

20. $\rho = \sqrt{x^2+y^2+z^2} = \sqrt{0+3+1} = 2$, $\cos\phi = \dfrac{z}{\rho} = \dfrac{1}{2} \Rightarrow \phi = \dfrac{\pi}{3}$, and $\cos\theta = \dfrac{x}{\rho\sin\phi} = \dfrac{0}{2\sin(\pi/3)} = 0 \Rightarrow \theta = \dfrac{\pi}{2}$ (since $y > 0$). Thus spherical coordinates are $\left(2, \dfrac{\pi}{2}, \dfrac{\pi}{3}\right)$.

21. $\rho = \sqrt{0+1+1} = \sqrt{2}$, $\cos\phi = \dfrac{-1}{\sqrt{2}} \Rightarrow \phi = \dfrac{3\pi}{4}$, and $\cos\theta = \dfrac{0}{\sqrt{2}\sin(3\pi/4)} = 0 \Rightarrow \theta = \dfrac{3\pi}{2}$ (since $y < 0$). Thus spherical coordinates are $\left(\sqrt{2}, \dfrac{3\pi}{2}, \dfrac{3\pi}{4}\right)$.

22. $\rho = \sqrt{1+1+6} = 2\sqrt{2}$, $\cos\phi = \dfrac{\sqrt{6}}{2\sqrt{2}} = \dfrac{\sqrt{3}}{2} \Rightarrow \phi = \dfrac{\pi}{6}$, and $\cos\theta = \dfrac{-1}{2\sqrt{2}\sin(\pi/6)} = -\dfrac{1}{\sqrt{2}} \Rightarrow \theta = \dfrac{3\pi}{4}$ (since $y > 0$). Thus spherical coordinates are $\left(2\sqrt{2}, \dfrac{3\pi}{4}, \dfrac{\pi}{6}\right)$.

23. $\rho = \sqrt{x^2+y^2+z^2} = \sqrt{r^2+z^2} = \sqrt{1+3} = 2$; $\theta = \dfrac{\pi}{6}$; $\cos\phi = \dfrac{z}{\rho} = \dfrac{\sqrt{3}}{2} \Rightarrow \phi = \dfrac{\pi}{6}$, thus in spherical coordinates the point is $\left(2, \dfrac{\pi}{6}, \dfrac{\pi}{6}\right)$.

24. $\rho = \sqrt{r^2+z^2} = \sqrt{6+2} = 2\sqrt{2}$; $\theta = \dfrac{\pi}{4}$; $\cos\phi = \dfrac{z}{\rho} = \dfrac{\sqrt{2}}{2\sqrt{2}} = \dfrac{1}{2} \Rightarrow \phi = \dfrac{\pi}{3}$, thus in spherical coordinates the point is $\left(2\sqrt{2}, \dfrac{\pi}{4}, \dfrac{\pi}{3}\right)$.

25. $\rho = \sqrt{r^2 + z^2} = \sqrt{3+1} = 2$; $\theta = \dfrac{\pi}{2}$; $\cos\phi = \dfrac{z}{\rho} = \dfrac{-1}{2} \quad\Rightarrow\quad \phi = \dfrac{2\pi}{3}$, so in spherical coordinates the point is $\left(2, \dfrac{\pi}{2}, \dfrac{2\pi}{3}\right)$.

26. $\rho = \sqrt{16+9} = 5$; $\theta = \frac{\pi}{8}$; $\cos\phi = \frac{3}{5} \quad\Rightarrow\quad \phi = \cos^{-1}\left(\frac{3}{5}\right)$, so in spherical coordinates the point is $\left(5, \frac{\pi}{8}, \cos^{-1}\left(\frac{3}{5}\right)\right) \approx \left(5, \frac{\pi}{8}, 0.927\right)$.

27. $z = \rho\cos\phi = 2\cos 0 = 2$, $\rho^2 = x^2 + y^2 + z^2 = r^2 + z^2 \quad\Rightarrow\quad r = \sqrt{\rho^2 - z^2} = \sqrt{2^2 - 2^2} = 0$, (or $r = 2\sin 0 = 0$), $\theta = 0$ and the point is $(0, 0, 2)$.

28. $z = 2\sqrt{2}\cos\frac{\pi}{2} = 0$, $r = 2\sqrt{2}\sin\frac{\pi}{2} = 2\sqrt{2}$, $\theta = \frac{3\pi}{2}$ and the point is $\left(2\sqrt{2}, \frac{3\pi}{2}, 0\right)$.

29. $z = 8\cos\frac{\pi}{2} = 0$, $r = 8\sin\frac{\pi}{2} = 8$, $\theta = \frac{\pi}{6}$ and the point is $\left(8, \frac{\pi}{6}, 0\right)$.

30. $z = 4\cos\frac{\pi}{3} = 2$, $r = 4\sin\frac{\pi}{3} = 2\sqrt{3}$, $\theta = \frac{\pi}{4}$ and the point is $\left(2\sqrt{3}, \frac{\pi}{4}, 2\right)$.

31. Since $r = 3$, $x^2 + y^2 = 9$ and the surface is a circular cylinder with radius 3 and axis the z-axis.

32. Since $\rho = 3$, $x^2 + y^2 + z^2 = 9$ and the surface is a sphere with center the origin and radius 3.

33. Since $\phi = 0$, $x = 0$ and $y = 0$ while $z = \rho \geq 0$. Thus the "surface" is the positive z-axis including the origin.

34. Since $\phi = \frac{\pi}{2}$, $z = 0$ but there are no restrictions on x and y ($x = \rho\cos\theta$, $y = \rho\sin\theta$). Thus the surface is the xy-plane.

35. Since $\phi = \frac{\pi}{3}$, the surface is the top half of the right circular cone with vertex at the origin and axis the positive z-axis.

36. Whether spherical or cylindrical coordinates, since $\theta = \frac{\pi}{3}$ the surface is a half-plane including the z-axis and intersecting the xy-plane in the half-line $y = \sqrt{3}\,x$, $x > 0$.

37. $z = r^2 = x^2 + y^2$, so the surface is a circular paraboloid with vertex at the origin and axis the positive z-axis.

38. Since $r = 4\sin\theta$ and $y = r\sin\theta$, $y = 4\sin^2\theta$. Also $r^2 = x^2 + y^2$ so $x^2 + y^2 = 16\sin^2\theta$. Thus $x^2 + y^2 - 4y = 16\sin^2\theta - 16\sin^2\theta = 0$ or $x^2 + (y-2)^2 = 4$, a circular cylinder of radius 2 and with axis parallel to the z-axis.

39. $2 = \rho\cos\phi = z$ is a plane through the point $(0, 0, 2)$ and parallel to the xy-plane.

40. Since $\rho\sin\phi = 2$ and $x = \rho\sin\phi\cos\theta$, $x = 2\cos\theta$. Also $y = \rho\sin\phi\sin\theta$ so $y = 2\sin\theta$. Then $x^2 + y^2 = 4\cos^2\theta + 4\sin^2\theta = 4$, a circular cylinder of radius 2 about the z-axis.

41. $r = 2\cos\theta \quad\Rightarrow\quad r^2 = x^2 + y^2 = 2r\cos\theta = 2x \quad\Leftrightarrow\quad (x-1)^2 + y^2 = 1$, which is the equation of a circular cylinder with radius 1, whose axis is the vertical line $x = 1$, $y = 0$, $z = z$.

42. $\rho = 2\cos\phi \quad \Rightarrow \quad \rho^2 = 2\rho\cos\phi = 2z \quad \Leftrightarrow \quad x^2 + y^2 + z^2 = 2z \quad \Leftrightarrow \quad x^2 + y^2 + (z-1)^2 = 1$. Therefore, the surface is a sphere of radius 1 centered at $(0,0,1)$.

43. Since $r^2 + z^2 = 25$ and $r^2 = x^2 + y^2$, we have $x^2 + y^2 + z^2 = 25$, a sphere with radius 5 and center at the origin.

44. Since $r^2 - 2z^2 = 4$ and $r^2 = x^2 + y^2$, we have $x^2 + y^2 - 2z^2 = 4$ or $\frac{1}{4}x^2 + \frac{1}{4}y^2 - \frac{1}{2}z^2 = 1$, a hyperboloid of one sheet with axis the z-axis.

45. Since $x^2 = \rho^2 \sin^2\phi \cos^2\theta$ and $z^2 = \rho^2\cos^2\phi$, the equation of the surface in rectangular coordinates is $x^2 + z^2 = 4$. Thus the surface is a circular cylinder of radius 2 about the y-axis.

46. Since $\rho^2(\sin^2\phi - 4\cos^2\phi) = 1$, $\rho^2(\sin^2\phi - 4\cos^2\phi) + \rho^2\cos^2\phi - \rho^2\cos^2\phi = 1$ or $\rho^2(\sin^2\phi + \cos^2\phi - 5\cos^2\phi) = 1$ or $\rho^2(1 - 5\cos^2\phi) = 1$. But $\rho^2 = x^2 + y^2 + z^2$ and $z^2 = \rho^2\cos^2\phi$, so we can rewrite the equation of the surface as $x^2 + y^2 + z^2 - 5z^2 = 1$ or $x^2 + y^2 - 4z^2 = 1$. Thus the surface is a hyperboloid of one sheet with axis the z-axis.

47. Since $r^2 - r = 0$, $r = 0$ or $r = 1$. But $x^2 + y^2 = r^2$. Thus the surface consists of the right circular cylinder of radius 1 and axis the z-axis along with the surface given by $x^2 + y^2 = 0$, that is, the z-axis.

48. Since $\rho^2 - 6\rho + 8 = 0$, either $\rho = 2$ or $\rho = 4$. Thus the surface consists of two concentric spheres (centered at the origin), one with radius 2 and the other with radius 4.

49. (a) $x^2 + y^2 = r^2$, so the equation becomes $z = r^2$.

(b) $x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$, and $z = \rho\cos\phi$, so the equation becomes $\rho\cos\phi = (\rho\sin\phi\cos\theta)^2 + (\rho\sin\phi\sin\theta)^2$ or $\rho\cos\phi = \rho^2\sin^2\phi$ or $\rho\sin^2\phi = \cos\phi$.

50. (a) $x^2 + y^2 = r^2$, so the equation becomes $r^2 + z^2 = 2$.

(b) $x^2 + y^2 + z^2 = \rho^2$, so the equation becomes $\rho^2 = 2$ or $\rho = \sqrt{2}$.

51. (a) $x = r\cos\theta$, so the equation becomes $r\cos\theta = 3$ or $r = 3\sec\theta$ (since $\cos\theta \neq 0$ here).

(b) $x = \rho\sin\phi\cos\theta$, so the equation becomes $\rho\sin\phi\cos\theta = 3$.

52. (a) $x^2 + y^2 = r^2$, so the equation becomes $r^2 + z^2 + 2z = 0$ or $r^2 + (z+1)^2 = 1$.

(b) $x^2 + y^2 + z^2 = \rho^2$ and $z = \rho\cos\phi$, so the equation becomes $\rho^2 + 2\rho\cos\phi = 0$ or $\rho = -2\cos\phi$.

53. (a) $r^2(\cos^2\theta - \sin^2\theta) - 2z^2 = 4$ or $2z^2 = r^2\cos 2\theta - 4$.

(b) $\rho^2(\sin^2\phi\cos^2\theta - \sin^2\phi\sin^2\theta - 2\cos^2\phi) = 4$ or $\rho^2(\sin^2\phi\cos 2\theta - 2\cos^2\phi) = 4$.

54. (a) $r^2\sin^2\theta + z^2 = 1$

(b) $\rho^2\sin^2\phi\sin^2\theta + \rho^2\cos^2\phi = 1$ or $\rho^2(\sin^2\phi\sin^2\theta + \cos^2\phi) = 1$.

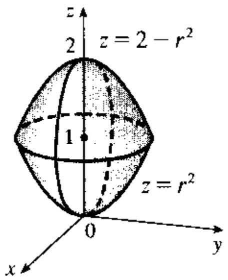
55. (a) $r^2 = 2r\sin\theta$ or $r = 2\sin\theta$.

(b) $\rho^2\sin^2\phi(\cos^2\theta + \sin^2\theta) = 2\rho\sin\phi\sin\theta$ or $\rho\sin^2\phi = 2\sin\phi\sin\theta$ or $\rho\sin\phi = 2\sin\theta$.

56. (a) $z = r^2(\cos^2\theta - \sin^2\theta)$ or $z = r^2\cos 2\theta$.

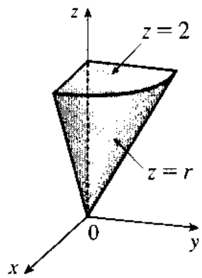
(b) $\rho\cos\phi = \rho^2\sin^2\phi(\cos^2\theta - \sin^2\theta)$ or $\cos\phi = \rho\sin^2\phi\cos 2\theta$.

57.



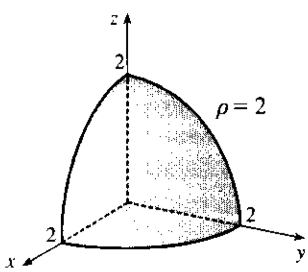
$z = r^2 = x^2 + y^2$ is a circular paraboloid with vertex $(0, 0, 0)$, opening upward. $z = 2 - r^2$ $\Rightarrow$ $z - 2 = -(x^2 + y^2)$ is a circular paraboloid with vertex $(0, 0, 2)$ opening downward. Thus $r^2 \le z \le 2 - r^2$ is the solid region enclosed by these two surfaces.

58.



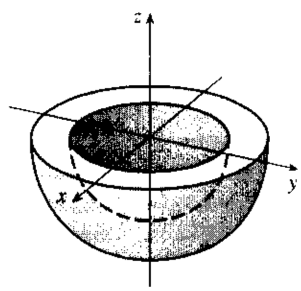
$z = r = \sqrt{x^2 + y^2}$ is a cone that opens upward. Thus $r \le z \le 2$ is the region above this cone and beneath the horizontal plane $z = 2$. $0 \le \theta \le \frac{\pi}{2}$ restricts the solid to that part of this region in the first octant.

59.



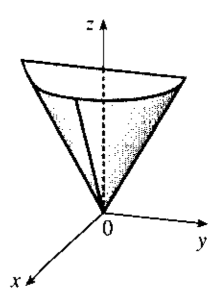
$\rho = 2$ represents a sphere of radius 2, centered at the origin, so $\rho \le 2$ is this sphere and its interior. $0 \le \phi \le \frac{\pi}{2}$ restricts the solid to that portion of the region that lies on or above the xy-plane, and $0 \le \theta \le \frac{\pi}{2}$ further restricts the solid to the first octant. Thus the solid is the portion in the first octant of the solid ball centered at the origin with radius 2.

60.

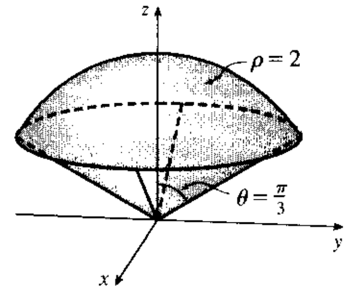


$2 \le \rho \le 3$ represents the solid region between and including the spheres of radii 2 and 3, centered at the origin. $\frac{\pi}{2} \le \phi \le \pi$ restricts the solid to that portion on or below the xy-plane.

61. $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ restricts the solid to the 4 octants in which x is positive. $\rho = \sec \phi \Rightarrow \rho \cos \phi = z = 1$, which is the equation of a horizontal plane. $0 \le \phi \le \frac{\pi}{6}$ describes a cone, opening upward. So the solid lies above the cone $\phi = \frac{\pi}{6}$ and below the plane $z = 1$.



62. $\rho = 2 \quad \Leftrightarrow \quad x^2 + y^2 + z^2 = 4$, which is a sphere of radius 2, centered at the origin. Hence $\rho \leq 2$ is this sphere and its interior. $0 \leq \phi \leq \frac{\pi}{3}$ restricts the solid to that section of this ball that lies above the cone $\phi = \frac{\pi}{3}$.



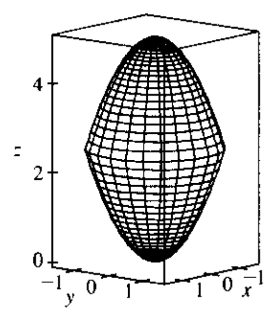
63. We can position the cylindrical shell vertically so that its axis coincides with the z-axis and its base lies in the xy-plane. If we use centimeters as the unit of measurement, then cylindrical coordinates conveniently describe the shell as $6 \leq r \leq 7, 0 \leq \theta \leq 2\pi, 0 \leq z \leq 20$.

64. (a) The hollow ball is a spherical shell with outer radius 15 cm and inner radius 14.5 cm. If we center the ball at the origin of the coordinate system and use centimeters as the unit of measurement, then spherical coordinates conveniently describe the hollow ball as $14.5 \leq \rho \leq 15, 0 \leq \theta \leq 2\pi, 0 \leq \phi \leq \pi$.

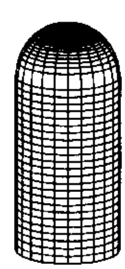
(b) If we position the ball as in part (a), one possibility is to take the half of the ball that is above the xy-plane which is described by $14.5 \leq \rho \leq 15, 0 \leq \theta \leq 2\pi, 0 \leq \phi \leq \pi/2$.

65. $z \geq \sqrt{x^2 + y^2}$ because the solid lies above the cone. Squaring both sides of this inequality gives $z^2 \geq x^2 + y^2$ $\Rightarrow \quad 2z^2 \geq x^2 + y^2 + z^2 = \rho^2 \quad \Rightarrow \quad z^2 = \rho^2 \cos^2 \phi \geq \frac{1}{2}\rho^2 \quad \Rightarrow \quad \cos^2 \phi \geq \frac{1}{2}$. The cone opens upward so that the inequality is $\cos \phi \geq \frac{1}{\sqrt{2}}$, or equivalently $0 \leq \phi \leq \frac{\pi}{4}$. In spherical coordinates the sphere $z = x^2 + y^2 + z^2$ is $\rho \cos \phi = \rho^2 \quad \Rightarrow \quad \rho = \cos \phi$. $0 \leq \rho \leq \cos \phi$ because the solid lies below the sphere. The solid can therefore be described as the region in spherical coordinates satisfying $0 \leq \rho \leq \cos \phi, 0 \leq \phi \leq \frac{\pi}{4}$.

66. In cylindrical coordinates, the equations are $z = r^2$ and $z = 5 - r^2$. The curve of intersection is $r^2 = 5 - r^2$ or $r = \sqrt{5/2}$. So we graph the surfaces in cylindrical coordinates, with $0 \leq r \leq \sqrt{5/2}$. In Maple, we can use either the `coords=cylindrical` option in a regular `plot` command, or the `plots[cylinderplot]` command. In Mathematica, we can use `ParametricPlot3d`.



67. In cylindrical coordinates, the equation of the cylinder is $r = 3$, $0 \leq z \leq 10$. The hemisphere is the upper part of the sphere radius 3, center $(0, 0, 10)$, equation $r^2 + (z - 10)^2 = 3^2$, $z \geq 10$. In Maple, we can use either the `coords=cylindrical` option in a regular `plot` command, or the `plots[cylinderplot]` command. In Mathematica, we can use `ParametricPlot3d`.



68. We begin by finding the positions of Los Angeles and Montréal in spherical coordinates, using the method described in the exercise:

Montréal	Los Angeles
$\rho = 3960$ mi	$\rho = 3960$ mi
$\theta = 360^\circ - 73.60^\circ = 286.40^\circ$	$\theta = 360^\circ - 118.25^\circ = 241.75^\circ$
$\phi = 90^\circ - 45.50^\circ = 44.50^\circ$	$\phi = 90^\circ - 34.06^\circ = 55.94^\circ$

Now we change the above to Cartesian coordinates using $x = \rho\cos\theta\sin\phi$, $y = \rho\sin\theta\sin\phi$ and $z = \rho\cos\phi$ to get two position vectors of length 3960 mi (since both cities must lie on the surface of the Earth). In particular:

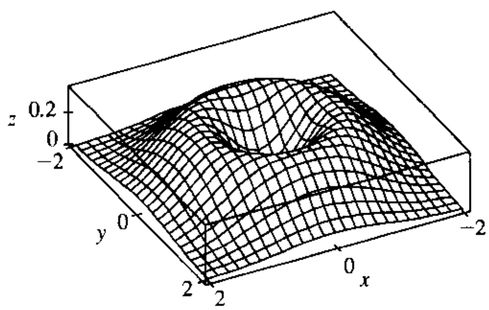
Montréal: $\langle 783.67, -2662.67, 2824.47\rangle$ Los Angeles: $\langle -1552.80, -2889.91, 2217.84\rangle$

To find the angle α between these two vectors we use the dot product:

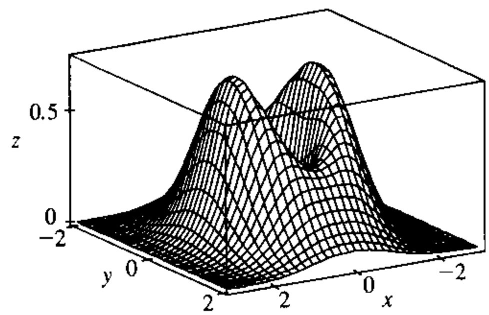
$\langle 783.67, -2662.67, 2824.47\rangle \cdot \langle -1552.80, -2889.91, 2217.84\rangle = (3960)^2\cos\alpha \quad\Rightarrow\quad \cos\alpha \approx 0.8126 \quad\Rightarrow$ $\alpha \approx 0.6223$ rad. The great circle distance between the cities is $s = \rho\theta \approx 3960(0.6223) \approx 2464$ mi.

LABORATORY PROJECT Families of Surfaces

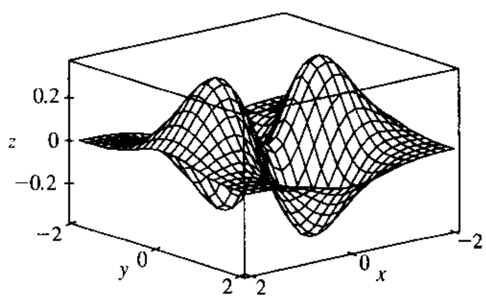
1. $z = (ax^2 + by^2)e^{-x^2-y^2}$. There are only three basic shapes which can be obtained (the fourth and fifth graphs are the reflections of the first and second ones in the xy-plane). Interchanging a and b rotates the graph by 90° about the z-axis.



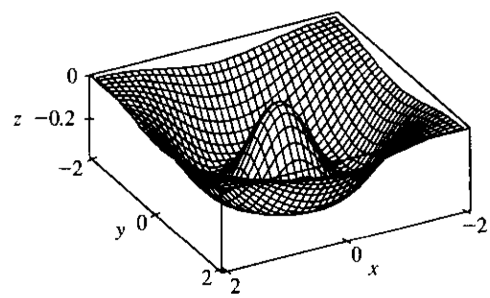
$a = 1, b = 1$



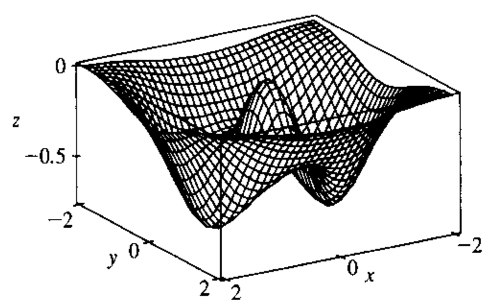
$a = 2, b = 1$



$a = 1, b = -1$



$a = -1, b = -1$

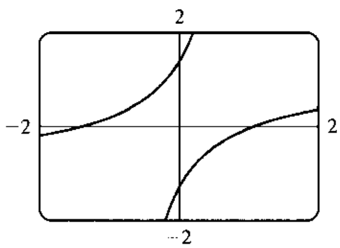


$a = -2, b = -1$

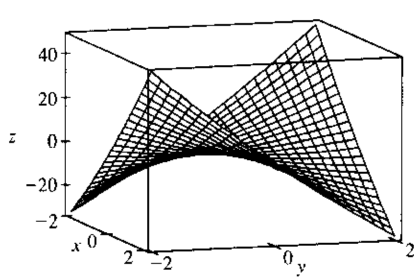
If a and b are both positive ($a \neq b$), we see that the graph has two maximum points whose height increases as a and b increase. If a and b have opposite signs, the graph has two maximum points and two minimum points, and if a and b are both negative, the graph has one maximum point and two minimum points.

2. $z = x^2 + y^2 + cxy$. When $c < -2$, the surface intersects the plane $z = k \neq 0$ in a hyperbola. (See graph below.) It intersects the plane $x = y$ in the parabola $z = (2 + c)x^2$, and the plane $x = -y$ in the parabola $z = (2 - c)x^2$. These parabolas open in opposite directions, so the surface is a hyperbolic paraboloid.

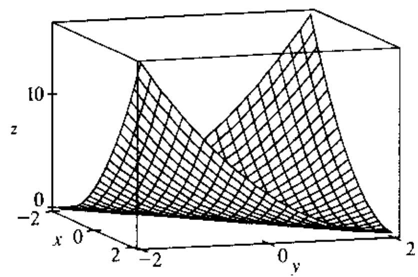
When $c = -2$ the surface is $z = x^2 + y^2 - 2xy = (x - y)^2$. So the surface is constant along each line $x - y = k$. That is, the surface is a cylinder with axis $x - y = 0$, $z = 0$. The shape of the cylinder is determined by its intersection with the plane $x + y = 0$, where $z = 4x^2$, and hence the cylinder is parabolic with minima of 0 on the line $y = x$.



$c = -5$, $z = 2$



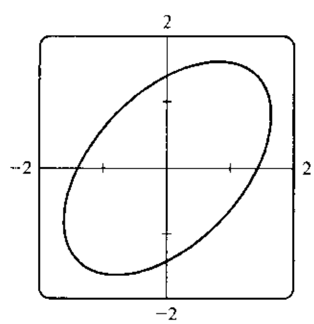
$c = -10$



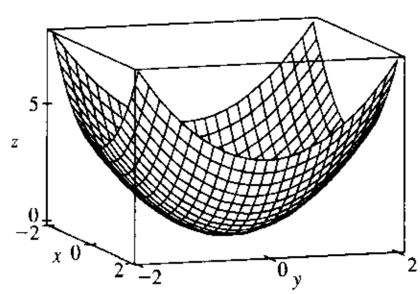
$c = -2$

When $-2 < c \leq 0$, $z \geq 0$ for all x and y. If x and y have the same sign, then $x^2 + y^2 + cxy \geq x^2 + y^2 - 2xy = (x - y)^2 \geq 0$. If they have opposite signs, then $cxy \geq 0$. The intersection with the surface and the plane $z = k > 0$ is an ellipse (see graph below). The intersection with the surface and the planes $x = 0$ and $y = 0$ are parabolas $z = y^2$ and $z = x^2$ respectively, so the surface is an elliptic paraboloid.

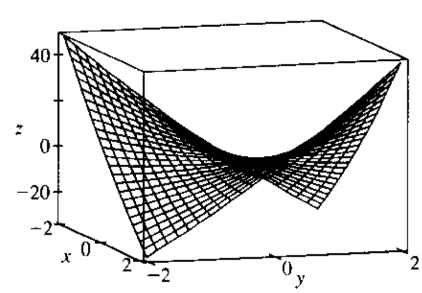
When $c > 0$ the graphs have the same shape, but are reflected in the plane $x = 0$, because $x^2 + y^2 + cxy = (-x)^2 + y^2 + (-c)(-x)y$. That is, the value of z is the same for c at (x, y) as it is for $-c$ at $(-x, y)$.



$c = -1$, $z = 2$



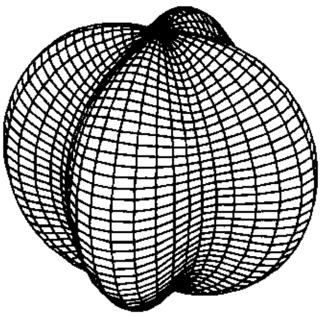
$c = 0$



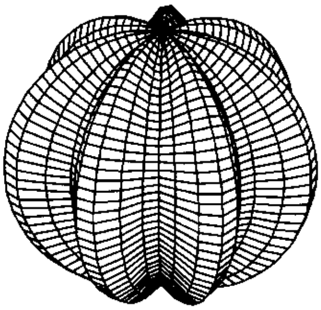
$c = 10$

So the surface is an elliptic paraboloid for $0 < c < 2$, a parabolic cylinder for $c = 2$, and a hyperbolic paraboloid for $c > 2$.

3. $\rho = 1 + 0.2 \sin m\theta \sin n\phi$. If we start with $m = 1, n = 1$ the equation is $\rho = 1 + 0.2 \sin \theta \sin \phi$, whose graph appears spherical or nearly spherical in shape. First we investigate varying just m. Values of $m > 1$ produce vertical ridges in the sphere, the number of ridges corresponding to the value of m. We graph two examples.

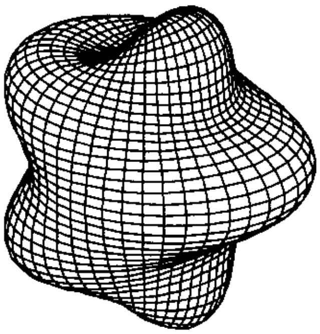


$m = 4, n = 1$

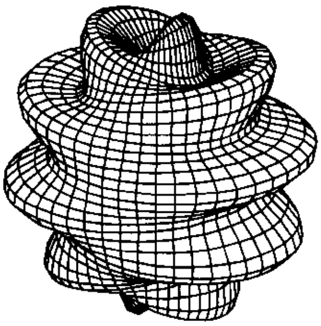


$m = 7, n = 1$

If we leave m fixed at 1 and vary n, we see horizontal ridges that span half the sphere arranged in a staggered fashion. Again, the number of "bumps" coincides with the value of n.

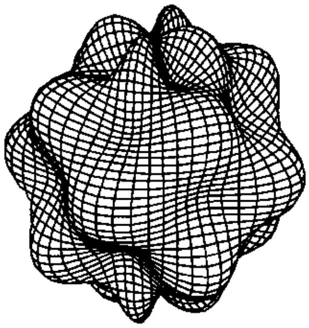


$m = 1, n = 5$

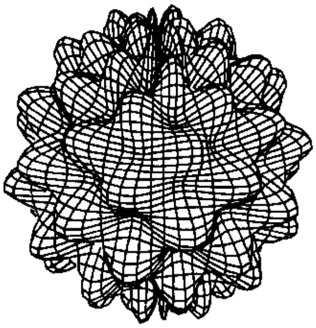


$m = 1, n = 10$

If we allow both m and n to vary, we get combinations of the vertical and horizontal bumps.



$m = 4, n = 5$



$m = 7, n = 10$

The graph on the left shows $m = 4, n = 5$. Looking at the top of the bumpy sphere, we can see the 4 vertical ridges which become perturbed horizontally as they progress down the sphere. We can also see the 5 horizontal rows of bumps. (Consequently, there are 20 bumps on the surface.) The graph on the right shows $m = 7, n = 10$ which should have 70 bumps.

13 Review

ET 12

CONCEPT CHECK

1. A scalar is a real number, while a vector is a quantity that has both a real-valued magnitude and a direction.

2. To add two vectors geometrically, we can use either the Triangle Law or the Parallelogram Law, as illustrated in Figures 3 and 4 in Section 13.2 [ET 12.2]. Algebraically, we add the corresponding components of the vectors.

3. For $c > 0$, $c\,\mathbf{a}$ is a vector with the same direction as $\mathbf{a}$ and length c times the length of $\mathbf{a}$. If $c < 0$, $c\mathbf{a}$ points in the opposite direction as $\mathbf{a}$ and has length $|c|$ times the length of $\mathbf{a}$. (See Figures 7 and 15 in Section 13.2 [ET 12.2].) Algebraically, to find $c\,\mathbf{a}$ we multiply each component of $\mathbf{a}$ by c.

4. See (1) in Section 13.2 [ET 12.2].

5. See Theorem 13.3.3 [ET 12.3.3] and Definition 13.3.1 [ET 12.3.1].

6. The dot product can be used to find the angle between two vectors and the scalar projection of one vector onto another. In particular, the dot product can determine if two vectors are orthogonal. Also, the dot product can be used to determine the work done moving an object given the force and displacement vectors.

7. See the boxed equations on page 847 [ET 811] as well as Figures 4 and 5 and the accompanying discussion on pages 846–47 [ET 810–11].

8. See Theorem 13.4.6 [ET 12.4.6] and the preceding discussion; use either (1) or (4) in Section 13.4 [ET 12.4].

9. The cross product can be used to create a vector orthogonal to two given vectors as well as to determine if two vectors are parallel. The cross product can also be used to find the area of a parallelogram determined by two vectors. In addition, the cross product can be used to determine torque if the force and position vectors are known.

10. (a) The area of the parallelogram determined by $\mathbf{a}$ and $\mathbf{b}$ is the length of the cross product: $|\mathbf{a}\times\mathbf{b}|$.

(b) The volume of the parallelepiped determined by $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ is the magnitude of their scalar triple product: $|\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})|$.

11. If an equation of the plane is known, it can be written as $ax + by + cz + d = 0$. A normal vector, which is perpendicular to the plane, is $\langle a, b, c\rangle$ (or any scalar multiple of $\langle a, b, c\rangle$). If an equation is not known, we can use points on the plane to find two non-parallel vectors which lie in the plane. The cross product of these vectors is a vector perpendicular to the plane.

12. The angle between two intersecting planes is defined as the acute angle between their normal vectors. We can find this angle using Corollary 13.3.6 [ET 12.3.6].

13. See (1), (2), and (3) in Section 13.5 [ET 12.5].

14. See (5), (6), and (7) in Section 13.5 [ET 12.5].

15. (a) Two (nonzero) vectors are parallel if and only if one is a scalar multiple of the other. In addition, two nonzero vectors are parallel if and only if their cross product is $\mathbf{0}$.

(b) Two vectors are perpendicular if and only if their dot product is 0.

(c) Two planes are parallel if and only if their normal vectors are parallel.

16. (a) Determine the vectors $\overrightarrow{PQ} = \langle a_1, a_2, a_3\rangle$ and $\overrightarrow{PR} = \langle b_1, b_2, b_3\rangle$. If there is a scalar t such that $\langle a_1, a_2, a_3\rangle = t\,\langle b_1, b_2, b_3\rangle$, then the vectors are parallel and the points must all lie on the same line.
Alternatively, if $\overrightarrow{PQ}\times\overrightarrow{PR} = \mathbf{0}$, then $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ are parallel, so P, Q, and R are collinear.
Thirdly, an algebraic method is to determine an equation of the line joining two of the points, and then check whether or not the third point satisfies this equation.

(b) Find the vectors $\overrightarrow{PQ} = \mathbf{a}$, $\overrightarrow{PR} = \mathbf{b}$, $\overrightarrow{PS} = \mathbf{c}$. $\mathbf{a} \times \mathbf{b}$ is normal to the plane formed by P, Q and R, and so S lies on this plane if $\mathbf{a} \times \mathbf{b}$ and $\mathbf{c}$ are orthogonal, that is, if $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c} = 0$. (Or use the reasoning in Example 5 in Section 13.4 [ET 12.4].)
Alternatively, find an equation for the plane determined by three of the points and check whether or not the fourth point satisfies this equation.

17. (a) See Exercise 13.4.39 [ET 12.4.39].
(b) See Example 8 in Section 13.5 [ET 12.5].
(c) See Example 10 in Section 13.5 [ET 12.5].

18. The traces of a surface are the curves of intersection of the surface with planes parallel to the coordinate planes. We can find the trace in the plane $x = k$ (parallel to the yz-plane) by setting $x = k$ and determining the curve represented by the resulting equation. Traces in the planes $y = k$ (parallel to the xz-plane) and $z = k$ (parallel to the xy-plane) are found similarly.

19. See Table 1 in Section 13.6 [ET 12.6].

20. (a) See (1) and the discussion accompanying Figure 3 in Section 13.7 [ET 12.7].
(b) See (3) and Figures 6–8, and the accompanying discussion, in Section 13.7 [ET 12.7].

TRUE-FALSE QUIZ

1. True, by Theorem 13.3.2 [ET 12.3.2] #2.

2. False. Theorem 13.4.8 [ET 12.4.8] #1 says that $\mathbf{u} \times \mathbf{v} = -\mathbf{v} \times \mathbf{u}$.

3. True. If θ is the angle between $\mathbf{u}$ and $\mathbf{v}$, then by Theorem 13.4.6 [ET 12.4.6],
$|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}|\,|\mathbf{v}|\sin\theta = |\mathbf{v}|\,|\mathbf{u}|\sin\theta = |\mathbf{v} \times \mathbf{u}|$.
(Or, by Theorem 13.4.8 [ET 12.4.8], $|\mathbf{u} \times \mathbf{v}| = |-\mathbf{v} \times \mathbf{u}| = |-1|\,|\mathbf{v} \times \mathbf{u}| = |\mathbf{v} \times \mathbf{u}|$.)

4. This is true by Theorem 13.3.2 [ET 12.3.2] #4.

5. Theorem 13.4.8 [ET 12.4.8] #2 tells us that this is true.

6. This is true by Theorem 13.4.8 [ET 12.4.8] #4.

7. This is true by Theorem 13.4.8 [ET 12.4.8] #5.

8. In general, this assertion is false; a counterexample is $\mathbf{i} \times (\mathbf{i} \times \mathbf{j}) \neq (\mathbf{i} \times \mathbf{i}) \times \mathbf{j}$. (See the paragraph preceding Theorem 13.4.8 [ET 12.4.8].)

9. This is true because $\mathbf{u} \times \mathbf{v}$ is orthogonal to $\mathbf{u}$ (see Theorem 13.4.5 [ET 12.4.5]), and the dot product of two orthogonal vectors is 0.

10.
$$\begin{aligned}(\mathbf{u} + \mathbf{v}) \times \mathbf{v} &= \mathbf{u} \times \mathbf{v} + \mathbf{v} \times \mathbf{v} && \text{(by Theorem 13.4.8 [ET 12.4.8] \#4)}\\ &= \mathbf{u} \times \mathbf{v} + \mathbf{0} && \text{(by Example 13.4.2 [ET 12.4.2])}\\ &= \mathbf{u} \times \mathbf{v}, \text{ so this is true.}\end{aligned}$$

11. If $|\mathbf{u}| = 1$, $|\mathbf{v}| = 1$ and θ is the angle between these two vectors (so $0 \leq \theta \leq \pi$), then by Theorem 13.4.6 [ET 12.4.6], $|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}|\,|\mathbf{v}|\sin\theta = \sin\theta$, which is equal to 1 if and only if $\theta = \frac{\pi}{2}$ (that is, if and only if the two vectors are orthogonal). Therefore, the assertion that the cross product of two unit vectors is a unit vector is false.

12. This is false, because according to Equation 13.5.8 [ET 12.5.8], $ax + by + cz + d = 0$ is the general equation of a plane.

13. This is false. In $\mathbb{R}^2$, $x^2 + y^2 = 1$ represents a circle, but $\{(x, y, z) \mid x^2 + y^2 = 1\}$ represents a *three-dimensional surface*, namely, a circular cylinder with axis the z-axis.

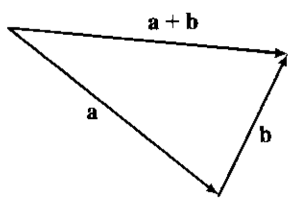
14. This is false, as the dot product of two vectors is a scalar, not a vector.

EXERCISES

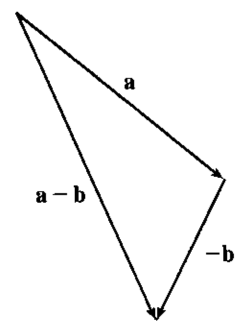
1. (a) By the formula for an equation of a sphere (see Section 13.1 [ET 12.1]), an equation of the sphere with center $(1, -1, 2)$ and radius 3 is $(x-1)^2 + (y+1)^2 + (z-2)^2 = 9$.

(b) Completing squares gives $(x+2)^2 + (y+3)^2 + (z-5)^2 = -2 + 4 + 9 + 25 = 36$. Thus, the sphere is centered at $(-2, -3, 5)$ and has radius 6.

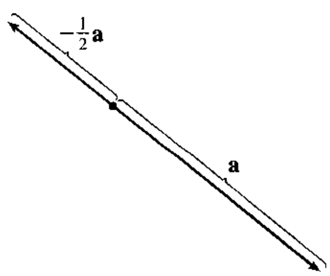
2. (a)



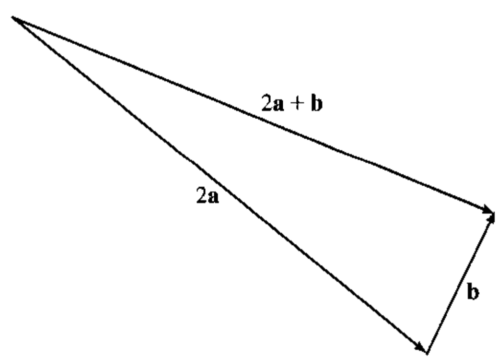
(b)



(c)



(d)



3. $\mathbf{u} \cdot \mathbf{v} = |\mathbf{u}|\,|\mathbf{v}| \cos 45^\circ = (2)(3)\frac{\sqrt{2}}{2} = 3\sqrt{2}$. $|\mathbf{u} \times \mathbf{v}| = |\mathbf{u}|\,|\mathbf{v}| \sin 45^\circ = (2)(3)\frac{\sqrt{2}}{2} = 3\sqrt{2}$. By the right-hand rule, $\mathbf{u} \times \mathbf{v}$ is directed out of the page.

4. (a) $2\mathbf{a} + 3\mathbf{b} = 2\mathbf{i} + 2\mathbf{j} - 4\mathbf{k} + 9\mathbf{i} - 6\mathbf{j} + 3\mathbf{k} = 11\mathbf{i} - 4\mathbf{j} - \mathbf{k}$

(b) $|\mathbf{b}| = \sqrt{9 + 4 + 1} = \sqrt{14}$

(c) $\mathbf{a} \cdot \mathbf{b} = (1)(3) + (1)(-2) + (-2)(1) = -1$

(d) $\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & -2 \\ 3 & -2 & 1 \end{vmatrix} = (1-4)\,\mathbf{i} - (1+6)\,\mathbf{j} + (-2-3)\,\mathbf{k} = -3\,\mathbf{i} - 7\,\mathbf{j} - 5\,\mathbf{k}$

(e) $\mathbf{b} \times \mathbf{c} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -2 & 1 \\ 0 & 1 & -5 \end{vmatrix} = 9\,\mathbf{i} + 15\,\mathbf{j} + 3\,\mathbf{k}$, $|\mathbf{b} \times \mathbf{c}| = 3\sqrt{9 + 25 + 1} = 3\sqrt{35}$

(f) $\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \begin{vmatrix} 1 & 1 & -2 \\ 3 & -2 & 1 \\ 0 & 1 & -5 \end{vmatrix} = \begin{vmatrix} -2 & 1 \\ 1 & -5 \end{vmatrix} - \begin{vmatrix} 3 & 1 \\ 0 & -5 \end{vmatrix} - 2\begin{vmatrix} 3 & -2 \\ 0 & 1 \end{vmatrix} = 9 + 15 - 6 = 18$

(g) $\mathbf{c}\times\mathbf{c} = \mathbf{0}$ for any $\mathbf{c}$.

(h) From part (e),

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = \mathbf{a}\times(9\,\mathbf{i}+15\,\mathbf{j}+3\,\mathbf{k}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & -2 \\ 9 & 15 & 3 \end{vmatrix} = (3+30)\,\mathbf{i} - (3+18)\,\mathbf{j} + (15-9)\,\mathbf{k}$$

$$= 33\,\mathbf{i} - 21\,\mathbf{j} + 6\,\mathbf{k}.$$

(i) The scalar projection is $\operatorname{comp}_{\mathbf{a}}\mathbf{b} = |\mathbf{b}|\cos\theta = \mathbf{a}\cdot\mathbf{b}/|\mathbf{a}| = -\frac{1}{\sqrt{6}}$.

(j) The vector projection is $\operatorname{proj}_{\mathbf{a}}\mathbf{b} = -\frac{1}{\sqrt{6}}(\mathbf{a}/|\mathbf{a}|) = -\frac{1}{6}(\mathbf{i}+\mathbf{j}-2\,\mathbf{k})$.

(k) $\cos\theta = \dfrac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|} = \dfrac{-1}{\sqrt{6}\sqrt{14}} = \dfrac{-1}{2\sqrt{21}}$ and $\theta = \cos^{-1}\left(\dfrac{-1}{2\sqrt{21}}\right) \approx 96^\circ$.

5. For the two vectors to be orthogonal, we need $\langle 3, 2, x\rangle\cdot\langle 2x, 4, x\rangle = 0 \quad\Leftrightarrow$ $(3)(2x) + (2)(4) + (x)(x) = 0 \quad\Leftrightarrow\quad x^2 + 6x + 8 = 0 \quad\Leftrightarrow\quad (x+2)(x+4) = 0 \quad\Leftrightarrow$ $x = -2$ or $x = -4$.

6. We know that the cross product of two vectors is orthogonal to both. So we calculate $(\mathbf{j}+2\,\mathbf{k})\times(\mathbf{i}-2\,\mathbf{j}+3\,\mathbf{k}) = [3-(-4)]\,\mathbf{i} - (0-2)\,\mathbf{j} + (0-1)\,\mathbf{k} = 7\,\mathbf{i}+2\,\mathbf{j}-\mathbf{k}$. Then two unit vectors orthogonal to both given vectors are $\pm\dfrac{7\,\mathbf{i}+2\,\mathbf{j}-\mathbf{k}}{\sqrt{7^2+2^2+(-1)^2}} = \pm\dfrac{1}{3\sqrt{6}}(7\,\mathbf{i}+2\,\mathbf{j}-\mathbf{k})$, that is, $\frac{7}{3\sqrt{6}}\,\mathbf{i} + \frac{2}{3\sqrt{6}}\,\mathbf{j} - \frac{1}{3\sqrt{6}}\,\mathbf{k}$ and $-\frac{7}{3\sqrt{6}}\,\mathbf{i} - \frac{2}{3\sqrt{6}}\,\mathbf{j} + \frac{1}{3\sqrt{6}}\,\mathbf{k}$.

7. (a) $(\mathbf{u}\times\mathbf{v})\cdot\mathbf{w} = \mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = 2$

(b) $\mathbf{u}\cdot(\mathbf{w}\times\mathbf{v}) = \mathbf{u}\cdot[-(\mathbf{v}\times\mathbf{w})] = -\mathbf{u}\cdot(\mathbf{v}\times\mathbf{w}) = -2$

(c) $\mathbf{v}\cdot(\mathbf{u}\times\mathbf{w}) = (\mathbf{v}\times\mathbf{u})\cdot\mathbf{w} = -(\mathbf{u}\times\mathbf{v})\cdot\mathbf{w} = -2$

(d) $(\mathbf{u}\times\mathbf{v})\cdot\mathbf{v} = \mathbf{u}\cdot(\mathbf{v}\times\mathbf{v}) = \mathbf{u}\cdot\mathbf{0} = 0$

8. $(\mathbf{a}\times\mathbf{b})\cdot[(\mathbf{b}\times\mathbf{c})\times(\mathbf{c}\times\mathbf{a})] = (\mathbf{a}\times\mathbf{b})\cdot([(\mathbf{b}\times\mathbf{c})\cdot\mathbf{a}]\,\mathbf{c} - [(\mathbf{b}\times\mathbf{c})\cdot\mathbf{c}]\,\mathbf{a})$

(see Exercise 13.4.42 [ET 12.4.42])

$$= (\mathbf{a}\times\mathbf{b})\cdot[(\mathbf{b}\times\mathbf{c})\cdot\mathbf{a}]\,\mathbf{c} = [\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})]\,(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}$$

$$= [\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})]\,[\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})] = [\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c})]^2$$

9. For simplicity, consider a unit cube positioned with its back left corner at the origin. Vector representations of the diagonals joining the points $(0,0,0)$ to $(1,1,1)$ and $(1,0,0)$ to $(0,1,1)$ are $\langle 1,1,1\rangle$ and $\langle -1,1,1\rangle$. Let θ be the angle between these two vectors. $\langle 1,1,1\rangle\cdot\langle -1,1,1\rangle = -1+1+1 = 1 = |\langle 1,1,1\rangle|\,|\langle -1,1,1\rangle|\cos\theta = 3\cos\theta$ $\Rightarrow \quad \cos\theta = \frac{1}{3} \quad\Rightarrow\quad \theta = \cos^{-1}\left(\frac{1}{3}\right) \approx 71^\circ$.

10. $\overrightarrow{AB} = \langle 1, 3, -1\rangle$, $\overrightarrow{AC} = \langle -2, 1, 3\rangle$ and $\overrightarrow{AD} = \langle -1, 3, 1\rangle$. By Equation 13.4.10 [ET 12.4.10],

$$\overrightarrow{AB}\cdot\left(\overrightarrow{AC}\times\overrightarrow{AD}\right) = \begin{vmatrix} 1 & 3 & -1 \\ -2 & 1 & 3 \\ -1 & 3 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix} - 3\begin{vmatrix} -2 & 3 \\ -1 & 1 \end{vmatrix} - \begin{vmatrix} -2 & 1 \\ -1 & 3 \end{vmatrix} = -8 - 3 + 5 = -6.$$

The volume is $\left|\overrightarrow{AB}\cdot\left(\overrightarrow{AC}\times\overrightarrow{AD}\right)\right| = 6$ cubic units.

11. $\overrightarrow{AB} = \langle 1, 0, -1 \rangle$, $\overrightarrow{AC} = \langle 0, 4, 3 \rangle$, so

(a) a vector perpendicular to the plane is $\overrightarrow{AB} \times \overrightarrow{AC} = \langle 0 + 4, -(3 + 0), 4 - 0 \rangle = \langle 4, -3, 4 \rangle$.

(b) $\frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right| = \frac{1}{2}\sqrt{16 + 9 + 16} = \frac{\sqrt{41}}{2}$.

12. $\mathbf{D} = 4\,\mathbf{i} + 3\,\mathbf{j} + 6\,\mathbf{k}$, $W = \mathbf{F} \cdot \mathbf{D} = 12 + 15 + 60 = 87$ joules

13. Let F_1 be the magnitude of the force directed $20°$ away from the direction of shore, and let F_2 be the magnitude of the other force. Separating these forces into components parallel to the direction of the resultant force and perpendicular to it gives $F_1 \cos 20° + F_2 \cos 30° = 255$ (1), and $F_1 \sin 20° - F_2 \sin 30° = 0 \Rightarrow$ $F_1 = F_2 \dfrac{\sin 30°}{\sin 20°}$ (2). Substituting (2) into (1) gives $F_2(\sin 30° \cot 20° + \cos 30°) = 255 \Rightarrow F_2 \approx 114$ N. Substituting this into (2) gives $F_1 \approx 166$ N.

14. $|\boldsymbol{\tau}| = |\mathbf{r}|\,|\mathbf{F}| \sin\theta = (0.40)(50)\sin(90° - 30°) \approx 17.3$ joules

15. The line has direction $\mathbf{v} = \langle -3, 2, 3 \rangle$. Letting $P_0 = (4, -1, 2)$, parametric equations are $x = 4 - 3t$, $y = -1 + 2t$, $z = 2 + 3t$.

16. A direction vector for the line is $\mathbf{v} = \langle 3, 2, 1 \rangle$, so parametric equations for the line are $x = 1 + 3t$, $y = 2t$, $z = -1 + t$.

17. A direction vector for the line is a normal vector for the plane, $\mathbf{n} = \langle 2, -1, 5 \rangle$, and parametric equations for the line are $x = -2 + 2t$, $y = 2 - t$, $z = 4 + 5t$.

18. Since the two planes are parallel, they will have the same normal vectors. Then we can take $\mathbf{n} = \langle 1, 4, -3 \rangle$ and an equation of the plane is $1(x - 2) + 4(y - 1) - 3(z - 0) = 0$ or $x + 4y - 3z = 6$.

19. Here the vectors $\mathbf{a} = \langle 4 - 3, 0 - (-1), 2 - 1 \rangle = \langle 1, 1, 1 \rangle$ and $\mathbf{b} = \langle 6 - 3, 3 - (-1), 1 - 1 \rangle = \langle 3, 4, 0 \rangle$ lie in the plane, so $\mathbf{n} = \mathbf{a} \times \mathbf{b} = \langle -4, 3, 1 \rangle$ is a normal vector to the plane and an equation of the plane is $-4(x - 3) + 3(y - (-1)) + 1(z - 1) = 0$ or $-4x + 3y + z = -14$.

20. If we first find two nonparallel vectors in the plane, their cross product will be a normal vector to the plane. Since the given line lies in the plane, its direction vector $\mathbf{a} = \langle 2, -1, 3 \rangle$ is one vector in the plane. We can verify that the given point $(1, 2, -2)$ does not lie on this line. The point $(0, 3, 1)$ is on the line (obtained by putting $t = 0$) and hence in the plane, so the vector $\mathbf{b} = \langle 0 - 1, 3 - 2, 1 - (-2) \rangle = \langle -1, 1, 3 \rangle$ lies in the plane, and a normal vector is $\mathbf{n} = \mathbf{a} \times \mathbf{b} = \langle -6, -9, 1 \rangle$. Thus an equation of the plane is $-6(x - 1) - 9(y - 2) + (z + 2) = 0$ or $6x + 93y - z = 26$.

21. Substitution of the parametric equations into the equation of the plane gives $2x - y + z = 2(2 - t) - (1 + 3t) + 4t = 2 \Rightarrow -t + 3 = 2 \Rightarrow t = 1$. When $t = 1$, the parametric equations give $x = 2 - 1 = 1$, $y = 1 + 3 = 4$ and $z = 4$. Therefore, the point of intersection is $(1, 4, 4)$.

22. Use the formula proven in Exercise 13.4.39 [ET 12.4.39]. In the notation used in that exercise, $\mathbf{a}$ is just the direction of the line; that is, $\mathbf{a} = \langle 1, -1, 2 \rangle$. A point on the line is $(1, 2, -1)$ (setting $t = 0$), and therefore $\mathbf{b} = \langle 1 - 0, 2 - 0, -1 - 0 \rangle = \langle 1, 2, -1 \rangle$. Hence

$$d = \frac{|\mathbf{a} \times \mathbf{b}|}{|\mathbf{a}|} = \frac{|\langle 1, -1, 2 \rangle \times \langle 1, 2, -1 \rangle|}{\sqrt{1 + 1 + 4}} = \frac{|\langle -3, 3, 3 \rangle|}{\sqrt{6}} = \sqrt{\frac{27}{6}} = \frac{3}{\sqrt{2}}.$$

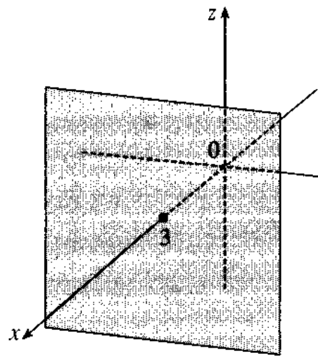
23. Since the direction vectors $\langle 2, 3, 4 \rangle$ and $\langle 6, -1, 2 \rangle$ aren't parallel, neither are the lines. For the lines to intersect, the three equations $1 + 2t = -1 + 6s$, $2 + 3t = 3 - s$, $3 + 4t = -5 + 2s$ must be satisfied simultaneously. Solving the first two equations gives $t = \frac{1}{5}$, $s = \frac{2}{5}$ and checking we see these values don't satisfy the third equation. Thus the lines aren't parallel and they don't intersect, so they must be skew.

24. (a) The normal vectors are $\langle 1, 1, -1\rangle$ and $\langle 2, -3, 4\rangle$. Since these vectors aren't parallel, neither are the planes parallel. Also $\langle 1, 1, -1\rangle \cdot \langle 2, -3, 4\rangle = 2 - 3 - 4 = -5 \neq 0$ so the normal vectors, and thus the planes, are not perpendicular.

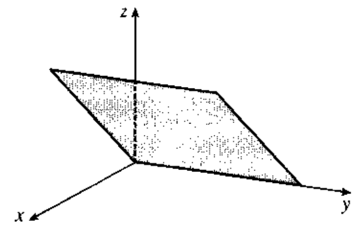
(b) $\cos\theta = \dfrac{\langle 1, 1, -1\rangle \cdot \langle 2, -3, 4\rangle}{\sqrt{3}\,\sqrt{29}} = -\dfrac{5}{\sqrt{87}}$ and $\theta = \cos^{-1}\left(-\frac{5}{\sqrt{87}}\right) \approx 122^\circ$ (or we can say $\approx 58^\circ$).

25. By Exercise 13.5.69 [ET 12.5.69], $D = \dfrac{|2 - 24|}{\sqrt{26}} = \dfrac{22}{\sqrt{26}}$.

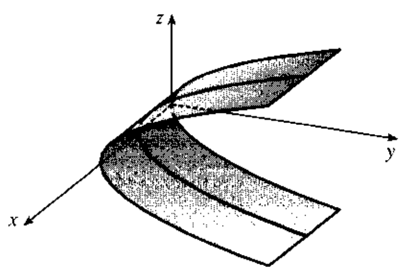
26. The equation $x = 3$ represents a plane parallel to the yz-plane and 3 units in front of it.



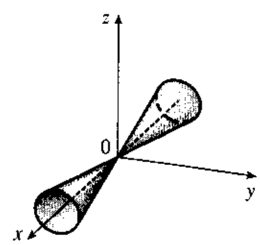
27. The equation $x = z$ represents a plane perpendicular to the xz-plane and intersecting the xz-plane in the line $x = z$, $y = 0$.



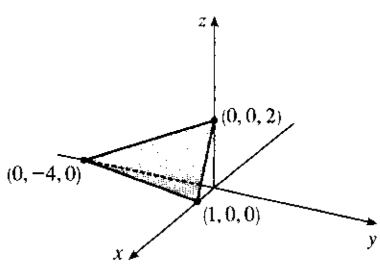
28. A parabolic cylinder whose trace in the xz-plane is the x-axis and which opens to the right.



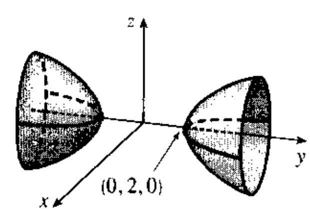
29. A (right elliptical) cone with vertex at the origin and axis the x-axis.



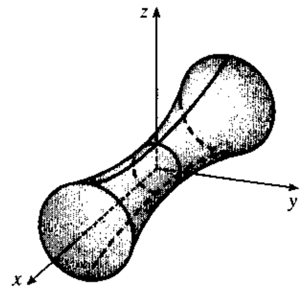
30. $4x - y + 2z = 4$ is a plane with intercepts $(1, 0, 0)$, $(0, -4, 0)$, and $(0, 0, 2)$.



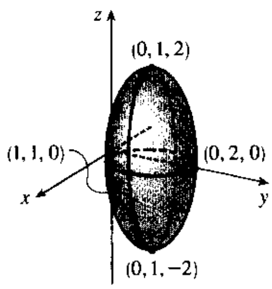
31. An equivalent equation is $-x^2 + \dfrac{y^2}{4} - z^2 = 1$, a hyperboloid of two sheets with axis the y-axis. For $|y| > 2$, traces parallel to the xz-plane are circles.



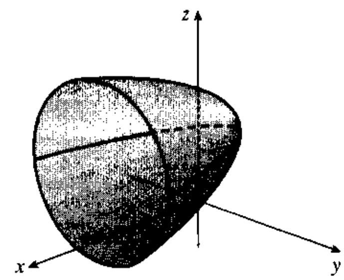
32. An equivalent equation is $-x^2 + y^2 + z^2 = 1$, a hyperboloid of one sheet with axis the x-axis.



33. Completing the square in y gives
$4x^2 + 4(y-1)^2 + z^2 = 4$ or
$x^2 + (y-1)^2 + \dfrac{z^2}{4} = 1$, an ellipsoid centered at $(0, 1, 0)$.



34. Completing the square in y and z gives
$x = (y-1)^2 + (z-2)^2$, a circular paraboloid with vertex $(0, 1, 2)$ and axis the horizontal line $y = 1$, $z = 2$.



35. $4x^2 + y^2 = 16 \quad \Leftrightarrow \quad \dfrac{x^2}{4} + \dfrac{y^2}{16} = 1$. The equation of the ellipsoid is $\dfrac{x^2}{4} + \dfrac{y^2}{16} + \dfrac{z^2}{c^2} = 1$, since the horizontal trace in the plane $z = 0$ must be the original ellipse. The traces of the ellipsoid in the yz-plane must be circles since the surface is obtained by rotation about the x-axis. Therefore, $c^2 = 16$ and the equation of the ellipsoid is
$\dfrac{x^2}{4} + \dfrac{y^2}{16} + \dfrac{z^2}{16} = 1 \quad \Leftrightarrow \quad 4x^2 + y^2 + z^2 = 16$.

36. The distance from a point $P(x, y, z)$ to the plane $y = 1$ is $|y - 1|$, so the given condition becomes
$|y-1| = 2\sqrt{(x-0)^2 + (y+1)^2 + (z-0)^2} \quad \Rightarrow \quad |y-1| = 2\sqrt{x^2 + (y+1)^2 + z^2} \quad \Rightarrow$
$(y-1)^2 = 4x^2 + 4(y+1)^2 + 4z^2 \quad \Leftrightarrow \quad -3 = 4x^2 + (3y^2 + 10y) + 4z^2 \quad \Leftrightarrow$
$\frac{16}{3} = 4x^2 + 3\left(y + \frac{5}{3}\right)^2 + 4z^2 \quad \Rightarrow \quad \frac{3}{4}x^2 + \frac{9}{16}\left(y + \frac{5}{3}\right)^2 + \frac{3}{4}z^2 = 1$. This is the equation of an ellipsoid whose center is $\left(0, -\frac{5}{3}, 0\right)$.

37. $x = r\cos\theta = 2\sqrt{3}\cos\frac{\pi}{3} = 2\sqrt{3}\cdot\frac{1}{2} = \sqrt{3}$, $y = r\sin\theta = 2\sqrt{3}\sin\frac{\pi}{3} = 2\sqrt{3}\cdot\frac{\sqrt{3}}{2} = 3$, $z = 2$, so in rectangular coordinates the point is $\left(\sqrt{3}, 3, 2\right)$. $\rho = \sqrt{r^2 + z^2} = \sqrt{12 + 4} = 4$, $\theta = \frac{\pi}{3}$, and $\cos\phi = \frac{z}{\rho} = \frac{1}{2}$, so $\phi = \frac{\pi}{3}$ and spherical coordinates are $\left(4, \frac{\pi}{3}, \frac{\pi}{3}\right)$.

38. $r = \sqrt{4 + 4} = 2\sqrt{2}$, $z = -1$, $\cos\theta = \frac{2}{2\sqrt{2}} = \frac{\sqrt{2}}{2}$ so $\theta = \frac{\pi}{4}$ and in cylindrical coordinates the point is $\left(2\sqrt{2}, \frac{\pi}{4}, -1\right)$. $\rho = \sqrt{4 + 4 + 1} = 3$, $\cos\phi = -\frac{1}{3}$, so the spherical coordinates are $\left(3, \frac{\pi}{4}, \cos^{-1}\left(-\frac{1}{3}\right)\right)$.

39. $x = \rho\sin\phi\cos\theta = 8\sin\frac{\pi}{6}\cos\frac{\pi}{4} = 8\cdot\frac{1}{2}\cdot\frac{\sqrt{2}}{2} = 2\sqrt{2}$, $y = \rho\sin\phi\sin\theta = 8\sin\frac{\pi}{6}\sin\frac{\pi}{4} = 2\sqrt{2}$, and
$z = \rho\cos\phi = 8\cos\frac{\pi}{6} = 8\cdot\frac{\sqrt{3}}{2} = 4\sqrt{3}$. Thus rectangular coordinates for the point are $\left(2\sqrt{2}, 2\sqrt{2}, 4\sqrt{3}\right)$.
$r^2 = x^2 + y^2 = 8 + 8 = 16 \quad \Rightarrow \quad r = 4$, $\theta = \frac{\pi}{4}$, and $z = 4\sqrt{3}$, so cylindrical coordinates are $\left(4, \frac{\pi}{4}, 4\sqrt{3}\right)$.

40. $\phi = \frac{\pi}{4}$. This is one frustum of a circular cone with vertex the origin and axis the positive z-axis.

41. $\theta = \frac{\pi}{4}$. In spherical coordinates, this is a half-plane including the z-axis and intersecting the xy-plane in the half-line $x = y$, $x > 0$.

42. Since $r = \cos\theta$ and $x = r\cos\theta$, $x = \cos^2\theta$. Also $r^2 = x^2 + y^2$ so $x^2 + y^2 = 2\cos^2\theta$. Thus $x^2 + y^2 - 2x = 0$ or $(x-1)^2 + y^2 = 1$. Thus the surface is a circular cylinder with axis the line $x = 1$, $y = 0$, $z = z$.

43. Since $\rho = 3\sec\phi$, $\rho\cos\phi = 3$ or $z = 3$. Thus the surface is a plane parallel to the xy-plane and through the point $(0, 0, 3)$.

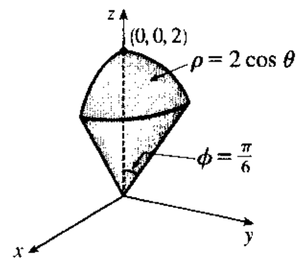
44. $x^2 + y^2 = 4$. In cylindrical coordinates: $r^2 = 4$. In spherical coordinates: $\rho^2 - z^2 = 4$ or $\rho^2 - \rho^2\cos^2\phi = 4$ or $\rho^2\sin^2\phi = 4$ or $\rho\sin\phi = 2$.

45. $x^2 + y^2 + z^2 = 4$. In cylindrical coordinates, this becomes $r^2 + z^2 = 4$. In spherical coordinates, it becomes $\rho^2 = 4$ or $\rho = 2$.

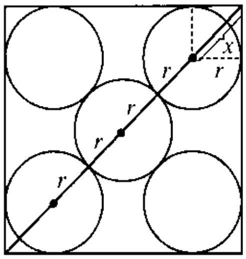
46. In cylindrical coordinates: $r^2 + z^2 = 2r\cos\theta$ or $z^2 = r(2\cos\theta - r)$.
In spherical coordinates: $\rho^2 = 2\rho\sin\phi\cos\theta$ or $\rho = 2\sin\phi\cos\theta$.

47. The resulting surface is a circular paraboloid with equation $z = 4x^2 + 4y^2$. Changing to cylindrical coordinates we have $z = 4(x^2 + y^2) = 4r^2$.

48. $\rho = 2\cos\phi \quad\Rightarrow\quad \rho^2 = 2\rho\cos\phi \quad\Rightarrow\quad x^2 + y^2 + z^2 = 2z \quad\Rightarrow$ $x^2 + y^2 + (z-1)^2 = 1$. This is the equation of a sphere with radius 1, centered at $(0, 0, 1)$. Therefore, $0 \le \rho \le 2\cos\phi$ is the solid ball whose boundary is this sphere. $0 \le \theta \le \frac{\pi}{2}$ and $0 \le \phi \le \frac{\pi}{6}$ restrict the solid to the section of this ball that lies above the cone $\phi = \frac{\pi}{6}$ and is in the first octant.



PROBLEMS PLUS

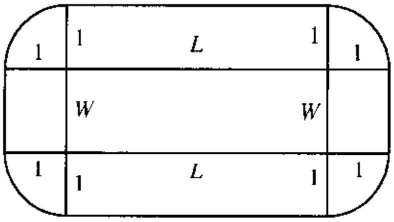


1. Since three-dimensional situations are often difficult to visualize and work with, let us first try to find an analogous problem in two dimensions. The analogue of a cube is a square and the analogue of a sphere is a circle. Thus a similar problem in two dimensions is the following: if five circles with the same radius r are contained in a square of side 1 m so that the circles touch each other and four of the circles touch two sides of the square, find r.

The diagonal of the square is $\sqrt{2}$. The diagonal is also $4r + 2x$. But x is the diagonal of a smaller square of side r. Therefore $x = \sqrt{2}\,r \quad \Rightarrow \quad \sqrt{2} = 4r + 2x = 4r + 2\sqrt{2}\,r = (4 + 2\sqrt{2})r \quad \Rightarrow \quad r = \frac{\sqrt{2}}{4 + 2\sqrt{2}}$.

Let us use these ideas to solve the original three-dimensional problem. The diagonal of the cube is $\sqrt{1^2 + 1^2 + 1^2} = \sqrt{3}$. The diagonal of the cube is also $4r + 2x$ where x is the diagonal of a smaller cube with edge r. Therefore $x = \sqrt{r^2 + r^2 + r^2} = \sqrt{3}\,r \quad \Rightarrow \quad \sqrt{3} = 4r + 2x = 4r + 2\sqrt{3}\,r = (4 + 2\sqrt{3})r$. Thus $r = \dfrac{\sqrt{3}}{4 + 2\sqrt{3}} = \dfrac{2\sqrt{3} - 3}{2}$. The radius of each ball is $\left(\sqrt{3} - \frac{3}{2}\right)$ m.

2. Try an analogous problem in two dimensions. Consider a rectangle with length L and width W and find the area of S in terms of L and W. Since S contains B, it has area



$$\begin{aligned} A(S) = LW &+ \text{the area of two } L \times 1 \text{ rectangles} \\ &+ \text{the area of two } 1 \times W \text{ rectangles} \\ &+ \text{the area of four quarter-circles of radius 1} \end{aligned}$$

as seen in the diagram. So $A(S) = LW + 2L + 2W + \pi \cdot 1^2$.

Now in three dimensions, the volume of S is

$$\begin{aligned} LWH &+ 2(L \times W \times 1) + 2(1 \times W \times H) + 2(L \times 1 \times H) \\ &+ \text{the volume of 4 quarter-cylinders with radius 1 and height } W \\ &+ \text{the volume of 4 quarter-cylinders with radius 1 and height } L \\ &+ \text{the volume of 4 quarter-cylinders with radius 1 and height } H \\ &+ \text{the volume of 8 eighths of a sphere of radius 1} \end{aligned}$$

So

$$\begin{aligned} V(S) &= LWH + 2LW + 2WH + 2LH + \pi \cdot 1^2 \cdot W + \pi \cdot 1^2 \cdot L + \pi \cdot 1^2 \cdot H + \tfrac{4}{3}\pi \cdot 1^3 \\ &= LWH + 2(LW + WH + LH) + \pi(L + W + H) + \tfrac{4}{3}\pi. \end{aligned}$$

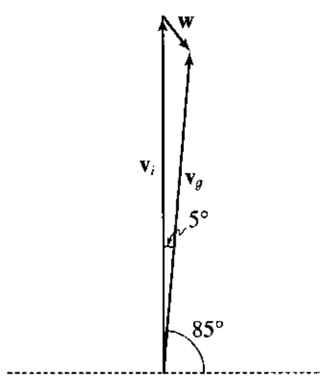
3. (a) We find the line of intersection L as in Example 13.5.7(b) [ET 12.5.7(b)]. Observe that the point $(-1, c, c)$ lies on both planes. Now since L lies in both planes, it is perpendicular to both of the normal vectors $\mathbf{n}_1$ and $\mathbf{n}_2$, and thus parallel to their cross product $\mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ c & 1 & 1 \\ 1 & -c & c \end{vmatrix} = \langle 2c, -c^2 + 1, -c^2 - 1 \rangle$. So symmetric equations of L can be written as $\dfrac{x+1}{-2c} = \dfrac{y-c}{c^2-1} = \dfrac{z-c}{c^2+1}$, provided that $c \neq 0, \pm 1$.

If $c = 0$, then the two planes are given by $y + z = 0$ and $x = -1$, so symmetric equations of L are $x = -1$, $y = -z$. If $c = -1$, then the two planes are given by $-x + y + z = -1$ and $x + y + z = -1$, and they intersect in the line $x = 0$, $y = -z - 1$. If $c = 1$, then the two planes are given by $x + y + z = 1$ and $x - y + z = 1$, and they intersect in the line $y = 0$, $x = 1 - z$.

(b) If we set $z = t$ in the symmetric equations and solve for x and y separately, we get $x + 1 = \dfrac{(t-c)(-2c)}{c^2+1}$, $y - c = \dfrac{(t-c)(c^2-1)}{c^2+1} \Rightarrow x = \dfrac{-2ct + (c^2-1)}{c^2+1}$, $y = \dfrac{(c^2-1)t + 2c}{c^2+1}$. Eliminating c from these equations, we have $x^2 + y^2 = t^2 + 1$. So the curve traced out by L in the plane $z = t$ is a circle with center at $(0, 0, t)$ and radius $\sqrt{t^2+1}$.

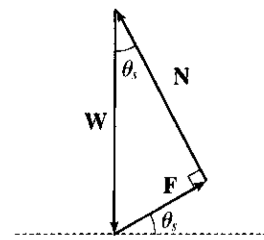
(c) The area of a horizontal cross-section of the solid is $A(z) = \pi(z^2 + 1)$, so $V = \int_0^1 A(z)dz = \pi\left[\frac{1}{3}z^3 + z\right]_0^1 = \frac{4\pi}{3}$.

4. (a) We consider velocity vectors for the plane and the wind. Let $\mathbf{v}_i$ be the initial, intended velocity for the plane and $\mathbf{v}_g$ the actual velocity relative to the ground. If $\mathbf{w}$ is the velocity of the wind, $\mathbf{v}_g$ is the resultant, that is, the vector sum $\mathbf{v}_i + \mathbf{w}$ as shown in the figure. We know $\mathbf{v}_i = 180\,\mathbf{j}$, and since the plane actually flew 80 km in $\frac{1}{2}$ hour, $|\mathbf{v}_g| = 160$. Thus $\mathbf{v}_g = (160\cos 85°)\,\mathbf{i} + (160 \sin 85°)\,\mathbf{j} \approx 13.9\,\mathbf{i} + 159.4\,\mathbf{j}$. Finally, $\mathbf{v}_i + \mathbf{w} = \mathbf{v}_g$, so $\mathbf{w} = \mathbf{v}_g - \mathbf{v}_i \approx 13.9\,\mathbf{i} - 20.6\,\mathbf{j}$. Thus, the wind velocity is about $13.9\,\mathbf{i} - 20.6\,\mathbf{j}$, and the wind speed is $|\mathbf{w}| \approx \sqrt{(13.9)^2 + (-20.6)^2} \approx 24.9$ km/h.



(b) Let $\mathbf{v}$ be the velocity the pilot should take. With the effect of wind, the actual velocity (with respect to the ground) will be $\mathbf{v} + \mathbf{w}$, which we want to be $\mathbf{v}_i$. Thus $\mathbf{v} = \mathbf{v}_i - \mathbf{w} \approx 180\,\mathbf{j} - (13.9\,\mathbf{i} - 20.6\,\mathbf{j}) \approx -13.9\,\mathbf{i} + 200.6\,\mathbf{j}$. The angle for this vector can be found by $\tan\theta \approx \frac{200.6}{-13.9} \Rightarrow \theta \approx 94.0°$, or $4.0°$ west of north.

5. (a) When $\theta = \theta_s$, the block is not moving, so the sum of the forces on the block must be $\mathbf{0}$, thus $\mathbf{N} + \mathbf{F} + \mathbf{W} = \mathbf{0}$. This relationship is illustrated geometrically in the figure. Since the vectors form a right triangle, we have $\tan(\theta_s) = \dfrac{|\mathbf{F}|}{|\mathbf{N}|} = \dfrac{\mu_s n}{n} = \mu_s$.



(b) We place the block at the origin and sketch the force vectors acting on the block, including the additional horizontal force $\mathbf{H}$, with initial points at the origin. We then rotate this system so that $\mathbf{F}$ lies along the positive x-axis and the inclined plane is parallel to the x-axis.

$|\mathbf{F}|$ is maximal, so $|\mathbf{F}| = \mu_s n$ for $\theta > \theta_s$. Then the vectors, in terms of components parallel and perpendicular to the inclined plane, are

$$\mathbf{N} = n\,\mathbf{j} \qquad \mathbf{F} = (\mu_s n)\,\mathbf{i}$$

$$\mathbf{W} = (-mg\sin\theta)\,\mathbf{i} + (-mg\cos\theta)\,\mathbf{j}$$

$$\mathbf{H} = (h_{\min}\cos\theta)\,\mathbf{i} + (-h_{\min}\sin\theta)\,\mathbf{j}$$

Equating components, we have

$$\mu_s n - mg\sin\theta + h_{\min}\cos\theta = 0 \quad\Rightarrow\quad h_{\min}\cos\theta + \mu_s n = mg\sin\theta \qquad (1)$$

$$n - mg\cos\theta - h_{\min}\sin\theta = 0 \quad\Rightarrow\quad h_{\min}\sin\theta + mg\cos\theta = n \qquad (2)$$

(c) Since (2) is solved for n, we substitute into (1):

$$h_{\min}\cos\theta + \mu_s(h_{\min}\sin\theta + mg\cos\theta) = mg\sin\theta \quad\Rightarrow$$

$$h_{\min}\cos\theta + h_{\min}\mu_s\sin\theta = mg\sin\theta - mg\mu_s\cos\theta \quad\Rightarrow$$

$$h_{\min} = mg\left(\frac{\sin\theta - \mu_s\cos\theta}{\cos\theta + \mu_s\sin\theta}\right) = mg\left(\frac{\tan\theta - \mu_s}{1 + \mu_s\tan\theta}\right)$$

From part (a) we know $\mu_s = \tan\theta_s$, so this becomes $h_{\min} = mg\left(\dfrac{\tan\theta - \tan\theta_s}{1 + \tan\theta_s\tan\theta}\right)$ and using a trigonometric identity, this is $mg\tan(\theta - \theta_s)$ as desired.

Note for $\theta = \theta_s$, $h_{\min} = mg\tan 0 = 0$, which makes sense since the block is at rest for θ_s, thus no additional force $\mathbf{H}$ is necessary to prevent it from moving. As θ increases, the factor $\tan(\theta - \theta_s)$, and hence the value of $h_{\min}$, increases slowly for small values of $\theta - \theta_s$ but much more rapidly as $\theta - \theta_s$ becomes significant. This seems reasonable, as the steeper the inclined plane, the less the horizontal components of the various forces affect the movement of the block, so we would need a much larger magnitude of horizontal force to keep the block motionless. If we allow $\theta \to 90^\circ$, corresponding to the inclined plane being placed vertically, the value of $h_{\min}$ is quite large; this is to be expected, as it takes a great amount of horizontal force to keep an object from moving vertically. In fact, without friction (so $\theta_s = 0$), we would have $\theta \to 90^\circ \Rightarrow h_{\min} \to \infty$, and it would be impossible to keep the block from slipping.

(d) Since $h_{\max}$ is the largest value of h that keeps the block from slipping, the force of friction is keeping the block from moving *up* the inclined plane; thus, $\mathbf{F}$ is directed *down* the plane. Our system of forces is similar to that in part (b), then, except that we have $\mathbf{F} = -(\mu_s n)\,\mathbf{i}$. (Note that $|\mathbf{F}|$ is again maximal.) Following our procedure in parts (b) and (c), we equate components:

$$-\mu_s n - mg\sin\theta + h_{\max}\cos\theta = 0 \quad\Rightarrow\quad h_{\max}\cos\theta - \mu_s n = mg\sin\theta$$

$$n - mg\cos\theta - h_{\max}\sin\theta = 0 \quad\Rightarrow\quad h_{\max}\sin\theta + mg\cos\theta = n$$

Then substituting,

$$h_{\max}\cos\theta - \mu_s(h_{\max}\sin\theta + mg\cos\theta) = mg\sin\theta \quad\Rightarrow$$

$$h_{\max}\cos\theta - h_{\max}\mu_s\sin\theta = mg\sin\theta + mg\mu_s\cos\theta \quad\Rightarrow$$

$$h_{\max} = mg\left(\frac{\sin\theta + \mu_s\cos\theta}{\cos\theta - \mu_s\sin\theta}\right) = mg\left(\frac{\tan\theta + \mu_s}{1 - \mu_s\tan\theta}\right)$$

$$= mg\left(\frac{\tan\theta + \tan\theta_s}{1 - \tan\theta_s\tan\theta}\right) = mg\tan(\theta + \theta_s)$$

We would expect $h_{\max}$ to increase as θ increases, with similar behavior as we established for $h_{\min}$, but with $h_{\max}$ values always larger than $h_{\min}$. We can see that this is the case if we graph $h_{\max}$ as a function of θ, as the curve is the graph of $h_{\min}$ translated $2\theta_s$ to the left, so the equation does seem reasonable. Notice that the equation predicts $h_{\max} \to \infty$ as $\theta \to (90^\circ - \theta_s)$. In fact, as $h_{\max}$ increases, the normal force increases as well. When $(90^\circ - \theta_s) \leq \theta \leq 90^\circ$, the horizontal force is completely counteracted by the sum of the normal and frictional forces, so no part of the horizontal force contributes to moving the block up the plane no matter how large its magnitude.