# Chapter 14

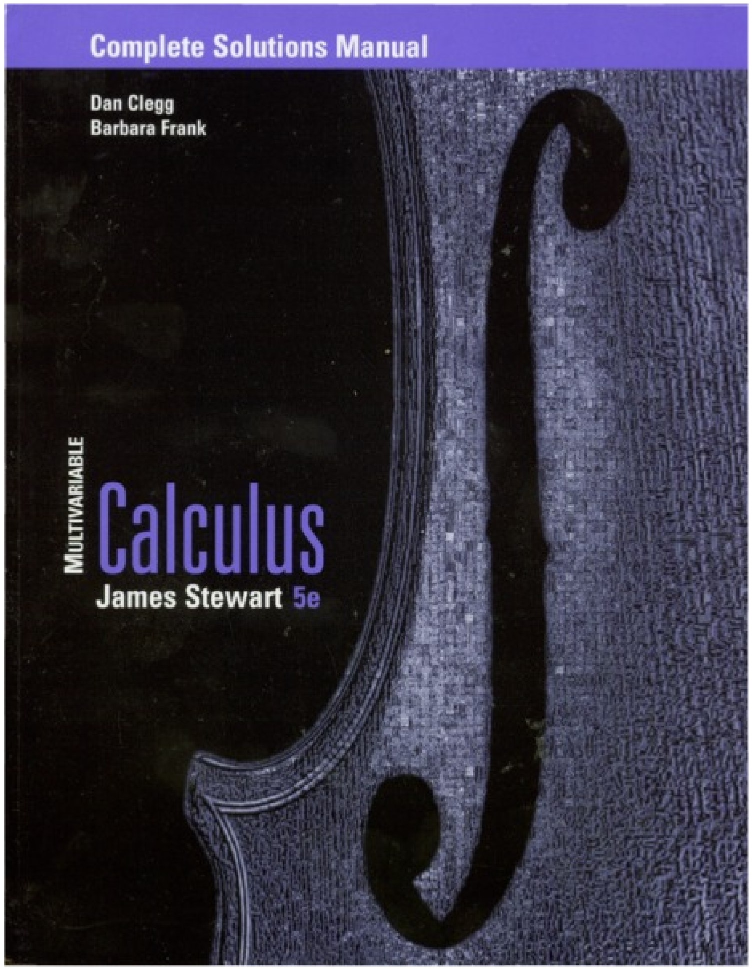
*Adapted from the*

## Complete Solutions Manual

*for*

## James Stewart's

## Calculus – 5th Edition

# 14 ☐ VECTOR FUNCTIONS ☐ ET 13

## 14.1 Vector Functions and Space Curves ET 13.1

**1.** The component functions $t^2$, $\sqrt{t-1}$, and $\sqrt{5-t}$ are all defined when $t-1 \geq 0 \Rightarrow t \geq 1$ and $5-t \geq 0 \Rightarrow t \leq 5$, so the domain of $\mathbf{r}(t)$ is $[1,5]$.

**2.** The component functions $\dfrac{t-2}{t+2}$, $\sin t$, and $\ln(9-t^2)$ are all defined when $t \neq -2$ and $9-t^2>0 \Rightarrow$ $-3<t<3$, so the domain of $\mathbf{r}(t)$ is $(-3,-2)\cup(-2,3)$.

**3.** $\lim\limits_{t\to 0^+}\cos t = \cos 0 = 1$, $\lim\limits_{t\to 0^+}\sin t = \sin 0 = 0$, $\lim\limits_{t\to 0^+} t\ln t = \lim\limits_{t\to 0^+}\dfrac{\ln t}{1/t} = \lim\limits_{t\to 0^+}\dfrac{1/t}{-1/t^2} = \lim\limits_{t\to 0^+} -t = 0$

[by l'Hospital's Rule]. Thus $\lim\limits_{t\to 0^+}\langle \cos t, \sin t, t\ln t\rangle = \left\langle \lim\limits_{t\to 0^+}\cos t, \lim\limits_{t\to 0^+}\sin t, \lim\limits_{t\to 0^+} t\ln t\right\rangle = \langle 1,0,0\rangle$.

**4.** $\lim\limits_{t\to 0}\dfrac{e^t-1}{t} = \lim\limits_{t\to 0}\dfrac{e^t}{1} = 1$ [using l'Hospital's Rule],

$\lim\limits_{t\to 0}\dfrac{\sqrt{1+t}-1}{t} = \lim\limits_{t\to 0}\dfrac{\sqrt{1+t}-1}{t}\cdot\dfrac{\sqrt{1+t}+1}{\sqrt{1+t}+1} = \lim\limits_{t\to 0}\dfrac{1}{\sqrt{1+t}+1} = \dfrac{1}{2}$, $\lim\limits_{t\to 0}\dfrac{3}{1+t} = 3$.

Thus the given limit equals $\langle 1, \frac{1}{2}, 3\rangle$.

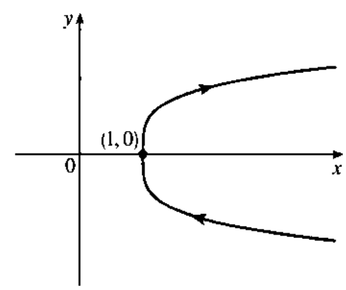
**5.** $\lim\limits_{t\to 1}\sqrt{t+3} = 2$, $\lim\limits_{t\to 1}\dfrac{t-1}{t^2-1} = \lim\limits_{t\to 1}\dfrac{1}{t+1} = \dfrac{1}{2}$, $\lim\limits_{t\to 1}\left(\dfrac{\tan t}{t}\right) = \tan 1$.

Thus the given limit equals $\langle 2, \frac{1}{2}, \tan 1\rangle$.

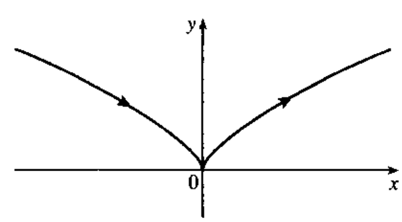
**6.** $\lim\limits_{t\to\infty}\arctan t = \frac{\pi}{2}$, $\lim\limits_{t\to\infty} e^{-2t} = 0$, $\lim\limits_{t\to\infty}\dfrac{\ln t}{t} = \lim\limits_{t\to\infty}\dfrac{1/t}{1} = 0$ [by l'Hospital's Rule].

Thus $\lim\limits_{t\to\infty}\left\langle \arctan t, e^{-2t}, \dfrac{\ln t}{t}\right\rangle = \langle \frac{\pi}{2}, 0, 0\rangle$.

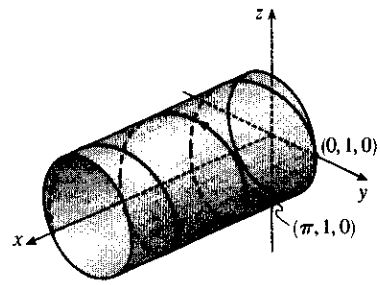
**7.** The corresponding parametric equations for this curve are $x = t^4+1$, $y=t$. We can make a table of values, or we can eliminate the parameter: $t = y \Rightarrow x = y^4+1$, with $y\in\mathbb{R}$. By comparing different values of $t$, we find the direction in which $t$ increases as indicated in the graph.



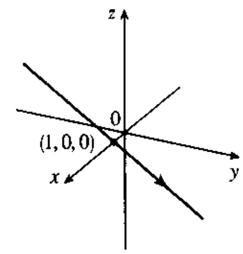

**8.** The corresponding parametric equations for this curve are $x = t^3, y = t^2$. We can make a table of values, or we can eliminate the parameter: $x = t^3 \Rightarrow t = \sqrt[3]{x} \Rightarrow$ $y = t^2 = (\sqrt[3]{x})^2 = x^{2/3}$, with $t\in\mathbb{R} \Rightarrow x\in\mathbb{R}$. By comparing different values of $t$, we find the direction in which $t$ increases as indicated in the graph.



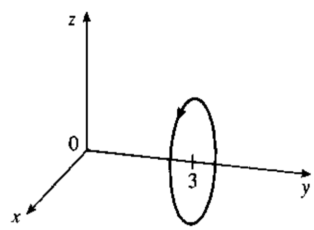
**9.** The corresponding parametric equations are $x = t$, $y = \cos 2t$, $z = \sin 2t$. Note that $y^2 + z^2 = \cos^2 2t + \sin^2 2t = 1$, so the curve lies on the circular cylinder $y^2 + z^2 = 1$. Since $x = t$, the curve is a helix.



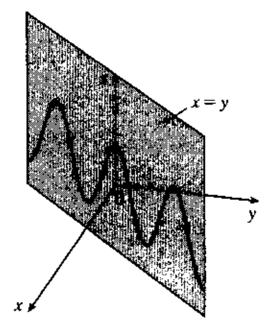

**10.** The corresponding parametric equations are $x = 1 + t$, $y = 3t$, $z = -t$, which are parametric equations of a line through the point $(1, 0, 0)$ and with direction vector $\langle 1, 3, -1 \rangle$.



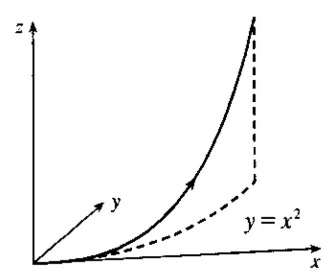

**11.** The parametric equations give $x^2 + z^2 = \sin^2 t + \cos^2 t = 1$, $y = 3$, which is a circle of radius 1, center $(0, 3, 0)$ in the plane $y = 3$.



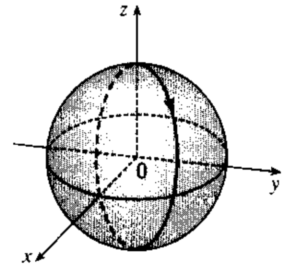

**12.** The parametric equations are $x = t$, $y = t$, $z = \cos t$. Thus $x = y$, so the curve must lie in the plane $x = y$. Combine this with $z = \cos t$ to determine that the curve traces out the cosine curve in the vertical plane $x = y$.




**13.** The parametric equations are $x = t^2$, $y = t^4$, $z = t^6$. These are positive for $t \neq 0$ and 0 when $t = 0$. So the curve lies entirely in the first quadrant. The projection of the graph onto the $xy$-plane is $y = x^2$, $y > 0$, a half parabola. On the $xz$-plane $z = x^3$, $z > 0$, a half cubic, and the $yz$-plane, $y^3 = z^2$.



**14.** The parametric equations give $x^2 + y^2 + z^2 = 2\sin^2 t + 2\cos^2 t = 2$, so the curve lies on the sphere with radius $\sqrt{2}$ and center $(0,0,0)$. Furthermore $x = y = \sin t$, so the curve is the intersection of this sphere with the plane $x = y$, that is, the curve is the circle of radius $\sqrt{2}$, center $(0,0,0)$ in the plane $x = y$.




**15.** Taking $\mathbf{r}_0 = \langle 0,0,0\rangle$ and $\mathbf{r}_1 = \langle 1,2,3\rangle$, we have from Equation 13.5.4 [ET 12.5.4]
$\mathbf{r}(t) = (1-t)\,\mathbf{r}_0 + t\,\mathbf{r}_1 = (1-t)\langle 0,0,0\rangle + t\langle 1,2,3\rangle, 0 \le t \le 1$ or $\mathbf{r}(t) = \langle t, 2t, 3t\rangle, 0 \le t \le 1$.
Parametric equations are $x = t$, $y = 2t$, $z = 3t$, $0 \le t \le 1$.

**16.** Taking $\mathbf{r}_0 = \langle 1,0,1\rangle$ and $\mathbf{r}_1 = \langle 2,3,1\rangle$, we have from Equation 13.5.4 [ET 12.5.4]
$\mathbf{r}(t) = (1-t)\,\mathbf{r}_0 + t\,\mathbf{r}_1 = (1-t)\langle 1,0,1\rangle + t\langle 2,3,1\rangle, 0 \le t \le 1$ or $\mathbf{r}(t) = \langle 1+t, 3t, 1\rangle, 0 \le t \le 1$.
Parametric equations are $x = 1+t$, $y = 3t$, $z = 1$, $0 \le t \le 1$.

**17.** Taking $\mathbf{r}_0 = \langle 1,-1,2\rangle$ and $\mathbf{r}_1 = \langle 4,1,7\rangle$, we have $\mathbf{r}(t) = (1-t)\,\mathbf{r}_0 + t\,\mathbf{r}_1 = (1-t)\langle 1,-1,2\rangle + t\langle 4,1,7\rangle$, $0 \le t \le 1$ or $\mathbf{r}(t) = \langle 1+3t, -1+2t, 2+5t\rangle, 0 \le t \le 1$. Parametric equations are $x = 1+3t$, $y = -1+2t$, $z = 2+5t$, $0 \le t \le 1$.

**18.** Taking $\mathbf{r}_0 = \langle -2,4,0\rangle$ and $\mathbf{r}_1 = \langle 6,-1,2\rangle$, we have $\mathbf{r}(t) = (1-t)\,\mathbf{r}_0 + t\,\mathbf{r}_1 = (1-t)\langle -2,4,0\rangle + t\langle 6,-1,2\rangle$, $0 \le t \le 1$ or $\mathbf{r}(t) = \langle -2+8t, 4-5t, 2t\rangle, 0 \le t \le 1$. Parametric equations are $x = -2+8t$, $y = 4-5t$, $z = 2t$, $0 \le t \le 1$.

**19.** $x = \cos 4t$, $y = t$, $z = \sin 4t$. At any point $(x,y,z)$ on the curve, $x^2 + z^2 = \cos^2 4t + \sin^2 4t = 1$. So the curve lies on a circular cylinder with axis the $y$-axis. Since $y = t$, this is a helix. So the graph is VI.

**20.** $x = t$, $y = t^2$, $z = e^{-t}$. At any point on the curve, $y = x^2$. So the curve lies on the parabolic cylinder $y = x^2$. Note that $y$ and $z$ are positive for all $t$, and the point $(0,0,1)$ is on the curve (when $t = 0$). As $t \to \infty$, $(x,y,z) \to (\infty,\infty,0)$, while as $t \to -\infty$, $(x,y,z) \to (-\infty,\infty,\infty)$, so the graph must be II.

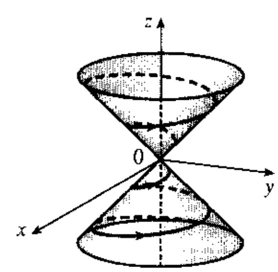
**21.** $x = t$, $y = 1/(1+t^2)$, $z = t^2$. Note that $y$ and $z$ are positive for all $t$. The curve passes through $(0,1,0)$ when $t = 0$. As $t \to \infty$, $(x,y,z) \to (\infty,0,\infty)$, and as $t \to -\infty$, $(x,y,z) \to (-\infty,0,\infty)$. So the graph is IV.

**22.** $x = e^{-t}\cos 10t$, $y = e^{-t}\sin 10t$, $z = e^{-t}$.
$x^2 + y^2 = e^{-2t}\cos^2 10t + e^{-2t}\sin^2 10t = e^{-2t}(\cos^2 10t + \sin^2 10t) = e^{-2t} = z^2$, so the curve lies on the cone $x^2 + y^2 = z^2$. Also, $z$ is always positive; the graph must be I.

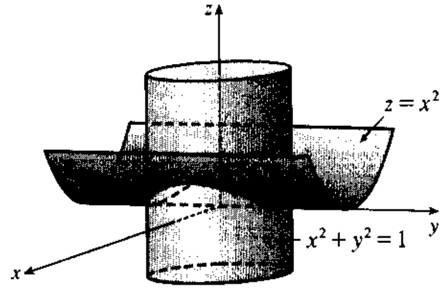
**23.** $x = \cos t$, $y = \sin t$, $z = \sin 5t$. $x^2 + y^2 = \cos^2 t + \sin^2 t = 1$, so the curve lies on a circular cylinder with axis the $z$-axis. Each of $x$, $y$ and $z$ is periodic, and at $t = 0$ and $t = 2\pi$ the curve passes through the same point, so the curve repeats itself and the graph is V.

**24.** $x = \cos t$, $y = \sin t$, $z = \ln t$. $x^2 + y^2 = \cos^2 t + \sin^2 t = 1$, so the curve lies on a circular cylinder with axis the $z$-axis. As $t \to 0$, $z \to -\infty$, so the graph is III.

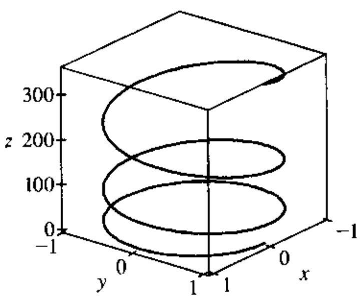
**25.** If $x = t\cos t$, $y = t\sin t$, and $z = t$, then $x^2 + y^2 = t^2\cos^2 t + t^2\sin^2 t = t^2 = z^2$, so the curve lies on the cone $z^2 = x^2 + y^2$. Since $z = t$, the curve is a spiral on this cone.



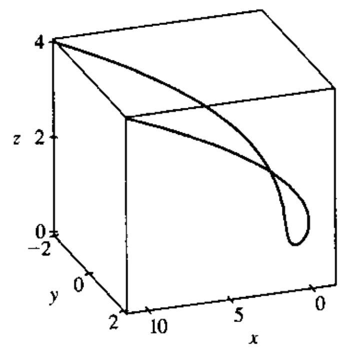
**26.** Here $x^2 = \sin^2 t = z$ and $x^2 + y^2 = \sin^2 t + \cos^2 t = 1$, so the curve is the intersection of the parabolic cylinder $z = x^2$ with the circular cylinder $x^2 + y^2 = 1$.



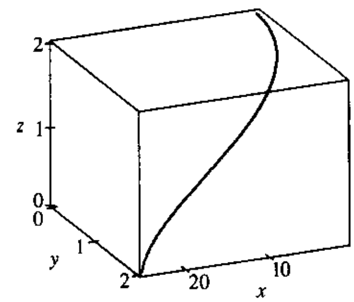

**27.** $\mathbf{r}(t) = \langle \sin t, \cos t, t^2 \rangle$



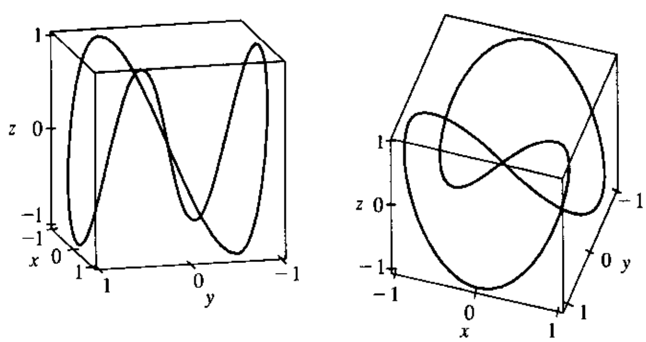

**28.** $\mathbf{r}(t) = \langle t^4 - t^2 + 1, t, t^2 \rangle$



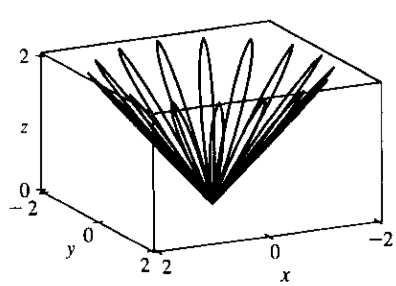

**29.** $\mathbf{r}(t) = \langle t^2, \sqrt{t-1}, \sqrt{5-t} \rangle$




**30.** We have the computer plot the parametric equations $x = \sin t$, $y = \sin 2t$, $z = \sin 3t$, $0 \le t \le 2\pi$. The shape of the curve is not clear from just one viewpoint, so we include a second plot drawn from a different angle.




**31.**

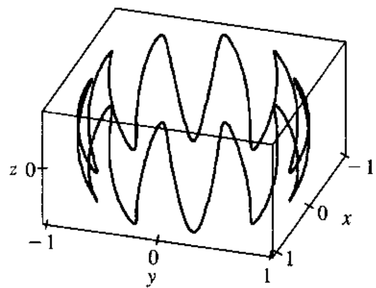



$x = (1 + \cos 16t)\cos t$, $y = (1 + \cos 16t)\sin t$, $z = 1 + \cos 16t$. At any point on the graph,

$$x^2 + y^2 = (1 + \cos 16t)^2 \cos^2 t + (1 + \cos 16t)^2 \sin^2 t$$
$$= (1 + \cos 16t)^2 = z^2,$$

so the graph lies on the cone $x^2 + y^2 = z^2$. From the graph at left, we see that this curve looks like the projection of a leaved two-dimensional curve onto a cone.

**32.**




$x = \sqrt{1 - 0.25\cos^2 10t}\,\cos t$, $y = \sqrt{1 - 0.25\cos^2 10t}\,\sin t$, $z = 0.5\cos 10t$. At any point on the graph,

$$x^2 + y^2 + z^2 = (1 - 0.25\cos^2 10t)\cos^2 t + (1 - 0.25\cos^2 10t)\sin^2 t + 0.25\cos^2 t$$
$$= 1 - 0.25\cos^2 10t + 0.25\cos^2 10t = 1,$$

so the graph lies on the sphere $x^2 + y^2 + z^2 = 1$, and since $z = 0.5\cos 10t$ the graph resembles a trigonometric curve with ten peaks projected onto the sphere. The graph is generated by $t \in [0, 2\pi]$.

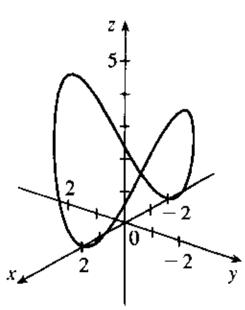
**33.** If $t = -1$, then $x = 1, y = 4, z = 0$, so the curve passes through the point $(1, 4, 0)$. If $t = 3$, then $x = 9, y = -8, z = 28$, so the curve passes through the point $(9, -8, 28)$. For the point $(4, 7, -6)$ to be on the curve, we require $y = 1 - 3t = 7 \;\Rightarrow\; t = -2$. But then $z = 1 + (-2)^3 = -7 \neq -6$, so $(4, 7, -6)$ is not on the curve.

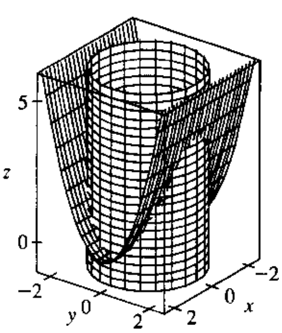
**34.** The projection of the curve $C$ of intersection onto the $xy$-plane is the circle $x^2 + y^2 = 4, z = 0$. Then we can write $x = 2\cos t, y = 2\sin t, 0 \leq t \leq 2\pi$. Since $C$ also lies on the surface $z = xy$, we have $z = xy = (2\cos t)(2\sin t) = 4\cos t\sin t$, or $2\sin(2t)$. Then parametric equations for $C$ are $x = 2\cos t, y = 2\sin t, z = 2\sin(2t), 0 \leq t \leq 2\pi$, and the corresponding vector function is $\mathbf{r}(t) = 2\cos t\,\mathbf{i} + 2\sin t\,\mathbf{j} + 2\sin(2t)\,\mathbf{k}, 0 \leq t \leq 2\pi$.

**35.** Both equations are solved for $z$, so we can substitute to eliminate $z$: $\sqrt{x^2 + y^2} = 1 + y \;\Rightarrow\; x^2 + y^2 = 1 + 2y + y^2 \;\Rightarrow\; x^2 = 1 + 2y \;\Rightarrow\; y = \frac{1}{2}(x^2 - 1)$. We can form parametric equations for the curve $C$ of intersection by choosing a parameter $x = t$, then $y = \frac{1}{2}(t^2 - 1)$ and $z = 1 + y = 1 + \frac{1}{2}(t^2 - 1) = \frac{1}{2}(t^2 + 1)$. Thus a vector function representing $C$ is $\mathbf{r}(t) = t\,\mathbf{i} + \frac{1}{2}(t^2 - 1)\,\mathbf{j} + \frac{1}{2}(t^2 + 1)\,\mathbf{k}$.

**36.** The projection of the curve $C$ of intersection onto the $xy$-plane is the parabola $y = x^2, z = 0$. Then we can choose the parameter $x = t \;\Rightarrow\; y = t^2$. Since $C$ also lies on the surface $z = 4x^2 + y^2$, we have $z = 4x^2 + y^2 = 4t^2 + (t^2)^2$. Then parametric equations for $C$ are $x = t, y = t^2, z = 4t^2 + t^4$, and the corresponding vector function is $\mathbf{r}(t) = t\,\mathbf{i} + t^2\,\mathbf{j} + (4t^2 + t^4)\,\mathbf{k}$.

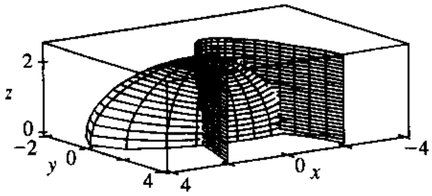
**37.**

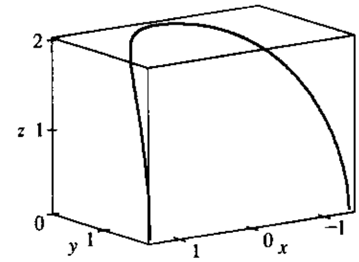



The projection of the curve $C$ of intersection onto the $xy$-plane is the circle $x^2 + y^2 = 4, z = 0$. Then we can write $x = 2\cos t, y = 2\sin t$, $0 \leq t \leq 2\pi$. Since $C$ also lies on the surface $z = x^2$, we have $z = x^2 = (2\cos t)^2 = 4\cos^2 t$. Then parametric equations for $C$ are $x = 2\cos t$, $y = 2\sin t, z = 4\cos^2 t, 0 \leq t \leq 2\pi$.

**38.**




$x = t \Rightarrow y = t^2 \Rightarrow 4z^2 = 16 - x^2 - 4y^2 = 16 - t^2 - 4t^4 \Rightarrow z = \sqrt{4 - \left(\frac{1}{2}t\right)^2 - t^4}$. Note that $z$ is positive because the intersection is with the top half of the ellipsoid. Hence the curve is given by $x = t$, $y = t^2$, $z = \sqrt{4 - \frac{1}{4}t^2 - t^4}$.

**39.** For the particles to collide, we require $\mathbf{r}_1(t) = \mathbf{r}_2(t) \Leftrightarrow \langle t^2, 7t - 12, t^2 \rangle = \langle 4t - 3, t^2, 5t - 6 \rangle$. Equating components gives $t^2 = 4t - 3$, $7t - 12 = t^2$, and $t^2 = 5t - 6$. From the first equation, $t^2 - 4t + 3 = 0 \Leftrightarrow (t - 3)(t - 1) = 0$ so $t = 1$ or $t = 3$. $t = 1$ does not satisfy the other two equations, but $t = 3$ does. The particles collide when $t = 3$, at the point $(9, 9, 9)$.

**40.** The particles collide provided $\mathbf{r}_1(t) = \mathbf{r}_2(t) \Leftrightarrow \langle t, t^2, t^3 \rangle = \langle 1 + 2t, 1 + 6t, 1 + 14t \rangle$. Equating components gives $t = 1 + 2t$, $t^2 = 1 + 6t$, and $t^3 = 1 + 14t$. The first equation gives $t = -1$, but this does not satisfy the other equations, so the particles do not collide. For the paths to intersect, we need to find a value for $t$ and a value for $s$ where $\mathbf{r}_1(t) = \mathbf{r}_2(s) \Leftrightarrow \langle t, t^2, t^3 \rangle = \langle 1 + 2s, 1 + 6s, 1 + 14s \rangle$. Equating components, $t = 1 + 2s$, $t^2 = 1 + 6s$, and $t^3 = 1 + 14s$. Substituting the first equation into the second gives $(1 + 2s)^2 = 1 + 6s \Rightarrow 4s^2 - 2s = 0 \Rightarrow 2s(2s - 1) = 0 \Rightarrow s = 0$ or $s = \frac{1}{2}$. From the first equation, $s = 0 \Rightarrow t = 1$ and $s = \frac{1}{2} \Rightarrow t = 2$. Checking, we see that both pairs of values satisfy the third equation. Thus the paths intersect twice, at the point $(1, 1, 1)$ when $s = 0$ and $t = 1$, and at $(2, 4, 8)$ when $s = \frac{1}{2}$ and $t = 2$.

**41.** Let $\mathbf{u}(t) = \langle u_1(t), u_2(t), u_3(t) \rangle$ and $\mathbf{v}(t) = \langle v_1(t), v_2(t), v_3(t) \rangle$. In each part of this problem the basic procedure is to use Equation 1 and then analyze the individual component functions using the limit properties we have already developed for real-valued functions.

(a) $\lim\limits_{t \to a} \mathbf{u}(t) + \lim\limits_{t \to a} \mathbf{v}(t) = \left\langle \lim\limits_{t \to a} u_1(t), \lim\limits_{t \to a} u_2(t), \lim\limits_{t \to a} u_3(t) \right\rangle + \left\langle \lim\limits_{t \to a} v_1(t), \lim\limits_{t \to a} v_2(t), \lim\limits_{t \to a} v_3(t) \right\rangle$ and the limits of these component functions must each exist since the vector functions both possess limits as $t \to a$. Then adding the two vectors and using the addition property of limits for real-valued functions, we have that

$$\begin{aligned}
\lim_{t \to a} \mathbf{u}(t) + \lim_{t \to a} \mathbf{v}(t) &= \left\langle \lim_{t \to a} u_1(t) + \lim_{t \to a} v_1(t), \lim_{t \to a} u_2(t) + \lim_{t \to a} v_2(t), \lim_{t \to a} u_3(t) + \lim_{t \to a} v_3(t) \right\rangle \\
&= \left\langle \lim_{t \to a} [u_1(t) + v_1(t)], \lim_{t \to a} [u_2(t) + v_2(t)], \lim_{t \to a} [u_3(t) + v_3(t)] \right\rangle \\
&= \lim_{t \to a} \langle u_1(t) + v_1(t), u_2(t) + v_2(t), u_3(t) + v_3(t) \rangle \quad \text{[using (1) backward]} \\
&= \lim_{t \to a} [\mathbf{u}(t) + \mathbf{v}(t)]
\end{aligned}$$

(b) $\lim_{t\to a} c\mathbf{u}(t) = \lim_{t\to a} \langle cu_1(t), cu_2(t), cu_3(t)\rangle = \left\langle \lim_{t\to a} cu_1(t), \lim_{t\to a} cu_2(t), \lim_{t\to a} cu_3(t)\right\rangle$

$$= \left\langle c\lim_{t\to a} u_1(t), c\lim_{t\to a} u_2(t), c\lim_{t\to a} u_3(t)\right\rangle = c\left\langle \lim_{t\to a} u_1(t), \lim_{t\to a} u_2(t), \lim_{t\to a} u_3(t)\right\rangle$$

$$= c\lim_{t\to a} \langle u_1(t), u_2(t), u_3(t)\rangle = c\lim_{t\to a} \mathbf{u}(t)$$

(c) $\lim_{t\to a} \mathbf{u}(t) \cdot \lim_{t\to a} \mathbf{v}(t) = \left\langle \lim_{t\to a} u_1(t), \lim_{t\to a} u_2(t), \lim_{t\to a} u_3(t)\right\rangle \cdot \left\langle \lim_{t\to a} v_1(t), \lim_{t\to a} v_2(t), \lim_{t\to a} v_3(t)\right\rangle$

$$= \left[\lim_{t\to a} u_1(t)\right]\left[\lim_{t\to a} v_1(t)\right] + \left[\lim_{t\to a} u_2(t)\right]\left[\lim_{t\to a} v_2(t)\right] + \left[\lim_{t\to a} u_3(t)\right]\left[\lim_{t\to a} v_3(t)\right]$$

$$= \lim_{t\to a} u_1(t)v_1(t) + \lim_{t\to a} u_2(t)v_2(t) + \lim_{t\to a} u_3(t)v_3(t)$$

$$= \lim_{t\to a} [u_1(t)v_1(t) + u_2(t)v_2(t) + u_3(t)v_3(t)] = \lim_{t\to a} [\mathbf{u}(t)\cdot\mathbf{v}(t)]$$

(d) $\lim_{t\to a} \mathbf{u}(t) \times \lim_{t\to a} \mathbf{v}(t) = \left\langle \lim_{t\to a} u_1(t), \lim_{t\to a} u_2(t), \lim_{t\to a} u_3(t)\right\rangle \times \left\langle \lim_{t\to a} v_1(t), \lim_{t\to a} v_2(t), \lim_{t\to a} v_3(t)\right\rangle$

$$= \left\langle \left[\lim_{t\to a} u_2(t)\right]\left[\lim_{t\to a} v_3(t)\right] - \left[\lim_{t\to a} u_3(t)\right]\left[\lim_{t\to a} v_2(t)\right],\right.$$

$$\left[\lim_{t\to a} u_3(t)\right]\left[\lim_{t\to a} v_1(t)\right] - \left[\lim_{t\to a} u_1(t)\right]\left[\lim_{t\to a} v_3(t)\right],$$

$$\left.\left[\lim_{t\to a} u_1(t)\right]\left[\lim_{t\to a} v_2(t)\right] - \left[\lim_{t\to a} u_2(t)\right]\left[\lim_{t\to a} v_1(t)\right]\right\rangle$$

$$= \left\langle \lim_{t\to a} [u_2(t)v_3(t) - u_3(t)v_2(t)], \lim_{t\to a} [u_3(t)v_1(t) - u_1(t)v_3(t)],\right.$$

$$\left.\lim_{t\to a} [u_1(t)v_2(t) - u_2(t)v_1(t)]\right\rangle$$

$$= \lim_{t\to a} \langle u_2(t)v_3(t) - u_3(t)v_2(t), u_3(t)v_1(t) - u_1(t)v_3(t),$$

$$u_1(t)v_2(t) - u_2(t)v_1(t)\rangle$$

$$= \lim_{t\to a} [\mathbf{u}(t)\times\mathbf{v}(t)]$$

**42.** The projection of the curve onto the $xy$-plane is given by the parametric equations $x = (2+\cos 1.5t)\cos t$, $y = (2+\cos 1.5t)\sin t$. If we convert to polar coordinates, we have

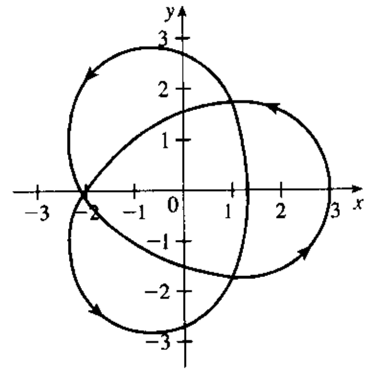
$$r^2 = x^2 + y^2 = [(2+\cos 1.5t)\cos t]^2 + [(2+\cos 1.5t)\sin t]^2$$

$$= (2+\cos 1.5t)^2(\cos^2 t + \sin^2 t)$$

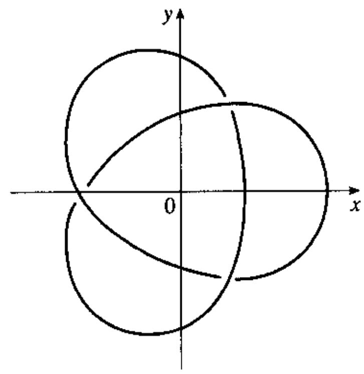
$$= (2+\cos 1.5t)^2$$

$$\Rightarrow \quad r = 2+\cos 1.5t. \text{ Also, } \tan\theta = \frac{y}{x} = \frac{(2+\cos 1.5t)\sin t}{(2+\cos 1.5t)\cos t} = \tan t \quad\Rightarrow\quad \theta = t.$$

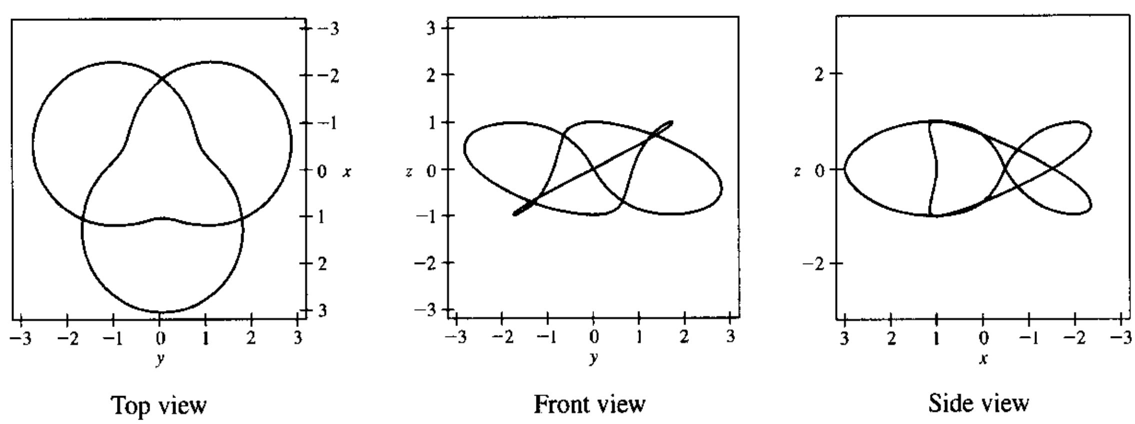
Thus the polar equation of the curve is $r = 2+\cos 1.5\theta$. At $\theta = 0$, we have $r = 3$, and $r$ decreases to 1 as $\theta$ increases to $\frac{2\pi}{3}$. For $\frac{2\pi}{3} \le \theta \le \frac{4\pi}{3}$, $r$ increases to 3; $r$ decreases to 1 again at $\theta = 2\pi$, increases to 3 at $\theta = \frac{8\pi}{3}$, decreases to 1 at $\theta = \frac{10\pi}{3}$, and completes the closed curve by increasing to 3 at $\theta = 4\pi$. We sketch an approximate graph as shown in the figure.



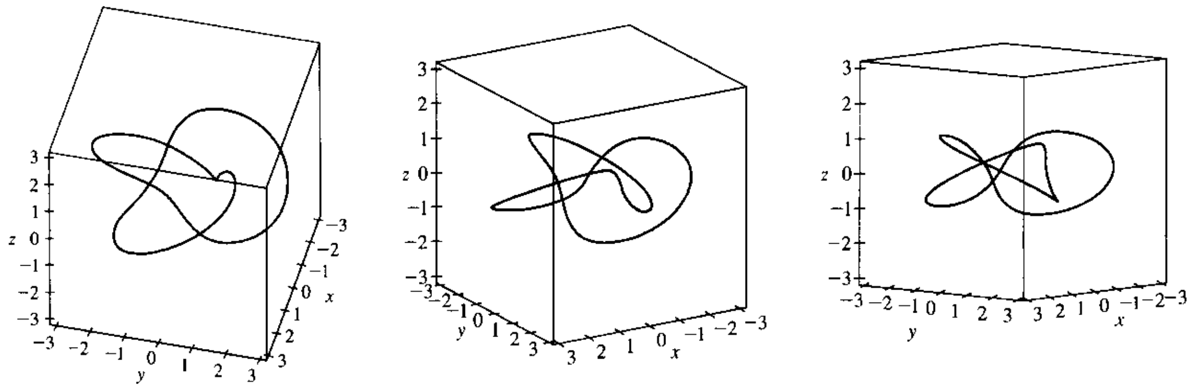

We can determine how the curve passes over itself by investigating the maximum and minimum values of $z$ for $t = \theta \in [0, 4\pi]$. Since $z = \sin 1.5t$, $z$ is maximized where $\sin 1.5t = 1 \Rightarrow 1.5t = \frac{\pi}{2}, \frac{5\pi}{2}$, or $\frac{9\pi}{2} \Rightarrow t = \frac{\pi}{3}, \frac{5\pi}{3}$, or $3\pi$. $z$ is minimized where $\sin 1.5t = -1 \Rightarrow 1.5t = \frac{3\pi}{2}, \frac{7\pi}{2}$, or $\frac{11\pi}{2} \Rightarrow t = \pi, \frac{7\pi}{3}$, or $\frac{11\pi}{3}$. Note that these are precisely the values for which $\cos 1.5t = 0 \Rightarrow r = 2$, and on the graph of the projection, these six points appear to be at the three self-intersections we see. Comparing the maximum and minimum values of $z$ at these intersections, we can determine where the curve passes over itself, as indicated in the figure.

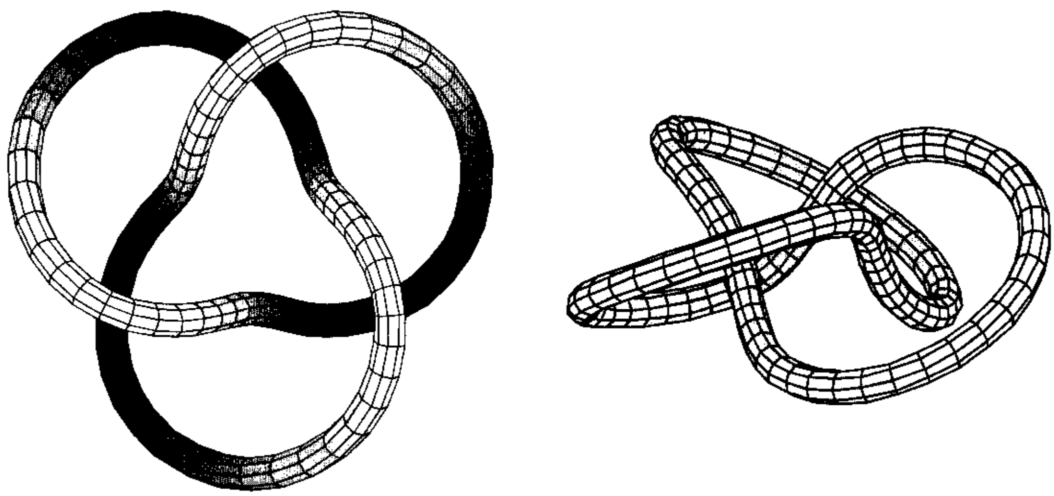



We show a computer-drawn graph of the curve from above, as well as views from the front and from the right side.




The top view graph shows a more accurate representation of the projection of the trefoil knot on the $xy$-plane (the axes are rotated 90°). Notice the indentations the graph exhibits at the points corresponding to $r = 1$. Finally, we graph several additional viewpoints of the trefoil knot, along with two plots showing a tube of radius 0.2 around the curve.



**43.** Let $\mathbf{r}(t) = \langle f(t), g(t), h(t) \rangle$ and $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$. If $\lim\limits_{t\to a} \mathbf{r}(t) = \mathbf{b}$, then $\lim\limits_{t\to a} \mathbf{r}(t)$ exists, so by (1),

$\mathbf{b} = \lim\limits_{t\to a} \mathbf{r}(t) = \left\langle \lim\limits_{t\to a} f(t), \lim\limits_{t\to a} g(t), \lim\limits_{t\to a} h(t) \right\rangle$. By the definition of equal vectors we have $\lim\limits_{t\to a} f(t) = b_1$, $\lim\limits_{t\to a} g(t) = b_2$ and $\lim\limits_{t\to a} h(t) = b_3$. But these are limits of real-valued functions, so by the definition of limits, for every $\epsilon > 0$ there exists $\delta_1 > 0$, $\delta_2 > 0$, $\delta_3 > 0$ so $|f(t) - b_1| < \epsilon/3$ whenever $0 < |t - a| < \delta_1$, $|g(t) - b_2| < \epsilon/3$ whenever $0 < |t - a| < \delta_2$, and $|h(t) - b_3| < \epsilon/3$ whenever $0 < |t - a| < \delta_3$. Letting $\delta =$ minimum of $\{\delta_1, \delta_2, \delta_3\}$, we have $|f(t) - b_1| + |g(t) - b_2| + |h(t) - b_3| < \epsilon/3 + \epsilon/3 + \epsilon/3 = \epsilon$ whenever $0 < |t - a| < \delta$. But $|\mathbf{r}(t) - \mathbf{b}| = |\langle f(t) - b_1, g(t) - b_2, h(t) - b_3 \rangle|$

$= \sqrt{(f(t) - b_1)^2 + (g(t) - b_2)^2 + (h(t) - b_3)^2} \le \sqrt{[f(t) - b_1]^2} + \sqrt{[g(t) - b_2]^2} + \sqrt{[h(t) - b_3]^2}$

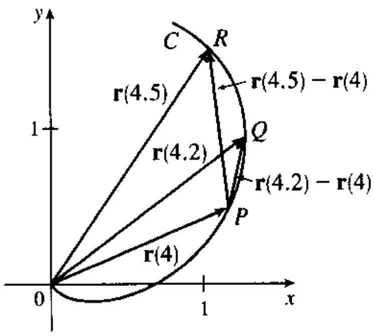
$= |f(t) - b_1| + |g(t) - b_2| + |h(t) - b_3|$. Thus for every $\epsilon > 0$ there exists $\delta > 0$ such that $|\mathbf{r}(t) - b| \le |f(t) - b_1| + |g(t) - b_2| + |h(t) - b_3| < \epsilon$ whenever $0 < |t - a| < \delta$. Conversely, if for every $\epsilon > 0$, there exists $\delta > 0$ such that $|\mathbf{r}(t) - \mathbf{b}| < \epsilon$ whenever $0 < |t - a| < \delta$, then

$|\langle f(t) - b_1, g(t) - b_2, h(t) - b_3 \rangle| < \epsilon \quad \Leftrightarrow \quad \sqrt{[f(t) - b_1]^2 + [g(t) - b_2]^2 + [h(t) - b_3]^2} < \epsilon \quad \Leftrightarrow$

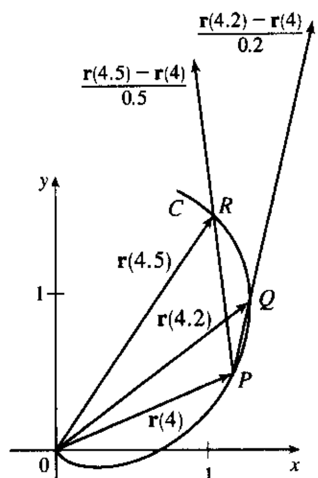
$[f(t) - b_1]^2 + [g(t) - b_2]^2 + [h(t) - b_3]^2 < \epsilon^2$ whenever $0 < |t - a| < \delta$. But each term on the left side of this inequality is positive so $[f(t) - b_1]^2 < \epsilon^2$, $[g(t) - b_2]^2 < \epsilon^2$ and $[h(t) - b_3]^2 < \epsilon^2$ whenever $0 < |t - a| < \delta$, or taking the square root of both sides in each of the above we have $|f(t) - b_1| < \epsilon$, $|g(t) - b_2| < \epsilon$ and $|h(t) - b_3| < \epsilon$ whenever $0 < |t - a| < \delta$. And by definition of limits of real-valued functions we have $\lim_{t \to a} f(t) = b_1$, $\lim_{t \to a} g(t) = b_2$ and $\lim_{t \to a} h(t) = b_3$. But by (1), $\lim_{t \to a} \mathbf{r}(t) = \left\langle \lim_{t \to a} f(t), \lim_{t \to a} g(t), \lim_{t \to a} h(t) \right\rangle$, so $\lim_{t \to a} \mathbf{r}(t) = \langle b_1, b_2, b_3 \rangle = \mathbf{b}$.

## 14.2 Derivatives and Integrals of Vector Functions ET 13.2

**1.** (a)



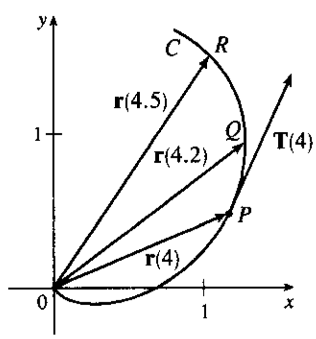

(b) $\dfrac{\mathbf{r}(4.5) - \mathbf{r}(4)}{0.5} = 2[\mathbf{r}(4.5) - \mathbf{r}(4)]$, so we draw a vector in the same direction but with twice the length of the vector $\mathbf{r}(4.5) - \mathbf{r}(4)$. $\dfrac{\mathbf{r}(4.2) - \mathbf{r}(4)}{0.2} = 5[\mathbf{r}(4.2) - \mathbf{r}(4)]$, so we draw a vector in the same direction but with 5 times the length of the vector $\mathbf{r}(4.2) - \mathbf{r}(4)$.



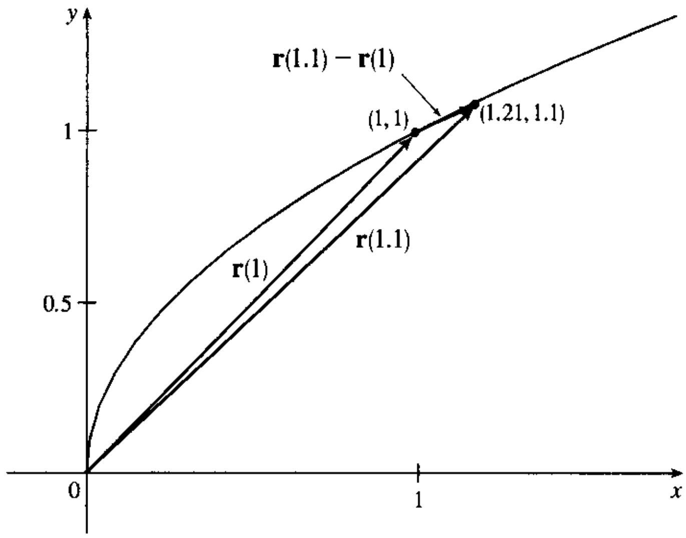

(c) By Definition 1, $\mathbf{r}'(4) = \lim_{h \to 0} \dfrac{\mathbf{r}(4 + h) - \mathbf{r}(4)}{h}$.

$\mathbf{T}(4) = \dfrac{\mathbf{r}'(4)}{|\mathbf{r}'(4)|}$.

(d) $\mathbf{T}(4)$ is a unit vector in the same direction as $\mathbf{r}'(4)$, that is, parallel to the tangent line to the curve at $\mathbf{r}(4)$ with length 1.

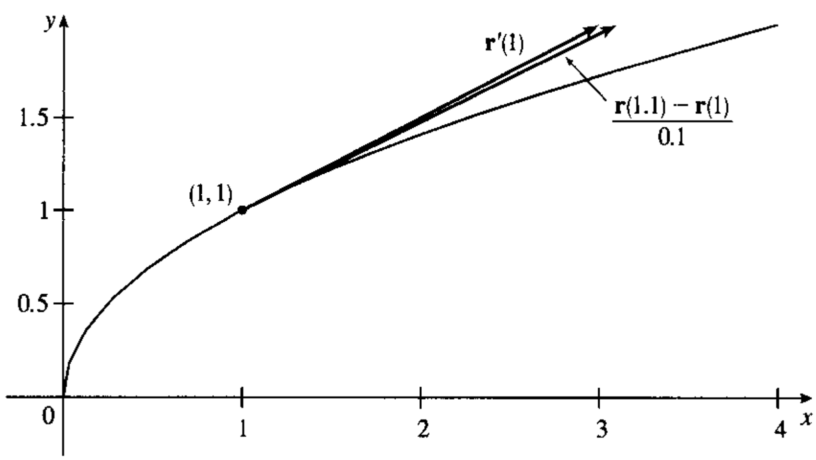


**2.** (a) The curve can be represented by the parametric equations $x = t^2$, $y = t$, $0 \le t \le 2$. Eliminating the parameter, we have $x = y^2$, $0 \le y \le 2$, a portion of which we graph here, along with the vectors $\mathbf{r}(1)$, $\mathbf{r}(1.1)$, and $\mathbf{r}(1.1) - \mathbf{r}(1)$.



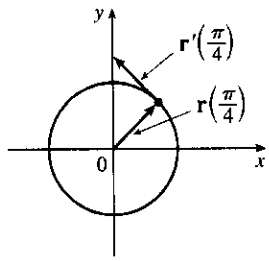

(b) Since $\mathbf{r}(t) = \langle t^2, t \rangle$, we differentiate components, giving $\mathbf{r}'(t) = \langle 2t, 1 \rangle$, so $\mathbf{r}'(1) = \langle 2, 1 \rangle$.

$$\frac{\mathbf{r}(1.1) - \mathbf{r}(1)}{0.1} = \frac{\langle 1.21, 1.1 \rangle - \langle 1, 1 \rangle}{0.1} = 10\,\langle 0.21, 0.1 \rangle = \langle 2.1, 1 \rangle.$$



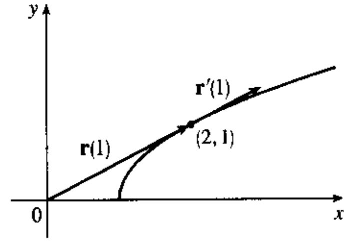

As we can see from the graph, these vectors are very close in length and direction. $\mathbf{r}'(1)$ is defined to be $\lim\limits_{h \to 0} \dfrac{\mathbf{r}(1+h) - \mathbf{r}(1)}{h}$, and we recognize $\dfrac{\mathbf{r}(1.1) - \mathbf{r}(1)}{0.1}$ as the expression after the limit sign with $h = 0.1$. Since $h$ is close to 0, we would expect $\dfrac{\mathbf{r}(1.1) - \mathbf{r}(1)}{0.1}$ to be a vector close to $\mathbf{r}'(1)$.

**3.** (a), (c)




(b) $\mathbf{r}'(t) = \langle -\sin t, \cos t\rangle$

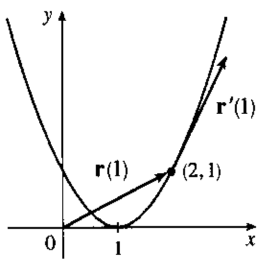
**4.** (a), (c)




(b) $\mathbf{r}'(t) = \left\langle 1, \dfrac{1}{2\sqrt{t}} \right\rangle$

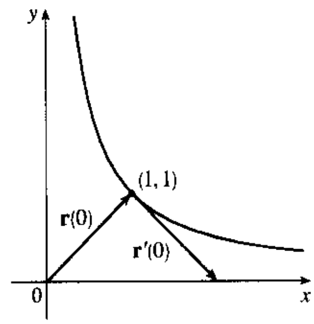
**5.** Since $(x-1)^2 = t^2 = y$, the curve is a parabola.

(a), (c)



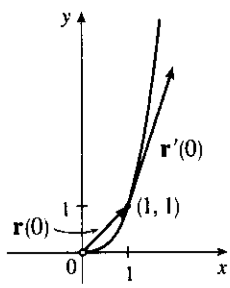

(b) $\mathbf{r}'(t) = \mathbf{i} + 2t\,\mathbf{j}$

**6.** (a), (c)




(b) $\mathbf{r}'(t) = e^t\,\mathbf{i} - e^{-t}\,\mathbf{j}$

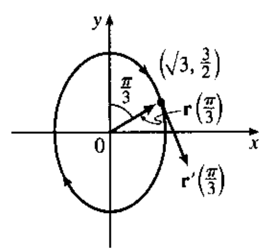
**7.** (a), (c)




(b) $\mathbf{r}'(t) = e^t\,\mathbf{i} + 3e^{3t}\,\mathbf{j}$

**8.** $x = 2\sin t$, $y = 3\cos t$, so $(x/2)^2 + (y/3)^2 = \sin^2 t + \cos^2 t = 1$ and the curve is an ellipse.

(a), (c)




(b) $\mathbf{r}'(t) = 2\cos t\,\mathbf{i} - 3\sin t\,\mathbf{j}$

**9.** $\mathbf{r}'(t) = \left\langle \dfrac{d}{dt}\left[t^2\right], \dfrac{d}{dt}\left[1-t\right], \dfrac{d}{dt}\left[\sqrt{t}\right] \right\rangle = \left\langle 2t, -1, \dfrac{1}{2\sqrt{t}} \right\rangle$

**10.** $\mathbf{r}(t) = \langle \cos 3t, t, \sin 3t\rangle \quad\Rightarrow\quad \mathbf{r}'(t) = \langle -3\sin 3t, 1, 3\cos 3t\rangle$

**11.** $\mathbf{r}(t) = \mathbf{i} - \mathbf{j} + e^{4t}\,\mathbf{k} \quad\Rightarrow\quad \mathbf{r}'(t) = 0\,\mathbf{i} + 0\,\mathbf{j} + 4e^{4t}\,\mathbf{k} = 4e^{4t}\,\mathbf{k}$

**12.** $\mathbf{r}(t) = \sin^{-1} t\,\mathbf{i} + \sqrt{1-t^2}\,\mathbf{j} + \mathbf{k} \quad\Rightarrow\quad \mathbf{r}'(t) = \dfrac{1}{\sqrt{1-t^2}}\,\mathbf{i} - \dfrac{t}{\sqrt{1-t^2}}\,\mathbf{j}$

**13.** $\mathbf{r}(t) = e^{t^2}\,\mathbf{i} - \mathbf{j} + \ln(1+3t)\,\mathbf{k} \quad\Rightarrow\quad \mathbf{r}'(t) = 2te^{t^2}\,\mathbf{i} + \dfrac{3}{1+3t}\,\mathbf{k}$

**14.** $\mathbf{r}'(t) = [at(-3\sin 3t) + a\cos 3t]\,\mathbf{i} + b\cdot 3\sin^2 t\cos t\,\mathbf{j} + c\cdot 3\cos^2 t(-\sin t)\,\mathbf{k}$

$= (a\cos 3t - 3at\sin 3t)\,\mathbf{i} + 3b\sin^2 t\cos t\,\mathbf{j} - 3c\cos^2 t\sin t\,\mathbf{k}$

**15.** $\mathbf{r}'(t) = \mathbf{0} + \mathbf{b} + 2t\,\mathbf{c} = \mathbf{b} + 2t\,\mathbf{c}$ by Formulas 1 and 3 of Theorem 3.

**16.** To find $\mathbf{r}'(t)$, we first expand $\mathbf{r}(t) = t\,\mathbf{a}\times(\mathbf{b}+t\,\mathbf{c}) = t(\mathbf{a}\times\mathbf{b}) + t^2(\mathbf{a}\times\mathbf{c})$, so $\mathbf{r}'(t) = \mathbf{a}\times\mathbf{b} + 2t(\mathbf{a}\times\mathbf{c})$.

**17.** $\mathbf{r}'(t) = \langle 30t^4, 12t^2, 2\rangle \Rightarrow \mathbf{r}'(1) = \langle 30, 12, 2\rangle$. So $|\mathbf{r}'(1)| = \sqrt{30^2+12^2+2^2} = \sqrt{1048} = 2\sqrt{262}$ and

$$\mathbf{T}(1) = \frac{\mathbf{r}'(1)}{|\mathbf{r}'(1)|} = \frac{1}{2\sqrt{262}}\langle 30, 12, 2\rangle = \left\langle \frac{15}{\sqrt{262}}, \frac{6}{\sqrt{262}}, \frac{1}{\sqrt{262}}\right\rangle.$$

**18.** $\mathbf{r}'(t) = \dfrac{2}{\sqrt{t}}\,\mathbf{i} + 2t\,\mathbf{j} + \mathbf{k} \Rightarrow \mathbf{r}'(1) = 2\,\mathbf{i} + 2\,\mathbf{j} + \mathbf{k}$. Thus

$$\mathbf{T}(1) = \frac{\mathbf{r}'(1)}{|\mathbf{r}'(1)|} = \frac{1}{\sqrt{2^2+2^2+1^2}}(2\,\mathbf{i}+2\,\mathbf{j}+\mathbf{k}) = \tfrac{1}{3}(2\,\mathbf{i}+2\,\mathbf{j}+\mathbf{k}) = \tfrac{2}{3}\,\mathbf{i} + \tfrac{2}{3}\,\mathbf{j} + \tfrac{1}{3}\,\mathbf{k}.$$

**19.** $\mathbf{r}'(t) = -\sin t\,\mathbf{i} + 3\,\mathbf{j} + 4\cos 2t\,\mathbf{k} \Rightarrow \mathbf{r}'(0) = 3\,\mathbf{j} + 4\,\mathbf{k}$. Thus

$$\mathbf{T}(0) = \frac{\mathbf{r}'(0)}{|\mathbf{r}'(0)|} = \frac{1}{\sqrt{0^2+3^2+4^2}}(3\,\mathbf{j}+4\,\mathbf{k}) = \tfrac{1}{5}(3\,\mathbf{j}+4\,\mathbf{k}) = \tfrac{3}{5}\,\mathbf{j} + \tfrac{4}{5}\,\mathbf{k}.$$

**20.** $\mathbf{r}'(t) = 2\cos t\,\mathbf{i} - 2\sin t\,\mathbf{j} + \sec^2 t\,\mathbf{k} \Rightarrow \mathbf{r}'\left(\frac{\pi}{4}\right) = \sqrt{2}\,\mathbf{i} - \sqrt{2}\,\mathbf{j} + 2\,\mathbf{k}$ and $\left|\mathbf{r}'\left(\frac{\pi}{4}\right)\right| = \sqrt{2+2+4} = 2\sqrt{2}$.

Thus $\mathbf{T}\left(\frac{\pi}{4}\right) = \dfrac{\mathbf{r}'\left(\frac{\pi}{4}\right)}{\left|\mathbf{r}'\left(\frac{\pi}{4}\right)\right|} = \dfrac{1}{2\sqrt{2}}\left(\sqrt{2}\,\mathbf{i} - \sqrt{2}\,\mathbf{j} + 2\,\mathbf{k}\right) = \dfrac{1}{2}\,\mathbf{i} - \dfrac{1}{2}\,\mathbf{j} + \dfrac{1}{\sqrt{2}}\,\mathbf{k}.$

**21.** $\mathbf{r}(t) = \langle t, t^2, t^3\rangle \Rightarrow \mathbf{r}'(t) = \langle 1, 2t, 3t^2\rangle$. Then $\mathbf{r}'(1) = \langle 1, 2, 3\rangle$ and $|\mathbf{r}'(1)| = \sqrt{1^2+2^2+3^2} = \sqrt{14}$, so

$\mathbf{T}(1) = \dfrac{\mathbf{r}'(1)}{|\mathbf{r}'(1)|} = \dfrac{1}{\sqrt{14}}\langle 1, 2, 3\rangle = \left\langle \dfrac{1}{\sqrt{14}}, \dfrac{2}{\sqrt{14}}, \dfrac{3}{\sqrt{14}}\right\rangle$. $\mathbf{r}''(t) = \langle 0, 2, 6t\rangle$, so

$$\mathbf{r}'(t)\times\mathbf{r}''(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2t & 3t^2 \\ 0 & 2 & 6t \end{vmatrix} = \begin{vmatrix} 2t & 3t^2 \\ 2 & 6t \end{vmatrix}\mathbf{i} - \begin{vmatrix} 1 & 3t^2 \\ 0 & 6t \end{vmatrix}\mathbf{j} + \begin{vmatrix} 1 & 2t \\ 0 & 2 \end{vmatrix}\mathbf{k}$$

$$= (12t^2 - 6t^2)\,\mathbf{i} - (6t - 0)\,\mathbf{j} + (2-0)\,\mathbf{k} = \langle 6t^2, -6t, 2\rangle.$$

**22.** $\mathbf{r}(t) = \langle e^{2t}, e^{-2t}, te^{2t}\rangle \Rightarrow \mathbf{r}'(t) = \langle 2e^{2t}, -2e^{-2t}, (2t+1)e^{2t}\rangle \Rightarrow$

$\mathbf{r}'(0) = \langle 2e^0, -2e^0, (0+1)e^0\rangle = \langle 2, -2, 1\rangle$ and $|\mathbf{r}'(0)| = \sqrt{2^2+(-2)^2+1^2} = 3$. Then

$\mathbf{T}(0) = \dfrac{\mathbf{r}'(0)}{|\mathbf{r}'(0)|} = \frac{1}{3}\langle 2, -2, 1\rangle = \left\langle \frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right\rangle$.

$\mathbf{r}''(t) = \langle 4e^{2t}, 4e^{-2t}, (4t+4)e^{2t}\rangle \Rightarrow \mathbf{r}''(0) = \langle 4e^0, 4e^0, (0+4)e^0\rangle = \langle 4, 4, 4\rangle$.

$$\begin{aligned}\mathbf{r}'(t)\cdot\mathbf{r}''(t) &= \langle 2e^{2t}, -2e^{-2t}, (2t+1)e^{2t}\rangle \cdot \langle 4e^{2t}, 4e^{-2t}, (4t+4)e^{2t}\rangle \\ &= (2e^{2t})(4e^{2t}) + (-2e^{-2t})(4e^{-2t}) + ((2t+1)e^{2t})((4t+4)e^{2t}) \\ &= 8e^{4t} - 8e^{-4t} + (8t^2+12t+4)e^{4t} = (8t^2+12t+12)e^{4t} - 8e^{-4t}\end{aligned}$$

**23.** The vector equation for the curve is $\mathbf{r}(t) = \langle t^5, t^4, t^3\rangle$, so $\mathbf{r}'(t) = \langle 5t^4, 4t^3, 3t^2\rangle$. The point $(1,1,1)$ corresponds to $t = 1$, so the tangent vector there is $\mathbf{r}'(1) = \langle 5, 4, 3\rangle$. Thus, the tangent line goes through the point $(1,1,1)$ and is parallel to the vector $\langle 5, 4, 3\rangle$. Parametric equations are $x = 1+5t$, $y = 1+4t$, $z = 1+3t$.

**24.** The vector equation for the curve is $\mathbf{r}(t) = \langle t^2-1, t^2+1, t+1\rangle$, so $\mathbf{r}'(t) = \langle 2t, 2t, 1\rangle$. The point $(-1,1,1)$ corresponds to $t = 0$, so the tangent vector there is $\mathbf{r}'(0) = \langle 0, 0, 1\rangle$. Thus, the tangent line is parallel to the vector $\langle 0, 0, 1\rangle$ and parametric equations are $x = -1 + 0\cdot t = -1$, $y = 1 + 0\cdot t = 1$, $z = 1 + 1\cdot t = 1+t$.

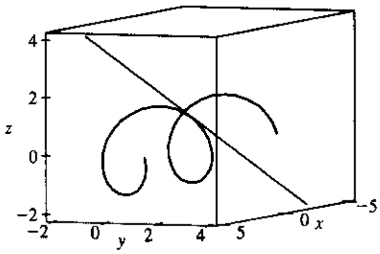
**25.** The vector equation for the curve is $\mathbf{r}(t) = \langle e^{-t}\cos t, e^{-t}\sin t, e^{-t}\rangle$, so

$$\mathbf{r}'(t) = \langle e^{-t}(-\sin t) + (\cos t)(-e^{-t}), e^{-t}\cos t + (\sin t)(-e^{-t}), (-e^{-t})\rangle$$
$$= \langle -e^{-t}(\cos t + \sin t), e^{-t}(\cos t - \sin t), -e^{-t}\rangle.$$

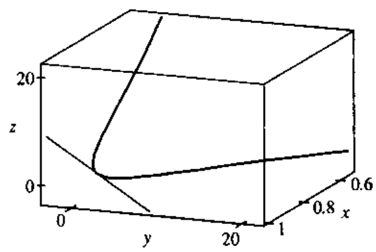
The point $(1,0,1)$ corresponds to $t = 0$, so the tangent vector there is $\mathbf{r}'(0) = \langle -e^0(\cos 0 + \sin 0), e^0(\cos 0 - \sin 0), -e^0\rangle = \langle -1, 1, -1\rangle$. Thus, the tangent line is parallel to the vector $\langle -1, 1, -1\rangle$ and parametric equations are $x = 1 + (-1)t = 1 - t$, $y = 0 + 1\cdot t = t$, $z = 1 + (-1)t = 1 - t$.

**26.** $\mathbf{r}(t) = \langle \ln t, 2\sqrt{t}, t^2\rangle$, $\mathbf{r}'(t) = \langle 1/t, 1/\sqrt{t}, 2t\rangle$. At $(0,2,1)$, $t = 1$ and $\mathbf{r}'(1) = \langle 1,1,2\rangle$. Thus, parametric equations of the tangent line are $x = t$, $y = 2 + t$, $z = 1 + 2t$.

**27.** $\mathbf{r}(t) = \langle t, \sqrt{2}\cos t, \sqrt{2}\sin t\rangle \;\Rightarrow$ $\mathbf{r}'(t) = \langle 1, -\sqrt{2}\sin t, \sqrt{2}\cos t\rangle$. At $\left(\frac{\pi}{4}, 1, 1\right)$, $t = \frac{\pi}{4}$ and $\mathbf{r}'\left(\frac{\pi}{4}\right) = \langle 1, -1, 1\rangle$. Thus, parametric equations of the tangent line are $x = \frac{\pi}{4} + t$, $y = 1 - t$, $z = 1 + t$.




**28.** $\mathbf{r}(t) = \langle \cos t, 3e^{2t}, 3e^{-2t}\rangle$, $\mathbf{r}'(t) = \langle -\sin t, 6e^{2t}, -6e^{-2t}\rangle$. At $(1,3,3)$, $t = 0$ and $\mathbf{r}'(0) = \langle 0, 6, -6\rangle$. Thus, parametric equations of the tangent line are $x = 1$, $y = 3 + 6t$, $z = 3 - 6t$.



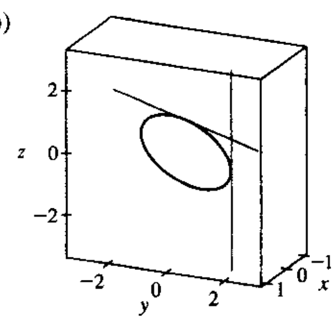

**29.** (a) $\mathbf{r}(t) = \langle t^3, t^4, t^5\rangle \;\Rightarrow\; \mathbf{r}'(t) = \langle 3t^2, 4t^3, 5t^4\rangle$, and since $\mathbf{r}'(0) = \langle 0,0,0\rangle = \mathbf{0}$, the curve is not smooth.

(b) $\mathbf{r}(t) = \langle t^3 + t, t^4, t^5\rangle \;\Rightarrow\; \mathbf{r}'(t) = \langle 3t^2 + 1, 4t^3, 5t^4\rangle$. $\mathbf{r}'(t)$ is continuous since its component functions are continuous. Also, $\mathbf{r}'(t) \neq \mathbf{0}$, as the $y$- and $z$-components are 0 only for $t = 0$, but $\mathbf{r}'(0) = \langle 1,0,0\rangle \neq \mathbf{0}$. Thus, the curve is smooth.

(c) $\mathbf{r}(t) = \langle \cos^3 t, \sin^3 t\rangle \;\Rightarrow\; \mathbf{r}'(t) = \langle -3\cos^2 t\sin t, 3\sin^2 t\cos t\rangle$. Since $\mathbf{r}'(0) = \langle -3\cos^2 0\sin 0, 3\sin^2 0\cos 0\rangle = \langle 0,0\rangle = \mathbf{0}$, the curve is not smooth.

**30.** (a) The tangent line at $t = 0$ is the line through the point with position vector $\mathbf{r}(0) = \langle \sin 0, 2\sin 0, \cos 0\rangle = \langle 0,0,1\rangle$, and in the direction of the tangent vector, $\mathbf{r}'(0) = \langle \pi\cos 0, 2\pi\cos 0, -\pi\sin 0\rangle = \langle \pi, 2\pi, 0\rangle$. So an equation of the line is $\langle x,y,z\rangle = \mathbf{r}(0) + u\,\mathbf{r}'(0) = \langle 0 + \pi u, 0 + 2\pi u, 1\rangle = \langle \pi u, 2\pi u, 1\rangle$.

(b)




$\mathbf{r}\left(\frac{1}{2}\right) = \langle \sin\frac{\pi}{2}, 2\sin\frac{\pi}{2}, \cos\frac{\pi}{2}\rangle = \langle 1,2,0\rangle$, $\mathbf{r}'\left(\frac{1}{2}\right) = \langle \pi\cos\frac{\pi}{2}, 2\pi\cos\frac{\pi}{2}, -\pi\sin\frac{\pi}{2}\rangle = \langle 0,0,-\pi\rangle$.

So the equation of the second line is $\langle x,y,z\rangle = \langle 1,2,0\rangle + v\,\langle 0,0,-\pi\rangle = \langle 1,2,-\pi v\rangle$. The lines intersect where $\langle \pi u, 2\pi u, 1\rangle = \langle 1,2,-\pi v\rangle$, so the point of intersection is $(1,2,1)$.

**31.** The angle of intersection of the two curves is the angle between the two tangent vectors to the curves at the point of intersection. Since $\mathbf{r}_1'(t) = \langle 1, 2t, 3t^2 \rangle$ and $t = 0$ at $(0,0,0)$, $\mathbf{r}_1'(0) = \langle 1,0,0 \rangle$ is a tangent vector to $\mathbf{r}_1$ at $(0,0,0)$. Similarly, $\mathbf{r}_2'(t) = \langle \cos t, 2\cos 2t, 1 \rangle$ and since $\mathbf{r}_2(0) = \langle 0,0,0 \rangle$, $\mathbf{r}_2'(0) = \langle 1,2,1 \rangle$ is a tangent vector to $\mathbf{r}_2$ at $(0,0,0)$. If $\theta$ is the angle between these two tangent vectors, then $\cos\theta = \frac{1}{\sqrt{1}\sqrt{6}} \langle 1,0,0 \rangle \cdot \langle 1,2,1 \rangle = \frac{1}{\sqrt{6}}$ and $\theta = \cos^{-1}\left(\frac{1}{\sqrt{6}}\right) \approx 66^\circ$.

**32.** To find the point of intersection, we must find the values of $t$ and $s$ which satisfy the following three equations simultaneously: $t = 3 - s$, $1 - t = s - 2$, $3 + t^2 = s^2$. Solving the last two equations gives $t = 1$, $s = 2$ (check these in the first equation). Thus the point of intersection is $(1,0,4)$. To find the angle $\theta$ of intersection, we proceed as in Exercise 31. The tangent vectors to the respective curves at $(1,0,4)$ are $\mathbf{r}_1'(1) = \langle 1,-1,2 \rangle$ and $\mathbf{r}_2'(2) = \langle -1,1,4 \rangle$. So $\cos\theta = \frac{1}{\sqrt{6}\sqrt{18}}(-1-1+8) = \frac{6}{6\sqrt{3}} = \frac{1}{\sqrt{3}}$ and $\theta = \cos^{-1}\left(\frac{1}{\sqrt{3}}\right) \approx 55^\circ$.

*Note:* In Exercise 31, the curves intersect when the value of both parameters is zero. However, as seen in this exercise, it is not necessary for the parameters to be of equal value at the point of intersection.

**33.** $$\int_0^1 (16t^3\,\mathbf{i} - 9t^2\,\mathbf{j} + 25t^4\,\mathbf{k})\,dt = \left(\int_0^1 16t^3\,dt\right)\mathbf{i} - \left(\int_0^1 9t^2\,dt\right)\mathbf{j} + \left(\int_0^1 25t^4\,dt\right)\mathbf{k}$$
$$= \left[4t^4\right]_0^1\mathbf{i} - \left[3t^3\right]_0^1\mathbf{j} + \left[5t^5\right]_0^1\mathbf{k} = 4\,\mathbf{i} - 3\,\mathbf{j} + 5\,\mathbf{k}$$

**34.** $$\int_0^1 \left(\frac{4}{1+t^2}\,\mathbf{j} + \frac{2t}{1+t^2}\,\mathbf{k}\right)dt = \left[4\tan^{-1}t\,\mathbf{j} + \ln(1+t^2)\,\mathbf{k}\right]_0^1$$
$$= \left[4\tan^{-1}1\,\mathbf{j} + \ln 2\,\mathbf{k}\right] - \left[4\tan^{-1}0\,\mathbf{j} + \ln 1\,\mathbf{k}\right] = 4\left(\tfrac{\pi}{4}\right)\mathbf{j} + \ln 2\,\mathbf{k} - 0\,\mathbf{j} - 0\,\mathbf{k} = \pi\,\mathbf{j} + \ln 2\,\mathbf{k}$$

**35.** $$\int_0^{\pi/2} (3\sin^2 t\cos t\,\mathbf{i} + 3\sin t\cos^2 t\,\mathbf{j} + 2\sin t\cos t\,\mathbf{k})\,dt$$
$$= \left(\int_0^{\pi/2} 3\sin^2 t\cos t\,dt\right)\mathbf{i} + \left(\int_0^{\pi/2} 3\sin t\cos^2 t\,dt\right)\mathbf{j} + \left(\int_0^{\pi/2} 2\sin t\cos t\,dt\right)\mathbf{k}$$
$$= \left[\sin^3 t\right]_0^{\pi/2}\mathbf{i} + \left[-\cos^3 t\right]_0^{\pi/2}\mathbf{j} + \left[\sin^2 t\right]_0^{\pi/2}\mathbf{k}$$
$$= (1-0)\,\mathbf{i} + (0+1)\,\mathbf{j} + (1-0)\,\mathbf{k} = \mathbf{i} + \mathbf{j} + \mathbf{k}$$

**36.** $$\int_1^4 (\sqrt{t}\,\mathbf{i} + te^{-t}\,\mathbf{j} + t^{-2}\,\mathbf{k})\,dt = \left[\tfrac{2}{3}t^{3/2}\,\mathbf{i} - t^{-1}\,\mathbf{k}\right]_1^4 + \left(\left[-te^{-t}\right]_1^4 + \int_1^4 e^{-t}\,dt\right)\mathbf{j}$$
$$= \left(\tfrac{16}{3} - \tfrac{2}{3}\right)\mathbf{i} - \left(\tfrac{1}{4} - 1\right)\mathbf{k} + \left(-4e^{-4} + e^{-1} - e^{-4} + e^{-1}\right)\mathbf{j} = \tfrac{14}{3}\,\mathbf{i} + e^{-1}(2 - 5e^{-3})\,\mathbf{j} + \tfrac{3}{4}\,\mathbf{k}$$

**37.** $$\int \left(e^t\,\mathbf{i} + 2t\,\mathbf{j} + \ln t\,\mathbf{k}\right)dt = \left(\int e^t\,dt\right)\mathbf{i} + \left(\int 2t\,dt\right)\mathbf{j} + \left(\int \ln t\,dt\right)\mathbf{k}$$
$= e^t\,\mathbf{i} + t^2\,\mathbf{j} + (t\ln t - t)\,\mathbf{k} + \mathbf{C}$, where $\mathbf{C}$ is a vector constant of integration.

**38.** $$\int(\cos\pi t\,\mathbf{i} + \sin\pi t\,\mathbf{j} + t\,\mathbf{k})\,dt = \left(\int\cos\pi t\,dt\right)\mathbf{i} + \left(\int\sin\pi t\,dt\right)\mathbf{j} + \left(\int t\,dt\right)\mathbf{k}$$
$$= \tfrac{1}{\pi}\sin\pi t\,\mathbf{i} - \tfrac{1}{\pi}\cos\pi t\,\mathbf{j} + \tfrac{1}{2}t^2\,\mathbf{k} + \mathbf{C}$$

**39.** $\mathbf{r}'(t) = t^2\,\mathbf{i} + 4t^3\,\mathbf{j} - t^2\,\mathbf{k} \quad\Rightarrow\quad \mathbf{r}(t) = \frac{1}{3}t^3\,\mathbf{i} + t^4\,\mathbf{j} - \frac{1}{3}t^3\,\mathbf{k} + \mathbf{C}$, where $\mathbf{C}$ is a constant vector.
But $\mathbf{j} = \mathbf{r}(0) = (0)\,\mathbf{i} + (0)\,\mathbf{j} - (0)\,\mathbf{k} + \mathbf{C}$. Thus $\mathbf{C} = \mathbf{j}$ and
$\mathbf{r}(t) = \frac{1}{3}t^3\,\mathbf{i} + t^4\,\mathbf{j} - \frac{1}{3}t^3\,\mathbf{k} + \mathbf{j} = \frac{1}{3}t^3\,\mathbf{i} + (t^4+1)\,\mathbf{j} - \frac{1}{3}t^3\,\mathbf{k}$.

**40.** $\mathbf{r}'(t) = \sin t\,\mathbf{i} - \cos t\,\mathbf{j} + 2t\,\mathbf{k} \quad\Rightarrow\quad \mathbf{r}(t) = (-\cos t)\,\mathbf{i} - (\sin t)\,\mathbf{j} + t^2\,\mathbf{k} + \mathbf{C}$.
But $\mathbf{i} + \mathbf{j} + 2\mathbf{k} = \mathbf{r}(0) = -\mathbf{i} + (0)\,\mathbf{j} + (0)\,\mathbf{k} + \mathbf{C}$. Thus $\mathbf{C} = 2\,\mathbf{i} + \mathbf{j} + 2\,\mathbf{k}$ and
$\mathbf{r}(t) = (2 - \cos t)\,\mathbf{i} + (1 - \sin t)\,\mathbf{j} + (2 + t^2)\,\mathbf{k}$.

For Exercises 41–44, let $\mathbf{u}(t) = \langle u_1(t), u_2(t), u_3(t)\rangle$ and $\mathbf{v}(t) = \langle v_1(t), v_2(t), v_3(t)\rangle$. In each of these exercises, the procedure is to apply Theorem 2 so that the corresponding properties of derivatives of real-valued functions can be used.

**41.** 
$$\begin{aligned}\frac{d}{dt}[\mathbf{u}(t)+\mathbf{v}(t)] &= \frac{d}{dt}\langle u_1(t)+v_1(t), u_2(t)+v_2(t), u_3(t)+v_3(t)\rangle \\ &= \left\langle \frac{d}{dt}[u_1(t)+v_1(t)], \frac{d}{dt}[u_2(t)+v_2(t)], \frac{d}{dt}[u_3(t)+v_3(t)]\right\rangle \\ &= \langle u_1'(t)+v_1'(t), u_2'(t)+v_2'(t), u_3'(t)+v_3'(t)\rangle \\ &= \langle u_1'(t), u_2'(t), u_3'(t)\rangle + \langle v_1'(t), v_2'(t), v_3'(t)\rangle = \mathbf{u}'(t)+\mathbf{v}'(t).\end{aligned}$$

**42.** 
$$\begin{aligned}\frac{d}{dt}[f(t)\,\mathbf{u}(t)] &= \frac{d}{dt}\langle f(t)u_1(t), f(t)u_2(t), f(t)u_3(t)\rangle \\ &= \left\langle \frac{d}{dt}[f(t)u_1(t)], \frac{d}{dt}[f(t)u_2(t)], \frac{d}{dt}[f(t)u_3(t)]\right\rangle \\ &= \langle f'(t)u_1(t)+f(t)u_1'(t), f'(t)u_2(t)+f(t)u_2'(t), f'(t)u_3(t)+f(t)u_3'(t)\rangle \\ &= f'(t)\langle u_1(t), u_2(t), u_3(t)\rangle + f(t)\langle u_1'(t), u_2'(t), u_3'(t)\rangle \\ &= f'(t)\,\mathbf{u}(t) + f(t)\,\mathbf{u}'(t)\end{aligned}$$

**43.** $\dfrac{d}{dt}[\mathbf{u}(t)\times\mathbf{v}(t)]$

$$= \frac{d}{dt}\langle u_2(t)v_3(t)-u_3(t)v_2(t), u_3(t)v_1(t)-u_1(t)v_3(t), u_1(t)v_2(t)-u_2(t)v_1(t)\rangle$$

$$\begin{aligned}= \langle &u_2'v_3(t)+u_2(t)v_3'(t)-u_3'(t)v_2(t)-u_3(t)v_2'(t), \\ &u_3'(t)v_1(t)+u_3(t)v_1'(t)-u_1'(t)v_3(t)-u_1(t)v_3'(t), \\ &u_1'(t)v_2(t)+u_1(t)v_2'(t)-u_2'(t)v_1(t)-u_2(t)v_1'(t)\rangle\end{aligned}$$

$$= \langle u_2'(t)v_3(t)-u_3'(t)v_2(t), u_3'(t)v_1(t)-u_1'(t)v_3(t), u_1'(t)v_2(t)-u_2'(t)v_1(t)\rangle$$

$$+ \langle u_2(t)v_3'(t)-u_3(t)v_2'(t), u_3(t)v_1'(t)-u_1(t)v_3'(t), u_1(t)v_2'(t)-u_2(t)v_1'(t)\rangle$$

$$= \mathbf{u}'(t)\times\mathbf{v}(t)+\mathbf{u}(t)\times\mathbf{v}'(t)$$

*Alternate solution:* Let $\mathbf{r}(t) = \mathbf{u}(t)\times\mathbf{v}(t)$. Then

$$\begin{aligned}\mathbf{r}(t+h)-\mathbf{r}(t) &= [\mathbf{u}(t+h)\times\mathbf{v}(t+h)] - [\mathbf{u}(t)\times\mathbf{v}(t)] \\ &= [\mathbf{u}(t+h)\times\mathbf{v}(t+h)] - [\mathbf{u}(t)\times\mathbf{v}(t)] + [\mathbf{u}(t+h)\times\mathbf{v}(t)] - [\mathbf{u}(t+h)\times\mathbf{v}(t)] \\ &= \mathbf{u}(t+h)\times[\mathbf{v}(t+h)-\mathbf{v}(t)] + [\mathbf{u}(t+h)-\mathbf{u}(t)]\times\mathbf{v}(t)\end{aligned}$$

(Be careful of the order of the cross product.)

Dividing through by $h$ and taking the limit as $h \to 0$ we have

$$\begin{aligned}\mathbf{r}'(t) &= \lim_{h\to 0}\frac{\mathbf{u}(t+h)\times[\mathbf{v}(t+h)-\mathbf{v}(t)]}{h} + \lim_{h\to 0}\frac{[\mathbf{u}(t+h)-\mathbf{u}(t)]\times\mathbf{v}(t)}{h} \\ &= \mathbf{u}(t)\times\mathbf{v}'(t)+\mathbf{u}'(t)\times\mathbf{v}(t)\end{aligned}$$

by Exercise 14.1.41(a) [ET 13.1.41(a)] and Definition 1.

**44.** $\dfrac{d}{dt}\,[\mathbf{u}(f(t))] = \dfrac{d}{dt}\,\langle u_1(f(t)), u_2(f(t)), u_3(f(t))\rangle = \left\langle \dfrac{d}{dt}\,[u_1(f(t))]\,, \dfrac{d}{dt}\,[u_2(f(t))]\,, \dfrac{d}{dt}\,[u_3(f(t))]\right\rangle$

$= \langle f'(t)u_1'(f(t)), f'(t)u_2'(f(t)), f'(t)u_3'(f(t))\rangle = f'(t)\,\mathbf{u}'(t)$

**45.** $\dfrac{d}{dt}\,[\mathbf{u}(t)\cdot\mathbf{v}(t)] = \mathbf{u}'(t)\cdot\mathbf{v}(t) + \mathbf{u}(t)\cdot\mathbf{v}'(t)$ [by Formula 4 of Theorem 3]

$= (-4t\,\mathbf{j} + 9t^2\,\mathbf{k})\cdot(t\,\mathbf{i} + \cos t\,\mathbf{j} + \sin t\,\mathbf{k}) + (\mathbf{i} - 2t^2\,\mathbf{j} + 3t^3\,\mathbf{k})\cdot(\mathbf{i} - \sin t\,\mathbf{j} + \cos t\,\mathbf{k})$

$= -4t\cos t + 9t^2\sin t + 1 + 2t^2\sin t + 3t^3\cos t$

$= 1 - 4t\cos t + 11t^2\sin t + 3t^3\cos t$

**46.** $\dfrac{d}{dt}\,[\mathbf{u}(t)\times\mathbf{v}(t)] = \mathbf{u}'(t)\times\mathbf{v}(t) + \mathbf{u}(t)\times\mathbf{v}'(t)$ [by Formula 5 of Theorem 3]

$= (-4t\,\mathbf{j} + 9t^2\,\mathbf{k})\times(t\,\mathbf{i} + \cos t\,\mathbf{j} + \sin t\,\mathbf{k}) + (\mathbf{i} - 2t^2\,\mathbf{j} + 3t^3\,\mathbf{k})\times(\mathbf{i} - \sin t\,\mathbf{j} + \cos t\,\mathbf{k})$

$= (-4t\sin t - 9t^2\cos t)\,\mathbf{i} + (9t^3 - 0)\,\mathbf{j} + (0 + 4t^2)\,\mathbf{k}$

$+ (-2t^2\cos t + 3t^3\sin t)\,\mathbf{i} + (3t^3 - \cos t)\,\mathbf{j} + (-\sin t + 2t^2)\,\mathbf{k}$

$= [(\sin t)(3t^3 - 4t) - 11t^2\cos t]\,\mathbf{i} + (12t^3 - \cos t)\,\mathbf{j} + (6t^2 - \sin t)\,\mathbf{k}$

**47.** $\dfrac{d}{dt}\,[\mathbf{r}(t)\times\mathbf{r}'(t)] = \mathbf{r}'(t)\times\mathbf{r}'(t) + \mathbf{r}(t)\times\mathbf{r}''(t)$ by Formula 5 of Theorem 3. But $\mathbf{r}'(t)\times\mathbf{r}'(t) = \mathbf{0}$ (see Example 13.4.2 [ET 12.4.2]). Thus, $\dfrac{d}{dt}\,[\mathbf{r}(t)\times\mathbf{r}'(t)] = \mathbf{r}(t)\times\mathbf{r}''(t)$.

**48.** $\dfrac{d}{dt}\,(\mathbf{u}(t)\cdot[\mathbf{v}(t)\times\mathbf{w}(t)]) = \mathbf{u}'(t)\cdot[\mathbf{v}(t)\times\mathbf{w}(t)] + \mathbf{u}(t)\cdot\dfrac{d}{dt}\,[\mathbf{v}(t)\times\mathbf{w}(t)]$

$= \mathbf{u}'(t)\cdot[\mathbf{v}(t)\times\mathbf{w}(t)] + \mathbf{u}(t)\cdot[\mathbf{v}'(t)\times\mathbf{w}(t) + \mathbf{v}(t)\times\mathbf{w}'(t)]$

$= \mathbf{u}'(t)\cdot[\mathbf{v}(t)\times\mathbf{w}(t)] + \mathbf{u}(t)\cdot[\mathbf{v}'(t)\times\mathbf{w}(t)] + \mathbf{u}(t)\cdot[\mathbf{v}(t)\times\mathbf{w}'(t)]$

$= \mathbf{u}'(t)\cdot[\mathbf{v}(t)\times\mathbf{w}(t)] - \mathbf{v}'(t)\cdot[\mathbf{u}(t)\times\mathbf{w}(t)] + \mathbf{w}'(t)\cdot[\mathbf{u}(t)\times\mathbf{v}(t)]$

**49.** $\dfrac{d}{dt}\,|\mathbf{r}(t)| = \dfrac{d}{dt}\,[\mathbf{r}(t)\cdot\mathbf{r}(t)]^{1/2} = \frac{1}{2}[\mathbf{r}(t)\cdot\mathbf{r}(t)]^{-1/2}\,[2\mathbf{r}(t)\cdot\mathbf{r}'(t)] = \dfrac{1}{|\mathbf{r}(t)|}\,\mathbf{r}(t)\cdot\mathbf{r}'(t)$

**50.** Since $\mathbf{r}(t)\cdot\mathbf{r}'(t) = 0$, we have $0 = 2\mathbf{r}(t)\cdot\mathbf{r}'(t) = \dfrac{d}{dt}\,[\mathbf{r}(t)\cdot\mathbf{r}(t)] = \dfrac{d}{dt}\,|\mathbf{r}(t)|^2$. Thus $|\mathbf{r}(t)|^2$, and so $|\mathbf{r}(t)|$, is a constant, and hence the curve lies on a sphere with center the origin.

**51.** Since $\mathbf{u}(t) = \mathbf{r}(t)\cdot[\mathbf{r}'(t)\times\mathbf{r}''(t)]$,

$$\mathbf{u}'(t) = \mathbf{r}'(t)\cdot[\mathbf{r}'(t)\times\mathbf{r}''(t)] + \mathbf{r}(t)\cdot\dfrac{d}{dt}\,[\mathbf{r}'(t)\times\mathbf{r}''(t)]$$

$$= 0 + \mathbf{r}(t)\cdot[\mathbf{r}''(t)\times\mathbf{r}''(t) + \mathbf{r}'(t)\times\mathbf{r}'''(t)] \qquad [\text{since } \mathbf{r}'(t)\perp\mathbf{r}'(t)\times\mathbf{r}''(t)]$$

$$= \mathbf{r}(t)\cdot[\mathbf{r}'(t)\times\mathbf{r}'''(t)] \qquad [\text{since } \mathbf{r}''(t)\times\mathbf{r}''(t) = \mathbf{0}]$$

## 14.3 Arc Length and Curvature ET 13.3

**1.** $\mathbf{r}'(t) = \langle 2\cos t, 5, -2\sin t\rangle \Rightarrow |\mathbf{r}'(t)| = \sqrt{(2\cos t)^2 + 5^2 + (-2\sin t)^2} = \sqrt{29}$. Then using Formula 3, we have $L = \int_{-10}^{10} |\mathbf{r}'(t)|\,dt = \int_{-10}^{10} \sqrt{29}\,dt = \sqrt{29}\,t\big]_{-10}^{10} = 20\sqrt{29}$.

**2.** $\mathbf{r}'(t) = \langle 2t, \cos t + t\sin t - \cos t, -\sin t + t\cos t + \sin t\rangle = \langle 2t, t\sin t, t\cos t\rangle \Rightarrow$
$|\mathbf{r}'(t)| = \sqrt{(2t)^2 + (t\sin t)^2 + (t\cos t)^2} = \sqrt{4t^2 + t^2(\sin^2 t + \cos^2 t)} = \sqrt{5}\,|t| = \sqrt{5}\,t$ for $0 \le t \le \pi$. Then using Formula 3, we have $L = \int_0^\pi |\mathbf{r}'(t)|\,dt = \int_0^\pi \sqrt{5}\,t\,dt = \sqrt{5}\,\frac{t^2}{2}\Big]_0^\pi = \frac{\sqrt{5}}{2}\pi^2$.

**3.** $\mathbf{r}'(t) = \sqrt{2}\,\mathbf{i} + e^t\mathbf{j} - e^{-t}\mathbf{k} \Rightarrow$
$|\mathbf{r}'(t)| = \sqrt{\left(\sqrt{2}\right)^2 + (e^t)^2 + (-e^{-t})^2} = \sqrt{2 + e^{2t} + e^{-2t}} = \sqrt{(e^t + e^{-t})^2} = e^t + e^{-t}$ (since $e^t + e^{-t} > 0$).
Then $L = \int_0^1 |\mathbf{r}'(t)|\,dt = \int_0^1 (e^t + e^{-t})\,dt = \left[e^t - e^{-t}\right]_0^1 = e - e^{-1}$.

**4.** $\mathbf{r}'(t) = \langle 2t, 2, 1/t\rangle$, $|\mathbf{r}'(t)| = \sqrt{4t^2 + 4 + (1/t)^2} = \dfrac{1 + 2t^2}{|t|} = \dfrac{1 + 2t^2}{t}$ for $1 \le t \le e$.

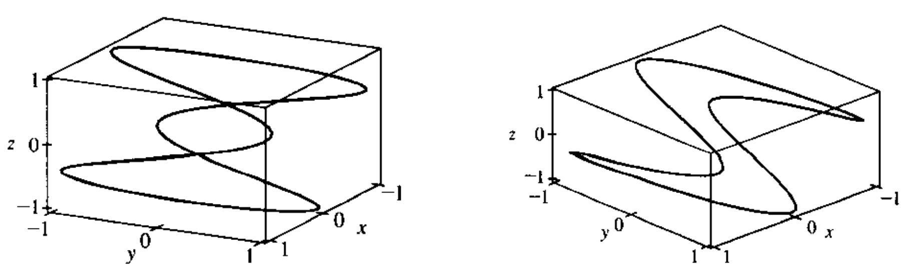
$$L = \int_1^e \frac{1 + 2t^2}{t}\,dt = \int_1^e \left(\frac{1}{t} + 2t\right) dt = \left[\ln t + t^2\right]_1^e = e^2$$

**5.** $\mathbf{r}'(t) = 2t\,\mathbf{j} + 3t^2\,\mathbf{k} \Rightarrow |\mathbf{r}'(t)| = \sqrt{4t^2 + 9t^4} = t\sqrt{4 + 9t^2}$ (since $t \ge 0$). Then
$L = \int_0^1 |\mathbf{r}'(t)|\,dt = \int_0^1 t\sqrt{4 + 9t^2}\,dt = \frac{1}{18}\cdot\frac{2}{3}(4 + 9t^2)^{3/2}\Big]_0^1 = \frac{1}{27}(13^{3/2} - 4^{3/2}) = \frac{1}{27}(13^{3/2} - 8)$.

**6.** $\mathbf{r}'(t) = 12\,\mathbf{i} + 12\sqrt{t}\,\mathbf{j} + 6t\,\mathbf{k} \Rightarrow |\mathbf{r}'(t)| = \sqrt{144 + 144t + 36t^2} = \sqrt{36(t+2)^2} = 6\,|t + 2| = 6(t + 2)$ for $0 \le t \le 1$. Then $L = \int_0^1 |\mathbf{r}'(t)|\,dt = \int_0^1 6(t + 2)\,dt = \left[3t^2 + 12t\right]_0^1 = 15$.

**7.** The point $(2, 4, 8)$ corresponds to $t = 2$, so by Equation 2, $L = \int_0^2 \sqrt{(1)^2 + (2t)^2 + (3t^2)^2}\,dt$.
If $f(t) = \sqrt{1 + 4t^2 + 9t^4}$, then Simpson's Rule gives
$L \approx \dfrac{2 - 0}{10 \cdot 3}\left[f(0) + 4f(0.2) + 2f(0.4) + \cdots + 4f(1.8) + f(2)\right] \approx 9.5706$.

**8.** Here are two views of the curve with parametric equations $x = \cos t$, $y = \sin 3t$, $z = \sin t$:




The complete curve is given by the parameter interval $[0, 2\pi]$, so
$L = \int_0^{2\pi} \sqrt{(-\sin t)^2 + (3\cos 3t)^2 + (\cos t)^2}\,dt = \int_0^{2\pi} \sqrt{1 + 9\cos^2 3t}\,dt \approx 13.9744$.

**9.** $\mathbf{r}'(t) = 2\,\mathbf{i} - 3\,\mathbf{j} + 4\,\mathbf{k}$ and $\frac{ds}{dt} = |\mathbf{r}'(t)| = \sqrt{4 + 9 + 16} = \sqrt{29}$. Then
$s = s(t) = \int_0^t |\mathbf{r}'(u)|\,du = \int_0^t \sqrt{29}\,du = \sqrt{29}\,t$. Therefore, $t = \frac{1}{\sqrt{29}}s$, and substituting for $t$ in the original equation, we have $\mathbf{r}(t(s)) = \frac{2}{\sqrt{29}}s\,\mathbf{i} + \left(1 - \frac{3}{\sqrt{29}}s\right)\mathbf{j} + \left(5 + \frac{4}{\sqrt{29}}s\right)\mathbf{k}$.

**10.** $\mathbf{r}'(t) = 2e^{2t}(\cos 2t - \sin 2t)\,\mathbf{i} + 2e^{2t}(\cos 2t + \sin 2t)\,\mathbf{k}$,

$\frac{ds}{dt} = |\mathbf{r}'(t)| = 2e^{2t}\sqrt{(\cos 2t - \sin 2t)^2 + (\cos 2t + \sin 2t)^2} = 2e^{2t}\sqrt{2\cos^2 2t + 2\sin^2 2t} = 2\sqrt{2}\,e^{2t}$.

$s = s(t) = \int_0^t |\mathbf{r}'(u)|\,du = \int_0^t 2\sqrt{2}\,e^{2u}\,du = \sqrt{2}\,e^{2u}\Big]_0^t = \sqrt{2}\,(e^{2t} - 1) \quad\Rightarrow$

$\frac{s}{\sqrt{2}} + 1 = e^{2t} \quad\Rightarrow\quad t = \frac{1}{2}\ln\!\left(\frac{s}{\sqrt{2}} + 1\right)$. Substituting, we have

$$\mathbf{r}(t(s)) = e^{2\left(\frac{1}{2}\ln\left(\frac{s}{\sqrt{2}}+1\right)\right)}\cos 2\left(\tfrac{1}{2}\ln\!\left(\tfrac{s}{\sqrt{2}}+1\right)\right)\mathbf{i} + 2\,\mathbf{j} + e^{2\left(\frac{1}{2}\ln\left(\frac{s}{\sqrt{2}}+1\right)\right)}\sin 2\left(\tfrac{1}{2}\ln\!\left(\tfrac{s}{\sqrt{2}}+1\right)\right)\mathbf{k}$$

$$= \left(\tfrac{s}{\sqrt{2}}+1\right)\cos\!\left(\ln\!\left(\tfrac{s}{\sqrt{2}}+1\right)\right)\mathbf{i} + 2\,\mathbf{j} + \left(\tfrac{s}{\sqrt{2}}+1\right)\sin\!\left(\ln\!\left(\tfrac{s}{\sqrt{2}}+1\right)\right)\mathbf{k}.$$

**11.** $|\mathbf{r}'(t)| = \sqrt{(3\cos t)^2 + 16 + (-3\sin t)^2} = \sqrt{9+16} = 5$ and $s(t) = \int_0^t |\mathbf{r}'(u)|\,du = \int_0^t 5\,du = 5t \quad\Rightarrow$

$t(s) = \frac{1}{5}s$. Therefore, $\mathbf{r}(t(s)) = 3\sin\!\left(\frac{1}{5}s\right)\mathbf{i} + \frac{4}{5}s\,\mathbf{j} + 3\cos\!\left(\frac{1}{5}s\right)\mathbf{k}$.

**12.** $\mathbf{r}'(t) = \dfrac{-4t}{(t^2+1)^2}\,\mathbf{i} + \dfrac{-2t^2+2}{(t^2+1)^2}\,\mathbf{j}$,

$$\frac{ds}{dt} = |\mathbf{r}'(t)| = \sqrt{\left[\frac{-4t}{(t^2+1)^2}\right]^2 + \left[\frac{-2t^2+2}{(t^2+1)^2}\right]^2} = \sqrt{\frac{4t^4+8t^2+4}{(t^2+1)^4}} = \sqrt{\frac{4(t^2+1)^2}{(t^2+1)^4}}$$

$$= \sqrt{\frac{4}{(t^2+1)^2}} = \frac{2}{t^2+1}$$

Since the initial point $(1,0)$ corresponds to $t=0$, the arc length function

$s(t) = \displaystyle\int_0^t |\mathbf{r}'(u)|\,du = \int_0^t \frac{2}{u^2+1}\,du = 2\arctan t$. Then $\arctan t = \frac{1}{2}s \quad\Rightarrow\quad t = \tan\frac{1}{2}s$. Substituting, we have

$$\mathbf{r}(t(s)) = \left[\frac{2}{\tan^2\left(\frac{1}{2}s\right)+1} - 1\right]\mathbf{i} + \frac{2\tan\left(\frac{1}{2}s\right)}{\tan^2\left(\frac{1}{2}s\right)+1}\,\mathbf{j} = \frac{1-\tan^2\left(\frac{1}{2}s\right)}{1+\tan^2\left(\frac{1}{2}s\right)}\,\mathbf{i} + \frac{2\tan\left(\frac{1}{2}s\right)}{\sec^2\left(\frac{1}{2}s\right)}\,\mathbf{j}$$

$$= \frac{1-\tan^2\left(\frac{1}{2}s\right)}{\sec^2\left(\frac{1}{2}s\right)}\,\mathbf{i} + 2\tan\left(\tfrac{1}{2}s\right)\cos^2\left(\tfrac{1}{2}s\right)\mathbf{j}$$

$$= \left[\cos^2\left(\tfrac{1}{2}s\right) - \sin^2\left(\tfrac{1}{2}s\right)\right]\mathbf{i} + 2\sin\left(\tfrac{1}{2}s\right)\cos\left(\tfrac{1}{2}s\right)\mathbf{j} = \cos s\,\mathbf{i} + \sin s\,\mathbf{j}$$

With this parametrization, we recognize the function as representing the unit circle. Note here that the curve approaches, but does not include, the point $(-1,0)$, since $\cos s = -1$ for $s = \pi + 2k\pi$ ($k$ an integer) but then $t = \tan\left(\frac{1}{2}s\right)$ is undefined.

**13.** (a) $\mathbf{r}'(t) = \langle 2\cos t, 5, -2\sin t\rangle \quad\Rightarrow\quad |\mathbf{r}'(t)| = \sqrt{4\cos^2 t + 25 + 4\sin^2 t} = \sqrt{29}$. Then

$\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{1}{\sqrt{29}}\langle 2\cos t, 5, -2\sin t\rangle$ or $\left\langle \frac{2}{\sqrt{29}}\cos t, \frac{5}{\sqrt{29}}, -\frac{2}{\sqrt{29}}\sin t\right\rangle$.

$\mathbf{T}'(t) = \frac{1}{\sqrt{29}}\langle -2\sin t, 0, -2\cos t\rangle \quad\Rightarrow\quad |\mathbf{T}'(t)| = \frac{1}{\sqrt{29}}\sqrt{4\sin^2 t + 0 + 4\cos^2 t} = \frac{2}{\sqrt{29}}$. Thus

$\mathbf{N}(t) = \dfrac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = \dfrac{1/\sqrt{29}}{2/\sqrt{29}}\langle -2\sin t, 0, -2\cos t\rangle = \langle -\sin t, 0, -\cos t\rangle$.

(b) $\kappa(t) = \dfrac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \dfrac{2/\sqrt{29}}{\sqrt{29}} = \dfrac{2}{29}$.

**14.** (a) $\mathbf{r}'(t) = \langle 2t, t\sin t, t\cos t\rangle \quad\Rightarrow\quad |\mathbf{r}'(t)| = \sqrt{4t^2 + t^2\sin^2 t + t^2\cos^2 t} = \sqrt{5t^2} = \sqrt{5}\,t$ (since $t > 0$). Then $\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \dfrac{1}{\sqrt{5}\,t}\langle 2t, t\sin t, t\cos t\rangle = \frac{1}{\sqrt{5}}\langle 2, \sin t, \cos t\rangle$.

$\mathbf{T}'(t) = \frac{1}{\sqrt{5}}\langle 0, \cos t, -\sin t\rangle \quad\Rightarrow\quad |\mathbf{T}'(t)| = \frac{1}{\sqrt{5}}\sqrt{0 + \cos^2 t + \sin^2 t} = \frac{1}{\sqrt{5}}$. Thus

$\mathbf{N}(t) = \dfrac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = \dfrac{1/\sqrt{5}}{1/\sqrt{5}}\langle 0, \cos t, -\sin t\rangle = \langle 0, \cos t, -\sin t\rangle$.

(b) $\kappa(t) = \dfrac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \dfrac{1/\sqrt{5}}{\sqrt{5}\,t} = \dfrac{1}{5t}$.

**15.** (a) $\mathbf{r}'(t) = \left\langle \sqrt{2}, e^t, -e^{-t}\right\rangle \quad\Rightarrow\quad |\mathbf{r}'(t)| = \sqrt{2 + e^{2t} + e^{-2t}} = \sqrt{(e^t + e^{-t})^2} = e^t + e^{-t}$. Then

$\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \dfrac{1}{e^t + e^{-t}}\left\langle \sqrt{2}, e^t, -e^{-t}\right\rangle = \dfrac{1}{e^{2t}+1}\left\langle \sqrt{2}e^t, e^{2t}, -1\right\rangle$ $\left(\text{after multiplying by } \dfrac{e^t}{e^t}\right)$ and

$$\begin{aligned}\mathbf{T}'(t) &= \frac{1}{e^{2t}+1}\left\langle \sqrt{2}e^t, 2e^{2t}, 0\right\rangle - \frac{2e^{2t}}{(e^{2t}+1)^2}\left\langle \sqrt{2}e^t, e^{2t}, -1\right\rangle \\ &= \frac{1}{(e^{2t}+1)^2}\left[(e^{2t}+1)\left\langle \sqrt{2}e^t, 2e^{2t}, 0\right\rangle - 2e^{2t}\left\langle \sqrt{2}e^t, e^{2t}, -1\right\rangle\right] \\ &= \frac{1}{(e^{2t}+1)^2}\left\langle \sqrt{2}e^t\left(1 - e^{2t}\right), 2e^{2t}, 2e^{2t}\right\rangle\end{aligned}$$

Then

$$\begin{aligned}|\mathbf{T}'(t)| &= \frac{1}{(e^{2t}+1)^2}\sqrt{2e^{2t}(1 - 2e^{2t} + e^{4t}) + 4e^{4t} + 4e^{4t}} = \frac{1}{(e^{2t}+1)^2}\sqrt{2e^{2t}(1 + 2e^{2t} + e^{4t})} \\ &= \frac{1}{(e^{2t}+1)^2}\sqrt{2e^{2t}\left(1 + e^{2t}\right)^2} = \frac{\sqrt{2}e^t(1 + e^{2t})}{(e^{2t}+1)^2} = \frac{\sqrt{2}\,e^t}{e^{2t}+1}\end{aligned}$$

Therefore

$$\begin{aligned}\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} &= \frac{e^{2t}+1}{\sqrt{2}\,e^t}\,\frac{1}{(e^{2t}+1)^2}\left\langle \sqrt{2}\,e^t(1 - e^{2t}), 2e^{2t}, 2e^{2t}\right\rangle \\ &= \frac{1}{\sqrt{2}\,e^t(e^{2t}+1)}\left\langle \sqrt{2}\,e^t(1 - e^{2t}), 2e^{2t}, 2e^{2t}\right\rangle \\ &= \frac{1}{e^{2t}+1}\left\langle 1 - e^{2t}, \sqrt{2}\,e^t, \sqrt{2}\,e^t\right\rangle\end{aligned}$$

(b) $\kappa(t) = \dfrac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \dfrac{\sqrt{2}\,e^t}{e^{2t}+1}\cdot\dfrac{1}{e^t + e^{-t}} = \dfrac{\sqrt{2}\,e^t}{e^{3t} + 2e^t + e^{-t}} = \dfrac{\sqrt{2}\,e^{2t}}{e^{4t} + 2e^{2t} + 1} = \dfrac{\sqrt{2}\,e^{2t}}{(e^{2t}+1)^2}$.

**16.** (a) $\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \dfrac{1}{\sqrt{4t^2 + 4 + (1/t)^2}}\langle 2t, 2, 1/t\rangle = \dfrac{|t|}{2t^2+1}\langle 2t, 2, 1/t\rangle$. But since the **k**-component is $\ln t$, $t$ is positive, $|t| = t$ and $\mathbf{T}(t) = \dfrac{1}{2t^2+1}\langle 2t^2, 2t, 1\rangle$. Then

$\mathbf{T}'(t) = \dfrac{1}{2t^2+1}\langle 4t, 2, 0\rangle - \left(2t^2+1\right)^{-2}(4t)\left\langle 2t^2, 2t, 1\right\rangle = \dfrac{1}{(2t^2+1)^2}\langle 4t, 2 - 4t^2, -4t\rangle$, so

$\mathbf{N}(t) = \dfrac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = \dfrac{\langle 4t, 2 - 4t^2, -4t\rangle}{\sqrt{(4t)^2 + (2 - 4t^2)^2 + (-4t)^2}} = \dfrac{1}{2t^2+1}\langle 2t, 1 - 2t^2, -2t\rangle$.

(b) $\kappa(t) = \dfrac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \dfrac{2}{2t^2+1}\left(\dfrac{t}{2t^2+1}\right) = \dfrac{2t}{(2t^2+1)^2}$

**17.** $\mathbf{r}'(t) = 2t\,\mathbf{i} + \mathbf{k}$, $\mathbf{r}''(t) = 2\,\mathbf{i}$, $|\mathbf{r}'(t)| = \sqrt{(2t)^2 + 0^2 + 1^2} = \sqrt{4t^2+1}$, $\mathbf{r}'(t) \times \mathbf{r}''(t) = 2\,\mathbf{j}$, $|\mathbf{r}'(t) \times \mathbf{r}''(t)| = 2$.

Then $\kappa(t) = \dfrac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} = \dfrac{2}{\left(\sqrt{4t^2+1}\right)^3} = \dfrac{2}{(4t^2+1)^{3/2}}$.

**18.** $\mathbf{r}'(t) = \mathbf{i} + \mathbf{j} + 2t\,\mathbf{k}$, $\mathbf{r}''(t) = 2\,\mathbf{k}$, $|\mathbf{r}'(t)| = \sqrt{1^2+1^2+(2t)^2} = \sqrt{4t^2+2}$,

$\mathbf{r}'(t) \times \mathbf{r}''(t) = 2\,\mathbf{i} - 2\,\mathbf{j}$, $|\mathbf{r}'(t) \times \mathbf{r}''(t)| = \sqrt{2^2+2^2+0^2} = \sqrt{8} = 2\sqrt{2}$. Then

$\kappa(t) = \dfrac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} = \dfrac{2\sqrt{2}}{\left(\sqrt{4t^2+2}\right)^3} = \dfrac{2\sqrt{2}}{\left(\sqrt{2}\,\sqrt{2t^2+1}\right)^3} = \dfrac{1}{(2t^2+1)^{3/2}}$.

**19.** $\mathbf{r}'(t) = 3\,\mathbf{i} + 4\cos t\,\mathbf{j} - 4\sin t\,\mathbf{k}$, $\mathbf{r}''(t) = -4\sin t\,\mathbf{j} - 4\cos t\,\mathbf{k}$,

$|\mathbf{r}'(t)| = \sqrt{9 + 16\cos^2 t + 16\sin^2 t} = \sqrt{9+16} = 5$, $\mathbf{r}'(t) \times \mathbf{r}''(t) = -16\,\mathbf{i} + 12\cos t\,\mathbf{j} - 12\sin t\,\mathbf{k}$,

$|\mathbf{r}'(t) \times \mathbf{r}''(t)| = \sqrt{256 + 144\cos^2 t + 144\sin^2 t} = \sqrt{400} = 20$. Then $\kappa(t) = \dfrac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} = \dfrac{20}{5^3} = \dfrac{4}{25}$.

**20.** $\mathbf{r}'(t) = \left\langle e^t\cos t - e^t\sin t, e^t\cos t + e^t\sin t, 1\right\rangle$. The point $(1,0,0)$ corresponds to $t=0$, and

$\mathbf{r}'(0) = \langle 1,1,1\rangle \quad\Rightarrow\quad |\mathbf{r}'(0)| = \sqrt{1^2+1^2+1^2} = \sqrt{3}$.

$\mathbf{r}''(t) = \left\langle e^t\cos t - e^t\sin t - e^t\cos t - e^t\sin t, e^t\cos t - e^t\sin t + e^t\cos t + e^t\sin t, 0\right\rangle$

$= \left\langle -2e^t\sin t, 2e^t\cos t, 0\right\rangle \quad\Rightarrow\quad \mathbf{r}''(0) = \langle 0,2,0\rangle$.

$\mathbf{r}'(0) \times \mathbf{r}''(0) = \langle -2,0,2\rangle$. $\quad |\mathbf{r}'(0) \times \mathbf{r}''(0)| = \sqrt{(-2)^2+0^2+2^2} = \sqrt{8} = 2\sqrt{2}$.

Then $\kappa(0) = \dfrac{|\mathbf{r}'(0) \times \mathbf{r}''(0)|}{|\mathbf{r}'(0)|^3} = \dfrac{2\sqrt{2}}{\left(\sqrt{3}\right)^3} = \dfrac{2\sqrt{2}}{3\sqrt{3}}$ or $\dfrac{2\sqrt{6}}{9}$.

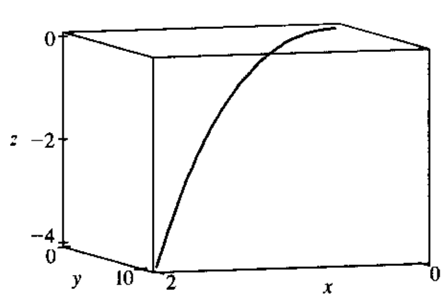
**21.** $\mathbf{r}'(t) = \left\langle 1, 2t, 3t^2\right\rangle$. The point $(1,1,1)$ corresponds to $t=1$, and

$\mathbf{r}'(1) = \langle 1,2,3\rangle \quad\Rightarrow\quad |\mathbf{r}'(1)| = \sqrt{1+4+9} = \sqrt{14}$. $\quad \mathbf{r}''(t) = \langle 0,2,6t\rangle \quad\Rightarrow\quad \mathbf{r}''(1) = \langle 0,2,6\rangle$.

$\mathbf{r}'(1) \times \mathbf{r}''(1) = \langle 6,-6,2\rangle$, so $|\mathbf{r}'(1) \times \mathbf{r}''(1)| = \sqrt{36+36+4} = \sqrt{76}$. Then

$\kappa(1) = \dfrac{|\mathbf{r}'(1) \times \mathbf{r}''(1)|}{|\mathbf{r}'(1)|^3} = \dfrac{\sqrt{76}}{\sqrt{14}^3} = \dfrac{1}{7}\sqrt{\dfrac{19}{14}}$.

**22.**




$\mathbf{r}(t) = \left\langle t, 4t^{3/2}, -t^2\right\rangle \quad\Rightarrow\quad \mathbf{r}'(t) = \left\langle 1, 6t^{1/2}, -2t\right\rangle$,

$\mathbf{r}''(t) = \left\langle 0, 3t^{-1/2}, -2\right\rangle$, $|\mathbf{r}'(t)|^3 = \left(1 + 36t + 4t^2\right)^{3/2}$,

$\mathbf{r}'(t) \times \mathbf{r}''(t) = \left\langle -12t^{1/2} + 6t^{1/2}, 2, 3t^{-1/2}\right\rangle \quad\Rightarrow$

$|\mathbf{r}'(t) \times \mathbf{r}''(t)| = \sqrt{36t + 4 + 9t^{-1}} = \left[\dfrac{36t^2+4t+9}{t}\right]^{1/2}$

$\kappa(t) = \dfrac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} = \left(\dfrac{36t^2+4t+9}{t}\right)^{1/2} \dfrac{1}{(1+36t+4t^2)^{3/2}} = \dfrac{\sqrt{36t^2+4t+9}}{t^{1/2}(1+36t+4t^2)^{3/2}}$.

The point $(1,4,-1)$ corresponds to $t=1$, so the curvature at this point is $\kappa(1) = \dfrac{\sqrt{36+4+9}}{(1+36+4)^{3/2}} = \dfrac{7}{41\sqrt{41}}$.

**23.** $f(x) = x^3$, $f'(x) = 3x^2$, $f''(x) = 6x$, $\kappa(x) = \dfrac{|f''(x)|}{\left[1 + (f'(x))^2\right]^{3/2}} = \dfrac{6\,|x|}{(1+9x^4)^{3/2}}$

**24.** $f(x) = \cos x$, $f'(x) = -\sin x$, $f''(x) = -\cos x$,

$$\kappa(x) = \frac{|f''(x)|}{[1 + (f'(x))^2]^{3/2}} = \frac{|-\cos x|}{[1 + (-\sin x)^2]^{3/2}} = \frac{|\cos x|}{(1 + \sin^2 x)^{3/2}}$$

**25.** $f(x) = 4x^{5/2}$, $f'(x) = 10x^{3/2}$, $f''(x) = 15x^{1/2}$,

$$\kappa(x) = \frac{|f''(x)|}{[1 + (f'(x))^2]^{3/2}} = \frac{\left|15x^{1/2}\right|}{[1 + (10x^{3/2})^2]^{3/2}} = \frac{15\sqrt{x}}{(1 + 100x^3)^{3/2}}$$

**26.** $y' = \dfrac{1}{x}$, $y'' = -\dfrac{1}{x^2}$,

$$\kappa(x) = \frac{|y''(x)|}{\left[1 + (y'(x))^2\right]^{3/2}} = \left|\frac{-1}{x^2}\right| \frac{1}{(1 + 1/x^2)^{3/2}} = \frac{1}{x^2}\frac{(x^2)^{3/2}}{(x^2 + 1)^{3/2}} = \frac{|x|}{(x^2 + 1)^{3/2}} = \frac{x}{(x^2 + 1)^{3/2}}$$

(since $x > 0$). To find the maximum curvature, we first find the critical numbers of $\kappa(x)$:

$$\kappa'(x) = \frac{(x^2 + 1)^{3/2} - x\left(\frac{3}{2}\right)(x^2 + 1)^{1/2}(2x)}{[(x^2 + 1)^{3/2}]^2} = \frac{(x^2 + 1)^{1/2}[(x^2 + 1) - 3x^2]}{(x^2 + 1)^3} = \frac{1 - 2x^2}{(x^2 + 1)^{5/2}};$$

$\kappa'(x) = 0 \;\Rightarrow\; 1 - 2x^2 = 0$, so the only critical number in the domain is $x = \frac{1}{\sqrt{2}}$. Since $\kappa'(x) > 0$ for $0 < x < \frac{1}{\sqrt{2}}$ and $\kappa'(x) < 0$ for $x > \frac{1}{\sqrt{2}}$, $\kappa(x)$ attains its maximum at $x = \frac{1}{\sqrt{2}}$. Thus, the maximum curvature occurs at $\left(\frac{1}{\sqrt{2}}, \ln\frac{1}{\sqrt{2}}\right)$. Since $\lim\limits_{x\to\infty} \dfrac{x}{(x^2 + 1)^{3/2}} = 0$, $\kappa(x)$ approaches 0 as $x \to \infty$.

**27.** Since $y' = y'' = e^x$, the curvature is $\kappa(x) = \dfrac{|y''(x)|}{[1 + (y'(x))^2]^{3/2}} = \dfrac{e^x}{(1 + e^{2x})^{3/2}} = e^x(1 + e^{2x})^{-3/2}$.

To find the maximum curvature, we first find the critical numbers of $\kappa(x)$:

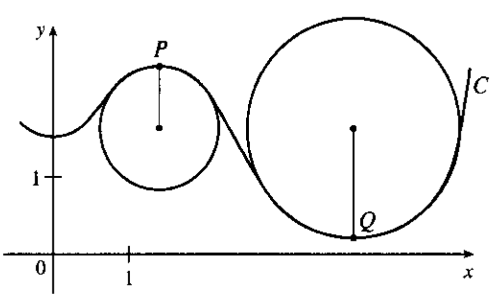
$$\kappa'(x) = e^x(1 + e^{2x})^{-3/2} + e^x\left(-\tfrac{3}{2}\right)(1 + e^{2x})^{-5/2}(2e^{2x}) = e^x\frac{1 + e^{2x} - 3e^{2x}}{(1 + e^{2x})^{5/2}} = e^x\frac{1 - 2e^{2x}}{(1 + e^{2x})^{5/2}}.$$

$\kappa'(x) = 0$ when $1 - 2e^{2x} = 0$, so $e^{2x} = \frac{1}{2}$ or $x = -\frac{1}{2}\ln 2$. And since $1 - 2e^{2x} > 0$ for $x < -\frac{1}{2}\ln 2$ and $1 - 2e^{2x} < 0$ for $x > -\frac{1}{2}\ln 2$, the maximum curvature is attained at the point $\left(-\frac{1}{2}\ln 2, e^{(-\ln 2)/2}\right) = \left(-\frac{1}{2}\ln 2, \frac{1}{\sqrt{2}}\right)$. Since $\lim\limits_{x\to\infty} e^x(1 + e^{2x})^{-3/2} = 0$, $\kappa(x)$ approaches 0 as $x \to \infty$.

**28.** We can take the parabola as having its vertex at the origin and opening upward, so the equation is $f(x) = ax^2, a > 0$. Then by Equation 11, $\kappa(x) = \dfrac{|f''(x)|}{[1 + (f'(x))^2]^{3/2}} = \dfrac{|2a|}{[1 + (2ax)^2]^{3/2}} = \dfrac{2a}{(1 + 4a^2x^2)^{3/2}}$, thus $\kappa(0) = 2a$. We want $\kappa(0) = 4$, so $a = 2$ and the equation is $y = 2x^2$.

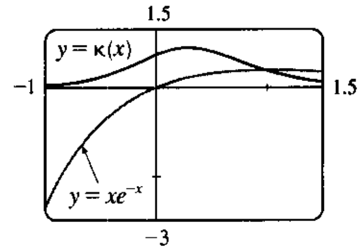
**29.** (a) $C$ appears to be changing direction more quickly at $P$ than $Q$, so we would expect the curvature to be greater at $P$.

(b) First we sketch approximate osculating circles at $P$ and $Q$. Using the axes scale as a guide, we measure the radius of the osculating circle at $P$ to be approximately 0.8 units, thus $\rho = \dfrac{1}{\kappa} \;\Rightarrow\; \kappa = \dfrac{1}{\rho} \approx \dfrac{1}{0.8} \approx 1.3$. Similarly, we estimate the radius of the osculating circle at $Q$ to be 1.4 units, so $\kappa = \dfrac{1}{\rho} \approx \dfrac{1}{1.4} \approx 0.7$.



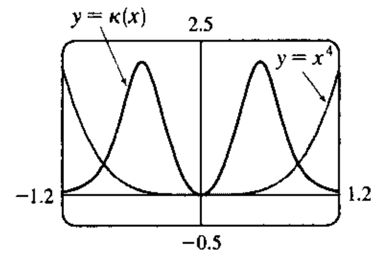
**30.** $y = xe^{-x} \Rightarrow y' = e^{-x}(1-x)$, $y'' = e^{-x}(x-2)$, and

$\kappa(x) = \dfrac{|y''|}{[1+(y')^2]^{3/2}} = \dfrac{e^{-x}\,|x-2|}{[1+e^{-2x}(1-x)^2]^{3/2}}$. The graph of the curvature here is what we would expect. The graph of $xe^{-x}$ is bending most sharply slightly to the right of the origin. As $x \to \infty$, the graph of $xe^{-x}$ is asymptotic to the $x$-axis, and so the curvature approaches zero.




**31.** $y = x^4 \Rightarrow y' = 4x^3$, $y'' = 12x^2$, and

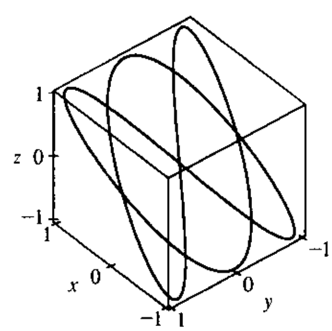
$\kappa(x) = \dfrac{|y''|}{[1+(y')^2]^{3/2}} = \dfrac{12x^2}{(1+16x^6)^{3/2}}$. The appearance of the two humps in this graph is perhaps a little surprising, but it is explained by the fact that $y = x^4$ is very flat around the origin, and so here the curvature is zero.




**32.** Notice that the curve $a$ is highest for the same $x$-values at which curve $b$ is turning more sharply, and $a$ is 0 or near 0 where $b$ is nearly straight. So, $a$ must be the graph of $y = \kappa(x)$, and $b$ is the graph of $y = f(x)$.

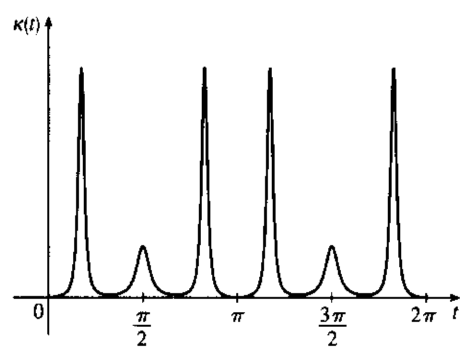
**33.** Notice that the curve $b$ has two inflection points at which the graph appears almost straight. We would expect the curvature to be 0 or nearly 0 at these values, but the curve $a$ isn't near 0 there. Thus, $a$ must be the graph of $y = f(x)$ rather than the graph of curvature, and $b$ is the graph of $y = \kappa(x)$.

**34.** (a) The complete curve is given by $0 \le t \le 2\pi$. Curvature appears to have a local (or absolute) maximum at 6 points. (Look at points where the curve appears to turn more sharply.)



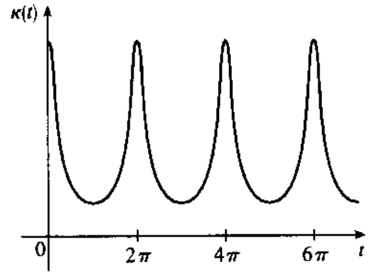

(b) Using a CAS, we find (after simplifying)

$\kappa(t) = \dfrac{3\sqrt{2}\sqrt{(5\sin t + \sin 5t)^2}}{(9\cos 6t + 2\cos 4t + 11)^{3/2}}$. (To compute cross products in Maple, use the `Linalg` package and the `crossprod(a,b)` command; in Mathematica, use `Cross[a,b]`.) The graph shows 6 local (or absolute) maximum points for $0 \le t \le 2\pi$, as observed in part (a).



**35.** Using a CAS, we find (after simplifying)

$\kappa(t) = \dfrac{6\sqrt{4\cos^2 t - 12\cos t + 13}}{(17 - 12\cos t)^{3/2}}$. (To compute cross products in Maple, use the `Linalg` package and the `crossprod(a,b)` command; in Mathematica, use `Cross[a,b]`.) Curvature is largest at integer multiples of $2\pi$.




**36.** Here $\mathbf{r}(t) = \langle f(t), g(t)\rangle$, $\mathbf{r}'(t) = \langle f'(t), g'(t)\rangle$, $\mathbf{r}''(t) = \langle f''(t), g''(t)\rangle$,

$|\mathbf{r}'(t)|^3 = \left[\sqrt{(f'(t))^2 + (g'(t))^2}\right]^3 = [(f'(t))^2 + (g'(t))^2]^{3/2} = (\dot{x}^2 + \dot{y}^2)^{3/2}$, and

$|\mathbf{r}'(t) \times \mathbf{r}''(t)| = |\langle 0, 0, f'(t)\,g''(t) - f''(t)\,g'(t)\rangle| = \left[(\dot{x}\ddot{y} - \ddot{x}\dot{y})^2\right]^{1/2} = |\dot{x}\ddot{y} - \dot{y}\ddot{x}|$.

Thus $\kappa(t) = \dfrac{|\dot{x}\ddot{y} - \dot{y}\ddot{x}|}{(\dot{x}^2 + \dot{y}^2)^{3/2}}$.

**37.** $x = e^t \cos t \;\Rightarrow\; \dot{x} = e^t(\cos t - \sin t) \;\Rightarrow\; \ddot{x} = e^t(-\sin t - \cos t) + e^t(\cos t - \sin t) = -2e^t \sin t$,
$y = e^t \sin t \;\Rightarrow\; \dot{y} = e^t(\cos t + \sin t) \;\Rightarrow\; \ddot{y} = e^t(-\sin t + \cos t) + e^t(\cos t + \sin t) = 2e^t \cos t$. Then

$$\kappa(t) = \frac{|\dot{x}\ddot{y} - \dot{y}\ddot{x}|}{(\dot{x}^2 + \dot{y}^2)^{3/2}} = \frac{\left|e^t(\cos t - \sin t)(2e^t \cos t) - e^t(\cos t + \sin t)(-2e^t \sin t)\right|}{([e^t(\cos t - \sin t)]^2 + [e^t(\cos t + \sin t)]^2)^{3/2}}$$

$$= \frac{\left|2e^{2t}(\cos^2 t - \sin t \cos t + \sin t \cos t + \sin^2 t)\right|}{\left[e^{2t}(\cos^2 t - 2\cos t \sin t + \sin^2 t + \cos^2 t + 2\cos t \sin t + \sin^2 t)\right]^{3/2}}$$

$$= \frac{\left|2e^{2t}(1)\right|}{\left[e^{2t}(1+1)\right]^{3/2}} = \frac{2e^{2t}}{e^{3t}(2)^{3/2}} = \frac{1}{\sqrt{2}\,e^t}$$

**38.** $x = 1 + t^3 \;\Rightarrow\; \dot{x} = 3t^2 \;\Rightarrow\; \ddot{x} = 6t$, $\quad y = t + t^2 \;\Rightarrow\; \dot{y} = 1 + 2t \;\Rightarrow\; \ddot{y} = 2$. Then

$$\kappa(t) = \frac{|\dot{x}\ddot{y} - \dot{y}\ddot{x}|}{(\dot{x}^2 + \dot{y}^2)^{3/2}} = \frac{\left|(3t^2)(2) - (1+2t)(6t)\right|}{[(3t^2)^2 + (1+2t)^2]^{3/2}} = \frac{\left|-6t^2 - 6t\right|}{(9t^4 + 4t^2 + 4t + 1)^{3/2}}$$

$$= \frac{6\left|t^2 + t\right|}{(9t^4 + 4t^2 + 4t + 1)^{3/2}}$$

**39.** $\left(1, \frac{2}{3}, 1\right)$ corresponds to $t = 1$. $\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \dfrac{\langle 2t, 2t^2, 1\rangle}{\sqrt{4t^2 + 4t^4 + 1}} = \dfrac{\langle 2t, 2t^2, 1\rangle}{2t^2 + 1}$, so $\mathbf{T}(1) = \left\langle \frac{2}{3}, \frac{2}{3}, \frac{1}{3}\right\rangle$.

$\mathbf{T}'(t) = -4t(2t^2+1)^{-2}\langle 2t, 2t^2, 1\rangle + (2t^2+1)^{-1}\langle 2, 4t, 0\rangle$ [by Theorem 14.2.3 [ET 13.2.3] #3]

$= (2t^2+1)^{-2}\langle -8t^2 + 4t^2 + 2, -8t^3 + 8t^3 + 4t, -4t\rangle = 2(2t^2+1)^{-2}\langle 1 - 2t^2, 2t, -2t\rangle$

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = \frac{2(2t^2+1)^{-2}\langle 1 - 2t^2, 2t, -2t\rangle}{2(2t^2+1)^{-2}\sqrt{(1-2t^2)^2 + (2t)^2 + (-2t)^2}} = \frac{\langle 1 - 2t^2, 2t, -2t\rangle}{\sqrt{1 - 4t^2 + 4t^4 + 8t^2}}$$

$$= \frac{\langle 1 - 2t^2, 2t, -2t\rangle}{1 + 2t^2}$$

$\mathbf{N}(1) = \left\langle -\frac{1}{3}, \frac{2}{3}, -\frac{2}{3}\right\rangle$ and $\mathbf{B}(1) = \mathbf{T}(1) \times \mathbf{N}(1) = \left\langle -\frac{4}{9} - \frac{2}{9}, -\left(-\frac{4}{9} + \frac{1}{9}\right), \frac{4}{9} + \frac{2}{9}\right\rangle = \left\langle -\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right\rangle$.

**40.** $(1,0,1)$ corresponds to $t=0$. $\mathbf{r}(t)=e^t\langle 1,\sin t,\cos t\rangle$, so

$\mathbf{r}'(t)=e^t\langle 1,\sin t,\cos t\rangle+e^t\langle 0,\cos t,-\sin t\rangle=e^t\langle 1,\sin t+\cos t,\cos t-\sin t\rangle$ and

$$\mathbf{T}(t)=\frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}=\frac{e^t\langle 1,\sin t+\cos t,\cos t-\sin t\rangle}{e^t\sqrt{1+\sin^2 t+2\sin t\cos t+\cos^2 t+\cos^2 t-2\sin t\cos t+\sin^2 t}}$$

$$=\frac{\langle 1,\sin t+\cos t,\cos t-\sin t\rangle}{\sqrt{3}},$$

$\mathbf{T}(0)=\left\langle\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right\rangle$. $\mathbf{T}'(t)=\frac{1}{\sqrt{3}}\langle 0,\cos t-\sin t,-\sin t-\cos t\rangle$, so

$$\mathbf{N}(t)=\frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}=\frac{\frac{1}{\sqrt{3}}\langle 0,\cos t-\sin t,-\sin t-\cos t\rangle}{\frac{1}{\sqrt{3}}\sqrt{0^2+\cos^2 t-2\cos t\sin t+\sin^2 t+\sin^2 t+2\sin t\cos t+\cos^2 t}}$$

$$=\frac{1}{\sqrt{2}}\langle 0,\cos t-\sin t,-\sin t-\cos t\rangle.$$

$\mathbf{N}(0)=\left\langle 0,\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}\right\rangle$ and $\mathbf{B}(0)=\mathbf{T}(0)\times\mathbf{N}(0)=\left\langle -\frac{2}{\sqrt{6}},\frac{1}{\sqrt{6}},\frac{1}{\sqrt{6}}\right\rangle$.

**41.** $(0,\pi,-2)$ corresponds to $t=\pi$. $\mathbf{r}(t)=\langle 2\sin 3t,t,2\cos 3t\rangle\ \Rightarrow$

$$\mathbf{T}(t)=\frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}=\frac{\langle 6\cos 3t,1,-6\sin 3t\rangle}{\sqrt{36\cos^2 3t+1+36\sin^2 3t}}=\frac{1}{\sqrt{37}}\langle 6\cos 3t,1,-6\sin 3t\rangle.$$

$\mathbf{T}(\pi)=\frac{1}{\sqrt{37}}\langle -6,1,0\rangle$ is a normal vector for the normal plane, and so $\langle -6,1,0\rangle$ is also normal. Thus an equation for the plane is $-6(x-0)+1(y-\pi)+0(z+2)=0$ or $y-6x=\pi$.

$$\mathbf{T}'(t)=\frac{1}{\sqrt{37}}\langle -18\sin 3t,0,-18\cos 3t\rangle\ \Rightarrow\ |\mathbf{T}'(t)|=\frac{\sqrt{18^2\sin^2 3t+18^2\cos^2 3t}}{\sqrt{37}}=\frac{18}{\sqrt{37}}\ \Rightarrow$$

$\mathbf{N}(t)=\dfrac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}=\langle -\sin 3t,0,-\cos 3t\rangle$. So $\mathbf{N}(\pi)=\langle 0,0,1\rangle$ and

$\mathbf{B}(\pi)=\frac{1}{\sqrt{37}}\langle -6,1,0\rangle\times\langle 0,0,1\rangle=\frac{1}{\sqrt{37}}\langle 1,6,0\rangle$. Since $\mathbf{B}(\pi)$ is a normal to the osculating plane, so is $\langle 1,6,0\rangle$ and an equation for the plane is $1(x-0)+6(y-\pi)+0(z+2)=0$ or $x+6y=6\pi$.

**42.** $t=1$ at $(1,1,1)$. $\mathbf{r}'(t)=\langle 1,2t,3t^2\rangle$. $\mathbf{r}'(1)=\langle 1,2,3\rangle$ is normal to the normal plane, so an equation for this plane is $1(x-1)+2(y-1)+3(z-1)=0$, or $x+2y+3z=6$.

$\mathbf{T}(t)=\dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}=\dfrac{1}{\sqrt{1+4t^2+9t^4}}\langle 1,2t,3t^2\rangle$. Using the product rule on each term of $\mathbf{T}(t)$ gives

$$\mathbf{T}'(t)=\frac{1}{(1+4t^2+9t^4)^{3/2}}\left\langle -\tfrac{1}{2}(8t+36t^3),2(1+4t^2+9t^4)-\tfrac{1}{2}(8t+36t^3)2t,\ 6t(1+4t^2+9t^4)-\tfrac{1}{2}(8t+36t^3)3t^2\right\rangle$$

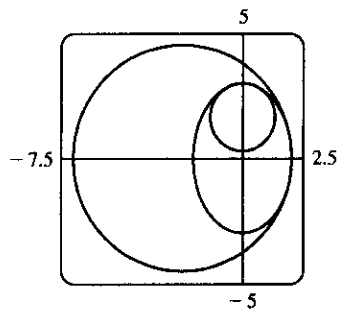
$$=\frac{1}{(1+4t^2+9t^4)^{3/2}}\left\langle -4t-18t^3,2-18t^4,6t+12t^3\right\rangle=\frac{-2}{(14)^{3/2}}\langle 11,8,-9\rangle \text{ when } t=1.$$

$\mathbf{N}(1)\parallel\mathbf{T}'(1)\parallel\langle 11,8,-9\rangle$ and $\mathbf{T}(1)\parallel\mathbf{r}'(1)=\langle 1,2,3\rangle\ \Rightarrow$ a normal vector to the osculating plane is $\langle 11,8,-9\rangle\times\langle 1,2,3\rangle=\langle 42,-42,14\rangle$ or equivalently $\langle 3,-3,1\rangle$. An equation for the plane is $3(x-1)-3(y-1)+(z-1)=0$ or $3x-3y+z=1$.

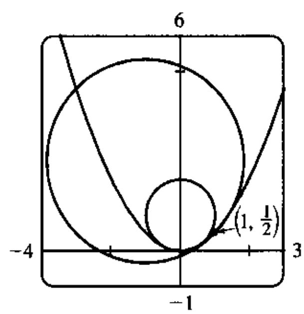
**43.** The ellipse is given by the parametric equations $x = 2\cos t$, $y = 3\sin t$, so using the result from Exercise 36,

$$\kappa(t) = \frac{|\dot{x}\ddot{y} - \ddot{x}\dot{y}|}{(\dot{x}^2 + \dot{y}^2)^{3/2}} = \frac{|(-2\sin t)(-3\sin t) - (3\cos t)(-2\cos t)|}{(4\sin^2 t + 9\cos^2 t)^{3/2}} = \frac{6}{(4\sin^2 t + 9\cos^2 t)^{3/2}}$$

At $(2, 0)$, $t = 0$. Now $\kappa(0) = \frac{6}{27} = \frac{2}{9}$, so the radius of the osculating circle is $1/\kappa(0) = \frac{9}{2}$ and its center is $\left(-\frac{5}{2}, 0\right)$. Its equation is therefore $\left(x + \frac{5}{2}\right)^2 + y^2 = \frac{81}{4}$. At $(0, 3)$, $t = \frac{\pi}{2}$, and $\kappa\left(\frac{\pi}{2}\right) = \frac{6}{8} = \frac{3}{4}$. So the radius of the osculating circle is $\frac{4}{3}$ and its center is $\left(0, \frac{5}{3}\right)$. Hence its equation is $x^2 + \left(y - \frac{5}{3}\right)^2 = \frac{16}{9}$.




**44.** $y = \frac{1}{2}x^2 \Rightarrow y' = x$ and $y'' = 1$, so Formula 11 gives $\kappa(x) = \dfrac{1}{(1 + x^2)^{3/2}}$. So the curvature at $(0, 0)$ is $\kappa(0) = 1$ and the osculating circle has radius 1 and center $(0, 1)$, and hence equation $x^2 + (y - 1)^2 = 1$. The curvature at $\left(1, \frac{1}{2}\right)$ is $\kappa(1) = \dfrac{1}{(1 + 1^2)^{3/2}} = \dfrac{1}{2\sqrt{2}}$. The tangent line to the parabola at $\left(1, \frac{1}{2}\right)$ has slope 1, so the normal line has slope $-1$. Thus the center of the osculating circle lies in the direction of the unit vector $\left\langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$. The circle has radius $2\sqrt{2}$, so its center has position vector $\left\langle 1, \frac{1}{2} \right\rangle + 2\sqrt{2}\left\langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle = \left\langle -1, \frac{5}{2} \right\rangle$. So the equation of the circle is $(x + 1)^2 + \left(y - \frac{5}{2}\right)^2 = 8$.




**45.** The tangent vector is normal to the normal plane, and the vector $\langle 6, 6, -8 \rangle$ is normal to the given plane. But $\mathbf{T}(t) \parallel \mathbf{r}'(t)$ and $\langle 6, 6, -8 \rangle \parallel \langle 3, 3, -4 \rangle$, so we need to find $t$ such that $\mathbf{r}'(t) \parallel \langle 3, 3, -4 \rangle$. $\mathbf{r}(t) = \langle t^3, 3t, t^4 \rangle \Rightarrow \mathbf{r}'(t) = \langle 3t^2, 3, 4t^3 \rangle \parallel \langle 3, 3, -4 \rangle$ when $t = -1$. So the planes are parallel at the point $\mathbf{r}(-1) = (-1, -3, 1)$.

**46.** To find the osculating plane, we first calculate the tangent and normal vectors.

In Maple, we set `x:=t^3; y:=3*t;` and `z:=t^4;` and then calculate the components of the tangent vector $\mathbf{T}(t)$ using the `diff` command. We find that $\mathbf{T}(t) = \dfrac{\langle 3t^2, 3, 4t^3 \rangle}{\sqrt{16t^6 + 9t^4 + 9}}$. Differentiating the components of $\mathbf{T}(t)$, we find that $\mathbf{N}(t) = \dfrac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = \dfrac{\langle -6t(8t^6 - 9), 3(48t^5 + 18t^3), 36t^2(t^4 + 3) \rangle}{\sqrt{144t(8t^6 - 9)^2 + 9(96t^5 + 36t^3)^2 + 5{,}184t^{12} + 31{,}104t^8 + 46{,}656t^4}}$.

In Maple, we can calculate $\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$ using the `linalg` package. First we define $\mathbf{T}$ and $\mathbf{N}$ using `T:=array([f,g,h]);` and `N:=array([F,G,H]);` where `f`, `g`, `h`, `F`, `G`, and `H` are the components of $\mathbf{T}$ and $\mathbf{N}$. Then we use the command `B:=crossprod(T,N);`. After normalization and simplification, we find that $\mathbf{B}(t) = b\langle 6t, -2t^3, -3 \rangle$, where

$$b = \frac{t\sqrt{16t^6 + 9t^4 + 9}}{\sqrt{16t^2(8t^6 - 9)^2 + (96t^5 + 36t^3)^2 + 576t^{12} + 3456t^8 + 5184t^4}}$$

In Mathematica, we use the command `Dt` to differentiate the components of $\mathbf{r}(t)$ and subsequently $\mathbf{T}(t)$, and then load the vector analysis package with the command `<<Calculus'VectorAnalysis'`. After setting `T={f,g,h}` and `N={F,G,H}`, we use `CrossProduct[T,N]` to find $\mathbf{B}$ (before normalization).

Now $\mathbf{B}(t)$ is parallel to $\langle 6t, -2t^3, -3\rangle$, so if $\mathbf{B}(t)$ is parallel to $\langle 1,1,1\rangle$ for some $t$, then $6t = 1 \Rightarrow t = \frac{1}{6}$, but $-2\left(\frac{1}{6}\right)^3 \neq 1$. So there is no such osculating plane.

**47.** $\kappa = \left|\dfrac{d\mathbf{T}}{ds}\right| = \left|\dfrac{d\mathbf{T}/dt}{ds/dt}\right| = \dfrac{|d\mathbf{T}/dt|}{ds/dt}$ and $\mathbf{N} = \dfrac{d\mathbf{T}/dt}{|d\mathbf{T}/dt|}$, so $\kappa\mathbf{N} = \dfrac{\left|\dfrac{d\mathbf{T}}{dt}\right| \dfrac{d\mathbf{T}}{dt}}{\left|\dfrac{d\mathbf{T}}{dt}\right| \dfrac{ds}{dt}} = \dfrac{d\mathbf{T}/dt}{ds/dt} = \dfrac{d\mathbf{T}}{ds}$ by the Chain Rule.

**48.** For a plane curve, $\mathbf{T} = |\mathbf{T}|\cos\phi\,\mathbf{i} + |\mathbf{T}|\sin\phi\,\mathbf{j} = \cos\phi\,\mathbf{i} + \sin\phi\,\mathbf{j}$. Then $\dfrac{d\mathbf{T}}{ds} = \left(\dfrac{d\mathbf{T}}{d\phi}\right)\left(\dfrac{d\phi}{ds}\right) = (-\sin\phi\,\mathbf{i} + \cos\phi\,\mathbf{j})\left(\dfrac{d\phi}{ds}\right)$ and $\left|\dfrac{d\mathbf{T}}{ds}\right| = |-\sin\phi\,\mathbf{i} + \cos\phi\,\mathbf{j}|\left|\dfrac{d\phi}{ds}\right| = \left|\dfrac{d\phi}{ds}\right|$. Hence for a plane curve, the curvature is $\kappa = |d\phi/ds|$.

**49.** (a) $|\mathbf{B}| = 1 \Rightarrow \mathbf{B}\cdot\mathbf{B} = 1 \Rightarrow \dfrac{d}{ds}(\mathbf{B}\cdot\mathbf{B}) = 0 \Rightarrow 2\dfrac{d\mathbf{B}}{ds}\cdot\mathbf{B} = 0 \Rightarrow \dfrac{d\mathbf{B}}{ds} \perp \mathbf{B}$

(b) $\mathbf{B} = \mathbf{T}\times\mathbf{N} \Rightarrow$

$$\frac{d\mathbf{B}}{ds} = \frac{d}{ds}(\mathbf{T}\times\mathbf{N}) = \frac{d}{dt}(\mathbf{T}\times\mathbf{N})\frac{1}{ds/dt} = \frac{d}{dt}(\mathbf{T}\times\mathbf{N})\frac{1}{|\mathbf{r}'(t)|}$$

$$= [(\mathbf{T}'\times\mathbf{N}) + (\mathbf{T}\times\mathbf{N}')]\frac{1}{|\mathbf{r}'(t)|} = \left[\left(\mathbf{T}'\times\frac{\mathbf{T}'}{|\mathbf{T}'|}\right) + (\mathbf{T}\times\mathbf{N}')\right]\frac{1}{|\mathbf{r}'(t)|} = \frac{\mathbf{T}\times\mathbf{N}'}{|\mathbf{r}'(t)|}$$

$$\Rightarrow \frac{d\mathbf{B}}{ds} \perp \mathbf{T}$$

(c) $\mathbf{B} = \mathbf{T}\times\mathbf{N} \Rightarrow \mathbf{T}\perp\mathbf{N}, \mathbf{B}\perp\mathbf{T}$ and $\mathbf{B}\perp\mathbf{N}$. So $\mathbf{B}$, $\mathbf{T}$ and $\mathbf{N}$ form an orthogonal set of vectors in the three-dimensional space $\mathbb{R}^3$. From parts (a) and (b), $d\mathbf{B}/ds$ is perpendicular to both $\mathbf{B}$ and $\mathbf{T}$, so $d\mathbf{B}/ds$ is parallel to $\mathbf{N}$. Therefore, $d\mathbf{B}/ds = -\tau(s)\mathbf{N}$, where $\tau(s)$ is a scalar.

(d) Since $\mathbf{B} = \mathbf{T}\times\mathbf{N}$, $\mathbf{T}\perp\mathbf{N}$ and both $\mathbf{T}$ and $\mathbf{N}$ are unit vectors, $\mathbf{B}$ is a unit vector mutually perpendicular to both $\mathbf{T}$ and $\mathbf{N}$. For a plane curve, $\mathbf{T}$ and $\mathbf{N}$ always lie in the plane of the curve, so that $\mathbf{B}$ is a constant unit vector always perpendicular to the plane. Thus $d\mathbf{B}/ds = \mathbf{0}$, but $d\mathbf{B}/ds = -\tau(s)\mathbf{N}$ and $\mathbf{N}\neq\mathbf{0}$, so $\tau(s) = 0$.

**50.** $\mathbf{N} = \mathbf{B}\times\mathbf{T} \Rightarrow$

$$\frac{d\mathbf{N}}{ds} = \frac{d}{ds}(\mathbf{B}\times\mathbf{T}) = \frac{d\mathbf{B}}{ds}\times\mathbf{T} + \mathbf{B}\times\frac{d\mathbf{T}}{ds} \qquad \text{[by Theorem 14.2.3 [ET 13.2.3] \#5]}$$

$$= -\tau\mathbf{N}\times\mathbf{T} + \mathbf{B}\times\kappa\mathbf{N} \qquad \text{[by Formulas 3 and 1]}$$

$$= -\tau(\mathbf{N}\times\mathbf{T}) + \kappa(\mathbf{B}\times\mathbf{N}) \qquad \text{[by Theorem 13.4.8 [ET 12.4.8] \#2]}$$

But $\mathbf{B}\times\mathbf{N} = \mathbf{B}\times(\mathbf{B}\times\mathbf{T}) = (\mathbf{B}\cdot\mathbf{T})\mathbf{B} - (\mathbf{B}\cdot\mathbf{B})\mathbf{T}$ [by Theorem 13.4.8 [ET 12.4.8] #6] $= -\mathbf{T} \Rightarrow$ $d\mathbf{N}/ds = \tau(\mathbf{T}\times\mathbf{N}) - \kappa\mathbf{T} = -\kappa\mathbf{T} + \tau\mathbf{B}$.

**51.** (a) $\mathbf{r}' = s'\,\mathbf{T} \Rightarrow \mathbf{r}'' = s''\,\mathbf{T} + s'\,\mathbf{T}' = s''\,\mathbf{T} + s'\,\dfrac{d\mathbf{T}}{ds}s' = s''\,\mathbf{T} + \kappa(s')^2\,\mathbf{N}$ by the first Serret-Frenet formula.

(b) Using part (a), we have

$$\begin{aligned}\mathbf{r}' \times \mathbf{r}'' &= (s'\,\mathbf{T}) \times [s''\,\mathbf{T} + \kappa(s')^2\,\mathbf{N}] \\ &= [(s'\,\mathbf{T}) \times (s''\,\mathbf{T})] + [(s'\mathbf{T}) \times (\kappa(s')^2\,\mathbf{N})] \qquad \text{[by Theorem 13.4.8 [ET 12.4.8] \#3]} \\ &= (s's'')(\mathbf{T} \times \mathbf{T}) + \kappa(s')^3(\mathbf{T} \times \mathbf{N}) = \mathbf{0} + \kappa(s')^3\,\mathbf{B} = \kappa(s')^3\,\mathbf{B}\end{aligned}$$

(c) Using part (a), we have

$$\begin{aligned}\mathbf{r}''' &= [s''\,\mathbf{T} + \kappa(s')^2\,\mathbf{N}]' = s'''\,\mathbf{T} + s''\,\mathbf{T}' + \kappa'(s')^2\,\mathbf{N} + 2\kappa s's''\,\mathbf{N} + \kappa(s')^2\,\mathbf{N}' \\ &= s'''\,\mathbf{T} + s''\frac{d\mathbf{T}}{ds}\,s' + \kappa'(s')^2\,\mathbf{N} + 2\kappa s's''\,\mathbf{N} + \kappa(s')^2\,\frac{d\mathbf{N}}{ds}\,s' \\ &= s'''\,\mathbf{T} + s''s'\kappa\,\mathbf{N} + \kappa'(s')^2\,\mathbf{N} + 2\kappa s's''\,\mathbf{N} + \kappa(s')^3(-\kappa\,\mathbf{T} + \tau\,\mathbf{B}) \qquad \text{[by the second formula]} \\ &= [s''' - \kappa^2(s')^3]\,\mathbf{T} + [3\kappa s's'' + \kappa'(s')^2]\,\mathbf{N} + \kappa\tau(s')^3\,\mathbf{B}\end{aligned}$$

(d) Using parts (b) and (c) and the facts that $\mathbf{B}\cdot\mathbf{T} = 0$, $\mathbf{B}\cdot\mathbf{N} = 0$, and $\mathbf{B}\cdot\mathbf{B} = 1$, we get

$$\begin{aligned}\frac{(\mathbf{r}' \times \mathbf{r}'')\cdot\mathbf{r}'''}{|\mathbf{r}' \times \mathbf{r}''|^2} &= \frac{\kappa(s')^3\,\mathbf{B}\cdot\{[s''' - \kappa^2(s')^3]\,\mathbf{T} + [3\kappa s's'' + \kappa'(s')^2]\,\mathbf{N} + \kappa\tau(s')^3\,\mathbf{B}\}}{|\kappa(s')^3\,\mathbf{B}|^2} \\ &= \frac{\kappa(s')^3\kappa\tau(s')^3}{[\kappa(s')^3]^2} = \tau\end{aligned}$$

**52.** First we find the quantities required to compute $\kappa$:

$$\mathbf{r}'(t) = \langle -a\sin t, a\cos t, b\rangle \Rightarrow \mathbf{r}''(t) = \langle -a\cos t, -a\sin t, 0\rangle \Rightarrow \mathbf{r}'''(t) = \langle a\sin t, -a\cos t, 0\rangle$$

$$|\mathbf{r}'(t)| = \sqrt{(-a\sin t)^2 + (a\cos t)^2 + b^2} = \sqrt{a^2 + b^2}$$

$$\mathbf{r}'(t) \times \mathbf{r}''(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -a\sin t & a\cos t & b \\ -a\cos t & -a\sin t & 0 \end{vmatrix} = ab\sin t\,\mathbf{i} - ab\cos t\,\mathbf{j} + a^2\,\mathbf{k}$$

$$|\mathbf{r}'(t) \times \mathbf{r}''(t)| = \sqrt{(ab\sin t)^2 + (-ab\cos t)^2 + (a^2)^2} = \sqrt{a^2b^2 + a^4}$$

$$(\mathbf{r}'(t) \times \mathbf{r}''(t))\cdot\mathbf{r}'''(t) = (ab\sin t)(a\sin t) + (-ab\cos t)(-a\cos t) + (a^2)(0) = a^2b$$

Then by Theorem 10,

$$\kappa(t) = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} = \frac{\sqrt{a^2b^2 + a^4}}{\left(\sqrt{a^2 + b^2}\right)^3} = \frac{a\sqrt{a^2 + b^2}}{\left(\sqrt{a^2 + b^2}\right)^3} = \frac{a}{a^2 + b^2}$$

which is a constant.

From Exercise 51(d), the torsion $\tau$ is given by

$$\tau = \frac{(\mathbf{r}' \times \mathbf{r}'')\cdot\mathbf{r}'''}{|\mathbf{r}' \times \mathbf{r}''|^2} = \frac{a^2b}{\left(\sqrt{a^2b^2 + a^4}\right)^2} = \frac{b}{a^2 + b^2}$$

which is also a constant.

**53.** $\mathbf{r} = \left\langle t, \frac{1}{2}t^2, \frac{1}{3}t^3 \right\rangle \Rightarrow \mathbf{r}' = \left\langle 1, t, t^2 \right\rangle, \mathbf{r}'' = \langle 0, 1, 2t \rangle, \mathbf{r}''' = \langle 0, 0, 2 \rangle \Rightarrow \mathbf{r}' \times \mathbf{r}'' = \left\langle t^2, -2t, 1 \right\rangle \Rightarrow$

$$\tau = \frac{(\mathbf{r}' \times \mathbf{r}'') \cdot \mathbf{r}'''}{|\mathbf{r}' \times \mathbf{r}''|^2} = \frac{\left\langle t^2, -2t, 1 \right\rangle \cdot \langle 0, 0, 2 \rangle}{t^4 + 4t^2 + 1} = \frac{2}{t^4 + 4t^2 + 1}$$

**54.** $\mathbf{r} = \langle \sinh t, \cosh t, t \rangle \Rightarrow \mathbf{r}' = \langle \cosh t, \sinh t, 1 \rangle, \mathbf{r}'' = \langle \sinh t, \cosh t, 0 \rangle, \mathbf{r}''' = \langle \cosh t, \sinh t, 0 \rangle \Rightarrow$

$\mathbf{r}' \times \mathbf{r}'' = \left\langle -\cosh t, \sinh t, \cosh^2 t - \sinh^2 t \right\rangle = \langle -\cosh t, \sinh t, 1 \rangle \Rightarrow$

$$\kappa = \frac{|\mathbf{r}' \times \mathbf{r}''|}{|\mathbf{r}'|^3} = \frac{|\langle -\cosh t, \sinh t, 1 \rangle|}{|\langle \cosh t, \sinh t, 1 \rangle|^3} = \frac{\sqrt{\cosh^2 t + \sinh^2 t + 1}}{\left(\cosh^2 t + \sinh^2 t + 1\right)^{3/2}} = \frac{1}{\cosh^2 t + \sinh^2 t + 1} = \frac{1}{2\cosh^2 t},$$

$$\tau = \frac{(\mathbf{r}' \times \mathbf{r}'') \cdot \mathbf{r}'''}{|\mathbf{r}' \times \mathbf{r}''|^2} = \frac{\langle -\cosh t, \sinh t, 1 \rangle \cdot \langle \cosh t, \sinh t, 0 \rangle}{\cosh^2 t + \sinh^2 t + 1} = \frac{-\cosh^2 t + \sinh^2 t}{2\cosh^2 t} = \frac{-1}{2\cosh^2 t}$$

So at the point $(0, 1, 0)$, $t = 0$, and $\kappa = \frac{1}{2}$ and $\tau = -\frac{1}{2}$.

**55.** For one helix, the vector equation is $\mathbf{r}(t) = \langle 10\cos t, 10\sin t, 34t/(2\pi) \rangle$ (measuring in angstroms), because the radius of each helix is 10 angstroms, and $z$ increases by 34 angstroms for each increase of $2\pi$ in $t$. Using the arc length formula, letting $t$ go from 0 to $2.9 \times 10^8 \times 2\pi$, we find the approximate length of each helix to be

$$L = \int_0^{2.9\times 10^8 \times 2\pi} |\mathbf{r}'(t)|\, dt = \int_0^{2.9\times 10^8 \times 2\pi} \sqrt{(-10\sin t)^2 + (10\cos t)^2 + \left(\tfrac{34}{2\pi}\right)^2}\, dt$$

$$= \sqrt{100 + \left(\tfrac{34}{2\pi}\right)^2}\, t \Big]_0^{2.9\times 10^8 \times 2\pi} = 2.9 \times 10^8 \times 2\pi \sqrt{100 + \left(\tfrac{34}{2\pi}\right)^2}$$

$\approx 2.07 \times 10^{10}$ Å — more than two meters!

**56.** (a) For the function $F(x) = \begin{cases} 0 & \text{if } x < 0 \\ P(x) & \text{if } 0 < x < 1 \\ 1 & \text{if } x \geq 1 \end{cases}$ to be continuous, we must have $P(0) = 0$ and $P(1) = 1$.

For $F'$ to be continuous, we must have $P'(0) = P'(1) = 0$. The curvature of the curve $y = F(x)$ at the point $(x, F(x))$ is $\kappa(x) = \dfrac{|F''(x)|}{\left(1 + [F'(x)]^2\right)^{3/2}}$. For $\kappa(x)$ to be continuous, we must have $P''(0) = P''(1) = 0$.

Write $P(x) = ax^5 + bx^4 + cx^3 + dx^2 + ex + f$. Then $P'(x) = 5ax^4 + 4bx^3 + 3cx^2 + 2dx + e$ and $P''(x) = 20ax^3 + 12bx^2 + 6cx + 2d$. Our six conditions are:

$$P(0) = 0 \quad\Rightarrow\quad f = 0 \qquad (1)$$

$$P(1) = 1 \quad\Rightarrow\quad a + b + c + d + e + f = 1 \qquad (2)$$

$$P'(0) = 0 \quad\Rightarrow\quad e = 0 \qquad (3)$$

$$P'(1) = 0 \quad\Rightarrow\quad 5a + 4b + 3c + 2d + e = 0 \qquad (4)$$

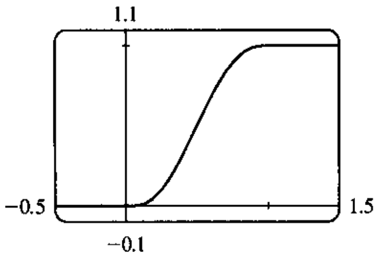
$$P''(0) = 0 \quad\Rightarrow\quad d = 0 \qquad (5)$$

$$P''(1) = 0 \quad\Rightarrow\quad 20a + 12b + 6c + 2d = 0 \qquad (6)$$

From (1), (3), and (5), we have $d = e = f = 0$. Thus (2), (4) and (6) become (7) $a + b + c = 1$,

(8) $5a + 4b + 3c = 0$, and (9) $10a + 6b + 3c = 0$. Subtracting (8) from (9) gives (10) $5a + 2b = 0$. Multiplying (7) by 3 and subtracting from (8) gives (11) $2a + b = -3$. Multiplying (11) by 2 and subtracting from (10) gives $a = 6$. By (10), $b = -15$. By (7), $c = 10$. Thus, $P(x) = 6x^5 - 15x^4 + 10x^3$.

(b)




## 14.4 Motion in Space: Velocity and Acceleration — ET 13.4

**1.** (a) If $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + z(t)\,\mathbf{k}$ is the position vector of the particle at time $t$, then the average velocity over the time interval [0, 1] is

$\mathbf{v}_{\text{ave}} = \dfrac{\mathbf{r}(1) - \mathbf{r}(0)}{1 - 0} = \dfrac{(4.5\,\mathbf{i} + 6.0\,\mathbf{j} + 3.0\,\mathbf{k}) - (2.7\,\mathbf{i} + 9.8\,\mathbf{j} + 3.7\,\mathbf{k})}{1} = 1.8\,\mathbf{i} - 3.8\,\mathbf{j} - 0.7\,\mathbf{k}$. Similarly, over the other intervals we have

$$[0.5, 1]: \quad \mathbf{v}_{\text{ave}} = \frac{\mathbf{r}(1) - \mathbf{r}(0.5)}{1 - 0.5} = \frac{(4.5\,\mathbf{i} + 6.0\,\mathbf{j} + 3.0\,\mathbf{k}) - (3.5\,\mathbf{i} + 7.2\,\mathbf{j} + 3.3\,\mathbf{k})}{0.5} = 2.0\,\mathbf{i} - 2.4\,\mathbf{j} - 0.6\,\mathbf{k}$$

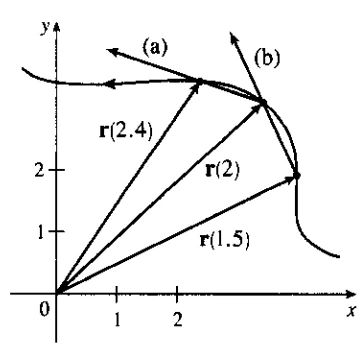
$$[1, 2]: \quad \mathbf{v}_{\text{ave}} = \frac{\mathbf{r}(2) - \mathbf{r}(1)}{2 - 1} = \frac{(7.3\,\mathbf{i} + 7.8\,\mathbf{j} + 2.7\,\mathbf{k}) - (4.5\,\mathbf{i} + 6.0\,\mathbf{j} + 3.0\,\mathbf{k})}{1} = 2.8\,\mathbf{i} + 1.8\,\mathbf{j} - 0.3\,\mathbf{k}$$

$$[1, 1.5]: \quad \mathbf{v}_{\text{ave}} = \frac{\mathbf{r}(1.5) - \mathbf{r}(1)}{1.5 - 1} = \frac{(5.9\,\mathbf{i} + 6.4\,\mathbf{j} + 2.8\,\mathbf{k}) - (4.5\,\mathbf{i} + 6.0\,\mathbf{j} + 3.0\,\mathbf{k})}{0.5} = 2.8\,\mathbf{i} + 0.8\,\mathbf{j} - 0.4\,\mathbf{k}$$

(b) We can estimate the velocity at $t = 1$ by averaging the average velocities over the time intervals [0.5, 1] and [1, 1.5]: $\mathbf{v}(1) \approx \frac{1}{2}[(2\,\mathbf{i} - 2.4\,\mathbf{j} - 0.6\,\mathbf{k}) + (2.8\,\mathbf{i} + 0.8\,\mathbf{j} - 0.4\,\mathbf{k})] = 2.4\,\mathbf{i} - 0.8\,\mathbf{j} - 0.5\,\mathbf{k}$. Then the speed is $|\mathbf{v}(1)| \approx \sqrt{(2.4)^2 + (-0.8)^2 + (-0.5)^2} \approx 2.58$.

**2.** (a) The average velocity over $2 \le t \le 2.4$ is

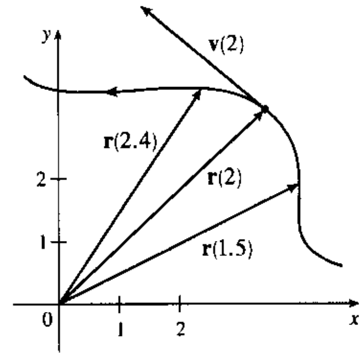
$\dfrac{\mathbf{r}(2.4) - \mathbf{r}(2)}{2.4 - 2} = 2.5\,[\mathbf{r}(2.4) - \mathbf{r}(2)]$, so we sketch a vector in the same direction but 2.5 times the length of $[\mathbf{r}(2.4) - \mathbf{r}(2)]$.



(b) The average velocity over $1.5 \le t \le 2$ is $\dfrac{\mathbf{r}(2) - \mathbf{r}(1.5)}{2 - 1.5} = 2[\mathbf{r}(2) - \mathbf{r}(1.5)]$, so we sketch a vector in the same direction but twice the length of $[\mathbf{r}(2) - \mathbf{r}(1.5)]$.

(c) Using Equation 2 we have $\mathbf{v}(2) = \lim\limits_{h \to 0} \dfrac{\mathbf{r}(2+h) - \mathbf{r}(2)}{h}$.

(d) $\mathbf{v}(2)$ is tangent to the curve at $\mathbf{r}(2)$ and points in the direction of increasing $t$. Its length is the speed of the particle at $t = 2$. We can estimate the speed by averaging the lengths of the vectors found in parts (a) and (b) which represent the average speed over $2 \le t \le 2.4$ and $1.5 \le t \le 2$ respectively. Using the axes scale as a guide, we estimate the vectors to have lengths 2.8 and 2.7. Thus, we estimate the speed at $t = 2$ to be $|\mathbf{v}(2)| \approx \frac{1}{2}(2.8 + 2.7) = 2.75$ and we draw the velocity vector $\mathbf{v}(2)$ with this length.

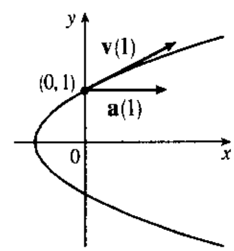



3. $\mathbf{r}(t) = \langle t^2 - 1, t \rangle \Rightarrow$ At $t = 1$:

$\mathbf{v}(t) = \mathbf{r}'(t) = \langle 2t, 1 \rangle$, $\quad \mathbf{v}(1) = \langle 2, 1 \rangle$

$\mathbf{a}(t) = \mathbf{r}''(t) = \langle 2, 0 \rangle$, $\quad \mathbf{a}(1) = \langle 2, 0 \rangle$

$|\mathbf{v}(t)| = \sqrt{4t^2 + 1}$

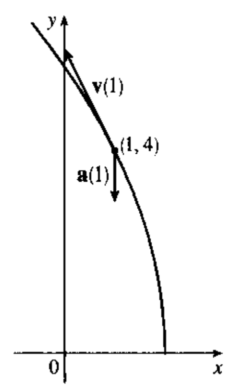



4. $\mathbf{r}(t) = \langle 2 - t, 4\sqrt{t} \rangle \Rightarrow$ At $t = 1$:

$\mathbf{v}(t) = \mathbf{r}'(t) = \langle -1, 2/\sqrt{t} \rangle$, $\quad \mathbf{v}(1) = \langle -1, 2 \rangle$

$\mathbf{a}(t) = \mathbf{r}''(t) = \left\langle 0, -1/t^{3/2} \right\rangle$, $\quad \mathbf{a}(1) = \langle 0, -1 \rangle$

$|\mathbf{v}(t)| = \sqrt{1 + 4/t}$



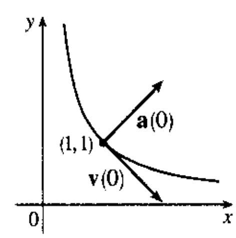

5. $\mathbf{r}(t) = e^t\,\mathbf{i} + e^{-t}\,\mathbf{j} \Rightarrow$ At $t = 0$:

$\mathbf{v}(t) = e^t\,\mathbf{i} - e^{-t}\,\mathbf{j}$, $\quad \mathbf{v}(0) = \mathbf{i} - \mathbf{j}$,

$\mathbf{a}(t) = e^t\,\mathbf{i} + e^{-t}\,\mathbf{j}$ $\quad \mathbf{a}(0) = \mathbf{i} + \mathbf{j}$

$|\mathbf{v}(t)| = \sqrt{e^{2t} + e^{-2t}} = e^{-t}\sqrt{e^{4t} + 1}$

Since $x = e^t$, $t = \ln x$ and $y = e^{-t} = e^{-\ln x} = 1/x$, and $x > 0$, $y > 0$.



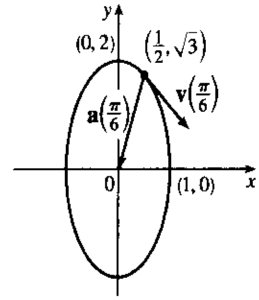
**6.** $\mathbf{r}(t) = \sin t\,\mathbf{i} + 2\cos t\,\mathbf{j} \quad\Rightarrow$

$\mathbf{v}(t) = \cos t\,\mathbf{i} - 2\sin t\,\mathbf{j},\ \mathbf{v}\left(\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2}\,\mathbf{i} - \mathbf{j}$

$\mathbf{a}(t) = -\sin t\,\mathbf{i} - 2\cos t\,\mathbf{j},\ \mathbf{a}\left(\frac{\pi}{6}\right) = -\frac{1}{2}\,\mathbf{i} - \sqrt{3}\,\mathbf{j}$

$|\mathbf{v}(t)| = \sqrt{\cos^2 t + 4\sin^2 t} = \sqrt{1 + 3\sin^2 t}$

And $x^2 + y^2/4 = \sin^2 t + \cos^2 t = 1$, an ellipse.



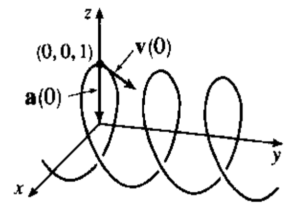

**7.** $\mathbf{r}(t) = \sin t\,\mathbf{i} + t\,\mathbf{j} + \cos t\,\mathbf{k} \quad\Rightarrow$

$\mathbf{v}(t) = \cos t\,\mathbf{i} + \mathbf{j} - \sin t\,\mathbf{k},\ \mathbf{v}(0) = \mathbf{i} + \mathbf{j}$

$\mathbf{a}(t) = -\sin t\,\mathbf{i} - \cos t\,\mathbf{k},\ \mathbf{a}(0) = -\mathbf{k}$

$|\mathbf{v}(t)| = \sqrt{\cos^2 t + 1 + \sin^2 t} = \sqrt{2}$

Since $x^2 + z^2 = 1$, $y = t$, the path of the particle is a helix about the $y$-axis.




**8.** $\mathbf{r}(t) = t\,\mathbf{i} + t^2\,\mathbf{j} + t^3\,\mathbf{k} \quad\Rightarrow$

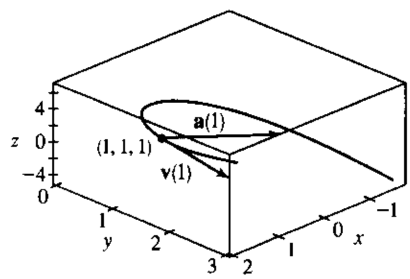
$\mathbf{v}(t) = \mathbf{i} + 2t\,\mathbf{j} + 3t^2\,\mathbf{k},\ \mathbf{v}(1) = \mathbf{i} + 2\,\mathbf{j} + 3\,\mathbf{k}$

$\mathbf{a}(t) = 2\,\mathbf{j} + 6t\,\mathbf{k},\ \mathbf{a}(1) = 2\,\mathbf{j} + 6\,\mathbf{k}$

$|\mathbf{v}(t)| = \sqrt{1 + 4t^2 + 9t^4}$

The path is a "twisted cubic"

(see Example 14.1.7 [ET 13.1.7]).




**9.** $\mathbf{r}(t) = \langle t^2 + 1, t^3, t^2 - 1\rangle \quad\Rightarrow\quad \mathbf{v}(t) = \mathbf{r}'(t) = \langle 2t, 3t^2, 2t\rangle,\ \mathbf{a}(t) = \mathbf{v}'(t) = \langle 2, 6t, 2\rangle,$
$|\mathbf{v}(t)| = \sqrt{(2t)^2 + (3t^2)^2 + (2t)^2} = \sqrt{9t^4 + 8t^2} = |t|\,\sqrt{9t^2 + 8}.$

**10.** $\mathbf{r}(t) = \langle 2\cos t, 3t, 2\sin t\rangle \quad\Rightarrow\quad \mathbf{v}(t) = \mathbf{r}'(t) = \langle -2\sin t, 3, 2\cos t\rangle,\ \mathbf{a}(t) = \mathbf{v}'(t) = \langle -2\cos t, 0, -2\sin t\rangle,$
$|\mathbf{v}(t)| = \sqrt{4\sin^2 t + 9 + 4\cos^2 t} = \sqrt{13}.$

**11.** $\mathbf{r}(t) = \sqrt{2}\,t\,\mathbf{i} + e^t\,\mathbf{j} + e^{-t}\,\mathbf{k} \quad\Rightarrow\quad \mathbf{v}(t) = \mathbf{r}'(t) = \sqrt{2}\,\mathbf{i} + e^t\,\mathbf{j} - e^{-t}\,\mathbf{k},\ \mathbf{a}(t) = \mathbf{v}'(t) = e^t\,\mathbf{j} + e^{-t}\,\mathbf{k},$
$|\mathbf{v}(t)| = \sqrt{2 + e^{2t} + e^{-2t}} = \sqrt{(e^t + e^{-t})^2} = e^t + e^{-t}.$

**12.** $\mathbf{r}(t) = t^2\,\mathbf{i} + \ln t\,\mathbf{j} + t\,\mathbf{k} \quad\Rightarrow\quad \mathbf{v}(t) = \mathbf{r}'(t) = 2t\,\mathbf{i} + t^{-1}\,\mathbf{j} + \mathbf{k},\ \mathbf{a}(t) = \mathbf{v}'(t) = 2\,\mathbf{i} - t^{-2}\,\mathbf{j},$
$|\mathbf{v}(t)| = \sqrt{4t^2 + t^{-2} + 1}.$

**13.** $\mathbf{r}(t) = e^t\langle \cos t, \sin t, t\rangle \quad\Rightarrow$

$\mathbf{v}(t) = \mathbf{r}'(t) = e^t\langle \cos t, \sin t, t\rangle + e^t\,\langle -\sin t, \cos t, 1\rangle = e^t\langle \cos t - \sin t, \sin t + \cos t, t + 1\rangle$

$$\mathbf{a}(t) = \mathbf{v}'(t) = e^t\langle \cos t - \sin t - \sin t - \cos t, \sin t + \cos t + \cos t - \sin t, t + 1 + 1\rangle$$
$$= e^t\langle -2\sin t, 2\cos t, t + 2\rangle$$

$$|\mathbf{v}(t)| = e^t\sqrt{\cos^2 t + \sin^2 t - 2\cos t\sin t + \sin^2 t + \cos^2 t + 2\sin t\cos t + t^2 + 2t + 1}$$
$$= e^t\sqrt{t^2 + 2t + 3}$$

**14.** $\mathbf{r}(t) = t\sin t\,\mathbf{i} + t\cos t\,\mathbf{j} + t^2\,\mathbf{k} \quad\Rightarrow\quad \mathbf{v}(t) = \mathbf{r}'(t) = (\sin t + t\cos t)\,\mathbf{i} + (\cos t - t\sin t)\,\mathbf{j} + 2t\,\mathbf{k},$
$\mathbf{a}(t) = \mathbf{v}'(t) = (2\cos t - t\sin t)\,\mathbf{i} + (-2\sin t - t\cos t)\,\mathbf{j} + 2\,\mathbf{k},$
$|\mathbf{v}(t)| = \sqrt{(\sin^2 t + 2t\sin t\cos t + t^2\cos^2 t) + (\cos^2 t - 2t\sin t\cos t + t^2\sin^2 t) + 4t^2} = \sqrt{5t^2 + 1}.$

**15.** $\mathbf{a}(t) = \mathbf{k} \Rightarrow \mathbf{v}(t) = \int \mathbf{a}(t)\,dt = \int \mathbf{k}\,dt = t\,\mathbf{k} + \mathbf{c}_1$ and $\mathbf{i} - \mathbf{j} = \mathbf{v}(0) = 0\,\mathbf{k} + \mathbf{c}_1$, so $\mathbf{c}_1 = \mathbf{i} - \mathbf{j}$ and $\mathbf{v}(t) = \mathbf{i} - \mathbf{j} + t\,\mathbf{k}$. $\mathbf{r}(t) = \int \mathbf{v}(t)\,dt = \int (\mathbf{i} - \mathbf{j} + t\,\mathbf{k})\,dt = t\,\mathbf{i} - t\,\mathbf{j} + \frac{1}{2}t^2\,\mathbf{k} + \mathbf{c}_2$. But $\mathbf{0} = \mathbf{r}(0) = \mathbf{0} + \mathbf{c}_2$, so $\mathbf{c}_2 = \mathbf{0}$ and $\mathbf{r}(t) = t\,\mathbf{i} - t\,\mathbf{j} + \frac{1}{2}t^2\,\mathbf{k}$.

**16.** $\mathbf{a}(t) = -10\,\mathbf{k} \Rightarrow \mathbf{v}(t) = \int (-10\mathbf{k})\,dt = -10t\,\mathbf{k} + \mathbf{c}_1$, and $\mathbf{i} + \mathbf{j} - \mathbf{k} = \mathbf{v}(0) = \mathbf{0} + \mathbf{c}_1$, so $\mathbf{c}_1 = \mathbf{i} + \mathbf{j} - \mathbf{k}$ and $\mathbf{v}(t) = \mathbf{i} + \mathbf{j} - (10t + 1)\,\mathbf{k}$.
$\mathbf{r}(t) = \int [\mathbf{i} + \mathbf{j} - (10t + 1)\,\mathbf{k}]\,dt = t\,\mathbf{i} + t\,\mathbf{j} - (5t^2 + t)\,\mathbf{k} + \mathbf{c}_2$. But $2\,\mathbf{i} + 3\,\mathbf{j} = \mathbf{r}(0) = \mathbf{0} + \mathbf{c}_2$, so $\mathbf{c}_2 = 2\,\mathbf{i} + 3\,\mathbf{j}$ and $\mathbf{r}(t) = (t + 2)\,\mathbf{i} + (t + 3)\,\mathbf{j} - (5t^2 + t)\,\mathbf{k}$.

**17.** (a) $\mathbf{a}(t) = \mathbf{i} + 2\,\mathbf{j} + 2t\,\mathbf{k} \Rightarrow$

$\mathbf{v}(t) = \int (\mathbf{i} + 2\,\mathbf{j} + 2t\,\mathbf{k})\,dt = t\,\mathbf{i} + 2t\,\mathbf{j} + t^2\,\mathbf{k} + \mathbf{c}_1$, and

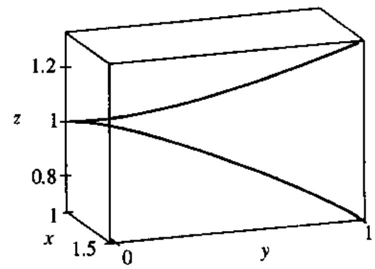
$\mathbf{0} = \mathbf{v}(0) = \mathbf{0} + \mathbf{c}_1$, so $\mathbf{c}_1 = \mathbf{0}$ and $\mathbf{v}(t) = \mathbf{i} + 2t\,\mathbf{j} + t^2\,\mathbf{k}$.

$\mathbf{r}(t) = \int (t\,\mathbf{i} + 2t\,\mathbf{j} + t^2\,\mathbf{k})\,dt = \frac{1}{2}t^2\,\mathbf{i} + t^2\,\mathbf{j} + \frac{1}{3}t^3\,\mathbf{k} + \mathbf{c}_2$.

But $\mathbf{i} + \mathbf{k} = \mathbf{r}(0) = \mathbf{0} + \mathbf{c}_2$, so $\mathbf{c}_2 = \mathbf{i} + \mathbf{k}$ and

$\mathbf{r}(t) = \left(1 + \frac{1}{2}t^2\right)\mathbf{i} + t^2\,\mathbf{j} + \left(1 + \frac{1}{3}t^3\right)\mathbf{k}$.

(b)




**18.** (a) $\mathbf{a}(t) = t\,\mathbf{i} + t^2\,\mathbf{j} + \cos 2t\,\mathbf{k} \Rightarrow$

$$\mathbf{v}(t) = \int (t\,\mathbf{i} + t^2\,\mathbf{j} + \cos 2t\,\mathbf{k})\,dt = \frac{t^2}{2}\,\mathbf{i} + \frac{t^3}{3}\,\mathbf{j} + \frac{\sin 2t}{2}\,\mathbf{k} + \mathbf{c}_1$$

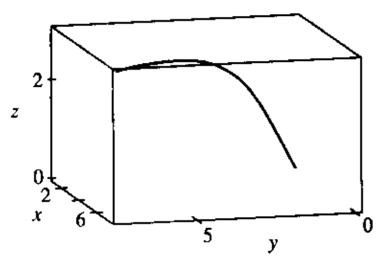
and $\mathbf{i} + \mathbf{k} = \mathbf{v}(0) = \mathbf{0} + \mathbf{c}_1$, so $\mathbf{c}_1 = \mathbf{i} + \mathbf{k}$ and

$\mathbf{v}(t) = \left(\frac{1}{2}t^2 + 1\right)\mathbf{i} + \frac{1}{3}t^3\,\mathbf{j} + \left(1 + \frac{1}{2}\sin 2t\right)\mathbf{k}$.

$\mathbf{r}(t) = \int \left[\left(\frac{1}{2}t^2 + 1\right)\mathbf{i} + \frac{1}{3}t^3\,\mathbf{j} + \left(1 + \frac{1}{2}\sin 2t\right)\mathbf{k}\right] dt = \left(\frac{1}{6}t^3 + t\right)\mathbf{i} + \frac{1}{12}t^4\,\mathbf{j} + \left(t - \frac{1}{4}\cos 2t\right)\mathbf{k} + \mathbf{c}_2$

But $\mathbf{j} = \mathbf{r}(0) = -\frac{1}{4}\,\mathbf{k} + \mathbf{c}_2$, so $\mathbf{c}_2 = \mathbf{j} + \frac{1}{4}\,\mathbf{k}$ and $\mathbf{r}(t) = \left(\frac{1}{6}t^3 + t\right)\mathbf{i} + \left(1 + \frac{1}{12}t^4\right)\mathbf{j} + \left(\frac{1}{4} + t - \frac{1}{4}\cos 2t\right)\mathbf{k}$.

(b)




**19.** $\mathbf{r}(t) = \langle t^2, 5t, t^2 - 16t \rangle \Rightarrow \mathbf{v}(t) = \langle 2t, 5, 2t - 16 \rangle$,
$|\mathbf{v}(t)| = \sqrt{4t^2 + 25 + 4t^2 - 64t + 256} = \sqrt{8t^2 - 64t + 281}$ and
$\frac{d}{dt}\,|\mathbf{v}(t)| = \frac{1}{2}(8t^2 - 64t + 281)^{-1/2}(16t - 64)$. This is zero if and only if the numerator is zero, that is, $16t - 64 = 0$ or $t = 4$. Since $\frac{d}{dt}\,|\mathbf{v}(t)| < 0$ for $t < 4$ and $\frac{d}{dt}\,|\mathbf{v}(t)| > 0$ for $t > 4$, the minimum speed of $\sqrt{153}$ is attained at $t = 4$ units of time.

**20.** Since $\mathbf{r}(t) = t^3\,\mathbf{i} + t^2\,\mathbf{j} + t^3\,\mathbf{k}$, $\mathbf{a}(t) = \mathbf{r}''(t) = 6t\,\mathbf{i} + 2\,\mathbf{j} + 6t\,\mathbf{k}$. By Newton's Second Law, $\mathbf{F}(t) = m\,\mathbf{a}(t) = 6mt\,\mathbf{i} + 2m\,\mathbf{j} + 6mt\,\mathbf{k}$ is the required force.

**21.** $|\mathbf{F}(t)| = 20$ N in the direction of the positive $z$-axis, so $\mathbf{F}(t) = 20\,\mathbf{k}$. Also $m = 4$ kg, $\mathbf{r}(0) = \mathbf{0}$ and $\mathbf{v}(0) = \mathbf{i} - \mathbf{j}$. Since $20\mathbf{k} = \mathbf{F}(t) = 4\,\mathbf{a}(t)$, $\mathbf{a}(t) = 5\,\mathbf{k}$. Then $\mathbf{v}(t) = 5t\,\mathbf{k} + \mathbf{c}_1$ where $\mathbf{c}_1 = \mathbf{i} - \mathbf{j}$ so $\mathbf{v}(t) = \mathbf{i} - \mathbf{j} + 5t\,\mathbf{k}$ and the speed is $|\mathbf{v}(t)| = \sqrt{1 + 1 + 25t^2} = \sqrt{25t^2 + 2}$. Also $\mathbf{r}(t) = t\,\mathbf{i} - t\,\mathbf{j} + \frac{5}{2}t^2\,\mathbf{k} + \mathbf{c}_2$ and $\mathbf{0} = \mathbf{r}(0)$, so $\mathbf{c}_2 = \mathbf{0}$ and $\mathbf{r}(t) = t\,\mathbf{i} - t\,\mathbf{j} + \frac{5}{2}t^2\,\mathbf{k}$.

**22.** The argument here is the same as that in Example 14.2.5 [ET 13.2.5] with $\mathbf{r}(t)$ replaced by $\mathbf{v}(t)$ and $\mathbf{r}'(t)$ replaced by $\mathbf{a}(t)$.

**23.** $|\mathbf{v}(0)| = 500$ m/s and since the angle of elevation is 30°, the direction of the velocity is $\frac{1}{2}(\sqrt{3}\,\mathbf{i} + \mathbf{j})$. Thus $\mathbf{v}(0) = 250(\sqrt{3}\,\mathbf{i} + \mathbf{j})$ and if we set up the axes so the projectile starts at the origin, then $\mathbf{r}(0) = \mathbf{0}$. Ignoring air resistance, the only force is that due to gravity, so $\mathbf{F}(t) = -mg\,\mathbf{j}$ where $g \approx 9.8$ m/s$^2$. Thus $\mathbf{a}(t) = -g\,\mathbf{j}$ and $\mathbf{v}(t) = -gt\,\mathbf{j} + \mathbf{c}_1$. But $250(\sqrt{3}\,\mathbf{i} + \mathbf{j}) = \mathbf{v}(0) = \mathbf{c}_1$, so $\mathbf{v}(t) = 250\sqrt{3}\,\mathbf{i} + (250 - gt)\,\mathbf{j}$ and $\mathbf{r}(t) = 250\sqrt{3}\,t\,\mathbf{i} + (250t - \frac{1}{2}gt^2)\,\mathbf{j} + \mathbf{c}_2$ where $\mathbf{0} = \mathbf{r}(0) = \mathbf{c}_2$. Thus $\mathbf{r}(t) = 250\sqrt{3}\,t\,\mathbf{i} + (250t - \frac{1}{2}gt^2)\,\mathbf{j}$.

(a) Setting $250t - \frac{1}{2}gt^2 = 0$ gives $t = 0$ or $t = \frac{500}{g} \approx 51.0$ s. So the range is $250\sqrt{3} \cdot \frac{500}{g} \approx 22$ km.

(b) $0 = \dfrac{d}{dt}(250t - \frac{1}{2}gt^2) = 250 - gt$ implies that the maximum height is attained when $t = 250/g \approx 25.5$ s. Thus, the maximum height is $(250)(250/g) - g(250/g)^2\frac{1}{2} = (250)^2/(2g) \approx 3.2$ km.

(c) From part (a), impact occurs at $t = 500/g \approx 51.0$. Thus, the velocity at impact is $\mathbf{v}(500/g) = 250\sqrt{3}\,\mathbf{i} + [250 - g(500/g)]\,\mathbf{j} = 250\sqrt{3}\,\mathbf{i} - 250\,\mathbf{j}$ and the speed is $|\mathbf{v}(500/g)| = 250\sqrt{3+1} = 500$ m/s.

**24.** As in Exercise 23, $\mathbf{v}(t) = 250\sqrt{3}\,\mathbf{i} + (250 - gt)\,\mathbf{j}$ and $\mathbf{r}(t) = 250\sqrt{3}\,t\,\mathbf{i} + (250t - \frac{1}{2}gt^2)\,\mathbf{j} + \mathbf{c}_2$. But $\mathbf{r}(0) = 200\,\mathbf{j}$, so $\mathbf{c}_2 = 200\,\mathbf{j}$ and $\mathbf{r}(t) = 250\sqrt{3}\,t\,\mathbf{i} + (200 + 250t - \frac{1}{2}gt^2)\,\mathbf{j}$.

(a) $200 + 250t - \frac{1}{2}gt^2 = 0$ implies that $gt^2 - 500t - 400 = 0$ or $t = \dfrac{500 \pm \sqrt{500^2 + 1600g}}{2g}$. Taking the positive $t$-value gives $t = \dfrac{500 + \sqrt{250{,}000 + 1600g}}{2g} \approx 51.8$ s. Thus the range is $(250\sqrt{3})\,\dfrac{500 + \sqrt{250{,}000 + 1600g}}{2g} \approx 22.4$ km.

(b) $0 = \dfrac{d}{dt}(200 + 250t - \frac{1}{2}gt^2) = 250 - gt$ implies that the maximum height is attained when $t = 250/g \approx 25.5$ s and thus the maximum height is

$$\left[200 + (250)\left(\frac{250}{g}\right) - \frac{g}{2}\left(\frac{250}{g}\right)^2\right] = 200 + \frac{(250)^2}{2g} \approx 3.4 \text{ km.}$$

*Alternate solution:* Because the projectile is fired in the same direction and with the same velocity as in Exercise 23, but from a point 200 m higher, the maximum height reached is 200 m higher than that found in Exercise 23, that is, 3.2 km + 200 m = 3.4 km.

(c) From part (a), impact occurs at $t = \dfrac{500 + \sqrt{250{,}000 + 1600g}}{2g}$. Thus the velocity at impact is

$$250\sqrt{3}\,\mathbf{i} + \left[250 - g\,\frac{500 + \sqrt{250{,}000 + 1600g}}{2g}\right]\mathbf{j}, \text{ so } |\mathbf{v}| \approx \sqrt{(250)^2(3) + (250 - 51.8g)^2} \approx 504 \text{ m/s.}$$

**25.** As in Example 5, $\mathbf{r}(t) = (v_0 \cos 45°)t\,\mathbf{i} + \left[(v_0 \sin 45°)t - \frac{1}{2}gt^2\right]\mathbf{j} = \frac{1}{2}\left[v_0\sqrt{2}\,t\,\mathbf{i} + (v_0\sqrt{2}\,t - gt^2)\,\mathbf{j}\right]$. Then the ball lands at $t = \dfrac{v_0\sqrt{2}}{g}$ s. Now since it lands 90 m away, $90 = \frac{1}{2}v_0\sqrt{2}\,\dfrac{v_0\sqrt{2}}{g}$ or $v_0^2 = 90g$ and the initial velocity is $v_0 = \sqrt{90g} \approx 30$ m/s.

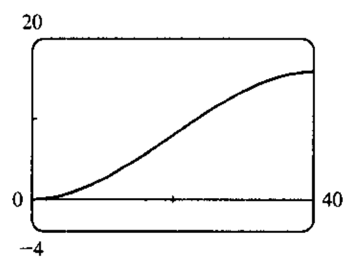
**26.** As in Example 5, $\mathbf{r}(t) = (v_0 \cos 30\,°)t\,\mathbf{i} + \left[(v_0 \sin 30\,°)t - \frac{1}{2}gt^2\right]\mathbf{j} = \frac{1}{2}\left[v_0\sqrt{3}\,t\,\mathbf{i} + (v_0\,t - gt^2)\,\mathbf{j}\right]$ and then $\mathbf{v}(t) = \mathbf{r}'(t) = \frac{1}{2}\left[v_0\sqrt{3}\,\mathbf{i} + (v_0 - 2gt)\,\mathbf{j}\right]$. The shell reaches its maximum height when the vertical component of velocity is zero, so $\frac{1}{2}(v_0 - 2gt) = 0 \;\Rightarrow\; t = \dfrac{v_0}{2g}$. The vertical height of the shell at that time is 500 m, so

$$\frac{1}{2}\left[v_0\left(\frac{v_0}{2g}\right) - g\left(\frac{v_0}{2g}\right)^2\right] = 500 \;\Rightarrow\; \frac{v_0^2}{8g} = 500 \;\Rightarrow\; v_0 = \sqrt{4000g} = \sqrt{4000(9.8)} \approx 198 \text{ m/s}.$$

**27.** Let $\alpha$ be the angle of elevation. Then $v_0 = 150$ m/s and from Example 5, the horizontal distance traveled by the projectile is $d = \dfrac{v_0^2 \sin 2\alpha}{g}$. Thus $\dfrac{150^2 \sin 2\alpha}{g} = 800 \;\Rightarrow\; \sin 2\alpha = \dfrac{800g}{150^2} \approx 0.3484 \;\Rightarrow\; 2\alpha \approx 20.4\,°$ or $180 - 20.4 = 159.6\,°$. Two angles of elevation then are $\alpha \approx 10.2\,°$ and $\alpha \approx 79.8\,°$.

**28.** Here $v_0 = 115$ ft/s, the angle of elevation is $\alpha = 50\,°$, and if we place the origin at home plate, then $\mathbf{r}(0) = 3\,\mathbf{j}$. As in Example 5, we have $\mathbf{r}\,(t) = -\frac{1}{2}gt^2\,\mathbf{j} + t\,\mathbf{v}_0 + \mathbf{D}$ where $\mathbf{D} = \mathbf{r}(0) = 3\,\mathbf{j}$ and $\mathbf{v}_0 = v_0 \cos\alpha\,\mathbf{i} + v_0 \sin\alpha\,\mathbf{j}$, so $\mathbf{r}(t) = (v_0 \cos\alpha)t\,\mathbf{i} + \left[(v_0 \sin\alpha)t - \frac{1}{2}gt^2 + 3\right]\mathbf{j}$. Thus, parametric equations for the trajectory of the ball are $x = (v_0 \cos\alpha)t$, $y = (v_0 \sin\alpha)t - \frac{1}{2}gt^2 + 3$. The ball reaches the fence when $x = 400 \;\Rightarrow\; (v_0 \cos\alpha)t = 400$ $\Rightarrow\; t = \dfrac{400}{v_0 \cos\alpha} = \dfrac{400}{115 \cos 50\,°} \approx 5.41$ s. At this time, the height of the ball is $y = (v_0 \sin\alpha)t - \frac{1}{2}gt^2 + 3 \approx (115 \sin 50\,°)(5.41) - \frac{1}{2}(32)(5.41)^2 + 3 \approx 11.2$ ft. Since the fence is 10 ft high, the ball clears the fence.

**29.** (a) After $t$ seconds, the boat will be $5t$ meters west of point $A$. The velocity of the water at that location is $\frac{3}{400}(5t)(40 - 5t)\,\mathbf{j}$. The velocity of the boat in still water is $5\,\mathbf{i}$, so the resultant velocity of the boat is $\mathbf{v}(t) = 5\,\mathbf{i} + \frac{3}{400}(5t)(40 - 5t)\,\mathbf{j} = 5\mathbf{i} + \left(\frac{3}{2}t - \frac{3}{16}t^2\right)\mathbf{j}$.




Integrating, we obtain $\mathbf{r}(t) = 5t\,\mathbf{i} + \left(\frac{3}{4}t^2 - \frac{1}{16}t^3\right)\mathbf{j} + \mathbf{C}$.
If we place the origin at $A$ (and consider $\mathbf{j}$ to coincide with the northern direction) then $\mathbf{r}(0) = \mathbf{0} \;\Rightarrow\; \mathbf{C} = \mathbf{0}$ and we have $\mathbf{r}(t) = 5t\,\mathbf{i} + \left(\frac{3}{4}t^2 - \frac{1}{16}t^3\right)\mathbf{j}$. The boat reaches the east bank after 8 s, and it is located at $\mathbf{r}(8) = 5(8)\mathbf{i} + \left(\frac{3}{4}(8)^2 - \frac{1}{16}(8)^3\right)\mathbf{j} = 40\,\mathbf{i} + 16\,\mathbf{j}$. Thus the boat is 16 m downstream.

(b) Let $\alpha$ be the angle north of east that the boat heads. Then the velocity of the boat in still water is given by $5(\cos\alpha)\,\mathbf{i} + 5(\sin\alpha)\,\mathbf{j}$. At $t$ seconds, the boat is $5(\cos\alpha)t$ meters from the west bank, at which point the velocity of the water is $\frac{3}{400}[5(\cos\alpha)t][40 - 5(\cos\alpha)t]\,\mathbf{j}$. The resultant velocity of the boat is given by

$$\begin{aligned}\mathbf{v}(t) &= 5(\cos\alpha)\,\mathbf{i} + \left[5\sin\alpha + \tfrac{3}{400}(5t\cos\alpha)(40 - 5t\cos\alpha)\right]\mathbf{j}\\ &= (5\cos\alpha)\,\mathbf{i} + \left(5\sin\alpha + \tfrac{3}{2}t\cos\alpha - \tfrac{3}{16}t^2\cos^2\alpha\right)\mathbf{j}\end{aligned}$$

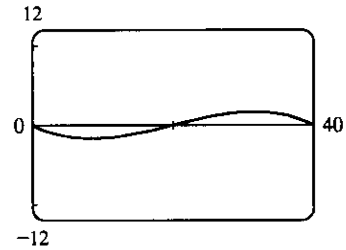
Integrating, $\mathbf{r}(t) = (5t\cos\alpha)\,\mathbf{i} + \left(5t\sin\alpha + \frac{3}{4}t^2\cos\alpha - \frac{1}{16}t^3\cos^2\alpha\right)\mathbf{j}$ (where we have again placed the origin at $A$). The boat will reach the east bank when $5t\cos\alpha = 40 \Rightarrow t = \dfrac{40}{5\cos\alpha} = \dfrac{8}{\cos\alpha}$.

In order to land at point $B\,(40,0)$ we need $5t\sin\alpha + \frac{3}{4}t^2\cos\alpha - \frac{1}{16}t^3\cos^2\alpha = 0 \Rightarrow$

$$5\left(\frac{8}{\cos\alpha}\right)\sin\alpha + \frac{3}{4}\left(\frac{8}{\cos\alpha}\right)^2\cos\alpha - \frac{1}{16}\left(\frac{8}{\cos\alpha}\right)^3\cos^2\alpha = 0 \Rightarrow \frac{1}{\cos\alpha}(40\sin\alpha + 48 - 32) = 0 \Rightarrow$$

$40\sin\alpha + 16 = 0 \Rightarrow \sin\alpha = -\frac{2}{5}$. Thus $\alpha = \sin^{-1}\left(-\frac{2}{5}\right) \approx -23.6\,^\circ$, so the boat should head $23.6\,^\circ$ south of east (upstream).

The path does seem realistic. The boat initially heads upstream to counteract the effect of the current. Near the center of the river, the current is stronger and the boat is pushed downstream. When the boat nears the eastern bank, the current is slower and the boat is able to progress upstream to arrive at point $B$.




**30.** As in Exercise 29(b), let $\alpha$ be the angle north of east that the boat heads, so the velocity of the boat in still water is given by $5(\cos\alpha)\,\mathbf{i} + 5(\sin\alpha)\,\mathbf{j}$. At $t$ seconds, the boat is $5(\cos\alpha)t$ meters from the west bank, at which point the velocity of the water is $3\sin(\pi x/40)\,\mathbf{j} = 3\sin[\pi\cdot 5(\cos\alpha)t/40]\,\mathbf{j} = 3\sin\left(\frac{\pi}{8}t\cos\alpha\right)\mathbf{j}$. The resultant velocity of the boat then is given by $\mathbf{v}(t) = 5(\cos\alpha)\,\mathbf{i} + \left[5\sin\alpha + 3\sin\left(\frac{\pi}{8}t\cos\alpha\right)\right]\mathbf{j}$. Integrating,

$$\mathbf{r}(t) = (5t\cos\alpha)\,\mathbf{i} + \left[5t\sin\alpha - \frac{24}{\pi\cos\alpha}\cos\left(\tfrac{\pi}{8}t\cos\alpha\right)\right]\mathbf{j} + \mathbf{C}.$$

If we place the origin at $A$ then $\mathbf{r}(0) = \mathbf{0} \Rightarrow -\dfrac{24}{\pi\cos\alpha}\mathbf{j} + \mathbf{C} = \mathbf{0} \Rightarrow \mathbf{C} = \dfrac{24}{\pi\cos\alpha}\mathbf{j}$ and

$$\mathbf{r}(t) = (5t\cos\alpha)\,\mathbf{i} + \left[5t\sin\alpha - \frac{24}{\pi\cos\alpha}\cos\left(\tfrac{\pi}{8}t\cos\alpha\right) + \frac{24}{\pi\cos\alpha}\right]\mathbf{j}.$$

The boat will reach the east bank when $5t\cos\alpha = 40 \Rightarrow t = \dfrac{8}{\cos\alpha}$. In order to land at point $B\,(40,0)$ we need $5t\sin\alpha - \dfrac{24}{\pi\cos\alpha}\cos\left(\frac{\pi}{8}t\cos\alpha\right) + \dfrac{24}{\pi\cos\alpha} = 0$

$$\Rightarrow 5\left(\frac{8}{\cos\alpha}\right)\sin\alpha - \frac{24}{\pi\cos\alpha}\cos\left[\frac{\pi}{8}\left(\frac{8}{\cos\alpha}\right)\cos\alpha\right] + \frac{24}{\pi\cos\alpha} = 0 \Rightarrow$$

$\dfrac{1}{\cos\alpha}\left(40\sin\alpha - \dfrac{24}{\pi}\cos\pi + \dfrac{24}{\pi}\right) = 0 \Rightarrow 40\sin\alpha + \dfrac{48}{\pi} = 0 \Rightarrow \sin\alpha = -\dfrac{6}{5\pi}$. Thus

$\alpha = \sin^{-1}\left(-\dfrac{6}{5\pi}\right) \approx -22.5\,^\circ$, so the boat should head $22.5\,^\circ$ south of east.

**31.** $\mathbf{r}(t) = (3t - t^3)\,\mathbf{i} + 3t^2\,\mathbf{j} \Rightarrow \mathbf{r}'(t) = (3 - 3t^2)\,\mathbf{i} + 6t\,\mathbf{j}$,

$|\mathbf{r}'(t)| = \sqrt{(3-3t^2)^2 + (6t)^2} = \sqrt{9 + 18t^2 + 9t^4} = \sqrt{(3-3t^2)^2} = 3 + 3t^2$,

$\mathbf{r}''(t) = -6t\,\mathbf{i} + 6\,\mathbf{j}$, $\mathbf{r}'(t)\times\mathbf{r}''(t) = (18 + 18t^2)\,\mathbf{k}$. Then Equation 9 gives

$$a_T = \frac{\mathbf{r}'(t)\cdot\mathbf{r}''(t)}{|\mathbf{r}'(t)|} = \frac{(3-3t^2)(-6t) + (6t)(6)}{3+3t^2} = \frac{18t + 18t^3}{3+3t^2} = \frac{18t(1+t^2)}{3(1+t^2)} = 6t \quad \left[\text{or by Equation 8, } a_T = v' = \frac{d}{dt}\left[3 + 3t^2\right] = 6t\right]$$

and Equation 10 gives $a_N = \dfrac{|\mathbf{r}'(t)\times\mathbf{r}''(t)|}{|\mathbf{r}'(t)|} = \dfrac{18 + 18t^2}{3 + 3t^2} = \dfrac{18(1+t^2)}{3(1+t^2)} = 6$.

**32.** $\mathbf{r}(t) = (1+t)\,\mathbf{i} + (t^2 - 2t)\,\mathbf{j} \Rightarrow \mathbf{r}'(t) = \mathbf{i} + (2t-2)\,\mathbf{j}$, $|\mathbf{r}'(t)| = \sqrt{1^2 + (2t-2)^2} = \sqrt{4t^2 - 8t + 5}$,

$\mathbf{r}''(t) = 2\,\mathbf{j}$, $\mathbf{r}'(t) \times \mathbf{r}''(t) = 2\,\mathbf{k}$. Then Equation 9 gives $a_T = \dfrac{\mathbf{r}'(t) \cdot \mathbf{r}''(t)}{|\mathbf{r}'(t)|} = \dfrac{2(2t-2)}{\sqrt{4t^2 - 8t + 5}}$ and Equation 10

gives $a_N = \dfrac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|} = \dfrac{2}{\sqrt{4t^2 - 8t + 5}}$.

**33.** $\mathbf{r}(t) = \cos t\,\mathbf{i} + \sin t\,\mathbf{j} + t\,\mathbf{k} \Rightarrow \mathbf{r}'(t) = -\sin t\,\mathbf{i} + \cos t\,\mathbf{j} + \mathbf{k}$, $|\mathbf{r}'(t)| = \sqrt{\sin^2 t + \cos^2 t + 1} = \sqrt{2}$,

$\mathbf{r}''(t) = -\cos t\,\mathbf{i} - \sin t\,\mathbf{j}$, $\mathbf{r}'(t) \times \mathbf{r}''(t) = \sin t\,\mathbf{i} - \cos t\,\mathbf{j} + \mathbf{k}$.

Then $a_T = \dfrac{\mathbf{r}'(t) \cdot \mathbf{r}''(t)}{|\mathbf{r}'(t)|} = \dfrac{\sin t \cos t - \sin t \cos t}{\sqrt{2}} = 0$ and

$$a_N = \frac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|} = \frac{\sqrt{\sin^2 t + \cos^2 t + 1}}{\sqrt{2}} = \frac{\sqrt{2}}{\sqrt{2}} = 1.$$

**34.** $\mathbf{r}(t) = t\,\mathbf{i} + t^2\,\mathbf{j} + 3t\,\mathbf{k} \Rightarrow \mathbf{r}'(t) = \mathbf{i} + 2t\,\mathbf{j} + 3\,\mathbf{k}$, $|\mathbf{r}'(t)| = \sqrt{1^2 + (2t)^2 + 3^2} = \sqrt{4t^2 + 10}$, $\mathbf{r}''(t) = 2\mathbf{j}$,

$\mathbf{r}'(t) \times \mathbf{r}''(t) = -6\mathbf{i} + 2\mathbf{k}$. Then $a_T = \dfrac{\mathbf{r}'(t) \cdot \mathbf{r}''(t)}{|\mathbf{r}'(t)|} = \dfrac{4t}{\sqrt{4t^2 + 10}}$ and $a_N = \dfrac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|} = \dfrac{2\sqrt{10}}{\sqrt{4t^2 + 10}}$.

**35.** $\mathbf{r}(t) = e^t\,\mathbf{i} + \sqrt{2}\,t\,\mathbf{j} + e^{-t}\,\mathbf{k} \Rightarrow \mathbf{r}'(t) = e^t\,\mathbf{i} + \sqrt{2}\,\mathbf{j} - e^{-t}\,\mathbf{k}$,

$|\mathbf{r}(t)| = \sqrt{e^{2t} + 2 + e^{-2t}} = \sqrt{(e^t + e^{-t})^2} = e^t + e^{-t}$, $\mathbf{r}''(t) = e^t\,\mathbf{i} + e^{-t}\,\mathbf{k}$.

Then $a_T = \dfrac{e^{2t} - e^{-2t}}{e^t + e^{-t}} = \dfrac{(e^t + e^{-t})(e^t - e^{-t})}{e^t + e^{-t}} = e^t - e^{-t} = 2\sinh t$ and

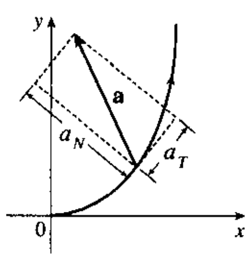
$$a_N = \frac{\left|\sqrt{2}e^{-t}\,\mathbf{i} - 2\,\mathbf{j} - \sqrt{2}e^t\,\mathbf{k}\right|}{e^t + e^{-t}} = \frac{\sqrt{2(e^{-2t} + 2 + e^{2t})}}{e^t + e^{-t}} = \sqrt{2}\,\frac{e^t + e^{-t}}{e^t + e^{-t}} = \sqrt{2}.$$

**36.** $\mathbf{r}(t) = t\,\mathbf{i} + \cos^2 t\,\mathbf{j} + \sin^2 t\,\mathbf{k} \Rightarrow \mathbf{r}'(t) = \mathbf{i} - 2\cos t \sin t\,\mathbf{j} + 2\sin t \cos t\,\mathbf{k} = \mathbf{i} - \sin 2t\,\mathbf{j} + \sin 2t\,\mathbf{k}$,

$|\mathbf{r}'(t)| = \sqrt{1 + 2\sin^2 2t}$, $\mathbf{r}''(t) = 2(\sin^2 t - \cos^2 t)\,\mathbf{j} + 2(\cos^2 t - \sin^2 t)\,\mathbf{k} = -2\cos 2t\,\mathbf{j} + 2\cos 2t\,\mathbf{k}$. So

$a_T = \dfrac{2\sin 2t \cos 2t + 2\sin 2t \cos 2t}{\sqrt{1 + 2\sin^2 2t}} = \dfrac{4\sin 2t \cos 2t}{\sqrt{1 + 2\sin^2 2t}}$ and $a_N = \dfrac{|-2\cos 2t\,\mathbf{j} - 2\cos 2t\,\mathbf{k}|}{\sqrt{1 + 2\sin^2 2t}} = \dfrac{2\sqrt{2}\,|\cos 2t|}{\sqrt{1 + 2\sin^2 t}}$.

**37.** The tangential component of $\mathbf{a}$ is the length of the projection of $\mathbf{a}$ onto $\mathbf{T}$, so we sketch the scalar projection of $\mathbf{a}$ in the tangential direction to the curve and estimate its length to be 4.5 (using the fact that $\mathbf{a}$ has length 10 as a guide). Similarly, the normal component of $\mathbf{a}$ is the length of the projection of $\mathbf{a}$ onto $\mathbf{N}$, so we sketch the scalar projection of $\mathbf{a}$ in the normal direction to the curve and estimate its length to be 9.0. Thus $a_T \approx 4.5\ \text{cm/s}^2$ and $a_N \approx 9.0\ \text{cm/s}^2$.




**38.** $\mathbf{L}(t) = m\,\mathbf{r}(t) \times \mathbf{v}(t) \Rightarrow$

$$\mathbf{L}'(t) = m[\mathbf{r}'(t) \times \mathbf{v}(t) + \mathbf{r}(t) \times \mathbf{v}'(t)] \qquad \text{[by Theorem 14.2.3 [ ET 13.2.3] \#5]}$$
$$= m[\mathbf{v}(t) \times \mathbf{v}(t) + \mathbf{r}(t) \times \mathbf{v}'(t)] = m[\mathbf{0} + \mathbf{r}(t) \times \mathbf{a}(t)] = \boldsymbol{\tau}(t)$$

So if the torque is always $\mathbf{0}$, then $\mathbf{L}'(t) = \mathbf{0}$ for all $t$, and so $\mathbf{L}(t)$ is constant.

**39.** If the engines are turned off at time $t$, then the spacecraft will continue to travel in the direction of $\mathbf{v}(t)$, so we need a $t$ such that for some scalar $s > 0$, $\mathbf{r}(t) + s\,\mathbf{v}(t) = \langle 6, 4, 9\rangle$. $\mathbf{v}(t) = \mathbf{r}'(t) = \mathbf{i} + \dfrac{1}{t}\mathbf{j} + \dfrac{8t}{(t^2+1)^2}\mathbf{k} \Rightarrow$

$$\mathbf{r}(t) + s\,\mathbf{v}(t) = \left\langle 3 + t + s, 2 + \ln t + \frac{s}{t}, 7 - \frac{4}{t^2+1} + \frac{8st}{(t^2+1)^2}\right\rangle \Rightarrow 3 + t + s = 6 \Rightarrow s = 3 - t,$$

so $7 - \dfrac{4}{t^2+1} + \dfrac{8(3-t)t}{(t^2+1)^2} = 9 \Leftrightarrow \dfrac{24t - 12t^2 - 4}{(t^2+1)^2} = 2 \Leftrightarrow t^4 + 8t^2 - 12t + 3 = 0$. It is easily seen that $t = 1$ is a root of this polynomial. Also $2 + \ln 1 + \dfrac{3-1}{1} = 4$, so $t = 1$ is the desired solution.

**40.** (a) $m\dfrac{d\mathbf{v}}{dt} = \dfrac{dm}{dt}\mathbf{v}_e \Leftrightarrow \dfrac{d\mathbf{v}}{dt} = \dfrac{1}{m}\dfrac{dm}{dt}\mathbf{v}_e$. Integrating both sides of this equation with respect to $t$ gives

$$\int_0^t \frac{d\mathbf{v}}{du}\,du = \mathbf{v}_e\int_0^t \frac{1}{m}\frac{dm}{du}\,du \Rightarrow \int_{\mathbf{v}(0)}^{\mathbf{v}(t)} d\mathbf{v} = \mathbf{v}_e\int_{m(0)}^{m(t)}\frac{dm}{m} \quad \text{[Substitution Rule]} \Rightarrow$$

$$\mathbf{v}(t) - \mathbf{v}(0) = \ln\left(\frac{m(t)}{m(0)}\right)\mathbf{v}_e \Rightarrow \mathbf{v}(t) = \mathbf{v}(0) - \ln\left(\frac{m(0)}{m(t)}\right)\mathbf{v}_e.$$

(b) $|\mathbf{v}(t)| = 2\,|\mathbf{v}_e|$, and $|\mathbf{v}(0)| = 0$. Therefore, by part (a), $2\,|\mathbf{v}_e| = \left|-\ln\left(\dfrac{m(0)}{m(t)}\right)\mathbf{v}_e\right| \Rightarrow$

$$2\,|\mathbf{v}_e| = \ln\left(\frac{m(0)}{m(t)}\right)|\mathbf{v}_e|. \left[\textit{Note: } m(0) > m(t) \text{ so that } \ln\left(\frac{m(0)}{m(t)}\right) > 0\right] \Rightarrow m(t) = e^{-2}m(0).$$

Thus $\dfrac{m(0) - e^{-2}m(0)}{m(0)} = 1 - e^{-2}$ is the fraction of the initial mass that is burned as fuel.

## APPLIED PROJECT Kepler's Laws

**1.** With $\mathbf{r} = (r\cos\theta)\,\mathbf{i} + (r\sin\theta)\,\mathbf{j}$ and $\mathbf{h} = \alpha\,\mathbf{k}$ where $\alpha > 0$,

(a) $\mathbf{h} = \mathbf{r}\times\mathbf{r}' = [(r\cos\theta)\,\mathbf{i} + (r\sin\theta)\,\mathbf{j}] \times \left[\left(r'\cos\theta - r\sin\theta\dfrac{d\theta}{dt}\right)\mathbf{i} + \left(r'\sin\theta + r\cos\theta\dfrac{d\theta}{dt}\right)\mathbf{j}\right]$

$$= \left[rr'\cos\theta\sin\theta + r^2\cos^2\theta\frac{d\theta}{dt} - rr'\cos\theta\sin\theta + r^2\sin^2\theta\frac{d\theta}{dt}\right]\mathbf{k} = r^2\frac{d\theta}{dt}\mathbf{k}$$

(b) Since $\mathbf{h} = \alpha\,\mathbf{k}$, $\alpha > 0$, $\alpha = |\mathbf{h}|$. But by part (a), $\alpha = |\mathbf{h}| = r^2\,(d\theta/dt)$.

(c) $A(t) = \frac{1}{2}\int_{\theta_0}^{\theta}|\mathbf{r}|^2\,d\theta = \frac{1}{2}\int_{t_0}^{t} r^2\,(d\theta/dt)\,dt$ in polar coordinates. Thus, by the Fundamental Theorem of Calculus, $\dfrac{dA}{dt} = \dfrac{r^2}{2}\dfrac{d\theta}{dt}$.

(d) $\dfrac{dA}{dt} = \dfrac{r^2}{2}\dfrac{d\theta}{dt} = \dfrac{h}{2} =$ constant since $\mathbf{h}$ is a constant vector and $h = |\mathbf{h}|$.

**2.** (a) Since $dA/dt = \frac{1}{2}h$, a constant, $A(t) = \frac{1}{2}ht + c_1$. But $A(0) = 0$, so $A(t) = \frac{1}{2}ht$. But $A(T) =$ area of the ellipse $= \pi ab$ and $A(T) = \frac{1}{2}hT$, so $T = 2\pi ab/h$.

(b) $h^2/(GM) = ed$ where $e$ is the eccentricity of the ellipse. But $a = ed/(1 - e^2)$ or $ed = a(1 - e^2)$ and $1 - e^2 = b^2/a^2$. Hence $h^2/(GM) = ed = b^2/a$.

(c) $T^2 = \dfrac{4\pi a^2b^2}{h^2} = 4\pi^2a^2b^2\dfrac{a}{GMb^2} = \dfrac{4\pi^2}{GM}a^3$.

**3.** From Problem 2, $T^2 = \dfrac{4\pi^2}{GM}a^3$. $T \approx 365.25$ days $\times\ 24 \cdot 60^2\ \dfrac{\text{seconds}}{\text{day}} \approx 3.1558 \times 10^7$ seconds. Therefore

$$a^3 = \frac{GMT^2}{4\pi^2} \approx \frac{(6.67 \times 10^{-11})(1.99 \times 10^{30})(3.1558 \times 10^7)^2}{4\pi^2} \approx 3.348 \times 10^{33}\ \text{m}^3 \quad \Rightarrow \quad a \approx 1.496 \times 10^{11}\ \text{m}.$$

Thus, the length of the major axis of Earth's orbit (that is, $2a$) is approximately $2.99 \times 10^{11}$ m $= 2.99 \times 10^8$ km.

**4.** We can adapt the equation $T^2 = \dfrac{4\pi^2}{GM}a^3$ from Problem 2(c) with Earth at the center of the system, so $T$ is the period of the satellite's orbit about Earth, $M$ is the mass of Earth, and $a$ is the length of the semimajor axis of the satellite's orbit (measured from Earth's center). Since we want the satellite to remain fixed above a particular point on Earth's equator, $T$ must coincide with the period of Earth's own rotation, so $T = 24$ h $= 86{,}400$ s. The mass of Earth is $M = 5.98 \times 10^{24}$ kg, so

$$a = \left(\frac{T^2GM}{4\pi^2}\right)^{1/3} \approx \left[\frac{(86{,}400)^2(6.67 \times 10^{-11})(5.98 \times 10^{24})}{4\pi^2}\right]^{1/3} \approx 4.23 \times 10^7\ \text{m}.$$

If we assume a circular orbit, the radius of the orbit is $a$, and since the radius of Earth is $6.37 \times 10^6$ m, the required altitude above Earth's surface for the satellite is $4.23 \times 10^7 - 6.37 \times 10^6 \approx 3.59 \times 10^7$ m, or 35,900 km.

## 14 Review ET 13

### CONCEPT CHECK

**1.** A vector function is a function whose domain is a set of real numbers and whose range is a set of vectors. To find the derivative or integral, we can differentiate or integrate each component of the vector function.

**2.** The tip of the moving vector $\mathbf{r}(t)$ of a continuous vector function traces out a space curve.

**3.** (a) A curve represented by the vector function $\mathbf{r}(t)$ is smooth if $\mathbf{r}'(t)$ is continuous and $\mathbf{r}'(t) \neq \mathbf{0}$ on its parametric domain (except possibly at the endpoints).

(b) The tangent vector to a smooth curve at a point $P$ with position vector $\mathbf{r}(t)$ is the vector $\mathbf{r}'(t)$. The tangent line at $P$ is the line through $P$ parallel to the tangent vector $\mathbf{r}'(t)$. The unit tangent vector is $\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|}$.

**4.** (a)–(f) See Theorem 14.2.3 [ET 13.2.3].

**5.** Use Formula 14.3.2 [ET 13.3.2], or equivalently 14.3.3 [ET 13.3.3].

**6.** (a) The curvature of a curve is $\kappa = \left|\dfrac{d\mathbf{T}}{ds}\right|$ where $\mathbf{T}$ is the unit tangent vector.

(b) $\kappa(t) = \left|\dfrac{\mathbf{T}'(t)}{\mathbf{r}'(t)}\right|$ (c) $\kappa(t) = \dfrac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$ (d) $\kappa(x) = \dfrac{|f''(x)|}{[1 + (f'(x))^2]^{3/2}}$

**7.** (a) The unit normal vector: $\mathbf{N}(t) = \dfrac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|}$. The binormal vector: $\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t)$.

(b) See the discussion preceding Example 7 in Section 14.3 [ET 13.3].

**8.** (a) If $\mathbf{r}(t)$ is the position vector of the particle on the space curve, the velocity $\mathbf{v}(t) = \mathbf{r}'(t)$, the speed is given by $|\mathbf{v}(t)|$, and the acceleration $\mathbf{a}(t) = \mathbf{v}'(t) = \mathbf{r}''(t)$.

(b) $\mathbf{a} = a_T\mathbf{T} + a_N\mathbf{N}$ where $a_T = v'$ and $a_N = \kappa v^2$.

**9.** See the statement of Kepler's Laws on page 912 [ET 876].

## TRUE-FALSE QUIZ

**1.** True. If we reparametrize the curve by replacing $u = t^3$, we have $\mathbf{r}(u) = u\,\mathbf{i} + 2u\,\mathbf{j} + 3u\,\mathbf{k}$, which is a line through the origin with direction vector $\mathbf{i} + 2\,\mathbf{j} + 3\,\mathbf{k}$.

**2.** True. $\mathbf{r}'(t) = \langle 1, 3t^2, 5t^4 \rangle$ is continuous for all $t$ (since its component functions are each continuous) and since $x'(t) = 1$, we have $\mathbf{r}'(t) \neq \mathbf{0}$ for all $t$.

**3.** False. $\mathbf{r}'(t) = \langle -\sin t, 2t, 4t^3 \rangle$, and since $\mathbf{r}'(0) = \langle 0, 0, 0 \rangle = \mathbf{0}$, the curve is not smooth.

**4.** True. See Theorem 14.2.2 [ET 13.2.2].

**5.** False. By Formula 5 of Theorem 14.2.3 [ET 13.2.3], $\dfrac{d}{dt}\left[\mathbf{u}(t) \times \mathbf{v}(t)\right] = \mathbf{u}'(t) \times \mathbf{v}(t) + \mathbf{u}(t) \times \mathbf{v}'(t)$.

**6.** False. For example, let $\mathbf{r}(t) = \langle \cos t, \sin t \rangle$. Then $|\mathbf{r}(t)| = \sqrt{\cos^2 t + \sin^2 t} = 1 \;\Rightarrow\; \dfrac{d}{dt}\,|\mathbf{r}(t)| = 0$, but $|\mathbf{r}'(t)| = |\langle -\sin t, \cos t \rangle| = \sqrt{(-\sin t)^2 + \cos^2 t} = 1$.

**7.** False. $\kappa$ is the magnitude of the rate of change of the unit tangent vector $\mathbf{T}$ with respect to arc length $s$, not with respect to $t$.

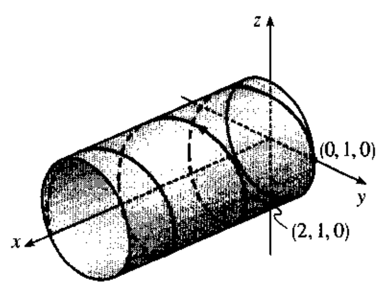
**8.** False. The binormal vector, by the definition given in Section 14.3 [ET 13.3], is $\mathbf{B}(t) = \mathbf{T}(t) \times \mathbf{N}(t) = -\left[\mathbf{N}(t) \times \mathbf{T}(t)\right]$.

**9.** True. See the discussion preceding Example 7 in Section 14.3 [ET 13.3].

**10.** False. For example, $\mathbf{r}_1(t) = \langle t, t \rangle$ and $\mathbf{r}_2(t) = \langle 2t, 2t \rangle$ both represent the same plane curve (the line $y = x$), but the tangent vector $\mathbf{r}_1'(t) = \langle 1, 1 \rangle$ for all $t$, while $\mathbf{r}_2'(t) = \langle 2, 2 \rangle$. In fact, different parametrizations give parallel tangent vectors at a point, but their magnitudes may differ.

## EXERCISES

**1.** (a) The corresponding parametric equations for the curve are $x = t$, $y = \cos \pi t$, $z = \sin \pi t$. Since $y^2 + z^2 = 1$, the curve is contained in a circular cylinder with axis the $x$-axis. Since $x = t$, the curve is a helix.




(b) $\mathbf{r}(t) = t\,\mathbf{i} + \cos \pi t\,\mathbf{j} + \sin \pi t\,\mathbf{k} \;\Rightarrow\; \mathbf{r}'(t) = \mathbf{i} - \pi \sin \pi t\,\mathbf{j} + \pi \cos \pi t\,\mathbf{k} \;\Rightarrow$
$\mathbf{r}''(t) = -\pi^2 \cos \pi t\,\mathbf{j} - \pi^2 \sin \pi t\,\mathbf{k}$

**2.** (a) The expressions $\sqrt{2-t}$, $(e^t - 1)/t$, and $\ln(t+1)$ are all defined when $2 - t \geq 0 \;\Rightarrow\; t \leq 2$, $t \neq 0$, and $t + 1 > 0 \;\Rightarrow\; t > -1$. Thus the domain of $\mathbf{r}$ is $(-1, 0) \cup (0, 2]$.

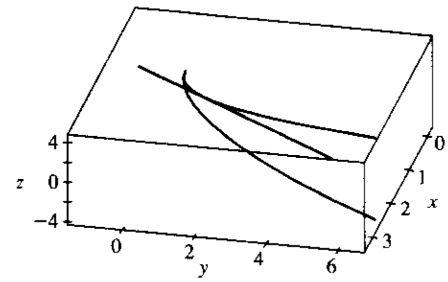
(b) $\displaystyle \lim_{t\to 0} \mathbf{r}(t) = \left\langle \lim_{t\to 0} \sqrt{2-t}, \lim_{t\to 0} \frac{e^t - 1}{t}, \lim_{t\to 0} \ln(t+1) \right\rangle = \left\langle \sqrt{2-0}, \lim_{t\to 0} \frac{e^t}{1}, \ln(0+1) \right\rangle = \langle \sqrt{2}, 1, 0 \rangle$

(using l'Hospital's Rule in the $y$-component).

(c) $\displaystyle \mathbf{r}'(t) = \left\langle \frac{d}{dt} \sqrt{2-t}, \frac{d}{dt} \frac{e^t - 1}{t}, \frac{d}{dt} \ln(t+1) \right\rangle = \left\langle -\frac{1}{2\sqrt{2-t}}, \frac{te^t - e^t + 1}{t^2}, \frac{1}{t+1} \right\rangle$

**3.** The projection of the curve $C$ of intersection onto the $xy$-plane is the circle $x^2 + y^2 = 16$, $z = 0$. So we can write $x = 4\cos t$, $y = 4\sin t$, $0 \le t \le 2\pi$. From the equation of the plane, we have $z = 5 - x = 5 - 4\cos t$, so parametric equations for $C$ are $x = 4\cos t$, $y = 4\sin t$, $z = 5 - 4\cos t$, $0 \le t \le 2\pi$, and the corresponding vector function is $\mathbf{r}(t) = 4\cos t\,\mathbf{i} + 4\sin t\,\mathbf{j} + (5 - 4\cos t)\,\mathbf{k}$, $0 \le t \le 2\pi$.

**4.** The curve is given by $\mathbf{r}(t) = \langle t^2, t^4, t^3\rangle$, so $\mathbf{r}'(t) = \langle 2t, 4t^3, 3t^2\rangle$. The point $(1, 1, 1)$ corresponds to $t = 1$, so the tangent vector there is $\mathbf{r}'(1) = \langle 2, 4, 3\rangle$. Then the tangent line has direction vector $\langle 2, 4, 3\rangle$ and includes the point $(1, 1, 1)$, so parametric equations are $x = 1 + 2t$, $y = 1 + 4t$, $z = 1 + 3t$.




**5.**
$$\int_0^1 (t^2\,\mathbf{i} + t\cos \pi t\,\mathbf{j} + \sin \pi t\,\mathbf{k})\,dt = \left(\int_0^1 t^2\,dt\right)\mathbf{i} + \left(\int_0^1 t\cos \pi t\,dt\right)\mathbf{j} + \left(\int_0^1 \sin \pi t\,dt\right)\mathbf{k}$$
$$= \left[\tfrac{1}{3}t^3\right]_0^1 \mathbf{i} + \left(\tfrac{t}{\pi}\sin \pi t\Big]_0^1 - \textstyle\int_0^1 \tfrac{1}{\pi}\sin \pi t\,dt\right)\mathbf{j} + \left[-\tfrac{1}{\pi}\cos \pi t\right]_0^1 \mathbf{k}$$
$$= \tfrac{1}{3}\,\mathbf{i} + \left[\tfrac{1}{\pi^2}\cos \pi t\right]_0^1 \mathbf{j} + \tfrac{2}{\pi}\,\mathbf{k} = \tfrac{1}{3}\,\mathbf{i} - \tfrac{2}{\pi^2}\,\mathbf{j} + \tfrac{2}{\pi}\,\mathbf{k}$$
where we integrated by parts in the $y$-component.

**6.** (a) $C$ intersects the $xz$-plane where $y = 0 \;\Rightarrow\; 2t - 1 = 0 \;\Rightarrow\; t = \frac{1}{2}$, so the point is $\left(2 - \left(\frac{1}{2}\right)^3, 0, \ln\frac{1}{2}\right) = \left(\frac{15}{8}, 0, -\ln 2\right)$.

(b) The curve is given by $\mathbf{r}(t) = \langle 2 - t^3, 2t - 1, \ln t\rangle$, so $\mathbf{r}'(t) = \langle -3t^2, 2, 1/t\rangle$. The point $(1, 1, 0)$ corresponds to $t = 1$, so the tangent vector there is $\mathbf{r}'(1) = \langle -3, 2, 1\rangle$. Then the tangent line has direction vector $\langle -3, 2, 1\rangle$ and includes the point $(1, 1, 0)$, so parametric equations are $x = 1 - 3t$, $y = 1 + 2t$, $z = t$.

(c) The normal plane has normal vector $\mathbf{r}'(1) = \langle -3, 2, 1\rangle$ and equation $-3(x - 1) + 2(y - 1) + z = 0$ or $3x - 2y - z = 1$.

**7.** $t = 1$ at $(1, 4, 2)$ and $t = 4$ at $(2, 1, 17)$, so

$$L = \int_1^4 \sqrt{\frac{1}{4t} + \frac{16}{t^4} + 4t^2}\,dt$$
$$\approx \frac{4-1}{3\cdot 4}\left[\sqrt{\tfrac{1}{4} + 16 + 4} \;+\; 4\cdot\sqrt{\frac{1}{4\cdot\frac{7}{4}} + \frac{16}{\left(\frac{7}{4}\right)^4} + 4\left(\tfrac{7}{4}\right)^2} \;+\; 2\cdot\sqrt{\frac{1}{4\cdot\frac{10}{4}} + \frac{16}{\left(\frac{10}{4}\right)^4} + 4\left(\tfrac{10}{4}\right)^2}\right.$$
$$\left. +\; 4\cdot\sqrt{\frac{1}{4\cdot\frac{13}{4}} + \frac{16}{\left(\frac{13}{4}\right)^4} + 4\left(\tfrac{13}{4}\right)^2} \;+\; \sqrt{\frac{1}{4\cdot 4} + \frac{16}{4^4} + 4\cdot 4^2}\,\right]$$
$$\approx 15.9241$$

**8.** $\mathbf{r}'(t) = \left\langle 3t^{1/2}, -2\sin 2t, 2\cos 2t\right\rangle$, $|\mathbf{r}'(t)| = \sqrt{9t + 4(\sin^2 2t + \cos^2 2t)} = \sqrt{9t + 4}$. Thus
$L = \int_0^1 \sqrt{9t + 4}\,dt = \int_4^{13} \frac{1}{9}u^{1/2}\,du = \frac{1}{9}\cdot\frac{2}{3}u^{3/2}\Big]_4^{13} = \frac{2}{27}(13^{3/2} - 8)$.

**9.** The angle of intersection of the two curves, $\theta$, is the angle between their respective tangents at the point of intersection. For both curves the point $(1,0,0)$ occurs when $t=0$. $\mathbf{r}_1'(t) = -\sin t\,\mathbf{i} + \cos t\,\mathbf{j} + \mathbf{k} \Rightarrow \mathbf{r}_1'(0) = \mathbf{j}+\mathbf{k}$ and $\mathbf{r}_2'(t) = \mathbf{i} + 2t\,\mathbf{j} + 3t^2\,\mathbf{k} \Rightarrow \mathbf{r}_2'(0) = \mathbf{i}$. $\mathbf{r}_1'(0)\cdot\mathbf{r}_2'(0) = (\mathbf{j}+\mathbf{k})\cdot\mathbf{i} = 0$. Therefore, the curves intersect in a right angle, that is, $\theta = \frac{\pi}{2}$.

**10.** The parametric value corresponding to the point $(1,0,1)$ is $t=0$.

$\mathbf{r}'(t) = e^t\,\mathbf{i} + e^t(\cos t + \sin t)\,\mathbf{j} + e^t(\cos t - \sin t)\,\mathbf{k}$

$\Rightarrow \quad |\mathbf{r}'(t)| = e^t\sqrt{1 + (\cos t + \sin t)^2 + (\cos t - \sin t)^2} = \sqrt{3}\,e^t$

and $s(t) = \int_0^t e^u\sqrt{3}\,du = \sqrt{3}(e^t - 1) \quad\Rightarrow\quad t = \ln\left(1 + \frac{1}{\sqrt{3}}s\right)$. Therefore,

$$\mathbf{r}(t(s)) = \left(1 + \tfrac{1}{\sqrt{3}}s\right)\mathbf{i} + \left(1 + \tfrac{1}{\sqrt{3}}s\right)\sin\ln\left(1 + \tfrac{1}{\sqrt{3}}s\right)\mathbf{j} + \left(1 + \tfrac{1}{\sqrt{3}}s\right)\cos\ln\left(1 + \tfrac{1}{\sqrt{3}}s\right)\mathbf{k}.$$

**11.** (a) $\mathbf{T}(t) = \dfrac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \dfrac{\langle t^2, t, 1\rangle}{|\langle t^2,t,1\rangle|} = \dfrac{\langle t^2,t,1\rangle}{\sqrt{t^4+t^2+1}}$

(b) $\mathbf{T}'(t) = -\frac{1}{2}(t^4+t^2+1)^{-3/2}(4t^3+2t)\,\langle t^2,t,1\rangle + (t^4+t^2+1)^{-1/2}\langle 2t,1,0\rangle$

$$= \frac{-2t^3 - t}{(t^4+t^2+1)^{3/2}}\langle t^2,t,1\rangle + \frac{1}{(t^4+t^2+1)^{1/2}}\langle 2t,1,0\rangle$$

$$= \frac{\langle -2t^5 - t^3, -2t^4 - t^2, -2t^3 - t\rangle + \langle 2t^5 + 2t^3 + 2t, t^4 + t^2 + 1, 0\rangle}{(t^4+t^2+1)^{3/2}}$$

$$= \frac{\langle 2t, -t^4 + 1, -2t^3 - t\rangle}{(t^4+t^2+1)^{3/2}}$$

$$|\mathbf{T}'(t)| = \frac{\sqrt{4t^2 + t^8 - 2t^4 + 1 + 4t^6 + 4t^4 + t^2}}{(t^4+t^2+1)^{3/2}} = \frac{\sqrt{t^8 + 4t^6 + 2t^4 + 5t^2}}{(t^4+t^2+1)^{3/2}}, \text{ and}$$

$$\mathbf{N}(t) = \frac{\langle 2t, 1 - t^4, -2t^3 - t\rangle}{\sqrt{t^8 + 4t^6 + 2t^4 + 5t^2}}.$$

(c) $\kappa(t) = \dfrac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \dfrac{\sqrt{t^8 + 4t^6 + 2t^4 + 5t^2}}{(t^4+t^2+1)^2}$

**12.** Using Exercise 14.3.36 [ET 13.3.36], we have $\mathbf{r}'(t) = \langle -3\sin t, 4\cos t\rangle$, $\mathbf{r}''(t) = \langle -3\cos t, -4\sin t\rangle$,

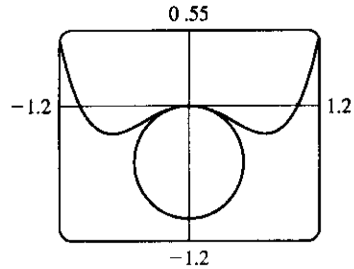
$|\mathbf{r}'(t)|^3 = \left(\sqrt{9\sin^2 t + 4\cos^2 t}\right)^3$ and then

$$\kappa(t) = \frac{|(-3\sin t)(-4\sin t) - (4\cos t)(-3\cos t)|}{(9\sin^2 t + 16\cos^2 t)^{3/2}} = \frac{12}{(9\sin^2 t + 16\cos^2 t)^{3/2}}.$$

At $(3,0)$, $t=0$ and $\kappa(0) = 12/(16)^{3/2} = \frac{12}{64} = \frac{3}{16}$. At $(0,4)$, $t = \frac{\pi}{2}$ and $\kappa\left(\frac{\pi}{2}\right) = 12/9^{3/2} = \frac{12}{27} = \frac{4}{9}$.

**13.** $y' = 4x^3$, $y'' = 12x^2$ and $\kappa(x) = \dfrac{|y''|}{[1+(y')^2]^{3/2}} = \dfrac{|12x^2|}{(1+16x^6)^{3/2}}$, so $\kappa(1) = \dfrac{12}{17^{3/2}}$.

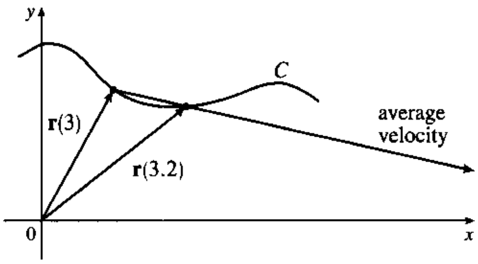
**14.** $\kappa(x) = \dfrac{|12x^2 - 2|}{[1 + (4x^3 - 2x)^2]^{3/2}} \Rightarrow \kappa(0) = 2$. So the osculating circle has radius $\frac{1}{2}$ and center $\left(0, -\frac{1}{2}\right)$. Thus its equation is $x^2 + \left(y + \frac{1}{2}\right)^2 = \frac{1}{4}$.




**15.** $\mathbf{r}(t) = \langle \sin 2t, t, \cos 2t \rangle \Rightarrow \mathbf{r}'(t) = \langle 2\cos 2t, 1, -2\sin 2t \rangle \Rightarrow \mathbf{T}(t) = \frac{1}{\sqrt{5}} \langle 2\cos 2t, 1, -2\sin 2t \rangle \Rightarrow$ $\mathbf{T}'(t) = \frac{1}{\sqrt{5}} \langle -4\sin 2t, 0, -4\cos 2t \rangle \Rightarrow \mathbf{N}(t) = \langle -\sin 2t, 0, -\cos 2t \rangle$. So $\mathbf{N} = \mathbf{N}(\pi) = \langle 0, 0, -1 \rangle$ and $\mathbf{B} = \mathbf{T} \times \mathbf{N} = \frac{1}{\sqrt{5}} \langle -1, 2, 0 \rangle$. So a normal to the osculating plane is $\langle -1, 2, 0 \rangle$ and an equation is $-1(x - 0) + 2(y - \pi) + 0(z - 1) = 0$ or $x - 2y + 2\pi = 0$.

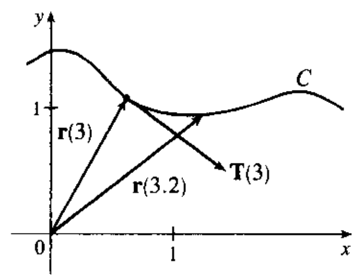
**16.** (a) The average velocity over $[3, 3.2]$ is given by

$$\frac{\mathbf{r}(3.2) - \mathbf{r}(3)}{3.2 - 3} = 5[\mathbf{r}(3.2) - \mathbf{r}(3)],$$ so we draw a vector with the same direction but 5 times the length of the vector $[\mathbf{r}(3.2) - \mathbf{r}(3)]$.




(b) $\mathbf{v}(3) = \mathbf{r}'(3) = \lim\limits_{h \to 0} \dfrac{\mathbf{r}(3 + h) - \mathbf{r}(3)}{h}$.

(c) $\mathbf{T}(3) = \dfrac{\mathbf{r}'(3)}{|\mathbf{r}'(3)|}$, a unit vector in the same direction as $\mathbf{r}'(3)$, that is, parallel to the tangent line to the curve at $\mathbf{r}(3)$, pointing in the direction corresponding to increasing $t$, and with length 1.




**17.** $\mathbf{r}(t) = t \ln t\, \mathbf{i} + t\, \mathbf{j} + e^{-t}\, \mathbf{k}$, $\mathbf{v}(t) = \mathbf{r}'(t) = (1 + \ln t)\, \mathbf{i} + \mathbf{j} - e^{-t}\, \mathbf{k}$,
$|\mathbf{v}(t)| = \sqrt{(1 + \ln t)^2 + 1^2 + (-e^{-t})^2} = \sqrt{2 + 2\ln t + (\ln t)^2 + e^{-2t}}$, $\mathbf{a}(t) = \mathbf{v}'(t) = \frac{1}{t}\, \mathbf{i} + e^{-t}\, \mathbf{k}$.

**18.** $\mathbf{v}(t) = \int \mathbf{a}(t)\, dt = \int (6t\, \mathbf{i} + 12t^2\, \mathbf{j} - 6t\, \mathbf{k})\, dt = 3t^2\, \mathbf{i} + 4t^3\, \mathbf{j} - 3t^2\, \mathbf{k} + \mathbf{C}$, but $\mathbf{i} - \mathbf{j} + 3\,\mathbf{k} = \mathbf{v}(0) = \mathbf{0} + \mathbf{C}$, so $\mathbf{C} = \mathbf{i} - \mathbf{j} + 3\,\mathbf{k}$ and $\mathbf{v}(t) = (3t^2 + 1)\, \mathbf{i} + (4t^3 - 1)\, \mathbf{j} + (3 - 3t^2)\, \mathbf{k}$. $\mathbf{r}(t) = \int \mathbf{v}(t)\, dt = (t^3 + t)\, \mathbf{i} + (t^4 - t)\, \mathbf{j} + (3t - t^3)\, \mathbf{k} + \mathbf{D}$. But $\mathbf{r}(0) = \mathbf{0}$, so $\mathbf{D} = \mathbf{0}$ and $\mathbf{r}(t) = (t^3 + t)\, \mathbf{i} + (t^4 - t)\, \mathbf{j} + (3t - t^3)\, \mathbf{k}$.

**19.** We set up the axes so that the shot leaves the athlete's hand 7 ft above the origin. Then we are given $\mathbf{r}(0) = 7\mathbf{j}$, $|\mathbf{v}(0)| = 43$ ft/s, and $\mathbf{v}(0)$ has direction given by a $45^\circ$ angle of elevation. Then a unit vector in the direction of $\mathbf{v}(0)$ is $\frac{1}{\sqrt{2}}(\mathbf{i} + \mathbf{j}) \Rightarrow \mathbf{v}(0) = \frac{43}{\sqrt{2}}(\mathbf{i} + \mathbf{j})$. Assuming air resistance is negligible, the only external force is due to gravity, so as in Example 14.4.5 [ET 13.4.5] we have $\mathbf{a} = -g\,\mathbf{j}$ where here $g \approx 32$ ft/s$^2$. Since $\mathbf{v}'(t) = \mathbf{a}(t)$, we integrate, giving $\mathbf{v}(t) = -gt\,\mathbf{j} + \mathbf{C}$ where $\mathbf{C} = \mathbf{v}(0) = \frac{43}{\sqrt{2}}(\mathbf{i} + \mathbf{j}) \;\Rightarrow\; \mathbf{v}(t) = \frac{43}{\sqrt{2}}\,\mathbf{i} + \left(\frac{43}{\sqrt{2}} - gt\right)\mathbf{j}$. Since $\mathbf{r}'(t) = \mathbf{v}(t)$ we integrate again, so $\mathbf{r}(t) = \frac{43}{\sqrt{2}}t\,\mathbf{i} + \left(\frac{43}{\sqrt{2}}t - \frac{1}{2}gt^2\right)\mathbf{j} + \mathbf{D}$. But $\mathbf{D} = \mathbf{r}(0) = 7\,\mathbf{j} \;\Rightarrow$ $\mathbf{r}(t) = \frac{43}{\sqrt{2}}t\,\mathbf{i} + \left(\frac{43}{\sqrt{2}}t - \frac{1}{2}gt^2 + 7\right)\mathbf{j}$.

(a) At 2 seconds, the shot is at $\mathbf{r}(2) = \frac{43}{\sqrt{2}}(2)\,\mathbf{i} + \left(\frac{43}{\sqrt{2}}(2) - \frac{1}{2}g(2)^2 + 7\right)\mathbf{j} \approx 60.8\,\mathbf{i} + 3.8\,\mathbf{j}$, so the shot is about 3.8 ft above the ground, at a horizontal distance of 60.8 ft from the athlete.

(b) The shot reaches its maximum height when the vertical component of velocity is 0: $\frac{43}{\sqrt{2}} - gt = 0 \;\Rightarrow$ $t = \dfrac{43}{\sqrt{2}\,g} \approx 0.95$ s. Then $\mathbf{r}(0.95) \approx 28.9\,\mathbf{i} + 21.4\,\mathbf{j}$, so the maximum height is approximately 21.4 ft.

(c) The shot hits the ground when the vertical component of $\mathbf{r}(t)$ is 0, so $\frac{43}{\sqrt{2}}t - \frac{1}{2}gt^2 + 7 = 0 \;\Rightarrow$ $-16t^2 + \frac{43}{\sqrt{2}}t + 7 = 0 \;\Rightarrow\; t \approx 2.11$ s. $\mathbf{r}(2.11) \approx 64.2\,\mathbf{i} - 0.08\,\mathbf{j}$, thus the shot lands approximately 64.2 ft from the athlete.

**20.** $\mathbf{r}'(t) = \mathbf{i} + 2\,\mathbf{j} + 2t\,\mathbf{k}$, $\mathbf{r}''(t) = 2\,\mathbf{k}$, $|\mathbf{r}'(t)| = \sqrt{1 + 4 + 4t^2} = \sqrt{4t^2 + 5}$.

Then $a_T = \dfrac{\mathbf{r}'(t) \cdot \mathbf{r}''(t)}{|\mathbf{r}'(t)|} = \dfrac{4t}{\sqrt{4t^2+5}}$ and $a_N = \dfrac{|\mathbf{r}'(t) \times \mathbf{r}''(t)|}{|\mathbf{r}'(t)|} = \dfrac{|4\mathbf{i} - 2\mathbf{j}|}{\sqrt{4t^2+5}} = \dfrac{2\sqrt{5}}{\sqrt{4t^2+5}}$.

**21.** (a) Instead of proceeding directly, we use Formula 3 of Theorem 14.2.3 [ET 13.2.3]:
$\mathbf{r}(t) = t\,\mathbf{R}(t) \;\Rightarrow\; \mathbf{v} = \mathbf{r}'(t) = \mathbf{R}(t) + t\,\mathbf{R}'(t) = \cos\omega t\,\mathbf{i} + \sin\omega t\,\mathbf{j} + t\,\mathbf{v}_d$.

(b) Using the same method as in part (a) and starting with $\mathbf{v} = \mathbf{R}(t) + t\,\mathbf{R}'(t)$, we have
$\mathbf{a} = \mathbf{v}' = \mathbf{R}'(t) + \mathbf{R}'(t) + t\,\mathbf{R}''(t) = 2\,\mathbf{R}'(t) + t\,\mathbf{R}''(t) = 2\,\mathbf{v}_d + t\,\mathbf{a}_d$.

(c) Here we have $\mathbf{r}(t) = e^{-t}\cos\omega t\,\mathbf{i} + e^{-t}\sin\omega t\,\mathbf{j} = e^{-t}\,\mathbf{R}(t)$. So, as in parts (a) and (b),

$$\mathbf{v} = \mathbf{r}'(t) = e^{-t}\,\mathbf{R}'(t) - e^{-t}\,\mathbf{R}(t) = e^{-t}[\mathbf{R}'(t) - \mathbf{R}(t)] \;\Rightarrow$$
$$\begin{aligned}\mathbf{a} = \mathbf{v}' &= e^{-t}[\mathbf{R}''(t) - \mathbf{R}'(t)] - e^{-t}[\mathbf{R}'(t) - \mathbf{R}(t)] = e^{-t}[\mathbf{R}''(t) - 2\,\mathbf{R}'(t) + \mathbf{R}(t)] \\ &= e^{-t}\,\mathbf{a}_d - 2e^{-t}\,\mathbf{v}_d + e^{-t}\,\mathbf{R}\end{aligned}$$

Thus, the Coriolis acceleration (the sum of the "extra" terms not involving $\mathbf{a}_d$) is $-2e^{-t}\,\mathbf{v}_d + e^{-t}\,\mathbf{R}$.

**22.** (a) $$F(x) = \begin{cases} 1 & \text{if } x \le 0 \\ \sqrt{1-x^2} & \text{if } 0 < x < \frac{1}{\sqrt{2}} \\ \sqrt{2} - x & \text{if } x \ge \frac{1}{\sqrt{2}} \end{cases} \;\Rightarrow\; F'(x) = \begin{cases} 0 & \text{if } x < 0 \\ -x/\sqrt{1-x^2} & \text{if } 0 < x < \frac{1}{\sqrt{2}} \\ -1 & \text{if } x > \frac{1}{\sqrt{2}} \end{cases} \;\Rightarrow$$

$$F''(x) = \begin{cases} 0 & \text{if } x < 0 \\ -1/(1-x^2)^{3/2} & \text{if } 0 < x < \frac{1}{\sqrt{2}} \\ 0 & \text{if } x > \frac{1}{\sqrt{2}} \end{cases}$$

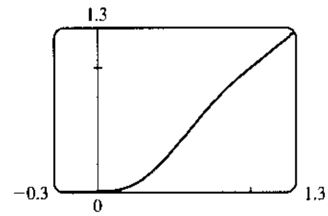
since $\dfrac{d}{dx}\left[-x(1-x^2)^{-1/2}\right] = -(1-x^2)^{-1/2} - x^2(1-x^2)^{-3/2} = -(1-x^2)^{-3/2}$.

Now $\lim\limits_{x\to 0^+} \sqrt{1-x^2} = 1 = F(0)$ and $\lim\limits_{x\to(1/\sqrt{2})^-} \sqrt{1-x^2} = \frac{1}{\sqrt{2}} = F\!\left(\frac{1}{\sqrt{2}}\right)$, so $F$ is continuous. Also, since

$\lim\limits_{x\to 0^+} F'(x) = 0 = \lim\limits_{x\to 0^-} F'(x)$ and $\lim\limits_{x\to (1/\sqrt{2})^-} F'(x) = -1 = \lim\limits_{x\to (1/\sqrt{2})^+} F'(x)$, $F'$ is continuous. But $\lim\limits_{x\to 0^+} F''(x) = -1 \neq 0 = \lim\limits_{x\to 0^-} F''(x)$, so $F''$ is not continuous at $x = 0$. (The same is true at $x = \frac{1}{\sqrt{2}}$.) So $F$ does not have continuous curvature.

(b) Set $P(x) = ax^5 + bx^4 + cx^3 + dx^2 + ex + f$. The continuity conditions on $P$ are $P(0) = 0$, $P(1) = 1$, $P'(0) = 0$ and $P'(1) = 1$. Also the curvature must be continuous. For $x \leq 0$ and $x \geq 1$, $\kappa(x) = 0$; elsewhere $\kappa(x) = \dfrac{|P''(x)|}{(1 + [P'(x)]^2)^{3/2}}$, so we need $P''(0) = 0$ and $P''(1) = 0$.

The conditions $P(0) = P'(0) = P''(0) = 0$ imply that $d = e = f = 0$. The other conditions imply that $a + b + c = 1$, $5a + 4b + 3c = 1$, and $10a + 6b + 3c = 0$. From these, we find that $a = 3$, $b = -8$, and $c = 6$. Therefore $P(x) = 3x^5 - 8x^4 + 6x^3$. Since there was no solution with $a = 0$, this could not have been done with a polynomial of degree 4.



# ☐ PROBLEMS PLUS

**1.** (a) $\mathbf{r}(t) = R\cos\omega t\,\mathbf{i} + R\sin\omega t\,\mathbf{j} \quad\Rightarrow\quad \mathbf{v} = \mathbf{r}'(t) = -\omega R\sin\omega t\,\mathbf{i} + \omega R\cos\omega t\,\mathbf{j}$, so $\mathbf{r} = R(\cos\omega t\,\mathbf{i} + \sin\omega t\,\mathbf{j})$ and $\mathbf{v} = \omega R(-\sin\omega t\,\mathbf{i} + \cos\omega t\,\mathbf{j})$. $\mathbf{v}\cdot\mathbf{r} = \omega R^2(-\cos\omega t\sin\omega t + \sin\omega t\cos\omega t) = 0$, so $\mathbf{v}\perp\mathbf{r}$. Since $\mathbf{r}$ points along a radius of the circle, and $\mathbf{v}\perp\mathbf{r}$, $\mathbf{v}$ is tangent to the circle. Because it is a velocity vector, $\mathbf{v}$ points in the direction of motion.

(b) In (a), we wrote $\mathbf{v}$ in the form $\omega R\,\mathbf{u}$, where $\mathbf{u}$ is the unit vector $-\sin\omega t\,\mathbf{i} + \cos\omega t\,\mathbf{j}$. Clearly $|\mathbf{v}| = \omega R\,|\mathbf{u}| = \omega R$. At speed $\omega R$, the particle completes one revolution, a distance $2\pi R$, in time $T = \dfrac{2\pi R}{\omega R} = \dfrac{2\pi}{\omega}$.

(c) $\mathbf{a} = \dfrac{d\mathbf{v}}{dt} = -\omega^2 R\cos\omega t\,\mathbf{i} - \omega^2 R\sin\omega t\,\mathbf{j} = -\omega^2 R(\cos\omega t\,\mathbf{i} + \sin\omega t\,\mathbf{j})$, so $\mathbf{a} = -\omega^2\mathbf{r}$. This shows that $\mathbf{a}$ is proportional to $\mathbf{r}$ and points in the opposite direction (toward the origin). Also, $|\mathbf{a}| = \omega^2\,|\mathbf{r}| = \omega^2 R$.

(d) By Newton's Second Law (see Section 14.4 [ET 13.4]), $\mathbf{F} = m\mathbf{a}$, so
$|\mathbf{F}| = m\,|\mathbf{a}| = mR\omega^2 = \dfrac{m\,(\omega R)^2}{R} = \dfrac{m\,|\mathbf{v}|^2}{R}$.

**2.** (a) Dividing the equation $|\mathbf{F}|\sin\theta = \dfrac{mv_R^2}{R}$ by the equation $|\mathbf{F}|\cos\theta = mg$, we obtain $\tan\theta = \dfrac{v_R^2}{Rg}$, so $v_R^2 = Rg\tan\theta$.

(b) $R = 400$ ft and $\theta = 12°$, so $v_R = \sqrt{Rg\tan\theta} \approx \sqrt{400\cdot 32\cdot\tan 12°} \approx 52.16$ ft/s $\approx 36$ mi/h.

(c) We want to choose a new radius $R_1$ for which the new rated speed is $\frac{3}{2}$ of the old one: $\sqrt{R_1 g\tan 12°} = \frac{3}{2}\sqrt{Rg\tan 12°}$. Squaring, we get $R_1 g\tan 12° = \frac{9}{4}Rg\tan 12°$, so $R_1 = \frac{9}{4}R = \frac{9}{4}(400) = 900$ ft.

**3.** (a) The projectile reaches maximum height when $0 = \dfrac{dy}{dt} = \dfrac{d}{dt}\left[(v_0\sin\alpha)t - \frac{1}{2}gt^2\right] = v_0\sin\alpha - gt$; that is, when $t = \dfrac{v_0\sin\alpha}{g}$ and $y = (v_0\sin\alpha)\left(\dfrac{v_0\sin\alpha}{g}\right) - \dfrac{1}{2}g\left(\dfrac{v_0\sin\alpha}{g}\right)^2 = \dfrac{v_0^2\sin^2\alpha}{2g}$. This is the maximum height attained when the projectile is fired with an angle of elevation $\alpha$. This maximum height is largest when $\alpha = \frac{\pi}{2}$. In that case, $\sin\alpha = 1$ and the maximum height is $\dfrac{v_0^2}{2g}$.

(b) Let $R = v_0^2/g$. We are asked to consider the parabola $x^2 + 2Ry - R^2 = 0$ which can be rewritten as $y = -\dfrac{1}{2R}x^2 + \dfrac{R}{2}$. The points on or inside this parabola are those for which $-R \le x \le R$ and $0 \le y \le \dfrac{-1}{2R}x^2 + \dfrac{R}{2}$. When the projectile is fired at angle of elevation $\alpha$, the points $(x, y)$ along its path satisfy the relations $x = (v_0\cos\alpha)\,t$ and $y = (v_0\sin\alpha)t - \frac{1}{2}gt^2$, where $0 \le t \le (2v_0\sin\alpha)/g$ (as in Example 14.4.5 [ET 13.4.5]). Thus $|x| \le \left|v_0\cos\alpha\left(\dfrac{2v_0\sin\alpha}{g}\right)\right| = \left|\dfrac{v_0^2}{g}\sin 2\alpha\right| \le \left|\dfrac{v_0^2}{g}\right| = |R|$. This shows that $-R \le x \le R$.

For $t$ in the specified range, we also have $y = t\left(v_0\sin\alpha - \frac{1}{2}gt\right) = \frac{1}{2}gt\left(\dfrac{2v_0\sin\alpha}{g} - t\right) \ge 0$ and

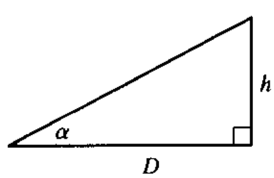
$$y = (v_0\sin\alpha)\frac{x}{v_0\cos\alpha} - \frac{g}{2}\left(\frac{x}{v_0\cos\alpha}\right)^2 = (\tan\alpha)\,x - \frac{g}{2v_0^2\cos^2\alpha}x^2 = -\frac{1}{2R\cos^2\alpha}x^2 + (\tan\alpha)\,x.$$

Thus

$$y-\left(\frac{-1}{2R}x^2+\frac{R}{2}\right)=\frac{-1}{2R\cos^2\alpha}x^2+\frac{1}{2R}x^2+(\tan\alpha)\,x-\frac{R}{2}$$
$$=\frac{x^2}{2R}\left(1-\frac{1}{\cos^2\alpha}\right)+(\tan\alpha)\,x-\frac{R}{2}=\frac{x^2(1-\sec^2\alpha)+2R\,(\tan\alpha)\,x-R^2}{2R}$$
$$=\frac{-(\tan^2\alpha)\,x^2+2R\,(\tan\alpha)\,x-R^2}{2R}=\frac{-\left[(\tan\alpha)\,x-R\right]^2}{2R}\leq 0$$

We have shown that every target that can be hit by the projectile lies on or inside the parabola $y=-\frac{1}{2R}x^2+\frac{R}{2}$. Now let $(a,b)$ be any point on or inside the parabola $y=-\frac{1}{2R}x^2+\frac{R}{2}$. Then $-R\leq a\leq R$ and $0\leq b\leq -\frac{1}{2R}a^2+\frac{R}{2}$. We seek an angle $\alpha$ such that $(a,b)$ lies in the path of the projectile; that is, we wish to find an angle $\alpha$ such that $b=-\frac{1}{2R\cos^2\alpha}a^2+(\tan\alpha)\,a$ or equivalently $b=\frac{-1}{2R}(\tan^2\alpha+1)a^2+(\tan\alpha)\,a$. Rearranging this equation we get $\frac{a^2}{2R}\tan^2\alpha-a\tan\alpha+\left(\frac{a^2}{2R}+b\right)=0$ or $a^2(\tan\alpha)^2-2aR(\tan\alpha)+(a^2+2bR)=0$ $(*)$. This quadratic equation for $\tan\alpha$ has real solutions exactly when the discriminant is nonnegative. Now $B^2-4AC\geq 0$ $\Leftrightarrow$ $(-2aR)^2-4a^2(a^2+2bR)\geq 0$ $\Leftrightarrow$ $4a^2(R^2-a^2-2bR)\geq 0$ $\Leftrightarrow$ $-a^2-2bR+R^2\geq 0$ $\Leftrightarrow$ $b\leq\frac{1}{2R}(R^2-a^2)$ $\Leftrightarrow$ $b\leq\frac{-1}{2R}a^2+\frac{R}{2}$. This condition is satisfied since $(a,b)$ is on or inside the parabola $y=-\frac{1}{2R}x^2+\frac{R}{2}$. It follows that $(a,b)$ lies in the path of the projectile when $\tan\alpha$ satisfies $(*)$, that is, when

$$\tan\alpha=\frac{2aR\pm\sqrt{4a^2(R^2-a^2-2bR)}}{2a^2}=\frac{R\pm\sqrt{R^2-2bR-a^2}}{a}.$$

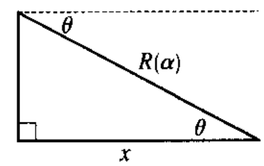
(c) 


If the gun is pointed at a target with height $h$ at a distance $D$ downrange, then $\tan\alpha=h/D$. When the projectile reaches a distance $D$ downrange (remember we are assuming that it doesn't hit the ground first), we have $D=x=(v_0\cos\alpha)t$, so $t=\frac{D}{v_0\cos\alpha}$ and $y=(v_0\sin\alpha)t-\frac{1}{2}gt^2=D\tan\alpha-\frac{gD^2}{2v_0^2\cos^2\alpha}$. Meanwhile, the target, whose $x$-coordinate is also $D$, has fallen from height $h$ to height $h-\frac{1}{2}gt^2=D\tan\alpha-\frac{gD^2}{2v_0^2\cos^2\alpha}$. Thus the projectile hits the target.

**4.** (a) As in Problem 3, $\mathbf{r}(t)=(v_0\cos\alpha)t\,\mathbf{i}+\left[(v_0\sin\alpha)t-\frac{1}{2}gt^2\right]\mathbf{j}$, so $x=(v_0\cos\alpha)t$ and $y=(v_0\sin\alpha)t-\frac{1}{2}gt^2$. The difference here is that the projectile travels until it reaches a point where $x>0$ and $y=-(\tan\theta)x$. (Here $0\leq\theta\leq\frac{\pi}{2}$.) From the parametric equations, we obtain $t=\frac{x}{v_0\cos\alpha}$ and $y=\frac{(v_0\sin\alpha)x}{v_0\cos\alpha}-\frac{gx^2}{2v_0^2\cos^2\alpha}=(\tan\alpha)x-\frac{gx^2}{2v_0^2\cos^2\alpha}$. Thus the projectile hits the inclined plane at the point where $(\tan\alpha)x-\frac{gx^2}{2v_0^2\cos^2\alpha}=-(\tan\theta)x$. Since $\frac{gx^2}{2v_0^2\cos^2\alpha}=(\tan\alpha+\tan\theta)x$ and $x>0$, we must have $\frac{gx}{2v_0^2\cos^2\alpha}=\tan\alpha+\tan\theta$. It follows that $x=\frac{2v_0^2\cos^2\alpha}{g}(\tan\alpha+\tan\theta)$ and

$t = \dfrac{x}{v_0 \cos\alpha} = \dfrac{2v_0\cos\alpha}{g}(\tan\alpha + \tan\theta)$. This means that the parametric equations are defined for $t$ in the interval $\left[0, \dfrac{2v_0\cos\alpha}{g}(\tan\alpha + \tan\theta)\right]$.

(b) The downhill range (that is, the distance to the projectile's landing point as measured along the inclined plane) is $R(\alpha) = x\sec\theta$, where $x$ is the coordinate of the landing point calculated in part (a). Thus




$$R(\alpha) = \frac{2v_0^2\cos^2\alpha}{g}(\tan\alpha + \tan\theta)\sec\theta = \frac{2v_0^2}{g}\left(\frac{\sin\alpha\cos\alpha}{\cos\theta} + \frac{\cos^2\alpha\sin\theta}{\cos^2\theta}\right)$$

$$= \frac{2v_0^2\cos\alpha}{g\cos^2\theta}(\sin\alpha\cos\theta + \cos\alpha\sin\theta) = \frac{2v_0^2\cos\alpha\sin(\alpha+\theta)}{g\cos^2\theta}$$

$R(\alpha)$ is maximized when

$$0 = R'(\alpha) = \frac{2v_0^2}{g\cos^2\theta}[-\sin\alpha\sin(\alpha+\theta) + \cos\alpha\cos(\alpha+\theta)]$$

$$= \frac{2v_0^2}{g\cos^2\theta}\cos[(\alpha+\theta)+\alpha] = \frac{2v_0^2\cos(2\alpha+\theta)}{g\cos^2\theta}$$

This condition implies that $\cos(2\alpha+\theta) = 0 \;\Rightarrow\; 2\alpha + \theta = \frac{\pi}{2} \;\Rightarrow\; \alpha = \frac{1}{2}\left(\frac{\pi}{2} - \theta\right)$.

(c) The solution is similar to the solutions to parts (a) and (b). This time the projectile travels until it reaches a point where $x > 0$ and $y = (\tan\theta)x$. Since $\tan\theta = -\tan(-\theta)$, we obtain the solution from the previous one by replacing $\theta$ with $-\theta$. The desired angle is $\alpha = \frac{1}{2}\left(\frac{\pi}{2} + \theta\right)$.

(d) As observed in part (c), firing the projectile up an inclined plane with angle of inclination $\theta$ involves the same equations as in parts (a) and (b) but with $\theta$ replaced by $-\theta$. So if $R$ is the distance up an inclined plane, we know from part (b) that $R = \dfrac{2v_0^2\cos\alpha\sin(\alpha-\theta)}{g\cos^2(-\theta)} \;\Rightarrow\; v_0^2 = \dfrac{Rg\cos^2\theta}{2\cos\alpha\sin(\alpha-\theta)}$. $v_0^2$ is minimized (and hence $v_0$ is minimized) with respect to $\alpha$ when

$$0 = \frac{d}{d\alpha}(v_0^2) = \frac{Rg\cos^2\theta}{2}\cdot\frac{-(\cos\alpha\cos(\alpha-\theta) - \sin\alpha\sin(\alpha-\theta))}{[\cos\alpha\sin(\alpha-\theta)]^2}$$

$$= \frac{-Rg\cos^2\theta}{2}\cdot\frac{\cos[\alpha+(\alpha-\theta)]}{[\cos\alpha\sin(\alpha-\theta)]^2} = \frac{-Rg\cos^2\theta}{2}\cdot\frac{\cos(2\alpha-\theta)}{[\cos\alpha\sin(\alpha-\theta)]^2}$$

Since $\theta < \alpha < \frac{\pi}{2}$, this implies $\cos(2\alpha - \theta) = 0 \;\Leftrightarrow\; 2\alpha - \theta = \frac{\pi}{2} \;\Rightarrow\; \alpha = \frac{1}{2}\left(\frac{\pi}{2} + \theta\right)$. Thus the initial speed, and hence the energy required, is minimized for $\alpha = \frac{1}{2}\left(\frac{\pi}{2} + \theta\right)$.

**5.** (a) $m\dfrac{d^2\mathbf{R}}{dt^2} = -mg\,\mathbf{j} - k\dfrac{d\mathbf{R}}{dt} \;\Rightarrow\; \dfrac{d}{dt}\left(m\dfrac{d\mathbf{R}}{dt} + k\mathbf{R} + mgt\,\mathbf{j}\right) = \mathbf{0} \;\Rightarrow\; m\dfrac{d\mathbf{R}}{dt} + k\mathbf{R} + mgt\,\mathbf{j} = \mathbf{c}$ ($\mathbf{c}$ is a constant vector in the $xy$-plane). At $t = 0$, this says that $m\,\mathbf{v}(0) + k\,\mathbf{R}(0) = \mathbf{c}$. Since $\mathbf{v}(0) = \mathbf{v}_0$ and $\mathbf{R}(0) = \mathbf{0}$, we have $\mathbf{c} = m\mathbf{v}_0$. Therefore $\dfrac{d\mathbf{R}}{dt} + \dfrac{k}{m}\mathbf{R} + gt\,\mathbf{j} = \mathbf{v}_0$, or $\dfrac{d\mathbf{R}}{dt} + \dfrac{k}{m}\mathbf{R} = \mathbf{v}_0 - gt\,\mathbf{j}$.

(b) Multiplying by $e^{(k/m)t}$ gives $e^{(k/m)t}\dfrac{d\mathbf{R}}{dt} + \dfrac{k}{m}e^{(k/m)t}\mathbf{R} = e^{(k/m)t}\mathbf{v}_0 - gte^{(k/m)t}\,\mathbf{j}$ or $\dfrac{d}{dt}\left(e^{(k/m)t}\mathbf{R}\right) = e^{(k/m)t}\mathbf{v}_0 - gte^{(k/m)t}\,\mathbf{j}$. Integrating gives

$$e^{(k/m)t}\,\mathbf{R} = \frac{m}{k}\,e^{(k/m)t}\,\mathbf{v}_0 - \left[\frac{mg}{k}\,te^{(k/m)t} - \frac{m^2g}{k^2}\,e^{(k/m)t}\right]\mathbf{j} + \mathbf{b} \text{ for some constant vector } \mathbf{b}.$$

Setting $t = 0$ yields the relation $\mathbf{R}(0) = \dfrac{m}{k}\,\mathbf{v}_0 + \dfrac{m^2g}{k^2}\,\mathbf{j} + \mathbf{b}$, so $\mathbf{b} = -\dfrac{m}{k}\,\mathbf{v}_0 - \dfrac{m^2g}{k^2}\,\mathbf{j}$. Thus

$$e^{(k/m)t}\,\mathbf{R} = \frac{m}{k}\left[e^{(k/m)t} - 1\right]\mathbf{v}_0 - \left[\frac{mg}{k}te^{(k/m)t} - \frac{m^2g}{k^2}\left(e^{(k/m)t} - 1\right)\right]\mathbf{j} \text{ and}$$

$$\mathbf{R}(t) = \frac{m}{k}\left[1 - e^{-kt/m}\right]\mathbf{v}_0 + \frac{mg}{k}\left[\frac{m}{k}(1 - e^{-kt/m}) - t\right]\mathbf{j}.$$

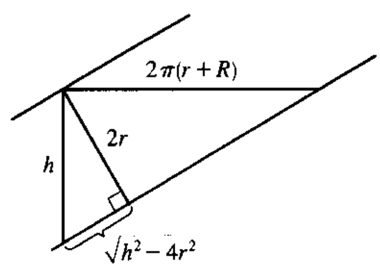
**6.** By the Fundamental Theorem of Calculus, $\mathbf{r}'(t) = \langle \sin(\pi t^2/2), \cos(\pi t^2/2)\rangle$, $|\mathbf{r}'(t)| = 1$ and so $\mathbf{T}(t) = \mathbf{r}'(t)$. Thus $\mathbf{T}'(t) = \pi t\,\langle \sin(\pi t^2/2), \cos(\pi t^2/2)\rangle$ and the curvature is $\kappa = |\mathbf{T}'(t)| = \sqrt{(\pi t)^2(1)} = \pi\,|t|$.

**7.** (a) $\mathbf{a} = -g\,\mathbf{j} \;\Rightarrow\; \mathbf{v} = \mathbf{v}_0 - gt\,\mathbf{j} = 2\,\mathbf{i} - gt\,\mathbf{j} \;\Rightarrow\; \mathbf{s} = \mathbf{s}_0 + 2t\,\mathbf{i} - \frac{1}{2}gt^2\,\mathbf{j} = 3.5\,\mathbf{j} + 2t\,\mathbf{i} - \frac{1}{2}gt^2\,\mathbf{j} \;\Rightarrow$ $\mathbf{s} = 2t\,\mathbf{i} + \left(3.5 - \frac{1}{2}gt^2\right)\mathbf{j}$. Therefore $y = 0$ when $t = \sqrt{7/g}$ seconds. At that instant, the ball is $2\sqrt{7/g} \approx 0.94$ ft to the right of the table top. Its coordinates (relative to an origin on the floor directly under the table's edge) are $(0.94, 0)$. At impact, the velocity is $\mathbf{v} = 2\,\mathbf{i} - \sqrt{7g}\,\mathbf{j}$, so the speed is $|\mathbf{v}| = \sqrt{4 + 7g} \approx 15$ ft/s.

(b) The slope of the curve when $t = \sqrt{\dfrac{7}{g}}$ is $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{-gt}{2} = \dfrac{-g\sqrt{7/g}}{2} = \dfrac{-\sqrt{7g}}{2}$. Thus $\cot\theta = \dfrac{\sqrt{7g}}{2}$ and $\theta \approx 7.6\,^\circ$.

(c) From (a), $|\mathbf{v}| = \sqrt{4 + 7g}$. So the ball rebounds with speed $0.8\sqrt{4 + 7g} \approx 12.08$ ft/s at angle of inclination $90\,^\circ - \theta \approx 82.3886\,^\circ$. By Example 14.4.5 [ET 13.4.5], the horizontal distance traveled between bounces is $d = \dfrac{v_0^2 \sin 2\alpha}{g}$, where $v_0 \approx 12.08$ ft/s and $\alpha \approx 82.3886\,^\circ$. Therefore, $d \approx 1.197$ ft. So the ball strikes the floor at about $2\sqrt{7/g} + 1.197 \approx 2.13$ ft to the right of the table's edge.

**8.** As the cable is wrapped around the spool, think of the top or bottom of the cable forming a helix of radius $R + r$. Let $h$ be the vertical distance between coils. Then, from similar triangles,




$$\frac{2r}{\sqrt{h^2 - 4r^2}} = \frac{2\pi(r + R)}{h} \;\Rightarrow\; h^2r^2 = \pi^2(r + R)^2(h^2 - 4r^2)$$

$$\Rightarrow\quad h = \frac{2\pi r(r + R)}{\sqrt{\pi^2(r + R)^2 - r^2}}.$$

If we parametrize the helix by $x(t) = (R + r)\cos t$, $y(t) = (R + r)\sin t$, then we must have $z(t) = [h/(2\pi)]t$. The length of one complete cycle is

$$\ell = \int_0^{2\pi} \sqrt{[x'(t)]^2 + [y'(t)]^2 + [z'(t)]^2}\,dt = \int_0^{2\pi} \sqrt{(R + r)^2 + \left(\frac{h}{2\pi}\right)^2}\,dt = 2\pi\sqrt{(R + r)^2 + \left(\frac{h}{2\pi}\right)^2}$$

$$= 2\pi\sqrt{(R + r)^2 + \frac{r^2(R + r)^2}{\pi^2(R + r)^2 - r^2}} = 2\pi(R + r)\sqrt{1 + \frac{r^2}{\pi^2(R + r)^2 - r^2}} = \frac{2\pi^2(R + r)^2}{\sqrt{\pi^2(R + r)^2 - r^2}}$$

The number of complete cycles is $[\![L/\ell]\!]$, and so the shortest length along the spool is

$$h\left[\!\!\left[\frac{L}{\ell}\right]\!\!\right] = \frac{2\pi r(R + r)}{\sqrt{\pi^2(R + r)^2 - r^2}}\left[\!\!\left[\frac{L\sqrt{\pi^2(R + r)^2 - r^2}}{2\pi^2(R + r)^2}\right]\!\!\right]$$