Chapter 15

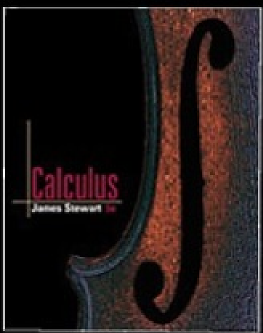
Adapted from the

Complete Solutions Manual

for

James Stewart's

Calculus – 5th Edition



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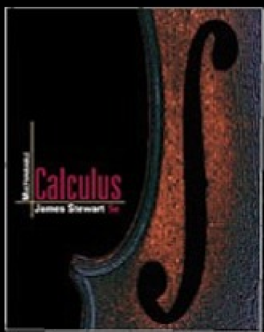

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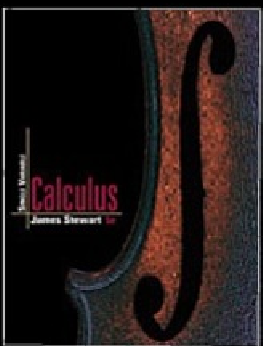

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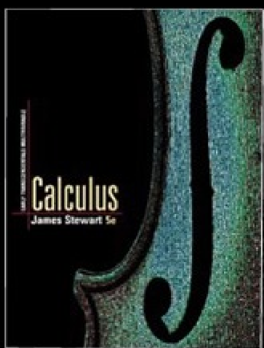

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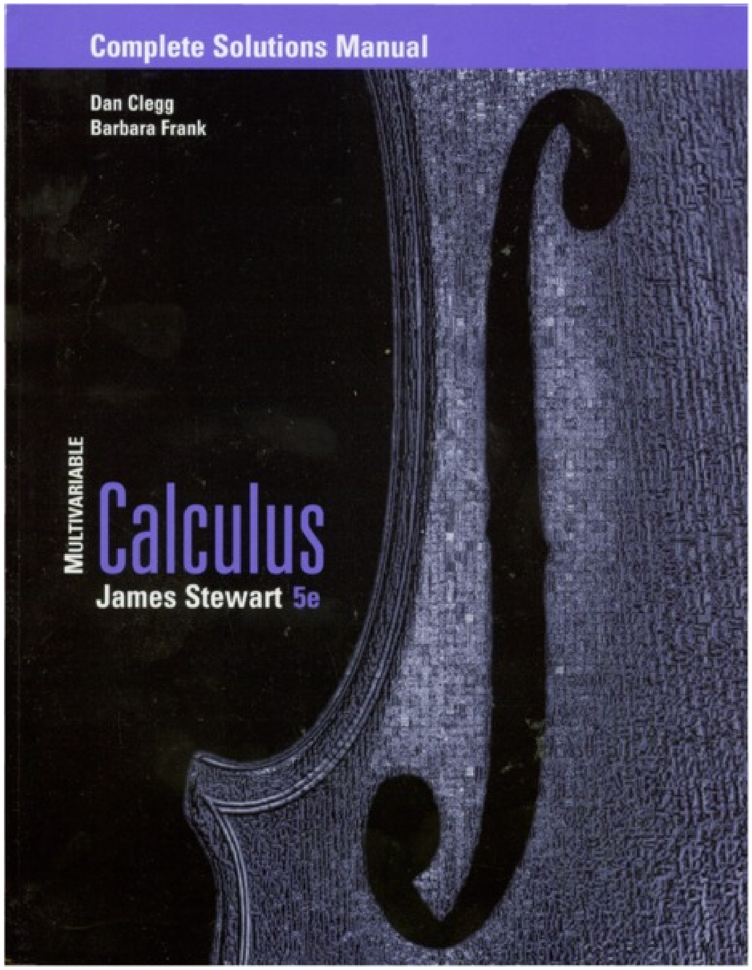


15 ☐ PARTIAL DERIVATIVES ☐ ET 14

15.1 Functions of Several Variables ET 14.1

1. (a) From Table 1, $f(-15, 40) = -27$, which means that if the temperature is $-15°$C and the wind speed is 40 km/h, then the air would feel equivalent to approximately $-27°$C without wind.

(b) The question is asking: when the temperature is $-20°$C, what wind speed gives a wind-chill index of $-30°$C? From Table 1, the speed is 20 km/h.

(c) The question is asking: when the wind speed is 20 km/h, what temperature gives a wind-chill index of $-49°$C? From Table 1, the temperature is $-35°$C.

(d) The function $W = f(-5, v)$ means that we fix T at -5 and allow v to vary, resulting in a function of one variable. In other words, the function gives wind-chill index values for different wind speeds when the temperature is $-5°$C. From Table 1 (look at the row corresponding to $T = -5$), the function decreases and appears to approach a constant value as v increases.

(e) The function $W = f(T, 50)$ means that we fix v at 50 and allow T to vary, again giving a function of one variable. In other words, the function gives wind-chill index values for different temperatures when the wind speed is 50 km/h . From Table 1 (look at the column corresponding to $v = 50$), the function increases almost linearly as T increases.

2. (a) From the table, $f(95, 70) = 124$, which means that when the actual temperature is $95°$F and the relative humidity is 70%, the perceived air temperature is approximately $124°$F.

(b) Looking at the row corresponding to $T = 90$, we see that $f(90, h) = 100$ when $h = 60$.

(c) Looking at the column corresponding to $h = 50$, we see that $f(T, 50) = 88$ when $T = 85$.

(d) $I = f(80, h)$ means that T is fixed at 80 and h is allowed to vary, resulting in a function of h that gives the humidex values for different relative humidities when the actual temperature is $80°$F. Similarly, $I = f(100, h)$ is a function of one variable that gives the humidex values for different relative humidities when the actual temperature is $100°$F. Looking at the rows of the table corresponding to $T = 80$ and $T = 100$, we see that $f(80, h)$ increases at a relatively constant rate of approximately $1°$F per 10% relative humidity, while $f(100, h)$ increases more quickly (at first with an average rate of change of $5°$F per 10% relative humidity) and at an increasing rate (approximately $12°$F per 10% relative humidity for larger values of h).

3. If the amounts of labor and capital are both doubled, we replace L, K in the function with $2L, 2K$, giving

$$P(2L, 2K) = 1.01(2L)^{0.75}(2K)^{0.25} = 1.01(2^{0.75})(2^{0.25})L^{0.75}K^{0.25} = (2^1)1.01L^{0.75}K^{0.25}$$
$$= 2P(L, K)$$

Thus, the production is doubled. It is also true for the general case $P(L, K) = bL^{\alpha}K^{1-\alpha}$:

$$P(2L, 2K) = b(2L)^{\alpha}(2K)^{1-\alpha} = b(2^{\alpha})(2^{1-\alpha})L^{\alpha}K^{1-\alpha} = (2^{\alpha+1-\alpha})bL^{\alpha}K^{1-\alpha} = 2P(L, K).$$

4. We compare the values for the wind-chill index given by Table 1 with those given by the model function:

Modeled Wind-Chill Index Values $W(T, v)$

Wind Speed (km/h)

Actual temperature (°C): T \ v	5	10	15	20	25	30	40	50	60	70	80
5	4.08	2.66	1.74	1.07	0.52	0.05	−0.71	−1.33	−1.85	−2.30	−2.70
0	−1.59	−3.31	−4.42	−5.24	−5.91	−6.47	−7.40	−8.14	−8.77	−9.32	−9.80
−5	−7.26	−9.29	−10.58	−11.55	−12.34	−13.00	−14.08	−14.96	−15.70	−16.34	−16.91
−10	−12.93	−15.26	−16.75	−17.86	−18.76	−19.52	−20.77	−21.77	−22.62	−23.36	−24.01
−15	−18.61	−21.23	−22.91	−24.17	−25.19	−26.04	−27.45	−28.59	−29.54	−30.38	−31.11
−20	−24.28	−27.21	−29.08	−30.48	−31.61	−32.57	−34.13	−35.40	−36.47	−37.40	−38.22
−25	−29.95	−33.18	−35.24	−36.79	−38.04	−39.09	−40.82	−42.22	−43.39	−44.42	−45.32
−30	−35.62	−39.15	−41.41	−43.10	−44.46	−45.62	−47.50	−49.03	−50.32	−51.44	−52.43
−35	−41.30	−45.13	−47.57	−49.41	−50.89	−52.14	−54.19	−55.84	−57.24	−58.46	−59.53
−40	−46.97	−51.10	−53.74	−55.72	−57.31	−58.66	−60.87	−62.66	−64.17	−65.48	−66.64

The values given by the function appear to be fairly close (within 0.5) to the values in Table 1.

5. (a) According to the table, $f(40, 15) = 25$, which means that if a 40-knot wind has been blowing in the open sea for 15 hours, it will create waves with estimated heights of 25 feet.

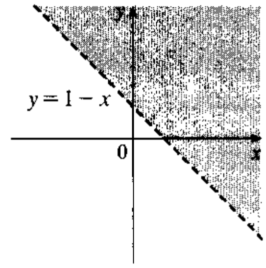
(b) $h = f(30, t)$ means we fix v at 30 and allow t to vary, resulting in a function of one variable. Thus here, $h = f(30, t)$ gives the wave heights produced by 30-knot winds blowing for t hours. From the table (look at the row corresponding to $v = 30$), the function increases but at a declining rate as t increases. In fact, the function values appear to be approaching a limiting value of approximately 19, which suggests that 30-knot winds cannot produce waves higher than about 19 feet.

(c) $h = f(v, 30)$ means we fix t at 30, again giving a function of one variable. So, $h = f(v, 30)$ gives the wave heights produced by winds of speed v blowing for 30 hours. From the table (look at the column corresponding to $t = 30$), the function appears to increase at an increasing rate, with no apparent limiting value. This suggests that faster winds (lasting 30 hours) always create higher waves.

6. (a) $f(1,1) = \ln(1+1-1) = \ln 1 = 0$

(b) $f(e,1) = \ln(e+1-1) = \ln e = 1$

(c) $\ln(x+y-1)$ is defined only when $x+y-1>0$, that is, $y > 1-x$. So the domain of f is $\{(x,y) \mid y > 1-x\}$.



(d) Since $\ln(x+y-1)$ can be any real number, the range is $\mathbb{R}$.

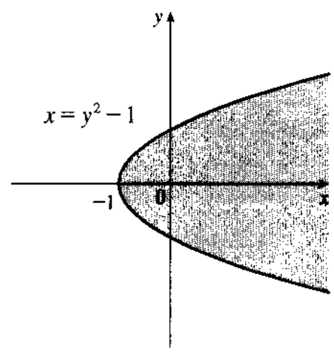
7. (a) $f(2,0) = 2^2 e^{3(2)(0)} = 4(1) = 4$

(b) Since both x^2 and the exponential function are defined everywhere, $x^2 e^{3xy}$ is defined for all choices of values for x and y. Thus, the domain of f is $\mathbb{R}^2$.

(c) Because the range of $g(x,y) = 3xy$ is $\mathbb{R}$, and the range of e^x is $(0,\infty)$, the range of $e^{g(x,y)} = e^{3xy}$ is $(0,\infty)$. The range of x^2 is $[0,\infty)$, so the range of the product $x^2 e^{3xy}$ is $[0,\infty)$.

8. $\sqrt{1+x-y^2}$ is defined only when $1+x-y^2 \geq 0 \;\Rightarrow\; x \geq y^2-1$, so the domain of f is $\{(x,y) \mid x \geq y^2-1\}$, all those points on or to the right of the parabola $x = y^2-1$.

The range of f is $[0,\infty)$.



9. (a) $f(2,-1,6) = e^{\sqrt{6-2^2-(-1)^2}} = e^{\sqrt{1}} = e.$

(b) $e^{\sqrt{z-x^2-y^2}}$ is defined when $z-x^2-y^2 \geq 0 \;\Rightarrow\; z \geq x^2+y^2$. Thus the domain of f is $\{(x,y,z) \mid z \geq x^2+y^2\}$.

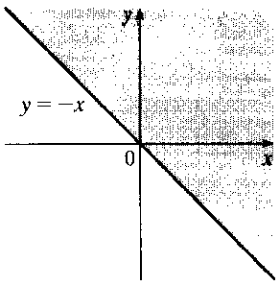
(c) Since $\sqrt{z-x^2-y^2} \geq 0$, we have $e^{\sqrt{z-x^2-y^2}} \geq 1$. Thus the range of f is $[1,\infty)$.

10. (a) $g(2,-2,4) = \ln\left(25-2^2-(-2)^2-4^2\right) = \ln 1 = 0.$

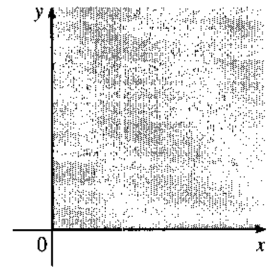
(b) For the logarithmic function to be defined, we need $25-x^2-y^2-z^2>0$. Thus the domain of g is $\{(x,y,z) \mid x^2+y^2+z^2<25\}$, the interior of the sphere $x^2+y^2+z^2=25$.

(c) Since $0 < 25-x^2-y^2-z^2 \leq 25$ for (x,y,z) in the domain of g, $\ln\left(25-x^2-y^2-z^2\right) \leq \ln 25$. Thus the range of g is $(-\infty, \ln 25]$.

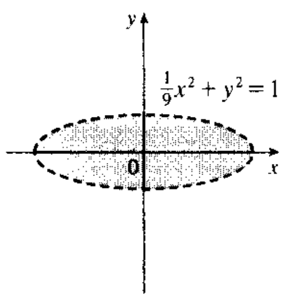
11. $\sqrt{x+y}$ is defined only when $x+y \geq 0$, or $y \geq -x$. So the domain of f is $\{(x,y) \mid y \geq -x\}$.



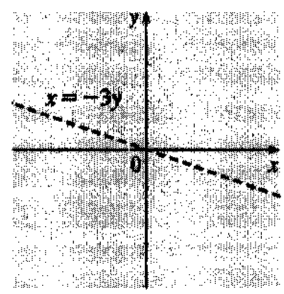
12. We need $x \geq 0$ and $y \geq 0$, so $D = \{(x,y) \mid x \geq 0 \text{ and } y \geq 0\}$, the first quadrant.



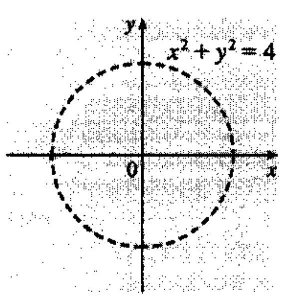
13. $\ln\left(9-x^2-9y^2\right)$ is defined only when $9-x^2-9y^2>0$, or $\frac{1}{9}x^2+y^2<1$. So the domain of f is $\left\{(x,y) \mid \frac{1}{9}x^2+y^2<1\right\}$, the interior of an ellipse.



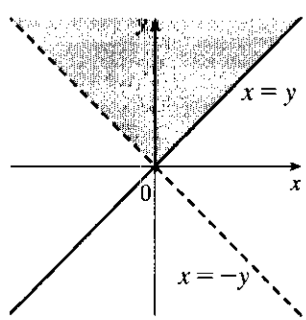
14. $\dfrac{x-3y}{x+3y}$ is defined only when $x+3y \neq 0$, or $x \neq -3y$. So the domain of f is $\{(x,y) \mid x \neq -3y\}$.



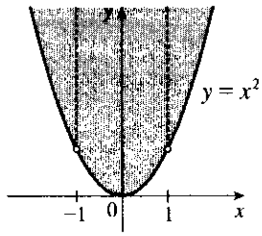
15. $\dfrac{3x+5y}{x^2+y^2-4}$ is defined only when $x^2+y^2-4 \neq 0$, or $x^2+y^2 \neq 4$. So the domain of f is $\left\{(x,y) \mid x^2+y^2 \neq 4\right\}$.



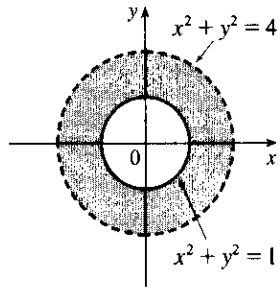
16. We need $y-x \geq 0$ or $y \geq x$ and $y+x>0$ or $x>-y$. Thus $D = \{(x,y) \mid -y<x \leq y, y>0\}$.



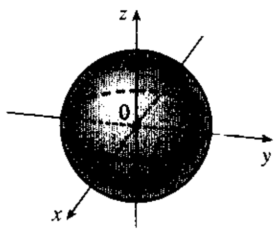
17. $\sqrt{y-x^2}$ is defined only when $y - x^2 \geq 0$, or $y \geq x^2$. In addition, f is not defined if $1 - x^2 = 0$ $\Rightarrow$ $x = \pm 1$. Thus the domain of f is $\{(x,y) \mid y \geq x^2, x \neq \pm 1\}$.



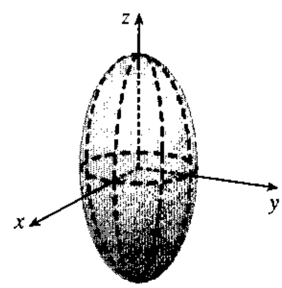
18. f is defined only when $x^2 + y^2 - 1 \geq 0$ $\Rightarrow$ $x^2 + y^2 \geq 1$ and $4 - x^2 - y^2 > 0$ $\Rightarrow$ $x^2 + y^2 < 4$. Thus $D = \{(x,y) \mid 1 \leq x^2 + y^2 < 4\}$.



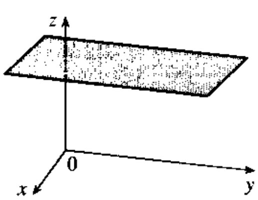
19. We need $1 - x^2 - y^2 - z^2 \geq 0$ or $x^2 + y^2 + z^2 \leq 1$, so $D = \{(x,y,z) \mid x^2 + y^2 + z^2 \leq 1\}$ (the points inside or on the sphere of radius 1, center the origin).



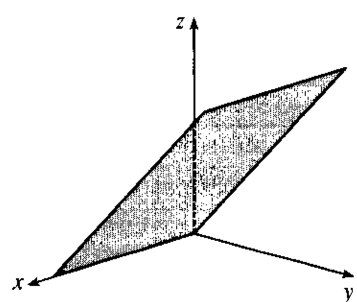
20. f is defined only when $16 - 4x^2 - 4y^2 - z^2 > 0$ $\Rightarrow$ $\dfrac{x^2}{4} + \dfrac{y^2}{4} + \dfrac{z^2}{16} < 1$. Thus, $D = \left\{(x,y,z) \,\middle|\, \dfrac{x^2}{4} + \dfrac{y^2}{4} + \dfrac{z^2}{16} < 1\right\}$, that is, the points inside the ellipsoid $\dfrac{x^2}{4} + \dfrac{y^2}{4} + \dfrac{z^2}{16} = 1$.



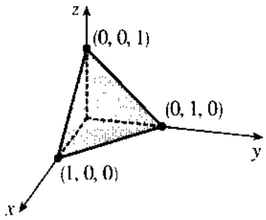
21. $z = 3$, a horizontal plane through the point $(0, 0, 3)$.



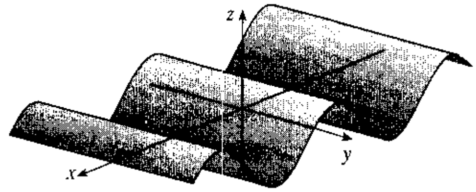
22. $z = y$, a plane which intersects the yz-plane in the line $z = y$, $x = 0$. The portion of this plane that lies in the first octant is shown.



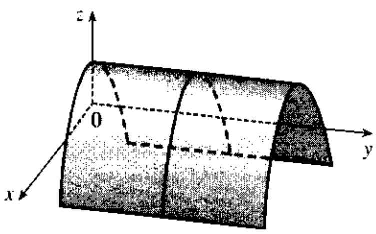
23. $z = 1 - x - y$ or $x + y + z = 1$, a plane with intercepts 1, 1, and 1.



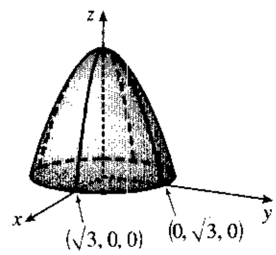
24. $z = \cos x$, a "wave."



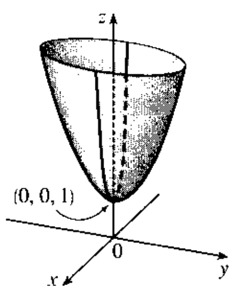
25. $z = 1 - x^2$, a parabolic cylinder.



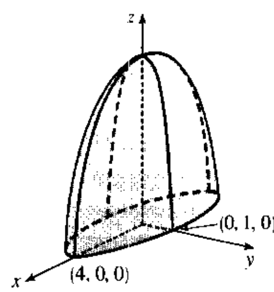
26. $z = 3 - x^2 - y^2$, a circular paraboloid with vertex at $(0, 0, 3)$.



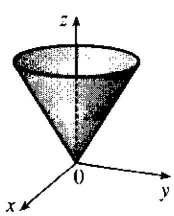
27. $z = 4x^2 + y^2 + 1$, an elliptic paraboloid with vertex at $(0, 0, 1)$.



28. $z = \sqrt{16 - x^2 - 16y^2}$ so $z \geq 0$ and $z^2 + x^2 + 16y^2 = 16$, the top half of an ellipsoid.



29. $z = \sqrt{x^2 + y^2}$ so $x^2 + y^2 = z^2$ and $z \geq 0$, the top half of a right circular cone.



30. All six graphs have different traces in the planes $x = 0$ and $y = 0$, so we investigate these for each function.

(a) $f(x, y) = |x| + |y|$. The trace in $x = 0$ is $z = |y|$, and in $y = 0$ is $z = |x|$, so it must be graph VI.

(b) $f(x, y) = |xy|$. The trace in $x = 0$ is $z = 0$, and in $y = 0$ is $z = 0$, so it must be graph V.

(c) $f(x, y) = \dfrac{1}{1 + x^2 + y^2}$. The trace in $x = 0$ is $z = \dfrac{1}{1 + y^2}$, and in $y = 0$ is $z = \dfrac{1}{1 + x^2}$. In addition, we can see that f is close to 0 for large values of x and y, so this is graph I.

(d) $f(x, y) = (x^2 - y^2)^2$. The trace in $x = 0$ is $z = y^4$, and in $y = 0$ is $z = x^4$. Both graph II and graph IV seem plausible; notice the trace in $z = 0$ is $0 = (x^2 - y^2)^2 \quad \Rightarrow \quad y = \pm x$, so it must be graph IV.

(e) $f(x, y) = (x - y)^2$. The trace in $x = 0$ is $z = y^2$, and in $y = 0$ is $z = x^2$. Both graph II and graph IV seem plausible; notice the trace in $z = 0$ is $0 = (x - y)^2 \quad \Rightarrow \quad y = x$, so it must be graph II.

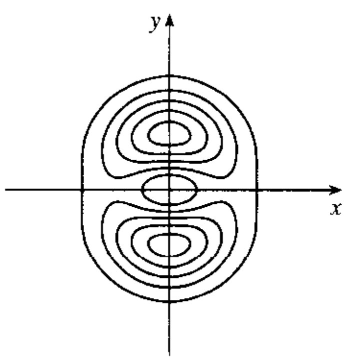
(f) $f(x, y) = \sin(|x| + |y|)$. The trace in $x = 0$ is $z = \sin|y|$, and in $y = 0$ is $z = \sin|x|$. In addition, notice that the oscillating nature of the graph is characteristic of trigonometric functions. So this is graph III.

31. The point $(-3, 3)$ lies between the level curves with z-values 50 and 60. Since the point is a little closer to the level curve with $z = 60$, we estimate that $f(-3, 3) \approx 56$. The point $(3, -2)$ appears to be just about halfway between the level curves with z-values 30 and 40, so we estimate $f(3, -2) \approx 35$. The graph rises as we approach the origin, gradually from above, steeply from below.

32. If we start at the origin and move along the x-axis, for example, the z-values of a cone centered at the origin increase at a constant rate, so we would expect its level curves to be equally spaced. A paraboloid with vertex the origin, on the other hand, has z-values which change slowly near the origin and more quickly as we move farther away. Thus, we would expect its level curves near the origin to be spaced more widely apart than those farther from the origin. Therefore contour map I must correspond to the paraboloid, and contour map II the cone.

33. Near A, the level curves are very close together, indicating that the terrain is quite steep. At B, the level curves are much farther apart, so we would expect the terrain to be much less steep than near A, perhaps almost flat.

34.



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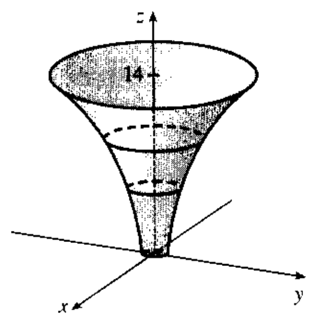
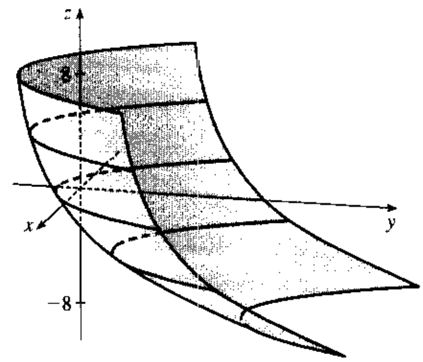
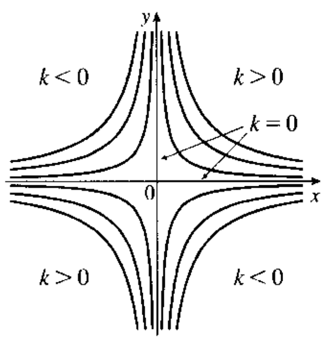


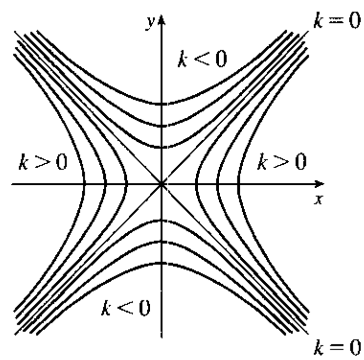
36.



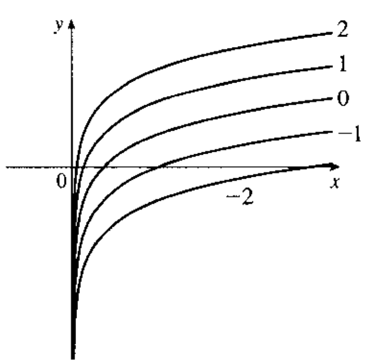
37. The level curves are $xy = k$. For $k = 0$ the curves are the coordinate axis; if $k > 0$, they are hyperbolas in the first and third quadrants; if $k < 0$, they are hyperbolas in the second and fourth quadrants.



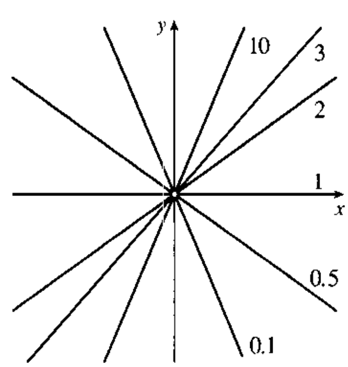
38. The level curves are $k = x^2 - y^2$. When $k = 0$, these are the lines $y = \pm x$. When $k > 0$, the curves are hyperbolas with axis the x-axis and when $k < 0$, they are hyperbolas with axis the y-axis.



39. The level curves are $y - \ln x = k$ or $y = \ln x + k$.

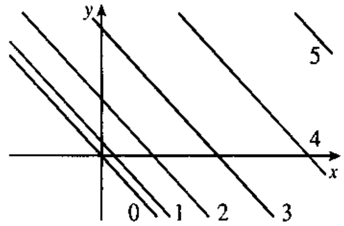


40. The level curves are $e^{y/x} = k$ or equivalently $y = x \ln k$ ($x \neq 0$), a family of lines with slope $\ln k$ ($k > 0$) without the origin.

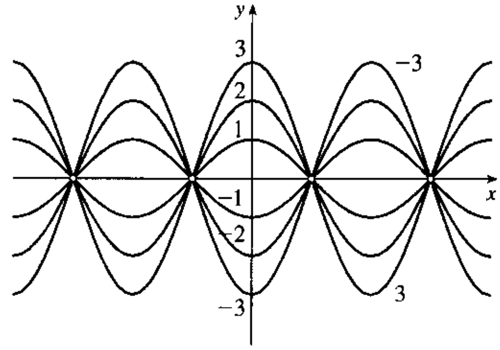


41. $k = \sqrt{x+y}$ or for $x + y \geq 0$, $k^2 = x + y$, or $y = -x + k^2$.

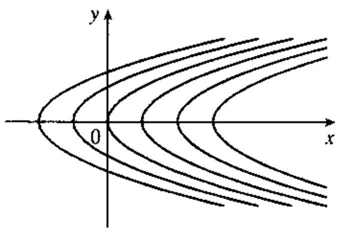
Note: $k \geq 0$ since $k = \sqrt{x+y}$.



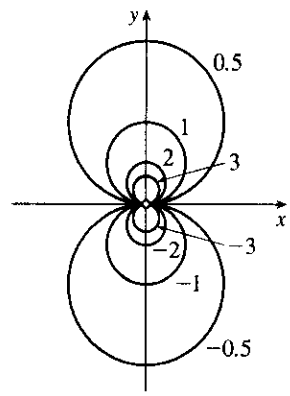
42. $k = y \sec x$ or $y = k \cos x$, $x \neq \frac{\pi}{2} + n\pi$ (n an integer).



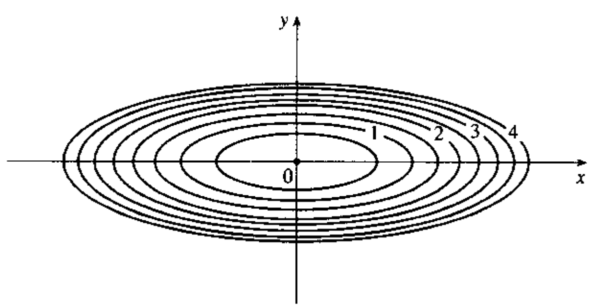
43. $k = x - y^2$, or $x - k = y^2$, a family of parabolas with vertex $(k, 0)$.

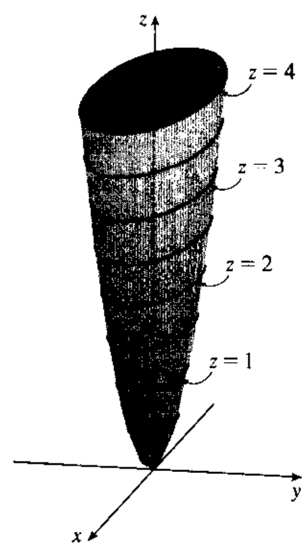


44. For $k \neq 0$ and $(x, y) \neq (0, 0)$, $k = \dfrac{y}{x^2 + y^2}$ $\Leftrightarrow$ $x^2 + y^2 - \dfrac{y}{k} = 0$ $\Leftrightarrow$ $x^2 + \left(y - \frac{1}{2k}\right)^2 = \dfrac{1}{4k^2}$, a family of circles with center $\left(0, \frac{1}{2k}\right)$ and radius $\frac{1}{2k}$ (without the origin). If $k = 0$, the level curve is the x-axis.



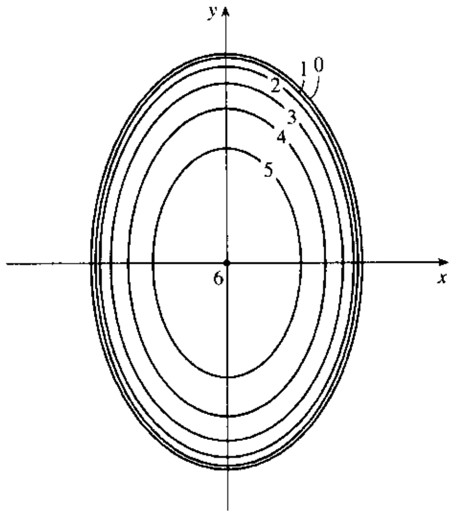
45. The contour map consists of the level curves $k = x^2 + 9y^2$, a family of ellipses with major axis the x-axis. (Or, if $k = 0$, the origin.) The graph of $f(x, y)$ is the surface $z = x^2 + 9y^2$, an elliptic paraboloid.



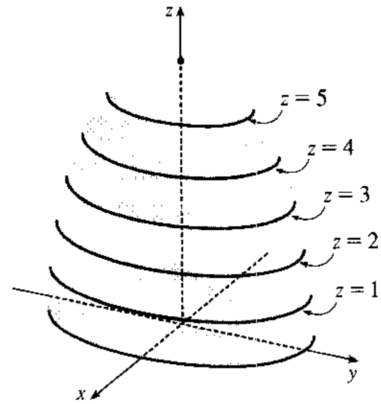


If we visualize lifting each ellipse $k = x^2 + 9y^2$ of the contour map to the plane $z = k$, we have horizontal traces that indicate the shape of the graph of f.

46.

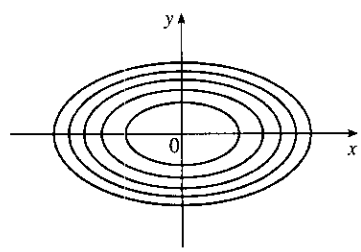


The contour map consists of the level curves $k = \sqrt{36 - 9x^2 - 4y^2}$ $\Rightarrow$ $9x^2 + 4y^2 = 36 - k^2$, $k \geq 0$, a family of ellipses with major axis the y-axis. (Or, if $k = 6$, the origin.)



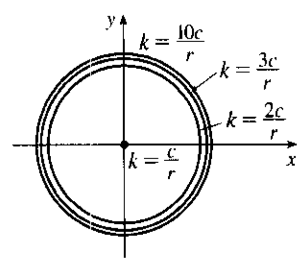
The graph of $f(x, y)$ is the surface $z = \sqrt{36 - 9x^2 - 4y^2}$, or equivalently the upper half of the ellipsoid $9x^2 + 4y^2 + z^2 = 36$. If we visualize lifting each ellipse $k = \sqrt{36 - 9x^2 - 4y^2}$ of the contour map to the plane $z = k$, we have horizontal traces that indicate the shape of the graph of f.

47. The isothermals are given by $k = 100/(1 + x^2 + 2y^2)$ or $x^2 + 2y^2 = (100 - k)/k$ $(0 < k \leq 100)$, a family of ellipses.

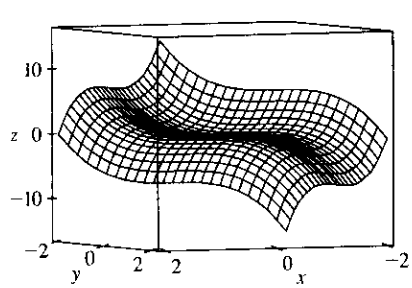


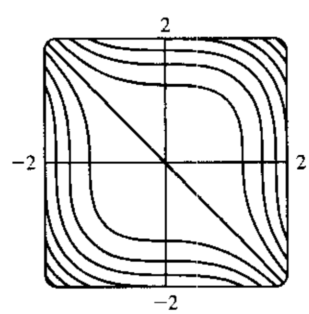
48. The equipotential curves are $k = \dfrac{c}{\sqrt{r^2 - x^2 - y^2}}$ or $x^2 + y^2 = r^2 - \left(\dfrac{c}{k}\right)^2$, a family of circles ($k \geq c/r$).

Note: As $k \to \infty$, the radius of the circle approaches r.



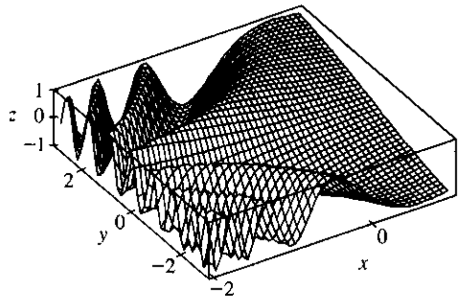
49. $f(x, y) = x^3 + y^3$

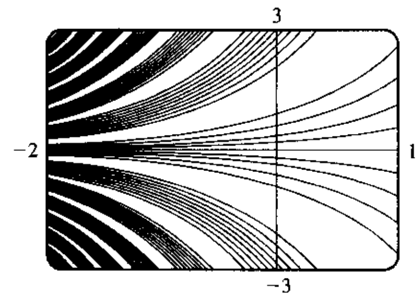




Note that the function is 0 along the line $y = -x$.

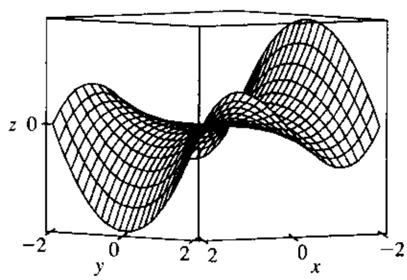
50. $f(x, y) = \sin(ye^{-x})$

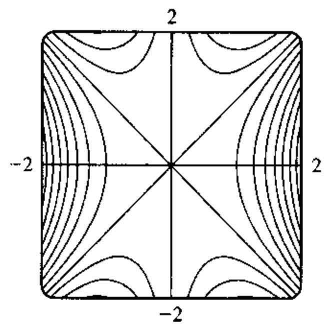




Cross-sections parallel to the yz-plane (such as the left-front trace in the first graph above) are sine-like curves. The periods of these curves decrease as x decreases.

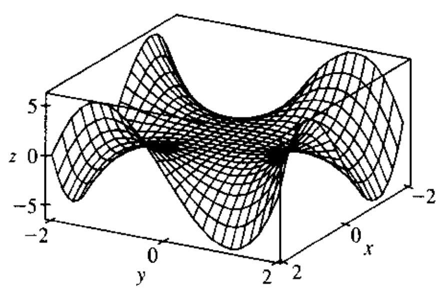
51. $f(x, y) = xy^2 - x^3$

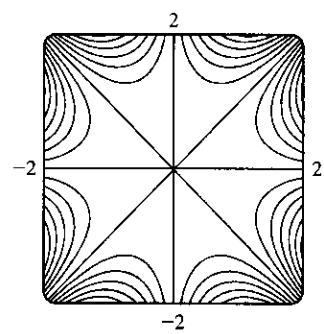




The traces parallel to the yz-plane (such as the left-front trace in the graph above) are parabolas; those parallel to the xz-plane (such as the right-front trace) are cubic curves. The surface is called a monkey saddle because a monkey sitting on the surface near the origin has places for both legs and tail to rest.

52. $f(x, y) = xy^3 - yx^3$





The traces parallel to either the yz-plane or the xz-plane are cubic curves.

53. (a) B *Reasons:* This function is constant on any circle centered at the origin, a description which matches only B and III.
(b) III

54. (a) C *Reasons:* This function is the same if x is interchanged with y, so its graph is symmetric about the plane $x = y$. Also, $z(0, 0) = 0$ and the values of z approach 0 as we use points farther from the origin. These conditions are satisfied only by C and II.
(b) II

55. (a) F *Reasons:* z increases without bound as we use points closer to the origin, a condition satisfied only
(b) V by F and V.

56. (a) A *Reasons:* Along the lines $y = \pm\frac{1}{\sqrt{3}}x$ and $x = 0$, this function is 0.
(b) VI

57. (a) D *Reasons:* This function is periodic in both x and y, with period 2π in each variable.
(b) IV

58. (a) E *Reasons:* This function is periodic along the x-axis, and increases as $|y|$ increases.
(b) I

59. $k = x + 3y + 5z$ is a family of parallel planes with normal vector $\langle 1, 3, 5\rangle$.

60. $k = x^2 + 3y^2 + 5z^2$ is a family of ellipsoids for $k > 0$ and the origin for $k = 0$.

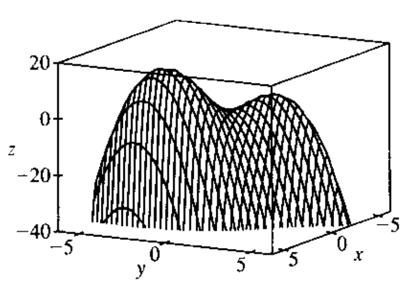
61. $k = x^2 - y^2 + z^2$ are the equations of the level surfaces. For $k = 0$, the surface is a right circular cone with vertex the origin and axis the y-axis. For $k > 0$, we have a family of hyperboloids of one sheet with axis the y-axis. For $k < 0$, we have a family of hyperboloids of two sheets with axis the y-axis.

62. $k = x^2 - y^2$ is a family of hyperbolic cylinders. The cross section of this family in the xy-plane has the same graph as the level curves in Exercise 38.

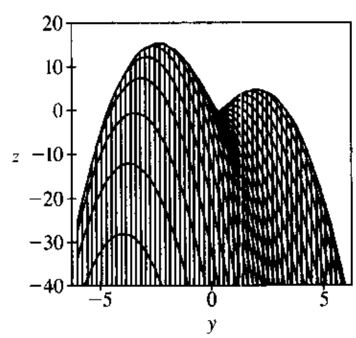
63. (a) The graph of g is the graph of f shifted upward 2 units.
(b) The graph of g is the graph of f stretched vertically by a factor of 2.
(c) The graph of g is the graph of f reflected about the xy-plane.
(d) The graph of $g(x, y) = -f(x, y) + 2$ is the graph of f reflected about the xy-plane and then shifted upward 2 units.

64. (a) The graph of g is the graph of f shifted 2 units in the positive x-direction.
(b) The graph of g is the graph of f shifted 2 units in the negative y-direction.
(c) The graph of g is the graph of f shifted 3 units in the negative x-direction and 4 units in the positive y-direction.

65. $f(x, y) = 3x - x^4 - 4y^2 - 10xy$



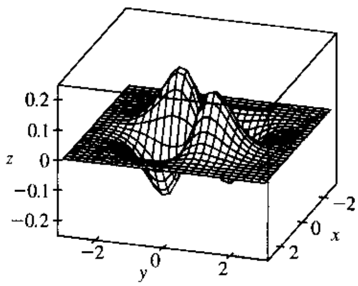
Three-dimensional view



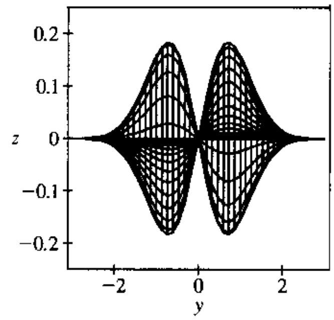
Front view

It does appear that the function has a maximum value, at the higher of the two "hilltops." From the front view graph, the maximum value appears to be approximately 15. Both hilltops could be considered local maximum points, as the values of f there are larger than at the neighboring points. There does not appear to be any local minimum point; although the valley shape between the two peaks looks like a minimum of some kind, some neighboring points have lower function values.

66. $f(x,y) = xye^{-x^2-y^2}$



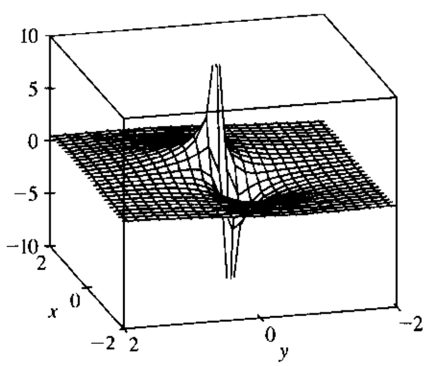
Three-dimensional view



Front view

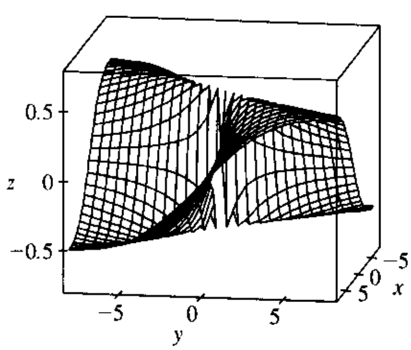
The function does have a maximum value, which it appears to achieve at two different points (the two "hilltops"). From the front view graph, we can estimate the maximum value to be approximately 0.18. These same two points can also be considered local maximum points. The two "valley bottoms" visible in the graph can be considered local minimum points, as all the neighboring points give greater values of f.

67.



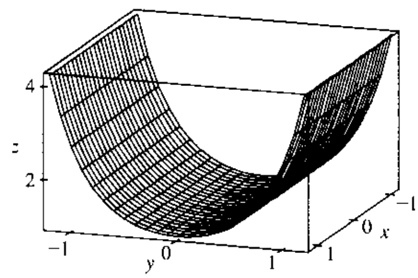
$f(x,y) = \dfrac{x+y}{x^2+y^2}$. As both x and y become large, the function values appear to approach 0, regardless of which direction is considered. As (x,y) approaches the origin, the graph exhibits asymptotic behavior. From some directions, $f(x,y) \to \infty$, while in others $f(x,y) \to -\infty$. (These are the vertical spikes visible in the graph.) If the graph is examined carefully, however, one can see that $f(x,y)$ approaches 0 along the line $y = -x$.

68.



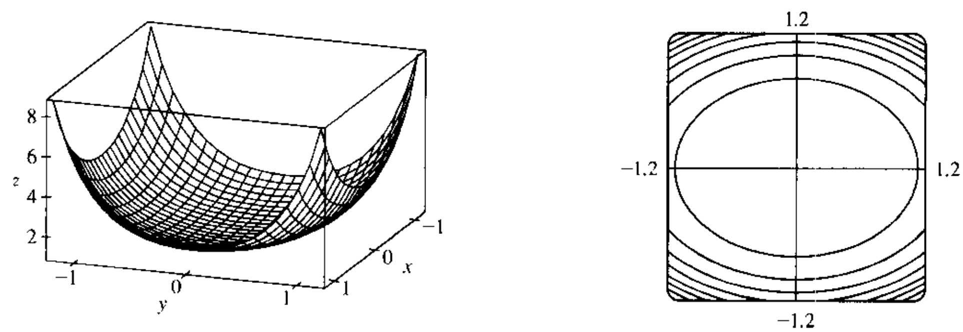
$f(x,y) = \dfrac{xy}{x^2+y^2}$. The graph exhibits different limiting values as x and y become large or as (x,y) approaches the origin, depending on the direction being examined. For example, although f is undefined at the origin, the function values appear to be $\frac{1}{2}$ along the line $y = x$, regardless of the distance from the origin. Along the line $y = -x$, the value is always $-\frac{1}{2}$. Along the axes, $f(x,y) = 0$ for all values of (x,y) except the origin. Other directions, heading toward the origin or away from the origin, give various limiting values between $-\frac{1}{2}$ and $\frac{1}{2}$.

69. $f(x,y) = e^{cx^2+y^2}$. First, if $c = 0$, the graph is the cylindrical surface $z = e^{y^2}$ (whose level curves are parallel lines). When $c > 0$, the vertical trace above the y-axis remains fixed while the sides of the surface in the x-direction "curl" upward, giving the graph a shape resembling an elliptic paraboloid. The level curves of the surface are ellipses centered at the origin.



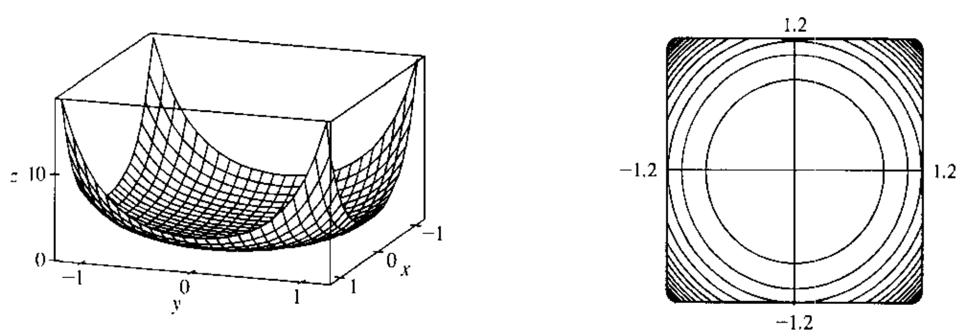
$c = 0$

For $0 < c < 1$, the ellipses have major axis the x-axis and the eccentricity increases as $c \to 0$.



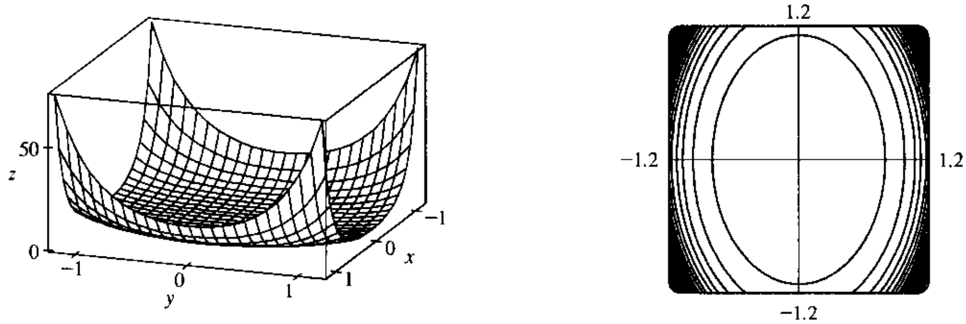
$c = 0.5$ (level curves in increments of 1)

For $c = 1$ the level curves are circles centered at the origin.



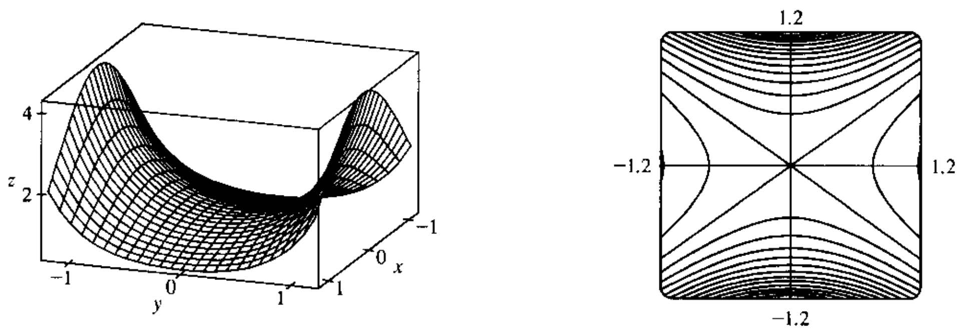
$c = 1$ (level curves in increments of 1)

When $c > 1$, the level curves are ellipses with major axis the y-axis, and the eccentricity increases as c increases.

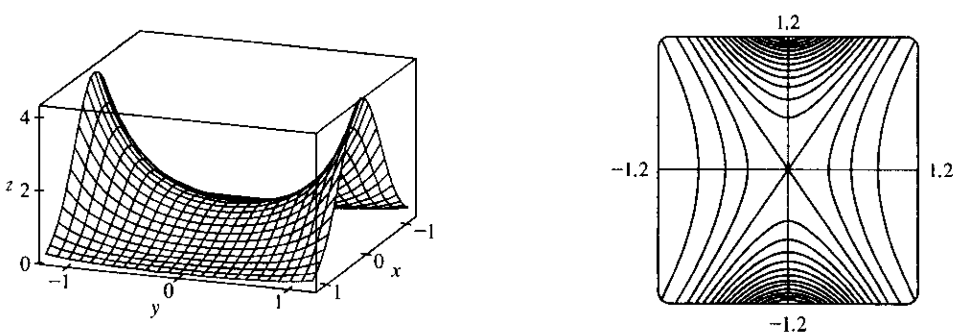


$c = 2$ (level curves in increments of 4)

For values of $c < 0$, the sides of the surface in the x-direction curl downward and approach the xy-plane (while the vertical trace $x = 0$ remains fixed), giving a saddle-shaped appearance to the graph near the point $(0, 0, 1)$. The level curves consist of a family of hyperbolas. As c decreases, the surface becomes flatter in the x-direction and the surface's approach to the curve in the trace $x = 0$ becomes steeper, as the graphs demonstrate.

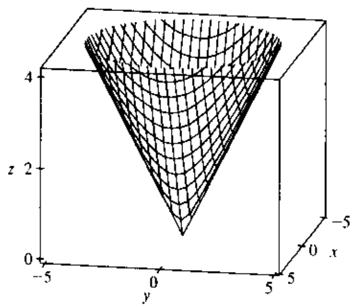


$c = -0.5$ (level curves in increments of 0.25)

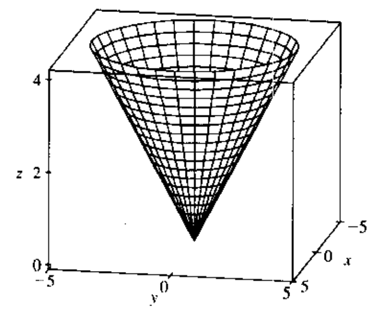


$c = -2$ (level curves in increments of 0.25)

70. First, we graph $f(x,y) = \sqrt{x^2 + y^2}$. As an alternative, the $x^2 + y^2$ expression suggests that cylindrical coordinates may be appropriate, giving the equivalent equation $z = \sqrt{r^2} = r$, $r \geq 0$ which we graph as well. Notice that the graph in cylindrical coordinates better demonstrates the symmetry of the surface.

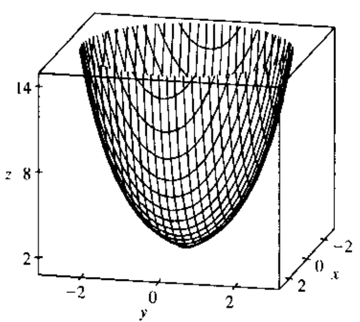


$f(x,y) = \sqrt{x^2 + y^2}$

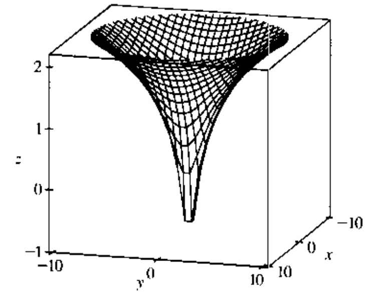


$z = r,\ r \geq 0$

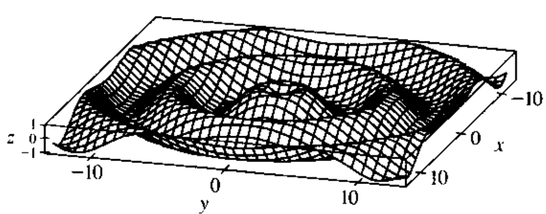
Graphs of the other four functions follow.



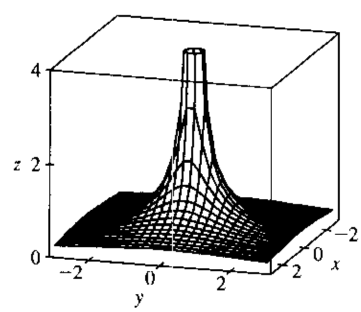
$f(x,y) = e^{\sqrt{x^2 + y^2}}$



$f(x,y) = \ln\sqrt{x^2 + y^2}$



$f(x,y) = \sin\left(\sqrt{x^2 + y^2}\right)$



$f(x,y) = \dfrac{1}{\sqrt{x^2 + y^2}}$

Notice that each graph $f(x,y) = g\left(\sqrt{x^2 + y^2}\right)$ exhibits radial symmetry about the z-axis and the trace in the xz-plane for $x \geq 0$ is the graph of $z = g(x)$, $x \geq 0$. This suggests that the graph of $f(x,y) = g\left(\sqrt{x^2 + y^2}\right)$ is obtained from the graph of g by graphing $z = g(x)$ in the xz-plane and rotating the curve about the z-axis.

71. (a) $P = bL^\alpha K^{1-\alpha} \;\Rightarrow\; \dfrac{P}{K} = bL^\alpha K^{-\alpha} \;\Rightarrow\; \dfrac{P}{K} = b\left(\dfrac{L}{K}\right)^\alpha \;\Rightarrow\; \ln\dfrac{P}{K} = \ln\left(b\left(\dfrac{L}{K}\right)^\alpha\right) \;\Rightarrow$

$\ln\dfrac{P}{K} = \ln b + \alpha\ln\left(\dfrac{L}{K}\right)$

(b) We list the values for $\ln(L/K)$ and $\ln(P/K)$ for the years 1899–1922. (Historically, these values were rounded to 2 decimal places.)

Year	$x = \ln(L/K)$	$y = \ln(P/K)$
1899	0	0
1900	−0.02	−0.06
1901	−0.04	−0.02
1902	−0.04	0
1903	−0.07	−0.05
1904	−0.13	−0.12
1905	−0.18	−0.04
1906	−0.20	−0.07
1907	−0.23	−0.15
1908	−0.41	−0.38
1909	−0.33	−0.24
1910	−0.35	−0.27

Year	$x = \ln(L/K)$	$y = \ln(P/K)$
1911	−0.38	−0.34
1912	−0.38	−0.24
1913	−0.41	−0.25
1914	−0.47	−0.37
1915	−0.53	−0.34
1916	−0.49	−0.28
1917	−0.53	−0.39
1918	−0.60	−0.50
1919	−0.68	−0.57
1920	−0.74	−0.57
1921	−1.05	−0.85
1922	−0.98	−0.59

After entering the (x, y) pairs into a calculator or CAS, the resulting least squares regression line through the points is approximately $y = 0.75136x + 0.01053$, which we round to $y = 0.75x + 0.01$.

(c) Comparing the regression line from part (b) to the equation $y = \ln b + \alpha x$ with $x = \ln(L/K)$ and $y = \ln(P/K)$, we have $\alpha = 0.75$ and $\ln b = 0.01 \;\Rightarrow\; b = e^{0.01} \approx 1.01$. Thus, the Cobb-Douglas production function is $P = bL^\alpha K^{1-\alpha} = 1.01L^{0.75}K^{0.25}$.

15.2 Limits and Continuity — ET 14.2

1. In general, we can't say anything about $f(3,1)$! $\lim\limits_{(x,y)\to(3,1)} f(x,y) = 6$ means that the values of $f(x,y)$ approach 6 as (x,y) approaches, but is not equal to, $(3,1)$. If f is continuous, we know that $\lim\limits_{(x,y)\to(a,b)} f(x,y) = f(a,b)$, so $\lim\limits_{(x,y)\to(3,1)} f(x,y) = f(3,1) = 6$.

2. (a) The outdoor temperature as a function of longitude, latitude, and time is continuous. Small changes in longitude, latitude, or time can produce only small changes in temperature, as the temperature doesn't jump abruptly from one value to another.

(b) Elevation is not necessarily continuous. If we think of a cliff with a sudden drop-off, a very small change in longitude or latitude can produce a comparatively large change in elevation, without all the intermediate values being attained. Elevation *can* jump from one value to another.

(c) The cost of a taxi ride is usually discontinuous. The cost normally increases in jumps, so small changes in distance traveled or time can produce a jump in cost. A graph of the function would show breaks in the surface.

3. We make a table of values of $f(x,y)=\dfrac{x^2y^3+x^3y^2-5}{2-xy}$ for a set of (x,y) points near the origin.

x \ y	−0.2	−0.1	−0.05	0	0.05	0.1	0.2
−0.2	−2.551	−2.525	−2.513	−2.500	−2.488	−2.475	−2.451
−0.1	−2.525	−2.513	−2.506	−2.500	−2.494	−2.488	−2.475
−0.05	−2.513	−2.506	−2.503	−2.500	−2.497	−2.494	−2.488
0	−2.500	−2.500	−2.500		−2.500	−2.500	−2.500
0.05	−2.488	−2.494	−2.497	−2.500	−2.503	−2.506	−2.513
0.1	−2.475	−2.488	−2.494	−2.500	−2.506	−2.513	−2.525
0.2	−2.451	−2.475	−2.488	−2.500	−2.513	−2.525	−2.551

As the table shows, the values of $f(x,y)$ seem to approach -2.5 as (x,y) approaches the origin from a variety of different directions. This suggests that $\lim\limits_{(x,y)\to(0,0)} f(x,y)=-2.5$.

Since f is a rational function, it is continuous on its domain. f is defined at $(0,0)$, so we can use direct substitution to establish that $\lim\limits_{(x,y)\to(0,0)} f(x,y)=\dfrac{0^20^3+0^30^2-5}{2-0\cdot 0}=-\dfrac{5}{2}$, verifying our guess.

4. We make a table of values of $f(x,y)=\dfrac{2xy}{x^2+2y^2}$ for a set of (x,y) points near the origin.

x \ y	−0.3	−0.2	−0.1	0	0.1	0.2	0.3
−0.3	0.667	0.706	0.545	0.000	−0.545	−0.706	−0.667
−0.2	0.545	0.667	0.667	0.000	−0.667	−0.667	−0.545
−0.1	0.316	0.444	0.667	0.000	−0.667	−0.444	−0.316
0	0.000	0.000	0.000		0.000	0.000	0.000
0.1	−0.316	−0.444	−0.667	0.000	0.667	0.444	0.316
0.2	−0.545	−0.667	−0.667	0.000	0.667	0.667	0.545
0.3	−0.667	−0.706	−0.545	0.000	0.545	0.706	0.667

It appears from the table that the values of $f(x,y)$ are not approaching a single value as (x,y) approaches the origin. For verification, if we first approach $(0,0)$ along the x-axis, we have $f(x,0)=0$, so $f(x,y)\to 0$. But if we approach $(0,0)$ along the line $y=x$, $f(x,x)=\dfrac{2x^2}{x^2+2x^2}=\dfrac{2}{3}$ $(x\neq 0)$, so $f(x,y)\to\frac{2}{3}$. Since f approaches different values along different paths to the origin, this limit does not exist.

5. $f(x,y)=x^5+4x^3y-5xy^2$ is a polynomial, and hence continuous, so $\lim\limits_{(x,y)\to(5,-2)} f(x,y)=f(5,-2)=5^5+4(5)^3(-2)-5(5)(-2)^2=2025$.

6. $x-2y$ is a polynomial and therefore continuous. Since $\cos t$ is a continuous function, the composition $\cos(x-2y)$ is also continuous. xy is also a polynomial, and hence continuous, so the product $f(x,y)=xy\cos(x-2y)$ is a continuous function. Then $\lim\limits_{(x,y)\to(6,3)} f(x,y)=f(6,3)=(6)(3)\cos(6-2\cdot 3)=18$.

7. $f(x,y) = x^2/(x^2+y^2)$. First approach $(0,0)$ along the x-axis. Then $f(x,0) = x^2/x^2 = 1$ for $x \neq 0$, so $f(x,y) \to 1$. Now approach $(0,0)$ along the y-axis. Then for $y \neq 0$, $f(0,y) = 0$, so $f(x,y) \to 0$. Since f has two different limits along two different lines, the limit does not exist.

8. $f(x,y) = (x^2 + \sin^2 y)/(2x^2+y^2)$. First approach $(0,0)$ along the x-axis. Then $f(x,0) = x^2/2x^2 = \frac{1}{2}$ for $x \neq 0$, so $f(x,y) \to \frac{1}{2}$. Next approach $(0,0)$ along the y-axis. For $y \neq 0$, $f(0,y) = \dfrac{\sin^2 y}{y^2} = \left(\dfrac{\sin y}{y}\right)^2$ and $\lim\limits_{y\to 0} \dfrac{\sin y}{y} = 1$, so $f(x,y) \to 1$. Since f has two different limits along two different lines, the limit does not exist.

9. $f(x,y) = (xy\cos y)/(3x^2+y^2)$. On the x-axis, $f(x,0) = 0$ for $x \neq 0$, so $f(x,y) \to 0$ as $(x,y) \to (0,0)$ along the x-axis. Approaching $(0,0)$ along the line $y = x$, $f(x,x) = (x^2\cos x)/4x^2 = \frac{1}{4}\cos x$ for $x \neq 0$, so $f(x,y) \to \frac{1}{4}$ along this line. Thus the limit does not exist.

10. $f(x,y) = 6x^3y/(2x^4+y^4)$. On the x-axis, $f(x,0) = 0$ for $x \neq 0$, so $f(x,y) \to 0$ as $(x,y) \to (0,0)$ along the x-axis. Approaching $(0,0)$ along the line $y = x$ gives $f(x,x) = 6x^4/(3x^4) = 2$ for $x \neq 0$, so along this line $f(x,y) \to 2$ as $(x,y) \to (0,0)$. Thus the limit does not exist.

11. $f(x,y) = \dfrac{xy}{\sqrt{x^2+y^2}}$. We can see that the limit along any line through $(0,0)$ is 0, as well as along other paths through $(0,0)$ such as $x = y^2$ and $y = x^2$. So we suspect that the limit exists and equals 0; we use the Squeeze Theorem to prove our assertion. $0 \leq \left|\dfrac{xy}{\sqrt{x^2+y^2}}\right| \leq |x|$ since $|y| \leq \sqrt{x^2+y^2}$, and $|x| \to 0$ as $(x,y) \to (0,0)$. So $\lim\limits_{(x,y)\to(0,0)} f(x,y) = 0$.

12. $f(x,y) = (x^4-y^4)/(x^2+y^2) = (x^2+y^2)(x^2-y^2)/(x^2+y^2) = x^2-y^2$ for $(x,y) \neq (0,0)$. Thus the limit as $(x,y) \to (0,0)$ is 0.

13. Let $f(x,y) = \dfrac{2x^2y}{x^4+y^2}$. Then $f(x,0) = 0$ for $x \neq 0$, so $f(x,y) \to 0$ as $(x,y) \to (0,0)$ along the x-axis. But $f(x,x^2) = \dfrac{2x^4}{2x^4} = 1$ for $x \neq 0$, so $f(x,y) \to 1$ as $(x,y) \to (0,0)$ along the parabola $y = x^2$. Thus the limit doesn't exist.

14. We can use the Squeeze Theorem to show that $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2\sin^2 y}{x^2+2y^2} = 0$:

$0 \leq \dfrac{x^2\sin^2 y}{x^2+2y^2} \leq \sin^2 y$ since $\dfrac{x^2}{x^2+2y^2} \leq 1$, and $\sin^2 y \to 0$ as $(x,y) \to (0,0)$, so $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^2\sin^2 y}{x^2+2y^2} = 0$.

15.

$$\begin{aligned}
\lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{\sqrt{x^2+y^2+1}-1} &= \lim_{(x,y)\to(0,0)} \frac{x^2+y^2}{\sqrt{x^2+y^2+1}-1} \cdot \frac{\sqrt{x^2+y^2+1}+1}{\sqrt{x^2+y^2+1}+1} \\
&= \lim_{(x,y)\to(0,0)} \frac{(x^2+y^2)\left(\sqrt{x^2+y^2+1}+1\right)}{x^2+y^2} \\
&= \lim_{(x,y)\to(0,0)} \left(\sqrt{x^2+y^2+1}+1\right) = 2
\end{aligned}$$

16. $f(x,y) = xy^4/(x^2+y^8)$. On the x-axis, $f(x,0) = 0$ for $x \neq 0$, so $f(x,y) \to 0$ as $(x,y) \to (0,0)$ along the x-axis. Approaching $(0,0)$ along the curve $x = y^4$ gives $f(y^4,y) = y^8/2y^8 = \frac{1}{2}$ for $y \neq 0$, so along this path $f(x,y) \to \frac{1}{2}$ as $(x,y) \to (0,0)$. Thus the limit does not exist.

17. e^{-xy} and $\sin(\pi z/2)$ are each compositions of continuous functions, and hence continuous, so their product $f(x,y,z) = e^{-xy}\sin(\pi z/2)$ is a continuous function. Then

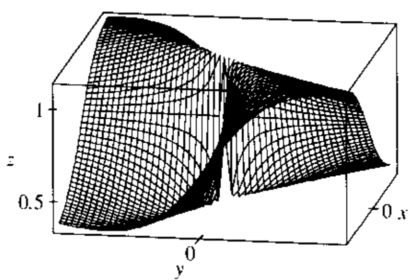
$$\lim_{(x,y,z)\to(3,0,1)} f(x,y,z) = f(3,0,1) = e^{-(3)(0)}\sin(\pi\cdot 1/2) = 1.$$

18. $f(x,y,z) = \dfrac{x^2+2y^2+3z^2}{x^2+y^2+z^2}$. Then $f(x,0,0) = \dfrac{x^2+0+0}{x^2+0+0} = 1$ for $x \neq 0$, so $f(x,y,z) \to 1$ as $(x,y,z) \to (0,0,0)$ along the x-axis. But $f(0,y,0) = \dfrac{0+2y^2+0}{0+y^2+0} = 2$ for $y \neq 0$, so $f(x,y,z) \to 2$ as $(x,y,z) \to (0,0,0)$ along the y-axis. Thus, the limit doesn't exist.

19. $f(x,y,z) = \dfrac{xy+yz^2+xz^2}{x^2+y^2+z^4}$. Then $f(x,0,0) = 0/x^2 = 0$ for $x \neq 0$, so as $(x,y,z) \to (0,0,0)$ along the x-axis, $f(x,y,z) \to 0$. But $f(x,x,0) = x^2/(2x^2) = \frac{1}{2}$ for $x \neq 0$, so as $(x,y,z) \to (0,0,0)$ along the line $y = x$, $z = 0$, $f(x,y,z) \to \frac{1}{2}$. Thus the limit doesn't exist.

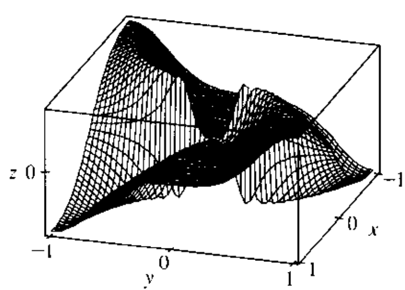
20. $f(x,y,z) = \dfrac{xy+yz+zx}{x^2+y^2+z^2}$. Then $f(x,0,0) = 0$ for $x \neq 0$, so as $(x,y,z) \to (0,0,0)$ along the x-axis, $f(x,y,z) \to 0$. But $f(x,x,0) = x^2/(2x^2) = \frac{1}{2}$ for $x \neq 0$, so as $(x,y,z) \to (0,0,0)$ along the line $y = x$, $z = 0$, $f(x,y,z) \to \frac{1}{2}$. Thus the limit doesn't exist.

21.



From the ridges on the graph, we see that as $(x,y) \to (0,0)$ along the lines under the two ridges, $f(x,y)$ approaches different values. So the limit does not exist.

22.

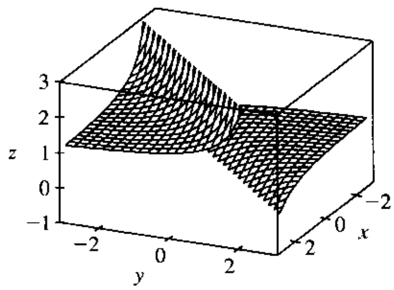


From the graph, it appears that as we approach the origin along the lines $x = 0$ or $y = 0$, the function is everywhere 0, whereas if we approach the origin along a certain curve it has a constant value of about $\frac{1}{2}$. [In fact, $f(y^3,y) = y^6/(2y^6) = \frac{1}{2}$ for $y \neq 0$, so $f(x,y) \to \frac{1}{2}$ as $(x,y) \to (0,0)$ along the curve $x = y^3$.] Since the function approaches different values depending on the path of approach, the limit does not exist.

23. $h(x,y) = g(f(x,y)) = (2x+3y-6)^2 + \sqrt{2x+3y-6}$. Since f is a polynomial, it is continuous on $\mathbb{R}^2$ and g is continuous on its domain $\{t \mid t \geq 0\}$. Thus h is continuous on its domain $D = \{(x,y) \mid 2x+3y-6 \geq 0\} = \left\{(x,y) \mid y \geq -\frac{2}{3}x + 2\right\}$, which consists of all points on or above the line $y = -\frac{2}{3}x + 2$.

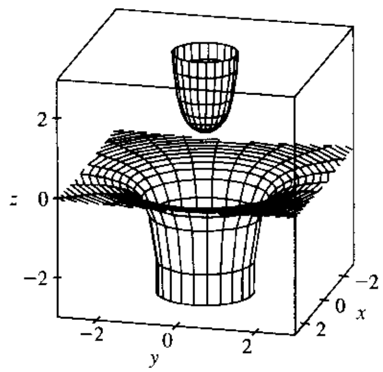
24. $h(x,y) = g(f(x,y)) = \left(\sqrt{x^2-y} - 1\right)\Big/\left(\sqrt{x^2-y} + 1\right)$. Since f is a polynomial, it is continuous on $\mathbb{R}^2$ and g is continuous on its domain $\{t \mid t \geq 0\}$. Thus h is continuous on its domain $D = \left\{(x,y) \mid x^2 - y \geq 0\right\} = \left\{(x,y) \mid y \leq x^2\right\}$ which consists of all points below or on the parabola $y = x^2$.

25.



From the graph, it appears that f is discontinuous along the line $y = x$. If we consider $f(x, y) = e^{1/(x-y)}$ as a composition of functions, $g(x, y) = 1/(x - y)$ is a rational function and therefore continuous except where $x - y = 0 \Rightarrow y = x$. Since the function $h(t) = e^t$ is continuous everywhere, the composition $h(g(x, y)) = e^{1/(x-y)} = f(x, y)$ is continuous except along the line $y = x$, as we suspected.

26.



We can see a circular break in the graph, corresponding approximately to the unit circle, where f is discontinuous. [*Note:* For a more accurate graph, try converting to cylindrical coordinates first.] Since $f(x, y) = \dfrac{1}{1 - x^2 - y^2}$ is a rational function, it is continuous except where $1 - x^2 - y^2 = 0 \Rightarrow x^2 + y^2 = 1$, confirming our observation that f is discontinuous on the circle $x^2 + y^2 = 1$.

27. The functions $\sin(xy)$ and $e^x - y^2$ are continuous everywhere, so $F(x, y) = \dfrac{\sin(xy)}{e^x - y^2}$ is continuous except where $e^x - y^2 = 0 \Rightarrow y^2 = e^x \Rightarrow y = \pm\sqrt{e^x} = \pm e^{\frac{1}{2}x}$. Thus F is continuous on its domain $\{(x, y) \mid y \neq \pm e^{x/2}\}$.

28. $F(x, y) = \dfrac{x - y}{1 + x^2 + y^2}$ is a rational function and thus is continuous on its domain $\mathbb{R}^2$ (since the denominator is never zero).

29. $F(x, y) = \arctan(x + \sqrt{y}) = g(f(x, y))$ where $f(x, y) = x + \sqrt{y}$, continuous on its domain $\{(x, y) \mid y \geq 0\}$, and $g(t) = \arctan t$ is continuous everywhere. Thus F is continuous on its domain $\{(x, y) \mid y \geq 0\}$.

30. $e^{x^2 y}$ is continuous on $\mathbb{R}^2$ and $\sqrt{x + y^2}$ is continuous on its domain $\{(x, y) \mid x + y^2 \geq 0\} = \{(x, y) \mid x \geq -y^2\}$, so $F(x, y) = e^{x^2 y} + \sqrt{x + y^2}$ is continuous on the set $\{(x, y) \mid x \geq -y^2\}$.

31. $G(x, y) = \ln(x^2 + y^2 - 4) = g(f(x, y))$ where $f(x, y) = x^2 + y^2 - 4$, continuous on $\mathbb{R}^2$, and $g(t) = \ln t$, continuous on its domain $\{t \mid t > 0\}$. Thus G is continuous on its domain $\{(x, y) \mid x^2 + y^2 - 4 > 0\} = \{(x, y) \mid x^2 + y^2 > 4\}$, the exterior of the circle $x^2 + y^2 = 4$.

32. $G(x, y) = g(f(x, y))$ where $f(x, y) = x^2 + y^2$, continuous on $\mathbb{R}^2$, and $g(t) = \sin^{-1} t$, continuous on its domain $\{t \mid -1 \leq t \leq 1\}$. Thus G is continuous on its domain $D = \{(x, y) \mid -1 \leq x^2 + y^2 \leq 1\} = \{(x, y) \mid x^2 + y^2 \leq 1\}$, inside and on the circle $x^2 + y^2 = 1$.

33. $\sqrt{y}$ is continuous on its domain $\{y \mid y \geq 0\}$ and $x^2 - y^2 + z^2$ is continuous everywhere, so $f(x, y, z) = \dfrac{\sqrt{y}}{x^2 - y^2 + z^2}$ is continuous for $y \geq 0$ and $x^2 - y^2 + z^2 \neq 0 \Rightarrow y^2 \neq x^2 + z^2$, that is, $\{(x, y, z) \mid y \geq 0, y \neq \sqrt{x^2 + z^2}\}$.

34. $f(x,y,z) = \sqrt{x+y+z} = h(g(x,y,z))$ where $g(x,y,z) = x+y+z$, continuous everywhere, and $h(t) = \sqrt{t}$ is continuous on its domain $\{t \mid t \geq 0\}$. Thus f is continuous on its domain $\{(x,y,z) \mid x+y+z \geq 0\}$, so f is continuous on and above the plane $z = -x-y$.

35. $f(x,y) = \begin{cases} \dfrac{x^2y^3}{2x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 1 & \text{if } (x,y) = (0,0) \end{cases}$ The first piece of f is a rational function defined everywhere except at the origin, so f is continuous on $\mathbb{R}^2$ except possibly at the origin. Since $x^2 \leq 2x^2 + y^2$, we have $\left|x^2y^3/(2x^2+y^2)\right| \leq \left|y^3\right|$. We know that $\left|y^3\right| \to 0$ as $(x,y) \to (0,0)$. So, by the Squeeze Theorem, $\lim\limits_{(x,y)\to(0,0)} f(x,y) = \lim\limits_{(x,y)\to(0,0)} \dfrac{x^2y^3}{2x^2+y^2} = 0$. But $f(0,0) = 1$, so f is discontinuous at $(0,0)$. Therefore, f is continuous on the set $\{(x,y) \mid (x,y) \neq (0,0)\}$.

36. $f(x,y) = \begin{cases} \dfrac{xy}{x^2+xy+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$ The first piece of f is a rational function defined everywhere except at the origin, so f is continuous on $\mathbb{R}^2$ except possibly at the origin. $f(x,0) = 0/x^2 = 0$ for $x \neq 0$, so $f(x,y) \to 0$ as $(x,y) \to (0,0)$ along the x-axis. But $f(x,x) = x^2/(3x^2) = \frac{1}{3}$ for $x \neq 0$, so $f(x,y) \to \frac{1}{3}$ as $(x,y) \to (0,0)$ along the line $y = x$. Thus $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ doesn't exist, so f is not continuous at $(0,0)$ and the largest set on which f is continuous is $\{(x,y) \mid (x,y) \neq (0,0)\}$.

37. $\lim\limits_{(x,y)\to(0,0)} \dfrac{x^3+y^3}{x^2+y^2} = \lim\limits_{r\to 0^+} \dfrac{(r\cos\theta)^3 + (r\sin\theta)^3}{r^2} = \lim\limits_{r\to 0^+} (r\cos^3\theta + r\sin^3\theta) = 0$

38. $\lim\limits_{(x,y)\to(0,0)} (x^2+y^2)\ln(x^2+y^2) = \lim\limits_{r\to 0^+} r^2 \ln r^2 = \lim\limits_{r\to 0^+} \dfrac{\ln r^2}{1/r^2}$

$$= \lim_{r\to 0^+} \frac{(1/r^2)(2r)}{-2/r^3} \quad \text{[using l'Hospital's Rule]} \quad = \lim_{r\to 0^+} (-r^2) = 0$$

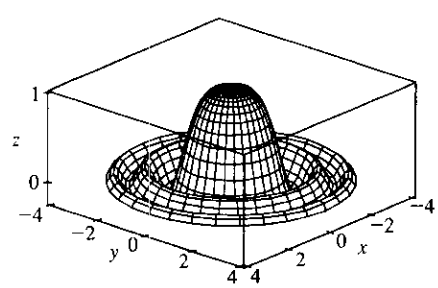
39. $\lim\limits_{(x,y,z)\to(0,0,0)} \dfrac{xyz}{x^2+y^2+z^2} = \lim\limits_{\rho\to 0^+} \dfrac{(\rho\sin\phi\cos\theta)(\rho\sin\phi\sin\theta)(\rho\cos\phi)}{\rho^2}$

$$= \lim_{\rho\to 0^+} (\rho\sin^2\phi\cos\phi\sin\theta\cos\theta) = 0$$

40. $\lim\limits_{(x,y)\to(0,0)} \dfrac{\sin(x^2+y^2)}{x^2+y^2} = \lim\limits_{r\to 0^+} \dfrac{\sin(r^2)}{r^2}$, which is an indeterminate form of type $0/0$. Using l'Hospital's Rule, we get

$$\lim_{r\to 0^+} \frac{\sin(r^2)}{r^2} \overset{\text{H}}{=} \lim_{r\to 0^+} \frac{2r\cos(r^2)}{2r} = \lim_{r\to 0^+} \cos(r^2) = 1.$$

Or: Use the fact that $\lim\limits_{\theta\to 0} \dfrac{\sin\theta}{\theta} = 1$.



41. Since $|\mathbf{x}-\mathbf{a}|^2 = |\mathbf{x}|^2 + |\mathbf{a}|^2 - 2|\mathbf{x}||\mathbf{a}|\cos\theta \geq |\mathbf{x}|^2 + |\mathbf{a}|^2 - 2|\mathbf{x}||\mathbf{a}| = (|\mathbf{x}| - |\mathbf{a}|)^2$, we have $\big||\mathbf{x}| - |\mathbf{a}|\big| \leq |\mathbf{x}-\mathbf{a}|$. Let $\epsilon > 0$ be given and set $\delta = \epsilon$. Then whenever $0 < |\mathbf{x}-\mathbf{a}| < \delta$, $\big||\mathbf{x}| - |\mathbf{a}|\big| \leq |\mathbf{x}-\mathbf{a}| < \delta = \epsilon$. Hence $\lim_{\mathbf{x}\to\mathbf{a}} |\mathbf{x}| = |\mathbf{a}|$ and $f(\mathbf{x}) = |\mathbf{x}|$ is continuous on $\mathbb{R}^n$.

42. Let $\epsilon > 0$ be given. We need to find $\delta > 0$ such that $|f(\mathbf{x}) - f(\mathbf{a})| < \epsilon$ whenever $|\mathbf{x} - \mathbf{a}| < \delta$ or $|\mathbf{c} \cdot \mathbf{x} - \mathbf{c} \cdot \mathbf{a}| < \epsilon$ whenever $|\mathbf{x} - \mathbf{a}| < \delta$. But $|\mathbf{c} \cdot \mathbf{x} - \mathbf{c} \cdot \mathbf{a}| = |\mathbf{c} \cdot (\mathbf{x} - \mathbf{a})|$ and $|\mathbf{c} \cdot (\mathbf{x} - \mathbf{a})| \le |\mathbf{c}|\,|\mathbf{x} - \mathbf{a}|$ by Exercise 13.3.57 [ET 12.3.57] (the Cauchy-Schwartz Inequality). Let $\epsilon > 0$ be given and set $\delta = \epsilon / |\mathbf{c}|$. Then whenever $0 < |\mathbf{x} - \mathbf{a}| < \delta$, $|f(\mathbf{x}) - f(\mathbf{a})| = |\mathbf{c} \cdot \mathbf{x} - \mathbf{c} \cdot \mathbf{a}| \le |\mathbf{c}|\,|\mathbf{x} - \mathbf{a}| < |\mathbf{c}|\,\delta = |\mathbf{c}|\,(\epsilon / |\mathbf{c}|) = \epsilon$. So f is continuous on $\mathbb{R}^n$.

15.3 Partial Derivatives ET 14.3

1. (a) $\partial T/\partial x$ represents the rate of change of T when we fix y and t and consider T as a function of the single variable x, which describes how quickly the temperature changes when longitude changes but latitude and time are constant. $\partial T/\partial y$ represents the rate of change of T when we fix x and t and consider T as a function of y, which describes how quickly the temperature changes when latitude changes but longitude and time are constant. $\partial T/\partial t$ represents the rate of change of T when we fix x and y and consider T as a function of t, which describes how quickly the temperature changes over time for a constant longitude and latitude.

(b) $f_x(158, 21, 9)$ represents the rate of change of temperature at longitude 158°W, latitude 21°N at 9:00 A.M. when only longitude varies. Since the air is warmer to the west than to the east, increasing longitude results in an increased air temperature, so we would expect $f_x(158, 21, 9)$ to be positive. $f_y(158, 21, 9)$ represents the rate of change of temperature at the same time and location when only latitude varies. Since the air is warmer to the south and cooler to the north, increasing latitude results in a decreased air temperature, so we would expect $f_y(158, 21, 9)$ to be negative. $f_t(158, 21, 9)$ represents the rate of change of temperature at the same time and location when only time varies. Since typically air temperature increases from the morning to the afternoon as the sun warms it, we would expect $f_t(158, 21, 9)$ to be positive.

2. By Definition 4, $f_T(92, 60) = \lim\limits_{h \to 0} \dfrac{f(92 + h, 60) - f(92, 60)}{h}$, which we can approximate by considering $h = 2$ and $h = -2$ and using the values given in Table 1: $f_T(92, 60) \approx \dfrac{f(94, 60) - f(92, 60)}{2} = \dfrac{111 - 105}{2} = 3$, $f_T(92, 60) \approx \dfrac{f(90, 60) - f(92, 60)}{-2} = \dfrac{100 - 105}{-2} = 2.5$. Averaging these values, we estimate $f_T(92, 60)$ to be approximately 2.75. Thus, when the actual temperature is 92°F and the relative humidity is 60%, the apparent temperature rises by about 2.75°F for every degree that the actual temperature rises.

Similarly, $f_H(92, 60) = \lim\limits_{h \to 0} \dfrac{f(92, 60 + h) - f(92, 60)}{h}$ which we can approximate by considering $h = 5$ and $h = -5$: $f_H(92, 60) \approx \dfrac{f(92, 65) - f(92, 60)}{5} = \dfrac{108 - 105}{5} = 0.6$, $f_H(92, 60) \approx \dfrac{f(92, 55) - f(92, 60)}{-5} = \dfrac{103 - 105}{-5} = 0.4$. Averaging these values, we estimate $f_H(92, 60)$ to be approximately 0.5. Thus, when the actual temperature is 92°F and the relative humidity is 60%, the apparent temperature rises by about 0.5°F for every percent that the relative humidity increases.

3. (a) By Definition 4, $f_T(-15,30) = \lim\limits_{h\to 0} \dfrac{f(-15+h,30) - f(-15,30)}{h}$, which we can approximate by considering $h = 5$ and $h = -5$ and using the values given in the table:

$f_T(-15,30) \approx \dfrac{f(-10,30) - f(-15,30)}{5} = \dfrac{-20-(-26)}{5} = \dfrac{6}{5} = 1.2$,

$f_T(-15,30) \approx \dfrac{f(-20,30) - f(-15,30)}{-5} = \dfrac{-33-(-26)}{-5} = \dfrac{-7}{-5} = 1.4$. Averaging these values, we estimate $f_T(-15,30)$ to be approximately 1.3. Thus, when the actual temperature is $-15°$C and the wind speed is 30 km/h, the apparent temperature rises by about $1.3°$C for every degree that the actual temperature rises.

Similarly, $f_v(-15,30) = \lim\limits_{h\to 0} \dfrac{f(-15,30+h) - f(-15,30)}{h}$ which we can approximate by considering $h = 10$ and $h = -10$: $f_v(-15,30) \approx \dfrac{f(-15,40) - f(-15,30)}{10} = \dfrac{-27-(-26)}{10} = \dfrac{-1}{10} = -0.1$,

$f_v(-15,30) \approx \dfrac{f(-15,20) - f(-15,30)}{-10} = \dfrac{-24-(-26)}{-10} = \dfrac{2}{-10} = -0.2$. Averaging these values, we estimate $f_v(-15,30)$ to be approximately -0.15. Thus, when the actual temperature is $-15°$C and the wind speed is 30 km/h, the apparent temperature decreases by about $0.15°$C for every km/h that the wind speed increases.

(b) For a fixed wind speed v, the values of the wind-chill index W increase as temperature T increases (look at a column of the table), so $\dfrac{\partial W}{\partial T}$ is positive. For a fixed temperature T, the values of W decrease (or remain constant) as v increases (look at a row of the table), so $\dfrac{\partial W}{\partial v}$ is negative (or perhaps 0).

(c) For fixed values of T, the function values $f(T,v)$ appear to become constant (or nearly constant) as v increases, so the corresponding rate of change is 0 or near 0 as v increases. This suggests that $\lim\limits_{v\to\infty} (\partial W/\partial v) = 0$.

4. (a) $\partial h/\partial v$ represents the rate of change of h when we fix t and consider h as a function of v, which describes how quickly the wave heights change when the wind speed changes for a fixed time duration. $\partial h/\partial t$ represents the rate of change of h when we fix v and consider h as a function of t, which describes how quickly the wave heights change when the duration of time changes, but the wind speed is constant.

(b) By Definition 4, $f_v(40,15) = \lim\limits_{h\to 0} \dfrac{f(40+h,15) - f(40,15)}{h}$ which we can approximate by considering $h = 10$ and $h = -10$ and using the values given in the table:

$f_v(40,15) \approx \dfrac{f(50,15) - f(40,15)}{10} = \dfrac{36-25}{10} = 1.1$,

$f_v(40,15) \approx \dfrac{f(30,15) - f(40,15)}{-10} = \dfrac{16-25}{-10} = 0.9$. Averaging these values, we have $f_v(40,15) \approx 1.0$.

Thus, when a 40-knot wind has been blowing for 15 hours, the wave heights should increase by about 1 foot for every knot that the wind speed increases (with the same time duration). Similarly,

$f_t(40,15) = \lim\limits_{h\to 0} \dfrac{f(40,15+h) - f(40,15)}{h}$ which we can approximate by considering

$h = 5$ and $h = -5$: $f_t(40,15) \approx \dfrac{f(40,20) - f(40,15)}{5} = \dfrac{28-25}{5} = 0.6$,

$f_t(40,15) \approx \dfrac{f(40,10) - f(40,15)}{-5} = \dfrac{21-25}{-5} = 0.8$. Averaging these values, we have $f_t(40,15) \approx 0.7$.

Thus, when a 40-knot wind has been blowing for 15 hours, the wave heights increase by about 0.7 feet for every additional hour that the wind blows.

(c) For fixed values of v, the function values $f(v,t)$ appear to increase in smaller and smaller increments, becoming nearly constant as t increases. Thus, the corresponding rate of change is nearly 0 as t increases, suggesting that $\lim\limits_{t\to\infty} (\partial h/\partial t) = 0$.

5. (a) If we start at $(1,2)$ and move in the positive x-direction, the graph of f increases. Thus $f_x(1,2)$ is positive.

(b) If we start at $(1,2)$ and move in the positive y-direction, the graph of f decreases. Thus $f_y(1,2)$ is negative.

6. (a) The graph of f decreases if we start at $(-1,2)$ and move in the positive x-direction, so $f_x(-1,2)$ is negative.

(b) The graph of f decreases if we start at $(-1,2)$ and move in the positive y-direction, so $f_y(-1,2)$ is negative.

(c) $f_{xx} = \frac{\partial}{\partial x}(f_x)$, so f_{xx} is the rate of change of f_x in the x-direction. f_x is negative at $(-1,2)$ and if we move in the positive x-direction, the surface becomes less steep. Thus the values of f_x are increasing and $f_{xx}(-1,2)$ is positive.

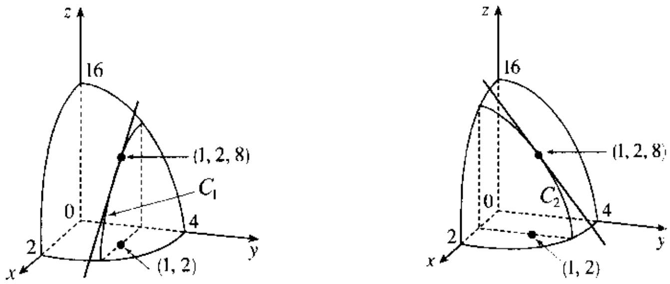
(d) f_{yy} is the rate of change of f_y in the y-direction. f_y is negative at $(-1,2)$ and if we move in the positive y-direction, the surface becomes steeper. Thus the values of f_y are decreasing, and $f_{yy}(-1,2)$ is negative.

7. First of all, if we start at the point $(3,-3)$ and move in the positive y-direction, we see that both b and c decrease, while a increases. Both b and c have a low point at about $(3,-1.5)$, while a is 0 at this point. So a is definitely the graph of f_y, and one of b and c is the graph of f. To see which is which, we start at the point $(-3,-1.5)$ and move in the positive x-direction. b traces out a line with negative slope, while c traces out a parabola opening downward. This tells us that b is the x-derivative of c. So c is the graph of f, b is the graph of f_x, and a is the graph of f_y.

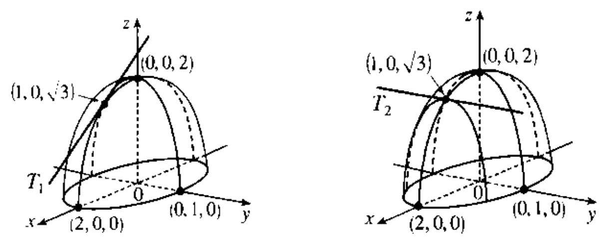
8. $f_x(2,1)$ is the rate of change of f at $(2,1)$ in the x-direction. If we start at $(2,1)$, where $f(2,1)=10$, and move in the positive x-direction, we reach the next contour line (where $f(x,y)=12$) after approximately 0.6 units. This represents an average rate of change of about $\frac{2}{0.6}$. If we approach the point $(2,1)$ from the left (moving in the positive x-direction) the output values increase from 8 to 10 with an increase in x of approximately 0.9 units, corresponding to an average rate of change of $\frac{2}{0.9}$. A good estimate for $f_x(2,1)$ would be the average of these two, so $f_x(2,1) \approx 2.8$. Similarly, $f_y(2,1)$ is the rate of change of f at $(2,1)$ in the y-direction. If we approach $(2,1)$ from below, the output values decrease from 12 to 10 with a change in y of approximately 1 unit, corresponding to an average rate of change of -2. If we start at $(2,1)$ and move in the positive y-direction, the output values decrease from 10 to 8 after approximately 0.9 units, a rate of change of $\frac{-2}{0.9}$. Averaging these two results, we estimate $f_y(2,1) \approx -2.1$.

9. $f(x,y) = 16 - 4x^2 - y^2 \Rightarrow f_x(x,y) = -8x$ and $f_y(x,y) = -2y \Rightarrow f_x(1,2) = -8$ and $f_y(1,2) = -4$. The graph of f is the paraboloid $z = 16 - 4x^2 - y^2$ and the vertical plane $y = 2$ intersects it in the parabola $z = 12 - 4x^2$, $y = 2$ (the curve C_1 in the first figure).

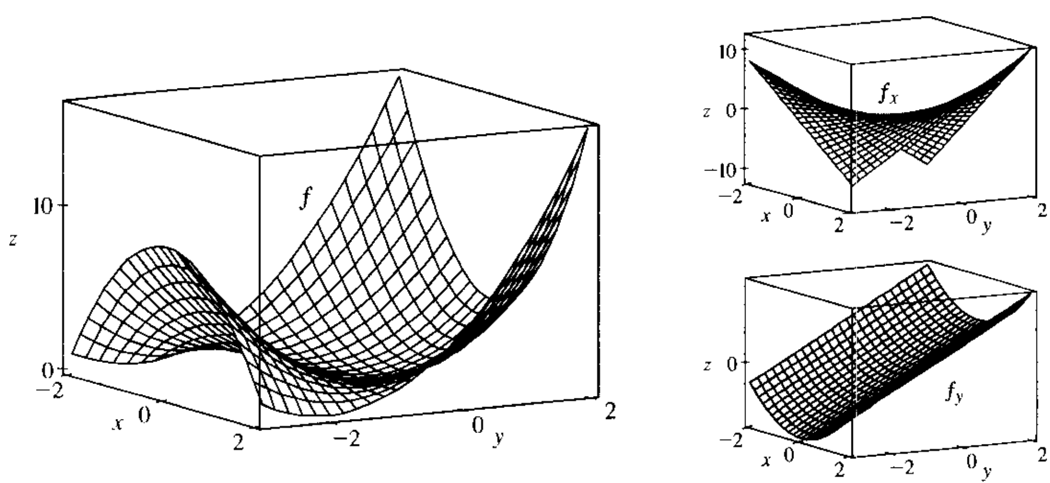
The slope of the tangent line to this parabola at $(1,2,8)$ is $f_x(1,2) = -8$. Similarly the plane $x = 1$ intersects the paraboloid in the parabola $z = 12 - y^2$, $x = 1$ (the curve C_2 in the second figure) and the slope of the tangent line at $(1,2,8)$ is $f_y(1,2) = -4$.



10. $f(x,y) = (4 - x^2 - 4y^2)^{1/2} \Rightarrow f_x(x,y) = -x(4 - x^2 - 4y^2)^{-1/2}$ and $f_y(x,y) = -4y(4 - x^2 - 4y^2)^{-1/2}$ $\Rightarrow f_x(1,0) = -\frac{1}{\sqrt{3}}$, $f_y(1,0) = 0$. The graph of f is the upper half of the ellipsoid $z^2 + x^2 + 4y^2 = 4$ and the plane $y = 0$ intersects the graph in the semicircle $x^2 + z^2 = 4$, $z \geq 0$ and the slope of the tangent line T_1 to this semicircle at $(1, 0, \sqrt{3})$ is $f_x(1,0) = -\frac{1}{\sqrt{3}}$. Similarly the plane $x = 1$ intersects the graph in the semi-ellipse $z^2 + 4y^2 = 3$, $z \geq 0$ and the slope of the tangent line T_2 to this semi-ellipse at $(1, 0, \sqrt{3})$ is $f_y(1,0) = 0$.

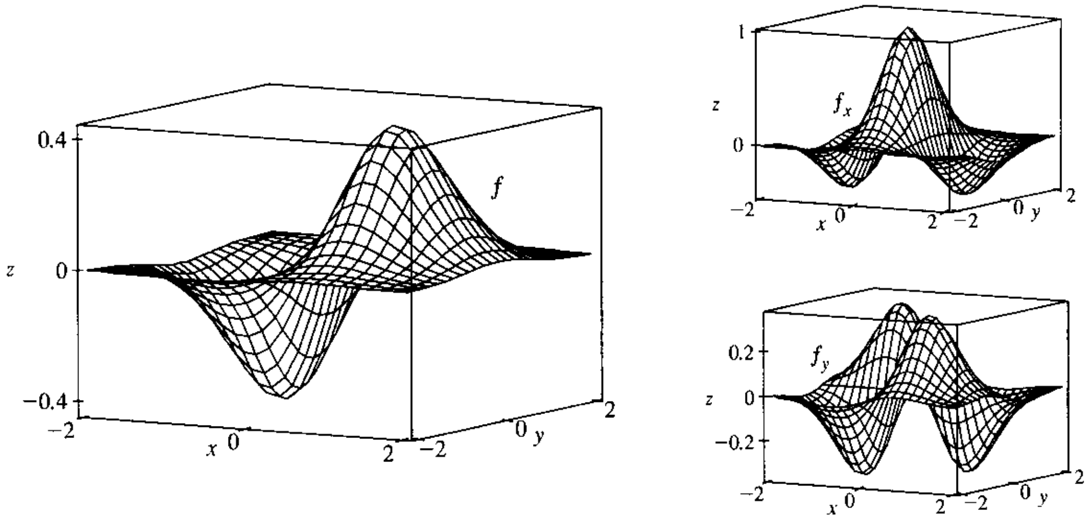


11. $f(x,y) = x^2 + y^2 + x^2y \Rightarrow f_x = 2x + 2xy$, $f_y = 2y + x^2$



Note that the traces of f in planes parallel to the xz-plane are parabolas which open downward for $y < -1$ and upward for $y > -1$, and the traces of f_x in these planes are straight lines, which have negative slopes for $y < -1$ and positive slopes for $y > -1$. The traces of f in planes parallel to the yz-plane are parabolas which always open upward, and the traces of f_y in these planes are straight lines with positive slopes.

12. $f(x,y) = xe^{-x^2-y^2} \quad \Rightarrow \quad f_x = x\left(-2xe^{-x^2-y^2}\right) + e^{-x^2-y^2} = e^{-x^2-y^2}(1-2x^2)$, $f_y = -2xye^{-x^2-y^2}$



Note that traces of f in planes parallel to the xz-plane have two extreme values, while traces of f_x in these planes have two zeros. Traces of f in planes parallel to the yz-plane have only one extreme value (a minimum if $x < 0$, a maximum if $x > 0$), and traces of f_y in these planes have only one zero (going from negative to positive if $x < 0$ and from positive to negative if $x > 0$).

13. $f(x,y) = 3x - 2y^4 \quad \Rightarrow \quad f_x(x,y) = 3 - 0 = 3$, $f_y(x,y) = 0 - 8y^3 = -8y^3$

14. $f(x,y) = x^5 + 3x^3y^2 + 3xy^4 \quad \Rightarrow \quad f_x(x,y) = 5x^4 + 3 \cdot 3x^2 \cdot y^2 + 3 \cdot 1 \cdot y^4 = 5x^4 + 9x^2y^2 + 3y^4$,
$f_y(x,y) = 0 + 3x^3 \cdot 2y + 3x \cdot 4y^3 = 6x^3y + 12xy^3$.

15. $z = xe^{3y} \quad \Rightarrow \quad \dfrac{\partial z}{\partial x} = e^{3y}, \ \dfrac{\partial z}{\partial y} = 3xe^{3y}$

16. $z = y \ln x \quad \Rightarrow \quad \dfrac{\partial z}{\partial x} = \dfrac{y}{x}, \ \dfrac{\partial z}{\partial y} = \ln x$

17. $f(x,y) = \dfrac{x-y}{x+y} \quad \Rightarrow \quad f_x(x,y) = \dfrac{(1)(x+y)-(x-y)(1)}{(x+y)^2} = \dfrac{2y}{(x+y)^2}$,

$f_y(x,y) = \dfrac{(-1)(x+y)-(x-y)(1)}{(x+y)^2} = -\dfrac{2x}{(x+y)^2}$

18. $f(x,y) = x^y \quad \Rightarrow \quad f_x(x,y) = yx^{y-1}$, $f_y(x,y) = x^y \ln x$

19. $w = \sin\alpha\cos\beta \quad \Rightarrow \quad \dfrac{\partial w}{\partial \alpha} = \cos\alpha\cos\beta, \ \dfrac{\partial w}{\partial \beta} = -\sin\alpha\sin\beta$

20. $f(s,t) = \dfrac{st^2}{s^2+t^2} \quad \Rightarrow$

$f_s(s,t) = \dfrac{t^2(s^2+t^2) - st^2(2s)}{(s^2+t^2)^2} = \dfrac{t^4 - s^2t^2}{(s^2+t^2)^2}, \ f_t(s,t) = \dfrac{2st(s^2+t^2) - st^2(2t)}{(s^2+t^2)^2} = \dfrac{2s^3t}{(s^2+t^2)^2}$

21. $f(r,s) = r\ln(r^2+s^2) \Rightarrow f_r(r,s) = r\cdot\dfrac{2r}{r^2+s^2} + \ln(r^2+s^2)\cdot 1 = \dfrac{2r^2}{r^2+s^2} + \ln(r^2+s^2)$,

$f_s(r,s) = r\cdot\dfrac{2s}{r^2+s^2} + 0 = \dfrac{2rs}{r^2+s^2}$

22. $f(x,t) = \arctan(x\sqrt{t}) \Rightarrow f_x(x,t) = \dfrac{1}{1+(x\sqrt{t})^2}\cdot\sqrt{t} = \dfrac{\sqrt{t}}{1+x^2t}$,

$f_t(x,t) = \dfrac{1}{1+(x\sqrt{t})^2}\cdot x\left(\dfrac{1}{2}t^{-1/2}\right) = \dfrac{x}{2\sqrt{t}\,(1+x^2t)}$

23. $u = te^{w/t} \Rightarrow \dfrac{\partial u}{\partial t} = t\cdot e^{w/t}(-wt^{-2}) + e^{w/t}\cdot 1 = e^{w/t} - \dfrac{w}{t}e^{w/t} = e^{w/t}\left(1-\dfrac{w}{t}\right)$, $\dfrac{\partial u}{\partial w} = te^{w/t}\cdot\dfrac{1}{t} = e^{w/t}$

24. $f(x,y) = \displaystyle\int_y^x \cos(t^2)\,dt \Rightarrow f_x(x,y) = \dfrac{\partial}{\partial x}\int_y^x \cos(t^2)\,dt = \cos(x^2)$ by the Fundamental Theorem of Calculus, Part 1; $f_y(x,y) = \dfrac{\partial}{\partial y}\displaystyle\int_y^x \cos(t^2)\,dt = -\dfrac{\partial}{\partial y}\cos(t^2)\,dt = -\cos(y^2)$.

25. $f(x,y,z) = xy^2z^3 + 3yz \Rightarrow f_x(x,y,z) = y^2z^3$, $f_y(x,y,z) = 2xyz^3 + 3z$, $f_z(x,y,z) = 3xy^2z^2 + 3y$

26. $f(x,y,z) = x^2e^{yz} \Rightarrow f_x(x,y,z) = 2xe^{yz}$, $f_y(x,y,z) = x^2e^{yz}(z) = x^2ze^{yz}$,

$f_z(x,y,z) = x^2e^{yz}(y) = x^2ye^{yz}$

27. $w = \ln(x+2y+3z) \Rightarrow \dfrac{\partial w}{\partial x} = \dfrac{1}{x+2y+3z}$, $\dfrac{\partial w}{\partial y} = \dfrac{2}{x+2y+3z}$, $\dfrac{\partial w}{\partial z} = \dfrac{3}{x+2y+3z}$

28. $w = \sqrt{r^2+s^2+t^2} \Rightarrow \dfrac{\partial w}{\partial r} = \dfrac{1}{2}(r^2+s^2+t^2)^{-1/2}(2r) = \dfrac{r}{\sqrt{r^2+s^2+t^2}}$, $\dfrac{\partial w}{\partial s} = \dfrac{s}{\sqrt{r^2+s^2+t^2}}$,

$\dfrac{\partial w}{\partial t} = \dfrac{t}{\sqrt{r^2+s^2+t^2}}$

29. $u = xe^{-t}\sin\theta \Rightarrow \dfrac{\partial u}{\partial x} = e^{-t}\sin\theta$, $\dfrac{\partial u}{\partial t} = -xe^{-t}\sin\theta$, $\dfrac{\partial u}{\partial \theta} = xe^{-t}\cos\theta$

30. $u = x^{y/z} \Rightarrow u_x = \dfrac{y}{z}x^{(y/z)-1}$, $u_y = x^{y/z}\ln x\cdot\dfrac{1}{z} = \dfrac{x^{y/z}}{z}\ln x$, $u_z = x^{y/z}\ln x\cdot\dfrac{-y}{z^2} = -\dfrac{yx^{y/z}}{z^2}\ln x$

31. $f(x,y,z,t) = xyz^2\tan(yt) \Rightarrow f_x(x,y,z,t) = yz^2\tan(yt)$,

$f_y(x,y,z,t) = xyz^2\cdot\sec^2(yt)\cdot t + xz^2\tan(yt) = xyz^2t\sec^2(yt) + xz^2\tan(yt)$,

$f_z(x,y,z,t) = 2xyz\tan(yt)$, $f_t(x,y,z,t) = xyz^2\sec^2(yt)\cdot y = xy^2z^2\sec^2(yt)$.

32. $f(x,y,z,t) = \dfrac{xy^2}{t+2z} \Rightarrow$

$f_x(x,y,z,t) = \dfrac{y^2}{t+2z}$, $f_y(x,y,z,t) = \dfrac{2xy}{t+2z}$,

$f_z(x,y,z,t) = xy^2(-1)(t+2z)^{-2}(2) = -\dfrac{2xy^2}{(t+2z)^2}$, $f_t(x,y,z,t) = xy^2(-1)(t+2z)^{-2}(1) = -\dfrac{xy^2}{(t+2z)^2}$.

33. $u = \sqrt{x_1^2+x_2^2+\cdots+x_n^2}$. For each $i = 1,\ldots,n$,

$u_{x_i} = \frac{1}{2}\left(x_1^2+x_2^2+\cdots+x_n^2\right)^{-1/2}(2x_i) = \dfrac{x_i}{\sqrt{x_1^2+x_2^2+\cdots+x_n^2}}$.

34. $u = \sin(x_1 + 2x_2 + \cdots + nx_n)$. For each $i = 1, \ldots, n$, $u_{x_i} = i\cos(x_1 + 2x_2 + \cdots + nx_n)$.

35. $f(x,y) = \sqrt{x^2+y^2} \quad\Rightarrow\quad f_x(x,y) = \frac{1}{2}(x^2+y^2)^{-1/2}(2x) = \dfrac{x}{\sqrt{x^2+y^2}}$, so $f_x(3,4) = \dfrac{3}{\sqrt{3^2+4^2}} = \dfrac{3}{5}$.

36. $f(x,y) = \sin(2x+3y) \quad\Rightarrow\quad f_y(x,y) = \cos(2x+3y)\cdot 3 = 3\cos(2x+3y)$, so
$f_y(-6,4) = 3\cos[2(-6)+3(4)] = 3\cos 0 = 3$.

37. $f(x,y,z) = \dfrac{x}{y+z} = x(y+z)^{-1} \quad\Rightarrow\quad f_z(x,y,z) = x(-1)(y+z)^{-2} = -\dfrac{x}{(y+z)^2}$, so

$f_z(3,2,1) = -\dfrac{3}{(2+1)^2} = -\dfrac{1}{3}$.

38. $f(u,v,w) = w\tan(uv) \quad\Rightarrow\quad f_v(u,v,w) = w\sec^2(uv)\cdot u = uw\sec^2(uv)$, so
$f_v(2,0,3) = (2)(3)\sec^2(2\cdot 0) = 6$.

39. $f(x,y) = x^2 - xy + 2y^2 \quad\Rightarrow$

$$f_x(x,y) = \lim_{h\to 0}\frac{f(x+h,y)-f(x,y)}{h} = \lim_{h\to 0}\frac{(x+h)^2-(x+h)y+2y^2-(x^2-xy+2y^2)}{h}$$
$$= \lim_{h\to 0}\frac{h(2x-y+h)}{h} = \lim_{h\to 0}(2x-y+h) = 2x-y$$

$$f_y(x,y) = \lim_{h\to 0}\frac{f(x,y+h)-f(x,y)}{h} = \lim_{h\to 0}\frac{x^2-x(y+h)+2(y+h)^2-(x^2-xy+2y^2)}{h}$$
$$= \lim_{h\to 0}\frac{h(4y-x+2h)}{h} = \lim_{h\to 0}(4y-x+2h) = 4y-x$$

40. $f(x,y) = \sqrt{3x-y} \quad\Rightarrow$

$$f_x(x,y) = \lim_{h\to 0}\frac{f(x+h,y)-f(x,y)}{h}$$
$$= \lim_{h\to 0}\frac{\sqrt{3(x+h)-y}-\sqrt{3x-y}}{h}\cdot\frac{\sqrt{3(x+h)-y}+\sqrt{3x-y}}{\sqrt{3(x+h)-y}+\sqrt{3x-y}}$$
$$= \lim_{h\to 0}\frac{3}{\sqrt{3(x+h)-y}+\sqrt{3x-y}} = \frac{3}{2\sqrt{3x-y}}$$

$$f_y(x,y) = \lim_{h\to 0}\frac{f(x,y+h)-f(x,y)}{h}$$
$$= \lim_{h\to 0}\frac{\sqrt{3x-(y+h)}-\sqrt{3x-y}}{h}\cdot\frac{\sqrt{3x-(y+h)}+\sqrt{3x-y}}{\sqrt{3x-(y+h)}+\sqrt{3x-y}}$$
$$= \lim_{h\to 0}\frac{-1}{\sqrt{3x-(y+h)}+\sqrt{3x-y}} = \frac{-1}{2\sqrt{3x-y}}$$

41. $x^2+y^2+z^2 = 3xyz \quad\Rightarrow\quad \dfrac{\partial}{\partial x}(x^2+y^2+z^2) = \dfrac{\partial}{\partial x}(3xyz) \quad\Rightarrow\quad 2x+0+2z\dfrac{\partial z}{\partial x} = 3y\left(x\dfrac{\partial z}{\partial x}+z\cdot 1\right)$

$\Leftrightarrow\quad 2z\dfrac{\partial z}{\partial x} - 3xy\dfrac{\partial z}{\partial x} = 3yz-2x \quad\Leftrightarrow\quad (2z-3xy)\dfrac{\partial z}{\partial x} = 3yz-2x$, so $\dfrac{\partial z}{\partial x} = \dfrac{3yz-2x}{2z-3xy}$.

$$\frac{\partial}{\partial y}(x^2+y^2+z^2) = \frac{\partial}{\partial y}(3xyz) \quad\Rightarrow\quad 0+2y+2z\frac{\partial z}{\partial y} = 3x\left(y\frac{\partial z}{\partial y}+z\cdot 1\right) \quad\Leftrightarrow$$

$$2z\frac{\partial z}{\partial y} - 3xy\frac{\partial z}{\partial y} = 3xz-2y \quad\Leftrightarrow\quad (2z-3xy)\frac{\partial z}{\partial y} = 3xz-2y, \text{ so } \frac{\partial z}{\partial y} = \frac{3xz-2y}{2z-3xy}.$$

42. $yz = \ln(x+z) \quad\Rightarrow\quad \dfrac{\partial}{\partial x}(yz) = \dfrac{\partial}{\partial x}(\ln(x+z)) \quad\Rightarrow\quad y\dfrac{\partial z}{\partial x} = \dfrac{1}{x+z}\left(1+\dfrac{\partial z}{\partial x}\right) \quad\Leftrightarrow$

$$\left(y-\frac{1}{x+z}\right)\frac{\partial z}{\partial x} = \frac{1}{x+z}, \text{ so } \frac{\partial z}{\partial x} = \frac{1/(x+z)}{y-1/(x+z)} = \frac{1}{y(x+z)-1}.$$

$$\frac{\partial}{\partial y}(yz) = \frac{\partial}{\partial y}(\ln(x+z)) \quad\Rightarrow\quad y\frac{\partial z}{\partial y}+z\cdot 1 = \frac{1}{x+z}\left(0+\frac{\partial z}{\partial y}\right) \quad\Leftrightarrow\quad \left(y-\frac{1}{x+z}\right)\frac{\partial z}{\partial y} = -z, \text{ so}$$

$$\frac{\partial z}{\partial y} = \frac{-z}{y-1/(x+z)} = \frac{z(x+z)}{1-y(x+z)}.$$

43. $x-z = \arctan(yz) \quad\Rightarrow\quad \dfrac{\partial}{\partial x}(x-z) = \dfrac{\partial}{\partial x}(\arctan(yz)) \quad\Rightarrow\quad 1-\dfrac{\partial z}{\partial x} = \dfrac{1}{1+(yz)^2}\cdot y\dfrac{\partial z}{\partial x} \quad\Leftrightarrow$

$$1 = \left(\frac{y}{1+y^2z^2}+1\right)\frac{\partial z}{\partial x} \quad\Leftrightarrow\quad 1 = \left(\frac{y+1+y^2z^2}{1+y^2z^2}\right)\frac{\partial z}{\partial x}, \text{ so } \frac{\partial z}{\partial x} = \frac{1+y^2z^2}{1+y+y^2z^2}.$$

$$\frac{\partial}{\partial y}(x-z) = \frac{\partial}{\partial y}(\arctan(yz)) \quad\Rightarrow\quad 0-\frac{\partial z}{\partial y} = \frac{1}{1+(yz)^2}\cdot\left(y\frac{\partial z}{\partial y}+z\cdot 1\right) \quad\Leftrightarrow$$

$$-\frac{z}{1+y^2z^2} = \left(\frac{y}{1+y^2z^2}+1\right)\frac{\partial z}{\partial y} \quad\Leftrightarrow\quad -\frac{z}{1+y^2z^2} = \left(\frac{y+1+y^2z^2}{1+y^2z^2}\right)\frac{\partial z}{\partial y} \quad\Leftrightarrow\quad \frac{\partial z}{\partial y} = -\frac{z}{1+y+y^2z^2}.$$

44. $\sin(xyz) = x+2y+3z \quad\Rightarrow\quad \dfrac{\partial}{\partial x}(\sin(xyz)) = \dfrac{\partial}{\partial x}(x+2y+3z) \quad\Rightarrow$

$$\cos(xyz)\cdot y\left(x\frac{\partial z}{\partial x}+z\right) = 1+3\frac{\partial z}{\partial x} \quad\Leftrightarrow\quad (xy\cos(xyz)-3)\frac{\partial z}{\partial x} = 1-yz\cos(xyz), \text{ so}$$

$$\frac{\partial z}{\partial x} = \frac{1-yz\cos(xyz)}{xy\cos(xyz)-3}.$$

$$\frac{\partial}{\partial y}(\sin(xyz)) = \frac{\partial}{\partial y}(x+2y+3z) \quad\Rightarrow\quad \cos(xyz)\cdot x\left(y\frac{\partial z}{\partial y}+z\right) = 2+3\frac{\partial z}{\partial y} \quad\Leftrightarrow$$

$$(xy\cos(xyz)-3)\frac{\partial z}{\partial y} = 2-xz\cos(xyz), \text{ so } \frac{\partial z}{\partial y} = \frac{2-xz\cos(xyz)}{xy\cos(xyz)-3}.$$

45. (a) $z = f(x)+g(y) \quad\Rightarrow\quad \dfrac{\partial z}{\partial x} = f'(x),\ \dfrac{\partial z}{\partial y} = g'(y)$

(b) $z = f(x+y)$. Let $u = x+y$. Then $\dfrac{\partial z}{\partial x} = \dfrac{df}{du}\dfrac{\partial u}{\partial x} = \dfrac{df}{du}(1) = f'(u) = f'(x+y)$,

$$\frac{\partial z}{\partial y} = \frac{df}{du}\frac{\partial u}{\partial y} = \frac{df}{du}(1) = f'(u) = f'(x+y).$$

46. (a) $z = f(x)g(y) \quad \Rightarrow \quad \dfrac{\partial z}{\partial x} = f'(x)g(y), \ \dfrac{\partial z}{\partial y} = f(x)g'(y)$

(b) $z = f(xy)$. Let $u = xy$. Then $\dfrac{\partial u}{\partial x} = y$ and $\dfrac{\partial u}{\partial y} = x$. Hence $\dfrac{\partial z}{\partial x} = \dfrac{df}{du}\dfrac{\partial u}{\partial x} = \dfrac{df}{du} \cdot y = yf'(u) = yf'(xy)$ and $\dfrac{\partial z}{\partial y} = \dfrac{df}{du}\dfrac{\partial u}{\partial y} = \dfrac{df}{du} \cdot x = xf'(u) = xf'(xy)$.

(c) $z = f\left(\dfrac{x}{y}\right)$. Let $u = \dfrac{x}{y}$. Then $\dfrac{\partial u}{\partial x} = \dfrac{1}{y}$ and $\dfrac{\partial u}{\partial y} = -\dfrac{x}{y^2}$. Hence $\dfrac{\partial z}{\partial x} = \dfrac{df}{du}\dfrac{\partial u}{\partial x} = f'(u)\dfrac{1}{y} = \dfrac{f'(x/y)}{y}$ and $\dfrac{\partial z}{\partial y} = \dfrac{df}{du}\dfrac{\partial u}{\partial y} = f'(u)\left(-\dfrac{x}{y^2}\right) = -\dfrac{xf'(x/y)}{y^2}$.

47. $f(x,y) = x^4 - 3x^2y^3 \quad \Rightarrow \quad f_x(x,y) = 4x^3 - 6xy^3$, $f_y(x,y) = -9x^2y^2$. Then $f_{xx}(x,y) = 12x^2 - 6y^3$, $f_{xy}(x,y) = -18xy^2$, $f_{yx}(x,y) = -18xy^2$, and $f_{yy}(x,y) = -18x^2y$.

48. $f(x,y) = \ln(3x+5y) \quad \Rightarrow \quad f_x(x,y) = \dfrac{3}{3x+5y}$, $f_y(x,y) = \dfrac{5}{3x+5y}$. Then

$f_{xx}(x,y) = 3(-1)(3x+5y)^{-2}(3) = -\dfrac{9}{(3x+5y)^2}$, $f_{xy}(x,y) = -\dfrac{15}{(3x+5y)^2}$, $f_{yx}(x,y) = -\dfrac{15}{(3x+5y)^2}$,

and $f_{yy}(x,y) = -\dfrac{25}{(3x+5y)^2}$.

49. $z = \dfrac{x}{x+y} = x(x+y)^{-1} \quad \Rightarrow \quad z_x = \dfrac{1(x+y) - 1(x)}{(x+y)^2} = \dfrac{y}{(x+y)^2}$, $z_y = x(-1)(x+y)^{-2} = -\dfrac{x}{(x+y)^2}$.

Then $z_{xx} = y(-2)(x+y)^{-3} = -\dfrac{2y}{(x+y)^3}$, $z_{xy} = \dfrac{1(x+y)^2 - y(2)(x+y)}{[(x+y)^2]^2} = \dfrac{x+y-2y}{(x+y)^3} = \dfrac{x-y}{(x+y)^3}$,

$z_{yx} = -\dfrac{1(x+y)^2 - x(2)(x+y)}{[(x+y)^2]^2} = -\dfrac{-x^2+xy+y^2}{(x+y)^2} = \dfrac{(x+y)(x-y)}{(x+y)^2} = \dfrac{x-y}{(x+y)^3}$, and

$z_{yy} = -x(-2)(x+y)^{-3} = \dfrac{2x}{(x+y)^3}$.

50. $z = y\tan 2x \quad \Rightarrow \quad z_x = y\sec^2(2x) \cdot 2 = 2y\sec^2(2x)$, $z_y = \tan 2x$. Then
$z_{xx} = 2y(2)\sec(2x) \cdot \sec(2x)\tan(2x) \cdot 2 = 8y\sec^2(2x)\tan(2x)$, $z_{xy} = 2\sec^2(2x)$,
$z_{yx} = \sec^2(2x) \cdot 2 = 2\sec^2(2x)$, and $z_{yy} = 0$.

51. $u = e^{-s}\sin t \quad \Rightarrow \quad u_s = -e^{-s}\sin t$, $u_t = e^{-s}\cos t$. Then $u_{ss} = e^{-s}\sin t$, $u_{st} = -e^{-s}\cos t$,
$u_{ts} = -e^{-s}\cos t$, and $u_{tt} = -e^{-s}\sin t$.

52. $v = \sqrt{x+y^2} \quad \Rightarrow \quad v_x = \frac{1}{2}(x+y^2)^{-1/2} = \dfrac{1}{2\sqrt{x+y^2}}$,

$v_y = \frac{1}{2}(x+y^2)^{-1/2}(2y) = \dfrac{y}{\sqrt{x+y^2}}$. Then $v_{xx} = \frac{1}{2}\left(-\frac{1}{2}\right)(x+y^2)^{-3/2} = -\dfrac{1}{4(x+y^2)^{3/2}}$,

$v_{xy} = \frac{1}{2}\left(-\frac{1}{2}\right)(x+y^2)^{-3/2}(2y) = -\dfrac{y}{2(x+y^2)^{3/2}}$, $v_{yx} = y\left(-\frac{1}{2}\right)(x+y^2)^{-3/2} = -\dfrac{y}{2(x+y^2)^{3/2}}$,

and $v_{yy} = \dfrac{1\sqrt{x+y^2} - y\left(\frac{1}{2}\right)(x+y^2)^{-1/2}(2y)}{\left(\sqrt{x+y^2}\right)^2} = \dfrac{(x+y^2) - y^2}{(x+y^2)^{3/2}} = \dfrac{x}{(x+y^2)^{3/2}}$.

53. $u = x\sin(x+2y) \quad\Rightarrow\quad u_x = x\cdot\cos(x+2y)(1) + \sin(x+2y)\cdot 1 = x\cos(x+2y) + \sin(x+2y)$,
$u_{xy} = x(-\sin(x+2y)(2)) + \cos(x+2y)(2) = 2\cos(x+2y) - 2x\sin(x+2y)$ and
$u_y = x\cos(x+2y)(2) = 2x\cos(x+2y)$,
$u_{yx} = 2x\cdot(-\sin(x+2y)(1)) + \cos(x+2y)\cdot 2 = 2\cos(x+2y) - 2x\sin(x+2y)$. Thus $u_{xy} = u_{yx}$.

54. $u = x^4y^2 - 2xy^5 \quad\Rightarrow\quad u_x = 4x^3y^2 - 2y^5$, $u_{xy} = 8x^3y - 10y^4$ and $u_y = 2x^4y - 10xy^4$, $u_{yx} = 8x^3y - 10y^4$.
Thus $u_{xy} = u_{yx}$.

55. $u = \ln\sqrt{x^2+y^2} = \ln\left(x^2+y^2\right)^{1/2} = \frac{1}{2}\ln\left(x^2+y^2\right) \quad\Rightarrow\quad u_x = \dfrac{1}{2}\dfrac{1}{x^2+y^2}\cdot 2x = \dfrac{x}{x^2+y^2}$,
$u_{xy} = x(-1)\left(x^2+y^2\right)^{-2}(2y) = -\dfrac{2xy}{(x^2+y^2)^2}$ and $u_y = \dfrac{1}{2}\dfrac{1}{x^2+y^2}\cdot 2y = \dfrac{y}{x^2+y^2}$,
$u_{yx} = y(-1)\left(x^2+y^2\right)^{-2}(2x) = -\dfrac{2xy}{(x^2+y^2)^2}$. Thus $u_{xy} = u_{yx}$.

56. $u = xye^y \quad\Rightarrow\quad u_x = ye^y$, $u_{xy} = ye^y + e^y = (y+1)e^y$ and $u_y = x(ye^y + e^y) = x(y+1)e^y$,
$u_{yx} = (y+1)e^y$. Thus $u_{xy} = u_{yx}$.

57. $f(x,y) = 3xy^4 + x^3y^2 \quad\Rightarrow\quad f_x = 3y^4 + 3x^2y^2$, $f_{xx} = 6xy^2$, $f_{xxy} = 12xy$ and $f_y = 12xy^3 + 2x^3y$,
$f_{yy} = 36xy^2 + 2x^3$, $f_{yyy} = 72xy$.

58. $f(x,t) = x^2e^{-ct} \quad\Rightarrow\quad f_t = x^2(-ce^{-ct})$, $f_{tt} = x^2(c^2e^{-ct})$, $f_{ttt} = x^2(-c^3e^{-ct}) = -c^3x^2e^{-ct}$ and
$f_{tx} = 2x(-ce^{-ct})$, $f_{txx} = 2(-ce^{-ct}) = -2ce^{-ct}$.

59. $f(x,y,z) = \cos(4x+3y+2z) \quad\Rightarrow$
$f_x = -\sin(4x+3y+2z)(4) = -4\sin(4x+3y+2z)$,
$f_{xy} = -4\cos(4x+3y+2z)(3) = -12\cos(4x+3y+2z)$,
$f_{xyz} = -12(-\sin(4x+3y+2z))(2) = 24\sin(4x+3y+2z)$ and
$f_y = -\sin(4x+3y+2z)(3) = -3\sin(4x+3y+2z)$,
$f_{yz} = -3\cos(4x+3y+2z)(2) = -6\cos(4x+3y+2z)$,
$f_{yzz} = -6(-\sin(4x+3y+2z))(2) = 12\sin(4x+3y+2z)$.

60. $f(r,s,t) = r\ln(rs^2t^3) \quad\Rightarrow$
$f_r = r\cdot\dfrac{1}{rs^2t^3}(s^2t^3) + \ln(rs^2t^3)\cdot 1 = \dfrac{rs^2t^3}{rs^2t^3} + \ln(rs^2t^3) = 1 + \ln(rs^2t^3)$,
$f_{rs} = \dfrac{1}{rs^2t^3}(2rst^3) = \dfrac{2}{s} = 2s^{-1}$, $f_{rss} = -2s^{-2} = -\dfrac{2}{s^2}$ and $f_{rst} = 0$.

61. $u = e^{r\theta}\sin\theta \quad\Rightarrow\quad \dfrac{\partial u}{\partial\theta} = e^{r\theta}\cos\theta + \sin\theta\cdot e^{r\theta}(r) = e^{r\theta}(\cos\theta + r\sin\theta)$,
$\dfrac{\partial^2 u}{\partial r\,\partial\theta} = e^{r\theta}(\sin\theta) + (\cos\theta + r\sin\theta)\,e^{r\theta}(\theta) = e^{r\theta}(\sin\theta + \theta\cos\theta + r\theta\sin\theta)$,
$\dfrac{\partial^3 u}{\partial r^2\,\partial\theta} = e^{r\theta}(\theta\sin\theta) + (\sin\theta + \theta\cos\theta + r\theta\sin\theta)\cdot e^{r\theta}(\theta) = \theta e^{r\theta}(2\sin\theta + \theta\cos\theta + r\theta\sin\theta)$.

62. $z = u\sqrt{v-w} = u(v-w)^{1/2} \quad\Rightarrow\quad \dfrac{\partial z}{\partial w} = u\left[\frac{1}{2}(v-w)^{-1/2}(-1)\right] = -\frac{1}{2}u(v-w)^{-1/2}$,
$\dfrac{\partial^2 z}{\partial v\,\partial w} = -\frac{1}{2}u\left(-\frac{1}{2}(v-w)^{-3/2}(1)\right) = \frac{1}{4}u(v-w)^{-3/2}$, $\dfrac{\partial^3 z}{\partial u\,\partial v\,\partial w} = \frac{1}{4}(v-w)^{-3/2}$.

63. $w = \dfrac{x}{y+2z} = x(y+2z)^{-1} \;\Rightarrow\; \dfrac{\partial w}{\partial x} = (y+2z)^{-1}, \dfrac{\partial^2 w}{\partial y\,\partial x} = -(y+2z)^{-2}(1) = -(y+2z)^{-2},$

$\dfrac{\partial^3 w}{\partial z\,\partial y\,\partial x} = -(-2)(y+2z)^{-3}(2) = 4(y+2z)^{-3} = \dfrac{4}{(y+2z)^3}$ and

$\dfrac{\partial w}{\partial y} = x(-1)(y+2z)^{-2}(1) = -x(y+2z)^{-2}, \dfrac{\partial^2 w}{\partial x\,\partial y} = -(y+2z)^{-2}, \dfrac{\partial^3 w}{\partial x^2\,\partial y} = 0.$

64. $u = x^a y^b z^c$. If $a = 0$, or if $b = 0$ or 1, or if $c = 0, 1$, or 2, then $\dfrac{\partial^6 u}{\partial x\,\partial y^2\,\partial z^3} = 0$. Otherwise $\dfrac{\partial u}{\partial z} = cx^a y^b z^{c-1}$,

$\dfrac{\partial^2 u}{\partial z^2} = c(c-1)x^a y^b z^{c-2}, \dfrac{\partial^3 u}{\partial z^3} = c(c-1)(c-2)x^a y^b z^{c-3}, \dfrac{\partial^4 u}{\partial y\,\partial z^3} = bc(c-1)(c-2)x^a y^{b-1} z^{c-3},$

$\dfrac{\partial^5 u}{\partial y^2\,\partial z^3} = b(b-1)c(c-1)(c-2)x^a y^{b-2} z^{c-3}$, and $\dfrac{\partial^6 u}{\partial x\,\partial y^2\,\partial z^3} = ab(b-1)c(c-1)(c-2)x^{a-1} y^{b-2} z^{c-3}$.

65. By Definition 4, $f_x(3,2) = \lim\limits_{h\to 0} \dfrac{f(3+h,2) - f(3,2)}{h}$ which we can approximate by considering $h = 0.5$ and $h = -0.5$: $f_x(3,2) \approx \dfrac{f(3.5,2) - f(3,2)}{0.5} = \dfrac{22.4 - 17.5}{0.5} = 9.8$,

$f_x(3,2) \approx \dfrac{f(2.5,2) - f(3,2)}{-0.5} = \dfrac{10.2 - 17.5}{-0.5} = 14.6$. Averaging these values, we estimate $f_x(3,2)$ to be approximately 12.2. Similarly, $f_x(3,2.2) = \lim\limits_{h\to 0} \dfrac{f(3+h,2.2) - f(3,2.2)}{h}$ which we can approximate by considering $h = 0.5$ and $h = -0.5$: $f_x(3,2.2) \approx \dfrac{f(3.5,2.2) - f(3,2.2)}{0.5} = \dfrac{26.1 - 15.9}{0.5} = 20.4$,

$f_x(3,2.2) \approx \dfrac{f(2.5,2.2) - f(3,2.2)}{-0.5} = \dfrac{9.3 - 15.9}{-0.5} = 13.2$. Averaging these values, we have $f_x(3,2.2) \approx 16.8$.

To estimate $f_{xy}(3,2)$, we first need an estimate for $f_x(3,1.8)$:

$f_x(3,1.8) \approx \dfrac{f(3.5,1.8) - f(3,1.8)}{0.5} = \dfrac{20.0 - 18.1}{0.5} = 3.8$,

$f_x(3,1.8) \approx \dfrac{f(2.5,1.8) - f(3,1.8)}{-0.5} = \dfrac{12.5 - 18.1}{-0.5} = 11.2$. Averaging these values, we get $f_x(3,1.8) \approx 7.5$.

Now $f_{xy}(x,y) = \dfrac{\partial}{\partial y}\,[f_x(x,y)]$ and $f_x(x,y)$ is itself a function of 2 variables, so Definition 4 says that

$f_{xy}(x,y) = \dfrac{\partial}{\partial y}\,[f_x(x,y)] = \lim\limits_{h\to 0} \dfrac{f_x(x,y+h) - f_x(x,y)}{h} \;\Rightarrow\; f_{xy}(3,2) = \lim\limits_{h\to 0} \dfrac{f_x(3,2+h) - f_x(3,2)}{h}$.

We can estimate this value using our previous work with $h = 0.2$ and $h = -0.2$:

$f_{xy}(3,2) \approx \dfrac{f_x(3,2.2) - f_x(3,2)}{0.2} = \dfrac{16.8 - 12.2}{0.2} = 23$,

$f_{xy}(3,2) \approx \dfrac{f_x(3,1.8) - f_x(3,2)}{-0.2} = \dfrac{7.5 - 12.2}{-0.2} = 23.5$. Averaging these values, we estimate $f_{xy}(3,2)$ to be approximately 23.25.

66. (a) If we fix y and allow x to vary, the level curves indicate that the value of f decreases as we move through P in the positive x-direction, so f_x is negative at P.

(b) If we fix x and allow y to vary, the level curves indicate that the value of f increases as we move through P in the positive y-direction, so f_y is positive at P.

(c) $f_{xx} = \dfrac{\partial}{\partial x}(f_x)$, so if we fix y and allow x to vary, f_{xx} is the rate of change of f_x as x increases. Note that at points to the right of P the level curves are spaced farther apart (in the x-direction) than at points to the left of P, demonstrating that f decreases less quickly with respect to x to the right of P. So as we move through P in the positive x-direction the (negative) value of f_x increases, hence $\dfrac{\partial}{\partial x}(f_x) = f_{xx}$ is positive at P.

(d) $f_{xy} = \dfrac{\partial}{\partial y}(f_x)$, so if we fix x and allow y to vary, f_{xy} is the rate of change of f_x as y increases. The level curves are closer together (in the x-direction) at points above P than at those below P, demonstrating that f decreases more quickly with respect to x for y-values above P. So as we move through P in the positive y-direction, the (negative) value of f_x decreases, hence f_{xy} is negative.

(e) $f_{yy} = \dfrac{\partial}{\partial y}(f_y)$, so if we fix x and allow y to vary, f_{yy} is the rate of change of f_y as y increases. The level curves are closer together (in the y-direction) at points above P than at those below P, demonstrating that f increases more quickly with respect to y above P. So as we move through P in the positive y-direction the (positive) value of f_y increases, hence $\dfrac{\partial}{\partial y}(f_y) = f_{yy}$ is positive at P.

67. $u = e^{-\alpha^2 k^2 t} \sin kx \quad \Rightarrow \quad u_x = ke^{-\alpha^2 k^2 t} \cos kx$, $u_{xx} = -k^2 e^{-\alpha^2 k^2 t} \sin kx$, and $u_t = -\alpha^2 k^2 e^{-\alpha^2 k^2 t} \sin kx$. Thus $\alpha^2 u_{xx} = u_t$.

68. (a) $u = x^2 + y^2 \quad \Rightarrow \quad u_x = 2x$, $u_{xx} = 2$; $u_y = 2y$, $u_{yy} = 2$. Thus $u_{xx} + u_{yy} \neq 0$ and $u = x^2 + y^2$ does not satisfy Laplace's Equation.

(b) $u = x^2 - y^2$ is a solution: $u_{xx} = 2$, $u_{yy} = -2$ so $u_{xx} + u_{yy} = 0$.

(c) $u = x^3 + 3xy^2$ is not a solution: $u_x = 3x^2 + 3y^2$, $u_{xx} = 6x$; $u_y = 6xy$, $u_{yy} = 6x$.

(d) $u = \ln \sqrt{x^2 + y^2}$ is a solution: $u_x = \dfrac{1}{\sqrt{x^2+y^2}} \left(\dfrac{1}{2}\right)(x^2+y^2)^{-1/2}(2x) = \dfrac{x}{x^2+y^2}$,

$u_{xx} = \dfrac{(x^2+y^2) - x(2x)}{(x^2+y^2)^2} = \dfrac{y^2 - x^2}{(x^2+y^2)^2}$. By symmetry, $u_{yy} = \dfrac{x^2 - y^2}{(x^2+y^2)^2}$, so $u_{xx} + u_{yy} = 0$.

(e) $u = \sin x \cosh y + \cos x \sinh y$ is a solution:
$u_x = \cos x \cosh y - \sin x \sinh y$, $u_{xx} = -\sin x \cosh y - \cos x \sinh y$, and $u_y = \sin x \sinh y + \cos x \cosh y$, $u_{yy} = \sin x \cosh y + \cos x \sinh y$.

(f) $u = e^{-x} \cos y - e^{-y} \cos x$ is a solution: $u_x = -e^{-x} \cos y + e^{-y} \sin x$, $u_{xx} = e^{-x} \cos y + e^{-y} \cos x$, and $u_y = -e^{-x} \sin y + e^{-y} \cos x$, $u_{yy} = -e^{-x} \cos y - e^{-y} \cos x$.

69. $u = \dfrac{1}{\sqrt{x^2+y^2+z^2}} \quad \Rightarrow \quad u_x = \left(-\tfrac{1}{2}\right)(x^2+y^2+z^2)^{-3/2}(2x) = -x(x^2+y^2+z^2)^{-3/2}$ and

$$u_{xx} = -(x^2+y^2+z^2)^{-3/2} - x\left(-\tfrac{3}{2}\right)(x^2+y^2+z^2)^{-5/2}(2x) = \frac{2x^2 - y^2 - z^2}{(x^2+y^2+z^2)^{5/2}}.$$

By symmetry, $u_{yy} = \dfrac{2y^2 - x^2 - z^2}{(x^2+y^2+z^2)^{5/2}}$ and $u_{zz} = \dfrac{2z^2 - x^2 - y^2}{(x^2+y^2+z^2)^{5/2}}$.

Thus $u_{xx} + u_{yy} + u_{zz} = \dfrac{2x^2 - y^2 - z^2 + 2y^2 - x^2 - z^2 + 2z^2 - x^2 - y^2}{(x^2+y^2+z^2)^{5/2}} = 0$.

70. (a) $u = \sin(kx)\sin(akt) \quad \Rightarrow \quad u_t = ak\sin(kx)\cos(akt)$, $u_{tt} = -a^2k^2\sin(kx)\sin(akt)$, $u_x = k\cos(kx)\sin(akt)$, $u_{xx} = -k^2\sin(kx)\sin(akt)$. Thus $u_{tt} = a^2 u_{xx}$.

(b) $u = \dfrac{t}{a^2t^2 - x^2} \quad \Rightarrow \quad u_t = \dfrac{(a^2t^2 - x^2) - t(2a^2t)}{(a^2t^2 - x^2)^2} = -\dfrac{a^2t^2 + x^2}{(a^2t^2 - x^2)^2}$,

$$u_{tt} = \frac{-2a^2t(a^2t^2 - x^2)^2 + (a^2t^2 - x^2)(2)(a^2t^2 - x^2)(2a^2t)}{(a^2t^2 - x^2)^4} = \frac{2a^4t^3 + 6a^2tx^2}{a^2t^2 - x^2},$$

$$u_x = t(-1)(a^2t^2 - x^2)^{-2}(2x) = \frac{2tx}{(a^2t^2 - x^2)^2},$$

$$u_{xx} = \frac{2t(a^2t^2 - x^2)^2 - 2tx\,(2)\,(a^2t^2 - x^2)(-2x)}{(a^2t^2 - x^2)^4} = \frac{2a^2t^3 - 2tx^2 + 8tx^2}{(a^2t^2 - x^2)^3} = \frac{2a^2t^3 + 6tx^2}{(a^2t^2 - x^2)^4}.$$

Thus $u_{tt} = a^2u_{xx}$.

(c) $u = (x - at)^6 + (x + at)^6 \quad \Rightarrow \quad u_t = -6a(x - at)^5 + 6a(x + at)^5$,

$u_{tt} = 30a^2(x - at)^4 + 30a^2(x + at)^4$, $u_x = 6(x - at)^5 + 6(x + at)^5$, $u_{xx} = 30(x - at)^4 + 30(x + at)^4$.

Thus $u_{tt} = a^2u_{xx}$.

(d) $u = \sin(x - at) + \ln(x + at) \quad \Rightarrow \quad u_t = -a\cos(x - at) + \dfrac{a}{x + at}$, $u_{tt} = -a^2\sin(x - at) - \dfrac{a^2}{(x + at)^2}$,

$u_x = \cos(x - at) + \dfrac{1}{x + at}$, $u_{xx} = -\sin(x - at) - \dfrac{1}{(x + at)^2}$. Thus $u_{tt} = a^2u_{xx}$.

71. Let $v = x + at$, $w = x - at$. Then $u_t = \dfrac{\partial[f(v) + g(w)]}{\partial t} = \dfrac{df(v)}{dv}\dfrac{\partial v}{\partial t} + \dfrac{dg(w)}{dw}\dfrac{\partial w}{\partial t} = af'(v) - ag'(w)$ and

$u_{tt} = \dfrac{\partial[af'(v) - ag'(w)]}{\partial t} = a[af''(v) + ag''(w)] = a^2[f''(v) + g''(w)]$. Similarly, by using the Chain Rule we have $u_x = f'(v) + g'(w)$ and $u_{xx} = f''(v) + g''(w)$. Thus $u_{tt} = a^2u_{xx}$.

72. For each i, $i = 1, \ldots, n$, $\partial u/\partial x_i = a_i e^{a_1x_1 + a_2x_2 + \cdots + a_nx_n}$ and $\partial^2 u/\partial x_i^2 = a_i^2 e^{a_1x_1 + a_2x_2 + \cdots + a_nx_n}$. Then

$\dfrac{\partial^2 u}{\partial x_1^2} + \dfrac{\partial^2 u}{\partial x_2^2} + \cdots + \dfrac{\partial^2 u}{\partial x_n^2} = (a_1^2 + a_2^2 + \cdots + a_n^2)e^{a_1x_1 + a_2x_2 + \cdots + a_nx_n} = e^{a_1x_1 + a_2x_2 + \cdots + a_nx_n} = u$ since $a_1^2 + a_2^2 + \cdots + a_n^2 = 1$.

73. $z_x = e^y + ye^x$, $z_{xx} = ye^x$, $\partial^3 z/\partial x^3 = ye^x$. By symmetry $z_y = xe^y + e^x$, $z_{yy} = xe^y$ and $\partial^3 z/\partial y^3 = xe^y$. Then $\partial^3 z/\partial x\partial y^2 = e^y$ and $\partial^3 z/\partial x^2\partial y = e^x$. Thus $z = xe^y + ye^x$ satisfies the given partial differential equation.

74. $P = bL^\alpha K^\beta$, so $\dfrac{\partial P}{\partial L} = \alpha bL^{\alpha-1}K^\beta$ and $\dfrac{\partial P}{\partial K} = \beta bL^\alpha K^{\beta-1}$. Then

$$L\frac{\partial P}{\partial L} + K\frac{\partial P}{\partial K} = L\left(\alpha bL^{\alpha-1}K^\beta\right) + K\left(\beta bL^\alpha K^{\beta-1}\right) = \alpha bL^{1+\alpha-1}K^\beta + \beta bL^\alpha K^{1+\beta-1}$$
$$= (\alpha + \beta)bL^\alpha K^\beta = (\alpha + \beta)P$$

75. If we fix $K = K_0$, $P(L, K_0)$ is a function of a single variable L, and $\dfrac{dP}{dL} = \alpha\dfrac{P}{L}$ is a separable differential equation. Then $\dfrac{dP}{P} = \alpha\dfrac{dL}{L} \quad \Rightarrow \quad \displaystyle\int \frac{dP}{P} = \int \alpha\frac{dL}{L} \quad \Rightarrow \quad \ln|P| = \alpha\ln|L| + C(K_0)$, where $C(K_0)$ can depend on K_0. Then $|P| = e^{\alpha\ln|L| + C(K_0)}$, and since $P > 0$ and $L > 0$, we have $P = e^{\alpha\ln L}e^{C(K_0)} = e^{C(K_0)}e^{\ln L^\alpha} = C_1(K_0)L^\alpha$ where $C_1(K_0) = e^{C(K_0)}$.

76. (a) $\partial T/\partial x = -60(2x)/(1+x^2+y^2)^2$, so at $(2,1)$, $T_x = -240/(1+4+1)^2 = -\frac{20}{3}$.

(b) $\partial T/\partial y = -60(2y)/(1+x^2+y^2)^2$, so at $(2,1)$, $T_y = -120/36 = -\frac{10}{3}$. Thus from the point $(2,1)$ the temperature is decreasing at a rate of $\frac{20}{3}$ °C/m in the x-direction and is decreasing at a rate of $\frac{10}{3}$ °C/m in the y-direction.

77. By the Chain Rule, taking the partial derivative of both sides with respect to R_1 gives

$$\frac{\partial R^{-1}}{\partial R}\frac{\partial R}{\partial R_1} = \frac{\partial\left[(1/R_1)+(1/R_2)+(1/R_3)\right]}{\partial R_1} \text{ or } -R^{-2}\frac{\partial R}{\partial R_1} = -R_1^{-2}. \text{ Thus } \frac{\partial R}{\partial R_1} = \frac{R^2}{R_1^2}.$$

78. $P = \dfrac{mRT}{V}$ so $\dfrac{\partial P}{\partial V} = \dfrac{-mRT}{V^2}$; $V = \dfrac{mRT}{P}$, so $\dfrac{\partial V}{\partial T} = \dfrac{mR}{P}$; $T = \dfrac{PV}{mR}$, so $\dfrac{\partial T}{\partial P} = \dfrac{V}{mR}$.

Thus $\dfrac{\partial P}{\partial V}\dfrac{\partial V}{\partial T}\dfrac{\partial T}{\partial P} = \dfrac{-mRT}{V^2}\dfrac{mR}{P}\dfrac{V}{mR} = \dfrac{-mRT}{PV} = -1$, since $PV = mRT$.

79. By Exercise 78, $PV = mRT \quad\Rightarrow\quad P = \dfrac{mRT}{V}$, so $\dfrac{\partial P}{\partial T} = \dfrac{mR}{V}$. Also, $PV = mRT \quad\Rightarrow\quad V = \dfrac{mRT}{P}$ and $\dfrac{\partial V}{\partial T} = \dfrac{mR}{P}$. Since $T = \dfrac{PV}{mR}$, we have $T\dfrac{\partial P}{\partial T}\dfrac{\partial V}{\partial T} = \dfrac{PV}{mR}\cdot\dfrac{mR}{V}\cdot\dfrac{mR}{P} = mR$.

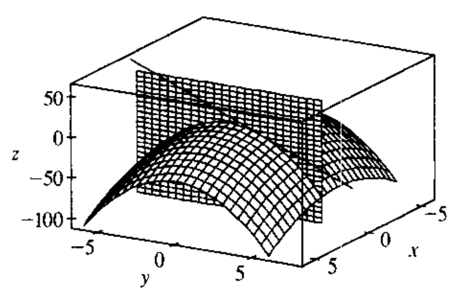
80. $\dfrac{\partial W}{\partial T} = 0.6215 + 0.3965v^{0.16}$. When $T = -15$°C and $v = 30$ km/h, $\dfrac{\partial W}{\partial T} = 0.6215 + 0.3965(30)^{0.16} \approx 1.3048$, so we would expect the apparent temperature to drop by approximately 1.3°C if the actual temperature decreases by 1°C. $\dfrac{\partial W}{\partial v} = -11.37(0.16)v^{-0.84} + 0.3965T(0.16)v^{-0.84}$ and when $T = -15$°C and $v = 30$ km/h, $\dfrac{\partial W}{\partial v} = -11.37(0.16)(30)^{-0.84} + 0.3965(-15)(0.16)(30)^{-0.84} \approx -0.1592$, so we would expect the apparent temperature to drop by approximately 0.16°C if the wind speed increases by 1 km/h.

81. $\dfrac{\partial K}{\partial m} = \frac{1}{2}V^2$, $\dfrac{\partial K}{\partial V} = mV$, $\dfrac{\partial^2 K}{\partial V^2} = m$. Thus $\dfrac{\partial K}{\partial m}\cdot\dfrac{\partial^2 K}{\partial V^2} = \frac{1}{2}V^2 m = K$.

82. The Law of Cosines says that $a^2 = b^2 + c^2 - 2bc\cos A$. Thus $\dfrac{\partial(a^2)}{\partial a} = \dfrac{\partial(b^2+c^2-2ab\cos A)}{\partial a}$ or $2a = -2bc(-\sin A)\dfrac{\partial A}{\partial a}$, implying that $\dfrac{\partial A}{\partial a} = \dfrac{a}{bc\sin A}$. Taking the partial derivative of both sides with respect to b gives $0 = 2b - 2c(\cos A) - 2bc(-\sin A)\dfrac{\partial A}{\partial b}$. Thus $\dfrac{\partial A}{\partial b} = \dfrac{c\cos A - b}{bc\sin A}$. By symmetry $\dfrac{\partial A}{\partial c} = \dfrac{b\cos A - c}{bc\sin A}$.

83. $f_x(x,y) = x + 4y \quad\Rightarrow\quad f_{xy}(x,y) = 4$ and $f_y(x,y) = 3x - y \quad\Rightarrow\quad f_{yx}(x,y) = 3$. Since f_{xy} and f_{yx} are continuous everywhere but $f_{xy}(x,y) \neq f_{yx}(x,y)$, Clairaut's Theorem implies that such a function $f(x,y)$ does not exist.

84. Setting $x = 1$, the equation of the parabola of intersection is $z = 6 - 1 - 1 - 2y^2 = 4 - 2y^2$. The slope of the tangent is $\partial z/\partial y = -4y$, so at $(1,2,-4)$ the slope is -8. Parametric equations for the line are therefore $x = 1$, $y = 2 + t$, $z = -4 - 8t$.



85. By the geometry of partial derivatives, the slope of the tangent line is $f_x(1,2)$. By implicit differentiation of $4x^2 + 2y^2 + z^2 = 16$, we get $8x + 2z\,(\partial z/\partial x) = 0 \Rightarrow \partial z/\partial x = -4x/z$, so when $x = 1$ and $z = 2$ we have $\partial z/\partial x = -2$. So the slope is $f_x(1,2) = -2$. Thus the tangent line is given by $z - 2 = -2(x-1)$, $y = 2$. Taking the parameter to be $t = x - 1$, we can write parametric equations for this line: $x = 1 + t$, $y = 2$, $z = 2 - 2t$.

86. $T(x,t) = T_0 + T_1 e^{-\lambda x} \sin(\omega t - \lambda x)$

(a) $$\begin{aligned}\partial T/\partial x &= T_1 e^{-\lambda x}\left[\cos(\omega t - \lambda x)(-\lambda)\right] + T_1(-\lambda e^{-\lambda x})\sin(\omega t - \lambda x)\\ &= -\lambda T_1 e^{-\lambda x}\left[\sin(\omega t - \lambda x) + \cos(\omega t - \lambda x)\right]\end{aligned}$$

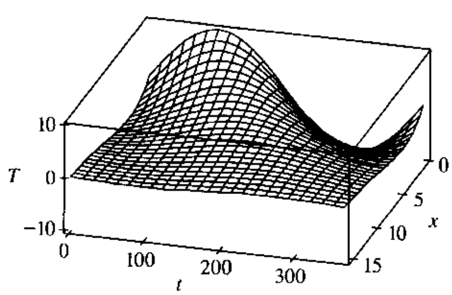
This quantity represents the rate of change of temperature with respect to depth below the surface, at a given time t.

(b) $\partial T/\partial t = T_1 e^{-\lambda x}\left[\cos(\omega t - \lambda x)(\omega)\right] = \omega T_1 e^{-\lambda x}\cos(\omega t - \lambda x)$. This quantity represents the rate of change of temperature with respect to time at a fixed depth x.

(c) $$\begin{aligned}T_{xx} &= \frac{\partial}{\partial x}\left(\frac{\partial T}{\partial x}\right)\\ &= -\lambda T_1\big(e^{-\lambda x}\left[\cos(\omega t - \lambda x)(-\lambda) - \sin(\omega t - \lambda x)(-\lambda)\right]\\ &\qquad\qquad + e^{-\lambda x}(-\lambda)\left[\sin(\omega t - \lambda x) + \cos(\omega t - \lambda x)\right]\big)\\ &= 2\lambda^2 T_1 e^{-\lambda x}\cos(\omega t - \lambda x)\end{aligned}$$

But from part (b), $T_t = \omega T_1 e^{-\lambda x}\cos(\omega t - \lambda x) = \dfrac{\omega}{2\lambda^2}T_{xx}$. So with $k = \dfrac{\omega}{2\lambda^2}$, the function T satisfies the heat equation.

(d)



Note that near the surface (that is, for small x) the temperature varies greatly as t changes, but deeper (for large x) the temperature is more stable.

(e) The term $-\lambda x$ is a phase shift: it represents the fact that since heat diffuses slowly through soil, it takes time for changes in the surface temperature to affect the temperature at deeper points. As x increases, the phase shift also increases. For example, at the surface the highest temperature is reached at $t \approx 100$, whereas at a depth of 5 feet the peak temperature is attained at $t \approx 150$, and at a depth of 10 feet, at $t \approx 220$.

87. By Clairaut's Theorem, $f_{xyy} = (f_{xy})_y = (f_{yx})_y = f_{yxy} = (f_y)_{xy} = (f_y)_{yx} = f_{yyx}$.

88. (a) Since we are differentiating n times, with two choices of variable at each differentiation, there are 2^n nth order partial derivatives.

(b) If these partial derivatives are all continuous, then the order in which the partials are taken doesn't affect the value of the result, that is, all nth order partial derivatives with p partials with respect to x and $n-p$ partials with respect to y are equal. Since the number of partials taken with respect to x for an nth order partial derivative can range from 0 to n, a function of two variables has $n+1$ distinct partial derivatives of order n if these partial derivatives are all continuous.

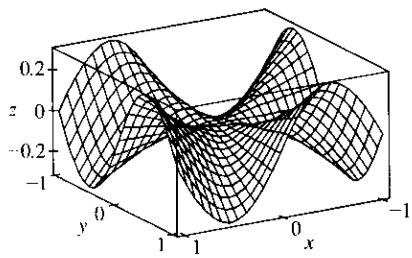
(c) Since n differentiations are to be performed with three choices of variable at each differentiation, there are 3^n nth order partial derivatives of a function of three variables.

89. Let $g(x) = f(x,0) = x(x^2)^{-3/2}e^0 = x\,|x|^{-3}$. But we are using the point $(1,0)$, so near $(1,0)$, $g(x) = x^{-2}$. Then $g'(x) = -2x^{-3}$ and $g'(1) = -2$, so using (1) we have $f_x(1,0) = g'(1) = -2$.

90. $f_x(0,0) = \lim\limits_{h\to 0} \dfrac{f(0+h,0)-f(0,0)}{h} = \lim\limits_{h\to 0} \dfrac{(h^3+0)^{1/3}-0}{h} = \lim\limits_{h\to 0} \dfrac{h}{h} = 1.$

Or: Let $g(x) = f(x,0) = \sqrt[3]{x^3+0} = x$. Then $g'(x) = 1$ and $g'(0) = 1$ so, by (1), $f_x(0,0) = g'(0) = 1$.

91. (a)



(b) For $(x,y) \neq (0,0)$, $f_x(x,y) = \dfrac{(3x^2y - y^3)(x^2+y^2) - (x^3y - xy^3)(2x)}{(x^2+y^2)^2} = \dfrac{x^4y + 4x^2y^3 - y^5}{(x^2+y^2)^2}$, and by symmetry $f_y(x,y) = \dfrac{x^5 - 4x^3y^2 - xy^4}{(x^2+y^2)^2}$.

(c) $f_x(0,0) = \lim\limits_{h\to 0} \dfrac{f(h,0)-f(0,0)}{h} = \lim\limits_{h\to 0} \dfrac{(0/h^2)-0}{h} = 0$ and $f_y(0,0) = \lim\limits_{h\to 0} \dfrac{f(0,h)-f(0,0)}{h} = 0.$

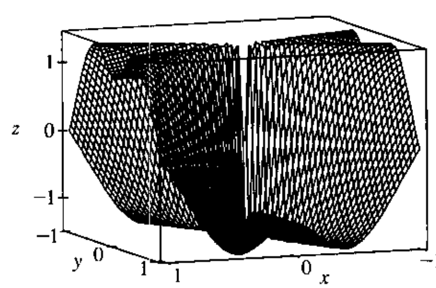
(d) By (3), $f_{xy}(0,0) = \dfrac{\partial f_x}{\partial y} = \lim\limits_{h\to 0} \dfrac{f_x(0,h)-f_x(0,0)}{h} = \lim\limits_{h\to 0} \dfrac{(-h^5-0)/h^4}{h} = -1$ while by (2),

$f_{yx}(0,0) = \dfrac{\partial f_y}{\partial x} = \lim\limits_{h\to 0} \dfrac{f_y(h,0)-f_y(0,0)}{h} = \lim\limits_{h\to 0} \dfrac{h^5/h^4}{h} = 1.$

(e) For $(x,y) \neq (0,0)$, we use a CAS to compute

$$f_{xy}(x,y) = \frac{x^6 + 9x^4y^2 - 4x^2y^4 + 4y^6}{(x^2+y^2)^3}.$$

Now as $(x,y) \to (0,0)$ along the x-axis, $f_{xy}(x,y) \to 1$ while as $(x,y) \to (0,0)$ along the y-axis, $f_{xy}(x,y) \to 4$. Thus f_{xy} isn't continuous at $(0,0)$ and Clairaut's Theorem doesn't apply, so there is no contradiction. The graphs of f_{xy} and f_{yx} are identical except at the origin, where we observe the discontinuity.



15.4 Tangent Planes and Linear Approximations — ET 14.4

1. $z = f(x,y) = 4x^2 - y^2 + 2y \quad \Rightarrow \quad f_x(x,y) = 8x$, $f_y(x,y) = -2y + 2$, so $f_x(-1,2) = -8$, $f_y(-1,2) = -2$. By Equation 2, an equation of the tangent plane is $z - 4 = f_x(-1,2)[x-(-1)] + f_y(-1,2)(y-2) \quad \Rightarrow$ $z - 4 = -8(x+1) - 2(y-2)$ or $z = -8x - 2y$.

2. $z = f(x,y) = 9x^2 + y^2 + 6x - 3y + 5 \quad \Rightarrow \quad f_x(x,y) = 18x + 6$, $f_y(x,y) = 2y - 3$, so $f_x(1,2) = 24$ and $f_y(1,2) = 1$. By Equation 2, an equation of the tangent plane is $z - 18 = f_x(1,2)(x-1) + f_y(1,2)(y-2) \quad \Rightarrow$ $z - 18 = 24(x-1) + 1(y-2)$ or $z = 24x + y - 8$.

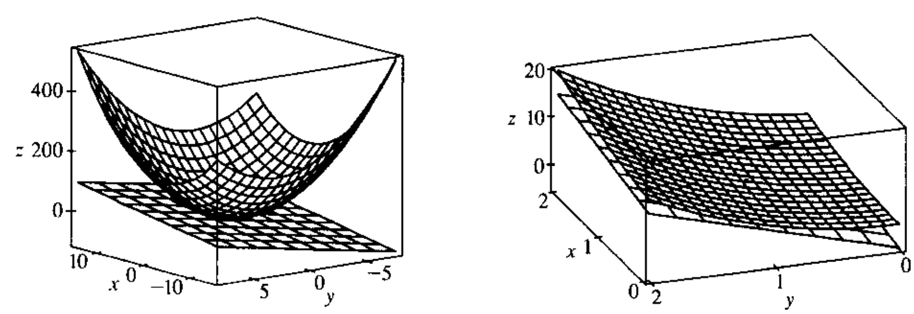
3. $z = f(x,y) = \sqrt{4 - x^2 - 2y^2} \quad \Rightarrow \quad f_x(x,y) = \frac{1}{2}(4 - x^2 - 2y^2)^{-1/2}(-2x) = -\dfrac{x}{\sqrt{4 - x^2 - 2y^2}}$, $f_y(x,y) = \frac{1}{2}(4 - x^2 - 2y^2)^{-1/2}(-4y) = -\dfrac{2y}{\sqrt{4 - x^2 - 2y^2}}$, so $f_x(1,-1) = -1$ and $f_y(1,-1) = 2$. Thus, an equation of the tangent plane is $z - 1 = f_x(1,-1)(x-1) + f_y(1,-1)[y-(-1)] \quad \Rightarrow$ $z - 1 = -1(x-1) + 2(y+1)$ or $x - 2y + z = 4$.

4. $z = f(x,y) = y \ln x \quad \Rightarrow \quad f_x(x,y) = y/x$, $f_y(x,y) = \ln x$, so $f_x(1,4) = 4$, $f_y(1,4) = 0$, and an equation of the tangent plane is $z - 0 = f_x(1,4)(x-1) + f_y(1,4)(y-4) \quad \Rightarrow \quad z = 4(x-1) + 0(y-4)$ or $z = 4x - 4$.

5. $z = f(x,y) = y\cos(x-y) \quad \Rightarrow \quad f_x = y(-\sin(x-y)(1)) = -y\sin(x-y)$, $f_y = y(-\sin(x-y)(-1)) + \cos(x-y) = y\sin(x-y) + \cos(x-y)$, so $f_x(2,2) = -2\sin(0) = 0$, $f_y(2,2) = 2\sin(0) + \cos(0) = 1$ and an equation of the tangent plane is $z - 2 = 0(x-2) + 1(y-2)$ or $z = y$.

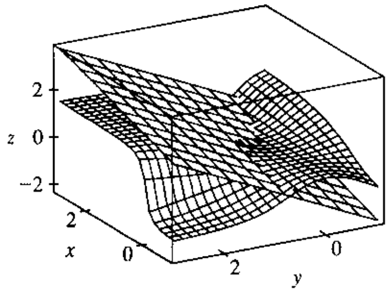
6. $z = f(x,y) = e^{x^2-y^2} \quad \Rightarrow \quad f_x(x,y) = 2xe^{x^2-y^2}$, $f_y(x,y) = -2ye^{x^2-y^2}$, so $f_x(1,-1) = 2$, $f_y(1,-1) = 2$. By Equation 2, an equation of the tangent plane is $z - 1 = f_x(1,-1)(x-1) + f_y(1,-1)[y-(-1)] \quad \Rightarrow$ $z - 1 = 2(x-1) + 2(y+1)$ or $z = 2x + 2y + 1$.

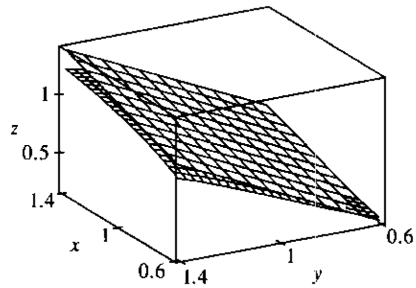
7. $z = f(x,y) = x^2 + xy + 3y^2$, so $f_x(x,y) = 2x + y \quad \Rightarrow \quad f_x(1,1) = 3$, $f_y(x,y) = x + 6y \quad \Rightarrow \quad f_y(1,1) = 7$ and an equation of the tangent plane is $z - 5 = 3(x-1) + 7(y-1)$ or $z = 3x + 7y - 5$. After zooming in, the surface and the tangent plane become almost indistinguishable. (Here, the tangent plane is below the surface.) If we zoom in farther, the surface and the tangent plane will appear to coincide.



8. $z = f(x,y) = \arctan(xy^2) \quad \Rightarrow \quad f_x = \dfrac{1}{1+(xy^2)^2}(y^2) = \dfrac{y^2}{1+x^2y^4}$, $f_y = \dfrac{1}{1+(xy^2)^2}(2xy) = \dfrac{2xy}{1+x^2y^4}$, $f_x(1,1) = \frac{1}{1+1} = \frac{1}{2}$, $f_y(1,1) = \frac{2}{1+1} = 1$, so an equation of the tangent plane is $z - \frac{\pi}{4} = \frac{1}{2}(x-1) + 1(y-1)$ or

$z = \frac{1}{2}x + y - \frac{3}{2} + \frac{\pi}{4}$. After zooming in, the surface and the tangent plane become almost indistinguishable. (Here the tangent plane is above the surface.) If we zoom in farther, the surface and the tangent plane will appear to coincide.

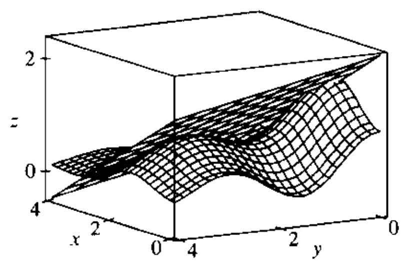


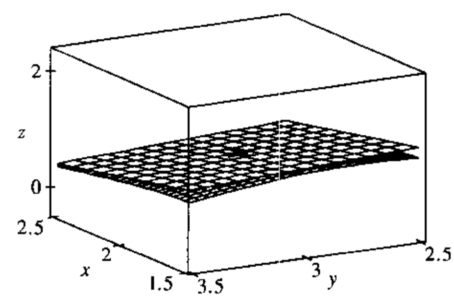


9. $f(x,y) = e^{-(x^2+y^2)/15}(\sin^2 x + \cos^2 y)$. A CAS gives

$f_x = -\frac{2}{15}e^{-(x^2+y^2)/15}(x\sin^2 x + x\cos^2 y - 15\sin x\cos x)$ and

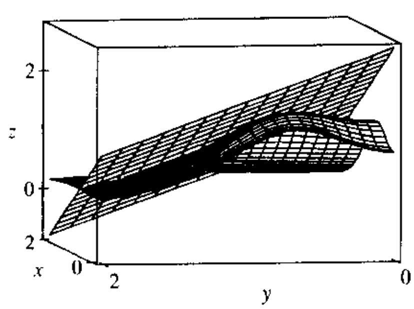
$f_y = -\frac{2}{15}e^{-(x^2+y^2)/15}(y\sin^2 x + y\cos^2 y + 15\sin y\cos y)$. We use the CAS to evaluate these at $(2,3)$, and then substitute the results into Equation 2 in order to plot the tangent plane. After zooming in, the surface and the tangent plane become almost indistinguishable. (Here, the tangent plane is above the surface.) If we zoom in farther, the surface and the tangent plane will appear to coincide.

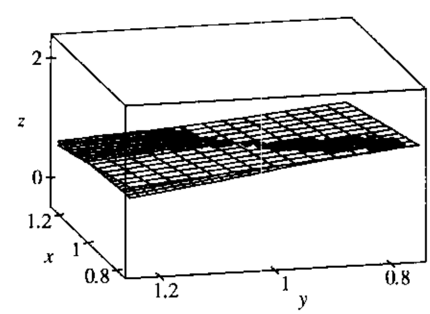




10. $f(x,y) = \dfrac{\sqrt{1+4x^2+4y^2}}{1+x^4+y^4}$. A CAS gives $f_x = \dfrac{4x(1-3x^4+y^4-x^2-4x^2y^2)}{\sqrt{1+4x^2+4y^2}\,(1+x^4+y^4)^2}$ and

$f_y = \dfrac{4y(1-3y^4+x^4-y^2-4x^2y^2)}{\sqrt{1+4x^2+4y^2}\,(1+x^4+y^4)^2}$. We use the CAS to evaluate these at $(1,1)$, and then substitute the results into Equation 2 to get an equation of the tangent plane: $z = \dfrac{25-8x-8y}{9}$. After zooming in, the surface and the tangent plane become almost indistinguishable. (Here, the tangent plane is shown with fewer traces than the surface.) If we zoom in farther, the surface and the tangent plane will appear to coincide.





11. $f(x,y)=x\sqrt{y}$. The partial derivatives are $f_x(x,y)=\sqrt{y}$ and $f_y(x,y)=\dfrac{x}{2\sqrt{y}}$, so $f_x(1,4)=2$ and $f_y(1,4)=\frac{1}{4}$. Both f_x and f_y are continuous functions for $y>0$, so by Theorem 8, f is differentiable at $(1,4)$. By Equation 3, the linearization of f at $(1,4)$ is given by
$L(x,y)=f(1,4)+f_x(1,4)(x-1)+f_y(1,4)(y-4)=2+2(x-1)+\frac{1}{4}(y-4)=2x+\frac{1}{4}y-1$.

12. $f(x,y)=\dfrac{x}{y}$. The partial derivatives are $f_x(x,y)=\dfrac{1}{y}$ and $f_y(x,y)=-\dfrac{x}{y^2}$, so $f_x(6,3)=\frac{1}{3}$ and $f_y(6,3)=-\frac{2}{3}$. Both f_x and f_y are continuous functions for $y\neq 0$, so f is differentiable at $(6,3)$ by Theorem 8. The linearization of f at $(6,3)$ is given by
$L(x,y)=f(6,3)+f_x(6,3)(x-6)+f_y(6,3)(y-3)=2+\frac{1}{3}(x-6)-\frac{2}{3}(y-3)=\frac{1}{3}x-\frac{2}{3}y+2$.

13. $f(x,y)=e^x\cos xy$. The partial derivatives are $f_x(x,y)=e^x(\cos xy-y\sin xy)$ and $f_y(x,y)=-xe^x\sin xy$, so $f_x(0,0)=1$ and $f_y(0,0)=0$. Both f_x and f_y are continuous functions, so f is differentiable at $(0,0)$ by Theorem 8. The linearization of f at $(0,0)$ is given by
$L(x,y)=f(0,0)+f_x(0,0)(x-0)+f_y(0,0)(y-0)=1+1(x-0)+0(y-0)=x+1$.

14. $f(x,y)=\sqrt{x+e^{4y}}=(x+e^{4y})^{1/2}$. The partial derivatives are $f_x(x,y)=\frac{1}{2}(x+e^{4y})^{-1/2}$ and $f_y(x,y)=\frac{1}{2}(x+e^{4y})^{-1/2}(4e^{4y})=2e^{4y}(x+e^{4y})^{-1/2}$, so $f_x(3,0)=\frac{1}{2}(3+e^0)^{-1/2}=\frac{1}{4}$ and $f_y(3,0)=2e^0(3+e^0)^{-1/2}=1$. Both f_x and f_y are continuous functions near $(3,0)$, so f is differentiable at $(3,0)$ by Theorem 8. The linearization of f at $(3,0)$ is
$L(x,y)=f(3,0)+f_x(3,0)(x-3)+f_y(3,0)(y-0)=2+\frac{1}{4}(x-3)+1(y-0)=\frac{1}{4}x+y+\frac{5}{4}$.

15. $f(x,y)=\tan^{-1}(x+2y)$. The partial derivatives are $f_x(x,y)=\dfrac{1}{1+(x+2y)^2}$ and $f_y(x,y)=\dfrac{2}{1+(x+2y)^2}$, so $f_x(1,0)=\frac{1}{2}$ and $f_y(1,0)=1$. Both f_x and f_y are continuous functions, so f is differentiable at $(1,0)$, and the linearization of f at $(1,0)$ is
$L(x,y)=f(1,0)+f_x(1,0)(x-1)+f_y(1,0)(y-0)=\frac{\pi}{4}+\frac{1}{2}(x-1)+1(y)=\frac{1}{2}x+y+\frac{\pi}{4}-\frac{1}{2}$.

16. $f(x,y)=\sin(2x+3y)$. The partial derivatives are $f_x(x,y)=2\cos(2x+3y)$ and $f_y(x,y)=3\cos(2x+3y)$, so $f_x(-3,2)=2$ and $f_y(-3,2)=3$. Both f_x and f_y are continuous functions, so f is differentiable at $(-3,2)$, and the linearization of f at $(-3,2)$ is
$L(x,y)=f(-3,2)+f_x(-3,2)(x+3)+f_y(-3,2)(y-2)=0+2(x+3)+3(y-2)=2x+3y$.

17. $f(x,y)=\sqrt{20-x^2-7y^2}\quad\Rightarrow\quad f_x(x,y)=-\dfrac{x}{\sqrt{20-x^2-7y^2}}$ and $f_y(x,y)=-\dfrac{7y}{\sqrt{20-x^2-7y^2}}$, so $f_x(2,1)=-\frac{2}{3}$ and $f_y(2,1)=-\frac{7}{3}$. Then the linear approximation of f at $(2,1)$ is given by

$$\begin{aligned} f(x,y) &\approx f(2,1)+f_x(2,1)(x-2)+f_y(2,1)(y-1)=3-\tfrac{2}{3}(x-2)-\tfrac{7}{3}(y-1)\\ &=-\tfrac{2}{3}x-\tfrac{7}{3}y+\tfrac{20}{3}\end{aligned}$$

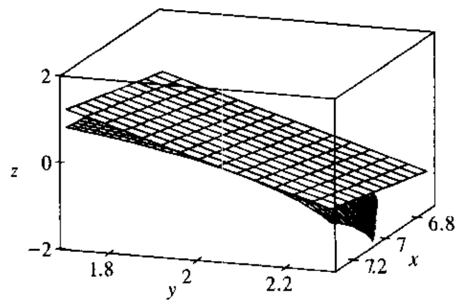
Thus $f(1.95,1.08)\approx-\frac{2}{3}(1.95)-\frac{7}{3}(1.08)+\frac{20}{3}=2.84\overline{6}$.

18. $f(x,y) = \ln(x-3y) \quad \Rightarrow \quad f_x(x,y) = \dfrac{1}{x-3y}$ and $f_y(x,y) = -\dfrac{3}{x-3y}$, so $f_x(7,2) = 1$ and $f_y(7,2) = -3$.

Then the linear approximation of f at $(7,2)$ is given by

$$\begin{aligned} f(x,y) &\approx f(7,2) + f_x(7,2)(x-7) + f_y(7,2)(y-2) \\ &= 0 + 1(x-7) - 3(y-2) = x - 3y - 1 \end{aligned}$$

Thus $f(6.9, 2.06) \approx 6.9 - 3(2.06) - 1 = -0.28$. The graph shows that our approximated value is slightly greater than the actual value.



19. $f(x,y,z) = \sqrt{x^2+y^2+z^2} \quad \Rightarrow \quad f_x(x,y,z) = \dfrac{x}{\sqrt{x^2+y^2+z^2}}$, $f_y(x,y,z) = \dfrac{y}{\sqrt{x^2+y^2+z^2}}$, and

$f_z(x,y,z) = \dfrac{z}{\sqrt{x^2+y^2+z^2}}$, so $f_x(3,2,6) = \frac{3}{7}$, $f_y(3,2,6) = \frac{2}{7}$, and $f_z(3,2,6) = \frac{6}{7}$. Then the linear approximation of f at $(3,2,6)$ is given by

$$\begin{aligned} f(x,y,z) &\approx f(3,2,6) + f_x(3,2,6)(x-3) + f_y(3,2,6)(y-2) + f_z(3,2,6)(z-6) \\ &= 7 + \tfrac{3}{7}(x-3) + \tfrac{2}{7}(y-2) + \tfrac{6}{7}(z-6) = \tfrac{3}{7}x + \tfrac{2}{7}y + \tfrac{6}{7}z \end{aligned}$$

Thus $\sqrt{(3.02)^2 + (1.97)^2 + (5.99)^2} = f(3.02, 1.97, 5.99) \approx \frac{3}{7}(3.02) + \frac{2}{7}(1.97) + \frac{6}{7}(5.99) \approx 6.9914$.

20. From the table, $f(40,20) = 28$. To estimate $f_v(40,20)$ and $f_t(40,20)$ we follow the procedure used in Exercise 15.3.4 [ET 14.3.4]. Since $f_v(40,20) = \lim\limits_{h\to 0} \dfrac{f(40+h,20) - f(40,20)}{h}$, we approximate this quantity with $h = \pm 10$ and use the values given in the table: $f_v(40,20) \approx \dfrac{f(50,20) - f(40,20)}{10} = \dfrac{40-28}{10} = 1.2$, $f_v(40,20) \approx \dfrac{f(30,20) - f(40,20)}{-10} = \dfrac{17-28}{-10} = 1.1$. Averaging these values gives $f_v(40,20) \approx 1.15$.

Similarly, $f_t(40,20) = \lim\limits_{h\to 0} \dfrac{f(40,20+h) - f(40,20)}{h}$, so we use $h = 10$ and $h = -5$:

$f_t(40,20) \approx \dfrac{f(40,30) - f(40,20)}{10} = \dfrac{31-28}{10} = 0.3$, $f_t(40,20) \approx \dfrac{f(40,15) - f(40,20)}{-5} = \dfrac{25-28}{-5} = 0.6$.

Averaging these values gives $f_t(40,15) \approx 0.45$. The linear approximation, then, is

$$\begin{aligned} f(v,t) &\approx f(40,20) + f_v(40,20)(v-40) + f_t(40,20)(t-20) \\ &\approx 28 + 1.15(v-40) + 0.45(t-20) \end{aligned}$$

When $v = 43$ and $t = 24$, we estimate $f(43,24) \approx 28 + 1.15(43-40) + 0.45(24-20) = 33.25$, so we would expect the wave heights to be approximately 33.25 ft.

21. From the table, $f(94,80) = 127$. To estimate $f_T(94,80)$ and $f_H(94,80)$ we follow the procedure used in Section 15.3 [ET 14.3]. Since $f_T(94,80) = \lim\limits_{h\to 0} \dfrac{f(94+h,80) - f(94,80)}{h}$, we approximate this quantity with $h = \pm 2$ and use the values given in the table: $f_T(94, 80) \approx \dfrac{f(96, 80) - f(94, 80)}{2} = \dfrac{135 - 127}{2} = 4$,

$f_T(94, 80) \approx \dfrac{f(92, 80) - f(94, 80)}{-2} = \dfrac{119 - 127}{-2} = 4$.

Averaging these values gives $f_T(94, 80) \approx 4$. Similarly, $f_H(94, 80) = \lim\limits_{h \to 0} \dfrac{f(94, 80 + h) - f(94, 80)}{h}$,

so we use $h = \pm 5$: $f_H(94, 80) \approx \dfrac{f(94, 85) - f(94, 80)}{5} = \dfrac{132 - 127}{5} = 1$,

$f_H(94, 80) \approx \dfrac{f(94, 75) - f(94, 80)}{-5} = \dfrac{122 - 127}{-5} = 1$. Averaging these values gives $f_H(94, 80) \approx 1$. The linear approximation, then, is

$$\begin{aligned} f(T, H) &\approx f(94, 80) + f_T(94, 80)(T - 94) + f_H(94, 80)(H - 80) \\ &\approx 127 + 4(T - 94) + 1(H - 80) \end{aligned}$$

Thus when $T = 95$ and $H = 78$, $f(95, 78) \approx 127 + 4(95 - 94) + 1(78 - 80) = 129$, so we estimate the heat index to be approximately 129 °F.

22. From the table, $f(-15, 50) = -29$. To estimate $f_T(-15, 50)$ and $f_v(-15, 50)$ we follow the procedure used in Section 15.3 [ET 14.3]. Since $f_T(-15, 50) = \lim\limits_{h \to 0} \dfrac{f(-15 + h, 50) - f(-15, 50)}{h}$, we approximate this quantity with $h = \pm 5$ and use the values given in the table:

$f_T(-15, 50) \approx \dfrac{f(-10, 50) - f(-15, 50)}{5} = \dfrac{-22 - (-29)}{5} = 1.4$,

$f_T(-15, 50) \approx \dfrac{f(-20, 50) - f(-15, 50)}{-5} = \dfrac{-35 - (-29)}{-5} = 1.2$.

Averaging these values gives $f_T(-15, 50) \approx 1.3$. Similarly $f_v(-15, 50) = \lim\limits_{h \to 0} \dfrac{f(-15, 50 + h) - f(-15, 50)}{h}$ so we use $h = \pm 10$: $f_v(-15, 50) \approx \dfrac{f(-15, 60) - f(-15, 50)}{10} = \dfrac{-30 - (-29)}{10} = -0.1$,

$f_v(-15, 50) \approx \dfrac{f(-15, 40) - f(-15, 50)}{-10} = \dfrac{-27 - (-29)}{-10} = -0.2$.

Averaging these values gives $f_v(-15, 50) \approx -0.15$. The linear approximation to the wind-chill index function, then, is

$$\begin{aligned} f(T, v) &\approx f(-15, 50) + f_T(-15, 50)(T - (-15)) + f_v(-15, 50)(v - 50) \\ &\approx -29 + (1.3)(T + 15) - (0.15)(v - 50) \end{aligned}$$

Thus when $T = -17°\text{C}$ and $v = 55$ km/h, $f(-17, 55) \approx -29 + (1.3)(-17 + 15) - (0.15)(55 - 50) = -32.35$, so we estimate the wind-chill index to be approximately $-32.35°\text{C}$.

23. $z = x^3 \ln(y^2) \quad \Rightarrow$

$$dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy = 3x^2 \ln(y^2)\, dx + x^3 \cdot \frac{1}{y^2}(2y)\, dy = 3x^2 \ln(y^2)\, dx + \frac{2x^3}{y} dy.$$

24. $v = y \cos xy \quad \Rightarrow$

$$dv = \frac{\partial v}{\partial x} dx + \frac{\partial v}{\partial y} dy = y(-\sin xy)y\, dx + [y(-\sin xy)x + \cos xy]\, dy = -y^2 \sin xy\, dx + (\cos xy - xy \sin xy)\, dy$$

25. $u = e^t \sin\theta \quad \Rightarrow \quad du = \dfrac{\partial u}{\partial t} dt + \dfrac{\partial u}{\partial \theta} d\theta = e^t \sin\theta\, dt + e^t \cos\theta\, d\theta$

26. $u = \dfrac{r}{s+2t} \quad \Rightarrow$

$$du = \frac{\partial u}{\partial r}\,dr + \frac{\partial u}{\partial s}\,ds + \frac{\partial u}{\partial t}\,dt = \frac{1}{s+2t}\,dr + r(-1)(s+2t)^{-2}\,ds + r(-1)(s+2t)^{-2}(2)\,dt$$

$$= \frac{1}{s+2t}\,dr - \frac{r}{(s+2t)^2}\,ds - \frac{2r}{(s+2t)^2}\,dt$$

27. $w = \ln\sqrt{x^2+y^2+z^2} \quad \Rightarrow$

$$dw = \frac{\partial w}{\partial x}\,dx + \frac{\partial w}{\partial y}\,dy + \frac{\partial w}{\partial z}\,dz$$

$$= \left(\frac{1}{2}\right)\frac{2x(x^2+y^2+z^2)^{-1/2}\,dx + 2y(x^2+y^2+z^2)^{-1/2}\,dy + 2z(x^2+y^2+z^2)^{-1/2}\,dz}{(x^2+y^2+z^2)^{1/2}}$$

$$= \frac{x\,dx + y\,dy + z\,dz}{x^2+y^2+z^2}$$

28. $w = xye^{xz} \quad \Rightarrow$

$$dw = \frac{\partial w}{\partial x}\,dx + \frac{\partial w}{\partial y}\,dy + \frac{\partial w}{\partial z}\,dz = (xyze^{xz} + ye^{xz})\,dx + xe^{xz}\,dy + x^2ye^{xz}\,dz$$

$$= (xz+1)ye^{xz}\,dx + xe^{xz}\,dy + x^2ye^{xz}\,dz.$$

29. $dx = \Delta x = 0.05$, $dy = \Delta y = 0.1$, $z = 5x^2 + y^2$, $z_x = 10x$, $z_y = 2y$. Thus when $x = 1$ and $y = 2$, $dz = z_x(1,2)\,dx + z_y(1,2)\,dy = (10)(0.05) + (4)(0.1) = 0.9$ while $\Delta z = f(1.05, 2.1) - f(1,2) = 5(1.05)^2 + (2.1)^2 - 5 - 4 = 0.9225$.

30. $dx = \Delta x = -0.04$, $dy = \Delta y = 0.05$, $z = x^2 - xy + 3y^2$, $z_x = 2x - y$, $z_y = 6y - x$. Thus when $x = 3$ and $y = -1$, $dz = (7)(-0.04) + (-9)(0.05) = -0.73$ while $\Delta z = (2.96)^2 - (2.96)(-0.95) + 3(-0.95)^2 - (9+3+3) = -0.7189$.

31. $dA = \dfrac{\partial A}{\partial x}\,dx + \dfrac{\partial A}{\partial y}\,dy = y\,dx + x\,dy$ and $|\Delta x| \le 0.1$, $|\Delta y| \le 0.1$. We use $dx = 0.1$, $dy = 0.1$ with $x = 30$, $y = 24$; then the maximum error in the area is about $dA = 24(0.1) + 30(0.1) = 5.4\ \text{cm}^2$.

32. Let S be surface area. Then $S = 2(xy + xz + yz)$ and $dS = 2(y+z)\,dx + 2(x+z)\,dy + 2(x+y)\,dz$. The maximum error occurs with $\Delta x = \Delta y = \Delta z = 0.2$. Using $dx = \Delta x$, $dy = \Delta y$, $dz = \Delta z$ we find the maximum error in calculated surface area to be about $dS = (220)(0.2) + (260)(0.2) + (280)(0.2) = 152\ \text{cm}^2$.

33. The volume of a can is $V = \pi r^2 h$ and $\Delta V \approx dV$ is an estimate of the amount of tin. Here $dV = 2\pi rh\,dr + \pi r^2\,dh$, so put $dr = 0.04$, $dh = 0.08$ (0.04 on top, 0.04 on bottom) and then $\Delta V \approx dV = 2\pi(48)(0.04) + \pi(16)(0.08) \approx 16.08\ \text{cm}^3$. Thus the amount of tin is about $16\ \text{cm}^3$.

34. Let V be the volume. Then $V = \pi r^2 h$ and $\Delta V \approx dV = 2\pi rh\,dr + \pi r^2\,dh$ is an estimate of the amount of metal. With $dr = 0.05$ and $dh = 0.2$ we get $dV = 2\pi(2)(10)(0.05) + \pi(2)^2(0.2) = 2.80\pi \approx 8.8\ \text{cm}^3$.

35. The area of the rectangle is $A = xy$, and $\Delta A \approx dA$ is an estimate of the area of paint in the stripe. Here $dA = y\,dx + x\,dy$, so with $dx = dy = \frac{3+3}{12} = \frac{1}{2}$, $\Delta A \approx dA = (100)\left(\frac{1}{2}\right) + (200)\left(\frac{1}{2}\right) = 150\ \text{ft}^2$. Thus there are approximately $150\ \text{ft}^2$ of paint in the stripe.

36. Here $dV = \Delta V = 0.3$, $dT = \Delta T = -5$, $P = 8.31\,\dfrac{T}{V}$, so

$$dP = \left(\frac{8.31}{V}\right) dT - \frac{8.31 \cdot T}{V^2}\, dV = 8.31\left[-\frac{5}{12} - \frac{310}{144} \cdot \frac{3}{10}\right] \approx -8.83.$$

Thus the pressure will drop by about 8.83 kPa.

37. First we find $\dfrac{\partial R}{\partial R_1}$ implicitly by taking partial derivatives of both sides with respect to R_1:

$$\frac{\partial}{\partial R_1}\left[\frac{1}{R}\right] = \frac{\partial\left[(1/R_1)+(1/R_2)+(1/R_3)\right]}{\partial R_1} \quad\Rightarrow\quad -R^{-2}\frac{\partial R}{\partial R_1} = -R_1^{-2} \quad\Rightarrow\quad \frac{\partial R}{\partial R_1} = \frac{R^2}{R_1^2}.$$ Then by symmetry, $\dfrac{\partial R}{\partial R_2} = \dfrac{R^2}{R_2^2}$, $\dfrac{\partial R}{\partial R_3} = \dfrac{R^2}{R_3^2}$. When $R_1 = 25$, $R_2 = 40$ and $R_3 = 50$, $\dfrac{1}{R} = \dfrac{17}{200} \;\Leftrightarrow\; R = \frac{200}{17}$ ohms. Since the possible error for each R_i is 0.5%, the maximum error of R is attained by setting $\Delta R_i = 0.005R_i$. So

$$\Delta R \approx dR = \frac{\partial R}{\partial R_1}\Delta R_1 + \frac{\partial R}{\partial R_2}\Delta R_2 + \frac{\partial R}{\partial R_3}\Delta R_3 = (0.005)R^2\left[\frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}\right]$$

$$= (0.005)R = \tfrac{1}{17} \approx 0.059 \text{ ohms}$$

38. Let x, y, z and w be the four numbers with $p(x,y,z,w) = xyzw$. Since the largest error due to rounding for each number is 0.05, the maximum error in the calculated product is approximated by $dp = (yzw)(0.05) + (xzw)(0.05) + (xyw)(0.05) + (xyz)(0.05)$. Furthermore, each of the numbers is positive but less than 50, so the product of any three is between 0 and $(50)^3$. Thus $dp \le 4(50)^3(0.05) = 25{,}000$.

39. $\Delta z = f(a+\Delta x, b+\Delta y) - f(a,b) = (a+\Delta x)^2 + (b+\Delta y)^2 - (a^2+b^2)$

$$= a^2 + 2a\,\Delta x + (\Delta x)^2 + b^2 + 2b\,\Delta y + (\Delta y)^2 - a^2 - b^2 = 2a\,\Delta x + (\Delta x)^2 + 2b\,\Delta y + (\Delta y)^2$$

But $f_x(a,b) = 2a$ and $f_y(a,b) = 2b$ and so $\Delta z = f_x(a,b)\,\Delta x + f_y(a,b)\,\Delta y + \Delta x\,\Delta x + \Delta y\,\Delta y$, which is Definition 7 with $\varepsilon_1 = \Delta x$ and $\varepsilon_2 = \Delta y$. Hence f is differentiable.

40. $\Delta z = f(a+\Delta x, b+\Delta y) - f(a,b) = (a+\Delta x)(b+\Delta y) - 5(b+\Delta y)^2 - (ab - 5b^2)$

$$= ab + a\,\Delta y + b\,\Delta x + \Delta x\,\Delta y - 5b^2 - 10b\,\Delta y - 5(\Delta y)^2 - ab + 5b^2$$

$$= (a - 10b)\,\Delta y + b\,\Delta x + \Delta x\,\Delta y - 5\,\Delta y\,\Delta y,$$

but $f_x(a,b) = b$ and $f_y(a,b) = a - 10b$ and so $\Delta z = f_x(a,b)\,\Delta x + f_y(a,b)\,\Delta y + \Delta x\,\Delta y - 5\Delta y\,\Delta y$, which is Definition 7 with $\varepsilon_1 = \Delta y$ and $\varepsilon_2 = -5\,\Delta y$. Hence f is differentiable.

41. To show that f is continuous at (a,b) we need to show that $\displaystyle\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$ or equivalently $\displaystyle\lim_{(\Delta x,\Delta y)\to(0,0)} f(a+\Delta x, b+\Delta y) = f(a,b)$. Since f is differentiable at (a,b), $f(a+\Delta x, b+\Delta y) - f(a,b) = \Delta z = f_x(a,b)\,\Delta x + f_y(a,b)\,\Delta y + \varepsilon_1\,\Delta x + \varepsilon_2\,\Delta y$, where ϵ_1 and $\epsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0,0)$. Thus $f(a+\Delta x, b+\Delta y) = f(a,b) + f_x(a,b)\,\Delta x + f_y(a,b)\,\Delta y + \varepsilon_1\,\Delta x + \varepsilon_2\,\Delta y$. Taking the limit of both sides as $(\Delta x, \Delta y) \to (0,0)$ gives $\displaystyle\lim_{(\Delta x,\Delta y)\to(0,0)} f(a+\Delta x, b+\Delta y) = f(a,b)$. Thus f is continuous at (a,b).

42. (a) $\displaystyle\lim_{h\to 0}\frac{f(h,0)-f(0,0)}{h} = \lim_{h\to 0}\frac{0-0}{h} = 0$ and $\displaystyle\lim_{h\to 0}\frac{f(0,h)-f(0,0)}{h} = \lim_{h\to 0}\frac{0-0}{h} = 0$. Thus $f_x(0,0) = f_y(0,0) = 0$. To show that f isn't differentiable at $(0,0)$ we need only show that f is not continuous at $(0,0)$ and apply Exercise 41. As $(x,y) \to (0,0)$ along the x-axis $f(x,y) = 0/x^2 = 0$ for $x \neq 0$ so

$f(x,y) \to 0$ as $(x,y) \to (0,0)$ along the x-axis. But as $(x,y) \to (0,0)$ along the line $y = x$, $f(x,x) = x^2/(2x^2) = \frac{1}{2}$ for $x \neq 0$ so $f(x,y) \to \frac{1}{2}$ as $(x,y) \to (0,0)$ along this line. Thus $\lim\limits_{(x,y)\to(0,0)} f(x,y)$ doesn't exist, so f is discontinuous at $(0,0)$ and thus not differentiable there.

(b) For $(x,y) \neq (0,0)$, $f_x(x,y) = \dfrac{(x^2+y^2)y - xy(2x)}{(x^2+y^2)^2} = \dfrac{y(y^2-x^2)}{(x^2+y^2)^2}$. If we approach $(0,0)$ along the y-axis, then $f_x(x,y) = f_x(0,y) = \dfrac{y^3}{y^4} = \dfrac{1}{y}$, so $f_x(x,y) \to \pm\infty$ as $(x,y) \to (0,0)$. Thus $\lim\limits_{(x,y)\to(0,0)} f_x(x,y)$ does not exist and $f_x(x,y)$ is not continuous at $(0,0)$. Similarly, $f_y(x,y) = \dfrac{(x^2+y^2)x - xy(2y)}{(x^2+y^2)^2} = \dfrac{x(x^2-y^2)}{(x^2+y^2)^2}$ for $(x,y) \neq (0,0)$, and if we approach $(0,0)$ along the x-axis, then $f_y(x,y) = f_x(x,0) = \dfrac{x^3}{x^4} = \dfrac{1}{x}$. Thus $\lim\limits_{(x,y)\to(0,0)} f_y(x,y)$ does not exist and $f_y(x,y)$ is not continuous at $(0,0)$.

15.5 The Chain Rule ET 14.5

1. $z = x^2y + xy^2$, $x = 2 + t^4$, $y = 1 - t^3 \Rightarrow$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = (2xy + y^2)(4t^3) + (x^2 + 2xy)(-3t^2) = 4(2xy + y^2)^3 - 3(x^2 + 2xy)t^2$$

2. $z = \sqrt{x^2 + y^2}$, $x = e^{2t}$, $y = e^{-2t} \Rightarrow$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = \tfrac{1}{2}(x^2+y^2)^{-1/2}(2x)\cdot e^{2t}(2) + \tfrac{1}{2}(x^2+y^2)^{-1/2}(2y)\cdot e^{-2t}(-2) = \frac{2xe^{2t} - 2ye^{2t}}{\sqrt{x^2+y^2}}$$

3. $z = \sin x \cos y$, $x = \pi t$, $y = \sqrt{t} \Rightarrow$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = \cos x\cos y\cdot\pi + \sin x\,(-\sin y)\cdot\tfrac{1}{2}t^{-1/2} = \pi\cos x\cos y - \frac{1}{2\sqrt{t}}\sin x\sin y$$

4. $z = x\ln(x + 2y)$, $x = \sin t$, $y = \cos t \Rightarrow$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt} = \left[x\cdot\frac{1}{x+2y} + 1\cdot\ln(x+2y)\right]\cos t + x\cdot\frac{1}{x+2y}\,(2)\cdot(-\sin t)$$
$$= \left[\frac{x}{x+2y} + \ln(x+2y)\right]\cos t - \frac{2x}{x+2y}\,(\sin t)$$

5. $w = xe^{y/z}$, $x = t^2$, $y = 1 - t$, $z = 1 + 2t \Rightarrow$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt} = e^{y/z}\cdot 2t + xe^{y/z}\left(\frac{1}{z}\right)\cdot(-1) + xe^{y/z}\left(-\frac{y}{z^2}\right)\cdot 2 = e^{y/z}\left(2t - \frac{x}{z} - \frac{2xy}{z^2}\right)$$

6. $w = xy + yz^2$, $x = e^t$, $y = e^t\sin t$, $z = e^t\cos t \Rightarrow$

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt} + \frac{\partial w}{\partial z}\frac{dz}{dt} = y\cdot e^t + (x + z^2)\cdot(e^t\cos t + e^t\sin t) + 2yz\cdot(-e^t\sin t + e^t\cos t)$$
$$= e^t\left[y + (x + z^2)(\cos t + \sin t) + 2yz(\cos t - \sin t)\right]$$

7. $z = x^2 + xy + y^2$, $x = s + t$, $y = st \Rightarrow$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} = (2x + y)(1) + (x + 2y)(t) = 2x + y + xt + 2yt$$
$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} = (2x + y)(1) + (x + 2y)(s) = 2x + y + xs + 2ys$$

8. $z = x/y,\ x = se^t,\ y = 1 + se^{-t} \quad \Rightarrow$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} = \frac{1}{y}(e^t) + \left(-\frac{x}{y^2}\right)(e^{-t}) = \frac{1}{y}\,e^t - \frac{x}{y^2}\,e^{-t}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} = \frac{1}{y}(se^t) + \left(-\frac{x}{y^2}\right)(-se^{-t}) = \frac{s}{y}\,e^t + \frac{xs}{y^2}\,e^{-t}$$

9. $z = \arctan(2x + y),\ x = s^2 t,\ y = s\ln t \quad \Rightarrow$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} = \frac{2}{1 + (2x + y)^2}\cdot 2st + \frac{1}{1 + (2x + y)^2}\cdot \ln t = \frac{4st + \ln t}{1 + (2x + y)^2}$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} = \frac{2}{1 + (2x + y)^2}\cdot s^2 + \frac{1}{1 + (2x + y)^2}\cdot \frac{s}{t} = \frac{2s^2 + s/t}{1 + (2x + y)^2}$$

10. $z = e^{xy}\tan y,\ x = s + 2t,\ y = s/t \quad \Rightarrow$

$$\frac{\partial z}{\partial s} = ye^{xy}\tan y\cdot 1 + (e^{xy}\sec^2 y + xe^{xy}\tan y)\cdot\frac{1}{t} = ye^{xy}\tan y + \frac{e^{xy}}{t}\,(\sec^2 y + x\tan y)$$

$$\frac{\partial z}{\partial t} = ye^{xy}\tan y\cdot 2 + (e^{xy}\sec^2 y + xe^{xy}\tan y)\left(\frac{-s}{t^2}\right) = 2ye^{xy}\tan y - \frac{se^{xy}}{t^2}\,(\sec^2 y + x\tan y)$$

11. $z = e^r\cos\theta,\ r = st,\ \theta = \sqrt{s^2 + t^2} \quad \Rightarrow$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial r}\frac{\partial r}{\partial s} + \frac{\partial z}{\partial \theta}\frac{\partial \theta}{\partial s} = e^r\cos\theta\cdot t + e^r(-\sin\theta)\cdot\tfrac{1}{2}(s^2 + t^2)^{-1/2}(2s)$$

$$= te^r\cos\theta - e^r\sin\theta\cdot\frac{s}{\sqrt{s^2 + t^2}} = e^r\left(t\cos\theta - \frac{s}{\sqrt{s^2 + t^2}}\sin\theta\right)$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial r}\frac{\partial r}{\partial t} + \frac{\partial z}{\partial \theta}\frac{\partial \theta}{\partial t} = e^r\cos\theta\cdot s + e^r(-\sin\theta)\cdot\tfrac{1}{2}(s^2 + t^2)^{-1/2}(2t)$$

$$= se^r\cos\theta - e^r\sin\theta\cdot\frac{t}{\sqrt{s^2 + t^2}} = e^r\left(s\cos\theta - \frac{t}{\sqrt{s^2 + t^2}}\sin\theta\right)$$

12. $z = \sin\alpha\tan\beta,\ \alpha = 3s + t,\ \beta = s - t \quad \Rightarrow$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial \alpha}\frac{\partial \alpha}{\partial s} + \frac{\partial z}{\partial \beta}\frac{\partial \beta}{\partial s} = \cos\alpha\tan\beta\cdot 3 + \sin\alpha\sec^2\beta\cdot 1 = 3\cos\alpha\tan\beta + \sin\alpha\sec^2\beta$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial \alpha}\frac{\partial \alpha}{\partial t} + \frac{\partial z}{\partial \beta}\frac{\partial \beta}{\partial t} = \cos\alpha\tan\beta\cdot 1 + \sin\alpha\sec^2\beta\cdot(-1) = \cos\alpha\tan\beta - \sin\alpha\sec^2\beta$$

13. When $t = 3$, $x = g(3) = 2$ and $y = h(3) = 7$. By the Chain Rule (2),

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = f_x(2,7)g'(3) + f_y(2,7)\,h'(3) = (6)(5) + (-8)(-4) = 62.$$

14. By the Chain Rule (3), $\dfrac{\partial W}{\partial s} = \dfrac{\partial W}{\partial u}\dfrac{\partial u}{\partial s} + \dfrac{\partial W}{\partial v}\dfrac{\partial v}{\partial s}$. Then

$$W_s(1,0) = F_u(u(1,0), v(1,0))\,u_s(1,0) + F_v(u(1,0), v(1,0))\,v_s(1,0)$$
$$= F_u(2,3)u_s(1,0) + F_v(2,3)v_s(1,0) = (-1)(-2) + (10)(5) = 52$$

Similarly, $\dfrac{\partial W}{\partial t} = \dfrac{\partial W}{\partial u}\dfrac{\partial u}{\partial t} + \dfrac{\partial W}{\partial v}\dfrac{\partial v}{\partial t} \quad \Rightarrow$

$$W_t(1,0) = F_u(u(1,0), v(1,0))\,u_t(1,0) + F_v(u(1,0), v(1,0))\,v_t(1,0)$$
$$= F_u(2,3)u_t(1,0) + F_v(2,3)v_t(1,0) = (-1)(6) + (10)(4) = 34$$

15. $g(u,v) = f(x(u,v), y(u,v))$ where $x = e^u + \sin v$, $y = e^u + \cos v$ $\Rightarrow$ $\dfrac{\partial x}{\partial u} = e^u$, $\dfrac{\partial x}{\partial v} = \cos v$, $\dfrac{\partial y}{\partial u} = e^u$, $\dfrac{\partial y}{\partial v} = -\sin v$. By the Chain Rule (3), $\dfrac{\partial g}{\partial u} = \dfrac{\partial f}{\partial x}\dfrac{\partial x}{\partial u} + \dfrac{\partial f}{\partial y}\dfrac{\partial y}{\partial u}$. Then

$$\begin{aligned} g_u(0,0) &= f_x(x(0,0), y(0,0))\, x_u(0,0) + f_y(x(0,0), y(0,0))\, y_u(0,0) \\ &= f_x(1,2)(e^0) + f_y(1,2)(e^0) = 2(1) + 5(1) = 7 \end{aligned}$$

Similarly $\dfrac{\partial g}{\partial v} = \dfrac{\partial f}{\partial x}\dfrac{\partial x}{\partial v} + \dfrac{\partial f}{\partial y}\dfrac{\partial y}{\partial v}$. Then

$$\begin{aligned} g_v(0,0) &= f_x(x(0,0), y(0,0))\, x_v(0,0) + f_y(x(0,0), y(0,0))\, y_v(0,0) \\ &= f_x(1,2)(\cos 0) + f_y(1,2)(-\sin 0) = 2(1) + 5(0) = 2 \end{aligned}$$

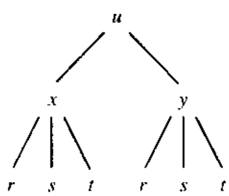
16. $g(r,s) = f(x(r,s), y(r,s))$ where $x = 2r - s$, $y = s^2 - 4r$ $\Rightarrow$ $\dfrac{\partial x}{\partial r} = 2$, $\dfrac{\partial x}{\partial s} = -1$, $\dfrac{\partial y}{\partial r} = -4$, $\dfrac{\partial y}{\partial s} = 2s$. By the Chain Rule (3) $\dfrac{\partial g}{\partial r} = \dfrac{\partial f}{\partial x}\dfrac{\partial x}{\partial r} + \dfrac{\partial f}{\partial y}\dfrac{\partial y}{\partial r}$. Then

$$\begin{aligned} g_r(1,2) &= f_x(x(1,2), y(1,2))\, x_r(1,2) + f_y(x(1,2), y(1,2))\, y_r(1,2) \\ &= f_x(0,0)(2) + f_y(0,0)(-4) = 4(2) + 8(-4) = -24 \end{aligned}$$

Similarly $\dfrac{\partial g}{\partial s} = \dfrac{\partial f}{\partial x}\dfrac{\partial x}{\partial s} + \dfrac{\partial f}{\partial y}\dfrac{\partial y}{\partial s}$. Then

$$\begin{aligned} g_s(1,2) &= f_x(x(1,2), y(1,2))\, x_s(1,2) + f_y(x(1,2), y(1,2))\, y_s(1,2) \\ &= f_x(0,0)(-1) + f_y(0,0)(4) = 4(-1) + 8(4) = 28 \end{aligned}$$

17.

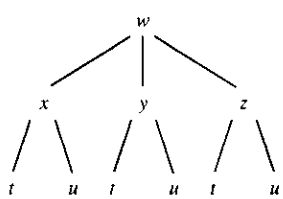


$u = f(x,y)$, $x = x(r,s,t)$, $y = y(r,s,t)$ $\Rightarrow$

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r},\quad \frac{\partial u}{\partial s} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial s},$$

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial t}$$

18.

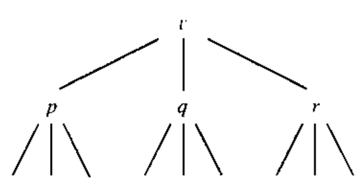


$w = f(x,y,z)$, $x = x(t,u)$, $y = y(t,u)$, $z = z(t,u)$ $\Rightarrow$

$$\frac{\partial w}{\partial t} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial t},$$

$$\frac{\partial w}{\partial u} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial u}$$

19.

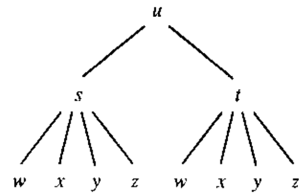


$v = f(p,q,r)$, $p = p(x,y,z)$, $q = q(x,y,z)$, $r = r(x,y,z)$ $\Rightarrow$

$$\frac{\partial v}{\partial x} = \frac{\partial v}{\partial p}\frac{\partial p}{\partial x} + \frac{\partial v}{\partial q}\frac{\partial q}{\partial x} + \frac{\partial v}{\partial r}\frac{\partial r}{\partial x},\quad \frac{\partial v}{\partial y} = \frac{\partial v}{\partial p}\frac{\partial p}{\partial y} + \frac{\partial v}{\partial q}\frac{\partial q}{\partial y} + \frac{\partial v}{\partial r}\frac{\partial r}{\partial y},$$

$$\frac{\partial v}{\partial z} = \frac{\partial v}{\partial p}\frac{\partial p}{\partial z} + \frac{\partial v}{\partial q}\frac{\partial q}{\partial z} + \frac{\partial v}{\partial r}\frac{\partial r}{\partial z}$$

20.



$u = f(s,t)$, $s = s(w,x,y,z)$, $t = t(w,x,y,z)$ $\Rightarrow$

$$\frac{\partial u}{\partial w} = \frac{\partial u}{\partial s}\frac{\partial s}{\partial w} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial w},\ \frac{\partial u}{\partial x} = \frac{\partial u}{\partial s}\frac{\partial s}{\partial x} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial x},$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial s}\frac{\partial s}{\partial y} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial y},\ \frac{\partial u}{\partial z} = \frac{\partial u}{\partial s}\frac{\partial s}{\partial z} + \frac{\partial u}{\partial t}\frac{\partial t}{\partial z}$$

21. $z = x^2 + xy^3$, $x = uv^2 + w^3$, $y = u + ve^w$ $\Rightarrow$ $\dfrac{\partial z}{\partial u} = \dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial u} + \dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial u} = (2x + y^3)(v^2) + (3xy^2)(1)$,

$\dfrac{\partial z}{\partial v} = \dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial v} + \dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial v} = (2x + y^3)(2uv) + (3xy^2)(e^w)$,

$\dfrac{\partial z}{\partial w} = \dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial w} + \dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial w} = (2x + y^3)(3w^2) + (3xy^2)(ve^w)$. When $u = 2$, $v = 1$, and $w = 0$, we have $x = 2$, $y = 3$, so $\dfrac{\partial z}{\partial u} = (31)(1) + (54)(1) = 85$, $\dfrac{\partial z}{\partial v} = (31)(4) + (54)(1) = 178$, $\dfrac{\partial z}{\partial w} = (31)(0) + (54)(1) = 54$.

22. $u = (r^2 + s^2)^{1/2}$, $r = y + x\cos t$, $s = x + y\sin t$ $\Rightarrow$

$\dfrac{\partial u}{\partial x} = \dfrac{\partial u}{\partial r}\dfrac{\partial r}{\partial x} + \dfrac{\partial u}{\partial s}\dfrac{\partial s}{\partial x} = \frac{1}{2}(r^2+s^2)^{-1/2}(2r)(\cos t) + \frac{1}{2}(r^2+s^2)^{-1/2}(2s)(1) = (r\cos t + s)/\sqrt{r^2+s^2}$,

$\dfrac{\partial u}{\partial y} = \dfrac{\partial u}{\partial r}\dfrac{\partial r}{\partial y} + \dfrac{\partial u}{\partial s}\dfrac{\partial s}{\partial y} = \frac{1}{2}(r^2+s^2)^{-1/2}(2r)(1) + \frac{1}{2}(r^2+s^2)^{-1/2}(2s)(\sin t) = (r + s\sin t)/\sqrt{r^2+s^2}$,

$\dfrac{\partial u}{\partial t} = \dfrac{\partial u}{\partial r}\dfrac{\partial r}{\partial t} + \dfrac{\partial u}{\partial s}\dfrac{\partial s}{\partial t} = \frac{1}{2}(r^2+s^2)^{-1/2}(2r)(-x\sin t) + \frac{1}{2}(r^2+s^2)^{-1/2}(2s)(y\cos t) = \dfrac{-rx\sin t + sy\cos t}{\sqrt{r^2+s^2}}$.

When $x = 1$, $y = 2$, and $t = 0$ we have $r = 3$ and $s = 1$, so $\dfrac{\partial u}{\partial x} = \dfrac{4}{\sqrt{10}}$, $\dfrac{\partial u}{\partial y} = \dfrac{3}{\sqrt{10}}$, and $\dfrac{\partial u}{\partial t} = \dfrac{2}{\sqrt{10}}$.

23. $R = \ln(u^2 + v^2 + w^2)$, $u = x + 2y$, $v = 2x - y$, $w = 2xy$ $\Rightarrow$

$$\frac{\partial R}{\partial x} = \frac{\partial R}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial R}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial R}{\partial w}\frac{\partial w}{\partial x} = \frac{2u}{u^2+v^2+w^2}(1) + \frac{2v}{u^2+v^2+w^2}(2) + \frac{2w}{u^2+v^2+w^2}(2y)$$
$$= \frac{2u + 4v + 4wy}{u^2+v^2+w^2},$$

$$\frac{\partial R}{\partial y} = \frac{\partial R}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial R}{\partial v}\frac{\partial v}{\partial y} + \frac{\partial R}{\partial w}\frac{\partial w}{\partial y} = \frac{2u}{u^2+v^2+w^2}(2) + \frac{2v}{u^2+v^2+w^2}(-1) + \frac{2w}{u^2+v^2+w^2}(2x)$$
$$= \frac{4u - 2v + 4wx}{u^2+v^2+w^2}.$$

When $x = y = 1$ we have $u = 3$, $v = 1$, and $w = 2$, so $\dfrac{\partial R}{\partial x} = \dfrac{9}{7}$ and $\dfrac{\partial R}{\partial y} = \dfrac{9}{7}$.

24. $M = xe^{y-z^2}$, $x = 2uv$, $y = u - v$, $z = u + v$ $\Rightarrow$

$$\frac{\partial M}{\partial u} = \frac{\partial M}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial M}{\partial y}\frac{\partial y}{\partial u} + \frac{\partial M}{\partial z}\frac{\partial z}{\partial u} = e^{y-z^2}(2v) + xe^{y-z^2}(1) + x(-2z)e^{y-z^2}(1)$$
$$= e^{y-z^2}(2v + x - 2xz),$$

$$\frac{\partial M}{\partial v} = \frac{\partial M}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial M}{\partial y}\frac{\partial y}{\partial v} + \frac{\partial M}{\partial z}\frac{\partial z}{\partial v} = e^{y-z^2}(2u) + xe^{y-z^2}(-1) + x(-2z)e^{y-z^2}(1)$$
$$= e^{y-z^2}(2u - x - 2xz).$$

When $u = 3$, $v = -1$ we have $x = -6$, $y = 4$, and $z = 2$, so $\dfrac{\partial M}{\partial u} = 16$ and $\dfrac{\partial M}{\partial v} = 36$.

25. $u = x^2 + yz$, $x = pr\cos\theta$, $y = pr\sin\theta$, $z = p + r \quad \Rightarrow$

$$\frac{\partial u}{\partial p} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial p} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial p} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial p} = (2x)(r\cos\theta) + (z)(r\sin\theta) + (y)(1) = 2xr\cos\theta + zr\sin\theta + y,$$

$$\frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial r} = (2x)(p\cos\theta) + (z)(p\sin\theta) + (y)(1) = 2xp\cos\theta + zp\sin\theta + y,$$

$$\frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial \theta} = (2x)(-pr\sin\theta) + (z)(pr\cos\theta) + (y)(0) = -2xpr\sin\theta + zpr\cos\theta.$$

When $p = 2$, $r = 3$, and $\theta = 0$ we have $x = 6$, $y = 0$, and $z = 5$, so $\frac{\partial u}{\partial p} = 36$, $\frac{\partial u}{\partial r} = 24$, and $\frac{\partial u}{\partial \theta} = 30$.

26. $Y = w\tan^{-1}(uv)$, $u = r + s$, $v = s + t$, $w = t + r \quad \Rightarrow$

$$\frac{\partial Y}{\partial r} = \frac{\partial Y}{\partial u}\frac{\partial u}{\partial r} + \frac{\partial Y}{\partial v}\frac{\partial v}{\partial r} + \frac{\partial Y}{\partial w}\frac{\partial w}{\partial r} = \frac{w}{1+(uv)^2}(v)(1) + \frac{w}{1+(uv)^2}(u)(0) + \tan^{-1}(uv)(1)$$
$$= \frac{vw}{1+u^2v^2} + \tan^{-1}(uv)$$

$$\frac{\partial Y}{\partial s} = \frac{\partial Y}{\partial u}\frac{\partial u}{\partial s} + \frac{\partial Y}{\partial v}\frac{\partial v}{\partial s} + \frac{\partial Y}{\partial w}\frac{\partial w}{\partial s} = \frac{wv}{1+u^2v^2}(1) + \frac{wu}{1+u^2v^2}(1) + \tan^{-1}(uv)(0)$$
$$= \frac{w(v+u)}{1+u^2v^2}$$

$$\frac{\partial Y}{\partial t} = \frac{\partial Y}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial Y}{\partial v}\frac{\partial v}{\partial t} + \frac{\partial Y}{\partial w}\frac{\partial w}{\partial t} = \frac{wv}{1+u^2v^2}(0) + \frac{wu}{1+u^2v^2}(1) + \tan^{-1}(uv)(1)$$
$$= \frac{wu}{1+u^2v^2} + \tan^{-1}(uv)$$

When $r = 1$, $s = 0$, and $t = 1$, we have $u = 1$, $v = 1$, and $w = 2$, so $\frac{\partial Y}{\partial r} = 1 + \frac{\pi}{4}$, $\frac{\partial Y}{\partial s} = 2$, and $\frac{\partial Y}{\partial t} = 1 + \frac{\pi}{4}$.

27. $\sqrt{xy} = 1 + x^2y$, so let $F(x,y) = (xy)^{1/2} - 1 - x^2y = 0$. Then by Equation 6

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{\frac{1}{2}(xy)^{-1/2}(y) - 2xy}{\frac{1}{2}(xy)^{-1/2}(x) - x^2} = -\frac{y - 4xy\sqrt{xy}}{x - 2x^2\sqrt{xy}} = \frac{4(xy)^{3/2} - y}{x - 2x^2\sqrt{xy}}.$$

28. $y^5 + x^2y^3 = 1 + ye^{x^2}$, so let $F(x,y) = y^5 + x^2y^3 - 1 - ye^{x^2} = 0$. Then

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{2xy^3 - 2xye^{x^2}}{5y^4 + 3x^2y^2 - e^{x^2}} = \frac{2xye^{x^2} - 2xy^3}{5y^4 + 3x^2y^2 - e^{x^2}}.$$

29. $\cos(x - y) = xe^y$, so let $F(x,y) = \cos(x-y) - xe^y = 0$.

Then $\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{-\sin(x-y) - e^y}{-\sin(x-y)(-1) - xe^y} = \frac{\sin(x-y) + e^y}{\sin(x-y) - xe^y}$.

30. $\sin x + \cos y = \sin x\cos y$, so let $F(x,y) = \sin x + \cos y - \sin x\cos y = 0$. Then

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = -\frac{\cos x - \cos x\cos y}{-\sin y + \sin x\sin y} = \frac{\cos x(\cos y - 1)}{\sin y(\sin x - 1)}.$$

31. $x^2 + y^2 + z^2 = 3xyz$, so let $F(x,y,z) = x^2 + y^2 + z^2 - 3xyz = 0$. Then by Equations 7

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{2x - 3yz}{2z - 3xy} = \frac{3yz - 2x}{2z - 3xy} \quad \text{and} \quad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{2y - 3xz}{2z - 3xy} = \frac{3xz - 2y}{2z - 3xy}.$$

32. $xyz = \cos(x + y + z)$. Let $F(x,y,z) = xyz - \cos(x+y+z) = 0$, so

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{yz + \sin(x+y+z)}{xy + \sin(x+y+z)}, \quad \frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{xz + \sin(x+y+z)}{xy + \sin(x+y+z)}.$$

33. $x - z = \arctan(yz)$, so let $F(x, y, z) = x - z - \arctan(yz) = 0$. Then

$$\frac{\partial z}{\partial x} = -\frac{F_x}{F_z} = -\frac{1}{-1 - \dfrac{1}{1 + (yz)^2}(y)} = \frac{1 + y^2z^2}{1 + y + y^2z^2} \text{ and}$$

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{-\dfrac{1}{1 + (yz)^2}(z)}{-1 - \dfrac{1}{1 + (yz)^2}(y)} = -\frac{\dfrac{z}{1 + y^2z^2}}{\dfrac{1 + y^2z^2 + y}{1 + y^2z^2}} = -\frac{z}{1 + y + y^2z^2}.$$

34. $yz = \ln(x + z)$, so let $F(x, y, z) = yz - \ln(x + z) = 0$. Then $\dfrac{\partial z}{\partial x} = -\dfrac{F_x}{F_z} = -\dfrac{-\dfrac{1}{x+z}(1)}{y - \dfrac{1}{x+z}(1)} = \dfrac{1}{y(x+z) - 1}$,

$$\frac{\partial z}{\partial y} = -\frac{F_y}{F_z} = -\frac{z}{y - \dfrac{1}{x+z}} = -\frac{z(x+z)}{y(x+z) - 1}.$$

35. Since x and y are each functions of t, $T(x, y)$ is a function of t, so by the Chain Rule, $\dfrac{dT}{dt} = \dfrac{\partial T}{\partial x}\dfrac{dx}{dt} + \dfrac{\partial T}{\partial y}\dfrac{dy}{dt}$.

After 3 seconds, $x = \sqrt{1+t} = \sqrt{1+3} = 2$, $y = 2 + \frac{1}{3}t = 2 + \frac{1}{3}(3) = 3$, $\dfrac{dx}{dt} = \dfrac{1}{2\sqrt{1+t}} = \dfrac{1}{2\sqrt{1+3}} = \dfrac{1}{4}$,

and $\dfrac{dy}{dt} = \dfrac{1}{3}$. Then $\dfrac{dT}{dt} = T_x(2, 3)\dfrac{dx}{dt} + T_y(2, 3)\dfrac{dy}{dt} = 4\left(\frac{1}{4}\right) + 3\left(\frac{1}{3}\right) = 2$. Thus the temperature is rising at a rate of 2°C/s.

36. (a) Since $\partial W/\partial T$ is negative, a rise in average temperature (while annual rainfall remains constant) causes a decrease in wheat production at the current production levels. Since $\partial W/\partial R$ is positive, an increase in annual rainfall (while the average temperature remains constant) causes an increase in wheat production.

(b) Since the average temperature is rising at a rate of 0.15°C/year, we know that $dT/dt = 0.15$. Since rainfall is decreasing at a rate of 0.1 cm/year, we know $dR/dt = -0.1$. Then, by the Chain Rule,
$\dfrac{dW}{dt} = \dfrac{\partial W}{\partial T}\dfrac{dT}{dt} + \dfrac{\partial W}{\partial R}\dfrac{dR}{dt} = (-2)(0.15) + (8)(-0.1) = -1.1$. Thus we estimate that wheat production will decrease at a rate of 1.1 units/year.

37. $C = 1449.2 + 4.6T - 0.055T^2 + 0.00029T^3 + 0.016D$, so $\dfrac{\partial C}{\partial T} = 4.6 - 0.11T + 0.00087T^2$ and $\dfrac{\partial C}{\partial D} = 0.016$. According to the graph, the diver is experiencing a temperature of approximately 12.5°C at $t = 20$ minutes, so $\dfrac{\partial C}{\partial T} = 4.6 - 0.11(12.5) + 0.00087(12.5)^2 \approx 3.36$. By sketching tangent lines at $t = 20$ to the graphs given, we estimate $\dfrac{dD}{dt} \approx \dfrac{1}{2}$ and $\dfrac{dT}{dt} \approx -\dfrac{1}{10}$. Then, by the Chain Rule,
$\dfrac{dC}{dt} = \dfrac{\partial C}{\partial T}\dfrac{dT}{dt} + \dfrac{\partial C}{\partial D}\dfrac{dD}{dt} \approx (3.36)\left(-\frac{1}{10}\right) + (0.016)\left(\frac{1}{2}\right) \approx -0.33$. Thus the speed of sound experienced by the diver is decreasing at a rate of approximately 0.33 m/s per minute.

38. $V = \pi r^2 h/3$, so $\dfrac{dV}{dt} = \dfrac{\partial V}{\partial r}\dfrac{dr}{dt} + \dfrac{\partial V}{\partial h}\dfrac{dh}{dt} = \dfrac{2\pi rh}{3}1.8 + \dfrac{\pi r^2}{3}(-2.5) = 20{,}160\pi - 12{,}000\pi = 8160\pi \text{ in}^3/\text{s}$.

39. (a) $V = \ell wh$, so by the Chain Rule,

$$\frac{dV}{dt} = \frac{\partial V}{\partial \ell}\frac{d\ell}{dt} + \frac{\partial V}{\partial w}\frac{dw}{dt} + \frac{\partial V}{\partial h}\frac{dh}{dt} = wh\frac{d\ell}{dt} + \ell h\frac{dw}{dt} + \ell w\frac{dh}{dt}$$
$$= 2 \cdot 2 \cdot 2 + 1 \cdot 2 \cdot 2 + 1 \cdot 2 \cdot (-3) = 6 \text{ m}^3/\text{s}$$

(b) $S = 2(\ell w + \ell h + wh)$, so by the Chain Rule,

$$\frac{dS}{dt} = \frac{\partial S}{\partial \ell}\frac{d\ell}{dt} + \frac{\partial S}{\partial w}\frac{dw}{dt} + \frac{\partial S}{\partial h}\frac{dh}{dt} = 2(w+h)\frac{d\ell}{dt} + 2(\ell+h)\frac{dw}{dt} + 2(\ell+w)\frac{dh}{dt}$$
$$= 2(2+2)2 + 2(1+2)2 + 2(1+2)(-3) = 10\text{ m}^2/\text{s}$$

(c) $L^2 = \ell^2 + w^2 + h^2 \Rightarrow 2L\frac{dL}{dt} = 2\ell\frac{d\ell}{dt} + 2w\frac{dw}{dt} + 2h\frac{dh}{dt} = 2(1)(2) + 2(2)(2) + 2(2)(-3) = 0 \Rightarrow$ $dL/dt = 0$ m/s.

40. $I = \dfrac{V}{R} \Rightarrow \dfrac{dI}{dt} = \dfrac{\partial I}{\partial V}\dfrac{dV}{dt} + \dfrac{\partial I}{\partial R}\dfrac{dR}{dt} = \dfrac{1}{R}\dfrac{dV}{dt} - \dfrac{V}{R^2}\dfrac{dR}{dt} = \dfrac{1}{R}\dfrac{dV}{dt} - \dfrac{I}{R}\dfrac{dR}{dt}$

$$= \tfrac{1}{400}(-0.01) - \tfrac{0.08}{400}(0.03) = -0.000031\text{ A/s}$$

41. $\dfrac{dP}{dt} = 0.05$, $\dfrac{dT}{dt} = 0.15$, $V = 8.31\dfrac{T}{P}$ and $\dfrac{dV}{dt} = \dfrac{8.31}{P}\dfrac{dT}{dt} - 8.31\dfrac{T}{P^2}\dfrac{dP}{dt}$. Thus when $P = 20$ and $T = 320$, $\dfrac{dV}{dt} = 8.31\left[\dfrac{0.15}{20} - \dfrac{(0.05)(320)}{400}\right] \approx -0.27$ L/s.

42. Let x and y be the respective distances of car A and car B from the intersection and let z be the distance between the two cars. Then $dx/dt = -90$, $dy/dt = -80$ and $z^2 = x^2 + y^2$. When $x = 0.3$ and $y = 0.4$, $z = \sqrt{0.25} = 0.5$ and $2z\,(dz/dt) = 2x\,(dx/dt) + 2y\,(dy/dt)$ or $dz/dt = 0.6(-90) + 0.8(-80) = -118$ km/h.

43. (a) By the Chain Rule, $\dfrac{\partial z}{\partial r} = \dfrac{\partial z}{\partial x}\cos\theta + \dfrac{\partial z}{\partial y}\sin\theta$, $\dfrac{\partial z}{\partial \theta} = \dfrac{\partial z}{\partial x}(-r\sin\theta) + \dfrac{\partial z}{\partial y}r\cos\theta$.

(b) $\left(\dfrac{\partial z}{\partial r}\right)^2 = \left(\dfrac{\partial z}{\partial x}\right)^2\cos^2\theta + 2\dfrac{\partial z}{\partial x}\dfrac{\partial z}{\partial y}\cos\theta\sin\theta + \left(\dfrac{\partial z}{\partial y}\right)^2\sin^2\theta$,

$\left(\dfrac{\partial z}{\partial \theta}\right)^2 = \left(\dfrac{\partial z}{\partial x}\right)^2 r^2\sin^2\theta - 2\dfrac{\partial z}{\partial x}\dfrac{\partial z}{\partial y}r^2\cos\theta\sin\theta + \left(\dfrac{\partial z}{\partial y}\right)^2 r^2\cos^2\theta$. Thus

$$\left(\frac{\partial z}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial z}{\partial \theta}\right)^2 = \left[\left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2\right](\cos^2\theta + \sin^2\theta) = \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2.$$

44. By the Chain Rule, $\dfrac{\partial u}{\partial s} = \dfrac{\partial u}{\partial x}e^s\cos t + \dfrac{\partial u}{\partial y}e^s\sin t$, $\dfrac{\partial u}{\partial t} = \dfrac{\partial u}{\partial x}(-e^s\sin t) + \dfrac{\partial u}{\partial y}e^s\cos t$. Then

$\left(\dfrac{\partial u}{\partial s}\right)^2 = \left(\dfrac{\partial u}{\partial x}\right)^2 e^{2s}\cos^2 t + 2\dfrac{\partial u}{\partial x}\dfrac{\partial u}{\partial y}e^{2s}\cos t\sin t + \left(\dfrac{\partial u}{\partial y}\right)^2 e^{2s}\sin^2 t$ and

$\left(\dfrac{\partial u}{\partial t}\right)^2 = \left(\dfrac{\partial u}{\partial x}\right)^2 e^{2s}\sin^2 t - 2\dfrac{\partial u}{\partial x}\dfrac{\partial u}{\partial y}e^{2s}\cos t\sin t + \left(\dfrac{\partial u}{\partial y}\right)^2 e^{2s}\sin^2 t$. Thus

$$\left[\left(\frac{\partial u}{\partial s}\right)^2 + \left(\frac{\partial u}{\partial t}\right)^2\right]e^{-2s} = \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial u}{\partial y}\right)^2.$$

45. Let $u = x - y$. Then $\dfrac{\partial z}{\partial x} = \dfrac{dz}{du}\dfrac{\partial u}{\partial x} = \dfrac{dz}{du}$ and $\dfrac{\partial z}{\partial y} = \dfrac{dz}{du}(-1)$. Thus $\dfrac{\partial z}{\partial x} + \dfrac{\partial z}{\partial y} = 0$.

46. $\dfrac{\partial z}{\partial s} = \dfrac{\partial z}{\partial x} + \dfrac{\partial z}{\partial y}$ and $\dfrac{\partial z}{\partial t} = \dfrac{\partial z}{\partial x} - \dfrac{\partial z}{\partial y}$. Thus $\dfrac{\partial z}{\partial s}\dfrac{\partial z}{\partial t} = \left(\dfrac{\partial z}{\partial x}\right)^2 - \left(\dfrac{\partial z}{\partial y}\right)^2$.

47. Let $u = x + at$, $v = x - at$. Then $z = f(u) + g(v)$, so $\partial z/\partial u = f'(u)$ and $\partial z/\partial v = g'(v)$.

Thus $\dfrac{\partial z}{\partial t} = \dfrac{\partial z}{\partial u}\dfrac{\partial u}{\partial t} + \dfrac{\partial z}{\partial v}\dfrac{\partial v}{\partial t} = af'(u) - ag'(v)$ and

$$\frac{\partial^2 z}{\partial t^2} = a\frac{\partial}{\partial t}\left[f'(u) - g'(v)\right] = a\left(\frac{df'(u)}{du}\frac{\partial u}{\partial t} - \frac{dg'(v)}{dv}\frac{\partial v}{\partial t}\right) = a^2 f''(u) + a^2 g''(v).$$

Similarly $\dfrac{\partial z}{\partial x} = f'(u) + g'(v)$ and $\dfrac{\partial^2 z}{\partial x^2} = f''(u) + g''(v)$. Thus $\dfrac{\partial^2 z}{\partial t^2} = a^2\dfrac{\partial^2 z}{\partial x^2}$.

48. By the Chain Rule, $\dfrac{\partial u}{\partial s} = e^s\cos t\,\dfrac{\partial u}{\partial x} + e^s\sin t\,\dfrac{\partial u}{\partial y}$ and $\dfrac{\partial u}{\partial t} = -e^s\sin t\,\dfrac{\partial u}{\partial x} + e^s\cos t\,\dfrac{\partial u}{\partial y}$. Then

$$\frac{\partial^2 u}{\partial s^2} = e^s\cos t\,\frac{\partial u}{\partial x} + e^s\cos t\,\frac{\partial}{\partial s}\left(\frac{\partial u}{\partial x}\right) + e^s\sin t\,\frac{\partial u}{\partial y} + e^s\sin t\,\frac{\partial}{\partial s}\left(\frac{\partial u}{\partial y}\right).$$

But $\dfrac{\partial}{\partial s}\left(\dfrac{\partial u}{\partial x}\right) = \dfrac{\partial^2 u}{\partial x^2}\dfrac{\partial x}{\partial s} + \dfrac{\partial^2 u}{\partial y\,\partial x}\dfrac{\partial y}{\partial s} = e^s\cos t\,\dfrac{\partial^2 u}{\partial x^2} + e^s\sin t\,\dfrac{\partial^2 u}{\partial y\,\partial x}$ and

$\dfrac{\partial}{\partial s}\left(\dfrac{\partial u}{\partial y}\right) = \dfrac{\partial^2 u}{\partial y^2}\dfrac{\partial y}{\partial s} + \dfrac{\partial^2 u}{\partial x\,\partial y}\dfrac{\partial x}{\partial s} = e^s\sin t\,\dfrac{\partial^2 u}{\partial y^2} + e^s\cos t\,\dfrac{\partial^2 u}{\partial x\,\partial y}$. Also, by continuity of the partials,

$\dfrac{\partial^2 u}{\partial x\,\partial y} = \dfrac{\partial^2 u}{\partial y\,\partial x}$. Thus

$$\begin{aligned}\frac{\partial^2 u}{\partial s^2} &= e^s\cos t\,\frac{\partial u}{\partial x} + e^s\cos t\left(e^s\cos t\,\frac{\partial^2 u}{\partial x^2} + e^s\sin t\,\frac{\partial^2 u}{\partial x\,\partial y}\right) + e^s\sin t\,\frac{\partial u}{\partial y}\\ &\qquad + e^s\sin t\left(e^s\sin t\,\frac{\partial^2 u}{\partial y^2} + e^s\cos t\,\frac{\partial^2 u}{\partial x\,\partial y}\right)\\ &= e^s\cos t\frac{\partial u}{\partial x} + e^s\sin t\,\frac{\partial u}{\partial y} + e^{2s}\cos^2 t\,\frac{\partial^2 u}{\partial x^2} + 2e^{2s}\cos t\sin t\,\frac{\partial^2 u}{\partial x\,\partial y} + e^{2s}\sin^2 t\,\frac{\partial^2 u}{\partial y^2}\end{aligned}$$

Similarly

$$\begin{aligned}\frac{\partial^2 u}{\partial t^2} &= -e^s\cos t\,\frac{\partial u}{\partial x} - e^s\sin t\,\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial x}\right) - e^s\sin t\,\frac{\partial u}{\partial y} + e^s\cos t\,\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial y}\right)\\ &= -e^s\cos t\,\frac{\partial u}{\partial x} - e^s\sin t\left(-e^s\sin t\,\frac{\partial^2 u}{\partial x^2} + e^s\cos t\,\frac{\partial^2 u}{\partial x\,\partial y}\right)\\ &\qquad - e^s\sin t\,\frac{\partial u}{\partial y} + e^s\cos t\left(e^s\cos t\,\frac{\partial^2 u}{\partial y^2} - e^s\sin t\,\frac{\partial^2 u}{\partial x\,\partial y}\right)\\ &= -e^s\cos t\,\frac{\partial u}{\partial x} - e^s\sin t\,\frac{\partial u}{\partial y} + e^{2s}\sin^2 t\,\frac{\partial^2 u}{\partial x^2} - 2e^{2s}\cos t\sin t\,\frac{\partial^2 u}{\partial x\,\partial y} + e^{2s}\cos^2 t\,\frac{\partial^2 u}{\partial y^2}\end{aligned}$$

Thus $e^{-2s}\left(\dfrac{\partial^2 u}{\partial s^2} + \dfrac{\partial^2 u}{\partial t^2}\right) = (\cos^2 t + \sin^2 t)\left(\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}\right) = \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}$, as desired.

49. $\dfrac{\partial z}{\partial s} = \dfrac{\partial z}{\partial x}\,2s + \dfrac{\partial z}{\partial y}\,2r$. Then

$$\begin{aligned}\frac{\partial^2 z}{\partial r\,\partial s} &= \frac{\partial}{\partial r}\left(\frac{\partial z}{\partial x}2s\right) + \frac{\partial}{\partial r}\left(\frac{\partial z}{\partial y}2r\right)\\ &= \frac{\partial^2 z}{\partial x^2}\frac{\partial x}{\partial r}\,2s + \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right)\frac{\partial y}{\partial r}\,2s + \frac{\partial z}{\partial x}\frac{\partial}{\partial r}\,2s + \frac{\partial^2 z}{\partial y^2}\frac{\partial y}{\partial r}\,2r + \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right)\frac{\partial x}{\partial r}\,2r + \frac{\partial z}{\partial y}\,2\\ &= 4rs\,\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y\,\partial x}\,4s^2 + 0 + 4rs\,\frac{\partial^2 z}{\partial y^2} + \frac{\partial^2 z}{\partial x\,\partial y}\,4r^2 + 2\,\frac{\partial z}{\partial y}\end{aligned}$$

By the continuity of the partials, $\dfrac{\partial^2 z}{\partial r\partial s} = 4rs\,\dfrac{\partial^2 z}{\partial x^2} + 4rs\,\dfrac{\partial^2 z}{\partial y^2} + (4r^2 + 4s^2)\,\dfrac{\partial^2 z}{\partial x\,\partial y} + 2\dfrac{\partial z}{\partial y}$.

50. By the Chain Rule,

(a) $\dfrac{\partial z}{\partial r} = \dfrac{\partial z}{\partial x}\cos\theta + \dfrac{\partial z}{\partial y}\sin\theta$

(b) $\dfrac{\partial z}{\partial \theta} = -\dfrac{\partial z}{\partial x}\,r\sin\theta + \dfrac{\partial z}{\partial y}\,r\cos\theta$

(c)
$$\begin{aligned}\frac{\partial^2 z}{\partial r\partial\theta} &= \frac{\partial^2 z}{\partial\theta\partial r} = \frac{\partial}{\partial\theta}\left(\frac{\partial z}{\partial x}\cos\theta + \frac{\partial z}{\partial y}\sin\theta\right)\\ &= -\sin\theta\,\frac{\partial z}{\partial x} + \cos\theta\,\frac{\partial}{\partial\theta}\left(\frac{\partial z}{\partial x}\right) + \cos\theta\,\frac{\partial z}{\partial y} + \sin\theta\,\frac{\partial}{\partial\theta}\left(\frac{\partial z}{\partial y}\right)\\ &= -\sin\theta\,\frac{\partial z}{\partial x} + \cos\theta\left(\frac{\partial^2 z}{\partial x^2}\frac{\partial x}{\partial\theta} + \frac{\partial^2 z}{\partial y\,\partial x}\frac{\partial y}{\partial\theta}\right) + \cos\theta\,\frac{\partial z}{\partial y} + \sin\theta\,\frac{\partial^2 z}{\partial y^2}\frac{\partial y}{\partial\theta} + \frac{\partial^2 z}{\partial x\,\partial y}\frac{\partial x}{\partial\theta}\\ &= -\sin\theta\,\frac{\partial z}{\partial x} + \cos\theta\left(-r\sin\theta\,\frac{\partial^2 z}{\partial x^2} + r\cos\theta\,\frac{\partial^2 z}{\partial y\,\partial x}\right) + \cos\theta\,\frac{\partial z}{\partial y}\\ &\qquad + \sin\theta\left(r\cos\theta\,\frac{\partial^2 z}{\partial y^2} - r\sin\theta\,\frac{\partial^2 z}{\partial x\,\partial y}\right)\\ &= -\sin\theta\,\frac{\partial z}{\partial x} - r\cos\theta\sin\theta\,\frac{\partial^2 z}{\partial x^2} + r\cos^2\theta\,\frac{\partial^2 z}{\partial y\,\partial x} + \cos\theta\,\frac{\partial z}{\partial y}\\ &\qquad + r\cos\theta\sin\theta\,\frac{\partial^2 z}{\partial y^2} - r\sin^2\theta\,\frac{\partial^2 z}{\partial y\,\partial x}\\ &= \cos\theta\,\frac{\partial z}{\partial y} - \sin\theta\,\frac{\partial z}{\partial x} + r\cos\theta\sin\theta\left(\frac{\partial^2 z}{\partial y^2} - \frac{\partial^2 z}{\partial x^2}\right) + r(\cos^2\theta - \sin^2\theta)\,\frac{\partial^2 z}{\partial y\,\partial x}\end{aligned}$$

51. $\dfrac{\partial z}{\partial r} = \dfrac{\partial z}{\partial x}\cos\theta + \dfrac{\partial z}{\partial y}\sin\theta$ and $\dfrac{\partial z}{\partial\theta} = -\dfrac{\partial z}{\partial x}\,r\sin\theta + \dfrac{\partial z}{\partial y}\,r\cos\theta$. Then

$$\begin{aligned}\frac{\partial^2 z}{\partial r^2} &= \cos\theta\left(\frac{\partial^2 z}{\partial x^2}\cos\theta + \frac{\partial^2 z}{\partial y\,\partial x}\sin\theta\right) + \sin\theta\left(\frac{\partial^2 z}{\partial y^2}\sin\theta + \frac{\partial^2 z}{\partial x\,\partial y}\cos\theta\right)\\ &= \cos^2\theta\,\frac{\partial^2 z}{\partial x^2} + 2\cos\theta\sin\theta\,\frac{\partial^2 z}{\partial x\,\partial y} + \sin^2\theta\,\frac{\partial^2 z}{\partial y^2}\end{aligned}$$

and

$$\begin{aligned}\frac{\partial^2 z}{\partial\theta^2} &= -r\cos\theta\,\frac{\partial z}{\partial x} + (-r\sin\theta)\left(\frac{\partial^2 z}{\partial x^2}(-r\sin\theta) + \frac{\partial^2 z}{\partial y\,\partial x}\,r\cos\theta\right)\\ &\qquad -r\sin\theta\,\frac{\partial z}{\partial y} + r\cos\theta\left(\frac{\partial^2 z}{\partial y^2}\,r\cos\theta + \frac{\partial^2 z}{\partial x\,\partial y}(-r\sin\theta)\right)\\ &= -r\cos\theta\,\frac{\partial z}{\partial x} - r\sin\theta\,\frac{\partial z}{\partial y} + r^2\sin^2\theta\,\frac{\partial^2 z}{\partial x^2} - 2r^2\cos\theta\sin\theta\,\frac{\partial^2 z}{\partial x\,\partial y} + r^2\cos^2\theta\,\frac{\partial^2 z}{\partial y^2}\end{aligned}$$

Thus

$$\begin{aligned}\frac{\partial^2 z}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 z}{\partial\theta^2} + \frac{1}{r}\frac{\partial z}{\partial r} &= (\cos^2\theta + \sin^2\theta)\,\frac{\partial^2 z}{\partial x^2} + \left(\sin^2\theta + \cos^2\theta\right)\frac{\partial^2 z}{\partial y^2} - \frac{1}{r}\cos\theta\,\frac{\partial z}{\partial x}\\ &\qquad -\frac{1}{r}\sin\theta\,\frac{\partial z}{\partial y} + \frac{1}{r}\left(\cos\theta\,\frac{\partial z}{\partial x} + \sin\theta\,\frac{\partial z}{\partial y}\right)\\ &= \frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} \text{ as desired.}\end{aligned}$$

52. (a) $\dfrac{\partial z}{\partial t} = \dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial t} + \dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial t}$. Then

$$\begin{aligned}\frac{\partial^2 z}{\partial t^2} &= \frac{\partial}{\partial t}\left(\frac{\partial z}{\partial x}\frac{\partial x}{\partial t}\right) + \frac{\partial}{\partial t}\left(\frac{\partial z}{\partial y}\frac{\partial y}{\partial t}\right)\\ &= \frac{\partial}{\partial t}\left(\frac{\partial z}{\partial x}\right)\frac{\partial x}{\partial t} + \frac{\partial^2 x}{\partial t^2}\frac{\partial z}{\partial x} + \frac{\partial}{\partial t}\left(\frac{\partial z}{\partial y}\right)\frac{\partial y}{\partial t} + \frac{\partial^2 y}{\partial t^2}\frac{\partial z}{\partial y}\\ &= \frac{\partial^2 z}{\partial x^2}\left(\frac{\partial x}{\partial t}\right)^2 + \frac{\partial^2 z}{\partial y\,\partial x}\frac{\partial x}{\partial t}\frac{\partial y}{\partial t} + \frac{\partial^2 x}{\partial t^2}\frac{\partial z}{\partial x} + \frac{\partial^2 z}{\partial y^2}\left(\frac{\partial y}{\partial t}\right)^2 + \frac{\partial^2 z}{\partial x\,\partial y}\frac{\partial y}{\partial t}\frac{\partial x}{\partial t} + \frac{\partial^2 y}{\partial t^2}\frac{\partial z}{\partial y}\\ &= \frac{\partial^2 z}{\partial x^2}\left(\frac{\partial x}{\partial t}\right)^2 + 2\frac{\partial^2 z}{\partial x\,\partial y}\frac{\partial x}{\partial t}\frac{\partial y}{\partial t} + \frac{\partial^2 z}{\partial y^2}\left(\frac{\partial y}{\partial t}\right)^2 + \frac{\partial^2 x}{\partial t^2}\frac{\partial z}{\partial x} + \frac{\partial^2 y}{\partial t^2}\frac{\partial z}{\partial y}\end{aligned}$$

(b)

$$\begin{aligned}\frac{\partial^2 z}{\partial s\,\partial t} &= \frac{\partial}{\partial s}\left(\frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}\right)\\ &= \left(\frac{\partial^2 z}{\partial x^2}\frac{\partial x}{\partial s} + \frac{\partial^2 z}{\partial y\,\partial x}\frac{\partial y}{\partial s}\right)\frac{\partial x}{\partial t} + \frac{\partial z}{\partial x}\frac{\partial^2 x}{\partial s\,\partial t} + \left(\frac{\partial^2 z}{\partial y^2}\frac{\partial y}{\partial s} + \frac{\partial^2 z}{\partial x\,\partial y}\frac{\partial x}{\partial s}\right)\frac{\partial y}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial^2 y}{\partial s\,\partial t}\\ &= \frac{\partial^2 z}{\partial x^2}\frac{\partial x}{\partial s}\frac{\partial x}{\partial t} + \frac{\partial^2 z}{\partial x\,\partial y}\left(\frac{\partial y}{\partial s}\frac{\partial x}{\partial t} + \frac{\partial y}{\partial t}\frac{\partial x}{\partial s}\right) + \frac{\partial z}{\partial x}\frac{\partial^2 x}{\partial s\,\partial t} + \frac{\partial z}{\partial y}\frac{\partial^2 y}{\partial s\,\partial t} + \frac{\partial^2 z}{\partial y^2}\frac{\partial y}{\partial s}\frac{\partial y}{\partial t}\end{aligned}$$

53. (a) Since f is a polynomial, it has continuous second-order partial derivatives, and

$$\begin{aligned}f(tx,ty) &= (tx)^2(ty) + 2(tx)(ty)^2 + 5(ty)^3 = t^3x^2y + 2t^3xy^2 + 5t^3y^3\\ &= t^3(x^2y + 2xy^2 + 5y^3) = t^3 f(x,y)\end{aligned}$$

Thus, f is homogeneous of degree 3.

(b) Differentiating both sides of $f(tx,ty) = t^n f(x,y)$ with respect to t using the Chain Rule, we get

$\dfrac{\partial}{\partial t} f(tx,ty) = \dfrac{\partial}{\partial t}\left[t^n f(x,y)\right] \quad \Leftrightarrow$

$$\frac{\partial}{\partial(tx)} f(tx,ty)\cdot\frac{\partial(tx)}{\partial t}+\frac{\partial}{\partial(ty)} f(tx,ty)\cdot\frac{\partial(ty)}{\partial t}=x\frac{\partial}{\partial(tx)} f(tx,ty)+y\frac{\partial}{\partial(ty)} f(tx,ty)=nt^{n-1}f(x,y).$$

Setting $t=1$: $x\frac{\partial}{\partial x} f(x,y)+y\frac{\partial}{\partial y} f(x,y)=nf(x,y)$.

54. Differentiating both sides of $f(tx,ty)=t^n f(x,y)$ with respect to t using the Chain Rule, we get

$$\frac{\partial}{\partial(tx)} f(tx,ty)\cdot\frac{\partial(tx)}{\partial t}+\frac{\partial}{\partial(ty)} f(tx,ty)\cdot\frac{\partial(ty)}{\partial t}=x\frac{\partial}{\partial(tx)} f(tx,ty)+y\frac{\partial}{\partial(ty)} f(tx,ty)=nt^{n-1}f(x,y) \text{ and}$$

differentiating again with respect to t gives

$$x\left[\frac{\partial^2}{\partial(tx)^2} f(tx,ty)\cdot\frac{\partial(tx)}{\partial t}+\frac{\partial^2}{\partial(ty)\,\partial(tx)} f(tx,ty)\cdot\frac{\partial(ty)}{\partial t}\right]$$

$$+\,y\left[\frac{\partial^2}{\partial(tx)\,\partial(ty)} f(tx,ty)\cdot\frac{\partial(tx)}{\partial t}+\frac{\partial^2}{\partial(ty)^2} f(tx,ty)\cdot\frac{\partial(ty)}{\partial t}\right]=n(n-1)t^{n-1}f(x,y).$$

Setting $t=1$ and using the fact that $f_{yx}=f_{xy}$, we have $x^2 f_{xx}+2xyf_{xy}+y^2 f_{yy}=n(n-1)f(x,y)$.

55. Differentiating both sides of $f(tx,ty)=t^n f(x,y)$ with respect to x using the Chain Rule, we get

$$\frac{\partial}{\partial x} f(tx,ty)=\frac{\partial}{\partial x}\left[t^n f(x,y)\right]\quad\Leftrightarrow$$

$$\frac{\partial}{\partial(tx)} f(tx,ty)\cdot\frac{\partial(tx)}{\partial x}+\frac{\partial}{\partial(ty)} f(tx,ty)\cdot\frac{\partial(ty)}{\partial x}=t^n\frac{\partial}{\partial x} f(x,y)\quad\Leftrightarrow\quad tf_x(tx,ty)=t^n f_x(x,y).$$

Thus $f_x(tx,ty)=t^{n-1}f_x(x,y)$.

56. $F(x,y,z)=0$ is assumed to define z as a function of x and y, that is, $z=f(x,y)$. So by (7), $\dfrac{\partial z}{\partial x}=-\dfrac{F_x}{F_z}$ since $F_z\neq 0$. Similarly, it is assumed that $F(x,y,z)=0$ defines x as a function of y and z, that is $x=h(x,z)$. Then $F(h(y,z),y,z)=0$ and by the Chain Rule, $F_x\dfrac{\partial x}{\partial y}+F_y\dfrac{\partial y}{\partial y}+F_z\dfrac{\partial z}{\partial y}=0$. But $\dfrac{\partial z}{\partial y}=0$ and $\dfrac{\partial y}{\partial y}=1$, so $F_x\dfrac{\partial x}{\partial y}+F_y=0\ \Rightarrow\ \dfrac{\partial x}{\partial y}=-\dfrac{F_y}{F_x}$. A similar calculation shows that $\dfrac{\partial y}{\partial z}=-\dfrac{F_z}{F_y}$. Thus

$$\frac{\partial z}{\partial x}\frac{\partial x}{\partial y}\frac{\partial y}{\partial z}=\left(-\frac{F_x}{F_z}\right)\left(-\frac{F_y}{F_x}\right)\left(-\frac{F_z}{F_y}\right)=-1.$$

15.6 Directional Derivatives and the Gradient Vector ET 14.6

1. First we draw a line passing through Raleigh and the eye of the hurricane. We can approximate the directional derivative at Raleigh in the direction of the eye of the hurricane by the average rate of change of pressure between the points where this line intersects the contour lines closest to Raleigh. In the direction of the eye of the hurricane, the pressure changes from 996 millibars to 992 millibars. We estimate the distance between these two points to be approximately 40 miles, so the rate of change of pressure in the direction given is approximately $\frac{992-996}{40}=-0.1$ millibar/mi.

2. First we draw a line passing through Muskegon and Ludington. We approximate the directional derivative at Muskegon in the direction of Ludington by the average rate of change of snowfall between the points where the line intersects the contour lines closest to Muskegon. In the direction of Ludington, the snowfall changes from 60 to 70 inches. We estimate the distance between these two points to be approximately 28 miles, so the rate of change of annual snowfall in the direction given is approximately $\frac{70-60}{28} \approx 0.36$ in/mi. [If we talk of snowfall (rather than annual snowfall), the units are (in/year)/mi.]

3. $D_{\mathbf{u}}\, f(-20,30) = \nabla f(-20,30) \cdot \mathbf{u} = f_T(-20,30)\left(\frac{1}{\sqrt{2}}\right) + f_v(-20,30)\left(\frac{1}{\sqrt{2}}\right)$.

$f_T(-20,30) = \lim\limits_{h\to 0} \dfrac{f(-20+h,30) - f(-20,30)}{h}$, so we can approximate $f_T(-20,30)$ by considering $h = \pm 5$ and using the values given in the table: $f_T(-20,30) \approx \dfrac{f(-15,30)-f(-20,30)}{5} = \dfrac{-26-(-33)}{5} = 1.4$,

$f_T(-20,30) \approx \dfrac{f(-25,30)-f(-20,30)}{-5} = \dfrac{-39-(-33)}{-5} = 1.2$. Averaging these values gives

$f_T(-20,30) \approx 1.3$. Similarly, $f_v(-20,30) = \lim\limits_{h\to 0} \dfrac{f(-20,30+h)-f(-20,30)}{h}$, so we can approximate $f_v(-20,30)$ with $h = \pm 10$:

$f_v(-20,30) \approx \dfrac{f(-20,40)-f(-20,30)}{10} = \dfrac{-34-(-33)}{10} = -0.1$,

$f_v(-20,30) \approx \dfrac{f(-20,20)-f(-20,30)}{-10} = \dfrac{-30-(-33)}{-10} = -0.3$. Averaging these values gives

$f_v(-20,30) \approx -0.2$. Then $D_{\mathbf{u}} f(-20,30) \approx 1.3\left(\frac{1}{\sqrt{2}}\right) + (-0.2)\left(\frac{1}{\sqrt{2}}\right) \approx 0.778$.

4. $f(x,y) = x^2y^3 - y^4 \quad\Rightarrow\quad f_x(x,y) = 2xy^3$ and $f_y(x,y) = 3x^2y^2 - 4y^3$. If $\mathbf{u}$ is a unit vector in the direction of $\theta = \frac{\pi}{4}$, then from Equation 6, $D_{\mathbf{u}} f(2,1) = f_x(2,1)\cos\left(\frac{\pi}{4}\right) + f_y(2,1)\sin\left(\frac{\pi}{4}\right) = 4 \cdot \frac{\sqrt{2}}{2} + 8 \cdot \frac{\sqrt{2}}{2} = 6\sqrt{2}$.

5. $f(x,y) = \sqrt{5x-4y} \quad\Rightarrow\quad f_x(x,y) = \frac{1}{2}(5x-4y)^{-1/2}(5) = \dfrac{5}{2\sqrt{5x-4y}}$ and

$f_y(x,y) = \frac{1}{2}(5x-4y)^{-1/2}(-4) = -\dfrac{2}{\sqrt{5x-4y}}$. If $\mathbf{u}$ is a unit vector in the direction of $\theta = -\frac{\pi}{6}$, then from Equation 6, $D_{\mathbf{u}}\, f(4,1) = f_x(4,1)\cos\left(-\frac{\pi}{6}\right) + f_y(4,1)\sin\left(-\frac{\pi}{6}\right) = \frac{5}{8} \cdot \frac{\sqrt{3}}{2} + \left(-\frac{1}{2}\right)\left(-\frac{1}{2}\right) = \frac{5\sqrt{3}}{16} + \frac{1}{4}$.

6. $f(x,y) = x\sin(xy) \quad\Rightarrow\quad f_x(x,y) = x\cos(xy)\cdot y + \sin(xy) = xy\cos(xy) + \sin(xy)$ and $f_y(x,y) = x\cos(xy)\cdot x = x^2\cos(xy)$. If $\mathbf{u}$ is a unit vector in the direction of $\theta = \frac{\pi}{3}$, then from Equation 6, $D_{\mathbf{u}}\, f(2,0) = f_x(2,0)\cos\frac{\pi}{3} + f_y(2,0)\sin\frac{\pi}{3} = 0 + 4\left(\frac{\sqrt{3}}{2}\right) = 2\sqrt{3}$.

7. $f(x,y) = 5xy^2 - 4x^3y$

(a) $\nabla f(x,y) = \langle f_x(x,y), f_y(x,y)\rangle = \langle 5y^2 - 12x^2y, 10xy - 4x^3\rangle$

(b) $\nabla f(1,2) = \langle 5(2)^2 - 12(1)^2(2), 10(1)(2) - 4(1)^3\rangle = \langle -4, 16\rangle$

(c) By Equation 9, $D_{\mathbf{u}}\, f(1,2) = \nabla f(1,2)\cdot \mathbf{u} = \langle -4,16\rangle \cdot \left\langle \frac{5}{13}, \frac{12}{13}\right\rangle = (-4)\left(\frac{5}{13}\right) + (16)\left(\frac{12}{13}\right) = \frac{172}{13}$.

8. $f(x,y) = y\ln x$

(a) $\nabla f(x,y) = \langle f_x(x,y), f_y(x,y)\rangle = \langle y/x, \ln x\rangle$ (b) $\nabla f(1,-3) = \left\langle \frac{-3}{1}, \ln 1\right\rangle = \langle -3, 0\rangle$

(c) By Equation 9, $D_{\mathbf{u}}\, f(1,-3) = \nabla f(1,-3)\cdot\mathbf{u} = \langle -3,0\rangle \cdot \left\langle -\frac{4}{5}, \frac{3}{5}\right\rangle = \frac{12}{5}$.

9. $f(x,y,z) = xe^{2yz}$

(a) $\nabla f(x,y,z) = \langle f_x(x,y,z), f_y(x,y,z), f_z(x,y,z)\rangle = \langle e^{2yz}, 2xze^{2yz}, 2xye^{2yz}\rangle$

(b) $\nabla f(3,0,2) = \langle 1,12,0\rangle$

(c) By Equation 14, $D_{\mathbf{u}}f(3,0,2) = \nabla f(3,0,2)\cdot\mathbf{u} = \langle 1,12,0\rangle\cdot\left\langle \frac{2}{3}, -\frac{2}{3}, \frac{1}{3}\right\rangle = \frac{2}{3} - \frac{24}{3} + 0 = -\frac{22}{3}$.

10. $f(x,y,z) = \sqrt{x+yz} = (x+yz)^{1/2}$

(a) $\nabla f(x,y,z) = \left\langle \frac{1}{2}(x+yz)^{-1/2}(1), \frac{1}{2}(x+yz)^{-1/2}(z), \frac{1}{2}(x+yz)^{-1/2}(y)\right\rangle$

$= \langle 1/\left(2\sqrt{x+yz}\right), z/\left(2\sqrt{x+yz}\right), y/\left(2\sqrt{x+yz}\right)\rangle$

(b) $\nabla f(1,3,1) = \left\langle \frac{1}{4}, \frac{1}{4}, \frac{3}{4}\right\rangle$

(c) $D_{\mathbf{u}}f(1,3,1) = \nabla f(1,3,1)\cdot\mathbf{u} = \left\langle \frac{1}{4}, \frac{1}{4}, \frac{3}{4}\right\rangle\cdot\left\langle \frac{2}{7}, \frac{3}{7}, \frac{6}{7}\right\rangle = \frac{2}{28} + \frac{3}{28} + \frac{18}{28} = \frac{23}{28}$.

11. $f(x,y) = 1 + 2x\sqrt{y} \Rightarrow \nabla f(x,y) = \left\langle 2\sqrt{y}, 2x\cdot\frac{1}{2}y^{-1/2}\right\rangle = \langle 2\sqrt{y}, x/\sqrt{y}\rangle$, $\nabla f(3,4) = \left\langle 4, \frac{3}{2}\right\rangle$,

and a unit vector in the direction of $\mathbf{v}$ is $\mathbf{u} = \frac{1}{\sqrt{4^2+(-3)^2}}\langle 4,-3\rangle = \left\langle \frac{4}{5}, -\frac{3}{5}\right\rangle$, so

$D_{\mathbf{u}}f(3,4) = \nabla f(3,4)\cdot\mathbf{u} = \left\langle 4, \frac{3}{2}\right\rangle\cdot\left\langle \frac{4}{5}, -\frac{3}{5}\right\rangle = \frac{23}{10}$.

12. $f(x,y) = \ln(x^2+y^2) \Rightarrow \nabla f(x,y) = \left\langle \dfrac{2x}{x^2+y^2}, \dfrac{2y}{x^2+y^2}\right\rangle$, $\nabla f(2,1) = \left\langle \frac{4}{5}, \frac{2}{5}\right\rangle$, and a unit

vector in the direction of $\mathbf{v} = \langle -1,2\rangle$ is $\mathbf{u} = \dfrac{1}{\sqrt{1+4}}\langle -1,2\rangle = \left\langle -\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right\rangle$, so

$D_{\mathbf{u}}f(2,1) = \nabla f(2,1)\cdot\mathbf{u} = \left\langle \frac{4}{5}, \frac{2}{5}\right\rangle\cdot\left\langle -\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right\rangle = -\frac{4}{5\sqrt{5}} + \frac{4}{5\sqrt{5}} = 0$.

13. $g(s,t) = s^2e^t \Rightarrow \nabla g(s,t) = 2se^t\,\mathbf{i} + s^2e^t\,\mathbf{j}$, $\nabla g(2,0) = 4\,\mathbf{i} + 4\,\mathbf{j}$, and a unit vector in the direction of $\mathbf{v}$ is $\mathbf{u} = \frac{1}{\sqrt{2}}(\mathbf{i}+\mathbf{j})$, so $D_{\mathbf{u}}\,g(2,0) = \nabla g(2,0)\cdot\mathbf{u} = (4\,\mathbf{i}+4\,\mathbf{j})\cdot\frac{1}{\sqrt{2}}(\mathbf{i}+\mathbf{j}) = \frac{8}{\sqrt{2}} = 4\sqrt{2}$.

14. $g(r,\theta) = e^{-r}\sin\theta \Rightarrow \nabla g(r,\theta) = \left(-e^{-r}\sin\theta\right)\mathbf{i} + \left(e^{-r}\cos\theta\right)\mathbf{j}$, $\nabla g\left(0,\frac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}\,\mathbf{i} + \frac{1}{2}\,\mathbf{j}$, and a unit vector in the direction of $\mathbf{v}$ is $\mathbf{u} = \frac{1}{\sqrt{13}}(3\,\mathbf{i} - 2\,\mathbf{j})$, so

$D_{\mathbf{u}}\,g\left(0,\frac{\pi}{3}\right) = \nabla g\left(0,\frac{\pi}{3}\right)\cdot\mathbf{u} = \left(-\frac{\sqrt{3}}{2}\,\mathbf{i} + \frac{1}{2}\,\mathbf{j}\right)\cdot\frac{1}{\sqrt{13}}(3\,\mathbf{i} - 2\,\mathbf{j}) = -\frac{3\sqrt{3}}{2\sqrt{13}} - \frac{1}{\sqrt{13}} = -\frac{3\sqrt{3}+2}{2\sqrt{13}}$.

15. $f(x,y,z) = \sqrt{x^2+y^2+z^2} \Rightarrow \nabla f(x,y,z) = \left\langle \dfrac{x}{\sqrt{x^2+y^2+z^2}}, \dfrac{y}{\sqrt{x^2+y^2+z^2}}, \dfrac{z}{\sqrt{x^2+y^2+z^2}}\right\rangle$,

$\nabla f(1,2,-2) = \left\langle \frac{1}{3}, \frac{2}{3}, -\frac{2}{3}\right\rangle$, and a unit vector in the direction of $\mathbf{v}$ is $\mathbf{u} = \frac{1}{9}\langle -6,6,-3\rangle = \left\langle -\frac{2}{3}, \frac{2}{3}, -\frac{1}{3}\right\rangle$, so

$D_{\mathbf{u}}f(1,2,-2) = \nabla f(1,2,-2)\cdot\mathbf{u} = \left\langle \frac{1}{3}, \frac{2}{3}, -\frac{2}{3}\right\rangle\cdot\left\langle -\frac{2}{3}, \frac{2}{3}, -\frac{1}{3}\right\rangle = \frac{4}{9}$.

16. $f(x,y,z) = \dfrac{x}{y+z} \Rightarrow \nabla f(x,y,z) = \left\langle \dfrac{1}{y+z}, -\dfrac{x}{(y+z)^2}, -\dfrac{x}{(y+z)^2}\right\rangle$,

$\nabla f(4,1,1) = \left\langle \frac{1}{2}, -1, -1\right\rangle$, and a unit vector in the direction of $\mathbf{v}$ is $\mathbf{u} = \frac{1}{\sqrt{14}}\langle 1,2,3\rangle$, so

$D_{\mathbf{u}}\,f(4,1,1) = \nabla f(4,1,1)\cdot\mathbf{u} = \left\langle \frac{1}{2}, -1, -1\right\rangle\cdot\frac{1}{\sqrt{14}}\langle 1,2,3\rangle = -\frac{9}{2\sqrt{14}}$.

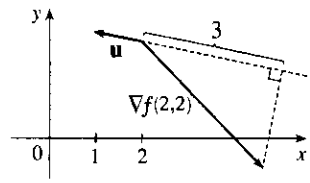
17. $g(x,y,z) = (x+2y+3z)^{3/2} \quad \Rightarrow$

$$\nabla g(x,y,z) = \left\langle \tfrac{3}{2}(x+2y+3z)^{1/2}(1), \tfrac{3}{2}(x+2y+3z)^{1/2}(2), \tfrac{3}{2}(x+2y+3z)^{1/2}(3) \right\rangle$$
$$= \left\langle \tfrac{3}{2}\sqrt{x+2y+3z}, 3\sqrt{x+2y+3z}, \tfrac{9}{2}\sqrt{x+2y+3z} \right\rangle, \nabla g(1,1,2) = \left\langle \tfrac{9}{2}, 9, \tfrac{27}{2} \right\rangle,$$

and a unit vector in the direction of $\mathbf{v} = 2\mathbf{j} - \mathbf{k}$ is $\mathbf{u} = \frac{2}{\sqrt{5}}\mathbf{j} - \frac{1}{\sqrt{5}}\mathbf{k}$, so

$$D_{\mathbf{u}}\, g(1,1,2) = \left\langle \tfrac{9}{2}, 9, \tfrac{27}{2} \right\rangle \cdot \left\langle 0, \tfrac{2}{\sqrt{5}}, -\tfrac{1}{\sqrt{5}} \right\rangle = \tfrac{18}{\sqrt{5}} - \tfrac{27}{2\sqrt{5}} = \tfrac{9}{2\sqrt{5}}.$$

18. $D_{\mathbf{u}} f(2,2) = \nabla f(2,2) \cdot \mathbf{u}$, the scalar projection of $\nabla f(2,2)$ onto $\mathbf{u}$, so we draw a perpendicular from the tip of $\nabla f(2,2)$ to the line containing $\mathbf{u}$. We can use the point $(2,2)$ to determine the scale of the axes, and we estimate the length of the projection to be approximately 3.0 units. Since the angle between $\nabla f(2,2)$ and $\mathbf{u}$ is greater than 90°, the scalar projection is negative. Thus $D_{\mathbf{u}}\, f(2,2) \approx -3$.



19. $f(x,y) = \sqrt{xy} \quad \Rightarrow \quad \nabla f(x,y) = \left\langle \tfrac{1}{2}(xy)^{-1/2}(y), \tfrac{1}{2}(xy)^{-1/2}(x) \right\rangle = \left\langle \dfrac{y}{2\sqrt{xy}}, \dfrac{x}{2\sqrt{xy}} \right\rangle$, so

$\nabla f(2,8) = \left\langle 1, \frac{1}{4} \right\rangle$. The unit vector in the direction of $\overrightarrow{PQ} = \langle 5-2, 4-8 \rangle = \langle 3, -4 \rangle$ is $\mathbf{u} = \left\langle \frac{3}{5}, -\frac{4}{5} \right\rangle$, so
$D_{\mathbf{u}}\, f(2,8) = \nabla f(2,8) \cdot \mathbf{u} = \left\langle 1, \frac{1}{4} \right\rangle \cdot \left\langle \frac{3}{5}, -\frac{4}{5} \right\rangle = \frac{2}{5}$.

20. $f(x,y,z) = x^2 + y^2 + z^2 \quad \Rightarrow \quad \nabla f(x,y,z) = \langle 2x, 2y, 2z \rangle$, so $\nabla f(2,1,3) = \langle 4, 2, 6 \rangle$. The unit vector in the direction of $\overrightarrow{PO} = \langle -2, -1, -3 \rangle$ is $\mathbf{u} = \frac{1}{\sqrt{14}}\langle -2, -1, -3 \rangle$, so
$D_{\mathbf{u}}\, f(2,1,3) = \nabla f(2,1,3) \cdot \mathbf{u} = \langle 4, 2, 6 \rangle \cdot \frac{1}{\sqrt{14}}\langle -2, -1, -3 \rangle = -\frac{28}{\sqrt{14}} = -2\sqrt{14}$.

21. $f(x,y) = y^2/x = y^2 x^{-1} \quad \Rightarrow \quad \nabla f(x,y) = \left\langle -y^2 x^{-2}, 2yx^{-1} \right\rangle = \left\langle -y^2/x^2, 2y/x \right\rangle$.
$\nabla f(2,4) = \langle -4, 4 \rangle$, or equivalently $\langle -1, 1 \rangle$, is the direction of maximum rate of change, and the maximum rate is $|\nabla f(2,4)| = \sqrt{16+16} = 4\sqrt{2}$.

22. $f(p,q) = qe^{-p} + pe^{-q} \quad \Rightarrow \quad \nabla f(p,q) = \left\langle -qe^{-p} + e^{-q}, e^{-p} - pe^{-q} \right\rangle$.
$\nabla f(0,0) = \langle 1, 1 \rangle$ is the direction of maximum rate of change and the maximum rate is $|\nabla f(0,0)| = \sqrt{2}$.

23. $f(x,y) = \sin(xy) \quad \Rightarrow \quad \nabla f(x,y) = \langle y\cos(xy), x\cos(xy) \rangle$, $\nabla f(1,0) = \langle 0, 1 \rangle$. Thus the maximum rate of change is $|\nabla f(1,0)| = 1$ in the direction $\langle 0, 1 \rangle$.

24. $f(x,y,z) = x^2y^3z^4 \quad \Rightarrow \quad \nabla f(x,y,z) = \left\langle 2xy^3z^4, 3x^2y^2z^4, 4x^2y^3z^3 \right\rangle$, $\nabla f(1,1,1) = \langle 2, 3, 4 \rangle$. Thus the maximum rate of change is $|\nabla f(1,1,1)| = \sqrt{29}$ in the direction $\langle 2, 3, 4 \rangle$.

25. $f(x,y,z) = \ln(xy^2z^3) \quad \Rightarrow \quad \nabla f(x,y,z) = \left\langle \dfrac{y^2z^3}{xy^2z^3}, \dfrac{2xyz^3}{xy^2z^3}, \dfrac{3xy^2z^2}{xy^2z^3} \right\rangle = \left\langle \dfrac{1}{x}, \dfrac{2}{y}, \dfrac{3}{z} \right\rangle$.
$\nabla f(1,-2,-3) = \langle 1, -1, -1 \rangle$ is the direction of maximum rate of change and the maximum rate is $|\nabla f(1,-2,-3)| = \sqrt{3}$.

26. $f(x,y,z) = \tan(x+2y+3z) \Rightarrow$

$\nabla f(x,y,z) = \left\langle \sec^2(x+2y+3z)(1), \sec^2(x+2y+3z)(2), \sec^2(x+2y+3z)(3) \right\rangle$.

$\nabla f(-5,1,1) = \left\langle \sec^2(0), 2\sec^2(0), 3\sec^2(0) \right\rangle = \langle 1,2,3 \rangle$ is the direction of maximum rate of change and the maximum rate is $|\nabla f(-5,1,1)| = \sqrt{14}$.

27. (a) As in the proof of Theorem 15, $D_{\mathbf{u}} f = |\nabla f| \cos\theta$. Since the minimum value of $\cos\theta$ is -1 occurring when $\theta = \pi$, the minimum value of $D_{\mathbf{u}} f$ is $-|\nabla f|$ occurring when $\theta = \pi$, that is when $\mathbf{u}$ is in the opposite direction of ∇f (assuming $\nabla f \neq \mathbf{0}$).

(b) $f(x,y) = x^4 y - x^2 y^3 \quad \Rightarrow \quad \nabla f(x,y) = \left\langle 4x^3 y - 2xy^3, x^4 - 3x^2 y^2 \right\rangle$, so f decreases fastest at the point $(2,-3)$ in the direction $-\nabla f(2,-3) = -\langle 12, -92 \rangle = \langle -12, 92 \rangle$.

28. $f(x,y) = x^2 + \sin xy \quad \Rightarrow \quad f_x(x,y) = 2x + y\cos xy$, $f_y(x,y) = x\cos xy$ and $f_x(1,0) = 2(1) + (0)\cos 0 = 2$, $f_y(1,0) = (1)\cos 0 = 1$. If $\mathbf{u}$ is a unit vector which makes an angle θ with the positive x-axis, then $D_{\mathbf{u}} f(1,0) = f_x(1,0)\cos\theta + f_y(1,0)\sin\theta = 2\cos\theta + \sin\theta$. We want $D_{\mathbf{u}} f(1,0) = 1$, so

$2\cos\theta + \sin\theta = 1 \quad \Rightarrow \quad \sin\theta = 1 - 2\cos\theta \quad \Rightarrow \quad \sin^2\theta = (1 - 2\cos\theta)^2 \quad \Rightarrow$

$1 - \cos^2\theta = 1 - 4\cos\theta + 4\cos^2\theta \quad \Rightarrow \quad 5\cos^2\theta - 4\cos\theta = 0 \quad \Rightarrow \quad \cos\theta(5\cos\theta - 4) = 0 \quad \Rightarrow$

$\cos\theta = 0$ or $\cos\theta = \frac{4}{5} \quad \Rightarrow \quad \theta = \frac{\pi}{2}$ or $\theta = 2\pi - \cos^{-1}\left(\frac{4}{5}\right) \approx 5.64$.

29. The direction of fastest change is $\nabla f(x,y) = (2x-2)\,\mathbf{i} + (2y-4)\,\mathbf{j}$, so we need to find all points (x,y) where $\nabla f(x,y)$ is parallel to $\mathbf{i} + \mathbf{j} \quad \Leftrightarrow \quad (2x-2)\,\mathbf{i} + (2y-4)\,\mathbf{j} = k\,(\mathbf{i} + \mathbf{j}) \quad \Leftrightarrow \quad k = 2x - 2$ and $k = 2y - 4$. Then $2x - 2 = 2y - 4 \quad \Rightarrow \quad y = x + 1$, so the direction of fastest change is $\mathbf{i} + \mathbf{j}$ at all points on the line $y = x + 1$.

30. The fisherman is traveling in the direction $\langle -80, -60 \rangle$. A unit vector in this direction is $\mathbf{u} = \frac{1}{100}\langle -80, -60 \rangle = \left\langle -\frac{4}{5}, -\frac{3}{5} \right\rangle$, and if the depth of the lake is given by $f(x,y) = 200 + 0.02x^2 - 0.001y^3$, then $\nabla f(x,y) = \left\langle 0.04x, -0.003y^2 \right\rangle$. $D_{\mathbf{u}} f(80,60) = \nabla f(80,60) \cdot \mathbf{u} = \langle 3.2, -10.8 \rangle \cdot \left\langle -\frac{4}{5}, -\frac{3}{5} \right\rangle = 3.92$. Since $D_{\mathbf{u}} f(80,60)$ is positive, the depth of the lake is increasing near $(80,60)$ in the direction toward the buoy.

31. $T = \dfrac{k}{\sqrt{x^2 + y^2 + z^2}}$ and $120 = T(1,2,2) = \dfrac{k}{3}$ so $k = 360$.

(a) $\mathbf{u} = \dfrac{\langle 1,-1,1 \rangle}{\sqrt{3}}$,

$$D_{\mathbf{u}} T(1,2,2) = \nabla T(1,2,2) \cdot \mathbf{u} = \left[-360\left(x^2 + y^2 + z^2\right)^{-3/2} \langle x,y,z \rangle \right]_{(1,2,2)} \cdot \mathbf{u}$$

$$= -\tfrac{40}{3}\langle 1,2,2 \rangle \cdot \tfrac{1}{\sqrt{3}}\langle 1,-1,1 \rangle = -\tfrac{40}{3\sqrt{3}}$$

(b) From (a), $\nabla T = -360\left(x^2 + y^2 + z^2\right)^{-3/2} \langle x,y,z \rangle$, and since $\langle x,y,z \rangle$ is the position vector of the point (x,y,z), the vector $-\langle x,y,z \rangle$, and thus ∇T, always points toward the origin.

32. $\nabla T = -400 e^{-x^2 - 3y^2 - 9z^2} \langle x, 3y, 9z \rangle$

(a) $\mathbf{u} = \frac{1}{\sqrt{6}}\langle 1,-2,1 \rangle$, $\nabla T(2,-1,2) = -400e^{-43}\langle 2,-3,18 \rangle$ and

$$D_{\mathbf{u}} T(2,-1,2) = \left(-\frac{400e^{-43}}{\sqrt{6}} \right)(26) = -\frac{5200\sqrt{6}}{3e^{43}} \ {}^\circ\mathrm{C/m}.$$

(b) $\nabla T(2,-1,2) = 400e^{-43}\langle -2,3,-18\rangle$ or equivalently $\langle -2,3,-18\rangle$.

(c) $|\nabla T| = 400e^{-x^2-3y^2-9z^2}\sqrt{x^2+9y^2+8z^2}$ °C /m is the maximum rate of increase. At $(2,-1,2)$ the maximum rate of increase is $400e^{-43}\sqrt{337}$ °C /m.

33. $\nabla V(x,y,z) = \langle 10x-3y+yz, xz-3x, xy\rangle$, $\nabla V(3,4,5) = \langle 38,6,12\rangle$

(a) $D_{\mathbf{u}}V(3,4,5) = \langle 38,6,12\rangle \cdot \frac{1}{\sqrt{3}}\langle 1,1,-1\rangle = \frac{32}{\sqrt{3}}$

(b) $\nabla V(3,4,5) = \langle 38,6,12\rangle$ or equivalently $\langle 19,3,6\rangle$.

(c) $|\nabla V(3,4,5)| = \sqrt{38^2+6^2+12^2} = \sqrt{1624} = 2\sqrt{406}$

34. $z = f(x,y) = 1000 - 0.01x^2 - 0.02y^2 \quad\Rightarrow\quad \nabla f(x,y) = \langle -0.02x, -0.04y\rangle$ and $\nabla f(50,80) = \langle -1,-3.2\rangle$

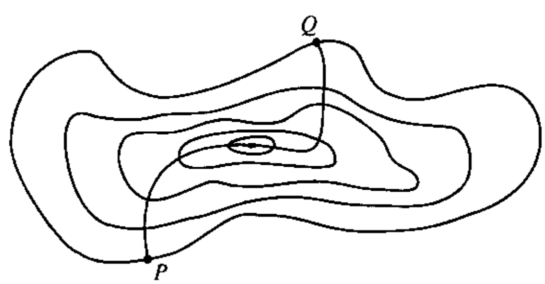
(a) Due south is in the direction of the unit vector $\mathbf{u} = -\mathbf{j}$ and $D_{\mathbf{u}}f(50,80) = \nabla f(50,80)\cdot\langle 0,-1\rangle = \langle -1,-3.2\rangle\cdot\langle 0,-1\rangle = 0 + 3.2 = 3.2$. Thus, if you walk due south from $(50,80,847)$ you will ascend at a rate of 3.2 vertical meters per horizontal meter.

(b) Northwest is in the direction of the unit vector $\mathbf{u} = \frac{1}{\sqrt{2}}\langle -1,1\rangle$ and $D_{\mathbf{u}}f(50,80) = \nabla f(50,80)\cdot\frac{1}{\sqrt{2}}\langle -1,1\rangle = \langle -1,-3.2\rangle\cdot\frac{1}{\sqrt{2}}\langle -1,1\rangle = -\frac{2.2}{\sqrt{2}} \approx -1.56$. Thus, if you walk northwest from $(50,80,847)$ you will descend at a rate of approximately 1.56 vertical meters per horizontal meter.

(c) $\nabla f(50,80) = \langle -1,-3.2\rangle$ is the direction of largest slope with a rate of ascent $|\nabla f(50,80)| = \sqrt{11.24} \approx 3.35$. The angle above the horizontal in which the path begins is given by $\tan\theta \approx 3.35 \quad\Rightarrow\quad \theta \approx \tan^{-1}(3.35) \approx 73.4°$.

35. A unit vector in the direction of $\overrightarrow{AB}$ is $\mathbf{i}$ and a unit vector in the direction of $\overrightarrow{AC}$ is $\mathbf{j}$. Thus $D_{\overrightarrow{AB}}f(1,3) = f_x(1,3) = 3$ and $D_{\overrightarrow{AC}}f(1,3) = f_y(1,3) = 26$. Therefore $\nabla f(1,3) = \langle f_x(1,3), f_y(1,3)\rangle = \langle 3,26\rangle$, and by definition, $D_{\overrightarrow{AD}}f(1,3) = \nabla f\cdot\mathbf{u}$ where $\mathbf{u}$ is a unit vector in the direction of $\overrightarrow{AD}$, which is $\left\langle \frac{5}{13},\frac{12}{13}\right\rangle$. Therefore, $D_{\overrightarrow{AD}}f(1,3) = \langle 3,26\rangle\cdot\left\langle\frac{5}{13},\frac{12}{13}\right\rangle = 3\cdot\frac{5}{13} + 26\cdot\frac{12}{13} = \frac{327}{13}$.

36. The curve of steepest ascent is perpendicular to all of the contour lines.



37. (a) $\nabla(au+bv) = \left\langle \frac{\partial(au+bv)}{\partial x}, \frac{\partial(au+bv)}{\partial y}\right\rangle = \left\langle a\frac{\partial u}{\partial x} + b\frac{\partial v}{\partial x}, a\frac{\partial u}{\partial y} + b\frac{\partial v}{\partial y}\right\rangle$

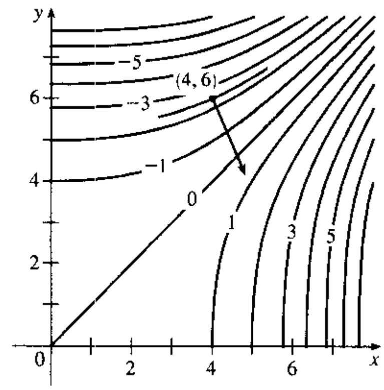
$$= a\left\langle\frac{\partial u}{\partial x},\frac{\partial u}{\partial y}\right\rangle + b\left\langle\frac{\partial v}{\partial x},\frac{\partial v}{\partial y}\right\rangle = a\,\nabla u + b\,\nabla v$$

(b) $\nabla(uv) = \left\langle v\frac{\partial u}{\partial x} + u\frac{\partial v}{\partial x}, v\frac{\partial u}{\partial y} + u\frac{\partial v}{\partial y}\right\rangle = v\left\langle\frac{\partial u}{\partial x},\frac{\partial u}{\partial y}\right\rangle + u\left\langle\frac{\partial v}{\partial x},\frac{\partial v}{\partial y}\right\rangle = v\,\nabla u + u\,\nabla v$

(c) $\nabla\left(\frac{u}{v}\right) = \left\langle \dfrac{v\,\frac{\partial u}{\partial x} - u\,\frac{\partial v}{\partial x}}{v^2}, \dfrac{v\,\frac{\partial u}{\partial y} - u\,\frac{\partial v}{\partial y}}{v^2} \right\rangle = \dfrac{v\left\langle \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}\right\rangle - u\left\langle \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}\right\rangle}{v^2} = \dfrac{v\,\nabla u - u\,\nabla v}{v^2}$

(d) $\nabla u^n = \left\langle \dfrac{\partial(u^n)}{\partial x}, \dfrac{\partial(u^n)}{\partial y} \right\rangle = \left\langle nu^{n-1}\,\dfrac{\partial u}{\partial x}, nu^{n-1}\,\dfrac{\partial u}{\partial y} \right\rangle = nu^{n-1}\,\nabla u.$

38. If we place the initial point of the gradient vector $\nabla f(4,6)$ at $(4,6)$, the vector is perpendicular to the level curve of f that includes $(4,6)$, so we sketch a portion of the level curve through $(4,6)$ (using the nearby level curves as a guideline) and draw a line perpendicular to the curve at $(4,6)$. The gradient vector is parallel to this line, pointing in the direction of increasing function values, and with length equal to the maximum value of the directional derivative of f at $(4,6)$. We can estimate this length by finding the average rate of change in the direction of the gradient. The line intersects the contour lines corresponding to -2 and -3 with an estimated distance of 0.5 units. Thus the rate of change is approximately $\dfrac{-2-(-3)}{0.5} = 2$, and we sketch the gradient vector with length 2.



39. Let $F(x,y,z) = x^2 + 2y^2 + 3z^2$. Then $x^2 + 2y^2 + 3z^2 = 21$ is a level surface of F. $F_x(x,y,z) = 2x \;\Rightarrow\; F_x(4,-1,1) = 8$, $F_y(x,y,z) = 4y \;\Rightarrow\; F_y(4,-1,1) = -4$, and $F_z(x,y,z) = 6z \;\Rightarrow\; F_z(4,-1,1) = 6$.

(a) Equation 19 gives an equation of the tangent plane at $(4,-1,1)$ as $8(x-4) - 4[y-(-1)] + 6(z-1) = 0$ or $4x - 2y + 3z = 21$.

(b) By Equation 20, the normal line has symmetric equations $\dfrac{x-4}{8} = \dfrac{y+1}{-4} = \dfrac{z-1}{6}$ or $\dfrac{x-4}{4} = \dfrac{y+1}{-2} = \dfrac{z-1}{3}$.

40. Let $F(x,y,z) = y^2 + z^2 - x$. Then $x = y^2 + z^2 - 2$ is the level surface $F(x,y,z) = 2$. $F_x(x,y,z) = -1 \;\Rightarrow\; F_x(-1,1,0) = -1$, $F_y(x,y,z) = 2y \;\Rightarrow\; F_y(-1,1,0) = 2$, and $F_z(x,y,z) = 2z \;\Rightarrow\; F_z(-1,1,0) = 0$.

(a) An equation of the tangent plane is $-1(x+1) + 2(y-1) + 0(z-0) = 0$ or $-x + 2y = 3$.

(b) The normal line has symmetric equations $\dfrac{x+1}{-1} = \dfrac{y-1}{2}$, $z = 0$.

41. Let $F(x,y,z) = x^2 - 2y^2 + z^2 + yz$. Then $x^2 - 2y^2 + z^2 + yz = 2$ is a level surface of F and $\nabla F(x,y,z) = \langle 2x, -4y+z, 2z+y \rangle$.

(a) $\nabla F(2,1,-1) = \langle 4,-5,-1 \rangle$ is a normal vector for the tangent plane at $(2,1,-1)$, so an equation of the tangent plane is $4(x-2) - 5(y-1) - 1(z+1) = 0$ or $4x - 5y - z = 4$.

(b) The normal line has direction $\langle 4,-5,-1 \rangle$, so parametric equations are $x = 2+4t$, $y = 1-5t$, $z = -1-t$, and symmetric equations are $\dfrac{x-2}{4} = \dfrac{y-1}{-5} = \dfrac{z+1}{-1}$.

42. Let $F(x,y,z) = x - z - 4\arctan(yz)$. Then $x - z = 4\arctan(yz)$ is the level surface $F(x,y,z) = 0$,

and $\nabla F(x,y,z) = \left\langle 1, -\dfrac{4z}{1+y^2z^2}, -1 - \dfrac{4y}{1+y^2z^2} \right\rangle$.

(a) $\nabla F(1+\pi, 1, 1) = \langle 1, -2, -3 \rangle$ and an equation of the tangent plane is $1(x - (1+\pi)) - 2(y-1) - 3(z-1) = 0$ or $x - 2y - 3z = -4 + \pi$.

(b) The normal line has direction $\langle 1, -2, -3 \rangle$, so parametric equations are $x = 1 + \pi + t$, $y = 1 - 2t$, $z = 1 - 3t$, and symmetric equations are $x - 1 - \pi = \dfrac{y-1}{-2} = \dfrac{z-1}{-3}$.

43. $F(x,y,z) = -z + xe^y \cos z \quad \Rightarrow \quad \nabla F(x,y,z) = \langle e^y \cos z, xe^y \cos z, -1 - xe^y \sin z \rangle$,

$\nabla F(1,0,0) = \langle 1, 1, -1 \rangle$

(a) $1(x-1) + 1(y-0) - 1(z-0) = 0$ or $x + y - z = 1$

(b) $x - 1 = y = -z$

44. $F(x,y,z) = yz - \ln(x+z) \quad \Rightarrow \quad \nabla F(x,y,z) = \left\langle -\dfrac{1}{x+z}, z, y - \dfrac{1}{x+z} \right\rangle$, $\nabla F(0,0,1) = \langle -1, 1, -1 \rangle$.

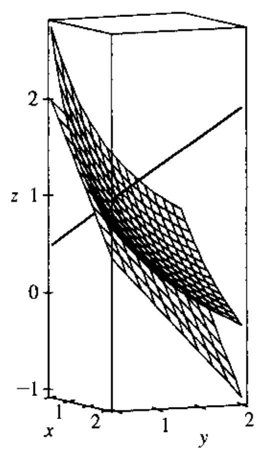
(a) $(-1)(x-0) + (1)(y-0) - 1(z-1) = 0$ or $x - y + z = 1$.

(b) Parametric equations are $x = -t$, $y = t$, $z = 1 - t$ and symmetric equations are $\dfrac{x}{-1} = \dfrac{y}{1} = \dfrac{z-1}{-1}$ or $-x = y = 1 - z$.

45. $F(x,y,z) = xy + yz + zx$,

$\nabla F(x,y,z) = \langle y+z, x+z, y+x \rangle$,

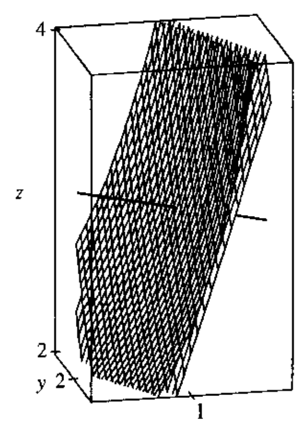
$\nabla F(1,1,1) = \langle 2,2,2 \rangle$, so an equation of the tangent plane is $2x + 2y + 2z = 6$ or $x + y + z = 3$, and the normal line is given by $x - 1 = y - 1 = z - 1$ or $x = y = z$.



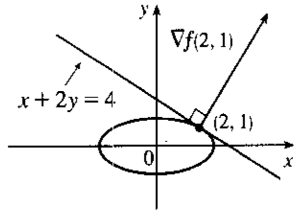
46. $F(x,y,z) = xyz$, $\nabla F(x,y,z) = \langle yz, xz, yx \rangle$,

$\nabla F(1,2,3) = \langle 6,3,2 \rangle$, so an equation of the tangent plane is $6x + 3y + 2z = 18$, and the normal line is given by

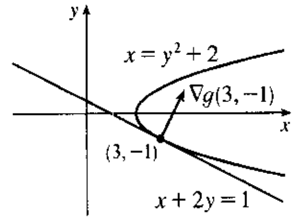
$\dfrac{x-1}{6} = \dfrac{y-2}{3} = \dfrac{z-3}{2}$ or $x = 1 + 6t$, $y = 2 + 3t$, $z = 3 + 2t$.



47. $\nabla f(x,y) = \langle 2x, 8y\rangle$, $\nabla f(2,1) = \langle 4, 8\rangle$. The tangent line has equation $\nabla f(2,1)\cdot\langle x-2, y-1\rangle = 0 \Rightarrow$ $4(x-2)+8(y-1)=0$, which simplifies to $x+2y=4$.



48. $\nabla g(x,y) = \langle 1, -2y\rangle$, $\nabla g(3,-1) = \langle 1, 2\rangle$. The tangent line has equation $\nabla g(3,-1)\cdot\langle x-3, y+1\rangle = 0 \Rightarrow$ $1(x-3)+2(y+1)=0$, which simplifies to $x+2y=1$.



49. $\nabla F(x_0,y_0,z_0) = \left\langle \dfrac{2x_0}{a^2}, \dfrac{2y_0}{b^2}, \dfrac{2z_0}{c^2}\right\rangle$. Thus an equation of the tangent plane at (x_0,y_0,z_0) is $\dfrac{2x_0}{a^2}x + \dfrac{2y_0}{b^2}y + \dfrac{2z_0}{c^2}z = 2\left(\dfrac{x_0^2}{a^2}+\dfrac{y_0^2}{b^2}+\dfrac{z_0^2}{c^2}\right) = 2(1) = 2$ since (x_0,y_0,z_0) is a point on the ellipsoid. Hence $\dfrac{x_0}{a^2}x + \dfrac{y_0}{b^2}y + \dfrac{z_0}{c^2}z = 1$ is an equation of the tangent plane.

50. $\nabla F(x_0,y_0,z_0) = \left\langle \dfrac{2x_0}{a^2}, \dfrac{2y_0}{b^2}, \dfrac{-2z_0}{c^2}\right\rangle$, so an equation of the tangent plane at (x_0,y_0,z_0) is $\dfrac{2x_0}{a^2}x + \dfrac{2y_0}{b^2}y - \dfrac{2z_0}{c^2}z = 2\left(\dfrac{x_0^2}{a^2}+\dfrac{y_0^2}{b^2}-\dfrac{z_0^2}{c^2}\right) = 2$ or $\dfrac{x_0}{a^2}x + \dfrac{y_0}{b^2}y - \dfrac{z_0}{c^2}z = 1$.

51. $\nabla F(x_0,y_0,z_0) = \left\langle \dfrac{2x_0}{a^2}, \dfrac{2y_0}{b^2}, \dfrac{-1}{c}\right\rangle$, so an equation of the tangent plane is $\dfrac{2x_0}{a^2}x + \dfrac{2y_0}{b^2}y - \dfrac{1}{c}z = \dfrac{2x_0^2}{a^2} + \dfrac{2y_0^2}{b^2} - \dfrac{z_0}{c}$ or $\dfrac{2x_0}{a^2}x + \dfrac{2y_0}{b^2}y = \dfrac{z}{c} + 2\left(\dfrac{x_0^2}{a^2}+\dfrac{y_0^2}{b^2}\right) - \dfrac{z_0}{c}$. But $\dfrac{z_0}{c} = \dfrac{x_0^2}{a^2}+\dfrac{y_0^2}{b^2}$, so the equation can be written as $\dfrac{2x_0}{a^2}x + \dfrac{2y_0}{b^2}y = \dfrac{z+z_0}{c}$.

52. Since $\nabla f(x_0,y_0,z_0) = \langle 2x_0, 4y_0, 6z_0\rangle$ and $\langle 3,-1,3\rangle$ are both normal vectors to the surface at (x_0,y_0,z_0), we need $\langle 2x_0, 4y_0, 6z_0\rangle = c\,\langle 3,-1,3\rangle$ or $\langle x_0, 2y_0, 3z_0\rangle = k\langle 3,-1,3\rangle$. Thus $x_0 = 3k$, $y_0 = -\frac{1}{2}k$ and $z_0 = k$. But $x_0^2 + 2y_0^2 + 3z_0^2 = 1$ or $(9+\frac{1}{2}+3)k^2 = 1$, so $k = \pm\frac{\sqrt{2}}{5}$ and there are two such points: $\left(\pm\frac{3\sqrt{2}}{5}, \mp\frac{1}{5\sqrt{2}}, \pm\frac{\sqrt{2}}{5}\right)$.

53. $\nabla f(x_0,y_0,z_0) = \langle 2x_0, -2y_0, 4z_0\rangle$ and the given line has direction numbers $2, 4, 6$, so $\langle 2x_0, -2y_0, 4z_0\rangle = k\langle 2,4,6\rangle$ or $x_0 = k$, $y_0 = -2k$ and $z_0 = \frac{3}{2}k$. But $x_0^2 - y_0^2 + 2z_0^2 = 1$ or $\left(1-4+\frac{9}{2}\right)k^2 = 1$, so $k = \pm\sqrt{\frac{2}{3}} = \pm\frac{\sqrt{6}}{3}$ and there are two such points: $\left(\pm\frac{\sqrt{6}}{3}, \mp\frac{2\sqrt{6}}{3}, \pm\frac{\sqrt{6}}{2}\right)$.

54. First note that the point $(1,1,2)$ is on both surfaces. For the ellipsoid, an equation of the tangent plane at $(1,1,2)$ is $6x+4y+4z = 18$ or $3x+2y+2z = 9$, and for the sphere, an equation of the tangent plane at $(1,1,2)$ is $(2-8)x + (2-6)y + (4-8)z = -18$ or $-6x-4y-4z = -18$ or $3x+2y+2z = 9$. Since these tangent planes are the same, the surfaces are tangent to each other at the point $(1,1,2)$.

55. Let (x_0, y_0, z_0) be a point on the cone [other than $(0,0,0)$]. Then an equation of the tangent plane to the cone at this point is $2x_0x + 2y_0y - 2z_0z = 2(x_0^2 + y_0^2 - z_0^2)$. But $x_0^2 + y_0^2 = z_0^2$ so the tangent plane is given by $x_0x + y_0y - z_0z = 0$, a plane which always contains the origin.

56. Let (x_0, y_0, z_0) be a point on the sphere. Then the normal line is given by $\dfrac{x - x_0}{2x_0} = \dfrac{y - y_0}{2y_0} = \dfrac{z - z_0}{2z_0}$. For the center $(0,0,0)$ to be on the line, we need $-\dfrac{x_0}{2x_0} = -\dfrac{y_0}{2y_0} = -\dfrac{z_0}{2z_0}$ or equivalently $1 = 1 = 1$, which is true.

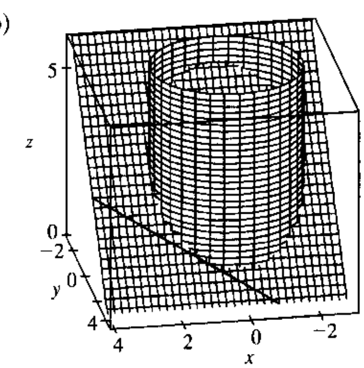
57. Let (x_0, y_0, z_0) be a point on the surface. Then an equation of the tangent plane at the point is $\dfrac{x}{2\sqrt{x_0}} + \dfrac{y}{2\sqrt{y_0}} + \dfrac{z}{2\sqrt{z_0}} = \dfrac{\sqrt{x_0} + \sqrt{y_0} + \sqrt{z_0}}{2}$. But $\sqrt{x_0} + \sqrt{y_0} + \sqrt{z_0} = \sqrt{c}$, so the equation is $\dfrac{x}{\sqrt{x_0}} + \dfrac{y}{\sqrt{y_0}} + \dfrac{z}{\sqrt{z_0}} = \sqrt{c}$. The x-, y-, and z-intercepts are $\sqrt{cx_0}$, $\sqrt{cy_0}$ and $\sqrt{cz_0}$ respectively. (The x-intercept is found by setting $y = z = 0$ and solving the resulting equation for x, and the y- and z-intercepts are found similarly.) So the sum of the intercepts is $\sqrt{c}\left(\sqrt{x_0} + \sqrt{y_0} + \sqrt{z_0}\right) = c$, a constant.

58. Here the equation of the tangent plane to the point (x_0, y_0, z_0) is $y_0z_0x + x_0z_0y + x_0y_0z = 3x_0y_0z_0$ or $\dfrac{x}{3x_0} + \dfrac{y}{3y_0} + \dfrac{z}{3z_0} = 1$. Then the x-, y-, and z-intercepts are $3x_0$, $3y_0$ and $3z_0$ respectively, and their product is $27x_0y_0z_0 = 27c^3$, a constant.

59. If $f(x,y,z) = z - x^2 - y^2$ and $g(x,y,z) = 4x^2 + y^2 + z^2$, then the tangent line is perpendicular to both ∇f and ∇g at $(-1,1,2)$. The vector $\mathbf{v} = \nabla f \times \nabla g$ will therefore be parallel to the tangent line. We have: $\nabla f(x,y,z) = \langle -2x, -2y, 1\rangle \Rightarrow \nabla f(-1,1,2) = \langle 2,-2,1\rangle$, and $\nabla g(x,y,z) = \langle 8x, 2y, 2z\rangle \Rightarrow$ $\nabla g(-1,1,2) = \langle -8,2,4\rangle$. Hence $\mathbf{v} = \nabla f \times \nabla g = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -2 & 1 \\ -8 & 2 & 4 \end{vmatrix} = -10\,\mathbf{i} - 16\,\mathbf{j} - 12\,\mathbf{k}$. Parametric equations are: $x = -1 - 10t$, $y = 1 - 16t$, $z = 2 - 12t$.

60. (a) Let $f(x,y,z) = y + z$ and $g(x,y,z) = x^2 + y^2$. Then the required tangent line is perpendicular to both ∇f and ∇g at $(1,2,1)$ and the vector $\mathbf{v} = \nabla f \times \nabla g$ is parallel to the tangent line. We have $\nabla f(x,y,z) = \langle 0,1,1\rangle \Rightarrow \nabla f(1,2,1) = \langle 0,1,1\rangle$, and $\nabla g(x,y,z) = \langle 2x, 2y, 0\rangle \Rightarrow \nabla g(1,2,1) = \langle 2,4,0\rangle$. Hence $\mathbf{v} = \nabla f \times \nabla g = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 1 \\ 2 & 4 & 0 \end{vmatrix} = -4\,\mathbf{i} + 2\,\mathbf{j} - 2\,\mathbf{k}$. So parametric equations of the desired tangent line are $x = 1 - 4t$, $y = 2 + 2t$, $z = 1 - 2t$.

(b)



61. (a) The direction of the normal line of F is given by ∇F, and that of G by ∇G. Assuming that $\nabla F \neq 0 \neq \nabla G$, the two normal lines are perpendicular at P if $\nabla F \cdot \nabla G = 0$ at P $\Leftrightarrow$ $\langle \partial F/\partial x, \partial F/\partial y, \partial F/\partial z\rangle \cdot \langle \partial G/\partial x, \partial G/\partial y, \partial G/\partial z\rangle = 0$ at P $\Leftrightarrow$ $F_xG_x + F_yG_y + F_zG_z = 0$ at P.

(b) Here $F = x^2 + y^2 - z^2$ and $G = x^2 + y^2 + z^2 - r^2$, so

$\nabla F \cdot \nabla G = \langle 2x, 2y, -2z\rangle \cdot \langle 2x, 2y, 2z\rangle = 4x^2 + 4y^2 - 4z^2 = 4F = 0$, since the point $\langle x, y, z\rangle$ lies on the graph of $F = 0$. To see that this is true without using calculus, note that $G = 0$ is the equation of a sphere centered at the origin and $F = 0$ is the equation of a right circular cone with vertex at the origin (which is generated by lines through the origin). At any point of intersection, the sphere's normal line (which passes through the origin) lies on the cone, and thus is perpendicular to the cone's normal line. So the surfaces with equations $F = 0$ and $G = 0$ are everywhere orthogonal.

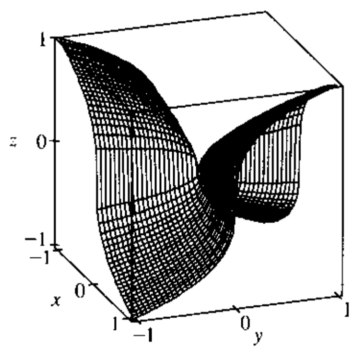
62. (a) The function $f(x,y) = (xy)^{1/3}$ is continuous on $\mathbb{R}^2$ since it is a composition of a polynomial and the cube root function, both of which are continuous. (See the text just after Example 15.2.8 [ET 14.2.8].)

$$f_x(0,0) = \lim_{h\to 0} \frac{f(0+h,0) - f(0,0)}{h} = \lim_{h\to 0} \frac{(h\cdot 0)^{1/3} - 0}{h} = 0,$$

$$f_y(0,0) = \lim_{h\to 0} \frac{f(0,0+h) - f(0,0)}{h} = \lim_{h\to 0} \frac{(0\cdot h)^{1/3} - 0}{h} = 0.$$ Therefore, $f_x(0,0)$ and $f_y(0,0)$ do exist and are equal to 0. Now let $\mathbf{u}$ be any unit vector other than $\mathbf{i}$ and $\mathbf{j}$ (these correspond to f_x and f_y respectively.) Then $\mathbf{u} = a\,\mathbf{i} + b\,\mathbf{j}$ where $a \neq 0$ and $b \neq 0$. Thus

$$D_{\mathbf{u}}\, f(0,0) = \lim_{h\to 0} \frac{f(0+ha, 0+hb) - f(0,0)}{h} = \lim_{h\to 0} \frac{\sqrt[3]{(ha)(hb)}}{h} = \lim_{h\to 0} \frac{\sqrt[3]{ab}}{h^{1/3}}$$ and this limit does not exist, so $D_{\mathbf{u}}\, f(0,0)$ does not exist.

(b)



Notice that if we start at the origin and proceed in the direction of the x- or y-axis, then the graph is flat. But if we proceed in any other direction, then the graph is extremely steep.

63. Let $\mathbf{u} = \langle a, b\rangle$ and $\mathbf{v} = \langle c, d\rangle$. Then we know that at the given point, $D_{\mathbf{u}}\, f = \nabla f \cdot \mathbf{u} = af_x + bf_y$ and $D_{\mathbf{v}}\, f = \nabla f \cdot \mathbf{v} = cf_x + df_y$. But these are just two linear equations in the two unknowns f_x and f_y, and since $\mathbf{u}$ and $\mathbf{v}$ are not parallel, we can solve the equations to find $\nabla f = \langle f_x, f_y\rangle$ at the given point. In fact,

$$\nabla f = \left\langle \frac{d\,D_{\mathbf{u}}\, f - b\,D_{\mathbf{v}}\, f}{ad - bc}, \frac{a\,D_{\mathbf{v}}\, f - c\,D_{\mathbf{u}}\, f}{ad - bc} \right\rangle.$$

64. Since $z = f(x,y)$ is differentiable at $\mathbf{x}_0 = (x_0, y_0)$, by Definition 15.4.7 [ET 14.4.7] we have $\Delta z = f_x(x_0, y_0)\,\Delta x + f_y(x_0, y_0)\,\Delta y + \varepsilon_1\,\Delta x + \varepsilon_2\,\Delta y$ where $\varepsilon_1, \varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0,0)$. Now $\Delta z = f(\mathbf{x}) - f(\mathbf{x}_0)$, $\langle \Delta x, \Delta y\rangle = \mathbf{x} - \mathbf{x}_0$ so $(\Delta x, \Delta y) \to (0,0)$ is equivalent to $\mathbf{x} \to \mathbf{x}_0$ and $\langle f_x(x_0, y_0), f_y(x_0, y_0)\rangle = \nabla f(\mathbf{x}_0)$. Substituting into (15.4.7 [ET 14.4.7]) gives

$f(\mathbf{x}) - f(\mathbf{x}_0) = \nabla f(\mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0) + \langle \varepsilon_1, \varepsilon_2\rangle \cdot \langle \Delta x, \Delta y\rangle$ or

$\langle \varepsilon_1, \varepsilon_2\rangle \cdot (\mathbf{x} - \mathbf{x}_0) = f(\mathbf{x}) - f(\mathbf{x}_0) - \nabla f(\mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0)$, and so

$$\frac{f(\mathbf{x}) - f(\mathbf{x}_0) - \nabla f(\mathbf{x}_0) \cdot (\mathbf{x} - \mathbf{x}_0)}{|\mathbf{x} - \mathbf{x}_0|} = \frac{\langle \varepsilon_1, \varepsilon_2\rangle \cdot (\mathbf{x} - \mathbf{x}_0)}{|\mathbf{x} - \mathbf{x}_0|}.$$ But $\dfrac{\mathbf{x} - \mathbf{x}_0}{|\mathbf{x} - \mathbf{x}_0|}$ is a unit vector so

$\lim\limits_{\mathbf{x}\to\mathbf{x}_0} \dfrac{\langle \varepsilon_1, \varepsilon_2\rangle \cdot (\mathbf{x}-\mathbf{x}_0)}{|\mathbf{x}-\mathbf{x}_0|} = 0$ since $\varepsilon_1, \varepsilon_2 \to 0$ as $\mathbf{x}\to\mathbf{x}_0$. Hence

$$\lim_{\mathbf{x}\to\mathbf{x}_0} \frac{f(\mathbf{x}) - f(\mathbf{x}_0) - \nabla f(\mathbf{x}_0)\cdot(\mathbf{x}-\mathbf{x}_0)}{|\mathbf{x}-\mathbf{x}_0|} = 0.$$

15.7 Maximum and Minimum Values ET 14.7

1. (a) First we compute $D(1,1) = f_{xx}(1,1)\,f_{yy}(1,1) - [f_{xy}(1,1)]^2 = (4)(2) - (1)^2 = 7$. Since $D(1,1) > 0$ and $f_{xx}(1,1) > 0$, f has a local minimum at $(1,1)$ by the Second Derivatives Test.

(b) $D(1,1) = f_{xx}(1,1)\,f_{yy}(1,1) - [f_{xy}(1,1)]^2 = (4)(2) - (3)^2 = -1$. Since $D(1,1) < 0$, f has a saddle point at $(1,1)$ by the Second Derivatives Test.

2. (a) $D = g_{xx}(0,2)\,g_{yy}(0,2) - [g_{xy}(0,2)]^2 = (-1)(1) - (6)^2 = -37$. Since $D < 0$, g has a saddle point at $(0,2)$ by the Second Derivatives Test.

(b) $D = g_{xx}(0,2)\,g_{yy}(0,2) - [g_{xy}(0,2)]^2 = (-1)(-8) - (2)^2 = 4$. Since $D > 0$ and $g_{xx}(0,2) < 0$, g has a local maximum at $(0,2)$ by the Second Derivatives Test.

(c) $D = g_{xx}(0,2)\,g_{yy}(0,2) - [g_{xy}(0,2)]^2 = (4)(9) - (6)^2 = 0$. In this case the Second Derivatives Test gives no information about g at the point $(0,2)$.

3. In the figure, a point at approximately $(1,1)$ is enclosed by level curves which are oval in shape and indicate that as we move away from the point in any direction the values of f are increasing. Hence we would expect a local minimum at or near $(1,1)$. The level curves near $(0,0)$ resemble hyperbolas, and as we move away from the origin, the values of f increase in some directions and decrease in others, so we would expect to find a saddle point there.

To verify our predictions, we have $f(x,y) = 4 + x^3 + y^3 - 3xy \;\Rightarrow\; f_x(x,y) = 3x^2 - 3y$, $f_y(x,y) = 3y^2 - 3x$. We have critical points where these partial derivatives are equal to 0: $3x^2 - 3y = 0$, $3y^2 - 3x = 0$. Substituting $y = x^2$ from the first equation into the second equation gives $3(x^2)^2 - 3x = 0 \;\Rightarrow\; 3x(x^3 - 1) = 0 \;\Rightarrow\; x = 0$ or $x = 1$. Then we have two critical points, $(0,0)$ and $(1,1)$. The second partial derivatives are $f_{xx}(x,y) = 6x$, $f_{xy}(x,y) = -3$, and $f_{yy}(x,y) = 6y$, so $D(x,y) = f_{xx}(x,y)\,f_{yy}(x,y) - [f_{xy}(x,y)]^2 = (6x)(6y) - (-3)^2 = 36xy - 9$. Then $D(0,0) = 36(0)(0) - 9 = -9$, and $D(1,1) = 36(1)(1) - 9 = 27$. Since $D(0,0) < 0$, f has a saddle point at $(0,0)$ by the Second Derivatives Test. Since $D(1,1) > 0$ and $f_{xx}(1,1) > 0$, f has a local minimum at $(1,1)$.

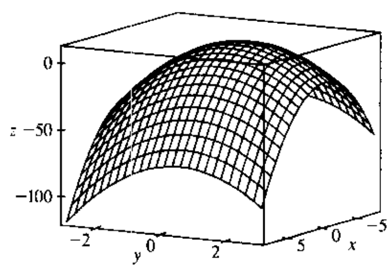
4. In the figure, points at approximately $(-1,1)$ and $(-1,-1)$ are enclosed by oval-shaped level curves which indicate that as we move away from either point in any direction, the values of f are increasing. Hence we would expect local minima at or near $(-1,\pm 1)$. Similarly, the point $(1,0)$ appears to be enclosed by oval-shaped level curves which indicate that as we move away from the point in any direction the values of f are decreasing, so we should have a local maximum there. We also show hyperbola-shaped level curves near the points $(-1,0)$, $(1,1)$, and $(1,-1)$. The values of f increase along some paths leaving these points and decrease in others, so we should have a saddle point at each of these points.

To confirm our predictions, we have $f(x,y) = 3x - x^3 - 2y^2 + y^4 \;\Rightarrow\; f_x(x,y) = 3 - 3x^2$, $f_y(x,y) = -4y + 4y^3$. Setting these partial derivatives equal to 0, we have $3 - 3x^2 = 0 \;\Rightarrow\; x = \pm 1$ and

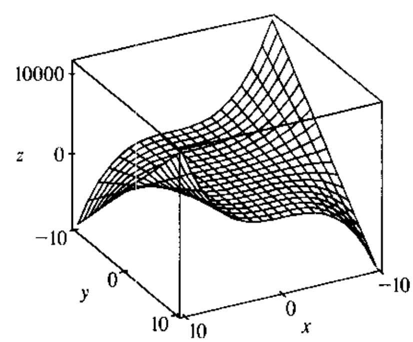
$-4y + 4y^3 = 0 \;\Rightarrow\; y(y^2 - 1) = 0 \;\Rightarrow\; y = 0, \pm 1$. So our critical points are $(\pm 1, 0)$, $(\pm 1, \pm 1)$. The second partial derivatives are $f_{xx}(x,y) = -6x$, $f_{xy}(x,y) = 0$, and $f_{yy}(x,y) = 12y^2 - 4$, so $D(x,y) = f_{xx}(x,y)\, f_{yy}(x,y) - [f_{xy}(x,y)]^2 = (-6x)(12y^2 - 4) - (0)^2 = -72xy^2 + 24x$. We use the Second Derivatives Test to classify the 6 critical points:

Critical Point	D	f_{xx}	Conclusion
$(1,0)$	24	-6	$D > 0,\ f_{xx} < 0 \;\Rightarrow\; f$ has a local maximum at $(1,0)$
$(1,1)$	-48		$D < 0 \;\Rightarrow\; f$ has a saddle point at $(1,1)$
$(1,-1)$	-48		$D < 0 \;\Rightarrow\; f$ has a saddle point at $(1,-1)$
$(-1,0)$	-24		$D < 0 \;\Rightarrow\; f$ has a saddle point at $(-1,0)$
$(-1,1)$	48	6	$D > 0,\ f_{xx} > 0 \;\Rightarrow\; f$ has a local minimum at $(-1,1)$
$(-1,-1)$	48	6	$D > 0,\ f_{xx} > 0 \;\Rightarrow\; f$ has a local minimum at $(-1,-1)$

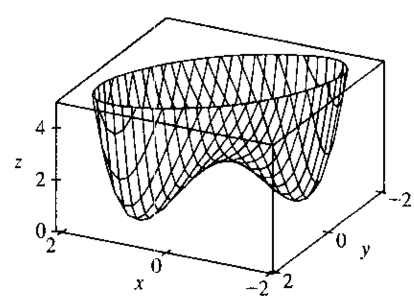
5. $f(x,y) = 9 - 2x + 4y - x^2 - 4y^2 \;\Rightarrow\; f_x = -2 - 2x$, $f_y = 4 - 8y$, $f_{xx} = -2$, $f_{xy} = 0$, $f_{yy} = -8$. Then $f_x = 0$ and $f_y = 0$ imply $x = -1$ and $y = \frac{1}{2}$, and the only critical point is $\left(-1, \frac{1}{2}\right)$. $D(x,y) = f_{xx} f_{yy} - (f_{xy})^2 = (-2)(-8) - 0^2 = 16$, and since $D\left(-1, \frac{1}{2}\right) = 16 > 0$ and $f_{xx}\left(-1, \frac{1}{2}\right) = -2 < 0$, $f\left(-1, \frac{1}{2}\right) = 11$ is a local maximum by the Second Derivatives Test.



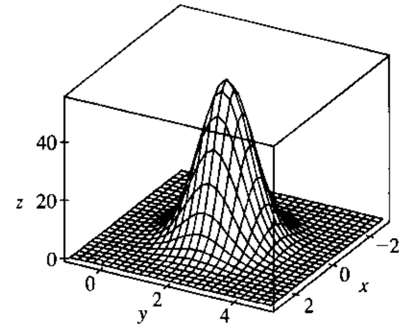
6. $f(x,y) = x^3 y + 12x^2 - 8y \;\Rightarrow\; f_x = 3x^2 y + 24x$, $f_y = x^3 - 8$, $f_{xx} = 6xy + 24$, $f_{xy} = 3x^2$, $f_{yy} = 0$. Then $f_y = 0$ implies $x = 2$, and substitution into $f_x = 0$ gives $12y + 48 = 0 \;\Rightarrow\; y = -4$. Thus, the only critical point is $(2, -4)$. $D(2,-4) = (-24)(0) - 12^2 = -144 < 0$, so $(2,-4)$ is a saddle point.



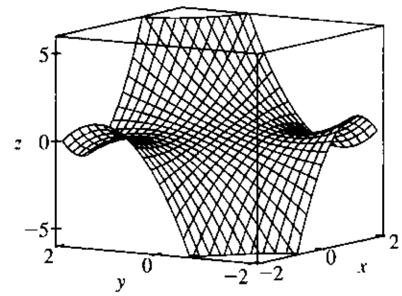
7. $f(x,y) = x^4 + y^4 - 4xy + 2 \;\Rightarrow\; f_x = 4x^3 - 4y$, $f_y = 4y^3 - 4x$, $f_{xx} = 12x^2$, $f_{xy} = -4$, $f_{yy} = 12y^2$. Then $f_x = 0$ implies $y = x^3$, and substitution into $f_y = 0 \;\Rightarrow\; x = y^3$ gives $x^9 - x = 0 \;\Rightarrow\; x(x^8 - 1) = 0 \;\Rightarrow\; x = 0$ or $x = \pm 1$. Thus the critical points are $(0,0)$, $(1,1)$, and $(-1,-1)$. Now $D(0,0) = 0 \cdot 0 - (-4)^2 = -16 < 0$, so $(0,0)$ is a saddle point. $D(1,1) = (12)(12) - (-4)^2 > 0$ and $f_{xx}(1,1) = 12 > 0$, so $f(1,1) = 0$ is a local minimum. $D(-1,-1) = (12)(12) - (-4)^2 > 0$ and $f_{xx} = (-1,-1) = 12 > 0$, so $f(-1,-1) = 0$ is also a local minimum.



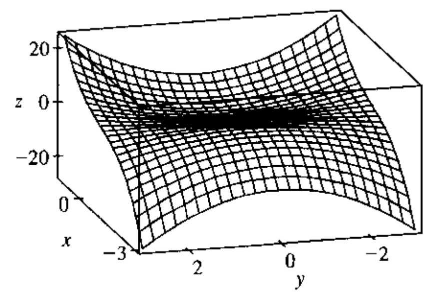
8. $f(x,y) = e^{4y-x^2-y^2} \quad \Rightarrow \quad f_x = -2xe^{4y-x^2-y^2}$,
$f_y = (4-2y)e^{4y-x^2-y^2}$, $f_{xx} = (4x^2-2)e^{4y-x^2-y^2}$,
$f_{xy} = -2x(4-2y)e^{4y-x^2-y^2}$,
$f_{yy} = (4y^2 - 16y + 14)e^{4y-x^2-y^2}$. Then $f_x = 0$ and $f_y = 0$ implies $x = 0$ and $y = 2$, so the only critical point is $(0,2)$.
$D(0,2) = (-2e^4)(-2e^4) - 0^2 = 4e^8 > 0$ and
$f_{xx}(0,2) = -2e^4 < 0$, so $f(0,2) = e^4$ is a local maximum.



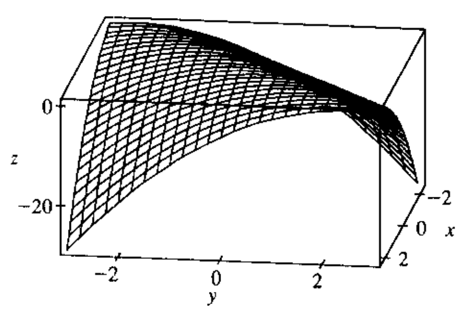
9. $f(x,y) = (1+xy)(x+y) = x + y + x^2y + xy^2 \quad \Rightarrow$
$f_x = 1 + 2xy + y^2$, $f_y = 1 + x^2 + 2xy$, $f_{xx} = 2y$, $f_{xy} = 2x + 2y$,
$f_{yy} = 2x$. Then $f_x = 0$ implies $1 + 2xy + y^2 = 0$ and $f_y = 0$ implies $1 + x^2 + 2xy = 0$. Subtracting the second equation from the first gives $y^2 - x^2 = 0 \quad \Rightarrow \quad y = \pm x$, but if $y = x$ then $1 + 2xy + y^2 = 0 \quad \Rightarrow \quad 1 + 3x^2 = 0$ which has no real solution. If $y = -x$ then $1 + 2xy + y^2 = 0 \quad \Rightarrow \quad 1 - x^2 = 0 \quad \Rightarrow \quad x = \pm 1$, so critical points are $(1,-1)$ and $(-1,1)$. $D(1,-1) = (-2)(2) - 0 < 0$ and $D(-1,1) = (2)(-2) - 0 < 0$, so $(-1,1)$ and $(1,-1)$ are saddle points.



10. $f(x,y) = 2x^3 + xy^2 + 5x^2 + y^2 \Rightarrow \quad f_x = 6x^2 + y^2 + 10x$,
$f_y = 2xy + 2y$, $f_{xx} = 12x + 10$, $f_{yy} = 2x + 2$, $f_{xy} = 2y$. Then $f_y = 0$ implies $y = 0$ or $x = -1$. Substituting into $f_x = 0$ gives the critical points $(0,0)$, $\left(-\frac{5}{3}, 0\right)$, $(-1, \pm 2)$. Now $D(0,0) = 20 > 0$ and $f_{xx}(0,0) = 10 > 0$, so $f(0,0) = 0$ is a local minimum. Also $f_{xx}\left(-\frac{5}{3}, 0\right) < 0$, $D\left(-\frac{5}{3}, 0\right) > 0$, and $D(-1, \pm 2) < 0$. Hence $f\left(-\frac{5}{3}, 0\right) = \frac{125}{27}$ is a local maximum while $(-1, \pm 2)$ are saddle points.



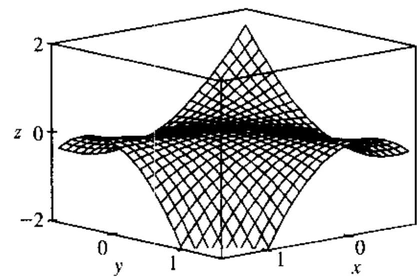
11. $f(x,y) = 1 + 2xy - x^2 - y^2 \quad \Rightarrow \quad f_x = 2y - 2x$,
$f_y = 2x - 2y$, $f_{xx} = f_{yy} = -2$, $f_{xy} = 2$. Then $f_x = 0$ and $f_y = 0$ implies $x = y$ so the critical points are all points of the form (x_0, x_0). But $D(x_0, x_0) = 4 - 4 = 0$ so the Second Derivatives Test gives no information. However $1 + 2xy - x^2 - y^2 = 1 - (x-y)^2$ and $1 - (x-y)^2 \leq 1$ for all (x,y), with equality if and only if $x = y$. Thus $f(x_0, x_0) = 1$ are local maxima.



12. $f(x,y) = xy(1-x-y) \;\Rightarrow\; f_x = y - 2xy - y^2$,

$f_y = x - x^2 - 2xy$, $f_{xx} = -2y$, $f_{yy} = -2x$, $f_{xy} = 1 - 2x - 2y$.

Then $f_x = 0$ implies $y = 0$ or $y = 1 - 2x$. Substituting $y = 0$ into $f_y = 0$ gives $x = 0$ or $x = 1$ and substituting $y = 1 - 2x$ into $f_y = 0$ gives $3x^2 - x = 0$ so $x = 0$ or $\frac{1}{3}$. Thus the critical points are $(0,0)$, $(1,0)$, $(0,1)$ and $\left(\frac{1}{3}, \frac{1}{3}\right)$.

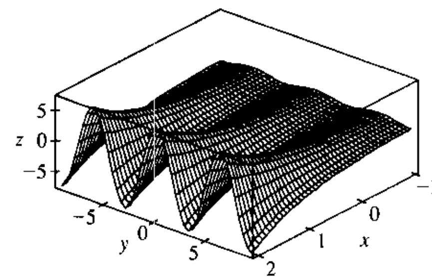


$D(0,0) = D(1,0) = D(0,1) = -1$ while $D\left(\frac{1}{3}, \frac{1}{3}\right) = \frac{1}{3}$ and $f_{xx}\left(\frac{1}{3}, \frac{1}{3}\right) = -\frac{2}{3} < 0$. Thus $(0,0)$, $(1,0)$ and $(0,1)$ are saddle points, and $f\left(\frac{1}{3}, \frac{1}{3}\right) = \frac{1}{27}$ is a local maximum.

13. $f(x,y) = e^x \cos y \;\Rightarrow\; f_x = e^x \cos y$, $f_y = -e^x \sin y$.

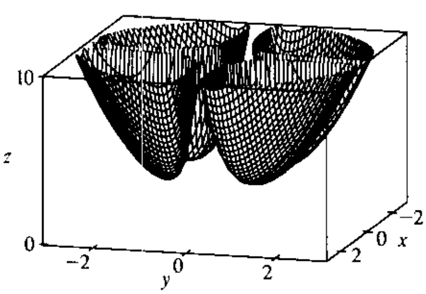
Now $f_x = 0$ implies $\cos y = 0$ or $y = \frac{\pi}{2} + n\pi$ for n an integer.

But $\sin\left(\frac{\pi}{2} + n\pi\right) \neq 0$, so there are no critical points.



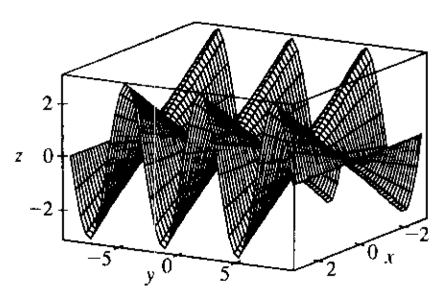
14. $f(x,y) = x^2 + y^2 + \dfrac{1}{x^2 y^2} \;\Rightarrow\; f_x = 2x - 2x^{-3}y^{-2}$,

$f_y = 2y - 2x^{-2}y^{-3}$, $f_{xx} = 2 + 6x^{-4}y^{-2}$, $f_{yy} = 2 + 6x^{-2}y^{-4}$, $f_{xy} = 4x^{-3}y^{-3}$. Then $f_x = 0$ implies $2x^4y^2 - 2 = 0$ or $x^4y^2 = 1$ or $y^2 = x^{-4}$. Note that neither x nor y can be zero. Now $f_y = 0$ implies $2x^2y^4 - 2 = 0$, and with $y^2 = x^{-4}$ this implies $2x^{-6} - 2 = 0$ or $x^6 = 1$. Thus $x = \pm 1$ and if $x = 1$, $y = \pm 1$; if $x = -1$, $y = \pm 1$. So the critical points are $(1,1)$, $(1,-1)$, $(-1,1)$ and $(-1,-1)$. Now $D(\pm 1, \pm 1) = D(\pm 1, \mp 1) = 64 - 16 > 0$ and $f_{xx} > 0$ always, so $f(\pm 1, \pm 1) = f(\pm 1, \mp 1) = 3$ are local minima.

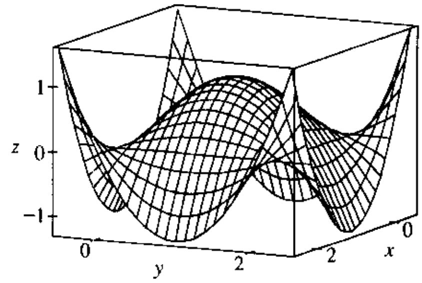


15. $f(x,y) = x \sin y \;\Rightarrow\; f_x = \sin y$, $f_y = x \cos y$, $f_{xx} = 0$,

$f_{yy} = -x \sin y$ and $f_{xy} = \cos y$. Then $f_x = 0$ if and only if $y = n\pi$, n an integer, and substituting into $f_y = 0$ requires $x = 0$ for each of these y-values. Thus the critical points are $(0, n\pi)$, n an integer. But $D(0, n\pi) = -\cos^2(n\pi) < 0$ so each critical point is a saddle point.



16. $f(x,y)=(2x-x^2)(2y-y^2) \Rightarrow \quad f_x=(2-2x)(2y-y^2)$,



$f_y=(2x-x^2)(2-2y)$, $f_{xx}=-2(2y-y^2)$, $f_{yy}=-2(2x-x^2)$ and $f_{xy}=(2-2x)(2-2y)$. Then $f_x=0$ implies $x=1$ or $y=0$ or $y=2$ and when $x=1$, $f_y=0$ implies $y=1$, when $y=0$, $f_y=0$ implies $x=0$ or $x=2$ and when $y=2$, $f_y=0$ implies $x=0$ or $x=2$. Thus the critical points are $(1,1)$, $(0,0)$, $(2,0)$, $(0,2)$ and $(2,2)$. Now $D(0,0)=D(2,0)=D(0,2)=D(2,2)=-16$ so these critical points are saddle points, and $D(1,1)=4$ with $f_{xx}(1,1)=-2$, so $f(1,1)=1$ is a local maximum.

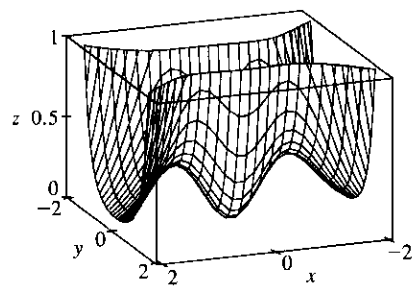
17. $f(x,y)=(x^2+y^2)e^{y^2-x^2} \quad\Rightarrow\quad f_x=(x^2+y^2)e^{y^2-x^2}(-2x)+2xe^{y^2-x^2}=2xe^{y^2-x^2}(1-x^2-y^2)$,

$f_y=(x^2+y^2)e^{y^2-x^2}(2y)+2ye^{y^2-x^2}=2ye^{y^2-x^2}(1+x^2+y^2)$,

$$f_{xx}=2xe^{y^2-x^2}(-2x)+(1-x^2-y^2)\left(2x\left(-2xe^{y^2-x^2}\right)+2e^{y^2-x^2}\right)$$
$$=2e^{y^2-x^2}((1-x^2-y^2)(1-2x^2)-2x^2),$$

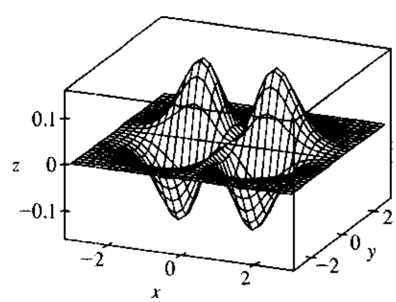
$$f_{xy}=2xe^{y^2-x^2}(-2y)+2x(2y)e^{y^2-x^2}(1-x^2-y^2)=-4xye^{y^2-x^2}(x^2+y^2),$$

$$f_{yy}=2ye^{y^2-x^2}(2y)+(1+x^2+y^2)\left(2y\left(2ye^{y^2-x^2}\right)+2e^{y^2-x^2}\right)$$
$$=2e^{y^2-x^2}((1+x^2+y^2)(1+2y^2)+2y^2).$$



$f_y=0$ implies $y=0$, and substituting into $f_x=0$ gives $2xe^{-x^2}(1-x^2)=0 \quad\Rightarrow\quad x=0$ or $x=\pm1$. Thus the critical points are $(0,0)$ and $(\pm1,0)$. $D(0,0)=(2)(2)-0>0$ and $f_{xx}(0,0)=2>0$, so $f(0,0)=0$ is a local minimum. $D(\pm1,0)=(-4e^{-1})(4e^{-1})-0<0$ so $(\pm1,0)$ are saddle points.

18. $f(x,y)=x^2ye^{-x^2-y^2} \quad\Rightarrow$



$f_x=x^2ye^{-x^2-y^2}(-2x)+2xye^{-x^2-y^2}=2xy(1-x^2)e^{-x^2-y^2}$,

$f_y=x^2ye^{-x^2-y^2}(-2y)+x^2e^{-x^2-y^2}=x^2(1-2y^2)e^{-x^2-y^2}$,

$f_{xx}=2y(2x^4-5x^2+1)e^{-x^2-y^2}$,

$f_{xy}=2x(1-x^2)(1-2y^2)e^{-x^2-y^2}$,

$f_{yy}=2x^2y(2y^2-3)e^{-x^2-y^2}$.

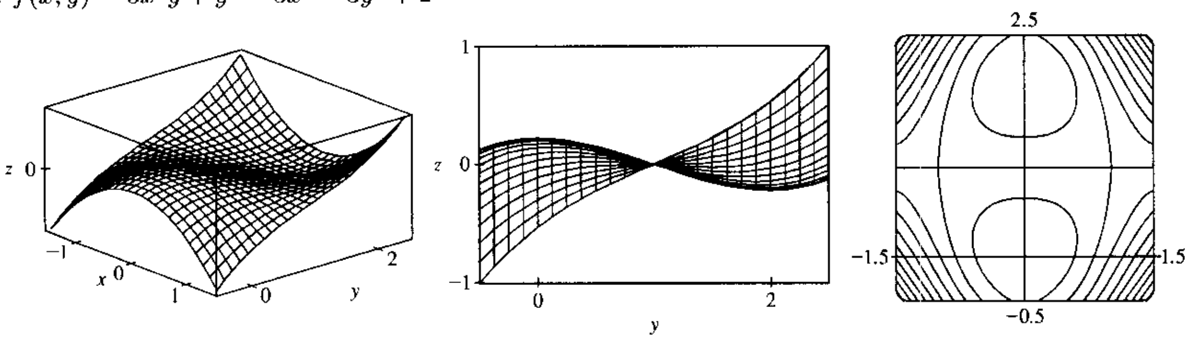
$f_x=0$ implies $x=0$, $y=0$, or $x=\pm1$. If $x=0$ then $f_y=0$ for any y-value, so all points of the form $(0,y)$ are critical points. If $y=0$ then $f_y=0 \quad\Rightarrow\quad x^2e^{-x^2}=0 \quad\Rightarrow\quad x=0$, so $(0,0)$ (already included above) is a critical point. If $x=\pm1$ then $(1-2y^2)e^{-1-y^2}=0 \quad\Rightarrow\quad y=\pm\frac{1}{\sqrt{2}}$, so $\left(1,\pm\frac{1}{\sqrt{2}}\right)$ and $\left(-1,\pm\frac{1}{\sqrt{2}}\right)$ are critical points. $D(0,y)=0$, so the Second Derivatives Test gives no information. However, if $y>0$ then

$x^2ye^{-x^2-y^2} \geq 0$ with equality only when $x = 0$, so we have local minimum values $f(0, y) = 0$, $y > 0$. Similarly, if $y < 0$ then $x^2ye^{-x^2-y^2} \leq 0$ with equality when $x = 0$ so $f(0, y) = 0$, $y < 0$ are local maximum values, and $(0, 0)$ is a saddle point.

$D\left(\pm 1, \frac{1}{\sqrt{2}}\right) = 8e^{-3} > 0$, $f_{xx}\left(\pm 1, \frac{1}{\sqrt{2}}\right) = -2\sqrt{2}e^{-3/2} < 0$ and

$D\left(\pm 1, -\frac{1}{\sqrt{2}}\right) = 8e^{-3} > 0$, $f_{xx}\left(\pm 1, -\frac{1}{\sqrt{2}}\right) = 2\sqrt{2}e^{-3/2} > 0$, so $f\left(\pm 1, \frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}e^{-3/2}$ are local maximum points while $f\left(\pm 1, -\frac{1}{\sqrt{2}}\right) = -\frac{1}{\sqrt{2}}e^{-3/2}$ are local minimum points.

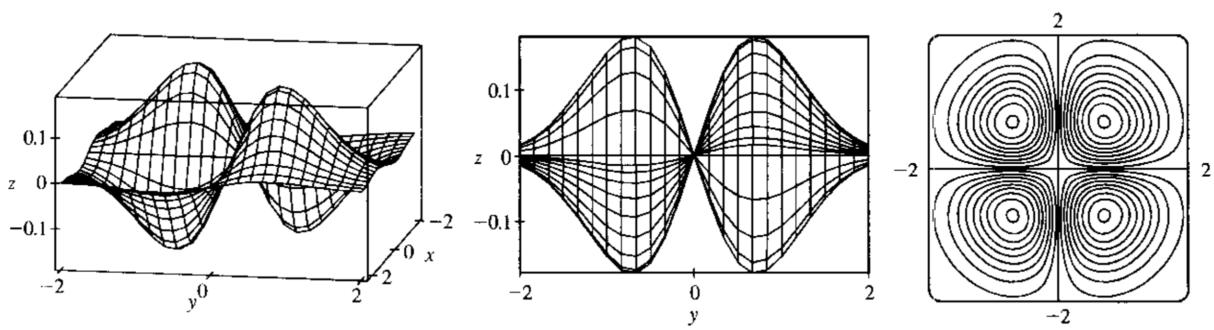
19. $f(x, y) = 3x^2y + y^3 - 3x^2 - 3y^2 + 2$



From the graphs, it appears that f has a local maximum $f(0, 0) \approx 2$ and a local minimum $f(0, 2) \approx -2$. There appear to be saddle points near $(\pm 1, 1)$.

$f_x = 6xy - 6x$, $f_y = 3x^2 + 3y^2 - 6y$. Then $f_x = 0$ implies $x = 0$ or $y = 1$ and when $x = 0$, $f_y = 0$ implies $y = 0$ or $y = 2$; when $y = 1$, $f_y = 0$ implies $x^2 = 1$ or $x = \pm 1$. Thus the critical points are $(0, 0)$, $(0, 2)$, $(\pm 1, 1)$. Now $f_{xx} = 6y - 6$, $f_{yy} = 6y - 6$ and $f_{xy} = 6x$, so $D(0, 0) = D(0, 2) = 36 > 0$ while $D(\pm 1, 1) = -36 < 0$ and $f_{xx}(0, 0) = -6$, $f_{xx}(0, 2) = 6$. Hence $(\pm 1, 1)$ are saddle points while $f(0, 0) = 2$ is a local maximum and $f(0, 2) = -2$ is a local minimum.

20. $f(x, y) = xye^{-x^2-y^2}$



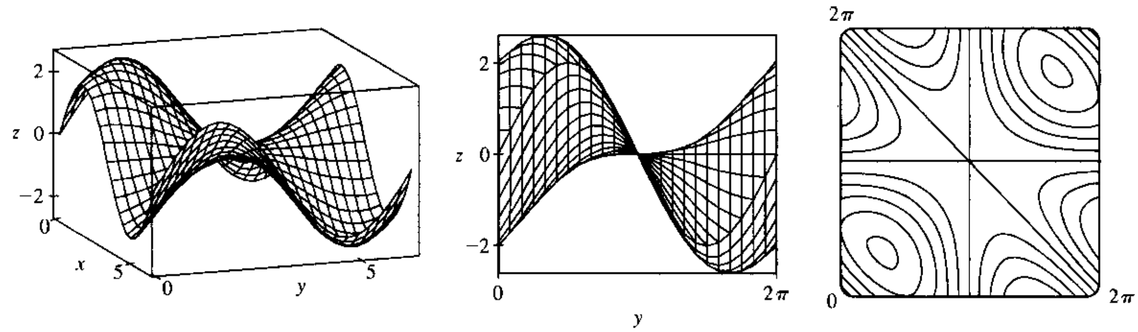
There appear to be local maxima of about $f(\pm 0.7, \pm 0.7) \approx 0.18$ and local minima of about $f(\pm 0.7, \mp 0.7) \approx -0.18$. Also, there seems to be a saddle point at the origin.

$f_x = ye^{-x^2-y^2}(1 - 2x^2)$, $f_y = xe^{-x^2-y^2}(1 - 2y^2)$, $f_{xx} = 2xye^{-x^2-y^2}(2x^2 - 3)$, $f_{yy} = 2xye^{-x^2-y^2}(2y^2 - 3)$, $f_{xy} = (1 - 2x^2)e^{-x^2-y^2}(1 - 2y^2)$. Then $f_x = 0$ implies $y = 0$ or $x = \pm\frac{1}{\sqrt{2}}$.

Substituting these values into $f_y = 0$ gives the critical points $(0, 0)$, $\left(\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}\right)$, $\left(-\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}\right)$. Then

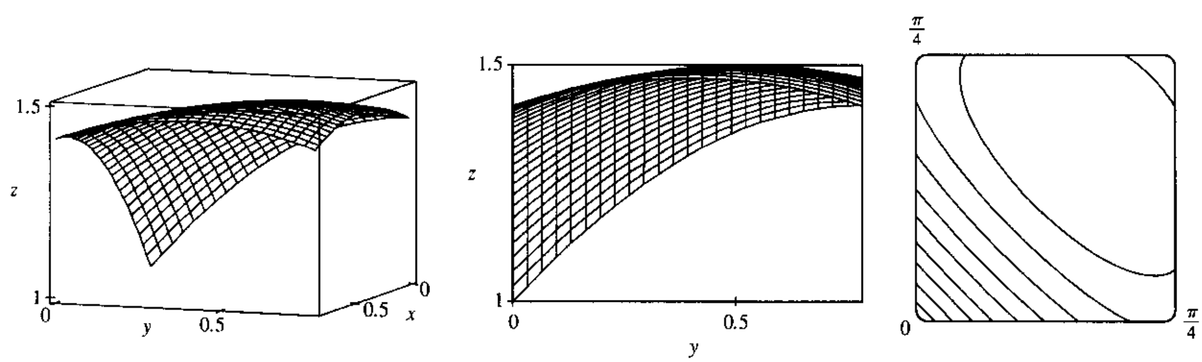
$D(x,y) = e^{2(-x^2-y^2)}\left[4x^2y^2(2x^2-3)(2y^2-3) - (1-2x^2)^2(1-2y^2)^2\right]$, so $D(0,0) = -1$, while $D\left(\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}\right) > 0$ and $D\left(-\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}\right) > 0$. But $f_{xx}\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) < 0$, $f_{xx}\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) > 0$, $f_{xx}\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) > 0$ and $f_{xx}\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) < 0$. Hence $(0,0)$ is a saddle point; $f\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = f\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = -\frac{1}{2e}$ are local minima and $f\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = f\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right) = \frac{1}{2e}$ are local maxima.

21. $f(x,y) = \sin x + \sin y + \sin(x+y)$, $0 \le x \le 2\pi$, $0 \le y \le 2\pi$



From the graphs it appears that f has a local maximum at about $(1,1)$ with value approximately 2.6, a local minimum at about $(5,5)$ with value approximately -2.6, and a saddle point at about $(3,3)$. $f_x = \cos x + \cos(x+y)$, $f_y = \cos y + \cos(x+y)$, $f_{xx} = -\sin x - \sin(x+y)$, $f_{yy} = -\sin y - \sin(x+y)$, $f_{xy} = -\sin(x+y)$. Setting $f_x = 0$ and $f_y = 0$ and subtracting gives $\cos x - \cos y = 0$ or $\cos x = \cos y$. Thus $x = y$ or $x = 2\pi - y$. If $x = y$, $f_x = 0$ becomes $\cos x + \cos 2x = 0$ or $2\cos^2 x + \cos x - 1 = 0$, a quadratic in $\cos x$. Thus $\cos x = -1$ or $\frac{1}{2}$ and $x = \pi$, $\frac{\pi}{3}$, or $\frac{5\pi}{3}$, yielding the critical points (π,π), $\left(\frac{\pi}{3}, \frac{\pi}{3}\right)$ and $\left(\frac{5\pi}{3}, \frac{5\pi}{3}\right)$. Similarly if $x = 2\pi - y$, $f_x = 0$ becomes $(\cos x) + 1 = 0$ and the resulting critical point is (π,π). Now $D(x,y) = \sin x \sin y + \sin x \sin(x+y) + \sin y \sin(x+y)$. So $D(\pi,\pi) = 0$ and the Second Derivatives Test doesn't apply. $D\left(\frac{\pi}{3}, \frac{\pi}{3}\right) = \frac{9}{4} > 0$ and $f_{xx}\left(\frac{\pi}{3}, \frac{\pi}{3}\right) < 0$ so $f\left(\frac{\pi}{3}, \frac{\pi}{3}\right) = \frac{3\sqrt{3}}{2}$ is a local maximum while $D\left(\frac{5\pi}{3}, \frac{5\pi}{3}\right) = \frac{9}{4} > 0$ and $f_{xx}\left(\frac{5\pi}{3}, \frac{5\pi}{3}\right) > 0$, so $f\left(\frac{5\pi}{3}, \frac{5\pi}{3}\right) = -\frac{3\sqrt{3}}{2}$ is a local minimum.

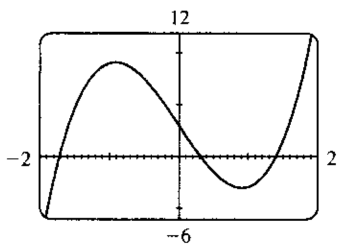
22. $f(x,y) = \sin x + \sin y + \cos(x+y)$, $0 \le x \le \frac{\pi}{4}$, $0 \le y \le \frac{\pi}{4}$

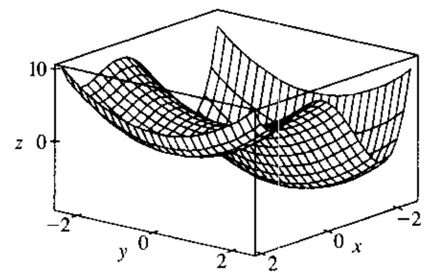


From the graphs, it seems that f has a local maximum at about $(0.5, 0.5)$. $f_x = \cos x - \sin(x+y)$, $f_y = \cos y - \sin(x+y)$, $f_{xx} = -\sin x - \cos(x+y)$, $f_{yy} = -\sin y - \cos(x+y)$,

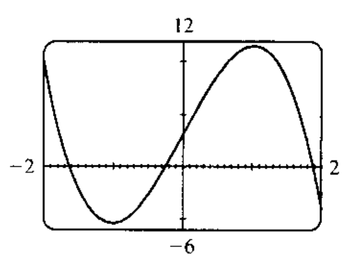
$f_{xy} = -\cos(x+y)$. Setting $f_x = 0$ and $f_y = 0$ and subtracting gives $\cos x = \cos y$. Thus $x = y$. Substituting $x = y$ into $f_x = 0$ gives $\cos x - \sin 2x = 0$ or $\cos x(1 - 2\sin x) = 0$. But $\cos x \neq 0$ for $0 \le x \le \frac{\pi}{4}$ and $1 - 2\sin x = 0$ implies $x = \frac{\pi}{6}$, so the only critical point is $\left(\frac{\pi}{6}, \frac{\pi}{6}\right)$. Here $f_{xx}\left(\frac{\pi}{6}, \frac{\pi}{6}\right) = -1 < 0$ and $D\left(\frac{\pi}{6}, \frac{\pi}{6}\right) = (-1)^2 - \frac{1}{4} > 0$. Thus $f\left(\frac{\pi}{6}, \frac{\pi}{6}\right) = \frac{3}{2}$ is a local maximum.

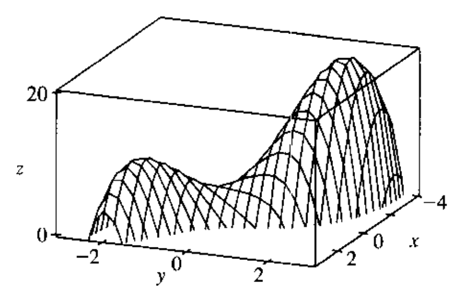
23. $f(x,y) = x^4 - 5x^2 + y^2 + 3x + 2 \quad \Rightarrow \quad f_x(x,y) = 4x^3 - 10x + 3$ and $f_y(x,y) = 2y$. $f_y = 0 \quad \Rightarrow \quad y = 0$, and the graph of f_x shows that the roots of $f_x = 0$ are approximately $x = -1.714, 0.312$ and 1.402. (Alternatively, we could have used a calculator or a CAS to find these roots.) So to three decimal places, the critical points are $(-1.714, 0)$, $(1.402, 0)$, and $(0.312, 0)$. Now since $f_{xx} = 12x^2 - 10$, $f_{xy} = 0$, $f_{yy} = 2$, and $D = 24x^2 - 20$, we have $D(-1.714, 0) > 0$, $f_{xx}(-1.714, 0) > 0$, $D(1.402, 0) > 0$, $f_{xx}(1.402, 0) > 0$, and $D(0.312, 0) < 0$. Therefore $f(-1.714, 0) \approx -9.200$ and $f(1.402, 0) \approx 0.242$ are local minima, and $(0.312, 0)$ is a saddle point. The lowest point on the graph is approximately $(-1.714, 0, -9.200)$.





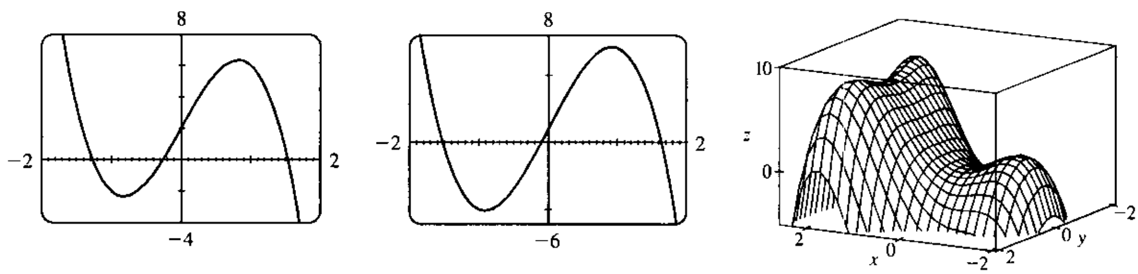
24. $f(x,y) = 5 - 10xy - 4x^2 + 3y - y^4 \quad \Rightarrow \quad f_x(x,y) = -10y - 8x$, $f_y(x,y) = -10x + 3 - 4y^3$. Now $f_x = 0 \quad \Rightarrow \quad x = -\frac{5}{4}y$, so using a graph, we find solutions to $0 = f_y\left(-\frac{5}{4}y, y\right) = -10\left(-\frac{5}{4}y\right) + 3 - 4y^3 = -4y^3 + \frac{25}{2}y + 3$. (Alternatively, we could have found the roots of $f_x = f_y = 0$ directly, using a calculator or a CAS.) To three decimal places, the solutions are $y \approx 1.877$, -0.245 and -1.633, so f has critical points at approximately $(-2.347, 1.877)$, $(0.306, -0.245)$, and $(2.041, -1.633)$. Now since $f_{xx} = -8$, $f_{xy} = -10$, $f_{yy} = -12y^2$, and $D = 96y^2 - 100$, we have $D(-2.347, 1.877) > 0$, $D(0.306, -0.245) < 0$, and $D(2.041, -1.633) > 0$. Therefore, since $f_{xx} < 0$ everywhere, $f(-2.347, 1.877) \approx 20.238$ and $f(2.041, -1.633) \approx 9.657$ are local maxima, and $(0.306, -0.245)$ is a saddle point. The highest point on the graph is approximately $(-2.347, 1.877, 20.238)$.



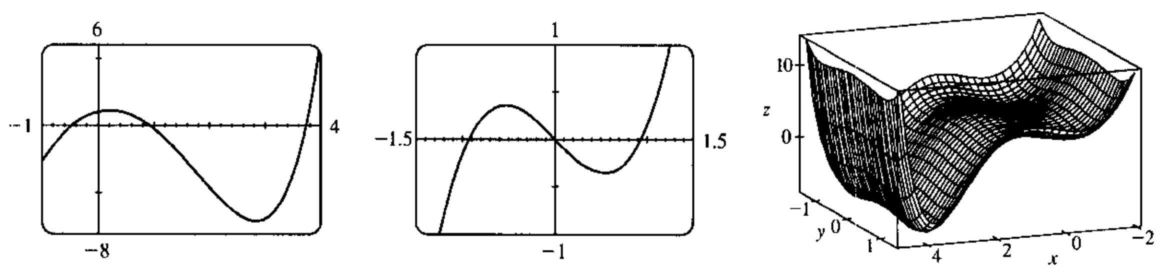


25. $f(x,y) = 2x + 4x^2 - y^2 + 2xy^2 - x^4 - y^4 \quad \Rightarrow \quad f_x(x,y) = 2 + 8x + 2y^2 - 4x^3$, $f_y(x,y) = -2y + 4xy - 4y^3$. Now $f_y = 0 \quad \Leftrightarrow \quad 2y(2y^2 - 2x + 1) = 0 \quad \Leftrightarrow \quad y = 0$ or $y^2 = x - \frac{1}{2}$.

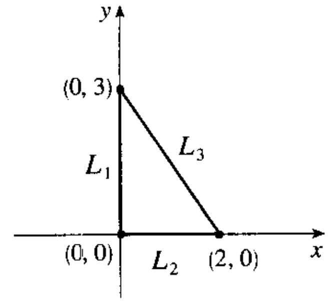
The first of these implies that $f_x = -4x^3 + 8x + 2$, and the second implies that $f_x = 2 + 8x + 2\left(x - \frac{1}{2}\right) - 4x^3 = -4x^3 + 10x + 1$. From the graphs, we see that the first possibility for f_x has roots at approximately -1.267, -0.259, and 1.526, and the second has a root at approximately 1.629 (the negative roots do not give critical points, since $y^2 = x - \frac{1}{2}$ must be positive). So to three decimal places, f has critical points at $(-1.267, 0)$, $(-0.259, 0)$, $(1.526, 0)$, and $(1.629, \pm1.063)$. Now since $f_{xx} = 8 - 12x^2$, $f_{xy} = 4y$, $f_{yy} = 4x - 12y^2$, and $D = (8 - 12x^2)(4x - 12y^2) - 16y^2$, we have $D(-1.267, 0) > 0$, $f_{xx}(-1.267, 0) > 0$, $D(-0.259, 0) < 0$, $D(1.526, 0) < 0$, $D(1.629, \pm1.063) > 0$, and $f_{xx}(1.629, \pm1.063) < 0$. Therefore, to three decimal places, $f(-1.267, 0) \approx 1.310$ and $f(1.629, \pm1.063) \approx 8.105$ are local maxima, and $(-0.259, 0)$ and $(1.526, 0)$ are saddle points. The highest points on the graph are approximately $(1.629, \pm1.063, 8.105)$.



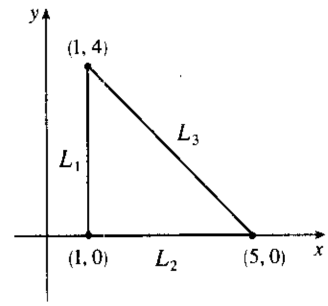
26. $f(x, y) = e^x + y^4 - x^3 + 4\cos y \;\Rightarrow\; f_x(x, y) = e^x - 3x^2$ and $f_y(x, y) = 4y^3 - 4\sin y$. From the graphs, we see that to three decimal places, $f_x = 0$ when $x \approx -0.459$, 0.910, or 3.733, and $f_y = 0$ when $y \approx 0$ or ±0.929. (Alternatively, we could have used a calculator or a CAS to find the roots of $f_x = 0$ and $f_y = 0$.) So, to three decimal places, f has critical points at $(-0.459, 0)$, $(-0.459, \pm0.929)$, $(0.910, 0)$, $(0.910, \pm0.929)$, $(3.733, 0)$, and $(3.733, \pm0.929)$. Now $f_{xx} = e^x - 6x$, $f_{xy} = 0$, $f_{yy} = 12y^2 - 4\cos y$, and $D = (e^x - 6x)(12y^2 - 4\cos y)$. Therefore $D(-0.459, 0) < 0$, $D(-0.459, \pm0.929) > 0$, $f_{xx}(-0.459, \pm0.929) > 0$, $D(0.910, 0) > 0$, $f_{xx}(0.910, 0) < 0$, $D(0.910, \pm0.929) < 0$, $D(3.733, 0) < 0$, $D(3.733, \pm0.929) > 0$, and $f_{xx}(3.733, \pm0.929) > 0$. So $f(-0.459, \pm0.929) \approx 3.868$ and $f(3.733, \pm0.929) \approx -7.077$ are local minima, $f(0.910, 0) \approx 5.731$ is a local maximum, and $(-0.459, 0)$, $(0.910, \pm0.929)$, and $(3.733, 0)$ are saddle points. The lowest points on the graph are approximately $(3.733, \pm0.929, -7.077)$.



27. Since f is a polynomial it is continuous on D, so an absolute maximum and minimum exist. Here $f_x = 4$, $f_y = -5$ so there are no critical points inside D. Thus the absolute extrema must both occur on the boundary. Along L_1, $x = 0$ and $f(0, y) = 1 - 5y$ for $0 \le y \le 3$, a decreasing function in y, so the maximum value is $f(0, 0) = 1$ and the minimum value is $f(0,3) = -14$. Along L_2, $y = 0$ and $f(x,0) = 1 + 4x$ for $0 \leq x \leq 2$, an increasing function in x, so the minimum value is $f(0,0) = 1$ and the maximum value is $f(2,0) = 9$. Along L_3, $y = -\frac{3}{2}x + 3$ and $f\left(x, -\frac{3}{2}x + 3\right) = \frac{23}{2}x - 14$ for $0 \leq x \leq 2$, an increasing function in x, so the minimum value is $f(0,3) = -14$ and the maximum value is $f(2,0) = 9$. Thus the absolue maximum of f on D is $f(2,0) = 9$ and the absolute minimum is $f(0,3) = -14$.



28. Since f is a polynomial it is continuous on D, so an absolute maximum and minimum exist. $f_x = y - 1$, $f_y = x - 2$, and setting $f_x = f_y = 0$ gives $(2,1)$ as the only critical point, where $f(2,1) = 1$. Along L_1: $x = 1$ and $f(1,y) = 2 - y$ for $0 \leq y \leq 4$, a decreasing function in y, so the maximum value is $f(1,0) = 2$ and the minimum value is $f(1,4) = -2$. Along L_2: $y = 0$ and $f(x,0) = 3 - x$ for $1 \leq x \leq 5$, a decreasing function in x, so the maximum value is $f(1,0) = 2$ and the minimum value is $f(5,0) = -2$. Along L_3, $y = 5 - x$ and $f(x, 5 - x) = -x^2 + 6x - 7 = -(x-3)^2 + 2$ for $1 \leq x \leq 5$, which has a maximum at $x = 3$ where $f(3,2) = 2$ and a minimum at both $x = 1$ and $x = 5$, where $f(1,4) = f(5,0) = -2$. Thus the absolute maximum of f on D is $f(1,0) = f(3,2) = 2$ and the absolute minimum is $f(1,4) = f(5,0) = -2$.



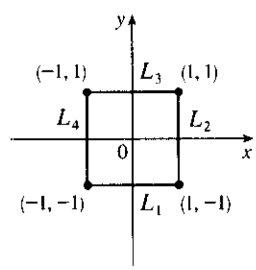
29. $f_x(x,y) = 2x + 2xy$, $f_y(x,y) = 2y + x^2$, and setting $f_x = f_y = 0$ gives $(0,0)$ as the only critical point in D, with $f(0,0) = 4$.

On L_1: $y = -1$, $f(x,-1) = 5$, a constant.

On L_2: $x = 1$, $f(1,y) = y^2 + y + 5$, a quadratic in y which attains its maximum at $(1,1)$, $f(1,1) = 7$ and its minimum at $\left(1, -\frac{1}{2}\right)$, $f\left(1, -\frac{1}{2}\right) = \frac{17}{4}$.

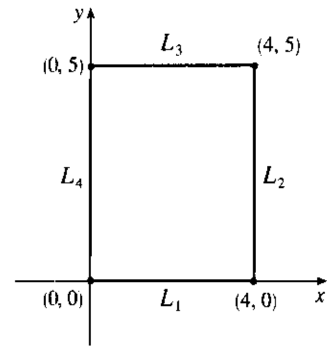
On L_3: $f(x,1) = 2x^2 + 5$ which attains its maximum at $(-1,1)$ and $(1,1)$ with $f(\pm 1, 1) = 7$ and its minimum at $(0,1)$, $f(0,1) = 5$.

On L_4: $f(-1,y) = y^2 + y + 5$ with maximum at $(-1,1)$, $f(-1,1) = 7$ and minimum at $\left(-1, -\frac{1}{2}\right)$, $f\left(-1, -\frac{1}{2}\right) = \frac{17}{4}$. Thus the absolute maximum is attained at both $(\pm 1, 1)$ with $f(\pm 1, 1) = 7$ and the absolute minimum on D is attained at $(0,0)$ with $f(0,0) = 4$.

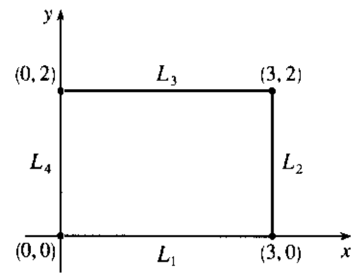


30. $f_x(x,y) = 4 - 2x$ and $f_y(x,y) = 6 - 2y$, so the only critical point is $(2,3)$ (which is in D) where $f(2,3) = 13$. Along L_1: $y = 0$, so $f(x,0) = 4x - x^2 = -(x-2)^2 + 4$, $0 \leq x \leq 4$, which has a maximum value when $x = 2$ where $f(2,0) = 4$ and a minimum value both when $x = 0$ and $x = 4$, where $f(0,0) = f(4,0) = 0$. Along L_2: $x = 4$, so $f(4,y) = 6y - y^2 = -(y-3)^2 + 9$, $0 \leq y \leq 5$, which has a maximum value when $y = 3$ where $f(4,3) = 9$ and a minimum value when $y = 0$ where $f(4,0) = 0$. Along L_3: $y = 5$, so $f(x,5) = -x^2 + 4x + 5 = -(x-2)^2 + 9$, $0 \leq x \leq 4$, which has a maximum value when $x = 2$ where $f(2,5) = 9$ and a minimum value both when $x = 0$ and $x = 4$, where $f(0,5) = f(4,5) = 5$.

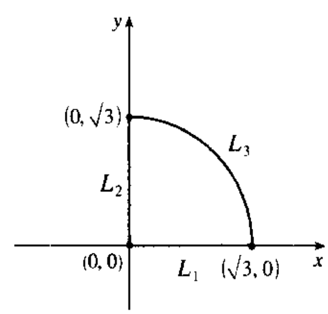
Along L_4: $x = 0$, so $f(0, y) = 6y - y^2 = -(y-3)^2 + 9$, $0 \le y \le 5$, which has a maximum value when $y = 3$ where $f(0,3) = 9$ and a minimum value when $y = 0$ where $f(0,0) = 0$. Thus the absolute maximum is $f(2,3) = 13$ and the absolute minimum is attained at both $(0,0)$ and $(4,0)$, where $f(0,0) = f(4,0) = 0$.



31. $f(x,y) = x^4 + y^4 - 4xy + 2$ is a polynomial and hence continuous on D, so it has an absolute maximum and minimum on D. In Exercise 7, we found the critical points of f; only $(1,1)$ with $f(1,1) = 0$ is inside D. On L_1: $y = 0$, $f(x,0) = x^4 + 2$, $0 \le x \le 3$, a polynomial in x which attains its maximum at $x = 3$, $f(3,0) = 83$, and its minimum at $x = 0$, $f(0,0) = 2$. On L_2: $x = 3$, $f(3,y) = y^4 - 12y + 83$, $0 \le y \le 2$, a polynomial in y which attains its minimum at $y = \sqrt[3]{3}$, $f(3, \sqrt[3]{3}) = 83 - 9\sqrt[3]{3} \approx 70.0$, and its maximum at $y = 0$, $f(3,0) = 83$. On L_3: $y = 2$, $f(x,2) = x^4 - 8x + 18$, $0 \le x \le 3$, a polynomial in x which attains its minimum at $x = \sqrt[3]{2}$, $f(\sqrt[3]{2}, 2) = 18 - 6\sqrt[3]{2} \approx 10.4$, and its maximum at $x = 3$, $f(3,2) = 75$. On L_4: $x = 0$, $f(0,y) = y^4 + 2$, $0 \le y \le 2$, a polynomial in y which attains its maximum at $y = 2$, $f(0,2) = 18$, and its minimum at $y = 0$, $f(0,0) = 2$. Thus the absolute maximum of f on D is $f(3,0) = 83$ and the absolute minimum is $f(1,1) = 0$.



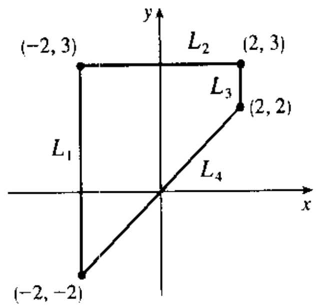
32. $f_x = y^2$ and $f_y = 2xy$, and since $f_x = 0 \Leftrightarrow y = 0$, there are no critical points in the interior of D. Along L_1, $y = 0$ and $f(x,0) = 0$. Along L_2, $x = 0$ and $f(0,y) = 0$. Along L_3, $y = \sqrt{3 - x^2}$, so let $g(x) = f(x, \sqrt{3 - x^2}) = 3x - x^3$ for $0 \le x \le \sqrt{3}$. Then $g'(x) = 3 - 3x^2 = 0 \Leftrightarrow x = 1$. The maximum value is $f(1, \sqrt{2}) = 2$ and the minimum occurs both at $x = 0$ and $x = \sqrt{3}$ where $f(0, \sqrt{3}) = f(\sqrt{3}, 0) = 0$. Thus the absolute maximum of f on D is $f(1, \sqrt{2}) = 2$, and the absolute minimum is 0 which occurs at all points along L_1 and L_2.



33. $f_x(x,y) = 6x^2$ and $f_y(x,y) = 4y^3$. And so $f_x = 0$ and $f_y = 0$ only occur when $x = y = 0$. Hence, the only critical point inside the disk is at $x = y = 0$ where $f(0,0) = 0$. Now on the circle $x^2 + y^2 = 1$, $y^2 = 1 - x^2$ so let $g(x) = f(x,y) = 2x^3 + (1 - x^2)^2 = x^4 + 2x^3 - 2x^2 + 1, -1 \le x \le 1$. Then $g'(x) = 4x^3 + 6x^2 - 4x = 0 \Rightarrow x = 0, -2$, or $\frac{1}{2}$. $f(0, \pm 1) = g(0) = 1$, $f\left(\frac{1}{2}, \pm\frac{\sqrt{3}}{2}\right) = g\left(\frac{1}{2}\right) = \frac{13}{16}$, and $(-2, -3)$ is not in D. Checking the endpoints, we get $f(-1,0) = g(-1) = -2$ and $f(1,0) = g(1) = 2$. Thus the absolute maximum and minimum of f on D are $f(1,0) = 2$ and $f(-1,0) = -2$.

Another method: On the boundary $x^2 + y^2 = 1$ we can write $x = \cos\theta$, $y = \sin\theta$, so $f(\cos\theta, \sin\theta) = 2\cos^3\theta + \sin^4\theta$, $0 \le \theta \le 2\pi$.

34. $f_x(x,y) = 3x^2 - 3$ and $f_y(x,y) = -3y^2 + 12$ and the critical points are $(1,2)$, $(1,-2)$, $(-1,2)$, and $(-1,-2)$. But only $(1,2)$ and $(-1,2)$ are in D and $f(1,2) = 14$, $f(-1,2) = 18$. Along L_1: $x = -2$ and $f(-2,y) = -2 - y^3 + 12y$, $-2 \le y \le 3$, which has a maximum at $y = 2$ where $f(-2,2) = 14$ and a minimum at $y = -2$ where $f(-2,-2) = -18$. Along L_2: $x = 2$ and $f(2,y) = 2 - y^3 + 12y$, $2 \le y \le 3$, which has a maximum at $y = 2$ where $f(2,2) = 18$ and a minimum at $y = 3$ where $f(2,3) = 11$. Along L_3: $y = 3$ and $f(x,3) = x^3 - 3x + 9$, $-2 \le x \le 2$, which has a maximum at $x = -1$ and $x = 2$ where $f(-1,3) = f(2,3) = 11$ and a minimum at $x = 1$ and $x = -2$ where $f(1,3) = f(-2,3) = 7$. Along L_4: $y = x$ and $f(x,x) = 9x$, $-2 \le x \le 2$, which has a maximum at $x = 2$ where $f(2,2) = 18$ and a minimum at $x = -2$ where $f(-2,-2) = -18$. So the absolute maximum value of f on D is $f(2,2) = 18$ and the minimum is $f(-2,-2) = -18$.



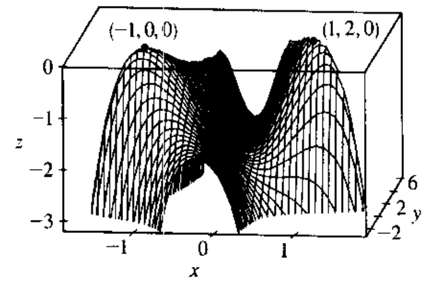
35. $f(x,y) = -(x^2-1)^2 - (x^2y - x - 1)^2 \quad\Rightarrow\quad f_x(x,y) = -2(x^2-1)(2x) - 2(x^2y - x - 1)(2xy - 1)$ and $f_y(x,y) = -2(x^2y - x - 1)x^2$. Setting $f_y(x,y) = 0$ gives either $x = 0$ or $x^2y - x - 1 = 0$. There are no critical points for $x = 0$, since $f_x(0,y) = -2$, so we set $x^2y - x - 1 = 0 \quad\Leftrightarrow\quad y = \dfrac{x+1}{x^2}$ $(x \ne 0)$, so

$$f_x\left(x, \frac{x+1}{x^2}\right) = -2(x^2-1)(2x) - 2\left(x^2\,\frac{x+1}{x^2} - x - 1\right)\left(2x\,\frac{x+1}{x^2} - 1\right) = -4x(x^2-1).$$ Therefore

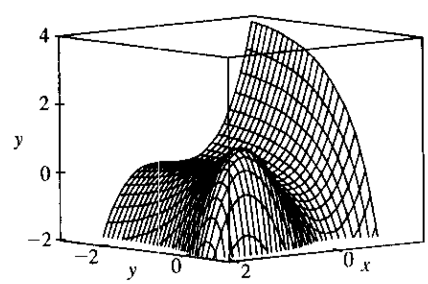
$f_x(x,y) = f_y(x,y) = 0$ at the points $(1,2)$ and $(-1,0)$. To classify these critical points, we calculate $f_{xx}(x,y) = -12x^2 - 12x^2y^2 + 12xy + 4y + 2$, $f_{yy}(x,y) = -2x^4$, and $f_{xy}(x,y) = -8x^3y + 6x^2 + 4x$. In order to use the Second Derivatives Test we calculate

$$D(-1,0) = f_{xx}(-1,0)\, f_{yy}(-1,0) - [f_{xy}(-1,0)]^2$$
$$= 16 > 0,$$

$f_{xx}(-1,0) = -10 < 0$, $D(1,2) = 16 > 0$, and $f_{xx}(1,2) = -26 < 0$, so both $(-1,0)$ and $(1,2)$ give local maxima.



36. $f(x,y) = 3xe^y - x^3 - e^{3y}$ is differentiable everywhere, so the requirement for critical points is that (1) $f_x = 3e^y - 3x^2 = 0$ and (2) $f_y = 3xe^y - 3e^{3y} = 0$. From (1) we obtain $e^y = x^2$, and then (2) gives $3x^3 - 3x^6 = 0 \quad\Rightarrow\quad x = 1$ or 0, but only $x = 1$ is valid, since $x = 0$ makes (1) impossible. So substituting $x = 1$ into (1) gives $y = 0$, and the only critical point is $(1,0)$.



The Second Derivatives Test shows that this gives a local maximum, since

$D(1,0) = \left[-6x(3xe^y - 9e^{3y}) - (3e^y)^2\right]_{(1,0)} = 27 > 0$ and $f_{xx}(1,0) = [-6x]_{(1,0)} = -6 < 0$. But $f(1,0) = 1$ is not an absolute maximum because, for instance, $f(-3,0) = 17$. This can also be seen from the graph.

37. Let d be the distance from $(2,1,-1)$ to any point (x,y,z) on the plane $x + y - z = 1$, so $d = \sqrt{(x-2)^2 + (y-1)^2 + (z+1)^2}$ where $z = x + y - 1$, and we minimize $d^2 = f(x,y) = (x-2)^2 + (y-1)^2 + (x+y)^2$. Then $f_x(x,y) = 2(x-2) + 2(x+y) = 4x + 2y - 4$, $f_y(x,y) = 2(y-1) + 2(x+y) = 2x + 4y - 2$. Solving $4x + 2y - 4 = 0$ and $2x + 4y - 2 = 0$ simultaneously gives $x = 1, y = 0$. An absolute minimum exists (since there is a minimum distance from the point to the plane) and it must occur at a critical point, so the shortest distance occurs for $x = 1, y = 0$ for which $d = \sqrt{(1-2)^2 + (0-1)^2 + (1+0)^2} = \sqrt{3}$.

38. Here the distance d from a point on the plane to the point $(1,2,3)$ is $d = \sqrt{(x-1)^2 + (y-2)^2 + (z-3)^2}$, where $z = 4 - x + y$. We can minimize $d^2 = f(x,y) = (x-1)^2 + (y-2)^2 + (1-x+y)^2$, so $f_x(x,y) = 2(x-1) + 2(1-x+y)(-1) = 4x - 2y - 4$ and $f_y(x,y) = 2(y-2) + 2(1-x+y) = 4y - 2x - 2$. Solving $4x - 2y - 4 = 0$ and $4y - 2x - 2 = 0$ simultaneously gives $x = \frac{5}{3}$ and $y = \frac{4}{3}$, so the only critical point is $\left(\frac{5}{3}, \frac{4}{3}\right)$. This point must correspond to the minimum distance, so the point on the plane closest to $(1,2,3)$ is $\left(\frac{5}{3}, \frac{4}{3}, \frac{11}{3}\right)$.

39. Minimize $d^2 = x^2 + y^2 + z^2 = x^2 + y^2 + xy + 1$. Then $f_x = 2x + y$, $f_y = 2y + x$ so the critical point is $(0,0)$ and $D(0,0) = 4 - 1 > 0$ with $f_{xx}(0,0) = 2$ so this is a minimum. Thus $z^2 = 1$ or $z = \pm 1$ and the points on the surface are $(0,0,\pm 1)$.

40. Since $z = 1/(x^2y^2)$ on the surface, we minimize $d^2 = x^2 + y^2 + z^2 = x^2 + y^2 + x^{-4}y^{-4} = f(x,y)$. $f_x = 2x - \dfrac{4}{x^5y^4}$, $f_y = 2y - \dfrac{4}{x^4y^5}$, so the critical points occur when $2x = \dfrac{4}{x^5y^4}$ and $2y = \dfrac{4}{x^4y^5}$ or $x^6y^4 = 2 = x^4y^6$, so $x^2 = y^2 \;\Rightarrow\; x = \pm y$ and $x^{10} = 2 \;\Rightarrow\; x = \pm 2^{1/10}$, $y = \pm 2^{1/10}$. The four critical points are $(\pm 2^{1/10}, \pm 2^{1/10})$. The absolute minimum must occur at these points (there is no maximum since the surface is infinite in extent). Thus the points on the surface closest to the origin are $(\pm 2^{1/10}, \pm 2^{1/10}, 2^{-2/5})$.

41. $x + y + z = 100$, so maximize $f(x,y) = xy(100 - x - y)$. $f_x = 100y - 2xy - y^2$, $f_y = 100x - x^2 - 2xy$, $f_{xx} = -2y$, $f_{yy} = -2x$, $f_{xy} = 100 - 2x - 2y$. Then $f_x = 0$ implies $y = 0$ or $y = 100 - 2x$. Substituting $y = 0$ into $f_y = 0$ gives $x = 0$ or $x = 100$ and substituting $y = 100 - 2x$ into $f_y = 0$ gives $3x^2 - 100x = 0$ so $x = 0$ or $\frac{100}{3}$. Thus the critical points are $(0,0)$, $(100,0)$, $(0,100)$ and $\left(\frac{100}{3}, \frac{100}{3}\right)$.
$D(0,0) = D(100,0) = D(0,100) = -10{,}000$ while $D\left(\frac{100}{3}, \frac{100}{3}\right) = \frac{10{,}000}{3}$ and $f_{xx}\left(\frac{100}{3}, \frac{100}{3}\right) = -\frac{200}{3} < 0$. Thus $(0,0)$, $(100,0)$ and $(0,100)$ are saddle points whereas $f\left(\frac{100}{3}, \frac{100}{3}\right)$ is a local maximum. Thus the numbers are $x = y = z = \frac{100}{3}$.

42. Maximize $f(x, y) = x^a y^b (100 - x - y)^c$.

$f_x = ax^{a-1}y^b(100 - x - y)^c - cx^a y^b(100 - x - y)^{c-1} = x^{a-1}y^b(100 - x - y)^{c-1}[a(100 - x - y) - cx]$ and $f_y = x^a y^{b-1}(100 - x - y)^{c-1}[b(100 - x - y) - cy]$. Since x, y and z are all positive, the only critical point occurs when $x = a\dfrac{100 - y}{a + c}$ and $y = \dfrac{100b}{a + b + c}$. Thus the point is $\left(\dfrac{100a}{a+b+c}, \dfrac{100b}{a+b+c}\right)$ and the numbers are $x = \dfrac{100a}{a+b+c}$, $y = \dfrac{100b}{a+b+c}$, $z = \dfrac{100c}{a+b+c}$.

43. Maximize $f(x, y) = xy(36 - 9x^2 - 36y^2)^{1/2}/2$ with (x, y, z) in first octant. Then

$f_x = \dfrac{y(36 - 9x^2 - 36y^2)^{1/2}}{2} + \dfrac{-9x^2y(36 - 9x^2 - 36y^2)^{-1/2}}{2} = \dfrac{(36y - 18x^2y - 36y^3)}{2(36 - 9x^2 - 36y^2)^{1/2}}$ and

$f_y = \dfrac{36x - 9x^3 - 72xy^2}{2(36 - 9x^2 - 36y^2)^{1/2}}$. Setting $f_x = 0$ gives $y = 0$ or $y^2 = \dfrac{2 - x^2}{2}$ but $y > 0$, so only the latter solution applies. Substituting this y into $f_y = 0$ gives $x^2 = \frac{4}{3}$ or $x = \frac{2}{\sqrt{3}}$, $y = \frac{1}{\sqrt{3}}$ and then $z^2 = (36 - 12 - 12)/4 = 3$. The fact that this gives a maximum volume follows from the geometry. This maximum volume is $V = (2x)(2y)(2z) = 8\left(\frac{2}{\sqrt{3}}\right)\left(\frac{1}{\sqrt{3}}\right)\left(\sqrt{3}\right) = \frac{16}{\sqrt{3}}$.

44. Here maximize $f(x, y) = xy\dfrac{(a^2b^2c^2 - b^2c^2x^2 - a^2c^2y^2)^{1/2}}{a^2b^2}$. Then

$f_x = yc^2 \dfrac{a^2b^2 - 2b^2x^2 - a^2y^2}{a^2b^2(a^2b^2c^2 - b^2c^2x^2 - a^2c^2y^2)^{1/2}}$ and $f_y = xc^2 \dfrac{a^2b^2 - 2a^2y^2 - b^2x^2}{a^2b^2(a^2b^2c^2 - b^2c^2x^2 - a^2c^2y^2)^{1/2}}$. Then $f_x = 0$ (with $x, y > 0$) implies $y^2 = \dfrac{a^2b^2 - 2b^2x^2}{a^2}$ and substituting into $f_y = 0$ implies $3b^2x^2 = a^2b^2$ or $x = \frac{1}{\sqrt{3}}a$, $y = \frac{1}{\sqrt{3}}b$ and then $z = \frac{1}{\sqrt{3}}c$. Thus the maximum volume of such a rectangle is $V = (2x)(2y)(2z) = \frac{8}{3\sqrt{3}}abc$.

45. Maximize $f(x, y) = \dfrac{xy}{3}(6 - x - 2y)$, then the maximum volume is $V = xyz$.

$f_x = \frac{1}{3}(6y - 2xy - y^2) = \frac{1}{3}y(6 - 2x - 2y)$ and $f_y = \frac{1}{3}x(6 - x - 4y)$. Setting $f_x = 0$ and $f_y = 0$ gives the critical point $(2, 1)$ which geometrically must yield a maximum. Thus the volume of the largest such box is $V = (2)(1)\left(\frac{2}{3}\right) = \frac{4}{3}$.

46. Surface area $= 2(xy + xz + yz) = 64\text{ cm}^2$, so $xy + xz + yz = 32$ or $z = \dfrac{32 - xy}{x + y}$. Maximize the volume $f(x, y) = xy\dfrac{32 - xy}{x + y}$. Then $f_x = \dfrac{32y^2 - 2xy^3 - x^2y^2}{(x + y)^2} = y^2\dfrac{32 - 2xy - x^2}{(x + y)^2}$ and $f_y = x^2\dfrac{32 - 2xy - y^2}{(x + y)^2}$. Setting $f_x = 0$ implies $y = \dfrac{32 - x^2}{2x}$ and substituting into $f_y = 0$ gives $32(4x^2) - (32 - x^2)(4x^2) - (32 - x^2)^2 = 0$ or $3x^4 + 64x^2 - (32)^2 = 0$. Thus $x^2 = \frac{64}{6}$ or $x = \frac{8}{\sqrt{6}}$, $y = \frac{64/3}{16/\sqrt{6}} = \frac{8}{\sqrt{6}}$ and $z = \frac{8}{\sqrt{6}}$. Thus the box is a cube with edge length $\frac{8}{\sqrt{6}}$ cm.

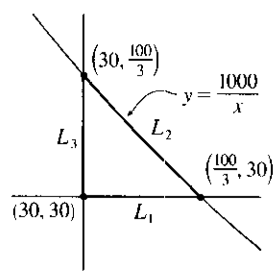
47. Let the dimensions be x, y, and z; then $4x + 4y + 4z = c$ and the volume is $V = xyz = xy\left(\frac{1}{4}c - x - y\right) = \frac{1}{4}cxy - x^2y - xy^2$, $x > 0$, $y > 0$. Then $V_x = \frac{1}{4}cy - 2xy - y^2$ and $V_y = \frac{1}{4}cx - x^2 - 2xy$, so $V_x = 0 = V_y$ when $2x + y = \frac{1}{4}c$ and $x + 2y = \frac{1}{4}c$. Solving, we get $x = \frac{1}{12}c$, $y = \frac{1}{12}c$ and $z = \frac{1}{4}c - x - y = \frac{1}{12}c$. From the geometrical nature of the problem, this critical point must give an absolute maximum. Thus the box is a cube with edge length $\frac{1}{12}c$.

48. The cost equals $5xy + 2(xz + yz)$ and $xyz = V$, so $C(x, y) = 5xy + 2V(x + y)/(xy) = 5xy + 2V(x^{-1} + y^{-1})$. Then $C_x = 5y - 2Vx^{-2}$, $C_y = 5x - 2Vy^{-2}$, $f_x = 0$ implies $y = 2V/(5x^2)$, $f_y = 0$ implies $x = \sqrt[3]{\frac{2}{5}V} = y$. Thus the dimensions of the aquarium which minimize the cost are $x = y = \sqrt[3]{\frac{2}{5}V}$ units, $z = V^{1/3}\left(\frac{5}{2}\right)^{2/3}$.

49. Let the dimensions be x, y and z, then minimize $xy + 2(xz + yz)$ if $xyz = 32{,}000\text{ m}^3$. Then $f(x, y) = xy + [64{,}000(x + y)/xy] = xy + 64{,}000(x^{-1} + y^{-1})$, $f_x = y - 64{,}000x^{-2}$, $f_y = x - 64{,}000y^{-2}$. And $f_x = 0$ implies $y = 64{,}000/x^2$; substituting into $f_y = 0$ implies $x^3 = 64{,}000$ or $x = 40$ and then $y = 40$. Now $D(x, y) = [(2)(64{,}000)]^2x^{-3}y^{-3} - 1 > 0$ for $(40, 40)$ and $f_{xx}(40, 40) > 0$ so this is indeed a minimum. Thus the dimensions of the box are $x = y = 40$ cm, $z = 20$ cm.

50. Let x be the length of the north and south walls, y the length of the east and west walls, and z the height of the building. The heat loss is given by $h = 10(2yz) + 8(2xz) + 1(xy) + 5(xy) = 6xy + 16xz + 20yz$. The volume is 4000 m^3, so $xyz = 4000$, and we substitute $z = \frac{4000}{xy}$ to obtain the heat loss function $h(x, y) = 6xy + 80{,}000/x + 64{,}000/y$.

(a) Since $z = \frac{4000}{xy} \geq 4$, $xy \leq 1000 \Rightarrow y \leq 1000/x$. Also $x \geq 30$ and $y \geq 30$, so the domain of h is $D = \{(x, y) \mid x \geq 30, 30 \leq y \leq 1000/x\}$.



(b) $h(x, y) = 6xy + 80{,}000x^{-1} + 64{,}000y^{-1} \Rightarrow h_x = 6y - 80{,}000x^{-2}$, $h_y = 6x - 64{,}000y^{-2}$. $h_x = 0$ implies $6x^2y = 80{,}000 \Rightarrow y = \dfrac{80{,}000}{6x^2}$ and substituting into $h_y = 0$ gives

$$6x = 64{,}000\left(\frac{6x^2}{80{,}000}\right)^2 \Rightarrow x^3 = \frac{80{,}000^2}{6 \cdot 64{,}000} = \frac{50{,}000}{3}, \text{ so } x = \sqrt[3]{\frac{50{,}000}{3}} = 10\sqrt[3]{\frac{50}{3}} \Rightarrow y = \frac{80}{\sqrt[3]{60}},$$

and the only critical point of h is $\left(10\sqrt[3]{\frac{50}{3}}, \frac{80}{\sqrt[3]{60}}\right) \approx (25.54, 20.43)$ which is not in D. Next we check the boundary of D. On L_1: $y = 30$, $h(x, 30) = 180x + 80{,}000/x + 6400/3$, $30 \leq x \leq \frac{100}{3}$. Since $h'(x, 30) = 180 - 80{,}000/x^2 > 0$ for $30 \leq x \leq \frac{100}{3}$, $h(x, 30)$ is an increasing function with minimum $h(30, 30) = 10{,}200$ and maximum $h\left(\frac{100}{3}, 30\right) \approx 10{,}533$. On L_2: $y = 1000/x$,

$h(x, 1000/x) = 6000 + 64x + 80{,}000/x$, $30 \le x \le \frac{100}{3}$. Since $h'(x, 1000/x) = 64 - 80{,}000/x^2 < 0$ for $30 \le x \le \frac{100}{3}$, $h(x, 1000/x)$ is a decreasing function with minimum $h\left(\frac{100}{3}, 30\right) \approx 10{,}533$ and maximum $h\left(30, \frac{100}{3}\right) \approx 10{,}587$. On L_3: $x = 30$, $h(30, y) = 180y + 64{,}000/y + 8000/3$, $30 \le y \le \frac{100}{3}$. $h'(30, y) = 180 - 64{,}000/y^2 > 0$ for $30 \le y \le \frac{100}{3}$, so $h(30, y)$ is an increasing function of y with minimum $h(30, 30) = 10{,}200$ and maximum $h\left(30, \frac{100}{3}\right) \approx 10{,}587$. Thus the absolute minimum of h is $h(30, 30) = 10{,}200$, and the dimensions of the building that minimize heat loss are walls 30 m in length and height $\frac{4000}{30^2} = \frac{40}{9} \approx 4.44$ m.

(c) From part (b), the only critical point of h, which gives a local (and absolute) minimum, is approximately $h(25.54, 20.43) \approx 9396$. So a building of volume 4000 m^2 with dimensions $x \approx 25.54$ m, $y \approx 20.43$ m, $z \approx \frac{4000}{(25.54)(20.43)} \approx 7.67$ m has the least amount of heat loss.

51. Let x, y, z be the dimensions of the rectangular box. Then the volume of the box is xyz and $L = \sqrt{x^2 + y^2 + z^2} \;\Rightarrow\; L^2 = x^2 + y^2 + z^2 \;\Rightarrow\; z = \sqrt{L^2 - x^2 - y^2}$. Substituting, we have volume $V(x, y) = xy\sqrt{L^2 - x^2 - y^2}$, $x, y > 0$.

$$V_x = xy \cdot \tfrac{1}{2}(L^2 - x^2 - y^2)^{-1/2}(-2x) + y\sqrt{L^2 - x^2 - y^2} = y\sqrt{L^2 - x^2 - y^2} - \frac{x^2 y}{\sqrt{L^2 - x^2 - y^2}},$$

$$V_y = x\sqrt{L^2 - x^2 - y^2} - \frac{xy^2}{\sqrt{L^2 - x^2 - y^2}},$$

$V_x = 0$ implies $y(L^2 - x^2 - y^2) = x^2 y \;\Rightarrow\; y(L^2 - 2x^2 - y^2) = 0 \;\Rightarrow\; 2x^2 + y^2 = L^2$ (since $y > 0$), and $V_y = 0$ implies $x(L^2 - x^2 - y^2) = xy^2 \;\Rightarrow\; x(L^2 - x^2 - 2y^2) = 0 \;\Rightarrow\; x^2 + 2y^2 = L^2$ (since $x > 0$). Substituting $y^2 = L^2 - 2x^2$ into $x^2 + 2y^2 = L^2$ gives $x^2 + 2L^2 - 4x^2 = L^2 \;\Rightarrow\; 3x^2 = L^2 \;\Rightarrow\; x = L/\sqrt{3}$ (since $x > 0$) and then $y = \sqrt{L^2 - 2\left(L/\sqrt{3}\right)^2} = L/\sqrt{3}$. So the only critical point is $\left(L/\sqrt{3}, L/\sqrt{3}\right)$ which, from the geometrical nature of the problem, must give an absolute maximum. Thus the maximum volume is $V\left(L/\sqrt{3}, L/\sqrt{3}\right) = \left(L/\sqrt{3}\right)^2 \sqrt{L^2 - \left(L/\sqrt{3}\right)^2 - \left(L/\sqrt{3}\right)^2} = L^3/\left(3\sqrt{3}\right)$ cubic units.

52. Since $p + q + r = 1$ we can substitute $p = 1 - r - q$ into P giving $P = P(q, r) = 2(1 - r - q)q + 2(1 - r - q)r + 2rq = 2q - 2q^2 + 2r - 2r^2 - 2rq$. Since p, q and r represent proportions and $p + q + r = 1$, we know $q \ge 0$, $r \ge 0$, and $q + r \le 1$. Thus, we want to find the absolute maximum of the continuous function $P(q, r)$ on the closed set D enclosed by the lines $q = 0$, $r = 0$, and $q + r = 1$. To find any critical points, we set the partial derivatives equal to zero: $P_q(q, r) = 2 - 4q - 2r = 0$ and $P_r(q, r) = 2 - 4r - 2q = 0$. The first equation gives $r = 1 - 2q$, and substituting into the second equation we have

$2 - 4(1-2q) - 2q = 0 \Rightarrow q = \frac{1}{3}$. Then we have one critical point, $\left(\frac{1}{3}, \frac{1}{3}\right)$, where $P\left(\frac{1}{3}, \frac{1}{3}\right) = \frac{2}{3}$. Next we find the maximum values of P on the boundary of D which consists of three line segments. For the segment given by $r = 0, 0 \le q \le 1$, $P(q,r) = P(q,0) = 2q - 2q^2, 0 \le q \le 1$. This represents a parabola with maximum value $P\left(\frac{1}{2}, 0\right) = \frac{1}{2}$. On the segment $q = 0, 0 \le r \le 1$ we have $P(0,r) = 2r - 2r^2, 0 \le r \le 1$. This represents a parabola with maximum value $P\left(0, \frac{1}{2}\right) = \frac{1}{2}$. Finally, on the segment $q + r = 1, 0 \le q \le 1$, $P(q,r) = P(q, 1-q) = 2q - 2q^2, 0 \le q \le 1$ which has a maximum value of $P\left(\frac{1}{2}, \frac{1}{2}\right) = \frac{1}{2}$. Comparing these values with the value of P at the critical point, we see that the absolute maximum value of $P(q,r)$ on D is $\frac{2}{3}$.

53. Note that here the variables are m and b, and $f(m,b) = \sum\limits_{i=1}^{n} [y_i - (mx_i + b)]^2$. Then

$f_m = \sum\limits_{i=1}^{n} -2x_i[y_i - (mx_i + b)] = 0$ implies $\sum\limits_{i=1}^{n} (x_i y_i - m x_i^2 - b x_i) = 0$ or $\sum\limits_{i=1}^{n} x_i y_i = m \sum\limits_{i=1}^{n} x_i^2 + b \sum\limits_{i=1}^{n} x_i$

and $f_b = \sum\limits_{i=1}^{n} -2[y_i - (mx_i + b)] = 0$ implies $\sum\limits_{i=1}^{n} y_i = m \sum\limits_{i=1}^{n} x_i + \sum\limits_{i=1}^{n} b = m\left(\sum\limits_{i=1}^{n} x_i\right) + nb$. Thus we have the two desired equations. Now $f_{mm} = \sum\limits_{i=1}^{n} 2x_i^2$, $f_{bb} = \sum\limits_{i=1}^{n} 2 = 2n$ and $f_{mb} = \sum\limits_{i=1}^{n} 2x_i$. And $f_{mm}(m,b) > 0$ always and $D(m,b) = 4n\left(\sum\limits_{i=1}^{n} x_i^2\right) - 4\left(\sum\limits_{i=1}^{n} x_i\right)^2 = 4\left[n\left(\sum\limits_{i=1}^{n} x_i^2\right) - \left(\sum\limits_{i=1}^{n} x_i\right)^2\right] > 0$ always so the solutions of these two equations do indeed minimize $\sum\limits_{i=1}^{n} d_i^2$.

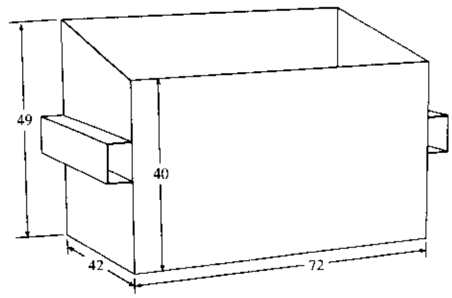
54. Any such plane must cut out a tetrahedron in the first octant. We need to minimize the volume of the tetrahedron that passes through the point $(1,2,3)$. Writing the equation of the plane as $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$, the volume of the tetrahedron is given by $V = \frac{abc}{6}$. But $(1,2,3)$ must lie on the plane, so we need $\frac{1}{a} + \frac{2}{b} + \frac{3}{c} = 1$ $(*)$ and thus can think of c as a function of a and b. Then $V_a = \frac{b}{6}\left(c + a\frac{\partial c}{\partial a}\right)$ and $V_b = \frac{a}{6}\left(c + b\frac{\partial c}{\partial b}\right)$. Differentiating $(*)$ with respect to a we get $-a^{-2} - 3c^{-2}\frac{\partial c}{\partial a} = 0 \Rightarrow \frac{\partial c}{\partial a} = \frac{-c^2}{3a^2}$, and differentiating $(*)$ with respect to b gives $-2b^{-2} - 3c^{-2}\frac{\partial c}{\partial b} = 0 \Rightarrow \frac{\partial c}{\partial b} = \frac{-2c^2}{3b^2}$. Then $V_a = \frac{b}{6}\left(c + a\frac{-c^2}{3a^2}\right) = 0 \Rightarrow c = 3a$, and $V_b = \frac{a}{6}\left(c + b\frac{-2c^2}{3b^2}\right) = 0 \Rightarrow c = \frac{3}{2}b$. Thus $3a = \frac{3}{2}b$ or $b = 2a$. Putting these into $(*)$ gives $\frac{3}{a} = 1$ or $a = 3$ and then $b = 6$, $c = 9$. Thus the equation of the required plane is $\frac{x}{3} + \frac{y}{6} + \frac{z}{9} = 1$ or $6x + 3y + 2z = 18$.

APPLIED PROJECT Designing a Dumpster

Note: The difficulty and results of this project vary widely with the type of container studied. In addition to the variation of basic shapes of containers, dumpsters may include additional constructed parts such as supports, lift pockets, wheels, etc. Also, a CAS or graphing utility may be needed to solve the resulting equations.

Here we present a typical solution for one particular trash dumpster.

1. The basic shape and dimensions (in inches) of an actual trash dumpster are as shown in the figure.



The front and back, as well as both sides, have an extra one-inch-wide flap that is folded under and welded to the base. In addition, the side panels each fold over one inch onto the front and back pieces where they are welded. Each side has a rectangular lift pocket, with cross-section 5 by 8 inches, made of the same material. These are attached with an extra one-inch width of steel on both top and bottom where each pocket is welded to the side sheet. All four sides have a "lip" at the top; the front and back panels have an extra 5 inches of steel at the top which is folded outward in three creases to form a rectangular tube. The edge is then welded back to the main sheet. The two sides form a top lip with separate sheets of steel 5 inches wide, similarly bent into three sides and welded to the main sheets (requiring two welds each). These extend beyond the main side sheets by 1.5 inches at each end in order to join with the lips on the front and back panels. The container has a hinged lid, extra steel supports on the base at each corner, metal "fins" serving as extra support for the side lift pockets, and wheels underneath. The volume of the container is $V = \frac{1}{2}(40 + 49) \times 42 \times 72 = 134{,}568 \text{ in}^3$ or 77.875 ft^3.

2. First, we assume that some aspects of the construction do not change with different dimensions, so they may be considered fixed costs. This includes the lid (with hinges), wheels, and extra steel supports. Also, the upper "lip" we previously described extends beyond the side width to connect to the other pieces. We can safely assume that this extra portion, including any associated welds, costs the same regardless of the container's dimensions, so we will consider just the portion matching the measurement of the side panels in our calculations. We will further assume that the angle of the top of the container should be preserved. Then to compute the variable costs, let x be the width, y the length, and z the height of the front of the container. The back of the container is 9 inches, or $\frac{3}{4}$ ft, taller than the front, so using similar triangles we can say the back panel has height $z + \frac{3}{14}x$. Measuring in feet, we want the volume to remain constant, so $V = \frac{1}{2}\left(z + z + \frac{3}{14}x\right)(x)(y) = xyz + \frac{3}{28}x^2y = 77.875$. To determine a function for the variable cost, we first find the area of each sheet of metal needed. The base has area xy ft^2. The front panel has visible area yz plus $\frac{1}{12}y$ for the portion folded onto the base and $\frac{5}{12}y$ for the steel at the top used to form the lip, so $\left(yz + \frac{1}{2}y\right)$ ft^2 in total. Similarly, the back sheet has area $y\left(z + \frac{3}{14}x\right) + \frac{1}{12}y + \frac{5}{12}y = yz + \frac{3}{14}xy + \frac{1}{2}y$.

Each side has visible area $\frac{1}{2}\left[z+\left(z+\frac{3}{14}x\right)\right](x)$, and the sheet includes one-inch flaps folding onto the front and back panels, so with area $\frac{1}{12}z$ and $\frac{1}{12}\left(z+\frac{3}{14}x\right)$, and a one-inch flap to fold onto the base with area $\frac{1}{12}x$. The lift pocket is constructed of a piece of steel 20 inches by x ft (including the 2 extra inches used by the welds). The additional metal used to make the lip at the top of the panel has width 5 inches and length that we can determine using the Pythagorean Theorem: $x^2+\left(\frac{3}{14}x\right)^2=\text{length}^2$, so length $=\frac{\sqrt{205}}{14}x\approx 1.0227x$. Thus the area of steel needed for each side panel is approximately

$$\frac{1}{2}\left[z+\left(z+\frac{3}{14}x\right)\right](x)+\frac{1}{12}z+\frac{1}{12}\left(z+\frac{3}{14}x\right)+\frac{1}{12}x+\frac{5}{3}x+\frac{5}{12}(1.0227x)\approx xz+\frac{3}{28}x^2+\frac{1}{6}z+2.194x$$

We also have the following welds:

Weld	Length
Front, back welded to base	$2y$
Sides welded to base	$2x$
Sides welded to front	$2z$
Sides welded to back	$2\left(z+\frac{3}{14}x\right)$
Weld on front and back lip	$2y$
Two welds on each side lip	$4(1.0227x)$
Two welds for each lift pocket	$4x$

Thus the total length of welds needed is

$$2y+2x+2z+2\left(z+\frac{3}{14}x\right)+2y+4(1.0227x)+4x\approx 10.519x+4y+4z$$

Finally, the total variable cost is approximately

$$\begin{aligned}0.90(xy)+0.70\Big[\left(yz+\tfrac{1}{2}y\right)+\left(yz+\tfrac{3}{14}xy+\tfrac{1}{2}y\right)+2\left(xz+\tfrac{3}{28}x^2+\tfrac{1}{6}z+2.194x\right)\Big]&\\ +0.18(10.519x+4y+4z)&\\ \approx 1.05xy+1.4yz+1.42y+1.4xz+0.15x^2+0.953z+4.965x&\end{aligned}$$

We would like to minimize this function while keeping volume constant, so since $xyz+\frac{3}{28}x^2y=77.875$ we can substitute $z=\dfrac{77.875}{xy}-\dfrac{3}{28}x$ giving variable cost as a function of x and y:

$C(x,y)\approx 0.9xy+\dfrac{109.0}{x}+1.42y+\dfrac{109.0}{y}+\dfrac{74.2}{xy}+4.86x$. Using a CAS, we solve the system of equations $C_x(x,y)=0$ and $C_y(x,y)=0$; the only critical point within an appropriate domain is approximately $(3.58, 5.29)$. From the nature of the function C (or from a graph) we can determine that C has an absolute minimum at $(3.58, 5.29)$, and so the minimum cost is attained for $x\approx 3.58$ ft (or 43.0 in), $y\approx 5.29$ ft (or 63.5 in), and $z\approx\frac{77.875}{3.58(5.29)}-\frac{3}{28}(3.58)\approx 3.73$ ft (or 44.8 in).

3. The fixed cost aspects of the container which we did not include in our calculations, such as the wheels and lid, don't affect the validity of our results. Some of our other assumptions, however, may influence the accuracy of our findings. We simplified the price of the steel sheets to include cuts and bends, and we simplified the price of welding to include the labor and materials. This may not be accurate for areas of the container, such as the lip and lift pockets, that require several cuts, bends, and welds in a relatively small surface area. Consequently, increasing some dimensions of the container may not increase the cost in the same manner as our computations predict. If we do not assume that the angle of the sloped top of the container must be preserved, it is likely that we could further improve our cost. Finally, our results show that the length of the container should be changed to minimize cost; this may not be possible if the two lift pockets must remain a fixed distance apart for handling by machinery.

4. The minimum variable cost using our values found in Problem 2 is $C(3.58, 5.29) \approx \$96.95$, while the current dimensions give an estimated variable cost of $C(3.5, 6.0) \approx \$97.30$. If we determine that our assumptions and simplifications are acceptable, our work shows that a slight savings can be gained by adjusting the dimensions of the container. However, the difference in cost is modest, and may not justify changes in the manufacturing process.

DISCOVERY PROJECT Quadratic Approximations and Critical Points

1. $$Q(x,y) = f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) + \tfrac{1}{2} f_{xx}(a,b)(x-a)^2 + f_{xy}(a,b)(x-a)(y-b) + \tfrac{1}{2} f_{yy}(a,b)(y-b)^2,$$

so

$$\begin{aligned} Q_x(x,y) &= f_x(a,b) + \tfrac{1}{2} f_{xx}(a,b)(2)(x-a) + f_{xy}(a,b)(y-b) \\ &= f_x(a,b) + f_{xx}(a,b)(x-a) + f_{xy}(a,b)(y-b) \end{aligned}$$

At (a,b) we have $Q_x(a,b) = f_x(a,b) + f_{xx}(a,b)\,(a-a) + f_{xy}(a,b)(b-b) = f_x(a,b)$.

Similarly, $Q_y(x,y) = f_y(a,b) + f_{xy}(a,b)(x-a) + f_{yy}(a,b)(y-b) \quad \Rightarrow$

$Q_y(a,b) = f_y(a,b) + f_{xy}(a,b)(a-a) + f_{yy}(a,b)(b-b) = f_y(a,b)$.

For the second-order partial derivatives we have

$$Q_{xx}(x,y) = \frac{\partial}{\partial x}\left[f_x(a,b) + f_{xx}(a,b)(x-a) + f_{xy}(a,b)(y-b)\right] = f_{xx}(a,b)$$

$$\Rightarrow \quad Q_{xx}(a,b) = f_{xx}(a,b)$$

$$Q_{xy}(x,y) = \frac{\partial}{\partial y}\left[f_x(a,b) + f_{xx}(a,b)(x-a) + f_{xy}(a,b)(y-b)\right] = f_{xy}(a,b)$$

$$\Rightarrow \quad Q_{xy}(a,b) = f_{xy}(a,b)$$

$$Q_{yy}(x,y) = \frac{\partial}{\partial y}\left[f_y(a,b) + f_{xy}(a,b)(x-a) + f_{yy}(a,b)(y-b)\right] = f_{yy}(a,b)$$

$$\Rightarrow \quad Q_{yy}(a,b) = f_{yy}(a,b)$$

2. (a) First we find the partial derivatives and values that will be needed:

$$f(x,y) = e^{-x^2-y^2} \qquad f(0,0) = 1$$
$$f_x(x,y) = -2xe^{-x^2-y^2} \qquad f_x(0,0) = 0$$
$$f_y(x,y) = -2ye^{-x^2-y^2} \qquad f_y(0,0) = 0$$
$$f_{xx}(x,y) = (4x^2-2)e^{-x^2-y^2} \qquad f_{xx}(0,0) = -2$$
$$f_{xy}(x,y) = 4xye^{-x^2-y^2} \qquad f_{xy}(0,0) = 0$$
$$f_{yy}(x,y) = (4y^2-2)e^{-x^2-y^2} \qquad f_{yy}(0,0) = -2$$

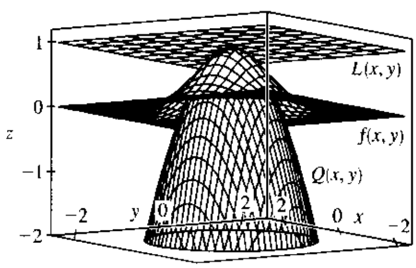
Then the first-degree Taylor polynomial of f at $(0,0)$ is

$$\begin{aligned} L(x,y) &= f(0,0) + f_x(0,0)(x-0) + f_y(0,0)(y-0) = 1 + (0)(x-0) + (0)(y-0) \\ &= 1 \end{aligned}$$

The second-degree Taylor polynomial is given by

$$\begin{aligned} Q(x,y) &= f(0,0) + f_x(0,0)(x-0) + f_y(0,0)(y-0) + \tfrac{1}{2}f_{xx}(0,0)(x-0)^2 \\ &\qquad + f_{xy}(0,0)(x-0)(y-0) + \tfrac{1}{2}f_{yy}(0,0)(y-0)^2 \\ &= 1 - x^2 - y^2 \end{aligned}$$

(b)



As we see from the graph, L approximates f well only for points (x,y) extremely close to the origin. Q is a much better approximation; the shape of its graph looks similar to that of the graph of f near the origin, and the values of Q appear to be good estimates for the values of f within a significant radius of the origin.

3. (a) First we find the partial derivatives and values that will be needed:

$$f(x,y) = xe^y \qquad f(1,0) = 1 \qquad f_{xx}(x,y) = 0 \qquad f_{xx}(1,0) = 0$$
$$f_x(x,y) = e^y \qquad f_x(1,0) = 1 \qquad f_{xy}(x,y) = e^y \qquad f_{xy}(1,0) = 1$$
$$f_y(x,y) = xe^y \qquad f_y(1,0) = 1 \qquad f_{yy}(x,y) = xe^y \qquad f_{yy}(1,0) = 1$$

Then the first-degree Taylor polynomial of f at $(1,0)$ is

$$\begin{aligned} L(x,y) &= f(1,0) + f_x(1,0)(x-1) + f_y(1,0)(y-0) \\ &= 1 + (1)(x-1) + (1)(y-0) \\ &= x + y \end{aligned}$$

The second-degree Taylor polynomial is given by

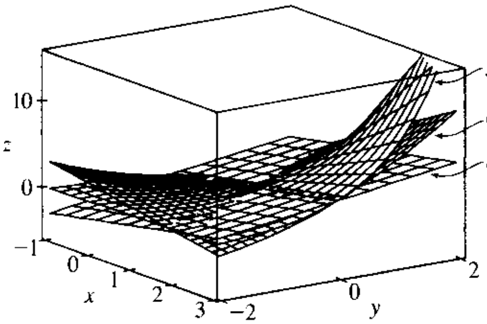
$$\begin{aligned} Q(x,y) &= f(1,0) + f_x(1,0)(x-1) + f_y(1,0)(y-0) + \tfrac{1}{2}f_{xx}(1,0)(x-1)^2 \\ &\qquad + f_{xy}(1,0)(x-1)(y-0) + \tfrac{1}{2}f_{yy}(1,0)(y-0)^2 \\ &= \tfrac{1}{2}y^2 + x + xy \end{aligned}$$

(b) $L(0.9, 0.1) = 0.9 + 0.1 = 1.0$

$Q(0.9, 0.1) = \frac{1}{2}(0.1)^2 + 0.9 + (0.9)(0.1) = 0.995$

$f(0.9, 0.1) = 0.9e^{0.1} \approx 0.9947$

(c)



As we see from the graph, L and Q both approximate f reasonably well near the point $(1, 0)$. As we venture farther from the point, the graph of Q follows the shape of the graph of f more closely than L.

4. (a) $f(x,y) = ax^2 + bxy + cy^2 = a\left[x^2 + \frac{b}{a}xy + \frac{c}{a}y^2\right]$

$$= a\left[x^2 + \frac{b}{a}xy + \left(\frac{b}{2a}y\right)^2 - \left(\frac{b}{2a}y\right)^2 + \frac{c}{a}y^2\right]$$

$$= a\left[\left(x + \frac{b}{2a}y\right)^2 - \frac{b^2}{4a^2}y^2 + \frac{c}{a}y^2\right] = a\left[\left(x + \frac{b}{2a}y\right)^2 + \left(\frac{4ac - b^2}{4a^2}\right)y^2\right]$$

(b) For $D = 4ac - b^2$, from part (a) we have $f(x,y) = a\left[\left(x + \frac{b}{2a}y\right)^2 + \left(\frac{D}{4a^2}\right)y^2\right]$. If $D > 0$, $\left(\frac{D}{4a^2}\right)y^2 \geq 0$ and $\left(x + \frac{b}{2a}y\right)^2 \geq 0$, so $\left[\left(x + \frac{b}{2a}y\right)^2 + \left(\frac{D}{4a^2}\right)y^2\right] \geq 0$. Here $a > 0$, thus $f(x,y) = a\left[\left(x + \frac{b}{2a}y\right)^2 + \left(\frac{D}{4a^2}\right)y^2\right] \geq 0$. We know $f(0,0) = 0$, so $f(0,0) \leq f(x,y)$ for all (x,y), and by definition f has a local minimum at $(0,0)$.

(c) As in part (b), $\left[\left(x + \frac{b}{2a}y\right)^2 + \left(\frac{D}{4a^2}\right)y^2\right] \geq 0$, and since $a < 0$ we have $f(x,y) = a\left[\left(x + \frac{b}{2a}y\right)^2 + \left(\frac{D}{4a^2}\right)y^2\right] \leq 0$. Since $f(0,0) = 0$, we must have $f(0,0) \geq f(x,y)$ for all (x,y), so by definition f has a local maximum at $(0,0)$.

(d) $f(x,y) = ax^2 + bxy + cy^2$, so $f_x(x,y) = 2ax + by \Rightarrow f_x(0,0) = 0$ and $f_y(x,y) = bx + 2cy \Rightarrow f_y(0,0) = 0$. Since $f(0,0) = 0$ and f and its partial derivatives are continuous, we know from Equation 15.4.2 [ET 14.4.2] that the tangent plane to the graph of f at $(0,0)$ is the plane $z = 0$. Then f has a saddle point at $(0,0)$ if the graph of f crosses the tangent plane at $(0,0)$, or equivalently, if some paths to the origin have positive function values while other paths have negative function values. Suppose we approach the origin along the x-axis; then we have $y = 0 \Rightarrow f(x,0) = ax^2$ which has the same sign as a. We must now find at least one path to the origin where $f(x,y)$ gives values with sign opposite that of a. Since

$f(x,y) = a\left[\left(x + \frac{b}{2a}\,y\right)^2 + \left(\frac{D}{4a^2}\right)y^2\right]$, if we approach the origin along the line $x = -\frac{b}{2a}\,y$, we have

$f\left(-\frac{b}{2a}\,y, y\right) = a\left[\left(-\frac{b}{2a}\,y + \frac{b}{2a}\,y\right)^2 + \left(\frac{D}{4a^2}\right)y^2\right] = \frac{D}{4a}\,y^2$. Since $D < 0$, these values have signs opposite that of a. Thus, f has a saddle point at $(0,0)$.

5. (a) Since the partial derivatives of f exist at $(0,0)$ and $(0,0)$ is a critical point, we know $f_x(0,0) = 0$ and $f_y(0,0) = 0$. Then the second-degree Taylor polynomial of f at $(0,0)$ can be expressed as

$$\begin{aligned} Q(x,y) &= f(0,0) + f_x(0,0)(x-0) + f_y(0,0)(y-0) + \tfrac{1}{2}f_{xx}(0,0)(x-0)^2 \\ &\qquad + f_{xy}(0,0)(x-0)(y-0) + \tfrac{1}{2}f_{yy}(0,0)(y-0)^2 \\ &= \tfrac{1}{2}f_{xx}(0,0)x^2 + f_{xy}(0,0)xy + \tfrac{1}{2}f_{yy}(0,0)y^2. \end{aligned}$$

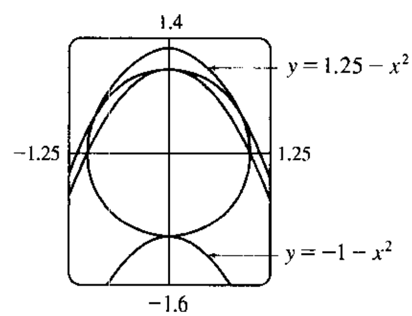
(b) $Q(x,y) = \frac{1}{2}\,f_{xx}(0,0)x^2 + f_{xy}(0,0)xy + \frac{1}{2}\,f_{yy}(0,0)y^2$ fits the form of the polynomial function in Problem 4 with $a = \frac{1}{2}\,f_{xx}(0,0)$, $b = f_{xy}(0,0)$, and $c = \frac{1}{2}\,f_{yy}(0,0)$. Then we know Q is a paraboloid, and that Q has a local maximum, local minimum, or saddle point at $(0,0)$. Here,
$D = 4ac - b^2 = 4\left(\frac{1}{2}\right)f_{xx}(0,0)\left(\frac{1}{2}\right)f_{yy}(0,0) - [f_{xy}(0,0)]^2 = f_{xx}(0,0)f_{yy}(0,0) - [f_{xy}(0,0)]^2$, and if $D > 0$ with $a = \frac{1}{2}\,f_{xx}(0,0) > 0 \;\Rightarrow\; f_{xx}(0,0) > 0$, we know from Problem 4 that Q has a local minimum at $(0,0)$. Similarly, if $D > 0$ and $a < 0 \;\Rightarrow\; f_{xx}(0,0) < 0$, Q has a local maximum at $(0,0)$, and if $D < 0$, Q has a saddle point at $(0,0)$.

(c) Since $f(x,y) \approx Q(x,y)$ near $(0,0)$, part (b) suggests that for $D = f_{xx}(0,0)f_{yy}(0,0) - [f_{xy}(0,0)]^2$, if $D > 0$ and $f_{xx}(0,0) > 0$, f has a local minimum at $(0,0)$. If $D > 0$ and $f_{xx}(0,0) < 0$, f has a local maximum at $(0,0)$, and if $D < 0$, f has a saddle point at $(0,0)$. Together with the conditions given in part (a), this is precisely the Second Derivatives Test from Section 15.7 [ET 14.7].

15.8 Lagrange Multipliers ET 14.8

1. At the extreme values of f, the level curves of f just touch the curve $g(x,y) = 8$ with a common tangent line. (See Figure 1 and the accompanying discussion.) We can observe several such occurrences on the contour map, but the level curve $f(x,y) = c$ with the largest value of c which still intersects the curve $g(x,y) = 8$ is approximately $c = 59$, and the smallest value of c corresponding to a level curve which intersects $g(x,y) = 8$ appears to be $c = 30$. Thus we estimate the maximum value of f subject to the constraint $g(x,y) = 8$ to be about 59 and the minimum to be 30.

2. (a) The values $c = \pm 1$ and $c = 1.25$ seem to give curves which are tangent to the circle. These values represent possible extreme values of the function $x^2 + y$ subject to the constraint $x^2 + y^2 = 1$.



(b) $\nabla f = \langle 2x, 1\rangle$, $\lambda\nabla g = \langle 2\lambda x, 2\lambda y\rangle$. So $2x = 2\lambda x \Rightarrow$ either $\lambda = 1$ or $x = 0$. If $\lambda = 1$, then $y = \frac{1}{2}$ and so $x = \pm\frac{\sqrt{3}}{2}$ (from the constraint). If $x = 0$, then $y = \pm 1$. Therefore f has possible extreme values at the points $(0, \pm 1)$ and $\left(\pm\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$. We calculate $f\left(\pm\frac{\sqrt{3}}{2}, \frac{1}{2}\right) = \frac{5}{4}$ (the maximum value), $f(0, 1) = 1$, and $f(0, -1) = -1$ (the minimum value). These are our answers from (a).

3. $f(x, y) = x^2 - y^2$, $g(x, y) = x^2 + y^2 = 1 \Rightarrow \nabla f = \langle 2x, -2y\rangle$, $\lambda\nabla g = \langle 2\lambda x, 2\lambda y\rangle$. Then $2x = 2\lambda x$ implies $x = 0$ or $\lambda = 1$. If $x = 0$, then $x^2 + y^2 = 1$ implies $y = \pm 1$ and if $\lambda = 1$, then $-2y = 2\lambda y$ implies $y = 0$ and thus $x = \pm 1$. Thus the possible points for the extreme values of f are $(\pm 1, 0)$, $(0, \pm 1)$. But $f(\pm 1, 0) = 1$ while $f(0, \pm 1) = -1$ so the maximum value of f on $x^2 + y^2 = 1$ is $f(\pm 1, 0) = 1$ and the minimum value is $f(0, \pm 1) = -1$.

4. $f(x, y) = 4x + 6y$, $g(x, y) = x^2 + y^2 = 13 \Rightarrow \nabla f = \langle 4, 6\rangle$, $\lambda\nabla g = \langle 2\lambda x, 2\lambda y\rangle$. Then $2\lambda x = 4$ and $2\lambda y = 6$ imply $x = \dfrac{2}{\lambda}$ and $y = \dfrac{3}{\lambda}$. But $13 = x^2 + y^2 = \left(\dfrac{2}{\lambda}\right)^2 + \left(\dfrac{3}{\lambda}\right)^2 \Rightarrow 13 = \dfrac{13}{\lambda^2} \Rightarrow \lambda = \pm 1$, so f has possible extreme values at the points $(2, 3)$, $(-2, -3)$. We compute $f(2, 3) = 26$ and $f(-2, -3) = -26$, so the maximum value of f on $x^2 + y^2 = 13$ is $f(2, 3) = 26$ and the minimum value is $f(-2, -3) = -26$.

5. $f(x, y) = x^2 y$, $g(x, y) = x^2 + 2y^2 = 6 \Rightarrow \nabla f = \langle 2xy, x^2\rangle$, $\lambda\nabla g = \langle 2\lambda x, 4\lambda y\rangle$. Then $2xy = 2\lambda x$ implies $x = 0$ or $\lambda = y$. If $x = 0$, then $x^2 = 4\lambda y$ implies $\lambda = 0$ or $y = 0$. However, if $y = 0$ then $g(x, y) = 0$, a contradiction. So $\lambda = 0$ and then $g(x, y) = 6 \Rightarrow y = \pm\sqrt{3}$. If $\lambda = y$, then $x^2 = 4\lambda y$ implies $x^2 = 4y^2$, and so $g(x, y) = 6 \Rightarrow 4y^2 + 2y^2 = 6 \Rightarrow y^2 = 1 \Rightarrow y = \pm 1$. Thus f has possible extreme values at the points $(0, \pm\sqrt{3})$, $(\pm 2, 1)$, and $(\pm 2, -1)$. After evaluating f at these points, we find the maximum value to be $f(\pm 2, 1) = 4$ and the minimum to be $f(\pm 2, -1) = -4$.

6. $f(x, y) = x^2 + y^2$, $g(x, y) = x^4 + y^4 = 1 \Rightarrow \nabla f = \langle 2x, 2y\rangle$, $\lambda\nabla g = \langle 4\lambda x^3, 4\lambda y^3\rangle$. Then $x = 2\lambda x^3$ implies $x = 0$ or $\lambda = \dfrac{1}{2x^2}$. If $x = 0$, then $x^4 + y^4 = 1$ implies $y = \pm 1$. But $y = 2\lambda y^3$ implies $y = 0$ so $x = \pm 1$ or $\lambda = \dfrac{1}{2y^2}$ and $x^2 = y^2$ and $2x^4 = 1$ so $x = \pm\frac{1}{\sqrt[4]{2}}$. Hence the possible points are $(0, \pm 1)$, $(\pm 1, 0)$, $\left(\pm\frac{1}{\sqrt[4]{2}}, \pm\frac{1}{\sqrt[4]{2}}\right)$, with the maximum value of f on $x^4 + y^4 = 1$ being $f\left(\pm\frac{1}{\sqrt[4]{2}}, \pm\frac{1}{\sqrt[4]{2}}\right) = \frac{2}{\sqrt{2}} = \sqrt{2}$ and the minimum value being $f(0, \pm 1) = f(\pm 1, 0) = 1$.

7. $f(x, y, z) = 2x + 6y + 10z$, $g(x, y, z) = x^2 + y^2 + z^2 = 35 \Rightarrow \nabla f = \langle 2, 6, 10\rangle$, $\lambda\nabla g = \langle 2\lambda x, 2\lambda y, 2\lambda z\rangle$. Then $2\lambda x = 2$, $2\lambda y = 6$, $2\lambda z = 10$ imply $x = \dfrac{1}{\lambda}$, $y = \dfrac{3}{\lambda}$, and $z = \dfrac{5}{\lambda}$. But

$$35 = x^2 + y^2 + z^2 = \left(\frac{1}{\lambda}\right)^2 + \left(\frac{3}{\lambda}\right)^2 + \left(\frac{5}{\lambda}\right)^2 \Rightarrow 35 = \frac{35}{\lambda^2} \Rightarrow \lambda = \pm 1,$$

so f has possible extreme values at the points $(1, 3, 5)$, $(-1, -3, -5)$. The maximum value of f on $x^2 + y^2 + z^2 = 35$ is $f(1, 3, 5) = 70$, and the minimum is $f(-1, -3, -5) = -70$.

8. $f(x, y, z) = 8x - 4z$, $g(x, y, z) = x^2 + 10y^2 + z^2 = 5 \Rightarrow \nabla f = \langle 8, 0, -4\rangle$, $\lambda\nabla g = \langle 2\lambda x, 20\lambda y, 2\lambda z\rangle$. Then $2\lambda x = 8$, $20\lambda y = 0$, $2\lambda z = -4$ imply $x = \dfrac{4}{\lambda}$, $y = 0$, and $z = -\dfrac{2}{\lambda}$. But

$$5 = x^2 + 10y^2 + z^2 = \left(\frac{4}{\lambda}\right)^2 + 10\,(0)^2 + \left(-\frac{2}{\lambda}\right)^2 \Rightarrow 5 = \frac{20}{\lambda^2} \Rightarrow \lambda = \pm 2,$$

so f has possible extreme values at the points $(2, 0, -1)$, $(-2, 0, 1)$. The maximum of f on $x^2 + 10y^2 + z^2 = 5$ is $f(2, 0, -1) = 20$, and the minimum is $f(-2, 0, 1) = -20$.

9. $f(x,y,z)=xyz$, $g(x,y,z)=x^2+2y^2+3z^2=6 \;\Rightarrow\; \nabla f=\langle yz,xz,xy\rangle$, $\lambda\nabla g=\langle 2\lambda x,4\lambda y,6\lambda z\rangle$. Then $\nabla f=\lambda\nabla g$ implies $\lambda=(yz)/(2x)=(xz)/(4y)=(xy)/(6z)$ or $x^2=2y^2$ and $z^2=\frac{2}{3}y^2$. Thus $x^2+2y^2+3z^2=6$ implies $6y^2=6$ or $y=\pm 1$. Then the possible points are $\left(\sqrt{2},\pm 1,\sqrt{\frac{2}{3}}\right)$, $\left(\sqrt{2},\pm 1,-\sqrt{\frac{2}{3}}\right)$, $\left(-\sqrt{2},\pm 1,\sqrt{\frac{2}{3}}\right)$, $\left(-\sqrt{2},\pm 1,-\sqrt{\frac{2}{3}}\right)$. The maximum value of f on the ellipsoid is $\frac{2}{\sqrt{3}}$, occurring when all coordinates are positive or exactly two are negative and the minimum is $-\frac{2}{\sqrt{3}}$ occurring when 1 or 3 of the coordinates are negative.

10. $f(x,y,z)=x^2y^2z^2$, $g(x,y,z)=x^2+y^2+z^2=1 \;\Rightarrow\; \nabla f=\langle 2xy^2z^2,2yx^2z^2,2zx^2y^2\rangle$, $\lambda\nabla g=\langle 2\lambda x,2\lambda y,2\lambda z\rangle$. Then $\nabla f=\lambda\nabla g$ implies (1) $\lambda=y^2z^2=x^2z^2=x^2y^2$ and $\lambda\neq 0$, or (2) $\lambda=0$ and one or two (but not three) of the coordinates are 0. If (1) then $x^2=y^2=z^2=\frac{1}{3}$. The minimum value of f on the sphere occurs in case (2) with a value of 0 and the maximum value is $\frac{1}{27}$ which arises from all the points from (1), that is, the points $\left(\pm\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)$, $\left(\pm\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)$, $\left(\pm\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}}\right)$.

11. $f(x,y,z)=x^2+y^2+z^2$, $g(x,y,z)=x^4+y^4+z^4=1 \;\Rightarrow$
$\nabla f=\langle 2x,2y,2z\rangle$, $\lambda\nabla g=\langle 4\lambda x^3,4\lambda y^3,4\lambda z^3\rangle$.

Case 1: If $x\neq 0$, $y\neq 0$ and $z\neq 0$, then $\nabla f=\lambda\nabla g$ implies $\lambda=1/(2x^2)=1/(2y^2)=1/(2z^2)$ or $x^2=y^2=z^2$ and $3x^4=1$ or $x=\pm\frac{1}{\sqrt[4]{3}}$ giving the points $\left(\pm\frac{1}{\sqrt[4]{3}},\frac{1}{\sqrt[4]{3}},\frac{1}{\sqrt[4]{3}}\right)$, $\left(\pm\frac{1}{\sqrt[4]{3}},-\frac{1}{\sqrt[4]{3}},\frac{1}{\sqrt[4]{3}}\right)$, $\left(\pm\frac{1}{\sqrt[4]{3}},\frac{1}{\sqrt[4]{3}},-\frac{1}{\sqrt[4]{3}}\right)$, $\left(\pm\frac{1}{\sqrt[4]{3}},-\frac{1}{\sqrt[4]{3}},-\frac{1}{\sqrt[4]{3}}\right)$ all with an f-value of $\sqrt{3}$.

Case 2: If one of the variables equals zero and the other two are not zero, then the squares of the two nonzero coordinates are equal with common value $\frac{1}{\sqrt{2}}$ and corresponding f value of $\sqrt{2}$.

Case 3: If exactly two of the variables are zero, then the third variable has value ± 1 with the corresponding f value of 1. Thus on $x^4+y^4+z^4=1$, the maximum value of f is $\sqrt{3}$ and the minimum value is 1.

12. $f(x,y,z)=x^4+y^4+z^4$, $g(x,y,z)=x^2+y^2+z^2=1 \;\Rightarrow$
$\nabla f=\langle 4x^3,4y^3,4z^3\rangle$, $\lambda\nabla g=\langle 2\lambda x,2\lambda y,2\lambda z\rangle$.

Case 1: If $x\neq 0$, $y\neq 0$ and $z\neq 0$ then $\nabla f=\lambda\nabla g$ implies $\lambda=2x^2=2y^2=2z^2$ or $x^2=y^2=z^2=\frac{1}{3}$ yielding 8 points each with an f-value of $\frac{1}{3}$.

Case 2: If one of the variables is 0 and the other two are not, then the squares of the two nonzero coordinates are equal with common value $\frac{1}{2}$ and the corresponding f-value is $\frac{1}{2}$.

Case 3: If exactly two of the variables are 0, then the third variable has value ± 1 with corresponding f-value of 1. Thus on $x^2+y^2+z^2=1$, the maximum value of f is 1 and the minimum value is $\frac{1}{3}$.

13. $f(x,y,z,t)=x+y+z+t$, $g(x,y,z,t)=x^2+y^2+z^2+t^2=1 \;\Rightarrow\; \langle 1,1,1,1\rangle=\langle 2\lambda x,2\lambda y,2\lambda z,2\lambda t\rangle$, so $\lambda=1/(2x)=1/(2y)=1/(2z)=1/(2t)$ and $x=y=z=t$. But $x^2+y^2+z^2+t^2=1$, so the possible points are $\left(\pm\frac{1}{2},\pm\frac{1}{2},\pm\frac{1}{2},\pm\frac{1}{2}\right)$. Thus the maximum value of f is $f\left(\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}\right)=2$ and the minimum value is $f\left(-\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\right)=-2$.

14. $f(x_1,x_2,\ldots,x_n)=x_1+x_2+\cdots+x_n$, $g(x_1,x_2,\ldots,x_n)=x_1^2+x_2^2+\cdots+x_n^2=1 \;\Rightarrow$
$\langle 1,1,\ldots,1\rangle=\langle 2\lambda x_1,2\lambda x_2,\ldots,2\lambda x_n\rangle$, so $\lambda=1/(2x_1)=1/(2x_2)=\cdots=1/(2x_n)$ and $x_1=x_2=\cdots=x_n$. But $x_1^2+x_2^2+\cdots+x_n^2=1$, so $x_i=\pm 1/\sqrt{n}$ for $i=1,\ldots,n$. Thus the maximum value of f is $f(1/\sqrt{n},1/\sqrt{n},\ldots,1/\sqrt{n})=\sqrt{n}$ and the minimum value is $f(-1/\sqrt{n},-1/\sqrt{n},\ldots,-1/\sqrt{n})=-\sqrt{n}$.

15. $f(x,y,z) = x + 2y$, $g(x,y,z) = x + y + z = 1$, $h(x,y,z) = y^2 + z^2 = 4 \quad \Rightarrow \quad \nabla f = \langle 1, 2, 0 \rangle$, $\lambda \nabla g = \langle \lambda, \lambda, \lambda \rangle$ and $\mu \nabla h = \langle 0, 2\mu y, 2\mu z \rangle$. Then $1 = \lambda$, $2 = \lambda + 2\mu y$ and $0 = \lambda + 2\mu z$ so $\mu y = \frac{1}{2} = -\mu z$ or $y = 1/(2\mu)$, $z = -1/(2\mu)$. Thus $x + y + z = 1$ implies $x = 1$ and $y^2 + z^2 = 4$ implies $\mu = \pm\frac{1}{2\sqrt{2}}$. Then the possible points are $\left(1, \pm\sqrt{2}, \mp\sqrt{2}\right)$ and the maximum value is $f\left(1, \sqrt{2}, -\sqrt{2}\right) = 1 + 2\sqrt{2}$ and the minimum value is $f\left(1, -\sqrt{2}, \sqrt{2}\right) = 1 - 2\sqrt{2}$.

16. $f(x,y,z) = 3x - y - 3z$, $g(x,y,z) = x + y - z = 0$, $h(x,y,z) = x^2 + 2z^2 = 1 \quad \Rightarrow \quad \nabla f = \langle 3, -1, -3 \rangle$, $\lambda \nabla g = \langle \lambda, \lambda, -\lambda \rangle$, $\mu \nabla h = (2\mu x, 0, 4\mu z)$. Then $3 = \lambda + 2\mu x$, $-1 = \lambda$ and $-3 = -\lambda + 4\mu z$, so $\lambda = -1$, $\mu z = -1$, $\mu x = 2$. Thus $h(x,y,z) = 1$ implies $\dfrac{4}{\mu^2} + 2\left(\dfrac{1}{\mu^2}\right) = 1$ or $\mu = \pm\sqrt{6}$, so $z = \mp\frac{1}{\sqrt{6}}$; $x = \pm\frac{2}{\sqrt{6}}$; and $g(x,y,z) = 0$ implies $y = \mp\frac{3}{\sqrt{6}}$. Hence the maximum of f subject to the constraints is $f\left(\frac{\sqrt{6}}{3}, -\frac{\sqrt{6}}{2}, -\frac{\sqrt{6}}{6}\right) = 2\sqrt{6}$ and the minimum is $f\left(-\frac{\sqrt{6}}{3}, \frac{\sqrt{6}}{2}, \frac{\sqrt{6}}{6}\right) = -2\sqrt{6}$.

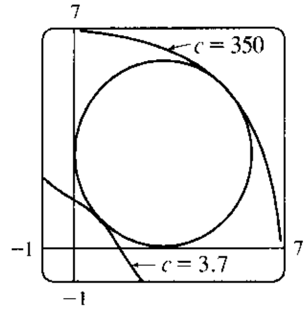
17. $f(x,y,z) = yz + xy$, $g(x,y,z) = xy = 1$, $h(x,y,z) = y^2 + z^2 = 1 \quad \Rightarrow \quad \nabla f = \langle y, x + z, y \rangle$, $\lambda \nabla g = \langle \lambda y, \lambda x, 0 \rangle$, $\mu \nabla h = \langle 0, 2\mu y, 2\mu z \rangle$. Then $y = \lambda y$ implies $\lambda = 1$ $[y \neq 0$ since $g(x,y,z) = 1]$, $x + z = \lambda x + 2\mu y$ and $y = 2\mu z$. Thus $\mu = z/(2y) = y/(2y)$ or $y^2 = z^2$, and so $y^2 + z^2 = 1$ implies $y = \pm\frac{1}{\sqrt{2}}$, $z = \pm\frac{1}{\sqrt{2}}$. Then $xy = 1$ implies $x = \pm\sqrt{2}$ and the possible points are $\left(\pm\sqrt{2}, \pm\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$, $\left(\pm\sqrt{2}, \pm\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$. Hence the maximum of f subject to the constraints is $f\left(\pm\sqrt{2}, \pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}\right) = \frac{3}{2}$ and the minimum is $f\left(\pm\sqrt{2}, \pm\frac{1}{\sqrt{2}}, \mp\frac{1}{\sqrt{2}}\right) = \frac{1}{2}$.

Note: Since $xy = 1$ is one of the constraints we could have solved the problem by solving $f(y,z) = yz + 1$ subject to $y^2 + z^2 = 1$.

18. $f(x,y) = 2x^2 + 3y^2 - 4x - 5 \quad \Rightarrow \quad \nabla f = \langle 4x - 4, 6y \rangle = \langle 0, 0 \rangle \quad \Rightarrow \quad x = 1, y = 0$. Thus $(1,0)$ is the only critical point of f, and it lies in the region $x^2 + y^2 < 16$. On the boundary, $g(x,y) = x^2 + y^2 = 16 \quad \Rightarrow$ $\lambda \nabla g = \langle 2\lambda x, 2\lambda y \rangle$, so $6y = 2\lambda y \quad \Rightarrow \quad$ either $y = 0$ or $\lambda = 3$. If $y = 0$, then $x = \pm 4$; if $\lambda = 3$, then $4x - 4 = 2\lambda x \quad \Rightarrow \quad x = -2$ and $y = \pm 2\sqrt{3}$. Now $f(1,0) = -7$, $f(4,0) = 11$, $f(-4,0) = 43$, and $f\left(-2, \pm 2\sqrt{3}\right) = 47$. Thus the maximum value of $f(x,y)$ on the disk $x^2 + y^2 \leq 16$ is $f\left(-2, \pm 2\sqrt{3}\right) = 47$, and the minimum value is $f(1,0) = -7$.

19. $f(x,y) = e^{-xy}$. For the interior of the region, we find the critical points: $f_x = -ye^{-xy}$, $f_y = -xe^{-xy}$, so the only critical point is $(0,0)$, and $f(0,0) = 1$. For the boundary, we use Lagrange multipliers. $g(x,y) = x^2 + 4y^2 = 1 \quad \Rightarrow \quad \lambda \nabla g = \langle 2\lambda x, 8\lambda y \rangle$, so setting $\nabla f = \lambda \nabla g$ we get $-ye^{-xy} = 2\lambda x$ and $-xe^{-xy} = 8\lambda y$. The first of these gives $e^{-xy} = -2\lambda x/y$, and then the second gives $-x(-2\lambda x/y) = 8\lambda y \quad \Rightarrow$ $x^2 = 4y^2$. Solving this last equation with the constraint $x^2 + 4y^2 = 1$ gives $x = \pm\frac{1}{\sqrt{2}}$ and $y = \pm\frac{1}{2\sqrt{2}}$. Now $f\left(\pm\frac{1}{\sqrt{2}}, \mp\frac{1}{2\sqrt{2}}\right) = e^{1/4} \approx 1.284$ and $f\left(\pm\frac{1}{\sqrt{2}}, \pm\frac{1}{2\sqrt{2}}\right) = e^{-1/4} \approx 0.779$. The former are the maxima on the region and the latter are the minima.

20. (a) The graphs of $f(x,y)=3.7$ and $f(x,y)=350$ seem to be tangent to the circle, and so 3.7 and 350 are the approximate minimum and maximum values of the function $f(x,y)$ subject to the constraint $(x-3)^2+(y-3)^2=9$.



(b) Let $g(x,y)=(x-3)^2+(y-3)^2$. We calculate $f_x(x,y)=3x^2+3y$, $f_y(x,y)=3y^2+3x$, $g_x(x,y)=2x-6$, and $g_y(x,y)=2y-6$, and use a CAS to search for solutions to the equations $g(x,y)=(x-3)^2+(y-3)^2=9$, $f_x=\lambda g_x$, and $f_y=\lambda g_y$.The solutions are $(x,y)=\left(3-\frac{3}{2}\sqrt{2},3-\frac{3}{2}\sqrt{2}\right)\approx(0.879,0.879)$ and $(x,y)=\left(3+\frac{3}{2}\sqrt{2},3+\frac{3}{2}\sqrt{2}\right)\approx(5.121,5.121)$. These give $f\left(3-\frac{3}{2}\sqrt{2},3-\frac{3}{2}\sqrt{2}\right)=\frac{351}{2}-\frac{243}{2}\sqrt{2}\approx3.673$ and $f\left(3+\frac{3}{2}\sqrt{2},3+\frac{3}{2}\sqrt{2}\right)=\frac{351}{2}+\frac{243}{2}\sqrt{2}\approx347.33$, in accordance with part (a).

21. $P(L,K)=bL^\alpha K^{1-\alpha}$, $g(L,K)=mL+nK=p\;\Rightarrow\;\nabla P=\langle\alpha bL^{\alpha-1}K^{1-\alpha},(1-\alpha)bL^\alpha K^{-\alpha}\rangle$, $\lambda\nabla g=\langle\lambda m,\lambda n\rangle$. Then $\alpha b(K/L)^{1-\alpha}=\lambda m$ and $(1-\alpha)b(L/K)^\alpha=\lambda n$ and $mL+nK=p$, so $\alpha b(K/L)^{1-\alpha}/m=(1-\alpha)b(L/K)^\alpha/n$ or $n\alpha/[m(1-\alpha)]=(L/K)^\alpha(L/K)^{1-\alpha}$ or $L=Kn\alpha/[m(1-\alpha)]$. Substituting into $mL+nK=p$ gives $K=(1-\alpha)p/n$ and $L=\alpha p/m$ for the maximum production.

22. $C(L,K)=mL+nK$, $g(L,K)=bL^\alpha K^{1-\alpha}=Q\;\Rightarrow\;\nabla C=\langle m,n\rangle$, $\lambda\nabla g=\langle\lambda\alpha bL^{\alpha-1}K^{1-\alpha},\lambda(1-\alpha)bL^\alpha K^{-\alpha}\rangle$. Then $\dfrac{m}{\alpha b}\left(\dfrac{L}{K}\right)^{1-\alpha}=\dfrac{n}{(1-\alpha)b}\left(\dfrac{K}{L}\right)^{\alpha}$ and $bL^\alpha K^{1-\alpha}=Q\;\Rightarrow\;\dfrac{n\alpha}{m(1-\alpha)}=\left(\dfrac{L}{K}\right)^{1-\alpha}\left(\dfrac{L}{K}\right)^{\alpha}\;\Rightarrow\;L=\dfrac{Kn\alpha}{m(1-\alpha)}$ and so $b\left[\dfrac{Kn\alpha}{m(1-\alpha)}\right]^\alpha K^{1-\alpha}=Q$.

Hence $K=\dfrac{Q}{b\left(n\alpha/[m(1-\alpha)]\right)^\alpha}=\dfrac{Qm^\alpha(1-\alpha)^\alpha}{bn^\alpha\alpha^\alpha}$ and $L=\dfrac{Qm^{\alpha-1}(1-\alpha)^{\alpha-1}}{bn^{\alpha-1}\alpha^{\alpha-1}}=\dfrac{Qn^{1-\alpha}\alpha^{1-\alpha}}{bm^{1-\alpha}(1-\alpha)^{1-\alpha}}$ minimizes cost.

23. Let the sides of the rectangle be x and y. Then $f(x,y)=xy$, $g(x,y)=2x+2y=p\;\Rightarrow\;\nabla f(x,y)=\langle y,x\rangle$, $\lambda\nabla g=\langle2\lambda,2\lambda\rangle$. Then $\lambda=\frac{1}{2}y=\frac{1}{2}x$ implies $x=y$ and the rectangle with maximum area is a square with side length $\frac{1}{4}p$.

24. Let $f(x,y,z)=s(s-x)(s-y)(s-z)$, $g(x,y,z)=x+y+z$. Then $\nabla f=\langle-s(s-y)(s-z),-s(s-x)(s-z),-s(s-x)(s-y)\rangle$, $\lambda\nabla g=\langle\lambda,\lambda,\lambda\rangle$. Thus (1) $(s-y)(s-z)=(s-x)(s-z)$ and (2) $(s-x)(s-z)=(s-x)(s-y)$. (1) implies $x=y$ while (2) implies $y=z$, so $x=y=z=p/3$ and the triangle with maximum area is equilateral.

25. Let $f(x,y,z)=d^2=(x-2)^2+(y-1)^2+(z+1)^2$, then we want to minimize f subject to the constraint $g(x,y,z)=x+y-z=1$. $\nabla f=\lambda\nabla g\;\Rightarrow\;\langle2(x-2),2(y-1),2(z+1)\rangle=\lambda\langle1,1,-1\rangle$, so $x=(\lambda+4)/2$, $y=(\lambda+2)/2$, $z=-(\lambda+2)/2$. Substituting into the constraint equation gives $\dfrac{\lambda+4}{2}+\dfrac{\lambda+2}{2}+\dfrac{\lambda+2}{2}=1\;\Rightarrow\;3\lambda+8=2\;\Rightarrow\;\lambda=-2$, so $x=1$, $y=0$, and $z=0$. This must correspond to a minimum, so the shortest distance is $d=\sqrt{(1-2)^2+(0-1)^2+(0+1)^2}=\sqrt{3}$.

26. Let $f(x,y,z) = d^2 = (x-1)^2 + (y-2)^2 + (z-3)^2$, then we want to minimize f subject to the constraint $g(x,y,z) = x - y + z = 4$. $\nabla f = \lambda \nabla g \Rightarrow \langle 2(x-1), 2(y-2), 2(z-3)\rangle = \lambda\langle 1,-1,1\rangle$, so $x = (\lambda+2)/2$, $y = (4-\lambda)/2$, $z = (\lambda+6)/2$. Substituting into the constraint equation gives $\dfrac{\lambda+2}{2} - \dfrac{4-\lambda}{2} + \dfrac{\lambda+6}{2} = 4 \Rightarrow$ $\lambda = \frac{4}{3}$, so $x = \frac{5}{3}$, $y = \frac{4}{3}$, and $z = \frac{11}{3}$. This must correspond to a minimum, so the point on the plane closest to the point $(1,2,3)$ is $\left(\frac{5}{3}, \frac{4}{3}, \frac{11}{3}\right)$.

27. $f(x,y,z) = x^2 + y^2 + z^2$, $g(x,y,z) = z^2 - xy - 1 = 0 \Rightarrow \nabla f = \langle 2x, 2y, 2z\rangle = \lambda\nabla g = \langle -\lambda y, -\lambda x, 2\lambda z\rangle$. Then $2z = 2\lambda z$ implies $z = 0$ or $\lambda = 1$. If $z = 0$ then $g(x,y,z) = 1$ implies $xy = -1$ or $x = -1/y$. Thus $2x = -\lambda y$ and $2y = -\lambda x$ imply $\lambda = 2/y^2 = 2y^2$ or $y = \pm 1$, $x = \pm 1$. If $\lambda = 1$, then $2x = -y$ and $2y = -x$ imply $x = y = 0$, so $z = \pm 1$. Hence the possible points are $(\pm 1, \mp 1, 0)$, $(0,0,\pm 1)$ and the minimum value of f is $f(0,0,\pm 1) = 1$, so the points closest to the origin are $(0,0,\pm 1)$.

28. $f(x,y,z) = x^2 + y^2 + z^2$, $g(x,y,z) = x^2y^2z = 1 \Rightarrow$ $\nabla f = \langle 2x, 2y, 2z\rangle = \lambda\nabla g = \langle 2\lambda xy^2z, 2\lambda x^2yz, \lambda x^2y^2\rangle$. Then $\lambda y^2 z = 1$, $\lambda x^2 z = 1$ and $\lambda x^2y^2 = 2z$ so $y^2z = x^2z$ and $x = \pm y$. Also $2z/1 = \lambda x^2y^2/(\lambda x^2 z)$ so $2z^2 = y^2$ and $y = \pm\sqrt{2}\,z$. But $x^2y^2z = 1$ implies $z > 0$ and $4z^5 = 1$. Thus the points are $(\pm 2^{1/10}, \pm 2^{1/10}, 2^{-2/5})$, and the minimum distance is attained at each of these.

29. $f(x,y,z) = xyz$, $g(x,y,z) = x + y + z = 100 \Rightarrow \nabla f = \langle yz, xz, xy\rangle = \lambda\nabla g = \langle\lambda,\lambda,\lambda\rangle$. Then $\lambda = yz = xz = xy$ implies $x = y = z = \frac{100}{3}$.

30. $f(x,y,z) = x^a y^b z^c$, $g(x,y,z) = x + y + z = 100 \Rightarrow$ $\nabla f = \left\langle ax^{a-1}y^bz^c, bx^ay^{b-1}z^c, cx^ay^bz^{c-1}\right\rangle = \lambda\nabla g = \langle\lambda,\lambda,\lambda\rangle$. Then $\lambda = ax^{a-1}y^bz^c = bx^ay^{b-1}z^c = cx^ay^bz^{c-1}$ or $ayz = bxz = cxy$. Thus $x = \dfrac{ay}{b}$, $z = \dfrac{cy}{b}$, and $\dfrac{ay}{b} + y + \dfrac{cy}{b} = 100$ implies that $y = \dfrac{100b}{a+b+c}$, $x = \dfrac{100a}{a+b+c}$ and $z = \dfrac{100c}{a+b+c}$ gives the maximum.

31. If the dimensions are $2x$, $2y$ and $2z$, then $f(x,y,z) = 8xyz$ and $g(x,y,z) = 9x^2 + 36y^2 + 4z^2 = 36 \Rightarrow$ $\nabla f = \langle 8yz, 8xz, 8xy\rangle = \lambda\nabla g = \langle 18\lambda x, 72\lambda y, 8\lambda z\rangle$. Thus $18\lambda x = 8yz$, $72\lambda y = 8xz$, $8\lambda z = 8xy$ so $x^2 = 4y^2$, $z^2 = 9y^2$ and $36y^2 + 36y^2 + 36y^2 = 36$ or $y = \frac{1}{\sqrt{3}}$ $(y > 0)$. Thus the volume of the largest such rectangle is $8\left(\frac{1}{\sqrt{3}}\right)\left(\frac{2}{\sqrt{3}}\right)\left(\frac{3}{\sqrt{3}}\right) = 16\sqrt{3}$.

32. $f(x,y,z) = 8xyz$, $g(x,y,z) = b^2c^2x + a^2c^2y^2 + a^2b^2z^2 = a^2b^2c^2 \Rightarrow$ $\nabla f = \langle 8yz, 8xz, 8xy\rangle = \lambda\nabla g = \left\langle 2\lambda b^2c^2x, 2\lambda a^2c^2y, 2\lambda a^2b^2z\right\rangle$. Then $4yz = \lambda b^2c^2x$, $4xz = \lambda a^2c^2y$, $4xy = \lambda a^2b^2z$ imply $\lambda = \dfrac{4yz}{b^2c^2x} = \dfrac{4xz}{a^2c^2y} = \dfrac{4xy}{a^2b^2z}$ or $\dfrac{y}{b^2x} = \dfrac{x}{a^2y}$ and $\dfrac{z}{c^2y} = \dfrac{y}{b^2z}$. Thus $x = \dfrac{ay}{b}$, $z = \dfrac{cy}{b}$, and $a^2c^2y^2 + c^2a^2y^2 + a^2c^2y^2 = a^2b^2c^2$, or $y = \dfrac{b}{\sqrt{3}}$, $x = \dfrac{a}{\sqrt{3}}$, $z = \dfrac{c}{\sqrt{3}}$ and the volume is $\frac{8}{3\sqrt{3}}\,abc$.

33. $f(x,y,z) = xyz$, $g(x,y,z) = x + 2y + 3z = 6 \quad \Rightarrow \quad \nabla f = \langle yz, xz, xy \rangle = \lambda \nabla g = \langle \lambda, 2\lambda, 3\lambda \rangle$.
Then $\lambda = yz = \frac{1}{2}xz = \frac{1}{3}xy$ implies $x = 2y$, $z = \frac{2}{3}y$. But $2y + 2y + 2y = 6$ so $y = 1$, $x = 2$, $z = \frac{2}{3}$ and the volume is $V = \frac{4}{3}$.

34. $f(x,y,z) = xyz$, $g(x,y,z) = xy + yz + xz = 32 \quad \Rightarrow$
$\nabla f = \langle yz, xz, xy \rangle = \lambda \nabla g = \langle \lambda(y+z), \lambda(x+z), \lambda(x+y) \rangle$. Then (1) $\lambda(y+z) = yz$, (2) $\lambda(x+z) = xz$ and (3) $\lambda(x+y) = xy$. And (1) minus (2) implies $\lambda(y-x) = z(y-x)$ so $x = y$ or $\lambda = z$. If $\lambda = z$, then (1) implies $z(y+z) = yz$ or $z = 0$ which is false. Thus $x = y$. Similarly (2) minus (3) implies $\lambda(z-y) = x(z-y)$ so $y = z$ or $\lambda = x$. As above, $\lambda \neq x$, so $x = y = z$ and $3x^2 = 32$ or $x = y = z = \frac{8}{\sqrt{6}}$ cm.

35. $f(x,y,z) = xyz$, $g(x,y,z) = 4(x+y+z) = c \quad \Rightarrow \quad \nabla f = \langle yz, xz, xy \rangle$, $\lambda \nabla g = \langle 4\lambda, 4\lambda, 4\lambda \rangle$. Thus $4\lambda = yz = xz = xy$ or $x = y = z = \frac{1}{12}c$ are the dimensions giving the maximum volume.

36. $C(x,y,z) = 5xy + 2xz + 2yz$, $g(x,y,z) = xyz = V \quad \Rightarrow$
$\nabla C = \langle 5y + 2z, 5x + 2z, 2x + 2y \rangle = \lambda \nabla g = \langle \lambda yz, \lambda xz, \lambda xy \rangle$. Then (1) $\lambda yz = 5y + 2z$, (2) $\lambda xz = 5x + 2z$, (3) $\lambda xy = 2(x+y)$ and (4) $xyz = V$. Now (1)–(2) implies $\lambda z(y-x) = 5(y-x)$, so $x = y$ or $\lambda = 5/z$, but z can't be 0, so $x = y$. Then twice (2) minus five times (3) together with $x = y$ implies $\lambda y(2x - 5y) = 2(2z - 5y)$ which gives $z = \frac{5}{2}y$ [again $\lambda \neq 2/y$ or else (3) implies $y = 0$]. Hence $\frac{5}{2}y^3 = V$ and the dimensions which minimize cost are $x = y = \sqrt[3]{\frac{2}{5}V}$ units, $z = V^{1/3}\left(\frac{5}{2}\right)^{2/3}$ units.

37. If the dimensions of the box are given by x, y, and z, then we need to find the maximum value of $f(x,y,z) = xyz$ $(x,y,z > 0)$ subject to the constraint $L = \sqrt{x^2 + y^2 + z^2}$ or $g(x,y,z) = x^2 + y^2 + z^2 = L^2$. $\nabla f = \lambda \nabla g \quad \Rightarrow$ $\langle yz, xz, xy \rangle = \lambda \langle 2x, 2y, 2z \rangle$, so $yz = 2\lambda x \quad \Rightarrow \quad \lambda = \dfrac{yz}{2x}$, $xz = 2\lambda y \quad \Rightarrow \quad \lambda = \dfrac{xz}{2y}$, and $xy = 2\lambda z \quad \Rightarrow$ $\lambda = \dfrac{xy}{2z}$. Thus $\lambda = \dfrac{yz}{2x} = \dfrac{xz}{2y} \quad \Rightarrow \quad x^2 = y^2$ [since $z \neq 0$] $\quad \Rightarrow \quad x = y$ and $\lambda = \dfrac{yz}{2x} = \dfrac{xy}{2z} \quad \Rightarrow \quad x = z$ [since $y \neq 0$]. Substituting into the constraint equation gives $x^2 + x^2 + x^2 = L^2 \quad \Rightarrow \quad x^2 = L^2/3 \quad \Rightarrow$ $x = L/\sqrt{3} = y = z$ and the maximum volume is $\left(L/\sqrt{3}\right)^3 = L^3/(3\sqrt{3})$.

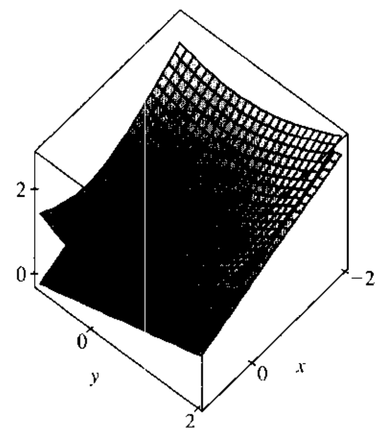
38. Let the dimensions of the box be x, y, and z, so its volume is $f(x,y,z) = xyz$, its surface area is $g(x,y,z) = xy + yz + xz = 750$ and its total edge length is $h(x,y,z) = x + y + z = 50$. Then $\nabla f = \langle yz, xz, xy \rangle = \lambda \nabla g + \mu \nabla h = \langle \lambda(y+z), \lambda(x+z), \lambda(x+y) \rangle + \langle \mu, \mu, \mu \rangle$. So (1) $yz = \lambda(y+z) + \mu$, (2) $xz = \lambda(x+z) + \mu$, and (3) $xy = \lambda(x+y) + \mu$. Notice that the box can't be a cube or else $x = y = z = \frac{50}{3}$ but then $xy + yz + xz = \frac{2500}{3} \neq 750$. Assume x is the distinct side, that is, $x \neq y$, $x \neq z$. Then (1) minus (2) implies $z(y-x) = \lambda(y-x)$ or $\lambda = z$, and (1) minus (3) implies $y(z-x) = \lambda(z-x)$ or $\lambda = y$. So $y = z = \lambda$ and $x + y + z = 50$ implies $x = 50 - 2\lambda$; also $xy + yz + xz = 750$ implies $x(2\lambda) + \lambda^2 = 750$. Hence

$50 - 2\lambda = \dfrac{750 - \lambda^2}{2\lambda}$ or $3\lambda^2 - 100\lambda + 750 = 0$ and $\lambda = \dfrac{50 \pm 5\sqrt{10}}{3}$, giving the points $\left(\frac{1}{3}(50 \mp 10\sqrt{10}), \frac{1}{3}(50 \pm 5\sqrt{10}), \frac{1}{3}(50 \pm 5\sqrt{10})\right)$. Thus the minimum of f is $f\left(\frac{1}{3}(50 - 10\sqrt{3}), \frac{1}{3}(50 + 5\sqrt{10}), \frac{1}{3}(50 + 5\sqrt{10})\right) = \frac{1}{27}(87{,}500 - 2500\sqrt{10})$, and its maximum is $f\left(\frac{1}{3}(50 + 10\sqrt{10}), \frac{1}{3}(50 - 5\sqrt{10}), \frac{1}{3}(50 - 5\sqrt{10})\right) = \frac{1}{27}(87{,}500 + 2500\sqrt{10})$.

Note: If either y or z is the distinct side, then symmetry gives the same result.

39. We need to find the extreme values of $f(x, y, z) = x^2 + y^2 + z^2$ subject to the two constraints $g(x, y, z) = x + y + 2z = 2$ and $h(x, y, z) = x^2 + y^2 - z = 0$. $\nabla f = \langle 2x, 2y, 2z\rangle$, $\lambda \nabla g = \langle \lambda, \lambda, 2\lambda\rangle$ and $\mu \nabla h = \langle 2\mu x, 2\mu y, -\mu\rangle$. Thus we need (1) $2x = \lambda + 2\mu x$, (2) $2y = \lambda + 2\mu y$, (3) $2z = 2\lambda - \mu$, (4) $x + y + 2z = 2$, and (5) $x^2 + y^2 - z = 0$. From (1) and (2), $2(x - y) = 2\mu(x - y)$, so if $x \neq y$, $\mu = 1$. Putting this in (3) gives $2z = 2\lambda - 1$ or $\lambda = z + \frac{1}{2}$, but putting $\mu = 1$ into (1) says $\lambda = 0$. Hence $z + \frac{1}{2} = 0$ or $z = -\frac{1}{2}$. Then (4) and (5) become $x + y - 3 = 0$ and $x^2 + y^2 + \frac{1}{2} = 0$. The last equation cannot be true, so this case gives no solution. So we must have $x = y$. Then (4) and (5) become $2x + 2z = 2$ and $2x^2 - z = 0$ which imply $z = 1 - x$ and $z = 2x^2$. Thus $2x^2 = 1 - x$ or $2x^2 + x - 1 = (2x - 1)(x + 1) = 0$ so $x = \frac{1}{2}$ or $x = -1$. The two points to check are $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$ and $(-1, -1, 2)$: $f\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right) = \frac{3}{4}$ and $f(-1, -1, 2) = 6$. Thus $\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$ is the point on the ellipse nearest the origin and $(-1, -1, 2)$ is the one farthest from the origin.

40. (a) Parametric equations for the ellipse are easiest to determine using cylindrical coordinates. The cone is given by $z = r$, and the plane is $4r\cos\theta - 3r\sin\theta + 8z = 5$. Substituting $z = r$ into the plane equation gives $4r\cos\theta - 3r\sin\theta + 8r = 5 \;\Rightarrow\; r = \dfrac{5}{4\cos\theta - 3\sin\theta + 8}$. Since $z = r$ on the ellipse, parametric equations (in cylindrical coordinates) are $\theta = t$, $r = z = \dfrac{5}{4\cos t - 3\sin t + 8}$, $0 \le t \le 2\pi$.



(b) We need to find the extreme values of $f(x, y, z) = z$ subject to the two constraints $g(x, y, z) = 4x - 3y + 8z = 5$ and $h(x, y, z) = x^2 + y^2 - z^2 = 0$. $\nabla f = \lambda \nabla g + \mu \nabla h \;\Rightarrow$ $\langle 0, 0, 1\rangle = \lambda\langle 4, -3, 8\rangle + \mu\langle 2x, 2y, -2z\rangle$, so we need (1) $4\lambda + 2\mu x = 0 \;\Rightarrow\; x = -\frac{2\lambda}{\mu}$, (2) $-3\lambda + 2\mu y = 0 \;\Rightarrow\; y = \frac{3\lambda}{2\mu}$, (3) $8\lambda - 2\mu z = 1 \;\Rightarrow\; z = \frac{8\lambda - 1}{2\mu}$, (4) $4x - 3y + 8z = 5$, and (5) $x^2 + y^2 = z^2$. [Note that $\mu \neq 0$, else $\lambda = 0$ from (1), but substitution into (3) gives a contradiction.] Substituting (1), (2), and (3) into (4) gives $4\left(-\frac{2\lambda}{\mu}\right) - 3\left(\frac{3\lambda}{2\mu}\right) + 8\left(\frac{8\lambda - 1}{2\mu}\right) = 5 \;\Rightarrow\; \mu = \frac{39\lambda - 8}{10}$ and into (5) gives $\left(-\frac{2\lambda}{\mu}\right)^2 + \left(\frac{3\lambda}{2\mu}\right)^2 = \left(\frac{8\lambda - 1}{2\mu}\right)^2 \;\Rightarrow\; 16\lambda^2 + 9\lambda^2 = (8\lambda - 1)^2 \;\Rightarrow\; 39\lambda^2 - 16\lambda + 1 = 0 \;\Rightarrow$

$\lambda = \frac{1}{13}$ or $\lambda = \frac{1}{3}$. If $\lambda = \frac{1}{13}$ then $\mu = -\frac{1}{2}$ and $x = \frac{4}{13}$, $y = -\frac{3}{13}$, $z = \frac{5}{13}$. If $\lambda = \frac{1}{3}$ then $\mu = \frac{1}{2}$ and $x = -\frac{4}{3}$, $y = 1$, $z = \frac{5}{3}$. Thus the highest point on the ellipse is $\left(-\frac{4}{3}, 1, \frac{5}{3}\right)$ and the lowest point is $\left(\frac{4}{13}, -\frac{3}{13}, \frac{5}{13}\right)$.

41. $f(x,y,z) = ye^{x-z}$, $g(x,y,z) = 9x^2 + 4y^2 + 36z^2 = 36$, $h(x,y,z) = xy + yz = 1$.

$\nabla f = \lambda \nabla g + \mu \nabla h \;\Rightarrow\; \langle ye^{x-z}, e^{x-z}, -ye^{x-z}\rangle = \lambda\langle 18x, 8y, 72z\rangle + \mu\langle y, x+z, y\rangle$, so $ye^{x-z} = 18\lambda x + \mu y$, $e^{x-z} = 8\lambda y + \mu(x+z)$, $-ye^{x-z} = 72\lambda z + \mu y$, $9x^2 + 4y^2 + 36z^2 = 36$, $xy + yz = 1$. Using a CAS to solve these 5 equations simultaneously for x, y, z, λ, and μ (in Maple, use the `allvalues` command), we get 4 real-valued solutions:

$$x \approx 0.222444, \quad y \approx -2.157012, \quad z \approx -0.686049, \quad \lambda \approx -0.200401, \quad \mu \approx 2.108584$$

$$x \approx -1.951921, \quad y \approx -0.545867, \quad z \approx 0.119973, \quad \lambda \approx 0.003141, \quad \mu \approx -0.076238$$

$$x \approx 0.155142, \quad y \approx 0.904622, \quad z \approx 0.950293, \quad \lambda \approx -0.012447, \quad \mu \approx 0.489938$$

$$x \approx 1.138731, \quad y \approx 1.768057, \quad z \approx -0.573138, \quad \lambda \approx 0.317141, \quad \mu \approx 1.862675$$

Substituting these values into f gives $f(0.222444, -2.157012, -0.686049) \approx -5.3506$, $f(-1.951921, -0.545867, 0.119973) \approx -0.0688$, $f(0.155142, 0.904622, 0.950293) \approx 0.4084$, $f(1.138731, 1.768057, -0.573138) \approx 9.7938$. Thus the maximum is approximately 9.7938, and the mininum is approximately -5.3506.

42. $f(x,y,z) = x + y + z$, $g(x,y,z) = x^2 - y^2 - z = 0$, $h(x,y,z) = x^2 + z^2 = 4$.

$\nabla f = \lambda \nabla g + \mu \nabla h \;\Rightarrow\; \langle 1,1,1\rangle = \lambda\langle 2x, -2y, -1\rangle + \mu\langle 2x, 0, 2z\rangle$, so $1 = 2\lambda x + 2\mu x$, $1 = -2\lambda y$, $1 = -\lambda + 2\mu z$, $x^2 - y^2 = z$, $x^2 + z^2 = 4$. Using a CAS to solve these 5 equations simultaneously for x, y, z, λ, and μ, we get 4 real-valued solutions:

$$x \approx -1.652878, \quad y \approx -1.964194, \quad z \approx -1.126052, \quad \lambda \approx 0.254557, \quad \mu \approx -0.557060$$

$$x \approx -1.502800, \quad y \approx 0.968872, \quad z \approx 1.319694, \quad \lambda \approx -0.516064, \quad \mu \approx 0.183352$$

$$x \approx -0.992513, \quad y \approx 1.649677, \quad z \approx -1.736352, \quad \lambda \approx -0.303090, \quad \mu \approx -0.200682$$

$$x \approx 1.895178, \quad y \approx 1.718347, \quad z \approx 0.638984, \quad \lambda \approx -0.290977, \quad \mu \approx 0.554805$$

Substituting these values into f gives $f(-1.652878, -1.964194, -1.126052) \approx -4.7431$, $f(-1.502800, 0.968872, 1.319694) \approx 0.7858$, $f(-0.992513, 1.649677, -1.736352) \approx -1.0792$, $f(1.895178, 1.718347, 0.638984) \approx 4.2525$. Thus the maximum is approximately 4.2525, and the mininum is approximately -4.7431.

43. (a) We wish to maximize $f(x_1, x_2, \ldots, x_n) = \sqrt[n]{x_1 x_2 \cdots x_n}$ subject to

$g(x_1, x_2, \ldots, x_n) = x_1 + x_2 + \cdots + x_n = c$ and $x_i > 0$.

$$\nabla f = \left\langle \tfrac{1}{n}(x_1 x_2 \cdots x_n)^{\frac{1}{n}-1}(x_2 \cdots x_n), \tfrac{1}{n}(x_1 x_2 \cdots x_n)^{\frac{1}{n}-1}(x_1 x_3 \cdots x_n), \ldots, \tfrac{1}{n}(x_1 x_2 \cdots x_n)^{\frac{1}{n}-1}(x_1 \cdots x_{n-1}) \right\rangle$$

and $\lambda\nabla g = \langle\lambda, \lambda, \ldots, \lambda\rangle$, so we need to solve the system of equations

$$\frac{1}{n}(x_1x_2\cdots x_n)^{\frac{1}{n}-1}(x_2\cdots x_n) = \lambda \quad\Rightarrow\quad x_1^{1/n}x_2^{1/n}\cdots x_n^{1/n} = n\lambda x_1$$

$$\frac{1}{n}(x_1x_2\cdots x_n)^{\frac{1}{n}-1}(x_1x_3\cdots x_n) = \lambda \quad\Rightarrow\quad x_1^{1/n}x_2^{1/n}\cdots x_n^{1/n} = n\lambda x_2$$

$$\vdots$$

$$\frac{1}{n}(x_1x_2\cdots x_n)^{\frac{1}{n}-1}(x_1\cdots x_{n-1}) = \lambda \quad\Rightarrow\quad x_1^{1/n}x_2^{1/n}\cdots x_n^{1/n} = n\lambda x_n$$

This implies $n\lambda x_1 = n\lambda x_2 = \cdots = n\lambda x_n$. Note $\lambda \neq 0$, otherwise we can't have all $x_i > 0$. Thus $x_1 = x_2 = \cdots = x_n$. But $x_1 + x_2 + \cdots + x_n = c \;\Rightarrow\; nx_1 = c \;\Rightarrow\; x_1 = \frac{c}{n} = x_2 = x_3 = \cdots = x_n$. Then the only point where f can have an extreme value is $\left(\frac{c}{n}, \frac{c}{n}, \ldots, \frac{c}{n}\right)$. Since we can choose values for $(x_1, x_2, \ldots, x_n)$ that make f as close to zero (but not equal) as we like, f has no minimum value. Thus the maximum value is $f\left(\frac{c}{n}, \frac{c}{n}, \ldots, \frac{c}{n}\right) = \sqrt[n]{\frac{c}{n}\cdot\frac{c}{n}\cdot\cdots\cdot\frac{c}{n}} = \frac{c}{n}$.

(b) From part (a), $\frac{c}{n}$ is the maximum value of f. Thus $f(x_1, x_2, \ldots, x_n) = \sqrt[n]{x_1x_2\cdots x_n} \leq \frac{c}{n}$. But $x_1 + x_2 + \cdots + x_n = c$, so $\sqrt[n]{x_1x_2\cdots x_n} \leq \frac{x_1 + x_2 + \cdots + x_n}{n}$. These two means are equal when f attains its maximum value $\frac{c}{n}$, but this can occur only at the point $\left(\frac{c}{n}, \frac{c}{n}, \ldots, \frac{c}{n}\right)$ we found in part (a). So the means are equal only when $x_1 = x_2 = x_3 = \cdots = x_n = \frac{c}{n}$.

44. (a) Let $f(x_1, \ldots, x_n, y_1, \ldots, y_n) = \sum_{i=1}^{n} x_iy_i$, $g(x_1, \ldots, x_n) = \sum_{i=1}^{n} x_i^2$, and $h(x_1, \ldots, x_n) = \sum_{i=1}^{n} y_i^2$. Then $\nabla f = \nabla\sum_{i=1}^{n} x_iy_i = \langle y_1, y_2, \ldots, y_n, x_1, x_2, \ldots, x_n\rangle$, $\nabla g = \nabla\sum_{i=1}^{n} x_i^2 = \langle 2x_1, 2x_2, \ldots, 2x_n, 0, 0, \ldots, 0\rangle$ and $\nabla h = \nabla\sum_{i=1}^{n} y_i^2 = \langle 0, 0, \ldots, 0, 2y_1, 2y_2, \ldots, 2y_n\rangle$. So $\nabla f = \lambda\nabla g + \mu\nabla h \;\Leftrightarrow\; y_i = 2\lambda x_i$ and $x_i = 2\mu y_i$, $1 \leq i \leq n$. Then $1 = \sum_{i=1}^{n} y_i^2 = \sum_{i=1}^{n} 4\lambda^2 x_i^2 = 4\lambda^2\sum_{i=1}^{n} x_i^2 = 4\lambda^2 \;\Rightarrow\; \lambda = \pm\frac{1}{2}$.

If $\lambda = \frac{1}{2}$ then $y_i = 2\left(\frac{1}{2}\right)x_i = x_i$, $1 \leq i \leq n$. Thus $\sum_{i=1}^{n} x_iy_i = \sum_{i=1}^{n} x_i^2 = 1$. Similarly if $\lambda = -\frac{1}{2}$ we get $y_i = -x_i$ and $\sum_{i=1}^{n} x_iy_i = -1$. Similarly we get $\mu = \pm\frac{1}{2}$ giving $y_i = \pm x_i$, $1 \leq i \leq n$, and $\sum_{i=1}^{n} x_iy_i = \pm 1$.

Thus the maximum value of $\sum_{i=1}^{n} x_iy_i$ is 1.

(b) Here we assume $\sum_{i=1}^{n} a_i^2 \neq 0$ and $\sum_{i=1}^{n} b_i^2 \neq 0$. (If $\sum_{i=1}^{n} a_i^2 = 0$, then each $a_i = 0$ and so the inequality is trivially true.) $x_i = \frac{a_i}{\sqrt{\sum a_j^2}} \;\Rightarrow\; \sum x_i^2 = \frac{\sum a_i^2}{\sum a_j^2} = 1$, and $y_i = \frac{b_i}{\sqrt{\sum b_j^2}} \;\Rightarrow\; \sum y_i^2 = \frac{\sum b_i^2}{\sum b_j^2} = 1$. Therefore, from (a), $\sum x_iy_i = \sum\frac{a_ib_i}{\sqrt{\sum a_j^2}\sqrt{\sum b_j^2}} \leq 1 \;\Leftrightarrow\; \sum a_ib_i \leq \sqrt{\sum a_j^2}\sqrt{\sum b_j^2}$.

APPLIED PROJECT Rocket Science

1. Initially the rocket engine has mass $M_r = M_1$ and payload mass $P = M_2 + M_3 + A$. Then the change in velocity resulting from the first stage is $\Delta V_1 = -c \ln\left(1 - \dfrac{(1-S)M_1}{M_2 + M_3 + A + M_1}\right)$. After the first stage is jettisoned we can consider the rocket engine to have mass $M_r = M_2$ and the payload to have mass $P = M_3 + A$. The resulting change in velocity from the second stage is $\Delta V_2 = -c \ln\left(1 - \dfrac{(1-S)M_2}{M_3 + A + M_2}\right)$. When only the third stage remains, we have $M_r = M_3$ and $P = A$, so the resulting change in velocity is $\Delta V_3 = -c \ln\left(1 - \dfrac{(1-S)M_3}{A + M_3}\right)$. Since the rocket started from rest, the final velocity attained is

$$\begin{aligned}
v_f &= \Delta V_1 + \Delta V_2 + \Delta V_3 \\
&= -c \ln\left(1 - \frac{(1-S)M_1}{M_2 + M_3 + A + M_1}\right) + (-c)\ln\left(1 - \frac{(1-S)M_2}{M_3 + A + M_2}\right) \\
&\qquad + (-c)\ln\left(1 - \frac{(1-S)M_3}{A + M_3}\right) \\
&= -c\left[\ln\left(\frac{M_1 + M_2 + M_3 + A - (1-S)M_1}{M_1 + M_2 + M_3 + A}\right) + \ln\left(\frac{M_2 + M_3 + A - (1-S)M_2}{M_2 + M_3 + A}\right)\right. \\
&\qquad \left. + \ln\left(\frac{M_3 + A - (1-S)M_3}{M_3 + A}\right)\right] \\
&= c\left[\ln\left(\frac{M_1 + M_2 + M_3 + A}{SM_1 + M_2 + M_3 + A}\right) + \ln\left(\frac{M_2 + M_3 + A}{SM_2 + M_3 + A}\right) + \ln\left(\frac{M_3 + A}{SM_3 + A}\right)\right]
\end{aligned}$$

2. Define $N_1 = \dfrac{M_1 + M_2 + M_3 + A}{SM_1 + M_2 + M_3 + A}$, $N_2 = \dfrac{M_2 + M_3 + A}{SM_2 + M_3 + A}$, and $N_3 = \dfrac{M_3 + A}{SM_3 + A}$. Then

$$\begin{aligned}
\frac{(1-S)N_1}{1 - SN_1} &= \frac{(1-S)\dfrac{M_1 + M_2 + M_3 + A}{SM_1 + M_2 + M_3 + A}}{1 - S\dfrac{M_1 + M_2 + M_3 + A}{SM_1 + M_2 + M_3 + A}} \\
&= \frac{(1-S)(M_1 + M_2 + M_3 + A)}{SM_1 + M_2 + M_3 + A - S(M_1 + M_2 + M_3 + A)} \\
&= \frac{(1-S)(M_1 + M_2 + M_3 + A)}{(1-S)(M_2 + M_3 + A)} = \frac{M_1 + M_2 + M_3 + A}{M_2 + M_3 + A}
\end{aligned}$$

as desired.

Similarly,

$$\frac{(1-S)N_2}{1 - SN_2} = \frac{(1-S)(M_2 + M_3 + A)}{SM_2 + M_3 + A - S(M_2 + M_3 + A)} = \frac{(1-S)(M_2 + M_3 + A)}{(1-S)(M_3 + A)} = \frac{M_2 + M_3 + A}{M_3 + A}$$

and

$$\frac{(1-S)N_3}{1 - SN_3} = \frac{(1-S)(M_3 + A)}{SM_3 + A - S(M_3 + A)} = \frac{(1-S)(M_3 + A)}{(1-S)(A)} = \frac{M_3 + A}{A}$$

Then

$$\frac{M+A}{A}=\frac{M_1+M_2+M_3+A}{A}=\frac{M_1+M_2+M_3+A}{M_2+M_3+A}\cdot\frac{M_2+M_3+A}{M_3+A}\cdot\frac{M_3+A}{A}$$
$$=\frac{(1-S)N_1}{1-SN_1}\cdot\frac{(1-S)N_2}{1-SN_2}\cdot\frac{(1-S)N_3}{1-SN_3}=\frac{(1-S)^3N_1N_2N_3}{(1-SN_1)(1-SN_2)(1-SN_3)}$$

3. Since $A>0$, $M+A$ and consequently $\dfrac{M+A}{A}$ is minimized for the same values as M. $\ln x$ is a strictly increasing function, so $\ln\left(\dfrac{M+A}{A}\right)$ must give a minimum for the same values as $\dfrac{M+A}{A}$ and hence M. We then wish to minimize $\ln\left(\dfrac{M+A}{A}\right)$ subject to the constraint $c\,(\ln N_1+\ln N_2+\ln N_3)=v_f$. From Problem 2,

$$\ln\left(\frac{M+A}{A}\right)=\ln\left(\frac{(1-S)^3\,N_1N_2N_3}{(1-SN_1)\,(1-SN_2)\,(1-SN_3)}\right)$$
$$=3\ln(1-S)+\ln N_1+\ln N_2+\ln N_3-\ln(1-SN_1)-\ln(1-SN_2)-\ln(1-SN_3)$$

Using the method of Lagrange multipliers, we need to solve $\nabla\left[\ln\left(\dfrac{M+A}{A}\right)\right]=\lambda\nabla[c(\ln N_1+\ln N_2+\ln N_3)]$ with $c(\ln N_1+\ln N_2+\ln N_3)=v_f$ in terms of N_1, N_2, and N_3. The resulting system is

$$\frac{1}{N_1}+\frac{S}{1-SN_1}=\lambda\frac{c}{N_1}\qquad\frac{1}{N_2}+\frac{S}{1-SN_2}=\lambda\frac{c}{N_2}\qquad\frac{1}{N_3}+\frac{S}{1-SN_3}=\lambda\frac{c}{N_3}$$
$$c\,(\ln N_1+\ln N_2+\ln N_3)=v_f$$

One approach to solving the system is isolating $c\lambda$ in the first three equations which gives $1+\dfrac{SN_1}{1-SN_1}=c\lambda=1+\dfrac{SN_2}{1-SN_2}=1+\dfrac{SN_3}{1-SN_3}\Rightarrow\dfrac{N_1}{1-SN_1}=\dfrac{N_2}{1-SN_2}=\dfrac{N_3}{1-SN_3}\Rightarrow$ $N_1=N_2=N_3$ (Verify!). This says the fourth equation can be expressed as $c(\ln N_1+\ln N_1+\ln N_1)=v_f\Rightarrow$ $3c\ln N_1=v_f\Rightarrow\ln N_1=\dfrac{v_f}{3c}$. Thus the minimum mass M of the rocket engine is attained for $N_1=N_2=N_3=e^{v_f/(3c)}$.

4. Using the previous results,

$$\frac{M+A}{A}=\frac{(1-S)^3N_1N_2N_3}{(1-SN_1)(1-SN_2)(1-SN_3)}=\frac{(1-S)^3\left[e^{v_f/(3c)}\right]^3}{\left[1-Se^{v_f/(3c)}\right]^3}=\frac{(1-S)^3e^{v_f/c}}{\left[1-Se^{v_f/(3c)}\right]^3}.$$ Then

$$M=\frac{A(1-S)^3e^{v_f/c}}{\left[1-Se^{v_f/(3c)}\right]^3}-A.$$

5. (a) From Problem 4, $M=\dfrac{A(1-0.2)^3e^{(17{,}500/6000)}}{\left(1-0.2e^{[17{,}500/(3\cdot 6000)]}\right)^3}-A\approx 90.4A-A=89.4A.$

(b) First, $N_3=\dfrac{M_3+A}{SM_3+A}\Rightarrow e^{[17{,}500/(3\cdot6000)]}=\dfrac{M_3+A}{0.2M_3+A}\Rightarrow M_3=\dfrac{A(1-e^{35/36})}{0.2e^{35/36}-1}\approx 3.49A$. Then

$N_2=\dfrac{M_2+M_3+A}{SM_2+M_3+A}=\dfrac{M_2+3.49A+A}{0.2M_2+3.49A+A}\Rightarrow M_2=\dfrac{4.49A(1-e^{35/36})}{0.2e^{35/36}-1}\approx 15.67A$

and $N_3=\dfrac{M_1+M_2+M_3+A}{SM_1+M_2+M_3+A}=\dfrac{M_1+15.67A+3.49A+A}{0.2M_1+15.67A+3.49A+A}\Rightarrow$

$M_1=\dfrac{20.16A(1-e^{35/36})}{0.2e^{35/36}-1}\approx 70.36A.$

6. As in Problem 5, $N_3 = \dfrac{M_3 + A}{SM_3 + A} \Rightarrow e^{24.700/(3 \cdot 6000)} = \dfrac{M_3 + A}{0.2M_3 + A} \Rightarrow$

$M_3 = \dfrac{A(1 - e^{247/180})}{0.2e^{247/180} - 1} \approx 13.9A$, $N_2 = \dfrac{M_2 + M_3 + A}{SM_2 + M_3 + A} = \dfrac{M_2 + 13.9A + A}{0.2M_2 + 13.9A + A} \Rightarrow$

$M_2 = \dfrac{14.9A(1 - e^{247/180})}{0.2e^{247/180} - 1} \approx 208A$, and $N_3 = \dfrac{M_1 + M_2 + M_3 + A}{SM_1 + M_2 + M_3 + A} = \dfrac{M_1 + 208A + 13.9A + A}{0.2M_1 + 208A + 13.9A + A} \Rightarrow$

$M_1 = \dfrac{222.9A(1 - e^{247/180})}{0.2e^{247/180} - 1} \approx 3110A$. Here $A = 500$, so the mass of each stage of the rocket engine is approximately $M_1 = 3110(500) = 1{,}550{,}000$ lb, $M_2 = 208(500) = 104{,}000$ lb, and $M_3 = 13.9(500) = 6950$ lb.

APPLIED PROJECT Hydro-Turbine Optimization

1. We wish to maximize the total energy production for a given total flow, so we can say Q_T is fixed and we want to maximize $KW_1 + KW_2 + KW_3$. Notice each KW_i has a constant factor $(170 - 1.6 \cdot 10^{-6} Q_T^2)$, so to simplify the computations we can equivalently maximize

$$\begin{aligned} f(Q_1, Q_2, Q_3) &= \frac{KW_1 + KW_2 + KW_3}{170 - 1.6 \cdot 10^{-6} Q_T^2} \\ &= (-18.89 + 0.1277Q_1 - 4.08 \cdot 10^{-5} Q_1^2) \\ &\quad + (-24.51 + 0.1358Q_2 - 4.69 \cdot 10^{-5} Q_2^2) \\ &\quad + (-27.02 + 0.1380Q_3 - 3.84 \cdot 10^{-5} Q_3^2) \end{aligned}$$

subject to the constraint $g(Q_1, Q_2, Q_3) = Q_1 + Q_2 + Q_3 = Q_T$. So first we find the values of Q_1, Q_2, Q_3 where $\nabla f(Q_1, Q_2, Q_3) = \lambda \nabla g(Q_1, Q_2, Q_3)$ and $Q_1 + Q_2 + Q_3 = Q_T$ which is equivalent to solving the system

$$\begin{aligned} 0.1277 - 2(4.08 \cdot 10^{-5})Q_1 &= \lambda \\ 0.1358 - 2(4.69 \cdot 10^{-5})Q_2 &= \lambda \\ 0.1380 - 2(3.84 \cdot 10^{-5})Q_3 &= \lambda \\ Q_1 + Q_2 + Q_3 &= Q_T \end{aligned}$$

Comparing the first and third equations, we have $0.1277 - 2(4.08 \cdot 10^{-5})Q_1 = 0.1380 - 2(3.84 \cdot 10^{-5})Q_3 \Rightarrow Q_1 = -126.2255 + 0.9412Q_3$. From the second and third equations, $0.1358 - 2(4.69 \cdot 10^{-5})Q_2 = 0.1380 - 2(3.84 \cdot 10^{-5})Q_3 \Rightarrow Q_2 = -23.4542 + 0.8188Q_3$. Substituting into $Q_1 + Q_2 + Q_3 = Q_T$ gives $(-126.2255 + 0.9412Q_3) + (-23.4542 + 0.8188Q_3) + Q_3 = Q_T \Rightarrow 2.76Q_3 = Q_T + 149.6797 \Rightarrow Q_3 = 0.3623Q_T + 54.23$. Then $Q_1 = -126.2255 + 0.9412Q_3 = -126.2255 + 0.9412(0.3623Q_T + 54.23) = 0.3410Q_T - 75.18$ and $Q_2 = -23.4542 + 0.8188(0.3623Q_T + 54.23) = 0.2967Q_T + 20.95$. As long as we maintain $250 \le Q_1 \le 1110$, $250 \le Q_2 \le 1110$, and $250 \le Q_3 \le 1225$, we can reason from the nature of the functions KW_i that these values give a maximum of f, and hence a maximum energy production, and not a minimum.

2. From Problem 1, the value of Q_1 that maximizes energy production is $0.3410Q_T - 75.18$, but since $250 \le Q_1 \le 1110$, we must have $250 \le 0.3410Q_T - 75.18 \le 1110 \quad \Rightarrow \quad 325.18 \le 0.3410Q_T \le 1185.18 \quad \Rightarrow$ $953.6 \le Q_T \le 3475.6$. Similarly, $250 \le Q_2 \le 1110 \quad \Rightarrow \quad 250 \le 0.2967Q_T + 20.95 \le 1110 \quad \Rightarrow$ $772.0 \le Q_T \le 3670.5$, and $250 \le Q_3 \le 1225 \quad \Rightarrow \quad 250 \le 0.3623Q_T + 54.23 \le 1225 \quad \Rightarrow$ $540.4 \le Q_T \le 3231.5$. Consolidating these results, we see that the values from Problem 1 are applicable only for $953.6 \le Q_T \le 3231.5$.

3. If $Q_T = 2500$, the results from Problem 1 show that the maximum energy production occurs for

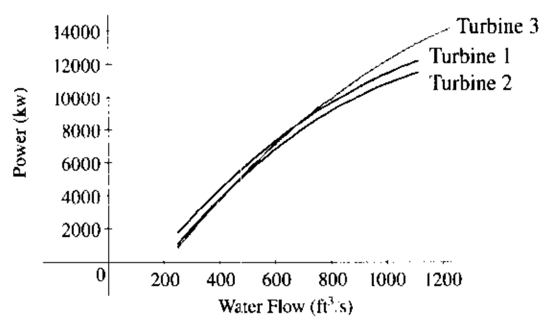
$$Q_1 = 0.3410Q_T - 75.18 = 0.3410(2500) - 75.18 = 777.3$$

$$Q_2 = 0.2967Q_T + 20.95 = 0.2967(2500) + 20.95 = 762.7$$

$$Q_3 = 0.3623Q_T + 54.23 = 0.3623(2500) + 54.23 = 960.0$$

The energy produced for these values is $KW_1 + KW_2 + KW_3 \approx 8915.2 + 8285.1 + 11{,}211.3 \approx 28{,}411.6$. We compute the energy production for a nearby distribution, $Q_1 = 770$, $Q_2 = 760$, and $Q_3 = 970$: $KW_1 + KW_2 + KW_3 \approx 8839.8 + 8257.4 + 11{,}313.5 = 28{,}410.7$. For another example, we take $Q_1 = 780$, $Q_2 = 765$, and $Q_3 = 955$: $KW_1 + KW_2 + KW_3 \approx 8942.9 + 8308.8 + 11{,}159.7 = 28{,}411.4$. These distributions are both close to the distribution from Problem 1 and both give slightly lower energy productions, suggesting that $Q_1 = 777.3$, $Q_2 = 762.7$, and $Q_3 = 960.0$ is indeed the optimal distribution.

4. First we graph each power function in its domain if all of the flow is directed to that turbine (so $Q_i = Q_T$).



If we use only one turbine, the graph indicates that for a water flow of $1000\ \text{ft}^3/\text{s}$, Turbine 3 produces the most power, approximately 12,200 kW. In comparison, if we use all three turbines, the results of Problem 1 with $Q_T = 1000$ give $Q_1 = 265.8$, $Q_2 = 317.7$, and $Q_3 = 416.5$, resulting in a total energy production of $KW_1 + KW_2 + KW_3 \approx 8397.4$ kW. Here, using only one turbine produces significantly more energy! If the flow is only $600\ \text{ft}^3/\text{s}$, we do not have the option of using all three turbines, as the domain restrictions require a minimum of $250\ \text{ft}^3/\text{s}$ in each turbine. We can use just one turbine, then, and from the graph Turbine 1 produces the most energy for a water flow of $600\ \text{ft}^3$.

5. If we examine the graph from Problem 4, we see that for water flows above approximately $450\ \text{ft}^3/\text{s}$, Turbine 2 produces the least amount of power. Therefore it seems reasonable to assume that we should distribute the incoming flow of $1500\ \text{ft}^3/\text{s}$ between Turbines 1 and 3. (This can be verified by computing the power produced with the other pairs of turbines for comparison.) So now we wish to maximize $KW_1 + KW_3$ subject to the constraint $Q_1 + Q_3 = Q_T$ where $Q_T = 1500$.

As in Problem 1, we can equivalently maximize

$$f(Q_1, Q_3) = \frac{KW_1 + KW_3}{170 - 1.6 \cdot 10^{-6} Q_T^2}$$

$$= \left(-18.89 + 0.1277Q_1 - 4.08 \cdot 10^{-5} Q_1^2\right) + \left(-27.02 + 0.1380Q_3 - 3.84 \cdot 10^{-5} Q_3^2\right)$$

subject to the constraint $g(Q_1, Q_3) = Q_1 + Q_3 = Q_T$.

Then we solve $\nabla f(Q_1, Q_3) = \lambda \nabla g(Q_1, Q_3) \Rightarrow 0.1277 - 2\left(4.08 \cdot 10^{-5}\right) Q_1 = \lambda$ and $0.1380 - 2(3.84 \cdot 10^{-5})Q_3 = \lambda$, thus $0.1277 - 2(4.08 \cdot 10^{-5})Q_1 = 0.1380 - 2(3.84 \cdot 10^{-5})Q_3 \Rightarrow Q_1 = -126.2255 + 0.9412Q_3$. Substituting into $Q_1 + Q_3 = Q_T$ gives $-126.2255 + 0.9412Q_3 + Q_3 = 1500 \Rightarrow Q_3 \approx 837.7$, and then $Q_1 = Q_T - Q_3 \approx 1500 - 837.7 = 662.3$. So we should apportion approximately 662.3 ft^3/s to Turbine 1 and the remaining 837.7 ft^3/s to Turbine 3. The resulting energy production is $KW_1 + KW_3 \approx 7952.1 + 10{,}256.2 = 18{,}208.3$ kW. (We can verify that this is indeed a maximum energy production by checking nearby distributions.) In comparison, if we use all three turbines with $Q_T = 1500$ we get $Q_1 = 436.3$, $Q_2 = 466.0$, and $Q_3 = 597.7$, resulting in a total energy production of $KW_1 + KW_2 + KW_3 \approx 16{,}538.7$ kW. Clearly, for this flow level it is beneficial to use only two turbines.

6. Note that an incoming flow of 3400 ft^3/s is not within the domain we established in Problem 2, so we cannot simply use our previous work to give the optimal distribution. We will need to use all three turbines, due to the capacity limitations of each individual turbine, but 3400 is less than the maximum combined capacity of 3445 ft^3/s, so we still must decide how to distribute the flows. From the graph in Problem 4, Turbine 3 produces the most power for the higher flows, so it seems reasonable to use Turbine 3 at its maximum capacity of 1225 and distribute the remaining 2175 ft^3/s flow between Turbines 1 and 2. We can again use the technique of Lagrange multipliers to determine the optimal distribution. Following the procedure we used in Problem 5, we wish to maximize $KW_1 + KW_2$ subject to the constraint $Q_1 + Q_2 = Q_T$ where $Q_T = 2175$. We can equivalently maximize

$$f(Q_1, Q_2) = \frac{KW_1 + KW_2}{170 - 1.6 \cdot 10^{-6} Q_T^2}$$

$$= \left(-18.89 + 0.1277Q_1 - 4.08 \cdot 10^{-5} Q_1^2\right) + \left(-24.51 + 0.1358Q_2 - 4.69 \cdot 10^{-5} Q_2^2\right)$$

subject to the constraint $g(Q_1, Q_2) = Q_1 + Q_2 = Q_T$. Then we solve $\nabla f(Q_1, Q_2) = \lambda \nabla g(Q_1, Q_2) \Rightarrow 0.1277 - 2(4.08 \cdot 10^{-5})Q_1 = \lambda$ and $0.1358 - 2(4.69 \cdot 10^{-5})Q_2 = \lambda$, thus $0.1277 - 2(4.08 \cdot 10^{-5})Q_1 = 0.1358 - 2(4.69 \cdot 10^{-5})Q_2 \Rightarrow Q_1 = -99.2647 + 1.1495Q_2$. Substituting into $Q_1 + Q_2 = Q_T$ gives $-99.2647 + 1.1495Q_2 + Q_2 = 2175 \Rightarrow Q_2 \approx 1058.0$, and then $Q_1 \approx 1117.0$. This value for Q_1 is larger than the allowable maximum flow to Turbine 1, but the result indicates that the flow to Turbine 1 should be maximized. Thus we should recommend that the company apportion the maximum allowable flows to Turbines 1 and 3, 1110 and 1225 ft^3/s, and the remaining 1065 ft^3/s to Turbine 2. Checking nearby distributions within the domain verifies that we have indeed found the optimal distribution.

15 Review ET 14

CONCEPT CHECK

1. (a) A function f of two variables is a rule that assigns to each ordered pair (x, y) of real numbers in its domain a unique real number denoted by $f(x, y)$.

   (b) One way to visualize a function of two variables is by graphing it, resulting in the surface $z = f(x, y)$. Another method for visualizing a function of two variables is a contour map. The contour map consists of level curves of the function which are horizontal traces of the graph of the function projected onto the xy-plane. Also, we can use an arrow diagram such as Figure 1 in Section 15.1 [ET 14.1].

2. A function f of three variables is a rule that assigns to each ordered triple (x, y, z) in its domain a unique real number $f(x, y, z)$. We can visualize a function of three variables by examining its level surfaces $f(x, y, z) = k$, where k is a constant.

3. $\lim_{(x,y)\to(a,b)} f(x, y) = L$ means the values of $f(x, y)$ approach the number L as the point (x, y) approaches the point (a, b) along any path that is within the domain of f. We can show that a limit at a point does not exist by finding two different paths approaching the point along which $f(x, y)$ has different limits.

4. (a) See Definition 15.2.4 [ET 14.2.4].

   (b) If f is continuous on $\mathbb{R}^2$, its graph will appear as a surface without holes or breaks.

5. (a) See (2) and (3) in Section 15.3 [ET 14.3].

   (b) See "Interpretations of Partial Derivatives" on page 948 [ET 912].

   (c) To find f_x, regard y as a constant and differentiate $f(x, y)$ with respect to x. To find f_y, regard x as a constant and differentiate $f(x, y)$ with respect to y.

6. See the statement of Clairaut's Theorem on page 952 [ET 916].

7. (a) See (2) in Section 15.4 [ET 14.4].

   (b) See (19) and the preceding discussion in Section 15.6 [ET 14.6].

8. See (3) and (4) and the accompanying discussion in Section 15.4 [ET 14.4]. We can interpret the linearization of f at (a, b) geometrically as the linear function whose graph is the tangent plane to the graph of f at (a, b). Thus it is the linear function which best approximates f near (a, b).

9. (a) See Definition 15.4.7 [ET 14.4.7].

   (b) Use Theorem 15.4.8 [ET 14.4.8].

10. See (10) and the associated discussion in Section 15.4 [ET 14.4].

11. See (2) and (3) in Section 15.5 [ET 14.5].

12. See (7) and the preceding discussion in Section 15.5 [ET 14.5].

13. (a) See Definition 15.6.2 [ET 14.6.2]. We can interpret it as the rate of change of f at (x_0, y_0) in the direction of $\mathbf{u}$. Geometrically, if P is the point $(x_0, y_0, f(x_0, y_0))$ on the graph of f and C is the curve of intersection of the graph of f with the vertical plane that passes through P in the direction $\mathbf{u}$, the directional derivative of f at (x_0, y_0) in the direction of $\mathbf{u}$ is the slope of the tangent line to C at P. (See Figure 5 in Section 15.6 [ET 14.6].)

    (b) See Theorem 15.6.3 [ET 14.6.3].

14. (a) See (8) and (13) in Section 15.6 [ET 14.6].

    (b) $D_{\mathbf{u}} f(x, y) = \nabla f(x, y) \cdot \mathbf{u}$ or $D_{\mathbf{u}} f(x, y, z) = \nabla f(x, y, z) \cdot \mathbf{u}$

    (c) The gradient vector of a function points in the direction of maximum rate of increase of the function. On a graph of the function, the gradient points in the direction of steepest ascent.

15. (a) f has a local maximum at (a, b) if $f(x, y) \leq f(a, b)$ when (x, y) is near (a, b).

(b) f has an absolute maximum at (a, b) if $f(x, y) \leq f(a, b)$ for all points (x, y) in the domain of f.

(c) f has a local minimum at (a, b) if $f(x, y) \geq f(a, b)$ when (x, y) is near (a, b).

(d) f has an absolute minimum at (a, b) if $f(x, y) \geq f(a, b)$ for all points (x, y) in the domain of f.

(e) f has a saddle point at (a, b) if $f(a, b)$ is a local maximum in one direction but a local minimum in another.

16. (a) By Theorem 15.7.2 [ET 14.7.2], if f has a local maximum at (a, b) and the first-order partial derivatives of f exist there, then $f_x(a, b) = 0$ and $f_y(a, b) = 0$.

(b) A critical point of f is a point (a, b) such that $f_x(a, b) = 0$ and $f_y(a, b) = 0$ or one of these partial derivatives does not exist.

17. See (3) in Section 15.7 [ET 14.7].

18. (a) See Figure 11 and the accompanying discussion in Section 15.7 [ET 14.7].

(b) See Theorem 15.7.8 [ET 14.7.8].

(c) See the procedure outlined in (9) in Section 15.7 [ET 14.7].

19. See the discussion beginning on page 1001 [ET 965]; see the discussion preceding Example 5 in Section 15.8 [ET 14.8].

TRUE-FALSE QUIZ

1. True. $f_y(a, b) = \lim_{h \to 0} \dfrac{f(a, b + h) - f(a, b)}{h}$ from Equation 15.3.3 [ET 14.3.3]. Let $h = y - b$. As $h \to 0$, $y \to b$. Then by substituting, we get $f_y(a, b) = \lim_{y \to b} \dfrac{f(a, y) - f(a, b)}{y - b}$.

2. False. If there were such a function, then $f_{xy} = 2y$ and $f_{yx} = 1$. So $f_{xy} \neq f_{yx}$, which contradicts Clairaut's Theorem.

3. False. $f_{xy} = \dfrac{\partial^2 f}{\partial y\, \partial x}$.

4. True. From Equation 15.6.14 [ET 14.6.14] we get $D_{\mathbf{k}}\, f(x, y, z) = \nabla f(x, y, z) \cdot \langle 0, 0, 1 \rangle = f_z(x, y, z)$.

5. False. See Example 15.2.3 [ET 14.2.3].

6. False. See Exercise 15.4.42(a) [ET 14.4.42(a)].

7. True. If f has a local minimum and f is differentiable at (a, b) then by Theorem 15.7.2 [ET 14.7.2], $f_x(a, b) = 0$ and $f_y(a, b) = 0$, so $\nabla f(a, b) = \langle f_x(a, b), f_y(a, b) \rangle = \langle 0, 0 \rangle = \mathbf{0}$.

8. False. If f is not continuous at $(2, 5)$, then we can have $\lim_{(x,y) \to (2,5)} f(x, y) \neq f(2, 5)$. (See Example 15.2.7 [ET 14.2.7].)

9. False. $\nabla f(x, y) = \langle 0, 1/y \rangle$.

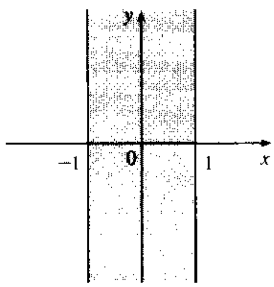
10. True. This is part (c) of the Second Derivatives Test (15.7.3 [ET 14.7.3]).

11. True. $\nabla f = \langle \cos x, \cos y \rangle$, so $|\nabla f| = \sqrt{\cos^2 x + \cos^2 y}$. But $|\cos \theta| \leq 1$, so $|\nabla f| \leq \sqrt{2}$. Now $D_{\mathbf{u}}\, f(x, y) = \nabla f \cdot \mathbf{u} = |\nabla f|\, |\mathbf{u}| \cos \theta$, but $\mathbf{u}$ is a unit vector, so $|D_{\mathbf{u}}\, f(x, y)| \leq \sqrt{2} \cdot 1 \cdot 1 = \sqrt{2}$.

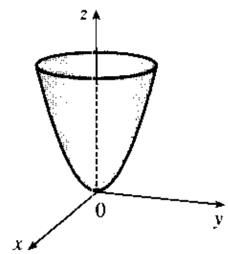
12. False. See Exercise 15.7.35 [14.7.35].

EXERCISES

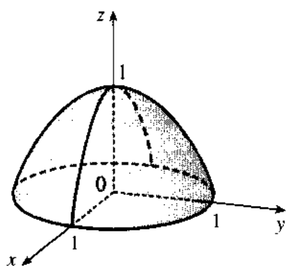
1. The domain of $\sin^{-1} x$ is $-1 \le x \le 1$ while the domain of $\tan^{-1} y$ is all real numbers, so the domain of $f(x,y) = \sin^{-1} x + \tan^{-1} y$ is $\{(x,y) \mid -1 \le x \le 1\}$.



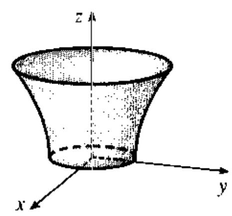
2. $D = \{(x,y,z) \mid z \ge x^2 + y^2\}$, the points on and above the paraboloid $z = x^2 + y^2$.



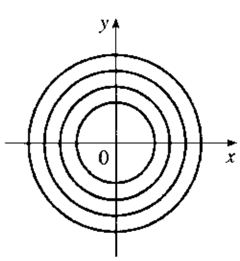
3. $z = f(x,y) = 1 - x^2 - y^2$, a paraboloid with vertex $(0,0,1)$.



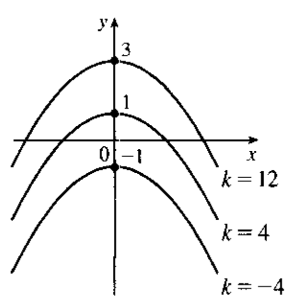
4. $z = f(x,y) = \sqrt{x^2 + y^2 - 1}$, so $z \ge 0$ and $1 = x^2 + y^2 - z^2$. Thus the graph is the upper half of a hyperboloid of one sheet.



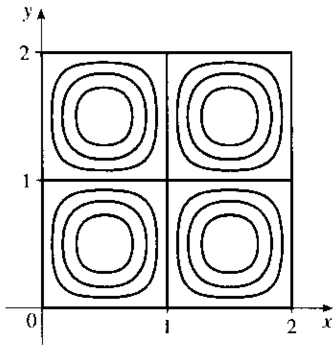
5. Let $k = e^{-c} = e^{-(x^2+y^2)}$ be the level curves. Then $-\ln k = c = x^2 + y^2$, so we have a family of concentric circles.



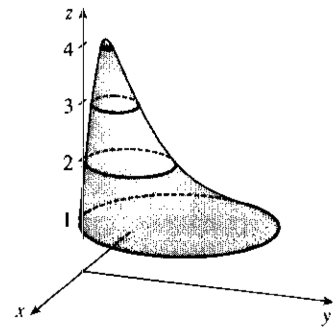
6. $k = x^2 + 4y$ or $4(y - k/4) = -x^2$, a family of parabolas with vertex at $(0, k/4)$.



7.



8.



9. f is a rational function, so it is continuous on its domain. Since f is defined at $(1, 1)$, we use direct substitution to evaluate the limit: $\displaystyle\lim_{(x,y)\to(1,1)} \frac{2xy}{x^2+2y^2} = \frac{2(1)(1)}{1^2+2(1)^2} = \frac{2}{3}$.

10. As $(x, y) \to (0, 0)$ along the x-axis, $f(x, 0) = 0/x^2 = 0$ for $x \neq 0$, so $f(x, y) \to 0$ along this line. But $f(x, x) = 2x^2/(3x^2) = \frac{2}{3}$, so as $(x, y) \to (0, 0)$ along the line $x = y$, $f(x, y) \to \frac{2}{3}$. Thus the limit doesn't exist.

11. (a) $T_x(6, 4) = \displaystyle\lim_{h\to 0} \frac{T(6+h, 4) - T(6, 4)}{h}$, so we can approximate $T_x(6, 4)$ by considering $h = \pm 2$ and using the values given in the table: $T_x(6, 4) \approx \dfrac{T(8, 4) - T(6, 4)}{2} = \dfrac{86 - 80}{2} = 3$,

$T_x(6, 4) \approx \dfrac{T(4, 4) - T(6, 4)}{-2} = \dfrac{72 - 80}{-2} = 4$. Averaging these values, we estimate $T_x(6, 4)$ to be approximately $3.5°\text{C}/\text{m}$. Similarly, $T_y(6, 4) = \displaystyle\lim_{h\to 0} \frac{T(6, 4+h) - T(6, 4)}{h}$, which we can approximate with $h = \pm 2$: $T_y(6, 4) \approx \dfrac{T(6, 6) - T(6, 4)}{2} = \dfrac{75 - 80}{2} = -2.5$,

$T_y(6, 4) \approx \dfrac{T(6, 2) - T(6, 4)}{-2} = \dfrac{87 - 80}{-2} = -3.5$. Averaging these values, we estimate $T_y(6, 4)$ to be approximately $-3.0°\text{C}/\text{m}$.

(b) Here $\mathbf{u} = \left\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle$, so by Equation 15.6.9 [ET 14.6.9],

$D_{\mathbf{u}}\, T(6, 4) = \nabla T(6, 4) \cdot \mathbf{u} = T_x(6, 4)\, \frac{1}{\sqrt{2}} + T_y(6, 4)\, \frac{1}{\sqrt{2}}$. Using our estimates from part (a), we have $D_{\mathbf{u}}\, T(6, 4) \approx (3.5)\, \frac{1}{\sqrt{2}} + (-3.0)\, \frac{1}{\sqrt{2}} = \frac{1}{2\sqrt{2}} \approx 0.35$. This means that as we move through the point $(6, 4)$ in the direction of $\mathbf{u}$, the temperature increases at a rate of approximately $0.35°\text{C}/\text{m}$.

Alternatively, we can use Definition 15.6.2 [ET 14.6.2]:

$D_{\mathbf{u}}\, T(6, 4) = \displaystyle\lim_{h\to 0} \frac{T\left(6 + h\,\frac{1}{\sqrt{2}}, 4 + h\,\frac{1}{\sqrt{2}}\right) - T(6, 4)}{h}$, which we can estimate with $h = \pm 2\sqrt{2}$. Then

$D_{\mathbf{u}}\, T(6, 4) \approx \dfrac{T(8, 6) - T(6, 4)}{2\sqrt{2}} = \dfrac{80 - 80}{2\sqrt{2}} = 0$, $D_{\mathbf{u}}\, T(6, 4) \approx \dfrac{T(4, 2) - T(6, 4)}{-2\sqrt{2}} = \dfrac{74 - 80}{-2\sqrt{2}} = \dfrac{3}{\sqrt{2}}$.

Averaging these values, we have $D_{\mathbf{u}}\, T(6, 4) \approx \frac{3}{2\sqrt{2}} \approx 1.1°\text{C}/\text{m}$.

(c) $T_{xy}(x,y) = \dfrac{\partial}{\partial y}[T_x(x,y)] = \lim\limits_{h\to 0}\dfrac{T_x(x,y+h)-T_x(x,y)}{h}$, so $T_{xy}(6,4) = \lim\limits_{h\to 0}\dfrac{T_x(6,4+h)-T_x(6,4)}{h}$

which we can estimate with $h=\pm 2$. We have $T_x(6,4)\approx 3.5$ from part (a), but we will also need values for $T_x(6,6)$ and $T_x(6,2)$. If we use $h=\pm 2$ and the values given in the table, we have

$T_x(6,6)\approx \dfrac{T(8,6)-T(6,6)}{2} = \dfrac{80-75}{2} = 2.5$, $T_x(6,6)\approx \dfrac{T(4,6)-T(6,6)}{-2} = \dfrac{68-75}{-2} = 3.5.$

Averaging these values, we estimate $T_x(6,6)\approx 3.0$. Similarly,

$T_x(6,2)\approx \dfrac{T(8,2)-T_x(6,2)}{2} = \dfrac{90-87}{2} = 1.5$, $T_x(6,2)\approx \dfrac{T(4,2)-T(6,2)}{-2} = \dfrac{74-87}{-2} = 6.5.$

Averaging these values, we estimate $T_x(6,2)\approx 4.0$. Finally, we estimate $T_{xy}(6,4)$:

$T_{xy}(6,4)\approx \dfrac{T_x(6,6)-T_x(6,4)}{2} = \dfrac{3.0-3.5}{2} = -0.25,$

$T_{xy}(6,4)\approx \dfrac{T_x(6,2)-T_x(6,4)}{-2} = \dfrac{4.0-3.5}{-2} = -0.25.$ Averaging these values, we have

$T_{xy}(6,4)\approx -0.25.$

12. From the table, $T(6,4)=80$, and from Exercise 11 we estimated $T_x(6,4)\approx 3.5$ and $T_y(6,4)\approx -3.0$. The linear approximation then is

$$\begin{aligned} T(x,y) &\approx T(6,4)+T_x(6,4)(x-6)+T_y(6,4)(y-4)\approx 80+3.5(x-6)-3(y-4) \\ &= 3.5x-3y+71 \end{aligned}$$

Thus at the point $(5,3.8)$, we can use the linear approximation to estimate $T(5,3.8)\approx 3.5(5)-3(3.8)+71\approx 77.1^\circ\text{C}$.

13. $f(x,y)=\sqrt{2x+y^2} \;\Rightarrow\; f_x=\frac{1}{2}(2x+y^2)^{-1/2}(2)=\dfrac{1}{\sqrt{2x+y^2}}$, $f_y=\frac{1}{2}(2x+y^2)^{-1/2}(2y)=\dfrac{y}{\sqrt{2x+y^2}}$

14. $u=e^{-r}\sin 2\theta \;\Rightarrow\; u_r=-e^{-r}\sin 2\theta$, $u_\theta = 2e^{-r}\cos 2\theta$

15. $g(u,v)=u\tan^{-1}v \;\Rightarrow\; g_u=\tan^{-1}v$, $g_v=\dfrac{u}{1+v^2}$

16. $w=\dfrac{x}{y-z} \;\Rightarrow\; w_x=\dfrac{1}{y-z}$, $w_y=x(-1)(y-z)^{-2}=-\dfrac{x}{(y-z)^2}$,
$w_z=x(-1)(y-z)^{-2}(-1)=\dfrac{x}{(y-z)^2}$

17. $T(p,q,r)=p\ln(q+e^r) \;\Rightarrow\; T_p=\ln(q+e^r)$, $T_q=\dfrac{p}{q+e^r}$, $T_r=\dfrac{pe^r}{q+e^r}$

18. $C=1449.2+4.6T-0.055T^2+0.00029T^3+(1.34-0.01T)(S-35)+0.016D \;\Rightarrow$ $\partial C/\partial T=4.6-0.11T+0.00087T^2-0.01(S-35)$, $\partial C/\partial S=1.34-0.01T$, and $\partial C/\partial D=0.016$. When $T=10$, $S=35$, and $D=100$ we have $\partial C/\partial T=4.6-0.11(10)+0.00087(10)^2-0.01(35-35)\approx 3.587$, thus in 10°C water with salinity 35 parts per thousand and a depth of 100 m, the speed of sound increases by about 3.59 m/s for every degree Celsius that the water temperature rises. Similarly, $\partial C/\partial S=1.34-0.01(10)=1.24$, so the speed of sound increases by about 1.24 m/s for every part per thousand the salinity of the water increases. $\partial C/\partial D=0.016$, so the speed of sound increases by about 0.016 m/s for every meter that the depth is increased.

19. $f(x,y)=4x^3-xy^2 \;\Rightarrow\; f_x=12x^2-y^2$, $f_y=-2xy$, $f_{xx}=24x$, $f_{yy}=-2x$, and $f_{xy}=f_{yx}=-2y$.

20. $z = xe^{-2y} \Rightarrow z_x = e^{-2y}$, $z_y = -2xe^{-2y}$, $z_{xx} = 0$, $z_{yy} = 4xe^{-2y}$, and $z_{xy} = z_{yx} = -2e^{-2y}$.

21. $f(x,y,z) = x^k y^l z^m \Rightarrow f_x = kx^{k-1}y^l z^m$, $f_y = lx^k y^{l-1} z^m$, $f_z = mx^k y^l z^{m-1}$,
$f_{xx} = k(k-1)x^{k-2}y^l z^m$, $f_{yy} = l(l-1)x^k y^{l-2} z^m$, $f_{zz} = m(m-1)x^k y^l z^{m-2}$, $f_{xy} = f_{yx} = klx^{k-1}y^{l-1}z^m$,
$f_{xz} = f_{zx} = kmx^{k-1}y^l z^{m-1}$, and $f_{yz} = f_{zy} = lmx^k y^{l-1} z^{m-1}$.

22. $v = r\cos(s+2t) \Rightarrow v_r = \cos(s+2t)$, $v_s = -r\sin(s+2t)$, $v_t = -2r\sin(s+2t)$, $v_{rr} = 0$,
$v_{ss} = -r\cos(s+2t)$, $v_{tt} = -4r\cos(s+2t)$, $v_{rs} = v_{sr} = -\sin(s+2t)$, $v_{rt} = v_{tr} = -2\sin(s+2t)$, and
$v_{st} = v_{ts} = -2r\cos(s+2t)$.

23. $u = x^y \Rightarrow u_x = yx^{y-1}$, $u_y = x^y \ln x$ and $(x/y)u_x + (\ln x)^{-1}u_y = x^y + x^y = 2u$.

24. $\rho = \sqrt{x^2+y^2+z^2} \Rightarrow \rho_x = \dfrac{x}{\sqrt{x^2+y^2+z^2}}$, $\rho_{xx} = \dfrac{y^2+z^2}{(x^2+y^2+z^2)^{3/2}}$.

By symmetry, $\rho_{yy} = \dfrac{x^2+z^2}{(x^2+y^2+z^2)^{3/2}}$ and $\rho_{zz} = \dfrac{x^2+y^2}{(x^2+y^2+z^2)^{3/2}}$. Thus

$\rho_{xx} + \rho_{yy} + \rho_{zz} = 2\dfrac{x^2+y^2+z^2}{(x^2+y^2+z^2)^{3/2}} = \dfrac{2}{(x^2+y^2+z^2)^{1/2}} = \dfrac{2}{\rho}$.

25. (a) $z_x = 6x + 2 \Rightarrow z_x(1,-2) = 8$ and $z_y = -2y \Rightarrow z_y(1,-2) = 4$, so an equation of the tangent plane is $z - 1 = 8(x-1) + 4(y+2)$ or $z = 8x + 4y + 1$.

(b) A normal vector to the tangent plane (and the surface) at $(1,-2,1)$ is $\langle 8,4,-1\rangle$. Then parametric equations for the normal line there are $x = 1+8t$, $y = -2+4t$, $z = 1-t$, and symmetric equations are $\dfrac{x-1}{8} = \dfrac{y+2}{4} = \dfrac{z-1}{-1}$.

26. (a) $z_x = e^x\cos y \Rightarrow z_x(0,0) = 1$ and $z_y = -e^x\sin y \Rightarrow z_y(0,0) = 0$, so an equation of the tangent plane is $z - 1 = 1(x-0) + 0(y-0)$ or $z = x + 1$.

(b) A normal vector to the tangent plane (and the surface) at $(0,0,1)$ is $\langle 1,0,-1\rangle$. Then parametric equations for the normal line there are $x = t$, $y = 0$, $z = 1-t$, and symmetric equations are $x = 1-z$, $y = 0$.

27. (a) Let $F(x,y,z) = x^2 + 2y^2 - 3z^2$. Then $F_x = 2x$, $F_y = 4y$, $F_z = -6z$, so $F_x(2,-1,1) = 4$, $F_y(2,-1,1) = -4$, $F_z(2,-1,1) = -6$. From Equation 15.6.19 [ET 14.6.19], an equation of the tangent plane is $4(x-2) - 4(y+1) - 6(z-1) = 0$ or equivalently $2x - 2y - 3z = 3$.

(b) From Equations 15.6.20 [ET 14.6.20], symmetric equations for the normal line are $\dfrac{x-2}{4} = \dfrac{y+1}{-4} = \dfrac{z-1}{-6}$.

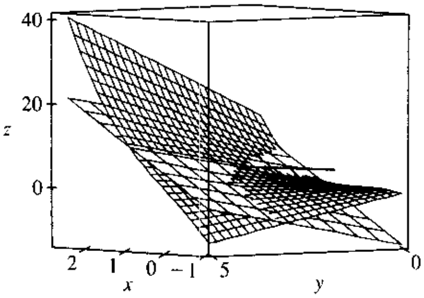
28. (a) Let $F(x,y,z) = xy + yz + zx$. Then $F_x = y+z$, $F_y = x+z$, $F_z = x+y$, so $F_x(1,1,1) = F_y(1,1,1) = F_z(1,1,1) = 2$. From Equation 15.6.19 [ET 14.6.19], an equation of the tangent plane is $2(x-1) + 2(y-1) + 2(z-1) = 0$ or equivalently $x + y + z = 3$.

(b) From Equations 15.6.20 [ET 14.6.20], symmetric equations for the normal line are $\dfrac{x-1}{2} = \dfrac{y-1}{2} = \dfrac{z-1}{2}$ or equivalently $x = y = z$.

29. (a) Let $F(x,y,z) = x + 2y + 3z - \sin(xyz)$. Then $F_x = 1 - yz\cos(xyz)$, $F_y = 2 - xz\cos(xyz)$, $F_z = 3 - xy\cos(xyz)$, so $F_x(2,-1,0) = 1$, $F_y(2,-1,0) = 2$, $F_z(2,-1,0) = 5$. From Equation 15.6.19 [ET 14.6.19], an equation of the tangent plane is $1(x-2) + 2(y+1) + 5(z-0) = 0$ or $x + 2y + 5z = 0$.

(b) From Equations 15.6.20 [ET 14.6.20], symmetric equations for the normal line are $\dfrac{x-2}{1} = \dfrac{y+1}{2} = \dfrac{z}{5}$.

30. Let $f(x,y)=x^3+2xy$. Then $f_x(x,y)=3x^2+2y$ and $f_y(x,y)=2x$, so $f_x(1,2)=7$, $f_y(1,2)=2$ and an equation of the tangent plane is $z-5=7(x-1)+2(y-2)$ or $7x+2y-z=6$. The normal line is given by $\dfrac{x-1}{7}=\dfrac{y-2}{2}=\dfrac{z-5}{-1}$ or $x=7t+1$, $y=2t+2$, $z=-t+5$.



31. $F(x,y,z)=x^2+y^2+z^2$, $\nabla F(x_0,y_0,z_0)=\langle 2x_0,2y_0,2z_0\rangle=k\langle 2,1,-3\rangle$ or $x_0=k$, $y_0=\frac{1}{2}k$ and $z_0=-\frac{3}{2}k$. But $x_0^2+y_0^2+z_0^2=1$, so $\frac{7}{2}k^2=1$ and $k=\pm\sqrt{\frac{2}{7}}$. Hence there are two such points: $\left(\pm\sqrt{\frac{2}{7}},\pm\frac{1}{\sqrt{14}},\mp\frac{3}{\sqrt{14}}\right)$.

32. $z=x^2\tan^{-1}y \quad\Rightarrow\quad dz=(2x\tan^{-1}y)\,dx+[x^2/(y^2+1)]\,dy$

33. $f(x,y,z)=x^3\sqrt{y^2+z^2} \quad\Rightarrow\quad f_x(x,y,z)=3x^2\sqrt{y^2+z^2}$, $f_y(x,y,z)=\dfrac{yx^3}{\sqrt{y^2+z^2}}$, and $f_z(x,y,z)=\dfrac{zx^3}{\sqrt{y^2+z^2}}$, so $f(2,3,4)=8(5)=40$, $f_x(2,3,4)=3(4)\sqrt{25}=60$, $f_y(2,3,4)=\frac{3(8)}{\sqrt{25}}=\frac{24}{5}$, and $f_z(2,3,4)=\frac{4(8)}{\sqrt{25}}=\frac{32}{5}$. Then the linear approximation of f at $(2,3,4)$ is

$$\begin{aligned}f(x,y,z)&\approx f(2,3,4)+f_x(2,3,4)(x-2)+f_y(2,3,4)(y-3)+f_z(2,3,4)(z-4)\\&=40+60(x-2)+\tfrac{24}{5}(y-3)+\tfrac{32}{5}(z-4)=60x+\tfrac{24}{5}y+\tfrac{32}{5}z-120\end{aligned}$$

Then

$$\begin{aligned}(1.98)^3\sqrt{(3.01)^2+(3.97)^2}&=f(1.98,3.01,3.97)\approx 60(1.98)+\tfrac{24}{5}(3.01)+\tfrac{32}{5}(3.97)-120\\&=38.656\end{aligned}$$

34. (a) $dA=\dfrac{\partial A}{\partial x}\,dx+\dfrac{\partial A}{\partial y}\,dy=\frac{1}{2}y\,dx+\frac{1}{2}x\,dy$ and $|\Delta x|\le 0.002$, $|\Delta y|\le 0.002$. Thus the maximum error in the calculated area is about $dA=6(0.002)+\frac{5}{2}(0.002)=0.017\text{ m}^2$ or 170 cm^2.

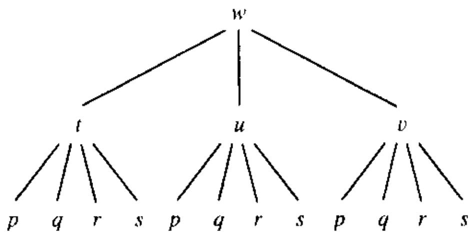
(b) $z=\sqrt{x^2+y^2}$, $dz=\dfrac{x}{\sqrt{x^2+y^2}}\,dx+\dfrac{y}{\sqrt{x^2+y^2}}\,dy$ and $|\Delta x|\le 0.002$, $|\Delta y|\le 0.002$. Thus the maximum error in the calculated hypotenuse length is about $dz=\frac{5}{13}(0.002)+\frac{12}{13}(0.002)=\frac{0.17}{65}\approx 0.0026$ m or 0.26 cm.

35. $\dfrac{dw}{dt}=\dfrac{1}{2\sqrt{x}}\,(2e^{2t})+\dfrac{2y}{z}\,(3t^2+4)+\dfrac{-y^2}{z^2}\,(2t)=e^t+\dfrac{2y}{z}\,(3t^2+4)-2t\,\dfrac{y^2}{z^2}$

36. $\dfrac{\partial z}{\partial u}=(-y\sin xy-y\sin x)(2u)+(-x\sin xy+\cos x)=\cos x-2uy\sin x-(\sin xy)(x+2uy)$,

$\dfrac{\partial z}{\partial v}=(-y\sin xy-y\sin x)(1)+(-x\sin xy+\cos x)(-2v)=-2v\cos x+(\sin xy)(2vx-y)-y\sin x$

37. By the Chain Rule, $\dfrac{\partial z}{\partial s}=\dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial s}+\dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial s}$. When $s=1$ and $t=2$, $x=g(1,2)=3$ and $y=h(1,2)=6$, so $\dfrac{\partial z}{\partial s}=f_x(3,6)g_s(1,2)+f_y(3,6)\,h_s(1,2)=(7)(-1)+(8)(-5)=-47$. Similarly, $\dfrac{\partial z}{\partial t}=\dfrac{\partial z}{\partial x}\dfrac{\partial x}{\partial t}+\dfrac{\partial z}{\partial y}\dfrac{\partial y}{\partial t}$, so $\dfrac{\partial z}{\partial t}=f_x(3,6)g_t(1,2)+f_y(3,6)\,h_t(1,2)=(7)(4)+(8)(10)=108$.

38. 

Using the tree diagram as a guide, we have

$$\frac{\partial w}{\partial p}=\frac{\partial w}{\partial t}\frac{\partial t}{\partial p}+\frac{\partial w}{\partial u}\frac{\partial u}{\partial p}+\frac{\partial w}{\partial v}\frac{\partial v}{\partial p}$$
$$\frac{\partial w}{\partial q}=\frac{\partial w}{\partial t}\frac{\partial t}{\partial q}+\frac{\partial w}{\partial u}\frac{\partial u}{\partial q}+\frac{\partial w}{\partial v}\frac{\partial v}{\partial q}$$
$$\frac{\partial w}{\partial r}=\frac{\partial w}{\partial t}\frac{\partial t}{\partial r}+\frac{\partial w}{\partial u}\frac{\partial u}{\partial r}+\frac{\partial w}{\partial v}\frac{\partial v}{\partial r}$$
$$\frac{\partial w}{\partial s}=\frac{\partial w}{\partial t}\frac{\partial t}{\partial s}+\frac{\partial w}{\partial u}\frac{\partial u}{\partial s}+\frac{\partial w}{\partial v}\frac{\partial v}{\partial s}$$

39. $\dfrac{\partial z}{\partial x}=2xf'(x^2-y^2)$, $\dfrac{\partial z}{\partial y}=1-2yf'(x^2-y^2)$ $\left[\text{where } f'=\dfrac{df}{d(x^2-y^2)}\right]$. Then

$y\dfrac{\partial z}{\partial x}+x\dfrac{\partial z}{\partial y}=2xyf'(x^2-y^2)+x-2xyf'(x^2-y^2)=x.$

40. $A=\frac{1}{2}xy\sin\theta$, $dx/dt=3$, $dy/dt=-2$, $d\theta/dt=0.05$, and

$\dfrac{dA}{dt}=\dfrac{1}{2}\left[(y\sin\theta)\dfrac{dx}{dt}+(x\sin\theta)\dfrac{dy}{dt}+(xy\cos\theta)\dfrac{d\theta}{dt}\right]$. So when $x=40$, $y=50$ and $\theta=\frac{\pi}{6}$,

$\dfrac{dA}{dt}=\dfrac{1}{2}\left[(25)(3)+(20)(-2)+(1000\sqrt{3})(0.05)\right]=\dfrac{35+50\sqrt{3}}{2}\approx 60.8\text{ in}^2/\text{s}.$

41. $\dfrac{\partial z}{\partial x}=\dfrac{\partial z}{\partial u}\,y+\dfrac{\partial z}{\partial v}\dfrac{-y}{x^2}$ and

$$\begin{aligned}\frac{\partial^2 z}{\partial x^2}&=y\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial u}\right)+\frac{2y}{x^3}\frac{\partial z}{\partial v}+\frac{-y}{x^2}\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial v}\right)\\&=\frac{2y}{x^3}\frac{\partial z}{\partial v}+y\left(\frac{\partial^2 z}{\partial u^2}\,y+\frac{\partial^2 z}{\partial v\,\partial u}\frac{-y}{x^2}\right)+\frac{-y}{x^2}\left(\frac{\partial^2 z}{\partial v^2}\frac{-y}{x^2}+\frac{\partial^2 z}{\partial u\,\partial v}\,y\right)\\&=\frac{2y}{x^3}\frac{\partial z}{\partial v}+y^2\frac{\partial^2 z}{\partial u^2}-\frac{2y^2}{x^2}\frac{\partial^2 z}{\partial u\,\partial v}+\frac{y^2}{x^4}\frac{\partial^2 z}{\partial v^2}\end{aligned}$$

Also $\dfrac{\partial z}{\partial y}=x\dfrac{\partial z}{\partial u}+\dfrac{1}{x}\dfrac{\partial z}{\partial v}$ and

$$\begin{aligned}\frac{\partial^2 z}{\partial y^2}&=x\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial u}\right)+\frac{1}{x}\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial v}\right)=x\left(\frac{\partial^2 z}{\partial u^2}x+\frac{\partial^2 z}{\partial v\,\partial u}\frac{1}{x}\right)+\frac{1}{x}\left(\frac{\partial^2 z}{\partial v^2}\frac{1}{x}+\frac{\partial^2 z}{\partial u\,\partial v}\,x\right)\\&=x^2\frac{\partial^2 z}{\partial u^2}+2\frac{\partial^2 z}{\partial u\,\partial v}+\frac{1}{x^2}\frac{\partial^2 z}{\partial v^2}\end{aligned}$$

Thus

$$\begin{aligned}x^2\frac{\partial^2 z}{\partial x^2}-y^2\frac{\partial^2 z}{\partial y^2}&=\frac{2y}{x}\frac{\partial z}{\partial v}+x^2y^2\frac{\partial^2 z}{\partial u^2}-2y^2\frac{\partial^2 z}{\partial u\,\partial v}+\frac{y^2}{x^2}\frac{\partial^2 z}{\partial v^2}-x^2y^2\frac{\partial^2 z}{\partial u^2}-2y^2\frac{\partial^2 z}{\partial u\,\partial v}-\frac{y^2}{x^2}\frac{\partial^2 z}{\partial v^2}\\&=\frac{2y}{x}\frac{\partial z}{\partial v}-4y^2\frac{\partial^2 z}{\partial u\,\partial v}=2v\frac{\partial z}{\partial v}-4uv\frac{\partial^2 z}{\partial u\,\partial v}\end{aligned}$$

since $y=xv=\dfrac{uv}{y}$ or $y^2=uv$.

42. $F(x,y,z)=e^{xyz}-yz^4-x^2z^3=0$, so $\dfrac{\partial z}{\partial x}=-\dfrac{F_x}{F_z}=-\dfrac{yze^{xyz}-2xz^3}{xye^{xyz}-4yz^3-3x^2z^2}=\dfrac{2xz^3-yze^{xyz}}{xye^{xyz}-4yz^3-3x^2z^2}$

and $\dfrac{\partial z}{\partial y}=-\dfrac{F_y}{F_z}=-\dfrac{xze^{xyz}-z^4}{xye^{xyz}-4yz^3-3x^2z^2}=\dfrac{z^4-xze^{xyz}}{xye^{xyz}-4yz^3-3x^2z^2}.$

43. $\nabla f = \left\langle z^2\sqrt{y}\,e^{x\sqrt{y}}, \dfrac{xz^2e^{x\sqrt{y}}}{2\sqrt{y}}, 2ze^{x\sqrt{y}} \right\rangle = ze^{x\sqrt{y}}\left\langle z\sqrt{y}, \dfrac{xz}{2\sqrt{y}}, 2\right\rangle$

44. (a) By Theorem 15.6.15 [ET 14.6.15], the maximum value of the directional derivative occurs when **u** has the same direction as the gradient vector.

(b) It is a minimum when **u** is in the direction opposite to that of the gradient vector (that is, **u** is in the direction of $-\nabla f$), since $D_{\mathbf{u}}\,f = |\nabla f|\cos\theta$ (see the proof of Theorem 15.6.15 [ET 14.6.15]) has a minimum when $\theta = \pi$.

(c) The directional derivative is 0 when **u** is perpendicular to the gradient vector, since then $D_{\mathbf{u}}\,f = \nabla f \cdot \mathbf{u} = 0$.

(d) The directional derivative is half of its maximum value when $D_{\mathbf{u}}\,f = |\nabla f|\cos\theta = \frac{1}{2}|\nabla f| \quad\Leftrightarrow$ $\cos\theta = \frac{1}{2} \quad\Leftrightarrow\quad \theta = \frac{\pi}{3}$.

45. $\nabla f = \langle 1/\sqrt{x}, -2y\rangle$, $\nabla f(1,5) = \langle 1, -10\rangle$, $\mathbf{u} = \frac{1}{5}\langle 3, -4\rangle$. Then $D_{\mathbf{u}}\,f(1,5) = \frac{43}{5}$.

46. $\nabla f = \left\langle 2xy + \sqrt{1+z}, x^2, x/(2\sqrt{1+z}\,)\right\rangle$, $\nabla f(1,2,3) = \left\langle 6, 1, \frac{1}{4}\right\rangle$, and $\mathbf{u} = \left\langle \frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\right\rangle$. Then $D_{\mathbf{u}}\,f(1,2,3) = \frac{25}{6}$.

47. $\nabla f = \left\langle 2xy, x^2 + 1/(2\sqrt{y}\,)\right\rangle$, $|\nabla f(2,1)| = \left|\left\langle 4, \frac{9}{2}\right\rangle\right|$. Thus the maximum rate of change of f at $(2,1)$ is $\frac{\sqrt{145}}{2}$ in the direction $\left\langle 4, \frac{9}{2}\right\rangle$.

48. $\nabla f = \langle zye^{xy}, zxe^{xy}, e^{xy}\rangle$, $\nabla f(0,1,2) = \langle 2,0,1\rangle$ is the direction of most rapid increase while the rate is $|\langle 2,0,1\rangle| = \sqrt{5}$.

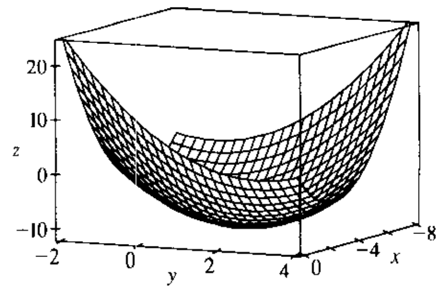
49. First we draw a line passing through Homestead and the eye of the hurricane. We can approximate the directional derivative at Homestead in the direction of the eye of the hurricane by the average rate of change of wind speed between the points where this line intersects the contour lines closest to Homestead. In the direction of the eye of the hurricane, the wind speed changes from 45 to 50 knots. We estimate the distance between these two points to be approximately 8 miles, so the rate of change of wind speed in the direction given is approximately $\frac{50-45}{8} = \frac{5}{8} = 0.625$ knot/mi.

50. The surfaces are $f(x,y,z) = z - 2x^2 + y^2 = 0$ and $g(x,y,z) = z - 4 = 0$. The tangent line is perpendicular to both ∇f and ∇g at $(-2,2,4)$. The vector $\mathbf{v} = \nabla f \times \nabla g$ is therefore parallel to the line.
$\nabla f(x,y,z) = \langle -4x, 2y, 1\rangle \quad\Rightarrow\quad \nabla f(-2,2,4) = \langle 8,4,1\rangle$, $\nabla g(x,y,z) = \langle 0,0,1\rangle \quad\Rightarrow$

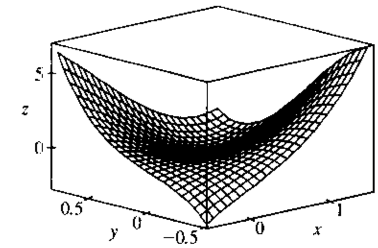
$\nabla g\langle -2,2,4\rangle = \langle 0,0,1\rangle$. Hence $\mathbf{v} = \nabla f \times \nabla g = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 8 & 4 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 4\,\mathbf{i} - 8\,\mathbf{j}$. Thus, parametric equations are:

$x = -2 + 4t$, $y = 2 - 8t$, and $z = 4$.

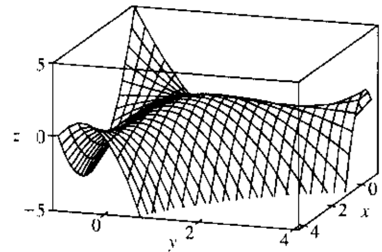
51. $f(x,y) = x^2 - xy + y^2 + 9x - 6y + 10 \quad\Rightarrow\quad f_x = 2x - y + 9$, $f_y = -x + 2y - 6$, $f_{xx} = 2 = f_{yy}$, $f_{xy} = -1$. Then $f_x = 0$ and $f_y = 0$ imply $y = 1$, $x = -4$. Thus the only critical point is $(-4,1)$ and $f_{xx}(-4,1) > 0$, $D(-4,1) = 3 > 0$, so $f(-4,1) = -11$ is a local minimum.



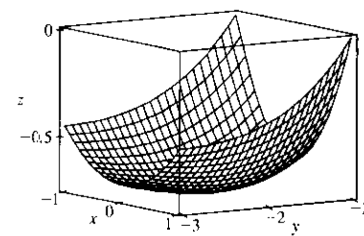
52. $f(x,y) = x^3 - 6xy + 8y^3 \quad \Rightarrow \quad f_x = 3x^2 - 6y,\ f_y = -6x + 24y^2$, $f_{xx} = 6x,\ f_{yy} = 48y,\ f_{xy} = -6$. Then $f_x = 0$ implies $y = x^2/2$, substituting into $f_y = 0$ implies $6x(x^3 - 1) = 0$, so the critical points are $(0,0)$, $\left(1, \frac{1}{2}\right)$. $D(0,0) = -36 < 0$ so $(0,0)$ is a saddle point while $f_{xx}\left(1, \frac{1}{2}\right) = 6 > 0$ and $D\left(1, \frac{1}{2}\right) = 108 > 0$ so $f\left(1, \frac{1}{2}\right) = -1$ is a local minimum.



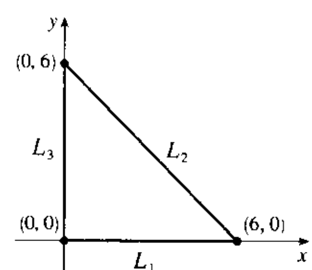
53. $f(x,y) = 3xy - x^2y - xy^2 \quad \Rightarrow \quad f_x = 3y - 2xy - y^2$, $f_y = 3x - x^2 - 2xy,\ f_{xx} = -2y,\ f_{yy} = -2x,\ f_{xy} = 3 - 2x - 2y$. Then $f_x = 0$ implies $y(3 - 2x - y) = 0$ so $y = 0$ or $y = 3 - 2x$. Substituting into $f_y = 0$ implies $x(3 - x) = 0$ or $3x(-1 + x) = 0$. Hence the critical points are $(0,0)$, $(3,0)$, $(0,3)$ and $(1,1)$.
$D(0,0) = D(3,0) = D(0,3) = -9 < 0$ so $(0,0)$, $(3,0)$, and $(0,3)$ are saddle points. $D(1,1) = 3 > 0$ and $f_{xx}(1,1) = -2 < 0$, so $f(1,1) = 1$ is a local maximum.



54. $f(x,y) = (x^2 + y)e^{y/2} \quad \Rightarrow \quad f_x = 2xe^{y/2},\ f_y = e^{y/2}(2 + x^2 + y)/2$, $f_{xx} = 2e^{y/2},\ f_{yy} = e^{y/2}(4 + x^2 + y)/4,\ f_{xy} = xe^{y/2}$. Then $f_x = 0$ implies $x = 0$, so $f_y = 0$ implies $y = -2$. But $f_{xx}(0,-2) > 0$, $D(0,-2) = e^{-2} - 0 > 0$ so $f(0,-2) = -2/e$ is a local minimum.

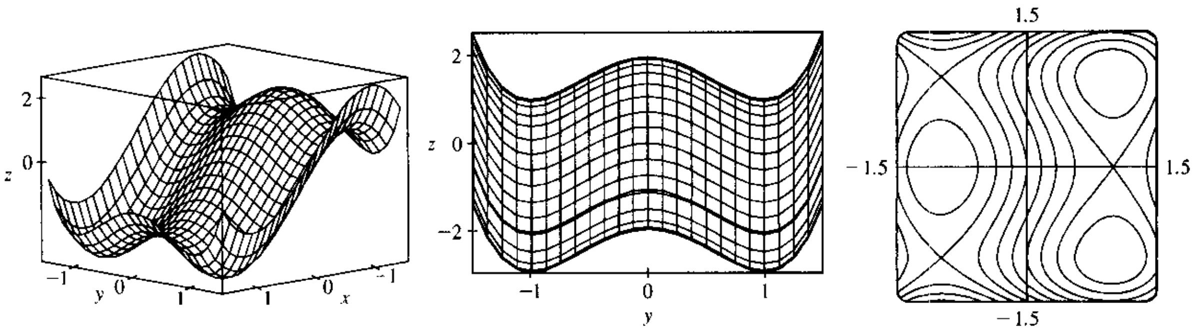


55. First solve inside D. Here $f_x = 4y^2 - 2xy^2 - y^3$, $f_y = 8xy - 2x^2y - 3xy^2$. Then $f_x = 0$ implies $y = 0$ or $y = 4 - 2x$, but $y = 0$ isn't inside D. Substituting $y = 4 - 2x$ into $f_y = 0$ implies $x = 0$, $x = 2$ or $x = 1$, but $x = 0$ isn't inside D, and when $x = 2$, $y = 0$ but $(2,0)$ isn't inside D. Thus the only critical point inside D is $(1,2)$ and $f(1,2) = 4$. Secondly we consider the boundary of D.
On L_1, $f(x,0) = 0$ and so $f = 0$ on L_1. On L_2, $x = -y + 6$ and $f(-y + 6, y) = y^2(6 - y)(-2) = -2(6y^2 - y^3)$ which has critical points at $y = 0$ and $y = 4$. Then $f(6,0) = 0$ while $f(2,4) = -64$. On L_3, $f(0,y) = 0$, so $f = 0$ on L_3. Thus on D the absolute maximum of f is $f(1,2) = 4$ while the absolute minimum is $f(2,4) = -64$.



56. Inside D: $f_x = 2xe^{-x^2-y^2}(1 - x^2 - 2y^2) = 0$ implies $x = 0$ or $x^2 + 2y^2 = 1$. Then if $x = 0$, $f_y = 2ye^{-x^2-y^2}(2 - x^2 - 2y^2) = 0$ implies $y = 0$ or $2 - 2y^2 = 0$ giving the critical points $(0,0)$, $(0,\pm 1)$. If $x^2 + 2y^2 = 1$, then $f_y = 0$ implies $y = 0$ giving the critical points $(\pm 1, 0)$. Now $f(0,0) = 0$, $f(\pm 1, 0) = e^{-1}$ and $f(0,\pm 1) = 2e^{-1}$. On the boundary of D: $x^2 + y^2 = 4$, so $f(x,y) = e^{-4}(4 + y^2)$ and f is smallest when $y = 0$ and largest when $y^2 = 4$. But $f(\pm 2, 0) = 4e^{-4}$, $f(0,\pm 2) = 8e^{-4}$. Thus on D the absolute maximum of f is $f(0,\pm 1) = 2e^{-1}$ and the absolute minimum is $f(0,0) = 0$.

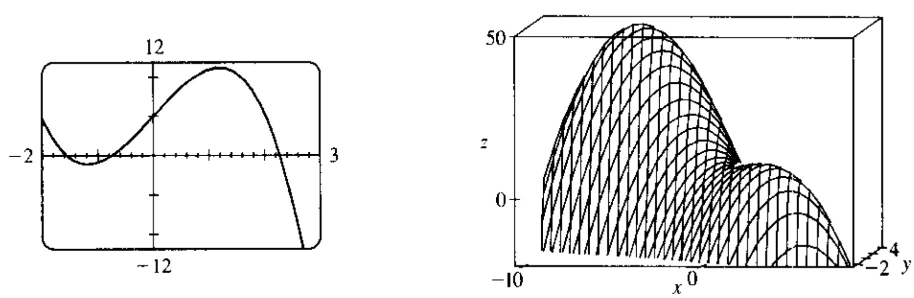
57. $f(x,y) = x^3 - 3x + y^4 - 2y^2$



From the graphs, it appears that f has a local maximum $f(-1,0) \approx 2$, local minima $f(1,\pm 1) \approx -3$, and saddle points at $(-1,\pm 1)$ and $(1,0)$.

To find the exact quantities, we calculate $f_x = 3x^2 - 3 = 0 \quad\Leftrightarrow\quad x = \pm 1$ and $f_y = 4y^3 - 4y = 0 \quad\Leftrightarrow$ $y = 0, \pm 1$, giving the critical points estimated above. Also $f_{xx} = 6x$, $f_{xy} = 0$, $f_{yy} = 12y^2 - 4$, so using the Second Derivatives Test, $D(-1,0) = 24 > 0$ and $f_{xx}(-1,0) = -6 < 0$ indicating a local maximum $f(-1,0) = 2$; $D(1,\pm 1) = 48 > 0$ and $f_{xx}(1,\pm 1) = 6 > 0$ indicating local minima $f(1,\pm 1) = -3$; and $D(-1,\pm 1) = -48$ and $D(1,0) = -24$, indicating saddle points.

58. $f(x,y) = 12 + 10y - 2x^2 - 8xy - y^4 \quad\Rightarrow\quad f_x(x,y) = -4x - 8y$, $f_y(x,y) = 10 - 8x - 4y^3$. Now $f_x(x,y) = 0 \quad\Rightarrow\quad x = -2x$, and substituting this into $f_y(x,y) = 0$ gives $10 + 16y - 4y^3 = 0 \quad\Leftrightarrow$ $5 + 8y - 2y^3 = 0$.



From the first graph, we see that this is true when $y \approx -1.542, -0.717$, or 2.260. (Alternatively, we could have found the solutions to $f_x = f_y = 0$ using a CAS.) So to three decimal places, the critical points are $(3.085, -1.542)$, $(1.434, -0.717)$, and $(-4.519, 2.260)$. Now in order to use the Second Derivatives Test, we calculate $f_{xx} = -4$, $f_{xy} = -8$, $f_{yy} = -12y^2$, and $D = 48y^2 - 64$. So since $D(3.085, -1.542) > 0$, $D(1.434, -0.717) < 0$, and $D(-4.519, 2.260) > 0$, and f_{xx} is always negative, $f(x,y)$ has local maxima $f(-4.519, 2.260) \approx 49.373$ and $f(3.085, -1.542) \approx 9.948$, and a saddle point at approximately $(1.434, -0.717)$. The highest point on the graph is approximately $(-4.519, 2.260, 49.373)$.

59. $f(x,y) = x^2 y$, $g(x,y) = x^2 + y^2 = 1 \quad\Rightarrow\quad \nabla f = \langle 2xy, x^2 \rangle = \lambda \nabla g = \langle 2\lambda x, 2\lambda y \rangle$. Then $2xy = 2\lambda x$ and $x^2 = 2\lambda y$ imply $\lambda = x^2/(2y)$ and $\lambda = y$ if $x \neq 0$ and $y \neq 0$. Hence $x^2 = 2y^2$. Then $x^2 + y^2 = 1$ implies $3y^2 = 1$ so $y = \pm\frac{1}{\sqrt{3}}$ and $x = \pm\sqrt{\frac{2}{3}}$. [Note if $x = 0$ then $x^2 = 2\lambda y$ implies $y = 0$ and $f(0,0) = 0$.] Thus the possible points are $\left(\pm\sqrt{\frac{2}{3}}, \pm\frac{1}{\sqrt{3}}\right)$ and the absolute maxima are $f\left(\pm\sqrt{\frac{2}{3}}, \frac{1}{\sqrt{3}}\right) = \frac{2}{3\sqrt{3}}$ while the absolute minima are $f\left(\pm\sqrt{\frac{2}{3}}, -\frac{1}{\sqrt{3}}\right) = -\frac{2}{3\sqrt{3}}$.

60. $f(x,y) = 1/x + 1/y$, $g(x,y) = 1/x^2 + 1/y^2 = 1 \Rightarrow \nabla f = \langle -x^{-2}, -y^{-2} \rangle = \lambda \nabla g = \langle -2\lambda x^{-3}, -2\lambda y^{-3} \rangle$. Then $-x^{-2} = -2\lambda x^3$ or $x = 2\lambda$ and $-y^{-2} = -2\lambda y^{-3}$ or $y = 2\lambda$. Thus $x = y$, so $1/x^2 + 1/y^2 = 2/x^2 = 1$ implies $x = \pm\sqrt{2}$ and the possible points are $(\pm\sqrt{2}, \pm\sqrt{2})$. The absolute maximum of f subject to $x^{-2} + y^{-2} = 1$ is then $f(\sqrt{2}, \sqrt{2}) = \sqrt{2}$ and the absolute minimum is $f(-\sqrt{2}, -\sqrt{2}) = -\sqrt{2}$.

61. $f(x,y,z) = xyz$, $g(x,y,z) = x^2 + y^2 + z^2 = 3$. $\nabla f = \lambda \nabla g \Rightarrow \langle yz, xz, xy \rangle = \lambda \langle 2x, 2y, 2z \rangle$. If any of x, y, or z is zero, then $x = y = z = 0$ which contradicts $x^2 + y^2 + z^2 = 3$. Then $\lambda = \dfrac{yz}{2x} = \dfrac{xz}{2y} = \dfrac{xy}{2z} \Rightarrow$ $2y^2z = 2x^2z \Rightarrow y^2 = x^2$, and similarly $2yz^2 = 2x^2y \Rightarrow z^2 = x^2$. Substituting into the constraint equation gives $x^2 + x^2 + x^2 = 3 \Rightarrow x^2 = 1 = y^2 = z^2$. Thus the possible points are $(1, 1, \pm 1)$, $(1, -1, \pm 1)$, $(-1, 1, \pm 1)$, $(-1, -1, \pm 1)$. The absolute maximum is $f(1,1,1) = f(1,-1,-1) = f(-1,1,-1) = f(-1,-1,1) = 1$ and the absolute minimum is $f(1,1,-1) = f(1,-1,1) = f(-1,1,1) = f(-1,-1,-1) = -1$.

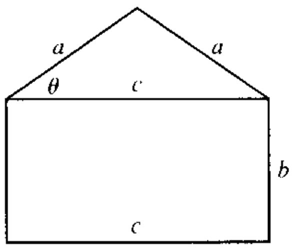
62. $f(x,y,z) = x^2 + 2y^2 + 3z^2$, $g(x,y,z) = x + y + z = 1$, $h(x,y,z) = x - y + 2z = 2 \Rightarrow$ $\nabla f = \langle 2x, 4y, 6z \rangle = \lambda \nabla g + \mu \nabla h = \langle \lambda + \mu, \lambda - \mu, \lambda + 2\mu \rangle$ and (1) $2x = \lambda + \mu$, (2) $4y = \lambda - \mu$, (3) $6z = \lambda + 2\mu$, (4) $x + y + z = 1$, (5) $x - y + 2z = 2$. Then six times (1) plus three times (2) plus two times (3) implies $12(x + y + z) = 11\lambda + 7\mu$, so (4) gives $11\lambda + 7\mu = 12$. Also six times (1) minus three times (2) plus four times (3) implies $12(x - y + 2z) = 7\lambda + 17\mu$, so (5) gives $7\lambda + 17\mu = 24$. Solving $11\lambda + 7\mu = 12$, $7\lambda + 17\mu = 24$ simultaneously gives $\lambda = \frac{6}{23}$, $\mu = \frac{30}{23}$. Substituting into (1), (2) and (3) implies $x = \frac{18}{23}$, $y = -\frac{6}{23}$, $z = \frac{11}{23}$ giving only one point. Then $f\left(\frac{18}{23}, -\frac{6}{23}, \frac{11}{23}\right) = \frac{33}{23}$. Now since $(0,0,1)$ satisfies both constraints and $f(0,0,1) = 3 > \frac{33}{23}$, $f\left(\frac{18}{23}, -\frac{6}{23}, \frac{11}{23}\right) = \frac{33}{23}$ is an absolute minimum, and there is no absolute maximum.

63. $f(x,y,z) = x^2 + y^2 + z^2$, $g(x,y,z) = xy^2z^3 = 2 \Rightarrow$ $\nabla f = \langle 2x, 2y, 2z \rangle = \lambda \nabla g = \langle \lambda y^2 z^3, 2\lambda xyz^3, 3\lambda xy^2z^2 \rangle$. Since $xy^2z^3 = 2$, $x \neq 0$, $y \neq 0$ and $z \neq 0$, so (1) $2x = \lambda y^2 z^3$, (2) $1 = \lambda xz^3$, (3) $2 = 3\lambda xy^2 z$. Then (2) and (3) imply $\dfrac{1}{xz^3} = \dfrac{2}{3xy^2z}$ or $y^2 = \frac{2}{3}z^2$ so $y = \pm z\sqrt{\frac{2}{3}}$. Similarly (1) and (3) imply $\dfrac{2x}{y^2z^3} = \dfrac{2}{3xy^2z}$ or $3x^2 = z^2$ so $x = \pm\frac{1}{\sqrt{3}}z$. But $xy^2z^3 = 2$ so x and z must have the same sign, that is, $x = \frac{1}{\sqrt{3}}z$. Thus $g(x,y,z) = 2$ implies $\frac{1}{\sqrt{3}}z\left(\frac{2}{3}z^2\right)z^3 = 2$ or $z = \pm 3^{1/4}$ and the possible points are $(\pm 3^{-1/4}, 3^{-1/4}\sqrt{2}, \pm 3^{1/4})$, $(\pm 3^{-1/4}, -3^{-1/4}\sqrt{2}, \pm 3^{1/4})$. However at each of these points f takes on the same value, $2\sqrt{3}$. But $(2,1,1)$ also satisfies $g(x,y,z) = 2$ and $f(2,1,1) = 6 > 2\sqrt{3}$. Thus f has an absolute minimum value of $2\sqrt{3}$ and no absolute maximum subject to the constraint $xy^2z^3 = 2$.

Alternate solution: $g(x,y,z) = xy^2z^3 = 2$ implies $y^2 = \dfrac{2}{xz^3}$, so minimize $f(x,z) = x^2 + \dfrac{2}{xz^3} + z^2$. Then $f_x = 2x - \dfrac{2}{x^2z^3}$, $f_z = -\dfrac{6}{xz^4} + 2z$, $f_{xx} = 2 + \dfrac{4}{x^3z^3}$, $f_{zz} = \dfrac{24}{xz^5} + 2$ and $f_{xz} = \dfrac{6}{x^2z^4}$. Now $f_x = 0$ implies $2x^3z^3 - 2 = 0$ or $z = 1/x$. Substituting into $f_y = 0$ implies $-6x^3 + 2x^{-1} = 0$ or $x = \frac{1}{\sqrt[4]{3}}$, so the two critical points are $\left(\pm\frac{1}{\sqrt[4]{3}}, \pm\sqrt[4]{3}\right)$. Then $D\left(\pm\frac{1}{\sqrt[4]{3}}, \pm\sqrt[4]{3}\right) = (2 + 4)\left(2 + \frac{24}{3}\right) - \left(\frac{6}{\sqrt{3}}\right)^2 > 0$ and $f_{xx}\left(\pm\frac{1}{\sqrt[4]{3}}, \pm\sqrt[4]{3}\right) = 6 > 0$, so each point is a minimum. Finally, $y^2 = \dfrac{2}{xz^3}$, so the four points closest to the origin are $\left(\pm\frac{1}{\sqrt[4]{3}}, \frac{\sqrt{2}}{\sqrt[4]{3}}, \pm\sqrt[4]{3}\right)$, $\left(\pm\frac{1}{\sqrt[4]{3}}, -\frac{\sqrt{2}}{\sqrt[4]{3}}, \pm\sqrt[4]{3}\right)$.

64. $V = xyz$, say x is the length and $x + 2y + 2z \leq 108$, $x > 0$, $y > 0$, $z > 0$. First maximize V subject to $x + 2y + 2z = 108$ with x, y, z all positive. Then $\langle yz, xz, xy \rangle = \langle \lambda, 2\lambda, 2\lambda \rangle$ implies $2yz = xz$ or $x = 2y$

and $xz = xy$ or $z = y$. Thus $g(x, y, z) = 108$ implies $6y = 108$ or $y = 18 = z$, $x = 36$, so the volume is $V = 11{,}664$ cubic units. Since $(104, 1, 1)$ also satisfies $g(x, y, z) = 108$ and $V(104, 1, 1) = 104$ cubic units, $(36, 18, 18)$ gives an absolute maximum of V subject to $g(x, y, z) = 108$. But if $x + 2y + 2z < 108$, there exists $\alpha > 0$ such that $x + 2y + 2z = 108 - \alpha$ and as above $6y = 108 - \alpha$ implies $y = (108 - \alpha)/6 = z$, $x = (108 - \alpha)/3$ with $V = (108 - \alpha)^3/(6^2 \cdot 3) < (108)^3/(6^2 \cdot 3) = 11{,}664$. Hence we have shown that the maximum of V subject to $g(x, y, z) \le 108$ is the maximum of V subject to $g(x, y, z) = 108$ (an intuitively obvious fact).

65. 

The area of the triangle is $\frac{1}{2}ca\sin\theta$ and the area of the rectangle is bc. Thus, the area of the whole object is $f(a, b, c) = \frac{1}{2}ca\sin\theta + bc$. The perimeter of the object is $g(a, b, c) = 2a + 2b + c = P$. To simplify $\sin\theta$ in terms of a, b, and c notice that $a^2\sin^2\theta + \left(\frac{1}{2}c\right)^2 = a^2 \Rightarrow$ $\sin\theta = \dfrac{1}{2a}\sqrt{4a^2 - c^2}$. Thus $f(a, b, c) = \dfrac{c}{4}\sqrt{4a^2 - c^2} + bc$.

(Instead of using θ, we could just have used the Pythagorean Theorem.) As a result, by Lagrange's method, we must find a, b, c, and λ by solving $\nabla f = \lambda \nabla g$ which gives the following equations: (1) $ca(4a^2 - c^2)^{-1/2} = 2\lambda$, (2) $c = 2\lambda$, (3) $\frac{1}{4}(4a^2 - c^2)^{1/2} - \frac{1}{4}c^2(4a^2 - c^2)^{-1/2} + b = \lambda$, and (4) $2a + 2b + c = P$. From (2), $\lambda = \frac{1}{2}c$ and so (1) produces $ca(4a^2 - c^2)^{-1/2} = c \;\Rightarrow\; (4a^2 - c^2)^{1/2} = a \;\Rightarrow\; 4a^2 - c^2 = a^2 \;\Rightarrow\;$ (5) $c = \sqrt{3}\,a$.

Similarly, since $\left(4a^2 - c^2\right)^{1/2} = a$ and $\lambda = \frac{1}{2}c$, (3) gives $\dfrac{a}{4} - \dfrac{c^2}{4a} + b = \dfrac{c}{2}$, so from (5), $\dfrac{a}{4} - \dfrac{3a}{4} + b = \dfrac{\sqrt{3}\,a}{2}$

$\Rightarrow\; -\dfrac{a}{2} - \dfrac{\sqrt{3}\,a}{2} = -b \;\Rightarrow\;$ (6) $b = \dfrac{a}{2}\left(1 + \sqrt{3}\right)$. Substituting (5) and (6) into (4) we get:

$2a + a\left(1 + \sqrt{3}\right) + \sqrt{3}\,a = P \;\Rightarrow\; 3a + 2\sqrt{3}\,a = P \;\Rightarrow\; a = \dfrac{P}{3 + 2\sqrt{3}} = \dfrac{2\sqrt{3} - 3}{3}P$ and thus

$b = \dfrac{(2\sqrt{3} - 3)(1 + \sqrt{3})}{6}P = \dfrac{3 - \sqrt{3}}{6}P$ and $c = \left(2 - \sqrt{3}\right)P$.

66. (a) $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j} + f(x(t), y(t))\,\mathbf{k} \;\Rightarrow\; \mathbf{v} = \dfrac{d\mathbf{r}}{dt} = \dfrac{dx}{dt}\,\mathbf{i} + \dfrac{dy}{dt}\,\mathbf{j} + \left(f_x\dfrac{dx}{dt} + f_y\dfrac{dy}{dt}\right)\mathbf{k}$

(by the Chain Rule). Therefore

$$K = \tfrac{1}{2}m\,|\mathbf{v}|^2 = \frac{m}{2}\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(f_x\frac{dx}{dt} + f_y\frac{dy}{dt}\right)^2\right]$$

$$= \frac{m}{2}\left[\left(1 + f_x^2\right)\left(\frac{dx}{dt}\right)^2 + 2f_xf_y\left(\frac{dx}{dt}\right)\left(\frac{dy}{dt}\right) + \left(1 + f_y^2\right)\left(\frac{dy}{dt}\right)^2\right]$$

(b) $\mathbf{a} = \dfrac{d\mathbf{v}}{dt} = \dfrac{d^2x}{dt^2}\,\mathbf{i} + \dfrac{d^2y}{dt^2}\,\mathbf{j} + \left[f_{xx}\left(\dfrac{dx}{dt}\right)^2 + 2f_{xy}\dfrac{dx}{dt}\dfrac{dy}{dt} + f_{yy}\left(\dfrac{dy}{dt}\right)^2 + f_x\dfrac{d^2x}{dt^2} + f_y\dfrac{d^2y}{dt^2}\right]\mathbf{k}$

(c) If $z = x^2 + y^2$, where $x = t\cos t$ and $y = t\sin t$, then $z = f(x, y) = t^2$.

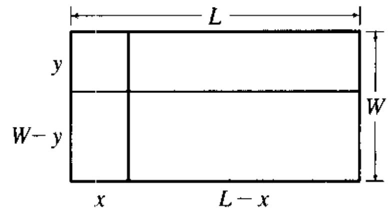
$\mathbf{r} = t\cos t\,\mathbf{i} + t\sin t\,\mathbf{j} + t^2\,\mathbf{k} \;\Rightarrow\; \mathbf{v} = (\cos t - t\sin t)\,\mathbf{i} + (\sin t + t\cos t)\,\mathbf{j} + 2t\,\mathbf{k}$,

$K = \dfrac{m}{2}[(\cos t - t\sin t)^2 + (\sin t + t\cos t)^2 + (2t)^2] = \dfrac{m}{2}(1 + t^2 + 4t^2) = \dfrac{m}{2}(1 + 5t^2)$, and

$\mathbf{a} = (-2\sin t - t\cos t)\,\mathbf{i} + (2\cos t - t\sin t)\,\mathbf{j} + 2\,\mathbf{k}$. Notice that it is easier not to use the formulas in (a) and (b).

PROBLEMS PLUS

1. The areas of the smaller rectangles are $A_1 = xy$, $A_2 = (L - x)y$, $A_3 = (L - x)(W - y)$, $A_4 = x(W - y)$. For $0 \le x \le L$, $0 \le y \le W$, let

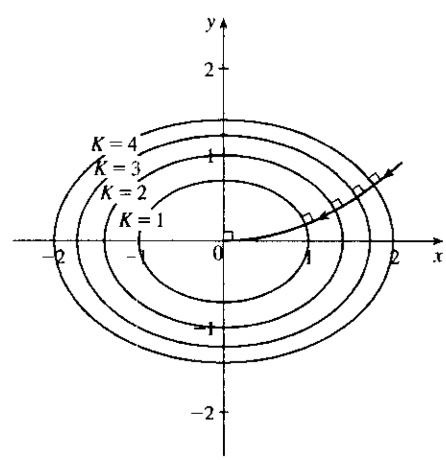


$$\begin{aligned} f(x, y) &= A_1^2 + A_2^2 + A_3^2 + A_4^2 \\ &= x^2y^2 + (L - x)^2y^2 + (L - x)^2(W - y)^2 + x^2(W - y)^2 \\ &= [x^2 + (L - x)^2][y^2 + (W - y)^2] \end{aligned}$$

Then we need to find the maximum and minimum values of $f(x, y)$. Here $f_x(x, y) = [2x - 2(L - x)][y^2 + (W - y)^2] = 0 \;\Rightarrow\; 4x - 2L = 0$ or $x = \frac{1}{2}L$, and $f_y(x, y) = [x^2 + (L - x)^2][2y - 2(W - y)] = 0 \;\Rightarrow\; 4y - 2W = 0$ or $y = W/2$. Also $f_{xx} = 4[y^2 + (W - y)^2]$, $f_{yy} = 4[x^2 + (L - x)^2]$, and $f_{xy} = (4x - 2L)(4y - 2W)$. Then $D = 16[y^2 + (W - y)^2][x^2 + (L - x)^2] - (4x - 2L)^2(4y - 2W)^2$. Thus when $x = \frac{1}{2}L$ and $y = \frac{1}{2}W$, $D > 0$ and $f_{xx} = 2W^2 > 0$. Thus a minimum of f occurs at $\left(\frac{1}{2}L, \frac{1}{2}W\right)$ and this minimum value is $f\left(\frac{1}{2}L, \frac{1}{2}W\right) = \frac{1}{4}L^2W^2$. There are no other critical points, so the maximum must occur on the boundary. Now along the width of the rectangle let $g(y) = f(0, y) = f(L, y) = L^2[y^2 + (W - y)^2]$, $0 \le y \le W$. Then $g'(y) = L^2[2y - 2(W - y)] = 0 \;\Leftrightarrow\; y = \frac{1}{2}W$. And $g\left(\frac{1}{2}\right) = \frac{1}{2}L^2W^2$. Checking the endpoints, we get $g(0) = g(W) = L^2W^2$. Along the length of the rectangle let $h(x) = f(x, 0) = f(x, W) = W^2[x^2 + (L - x)^2]$, $0 \le x \le L$. By symmetry $h'(x) = 0 \;\Leftrightarrow\; x = \frac{1}{2}L$ and $h\left(\frac{1}{2}L\right) = \frac{1}{2}L^2W^2$. At the endpoints we have $h(0) = h(L) = L^2W^2$. Therefore L^2W^2 is the maximum value of f. This maximum value of f occurs when the "cutting" lines correspond to sides of the rectangle.

2. (a) The level curves of the function $C(x, y) = e^{-(x^2 + 2y^2)/10^4}$ are the curves $e^{-(x^2 + 2y^2)/10^4} = k$ (k is a positive constant). This equation is equivalent to $x^2 + 2y^2 = K \;\Rightarrow$

$$\frac{x^2}{\left(\sqrt{K}\right)^2} + \frac{y^2}{\left(\sqrt{K/2}\right)^2} = 1, \text{ where } K = -10^4 \ln k,$$



a family of ellipses. We sketch level curves for $K = 1, 2, 3$, and 4. If the shark always swims in the direction of maximum increase of blood concentration, its direction at any point would coincide with the gradient vector. Then we know the shark's path is perpendicular to the level curves it intersects. We sketch one example of such a path.

(b) $\nabla C = -\frac{2}{10^4} e^{-(x^2+2y^2)/10^4}(x\,\mathbf{i} + 2y\,\mathbf{j})$. And ∇C points in the direction of most rapid increase in concentration; that is, ∇C is tangent to the most rapid increase curve. If $\mathbf{r}(t) = x(t)\,\mathbf{i} + y(t)\,\mathbf{j}$ is a parametrization of the most rapid increase curve, then $\dfrac{d\mathbf{r}}{dt} = \dfrac{dx}{dt}\,\mathbf{i} + \dfrac{dy}{dt}\,\mathbf{j}$ is tangent to the curve, so $\dfrac{d\mathbf{r}}{dt} = \lambda\nabla C$

$\Rightarrow \quad \dfrac{dx}{dt} = \lambda\left[-\dfrac{2}{10^4} e^{-(x^2+2y^2)/10^4}\right] x$ and $\dfrac{dy}{dt} = \lambda\left[-\dfrac{2}{10^4} e^{-(x^2+2y^2)/10^4}\right](2y)$. Therefore

$\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = 2\dfrac{y}{x} \quad \Rightarrow \quad \dfrac{dy}{y} = 2\dfrac{dx}{x} \quad \Rightarrow \quad \ln|y| = 2\ln|x|$ so that $y = kx^2$ for some constant k. But

$y(x_0) = y_0 \quad \Rightarrow \quad y_0 = kx_0^2 \quad \Rightarrow \quad k = y_0/x_0^2 \quad (x_0 = 0 \quad \Rightarrow \quad y_0 = 0 \quad \Rightarrow$ the shark is already at the origin, so we can assume $x_0 \neq 0$.) Therefore the path the shark will follow is along the parabola $y = y_0(x/x_0)^2$.

3. (a) The area of a trapezoid is $\frac{1}{2}h(b_1 + b_2)$, where h is the height (the distance between the two parallel sides) and b_1, b_2 are the lengths of the bases (the parallel sides). From the figure in the text, we see that $h = x\sin\theta$, $b_1 = w - 2x$, and $b_2 = w - 2x + 2x\cos\theta$. Therefore the cross-sectional area of the rain gutter is

$$\begin{aligned} A(x,\theta) &= \tfrac{1}{2}x\sin\theta\,[(w-2x) + (w - 2x + 2x\cos\theta)] = (x\sin\theta)(w - 2x + x\cos\theta) \\ &= wx\sin\theta - 2x^2\sin\theta + x^2\sin\theta\cos\theta,\ 0 < x \le \tfrac{1}{2}w, 0 < \theta \le \tfrac{\pi}{2} \end{aligned}$$

We look for the critical points of A: $\partial A/\partial x = w\sin\theta - 4x\sin\theta + 2x\sin\theta\cos\theta$ and $\partial A/\partial\theta = wx\cos\theta - 2x^2\cos\theta + x^2(\cos^2\theta - \sin^2\theta)$, so $\partial A/\partial x = 0 \quad \Leftrightarrow \quad \sin\theta\,(w - 4x + 2x\cos\theta) = 0$

$\Leftrightarrow \quad \cos\theta = \dfrac{4x - w}{2x} = 2 - \dfrac{w}{2x} \quad (0 < \theta \le \frac{\pi}{2} \quad \Rightarrow \quad \sin\theta > 0)$. If, in addition, $\partial A/\partial\theta = 0$, then

$$\begin{aligned} 0 &= wx\cos\theta - 2x^2\cos\theta + x^2(2\cos^2\theta - 1) \\ &= wx\left(2 - \frac{w}{2x}\right) - 2x^2\left(2 - \frac{w}{2x}\right) + x^2\left[2\left(2 - \frac{w}{2x}\right)^2 - 1\right] \\ &= 2wx - \tfrac{1}{2}w^2 - 4x^2 + wx + x^2\left[8 - \frac{4w}{x} + \frac{w^2}{2x^2} - 1\right] = -wx + 3x^2 = x(3x - w) \end{aligned}$$

Since $x > 0$, we must have $x = \frac{1}{3}w$, in which case $\cos\theta = \frac{1}{2}$, so $\theta = \frac{\pi}{3}$, $\sin\theta = \frac{\sqrt{3}}{2}$, $k = \frac{\sqrt{3}}{6}w$, $b_1 = \frac{1}{3}w$, $b_2 = \frac{2}{3}w$, and $A = \frac{\sqrt{3}}{12}w^2$. As in Example 15.7.6 [ET 14.7.6], we can argue from the physical nature of this problem that we have found a local maximum of A. Now checking the boundary of A, let $g(\theta) = A(w/2, \theta) = \frac{1}{2}w^2\sin\theta - \frac{1}{2}w^2\sin\theta + \frac{1}{4}w^2\sin\theta\cos\theta = \frac{1}{8}w^2\sin 2\theta$, $0 < \theta \le \frac{\pi}{2}$. Clearly g is maximized when $\sin 2\theta = 1$ in which case $A = \frac{1}{8}w^2$. Also along the line $\theta = \frac{\pi}{2}$, let $h(x) = A\left(x, \frac{\pi}{2}\right) = wx - 2x^2$, $0 < x < \frac{1}{2}w \quad \Rightarrow \quad h'(x) = w - 4x = 0 \quad \Leftrightarrow \quad x = \frac{1}{4}w$, and $h\left(\frac{1}{4}w\right) = w\left(\frac{1}{4}w\right) - 2\left(\frac{1}{4}w\right)^2 = \frac{1}{8}w^2$. Since $\frac{1}{8}w^2 < \frac{\sqrt{3}}{12}w^2$, we conclude that the local maximum found earlier was an absolute maximum.

(b) If the metal were bent into a semi-circular gutter of radius r, we would have $w = \pi r$ and $A = \frac{1}{2}\pi r^2 = \frac{1}{2}\pi\left(\dfrac{w}{\pi}\right)^2 = \dfrac{w^2}{2\pi}$. Since $\dfrac{w^2}{2\pi} > \dfrac{\sqrt{3}\,w^2}{12}$, it *would* be better to bend the metal into a gutter with a semicircular cross-section.

4. Since $(x+y+z)^r/(x^2+y^2+z^2)$ is a rational function with domain $\{(x,y,z) \mid (x,y,z) \neq (0,0,0)\}$, f is continuous on $\mathbb{R}^3$ if and only if $\lim\limits_{(x,y,z)\to(0,0,0)} f(x,y,z) = f(0,0,0) = 0$. Recall that

$(a+b)^2 \leq 2a^2 + 2b^2$ and a double application of this inequality to $(x+y+z)^2$

gives $(x+y+z)^2 \leq 4x^2 + 4y^2 + 2z^2 \leq 4(x^2+y^2+z^2)$. Now for each r,

$$|(x+y+z)^r| = \left(|x+y+z|^2\right)^{r/2} = \left[(x+y+z)^2\right]^{r/2} \leq \left[4(x^2+y^2+z^2)\right]^{r/2} = 2^r(x^2+y^2+z^2)^{r/2}$$

for $(x,y,z) \neq (0,0,0)$. Thus

$$|f(x,y,z) - 0| = \left|\frac{(x+y+z)^r}{x^2+y^2+z^2}\right| = \frac{|(x+y+z)^r|}{x^2+y^2+z^2} \leq 2^r \frac{(x^2+y^2+z^2)^{r/2}}{x^2+y^2+z^2} = 2^r(x^2+y^2+z^2)^{(r/2)-1}$$

for $(x,y,z) \neq (0,0,0)$. Thus if $(r/2) - 1 > 0$, that is $r > 2$, then $2^r(x^2+y^2+z^2)^{(r/2)-1} \to 0$ as $(x,y,z) \to (0,0,0)$ and so $\lim\limits_{(x,y,z)\to(0,0,0)} (x+y+z)^r/(x^2+y^2+z^2) = 0$. Hence for $r > 2$, f is continuous on $\mathbb{R}^3$. Now if $r \leq 2$, then as $(x,y,z) \to (0,0,0)$ along the x-axis, $f(x,0,0) = x^r/x^2 = x^{r-2}$ for $x \neq 0$. So when $r = 2$, $f(x,y,z) \to 1 \neq 0$ as $(x,y,z) \to (0,0,0)$ along the x-axis and when $r < 2$ the limit of $f(x,y,z)$ as $(x,y,z) \to (0,0,0)$ along the x-axis doesn't exist and thus can't be zero. Hence for $r \leq 2$ f isn't continuous at $(0,0,0)$ and thus is not continuous on $\mathbb{R}^3$.

5. Let $g(x,y) = xf\left(\frac{y}{x}\right)$. Then $g_x(x,y) = f\left(\frac{y}{x}\right) + xf'\left(\frac{y}{x}\right)\left(-\frac{y}{x^2}\right) = f\left(\frac{y}{x}\right) - \frac{y}{x} f'\left(\frac{y}{x}\right)$ and

$g_y(x,y) = xf'\left(\frac{y}{x}\right)\left(\frac{1}{x}\right) = f'\left(\frac{y}{x}\right)$. Thus the tangent plane at (x_0, y_0, z_0) on the surface has equation

$$z - x_0 f\left(\frac{y_0}{x_0}\right) = \left[f\left(\frac{y_0}{x_0}\right) - y_0 x_0^{-1} f'\left(\frac{y_0}{x_0}\right)\right](x - x_0) + f'\left(\frac{y_0}{x_0}\right)(y - y_0) \quad \Rightarrow$$

$\left[f\left(\frac{y_0}{x_0}\right) - y_0 x_0^{-1} f'\left(\frac{y_0}{x_0}\right)\right]x + \left[f'\left(\frac{y_0}{x_0}\right)\right]y - z = 0$. But any plane whose equation is of the form $ax + by + cz = 0$ passes through the origin. Thus the origin is the common point of intersection.

6. (a) At $(x_1, y_1, 0)$ the equations of the tangent planes to $z = f(x,y)$ and $z = g(x,y)$ are

$$P_1\colon\ z - f(x_1,y_1) = f_x(x_1,y_1)(x - x_1) + f_y(x_1,y_1)(y - y_1)$$

and

$$P_2\colon\ z - g(x_1,y_1) = g_x(x_1,y_1)(x - x_1) + g_y(x_1,y_1)(y - y_1)$$

respectively. P_1 intersects the xy-plane in the line given by $f_x(x_1,y_1)(x-x_1) + f_y(x_1,y_1)(y-y_1) = -f(x_1,y_1)$, $z = 0$; and P_2 intersects the xy-plane in the line given by $g_x(x_1,y_1)(x-x_1) + g_y(x_1,y_1)(y-y_1) = -g(x_1,y_1)$, $z = 0$. The point $(x_2, y_2, 0)$ is the point of intersection of these two lines, since $(x_2, y_2, 0)$ is the point where the line of intersection of the two tangent planes intersects the xy-plane. Thus (x_2, y_2) is the solution of the simultaneous equations

$$f_x(x_1,y_1)(x_2 - x_1) + f_y(x_1,y_1)(y_2 - y_1) = -f(x_1,y_1)$$

and

$$g_x(x_1,y_1)(x_2 - x_1) + g_y(x_1,y_1)(y_2 - y_1) = -g(x_1,y_1)$$

For simplicity, rewrite $f_x(x_1,y_1)$ as f_x and similarly for f_y, g_x, g_y, f and g and solve the equations

$(f_x)(x_2 - x_1) + (f_y)(y_2 - y_1) = -f$ and $(g_x)(x_2 - x_1) + (g_y)(y_2 - y_1) = -g$

simultaneously for $(x_2 - x_1)$ and $(y_2 - y_1)$. Then $y_2 - y_1 = \dfrac{gf_x - fg_x}{g_x f_y - f_x g_y}$ or

$y_2 = y_1 - \dfrac{gf_x - fg_x}{f_x g_y - g_x f_y}$ and $(f_x)(x_2 - x_1) + \dfrac{(f_y)(gf_x - fg_x)}{g_x f_y - f_x g_y} = -f$ so

$x_2 - x_1 = \dfrac{-f - [(f_y)(gf_x - fg_x)/(g_x f_y - f_x g_y)]}{f_x} = \dfrac{fg_y - f_y g}{g_x f_y - f_x g_y}$. Hence $x_2 = x_1 - \dfrac{fg_y - f_y g}{f_x g_y - g_x f_y}$.

(b) Let $f(x, y) = x^x + y^y - 1000$ and $g(x, y) = x^y + y^x - 100$. Then we wish to solve the system of equations $f(x, y) = 0$, $g(x, y) = 0$. Recall $\dfrac{d}{dx}[x^x] = x^x(1 + \ln x)$ (differentiate logarithmically), so $f_x(x, y) = x^x(1 + \ln x)$, $f_y(x, y) = y^y(1 + \ln y)$, $g_x(x, y) = yx^{y-1} + y^x \ln y$, and $g_y(x, y) = x^y \ln x + xy^{x-1}$. Looking at the graph, we estimate the first point of intersection of the curves, and thus the solution to the system, to be approximately $(2.5, 4.5)$. Then following the method of part (a), $x_1 = 2.5$, $y_1 = 4.5$ and

$$x_2 = 2.5 - \frac{f(2.5, 4.5)\, g_y(2.5, 4.5) - f_y(2.5, 4.5)\, g(2.5, 4.5)}{f_x(2.5, 4.5)\, g_y(2.5, 4.5) - f_y(2.5, 4.5)\, g_x(2.5, 4.5)} \approx 2.447674117$$

$$y_2 = 4.5 - \frac{f_x(2.5, 4.5)\, g(2.5, 4.5) - f(2.5, 4.5)\, g_x(2.5, 4.5)}{f_x(2.5, 4.5)\, g_y(2.5, 4.5) - f_y(2.5, 4.5)\, g_x(2.5, 4.5)} \approx 4.555657467$$

Continuing this procedure, we arrive at the following values. (If you use a CAS, you may need to increase its computational precision.)

$x_1 = 2.5$	$y_1 = 4.5$
$x_2 = 2.447674117$	$y_2 = 4.555657467$
$x_3 = 2.449614877$	$y_3 = 4.551969333$
$x_4 = 2.449624628$	$y_4 = 4.551951420$
$x_5 = 2.449624628$	$y_5 = 4.551951420$

Thus, to six decimal places, the point of intersection is $(2.449625, 4.551951)$. The second point of intersection can be found similarly, or, by symmetry it is approximately $(4.551951, 2.449625)$.

7. (a) $x = r\cos\theta$, $y = r\sin\theta$, $z = z$. Then $\dfrac{\partial u}{\partial r} = \dfrac{\partial u}{\partial x}\dfrac{\partial x}{\partial r} + \dfrac{\partial u}{\partial y}\dfrac{\partial y}{\partial r} + \dfrac{\partial u}{\partial z}\dfrac{\partial z}{\partial r} = \dfrac{\partial u}{\partial x}\cos\theta + \dfrac{\partial u}{\partial y}\sin\theta$ and

$$\frac{\partial^2 u}{\partial r^2} = \cos\theta\left[\frac{\partial^2 u}{\partial x^2}\frac{\partial x}{\partial r} + \frac{\partial^2 u}{\partial y\,\partial x}\frac{\partial y}{\partial r} + \frac{\partial^2 u}{\partial z\,\partial x}\frac{\partial z}{\partial r}\right] + \sin\theta\left[\frac{\partial^2 u}{\partial y^2}\frac{\partial y}{\partial r} + \frac{\partial^2 u}{\partial x\,\partial y}\frac{\partial x}{\partial r} + \frac{\partial^2 u}{\partial z\,\partial y}\frac{\partial z}{\partial r}\right]$$

$$= \frac{\partial^2 u}{\partial x^2}\cos^2\theta + \frac{\partial^2 u}{\partial y^2}\sin^2\theta + 2\,\frac{\partial^2 u}{\partial y\,\partial x}\cos\theta\sin\theta$$

Similarly $\dfrac{\partial u}{\partial \theta} = -\dfrac{\partial u}{\partial x} r\sin\theta + \dfrac{\partial u}{\partial y} r\cos\theta$ and

$$\frac{\partial^2 u}{\partial \theta^2} = \frac{\partial^2 u}{\partial x^2} r^2 \sin^2\theta + \frac{\partial^2 u}{\partial y^2} r^2 \cos^2\theta - 2\frac{\partial^2 u}{\partial y\,\partial x} r^2 \sin\theta\cos\theta - \frac{\partial u}{\partial x} r\cos\theta - \frac{\partial u}{\partial y} r\sin\theta.$$ So

$$\begin{aligned}\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial z^2} &= \frac{\partial^2 u}{\partial x^2}\cos^2\theta + \frac{\partial^2 u}{\partial y^2}\sin^2\theta + 2\frac{\partial^2 u}{\partial y\,\partial x}\cos\theta\sin\theta + \frac{\partial u}{\partial x}\frac{\cos\theta}{r} + \frac{\partial u}{\partial y}\frac{\sin\theta}{r} \\ &\quad + \frac{\partial^2 u}{\partial x^2}\sin^2\theta + \frac{\partial^2 u}{\partial y^2}\cos^2\theta - 2\frac{\partial^2 u}{\partial y\,\partial x}\sin\theta\cos\theta - \frac{\partial u}{\partial x}\frac{\cos\theta}{r} - \frac{\partial u}{\partial y}\frac{\sin\theta}{r} + \frac{\partial^2 u}{\partial z^2} \\ &= \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\end{aligned}$$

(b) $x = \rho\sin\phi\cos\theta$, $y = \rho\sin\phi\sin\theta$, $z = \rho\cos\phi$. Then

$$\frac{\partial u}{\partial \rho} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \rho} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \rho} + \frac{\partial u}{\partial z}\frac{\partial z}{\partial \rho} = \frac{\partial u}{\partial x}\sin\phi\cos\theta + \frac{\partial u}{\partial y}\sin\phi\sin\theta + \frac{\partial u}{\partial z}\cos\phi,$$ and

$$\begin{aligned}\frac{\partial^2 u}{\partial \rho^2} &= \sin\phi\cos\theta\left[\frac{\partial^2 u}{\partial x^2}\frac{\partial x}{\partial \rho} + \frac{\partial^2 u}{\partial y\,\partial x}\frac{\partial y}{\partial \rho} + \frac{\partial^2 u}{\partial z\,\partial x}\frac{\partial z}{\partial \rho}\right] \\ &\quad + \sin\phi\sin\theta\left[\frac{\partial^2 u}{\partial y^2}\frac{\partial y}{\partial \rho} + \frac{\partial^2 u}{\partial x\,\partial y}\frac{\partial x}{\partial \rho} + \frac{\partial^2 u}{\partial z\,\partial y}\frac{\partial z}{\partial \rho}\right] \\ &\quad + \cos\phi\left[\frac{\partial^2 u}{\partial z^2}\frac{\partial z}{\partial \rho} + \frac{\partial^2 u}{\partial x\,\partial z}\frac{\partial x}{\partial \rho} + \frac{\partial^2 u}{\partial y\,\partial z}\frac{\partial y}{\partial \rho}\right] \\ &= 2\frac{\partial^2 u}{\partial y\,\partial x}\sin^2\phi\sin\theta\cos\theta + 2\frac{\partial^2 u}{\partial z\,\partial x}\sin\phi\cos\phi\cos\theta + 2\frac{\partial^2 u}{\partial y\,\partial z}\sin\phi\cos\phi\sin\theta \\ &\quad + \frac{\partial^2 u}{\partial x^2}\sin^2\phi\cos^2\theta + \frac{\partial^2 u}{\partial y^2}\sin^2\phi\sin^2\theta + \frac{\partial^2 u}{\partial z^2}\cos^2\phi\end{aligned}$$

Similarly $\dfrac{\partial u}{\partial \phi} = \dfrac{\partial u}{\partial x}\rho\cos\phi\cos\theta + \dfrac{\partial u}{\partial y}\rho\cos\phi\sin\theta - \dfrac{\partial u}{\partial z}\rho\sin\phi$, and

$$\begin{aligned}\frac{\partial^2 u}{\partial \phi^2} &= 2\frac{\partial^2 u}{\partial y\,\partial x}\rho^2\cos^2\phi\sin\theta\cos\theta - 2\frac{\partial^2 u}{\partial x\,\partial z}\rho^2\sin\phi\cos\phi\cos\theta \\ &\quad - 2\frac{\partial^2 u}{\partial y\,\partial z}\rho^2\sin\phi\cos\phi\sin\theta + \frac{\partial^2 u}{\partial x^2}\rho^2\cos^2\phi\cos^2\theta + \frac{\partial^2 u}{\partial y^2}\rho^2\cos^2\phi\sin^2\theta \\ &\quad + \frac{\partial^2 u}{\partial z^2}\rho^2\sin^2\phi - \frac{\partial u}{\partial x}\rho\sin\phi\cos\theta - \frac{\partial u}{\partial y}\rho\sin\phi\sin\theta - \frac{\partial u}{\partial z}\rho\cos\phi\end{aligned}$$

And $\dfrac{\partial u}{\partial \theta} = -\dfrac{\partial u}{\partial x}\rho\sin\phi\sin\theta + \dfrac{\partial u}{\partial y}\rho\sin\phi\cos\theta$, while

$$\begin{aligned}\frac{\partial^2 u}{\partial \theta^2} &= -2\frac{\partial^2 u}{\partial y\,\partial x}\rho^2\sin^2\phi\cos\theta\sin\theta + \frac{\partial^2 u}{\partial x^2}\rho^2\sin^2\phi\sin^2\theta \\ &\quad + \frac{\partial^2 u}{\partial y^2}\rho^2\sin^2\phi\cos^2\theta - \frac{\partial u}{\partial x}\rho\sin\phi\cos\theta - \frac{\partial u}{\partial y}\rho\sin\phi\sin\theta\end{aligned}$$

Therefore

$$\frac{\partial^2 u}{\partial \rho^2} + \frac{2}{\rho}\frac{\partial u}{\partial \rho} + \frac{\cot\phi}{\rho^2}\frac{\partial u}{\partial \phi} + \frac{1}{\rho^2}\frac{\partial^2 u}{\partial \phi^2} + \frac{1}{\rho^2 \sin^2\phi}\frac{\partial^2 u}{\partial \theta^2}$$

$$= \frac{\partial^2 u}{\partial x^2}\left[(\sin^2\phi\cos^2\theta) + (\cos^2\phi\cos^2\theta) + \sin^2\theta\right]$$

$$+ \frac{\partial^2 u}{\partial y^2}\left[(\sin^2\phi\sin^2\theta) + (\cos^2\phi\sin^2\theta) + \cos^2\theta\right] + \frac{\partial^2 u}{\partial z^2}\left[\cos^2\phi + \sin^2\phi\right]$$

$$+ \frac{\partial u}{\partial x}\left[\frac{2\sin^2\phi\cos\theta + \cos^2\phi\cos\theta - \sin^2\phi\cos\theta - \cos\theta}{\rho\sin\phi}\right]$$

$$+ \frac{\partial u}{\partial y}\left[\frac{2\sin^2\phi\sin\theta + \cos^2\phi\sin\theta - \sin^2\phi\sin\theta - \sin\theta}{\rho\sin\phi}\right]$$

But $2\sin^2\phi\cos\theta + \cos^2\phi\cos\theta - \sin^2\phi\cos\theta - \cos\theta = (\sin^2\phi + \cos^2\phi - 1)\cos\theta = 0$ and similarly the coefficient of $\partial u/\partial y$ is 0. Also $\sin^2\phi\cos^2\theta + \cos^2\phi\cos^2\theta + \sin^2\theta = \cos^2\theta(\sin^2\phi + \cos^2\phi) + \sin^2\theta = 1$, and similarly the coefficient of $\partial^2 u/\partial y^2$ is 1. So Laplace's Equation in spherical coordinates is as stated.

8. The tangent plane to the surface $xy^2z^2 = 1$, at the point (x_0, y_0, z_0) is

$y_0^2z_0^2(x - x_0) + 2x_0y_0z_0^2(y - y_0) + 2x_0y_0^2z_0(z - z_0) = 0 \quad \Rightarrow$

$(y_0^2z_0^2)x + (2x_0y_0z_0^2)y + (2x_0y_0^2z_0)z = 5x_0y_0^2z_0^2 = 5$. Using the formula derived in

Example 13.5.8 [ET 12.5.8], we find that the distance from $(0, 0, 0)$ to this tangent plane is

$$D(x_0, y_0, z_0) = \frac{|5x_0y_0^2z_0^2|}{\sqrt{(y_0^2z_0^2)^2 + (2x_0y_0z_0^2)^2 + (2x_0y_0^2z_0)^2}}.$$

When D is a maximum, D^2 is a maximum and $\nabla D^2 = \mathbf{0}$. Dropping the subscripts, let

$f(x, y, z) = D^2 = \dfrac{25(xyz)^2}{y^2z^2 + 4x^2z^2 + 4x^2y^2}$. Now use the fact that for points on the surface $xy^2z^2 = 1$ we have

$z^2 = \dfrac{1}{xy^2}$, to get $f(x, y) = D^2 = \dfrac{25x}{\dfrac{1}{x} + \dfrac{4x}{y^2} + 4x^2y^2} = \dfrac{25x^2y^2}{y^2 + 4x^2 + 4x^3y^4}$. Now $\nabla D^2 = \mathbf{0} \quad \Rightarrow \quad f_x = 0$

and $f_y = 0$. $f_x = 0 \quad \Rightarrow \quad \dfrac{50xy^2(y^2 + 4x^2 + 4x^3y^4) - (8x + 12x^2y^4)(25x^2y^2)}{(y^2 + 4x^2 + 4x^3y^4)^2} = 0 \quad \Rightarrow$

$xy^2(y^2 + 4x^2 + 4x^3y^4) - (4x + 6x^2y^4)x^2y^2 = 0 \quad \Rightarrow \quad xy^4 - 2x^4y^6 = 0 \quad \Rightarrow \quad xy^4(1 - 2x^3y^2) = 0 \quad \Rightarrow$

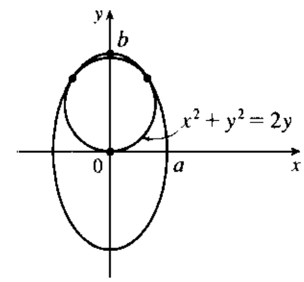
$1 = 2y^2x^3$ (since $x = 0$, $y = 0$ both give a minimum distance of 0). Also $f_y = 0 \quad \Rightarrow$

$\dfrac{50x^2y(y^2 + 4x^2 + 4x^3y^4) - (2y + 16x^3y^3)25x^2y^2}{(y^2 + 4x^2 + 4x^3y^4)^2} = 0 \quad \Rightarrow \quad 4x^4y - 4x^5y^5 = 0 \quad \Rightarrow \quad x^4y(1 - xy^4) = 0$

$\Rightarrow \quad 1 = xy^4$. Now substituting $x = 1/y^4$ into $1 = 2y^2x^3$, we get $1 = 2y^{-10} \quad \Rightarrow \quad y = \pm 2^{1/10} \quad \Rightarrow$

$x = 2^{-2/5} \quad \Rightarrow \quad z^2 = \dfrac{1}{xy^2} = \dfrac{1}{(2^{-2/5})(2^{1/5})} = 2^{1/5} \quad \Rightarrow \quad z = \pm 2^{1/10}$.

Therefore the tangent planes that are farthest from the origin are at the four points $(2^{-2/5}, \pm 2^{1/10}, \pm 2^{1/10})$. These points all give a maximum since the minimum distance occurs when $x_0 = 0$ or $y_0 = 0$ in which case $D = 0$. The equations are $(2^{1/5} 2^{1/5})x \pm [(2)(2^{-2/5})(2^{1/10})(2^{1/5})]y \pm [(2)(2^{-2/5})(2^{1/5})(2^{1/10})]z = 5 \Rightarrow$ $(2^{2/5})x \pm (2^{9/10})y \pm (2^{9/10})z = 5$.



9. Since we are minimizing the area of the ellipse, and the circle lies above the x-axis, the ellipse will intersect the circle for only one value of y. This y-value must satisfy both the equation of the circle and the equation of the ellipse. Now $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1 \Rightarrow x^2 = \dfrac{a^2}{b^2}(b^2 - y^2)$. Substituting into the equation of the circle gives $\dfrac{a^2}{b^2}(b^2 - y^2) + y^2 - 2y = 0 \Rightarrow$ $\left(\dfrac{b^2 - a^2}{b^2}\right)y^2 - 2y + a^2 = 0$. In order for there to be only one solution to this quadratic equation, the discriminant must be 0, so $4 - 4a^2\dfrac{b^2 - a^2}{b^2} = 0 \Rightarrow b^2 - a^2 b^2 + a^4 = 0$. The area of the ellipse is $A(a, b) = \pi ab$, and we minimize this function subject to the constraint $g(a, b) = b^2 - a^2 b^2 + a^4 = 0$.

Now $\nabla A = \lambda \nabla g \Leftrightarrow \pi b = \lambda(4a^3 - 2ab^2), \pi a = \lambda(2b - 2ba^2) \Rightarrow$ (1) $\lambda = \dfrac{\pi b}{2a(2a^2 - b^2)}$,

(2) $\lambda = \dfrac{\pi a}{2b(1 - a^2)}$, (3) $b^2 - a^2 b^2 + a^4 = 0$. Comparing (1) and (2) gives $\dfrac{\pi b}{2a(2a^2 - b^2)} = \dfrac{\pi a}{2b(1 - a^2)} \Rightarrow$

$2\pi b^2 = 4\pi a^4 \Leftrightarrow a^2 = \frac{1}{\sqrt{2}} b$. Substitute this into (3) to get $b = \frac{3}{\sqrt{2}} \Rightarrow a = \sqrt{\frac{3}{2}}$.