Chapter 16

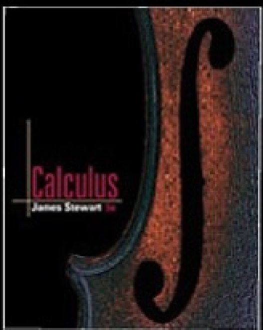
Adapted from the

Complete Solutions Manual

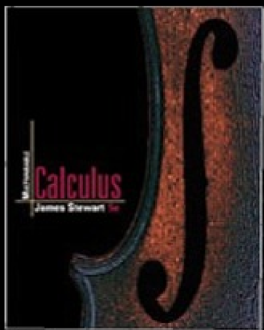
for

James Stewart's

Calculus – 5th Edition



Calculus 5e
James Stewart
ISBN 0-534-39339-X



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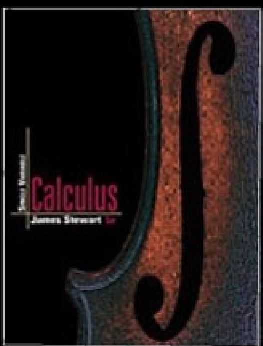

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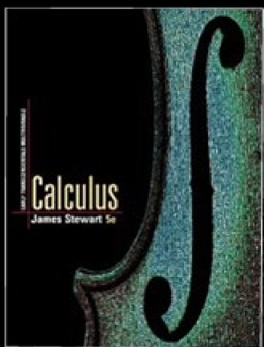

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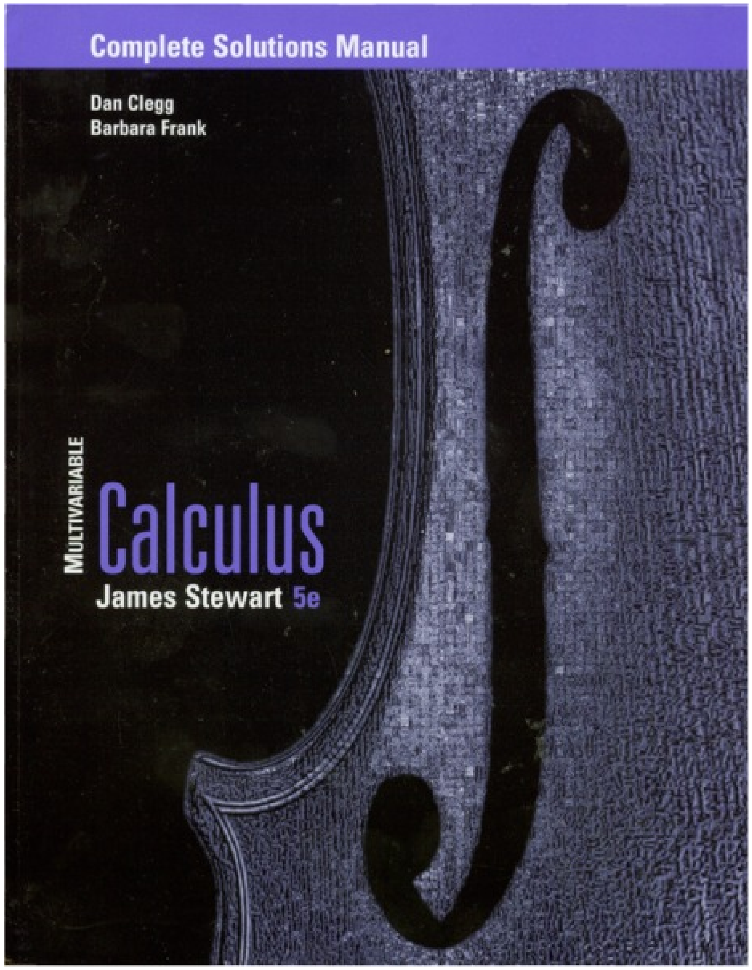
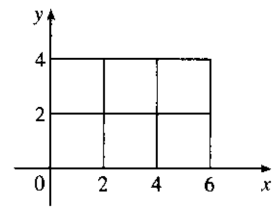


16 □ MULTIPLE INTEGRALS □ ET 15

16.1 Double Integrals over Rectangles ET 15.1

1. (a) The subrectangles are shown in the figure.



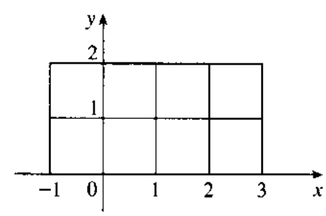
The surface is the graph of $f(x,y)=xy$ and $\Delta A=4$,

so we estimate

$$V\approx \sum_{i=1}^{3}\sum_{j=1}^{2} f(x_i,y_j)\,\Delta A$$
$$= f(2,2)\,\Delta A + f(2,4)\,\Delta A + f(4,2)\,\Delta A + f(4,4)\,\Delta A + f(6,2)\,\Delta A + f(6,4)\,\Delta A$$
$$= 4(4)+8(4)+8(4)+16(4)+12(4)+24(4)=288$$

(b) $$V\approx \sum_{i=1}^{3}\sum_{j=1}^{2} f(\overline{x}_i,\overline{y}_j)\,\Delta A$$
$$= f(1,1)\,\Delta A + f(1,3)\,\Delta A + f(3,1)\,\Delta A + f(3,3)\,\Delta A + f(5,1)\,\Delta A + f(5,3)\,\Delta A$$
$$= 1(4)+3(4)+3(4)+9(4)+5(4)+15(4)=144$$

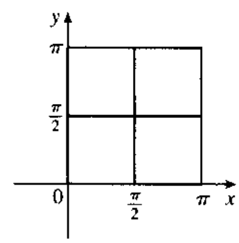
2. The subrectangles are shown in the figure.



Since $\Delta A=1$, we estimate

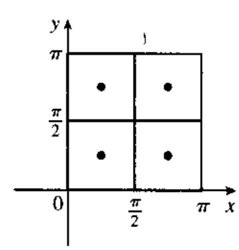
$$\iint_R (y^2-2x^2)\,dA\approx \sum_{i=1}^{4}\sum_{j=1}^{2} f(x_{ij}^*,y_{ij}^*)\,\Delta A$$
$$= f(-1,1)\,\Delta A + f(-1,2)\,\Delta A + f(0,1)\,\Delta A + f(0,2)\,\Delta A$$
$$+ f(1,1)\,\Delta A + f(1,2)\,\Delta A + f(2,1)\,\Delta A + f(2,2)\,\Delta A$$
$$= -1(1)+2(1)+1(1)+4(1)-1(1)+2(1)-7(1)-4(1)=-4$$

3. (a) The subrectangles are shown in the figure. Since $\Delta A=\pi^2/4$, we estimate



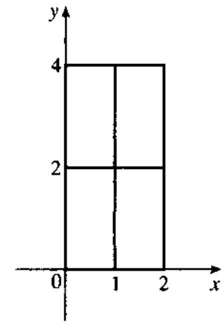
$$\iint_R \sin(x+y)\,dA\approx \sum_{i=1}^{2}\sum_{j=1}^{2} f(x_{ij}^*,y_{ij}^*)\,\Delta A$$
$$= f(0,0)\,\Delta A + f(0,\tfrac{\pi}{2})\,\Delta A + f(\tfrac{\pi}{2},0)\,\Delta A + f(\tfrac{\pi}{2},\tfrac{\pi}{2})\,\Delta A$$
$$= 0\left(\tfrac{\pi^2}{4}\right)+1\left(\tfrac{\pi^2}{4}\right)+1\left(\tfrac{\pi^2}{4}\right)+0\left(\tfrac{\pi^2}{4}\right)=\tfrac{\pi^2}{2}\approx 4.935$$

(b) $$\iint_R \sin(x+y)\,dA\approx \sum_{i=1}^{2}\sum_{j=1}^{2} f(\overline{x}_i,\overline{y}_j)\,\Delta A$$



$$= f(\tfrac{\pi}{4},\tfrac{\pi}{4})\,\Delta A + f(\tfrac{\pi}{4},\tfrac{3\pi}{4})\,\Delta A$$
$$+ f(\tfrac{3\pi}{4},\tfrac{\pi}{4})\,\Delta A + f(\tfrac{3\pi}{4},\tfrac{3\pi}{4})\,\Delta A$$
$$= 1\left(\tfrac{\pi^2}{4}\right)+0\left(\tfrac{\pi^2}{4}\right)+0\left(\tfrac{\pi^2}{4}\right)+(-1)\left(\tfrac{\pi^2}{4}\right)=0$$

4. (a) The subrectangles are shown in the figure.



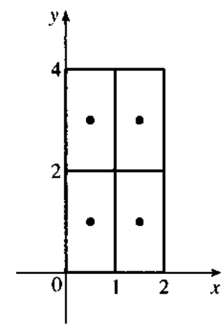
The surface is the graph of $f(x,y) = x + 2y^2$ and $\Delta A = 2$, so we estimate

$$V = \iint_R (x + 2y^2)\,dA \approx \sum_{i=1}^{2}\sum_{j=1}^{2} f\left(x_{ij}^*, y_{ij}^*\right)\Delta A$$

$$= f(1,0)\,\Delta A + f(1,2)\,\Delta A + f(2,0)\,\Delta A + f(2,2)\,\Delta A$$

$$= 1(2) + 9(2) + 2(2) + 10(2) = 44$$

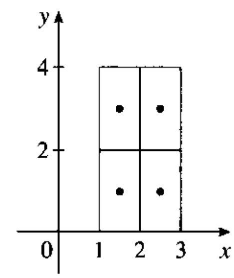
(b) $$V = \iint_R (x + 2y^2)\,dA \approx \sum_{i=1}^{2}\sum_{j=1}^{2} f(\overline{x}_i, \overline{y}_j)\,\Delta A$$



$$= f\left(\tfrac{1}{2},1\right)\Delta A + f\left(\tfrac{1}{2},3\right)\Delta A + f\left(\tfrac{3}{2},1\right)\Delta A + f\left(\tfrac{3}{2},3\right)\Delta A$$

$$= \tfrac{5}{2}(2) + \tfrac{37}{2}(2) + \tfrac{7}{2}(2) + \tfrac{39}{2}(2) = 88$$

5. (a) Each subrectangle and its midpoint are shown in the figure. The area of each subrectangle is $\Delta A = 2$, so we evaluate f at each midpoint and estimate

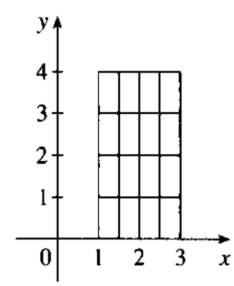


$$\iint_R f(x,y)\,dA \approx \sum_{i=1}^{2}\sum_{j=1}^{2} f(\overline{x}_i, \overline{y}_j)\,\Delta A$$

$$= f(1.5,1)\,\Delta A + f(1.5,3)\,\Delta A + f(2.5,1)\,\Delta A + f(2.5,3)\,\Delta A$$

$$= 1(2) + (-8)(2) + 5(2) + (-1)(2) = -6$$

(b) The subrectangles are shown in the figure. In each subrectangle, the sample point farthest from the origin is the upper right corner, and the area of each subrectangle is $\Delta A = \frac{1}{2}$. Thus we estimate



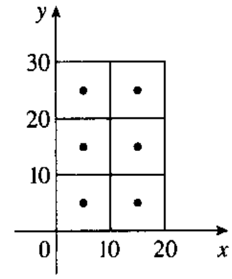
$$\iint_R f(x,y)\,dA \approx \sum_{i=1}^{4}\sum_{j=1}^{4} f(x_i, y_j)\,\Delta A$$

$$= f(1.5,1)\,\Delta A + f(1.5,2)\,\Delta A + f(1.5,3)\,\Delta A + f(1.5,4)\,\Delta A$$
$$+ f(2,1)\,\Delta A + f(2,2)\,\Delta A + f(2,3)\,\Delta A + f(2,4)\,\Delta A$$
$$+ f(2.5,1)\,\Delta A + f(2.5,2)\,\Delta A + f(2.5,3)\,\Delta A + f(2.5,4)\,\Delta A$$
$$+ f(3,1)\,\Delta A + f(3,2)\,\Delta A + f(3,3)\,\Delta A + f(3,4)\,\Delta A$$

$$= 1\left(\tfrac{1}{2}\right) + (-4)\left(\tfrac{1}{2}\right) + (-8)\left(\tfrac{1}{2}\right) + (-6)\left(\tfrac{1}{2}\right) + 3\left(\tfrac{1}{2}\right) + 0\left(\tfrac{1}{2}\right) + (-5)\left(\tfrac{1}{2}\right) + (-8)\left(\tfrac{1}{2}\right)$$
$$+ 5\left(\tfrac{1}{2}\right) + 3\left(\tfrac{1}{2}\right) + (-1)\left(\tfrac{1}{2}\right) + (-4)\left(\tfrac{1}{2}\right) + 8\left(\tfrac{1}{2}\right) + 6\left(\tfrac{1}{2}\right) + 3\left(\tfrac{1}{2}\right) + 0\left(\tfrac{1}{2}\right)$$

$$= -3.5$$

6. To approximate the volume, let R be the planar region corresponding to the surface of the water in the pool, and place R on coordinate axes so that x and y correspond to the dimensions given. Then we define $f(x, y)$ to be the depth of the water at (x, y), so the volume of water in the pool is the volume of the solid that lies above the rectangle $R = [0, 20] \times [0, 30]$ and below the graph of $f(x, y)$. We can estimate this volume using the Midpoint Rule with $m = 2$ and $n = 3$, so $\Delta A = 100$. Each subrectangle with its midpoint is shown in the figure. Then



$$
\begin{aligned}
V &\approx \sum_{i=1}^{2} \sum_{j=1}^{3} f(\overline{x}_i, \overline{y}_j)\, \Delta A \\
&= \Delta A[f(5,5) + f(5,15) + f(5,25) + f(15,5) + f(15,15) + f(15,25)] \\
&= 100(3 + 7 + 10 + 3 + 5 + 8) = 3600
\end{aligned}
$$

Thus, we estimate that the pool contains 3600 cubic feet of water.

Alternatively, we can approximate the volume with a Riemann sum where $m = 4$, $n = 6$ and the sample points are taken to be, for example, the upper right corner of each subrectangle. Then $\Delta A = 25$ and

$$
\begin{aligned}
V &\approx \sum_{i=1}^{4} \sum_{j=1}^{6} f(x_i, y_j)\, \Delta A \\
&= 25[3 + 4 + 7 + 8 + 10 + 8 + 4 + 6 + 8 + 10 + 12 + 10 + 3 + 4 \\
&\qquad + 5 + 6 + 8 + 7 + 2 + 2 + 2 + 3 + 4 + 4] \\
&= 25(140) = 3500
\end{aligned}
$$

So we estimate that the pool contains 3500 ft^3 of water.

7. The values of $f(x, y) = \sqrt{52 - x^2 - y^2}$ get smaller as we move farther from the origin, so on any of the subrectangles in the problem, the function will have its largest value at the lower left corner of the subrectangle and its smallest value at the upper right corner, and any other value will lie between these two. So using these subrectangles we have $U < V < L$. (Note that this is true no matter how R is divided into subrectangles.)

8. From the level curves we see that $f\left(\frac{1}{2}, \frac{1}{2}\right) \approx 11$. So, using the Midpoint Rule with only one subrectangle, we get $\iint_R f(x, y)\, dA \approx 1 \cdot f\left(\frac{1}{2}, \frac{1}{2}\right) \approx 11$. Dividing R into four squares of equal size, we get $\iint_R f(x, y)\, dA \approx \frac{1}{4}\left[f\left(\frac{1}{4}, \frac{1}{4}\right) + f\left(\frac{1}{4}, \frac{3}{4}\right) + f\left(\frac{3}{4}, \frac{1}{4}\right) + f\left(\frac{3}{4}, \frac{3}{4}\right)\right] \approx \frac{1}{4}(11 + 13 + 9.5 + 11) \approx 11$. Using sixteen squares we get the same result. So $\iint_R f(x, y)\, dA \approx 11$.

9. (a) With $m = n = 2$, we have $\Delta A = 4$. Using the contour map to estimate the value of f at the center of each subrectangle, we have

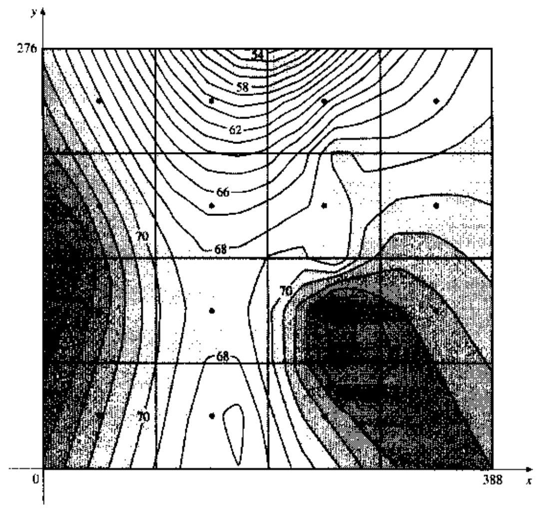
$$
\begin{aligned}
\iint_R f(x, y)\, dA &\approx \sum_{i=1}^{2} \sum_{j=1}^{2} f(\overline{x}_i, \overline{y}_j)\, \Delta A = \Delta A[f(1,1) + f(1,3) + f(3,1) + f(3,3)] \\
&\approx 4(27 + 4 + 14 + 17) = 248
\end{aligned}
$$

(b) $f_{\text{ave}} = \frac{1}{A(R)} \iint_R f(x, y)\, dA \approx \frac{1}{16}(248) = 15.5$

10. As in Example 4, we place the origin at the southwest corner of the state. Then $R = [0, 388] \times [0, 276]$ (in miles) is the rectangle corresponding to Colorado and we define $f(x, y)$ to be the temperature at the location (x, y). The average temperature is given by

$$f_{\text{ave}} = \frac{1}{A(R)} \iint_R f(x, y)\, dA = \frac{1}{388 \cdot 276} \iint_R f(x, y)\, dA$$

We can use the Midpoint Rule with $m = n = 4$ to give a reasonable estimate of the value of the double integral.



Thus, we divide R into 16 regions of equal size, as shown in the figure, with the center of each subrectangle indicated. The area of each subrectangle is $\Delta A = \frac{388}{4} \cdot \frac{276}{4} = 6693$, so using the contour map to estimate the function values at each midpoint, we have

$$\begin{aligned}\iint_R f(x, y)\, dA &\approx \sum_{i=1}^{4} \sum_{j=1}^{4} f(\overline{x}_i, \overline{y}_j)\, \Delta A \\ &\approx \Delta A\,[72.2 + 73.6 + 72.1 + 68.2 + 67.4 + 68.5 + 66.7 + 60.3 \\ &\qquad + 72.0 + 74.9 + 68.4 + 63.7 + 73.2 + 72.3 + 70.3 + 67.7] \\ &= 6693(1111.5)\end{aligned}$$

Therefore, $f_{\text{ave}} \approx \dfrac{6693 \cdot 1111.5}{388 \cdot 276} \approx 69.5$, so the average temperature in Colorado on May 1, 1996, was approximately 69.5°F.

Alternatively, we can use the Midpoint Rule with $m = n = 2$ which is easier computationally but will most likely be less accurate since we have fewer subrectangles. In this case, $\Delta A = \frac{388}{2} \cdot \frac{276}{2} = 26{,}772$ and we can use the same grid to estimate the function values at the midpoints of the four subrectangles. Then

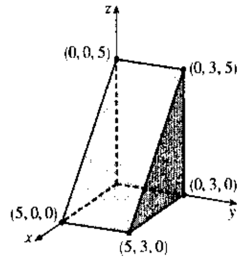
$$\begin{aligned}\iint_R f(x, y)\, dA &\approx \sum_{i=1}^{2} \sum_{j=1}^{2} f(\overline{x}_i, \overline{y}_j)\, \Delta A \approx 26{,}772[70.0 + 66.5 + 74.3 + 68.5] \\ &= 26{,}772 \cdot 279.3\end{aligned}$$

and $f_{\text{ave}} \approx \dfrac{26{,}772 \cdot 279.3}{388 \cdot 276} \approx 69.8$°F.

11. $z = 3 > 0$, so we can interpret the integral as the volume of the solid S that lies below the plane $z = 3$ and above the rectangle $[-2, 2] \times [1, 6]$. S is a rectangular solid, thus $\iint_R 3\, dA = 4 \cdot 5 \cdot 3 = 60$.

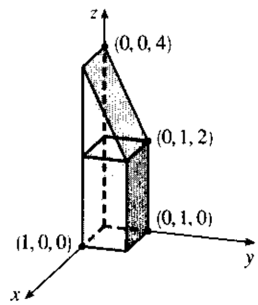
12. $z = 5 - x \geq 0$ for $0 \leq x \leq 5$, so we can interpret the integral as the volume of the solid S that lies below the plane $z = 5 - x$ and above the rectangle $[0, 5] \times [0, 3]$. S is a triangular cylinder whose volume is $3(\text{area of triangle}) = 3\left(\frac{1}{2} \cdot 5 \cdot 5\right) = 37.5$. Thus,

$$\iint_R (5 - x)\, dA = 37.5$$

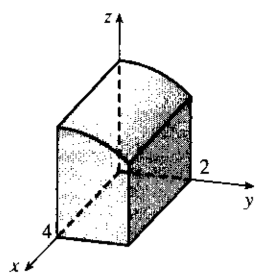


13. $z = f(x, y) = 4 - 2y \geq 0$ for $0 \leq y \leq 1$. Thus the integral represents the volume of that part of the rectangular solid $[0, 1] \times [0, 1] \times [0, 4]$ which lies below the plane $z = 4 - 2y$. So

$$\iint_R (4 - 2y)\, dA = (1)(1)(2) + \tfrac{1}{2}(1)(1)(2) = 3$$



14. Here $z = \sqrt{9 - y^2}$, so $z^2 + y^2 = 9$, $z \geq 0$. Thus the integral represents the volume of the top half of the part of the circular cylinder $z^2 + y^2 = 9$ that lies above the rectangle $[0, 4] \times [0, 2]$.



15. To calculate the estimates using a programmable calculator, we can use an algorithm similar to that of Exercise 5.1.7 [ET 5.1.7]. In Maple, we can define the function $f(x, y) = e^{-x^2 - y^2}$ (calling it `f`), load the `student` package, and then use the command

```
middlesum(middlesum(f,x=0..1,m),
                    y=0..1,m);
```

to get the estimate with $n = m^2$ squares of equal size. Mathematica has no special Riemann sum command, but we can define `f` and then use nested `Sum` commands to calculate the estimates.

n	estimate
1	0.6065
4	0.5694
16	0.5606
64	0.5585
256	0.5579
1024	0.5578

16.

n	estimate
1	0.9922
4	0.9262
16	0.8797

n	estimate
64	0.8660
256	0.8625
1024	0.8616

17. If we divide R into mn subrectangles, $\iint_R k\,dA \approx \sum\limits_{i=1}^{m}\sum\limits_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\,\Delta A$ for any choice of sample points (x_{ij}^*, y_{ij}^*). But $f(x_{ij}^*, y_{ij}^*) = k$ always and $\sum\limits_{i=1}^{m}\sum\limits_{j=1}^{n}\Delta A =$ area of $R = (b-a)(d-c)$. Thus, no matter how we choose the sample points, $\sum\limits_{i=1}^{m}\sum\limits_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\,\Delta A = k\sum\limits_{i=1}^{m}\sum\limits_{j=1}^{n}\Delta A = k(b-a)(d-c)$ and so

$$\iint_R k\,dA = \lim_{m,n\to\infty}\sum_{i=1}^{m}\sum_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\,\Delta A = \lim_{m,n\to\infty} k\sum_{i=1}^{m}\sum_{j=1}^{n}\Delta A$$
$$= \lim_{m,n\to\infty} k(b-a)(d-c) = k(b-a)(d-c)$$

18. On R, $0 \le x+y \le 2 < \pi$ and $\sin\theta \ge 0$ for $0 \le \theta \le \pi$. Thus $f(x,y) = \sin(x+y) \ge 0$ for all $(x,y) \in R$. Since $0 \le \sin(x+y) \le 1$, Property (9) gives $\iint_R 0\,dA \le \iint_R \sin(x+y)\,dA \le \iint_R 1\,dA$, so by Exercise 17 we have $0 \le \iint_R \sin(x+y)\,dA \le 1$.

16.2 Iterated Integrals ET 15.2

1. $\int_0^3 (2x + 3x^2 y)\,dx = \left[x^2 + x^3 y\right]_{x=0}^{x=3} = (9 + 27y) - (0+0) = 9 + 27y,$

$\int_0^4 (2x + 3x^2 y)\,dy = \left[2xy + 3x^2\,\dfrac{y^2}{2}\right]_{y=0}^{y=4} = \left(8x + 3x^2 \cdot \dfrac{16}{2}\right) - (0+0) = 8x + 24x^2$

2. $\displaystyle\int_0^3 \frac{y}{x+2}\,dx = y\ln|x+2|\,\Big]_{x=0}^{x=3} = y\ln 5 - y\ln 2 = y\ln\frac{5}{2},$

$\displaystyle\int_0^4 \frac{y}{x+2}\,dy = \frac{1}{x+2}\left[\frac{y^2}{2}\right]_{y=0}^{y=4} = \frac{1}{x+2}\left(\frac{16}{2} - 0\right) = \frac{8}{x+2}$

3. $\int_1^3\int_0^1 (1 + 4xy)\,dx\,dy = \int_1^3 \left[x + 2x^2 y\right]_{x=0}^{x=1} dy = \int_1^3 (1 + 2y)\,dy = \left[y + y^2\right]_1^3 = (3+9) - (1+1) = 10$

4. $\int_2^4\int_{-1}^1 (x^2 + y^2)\,dy\,dx = \int_2^4 \left[x^2 y + \frac{1}{3}y^3\right]_{y=-1}^{y=1} dx = \int_2^4 \left[\left(x^2 + \frac{1}{3}\right) - \left(-x^2 - \frac{1}{3}\right)\right] dx$

$= \int_2^4 \left(2x^2 + \frac{2}{3}\right) dx = \left[\frac{2}{3}x^3 + \frac{2}{3}x\right]_2^4 = \left(\frac{128}{3} + \frac{8}{3}\right) - \left(\frac{16}{3} + \frac{4}{3}\right) = \frac{116}{3}$

5. $\int_0^2\int_0^{\pi/2} x\sin y\,dy\,dx = \int_0^2 x\,dx\int_0^{\pi/2}\sin y\,dy$ [as in Example 5]

$= \left[\dfrac{x^2}{2}\right]_0^2 \Big[-\cos y\Big]_0^{\pi/2} = (2-0)(0+1) = 2.$

6. $\int_1^4\int_0^2 (x + \sqrt{y})\,dx\,dy = \int_1^4 \left[\frac{1}{2}x^2 + x\sqrt{y}\right]_{x=0}^{x=2} dy = \int_1^4 (2 + 2\sqrt{y})\,dy$

$= \left[2y + 2\cdot\frac{2}{3}y^{3/2}\right]_1^4 = \left(8 + \frac{4}{3}\cdot 8\right) - \left(2 + \frac{4}{3}\right) = \frac{46}{3}$

7. $\int_0^2 \int_0^1 (2x+y)^8 \, dx \, dy = \int_0^2 \left[\frac{1}{2}\frac{(2x+y)^9}{9}\right]_{x=0}^{x=1} dy$ [substitute $u = 2x + y \Rightarrow dx = \frac{1}{2} du$]

$$= \frac{1}{18}\int_0^2 [(2+y)^9 - (0+y)^9] \, dy = \frac{1}{18}\left[\frac{(2+y)^{10}}{10} - \frac{y^{10}}{10}\right]_0^2$$

$$= \tfrac{1}{180}[(4^{10} - 2^{10}) - (2^{10} - 0^{10})] = \tfrac{1{,}046{,}528}{180} = \tfrac{261{,}632}{45}$$

8. $\int_0^1 \int_1^2 \frac{xe^x}{y} \, dy \, dx = \int_0^1 xe^x \, dx \int_1^2 \frac{1}{y} \, dy$ [as in Example 5]

$= \left[xe^x - e^x\right]_0^1 \left[\ln|y|\right]_1^2$ [by integrating by parts]

$$= [(e - e) - (0 - 1)](\ln 2 - 0) = \ln 2$$

9. $\int_1^4 \int_1^2 \left(\frac{x}{y} + \frac{y}{x}\right) dy \, dx = \int_1^4 \left[x \ln|y| + \frac{1}{x}\cdot\frac{1}{2}y^2\right]_{y=1}^{y=2} dx = \int_1^4 \left(x \ln 2 + \frac{3}{2x}\right) dx$

$$= \left[\tfrac{1}{2}x^2 \ln 2 + \tfrac{3}{2}\ln|x|\right]_1^4 = 8\ln 2 + \tfrac{3}{2}\ln 4 - \tfrac{1}{2}\ln 2$$

$$= \tfrac{15}{2}\ln 2 + 3\ln 4^{1/2} = \tfrac{21}{2}\ln 2$$

10. $\int_1^2 \int_0^1 (x+y)^{-2} \, dx \, dy = \int_1^2 \left[-(x+y)^{-1}\right]_{x=0}^{x=1} dy = \int_1^2 \left[y^{-1} - (1+y)^{-1}\right] dy$

$$= \left[\ln y - \ln(1+y)\right]_1^2 = \ln 2 - \ln 3 - 0 + \ln 2 = \ln \tfrac{4}{3}$$

11. $\int_0^{\ln 2} \int_0^{\ln 5} e^{2x-y} \, dx \, dy = \left(\int_0^{\ln 5} e^{2x} \, dx\right)\left(\int_0^{\ln 2} e^{-y} \, dy\right) = \left[\tfrac{1}{2}e^{2x}\right]_0^{\ln 5} \left[-e^{-y}\right]_0^{\ln 2}$

$$= \left(\tfrac{25}{2} - \tfrac{1}{2}\right)\left(-\tfrac{1}{2} + 1\right) = 6$$

12. $\int_0^1 \int_0^1 \frac{xy}{\sqrt{x^2+y^2+1}} \, dy \, dx = \int_0^1 \left[x\sqrt{x^2+y^2+1}\right]_{y=0}^{y=1} dx = \int_0^1 x\left(\sqrt{x^2+2} - \sqrt{x^2+1}\right) dx$

$$= \tfrac{1}{3}\left[(x^2+2)^{3/2} - (x^2+1)^{3/2}\right]_0^1 = \tfrac{1}{3}\left[(3^{3/2} - 2^{3/2}) - (2^{3/2} - 1)\right]$$

$$= \tfrac{1}{3}\left(3\sqrt{3} - 4\sqrt{2} + 1\right)$$

13. $\iint_R (6x^2y^3 - 5y^4) \, dA = \int_0^3 \int_0^1 (6x^2y^3 - 5y^4) \, dy \, dx = \int_0^3 \left[\tfrac{3}{2}x^2y^4 - y^5\right]_{y=0}^{y=1} dx$

$$= \int_0^3 \left(\tfrac{3}{2}x^2 - 1\right) dx = \left[\tfrac{1}{2}x^3 - x\right]_0^3 = \tfrac{27}{2} - 3 = \tfrac{21}{2}$$

14. $\iint_R \cos(x+2y) \, dA = \int_0^\pi \int_0^{\pi/2} \cos(x+2y) \, dy \, dx$

$$= \int_0^\pi \left[\tfrac{1}{2}\sin(x+2y)\right]_{y=0}^{y=\pi/2} dx = \tfrac{1}{2}\int_0^\pi (\sin(x+\pi) - \sin x) \, dx$$

$$= \tfrac{1}{2}\left[-\cos(x+\pi) + \cos x\right]_0^\pi = \tfrac{1}{2}\left[-\cos 2\pi + \cos\pi - (-\cos\pi + \cos 0)\right]$$

$$= \tfrac{1}{2}(-1 - 1 - (1+1)) = -2$$

15. $\iint_R \frac{xy^2}{x^2+1} \, dA = \int_0^1 \int_{-3}^3 \frac{xy^2}{x^2+1} \, dy \, dx = \int_0^1 \frac{x}{x^2+1} \, dx \int_{-3}^3 y^2 \, dy$

$$= \left[\tfrac{1}{2}\ln(x^2+1)\right]_0^1 \left[\tfrac{1}{3}y^3\right]_{-3}^3 = \tfrac{1}{2}(\ln 2 - \ln 1)\cdot\tfrac{1}{3}(27 + 27) = 9\ln 2$$

16. $\displaystyle\iint_R \frac{1+x^2}{1+y^2}\,dA = \int_0^1\int_0^1 \frac{1+x^2}{1+y^2}\,dy\,dx = \int_0^1 (1+x^2)\,dx \int_0^1 \frac{1}{1+y^2}\,dy$

$= \left[x + \tfrac{1}{3}x^3\right]_0^1 \left[\tan^{-1} y\right]_0^1 = \left(1 + \tfrac{1}{3} - 0\right)\left(\tfrac{\pi}{4} - 0\right) = \tfrac{\pi}{3}$

17. $\int_0^{\pi/6}\int_0^{\pi/3} x\sin(x+y)\,dy\,dx$

$= \int_0^{\pi/6}\left[-x\cos(x+y)\right]_{y=0}^{y=\pi/3}\,dx = \int_0^{\pi/6}\left[x\cos x - x\cos\left(x+\tfrac{\pi}{3}\right)\right]dx$

$= x\left[\sin x - \sin\left(x+\tfrac{\pi}{3}\right)\right]_0^{\pi/6} - \int_0^{\pi/6}\left[\sin x - \sin\left(x+\tfrac{\pi}{3}\right)\right]dx$

[by integrating by parts separately for each term]

$= \tfrac{\pi}{6}\left[\tfrac{1}{2} - 1\right] - \left[-\cos x + \cos\left(x+\tfrac{\pi}{3}\right)\right]_0^{\pi/6} = -\tfrac{\pi}{12} - \left[-\tfrac{\sqrt{3}}{2} + 0 - \left(-1 + \tfrac{1}{2}\right)\right]$

$= \tfrac{\sqrt{3}-1}{2} - \tfrac{\pi}{12}$

18. $\displaystyle\iint_R \frac{x}{1+xy}\,dA = \int_0^1\int_0^1 \frac{x}{1+xy}\,dy\,dx$

$= \int_0^1 \left[\ln(1+xy)\right]_{y=0}^{y=1}\,dx = \int_0^1 \left[\ln(1+x) - \ln 1\right]dx$

$= \int_0^1 \ln(1+x)\,dx = \left[(1+x)\ln(1+x) - x\right]_0^1$ [by integrating by parts]

$= (2\ln 2 - 1) - (\ln 1 - 0) = 2\ln 2 - 1$

19. $\iint_R xye^{x^2y}\,dA = \int_0^2\int_0^1 xye^{x^2y}\,dx\,dy = \int_0^2 \left[\tfrac{1}{2}e^{x^2y}\right]_{x=0}^{x=1} dy = \tfrac{1}{2}\int_0^2 (e^y - 1)\,dy$

$= \tfrac{1}{2}\left[e^y - y\right]_0^2 = \tfrac{1}{2}\left[(e^2 - 2) - (1 - 0)\right] = \tfrac{1}{2}(e^2 - 3)$

20. $\displaystyle\int_0^1\int_1^2 \frac{x}{x^2+y^2}\,dx\,dy = \int_0^1 \left[\tfrac{1}{2}\ln(x^2+y^2)\right]_{x=1}^{x=2} dy = \tfrac{1}{2}\int_0^1 \left[\ln(4+y^2) - \ln(1+y^2)\right]dy$

To evaluate the first term, we integrate by parts with $u = \ln(4+y^2) \;\Rightarrow\; du = \dfrac{2y}{4+y^2}\,dy$ and $dv = dy \;\Rightarrow\; v = y$. Then

$$\int \ln(4+y^2)\,dy = y\ln(4+y^2) - \int \frac{2y^2}{4+y^2}\,dy = y\ln(4+y^2) - \int\left(2 - \frac{8}{4+y^2}\right)dy$$

$$= y\ln(4+y^2) - 2y + 8\cdot\frac{1}{2}\tan^{-1}\left(\frac{y}{2}\right) = y\ln(4+y^2) - 2y + 4\tan^{-1}\left(\frac{y}{2}\right)$$

Similarly, $\int \ln(1+y^2)\,dy = y\ln(1+y^2) - 2y + 2\tan^{-1} y$. Thus,

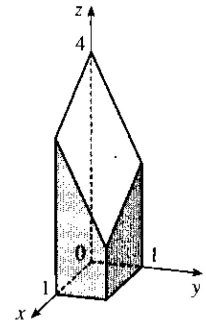
$$\int_0^1\int_1^2 \frac{x}{x^2+y^2}\,dx\,dy = \frac{1}{2}\int_0^1 \left[\ln(4+y^2) - \ln(1+y^2)\right]dy$$

$$= \frac{1}{2}\left[y\ln(4+y^2) - 2y + 4\tan^{-1}\left(\frac{y}{2}\right) - y\ln(1+y^2) + 2y - 2\tan^{-1} y\right]_0^1$$

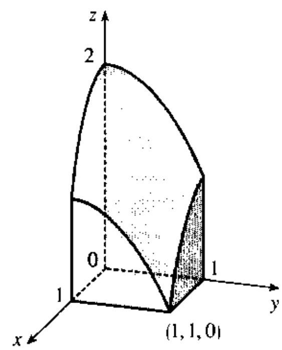
$$= \tfrac{1}{2}\left[\left(\ln 5 + 4\tan^{-1}\left(\tfrac{1}{2}\right) - \ln 2 - 2\tan^{-1} 1\right) - 0\right]$$

$$= \tfrac{1}{2}\left[\ln 5 - \ln 2 + 4\tan^{-1}\left(\tfrac{1}{2}\right) - 2\left(\tfrac{\pi}{4}\right)\right] = \tfrac{1}{2}\ln\tfrac{5}{2} + 2\tan^{-1}\left(\tfrac{1}{2}\right) - \tfrac{\pi}{4}$$

21. $z = f(x, y) = 4 - x - 2y \geq 0$ for $0 \leq x \leq 1$ and $0 \leq y \leq 1$. So the solid is the region in the first octant which lies below the plane $z = 4 - x - 2y$ and above $[0, 1] \times [0, 1]$.



22. $z = 2 - x^2 - y^2 \geq 0$ for $0 \leq x \leq 1$ and $0 \leq y \leq 1$. So the solid is the region in the first octant which lies below the circular paraboloid $z = 2 - x^2 - y^2$ and above $[0, 1] \times [0, 1]$.



23. $V = \iint_R (12 - 3x - 2y)\, dA = \int_{-2}^{3} \int_0^1 (12 - 3x - 2y)\, dx\, dy = \int_{-2}^{3} \left[12x - \frac{3}{2}x^2 - 2xy\right]_{x=0}^{x=1} dy$
$= \int_{-2}^{3} \left(\frac{21}{2} - 2y\right) dy = \left[\frac{21}{2}y - y^2\right]_{-2}^{3} = \frac{95}{2}$

24. $V = \iint_R (4 + x^2 - y^2)\, dA = \int_{-1}^{1} \int_0^2 (4 + x^2 - y^2)\, dy\, dx = \int_{-1}^{1} \left[4y + x^2 y - \frac{1}{3}y^3\right]_{y=0}^{y=2} dx$
$= \int_{-1}^{1} \left(2x^2 + \frac{16}{3}\right) dx = \left[\frac{2}{3}x^3 + \frac{16}{3}x\right]_{-1}^{1} = \frac{2}{3} + \frac{16}{3} + \frac{2}{3} + \frac{16}{3} = 12$

25. $V = \int_{-2}^{2} \int_{-1}^{1} \left(1 - \frac{1}{4}x^2 - \frac{1}{9}y^2\right) dx\, dy = 4 \int_0^2 \int_0^1 \left(1 - \frac{1}{4}x^2 - \frac{1}{9}y^2\right) dx\, dy$
$= 4 \int_0^2 \left[x - \frac{1}{12}x^3 - \frac{1}{9}y^2 x\right]_{x=0}^{x=1} dy = 4 \int_0^2 \left(\frac{11}{12} - \frac{1}{9}y^2\right) dy = 4\left[\frac{11}{12}y - \frac{1}{27}y^3\right]_0^2 = 4 \cdot \frac{83}{54} = \frac{166}{27}$

26. $V = \int_{-1}^{1} \int_0^{\pi} (1 + e^x \sin y)\, dy\, dx = \int_{-1}^{1} \left[y - e^x \cos y\right]_{y=0}^{y=\pi} dx = \int_{-1}^{1} (\pi + e^x - 0 + e^x)\, dx$
$= \int_{-1}^{1} (\pi + 2e^x)\, dx = \left[\pi x + 2e^x\right]_{-1}^{1} = 2\pi + 2e - \frac{2}{e}$

27. Here we need the volume of the solid lying under the surface $z = x\sqrt{x^2 + y}$ and above the square $R = [0, 1] \times [0, 1]$ in the xy-plane.

$$V \int_0^1 \int_0^1 x\sqrt{x^2 + y}\, dx\, dy = \int_0^1 \frac{1}{3}\left[(x^2 + y)^{3/2}\right]_{x=0}^{x=1} dy = \frac{1}{3} \int_0^1 \left[(1 + y)^{3/2} - y^{3/2}\right] dy$$
$$= \frac{1}{3} \cdot \frac{2}{5}\left[(1 + y)^{5/2} - y^{5/2}\right]_0^1 = \frac{4}{15}\left(2\sqrt{2} - 1\right)$$

28. Here we need the volume of the solid lying under the surface $z = 1 + (x - 1)^2 + 4y^2$ and above the rectangle $R = [0, 3] \times [0, 2]$ in the xy-plane.

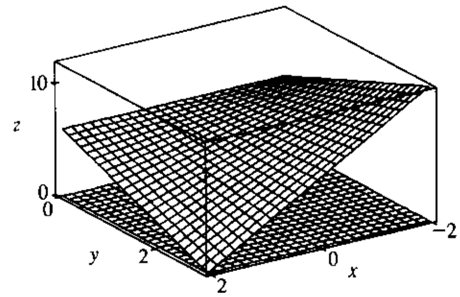
$$V = \int_0^3 \int_0^2 \left[1 + (x - 1)^2 + 4y^2\right] dy\, dx = \int_0^3 \left[y + (x - 1)^2 y + \frac{4}{3}y^3\right]_{y=0}^{y=2} dx$$
$$= \int_0^3 \left[2 + 2(x - 1)^2 + \frac{32}{3}\right] dx = \left[\frac{38}{3}x + \frac{2}{3}(x - 1)^3\right]_0^3 = 44$$

29. In the first octant, $z \geq 0 \quad \Rightarrow \quad y \leq 3$, so
$V = \int_0^3 \int_0^2 (9 - y^2)\, dx\, dy = \int_0^3 \left[9x - y^2 x\right]_{x=0}^{x=2} dy = \int_0^3 (18 - 2y^2)\, dy = \left[18y - \frac{2}{3}y^3\right]_0^3 = 36$

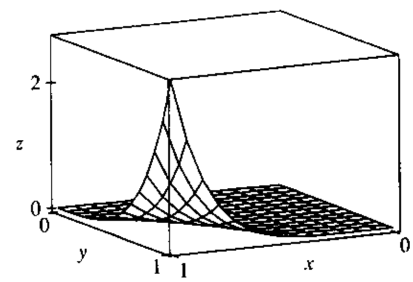
30. (a) Here we need the volume of the solid lying under the surface $z = 6 - xy$ and above the rectangle $R = [-2, 2] \times [0, 3]$ in the xy-plane.

$$\begin{aligned} V &= \int_{-2}^{2} \int_{0}^{3} (6 - xy)\, dy\, dx \\ &= \int_{-2}^{2} \left[6y - \tfrac{1}{2}xy^2\right]_{y=0}^{y=3} dx \\ &= \int_{-2}^{2} \left(18 - \tfrac{9}{2}x\right) dx \\ &= \left[18x - \tfrac{9}{4}x^2\right]_{-2}^{2} = 72 \end{aligned}$$

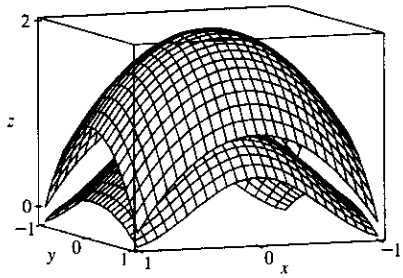
(b) The solid occupies the region between the two surfaces shown.



31. In Maple, we can calculate the integral by defining the integrand as `f` and then using the command `int(int(f,x=0..1),y=0..1);`. In Mathematica, we can use the command `Integrate[Integrate[f,{x,0,1}],{y,0,1}]`. We find that $\iint_R x^5 y^3 e^{xy}\, dA = 21e - 57 \approx 0.0839$. We can use `plot3d` (in Maple) or `Plot3d` (in Mathematica) to graph the function.



32. In Maple, we can calculate the integral by defining `f:=exp(-x^2)*cos(x^2+y^2);` and `g:=2-x^2-y^2;` and then [since $2 - x^2 - y^2 > e^{-x^2} \cos(x^2 + y^2)$ for $-1 \le x \le 1$, $-1 \le y \le 1$] using the command `evalf(int(int(g-f,x=-1..1),y=-1..1),5);`. In Mathematica, we can use the command `N[Integrate[Integrate[f,{x,0,1}],{y,0,1}],5]`.



In each of these commands, the `5` indicates that we want only five significant digits; this speeds up the calculation considerably. We find that $\iint_R \left[(2 - x^2 - y^2) - \left(e^{-x^2} \cos(x^2 + y^2)\right)\right] dA \approx 3.0271$. We can use the `plot3d` command (in Maple) or `Plot3d` (in Mathematica) to graph both functions on the same screen.

33. R is the rectangle $[-1, 1] \times [0, 5]$. Thus, $A(R) = 2 \cdot 5 = 10$ and

$$f_{\text{ave}} = \frac{1}{A(R)} \iint_R f(x, y)\, dA = \tfrac{1}{10} \int_0^5 \int_{-1}^{1} x^2 y\, dx\, dy = \tfrac{1}{10} \int_0^5 \left[\tfrac{1}{3}x^3 y\right]_{x=-1}^{x=1} dy = \tfrac{1}{10} \int_0^5 \tfrac{2}{3} y\, dy$$

$$= \tfrac{1}{10} \left[\tfrac{1}{3}y^2\right]_0^5 = \tfrac{5}{6}$$

34. $A(R) = 4 \cdot 1 = 4$, so

$$f_{\text{ave}} = \frac{1}{A(R)} \iint_R f(x, y)\, dA = \tfrac{1}{4} \int_0^4 \int_0^1 e^y \sqrt{x + e^y}\, dy\, dx = \tfrac{1}{4} \int_0^4 \left[\tfrac{2}{3}(x + e^y)^{3/2}\right]_{y=0}^{y=1} dx$$

$$= \tfrac{1}{4} \cdot \tfrac{2}{3} \int_0^4 \left[(x + e)^{3/2} - (x + 1)^{3/2}\right] dx = \tfrac{1}{6} \left[\tfrac{2}{5}(x + e)^{5/2} - \tfrac{2}{5}(x + 1)^{5/2}\right]_0^4$$

$$= \tfrac{1}{6} \cdot \tfrac{2}{5} \left[(4 + e)^{5/2} - 5^{5/2} - e^{5/2} + 1\right] = \tfrac{1}{15} \left[(4 + e)^{5/2} - e^{5/2} - 5^{5/2} + 1\right] \approx 3.327$$

35. Let $f(x,y) = \dfrac{x-y}{(x+y)^3}$. Then a CAS gives $\int_0^1 \int_0^1 f(x,y)\,dy\,dx = \frac{1}{2}$ and $\int_0^1 \int_0^1 f(x,y)\,dx\,dy = -\frac{1}{2}$.

To explain the seeming violation of Fubini's Theorem, note that f has an infinite discontinuity at $(0,0)$ and thus does not satisfy the conditions of Fubini's Theorem. In fact, both iterated integrals involve improper integrals which diverge at their lower limits of integration.

36. (a) Loosely speaking, Fubini's Theorem says that the order of integration of a function of two variables does not affect the value of the double integral, while Clairaut's Theorem says that the order of differentiation of such a function does not affect the value of the second-order derivative. Also, both theorems require continuity (though Fubini's allows a finite number of smooth curves to contain discontinuities).

(b) To find g_{xy}, we first hold y constant and use the single-variable Fundamental Theorem of Calculus, Part 1: $g_x = \dfrac{d}{dx}\,g(x,y) = \dfrac{d}{dx}\displaystyle\int_a^x \left(\int_c^y f(s,t)\,dt\right) ds = \int_c^y f(x,t)\,dt$. Now we use the Fundamental Theorem again: $g_{xy} = \dfrac{d}{dy}\displaystyle\int_c^y f(x,t)\,dt = f(x,y)$.

To find g_{yx}, we first use Fubini's Theorem to find that $\int_a^x \int_c^y f(s,t)\,dt\,ds = \int_c^y \int_a^x f(s,t)\,dt\,ds$, and then use the Fundamental Theorem twice, as above, to get $g_{yx} = f(x,y)$. So $g_{xy} = g_{yx} = f(x,y)$.

16.3 Double Integrals over General Regions ET 15.3

1. $\int_0^1 \int_0^{x^2} (x+2y)\,dy\,dx = \int_0^1 \left[xy + y^2\right]_{y=0}^{y=x^2} dx = \int_0^1 \left[x(x^2) + (x^2)^2 - 0 - 0\right] dx$
$= \int_0^1 (x^3 + x^4)\,dx = \left[\frac{1}{4}x^4 + \frac{1}{5}x^5\right]_0^1 = \frac{9}{20}$

2. $\int_1^2 \int_y^2 xy\,dx\,dy = \int_1^2 \left[\frac{1}{2}x^2 y\right]_{x=y}^{x=2} dy = \int_1^2 \frac{1}{2}y(4-y^2)\,dy = \frac{1}{2}\int_1^2 (4y - y^3)\,dy$
$= \frac{1}{2}\left[2y^2 - \frac{1}{4}y^4\right]_1^2 = \frac{1}{2}\left(8 - 4 - 2 + \frac{1}{4}\right) = \frac{9}{8}$

3. $\int_0^1 \int_y^{e^y} \sqrt{x}\,dx\,dy = \int_0^1 \left[\frac{2}{3}x^{3/2}\right]_{x=y}^{x=e^y} dy = \frac{2}{3}\int_0^1 (e^{3y/2} - y^{3/2})\,dy = \frac{2}{3}\left[\frac{2}{3}e^{3y/2} - \frac{2}{5}y^{5/2}\right]_0^1$
$= \frac{2}{3}\left(\frac{2}{3}e^{3/2} - \frac{2}{5} - \frac{2}{3}e^0 + 0\right) = \frac{4}{9}e^{3/2} - \frac{32}{45}$

4. $\int_0^1 \int_x^{2-x} (x^2 - y)\,dy\,dx = \int_0^1 \left[x^2 y - \frac{1}{2}y^2\right]_{y=x}^{y=2-x} dx = \int_0^1 \left[x^2(2-x) - \frac{1}{2}(2-x)^2 - x^2(x) + \frac{1}{2}x^2\right] dx$
$= \int_0^1 (-2x^3 + 2x^2 + 2x - 2)\,dx = \left[-\frac{1}{2}x^4 + \frac{2}{3}x^3 + x^2 - 2x\right]_0^1 = -\frac{5}{6}$

5. $\int_0^{\pi/2} \int_0^{\cos\theta} e^{\sin\theta}\,dr\,d\theta = \int_0^{\pi/2} \left[re^{\sin\theta}\right]_{r=0}^{r=\cos\theta} d\theta = \int_0^{\pi/2} (\cos\theta)\,e^{\sin\theta}\,d\theta = e^{\sin\theta}\Big]_0^{\pi/2}$
$= e^{\sin(\pi/2)} - e^0 = e - 1$

6. $\int_0^1 \int_0^v \sqrt{1-v^2}\,du\,dv = \int_0^1 \left[u\sqrt{1-v^2}\,\right]_{u=0}^{u=v} dv = \int_0^1 v\sqrt{1-v^2}\,dv = -\frac{1}{3}(1-v^2)^{3/2}\Big]_0^1$
$= -\frac{1}{3}(0-1) = \frac{1}{3}$

7. $\iint_D x^3 y^2\,dA = \int_0^2 \int_{-x}^{x} x^3 y^2\,dy\,dx = \int_0^2 \left[\frac{1}{3}x^3 y^3\right]_{y=-x}^{y=x} dx = \frac{1}{3}\int_0^2 2x^6\,dx$
$= \frac{2}{3}\left[\frac{1}{7}x^7\right]_0^2 = \frac{2}{21}\left[2^7 - 0\right] = \frac{256}{21}$

8. $\displaystyle\iint_D \frac{4y}{x^3+2}\,dA = \int_1^2\int_0^{2x} \frac{4y}{x^3+2}\,dy\,dx = \int_1^2 \left[\frac{2y^2}{x^3+2}\right]_{y=0}^{y=2x} dx = \int_1^2 \frac{8x^2}{x^3+2}\,dx$

$$= \tfrac{8}{3}\ln\left|x^3+2\right|\Big]_1^2 = \tfrac{8}{3}(\ln 10 - \ln 3) = \tfrac{8}{3}\ln\tfrac{10}{3}$$

9. $\displaystyle\int_0^1\int_0^{\sqrt{x}} \frac{2y}{x^2+1}\,dy\,dx = \int_0^1 \left[\frac{y^2}{x^2+1}\right]_{y=0}^{y=\sqrt{x}} dx = \int_0^1 \frac{x}{x^2+1}\,dx$

$$= \tfrac{1}{2}\ln\left|x^2+1\right|\Big]_0^1 = \tfrac{1}{2}(\ln 2 - \ln 1) = \tfrac{1}{2}\ln 2$$

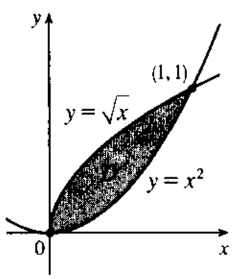
10. $\int_0^1\int_0^y e^{y^2}\,dx\,dy = \int_0^1 \left[xe^{y^2}\right]_{x=0}^{x=y} dy = \int_0^1 ye^{y^2}\,dy = \tfrac{1}{2}e^{y^2}\Big]_0^1 = \tfrac{1}{2}(e-1)$

11. $\int_1^2\int_y^{y^3} e^{x/y}\,dx\,dy = \int_1^2 \left[ye^{x/y}\right]_{x=y}^{x=y^3} dy = \int_1^2 \left(ye^{y^2} - ey\right) dy = \left[\tfrac{1}{2}e^{y^2} - \tfrac{1}{2}ey^2\right]_1^2 = \tfrac{1}{2}(e^4 - 4e)$

12. $\int_0^1\int_0^y x\sqrt{y^2-x^2}\,dx\,dy = \int_0^1 \left[-\tfrac{1}{3}(y^2-x^2)^{3/2}\right]_{x=0}^{x=y} dy = \tfrac{1}{3}\int_0^1 y^3\,dy = \tfrac{1}{3}\cdot\tfrac{1}{4}y^4\Big]_0^1 = \tfrac{1}{12}$

13. $\int_0^1\int_0^{x^2} x\cos y\,dy\,dx = \int_0^1 \left[x\sin y\right]_{y=0}^{y=x^2} dx = \int_0^1 x\sin x^2\,dx = -\tfrac{1}{2}\cos x^2\Big]_0^1 = \tfrac{1}{2}(1-\cos 1)$

14.

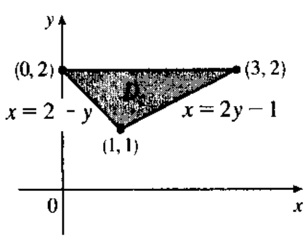


$$\int_0^1\int_{x^2}^{\sqrt{x}} (x+y)\,dy\,dx = \int_0^1 \left[xy + \tfrac{1}{2}y^2\right]_{y=x^2}^{y=\sqrt{x}} dx$$

$$= \int_0^1 \left(x^{3/2} + \tfrac{1}{2}x - x^3 - \tfrac{1}{2}x^4\right) dx$$

$$= \left[\tfrac{2}{5}x^{5/2} + \tfrac{1}{4}x^2 - \tfrac{1}{4}x^4 - \tfrac{1}{10}x^5\right]_0^1 = \tfrac{3}{10}$$

15.



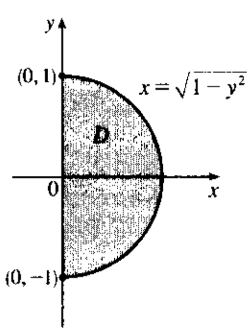
$$\int_1^2\int_{2-y}^{2y-1} y^3\,dx\,dy = \int_1^2 \left[xy^3\right]_{x=2-y}^{x=2y-1} dy$$

$$= \int_1^2 \left[(2y-1)-(2-y)\right]y^3\,dy$$

$$= \int_1^2 (3y^4 - 3y^3)\,dy = \left[\tfrac{3}{5}y^5 - \tfrac{3}{4}y^4\right]_1^2$$

$$= \tfrac{96}{5} - 12 - \tfrac{3}{5} + \tfrac{3}{4} = \tfrac{147}{20}$$

16.



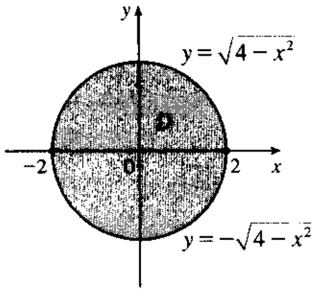
$$\iint_D xy^2\,dA = \int_{-1}^1\int_0^{\sqrt{1-y^2}} xy^2\,dx\,dy$$

$$= \int_{-1}^1 y^2\left[\tfrac{1}{2}x^2\right]_{x=0}^{x=\sqrt{1-y^2}} dy = \tfrac{1}{2}\int_{-1}^1 y^2(1-y^2)\,dy$$

$$= \tfrac{1}{2}\int_{-1}^1 (y^2-y^4)\,dy = \tfrac{1}{2}\left[\tfrac{1}{3}y^3 - \tfrac{1}{5}y^5\right]_{-1}^1$$

$$= \tfrac{1}{2}\left(\tfrac{1}{3} - \tfrac{1}{5} + \tfrac{1}{3} - \tfrac{1}{5}\right) = \tfrac{2}{15}$$

17.



$$\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}(2x-y)\,dy\,dx$$

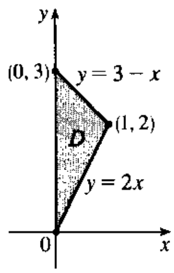
$$=\int_{-2}^{2}\left[2xy-\frac{1}{2}y^2\right]_{y=-\sqrt{4-x^2}}^{y=\sqrt{4-x^2}}dx$$

$$=\int_{-2}^{2}\left[2x\sqrt{4-x^2}-\tfrac{1}{2}\left(4-x^2\right)+2x\sqrt{4-x^2}+\tfrac{1}{2}\left(4-x^2\right)\right]dx$$

$$=\int_{-2}^{2}4x\sqrt{4-x^2}\,dx=-\tfrac{4}{3}\left(4-x^2\right)^{3/2}\Big]_{-2}^{2}=0$$

(Or, note that $4x\sqrt{4-x^2}$ is an odd function, so $\int_{-2}^{2}4x\sqrt{4-x^2}\,dx=0$.)

18.



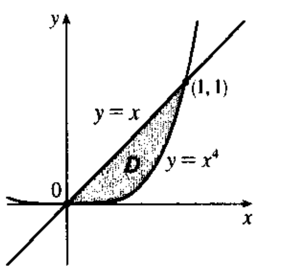
$$\iint_D 2xy\,dA=\int_0^1\int_{2x}^{3-x}2xy\,dy\,dx=\int_0^1\left[xy^2\right]_{y=2x}^{y=3-x}dx$$

$$=\int_0^1 x\left[(3-x)^2-(2x)^2\right]dx$$

$$=\int_0^1\left(-3x^3-6x^2+9x\right)dx$$

$$=\left[-\tfrac{3}{4}x^4-2x^3+\tfrac{9}{2}x^2\right]_0^1=-\tfrac{3}{4}-2+\tfrac{9}{2}=\tfrac{7}{4}$$

19.

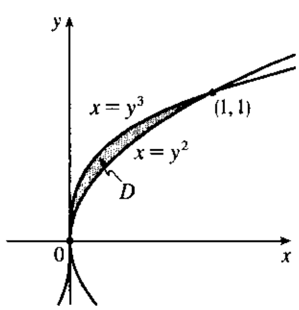


$$V=\int_0^1\int_{x^4}^{x}(x+2y)\,dy\,dx$$

$$=\int_0^1\left[xy+y^2\right]_{y=x^4}^{y=x}dx=\int_0^1\left(2x^2-x^5-x^8\right)dx$$

$$=\left[\tfrac{2}{3}x^3-\tfrac{1}{6}x^6-\tfrac{1}{9}x^9\right]_0^1=\tfrac{2}{3}-\tfrac{1}{6}-\tfrac{1}{9}=\tfrac{7}{18}$$

20.

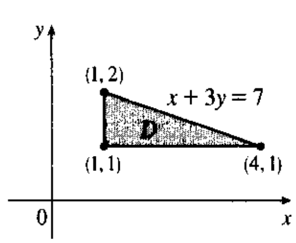


$$V=\int_0^1\int_{y^3}^{y^2}\left(2x+y^2\right)dx\,dy$$

$$=\int_0^1\left[x^2+xy^2\right]_{x=y^3}^{x=y^2}dy=\int_0^1\left(2y^4-y^6-y^5\right)dy$$

$$=\left[\tfrac{2}{5}y^5-\tfrac{1}{7}y^7-\tfrac{1}{6}y^6\right]_0^1=\tfrac{19}{210}$$

21.

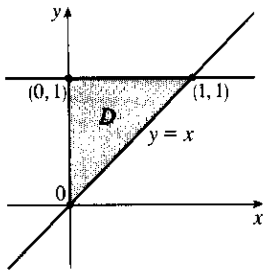


$$V=\int_1^2\int_1^{7-3y}xy\,dx\,dy=\int_1^2\left[\tfrac{1}{2}x^2y\right]_{x=1}^{x=7-3y}dy$$

$$=\tfrac{1}{2}\int_1^2\left(48y-42y^2+9y^3\right)dy$$

$$=\tfrac{1}{2}\left[24y^2-14y^3+\tfrac{9}{4}y^4\right]_1^2=\tfrac{31}{8}$$

22.

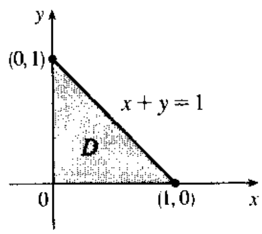


$$V = \int_0^1 \int_x^1 (x^2 + 3y^2)\,dy\,dx$$

$$= \int_0^1 \left[x^2 y + y^3\right]_{y=x}^{y=1} dx = \int_0^1 (x^2 + 1 - 2x^3)\,dx$$

$$= \left[\tfrac{1}{3}x^3 + x - \tfrac{1}{2}x^4\right]_0^1 = \tfrac{5}{6}$$

23.



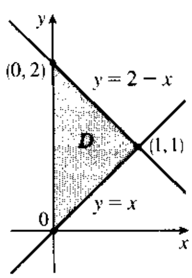
$$V = \int_0^1 \int_0^{1-x} (1 - x - y)\,dy\,dx$$

$$= \int_0^1 \left[y - xy - \frac{y^2}{2}\right]_{y=0}^{y=1-x} dx$$

$$= \int_0^1 \left[(1-x)^2 - \tfrac{1}{2}(1-x)^2\right] dx$$

$$= \int_0^1 \tfrac{1}{2}(1-x)^2\,dx = \left[-\tfrac{1}{6}(1-x)^3\right]_0^1 = \tfrac{1}{6}$$

24.

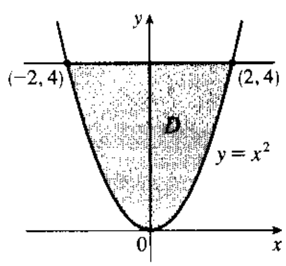


$$V = \int_0^1 \int_x^{2-x} x\,dy\,dx$$

$$= \int_0^1 x\left[y\right]_{y=x}^{y=2-x} dx = \int_0^1 (2x - 2x^2)\,dx$$

$$= \left[x^2 - \tfrac{2}{3}x^3\right]_0^1 = \tfrac{1}{3}$$

25.

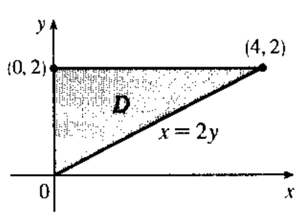


$$V = \int_{-2}^2 \int_{x^2}^4 x^2\,dy\,dx$$

$$= \int_{-2}^2 x^2\left[y\right]_{y=x^2}^{y=4} dx = \int_{-2}^2 (4x^2 - x^4)\,dx$$

$$= \left[\tfrac{4}{3}x^3 - \tfrac{1}{5}x^5\right]_{-2}^2 = \tfrac{32}{3} - \tfrac{32}{5} + \tfrac{32}{3} - \tfrac{32}{5} = \tfrac{128}{15}$$

26.

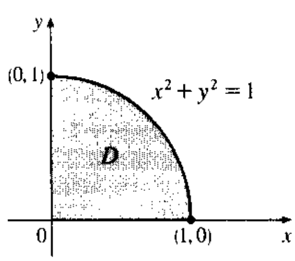


$$V = \int_0^2 \int_0^{2y} \sqrt{4 - y^2}\,dx\,dy$$

$$= \int_0^2 \left[x\sqrt{4 - y^2}\right]_{x=0}^{x=2y} dy = \int_0^2 2y\sqrt{4 - y^2}\,dy$$

$$= \left[-\tfrac{2}{3}(4 - y^2)^{3/2}\right]_0^2 = 0 + \tfrac{16}{3} = \tfrac{16}{3}$$

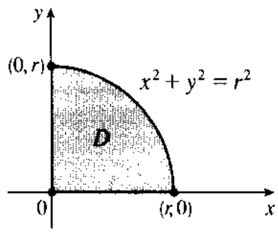
27.



$$V = \int_0^1 \int_0^{\sqrt{1-x^2}} y\,dy\,dx = \int_0^1 \left[\frac{y^2}{2}\right]_{y=0}^{y=\sqrt{1-x^2}} dx$$

$$= \int_0^1 \frac{1 - x^2}{2}\,dx = \tfrac{1}{2}\left[x - \tfrac{1}{3}x^3\right]_0^1 = \tfrac{1}{3}$$

28.



By symmetry, the desired volume V is 8 times the volume V_1 in the first octant. Now

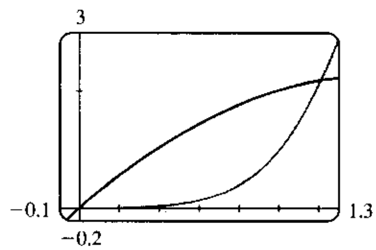
$$V_1 = \int_0^r \int_0^{\sqrt{r^2-y^2}} \sqrt{r^2-y^2}\,dx\,dy$$

$$= \int_0^r \left[x\sqrt{r^2-y^2}\right]_{x=0}^{x=\sqrt{r^2-y^2}} dy$$

$$= \int_0^r (r^2-y^2)\,dy = \left[r^2y - \tfrac{1}{3}y^3\right]_0^r = \tfrac{2}{3}r^3$$

Thus $V = \frac{16}{3}r^3$.

29.



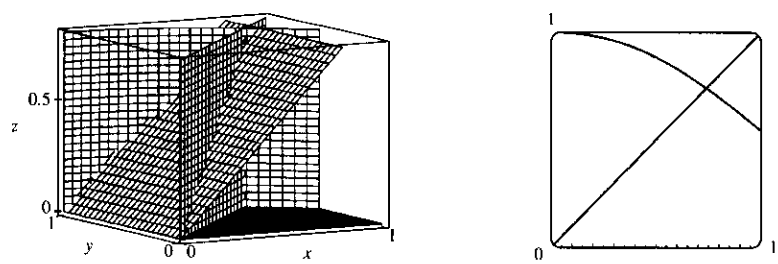
From the graph, it appears that the two curves intersect at $x = 0$ and at $x \approx 1.213$. Thus the desired integral is

$$\iint_D x\,dA \approx \int_0^{1.213}\int_{x^4}^{3x-x^2} x\,dy\,dx = \int_0^{1.213}\left[xy\right]_{y=x^4}^{y=3x-x^2} dx$$

$$= \int_0^{1.213}(3x^2 - x^3 - x^5)\,dx = \left[x^3 - \tfrac{1}{4}x^4 - \tfrac{1}{6}x^6\right]_0^{1.213}$$

$$\approx 0.713$$

30.



The desired solid is shown in the first graph. From the second graph, we estimate that $y = \cos x$ intersects $y = x$ at $x \approx 0.7391$. Therefore the volume of the solid is

$$V \approx \int_0^{0.7391}\int_x^{\cos x} z\,dy\,dx = \int_0^{0.7391}\int_x^{\cos x} x\,dy\,dx$$

$$= \int_0^{0.7391}\left[xy\right]_{y=x}^{y=\cos x} dx = \int_0^{0.7391}(x\cos x - x^2)\,dx$$

$$= \left[\cos x + x\sin x - \tfrac{1}{3}x^3\right]_0^{0.7391} \approx 0.1024$$

Note: There is a different solid which can also be construed to satisfy the conditions stated in the exercise. This is the solid bounded by all of the given surfaces, as well as the plane $y = 0$. In case you calculated the volume of this solid and want to check your work, its volume is $V \approx \int_0^{0.7391}\int_0^x x\,dy\,dx + \int_{0.7391}^{\pi/2}\int_0^{\cos x} x\,dy\,dx \approx 0.4684$.

31. The two bounding curves $y = 1 - x^2$ and $y = x^2 - 1$ intersect at $(\pm 1, 0)$ with $1 - x^2 \geq x^2 - 1$ on $[-1, 1]$. Within this region, the plane $z = 2x + 2y + 10$ is above the plane $z = 2 - x - y$, so

$$V = \int_{-1}^{1}\int_{x^2-1}^{1-x^2} (2x + 2y + 10)\, dy\, dx - \int_{-1}^{1}\int_{x^2-1}^{1-x^2} (2 - x - y)\, dy\, dx$$

$$= \int_{-1}^{1}\int_{x^2-1}^{1-x^2} (2x + 2y + 10 - (2 - x - y))\, dy\, dx = \int_{-1}^{1}\int_{x^2-1}^{1-x^2} (3x + 3y + 8)\, dy\, dx$$

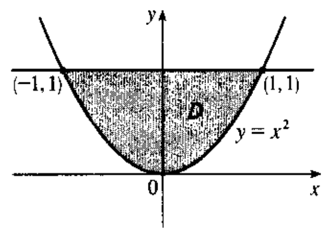
$$= \int_{-1}^{1} \left[3xy + \tfrac{3}{2}y^2 + 8y\right]_{y=x^2-1}^{y=1-x^2} dx$$

$$= \int_{-1}^{1} \left[3x(1 - x^2) + \tfrac{3}{2}(1 - x^2)^2 + 8(1 - x^2) - 3x(x^2 - 1) - \tfrac{3}{2}(x^2 - 1)^2 - 8(x^2 - 1)\right] dx$$

$$= \int_{-1}^{1} (-6x^3 - 16x^2 + 6x + 16)\, dx = \left[-\tfrac{3}{2}x^4 - \tfrac{16}{3}x^3 + 3x^2 + 16x\right]_{-1}^{1}$$

$$= -\tfrac{3}{2} - \tfrac{16}{3} + 3 + 16 + \tfrac{3}{2} - \tfrac{16}{3} - 3 + 16 = \tfrac{64}{3}$$

32. The two planes intersect in the line $y = 1$, $z = 3$, so the region of integration is the plane region enclosed by the parabola $y = x^2$ and the line $y = 1$. We have $2 + y \geq 3y$ for $0 \leq y \leq 1$, so the solid region is bounded above by $z = 2 + y$ and bounded below by $z = 3y$.



$$V = \int_{-1}^{1}\int_{x^2}^{1} (2 + y)\, dy\, dx - \int_{-1}^{1}\int_{x^2}^{1} (3y)\, dy\, dx = \int_{-1}^{1}\int_{x^2}^{1} (2 + y - 3y)\, dy\, dx = \int_{-1}^{1}\int_{x^2}^{1} (2 - 2y)\, dy\, dx$$

$$= \int_{-1}^{1} \left[2y - y^2\right]_{y=x^2}^{y=1} dx = \int_{-1}^{1} (1 - 2x^2 + x^4)\, dx = x - \tfrac{2}{3}x^3 + \tfrac{1}{5}x^5\Big]_{-1}^{1} = \tfrac{16}{15}$$

33. The two bounding curves $y = x^3 - x$ and $y = x^2 + x$ intersect at the origin and at $x = 2$, with $x^2 + x > x^3 - x$ on $(0, 2)$. Using a CAS, we find that the volume is

$$V = \int_0^2 \int_{x^3 - x}^{x^2 + x} z\, dy\, dx = \int_0^2 \int_{x^3 - x}^{x^2 + x} (x^3 y^4 + xy^2)\, dy\, dx = \frac{13{,}984{,}735{,}616}{14{,}549{,}535}$$

34. For $|x| \leq 1$ and $|y| \leq 1$, $2x^2 + y^2 < 8 - x^2 - 2y^2$. Also, the cylinder is described by the inequalities $-1 \leq x \leq 1$, $-\sqrt{1 - x^2} \leq y \leq \sqrt{1 - x^2}$. So the volume is given by

$$V = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \left[(8 - x^2 - 2y^2) - (2x^2 + y^2)\right] dy\, dx = \frac{13\pi}{2} \qquad \text{[using a CAS]}$$

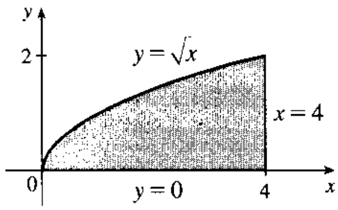
35. The two surfaces intersect in the circle $x^2 + y^2 = 1$, $z = 0$ and the region of integration is the disk D: $x^2 + y^2 \leq 1$.

Using a CAS, the volume is $\iint_D (1 - x^2 - y^2)\, dA = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (1 - x^2 - y^2)\, dy\, dx = \frac{\pi}{2}$.

36. The projection onto the xy-plane of the intersection of the two surfaces is the circle $x^2 + y^2 = 2y \Rightarrow x^2 + y^2 - 2y = 0 \Rightarrow x^2 + (y-1)^2 = 1$, so the region of integration is given by $-1 \le x \le 1$, $1 - \sqrt{1-x^2} \le y \le 1 + \sqrt{1-x^2}$. In this region, $2y \ge x^2 + y^2$ so, using a CAS, the volume is

$$V = \int_{-1}^{1} \int_{1-\sqrt{1-x^2}}^{1+\sqrt{1-x^2}} [2y - (x^2+y^2)]\, dy\, dx = \frac{\pi}{2}.$$

37.



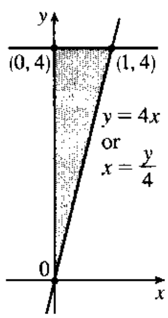
Because the region of integration is

$$D = \{(x,y) \mid 0 \le y \le \sqrt{x}, 0 \le x \le 4\} = \{(x,y) \mid y^2 \le x \le 4, 0 \le y \le 2\}$$

we have

$$\int_0^4 \int_0^{\sqrt{x}} f(x,y)\, dy\, dx = \iint_D f(x,y)\, dA = \int_0^2 \int_{y^2}^4 f(x,y)\, dx\, dy.$$

38.



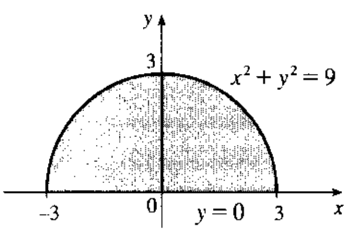
Because the region of integration is

$$D = \{(x,y) \mid 4x \le y \le 4, 0 \le x \le 1\} = \{(x,y) \mid 0 \le x \le \tfrac{y}{4}, 0 \le y \le 4\}$$

we have

$$\int_0^1 \int_{4x}^4 f(x,y)\, dy\, dx = \iint_D f(x,y)\, dA = \int_0^4 \int_0^{y/4} f(x,y)\, dx\, dy$$

39.



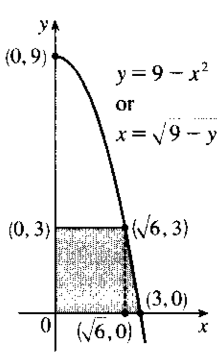
Because the region of integration is

$$D = \left\{(x,y) \mid -\sqrt{9-y^2} \le x \le \sqrt{9-y^2}, 0 \le y \le 3\right\} = \left\{(x,y) \mid 0 \le y \le \sqrt{9-x^2}, -3 \le x \le 3\right\}$$

we have

$$\int_0^3 \int_{-\sqrt{9-y^2}}^{\sqrt{9-y^2}} f(x,y)\, dx\, dy = \iint_D f(x,y)\, dA = \int_{-3}^3 \int_0^{\sqrt{9-x^2}} f(x,y)\, dy\, dx$$

40.



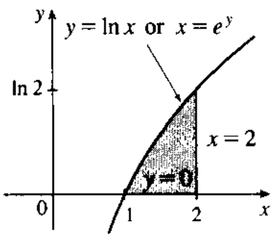
To reverse the order, we must break the region into two separate type I regions. Because the region of integration is

$$D = \left\{(x,y) \mid 0 \le x \le \sqrt{9-y}, 0 \le y \le 3\right\} = \left\{(x,y) \mid 0 \le y \le 3, 0 \le x \le \sqrt{6}\right\} \cup \left\{(x,y) \mid 0 \le y \le 9 - x^2, \sqrt{6} \le x \le 3\right\}$$

we have

$$\int_0^3 \int_0^{\sqrt{9-y}} f(x,y)\, dx\, dy = \iint_D f(x,y)\, dA = \int_0^{\sqrt{6}} \int_0^3 f(x,y)\, dy\, dx + \int_{\sqrt{6}}^3 \int_0^{9-x^2} f(x,y)\, dy\, dx$$

41.



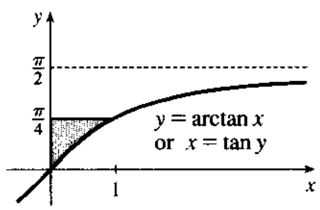
Because the region of integration is

$$D = \{(x,y) \mid 0 \le y \le \ln x, 1 \le x \le 2\}$$
$$= \{(x,y) \mid e^y \le x \le 2, 0 \le y \le \ln 2\}$$

we have

$$\int_1^2 \int_0^{\ln x} f(x,y)\,dy\,dx = \iint_D f(x,y)\,dA$$
$$= \int_0^{\ln 2} \int_{e^y}^2 f(x,y)\,dx\,dy$$

42.



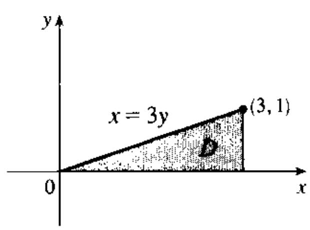
Because the region of integration is

$$D = \left\{(x,y) \mid \arctan x \le y \le \tfrac{\pi}{4}, 0 \le x \le 1\right\}$$
$$= \left\{(x,y) \mid 0 \le x \le \tan y, 0 \le y \le \tfrac{\pi}{4}\right\}$$

we have

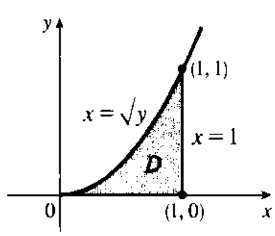
$$\int_0^1 \int_{\arctan x}^{\pi/4} f(x,y)\,dy\,dx = \iint_D f(x,y)\,dA$$
$$= \int_0^{\pi/4} \int_0^{\tan y} f(x,y)\,dx\,dy$$

43.



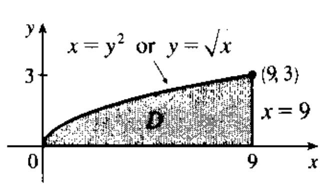
$$\int_0^1 \int_{3y}^3 e^{x^2}\,dx\,dy = \int_0^3 \int_0^{x/3} e^{x^2}\,dy\,dx$$
$$= \int_0^3 \left[e^{x^2} y\right]_{y=0}^{y=x/3} dx = \int_0^3 \left(\frac{x}{3}\right) e^{x^2}\,dx$$
$$= \tfrac{1}{6}\, e^{x^2}\Big]_0^3 = \frac{e^9 - 1}{6}$$

44.



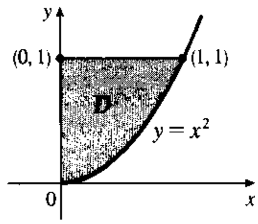
$$\int_0^1 \int_{\sqrt{y}}^1 \sqrt{x^3+1}\,dx\,dy = \int_0^1 \int_0^{x^2} \sqrt{x^3+1}\,dy\,dx$$
$$= \int_0^1 \left[\sqrt{x^3+1}\,y\right]_{y=0}^{y=x^2} dx = \int_0^1 x^2\sqrt{x^3+1}\,dx$$
$$= \tfrac{2}{9}(x^3+1)^{3/2}\Big]_0^1 = \tfrac{2}{9}\left(2^{3/2} - 1\right)$$

45.



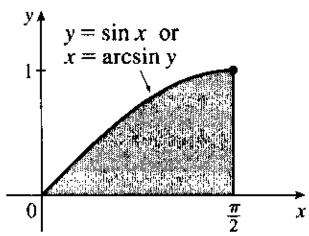
$$\textstyle\int_0^3 \int_{y^2}^9 y\cos x^2\,dx\,dy = \int_0^9 \int_0^{\sqrt{x}} y\cos x^2\,dy\,dx$$
$$= \int_0^9 \cos x^2 \left[\frac{y^2}{2}\right]_{y=0}^{y=\sqrt{x}} dx = \textstyle\int_0^9 \tfrac{1}{2}x\cos x^2\,dx$$
$$= \tfrac{1}{4}\sin x^2\Big]_0^9 = \tfrac{1}{4}\sin 81$$

46.



$$\int_0^1 \int_{x^2}^1 x^3 \sin(y^3)\, dy\, dx = \int_0^1 \int_0^{\sqrt{y}} x^3 \sin(y^3)\, dx\, dy$$
$$= \int_0^1 \left[\frac{x^4}{4} \sin(y^3)\right]_{x=0}^{x=\sqrt{y}} dy$$
$$= \int_0^1 \tfrac{1}{4} y^2 \sin(y^3)\, dy$$
$$= -\tfrac{1}{12} \cos(y^3)\Big]_0^1 = \tfrac{1}{12}(1 - \cos 1)$$

47.

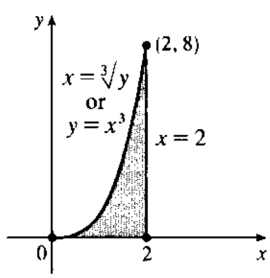


$$\int_0^1 \int_{\arcsin y}^{\pi/2} \cos x \sqrt{1 + \cos^2 x}\, dx\, dy$$
$$= \int_0^{\pi/2} \int_0^{\sin x} \cos x \sqrt{1 + \cos^2 x}\, dy\, dx$$
$$= \int_0^{\pi/2} \cos x \sqrt{1 + \cos^2 x}\, \big[y\big]_{y=0}^{y=\sin x}\, dx$$
$$= \int_0^{\pi/2} \cos x \sqrt{1 + \cos^2 x}\, \sin x\, dx$$

[Let $u = \cos x$, $du = -\sin x\, dx$, $dx = du/(-\sin x)$]

$$= \int_1^0 -u \sqrt{1 + u^2}\, du = -\tfrac{1}{3}\left(1 + u^2\right)^{3/2}\Big]_1^0$$
$$= \tfrac{1}{3}\left(\sqrt{8} - 1\right) = \tfrac{1}{3}\left(2\sqrt{2} - 1\right)$$

48.



$$\int_0^8 \int_{\sqrt[3]{y}}^2 e^{x^4}\, dx\, dy = \int_0^2 \int_0^{x^3} e^{x^4}\, dy\, dx$$
$$= \int_0^2 e^{x^4} \big[y\big]_{y=0}^{y=x^3}\, dx = \int_0^2 x^3 e^{x^4}\, dx$$
$$= \tfrac{1}{4} e^{x^4}\Big]_0^2 = \tfrac{1}{4}(e^{16} - 1)$$

49. $D = \{(x,y) \mid 0 \le x \le 1,\ -x + 1 \le y \le 1\} \cup \{(x,y) \mid -1 \le x \le 0, x + 1 \le y \le 1\}$

$\cup \{(x,y) \mid 0 \le x \le 1,\ -1 \le y \le x - 1\} \cup \{(x,y) \mid -1 \le x \le 0,\ -1 \le y \le -x - 1\}$,

all type I.

$$\iint_D x^2\, dA = \int_0^1 \int_{1-x}^1 x^2\, dy\, dx + \int_{-1}^0 \int_{x+1}^1 x^2\, dy\, dx + \int_0^1 \int_{-1}^{x-1} x^2\, dy\, dx + \int_{-1}^0 \int_{-1}^{-x-1} x^2\, dy\, dx$$

$$= 4 \int_0^1 \int_{1-x}^1 x^2\, dy\, dx$$ [by symmetry of the regions and because $f(x,y) = x^2 \ge 0$]

$$= 4 \int_0^1 x^3\, dx = 4\left[\tfrac{1}{4} x^4\right]_0^1 = 1$$

50. $D = \{(x,y) \mid -1 \le x \le 0,\ -1 \le y \le 1 + x^2\} \cup \{(x,y) \mid 0 \le x \le 1,\ \sqrt{x} \le y \le 1 + x^2\}$

$\cup \{(x,y) \mid 0 \le x \le 1,\ -1 \le y \le -\sqrt{x}\}$,

all type I.

$$\begin{aligned}
\iint_D xy\,dA &= \int_{-1}^{0}\int_{-1}^{1+x^2} xy\,dy\,dx + \int_0^1\int_{\sqrt{x}}^{1+x^2} xy\,dy\,dx + \int_0^1\int_{-1}^{-\sqrt{x}} xy\,dy\,dx \\
&= \int_{-1}^{0}\left[\tfrac{1}{2}xy^2\right]_{y=-1}^{y=1+x^2} dx + \int_0^1\left[\tfrac{1}{2}xy^2\right]_{y=\sqrt{x}}^{y=1+x^2} dx + \int_0^1\left[\tfrac{1}{2}xy^2\right]_{y=-1}^{y=-\sqrt{x}} dx \\
&= \int_{-1}^{0}\left(x^3 + \tfrac{1}{2}x^5\right) dx + \int_0^1 \tfrac{1}{2}(x^5 + 2x^3 - x^2 + x)\,dx + \int_0^1 \tfrac{1}{2}(x^2 - x)\,dx \\
&= \left[\tfrac{1}{4}x^4 + \tfrac{1}{12}x^6\right]_{-1}^{0} + \tfrac{1}{2}\left[\tfrac{1}{6}x^6 + \tfrac{1}{2}x^4 - \tfrac{1}{3}x^3 + \tfrac{1}{2}x^2\right]_0^1 + \tfrac{1}{2}\left[\tfrac{1}{3}x^3 - \tfrac{1}{2}x^2\right]_0^1 \\
&= -\tfrac{1}{3} + \tfrac{5}{12} - \tfrac{1}{12} = 0
\end{aligned}$$

51. For $D = [0,1] \times [0,1]$, $0 \le \sqrt{x^3 + y^3} \le \sqrt{2}$ and $A(D) = 1$, so $0 \le \iint_D \sqrt{x^3 + y^3}\,dA \le \sqrt{2}$.

52. Since $D = \{(x,y) \mid x^2 + y^2 \le \frac{1}{4}\}$, $1 = e^0 \le e^{x^2+y^2} \le e^{1/4}$ and $A(D) = \frac{\pi}{4}$, so

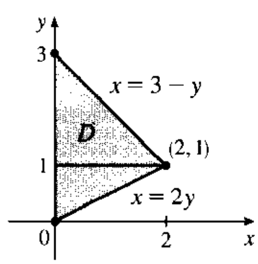
$\frac{\pi}{4} \le \iint_D e^{x^2+y^2}\,dA \le (e^{1/4})\frac{\pi}{4}$.

53. Since $m \le f(x,y) \le M$, $\iint_D m\,dA \le \iint_D f(x,y)\,dA \le \iint_D M\,dA$ by (8) $\Rightarrow$

$m\iint_D 1\,dA \le \iint_D f(x,y)\,dA \le M\iint_D 1\,dA$ by (7) $\Rightarrow$

$mA(D) \le \iint_D f(x,y)\,dA \le MA(D)$ by (10).

54.



$$\iint_D f(x,y)\,dA = \int_0^1\int_0^{2y} f(x,y)\,dx\,dy + \int_1^3\int_0^{3-y} f(x,y)\,dx\,dy$$

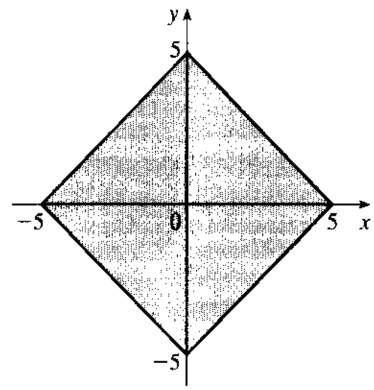
$$= \int_0^2\int_{x/2}^{3-x} f(x,y)\,dy\,dx$$

55. $\iint_D (x^2\tan x + y^3 + 4)\,dA = \iint_D x^2\tan x\,dA + \iint_D y^3\,dA + \iint_D 4\,dA$. But $x^2\tan x$ is an odd function of x and D is symmetric with respect to the y-axis, so $\iint_D x^2\tan x\,dA = 0$. Similarly, y^3 is an odd function of y and D is symmetric with respect to the x-axis, so $\iint_D y^3\,dA = 0$. Thus

$$\iint_D (x^2\tan x + y^3 + 4)\,dA = 4\iint_D dA = 4(\text{area of } D) = 4\cdot\pi\left(\sqrt{2}\right)^2 = 8\pi$$

56. First,

$$\iint_D (2-3x+4y)\,dA = \iint_D 2\,dA - \iint_D 3x\,dA + \iint_D 4y\,dA$$

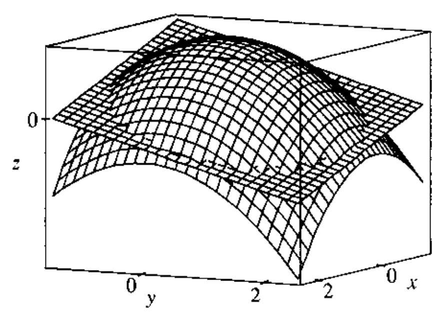


The region D, shown in the figure, is symmetric with respect to the y-axis and $3x$ is an odd function of x, so $\iint_D 3x\,dA = 0$. Similarly, $4y$ is an odd function of y and D is symmetric with respect to the x-axis, so $\iint_D 4y\,dA = 0$. Then

$$\begin{aligned}\iint_D (2-3x+4y)\,dA &= \iint_D 2\,dA = 2\iint_D dA \\ &= 2(\text{area of } D) = 2(50) \\ &= 100\end{aligned}$$

57. Since $\sqrt{1-x^2-y^2} \geq 0$, we can interpret $\iint_D \sqrt{1-x^2-y^2}\,dA$ as the volume of the solid that lies below the graph of $z=\sqrt{1-x^2-y^2}$ and above the region D in the xy-plane. $z=\sqrt{1-x^2-y^2}$ is equivalent to $x^2+y^2+z^2=1$, $z \geq 0$ which meets the xy-plane in the circle $x^2+y^2=1$, the boundary of D. Thus, the solid is an upper hemisphere of radius 1 which has volume $\frac{1}{2}\left[\frac{4}{3}\pi(1)^3\right] = \frac{2}{3}\pi$.

58. To find the equations of the boundary curves, we require that the z-values of the two surfaces be the same. In Maple, we use the command `solve(4-x^2-y^2=1-x-y,y);` and in Mathematica, we use `Solve[4-x^2-y^2==1-x-y,y]`. We find that the curves have equations $y = \dfrac{1 \pm \sqrt{13+4x-4x^2}}{2}$.



To find the two points of intersection of these curves, we use the CAS to solve $13+4x-4x^2=0$, finding that $x=\frac{1\pm\sqrt{14}}{2}$. So, using the CAS to evaluate the integral, the volume of intersection is

$$V = \int_{(1-\sqrt{14})/2}^{(1+\sqrt{14})/2} \int_{\left(1-\sqrt{13+4x-4x^2}\right)/2}^{\left(1+\sqrt{13+4x-4x^2}\right)/2} \left[(4-x^2-y^2)-(1-x-y)\right] dy\,dx = \frac{49\pi}{8}.$$

16.4 Double Integrals in Polar Coordinates ET 15.4

1. The region R is more easily described by polar coordinates: $R=\{(r,\theta) \mid 0 \leq r \leq 2, 0 \leq \theta \leq 2\pi\}$. Thus $\iint_R f(x,y)\,dA = \int_0^{2\pi}\int_0^2 f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$.

2. The region R is more easily described by rectangular coordinates: $R=\{(x,y) \mid 0 \leq x \leq 2, 0 \leq y \leq 2-x\}$. Thus $\iint_R f(x,y)\,dA = \int_0^2\int_0^{2-x} f(x,y)\,dy\,dx$.

3. The region R is more easily described by rectangular coordinates: $R=\{(x,y) \mid -2 \leq x \leq 2, x \leq y \leq 2\}$. Thus $\iint_R f(x,y)\,dA = \int_{-2}^2\int_x^2 f(x,y)\,dy\,dx$.

4. The region R is more easily described by polar coordinates: $R = \left\{(r,\theta) \mid 1 \le r \le 3, 0 \le \theta \le \frac{\pi}{2}\right\}$.

Thus $\iint_R f(x,y)\,dA = \int_0^{\pi/2}\int_1^3 f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$.

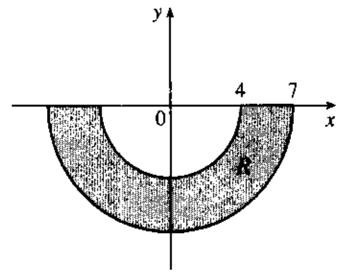
5. The region R is more easily described by polar coordinates: $R = \{(r,\theta) \mid 2 \le r \le 5, 0 \le \theta \le 2\pi\}$.

Thus $\iint_R f(x,y)\,dA = \int_0^{2\pi}\int_2^5 f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$.

6. The region R is more easily described by polar coordinates: $R = \left\{(r,\theta) \mid 0 \le r \le 2\sqrt{2}, \frac{\pi}{4} \le \theta \le \frac{5\pi}{4}\right\}$.

Thus $\iint_R f(x,y)\,dA = \int_{\pi/4}^{5\pi/4}\int_0^{2\sqrt{2}} f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$.

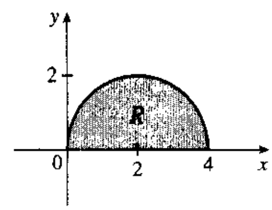
7. The integral $\int_\pi^{2\pi}\int_4^7 r\,dr\,d\theta$ represents the area of the region $R = \{(r,\theta) \mid 4 \le r \le 7, \pi \le \theta \le 2\pi\}$, the lower half of a ring.



$$\int_\pi^{2\pi}\int_4^7 r\,dr\,d\theta = \left(\int_\pi^{2\pi} d\theta\right)\left(\int_4^7 r\,dr\right)$$

$$= \left[\theta\right]_\pi^{2\pi}\left[\tfrac{1}{2}r^2\right]_4^7 = \pi\cdot\frac{1}{2}(49-16) = \frac{33\pi}{2}$$

8. The integral $\int_0^{\pi/2}\int_0^{4\cos\theta} r\,dr\,d\theta$ represents the area of the region $R = \{(r,\theta) \mid 0 \le r \le 4\cos\theta, 0 \le \theta \le \pi/2\}$. Since $r = 4\cos\theta \Leftrightarrow r^2 = 4r\cos\theta \Leftrightarrow x^2+y^2 = 4x \Leftrightarrow (x-2)^2+y^2 = 4$, R is the portion in the first quadrant of a circle of radius 2 with center $(2,0)$.



$$\int_0^{\pi/2}\int_0^{4\cos\theta} r\,dr\,d\theta = \int_0^{\pi/2}\left[\tfrac{1}{2}r^2\right]_{r=0}^{r=4\cos\theta}d\theta = \int_0^{\pi/2} 8\cos^2\theta\,d\theta$$

$$= \int_0^{\pi/2} 4(1+\cos 2\theta)\,d\theta = 4\left[\theta + \tfrac{1}{2}\sin 2\theta\right]_0^{\pi/2} = 2\pi$$

9. The disk D can be described in polar coordinates as $D = \{(r,\theta) \mid 0 \le r \le 3, 0 \le \theta \le 2\pi\}$. Then

$$\iint_D xy\,dA = \int_0^{2\pi}\int_0^3 (r\cos\theta)(r\sin\theta)\,r\,dr\,d\theta = \left(\int_0^{2\pi}\sin\theta\cos\theta\,d\theta\right)\left(\int_0^3 r^3\,dr\right)$$

$$= \left[\tfrac{1}{2}\sin^2\theta\right]_0^{2\pi}\left[\tfrac{1}{4}r^4\right]_0^3 = 0$$

10. $\iint_R (x+y)\,dA = \int_{\pi/2}^{3\pi/2}\int_1^2 (r\cos\theta + r\sin\theta)\,r\,dr\,d\theta = \int_{\pi/2}^{3\pi/2}\int_1^2 r^2(\cos\theta+\sin\theta)\,dr\,d\theta$

$$= \left(\int_{\pi/2}^{3\pi/2}(\cos\theta+\sin\theta)\,d\theta\right)\left(\int_1^2 r^2\,dr\right) = \left[\sin\theta - \cos\theta\right]_{\pi/2}^{3\pi/2}\left[\tfrac{1}{3}r^3\right]_1^2$$

$$= (-1-0-1+0)\left(\tfrac{8}{3}-\tfrac{1}{3}\right) = -\tfrac{14}{3}$$

11. $\iint_R \cos(x^2+y^2)\,dA = \int_0^\pi\int_0^3 \cos(r^2)\,r\,dr\,d\theta = \left(\int_0^\pi d\theta\right)\left(\int_0^3 r\cos(r^2)\,dr\right)$

$$= \left[\theta\right]_0^\pi\left[\tfrac{1}{2}\sin(r^2)\right]_0^3 = \pi\cdot\tfrac{1}{2}(\sin 9 - \sin 0) = \tfrac{\pi}{2}\sin 9$$

12. $\iint_R \sqrt{4-x^2-y^2}\,dA = \int_{-\pi/2}^{\pi/2}\int_0^2 \sqrt{4-r^2}\,r\,dr\,d\theta = \left(\int_{-\pi/2}^{\pi/2} d\theta\right)\left(\int_0^2 r\sqrt{4-r^2}\,dr\right)$

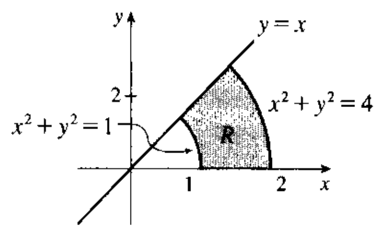
$$= [\theta]_{-\pi/2}^{\pi/2}\left[-\tfrac12\cdot\tfrac23(4-r^2)^{3/2}\right]_0^2 = \left(\tfrac\pi2+\tfrac\pi2\right)\left(-\tfrac13\left(0-4^{3/2}\right)\right) = \tfrac83\pi$$

13. $\iint_D e^{-x^2-y^2}\,dA = \int_{-\pi/2}^{\pi/2}\int_0^2 e^{-r^2}\,r\,dr\,d\theta = \left(\int_{-\pi/2}^{\pi/2} d\theta\right)\left(\int_0^2 re^{-r^2}\,dr\right)$

$$= [\theta]_{-\pi/2}^{\pi/2}\left[-\tfrac12 e^{-r^2}\right]_0^2 = \pi\left(-\tfrac12\right)\left(e^{-4}-e^0\right) = \tfrac\pi2\left(1-e^{-4}\right)$$

14. $\iint_R ye^x\,dA = \int_0^{\pi/2}\int_0^5 (r\sin\theta)\,e^{r\cos\theta}\,r\,dr\,d\theta = \int_0^5\int_0^{\pi/2} r^2\sin\theta\,e^{r\cos\theta}\,d\theta\,dr$. First we integrate $\int_0^{\pi/2} r^2\sin\theta\,e^{r\cos\theta}\,d\theta$: Let $u = r\cos\theta \;\Rightarrow\; du = -r\sin\theta\,d\theta$,
and $\int_0^{\pi/2} r^2\sin\theta\,e^{r\cos\theta}\,d\theta = \int_{u=r}^{u=0} -r\,e^u\,du = -r[e^0-e^r] = re^r - r$. Then
$\int_0^5\int_0^{\pi/2} r^2\sin\theta\,e^{r\cos\theta}\,d\theta\,dr = \int_0^5 (re^r-r)\,dr = \left[re^r - e^r - \tfrac12 r^2\right]_0^5 = 4e^5 - \tfrac{23}{2}$, where we integrated by parts in the first term.

15. R is the region shown in the figure, and can be described by $R = \{(r,\theta) \mid 0 \le \theta \le \pi/4,\, 1 \le r \le 2\}$. Thus

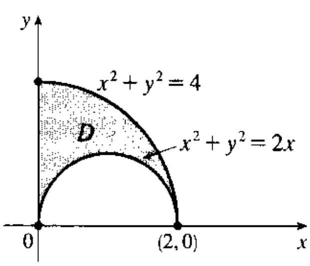


$\iint_R \arctan(y/x)\,dA = \int_0^{\pi/4}\int_1^2 \arctan(\tan\theta)\,r\,dr\,d\theta$

since $y/x = \tan\theta$. Also, $\arctan(\tan\theta) = \theta$ for $0 \le \theta \le \pi/4$, so the integral becomes

$$\int_0^{\pi/4}\int_1^2 \theta\,r\,dr\,d\theta = \int_0^{\pi/4}\theta\,d\theta\int_1^2 r\,dr = \left[\tfrac12\theta^2\right]_0^{\pi/4}\left[\tfrac12 r^2\right]_1^2 = \tfrac{\pi^2}{32}\cdot\tfrac32 = \tfrac{3}{64}\pi^2.$$

16.



$$\iint_D x\,dA = \iint_{\substack{x^2+y^2\le 4\\ x\ge 0,\, y\ge 0}} x\,dA - \iint_{\substack{(x-1)^2+y^2\le 1\\ y\ge 0}} x\,dA$$

$$= \int_0^{\pi/2}\int_0^2 r^2\cos\theta\,dr\,d\theta - \int_0^{\pi/2}\int_0^{2\cos\theta} r^2\cos\theta\,dr\,d\theta$$

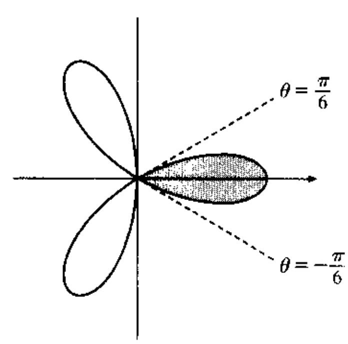
$$= \int_0^{\pi/2}\tfrac13(8\cos\theta)\,d\theta - \int_0^{\pi/2}\tfrac13(8\cos^4\theta)\,d\theta$$

$$= \tfrac83 - \tfrac{8}{12}\left[\cos^3\theta\sin\theta + \tfrac32(\theta+\sin\theta\cos\theta)\right]_0^{\pi/2}$$

$$= \tfrac83 - \tfrac23\left[0 + \tfrac32\left(\tfrac\pi2\right)\right] = \tfrac{16-3\pi}{6}$$

17. One loop is given by the region

$D = \{(r,\theta) \mid -\pi/6 \le \theta \le \pi/6,\, 0 \le r \le \cos 3\theta\}$, so the area is



$$\iint_D dA = \int_{-\pi/6}^{\pi/6}\int_0^{\cos 3\theta} r\,dr\,d\theta = \int_{-\pi/6}^{\pi/6}\left[\frac12 r^2\right]_{r=0}^{r=\cos 3\theta} d\theta$$

$$= \int_{-\pi/6}^{\pi/6}\frac12\cos^2 3\theta\,d\theta = 2\int_0^{\pi/6}\frac12\left(\frac{1+\cos 6\theta}{2}\right)d\theta$$

$$= \frac12\left[\theta + \frac16\sin 6\theta\right]_0^{\pi/6} = \frac{\pi}{12}$$

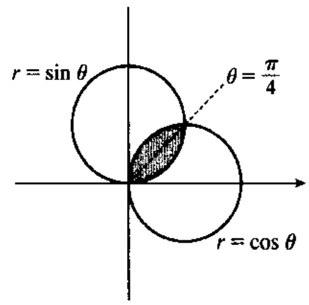
18. $D = \{(r,\theta) \mid 0 \le \theta \le 2\pi, 0 \le r \le 4 + 3\cos\theta\}$, so

$$A(D) = \iint_D dA = \int_0^{2\pi}\int_0^{4+3\cos\theta} r\,dr\,d\theta = \int_0^{2\pi}\left[\tfrac{1}{2}r^2\right]_{r=0}^{r=4+3\cos\theta} d\theta = \tfrac{1}{2}\int_0^{2\pi}(4+3\cos\theta)^2 d\theta$$

$$= \tfrac{1}{2}\int_0^{2\pi}(16 + 24\cos\theta + 9\cos^2\theta)\,d\theta = \tfrac{1}{2}\int_0^{2\pi}\left(16 + 24\cos\theta + 9\cdot\tfrac{1+\cos 2\theta}{2}\right)d\theta$$

$$= \tfrac{1}{2}\left[16\theta + 24\sin\theta + \tfrac{9}{2}\theta + \tfrac{9}{4}\sin 2\theta\right]_0^{2\pi} = \tfrac{41}{2}\pi$$

19. By symmetry,



$$A = 2\int_0^{\pi/4}\int_0^{\sin\theta} r\,dr\,d\theta = 2\int_0^{\pi/4}\left[\tfrac{1}{2}r^2\right]_{r=0}^{r=\sin\theta} d\theta$$

$$= \int_0^{\pi/4}\sin^2\theta\,d\theta = \int_0^{\pi/4}\tfrac{1}{2}(1-\cos 2\theta)\,d\theta$$

$$= \tfrac{1}{2}\left[\theta - \tfrac{1}{2}\sin 2\theta\right]_0^{\pi/4}$$

$$= \tfrac{1}{2}\left[\tfrac{\pi}{4} - \tfrac{1}{2}\sin\tfrac{\pi}{2} - 0 + \tfrac{1}{2}\sin 0\right] = \tfrac{1}{8}(\pi - 2)$$

20. $2 = 4\sin\theta$ implies that $\theta = \frac{\pi}{6}$ or $\frac{5\pi}{6}$, so

$$A = \int_{\pi/6}^{5\pi/6}\int_2^{4\sin\theta} r\,dr\,d\theta = \int_{\pi/6}^{5\pi/6}\left[\tfrac{1}{2}r^2\right]_{r=2}^{r=4\sin\theta} d\theta = \int_{\pi/6}^{5\pi/6}(8\sin^2\theta - 2)\,d\theta$$

$$= \int_{\pi/6}^{5\pi/6}[4(1-\cos 2\theta) - 2]\,d\theta = \left[2\theta - 2\sin 2\theta\right]_{\pi/6}^{5\pi/6} = \tfrac{4\pi}{3} + 2\sqrt{3}.$$

21. $V = \displaystyle\iint_{x^2+y^2\le 9}(x^2+y^2)\,dA = \int_0^{2\pi}\int_0^3 (r^2)\,r\,dr\,d\theta = \int_0^{2\pi}d\theta\int_0^3 r^3\,dr = \left[\theta\right]_0^{2\pi}\left[\tfrac{1}{4}r^4\right]_0^3 = 2\pi\left(\tfrac{81}{4}\right) = \tfrac{81\pi}{2}$

22. The sphere $x^2+y^2+z^2 = 16$ intersects the xy-plane in the circle $x^2+y^2 = 16$, so

$$V = 2\iint_{4\le x^2+y^2\le 16}\sqrt{16 - x^2 - y^2}\,dA \qquad \text{[by symmetry]}$$

$$= 2\int_0^{2\pi}\int_2^4\sqrt{16-r^2}\,r\,dr\,d\theta = 2\int_0^{2\pi}d\theta\int_2^4 r(16-r^2)^{1/2}dr$$

$$= 2\left[\theta\right]_0^{2\pi}\left[-\tfrac{1}{3}(16-r^2)^{3/2}\right]_2^4 = -\tfrac{2}{3}(2\pi)(0 - 12^{3/2}) = \tfrac{4\pi}{3}\left(12\sqrt{12}\right) = 32\sqrt{3}\,\pi$$

23. By symmetry,

$$V = 2\iint_{x^2+y^2\le a^2}\sqrt{a^2-x^2-y^2}\,dA = 2\int_0^{2\pi}\int_0^a\sqrt{a^2-r^2}\,r\,dr\,d\theta = 2\int_0^{2\pi}d\theta\int_0^a r\sqrt{a^2-r^2}\,dr$$

$$= 2\left[\theta\right]_0^{2\pi}\left[-\tfrac{1}{3}(a^2-r^2)^{3/2}\right]_0^a = 2(2\pi)\left(0 + \tfrac{1}{3}a^3\right) = \tfrac{4\pi}{3}a^3$$

24. The paraboloid $z = 10 - 3x^2 - 3y^2$ intersects the plane $z = 4$ when $4 = 10 - 3x^2 - 3y^2$ or $x^2 + y^2 = 2$. So

$$V = \iint_{x^2+y^2\le 2}\left[(10 - 3x^2 - 3y^2) - 4\right]dA = \int_0^{2\pi}\int_0^{\sqrt{2}}(6 - 3r^2)\,r\,dr\,d\theta$$

$$= \int_0^{2\pi}d\theta\int_0^{\sqrt{2}}(6r - 3r^3)\,dr = \left[\theta\right]_0^{2\pi}\left[3r^2 - \tfrac{3}{4}r^4\right]_0^{\sqrt{2}} = 6\pi$$

25. The cone $z = \sqrt{x^2 + y^2}$ intersects the sphere $x^2 + y^2 + z^2 = 1$ when $x^2 + y^2 + \left(\sqrt{x^2 + y^2}\right)^2 = 1$ or $x^2 + y^2 = \frac{1}{2}$. So

$$V = \iint\limits_{x^2+y^2 \le 1/2} \left(\sqrt{1 - x^2 - y^2} - \sqrt{x^2 + y^2}\right) dA = \int_0^{2\pi} \int_0^{1/\sqrt{2}} \left(\sqrt{1 - r^2} - r\right) r \, dr \, d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^{1/\sqrt{2}} \left(r\sqrt{1 - r^2} - r^2\right) dr = \left[\,\theta\,\right]_0^{2\pi} \left[-\tfrac{1}{3}(1 - r^2)^{3/2} - \tfrac{1}{3}r^3\right]_0^{1/\sqrt{2}}$$

$$= 2\pi\left(-\tfrac{1}{3}\right)\left(\tfrac{1}{\sqrt{2}} - 1\right) = \tfrac{\pi}{3}\left(2 - \sqrt{2}\right)$$

26. The two paraboloids intersect when $3x^2 + 3y^2 = 4 - x^2 - y^2$ or $x^2 + y^2 = 1$. So

$$V = \iint\limits_{x^2+y^2 \le 1} [(4 - x^2 - y^2) - 3(x^2 + y^2)] \, dA = \int_0^{2\pi} \int_0^1 4(1 - r^2) \, r \, dr \, d\theta$$

$$= \int_0^{2\pi} d\theta \int_0^1 (4r - 4r^3) \, dr = \left[\,\theta\,\right]_0^{2\pi} \left[2r^2 - r^4\right]_0^1 = 2\pi$$

27. The given solid is the region inside the cylinder $x^2 + y^2 = 4$ between the surfaces $z = \sqrt{64 - 4x^2 - 4y^2}$ and $z = -\sqrt{64 - 4x^2 - 4y^2}$. So

$$V = \iint\limits_{x^2+y^2 \le 4} \left[\sqrt{64 - 4x^2 - 4y^2} - \left(-\sqrt{64 - 4x^2 - 4y^2}\right)\right] dA$$

$$= \iint\limits_{x^2+y^2 \le 4} 2\sqrt{64 - 4x^2 - 4y^2} \, dA = 4\int_0^{2\pi} \int_0^2 \sqrt{16 - r^2} \, r \, dr \, d\theta$$

$$= 4\int_0^{2\pi} d\theta \int_0^2 r\sqrt{16 - r^2} \, dr = 4\left[\,\theta\,\right]_0^{2\pi} \left[-\tfrac{1}{3}(16 - r^2)^{3/2}\right]_0^2$$

$$= 8\pi\left(-\tfrac{1}{3}\right)\left(12^{3/2} - 16^{2/3}\right) = \tfrac{8\pi}{3}\left(64 - 24\sqrt{3}\right)$$

28. (a) Here the region in the xy-plane is the annular region $r_1^2 \le x^2 + y^2 \le r_2^2$ and the desired volume is twice that above the xy-plane. Hence

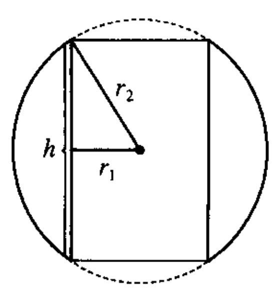
$$V = 2\iint\limits_{r_1^2 \le x^2+y^2 \le r_2^2} \sqrt{r_2^2 - x^2 - y^2} \, dA = 2\int_0^{2\pi} \int_{r_1}^{r_2} \sqrt{r_2^2 - r^2} \, r \, dr \, d\theta$$

$$= 2\int_0^{2\pi} d\theta \int_{r_1}^{r_2} \sqrt{r_2^2 - r^2} \, r \, dr = \tfrac{4\pi}{3}\left[-(r_2^2 - r^2)^{3/2}\right]_{r_1}^{r_2} = \tfrac{4\pi}{3}(r_2^2 - r_1^2)^{3/2}$$

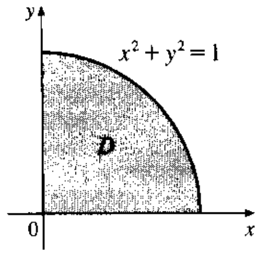
(b) A cross-sectional cut is shown in the figure.

So $r_2^2 = \left(\frac{1}{2}h\right)^2 + r_1^2$ or $\frac{1}{4}h^2 = r_2^2 - r_1^2$.

Thus the volume in terms of h is $V = \frac{4\pi}{3}\left(\frac{1}{4}h^2\right)^{3/2} = \frac{\pi}{6}h^3$.

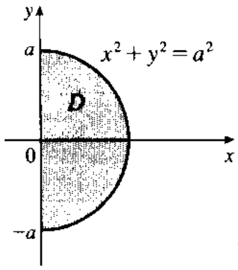


29.



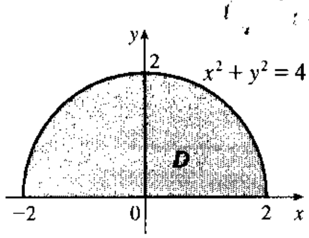
$$\int_0^1\int_0^{\sqrt{1-x^2}} e^{x^2+y^2}\,dy\,dx = \int_0^{\pi/2}\int_0^1 e^{r^2}\,r\,dr\,d\theta$$
$$= \int_0^{\pi/2} d\theta \int_0^1 re^{r^2}\,dr$$
$$= \left[\theta\right]_0^{\pi/2}\left[\tfrac{1}{2}e^{r^2}\right]_0^1 = \tfrac{1}{4}\pi(e-1)$$

30.



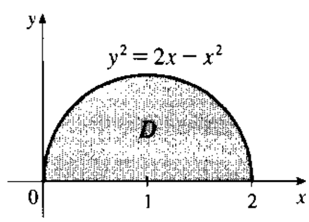
$$\int_{-\pi/2}^{\pi/2}\int_0^a (r^2)^{3/2}\,r\,dr\,d\theta = \int_{-\pi/2}^{\pi/2} d\theta \int_0^a r^4\,dr$$
$$= \left[\theta\right]_{-\pi/2}^{\pi/2}\left[\tfrac{1}{5}r^5\right]_0^a$$
$$= \tfrac{1}{5}\pi a^5$$

31.



$$\int_0^{\pi}\int_0^2 (r\cos\theta)^2(r\sin\theta)^2 r\,dr\,d\theta = \int_0^{\pi}(\sin\theta\cos\theta)^2 d\theta \int_0^2 r^5\,dr$$
$$= \int_0^{\pi}\left(\tfrac{1}{2}\sin 2\theta\right)^2 d\theta \int_0^2 r^5\,dr$$
$$= \tfrac{1}{4}\left[\tfrac{1}{2}\theta - \tfrac{1}{8}\sin 4\theta\right]_0^{\pi}\left[\tfrac{1}{6}r^6\right]_0^2$$
$$= \tfrac{1}{4}\left(\tfrac{\pi}{2}\right)\left(\tfrac{64}{6}\right) = \tfrac{4\pi}{3}$$

32.



$$\int_0^{\pi/2}\int_0^{2\cos\theta} r^2\,dr\,d\theta = \int_0^{\pi/2}\left[\tfrac{1}{3}r^3\right]_{r=0}^{r=2\cos\theta} d\theta$$
$$= \int_0^{\pi/2}\left(\tfrac{8}{3}\cos^3\theta\right) d\theta$$
$$= \tfrac{8}{3}\left[\sin\theta - \tfrac{1}{3}\sin^3\theta\right]_0^{\pi/2} = \tfrac{16}{9}$$

33. The surface of the water in the pool is a circular disk D with radius 20 ft. If we place D on coordinate axes with the origin at the center of D and define $f(x,y)$ to be the depth of the water at (x,y), then the volume of water in the pool is the volume of the solid that lies above $D = \{(x,y) \mid x^2+y^2 \le 400\}$ and below the graph of $f(x,y)$. We can associate north with the positive y-direction, so we are given that the depth is constant in the x-direction and the depth increases linearly in the y-direction from $f(0,-20)=2$ to $f(0,20)=7$. The trace in the yz-plane is a line segment from $(0,-20,2)$ to $(0,20,7)$. The slope of this line is $\frac{7-2}{20-(-20)} = \frac{1}{8}$, so an equation of the line is $z-7 = \frac{1}{8}(y-20) \;\Rightarrow\; z = \frac{1}{8}y + \frac{9}{2}$. Since $f(x,y)$ is independent of x, $f(x,y) = \frac{1}{8}y + \frac{9}{2}$. Thus the volume is given by $\iint_D f(x,y)\,dA$, which is most conveniently evaluated using polar coordinates. Then $D = \{(r,\theta) \mid 0 \le r \le 20, 0 \le \theta \le 2\pi\}$ and substituting $x = r\cos\theta$, $y = r\sin\theta$ the integral becomes

$$\int_0^{2\pi}\int_0^{20}\left(\tfrac{1}{8}r\sin\theta + \tfrac{9}{2}\right) r\,dr\,d\theta = \int_0^{2\pi}\left[\tfrac{1}{24}r^3\sin\theta + \tfrac{9}{4}r^2\right]_{r=0}^{r=20} d\theta = \int_0^{2\pi}\left(\tfrac{1000}{3}\sin\theta + 900\right) d\theta$$
$$= \left[-\tfrac{1000}{3}\cos\theta + 900\theta\right]_0^{2\pi} = 1800\pi$$

Thus the pool contains $1800\pi \approx 5655$ ft^3 of water.

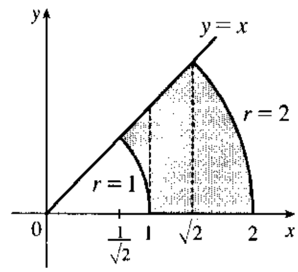
34. (a) The total amount of water supplied each hour to the region within R feet of the sprinkler is

$$V = \int_0^{2\pi}\int_0^R e^{-r} r\,dr\,d\theta = \int_0^{2\pi} d\theta \int_0^R re^{-r}\,dr = \big[\,\theta\,\big]_0^{2\pi}\left[-re^{-r} - e^{-r}\right]_0^R$$
$$= 2\pi[-Re^{-R} - e^{-R} + 0 + 1] = 2\pi(1 - Re^{-R} - e^{-R})\text{ ft}^3$$

(b) The average amount of water per hour per square foot supplied to the region within R feet of the sprinkler is $\dfrac{V}{\text{area of region}} = \dfrac{V}{\pi R^2} = \dfrac{2\left(1 - Re^{-R} - e^{-R}\right)}{R^2}$ ft^3 (per hour per square foot). See the definition of the average value of a function on page 1022 [ET 986].

35. $\displaystyle\int_{1/\sqrt{2}}^{1}\int_{\sqrt{1-x^2}}^{x} xy\,dy\,dx + \int_1^{\sqrt{2}}\int_0^x xy\,dy\,dx + \int_{\sqrt{2}}^{2}\int_0^{\sqrt{4-x^2}} xy\,dy\,dx$

$$= \int_0^{\pi/4}\int_1^2 r^3\cos\theta\sin\theta\,dr\,d\theta = \int_0^{\pi/4}\left[\frac{r^4}{4}\cos\theta\sin\theta\right]_{r=1}^{r=2} d\theta$$
$$= \frac{15}{4}\int_0^{\pi/4}\sin\theta\cos\theta\,d\theta = \frac{15}{4}\left[\frac{\sin^2\theta}{2}\right]_0^{\pi/4} = \frac{15}{16}$$



36. (a) $\iint_{D_a} e^{-(x^2+y^2)}dA = \int_0^{2\pi}\int_0^a re^{-r^2}\,dr\,d\theta = 2\pi\left[-\frac{1}{2}e^{-r^2}\right]_0^a = \pi\left(1 - e^{-a^2}\right)$ for each a. Then $\lim\limits_{a\to\infty} \pi\left(1 - e^{-a^2}\right) = \pi$ since $e^{-a^2} \to 0$ as $a \to \infty$. Hence $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,dA = \pi$.

(b) $\iint_{S_a} e^{-(x^2+y^2)}\,dA = \int_{-a}^{a}\int_{-a}^{a} e^{-x^2}e^{-y^2}\,dx\,dy = \left(\int_{-a}^{a} e^{-x^2}\,dx\right)\left(\int_{-a}^{a} e^{-y^2}\,dy\right)$ for each a.

Then, from (a), $\pi = \iint_{\mathbb{R}^2} -(x^2+y^2)\,dA$, so

$\pi = \lim\limits_{a\to\infty}\iint_{S_a} e^{-(x^2+y^2)}\,dA = \lim\limits_{a\to\infty}\left(\int_{-a}^{a} e^{-x^2}\,dx\right)\left(\int_{-a}^{a} e^{-y^2}\,dy\right) = \left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2}\,dy\right)$.

To evaluate $\lim\limits_{a\to\infty}\left(\int_{-a}^{a} e^{-x^2}\,dx\right)\left(\int_{-a}^{a} e^{-y^2}\,dy\right)$, we are using the fact that these integrals are bounded. This is true since on $[-1,1]$, $0 < e^{-x^2} \le 1$ while on $(-\infty,-1)$, $0 < e^{-x^2} \le e^x$ and on $(1,\infty)$, $0 < e^{-x^2} < e^{-x}$. Hence $0 \le \int_{-\infty}^{\infty} e^{-x^2}\,dx \le \int_{-\infty}^{-1} e^x\,dx + \int_{-1}^{1} dx + \int_1^{\infty} e^{-x}\,dx = 2(e^{-1}+1)$.

(c) Since $\left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2}\,dy\right) = \pi$ and y can be replaced by x, $\left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)^2 = \pi$ implies that $\int_{-\infty}^{\infty} e^{-x^2}\,dx = \pm\sqrt{\pi}$. But $e^{-x^2} \ge 0$ for all x, so $\int_{-\infty}^{\infty} e^{-x^2}\,dx = \sqrt{\pi}$.

(d) Letting $t = \sqrt{2}x$, $\int_{-\infty}^{\infty} e^{-x^2}\,dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2}}\left(e^{-t^2/2}\right)dt$, so that $\sqrt{\pi} = \frac{1}{\sqrt{2}}\int_{-\infty}^{\infty} e^{-t^2/2}\,dt$ or $\int_{-\infty}^{\infty} e^{-t^2/2}\,dt = \sqrt{2\pi}$.

37. (a) We integrate by parts with $u = x$ and $dv = xe^{-x^2}\,dx$. Then $du = dx$ and $v = -\frac{1}{2}e^{-x^2}$, so

$$\int_0^{\infty} x^2e^{-x^2}\,dx = \lim_{t\to\infty}\int_0^t x^2e^{-x^2}\,dx = \lim_{t\to\infty}\left(\left[-\tfrac{1}{2}xe^{-x^2}\right]_0^t + \int_0^t \tfrac{1}{2}e^{-x^2}\,dx\right)$$
$$= \lim_{t\to\infty}\left(-\tfrac{1}{2}te^{-t^2}\right) + \tfrac{1}{2}\int_0^{\infty} e^{-x^2}\,dx = 0 + \tfrac{1}{2}\int_0^{\infty} e^{-x^2}\,dx \qquad \text{[by l'Hospital's Rule]}$$
$$= \tfrac{1}{4}\int_{-\infty}^{\infty} e^{-x^2}\,dx \qquad \text{[since } e^{-x^2} \text{ is an even function]}$$
$$= \tfrac{1}{4}\sqrt{\pi} \qquad \text{[by Exercise 36(c)]}$$

(b) Let $u = \sqrt{x}$. Then $u^2 = x \Rightarrow dx = 2u\,du \Rightarrow$

$$\int_0^\infty \sqrt{x}e^{-x}\,dx = \lim_{t\to\infty}\int_0^t \sqrt{x}\,e^{-x}\,dx = \lim_{t\to\infty}\int_0^{\sqrt{t}} ue^{-u^2}2u\,du = 2\int_0^\infty u^2e^{-u^2}\,du$$
$$= 2\left(\tfrac{1}{4}\sqrt{\pi}\right)\ \text{[by part(a)]} = \tfrac{1}{2}\sqrt{\pi}$$

16.5 Applications of Double Integrals ET 15.5

1. $Q = \iint_D \sigma(x,y)\,dA = \int_1^3\int_0^2 (2xy + y^2)\,dy\,dx = \int_1^3 \left[xy^2 + \tfrac{1}{3}y^3\right]_{y=0}^{y=2} dx$
$= \int_1^3 \left(4x + \tfrac{8}{3}\right) dx = \left[2x^2 + \tfrac{8}{3}x\right]_1^3 = 16 + \tfrac{16}{3} = \tfrac{64}{3}$ C

2. $Q = \iint_D \sigma(x,y)\,dA = \iint_D (x + y + x^2 + y^2)\,dA = \int_0^{2\pi}\int_0^2 (r\cos\theta + r\sin\theta + r^2)\,r\,dr\,d\theta$
$= \int_0^{2\pi}\int_0^2 \left[r^2(\cos\theta + \sin\theta) + r^3\right] dr\,d\theta = \int_0^{2\pi}\left[\tfrac{1}{3}r^3(\cos\theta + \sin\theta) + \tfrac{1}{4}r^4\right]_{r=0}^{r=2} d\theta$
$= \int_0^{2\pi}\left[\tfrac{8}{3}(\cos\theta + \sin\theta) + 4\right] d\theta = \left[\tfrac{8}{3}(\sin\theta - \cos\theta) + 4\theta\right]_0^{2\pi} = 8\pi$ C

3. $m = \iint_D \rho(x,y)\,dA = \int_0^2\int_{-1}^1 xy^2\,dy\,dx = \int_0^2 x\,dx \int_{-1}^1 y^2\,dy = \left[\tfrac{1}{2}x^2\right]_0^2 \left[\tfrac{1}{3}y^3\right]_{-1}^1 = 2\cdot\tfrac{2}{3} = \tfrac{4}{3}$,

$\overline{x} = \tfrac{1}{m}\iint_D x\rho(x,y)\,dA = \tfrac{3}{4}\int_0^2\int_{-1}^1 x^2y^2\,dy\,dx = \tfrac{3}{4}\int_0^2 x^2\,dx\int_{-1}^1 y^2\,dy = \tfrac{3}{4}\left[\tfrac{1}{3}x^3\right]_0^2\left[\tfrac{1}{3}y^3\right]_{-1}^1 = \tfrac{3}{4}\cdot\tfrac{8}{3}\cdot\tfrac{2}{3} = \tfrac{4}{3}$,

$\overline{y} = \tfrac{1}{m}\iint_D y\rho(x,y)\,dA = \tfrac{3}{4}\int_0^2\int_{-1}^1 xy^3\,dy\,dx = \tfrac{3}{4}\int_0^2 x\,dx\int_{-1}^1 y^3\,dy = \tfrac{3}{4}\left[\tfrac{1}{2}x^2\right]_0^2\left[\tfrac{1}{4}y^4\right]_{-1}^1 = \tfrac{3}{4}\cdot 2\cdot 0 = 0$.

Hence, $(\overline{x},\overline{y}) = \left(\tfrac{4}{3}, 0\right)$.

4. $m = \iint_D \rho(x,y)\,dA = \int_0^a\int_0^b cxy\,dy\,dx = c\int_0^a x\,dx\int_0^b y\,dy = c\left[\tfrac{1}{2}x^2\right]_0^a\left[\tfrac{1}{2}y^2\right]_0^b = \tfrac{1}{4}a^2b^2c$,

$M_y = \iint_D x\rho(x,y)\,dA = \int_0^a\int_0^b cx^2y\,dy\,dx = c\int_0^a x^2\,dx\int_0^b y\,dy = c\left[\tfrac{1}{3}x^3\right]_0^a\left[\tfrac{1}{2}y^2\right]_0^b = \tfrac{1}{6}a^3b^2c$, and

$M_x = \iint_D y\rho(x,y)\,dA = \int_0^a\int_0^b cxy^2\,dy\,dx = c\int_0^a x\,dx\int_0^b y^2\,dy = c\left[\tfrac{1}{2}x^2\right]_0^a\left[\tfrac{1}{3}y^3\right]_0^b = \tfrac{1}{6}a^2b^3c$.

Hence, $(\overline{x},\overline{y}) = \left(\dfrac{M_y}{m}, \dfrac{M_x}{m}\right) = \left(\dfrac{2}{3}a, \dfrac{2}{3}b\right)$.

5. $m = \int_0^2\int_{x/2}^{3-x}(x + y)\,dy\,dx = \int_0^2\left[xy + \tfrac{1}{2}y^2\right]_{y=x/2}^{y=3-x} dx = \int_0^2\left[x\left(3 - \tfrac{3}{2}x\right) + \tfrac{1}{2}(3 - x)^2 - \tfrac{1}{8}x^2\right] dx$
$= \int_0^2\left(-\tfrac{9}{8}x^2 + \tfrac{9}{2}\right) dx = \left[-\tfrac{9}{8}\left(\tfrac{1}{3}x^3\right) + \tfrac{9}{2}x\right]_0^2 = 6$,

$M_y = \int_0^2\int_{x/2}^{3-x}(x^2 + xy)\,dy\,dx = \int_0^2\left[x^2y + \tfrac{1}{2}xy^2\right]_{y=x/2}^{y=3-x} dx = \int_0^2\left(\tfrac{9}{2}x - \tfrac{9}{8}x^3\right) dx = \tfrac{9}{2}$, and

$M_x = \int_0^2\int_{x/2}^{3-y}(xy + y^2)\,dy\,dx = \int_0^2\left[\tfrac{1}{2}xy^2 + \tfrac{1}{3}y^3\right]_{y=x/2}^{y=3-x} dx = \int_0^2\left(9 - \tfrac{9}{2}x\right) dx = 9$.

Hence $m = 6$, $(\overline{x},\overline{y}) = \left(\dfrac{M_y}{m}, \dfrac{M_x}{m}\right) = \left(\dfrac{3}{4}, \dfrac{3}{2}\right)$.

6. $m = \int_0^1\int_y^{4-3y} x\,dx\,dy = \int_0^1\left[\tfrac{1}{2}(4 - 3y)^2 - \tfrac{1}{2}y^2\right] dy = \left[-\tfrac{1}{18}(4 - 3y)^3 - \tfrac{1}{6}y^3\right]_0^1 = \tfrac{10}{3}$,

$M_y = \int_0^1\int_y^{4-3y} x^2\,dx\,dy = \int_0^1\left[\tfrac{1}{3}(4 - 3y)^3 - \tfrac{1}{3}y^3\right] dy = \left[-\tfrac{1}{36}(4 - 3y)^4 - \tfrac{1}{12}y^4\right]_0^1 = 7$,

$M_x = \int_0^1\int_y^{4-3y} xy\,dx\,dy = \int_0^1\left[\tfrac{1}{2}y(4 - 3y)^2 - \tfrac{1}{2}y^3\right] dy = \int_0^1 (8y - 12y^2 + 4y^3)\,dy = 1$.

Hence $m = \tfrac{10}{3}$, $(\overline{x},\overline{y}) = (2.1, 0.3)$.

7. $m = \int_0^1 \int_0^{e^x} y\,dy\,dx = \int_0^1 \left[\frac{1}{2}y^2\right]_{y=0}^{y=e^x} dx = \frac{1}{2}\int_0^1 e^{2x}\,dx = \frac{1}{4}e^{2x}\Big]_0^1 = \frac{1}{4}(e^2-1)$,

$M_y = \int_0^1 \int_0^{e^x} xy\,dy\,dx = \frac{1}{2}\int_0^1 xe^{2x}\,dx = \frac{1}{2}\left[\frac{1}{2}xe^{2x} - \frac{1}{4}e^{2x}\right]_0^1 = \frac{1}{8}(e^2+1)$, and

$M_x = \int_0^1 \int_0^{e^x} y^2\,dy\,dx = \int_0^1 \left[\frac{1}{3}y^3\right]_{y=0}^{y=e^x} dx = \frac{1}{3}\int_0^1 e^{3x}\,dx = \frac{1}{3}\left[\frac{1}{3}e^{3x}\right]_0^1 = \frac{1}{9}(e^3-1)$.

Hence $m = \frac{1}{4}(e^2-1)$, $(\overline{x},\overline{y}) = \left(\dfrac{\frac{1}{8}(e^2+1)}{\frac{1}{4}(e^2-1)}, \dfrac{\frac{1}{9}(e^3-1)}{\frac{1}{4}(e^2-1)}\right) = \left(\dfrac{e^2+1}{2(e^2-1)}, \dfrac{4(e^3-1)}{9(e^2-1)}\right)$.

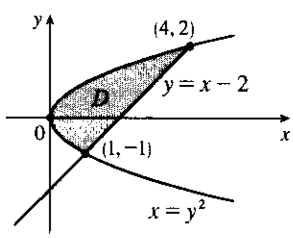
8. $m = \int_0^1 \int_0^{\sqrt{x}} x\,dy\,dx = \int_0^1 x\left[y\right]_{y=0}^{y=\sqrt{x}} dx = \int_0^1 x^{3/2}\,dx = \frac{2}{5}x^{5/2}\Big]_0^1 = \frac{2}{5}$,

$M_y = \int_0^1 \int_0^{\sqrt{x}} x^2\,dy\,dx = \int_0^1 x\left[y\right]_{y=0}^{y=\sqrt{x}} dx = \int_0^1 x^{5/2}\,dx = \frac{2}{7}x^{7/2}\Big]_0^1 = \frac{2}{7}$, and

$M_x = \int_0^1 \int_0^{\sqrt{x}} yx\,dy\,dx = \int_0^1 x\left[\frac{1}{2}y^2\right]_{y=0}^{y=\sqrt{x}} dx = \frac{1}{2}\int_0^1 x^2\,dx = \frac{1}{2}\left[\frac{1}{3}x^3\right]_0^1 = \frac{1}{6}$.

Hence $m = \frac{2}{5}$, $(\overline{x},\overline{y}) = \left(\frac{2/7}{2/5}, \frac{1/6}{2/5}\right) = \left(\frac{5}{7}, \frac{5}{12}\right)$.

9.



$m = \int_{-1}^2 \int_{y^2}^{y+2} 3\,dx\,dy = \int_{-1}^2 (3y+6-3y^2)\,dy = \frac{27}{2}$,

$M_y = \int_{-1}^2 \int_{y^2}^{y+2} 3x\,dx\,dy = \int_{-1}^2 \frac{3}{2}\left[(y+2)^2 - y^4\right] dy$

$= \left[\frac{1}{2}(y+2)^3 - \frac{3}{10}y^5\right]_{-1}^2 = \frac{108}{5}$

and

$M_x = \int_{-1}^2 \int_{y^2}^{y+2} 3y\,dx\,dy = \int_{-1}^2 (3y^2+6y-3y^3)\,dy$

$= \left[y^3 + 3y^2 - \frac{3}{4}y^4\right]_{-1}^2 = \frac{27}{4}$

Hence $m = \frac{27}{2}$, $(\overline{x},\overline{y}) = \left(\frac{8}{5}, \frac{1}{2}\right)$.

10. $m = \int_0^{\pi/2} \int_0^{\cos x} x\,dy\,dx = \int_0^{\pi/2} x\cos x\,dx = \left[x\sin x + \cos x\right]_0^{\pi/2} = \frac{\pi}{2} - 1$,

$M_y = \int_0^{\pi/2} \int_0^{\cos x} x^2\,dy\,dx = \int_0^{\pi/2} x^2\cos x\,dx = \left[x^2\sin x + 2x\cos x - 2\sin x\right]_0^{\pi/2} = \frac{\pi^2}{4} - 2$, and

$M_x = \int_0^{\pi/2} \int_0^{\cos x} xy\,dy\,dx = \int_0^{\pi/2} \frac{1}{2}x\cos^2 x\,dx = \frac{1}{2}\left[\frac{1}{4}x^2 + \frac{1}{4}x\sin 2x + \frac{1}{8}\cos 2x\right]_0^{\pi/2} = \frac{\pi^2}{32} - \frac{1}{8}$.

Hence $m = \dfrac{\pi-2}{2}$, $(\overline{x},\overline{y}) = \left(\dfrac{\pi^2-8}{2(\pi-2)}, \dfrac{\pi+2}{16}\right)$.

11. $\rho(x,y) = ky = kr\sin\theta$, $m = \int_0^{\pi/2} \int_0^1 kr^2\sin\theta\,dr\,d\theta = \frac{1}{3}k\int_0^{\pi/2} \sin\theta\,d\theta = \frac{1}{3}k\left[-\cos\theta\right]_0^{\pi/2} = \frac{1}{3}k$,

$M_y = \int_0^{\pi/2} \int_0^1 kr^3\sin\theta\cos\theta\,dr\,d\theta = \frac{1}{4}k\int_0^{\pi/2} \sin\theta\cos\theta\,d\theta = \frac{1}{8}k\left[-\cos 2\theta\right]_0^{\pi/2} = \frac{1}{8}k$,

$M_x = \int_0^{\pi/2} \int_0^1 kr^3\sin^2\theta\,dr\,d\theta = \frac{1}{4}k\int_0^{\pi/2} \sin^2\theta\,d\theta = \frac{1}{8}k\left[\theta + \sin 2\theta\right]_0^{\pi/2} = \frac{\pi}{16}k$.

Hence $(\overline{x},\overline{y}) = \left(\frac{3}{8}, \frac{3\pi}{16}\right)$.

12. $\rho(x,y) = k(x^2+y^2) = kr^2$, $m = \int_0^{\pi/2}\int_0^1 kr^3\,dr\,d\theta = \frac{\pi}{8}k$,

$M_y = \int_0^{\pi/2}\int_0^1 kr^4\cos\theta\,dr\,d\theta = \frac{1}{5}k\int_0^{\pi/2}\cos\theta\,d\theta = \frac{1}{5}k\big[\sin\theta\big]_0^{\pi/2} = \frac{1}{5}k$,

$M_x = \int_0^{\pi/2}\int_0^1 kr^4\sin\theta\,dr\,d\theta = \frac{1}{5}k\int_0^{\pi/2}\sin\theta\,d\theta = \frac{1}{5}k\big[-\cos\theta\big]_0^{\pi/2} = \frac{1}{5}k$.

Hence $(\overline{x},\overline{y}) = \left(\frac{8}{5\pi},\frac{8}{5\pi}\right)$.

13. Placing the vertex opposite the hypotenuse at $(0,0)$, $\rho(x,y) = k(x^2+y^2)$. Then

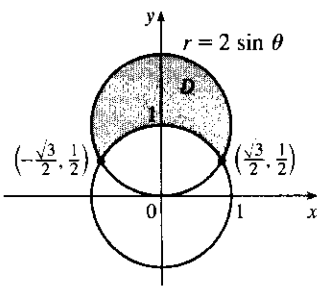
$$m = \int_0^a\int_0^{a-x} k(x^2+y^2)\,dy\,dx = k\int_0^a\left[ax^2 - x^3 + \tfrac{1}{3}(a-x)^3\right]dx$$
$$= k\left[\tfrac{1}{3}ax^3 - \tfrac{1}{4}x^4 - \tfrac{1}{12}(a-x)^4\right]_0^a = \tfrac{1}{6}ka^4$$

By symmetry,

$$M_y = M_x = \int_0^a\int_0^{a-x} ky(x^2+y^2)\,dy\,dx = k\int_0^a\left[\tfrac{1}{2}(a-x)^2x^2 + \tfrac{1}{4}(a-x)^4\right]dx$$
$$= k\left[\tfrac{1}{6}a^2x^3 - \tfrac{1}{4}ax^4 + \tfrac{1}{10}x^5 - \tfrac{1}{20}(a-x)^5\right]_0^a = \tfrac{1}{15}ka^5$$

Hence $(\overline{x},\overline{y}) = \left(\frac{2}{5}a, \frac{2}{5}a\right)$.

14.



$\rho(x,y) = k/\sqrt{x^2+y^2} = k/r$,

$$m = \int_{\pi/6}^{5\pi/6}\int_1^{2\sin\theta}\frac{k}{r}\,r\,dr\,d\theta = k\int_{\pi/6}^{5\pi/6}[(2\sin\theta)-1]\,d\theta$$
$$= k\big[-2\cos\theta - \theta\big]_{\pi/6}^{5\pi/6} = 2k\left(\sqrt{3}-\tfrac{\pi}{3}\right)$$

By symmetry of D and $f(x) = x$, $M_y = 0$, and

$$M_x = \int_{\pi/6}^{5\pi/6}\int_1^{2\sin\theta} kr\sin\theta\,dr\,d\theta = \tfrac{1}{2}k\int_{\pi/6}^{5\pi/6}(4\sin^3\theta - \sin\theta)\,d\theta$$
$$= \tfrac{1}{2}k\left[-3\cos\theta + \tfrac{4}{3}\cos^3\theta\right]_{\pi/6}^{5\pi/6} = \sqrt{3}\,k$$

Hence $(\overline{x},\overline{y}) = \left(0, \dfrac{3\sqrt{3}}{2(3\sqrt{3}-\pi)}\right)$.

15. $I_x = \iint_D y^2\rho(x,y)\,dA = \int_0^1\int_0^{e^x} y^2\cdot y\,dy\,dx = \int_0^1\left[\frac{1}{4}y^4\right]_{y=0}^{y=e^x}dx = \frac{1}{4}\int_0^1 e^{4x}\,dx$

$= \frac{1}{4}\left[\frac{1}{4}e^{4x}\right]_0^1 = \frac{1}{16}(e^4-1)$,

$I_y = \iint_D x^2\rho(x,y)\,dA = \int_0^1\int_0^{e^x} x^2y\,dy\,dx = \int_0^1 x^2\left[\frac{1}{2}y^2\right]_{y=0}^{y=e^x}dx = \frac{1}{2}\int_0^1 x^2e^{2x}\,dx$

$= \frac{1}{2}\left[\left(\frac{1}{2}x^2 - \frac{1}{2}x + \frac{1}{4}\right)e^{2x}\right]_0^1$ [integrate by parts twice]

$= \frac{1}{8}(e^2-1)$,

and $I_0 = I_x + I_y = \frac{1}{16}(e^4-1) + \frac{1}{8}(e^2-1) = \frac{1}{16}(e^4+2e^2-3)$.

16. $I_x = \int_0^{\pi/2}\int_0^1 (r^2\sin^2\theta)(kr^2)\,r\,dr\,d\theta = \frac{1}{6}k\int_0^{\pi/2}\sin^2\theta\,d\theta = \frac{1}{6}k\left[\frac{1}{4}(2\theta - \sin 2\theta)\right]_0^{\pi/2} = \frac{\pi}{24}k$,

$I_y = \int_0^{\pi/2}\int_0^1 (r^2\cos^2\theta)(kr^2)\,r\,dr\,d\theta = \frac{1}{6}k\int_0^{\pi/6}\cos^2\theta\,d\theta = \frac{1}{6}k\left[\frac{1}{4}(2\theta + \sin 2\theta)\right]_0^{\pi/2} = \frac{\pi}{24}k$, and

$I_0 = I_x + I_y = \frac{\pi}{12}k$.

17. $I_x = \int_{-1}^{2}\int_{y^2}^{y+2} 3y^2\,dx\,dy = \int_{-1}^{2}(3y^3+6y^2-3y^4)\,dy = \left[\frac{3}{4}y^4+2y^3-\frac{3}{5}y^5\right]_{-1}^{2} = \frac{189}{20}$,

$I_y = \int_{-1}^{2}\int_{y^2}^{y+2} 3x^2\,dx\,dy = \int_{-1}^{2}\left[(y+2)^3-y^6\right]dy = \left[\frac{1}{4}(y+2)^4-\frac{1}{7}y^7\right]_{-1}^{2} = \frac{1269}{28}$, and

$I_0 = I_x + I_y = \frac{1917}{35}$.

18. If we find the moments of inertia about the x- and y-axes, we can determine in which direction rotation will be more difficult. (See the explanation following Example 4.) The moment of inertia about the x-axis is given by

$$I_x = \iint_D y^2\rho(x,y)\,dA = \int_0^2\int_0^2 y^2(1+0.1x)\,dy\,dx = \int_0^2(1+0.1x)\left[\tfrac{1}{3}y^3\right]_{y=0}^{y=2}dx$$

$$= \tfrac{8}{3}\int_0^2(1+0.1x)\,dx = \tfrac{8}{3}\left[x+0.1\cdot\tfrac{1}{2}x^2\right]_0^2 = \tfrac{8}{3}(2.2) \approx 5.87$$

Similarly, the moment of inertia about the y-axis is given by

$$I_y = \iint_D x^2\rho(x,y)\,dA = \int_0^2\int_0^2 x^2(1+0.1x)\,dy\,dx = \int_0^2 x^2(1+0.1x)\left[\,y\,\right]_{y=0}^{y=2}dx$$

$$= 2\int_0^2(x^2+0.1x^3)\,dx = 2\left[\tfrac{1}{3}x^3+0.1\cdot\tfrac{1}{4}x^4\right]_0^2 = 2\left(\tfrac{8}{3}+0.4\right) \approx 6.13$$

Since $I_y > I_x$, more force is required to rotate the fan blade about the y-axis.

19. Using a CAS, we find $m = \iint_D \rho(x,y)\,dA = \int_0^\pi\int_0^{\sin x} xy\,dy\,dx = \dfrac{\pi^2}{8}$. Then

$\overline{x} = \dfrac{1}{m}\displaystyle\iint_D x\rho(x,y)\,dA = \dfrac{8}{\pi^2}\int_0^\pi\int_0^{\sin x} x^2y\,dy\,dx = \dfrac{2\pi}{3} - \dfrac{1}{\pi}$ and

$\overline{y} = \dfrac{1}{m}\displaystyle\iint_D y\rho(x,y)\,dA = \dfrac{8}{\pi^2}\int_0^\pi\int_0^{\sin x} xy^2\,dy\,dx = \dfrac{16}{9\pi}$, so $(\overline{x},\overline{y}) = \left(\dfrac{2\pi}{3}-\dfrac{1}{\pi}, \dfrac{16}{9\pi}\right)$.

The moments of inertia are $I_x = \iint_D y^2\rho(x,y)\,dA = \int_0^\pi\int_0^{\sin x} xy^3\,dy\,dx = \dfrac{3\pi^2}{64}$,

$I_y = \iint_D x^2\rho(x,y)\,dA = \int_0^\pi\int_0^{\sin x} x^3y\,dy\,dx = \dfrac{\pi^2}{16}(\pi^2-3)$, and $I_0 = I_x + I_y = \dfrac{\pi^2}{64}(4\pi^2-9)$.

20. Using a CAS, we find $m = \iint_D \sqrt{x^2+y^2}\,dA = \int_0^{2\pi}\int_0^{1+\cos\theta} r^2\,dr\,d\theta = \frac{5}{3}\pi$,

$\overline{x} = \frac{1}{m}\iint_D x\sqrt{x^2+y^2}\,dA = \frac{3}{5\pi}\int_0^{2\pi}\int_0^{1+\cos\theta} r^3\cos\theta\,dr\,d\theta = \frac{21}{20}$ and

$\overline{y} = \frac{1}{m}\iint_D y\sqrt{x^2+y^2}\,dA = \frac{3}{5\pi}\int_0^{2\pi}\int_0^{1+\cos\theta} r^3\sin\theta\,dr\,d\theta = 0$, so $(\overline{x},\overline{y}) = \left(\frac{21}{20},0\right)$.

The moments of inertia are $I_x = \iint_D y^2\sqrt{x^2+y^2}\,dA = \int_0^{2\pi}\int_0^{1+\cos\theta} r^4\sin^2\theta\,dr\,d\theta = \frac{33}{40}\pi$,

$I_y = \iint_D x^2\sqrt{x^2+y^2}\,dA = \int_0^{2\pi}\int_0^{1+\cos\theta} r^4\cos^2\theta\,dr\,d\theta = \frac{93}{40}\pi$, and $I_0 = I_x + I_y = \frac{63}{20}\pi$.

21. $I_x = \int_0^a\int_0^a \rho y^2\,dx\,dy = \rho\int_0^a dx\int_0^a y^2\,dy = \rho\left[\,x\,\right]_0^a\left[\frac{1}{3}y^3\right]_0^a = \rho a\left(\frac{1}{3}a^3\right) = \frac{1}{3}\rho a^4 = I_y$ by symmetry, and

$m = \rho a^2$ since the lamina is homogeneous. Hence $\overline{\overline{x}}^2 = \dfrac{I_y}{m} \;\Rightarrow\; \overline{\overline{x}} = \left[\left(\frac{1}{3}\rho a^4\right)/(\rho a^2)\right]^{1/2} = \frac{1}{\sqrt{3}}a$ and

$\overline{\overline{y}}^2 = \dfrac{I_x}{m} \;\Rightarrow\; \overline{\overline{y}} = \frac{1}{\sqrt{3}}a$.

22. $m = \int_0^\pi \int_0^{\sin x} \rho \, dy \, dx = \rho \int_0^\pi \sin x \, dx = \rho\big[-\cos x\big]_0^\pi = 2\rho,$

$I_x = \int_0^\pi \int_0^{\sin x} \rho y^2 \, dy \, dx = \frac{1}{3}\rho \int_0^\pi \sin^3 x \, dx = \frac{1}{3}\rho \int_0^\pi (1 - \cos^2 x) \sin x \, dx$

$= \frac{1}{3}\rho\big[-\cos\theta + \frac{1}{3}\cos^3\theta\big]_0^\pi = \frac{4}{9}\rho,$

$I_y = \int_0^\pi \int_0^{\sin x} \rho x^2 \, dy \, dx = \rho \int_0^\pi x^2 \sin x \, dx$

$= \rho\big[-x^2 \cos x + 2x \sin x + 2\cos x\big]_0^\pi$ [by integrating by parts twice]

$= \rho(\pi^2 - 4).$

Then $\overline{\overline{y}}^2 = \dfrac{I_x}{m} = \dfrac{2}{9}$, so $\overline{\overline{y}} = \dfrac{\sqrt{2}}{3}$ and $\overline{\overline{x}}^2 = \dfrac{I_y}{m} = \dfrac{\pi^2 - 4}{2}$, so $\overline{\overline{x}} = \sqrt{\dfrac{\pi^2 - 4}{2}}$.

23. (a) $f(x, y)$ is a joint density function, so we know $\iint_{\mathbb{R}^2} f(x, y) \, dA = 1$. Since $f(x, y) = 0$ outside the rectangle $[0, 1] \times [0, 2]$, we can say

$$\iint_{\mathbb{R}^2} f(x, y) \, dA = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) \, dy \, dx = \int_0^1 \int_0^2 Cx(1 + y) \, dy \, dx$$

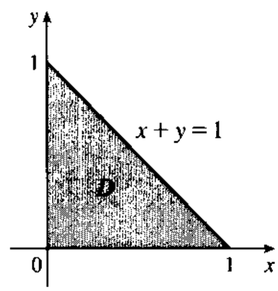
$$= C \int_0^1 x\big[y + \tfrac{1}{2}y^2\big]_{y=0}^{y=2} \, dx = C \int_0^1 4x \, dx = C\Big[2x^2\Big]_0^1 = 2C$$

Then $2C = 1 \;\Rightarrow\; C = \frac{1}{2}$.

(b) $P(X \le 1, Y \le 1) = \int_{-\infty}^1 \int_{-\infty}^1 f(x, y) \, dy \, dx = \int_0^1 \int_0^1 \frac{1}{2}x(1 + y) \, dy \, dx$

$$= \int_0^1 \tfrac{1}{2}x\big[y + \tfrac{1}{2}y^2\big]_{y=0}^{y=1} \, dx = \int_0^1 \tfrac{1}{2}x\big(\tfrac{3}{2}\big) \, dx = \tfrac{3}{4}\big[\tfrac{1}{2}x^2\big]_0^1 = \tfrac{3}{8} \text{ or } 0.375$$

(c) $P(X + Y \le 1) = P((X, Y) \in D)$ where D is the triangular region shown in the figure. Thus



$P(X + Y \le 1) = \iint_D f(x, y) \, dA = \int_0^1 \int_0^{1-x} \frac{1}{2}x(1 + y) \, dy \, dx$

$= \int_0^1 \frac{1}{2}x\big[y + \frac{1}{2}y^2\big]_{y=0}^{y=1-x} \, dx = \int_0^1 \frac{1}{2}x\big(\frac{1}{2}x^2 - 2x + \frac{3}{2}\big) \, dx$

$= \frac{1}{4} \int_0^1 (x^3 - 4x^2 + 3x) \, dx = \frac{1}{4}\Big[\frac{x^4}{4} - 4\frac{x^3}{3} + 3\frac{x^2}{2}\Big]_0^1$

$= \frac{5}{48} \approx 0.1042$

24. (a) $f(x, y) \ge 0$, so f is a joint density function if $\iint_{\mathbb{R}^2} f(x, y) \, dA = 1$. Here, $f(x, y) = 0$ outside the square $[0, 1] \times [0, 1]$, so $\iint_{\mathbb{R}^2} f(x, y) \, dA = \int_0^1 \int_0^1 4xy \, dy \, dx = \int_0^1 \big[2xy^2\big]_{y=0}^{y=1} \, dx = \int_0^1 2x \, dx = x^2\big]_0^1 = 1$. Thus, $f(x, y)$ is a joint density function.

(b) (i) No restriction is placed on Y, so

$$P\big(X \ge \tfrac{1}{2}\big) = \int_{1/2}^{\infty} \int_{-\infty}^{\infty} f(x, y) \, dy \, dx = \int_{1/2}^1 \int_0^1 4xy \, dy \, dx$$

$$= \int_{1/2}^1 \big[2xy^2\big]_{y=0}^{y=1} \, dx = \int_{1/2}^1 2x \, dx = x^2\big]_{1/2}^1 = \tfrac{3}{4}$$

(ii) $P\big(X \ge \frac{1}{2}, Y \le \frac{1}{2}\big) = \int_{1/2}^{\infty} \int_{-\infty}^{1/2} f(x, y) \, dy \, dx = \int_{1/2}^1 \int_0^{1/2} 4xy \, dy \, dx$

$$= \int_{1/2}^1 \big[2xy^2\big]_{y=0}^{y=1/2} \, dx = \int_{1/2}^1 \tfrac{1}{2}x \, dx = \tfrac{1}{2} \cdot \tfrac{1}{2}x^2\big]_{1/2}^1 = \tfrac{3}{16}$$

(c) The expected value of X is given by

$$\mu_1 = \iint_{\mathbb{R}^2} x\,f(x,y)\,dA = \int_0^1\int_0^1 x(4xy)\,dy\,dx = \int_0^1 2x^2\left[y^2\right]_{y=0}^{y=1}dx$$
$$= 2\int_0^1 x^2\,dx = 2\left[\tfrac{1}{3}x^3\right]_0^1 = \tfrac{2}{3}$$

The expected value of Y is

$$\mu_2 = \iint_{\mathbb{R}^2} y\,f(x,y)\,dA = \int_0^1\int_0^1 y(4xy)\,dy\,dx = \int_0^1 4x\left[\tfrac{1}{3}y^3\right]_{y=0}^{y=1}dx$$
$$= \tfrac{4}{3}\int_0^1 x\,dx = \tfrac{4}{3}\left[\tfrac{1}{2}x^2\right]_0^1 = \tfrac{2}{3}$$

25. (a) $f(x,y) \ge 0$, so f is a joint density function if $\iint_{\mathbb{R}^2} f(x,y)\,dA = 1$. Here, $f(x,y) = 0$ outside the first quadrant, so

$$\iint_{\mathbb{R}^2} f(x,y)\,dA = \int_0^\infty\int_0^\infty 0.1e^{-(0.5x+0.2y)}\,dy\,dx = 0.1\int_0^\infty\int_0^\infty e^{-0.5x}e^{-0.2y}\,dy\,dx$$
$$= 0.1\int_0^\infty e^{-0.5x}\,dx\int_0^\infty e^{-0.2y}\,dy = 0.1\lim_{t\to\infty}\int_0^t e^{-0.5x}\,dx\lim_{t\to\infty}\int_0^t e^{-0.2y}\,dy$$
$$= 0.1\lim_{t\to\infty}\left[-2e^{-0.5x}\right]_0^t\lim_{t\to\infty}\left[-5e^{-0.2y}\right]_0^t$$
$$= 0.1\lim_{t\to\infty}\left[-2(e^{-0.5t}-1)\right]\lim_{t\to\infty}\left[-5(e^{-0.2t}-1)\right]$$
$$= (0.1)\cdot(-2)(0-1)\cdot(-5)(0-1) = 1$$

Thus $f(x,y)$ is a joint density function.

(b) (i) No restriction is placed on X, so

$$P(Y \ge 1) = \int_{-\infty}^\infty\int_1^\infty f(x,y)\,dy\,dx = \int_0^\infty\int_1^\infty 0.1e^{-(0.5x+0.2y)}\,dy\,dx$$
$$= 0.1\int_0^\infty e^{-0.5x}\,dx\int_1^\infty e^{-0.2y}\,dy = 0.1\lim_{t\to\infty}\int_0^t e^{-0.5x}\,dx\lim_{t\to\infty}\int_1^t e^{-0.2y}\,dy$$
$$= 0.1\lim_{t\to\infty}\left[-2e^{-0.5x}\right]_0^t\lim_{t\to\infty}\left[-5e^{-0.2y}\right]_1^t$$
$$= 0.1\lim_{t\to\infty}\left[-2(e^{-0.5t}-1)\right]\lim_{t\to\infty}\left[-5(e^{-0.2t}-e^{-0.2})\right]$$
$$= (0.1)\cdot(-2)(0-1)\cdot(-5)(0-e^{-0.2}) = e^{-0.2} \approx 0.8187$$

(ii) $P(X \le 2, Y \le 4) = \int_{-\infty}^2\int_{-\infty}^4 f(x,y)\,dy\,dx = \int_0^2\int_0^4 0.1e^{-(0.5x+0.2y)}\,dy\,dx$

$$= 0.1\int_0^2 e^{-0.5x}\,dx\int_0^4 e^{-0.2y}\,dy = 0.1\left[-2e^{-0.5x}\right]_0^2\left[-5e^{-0.2y}\right]_0^4$$
$$= (0.1)\cdot(-2)(e^{-1}-1)\cdot(-5)(e^{-0.8}-1)$$
$$= (e^{-1}-1)(e^{-0.8}-1) = 1 + e^{-1.8} - e^{-0.8} - e^{-1} \approx 0.3481$$

(c) The expected value of X is given by

$$\mu_1 = \iint_{\mathbb{R}^2} x\,f(x,y)\,dA = \int_0^\infty\int_0^\infty x\left[0.1e^{-(0.5x+0.2y)}\right]dy\,dx$$
$$= 0.1\int_0^\infty xe^{-0.5x}\,dx\int_0^\infty e^{-0.2y}\,dy = 0.1\lim_{t\to\infty}\int_0^t xe^{-0.5x}\,dx\lim_{t\to\infty}\int_0^t e^{-0.2y}\,dy$$

To evaluate the first integral, we integrate by parts with $u = x$ and $dv = e^{-0.5x}\,dx$ (or we can use Formula 96 in the Table of Integrals):
$\int xe^{-0.5x}\,dx = -2xe^{-0.5x} - \int -2e^{-0.5x}\,dx = -2xe^{-0.5x} - 4e^{-0.5x} = -2(x+2)e^{-0.5x}$. Thus

$$\begin{aligned}
\mu_1 &= 0.1 \lim_{t\to\infty} \left[-2(x+2)e^{-0.5x}\right]_0^t \lim_{t\to\infty} \left[-5e^{-0.2y}\right]_0^t \\
&= 0.1 \lim_{t\to\infty} (-2)\left[(t+2)e^{-0.5t} - 2\right] \lim_{t\to\infty} (-5)\left[e^{-0.2t} - 1\right] \\
&= 0.1(-2)\left(\lim_{t\to\infty} \frac{t+2}{e^{0.5t}} - 2\right)(-5)(-1) = 2 \qquad \text{[by l'Hospital's Rule]}
\end{aligned}$$

The expected value of Y is given by

$$\begin{aligned}
\mu_2 &= \iint_{\mathbb{R}^2} y\,f(x,y)\,dA = \int_0^\infty \int_0^\infty y\left[0.1e^{-(0.5x+0.2y)}\right] dy\,dx \\
&= 0.1 \int_0^\infty e^{-0.5x}\,dx \int_0^\infty ye^{-0.2y}\,dy = 0.1 \lim_{t\to\infty} \int_0^t e^{-0.5x}\,dx \lim_{t\to\infty} \int_0^t ye^{-0.2y}\,dy
\end{aligned}$$

To evaluate the second integral, we integrate by parts with $u = y$ and $dv = e^{-0.2y}\,dy$ (or again we can use Formula 96 in the Table of Integrals) which gives
$\int ye^{-0.2y}\,dy = -5ye^{-0.2y} + \int 5e^{-0.2y}\,dy = -5(y+5)e^{-0.2y}$. Then

$$\begin{aligned}
\mu_2 &= 0.1 \lim_{t\to\infty} \left[-2e^{-0.5x}\right]_0^t \lim_{t\to\infty} \left[-5(y+5)e^{-0.2y}\right]_0^t \\
&= 0.1 \lim_{t\to\infty} \left[-2(e^{-0.5t} - 1)\right] \lim_{t\to\infty} \left(-5\left[(t+5)e^{-0.2t} - 5\right]\right) \\
&= 0.1(-2)(-1)\cdot(-5)\left(\lim_{t\to\infty} \frac{t+5}{e^{0.2t}} - 5\right) = 5 \qquad \text{[by l'Hospital's Rule]}
\end{aligned}$$

26. (a) Each lamp has exponential density function

$$f(t) = \begin{cases} 0 & \text{if } t < 0 \\ \frac{1}{1000}e^{-t/1000} & \text{if } t \geq 0 \end{cases}$$

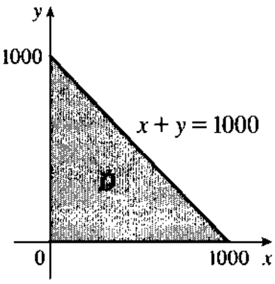
If X and Y are the lifetimes of the individual bulbs, then X and Y are independent, so the joint density function is the product of the individual density functions:

$$f(x,y) = \begin{cases} 10^{-6}e^{-(x+y)/1000} & \text{if } x \geq 0,\, y \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

The probability that both of the bulbs fail within 1000 hours is

$$\begin{aligned}
P(X \leq 1000, Y \leq 1000) &= \int_{-\infty}^{1000} \int_{-\infty}^{1000} f(x,y)\,dy\,dx \\
&= \int_0^{1000} \int_0^{1000} 10^{-6}e^{-(x+y)/1000}\,dy\,dx \\
&= 10^{-6} \int_0^{1000} e^{-x/1000}\,dx \int_0^{1000} e^{-y/1000}\,dy \\
&= 10^{-6}\left[-1000e^{-x/1000}\right]_0^{1000} \left[-1000e^{-y/1000}\right]_0^{1000} \\
&= \left(e^{-1} - 1\right)^2 \approx 0.3996
\end{aligned}$$

(b) Now we are asked for the probability that the combined lifetimes of both bulbs is 1000 hours or less. Thus we want to find $P(X + Y \leq 1000)$, or equivalently $P((X,Y) \in D)$ where D is the triangular region shown in the figure.



Then

$$\begin{aligned} P(X+Y \leq 1000) &= \iint_D f(x,y)\, dA = \int_0^{1000} \int_0^{1000-x} 10^{-6} e^{-(x+y)/1000}\, dy\, dx \\ &= 10^{-6} \int_0^{1000} \left[-1000 e^{-(x+y)/1000}\right]_{y=0}^{y=1000-x} dx \\ &= -10^{-3} \int_0^{1000} \left(e^{-1} - e^{-x/1000}\right) dx \\ &= -10^{-3} \left[e^{-1}x + 1000 e^{-x/1000}\right]_0^{1000} = 1 - 2e^{-1} \approx 0.2642 \end{aligned}$$

27. (a) The random variables X and Y are normally distributed with $\mu_1 = 45$, $\mu_2 = 20$, $\sigma_1 = 0.5$, and $\sigma_2 = 0.1$. The individual density functions for X and Y, then, are $f_1(x) = \dfrac{1}{0.5\sqrt{2\pi}} e^{-(x-45)^2/0.5}$ and $f_2(y) = \dfrac{1}{0.1\sqrt{2\pi}} e^{-(y-20)^2/0.02}$. Since X and Y are independent, the joint density function is the product

$$\begin{aligned} f(x,y) = f_1(x) f_2(y) &= \frac{1}{0.5\sqrt{2\pi}} e^{-(x-45)^2/0.5} \frac{1}{0.1\sqrt{2\pi}} e^{-(y-20)^2/0.02} \\ &= \tfrac{10}{\pi} e^{-2(x-45)^2 - 50(y-20)^2} \end{aligned}$$

Then

$$\begin{aligned} P(40 \leq X \leq 50, 20 \leq Y \leq 25) &= \int_{40}^{50} \int_{20}^{25} f(x,y)\, dy\, dx \\ &= \tfrac{10}{\pi} \int_{40}^{50} \int_{20}^{25} e^{-2(x-45)^2 - 50(y-20)^2}\, dy\, dx \end{aligned}$$

Using a CAS or calculator to evaluate the integral, we get $P(40 \leq X \leq 50, 20 \leq Y \leq 25) \approx 0.500$.

(b) $P(4(X-45)^2 + 100(Y-20)^2 \leq 2) = \iint_D \frac{10}{\pi} e^{-2(x-45)^2 - 50(y-20)^2}\, dA$, where D is the region enclosed by the ellipse $4(x-45)^2 + 100(y-20)^2 = 2$. Solving for y gives $y = 20 \pm \frac{1}{10}\sqrt{2 - 4(x-45)^2}$, the upper and lower halves of the ellipse, and these two halves meet where $y = 20$ [since the ellipse is centered at $(45, 20)$] $\Rightarrow$ $4(x-45)^2 = 2$ $\Rightarrow$ $x = 45 \pm \frac{1}{\sqrt{2}}$. Thus

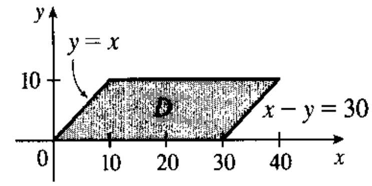
$$\iint_D \tfrac{10}{\pi} e^{-2(x-45)^2 - 50(y-20)^2}\, dA = \tfrac{10}{\pi} \int_{45-1/\sqrt{2}}^{45+1/\sqrt{2}} \int_{20 - \frac{1}{10}\sqrt{2-4(x-45)^2}}^{20 + \frac{1}{10}\sqrt{2-4(x-45)^2}} e^{-2(x-45)^2 - 50(y-20)^2}\, dy\, dx.$$

Using a CAS or calculator to evaluate the integral, we get $P(4(X-45)^2 + 100(Y-20)^2 \leq 2) \approx 0.632$.

28. Because X and Y are independent, the joint density function for Xavier's and Yolanda's arrival times is the product of the individual density functions:

$$f(x,y) = f_1(x) f_2(y) = \begin{cases} \frac{1}{50} e^{-x} y & \text{if } x \geq 0, 0 \leq y \leq 10 \\ 0 & \text{otherwise} \end{cases}$$

Since Xavier won't wait for Yolanda, they won't meet unless $X \geq Y$. Additionally, Yolanda will wait up to half an hour but no longer, so they won't meet unless $X - Y \le 30$. Thus the probability that they meet is $P((X, Y) \in D)$ where D is the parallelogram shown in the figure. The integral is simpler to evaluate if we consider D as a type II region, so



$$P((X,Y)\in D) = \iint_D f(x,y)\,dx\,dy = \int_0^{10}\int_y^{y+30} \tfrac{1}{50}e^{-x}y\,dx\,dy = \tfrac{1}{50}\int_0^{10} y\left[-e^{-x}\right]_{x=y}^{x=y+30} dy$$

$$= \tfrac{1}{50}\int_0^{10} y\left(-e^{-(y+30)} + e^{-y}\right) dy = \tfrac{1}{50}\left(1-e^{-30}\right)\int_0^{10} ye^{-y}\,dy$$

By integration by parts (or Formula 96 in the Table of Integrals), this is $\frac{1}{50}(1-e^{-30})\left[-(y+1)e^{-y}\right]_0^{10} = \frac{1}{50}(1-e^{-30})(1-11e^{-10}) \approx 0.020$. Thus there is only about a 2% chance they will meet. Such is student life!

29. (a) If $f(P, A)$ is the probability that an individual at A will be infected by an individual at P, and $k\,dA$ is the number of infected individuals in an element of area dA, then $f(P, A)k\,dA$ is the number of infections that should result from exposure of the individual at A to infected people in the element of area dA. Integration over D gives the number of infections of the person at A due to all the infected people in D. In rectangular coordinates (with the origin at the city's center), the exposure of a person at A is

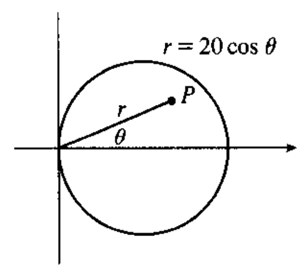
$$E = \iint_D kf(P,A)\,dA = k\iint_D \frac{20 - d(P,A)}{20}\,dA = k\iint_D \left[1 - \frac{\sqrt{(x-x_0)^2 + (y-y_0)^2}}{20}\right] dx\,dy.$$

(b) If $A = (0, 0)$, then

$$E = k\iint_D \left[1 - \frac{1}{20}\sqrt{x^2+y^2}\right] dx\,dy$$

$$= k\int_0^{2\pi}\int_0^{10}\left(1 - \frac{r}{20}\right) r\,dr\,d\theta = 2\pi k\left[\frac{r^2}{2} - \frac{r^3}{60}\right]_0^{10}$$

$$= 2\pi k\left(50 - \tfrac{50}{3}\right) = \tfrac{200}{3}\pi k \approx 209k$$



For A at the edge of the city, it is convenient to use a polar coordinate system centered at A. Then the polar equation for the circular boundary of the city becomes $r = 20\cos\theta$ instead of $r = 10$, and the distance from A to a point P in the city is again r (see the figure). So

$$E = k\int_{-\pi/2}^{\pi/2}\int_0^{20\cos\theta}\left(1 - \frac{r}{20}\right) r\,dr\,d\theta = k\int_{-\pi/2}^{\pi/2}\left[\frac{r^2}{2} - \frac{r^3}{60}\right]_{r=0}^{r=20\cos\theta} d\theta$$

$$= k\int_{-\pi/2}^{\pi/2}\left(200\cos^2\theta - \tfrac{400}{3}\cos^3\theta\right) d\theta = 200k\int_{-\pi/2}^{\pi/2}\left[\tfrac{1}{2} + \tfrac{1}{2}\cos 2\theta - \tfrac{2}{3}\left(1-\sin^2\theta\right)\cos\theta\right] d\theta$$

$$= 200k\left[\tfrac{1}{2}\theta + \tfrac{1}{4}\sin 2\theta - \tfrac{2}{3}\sin\theta + \tfrac{2}{3}\cdot\tfrac{1}{3}\sin^3\theta\right]_{-\pi/2}^{\pi/2} = 200k\left[\tfrac{\pi}{4} + 0 - \tfrac{2}{3} + \tfrac{2}{9} + \tfrac{\pi}{4} + 0 - \tfrac{2}{3} + \tfrac{2}{9}\right]$$

$$= 200k\left(\tfrac{\pi}{2} - \tfrac{8}{9}\right) \approx 136k$$

Therefore the risk of infection is much lower at the edge of the city than in the middle, so it is better to live at the edge.

16.6 Surface Area

ET 15.6

1. Here $z = f(x,y) = 2 + 3x + 4y$ and D is the rectangle $[0,5] \times [1,4]$, so by Formula 2 the area of the surface is

$$A(S) = \iint_D \sqrt{[f_x(x,y)]^2 + [f_y(x,y)]^2 + 1}\, dA = \iint_D \sqrt{3^2 + 4^2 + 1}\, dA = \sqrt{26} \iint_D dA$$
$$= \sqrt{26}\, A(D) = \sqrt{26}\,(5)(3) = 15\sqrt{26}$$

2. $z = f(x,y) = 10 - 2x - 5y$ and D is the disk $x^2 + y^2 \le 9$, so by Formula 2

$$A(S) = \iint_D \sqrt{(-2)^2 + (-5)^2 + 1}\, dA = \sqrt{30} \iint_D dA = \sqrt{30}\, A(D)$$
$$= \sqrt{30}\,(\pi \cdot 3^2) = 9\sqrt{30}\,\pi$$

3. $z = f(x,y) = 6 - 3x - 2y$ which intersects the xy-plane in the line $3x + 2y = 6$, so D is the triangular region given by $\left\{(x,y) \mid 0 \le x \le 2, 0 \le y \le 3 - \frac{3}{2}x\right\}$. Thus

$$A(S) = \iint_D \sqrt{(-3)^2 + (-2)^2 + 1}\, dA = \sqrt{14} \iint_D dA = \sqrt{14}\, A(D)$$
$$= \sqrt{14}\left(\tfrac{1}{2} \cdot 2 \cdot 3\right) = 3\sqrt{14}$$

4. $z = f(x,y) = 1 + 3x + 2y^2$ with $0 \le x \le 2y$, $0 \le y \le 1$. Thus by Formula 2,

$$A(S) = \iint_D \sqrt{(3)^2 + (4y)^2 + 1}\, dA = \int_0^1 \int_0^{2y} \sqrt{10 + 16y^2}\, dx\, dy = \int_0^1 \sqrt{10 + 16y^2}\, \Big[\, x \,\Big]_{x=0}^{x=2y}\, dy$$
$$= \int_0^1 2y\sqrt{10 + 16y^2}\, dy = 2 \cdot \tfrac{1}{32} \cdot \tfrac{2}{3}(10 + 16y^2)^{3/2}\Big]_0^1 = \tfrac{1}{24}(26^{3/2} - 10^{3/2})$$

5. $y^2 + z^2 = 9 \;\Rightarrow\; z = \sqrt{9 - y^2}$. $f_x = 0$, $f_y = -y(9 - y^2)^{-1/2} \;\Rightarrow$

$$A(S) = \int_0^4 \int_0^2 \sqrt{0^2 + [-y(9 - y^2)^{-1/2}]^2 + 1}\, dy\, dx = \int_0^4 \int_0^2 \sqrt{\frac{y^2}{9 - y^2} + 1}\, dy\, dx$$
$$= \int_0^4 \int_0^2 \frac{3}{\sqrt{9 - y^2}}\, dy\, dx = 3 \int_0^4 \left[\sin^{-1} \frac{y}{3}\right]_{y=0}^{y=2} dx = 3\left[\left(\sin^{-1}\left(\tfrac{2}{3}\right)\right)x\right]_0^4 = 12 \sin^{-1}\left(\tfrac{2}{3}\right)$$

6. $z = f(x,y) = 4 - x^2 - y^2$ and D is the projection of the paraboloid $z = 4 - x^2 - y^2$ onto the xy-plane, that is, $D = \left\{(x,y) \mid x^2 + y^2 \le 4\right\}$. So $f_x = -2x$, $f_y = -2y \;\Rightarrow$

$$A(S) = \iint_D \sqrt{(-2x)^2 + (-2y)^2 + 1}\, dA = \iint_D \sqrt{4(x^2 + y^2) + 1}\, dA = \int_0^{2\pi} \int_0^2 \sqrt{4r^2 + 1}\, r\, dr\, d\theta$$
$$= \int_0^{2\pi} \left[\tfrac{1}{12}(4r^2 + 1)^{3/2}\right]_{r=0}^{r=2} d\theta = \int_0^{2\pi} \tfrac{1}{12}\left(17\sqrt{17} - 1\right) d\theta = \tfrac{\pi}{6}\left(17\sqrt{17} - 1\right)$$

7. $z = f(x,y) = y^2 - x^2$ with $1 \le x^2 + y^2 \le 4$. Then

$$A(S) = \iint_D \sqrt{1 + 4x^2 + 4y^2}\, dA = \int_0^{2\pi} \int_1^2 \sqrt{1 + 4r^2}\, r\, dr\, d\theta = \int_0^{2\pi} d\theta \int_1^2 r\sqrt{1 + 4r^2}\, dr$$
$$= \Big[\,\theta\,\Big]_0^{2\pi} \left[\tfrac{1}{12}(1 + 4r^2)^{3/2}\right]_1^2 = \tfrac{\pi}{6}\left(17\sqrt{17} - 5\sqrt{5}\right)$$

8. $z = f(x,y) = \frac{2}{3}(x^{3/2} + y^{3/2})$ and $D = \{(x,y) \mid 0 \le x \le 1, 0 \le y \le 1\}$. Then $f_x = x^{1/2}$, $f_y = y^{1/2}$ and

$$A(S) = \iint_D \sqrt{\left(\sqrt{x}\right)^2 + \left(\sqrt{y}\right)^2 + 1}\, dA = \int_0^1 \int_0^1 \sqrt{x+y+1}\, dy\, dx$$

$$= \int_0^1 \left[\tfrac{2}{3}(x+y+1)^{3/2}\right]_{y=0}^{y=1} dx = \tfrac{2}{3}\int_0^1 \left[(x+2)^{3/2} - (x+1)^{3/2}\right] dx$$

$$= \tfrac{2}{3}\left[\tfrac{2}{5}(x+2)^{5/2} - \tfrac{2}{5}(x+1)^{5/2}\right]_0^1 = \tfrac{4}{15}(3^{5/2} - 2^{5/2} - 2^{5/2} + 1)$$

$$= \tfrac{4}{15}(3^{5/2} - 2^{7/2} + 1)$$

9. $z = f(x,y) = xy$ with $0 \le x^2 + y^2 \le 1$, so $f_x = y$, $f_y = x \Rightarrow$

$$A(S) = \iint_D \sqrt{y^2 + x^2 + 1}\, dA = \int_0^{2\pi}\int_0^1 \sqrt{r^2+1}\, r\, dr\, d\theta = \int_0^{2\pi} \left[\tfrac{1}{3}(r^2+1)^{3/2}\right]_{r=0}^{r=1} d\theta$$
$$= \int_0^{2\pi} \tfrac{1}{3}(2\sqrt{2} - 1)\, d\theta = \tfrac{2\pi}{3}(2\sqrt{2} - 1)$$

10. Given the sphere $x^2 + y^2 + z^2 = 4$, when $z = 1$, we get $x^2 + y^2 = 3$ so $D = \{(x,y) \mid x^2 + y^2 \le 3\}$ and $z = f(x,y) = \sqrt{4 - x^2 - y^2}$. Thus

$$A(S) = \iint_D \sqrt{[(-x)(4 - x^2 - y^2)^{-1/2}]^2 + [(-y)(4 - x^2 - y^2)^{-1/2}]^2 + 1}\, dA$$

$$= \int_0^{2\pi}\int_0^{\sqrt{3}} \sqrt{\frac{r^2}{4 - r^2} + 1}\, r\, dr\, d\theta = \int_0^{2\pi}\int_0^{\sqrt{3}} \frac{2r}{\sqrt{4 - r^2}}\, dr\, d\theta$$

$$= \int_0^{2\pi} \left[-2(4 - r^2)^{1/2}\right]_{r=0}^{r=\sqrt{3}} d\theta = \int_0^{2\pi}(-2+4)\, d\theta = 2\theta\Big]_0^{2\pi} = 4\pi$$

11. $z = \sqrt{a^2 - x^2 - y^2}$, $z_x = -x(a^2 - x^2 - y^2)^{-1/2}$, $z_y = -y(a^2 - x^2 - y^2)^{-1/2}$,

$$A(S) = \iint_D \sqrt{\frac{x^2 + y^2}{a^2 - x^2 - y^2} + 1}\, dA$$

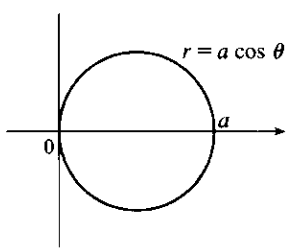
$$= \int_{-\pi/2}^{\pi/2}\int_0^{a\cos\theta} \sqrt{\frac{r^2}{a^2 - r^2} + 1}\, r\, dr\, d\theta$$

$$= \int_{-\pi/2}^{\pi/2}\int_0^{a\cos\theta} \frac{ar}{\sqrt{a^2 - r^2}}\, dr\, d\theta$$

$$= \int_{-\pi/2}^{\pi/2} \left[-a\sqrt{a^2 - r^2}\right]_{r=0}^{r=a\cos\theta} d\theta$$

$$= \int_{-\pi/2}^{\pi/2} -a\left(\sqrt{a^2 - a^2\cos^2\theta} - a\right) d\theta = 2a^2 \int_0^{\pi/2} \left(1 - \sqrt{1 - \cos^2\theta}\right) d\theta$$

$$= 2a^2 \int_0^{\pi/2} d\theta - 2a^2 \int_0^{\pi/2} \sqrt{\sin^2\theta}\, d\theta = a^2\pi - 2a^2 \int_0^{\pi/2} \sin\theta\, d\theta = a^2(\pi - 2)$$



12. To find the region D: $z = x^2 + y^2$ implies $z + z^2 = 4z$ or $z^2 - 3z = 0$. Thus $z = 0$ or $z = 3$ are the planes where the surfaces intersect. But $x^2 + y^2 + z^2 = 4z$ implies $x^2 + y^2 + (z-2)^2 = 4$, so $z = 3$ intersects the upper hemisphere. Thus $(z-2)^2 = 4 - x^2 - y^2$ or $z = 2 + \sqrt{4 - x^2 - y^2}$. Therefore D is the region inside the circle $x^2 + y^2 + (3-2)^2 = 4$, that is, $D = \{(x, y) \mid x^2 + y^2 \le 3\}$.

$$A(S) = \iint_D \sqrt{1 + [(-x)(4 - x^2 - y^2)^{-1/2}]^2 + [(-y)(4 - x^2 - y^2)^{-1/2}]^2}\, dA$$

$$= \int_0^{2\pi} \int_0^{\sqrt{3}} \sqrt{1 + \frac{r^2}{4 - r^2}}\, r\, dr\, d\theta = \int_0^{2\pi} \int_0^{\sqrt{3}} \frac{2r\, dr}{\sqrt{4 - r^2}}\, d\theta = \int_0^{2\pi} \left[-2(4 - r^2)^{1/2}\right]_{r=0}^{r=\sqrt{3}} d\theta$$

$$= \int_0^{2\pi} (-2 + 4)\, d\theta = 2\theta\Big]_0^{2\pi} = 4\pi$$

13. $z = f(x, y) = e^{-x^2 - y^2}$, $f_x = -2xe^{-x^2-y^2}$, $f_y = -2ye^{-x^2-y^2}$. Then

$$A(S) = \iint\limits_{x^2+y^2 \le 4} \sqrt{(-2xe^{-x^2-y^2})^2 + (-2ye^{-x^2-y^2})^2 + 1}\, dA = \iint\limits_{x^2+y^2 \le 4} \sqrt{4(x^2 + y^2)e^{-2(x^2+y^2)} + 1}\, dA.$$

Converting to polar coordinates we have

$$A(S) = \int_0^{2\pi} \int_0^2 \sqrt{4r^2 e^{-2r^2} + 1}\, r\, dr\, d\theta = \int_0^{2\pi} d\theta \int_0^2 r \sqrt{4r^2 e^{-2r^2} + 1}\, dr$$

$$= 2\pi \int_0^2 r \sqrt{4r^2 e^{-2r^2} + 1}\, dr \approx 13.9783 \text{ using a calculator.}$$

14. $z = f(x, y) = \cos(x^2 + y^2)$, $f_x = -2x \sin(x^2 + y^2)$, $f_y = -2y \sin(x^2 + y^2)$.

$$A(S) = \iint\limits_{x^2+y^2 \le 1} \sqrt{4x^2 \sin^2(x^2 + y^2) + 4y^2 \sin^2(x^2 + y^2) + 1}\, dA$$

$$= \iint\limits_{x^2+y^2 \le 1} \sqrt{4(x^2 + y^2) \sin^2(x^2 + y^2) + 1}\, dA$$

Converting to polar coordinates gives

$$A(S) = \int_0^{2\pi} \int_0^1 \sqrt{4r^2 \sin^2(r^2) + 1}\, r\, dr\, d\theta = \int_0^{2\pi} d\theta \int_0^1 r \sqrt{4r^2 \sin^2(r^2) + 1}\, dr$$

$$= 2\pi \int_0^1 r \sqrt{4r^2 \sin^2(r^2) + 1}\, dr \approx 4.1073 \quad \text{using a calculator.}$$

15. (a) The midpoints of the four squares are $\left(\frac{1}{4}, \frac{1}{4}\right)$, $\left(\frac{1}{4}, \frac{3}{4}\right)$, $\left(\frac{3}{4}, \frac{1}{4}\right)$, and $\left(\frac{3}{4}, \frac{3}{4}\right)$. Here $f(x, y) = x^2 + y^2$, so the Midpoint Rule gives

$$A(S) = \iint_D \sqrt{[f_x(x, y)]^2 + [f_y(x, y)]^2 + 1}\, dA = \iint_D \sqrt{(2x)^2 + (2y)^2 + 1}\, dA$$

$$\approx \tfrac{1}{4}\Bigg(\sqrt{\left[2\left(\tfrac{1}{4}\right)\right]^2 + \left[2\left(\tfrac{1}{4}\right)\right]^2 + 1} + \sqrt{\left[2\left(\tfrac{1}{4}\right)\right]^2 + \left[2\left(\tfrac{3}{4}\right)\right]^2 + 1}$$

$$+ \sqrt{\left[2\left(\tfrac{3}{4}\right)\right]^2 + \left[2\left(\tfrac{1}{4}\right)\right]^2 + 1} + \sqrt{\left[2\left(\tfrac{3}{4}\right)\right]^2 + \left[2\left(\tfrac{3}{4}\right)\right]^2 + 1}\Bigg)$$

$$= \tfrac{1}{4}\left(\sqrt{\tfrac{3}{2}} + 2\sqrt{\tfrac{7}{2}} + \sqrt{\tfrac{11}{2}}\right) \approx 1.8279$$

(b) A CAS estimates the integral to be
$A(S) = \iint_D \sqrt{1 + (2x)^2 + (2y)^2}\, dA = \int_0^1 \int_0^1 \sqrt{1 + 4x^2 + 4y^2}\, dy\, dx \approx 1.8616$. This agrees with the Midpoint estimate only in the first decimal place.

16.7 Triple Integrals ET 15.7

1. $\iiint_B xyz^2\,dV = \int_0^1\int_{-1}^2\int_0^3 xyz^2\,dz\,dx\,dy = \int_0^1\int_{-1}^2 xy\left[\frac{1}{3}z^3\right]_{z=0}^{z=3}dx\,dy = \int_0^1\int_{-1}^2 9xy\,dx\,dy$

$= \int_0^1\left[\frac{9}{2}x^2y\right]_{x=-1}^{x=2}dy = \int_0^1 \frac{27}{2}\,y\,dy = \frac{27}{4}y^2\Big]_0^1 = \frac{27}{4}$

2. There are six different possible orders of integration.

$$\iiint_E (xz-y^3)\,dV = \int_{-1}^1\int_0^2\int_0^1 (xz-y^3)\,dz\,dy\,dx = \int_{-1}^1\int_0^2\left[\tfrac{1}{2}xz^2-y^3z\right]_{z=0}^{z=1}dy\,dx$$
$$= \int_{-1}^1\int_0^2\left(\tfrac{1}{2}x-y^3\right)dy\,dx = \int_{-1}^1\left[\tfrac{1}{2}xy-\tfrac{1}{4}y^4\right]_{y=0}^{y=2}dx$$
$$= \int_{-1}^1(x-4)\,dx = \left[\tfrac{1}{2}x^2-4x\right]_{-1}^1 = -8$$

$$\iiint_E (xz-y^3)\,dV = \int_0^2\int_{-1}^1\int_0^1 (xz-y^3)\,dz\,dx\,dy = \int_0^2\int_{-1}^1\left[\tfrac{1}{2}xz^2-y^3z\right]_{z=0}^{z=1}dx\,dy$$
$$= \int_0^2\int_{-1}^1\left(\tfrac{1}{2}x-y^3\right)dx\,dy = \int_0^2\left[\tfrac{1}{4}x^2-xy^3\right]_{x=-1}^{x=1}dy$$
$$= \int_0^2 -2y^3\,dy = -\tfrac{1}{2}y^4\Big]_0^2 = -8$$

$$\iiint_E (xz-y^3)\,dV = \int_{-1}^1\int_0^1\int_0^2 (xz-y^3)\,dy\,dz\,dx = \int_{-1}^1\int_0^1\left[xyz-\tfrac{1}{4}y^4\right]_{y=0}^{y=2}dz\,dx$$
$$= \int_{-1}^1\int_0^1(2xz-4)\,dz\,dx = \int_{-1}^1\left[xz^2-4z\right]_{z=0}^{z=1}dx$$
$$= \int_{-1}^1(x-4)\,dx = \left[\tfrac{1}{2}x^2-4x\right]_{-1}^1 = -8$$

$$\iiint_E (xz-y^3)\,dV = \int_0^1\int_{-1}^1\int_0^2 (xz-y^3)\,dy\,dx\,dz = \int_0^1\int_{-1}^1\left[xyz-\tfrac{1}{4}y^4\right]_{y=0}^{y=2}dx\,dz$$
$$= \int_0^1\int_{-1}^1(2xz-4)\,dx\,dz = \int_0^1\left[x^2z-4x\right]_{x=-1}^{x=1}dz$$
$$= \int_0^1 -8\,dz = -8z\Big]_0^1 = -8$$

$$\iiint_E (xz-y^3)\,dV = \int_0^2\int_0^1\int_{-1}^1 (xz-y^3)\,dx\,dz\,dy = \int_0^2\int_0^1\left[\tfrac{1}{2}x^2z-xy^3\right]_{x=-1}^{x=1}dz\,dy$$
$$= \int_0^2\int_0^1 -2y^3\,dz\,dy = \int_0^2\left[-2y^3z\right]_{z=0}^{z=1}dy = \int_0^2 -2y^3\,dy = -\tfrac{1}{2}y^4\Big]_0^2 = -8$$

$$\iiint_E (xz-y^3)\,dV = \int_0^1\int_0^2\int_{-1}^1 (xz-y^3)\,dx\,dy\,dz = \int_0^1\int_0^2\left[\tfrac{1}{2}x^2z-xy^3\right]_{x=-1}^{x=1}dy\,dz$$
$$= \int_0^1\int_0^2 -2y^3\,dy\,dz = \int_0^1\left[-\tfrac{1}{2}y^4\right]_{y=0}^{y=2}dz = \int_0^1 -8\,dz = -8z\Big]_0^1 = -8$$

3. $\int_0^1\int_0^z\int_0^{x+z} 6xz\,dy\,dx\,dz = \int_0^1\int_0^z\left[6xyz\right]_{y=0}^{y=x+z}dx\,dz = \int_0^1\int_0^z 6xz(x+z)\,dx\,dz$

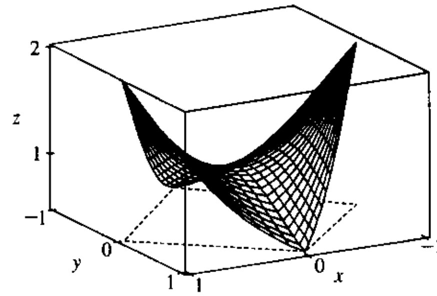
$= \int_0^1\left[2x^3z+3x^2z^2\right]_{x=0}^{x=z}dz = \int_0^1(2z^4+3z^4)\,dz = \int_0^1 5z^4\,dz = z^5\Big]_0^1 = 1$

4. $\int_0^1\int_x^{2x}\int_0^y 2xyz\,dz\,dy\,dx = \int_0^1\int_x^{2x}\left[xyz^2\right]_{z=0}^{z=y}dy\,dx = \int_0^1\int_x^{2x} xy^3\,dy\,dx$

$= \int_0^1\left[\frac{1}{4}xy^4\right]_{y=x}^{y=2x}dx = \int_0^1 \frac{15}{4}x^5\,dx = \frac{5}{8}x^6\Big]_0^1 = \frac{5}{8}$

20. Let $f(x,y) = \dfrac{1+x^2}{1+y^2}$. Then $f_x = \dfrac{2x}{1+y^2}$,

$f_y = (1+x^2)\left[-\dfrac{2y}{(1+y^2)^2}\right] = -\dfrac{2y(1+x^2)}{(1+y^2)^2}$. We use a CAS to estimate $\int_{-1}^{1}\int_{-(1-|x|)}^{1-|x|}\sqrt{f_x^2+f_y^2+1}\,dy\,dx \approx 2.6959$. In order to graph only the part of the surface above the square, we use $-(1-|x|) \le y \le 1-|x|$ as the y-range in our plot command.



21. Here $z = f(x,y) = ax + by + c$, $f_x(x,y) = a$, $f_y(x,y) = b$, so
$A(S) = \iint_D \sqrt{a^2+b^2+1}\,dA = \sqrt{a^2+b^2+1}\iint_D dA = \sqrt{a^2+b^2+1}\,A(D)$.

22. Let S be the upper hemisphere. Then $z = f(x,y) = \sqrt{a^2-x^2-y^2}$, so

$$A(S) = \iint_D \sqrt{[-x(a^2-x^2-y^2)^{-1/2}]^2 + [-y(a^2-x^2-y^2)^{-1/2}]^2 + 1}\,dA$$
$$= \iint_D \sqrt{\frac{x^2+y^2}{a^2-x^2-y^2}+1}\,dA = \lim_{t\to a^-}\int_0^{2\pi}\int_0^t \sqrt{\frac{r^2}{a^2-r^2}+1}\,r\,dr\,d\theta$$
$$= \lim_{t\to a^-}\int_0^{2\pi}\int_0^t \frac{ar}{\sqrt{a^2-r^2}}\,dr\,d\theta = 2\pi\lim_{t\to a^-}\left[-a\sqrt{a^2-r^2}\right]_0^t = 2\pi\lim_{t\to a^-} -a\left[\sqrt{a^2-t^2}-a\right]$$
$= 2\pi(-a)(-a) = 2\pi a^2$. Thus the surface area of the entire sphere is $4\pi a^2$.

23. If we project the surface onto the xz-plane, then the surface lies "above" the disk $x^2+z^2 \le 25$ in the xz-plane. We have $y = f(x,z) = x^2+z^2$ and, adapting Formula 2, the area of the surface is

$$A(S) = \iint_{x^2+z^2\le 25}\sqrt{[f_x(x,z)]^2+[f_z(x,z)]^2+1}\,dA = \iint_{x^2+z^2\le 25}\sqrt{4x^2+4z^2+1}\,dA$$

Converting to polar coordinates $x = r\cos\theta$, $z = r\sin\theta$ we have

$$A(S) = \int_0^{2\pi}\int_0^5 \sqrt{4r^2+1}\,r\,dr\,d\theta = \int_0^{2\pi}d\theta\int_0^5 r(4r^2+1)^{1/2}\,dr = \Big[\,\theta\,\Big]_0^{2\pi}\left[\tfrac{1}{12}(4r^2+1)^{3/2}\right]_0^5$$
$$= \tfrac{\pi}{6}\left(101\sqrt{101}-1\right)$$

24. First we find the area of the face of the surface that intersects the positive y-axis. As in Exercise 23, we can project the face onto the xz-plane, so the surface lies "above" the disk $x^2+z^2 \le 1$. Then $z = f(x,z) = \sqrt{1-z^2}$ and the area is

$$A(S) = \iint_{x^2+z^2\le 1}\sqrt{[f_x(x,z)]^2+[f_z(x,z)]^2+1}\,dA = \iint_{x^2+z^2\le 1}\sqrt{0+\left(\frac{-z}{\sqrt{1-z^2}}\right)^2+1}\,dA$$
$$= \iint_{x^2+z^2\le 1}\sqrt{\frac{z^2}{1-z^2}+1}\,dA = \int_{-1}^{1}\int_{-\sqrt{1-z^2}}^{\sqrt{1-z^2}}\frac{1}{\sqrt{1-z^2}}\,dx\,dz$$
$$= 4\int_0^1\int_0^{\sqrt{1-z^2}}\frac{1}{\sqrt{1-z^2}}\,dx\,dz \qquad \text{[by the symmetry of the surface]}$$

This integral is improper (when $z = 1$), so

$$A(S) = \lim_{t\to 1^-} 4\int_0^t\int_0^{\sqrt{1-z^2}}\frac{1}{\sqrt{1-z^2}}\,dx\,dz = \lim_{t\to 1^-} 4\int_0^t \frac{\sqrt{1-z^2}}{\sqrt{1-z^2}}\,dz = \lim_{t\to 1^-} 4\int_0^t dz = \lim_{t\to 1^-} 4t = 4.$$

Since the complete surface consists of four congruent faces, the total surface area is $4(4) = 16$.

16. (a) With $m = n = 2$ we have four squares with midpoints $\left(\frac{1}{2}, \frac{1}{2}\right)$, $\left(\frac{1}{2}, \frac{3}{2}\right)$, $\left(\frac{3}{2}, \frac{1}{2}\right)$, and $\left(\frac{3}{2}, \frac{3}{2}\right)$. Since $z = xy + x^2 + y^2$, the Midpoint Rule gives

$$A(S) = \iint_D \sqrt{[f_x(x,y)]^2 + [f_y(x,y)]^2 + 1}\, dA = \iint_D \sqrt{(y+2x)^2 + (x+2y)^2 + 1}\, dA$$

$$\approx 1\left(\sqrt{\left(\tfrac{3}{2}\right)^2 + \left(\tfrac{3}{2}\right)^2 + 1} + \sqrt{\left(\tfrac{5}{2}\right)^2 + \left(\tfrac{7}{2}\right)^2 + 1} + \sqrt{\left(\tfrac{7}{2}\right)^2 + \left(\tfrac{5}{2}\right)^2 + 1} + \sqrt{\left(\tfrac{9}{2}\right)^2 + \left(\tfrac{9}{2}\right)^2 + 1}\right)$$

$$= \frac{\sqrt{22}}{2} + \frac{\sqrt{78}}{2} + \frac{\sqrt{78}}{2} + \frac{\sqrt{166}}{2} \approx 17.619$$

(b) Using a CAS, we have

$$A(S) = \iint_D \sqrt{(y+2x)^2 + (x+2y)^2 + 1}\, dA = \int_0^2 \int_0^2 \sqrt{1 + (y+2x)^2 + (x+2y)^2}\, dy\, dx \approx 17.7165.$$

This is within about 0.1 of the Midpoint Rule estimate.

17. $z = 1 + 2x + 3y + 4y^2$, so

$$A(S) = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\, dA = \int_1^4 \int_0^1 \sqrt{1 + 4 + (3+8y)^2}\, dy\, dx$$

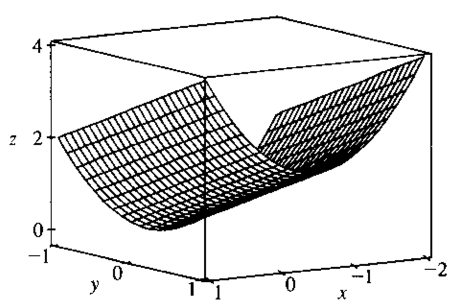
$$= \int_1^4 \int_0^1 \sqrt{14 + 48y + 64y^2}\, dy\, dx.$$

Using a CAS, we have

$$\int_1^4 \int_0^1 \sqrt{14 + 48y + 64y^2}\, dy\, dx = \tfrac{45}{8}\sqrt{14} + \tfrac{15}{16}\ln\left(11\sqrt{5} + 3\sqrt{14}\sqrt{5}\right) - \tfrac{15}{16}\ln\left(3\sqrt{5} + \sqrt{14}\sqrt{5}\right)$$

or $\frac{45}{8}\sqrt{14} + \frac{15}{16}\ln\dfrac{11\sqrt{5} + 3\sqrt{70}}{3\sqrt{5} + \sqrt{70}}$.

18. $f(x,y) = 1 + x + y + x^2 \quad \Rightarrow \quad f_x = 1 + 2x$, $f_y = 1$. We use a CAS to calculate the integral $A(S) = \int_{-2}^{1}\int_{-1}^{1} \sqrt{f_x^2 + f_y^2 + 1}\, dy\, dx = \int_{-2}^{1}\int_{-1}^{1} \sqrt{(1+2x)^2 + 2}\, dy\, dx = 2\int_{-2}^{1} \sqrt{4x^2 + 4x + 3}\, dx$ and find that $A(S) = 3\sqrt{11} + 2\sinh^{-1}\left(\frac{3\sqrt{2}}{2}\right)$ or $A(S) = 3\sqrt{11} + \ln\left(10 + 3\sqrt{11}\right)$.



19. $f(x,y) = 1 + x^2y^2 \quad \Rightarrow \quad f_x = 2xy^2$, $f_y = 2x^2y$. We use a CAS (with precision reduced to five significant digits, to speed up the calculation) to estimate the integral

$$A(S) = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sqrt{f_x^2 + f_y^2 + 1}\, dy\, dx = \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \sqrt{4x^2y^4 + 4x^4y^2 + 1}\, dy\, dx,$$ and find that

$A(S) \approx 3.3213$.

5. $\int_0^3 \int_0^1 \int_0^{\sqrt{1-z^2}} ze^y \, dx \, dz \, dy = \int_0^3 \int_0^1 \left[xze^y \right]_{x=0}^{x=\sqrt{1-z^2}} dz \, dy = \int_0^3 \int_0^1 ze^y \sqrt{1-z^2} \, dz \, dy$

$= \int_0^3 \left[-\frac{1}{3}(1-z^2)^{3/2} e^y \right]_{z=0}^{z=1} dy = \int_0^3 \frac{1}{3} e^y \, dy = \frac{1}{3} e^y \Big]_0^3 = \frac{1}{3}(e^3 - 1)$

6. $\int_0^1 \int_0^z \int_0^y ze^{-y^2} \, dx \, dy \, dz = \int_0^1 \int_0^z \left[xze^{-y^2} \right]_{x=0}^{x=y} dy \, dz = \int_0^1 \int_0^z yze^{-y^2} \, dy \, dz = \int_0^1 \left[-\frac{1}{2} ze^{-y^2} \right]_{y=0}^{y=z} dz$

$= \int_0^1 -\frac{1}{2} z \left(e^{-z^2} - 1 \right) dz = \frac{1}{2} \int_0^1 \left(z - ze^{-z^2} \right) dz$

$= \frac{1}{2} \left[\frac{1}{2} z^2 + \frac{1}{2} e^{-z^2} \right]_0^1 = \frac{1}{4}(1 + e^{-1} - 0 - 1) = \frac{1}{4e}$

7. $\iiint_E 2x \, dV = \int_0^2 \int_0^{\sqrt{4-y^2}} \int_0^y 2x \, dz \, dx \, dy = \int_0^2 \int_0^{\sqrt{4-y^2}} \left[2xz \right]_{z=0}^{z=y} dx \, dy = \int_0^2 \int_0^{\sqrt{4-y^2}} 2xy \, dx \, dy$

$= \int_0^2 \left[x^2 y \right]_{x=0}^{x=\sqrt{4-y^2}} dy = \int_0^2 (4-y^2) y \, dy = \left[2y^2 - \frac{1}{4} y^4 \right]_0^2 = 4$

8. $\iiint_E yz\cos(x^5) \, dV = \int_0^1 \int_0^x \int_x^{2x} yz\cos(x^5) \, dz \, dy \, dx = \int_0^1 \int_0^x \left[\frac{1}{2} yz^2 \cos(x^5) \right]_{z=x}^{z=2x} dy \, dx$

$= \frac{1}{2} \int_0^1 \int_0^x 3x^2 y \cos(x^5) \, dy \, dx = \frac{1}{2} \int_0^1 \left[\frac{3}{2} x^2 y^2 \cos(x^5) \right]_{y=0}^{y=x} dx$

$= \frac{3}{4} \int_0^1 x^4 \cos(x^5) \, dx = \frac{3}{4} \left[\frac{1}{5} \sin(x^5) \right]_0^1 = \frac{3}{20}(\sin 1 - \sin 0) = \frac{3}{20} \sin 1$

9. Here $E = \{(x,y,z) \mid 0 \le x \le 1, 0 \le y \le \sqrt{x}, 0 \le z \le 1 + x + y\}$, so

$\iiint_E 6xy \, dV = \int_0^1 \int_0^{\sqrt{x}} \int_0^{1+x+y} 6xy \, dz \, dy \, dx = \int_0^1 \int_0^{\sqrt{x}} \left[6xyz \right]_{z=0}^{z=1+x+y} dy \, dx$

$= \int_0^1 \int_0^{\sqrt{x}} 6xy(1+x+y) \, dy \, dx = \int_0^1 \left[3xy^2 + 3x^2y^2 + 2xy^3 \right]_{y=0}^{y=\sqrt{x}} dx$

$= \int_0^1 (3x^2 + 3x^3 + 2x^{5/2}) \, dx = \left[x^3 + \frac{3}{4} x^4 + \frac{4}{7} x^{7/2} \right]_0^1 = \frac{65}{28}$

10. Here E is the region in the first octant that lies below the plane $2x + 2y + z = 4$ (and above the region in the xy-plane bounded by the lines $x = 0$, $y = 0$, $x + y = 2$). So

$\iiint_E y \, dV = \int_0^2 \int_0^{2-x} \int_0^{4-2x-2y} y \, dz \, dy \, dx = \int_0^2 \int_0^{2-x} y(4 - 2x - 2y) \, dy \, dx$

$= \int_0^2 \int_0^{2-x} (4y - 2xy - 2y^2) \, dy \, dx = \int_0^2 \left[2y^2 - xy^2 - \frac{2}{3} y^3 \right]_{y=0}^{y=2-x} dx$

$= \int_0^2 \left[2(2-x)^2 - x(2-x)^2 - \frac{2}{3}(2-x)^3 \right] dx$

$= \int_0^2 \left[(2-x)(2-x)^2 - \frac{2}{3}(2-x)^3 \right] dx = \frac{1}{3} \int_0^2 (2-x)^3 \, dx$

$= \frac{1}{3} \left[-\frac{1}{4}(2-x)^4 \right]_0^2 = -\frac{1}{12}(0 - 16) = \frac{4}{3}$

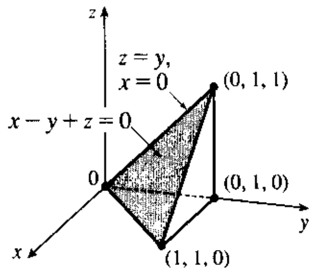
11. Here E is the region that lies below the plane with x-, y-, and z-intercepts 1, 2, and 3 respectively, that is, below the plane $2z + 6x + 3y = 6$ and above the region in the xy-plane bounded by the lines $x = 0$, $y = 0$ and $6x + 3y = 6$. So

$\iiint_E xy \, dV = \int_0^1 \int_0^{2-2x} \int_0^{3-3x-3y/2} xy \, dz \, dy \, dx = \int_0^1 \int_0^{2-2x} \left(3xy - 3x^2 y - \frac{3}{2} xy^2 \right) dy \, dx$

$= \int_0^1 \left[\frac{3}{2} xy^2 - \frac{3}{2} x^2 y^2 - \frac{1}{2} xy^3 \right]_{y=0}^{y=2-2x} dx = \int_0^1 (2x - 6x^2 + 6x^3 - 2x^4) \, dx$

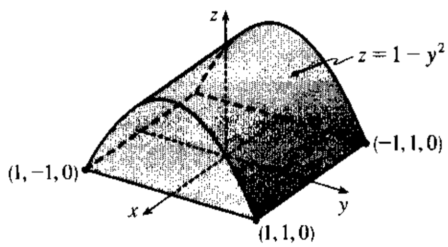
$= \left[x^2 - 2x^3 + \frac{3}{2} x^4 - \frac{2}{5} x^5 \right]_0^1 = \frac{1}{10}.$

12.



$$\int_0^1\int_0^y\int_0^{y-z} xz\,dx\,dz\,dy = \int_0^1\int_0^y \tfrac{1}{2}(y-z)^2 z\,dz\,dy$$
$$= \tfrac{1}{2}\int_0^1 \left[\tfrac{1}{2}y^2z^2 - \tfrac{2}{3}yz^3 + \tfrac{1}{4}z^4\right]_{z=0}^{z=y} dy$$
$$= \tfrac{1}{24}\int_0^1 y^4\,dy = \tfrac{1}{24}\left[\tfrac{1}{5}y^5\right]_0^1 = \tfrac{1}{120}$$

13.



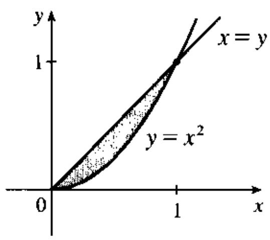
E is the region below the parabolic cylinder $z = 1 - y^2$ and above the square $[-1,1]\times[-1,1]$ in the xy-plane.

$$\iiint_E x^2e^y\,dV = \int_{-1}^1\int_{-1}^1\int_0^{1-y^2} x^2e^y\,dz\,dy\,dx$$
$$= \int_{-1}^1\int_{-1}^1 x^2e^y(1-y^2)\,dy\,dx$$
$$= \int_{-1}^1 x^2\,dx\,\int_{-1}^1(e^y - y^2e^y)\,dy$$
$$= \left[\tfrac{1}{3}x^3\right]_{-1}^1\left[e^y - (y^2-2y+2)e^y\right]_{-1}^1$$

[integrate by parts twice]

$$= \tfrac{1}{3}(2)[e - e - e^{-1} + 5e^{-1}] = \tfrac{8}{3e}$$

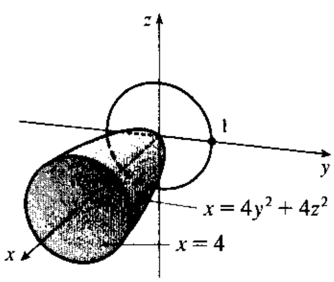
14.



E is the solid above the region shown in the xy-plane and below the plane $z = x$. Thus,

$$\iiint_E (x+2y)\,dV = \int_0^1\int_{x^2}^x\int_0^x (x+2y)\,dz\,dy\,dx$$
$$= \int_0^1\int_{x^2}^x (x^2+2yx)\,dy\,dx = \int_0^1 \left[x^2y + xy^2\right]_{y=x^2}^{y=x}\,dx$$
$$= \int_0^1 (2x^3 - x^4 - x^5)\,dx = \left[\tfrac{1}{2}x^4 - \tfrac{1}{5}x^5 - \tfrac{1}{6}x^6\right]_0^1 = \tfrac{2}{15}$$

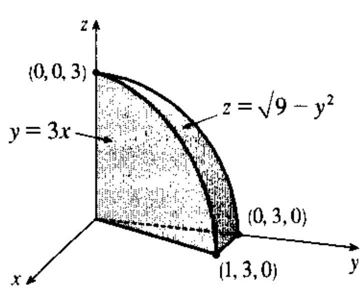
15.



The projection E on the yz-plane is the disk $y^2 + z^2 \leq 1$. Using polar coordinates $y = r\cos\theta$ and $z = r\sin\theta$, we get

$$\iiint_E x\,dV = \iint_D \left[\int_{4y^2+4z^2}^4 x\,dx\right] dA$$
$$= \tfrac{1}{2}\iint_D \left[4^2 - (4y^2+4z^2)^2\right] dA = 8\int_0^{2\pi}\int_0^1 (1-r^4)\,r\,dr\,d\theta$$
$$= 8\int_0^{2\pi} d\theta\int_0^1 (r - r^5)\,dr = 8(2\pi)\left[\tfrac{1}{2}r^2 - \tfrac{1}{6}r^6\right]_0^1 = \tfrac{16\pi}{3}$$

16.



$$\int_0^1\int_{3x}^3\int_0^{\sqrt{9-y^2}} z\,dz\,dy\,dx = \int_0^1\int_{3x}^3 \tfrac{1}{2}(9-y^2)\,dy\,dx$$
$$= \int_0^1 \left[\tfrac{9}{2}y - \tfrac{1}{6}y^3\right]_{y=3x}^{y=3} dx$$
$$= \int_0^1 \left[9 - \tfrac{27}{2}x + \tfrac{9}{2}x^3\right] dx$$
$$= \left[9x - \tfrac{27}{4}x^2 + \tfrac{9}{8}x^4\right]_0^1 = \tfrac{27}{8}$$

17. The plane $2x + y + z = 4$ intersects the xy-plane when

$2x + y + 0 = 4 \quad \Rightarrow \quad y = 4 - 2x$, so

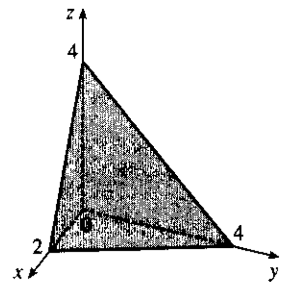
$E = \{(x, y, z) \mid 0 \le x \le 2, 0 \le y \le 4 - 2x, 0 \le z \le 4 - 2x - y\}$ and

$$V = \int_0^2 \int_0^{4-2x} \int_0^{4-2x-y} dz\, dy\, dx = \int_0^2 \int_0^{4-2x} (4 - 2x - y)\, dy\, dx$$

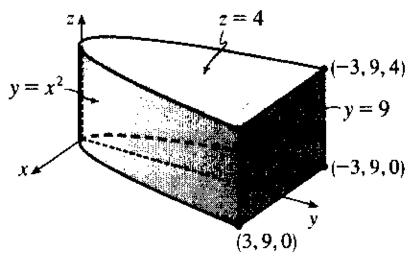
$$= \int_0^2 \left[4y - 2xy - \tfrac{1}{2}y^2\right]_{y=0}^{y=4-2x} dx$$

$$= \int_0^2 \left[4(4 - 2x) - 2x(4 - 2x) - \tfrac{1}{2}(4 - 2x)^2\right] dx$$

$$= \int_0^2 (2x^2 - 8x + 8)\, dx = \left[\tfrac{2}{3}x^3 - 4x^2 + 8x\right]_0^2 = \tfrac{16}{3}$$



18.



$$V = \iiint_E dV = \int_{-3}^{3} \int_{x^2}^{9} \int_0^4 dz\, dy\, dx$$

$$= 4 \int_{-3}^{3} \int_{x^2}^{9} dy\, dx = 4 \int_{-3}^{3} \left(9 - x^2\right) dx$$

$$= 4\left[9x - \tfrac{1}{3}x^3\right]_{-3}^{3} = 4(27 - 9 + 27 - 9) = 144$$

19. $$V = \int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_1^{5-y} dz\, dy\, dx = \int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} (5 - y - 1)\, dy\, dx = \int_{-3}^{3} \left[4y - \tfrac{1}{2}y^2\right]_{y=-\sqrt{9-x^2}}^{y=\sqrt{9-x^2}} dx$$

$$= \int_{-3}^{3} 8\sqrt{9 - x^2}\, dx = 8\left[\tfrac{x}{2}\sqrt{9 - x^2} + \tfrac{9}{2}\sin^{-1}\left(\tfrac{x}{3}\right)\right]_{-3}^{3}$$ [using trigonometric substitution or Formula 30 in the Table of Integrals]

$$= 8\left[\tfrac{9}{2}\sin^{-1}(1) - \tfrac{9}{2}\sin^{-1}(-1)\right] = 36\left(\tfrac{\pi}{2} - \left(-\tfrac{\pi}{2}\right)\right) = 36\pi$$

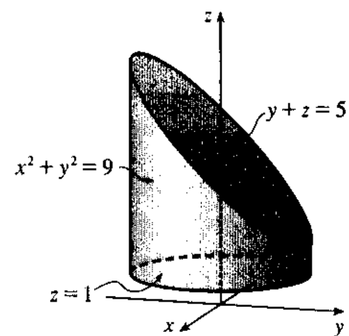
Alternatively, use polar coordinates to evaluate the double integral:

$$\int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} (4 - y)\, dy\, dx = \int_0^{2\pi} \int_0^3 (4 - r\sin\theta)\, r\, dr\, d\theta$$

$$= \int_0^{2\pi} \left[2r^2 - \tfrac{1}{3}r^3 \sin\theta\right]_{r=0}^{r=3} d\theta$$

$$= \int_0^{2\pi} (18 - 9\sin\theta)\, d\theta$$

$$= 18\theta + 9\cos\theta\Big]_0^{2\pi} = 36\pi$$



20. The paraboloid $x = y^2 + z^2$ intersects the plane $x = 16$ in the circle $y^2 + z^2 = 16$, $x = 16$.

Thus, $E = \left\{(x, y, z) \mid y^2 + z^2 \le x \le 16, y^2 + z^2 \le 16\right\}$.

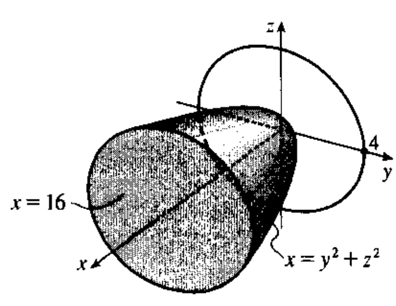
Let $D = \left\{(y, z) \mid y^2 + z^2 \le 16\right\}$. Then using polar coordinates

$y = r\cos\theta$ and $z = r\sin\theta$, we have

$$V = \iint_D \left(\int_{y^2+z^2}^{16} dx\right) dA = \iint_D (16 - (y^2 + z^2))\, dA$$

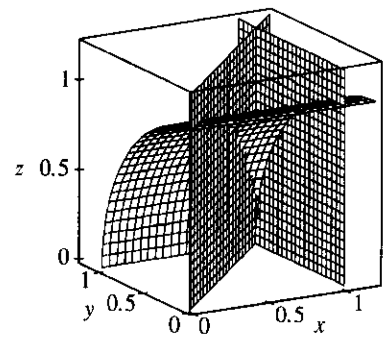
$$= \int_0^{2\pi} \int_0^4 (16 - r^2)\, r\, dr\, d\theta = \int_0^{2\pi} d\theta \int_0^4 (16r - r^3)\, dr$$

$$= \left[\,\theta\,\right]_0^{2\pi} \left[8r^2 - \tfrac{1}{4}r^4\right]_0^4 = 2\pi(128 - 64) = 128\pi$$



21. (a) The wedge can be described as the region

$$D = \{(x,y,z) \mid y^2 + z^2 \le 1, 0 \le x \le 1, 0 \le y \le x\}$$
$$= \left\{(x,y,z) \mid 0 \le x \le 1, 0 \le y \le x, 0 \le z \le \sqrt{1-y^2}\right\}$$



So the integral expressing the volume of the wedge is

$$\iiint_D dV = \int_0^1 \int_0^x \int_0^{\sqrt{1-y^2}} dz\,dy\,dx.$$

(b) A CAS gives $\int_0^1 \int_0^x \int_0^{\sqrt{1-y^2}} dz\,dy\,dx = \frac{\pi}{4} - \frac{1}{3}$.

(Or use Formulas 30 and 87 from the Table of Integrals.)

22. (a) Note that $\Delta V = \left(\frac{1}{2}\right)^3 = \frac{1}{8}$, so the Midpoint Rule gives

$$\begin{aligned}\iiint_B f(x,y,z)\,dV &\approx \tfrac{1}{8}\left[f\left(\tfrac14,\tfrac14,\tfrac14\right) + f\left(\tfrac14,\tfrac14,\tfrac34\right) + f\left(\tfrac14,\tfrac34,\tfrac14\right) + f\left(\tfrac34,\tfrac14,\tfrac14\right)\right.\\ &\quad \left. + f\left(\tfrac14,\tfrac34,\tfrac34\right) + f\left(\tfrac34,\tfrac14,\tfrac34\right) + f\left(\tfrac34,\tfrac34,\tfrac14\right) + f\left(\tfrac34,\tfrac34,\tfrac34\right)\right]\\ &= \tfrac{1}{8}\left[e^{-3(1/4)^2} + 3e^{-2(1/4)^2-(3/4)^2} + 3e^{-(1/4)^2-2(3/4)^2} + e^{-3(3/4)^2}\right] \approx 0.42968\end{aligned}$$

(b) A CAS estimates the integral to be $\iiint_B e^{-x^2-y^2-z^2}\,dV \approx 0.42$. The estimate in part (a) is correct to one decimal place, and is larger than the actual value of the integral.

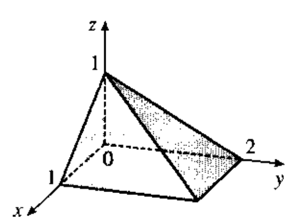
23. Here $f(x,y,z) = \dfrac{1}{\ln(1+x+y+z)}$ and $\Delta V = 2 \cdot 4 \cdot 2 = 16$, so the Midpoint Rule gives

$$\begin{aligned}\iiint_B f(x,y,z)\,dV &\approx \sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f(\overline{x}_i,\overline{y}_j,\overline{z}_k)\,\Delta V\\ &= 16\,[f(1,2,1) + f(1,2,3) + f(1,6,1) + f(1,6,3)\\ &\quad + f(3,2,1) + f(3,2,3) + f(3,6,1) + f(3,6,3)]\\ &= 16\left[\frac{1}{\ln 5} + \frac{1}{\ln 7} + \frac{1}{\ln 9} + \frac{1}{\ln 11} + \frac{1}{\ln 7} + \frac{1}{\ln 9} + \frac{1}{\ln 11} + \frac{1}{\ln 13}\right] \approx 60.533\end{aligned}$$

24. Here $f(x,y,z) = \sin(xy^2z^3)$ and $\Delta V = 2 \cdot 1 \cdot \frac{1}{2} = 1$, so the Midpoint Rule gives

$$\begin{aligned}\iiint_B f(x,y,z)\,dV &\approx \sum_{i=1}^{l}\sum_{j=1}^{m}\sum_{k=1}^{n} f(\overline{x}_i,\overline{y}_j,\overline{z}_k)\,\Delta V\\ &= 1\left[f\left(1,\tfrac12,\tfrac14\right) + f\left(1,\tfrac12,\tfrac34\right) + f\left(1,\tfrac32,\tfrac14\right) + f\left(1,\tfrac32,\tfrac34\right)\right.\\ &\quad \left. + f\left(3,\tfrac12,\tfrac14\right) + f\left(3,\tfrac12,\tfrac34\right) + f\left(3,\tfrac32,\tfrac14\right) + f\left(3,\tfrac32,\tfrac34\right)\right]\\ &= \sin\tfrac{1}{256} + \sin\tfrac{27}{256} + \sin\tfrac{9}{256} + \sin\tfrac{243}{256} + \sin\tfrac{3}{256} + \sin\tfrac{81}{256} + \sin\tfrac{27}{256} + \sin\tfrac{729}{256} \approx 1.675\end{aligned}$$

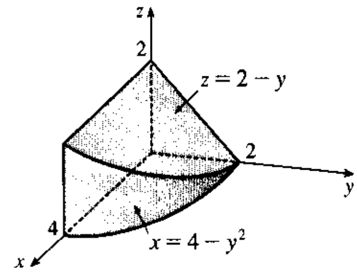
25. $E = \{(x,y,z) \mid 0 \le x \le 1, 0 \le z \le 1-x, 0 \le y \le 2-2z\}$, the solid bounded by the three coordinate planes and the planes $z = 1-x$, $y = 2-2z$.



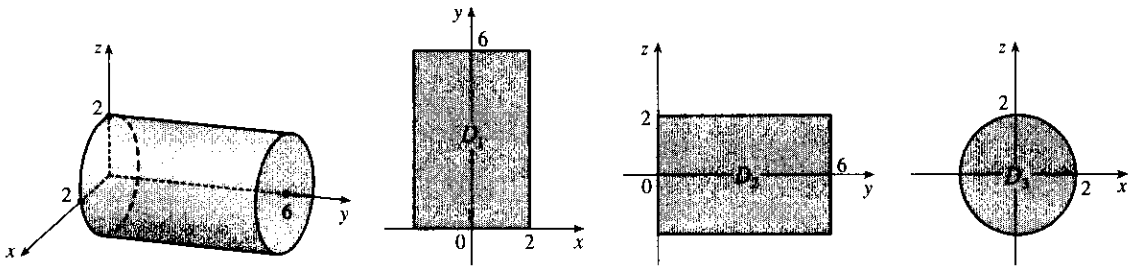
26. $E = \{(x,y,z) \mid 0 \le y \le 2, 0 \le z \le 2 - y, 0 \le x \le 4 - y^2\}$,

the solid bounded by the three coordinate planes, the plane $z = 2 - y$,

and the cylindrical surface $x = 4 - y^2$.



27.



If D_1, D_2, D_3 are the projections of E on the xy-, yz-, and xz-planes, then

$$D_1 = \{(x,y) \mid -2 \le x \le 2, 0 \le y \le 6\}$$

$$D_2 = \{(y,z) \mid -2 \le z \le 2, 0 \le y \le 6\}$$

$$D_3 = \{(x,z) \mid x^2 + z^2 \le 4\}$$

Therefore

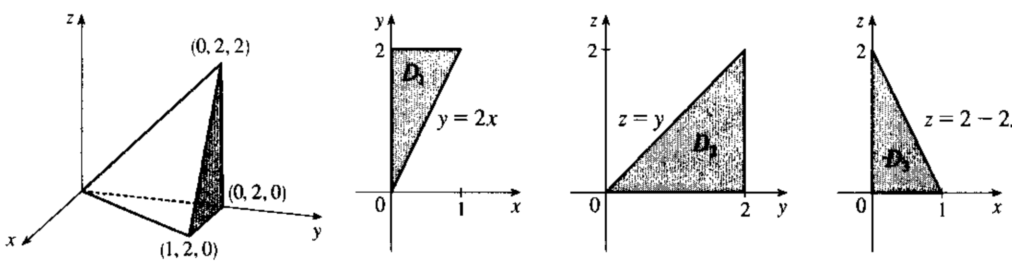
$$E = \left\{(x,y,z) \mid -\sqrt{4-x^2} \le z \le \sqrt{4-x^2},\ -2 \le x \le 2, 0 \le y \le 6\right\}$$
$$= \left\{(x,y,z) \mid -\sqrt{4-z^2} \le x \le \sqrt{4-z^2},\ -2 \le z \le 2, 0 \le y \le 6\right\}$$

$$\iiint_E f(x,y,z)\,dV = \int_{-2}^{2}\int_0^6\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} f(x,y,z)\,dz\,dy\,dx = \int_0^6\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} f(x,y,z)\,dz\,dx\,dy$$

$$= \int_0^6\int_{-2}^{2}\int_{-\sqrt{4-z^2}}^{\sqrt{4-z^2}} f(x,y,z)\,dx\,dz\,dy = \int_{-2}^{2}\int_0^6\int_{-\sqrt{4-z^2}}^{\sqrt{4-z^2}} f(x,y,z)\,dx\,dy\,dz$$

$$= \int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}}\int_0^6 f(x,y,z)\,dy\,dz\,dx = \int_{-2}^{2}\int_{-\sqrt{4-z^2}}^{\sqrt{4-z^2}}\int_0^6 f(x,y,z)\,dy\,dx\,dz$$

28.



If D_1, D_2, and D_3 are the projections of E on the xy-, yz-, and xz-planes, then

$$D_1 = \{(x,y) \mid 0 \le x \le 1, 2x \le y \le 2\} = \{(x,y) \mid 0 \le y \le 2, 0 \le x \le y/2\},$$

$D_2 = \{(y,z) \mid 0 \le y \le 2, 0 \le z \le y\} = \{(y,z) \mid 0 \le z \le 2, z \le y \le 2\}$, and

$D_3 = \{(x,z) \mid 0 \le x \le 1, 0 \le z \le 2-2x\} = \{(x,z) \mid 0 \le z \le 2, 0 \le x \le (2-z)/2\}$

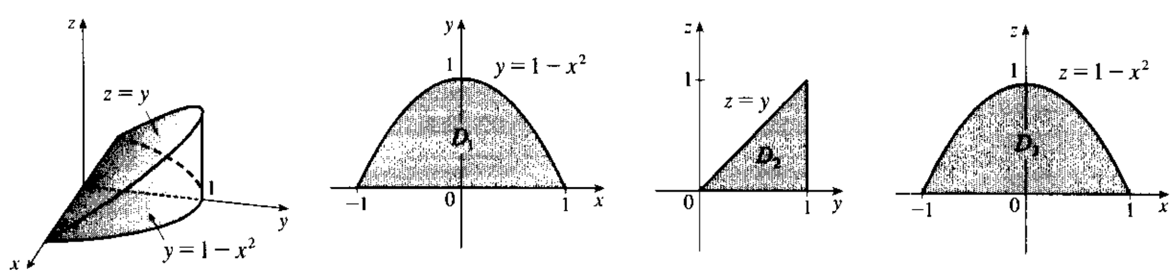
Therefore

$$\begin{aligned}
E &= \{(x,y,z) \mid 0 \le x \le 1, 2x \le y \le 2, 0 \le z \le y-2x\} \\
&= \{(x,y,z) \mid 0 \le y \le 2, 0 \le x \le y/2, 0 \le z \le y-2x\} \\
&= \{(x,y,z) \mid 0 \le y \le 2, 0 \le z \le y, 0 \le x \le (y-z)/2\} \\
&= \{(x,y,z) \mid 0 \le z \le 2, z \le y \le 2, 0 \le x \le (y-z)/2\} \\
&= \{(x,y,z) \mid 0 \le x \le 1, 0 \le z \le 2-2x, z+2x \le y \le 2\} \\
&= \{(x,y,z) \mid 0 \le z \le 2, 0 \le x \le (2-z)/2, z+2x \le y \le 2\}
\end{aligned}$$

Then

$$\begin{aligned}
\iiint_E f(x,y,z)\,dV &= \int_0^1 \int_{2x}^2 \int_0^{y-2x} f(x,y,z)\,dz\,dy\,dx \\
&= \int_0^2 \int_0^{y/2} \int_0^{y-2x} f(x,y,z)\,dz\,dx\,dy \\
&= \int_0^2 \int_0^y \int_0^{(y-z)/2} f(x,y,z)\,dx\,dz\,dy \\
&= \int_0^2 \int_z^2 \int_0^{(y-z)/2} f(x,y,z)\,dx\,dy\,dz \\
&= \int_0^1 \int_0^{2-2x} \int_{z+2x}^2 f(x,y,z)\,dy\,dz\,dx \\
&= \int_0^2 \int_0^{(2-z)/2} \int_{z+2x}^2 f(x,y,z)\,dy\,dx\,dz
\end{aligned}$$

29.



If D_1, D_2, and D_3 are the projections of E on the xy-, yz-, and xz-planes, then

$D_1 = \{(x,y) \mid -1 \le x \le 1, 0 \le y \le 1-x^2\} = \left\{(x,y) \mid 0 \le y \le 1, -\sqrt{1-y} \le x \le \sqrt{1-y}\right\}$,

$D_2 = \{(y,z) \mid 0 \le y \le 1, 0 \le z \le y\} = \{(y,z) \mid 0 \le z \le 1, z \le y \le 1\}$, and

$D_3 = \{(x,z) \mid -1 \le x \le 1, 0 \le z \le 1-x^2\} = \{(x,z) \mid 0 \le z \le 1, -\sqrt{1-z} \le x \le \sqrt{1-z}\}$

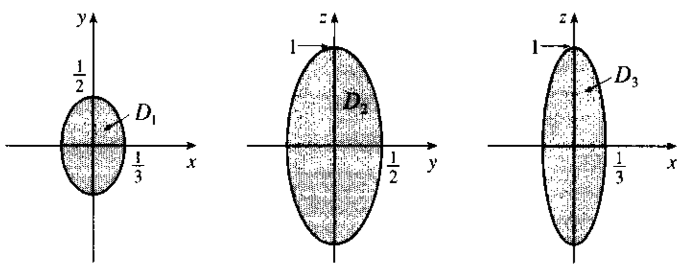
Therefore

$$
\begin{aligned}
E &= \{(x,y,z) \mid -1 \le x \le 1, 0 \le y \le 1-x^2, 0 \le z \le y\} \\
&= \{(x,y,z) \mid 0 \le y \le 1,\ -\sqrt{1-y} \le x \le \sqrt{1-y}, 0 \le z \le y\} \\
&= \{(x,y,z) \mid 0 \le y \le 1, 0 \le z \le y,\ -\sqrt{1-y} \le x \le \sqrt{1-y}\} \\
&= \{(x,y,z) \mid 0 \le z \le 1, z \le y \le 1,\ -\sqrt{1-y} \le x \le \sqrt{1-y}\} \\
&= \{(x,y,z) \mid -1 \le x \le 1, 0 \le z \le 1-x^2, z \le y \le 1-x^2\} \\
&= \{(x,y,z) \mid 0 \le z \le 1,\ -\sqrt{1-z} \le x \le \sqrt{1-z}, z \le y \le 1-x^2\}
\end{aligned}
$$

Then

$$
\begin{aligned}
\iiint_E f(x,y,z)\,dV &= \int_{-1}^{1}\int_0^{1-x^2}\int_0^y f(x,y,z)\,dz\,dy\,dx = \int_0^1\int_{-\sqrt{1-y}}^{\sqrt{1-y}}\int_0^y f(x,y,z)\,dz\,dx\,dy \\
&= \int_0^1\int_0^y\int_{-\sqrt{1-y}}^{\sqrt{1-y}} f(x,y,z)\,dx\,dz\,dy = \int_0^1\int_z^1\int_{-\sqrt{1-y}}^{\sqrt{1-y}} f(x,y,z)\,dx\,dy\,dz \\
&= \int_{-1}^{1}\int_0^{1-x^2}\int_z^{1-x^2} f(x,y,z)\,dy\,dz\,dx = \int_0^1\int_{-\sqrt{1-z}}^{\sqrt{1-z}}\int_z^{1-x^2} f(x,y,z)\,dy\,dx\,dz
\end{aligned}
$$

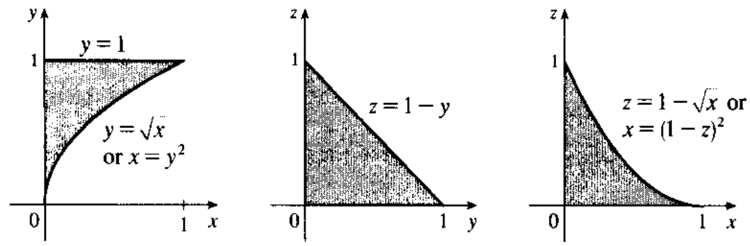
30.



If D_1, D_2 and D_3 are the projections of E on the xy-, yz-, and xz-planes, then $D_1 = \{(x,y) \mid 9x^2 + 4y^2 \le 1\}$, $D_2 = \{(y,z) \mid 4y^2 + z^2 \le 1\}$, $D_3 = \{(x,z) \mid 9x^2 + z^2 \le 1\}$. Therefore

$$
\begin{aligned}
\iiint_E f(x,y,z)\,dV &= \int_{-1/3}^{1/3}\int_{-\sqrt{1-9x^2}/2}^{\sqrt{1-9x^2}/2}\int_{-\sqrt{1-9x^2-4y^2}}^{\sqrt{1-9x^2-4y^2}} f(x,y,z)\,dz\,dy\,dx \\
&= \int_{-1/2}^{1/2}\int_{-\sqrt{1-4y^2}/3}^{\sqrt{1-4y^2}/3}\int_{-\sqrt{1-9x^2-4y^2}}^{\sqrt{1-9x^2-4y^2}} f(x,y,z)\,dz\,dx\,dy \\
&= \int_{-1/2}^{1/2}\int_{-\sqrt{1-4y^2}}^{\sqrt{1-4y^2}}\int_{-\sqrt{1-4y^2-z^2}/3}^{\sqrt{1-4y^2-z^2}/3} f(x,y,z)\,dx\,dz\,dy \\
&= \int_{-1}^{1}\int_{-\sqrt{1-z^2}/2}^{\sqrt{1-z^2}/2}\int_{-\sqrt{1-4y^2-z^2}/3}^{\sqrt{1-4y^2-z^2}/3} f(x,y,z)\,dx\,dy\,dz \\
&= \int_{-1/3}^{1/3}\int_{-\sqrt{1-9x^2}}^{\sqrt{1-9x^2}}\int_{-\sqrt{1-9x^2-z^2}/2}^{\sqrt{1-9x^2-z^2}/2} f(x,y,z)\,dy\,dz\,dx \\
&= \int_{-1}^{1}\int_{-\sqrt{1-z^2}/3}^{\sqrt{1-z^2}/3}\int_{-\sqrt{1-9x^2-z^2}/2}^{\sqrt{1-9x^2-z^2}/2} f(x,y,z)\,dy\,dx\,dz
\end{aligned}
$$

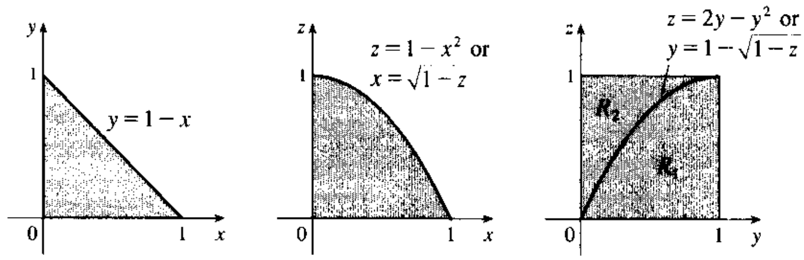
31.



The diagrams show the projections of E on the xy-, yz-, and xz-planes. Therefore

$$\begin{aligned}\int_0^1\int_{\sqrt{x}}^1\int_0^{1-y} f(x,y,z)\,dz\,dy\,dx &= \int_0^1\int_0^{y^2}\int_0^{1-y} f(x,y,z)\,dz\,dx\,dy\\ &= \int_0^1\int_0^{1-z}\int_0^{y^2} f(x,y,z)\,dx\,dy\,dz\\ &= \int_0^1\int_0^{1-y}\int_0^{y^2} f(x,y,z)\,dx\,dz\,dy\\ &= \int_0^1\int_0^{1-\sqrt{x}}\int_{\sqrt{x}}^{1-z} f(x,y,z)\,dy\,dz\,dx\\ &= \int_0^1\int_0^{(1-z)^2}\int_{\sqrt{x}}^{1-z} f(x,y,z)\,dy\,dx\,dz\end{aligned}$$

32.



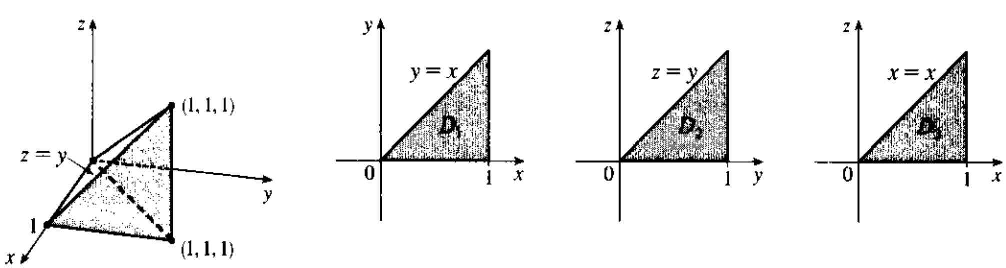
The projections of E onto the xy- and xz-planes are as in the first two diagrams and so

$$\begin{aligned}\int_0^1\int_0^{1-x^2}\int_0^{1-x} f(x,y,z)\,dy\,dz\,dx &= \int_0^1\int_0^{\sqrt{1-z}}\int_0^{1-x} f(x,y,z)\,dy\,dx\,dz\\ &= \int_0^1\int_0^{1-y}\int_0^{1-x^2} f(x,y,z)\,dz\,dx\,dy = \int_0^1\int_0^{1-x}\int_0^{1-x^2} f(x,y,z)\,dz\,dy\,dx\end{aligned}$$

Now the surface $z = 1 - x^2$ intersects the plane $y = 1 - x$ in a curve whose projection in the yz-plane is $z = 1 - (1-y)^2$ or $z = 2y - y^2$. So we must split up the projection of E on the yz-plane into two regions as in the third diagram. For (y,z) in R_1, $0 \le x \le 1 - y$ and for (y,z) in R_2, $0 \le x \le \sqrt{1-z}$, and so the given integral is also equal to

$$\begin{aligned}&\int_0^1\int_0^{1-\sqrt{1-z}}\int_0^{\sqrt{1-z}} f(x,y,z)\,dx\,dy\,dz + \int_0^1\int_{1-\sqrt{1-z}}^1\int_0^{1-y} f(x,y,z)\,dx\,dy\,dz\\ &\qquad= \int_0^1\int_0^{2y-y^2}\int_0^{1-y} f(x,y,z)\,dx\,dz\,dy + \int_0^1\int_{2y-y^2}^1\int_0^{\sqrt{1-z}} f(x,y,z)\,dx\,dz\,dy.\end{aligned}$$

33.



$\int_0^1\int_y^1\int_0^y f(x,y,z)\,dz\,dx\,dy = \iiint_E f(x,y,z)\,dV$ where $E = \{(x,y,z) \mid 0 \le z \le y, y \le x \le 1, 0 \le y \le 1\}$.

If D_1, D_2, and D_3 are the projections of E on the xy-, yz- and xz-planes then

$$D_1 = \{(x,y) \mid 0 \le y \le 1, y \le x \le 1\} = \{(x,y) \mid 0 \le x \le 1, 0 \le y \le x\},$$

$$D_2 = \{(y,z) \mid 0 \le y \le 1, 0 \le z \le y\} = \{(y,z) \mid 0 \le z \le 1, z \le y \le 1\}, \text{ and}$$

$$D_3 = \{(x,z) \mid 0 \le x \le 1, 0 \le z \le x\} = \{(x,z) \mid 0 \le z \le 1, z \le x \le 1\}.$$

Thus we also have

$$E = \{(x,y,z) \mid 0 \le x \le 1, 0 \le y \le x, 0 \le z \le y\} = \{(x,y,z) \mid 0 \le y \le 1, 0 \le z \le y, y \le x \le 1\}$$

$$= \{(x,y,z) \mid 0 \le z \le 1, z \le y \le 1, y \le x \le 1\} = \{(x,y,z) \mid 0 \le x \le 1, 0 \le z \le x, z \le y \le x\}$$

$$= \{(x,y,z) \mid 0 \le z \le 1, z \le x \le 1, z \le y \le x\}.$$

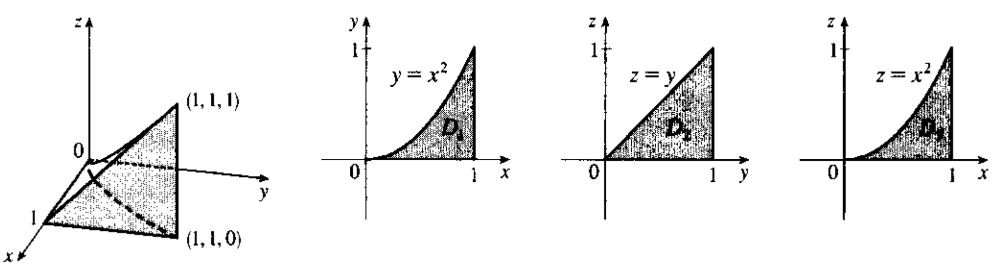
Then

$$\int_0^1 \int_y^1 \int_0^y f(x,y,z)\,dz\,dx\,dy = \int_0^1 \int_0^x \int_0^y f(x,y,z)\,dz\,dy\,dx = \int_0^1 \int_0^y \int_y^1 f(x,y,z)\,dx\,dz\,dy$$

$$= \int_0^1 \int_z^1 \int_y^1 f(x,y,z)\,dx\,dy\,dz = \int_0^1 \int_0^x \int_z^x f(x,y,z)\,dy\,dz\,dx$$

$$= \int_0^1 \int_z^1 \int_z^x f(x,y,z)\,dy\,dx\,dz$$

34.



$\int_0^1 \int_0^{x^2} \int_0^y f(x,y,z)\,dz\,dy\,dx = \iiint_E f(x,y,z)\,dV$ where $E = \left\{(x,y,z) \mid 0 \le x \le 1, 0 \le y \le x^2, 0 \le z \le y\right\}$. If D_1, D_2, D_3 are the projections of E on the xy-, yz-, and xz-planes, then $D_1 = \left\{(x,y) \mid 0 \le x \le 1, 0 \le y \le x^2\right\} = \left\{(x,y) \mid 0 \le y \le 1, \sqrt{y} \le x \le 1\right\}$, $D_2 = \{(y,z) \mid 0 \le y \le 1, 0 \le z \le y\} = \{(y,z) \mid 0 \le z \le 1, z \le y \le 1\}$, $D_3 = \left\{(x,z) \mid 0 \le x \le 1, 0 \le z \le x^2\right\} = \{(x,z) \mid 0 \le z \le 1, \sqrt{z} \le x \le 1\}$. Thus we also have

$$E = \{(x,y,z) \mid 0 \le y \le 1, \sqrt{y} \le x \le 1, 0 \le z \le y\}$$

$$= \{(x,y,z) \mid 0 \le y \le 1, 0 \le z \le y, \sqrt{y} \le x \le 1\}$$

$$= \{(x,y,z) \mid 0 \le z \le 1, z \le y \le 1, \sqrt{y} \le x \le 1\}$$

$$= \left\{(x,y,z) \mid 0 \le x \le 1, 0 \le z \le x^2, z \le y \le x^2\right\}$$

$$= \left\{(x,y,z) \mid 0 \le z \le 1, \sqrt{z} \le x \le 1, z \le y \le x^2\right\}$$

Then

$$\int_0^1 \int_0^{x^2} \int_0^y f(x,y,z)\,dz\,dy\,dx = \int_0^1 \int_{\sqrt{y}}^1 \int_0^y f(x,y,z)\,dz\,dx\,dy$$

$$= \int_0^1 \int_0^y \int_{\sqrt{y}}^1 f(x,y,z)\,dx\,dz\,dy$$

$$= \int_0^1 \int_z^1 \int_{\sqrt{y}}^1 f(x,y,z)\,dx\,dy\,dz$$

$$= \int_0^1 \int_0^{x^2} \int_z^{x^2} f(x,y,z)\,dy\,dz\,dx$$

$$= \int_0^1 \int_{\sqrt{z}}^1 \int_z^{x^2} f(x,y,z)\,dy\,dx\,dz$$

35. $m = \iiint_E \rho(x,y,z)\,dV = \int_0^1 \int_0^{\sqrt{x}} \int_0^{1+x+y} 2\,dz\,dy\,dx$

$= \int_0^1 \int_0^{\sqrt{x}} 2(1+x+y)\,dy\,dx = \int_0^1 \left[2y + 2xy + y^2\right]_{y=0}^{y=\sqrt{x}}\,dx$

$= \int_0^1 \left(2\sqrt{x} + 2x^{3/2} + x\right)dx = \left[\frac{4}{3}x^{3/2} + \frac{4}{5}x^{5/2} + \frac{1}{2}x^2\right]_0^1 = \frac{79}{30}$

$M_{yz} = \iiint_E x\rho(x,y,z)\,dV = \int_0^1 \int_0^{\sqrt{x}} \int_0^{1+x+y} 2x\,dz\,dy\,dx$

$= \int_0^1 \int_0^{\sqrt{x}} 2x(1+x+y)\,dy\,dx = \int_0^1 \left[2xy + 2x^2y + xy^2\right]_{y=0}^{y=\sqrt{x}}\,dx$

$= \int_0^1 (2x^{3/2} + 2x^{5/2} + x^2)\,dx = \left[\frac{4}{5}x^{5/2} + \frac{4}{7}x^{7/2} + \frac{1}{3}x^3\right]_0^1 = \frac{179}{105}$

$M_{xz} = \iiint_E y\rho(x,y,z)\,dV = \int_0^1 \int_0^{\sqrt{x}} \int_0^{1+x+y} 2y\,dz\,dy\,dx$

$= \int_0^1 \int_0^{\sqrt{x}} 2y(1+x+y)\,dy\,dx = \int_0^1 \left[y^2 + xy^2 + \frac{2}{3}y^3\right]_{y=0}^{y=\sqrt{x}}\,dx$

$= \int_0^1 \left(x + x^2 + \frac{2}{3}x^{3/2}\right)dx = \left[\frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{4}{15}x^{5/2}\right]_0^1 = \frac{11}{10}$

$M_{xy} = \iiint_E z\rho(x,y,z)\,dV = \int_0^1 \int_0^{\sqrt{x}} \int_0^{1+x+y} 2z\,dz\,dy\,dx$

$= \int_0^1 \int_0^{\sqrt{x}} \left[z^2\right]_{z=0}^{z=1+x+y}\,dy\,dx = \int_0^1 \int_0^{\sqrt{x}} (1+x+y)^2\,dy\,dx$

$= \int_0^1 \int_0^{\sqrt{x}} (1 + 2x + 2y + 2xy + x^2 + y^2)\,dy\,dx$

$= \int_0^1 \left[y + 2xy + y^2 + xy^2 + x^2y + \frac{1}{3}y^3\right]_{y=0}^{y=\sqrt{x}}\,dx = \int_0^1 \left(\sqrt{x} + \frac{7}{3}x^{3/2} + x + x^2 + x^{5/2}\right)dx$

$= \left[\frac{2}{3}x^{3/2} + \frac{14}{15}x^{5/2} + \frac{1}{2}x^2 + \frac{1}{3}x^3 + \frac{2}{7}x^{7/2}\right]_0^1 = \frac{571}{210}$

Thus the mass is $\frac{79}{30}$ and the center of mass is $(\overline{x},\overline{y},\overline{z}) = \left(\dfrac{M_{yz}}{m}, \dfrac{M_{xz}}{m}, \dfrac{M_{xy}}{m}\right) = \left(\dfrac{358}{553}, \dfrac{33}{79}, \dfrac{571}{553}\right)$.

36. $m = \int_{-1}^1 \int_0^{1-y^2} \int_0^{1-z} 4\,dx\,dz\,dy = 4\int_{-1}^1 \int_0^{1-y^2} (1-z)\,dz\,dy$

$= 4\int_{-1}^1 \left[z - \frac{1}{2}z^2\right]_{z=0}^{z=1-y^2}\,dy = 2\int_{-1}^1 (1-y^4)\,dy = \frac{16}{5}$,

$M_{yz} = \int_{-1}^1 \int_0^{1-y^2} \int_0^{1-z} 4x\,dx\,dz\,dy = 2\int_{-1}^1 \int_0^{1-y^2} (1-z)^2\,dz\,dy = 2\int_{-1}^1 \left[-\frac{1}{3}(1-z)^3\right]_{z=0}^{z=1-y^2}\,dy$

$= \frac{2}{3}\int_{-1}^1 (1-y^6)\,dy = \left(\frac{4}{3}\right)\left(\frac{6}{7}\right) = \frac{24}{21}$

$$M_{xz} = \int_{-1}^{1}\int_0^{1-y^2}\int_0^{1-z} 4y\,dx\,dz\,dy = \int_{-1}^{1}\int_0^{1-y^2} 4y(1-z)\,dz\,dy$$

$$= \int_{-1}^{1}\left[4y(1-y^2) - 2y(1-y^2)^2\right]dy = \int_{-1}^{1}(2y - 2y^5)\,dy = 0 \quad \text{[the integrand is odd]}$$

$$M_{xy} = \int_{-1}^{1}\int_0^{1-y^2}\int_0^{1-z} 4z\,dx\,dz\,dy = \int_{-1}^{1}\int_0^{1-y^2}(4z - 4z^2)\,dz\,dy = 2\int_{-1}^{1}\left[(1-y^2)^2 - \tfrac{2}{3}(1-y^2)^3\right]dy$$

$$= 2\int_{-1}^{1}\left[\tfrac{1}{3} - y^4 + \tfrac{2}{3}y^6\right]dy = \left[\tfrac{4}{3}y - \tfrac{4}{5}y^5 + \tfrac{8}{21}y^7\right]_0^1 = \tfrac{96}{105} = \tfrac{32}{35}$$

Thus, $(\overline{x}, \overline{y}, \overline{z}) = \left(\frac{5}{14}, 0, \frac{2}{7}\right)$

37. $m = \int_0^a\int_0^a\int_0^a (x^2 + y^2 + z^2)\,dx\,dy\,dz = \int_0^a\int_0^a \left[\tfrac{1}{3}x^3 + xy^2 + xz^2\right]_{x=0}^{x=a} dy\,dz$

$$= \int_0^a\int_0^a \left(\tfrac{1}{3}a^3 + ay^2 + az^2\right)dy\,dz = \int_0^a \left[\tfrac{1}{3}a^3y + \tfrac{1}{3}ay^3 + ayz^2\right]_{y=0}^{y=a} dz$$

$$= \int_0^a \left(\tfrac{2}{3}a^4 + a^2z^2\right)dz = \left[\tfrac{2}{3}a^4z + \tfrac{1}{3}a^2z^3\right]_0^a = \tfrac{2}{3}a^5 + \tfrac{1}{3}a^5 = a^5$$

$$M_{yz} = \int_0^a\int_0^a\int_0^a \left[x^3 + x(y^2 + z^2)\right]dx\,dy\,dz = \int_0^a\int_0^a \left[\tfrac{1}{4}a^4 + \tfrac{1}{2}a^2(y^2 + z^2)\right]dy\,dz$$

$$= \int_0^a \left(\tfrac{1}{4}a^5 + \tfrac{1}{6}a^5 + \tfrac{1}{2}a^3z^2\right)dz = \tfrac{1}{4}a^6 + \tfrac{1}{3}a^6 = \tfrac{7}{12}a^6$$

$= M_{xz} = M_{xy}$ by symmetry of E and $\rho(x, y, z)$

Hence $(\overline{x}, \overline{y}, \overline{z}) = \left(\frac{7}{12}a, \frac{7}{12}a, \frac{7}{12}a\right)$.

38. $m = \int_0^1\int_0^{1-x}\int_0^{1-x-y} y\,dz\,dy\,dx = \int_0^1\int_0^{1-x}\left[(1-x)y - y^2\right]dy\,dx$

$$= \int_0^1 \left[\tfrac{1}{2}(1-x)^3 - \tfrac{1}{3}(1-x)^3\right]dx = \tfrac{1}{6}\int_0^1 (1-x)^3\,dx = \tfrac{1}{24}$$

$$M_{yz} = \int_0^1\int_0^{1-x}\int_0^{1-x-y} xy\,dz\,dy\,dx = \int_0^1\int_0^{1-x}\left[(x - x^2)y - xy^2\right]dy\,dx$$

$$= \int_0^1 \left[\tfrac{1}{2}x(1-x)^3 - \tfrac{1}{3}x(1-x)^3\right]dx = \tfrac{1}{6}\int_0^1 \left(x - 3x^2 + 3x^3 - x^4\right)dx$$

$$= \tfrac{1}{6}\left(\tfrac{1}{2} - 1 + \tfrac{3}{4} - \tfrac{1}{5}\right) = \tfrac{1}{120}$$

$$M_{xz} = \int_0^1\int_0^{1-x}\int_0^{1-x-y} y^2\,dz\,dy\,dx = \int_0^1\int_0^{1-x}\left[(1-x)y^2 - y^3\right]dy\,dx$$

$$= \int_0^1 \left[\tfrac{1}{3}(1-x)^4 - \tfrac{1}{4}(1-x)^4\right]dx = \tfrac{1}{12}\left[-\tfrac{1}{5}(1-x)^5\right]_0^1 = \tfrac{1}{60}$$

$$M_{xy} = \int_0^1\int_0^{1-x}\int_0^{1-x-y} yz\,dz\,dy\,dx = \int_0^1\int_0^{1-x}\left[\tfrac{1}{2}y(1-x-y)^2\right]dy\,dx$$

$$= \tfrac{1}{2}\int_0^1\int_0^{1-x}\left[(1-x)^2y - 2(1-x)y^2 + y^3\right]dy\,dx$$

$$= \tfrac{1}{2}\int_0^1 \left[\tfrac{1}{2}(1-x)^4 - \tfrac{2}{3}(1-x)^4 + \tfrac{1}{4}(1-x)^4\right]dx$$

$$= \tfrac{1}{24}\int_0^1 (1-x)^4\,dx = -\tfrac{1}{24}\left[\tfrac{1}{5}(1-x)^5\right]_0^1 = \tfrac{1}{120}$$

Hence $(\overline{x}, \overline{y}, \overline{z}) = \left(\frac{1}{5}, \frac{2}{5}, \frac{1}{5}\right)$.

39. (a) $m = \int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_{1}^{5-y} \sqrt{x^2+y^2}\, dz\, dy\, dx$

(b) $(\overline{x}, \overline{y}, \overline{z}) = \left(\dfrac{M_{yz}}{m}, \dfrac{M_{xz}}{m}, \dfrac{M_{xy}}{m} \right)$ where

$M_{yz} = \int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_{1}^{5-y} x\sqrt{x^2+y^2}\, dz\, dy\, dx$, $M_{xz} = \int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_{1}^{5-y} y\sqrt{x^2+y^2}\, dz\, dy\, dx$, and

$M_{xy} = \int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_{1}^{5-y} z\sqrt{x^2+y^2}\, dz\, dy\, dx$.

(c) $I_z = \int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_{1}^{5-y} (x^2+y^2)\sqrt{x^2+y^2}\, dz\, dy\, dx = \int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_{1}^{5-y} (x^2+y^2)^{3/2}\, dz\, dy\, dx$

40. (a) $m = \int_{-1}^{1} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \int_{0}^{\sqrt{1-x^2-y^2}} \sqrt{x^2+y^2+z^2}\, dz\, dx\, dy$

(b) $(\overline{x}, \overline{y}, \overline{z})$ where $\overline{x} = m^{-1} \int_{-1}^{1} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \int_{0}^{\sqrt{1-x^2-y^2}} x\sqrt{x^2+y^2+z^2}\, dz\, dx\, dy$,

$\overline{y} = m^{-1} \int_{-1}^{1} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \int_{0}^{\sqrt{1-x^2-y^2}} y\sqrt{x^2+y^2+z^2}\, dz\, dx\, dy$,

$\overline{z} = m^{-1} \int_{-1}^{1} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \int_{0}^{\sqrt{1-x^2-y^2}} z\sqrt{x^2+y^2+z^2}\, dz\, dx\, dy$

(c) $I_z = \int_{-1}^{1} \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \int_{0}^{\sqrt{1-x^2-y^2}} (x^2+y^2)(1+x+y+z)\, dz\, dx\, dy$

41. (a) $m = \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^y (1+x+y+z)\, dz\, dy\, dx = \frac{3\pi}{32} + \frac{11}{24}$

(b) $(\overline{x}, \overline{y}, \overline{z}) = \Big(m^{-1} \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^y x(1+x+y+z)\, dz\, dy\, dx,$

$m^{-1} \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^y y(1+x+y+z)\, dz\, dy\, dx,$

$m^{-1} \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^y z(1+x+y+z)\, dz\, dy\, dx \Big)$

$$= \left(\frac{28}{9\pi+44}, \frac{30\pi+128}{45\pi+220}, \frac{45\pi+208}{135\pi+660} \right)$$

(c) $I_z = \displaystyle\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^y (x^2+y^2)(1+x+y+z)\, dz\, dy\, dx = \frac{68+15\pi}{240}$

42. (a) $m = \int_0^1 \int_{3x}^3 \int_0^{\sqrt{9-y^2}} (x^2+y^2)\, dz\, dy\, dx = \frac{56}{5} = 11.2$

(b) $(\overline{x}, \overline{y}, \overline{z})$ where $\overline{x} = m^{-1} \int_0^1 \int_{3x}^3 \int_0^{\sqrt{9-y^2}} x(x^2+y^2)\, dz\, dy\, dx \approx 0.375$,

$\overline{y} = m^{-1} \int_0^1 \int_{3x}^3 \int_0^{\sqrt{9-y^2}} y(x^2+y^2)\, dz\, dy\, dx = \frac{45\pi}{64} \approx 2.209$,

$\overline{z} = m^{-1} \int_0^1 \int_{3x}^3 \int_0^{\sqrt{9-y^2}} z(x^2+y^2)\, dz\, dy\, dx = \frac{15}{16} = 0.9375$.

(c) $I_z = \int_0^1 \int_{3x}^3 \int_0^{\sqrt{9-y^2}} (x^2+y^2)^2\, dz\, dy\, dx = \frac{10{,}464}{175} \approx 59.79$

43. $I_x = \int_0^L \int_0^L \int_0^L k(y^2 + z^2)\,dz\,dy\,dx = k\int_0^L \int_0^L \left(Ly^2 + \frac{1}{3}L^3\right)dy\,dx = k\int_0^L \frac{2}{3}L^4\,dx = \frac{2}{3}kL^5$.

By symmetry, $I_x = I_y = I_z = \frac{2}{3}kL^5$.

44. Let k be the density. Then

$$I_x = \int_{-c/2}^{c/2} \int_{-b/2}^{b/2} \int_{-a/2}^{a/2} k(y^2 + z^2)\,dx\,dy\,dz = ka\int_{-c/2}^{c/2} \int_{-b/2}^{b/2} (y^2 + z^2)\,dy\,dz$$

$$= ak\int_{-c/2}^{c/2} \left[\tfrac{1}{3}y^3 + z^2 y\right]_{y=-b/2}^{y=b/2} dz = ak\int_{-c/2}^{c/2} \left(\tfrac{1}{12}b^3 + bz^2\right)dz = ak\left[\tfrac{1}{12}b^3 z + \tfrac{1}{3}bz^3\right]_{-c/2}^{c/2}$$

$$= ak\left(\tfrac{1}{12}b^3 c + \tfrac{1}{12}bc^3\right) = \tfrac{1}{12}kabc(b^2 + c^2)$$

By symmetry, $I_y = \frac{1}{12}kabc(a^2 + c^2)$ and $I_z = \frac{1}{12}kabc(a^2 + b^2)$.

45. (a) $f(x, y, z)$ is a joint density function, so we know $\iiint_{\mathbb{R}^3} f(x, y, z)\,dV = 1$. Here we have

$$\iiint_{\mathbb{R}^3} f(x, y, z)\,dV = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y, z)\,dz\,dy\,dx = \int_0^2 \int_0^2 \int_0^2 Cxyz\,dz\,dy\,dx$$

$$= C\int_0^2 x\,dx \int_0^2 y\,dy \int_0^2 z\,dz = C\left[\frac{x^2}{2}\right]_0^2 \left[\frac{y^2}{2}\right]_0^2 \left[\frac{z^2}{2}\right]_0^2$$

$$= 8C$$

Then we must have $8C = 1 \quad \Rightarrow \quad C = \frac{1}{8}$.

(b) $P(X \leq 1, Y \leq 1, Z \leq 1) = \int_{-\infty}^{1} \int_{-\infty}^{1} \int_{-\infty}^{1} f(x, y, z)\,dz\,dy\,dx$

$$= \int_0^1 \int_0^1 \int_0^1 \tfrac{1}{8}xyz\,dz\,dy\,dx = \tfrac{1}{8}\int_0^1 x\,dx \int_0^1 y\,dy \int_0^1 z\,dz$$

$$= \frac{1}{8}\left[\frac{x^2}{2}\right]_0^1 \left[\frac{y^2}{2}\right]_0^1 \left[\frac{z^2}{2}\right]_0^1 = \tfrac{1}{8}\left(\tfrac{1}{2}\right)^3 = \tfrac{1}{64}$$

(c) $P(X + Y + Z \leq 1) = P((X, Y, Z) \in E)$ where E is the solid region in the first octant bounded by the coordinate planes and the plane $x + y + z = 1$. The plane $x + y + z = 1$ meets the xy-plane in the line $x + y = 1$, so we have

$$P(X + Y + Z \leq 1) = \iiint_E f(x, y, z)\,dV = \int_0^1 \int_0^{1-x} \int_0^{1-x-y} \tfrac{1}{8}xyz\,dz\,dy\,dx$$

$$= \tfrac{1}{8}\int_0^1 \int_0^{1-x} xy\left[\tfrac{1}{2}z^2\right]_{z=0}^{z=1-x-y} dy\,dx$$

$$= \tfrac{1}{16}\int_0^1 \int_0^{1-x} xy(1 - x - y)^2\,dy\,dx$$

$$= \tfrac{1}{16}\int_0^1 \int_0^{1-x} \left[(x^3 - 2x^2 + x)y + (2x^2 - 2x)y^2 + xy^3\right]dy\,dx$$

$$= \tfrac{1}{16}\int_0^1 \left[(x^3 - 2x^2 + x)\tfrac{1}{2}y^2 + (2x^2 - 2x)\tfrac{1}{3}y^3 + x\left(\tfrac{1}{4}y^4\right)\right]_{y=0}^{y=1-x} dx$$

$$= \tfrac{1}{192}\int_0^1 (x - 4x^2 + 6x^3 - 4x^4 + x^5)\,dx = \tfrac{1}{192}\left(\tfrac{1}{30}\right) = \tfrac{1}{5760}$$

46. (a) $f(x,y,z)$ is a joint density function, so we know $\iiint_{\mathbb{R}^3} f(x,y,z)\,dV = 1$. Here we have

$$\begin{aligned}
\iiint_{\mathbb{R}^3} f(x,y,z)\,dV &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y,z)\,dz\,dy\,dx \\
&= \int_0^{\infty}\int_0^{\infty}\int_0^{\infty} Ce^{-(0.5x+0.2y+0.1z)}\,dz\,dy\,dx \\
&= C\int_0^{\infty} e^{-0.5x}\,dx \int_0^{\infty} e^{-0.2y}\,dy \int_0^{\infty} e^{-0.1z}\,dz \\
&= C\lim_{t\to\infty}\int_0^t e^{-0.5x}\,dx \lim_{t\to\infty}\int_0^t e^{-0.2y}\,dy \lim_{t\to\infty}\int_0^t e^{-0.1z}\,dz \\
&= C\lim_{t\to\infty}\left[-2e^{-0.5x}\right]_0^t \lim_{t\to\infty}\left[-5e^{-0.2y}\right]_0^t \lim_{t\to\infty}\left[-10e^{-0.1z}\right]_0^t \\
&= C\lim_{t\to\infty}\left[-2(e^{-0.5t}-1)\right] \lim_{t\to\infty}\left[-5(e^{-0.2t}-1)\right] \lim_{t\to\infty}\left[-10(e^{-0.1t}-1)\right] \\
&= C\cdot(-2)(0-1)\cdot(-5)(0-1)\cdot(-10)(0-1) = 100C
\end{aligned}$$

So we must have $100C = 1 \Rightarrow C = \frac{1}{100}$.

(b) We have no restriction on Z, so

$$\begin{aligned}
P(X\le 1, Y\le 1) &= \int_{-\infty}^{1}\int_{-\infty}^{1}\int_{-\infty}^{\infty} f(x,y,z)\,dz\,dy\,dx \\
&= \int_0^1\int_0^1\int_0^{\infty} \tfrac{1}{100}e^{-(0.5x+0.2y+0.1z)}\,dz\,dy\,dx \\
&= \tfrac{1}{100}\int_0^1 e^{-0.5x}\,dx \int_0^1 e^{-0.2y}\,dy \int_0^{\infty} e^{-0.1z}\,dz \\
&= \tfrac{1}{100}\left[-2e^{-0.5x}\right]_0^1 \left[-5e^{-0.2y}\right]_0^1 \lim_{t\to\infty}\left[-10e^{-0.1z}\right]_0^t \qquad \text{[by part (a)]} \\
&= \tfrac{1}{100}(2-2e^{-0.5})(5-5e^{-0.2})(10) = (1-e^{-0.5})(1-e^{-0.2}) \approx 0.07132
\end{aligned}$$

(c)
$$\begin{aligned}
P(X\le 1, Y\le 1, Z\le 1) &= \int_{-\infty}^{1}\int_{-\infty}^{1}\int_{-\infty}^{1} f(x,y,z)\,dz\,dy\,dx \\
&= \int_0^1\int_0^1\int_0^1 \tfrac{1}{100}e^{-(0.5x+0.2y+0.1z)}\,dz\,dy\,dx \\
&= \tfrac{1}{100}\int_0^1 e^{-0.5x}\,dx \int_0^1 e^{-0.2y}\,dy \int_0^1 e^{-0.1z}\,dz \\
&= \tfrac{1}{100}\left[-2e^{-0.5x}\right]_0^1 \left[-5e^{-0.2y}\right]_0^1 \left[-10e^{-0.1z}\right]_0^1 \\
&= (1-e^{-0.5})(1-e^{-0.2})(1-e^{-0.1}) \approx 0.006787
\end{aligned}$$

47. $V(E) = L^3$,

$$\begin{aligned}
f_{\text{ave}} &= \frac{1}{L^3}\int_0^L\int_0^L\int_0^L xyz\,dx\,dy\,dz = \frac{1}{L^3}\int_0^L x\,dx \int_0^L y\,dy \int_0^L z\,dz \\
&= \frac{1}{L^3}\left[\frac{x^2}{2}\right]_0^L \left[\frac{y^2}{2}\right]_0^L \left[\frac{z^2}{2}\right]_0^L = \frac{1}{L^3}\,\frac{L^2}{2}\,\frac{L^2}{2}\,\frac{L^2}{2} = \frac{L^3}{8}
\end{aligned}$$

48. $V(E) = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}\int_0^{1-x^2-y^2} dz\,dy\,dx = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}(1-x^2-y^2)\,dy\,dx$

$$= \int_0^{2\pi}\int_0^1 (1-r^2)\,r\,dr\,d\theta = \int_0^{2\pi} d\theta \int_0^1 (r-r^3)\,dr = 2\pi\left(\tfrac{r^2}{2}-\tfrac{r^4}{4}\right)\Big]_0^1 = \tfrac{\pi}{2}.$$

Then $f_{\text{ave}} = \frac{1}{\pi/2} \iiint_E (x^2 z + y^2 z)\, dV = \frac{2}{\pi} \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_0^{1-x^2-y^2} (x^2 + y^2)\, z\, dz\, dy\, dx$

$$= \tfrac{2}{\pi} \int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} (x^2 + y^2) \cdot \tfrac{1}{2}(1 - x^2 - y^2)^2\, dy\, dx = \tfrac{1}{\pi} \int_0^{2\pi} \int_0^1 r^2 (1 - r^2)^2\, r\, dr\, d\theta$$

$$= \tfrac{1}{\pi} \int_0^{2\pi} d\theta \int_0^1 (r^3 - 2r^5 + r^7)\, dr = \tfrac{1}{\pi}(2\pi) \left[\tfrac{1}{4}r^4 - \tfrac{1}{3}r^6 + \tfrac{1}{8}r^8\right]_0^1$$

$$= 2\left(\tfrac{1}{24}\right) = \tfrac{1}{12}$$

49. The triple integral will attain its maximum when the integrand $1 - x^2 - 2y^2 - 3z^2$ is positive in the region E and negative everywhere else. For if E contains some region F where the integrand is negative, the integral could be increased by excluding F from E, and if E fails to contain some part G of the region where the integrand is positive, the integral could be increased by including G in E. So we require that $x^2 + 2y^2 + 3z^2 \le 1$. This describes the region bounded by the ellipsoid $x^2 + 2y^2 + 3z^2 = 1$.

DISCOVERY PROJECT Volumes of Hyperspheres

In this project we use V_n to denote the n-dimensional volume of an n-dimensional hypersphere.

1. The interior of the circle is the set of points $\left\{(x, y) \mid -r \le y \le r,\ -\sqrt{r^2 - y^2} \le x \le \sqrt{r^2 - y^2}\right\}$. So, substituting $y = r\sin\theta$ and then using Formula 64 to evaluate the integral, we get

$$V_2 = \int_{-r}^{r} \int_{-\sqrt{r^2-y^2}}^{\sqrt{r^2-y^2}} dx\, dy = \int_{-r}^{r} 2\sqrt{r^2 - y^2}\, dy = \int_{-\pi/2}^{\pi/2} 2r\sqrt{1 - \sin^2\theta}\, (r\cos\theta\, d\theta)$$

$$= 2r^2 \int_{-\pi/2}^{\pi/2} \cos^2\theta\, d\theta = 2r^2 \left[\tfrac{1}{2}\theta + \tfrac{1}{4}\sin 2\theta\right]_{-\pi/2}^{\pi/2} = 2r^2\left(\tfrac{\pi}{2}\right) = \pi r^2$$

2. The region of integration is

$\left\{(x, y, z) \mid -r \le z \le r, -\sqrt{r^2 - z^2} \le y \le \sqrt{r^2 - z^2}, -\sqrt{r^2 - z^2 - y^2} \le x \le \sqrt{r^2 - z^2 - y^2}\right\}$. Substituting $y = \sqrt{r^2 - z^2}\sin\theta$ and using Formula 64 to integrate $\cos^2\theta$, we get

$$V_3 = \int_{-r}^{r} \int_{-\sqrt{r^2-z^2}}^{\sqrt{r^2-z^2}} \int_{-\sqrt{r^2-z^2-y^2}}^{\sqrt{r^2-z^2-y^2}} dx\, dy\, dz = \int_{-r}^{r} \int_{-\sqrt{r^2-z^2}}^{\sqrt{r^2-z^2}} 2\sqrt{r^2 - z^2 - y^2}\, dy\, dz$$

$$= \int_{-r}^{r} \int_{-\pi/2}^{\pi/2} 2\sqrt{r^2 - z^2}\, \sqrt{1 - \sin^2\theta}\, \left(\sqrt{r^2 - z^2}\cos\theta\, d\theta\right) dz$$

$$= 2\left[\int_{-r}^{r} (r^2 - z^2)\, dz\right] \left[\int_{-\pi/2}^{\pi/2} \cos^2\theta\, d\theta\right] = 2\left(\frac{4r^3}{3}\right)\left(\frac{\pi}{2}\right) = \frac{4\pi r^3}{3}$$

3. Here we substitute $y = \sqrt{r^2 - w^2 - z^2}\sin\theta$ and, later, $w = r\sin\phi$. Because $\int_{-\pi/2}^{\pi/2} \cos^p\theta\, d\theta$ seems to occur frequently in these calculations, it is useful to find a general formula for that integral. From Exercises 43 and 44 in Section 8.1 [ET 7.1], we have

$$\int_0^{\pi/2} \sin^{2k} x\, dx = \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2k - 1)}{2 \cdot 4 \cdot 6 \cdot \cdots \cdot 2k} \frac{\pi}{2} \text{ and } \int_0^{\pi/2} \sin^{2k+1} x\, dx = \frac{2 \cdot 4 \cdot 6 \cdot \cdots \cdot 2k}{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2k + 1)}$$

and from the symmetry of the sine and cosine functions, we can conclude that

$$\int_{-\pi/2}^{\pi/2} \cos^{2k} x\,dx = 2\int_0^{\pi/2} \sin^{2k} x\,dx = \frac{1\cdot 3\cdot 5\cdot \cdots \cdot (2k-1)\pi}{2\cdot 4\cdot 6\cdot \cdots \cdot 2k} \tag{1}$$

$$\int_{-\pi/2}^{\pi/2} \cos^{2k+1} x\,dx = 2\int_0^{\pi/2} \sin^{2k+1} x\,dx = \frac{2\cdot 2\cdot 4\cdot 6\cdot \cdots \cdot 2k}{1\cdot 3\cdot 5\cdot \cdots \cdot (2k+1)} \tag{2}$$

Thus
$$\begin{aligned}
V_4 &= \int_{-r}^{r}\int_{-\sqrt{r^2-w^2}}^{\sqrt{r^2-w^2}}\int_{-\sqrt{r^2-w^2-z^2}}^{\sqrt{r^2-w^2-z^2}}\int_{-\sqrt{r^2-w^2-z^2-y^2}}^{\sqrt{r^2-w^2-z^2-y^2}} dx\,dy\,dz\,dw \\
&= 2\int_{-r}^{r}\int_{-\sqrt{r^2-w^2}}^{\sqrt{r^2-w^2}}\int_{-\sqrt{r^2-w^2-z^2}}^{\sqrt{r^2-w^2-z^2}} \sqrt{r^2-w^2-z^2-y^2}\,dy\,dz\,dw \\
&= 2\int_{-r}^{r}\int_{-\sqrt{r^2-w^2}}^{\sqrt{r^2-w^2}} \textstyle\int_{-\pi/2}^{\pi/2}(r^2-w^2-z^2)\cos^2\theta\,d\theta\,dz\,dw \\
&= 2\left[\int_{-r}^{r}\int_{-\sqrt{r^2-w^2}}^{\sqrt{r^2-w^2}}(r^2-w^2-z^2)\,dz\,dw\right]\left[\int_{-\pi/2}^{\pi/2}\cos^2\theta\,d\theta\right] \\
&= 2\left(\tfrac{\pi}{2}\right)\left[\int_{-r}^{r}\tfrac{4}{3}(r^2-w^2)^{3/2}\,dw\right] = \pi\left(\tfrac{4}{3}\right)\textstyle\int_{-\pi/2}^{\pi/2} r^4\cos^4\phi\,d\phi = \dfrac{4\pi}{3}r^4\cdot\dfrac{1\cdot 3\cdot \pi}{2\cdot 4} = \dfrac{\pi^2 r^4}{2}
\end{aligned}$$

4. By using the substitutions $x_i = \sqrt{r^2 - x_n^2 - x_{n-1}^2 - \cdots - x_{i+1}^2}\cos\theta_i$ and then applying Formulas 1 and 2 from Problem 3, we can write

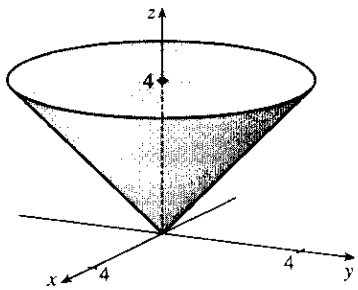
$$\begin{aligned}
V_n &= \int_{-r}^{r}\int_{-\sqrt{r^2-x_n^2}}^{\sqrt{r^2-x_n^2}}\cdots\int_{-\sqrt{r^2-x_n^2-x_{n-1}^2-\cdots-x_3^2}}^{\sqrt{r^2-x_n^2-x_{n-1}^2-\cdots-x_3^2}}\int_{-\sqrt{r^2-x_n^2-x_{n-1}^2-\cdots-x_3^2-x_2^2}}^{\sqrt{r^2-x_n^2-x_{n-1}^2-\cdots-x_3^2-x_2^2}} dx_1\,dx_2\cdots dx_{n-1}\,dx_n \\
&= 2\left[\int_{-\pi/2}^{\pi/2}\cos^2\theta_2\,d\theta_2\right]\left[\int_{-\pi/2}^{\pi/2}\cos^3\theta_3\,d\theta_3\right]\cdots\left[\int_{-\pi/2}^{\pi/2}\cos^{n-1}\theta_{n-1}\,d\theta_{n-1}\right]\left[\int_{-\pi/2}^{\pi/2}\cos^n\theta_n\,d\theta_n\right]r^n \\
&= \begin{cases}
\left[2\cdot\dfrac{\pi}{2}\right]\left[\dfrac{2\cdot 2}{1\cdot 3}\cdot\dfrac{1\cdot 3\pi}{2\cdot 4}\right]\left[\dfrac{2\cdot 2\cdot 4}{1\cdot 3\cdot 5}\cdot\dfrac{1\cdot 3\cdot 5\pi}{2\cdot 4\cdot 6}\right]\cdots\left[\dfrac{2\cdot\cdots\cdot(n-2)}{1\cdot\cdots\cdot(n-1)}\cdot\dfrac{1\cdot\cdots\cdot(n-1)\,\pi}{2\cdot\cdots\cdot n}\right]r^n & n \text{ even} \\
2\left[\dfrac{\pi}{2}\cdot\dfrac{2\cdot 2}{1\cdot 3}\right]\left[\dfrac{1\cdot 3\pi}{2\cdot 4}\cdot\dfrac{2\cdot 2\cdot 4}{1\cdot 3\cdot 5}\right]\cdots\left[\dfrac{1\cdot\cdots\cdot(n-2)\,\pi}{2\cdot\cdots\cdot(n-1)}\cdot\dfrac{2\cdot\cdots\cdot(n-1)}{1\cdot\cdots\cdot n}\right]r^n & n \text{ odd}
\end{cases}
\end{aligned}$$

By canceling within each set of brackets, we find that

$$V_n = \begin{cases}
\dfrac{2\pi}{2}\cdot\dfrac{2\pi}{4}\cdot\dfrac{2\pi}{6}\cdot\cdots\cdot\dfrac{2\pi}{n}r^n = \dfrac{(2\pi)^{n/2}}{2\cdot 4\cdot 6\cdot\cdots\cdot n}r^n = \dfrac{\pi^{n/2}}{\left(\frac{1}{2}n\right)!}r^n & n \text{ even} \\
2\cdot\dfrac{2\pi}{3}\cdot\dfrac{2\pi}{5}\cdot\dfrac{2\pi}{7}\cdot\cdots\cdot\dfrac{2\pi}{n}r^n = \dfrac{2(2\pi)^{(n-1)/2}}{3\cdot 5\cdot 7\cdot\cdots\cdot n}r^n = \dfrac{2^n\left[\frac{1}{2}(n-1)\right]!\,\pi^{(n-1)/2}}{n!}r^n & n \text{ odd}
\end{cases}$$

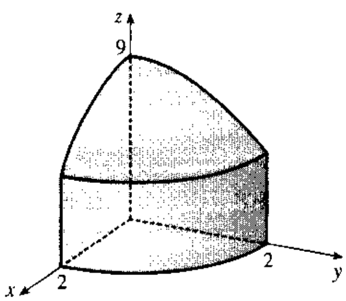
16.8 Triple Integrals in Cylindrical and Spherical Coordinates ET 15.8

1. The region of integration is given in cylindrical coordinates by $E = \{(r, \theta, z) \mid 0 \le \theta \le 2\pi, 0 \le r \le 4, r \le z \le 4\}$. This represents the solid region bounded below by the cone $z = r$ and above by the horizontal plane $z = 4$.



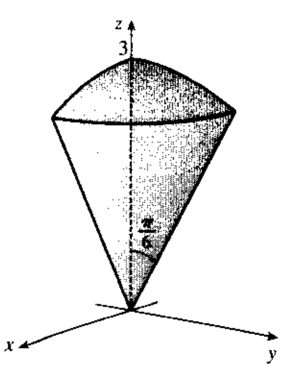
$$\begin{aligned}\int_0^4 \int_0^{2\pi} \int_r^4 r\,dz\,d\theta\,dr &= \int_0^4 \int_0^{2\pi} \big[rz\big]_{z=r}^{z=4}\,d\theta\,dr \\ &= \int_0^4 \int_0^{2\pi} r(4-r)\,d\theta\,dr \\ &= \int_0^4 (4r - r^2)\,dr \int_0^{2\pi} d\theta \\ &= \left[2r^2 - \tfrac{1}{3}r^3\right]_0^4 \big[\theta\big]_0^{2\pi} \\ &= \left(32 - \tfrac{64}{3}\right)(2\pi) = \tfrac{64\pi}{3}\end{aligned}$$

2. The region of integration is given in cylindrical coordinates by $E = \{(r, \theta, z) \mid 0 \le \theta \le \pi/2, 0 \le r \le 2, 0 \le z \le 9 - r^2\}$. This represents the solid region in the first octant enclosed by the circular cylinder $r = 2$, bounded above by $z = 9 - r^2$, a circular paraboloid, and bounded below by the xy-plane.



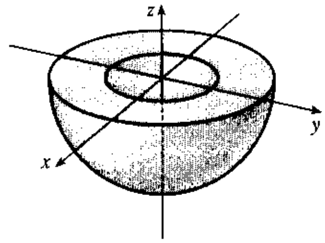
$$\begin{aligned}\int_0^{\pi/2} \int_0^2 \int_0^{9-r^2} r\,dz\,dr\,d\theta &= \int_0^{\pi/2} \int_0^2 \big[rz\big]_{z=0}^{z=9-r^2}\,dr\,d\theta \\ &= \int_0^{\pi/2} \int_0^2 r(9-r^2)\,dr\,d\theta \\ &= \int_0^{\pi/2} d\theta \int_0^2 (9r - r^3)\,dr \\ &= \big[\theta\big]_0^{\pi/2} \left[\tfrac{9}{2}r^2 - \tfrac{1}{4}r^4\right]_0^2 \\ &= \tfrac{\pi}{2}(18 - 4) = 7\pi\end{aligned}$$

3. The region of integration is given in spherical coordinates by $E = \{(\rho, \theta, \phi) \mid 0 \le \rho \le 3, 0 \le \theta \le \pi/2, 0 \le \phi \le \pi/6\}$. This represents the solid region in the first octant bounded above by the sphere $\rho = 3$ and below by the cone $\phi = \pi/6$.



$$\begin{aligned}\int_0^{\pi/6} \int_0^{\pi/2} \int_0^3 \rho^2 \sin\phi\,d\rho\,d\theta\,d\phi &= \int_0^{\pi/6} \sin\phi\,d\phi \int_0^{\pi/2} d\theta \int_0^3 \rho^2\,d\rho \\ &= \big[-\cos\phi\big]_0^{\pi/6} \big[\theta\big]_0^{\pi/2} \left[\tfrac{1}{3}\rho^3\right]_0^3 \\ &= \left(1 - \frac{\sqrt{3}}{2}\right)\left(\frac{\pi}{2}\right)(9) \\ &= \frac{9\pi}{4}\left(2 - \sqrt{3}\right)\end{aligned}$$

4. The region of integration is given in spherical coordinates by $E = \{(\rho, \theta, \phi) \mid 1 \le \rho \le 2, 0 \le \theta \le 2\pi, \pi/2 \le \phi \le \pi\}$. This represents the solid region between the spheres $\rho = 1$ and $\rho = 2$ and below the xy-plane.



$$\int_0^{2\pi}\int_{\pi/2}^{\pi}\int_1^2 \rho^2 \sin\phi \, d\rho\, d\phi\, d\theta = \int_0^{2\pi} d\theta \int_{\pi/2}^{\pi} \sin\phi\, d\phi \int_1^2 \rho^2\, d\rho = \big[\theta\big]_0^{2\pi}\big[-\cos\phi\big]_{\pi/2}^{\pi}\big[\tfrac{1}{3}\rho^3\big]_1^2 = 2\pi(1)\left(\tfrac{7}{3}\right) = \tfrac{14\pi}{3}$$

5. The solid E is most conveniently described if we use cylindrical coordinates: $E = \left\{(r, \theta, z) \mid 0 \le \theta \le \frac{\pi}{2}, 0 \le r \le 3, 0 \le z \le 2\right\}$. Then

$$\iiint_E f(x,y,z)\, dV = \int_0^{\pi/2}\int_0^3\int_0^2 f(r\cos\theta, r\sin\theta, z)\, r\, dz\, dr\, d\theta.$$

6. The solid E is most conveniently described if we use spherical coordinates: $E = \left\{(\rho, \theta, \phi) \mid 1 \le \rho \le 2, \frac{\pi}{2} \le \theta \le 2\pi, 0 \le \phi \le \frac{\pi}{2}\right\}$. Then

$$\iiint_E f(x,y,z)\, dV = \int_0^{\pi/2}\int_{\pi/2}^{2\pi}\int_1^2 f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\, \rho^2\sin\phi\, d\rho\, d\theta\, d\phi.$$

7. In cylindrical coordinates, E is given by $\{(r, \theta, z) \mid 0 \le \theta \le 2\pi, 0 \le r \le 4, -5 \le z \le 4\}$. So

$$\iiint_E \sqrt{x^2+y^2}\, dV = \int_0^{2\pi}\int_0^4\int_{-5}^4 \sqrt{r^2}\, r\, dz\, dr\, d\theta = \int_0^{2\pi} d\theta \int_0^4 r^2\, dr \int_{-5}^4 dz = \big[\theta\big]_0^{2\pi}\big[\tfrac{1}{3}r^3\big]_0^4\big[z\big]_{-5}^4 = (2\pi)\left(\tfrac{64}{3}\right)(9) = 384\pi$$

8. The paraboloid $z = 1 - x^2 - y^2$ intersects the xy-plane in the circle $x^2 + y^2 = r^2 = 1$ or $r = 1$, so in cylindrical coordinates, E is given by $\left\{(r, \theta, z) \,\middle|\, 0 \le \theta \le \frac{\pi}{2}, 0 \le r \le 1, 0 \le z \le 1 - r^2\right\}$. Thus

$$\begin{aligned}\iiint_E (x^3 + xy^2)\, dV &= \int_0^{\pi/2}\int_0^1\int_0^{1-r^2} (r^3\cos^3\theta + r^3\cos\theta\sin^2\theta)\, r\, dz\, dr\, d\theta \\ &= \int_0^{\pi/2}\int_0^1\int_0^{1-r^2} r^4\cos\theta\, dz\, dr\, d\theta = \int_0^{\pi/2}\int_0^1 r^4\cos\theta\,\big[z\big]_{z=0}^{z=1-r^2} dr\, d\theta \\ &= \int_0^{\pi/2}\int_0^1 r^4(1-r^2)\cos\theta\, dr\, d\theta = \int_0^{\pi/2}\cos\theta\left[\tfrac{1}{5}r^5 - \tfrac{1}{7}r^7\right]_{r=0}^{r=1} d\theta \\ &= \int_0^{\pi/2} \tfrac{2}{35}\cos\theta\, d\theta = \tfrac{2}{35}\big[\sin\theta\big]_0^{\pi/2} = \tfrac{2}{35}\end{aligned}$$

9. In cylindrical coordinates E is bounded by the paraboloid $z = 1 + r^2$, the cylinder $r^2 = 5$ or $r = \sqrt{5}$, and the xy-plane, so E is given by $\left\{(r, \theta, z) \mid 0 \le \theta \le 2\pi, 0 \le r \le \sqrt{5}, 0 \le z \le 1 + r^2\right\}$. Thus

$$\begin{aligned}\iiint_E e^z\, dV &= \int_0^{2\pi}\int_0^{\sqrt{5}}\int_0^{1+r^2} e^z\, r\, dz\, dr\, d\theta = \int_0^{2\pi}\int_0^{\sqrt{5}} r\big[e^z\big]_{z=0}^{z=1+r^2} dr\, d\theta = \int_0^{2\pi}\int_0^{\sqrt{5}} r(e^{1+r^2} - 1)\, dr\, d\theta \\ &= \int_0^{2\pi} d\theta \int_0^{\sqrt{5}} \left(re^{1+r^2} - r\right) dr = 2\pi\left[\tfrac{1}{2}e^{1+r^2} - \tfrac{1}{2}r^2\right]_0^{\sqrt{5}} = \pi(e^6 - e - 5)\end{aligned}$$

10. In cylindrical coordinates E is bounded by the planes $z = 0$, $z = r\cos\theta + r\sin\theta + 3$ and the cylinders $r = 2$ and $r = 3$, so E is given by $\{(r,\theta,z) \mid 0 \le \theta \le 2\pi, 2 \le r \le 3, 0 \le z \le r\cos\theta + r\sin\theta + 3\}$. Thus

$$\begin{aligned}
\iiint_E x\,dV &= \int_0^{2\pi}\int_2^3\int_0^{r\cos\theta + r\sin\theta+3}(r\cos\theta)\,r\,dz\,dr\,d\theta \\
&= \int_0^{2\pi}\int_2^3 (r^2\cos\theta)\big[z\big]_{z=0}^{z=r\cos\theta+r\sin\theta+3}\,dr\,d\theta = \int_0^{2\pi}\int_2^3 (r^2\cos\theta)(r\cos\theta + r\sin\theta + 3)\,dr\,d\theta \\
&= \int_0^{2\pi}\int_2^3 (r^3(\cos^2\theta + \cos\theta\sin\theta) + 3r^2\cos\theta)\,dr\,d\theta \\
&= \int_0^{2\pi}\left[\tfrac{1}{4}r^4(\cos^2\theta + \cos\theta\sin\theta) + r^3\cos\theta\right]_{r=2}^{r=3} d\theta \\
&= \int_0^{2\pi}\left[\left(\tfrac{81}{4} - \tfrac{16}{4}\right)(\cos^2\theta + \cos\theta\sin\theta) + (27-8)\cos\theta\right] d\theta \\
&= \int_0^{2\pi}\left(\tfrac{65}{4}\left(\tfrac{1}{2}(1+\cos 2\theta) + \cos\theta\sin\theta\right) + 19\cos\theta\right) d\theta \\
&= \left[\tfrac{65}{8}\theta + \tfrac{65}{16}\sin 2\theta + \tfrac{65}{8}\sin^2\theta + 19\sin\theta\right]_0^{2\pi} = \tfrac{65}{4}\pi
\end{aligned}$$

11. In cylindrical coordinates, E is bounded by the cylinder $r = 1$, the plane $z = 0$, and the cone $z = 2r$. So $E = \{(r,\theta,z) \mid 0 \le \theta \le 2\pi, 0 \le r \le 1, 0 \le z \le 2r\}$ and

$$\begin{aligned}
\iiint_E x^2\,dV &= \int_0^{2\pi}\int_0^1\int_0^{2r} r^2\cos^2\theta\,r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^1\left[r^3\cos^2\theta\,z\right]_{z=0}^{z=2r} dr\,d\theta \\
&= \int_0^{2\pi}\int_0^1 2r^4\cos^2\theta\,dr\,d\theta = \int_0^{2\pi}\left[\tfrac{2}{5}r^5\cos^2\theta\right]_{r=0}^{r=1} d\theta = \tfrac{2}{5}\int_0^{2\pi}\cos^2\theta\,d\theta \\
&= \frac{2}{5}\int_0^{2\pi}\frac{1+\cos 2\theta}{2}\,d\theta = \frac{1}{5}\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{2\pi} = \frac{2\pi}{5}
\end{aligned}$$

12. In cylindrical coordinates E is the solid region within the cylinder $r = 1$ bounded above and below by the sphere $r^2 + z^2 = 4$, so $E = \left\{(r,\theta,z) \mid 0 \le \theta \le 2\pi, 0 \le r \le 1, -\sqrt{4-r^2} \le z \le \sqrt{4-r^2}\right\}$. Thus the volume is

$$\begin{aligned}
\iiint_E dV &= \int_0^{2\pi}\int_0^1\int_{-\sqrt{4-r^2}}^{\sqrt{4-r^2}} r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^1 2r\sqrt{4-r^2}\,dr\,d\theta \\
&= \int_0^{2\pi} d\theta \int_0^1 2r\sqrt{4-r^2}\,dr = 2\pi\left[-\tfrac{2}{3}(4-r^2)^{3/2}\right]_0^1 = \tfrac{4}{3}\pi(8 - 3^{3/2})
\end{aligned}$$

13. (a) The paraboloids intersect when $x^2 + y^2 = 36 - 3x^2 - 3y^2 \;\Rightarrow\; x^2 + y^2 = 9$, so the region of integration is $D = \{(x,y) \mid x^2 + y^2 \le 9\}$. Then, in cylindrical coordinates, $E = \{(r,\theta,z) \mid r^2 \le z \le 36 - 3r^2, 0 \le r \le 3, 0 \le \theta \le 2\pi\}$ and

$$\begin{aligned}
V &= \int_0^{2\pi}\int_0^3\int_{r^2}^{36-3r^2} r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^3 (36r - 4r^3)\,dr\,d\theta \\
&= \int_0^{2\pi}\left[18r^2 - r^4\right]_{r=0}^{r=3} d\theta = \int_0^{2\pi} 81\,d\theta = 162\pi
\end{aligned}$$

(b) For constant density K, $m = KV = 162\pi K$ from part (a). Since the region is homogeneous and symmetric, $M_{yz} = M_{xz} = 0$ and

$$\begin{aligned}
M_{xy} &= \int_0^{2\pi}\int_0^3\int_{r^2}^{36-3r^2}(zK)\,r\,dz\,dr\,d\theta = K\int_0^{2\pi}\int_0^3 r\left[\tfrac{1}{2}z^2\right]_{z=r^2}^{z=36-3r^2} dr\,d\theta \\
&= \tfrac{K}{2}\int_0^{2\pi}\int_0^3 r((36-3r^2)^2 - r^4)\,dr\,d\theta = \tfrac{K}{2}\int_0^{2\pi} d\theta\int_0^3(8r^5 - 216r^3 + 1296r)\,dr \\
&= \tfrac{K}{2}(2\pi)\left[\tfrac{8}{6}r^6 - \tfrac{216}{4}r^4 + \tfrac{1296}{2}r^2\right]_0^3 = \pi K(2430) = 2430\pi K
\end{aligned}$$

Thus $(\bar{x},\bar{y},\bar{z}) = \left(\dfrac{M_{yz}}{m}, \dfrac{M_{xz}}{m}, \dfrac{M_{xy}}{m}\right) = \left(0, 0, \tfrac{2430\pi K}{162\pi K}\right) = (0,0,15)$.

14. (a) $V = \int_{-\pi/2}^{\pi/2}\int_0^{a\cos\theta}\int_{-\sqrt{a^2-r^2}}^{\sqrt{a^2-r^2}} r\,dz\,dr\,d\theta$

$= 4\int_0^{\pi/2}\int_0^{a\cos\theta}\int_0^{\sqrt{a^2-r^2}} r\,dz\,dr\,d\theta$

$= 4\int_0^{\pi/2}\int_0^{a\cos\theta} r\sqrt{a^2-r^2}\,dr\,d\theta$

$= -\frac{4}{3}\int_0^{\pi/2}\left[\left(a^2-r^2\right)^{3/2}\right]_{r=0}^{r=a\cos\theta}\,d\theta$

$= -\frac{4}{3}\int_0^{\pi/2}\left[\left(a^2-a^2\cos^2\theta\right)^{3/2}-a^3\right]d\theta$

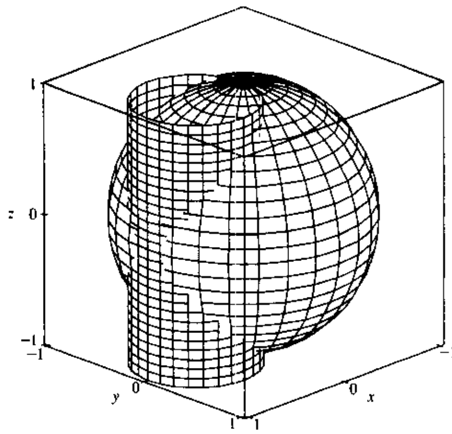
$= -\frac{4}{3}\int_0^{\pi/2}\left[\left(a^2\sin^2\theta\right)^{3/2}-a^3\right]d\theta$

$= -\frac{4}{3}\int_0^{\pi/2}\left(a^3\sin^3\theta-a^3\right)d\theta$

$= -\dfrac{4a^3}{3}\displaystyle\int_0^{\pi/2}\left[\sin\theta\left(1-\cos^2\theta\right)-1\right]d\theta$

$= -\dfrac{4a^3}{3}\left[-\cos\theta+\frac{1}{3}\cos^3\theta-\theta\right]_0^{\pi/2} = -\dfrac{4a^3}{3}\left(-\frac{\pi}{2}+\frac{2}{3}\right) = \frac{2}{9}a^3(3\pi-4)$

(b)



To plot the cylinder and the sphere on the same screen in Maple, we can use the sequence of commands

```
sphere:=plot3d(1,theta=0..2*Pi,phi=0..Pi,coords=spherical):
cylinder:=plot3d([cos(theta),theta,z],
theta=0..2*Pi,z=-1..1,coords=cylindrical):
with(plots):  display3d({sphere,cylinder});
```

In Mathematica, we can use

```
sphere=SphericalPlot3d[1,{theta,0,2Pi},{phi,0,Pi}]
cylinder=ParametricPlot3d[{Sin[theta],Cos[theta],z},
{theta,0,2Pi},{z,-1,1}]
Show[{sphere,cylinder}]
```

15. The paraboloid $z = 4x^2+4y^2$ intersects the plane $z = a$ when $a = 4x^2+4y^2$ or $x^2+y^2 = \frac{1}{4}a$. So, in cylindrical coordinates, $E = \left\{(r,\theta,z) \mid 0 \le r \le \frac{1}{2}\sqrt{a}, 0 \le \theta \le 2\pi, 4r^2 \le z \le a\right\}$. Thus

$$m = \int_0^{2\pi}\int_0^{\sqrt{a}/2}\int_{4r^2}^{a} Kr\,dz\,dr\,d\theta = K\int_0^{2\pi}\int_0^{\sqrt{a}/2}\left(ar-4r^3\right)dr\,d\theta$$

$$= K\int_0^{2\pi}\left[\tfrac{1}{2}ar^2-r^4\right]_{r=0}^{r=\sqrt{a}/2}d\theta = K\int_0^{2\pi}\tfrac{1}{16}a^2\,d\theta = \tfrac{1}{8}a^2\pi K$$

Since the region is homogeneous and symmetric, $M_{yz} = M_{xz} = 0$ and

$$M_{xy} = \int_0^{2\pi}\int_0^{\sqrt{a}/2}\int_{4r^2}^{a} Krz\,dz\,dr\,d\theta = K\int_0^{2\pi}\int_0^{\sqrt{a}/2}\left(\tfrac{1}{2}a^2r-8r^5\right)dr\,d\theta$$

$$= K\int_0^{2\pi}\left[\tfrac{1}{4}a^2r^2-\tfrac{4}{3}r^6\right]_{r=0}^{r=\sqrt{a}/2}d\theta = K\int_0^{2\pi}\tfrac{1}{24}a^3\,d\theta = \tfrac{1}{12}a^3\pi K$$

Hence $(\overline{x},\overline{y},\overline{z}) = \left(0,0,\frac{2}{3}a\right)$.

16. Since density is proportional to the distance from the z-axis, we can say $\rho(x,y,z) = K\sqrt{x^2+y^2}$. Then

$m = 2\int_0^{2\pi}\int_0^{a}\int_0^{\sqrt{a^2-r^2}} Kr^2\,dz\,dr\,d\theta = 2K\int_0^{2\pi}\int_0^{a} r^2\sqrt{a^2-r^2}\,dr\,d\theta$

$= 2K\int_0^{2\pi}\left[\frac{1}{8}r\left(2r^2-a^2\right)\sqrt{a^2-r^2}+\frac{1}{8}a^4\sin^{-1}(r/a)\right]_{r=0}^{r=a}d\theta$

$= 2K\int_0^{2\pi}\left[\left(\frac{1}{8}a^4\right)\left(\frac{\pi}{2}\right)\right]d\theta = \frac{1}{4}a^4\pi^2K.$

17. In spherical coordinates, B is represented by $\{(\rho,\theta,\phi)\mid 0\le\rho\le 1, 0\le\theta\le 2\pi, 0\le\phi\le\pi\}$. Thus

$$\iiint_B (x^2+y^2+z^2)\,dV = \int_0^\pi\int_0^{2\pi}\int_0^1 (\rho^2)\rho^2\sin\phi\,d\rho\,d\theta\,d\phi = \int_0^\pi \sin\phi\,d\phi\,\int_0^{2\pi}d\theta\,\int_0^1\rho^4\,d\rho$$
$$= \big[-\cos\phi\big]_0^\pi\,\big[\theta\big]_0^{2\pi}\,\big[\tfrac15\rho^5\big]_0^1 = (2)(2\pi)\big(\tfrac15\big) = \tfrac{4\pi}{5}$$

18. In spherical coordinates, H is represented by $\{(\rho,\theta,\phi)\mid 0\le\rho\le 1, 0\le\theta\le 2\pi, 0\le\phi\le\frac{\pi}{2}\}$. Thus

$$\iiint_H (x^2+y^2)\,dV = \int_0^{2\pi}\int_0^{\pi/2}\int_0^1 (\rho^2\sin^2\phi)\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta = \int_0^{2\pi}d\theta\,\int_0^{\pi/2}\sin^3\phi\,d\phi\,\int_0^1\rho^4\,d\rho$$
$$= \big[\theta\big]_0^{2\pi}\,\big[-\cos\phi+\tfrac13\cos^3\phi\big]_0^{\pi/2}\,\big[\tfrac15\rho^5\big]_0^1 = \tfrac{4\pi}{15}$$

19. In spherical coordinates, E is represented by $\{(\rho,\theta,\phi)\mid 1\le\rho\le 2, 0\le\theta\le\frac{\pi}{2}, 0\le\phi\le\frac{\pi}{2}\}$. Thus

$$\iiint_E z\,dV = \int_0^{\pi/2}\int_0^{\pi/2}\int_1^2 (\rho\cos\phi)\,\rho^2\sin\phi\,d\rho\,d\theta\,d\phi$$
$$= \int_0^{\pi/2}\cos\phi\sin\phi\,d\phi\int_0^{\pi/2}d\theta\int_1^2\rho^3\,d\rho = \big[\tfrac12\sin^2\phi\big]_0^{\pi/2}\,\big[\theta\big]_0^{\pi/2}\,\big[\tfrac14\rho^4\big]_1^2$$
$$= \big(\tfrac12\big)\big(\tfrac{\pi}{2}\big)\big(\tfrac{15}{4}\big) = \tfrac{15\pi}{16}$$

20. $$\iiint_E e^{\sqrt{x^2+y^2+z^2}}\,dV = \int_0^{\pi/2}\int_0^{\pi/2}\int_0^3 e^{\sqrt{\rho^2}}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta = \int_0^{\pi/2}\int_0^{\pi/2}\int_0^3\rho^2e^\rho\sin\phi\,d\rho\,d\phi\,d\theta$$
$$= \int_0^{\pi/2}d\theta\int_0^{\pi/2}\sin\phi\,d\phi\int_0^3\rho^2e^\rho\,d\rho = \big[\theta\big]_0^{\pi/2}\,\big[-\cos\phi\big]_0^{\pi/2}\,\big[(\rho^2-2\rho+2)e^\rho\big]_0^3$$
[integrate by parts twice]
$$= \tfrac{\pi}{2}(0+1)(5e^3-2) = \tfrac{\pi}{2}(5e^3-2)$$

21. $$\iiint_E x^2\,dV = \int_0^\pi\int_0^\pi\int_3^4(\rho\sin\phi\cos\theta)^2\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$
$$= \int_0^\pi\cos^2\theta\,d\theta\int_0^\pi\sin^3\phi\,d\phi\int_3^4\rho^4\,d\rho$$
$$= \big[\tfrac12\theta+\tfrac14\sin2\theta\big]_0^\pi\,\big[-\tfrac13(2+\sin^2\phi)\cos\phi\big]_0^\pi\,\big[\tfrac15\rho^5\big]_3^4$$
$$= \big(\tfrac{\pi}{2}\big)\big(\tfrac23+\tfrac23\big)\tfrac15(4^5-3^5) = \tfrac{1562}{15}\pi$$

22. $$\iiint_E xyz\,dV = \int_0^{\pi/3}\int_0^{2\pi}\int_2^4(\rho\sin\phi\cos\theta)(\rho\sin\phi\sin\theta)(\rho\cos\phi)\,\rho^2\sin\phi\,d\rho\,d\theta\,d\phi$$
$$= \int_0^{\pi/3}\sin^3\phi\cos\phi\,d\phi\int_0^{2\pi}\sin\theta\cos\theta\,d\theta\int_2^4\rho^5\,d\rho = \big[\tfrac14\sin^4\phi\big]_0^{\pi/3}\,\big[\tfrac12\sin^2\theta\big]_0^{2\pi}\,\big[\tfrac16\rho^6\big]_2^4 = 0$$

23. Since $\rho = 4\cos\phi$ implies $\rho^2 = 4\rho\cos\phi$, the equation is that of a sphere of radius 2 with center at $(0,0,2)$. Thus

$$V = \int_0^{2\pi}\int_0^{\pi/3}\int_0^{4\cos\phi}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta = \int_0^{2\pi}\int_0^{\pi/3}\big[\tfrac13\rho^3\big]_{\rho=0}^{\rho=4\cos\phi}\sin\phi\,d\phi\,d\theta$$
$$= \int_0^{2\pi}\int_0^{\pi/3}\big(\tfrac{64}{3}\cos^3\phi\big)\sin\phi\,d\phi\,d\theta = \int_0^{2\pi}\big[-\tfrac{16}{3}\cos^4\phi\big]_{\phi=0}^{\phi=\pi/3}\,d\theta$$
$$= \int_0^{2\pi} -\tfrac{16}{3}\big(\tfrac{1}{16}-1\big)\,d\theta = 5\theta\Big]_0^{2\pi} = 10\pi$$

24. In spherical coordinates, the sphere $x^2+y^2+z^2=4$ is equivalent to $\rho=2$ and the cone $z=\sqrt{x^2+y^2}$ is represented by $\phi=\frac{\pi}{4}$. Thus, the solid is given by $\{(\rho,\theta,\phi)\mid 0\le\rho\le 2, 0\le\theta\le 2\pi, \frac{\pi}{4}\le\phi\le\frac{\pi}{2}\}$ and

$$V = \int_{\pi/4}^{\pi/2}\int_0^{2\pi}\int_0^2\rho^2\sin\phi\,d\rho\,d\theta\,d\phi = \int_{\pi/4}^{\pi/2}\sin\phi\,d\phi\int_0^{2\pi}d\theta\int_0^2\rho^2\,d\rho$$
$$= \big[-\cos\phi\big]_{\pi/4}^{\pi/2}\,\big[\theta\big]_0^{2\pi}\,\big[\tfrac13\rho^3\big]_0^2 = \Big(\tfrac{\sqrt2}{2}\Big)(2\pi)\big(\tfrac83\big) = \tfrac{8\sqrt2\,\pi}{3}$$

25. By the symmetry of the region, $M_{xy} = 0$ and $M_{yz} = 0$. Assuming constant density K,

$$m = \iiint_E KV = K\int_0^\pi \int_0^\pi \int_3^4 \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = K\int_0^\pi d\theta \int_0^\pi \sin\phi \, d\phi \int_3^4 \rho^2 \, d\rho$$
$$= K\pi \left[-\cos\phi\right]_0^\pi \left[\tfrac{1}{3}\rho^3\right]_3^4 = 2K\pi \cdot \tfrac{37}{3} = \tfrac{74}{3}\pi K$$

and

$$M_{xz} = \iiint_E y\,K\,dV = K\int_0^\pi \int_0^\pi \int_3^4 (\rho\sin\phi\sin\theta)\,\rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$$
$$= K\int_0^\pi \sin\theta \, d\theta \int_0^\pi \sin^2\phi \, d\phi \int_3^4 \rho^3 \, d\rho$$
$$= K\left[-\cos\theta\right]_0^\pi \left[\tfrac{1}{2}\phi - \tfrac{1}{4}\sin 2\phi\right]_0^\pi \left[\tfrac{1}{4}\rho^4\right]_3^4$$
$$= K(2)\left(\tfrac{\pi}{2}\right)\tfrac{1}{4}(256 - 81) = \tfrac{175}{4}\pi K$$

Thus the centroid is $(\overline{x}, \overline{y}, \overline{z}) = \left(\frac{M_{yz}}{m}, \frac{M_{xz}}{m}, \frac{M_{xy}}{m}\right) = \left(0, \frac{175\pi K/4}{74\pi K/3}, 0\right) = \left(0, \frac{525}{296}, 0\right)$.

26. (a) Placing the center of the base at $(0,0,0)$, $\rho(x,y,z) = K\sqrt{x^2+y^2+z^2}$ is the density function. So

$$m = \int_0^{2\pi}\int_0^{\pi/2}\int_0^a K\rho^3 \sin\phi \, d\rho \, d\phi \, d\theta = K\int_0^{2\pi} d\theta \int_0^{\pi/2} \sin\phi \, d\phi \int_0^a \rho^3 \, d\rho$$
$$= K\left[\theta\right]_0^{2\pi} \left[-\cos\phi\right]_0^{\pi/2} \left[\tfrac{1}{4}\rho^4\right]_0^a = K(2\pi)(1)\left(\tfrac{1}{4}a^4\right) = \tfrac{1}{2}\pi K a^4$$

(b) By the symmetry of the problem $M_{yz} = M_{xz} = 0$. Then

$$M_{xy} = \int_0^{2\pi}\int_0^{\pi/2}\int_0^a K\rho^4 \sin\phi\cos\phi \, d\rho \, d\phi \, d\theta = K\int_0^{2\pi} d\theta \int_0^{\pi/2} \sin\phi\cos\phi \, d\phi \int_0^a \rho^4 \, d\rho$$
$$= K\left[\theta\right]_0^{2\pi} \left[\tfrac{1}{2}\sin^2\phi\right]_0^{\pi/2} \left[\tfrac{1}{5}\rho^5\right]_0^a = K(2\pi)\left(\tfrac{1}{2}\right)\left(\tfrac{1}{5}a^5\right) = \tfrac{1}{5}\pi K a^5$$

Hence $(\overline{x}, \overline{y}, \overline{z}) = \left(0, 0, \tfrac{2}{5}a\right)$.

(c) $I_z = \int_0^{2\pi}\int_0^{\pi/2}\int_0^a (K\rho^3\sin\phi)(\rho^2\sin^2\phi) \, d\rho \, d\phi \, d\theta = K\int_0^{2\pi} d\theta \int_0^{\pi/2} \sin^3\phi \, d\phi \int_0^a \rho^5 \, d\rho$

$$= K\left[\theta\right]_0^{2\pi} \left[-\cos\phi + \tfrac{1}{3}\cos^3\phi\right]_0^{\pi/2} \left[\tfrac{1}{6}\rho^6\right]_0^a = K(2\pi)\left(\tfrac{2}{3}\right)\left(\tfrac{1}{6}a^6\right) = \tfrac{2}{9}\pi K a^6$$

27. (a) The density function is $\rho(x,y,z) = K$, a constant, and by the symmetry of the problem $M_{xz} = M_{yz} = 0$. Then $M_{xy} = \int_0^{2\pi}\int_0^{\pi/2}\int_0^a K\rho^3 \sin\phi\cos\phi \, d\rho \, d\phi \, d\theta = \frac{1}{2}\pi K a^4 \int_0^{\pi/2} \sin\phi\cos\phi \, d\phi = \frac{1}{8}\pi K a^4$. But the mass is K(volume of the hemisphere) $= \frac{2}{3}\pi K a^3$, so the centroid is $\left(0, 0, \frac{3}{8}a\right)$.

(b) Place the center of the base at $(0,0,0)$; the density function is $\rho(x,y,z) = K$. By symmetry, the moments of inertia about any two such diameters will be equal, so we just need to find I_x:

$$I_x = \int_0^{2\pi}\int_0^{\pi/2}\int_0^a \left(K\rho^2\sin\phi\right)\rho^2\left(\sin^2\phi\sin^2\theta + \cos^2\phi\right) d\rho \, d\phi \, d\theta$$
$$= K\int_0^{2\pi}\int_0^{\pi/2} \left(\sin^3\phi\sin^2\theta + \sin\phi\cos^2\phi\right)\left(\tfrac{1}{5}a^5\right) d\phi \, d\theta$$
$$= \tfrac{1}{5}Ka^5 \int_0^{2\pi} \left[\sin^2\theta\left(-\cos\phi + \tfrac{1}{3}\cos^3\phi\right) + \left(-\tfrac{1}{3}\cos^3\phi\right)\right]_{\phi=0}^{\phi=\pi/2} d\theta$$
$$= \tfrac{1}{5}Ka^5 \int_0^{2\pi} \left[\tfrac{2}{3}\sin^2\theta + \tfrac{1}{3}\right] d\theta = \tfrac{1}{5}Ka^5 \left[\tfrac{2}{3}\left(\tfrac{1}{2}\theta - \tfrac{1}{4}\sin 2\theta\right) + \tfrac{1}{3}\theta\right]_0^{2\pi}$$
$$= \tfrac{1}{5}Ka^5 \left[\tfrac{2}{3}(\pi - 0) + \tfrac{1}{3}(2\pi - 0)\right] = \tfrac{4}{15}Ka^5\pi$$

28. Place the center of the base at $(0,0,0)$, then the density is $\rho(x,y,z) = Kz$, K a constant. Then

$$\begin{aligned} m &= \int_0^{2\pi}\int_0^{\pi/2}\int_0^a (K\rho\cos\phi)\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta \\ &= 2\pi K\int_0^{\pi/2}\cos\phi\sin\phi\cdot\tfrac{1}{4}a^4\,d\phi \\ &= \tfrac{1}{2}\pi K a^4\left[-\tfrac{1}{4}\cos 2\phi\right]_0^{\pi/2} = \tfrac{\pi}{4}Ka^4 \end{aligned}$$

By the symmetry of the problem $M_{xz} = M_{yz} = 0$, and

$$\begin{aligned} M_{xy} &= \int_0^{2\pi}\int_0^{\pi/2}\int_0^a K\rho^4\cos^2\phi\sin\phi\,d\rho\,d\phi\,d\theta \\ &= \tfrac{2}{5}\pi K a^5\int_0^{\pi/2}\cos^2\phi\sin\phi\,d\phi \\ &= \tfrac{2}{5}\pi K a^5\left[-\tfrac{1}{3}\cos^3\theta\right]_0^{\pi/2} = \tfrac{2}{15}\pi K a^5 \end{aligned}$$

Hence $(\overline{x},\overline{y},\overline{z}) = \left(0,0,\frac{8}{15}a\right)$.

29. In spherical coordinates $z = \sqrt{x^2+y^2}$ becomes $\cos\phi = \sin\phi$ or $\phi = \frac{\pi}{4}$. Then

$V = \int_0^{2\pi}\int_0^{\pi/4}\int_0^1 \rho^2\sin\phi\,d\rho\,d\phi\,d\theta = \int_0^{2\pi}d\theta\int_0^{\pi/4}\sin\phi\,d\phi\int_0^1\rho^2\,d\rho = \frac{1}{3}\pi(2-\sqrt{2})$,

$M_{xy} = \int_0^{2\pi}\int_0^{\pi/4}\int_0^1 \rho^3\sin\phi\cos\phi\,d\rho\,d\phi\,d\theta = 2\pi\left[-\frac{1}{4}\cos 2\phi\right]_0^{\pi/4}\left(\frac{1}{4}\right) = \frac{\pi}{8}$ and by symmetry $M_{yz} = M_{xz} = 0$.

Hence $(\overline{x},\overline{y},\overline{z}) = \left(0,0,\dfrac{3}{8(2-\sqrt{2})}\right)$.

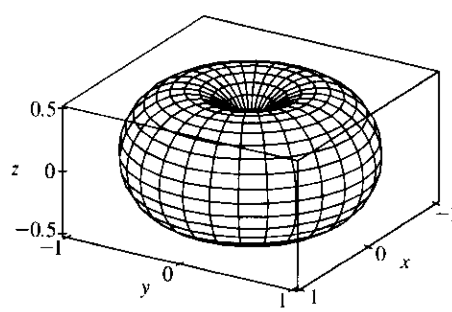
30. Place the center of the sphere at $(0,0,0)$, let the diameter of intersection be along the z-axis, one of the planes be the xz-plane and the other be the plane whose angle with the xz-plane is $\theta = \frac{\pi}{6}$. Then in spherical coordinates the volume is given by $V = \int_0^{\pi/6}\int_0^{\pi}\int_0^a \rho^2\sin\phi\,d\rho\,d\phi\,d\theta = \int_0^{\pi/6}d\theta\int_0^{\pi}\sin\phi\,d\phi\int_0^a\rho^2\,d\rho = \frac{\pi}{6}(2)\left(\frac{1}{3}a^3\right) = \frac{1}{9}\pi a^3$.

31. In cylindrical coordinates the paraboloid is given by $z = r^2$ and the plane by $z = 2r\sin\theta$ and they intersect in the circle $r = 2\sin\theta$. Then $\iiint_E z\,dV = \int_0^{\pi}\int_0^{2\sin\theta}\int_{r^2}^{2r\sin\theta} rz\,dz\,dr\,d\theta = \frac{5\pi}{6}$ [using a CAS].

32. (a) The region enclosed by the torus is $\{(\rho,\theta,\phi) \mid 0 \le \theta \le 2\pi, 0 \le \phi \le \pi, 0 \le \rho \le \sin\phi\}$, so its volume is

$$V = \int_0^{2\pi}\int_0^{\pi}\int_0^{\sin\phi}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta = 2\pi\int_0^{\pi}\tfrac{1}{3}\sin^4\phi\,d\phi = \tfrac{2}{3}\pi\left[\tfrac{3}{8}\phi - \tfrac{1}{4}\sin 2\phi + \tfrac{1}{16}\sin 4\phi\right]_0^{\pi} = \tfrac{1}{4}\pi^2.$$

(b) In Maple, we can plot the torus using the `plots[sphereplot]` command, or with the `coords=spherical` option in a regular `plot` command. In Mathematica, use `ParametricPlot3d`.



33. The region E of integration is the region above the paraboloid $z = x^2 + y^2$, or $z = r^2$, and below the paraboloid $z = 2 - x^2 - y^2$, or $z = 2 - r^2$. Also, we have $-1 \le x \le 1$ with $-\sqrt{1 - x^2} \le y \le \sqrt{1 - x^2}$ which describes the unit circle in the xy-plane. Thus,

$$\int_{-1}^{1} \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \int_{x^2+y^2}^{2-x^2-y^2} (x^2 + y^2)^{3/2}\, dz\, dy\, dx = \int_0^{2\pi} \int_0^1 \int_{r^2}^{2-r^2} (r^2)^{3/2}\, r\, dz\, dr\, d\theta$$

$$= \int_0^{2\pi} \int_0^1 \left[r^4 z\right]_{z=r^2}^{z=2-r^2} dr\, d\theta = \int_0^{2\pi} \int_0^1 (2r^4 - r^6 - r^6)\, dr\, d\theta = \int_0^{2\pi} \left(\tfrac{2}{5} - \tfrac{2}{7}\right) d\theta = \tfrac{8\pi}{35}$$

34. The region E of integration is the region above the paraboloid $z = x^2 + y^2 = r^2$ and below the cone $z = \sqrt{x^2 + y^2} = r$. Also, we have $0 \le y \le 1, 0 \le x \le \sqrt{1 - y^2}$ which is equivalent to $0 \le \theta \le \frac{\pi}{2}, 0 \le r \le 1$. Thus

$$\int_0^1 \int_0^{\sqrt{1-y^2}} \int_{x^2+y^2}^{\sqrt{x^2+y^2}} xyz\, dz\, dx\, dy = \int_0^{\pi/2} \int_0^1 \int_{r^2}^{r} r^2 \cos\theta \sin\theta\, zr\, dz\, dr\, d\theta$$

$$= \tfrac{1}{2} \int_0^{\pi/2} \int_0^1 r^3 \cos\theta \sin\theta \left[z^2\right]_{z=r^2}^{z=r} dr\, d\theta = \tfrac{1}{2} \int_0^{\pi/2} \int_0^1 (r^5 - r^7) \cos\theta \sin\theta\, dr\, d\theta$$

$$= \tfrac{1}{2} \int_0^{\pi/2} \left[\tfrac{1}{6}r^6 - \tfrac{1}{8}r^8\right]_{r=0}^{r=1} \cos\theta \sin\theta\, d\theta = \tfrac{1}{2} \int_0^{\pi/2} \tfrac{1}{24} \cos\theta \sin\theta\, d\theta$$

$$= \tfrac{1}{48} \int_0^{\pi/2} \tfrac{1}{2} \sin 2\theta\, d\theta = \tfrac{1}{96} \left[-\tfrac{1}{2} \cos 2\theta\right]_0^{\pi/2} = \tfrac{1}{96}$$

35. The region of integration E is the top half of the sphere $x^2 + y^2 + z^2 = 9$. So

$$\int_{-3}^{3} \int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}} \int_0^{\sqrt{9-x^2-y^2}} z \sqrt{x^2 + y^2 + z^2}\, dz\, dy\, dx = \iiint_E z \sqrt{x^2 + y^2 + z^2}\, dV$$

$$= \int_0^{2\pi} \int_0^{\pi/2} \int_0^3 (\rho^2 \cos\phi)(\rho^2 \sin\phi)\, d\rho\, d\phi\, d\theta = \int_0^{2\pi} d\theta \int_0^{\pi/2} \cos\phi \sin\phi\, d\phi \int_0^3 \rho^4\, d\rho$$

$$= \left[\theta\right]_0^{2\pi} \left[\tfrac{1}{2} \sin^2\phi\right]_0^{\pi/2} \left[\tfrac{1}{5}\rho^5\right]_0^3 = (2\pi)\left(\tfrac{1}{2}\right)\left(\tfrac{243}{5}\right) = \tfrac{243}{5}\pi$$

36. The region of integration E is the region above the cone $z = \sqrt{x^2 + y^2}$ and below the sphere $x^2 + y^2 + z^2 = 18$ in the first octant. Because E is in the first octant we have $0 \le \theta \le \frac{\pi}{2}$. The cone has equation $\phi = \frac{\pi}{4}$ (as in Example 4) and so $0 \le \phi \le \frac{\pi}{4}$. Also $0 \le \rho \le \sqrt{18} = 3\sqrt{2}$. So the integral becomes

$$\int_0^{\pi/2} \int_0^{\pi/4} \int_0^{3\sqrt{2}} \rho^4 \sin\phi\, d\rho\, d\phi\, d\theta = \int_0^{\pi/2} d\theta \int_0^{\pi/4} \sin\phi\, d\phi \int_0^{3\sqrt{2}} \rho^4\, d\rho$$

$$= \left[\theta\right]_0^{\pi/2} \left[-\cos\phi\right]_0^{\pi/4} \left[\tfrac{1}{5}\rho^5\right]_0^{3\sqrt{2}}$$

$$= \left(\tfrac{\pi}{2}\right)\left(1 - \tfrac{\sqrt{2}}{2}\right)\left(\tfrac{972\sqrt{2}}{5}\right) = 486\pi\left(\tfrac{\sqrt{2}-1}{5}\right)$$

37. If E is the solid enclosed by the surface $\rho = 1 + \frac{1}{5} \sin 6\theta \sin 5\phi$, it can be described in spherical coordinates as $E = \left\{(\rho, \theta, \phi) \mid 0 \le \rho \le 1 + \frac{1}{5} \sin 6\theta \sin 5\phi, 0 \le \theta \le 2\pi, 0 \le \phi \le \pi\right\}$. Its volume is given by $V(E) = \iiint_E dV = \int_0^{\pi} \int_0^{2\pi} \int_0^{1 + (\sin 6\theta \sin 5\phi)/5} \rho^2 \sin\phi\, d\rho\, d\theta\, d\phi = \frac{136\pi}{99}$ [using a CAS].

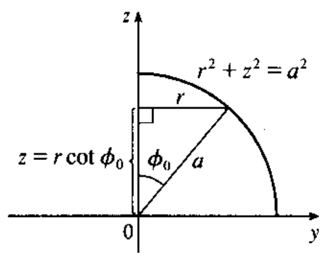
38. The given integral is equal to

$\lim\limits_{R\to\infty}\int_0^{2\pi}\int_0^{\pi}\int_0^{R}\rho e^{-\rho^2}\rho^2\sin\phi\,d\rho\,d\phi\,d\theta=\lim\limits_{R\to\infty}\left(\int_0^{2\pi}d\theta\right)\left(\int_0^{\pi}\sin\phi\,d\phi\right)\left(\int_0^{R}\rho^3e^{-\rho^2}\,d\rho\right)$. Now use integration by parts with $u=\rho^2$, $dv=\rho e^{-\rho^2}\,d\rho$ to get

$$\begin{aligned}\lim_{R\to\infty}2\pi(2)\left(\rho^2\left(-\tfrac12\right)e^{-\rho^2}\Big]_0^R-\int_0^R2\rho\left(-\tfrac12\right)e^{-\rho^2}\,d\rho\right)&=\lim_{R\to\infty}4\pi\left(-\tfrac12R^2e^{-R^2}+\left[-\tfrac12e^{-\rho^2}\right]_0^R\right)\\&=4\pi\lim_{R\to\infty}\left[-\tfrac12R^2e^{-R^2}-\tfrac12e^{-R^2}+\tfrac12\right]\\&=4\pi\left(\tfrac12\right)=2\pi\end{aligned}$$

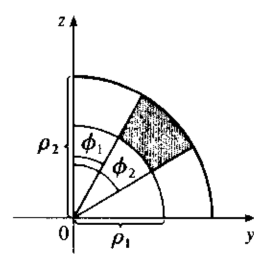
(Note that $R^2e^{-R^2}\to0$ as $R\to\infty$ by l'Hospital's Rule.)

39. (a) From the diagram, $z=r\cot\phi_0$ to $z=\sqrt{a^2-r^2}$, $r=0$ to $r=a\sin\phi_0$ (or use $a^2-r^2=r^2\cot^2\phi_0$). Thus



$$\begin{aligned}V&=\int_0^{2\pi}\int_0^{a\sin\phi_0}\int_{r\cot\phi_0}^{\sqrt{a^2-r^2}}r\,dz\,dr\,d\theta\\&=2\pi\int_0^{a\sin\phi_0}\left(r\sqrt{a^2-r^2}-r^2\cot\phi_0\right)dr\\&=\tfrac{2\pi}{3}\left[-(a^2-r^2)^{3/2}-r^3\cot\phi_0\right]_0^{a\sin\phi_0}\\&=\tfrac{2\pi}{3}\left[-\left(a^2-a^2\sin^2\phi_0\right)^{3/2}-a^3\sin^3\phi_0\cot\phi_0+a^3\right]\\&=\tfrac23\pi a^3\left[1-\left(\cos^3\phi_0+\sin^2\phi_0\cos\phi_0\right)\right]=\tfrac23\pi a^3(1-\cos\phi_0)\end{aligned}$$

(b) The wedge in question is the shaded area rotated from $\theta=\theta_1$ to $\theta=\theta_2$. Letting



$V_{ij}=$ volume of the region bounded by the sphere of radius ρ_i and the cone with angle ϕ_j ($\theta=\theta_1$ to θ_2)

and letting V be the volume of the wedge, we have

$$\begin{aligned}V&=(V_{22}-V_{21})-(V_{12}-V_{11})\\&=\tfrac13(\theta_2-\theta_1)\left[\rho_2^3(1-\cos\phi_2)-\rho_2^3(1-\cos\phi_1)-\rho_1^3(1-\cos\phi_2)+\rho_1^3(1-\cos\phi_1)\right]\\&=\tfrac13(\theta_2-\theta_1)\left[\left(\rho_2^3-\rho_1^3\right)(1-\cos\phi_2)-\left(\rho_2^3-\rho_1^3\right)(1-\cos\phi_1)\right]\\&=\tfrac13(\theta_2-\theta_1)\left[\left(\rho_2^3-\rho_1^3\right)(\cos\phi_1-\cos\phi_2)\right]\end{aligned}$$

Or: Show that $V=\displaystyle\int_{\theta_1}^{\theta_2}\int_{\rho_1\sin\phi_1}^{\rho_2\sin\phi_2}\int_{r\cot\phi_2}^{r\cot\phi_1}r\,dz\,dr\,d\theta$.

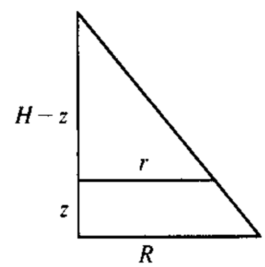
(c) By the Mean Value Theorem with $f(\rho)=\rho^3$ there exists some $\tilde\rho$ with $\rho_1\le\tilde\rho\le\rho_2$ such that $f(\rho_2)-f(\rho_1)=f'(\tilde\rho)(\rho_2-\rho_1)$ or $\rho_1^3-\rho_2^3=3\tilde\rho^2\Delta\rho$. Similarly there exists ϕ with $\phi_1\le\tilde\phi\le\phi_2$ such that $\cos\phi_2-\cos\phi_1=\left(-\sin\tilde\phi\right)\Delta\phi$. Substituting into the result from (b) gives

$\Delta V=(\tilde\rho^2\,\Delta\rho)(\theta_2-\theta_1)(\sin\tilde\phi)\,\Delta\phi=\tilde\rho^2\sin\tilde\phi\,\Delta\rho\,\Delta\phi\,\Delta\theta$.

40. (a) The mountain comprises a solid conical region C. The work done in lifting a small volume of material ΔV with density $g(P)$ to a height $h(P)$ above sea level is $h(P)g(P)\,\Delta V$. Summing over the whole mountain we get $W = \iiint_C h(P)g(P)\,\Delta V$.

(b) Here C is a solid right circular cone with radius $R = 62{,}000$ ft, height $H = 12{,}400$ ft, and density $g(P) = 200\text{ lb/ft}^3$ at all points P in C. We use cylindrical coordinates:

$$\begin{aligned} W &= \int_0^{2\pi}\int_0^H\int_0^{R(1-z/H)} z\cdot 200r\,dr\,dz\,d\theta \\ &= 2\pi\int_0^H 200z\left[\tfrac{1}{2}r^2\right]_{r=0}^{r=R(1-z/H)}dz \\ &= 400\pi\int_0^H z\,\frac{R^2}{2}\left(1-\frac{z}{H}\right)^2 dz \\ &= 200\pi R^2\int_0^H\left(z-\frac{2z^2}{H}+\frac{z^3}{H^2}\right)dz \\ &= 200\pi R^2\left[\frac{z^2}{2}-\frac{2z^3}{3H}+\frac{z^4}{4H^2}\right]_0^H \\ &= 200\pi R^2\left(\frac{H^2}{2}-\frac{2H^2}{3}+\frac{H^2}{4}\right) = \tfrac{50}{3}\pi R^2H^2 \\ &= \tfrac{50}{3}\pi(62{,}000)^2(12{,}400)^2 \approx 3.1\times 10^{19}\text{ ft-lb} \end{aligned}$$



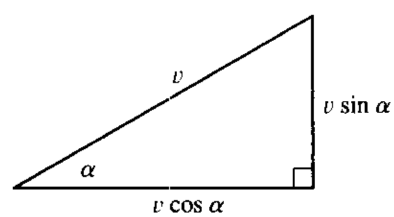
$$\frac{r}{R} = \frac{H-z}{H} = 1-\frac{z}{H}$$

APPLIED PROJECT Roller Derby

1. $mgh = \frac{1}{2}mv^2 + \frac{1}{2}I\omega^2 = \frac{1}{2}(m + I/r^2)v^2$, so $v^2 = \dfrac{2mgh}{m+I/r^2} = \dfrac{2gh}{1+I^*}$.

2. The vertical component of the speed is $v\sin\alpha$, so

$$\frac{dy}{dt} = \sqrt{\frac{2gy}{1+I^*}}\sin\alpha = \sqrt{\frac{2g}{1+I^*}}\sin\alpha\,\sqrt{y}.$$



3. Solving the separable differential equation, we get $\dfrac{dy}{\sqrt{y}} = \sqrt{\dfrac{2g}{1+I^*}}\sin\alpha\,dt \;\Rightarrow$

$2\sqrt{y} = \sqrt{\dfrac{2g}{1+I^*}}(\sin\alpha)t + C$. But $y = 0$ when $t = 0$, so $C = 0$ and we have $2\sqrt{y} = \sqrt{\dfrac{2g}{1+I^*}}(\sin\alpha)t$.

Solving for t when $y = h$ gives $T = \dfrac{2\sqrt{h}}{\sin\alpha}\sqrt{\dfrac{1+I^*}{2g}} = \sqrt{\dfrac{2h(1+I^*)}{g\sin^2\alpha}}$.

4. Assume that the length of each cylinder is ℓ. Then the density of the solid cylinder is $\dfrac{m}{\pi r^2\ell}$, and from Formulas 16.7.16 [ET 15.7.16], its moment of inertia (using cylindrical coordinates) is

$$I_z = \iiint \frac{m}{\pi r^2\ell}(x^2+y^2)\,dV = \int_0^\ell\int_0^{2\pi}\int_0^r \frac{m}{\pi r^2\ell}R^2R\,dR\,d\theta\,dz = \frac{m}{\pi r^2\ell}2\pi\ell\left[\tfrac{1}{4}R^4\right]_0^r = \frac{mr^2}{2}$$

and so $I^* = \dfrac{I_z}{mr^2} = \dfrac{1}{2}$.

For the hollow cylinder, we consider its entire mass to lie a distance r from the axis of rotation, so $x^2 + y^2 = r^2$ is a constant. We express the density in terms of mass per unit area as $\rho = \dfrac{m}{2\pi r\ell}$, and then the moment of inertia is calculated as a double integral:

$$I_z = \iint (x^2 + y^2)\,\frac{m}{2\pi r\ell}\,dA = \frac{mr^2}{2\pi r\ell}\iint dA = mr^2$$

so $I^* = \dfrac{I_z}{mr^2} = 1$.

5. The volume of such a ball is $\frac{4}{3}\pi(r^3 - a^3) = \frac{4}{3}\pi r(1 - b^3)$, and so its density is $\dfrac{m}{\frac{4}{3}\pi r^3(1 - b^3)}$.

Using Formula 16.8.4 [ET 15.8.4], we get

$$\begin{aligned} I_z &= \iiint (x^2 + y^2)\,\frac{m}{\frac{4}{3}\pi r^3(1-b^3)}\,dV \\ &= \frac{m}{\frac{4}{3}\pi r^3(1-b^3)}\int_a^r\int_0^{2\pi}\int_0^{\pi}(\rho^2\sin^2\phi)(\rho^2\sin\phi)\,d\phi\,d\theta\,d\rho \\ &= \frac{m}{\frac{4}{3}\pi r^3(1-b^3)}\cdot 2\pi\left[-\frac{(2+\sin^2\phi)\cos\phi}{3}\right]_0^{\pi}\left[\frac{\rho^5}{5}\right]_a^r \qquad \text{[from the Table of Integrals]} \\ &= \frac{m}{\frac{4}{3}\pi r^3(1-b^3)}\cdot 2\pi\cdot\frac{4}{3}\cdot\frac{r^5-a^5}{5} \\ &= \frac{2mr^5(1-b^5)}{5r^3(1-b^3)} = \frac{2(1-b^5)mr^2}{5(1-b^3)} \end{aligned}$$

Therefore $I^* = \dfrac{2(1-b^5)}{5(1-b^3)}$.

Since a represents the inner radius, $a \to 0$ corresponds to a solid ball, and $a \to r$ corresponds to a hollow ball.

6. For a solid ball, $a \to 0 \;\Rightarrow\; b \to 0$, so $I^* = \displaystyle\lim_{b\to 0}\frac{2(1-b^5)}{5(1-b^3)} = \frac{2}{5}$. For a hollow ball, $a \to r \;\Rightarrow\; b \to 1$, so

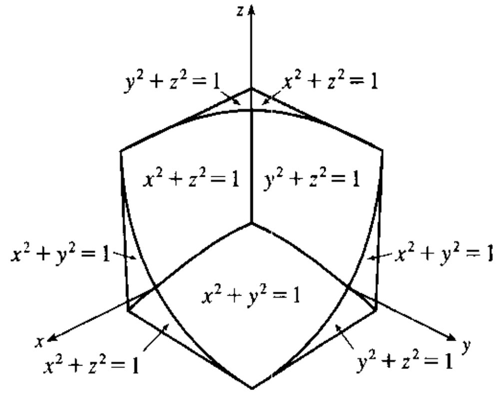
$$I^* = \lim_{b\to 1}\frac{2(1-b^5)}{5(1-b^3)} = \frac{2}{5}\lim_{b\to 1}\frac{-5b^4}{-3b^2} \overset{\text{H}}{=} \frac{2}{5}\left(\frac{5}{3}\right) = \frac{2}{3} \qquad \text{[by l'Hospital's Rule]}$$

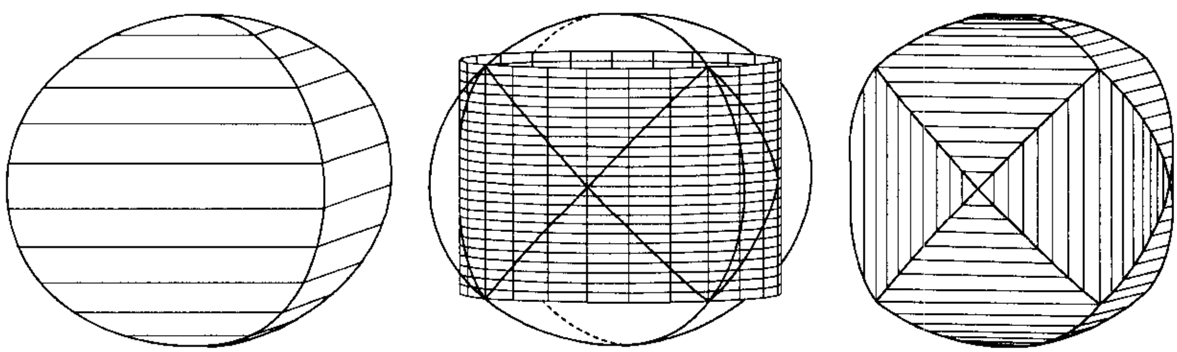
Note: We could instead have calculated $I^* = \displaystyle\lim_{b\to 1}\frac{2(1-b)(1+b+b^2+b^3+b^4)}{5(1-b)(1+b+b^2)} = \frac{2\cdot 5}{5\cdot 3} = \frac{2}{3}$.

Thus the objects finish in the following order: solid ball ($I^* = \frac{2}{5}$), solid cylinder ($I^* = \frac{1}{2}$), hollow ball ($I^* = \frac{2}{3}$), hollow cylinder ($I^* = 1$).

DISCOVERY PROJECT The Intersection of Three Cylinders

1. The three cylinders in the illustration in the text can be visualized as representing the surfaces $x^2 + y^2 = 1$, $x^2 + z^2 = 1$, and $y^2 + z^2 = 1$. Then we sketch the solid of intersection with the coordinate axes and equations indicated. To be more precise, we start by finding the bounding curves of the solid (shown in the first graph below) enclosed by the two cylinders $x^2 + z^2 = 1$ and $y^2 + z^2 = 1$: $x = \pm y = \pm\sqrt{1 - z^2}$ are the symmetric equations, and these can be expressed parametrically as $x = s$, $y = \pm s$, $z = \pm\sqrt{1 - s^2}$, $-1 \le s \le 1$. Now the cylinder $x^2 + y^2 = 1$ intersects these curves at the eight points $\left(\pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}, \pm\frac{1}{\sqrt{2}}\right)$. The resulting solid has twelve curved faces bounded by "edges" which are arcs of circles, as shown in the third diagram. Each cylinder defines four of the twelve faces.





2. To find the volume, we split the solid into sixteen congruent pieces, one of which lies in the part of the first octant with $0 \le \theta \le \frac{\pi}{4}$. (Naturally, we use cylindrical coordinates!) This piece is described by

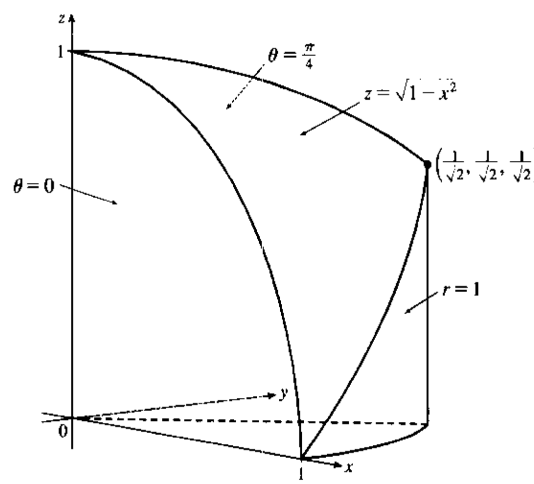
$\left\{(r, \theta, z) \mid 0 \le r \le 1, 0 \le \theta \le \frac{\pi}{4}, 0 \le z \le \sqrt{1 - x^2}\right\}$,

and so, substituting $x = r\cos\theta$, the volume of the entire solid is

$$V = 16\int_0^{\pi/4}\int_0^1\int_0^{\sqrt{1-x^2}} r\,dz\,dr\,d\theta$$

$$= 16\int_0^{\pi/4}\int_0^1 r\sqrt{1 - r^2\cos^2\theta}\,dr\,d\theta$$

$$= 16 - 8\sqrt{2} \approx 4.6863$$

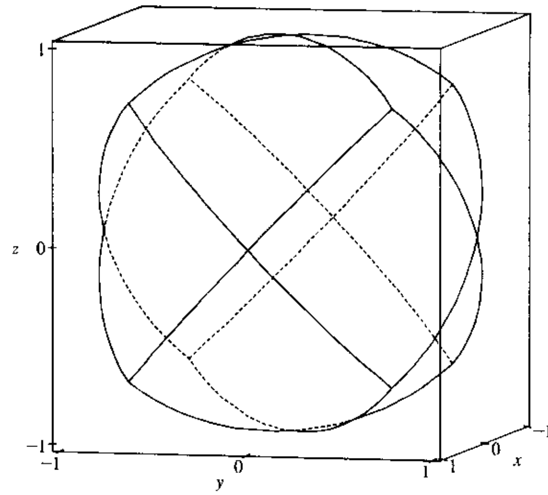


3. To graph the edges of the solid, we use parametrized curves similar to those found in Problem 1 for the intersection of two cylinders. We must restrict the parameter intervals so that each arc extends exactly to the desired vertex. One possible set of parametric equations (with all sign choices allowed) is

$x = r,\ y = \pm r,\ z = \pm\sqrt{1 - r^2},\ -\frac{1}{\sqrt{2}} \le r \le \frac{1}{\sqrt{2}};$

$x = \pm s,\ y = \pm\sqrt{1 - s^2},\ z = s,\ -\frac{1}{\sqrt{2}} \le s \le \frac{1}{\sqrt{2}};$

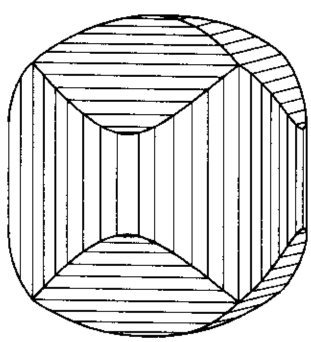
$x = \pm\sqrt{1 - t^2},\ y = t,\ z = \pm t,\ -\frac{1}{\sqrt{2}} \le t \le \frac{1}{\sqrt{2}}.$



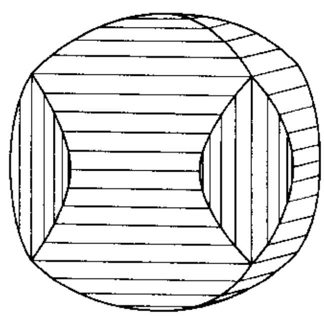
4. Let the three cylinders be $x^2 + y^2 = a^2$, $x^2 + z^2 = 1$, and $y^2 + z^2 = 1$.

If $a < 1$, then the four faces defined by the cylinder $x^2 + y^2 = 1$ in Problem 1 collapse into a single face, as in the first graph. If $1 < a < \sqrt{2}$, then each pair of vertically opposed faces, defined by one of the other two cylinders, collapse into a single face, as in the second graph. If $a \ge \sqrt{2}$, then the vertical cylinder encloses the solid of intersection of the other two cylinders completely, so the solid of intersection coincides with the solid of intersection of the two cylinders $x^2 + z^2 = 1$ and $y^2 + z^2 = 1$, as illustrated in Problem 1.

If we were to vary b or c instead of a, we would get solids with the same shape, but differently oriented.



$a = 0.95, b = c = 1$

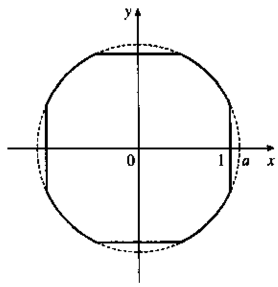


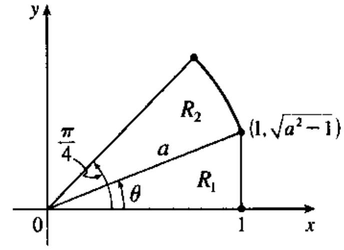
$a = 1.1, b = c = 1$

5. If $a < 1$, the solid looks similar to the first graph in Problem 4. As in Problem 2, we split the solid into sixteen congruent pieces, one of which can be described as the solid above the polar region $\{(r, \theta) \mid 0 \le r \le a, 0 \le \theta \le \frac{\pi}{4}\}$ in the xy-plane and below the surface $z = \sqrt{1 - x^2} = \sqrt{1 - r^2 \cos^2 \theta}$. Thus, the total volume is

$$V = 16 \int_0^{\pi/4} \int_0^a \sqrt{1 - r^2 \cos^2 \theta}\, r\, dr\, d\theta$$

If $a > 1$ and $a < \sqrt{2}$, we have a solid similar to the second graph in Problem 4. Its intersection with the xy-plane is graphed below. Again we split the solid into sixteen congruent pieces, one of which is the solid above the region shown in the second figure and below the surface $z = \sqrt{1 - x^2} = \sqrt{1 - r^2 \cos^2 \theta}$.





We split the region of integration where the outside boundary changes from the vertical line $x = 1$ to the circle $x^2 + y^2 = a^2$ or $r = 1$. R_1 is a right triangle, so $\cos\theta = \frac{1}{a}$. Thus, the boundary between R_1 and R_2 is $\theta = \cos^{-1}\left(\frac{1}{a}\right)$ in polar coordinates, or $y = \sqrt{a^2 - 1}\,x$ in rectangular coordinates. Using rectangular coordinates for the region R_1 and polar coordinates for R_2, we find the total volume of the solid to be

$$V = 16\left[\int_0^1 \int_0^{\sqrt{a^2-1}\,x} \sqrt{1 - x^2}\,dy\,dx + \int_{\cos^{-1}(1/a)}^{\pi/4} \int_0^a \sqrt{1 - r^2\cos^2\theta}\,r\,dr\,d\theta\right]$$

If $a \geq \sqrt{2}$, the cylinder $x^2 + y^2 = 1$ completely encloses the intersection of the other two cylinders, so the solid of intersection of the three cylinders coincides with the intersection of $x^2 + z^2 = 1$ and $y^2 + z^2 = 1$ as illustrated in Exercise 16.6.24 [ET 15.6.24]. Its volume is

$$V = 16\int_0^1 \int_0^x \sqrt{1 - x^2}\,dy\,dx$$

16.9 Change of Variables in Multiple Integrals ET 15.9

1. $x = u + 4v$, $y = 3u - 2v$.

The Jacobian is $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix} = \begin{vmatrix} 1 & 4 \\ 3 & -2 \end{vmatrix} = 1(-2) - 4(3) = -14$.

2. $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix} = \begin{vmatrix} 2u & -2v \\ 2u & 2v \end{vmatrix} = 4uv - (-4uv) = 8uv$

3. $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \dfrac{v}{(u+v)^2} & -\dfrac{u}{(u+v)^2} \\ -\dfrac{v}{(u-v)^2} & \dfrac{u}{(u-v)^2} \end{vmatrix} = \dfrac{uv}{(u+v)^2(u-v)^2} - \dfrac{uv}{(u+v)^2(u-v)^2} = 0$

4. $\dfrac{\partial(x,y)}{\partial(\alpha,\beta)} = \begin{vmatrix} \partial x/\partial\alpha & \partial x/\partial\beta \\ \partial y/\partial\alpha & \partial y/\partial\beta \end{vmatrix} = \begin{vmatrix} \sin\beta & \alpha\cos\beta \\ \cos\beta & -\alpha\sin\beta \end{vmatrix} = -\alpha\sin^2\beta - \alpha\cos^2\beta = -\alpha$

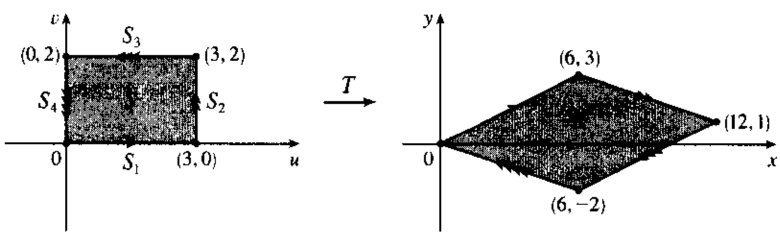
5. $$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v & \partial x/\partial w \\ \partial y/\partial u & \partial y/\partial v & \partial y/\partial w \\ \partial z/\partial u & \partial z/\partial v & \partial z/\partial w \end{vmatrix} = \begin{vmatrix} v & u & 0 \\ 0 & w & v \\ w & 0 & u \end{vmatrix}$$

$$= v\begin{vmatrix} w & v \\ 0 & u \end{vmatrix} - u\begin{vmatrix} 0 & v \\ w & u \end{vmatrix} + 0\begin{vmatrix} 0 & w \\ w & 0 \end{vmatrix} = v(uw - 0) - u(0 - vw) = 2uvw$$

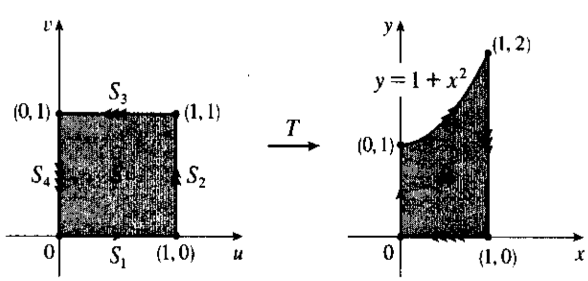
6. $$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} e^{u-v} & -e^{u-v} & 0 \\ e^{u+v} & e^{u+v} & 0 \\ e^{u+v+w} & e^{u+v+w} & e^{u+v+w} \end{vmatrix} = e^{u+v+w}\begin{vmatrix} e^{u-v} & -e^{u-v} \\ e^{u+v} & e^{u+v} \end{vmatrix}$$

$$= e^{u+v+w}\left(e^{u-v}e^{u+v} + e^{u-v}e^{u+v}\right) = e^{u+v+w}\left(2e^{2u}\right) = 2e^{3u+v+w}$$

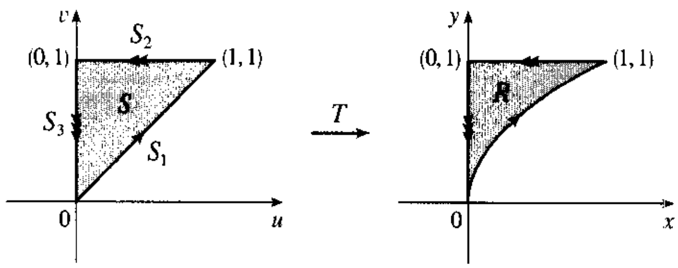
7. The transformation maps the boundary of S to the boundary of the image R, so we first look at side S_1 in the uv-plane. S_1 is described by $v = 0$ $(0 \le u \le 3)$, so $x = 2u + 3v = 2u$ and $y = u - v = u$. Eliminating u, we have $x = 2y$, $0 \le x \le 6$. S_2 is the line segment $u = 3$, $0 \le v \le 2$, so $x = 6 + 3v$ and $y = 3 - v$. Then $v = 3 - y \Rightarrow x = 6 + 3(3 - y) = 15 - 3y$, $6 \le x \le 12$. S_3 is the line segment $v = 2$, $0 \le u \le 3$, so $x = 2u + 6$ and $y = u - 2$, giving $u = y + 2 \Rightarrow x = 2y + 10$, $6 \le x \le 12$. Finally, S_4 is the segment $u = 0$, $0 \le v \le 2$, so $x = 3v$ and $y = -v \Rightarrow x = -3y$, $0 \le x \le 6$. The image of set S is the region R shown in the xy-plane, a parallelogram bounded by these four segments.



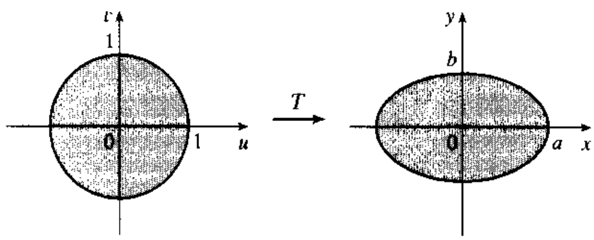
8. S_1 is the line segment $v = 0$, $0 \le u \le 1$, so $x = v = 0$ and $y = u(1 + v^2) = u$. Since $0 \le u \le 1$, the image is the line segment $x = 0$, $0 \le y \le 1$. S_2 is the segment $u = 1$, $0 \le v \le 1$, so $x = v$ and $y = u(1 + v^2) = 1 + x^2$. Thus the image is the portion of the parabola $y = 1 + x^2$ for $0 \le x \le 1$. S_3 is the segment $v = 1$, $0 \le u \le 1$, so $x = 1$ and $y = 2u$. The image is the segment $x = 1$, $0 \le y \le 2$. S_4 is described by $u = 0$, $0 \le v \le 1$, so $0 \le x = v \le 1$ and $y = u(1 + v^2) = 0$. The image is the line segment $y = 0$, $0 \le x \le 1$. Thus, the image of S is the region R bounded by the parabola $y = 1 + x^2$, the x-axis, and the lines $x = 0$, $x = 1$.



9. S_1 is the line segment $u = v$, $0 \le u \le 1$, so $y = v = u$ and $x = u^2 = y^2$. Since $0 \le u \le 1$, the image is the portion of the parabola $x = y^2$, $0 \le y \le 1$. S_2 is the segment $v = 1$, $0 \le u \le 1$, thus $y = v = 1$ and $x = u^2$, so $0 \le x \le 1$. The image is the line segment $y = 1$, $0 \le x \le 1$. S_3 is the segment $u = 0$, $0 \le v \le 1$, so $x = u^2 = 0$ and $y = v \Rightarrow 0 \le y \le 1$. The image is the segment $x = 0$, $0 \le y \le 1$. Thus, the image of S is the region R in the first quadrant bounded by the parabola $x = y^2$, the y-axis, and the line $y = 1$.



10. Substituting $u = \frac{x}{a}$, $v = \frac{y}{b}$ into $u^2 + v^2 \le 1$ gives $\frac{x^2}{a^2} + \frac{y^2}{b^2} \le 1$, so the image of $u^2 + v^2 \le 1$ is the elliptical region $\frac{x^2}{a^2} + \frac{y^2}{b^2} \le 1$.



11. $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} = 3$ and $x - 3y = (2u + v) - 3(u + 2v) = -u - 5v$. To find the region S in the uv-plane that corresponds to R we first find the corresponding boundary under the given transformation. The line through $(0, 0)$ and $(2, 1)$ is $y = \frac{1}{2}x$ which is the image of $u + 2v = \frac{1}{2}(2u + v) \Rightarrow v = 0$; the line through $(2, 1)$ and $(1, 2)$ is $x + y = 3$ which is the image of $(2u + v) + (u + 2v) = 3 \Rightarrow u + v = 1$; the line through $(0, 0)$ and $(1, 2)$ is $y = 2x$ which is the image of $u + 2v = 2(2u + v) \Rightarrow u = 0$. Thus S is the triangle $0 \le v \le 1 - u$, $0 \le u \le 1$ in the uv-plane and

$$\begin{aligned}\iint_R (x - 3y)\, dA &= \int_0^1 \int_0^{1-u} (-u - 5v)\, |3|\, dv\, du \\ &= -3 \int_0^1 \left[uv + \tfrac{5}{2}v^2\right]_{v=0}^{v=1-u} du = -3 \int_0^1 \left(u - u^2 + \tfrac{5}{2}(1-u)^2\right) du \\ &= -3\left[\tfrac{1}{2}u^2 - \tfrac{1}{3}u^3 - \tfrac{5}{6}(1-u)^3\right]_0^1 = -3\left(\tfrac{1}{2} - \tfrac{1}{3} + \tfrac{5}{6}\right) = -3\end{aligned}$$

12. $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1/4 & 1/4 \\ -3/4 & 1/4 \end{vmatrix} = \dfrac{1}{4}$, $4x + 8y = 4 \cdot \frac{1}{4}(u + v) + 8 \cdot \frac{1}{4}(v - 3u) = 3v - 5u$. R is a parallelogram bounded by the lines $x - y = -4$, $x - y = 4$, $3x + y = 0$, $3x + y = 8$. Since $u = x - y$ and $v = 3x + y$, R is the image of the rectangle enclosed by the lines $u=-4$, $u=4$, $v=0$, and $v=8$. Thus

$$\iint_R(4x+8y)\,dA=\int_{-4}^{4}\int_0^8(3v-5u)\left|\tfrac{1}{4}\right|\,dv\,du=\tfrac{1}{4}\int_{-4}^{4}\left[\tfrac{3}{2}v^2-5uv\right]_{v=0}^{v=8}du$$
$$=\tfrac{1}{4}\int_{-4}^{4}(96-40u)\,du=\tfrac{1}{4}\left[96u-20u^2\right]_{-4}^{4}=192$$

13. $\dfrac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix}2&0\\0&3\end{vmatrix}=6$, $x^2=4u^2$ and the planar ellipse $9x^2+4y^2\le 36$ is the image of the disk $u^2+v^2\le 1$. Thus

$$\iint_R x^2\,dA=\iint\limits_{u^2+v^2\le 1}(4u^2)(6)\,du\,dv=\int_0^{2\pi}\int_0^1(24r^2\cos^2\theta)\,r\,dr\,d\theta$$
$$=24\int_0^{2\pi}\cos^2\theta\,d\theta\int_0^1 r^3\,dr=24\left[\tfrac{1}{2}x+\tfrac{1}{4}\sin 2x\right]_0^{2\pi}\left[\tfrac{1}{4}r^4\right]_0^1$$
$$=24(\pi)\left(\tfrac{1}{4}\right)=6\pi$$

14. $\dfrac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix}\sqrt{2}&-\sqrt{2/3}\\\sqrt{2}&\sqrt{2/3}\end{vmatrix}=\dfrac{4}{\sqrt{3}}$, $x^2-xy+y^2=2u^2+2v^2$ and the planar ellipse $x^2-xy+y^2\le 2$ is the image of the disk $u^2+v^2\le 1$. Thus

$$\iint_R(x^2-xy+y^2)\,dA=\iint\limits_{u^2+v^2\le 1}(2u^2+2v^2)\left(\tfrac{4}{\sqrt{3}}\,du\,dv\right)=\int_0^{2\pi}\int_0^1\tfrac{8}{\sqrt{3}}r^3\,dr\,d\theta=\tfrac{4\pi}{\sqrt{3}}$$

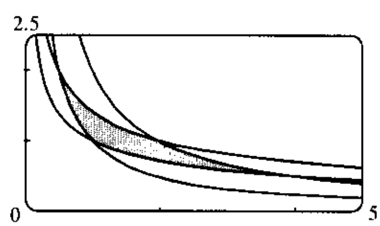
15. $\dfrac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix}1/v&-u/v^2\\0&1\end{vmatrix}=\dfrac{1}{v}$, $xy=u$, $y=x$ is the image of the parabola $v^2=u$, $y=3x$ is the image of the parabola $v^2=3u$, and the hyperbolas $xy=1$, $xy=3$ are the images of the lines $u=1$ and $u=3$ respectively. Thus

$$\iint_R xy\,dA=\int_1^3\int_{\sqrt{u}}^{\sqrt{3u}}u\left(\frac{1}{v}\right)dv\,du=\int_1^3 u\left(\ln\sqrt{3u}-\ln\sqrt{u}\right)du$$
$$=\int_1^3 u\ln\sqrt{3}\,du=4\ln\sqrt{3}=2\ln 3$$

16. Here $y=\dfrac{v}{u}$, $x=\dfrac{u^2}{v}$ so $\dfrac{\partial(x,y)}{\partial(u,v)}=\begin{vmatrix}2u/v&-u^2/v^2\\-v/u^2&1/u\end{vmatrix}=\dfrac{1}{v}$

and R is the image of the square with vertices $(1,1)$, $(2,1)$, $(2,2)$, and $(1,2)$. So

$$\iint_R y^2\,dA=\int_1^2\int_1^2\frac{v^2}{u^2}\left(\frac{1}{v}\right)du\,dv=\int_1^2\frac{v}{2}\,dv=\frac{3}{4}$$



17. (a) $\dfrac{\partial(x,y,z)}{\partial(u,v,w)}=\begin{vmatrix}a&0&0\\0&b&0\\0&0&c\end{vmatrix}=abc$ and since $u=\dfrac{x}{a}$, $v=\dfrac{y}{b}$, $w=\dfrac{z}{c}$ the solid enclosed by the ellipsoid is the image of the ball $u^2+v^2+w^2\le 1$. So

$$\iiint_E dV=\iiint\limits_{u^2+v^2+w^2\le 1}abc\,du\,dv\,dw=(abc)(\text{volume of the ball})=\tfrac{4}{3}\pi abc$$

(b) If we approximate the surface of Earth by the ellipsoid $\dfrac{x^2}{6378^2} + \dfrac{y^2}{6378^2} + \dfrac{z^2}{6356^2} = 1$, then we can estimate the volume of Earth by finding the volume of the solid E enclosed by the ellipsoid. From part (a), this is $\iiint_E dV = \frac{4}{3}\pi(6378)(6378)(6356) \approx 1.083 \times 10^{12}\ \text{km}^3$.

18. $\dfrac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{vmatrix} = abc$ and the solid enclosed by the ellipsoid is the image of the ball $u^2 + v^2 + w^2 \le 1$.

Now $x^2 y = (a^2u^2)(bv)$, so

$$\begin{aligned}
\iiint_E x^2 y\, dV &= \iiint_{u^2+v^2+w^2 \le 1} (a^2bu^2v)(abc)\, du\, dv\, dw \\
&= \int_0^{2\pi}\int_0^{\pi}\int_0^1 (a^3b^2c)(\rho^2 \sin^2\phi \cos^2\theta)(\rho \sin\phi \sin\theta)\, \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta \\
&= a^3b^2c \int_0^{2\pi}\int_0^{\pi}\int_0^1 (\rho^5 \sin^4\phi \cos^2\theta \sin\theta)\, d\rho\, d\phi\, d\theta \\
&= a^3b^2c \int_0^{2\pi} \cos^2\theta \sin\theta\, d\theta \int_0^{\pi} \sin^4\phi\, d\phi \int_0^1 \rho^5\, d\rho \\
&= 0 \quad \text{since } \int_0^{2\pi} \cos^2\theta \sin\theta\, d\theta = 0
\end{aligned}$$

19. Letting $u = x - 2y$ and $v = 3x - y$, we have $x = \frac{1}{5}(2v - u)$ and $y = \frac{1}{5}(v - 3u)$. Then

$\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} -1/5 & 2/5 \\ -3/5 & 1/5 \end{vmatrix} = \dfrac{1}{5}$ and R is the image of the rectangle enclosed by the lines $u = 0$, $u = 4$, $v = 1$, and $v = 8$. Thus $\displaystyle\iint_R \frac{x - 2y}{3x - y}\, dA = \int_0^4 \int_1^8 \frac{u}{v} \left|\frac{1}{5}\right| dv\, du = \frac{1}{5}\int_0^4 u\, du \int_1^8 \frac{1}{v}\, dv = \tfrac{1}{5}\left[\tfrac{1}{2}u^2\right]_0^4 \left[\ln|v|\right]_1^8 = \tfrac{8}{5}\ln 8.$

20. Letting $u = x + y$ and $v = x - y$, we have $x = \frac{1}{2}(u + v)$ and $y = \frac{1}{2}(u - v)$. Then

$\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1/2 & 1/2 \\ 1/2 & -1/2 \end{vmatrix} = -\dfrac{1}{2}$ and R is the image of the rectangle enclosed by the lines $u = 0$, $u = 3$, $v = 0$, and $v = 2$. Thus

$$\begin{aligned}
\iint_R (x+y)\, e^{x^2-y^2}\, dA &= \int_0^3\int_0^2 ue^{uv}\left|-\tfrac{1}{2}\right| dv\, du = \tfrac{1}{2}\int_0^3 \left[e^{uv}\right]_{v=0}^{v=2} du = \tfrac{1}{2}\int_0^3 (e^{2u} - 1)\, du \\
&= \tfrac{1}{2}\left[\tfrac{1}{2}e^{2u} - u\right]_0^3 = \tfrac{1}{2}\left(\tfrac{1}{2}e^6 - 3 - \tfrac{1}{2}\right) = \tfrac{1}{4}(e^6 - 7)
\end{aligned}$$

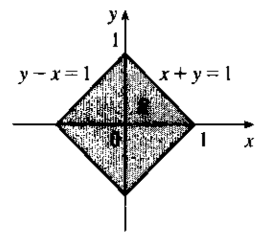
21. Letting $u = y - x$, $v = y + x$, we have $y = \frac{1}{2}(u + v)$, $x = \frac{1}{2}(v - u)$. Then $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} -1/2 & 1/2 \\ 1/2 & 1/2 \end{vmatrix} = -\dfrac{1}{2}$ and R is the image of the trapezoidal region with vertices $(-1,1)$, $(-2,2)$, $(2,2)$, and $(1,1)$. Thus

$$\begin{aligned}
\iint_R \cos\frac{y-x}{y+x}\, dA &= \int_1^2\int_{-v}^{v} \cos\frac{u}{v}\left|-\frac{1}{2}\right| du\, dv = \frac{1}{2}\int_1^2 \left[v\sin\frac{u}{v}\right]_{u=-v}^{u=v} dv \\
&= \tfrac{1}{2}\int_1^2 2v\sin(1)\, dv = \tfrac{3}{2}\sin 1
\end{aligned}$$

22. Letting $u = 3x$, $v = 2y$, we have $9x^2 + 4y^2 = u^2 + v^2$, $x = \frac{1}{3}u$, and $y = \frac{1}{2}v$. Then $\dfrac{\partial(x,y)}{\partial(u,v)} = \dfrac{1}{6}$ and R is the image of the quarter-disk D given by $u^2 + v^2 \le 1$, $u \ge 0$, $v \ge 0$. Thus

$$\iint_R \sin(9x^2 + 4y^2)\,dA = \iint_D \tfrac{1}{6}\sin(u^2 + v^2)\,du\,dv = \int_0^{\pi/2}\int_0^1 \tfrac{1}{6}\sin(r^2)\,r\,dr\,d\theta$$
$$= \tfrac{\pi}{12}\left[-\tfrac{1}{2}\cos r^2\right]_0^1 = \tfrac{\pi}{24}(1 - \cos 1)$$

23. Let $u = x + y$ and $v = -x + y$. Then $u + v = 2y \;\Rightarrow\; y = \frac{1}{2}(u + v)$ and $u - v = 2x \;\Rightarrow\; x = \frac{1}{2}(u - v)$. $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1/2 & -1/2 \\ 1/2 & 1/2 \end{vmatrix} = \dfrac{1}{2}$.



Now $|u| = |x + y| \le |x| + |y| \le 1 \;\Rightarrow\; -1 \le u \le 1$, and $|v| = |-x + y| \le |x| + |y| \le 1 \Rightarrow \; -1 \le v \le 1$. R is the image of the square region with vertices $(1, 1)$, $(1, -1)$, $(-1, -1)$, and $(-1, 1)$.
So $\iint_R e^{x+y}\,dA = \frac{1}{2}\int_{-1}^1\int_{-1}^1 e^u\,du\,dv = \frac{1}{2}\left[e^u\right]_{-1}^1\left[v\right]_{-1}^1 = e - e^{-1}$.

24. Let $u = x + y$ and $v = y$, then $x = u - v$, $y = v$, $\dfrac{\partial(x,y)}{\partial(u,v)} = 1$ and R is the image under T of the triangular region with vertices $(0, 0)$, $(1, 0)$ and $(1, 1)$. Thus

$$\iint_R f(x + y)\,dA = \int_0^1\int_0^u (1)\,f(u)\,dv\,du = \int_0^1 f(u)\left[\,v\,\right]_{v=0}^{v=u} du = \int_0^1 u f(u)\,du \quad \text{as desired.}$$

16 Review — ET 15

CONCEPT CHECK

1. (a) A double Riemann sum of f is $\sum\limits_{i=1}^{m}\sum\limits_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\,\Delta A$, where ΔA is the area of each subrectangle and (x_{ij}^*, y_{ij}^*) is a sample point in each subrectangle. If $f(x, y) \ge 0$, this sum represents an approximation to the volume of the solid that lies above the rectangle R and below the graph of f.

(b) $\iint_R f(x, y)\,dA = \lim\limits_{m,n\to\infty} \sum\limits_{i=1}^{m}\sum\limits_{j=1}^{n} f(x_{ij}^*, y_{ij}^*)\,\Delta A$

(c) If $f(x, y) \ge 0$, $\iint_R f(x, y)\,dA$ represents the volume of the solid that lies above the rectangle R and below the surface $z = f(x, y)$. If f takes on both positive and negative values, $\iint_R f(x, y)\,dA$ is the difference of the volume above R but below the surface $z = f(x, y)$ and the volume below R but above the surface $z = f(x, y)$.

(d) We usually evaluate $\iint_R f(x, y)\,dA$ as an iterated integral according to Fubini's Theorem (see Theorem 16.2.4 [ET 15.2.4]).

(e) The Midpoint Rule for Double Integrals says that we approximate the double integral $\iint_R f(x, y)\,dA$ by the double Riemann sum $\sum\limits_{i=1}^{m}\sum\limits_{j=1}^{n} f(\overline{x}_i, \overline{y}_j)\,\Delta A$ where the sample points $(\overline{x}_i, \overline{y}_j)$ are the centers of the subrectangles.

(f) $f_{\text{ave}} = \dfrac{1}{A(R)}\displaystyle\iint_R f(x, y)\,dA$ where $A(R)$ is the area of R.

2. (a) See (1) and (2) and the accompanying discussion in Section 16.3 [ET 15.3].

(b) See (3) and the accompanying discussion in Section 16.3 [ET 15.3].

(c) See (5) and the preceding discussion in Section 16.3 [ET 15.3].

(d) See (6)–(11) in Section 16.3 [ET 15.3].

3. We may want to change from rectangular to polar coordinates in a double integral if the region R of integration is more easily described in polar coordinates. To accomplish this, we use
$\iint_R f(x,y)\,dA = \int_\alpha^\beta \int_a^b f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$ where R is given by $0 \le a \le r \le b$, $\alpha \le \theta \le \beta$.

4. (a) $m = \iint_D \rho(x,y)\,dA$

(b) $M_x = \iint_D y\rho(x,y)\,dA$, $M_y = \iint_D x\rho(x,y)\,dA$

(c) The center of mass is $(\overline{x}, \overline{y})$ where $\overline{x} = \dfrac{M_y}{m}$ and $\overline{y} = \dfrac{M_x}{m}$.

(d) $I_x = \iint_D y^2\rho(x,y)\,dA$, $I_y = \iint_D x^2\rho(x,y)\,dA$, $I_0 = \iint_D (x^2+y^2)\rho(x,y)\,dA$

5. (a) $P(a \le X \le b, c \le Y \le d) = \int_a^b \int_c^d f(x,y)\,dy\,dx$

(b) $f(x,y) \ge 0$ and $\iint_{\mathbb{R}^2} f(x,y)\,dA = 1$.

(c) The expected value of X is $\mu_1 = \iint_{\mathbb{R}^2} xf(x,y)\,dA$; the expected value of Y is $\mu_2 = \iint_{\mathbb{R}^2} yf(x,y)\,dA$.

6. $A(S) = \iint_D \sqrt{[f_x(x,y)]^2 + [f_y(x,y)]^2 + 1}\,dA$

7. (a) $\iiint_B f(x,y,z)\,dV = \lim\limits_{l,m,n\to\infty} \sum\limits_{i=1}^{l} \sum\limits_{j=1}^{m} \sum\limits_{k=1}^{n} f(x_{ijk}^*, y_{ijk}^*, z_{ijk}^*)\,\Delta V$

(b) We usually evaluate $\iiint_B f(x,y,z)\,dV$ as an iterated integral according to Fubini's Theorem for Triple Integrals (see Theorem 16.7.4 [ET 15.7.4]).

(c) See the paragraph following Example 16.7.1 [ET 15.7.1].

(d) See (5) and (6) and the accompanying discussion in Section 16.7 [ET 15.7].

(e) See (10) and the accompanying discussion in Section 16.7 [ET 15.7].

(f) See (11) and the preceding discussion in Section 16.7 [ET 15.7].

8. (a) $m = \iiint_E \rho(x,y,z)\,dV$

(b) $M_{yz} = \iiint_E x\rho(x,y,z)\,dV$, $M_{xz} = \iiint_E y\rho(x,y,z)\,dV$, $M_{xy} = \iiint_E z\rho(x,y,z)\,dV$.

(c) The center of mass is $(\overline{x}, \overline{y}, \overline{z})$ where $\overline{x} = \dfrac{M_{yz}}{m}$, $\overline{y} = \dfrac{M_{xz}}{m}$, and $\overline{z} = \dfrac{M_{xy}}{m}$.

(d) $I_x = \iiint_E (y^2+z^2)\rho(x,y,z)\,dV$, $I_y = \iiint_E (x^2+z^2)\rho(x,y,z)\,dV$, $I_z = \iiint_E (x^2+y^2)\rho(x,y,z)\,dV$.

9. (a) See Formula 16.8.2 [ET 15.8.2] and the accompanying discussion.

(b) See Formula 16.8.4 [ET 15.8.4] and the accompanying discussion.

(c) We may want to change from rectangular to cylindrical or spherical coordinates in a triple integral if the region E of integration is more easily described in cylindrical or spherical coordinates or if the triple integral is easier to evaluate using cylindrical or spherical coordinates.

10. (a) $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix} = \dfrac{\partial x}{\partial u}\dfrac{\partial y}{\partial v} - \dfrac{\partial x}{\partial v}\dfrac{\partial y}{\partial u}$

(b) See (9) and the accompanying discussion in Section 16.9 [ET 15.9].

(c) See (13) and the accompanying discussion in Section 16.9 [ET 15.9].

TRUE-FALSE QUIZ

1. This is true by Fubini's Theorem.

2. False. $\int_0^1 \int_0^x \sqrt{x+y^2}\,dy\,dx$ describes the region of integration as a Type I region. To reverse the order of integration, we must consider the region as a Type II region: $\int_0^1 \int_y^1 \sqrt{x+y^2}\,dx\,dy$.

3. True. See the discussion following Example 4 on page 1029 [ET page 993].

4. $\int_{-1}^1 \int_0^1 e^{x^2+y^2} \sin y\,dx\,dy = \left(\int_0^1 e^{x^2}\,dx\right)\left(\int_{-1}^1 e^{y^2}\sin y\,dy\right) = \left(\int_0^1 e^{x^2}\,dx\right)(0) = 0$, since $e^{y^2}\sin y$ is an odd function. Therefore the statement is true.

5. True:

$$\begin{aligned}\iint_D \sqrt{4-x^2-y^2}\,dA &= \text{the volume under the surface } x^2+y^2+z^2=4 \text{ and above the } xy\text{-plane} \\ &= \tfrac{1}{2}\left(\text{the volume of the sphere } x^2+y^2+z^2=4\right) = \tfrac{1}{2}\cdot\tfrac{4}{3}\pi(2)^3 = \tfrac{16}{3}\pi\end{aligned}$$

6. This statement is true because in the given region, $(x^2+\sqrt{y})\sin(x^2y^2) \le (1+2)(1) = 3$, so $\int_1^4 \int_0^1 (x^2+\sqrt{y})\sin(x^2y^2)\,dx\,dy \le \int_1^4 \int_0^1 3\,dA = 3A(D) = 3(3) = 9$.

7. The volume enclosed by the cone $z = \sqrt{x^2+y^2}$ and the plane $z = 2$ is, in cylindrical coordinates, $V = \int_0^{2\pi} \int_0^2 \int_r^2 r\,dz\,dr\,d\theta \neq \int_0^{2\pi} \int_0^2 \int_r^2 dz\,dr\,d\theta$, so the assertion is false.

8. True. The moment of inertia about the z-axis of a solid E with constant density k is $I_z = \iiint_E (x^2+y^2)\rho(x,y,z)\,dV = \iiint_E (kr^2)\,r\,dz\,dr\,d\theta = \iiint_E kr^3\,dz\,dr\,d\theta$.

EXERCISES

1. As shown in the contour map, we divide R into 9 equally sized subsquares, each with area $\Delta A = 1$. Then we approximate $\iint_R f(x,y)\,dA$ by a Riemann sum with $m = n = 3$ and the sample points the upper right corners of each square, so

$$\begin{aligned}\iint_R f(x,y)\,dA &\approx \sum_{i=1}^{3}\sum_{j=1}^{3} f(x_i,y_j)\,\Delta A \\ &= \Delta A\,[f(1,1)+f(1,2)+f(1,3)+f(2,1)+f(2,2) \\ &\qquad + f(2,3)+f(3,1)+f(3,2)+f(3,3)]\end{aligned}$$

Using the contour lines to estimate the function values, we have

$$\iint_R f(x,y)\,dA \approx 1[2.7+4.7+8.0+4.7+6.7+10.0+6.7+8.6+11.9] \approx 64.0$$

2. As in Exercise 1, we have $m = n = 3$ and $\Delta A = 1$. Using the contour map to estimate the value of f at the center of each subsquare, we have

$$\begin{aligned}\iint_R f(x,y)\,dA &\approx \sum_{i=1}^{3}\sum_{j=1}^{3} f(\overline{x}_i,\overline{y}_j)\,\Delta A \\ &= \Delta A\,[f(0.5,0.5) + (0.5,1.5) + (0.5,2.5) + (1.5,0.5) + f(1.5,1.5) \\ &\qquad + f(1.5,2.5) + (2.5,0.5) + f(2.5,1.5) + f(2.5,2.5)] \\ &\approx 1[1.2 + 2.5 + 5.0 + 3.2 + 4.5 + 7.1 + 5.2 + 6.5 + 9.0] = 44.2\end{aligned}$$

3. $\int_1^2\int_0^2 (y + 2xe^y)\,dx\,dy = \int_1^2 \left[xy + x^2e^y\right]_{x=0}^{x=2} dy = \int_1^2 (2y + 4e^y)\,dy = \left[y^2 + 4e^y\right]_1^2$
$= 4 + 4e^2 - 1 - 4e = 4e^2 - 4e + 3$

4. $\int_0^1\int_0^1 ye^{xy}\,dx\,dy = \int_0^1 \left[e^{xy}\right]_{x=0}^{x=1} dy = \int_0^1 (e^y - 1)\,dy = \left[e^y - y\right]_0^1 = e - 2$

5. $\int_0^1\int_0^x \cos(x^2)\,dy\,dx = \int_0^1 \left[\cos(x^2)y\right]_{y=0}^{y=x} dx = \int_0^1 x\cos(x^2)\,dx = \frac{1}{2}\sin(x^2)\Big]_0^1 = \frac{1}{2}\sin 1$

6. $\int_0^1\int_x^{e^x} 3xy^2\,dy\,dx = \int_0^1 \left[xy^3\right]_{y=x}^{y=e^x} dx = \int_0^1 (xe^{3x} - x^4)\,dx$
$= \frac{1}{3}xe^{3x}\Big]_0^1 - \int_0^1 \frac{1}{3}e^{3x}\,dx - \left[\frac{1}{5}x^5\right]_0^1$ [integrating by parts in the first term]
$= \frac{1}{3}e^3 - \left[\frac{1}{9}e^{3x}\right]_0^1 - \frac{1}{5} = \frac{2}{9}e^3 - \frac{4}{45}$

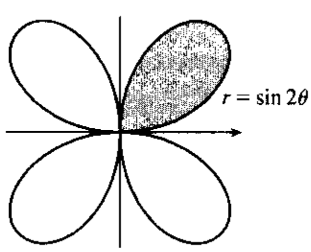
7. $\int_0^\pi\int_0^1\int_0^{\sqrt{1-y^2}} y\sin x\,dz\,dy\,dx = \int_0^\pi\int_0^1 \left[(y\sin x)z\right]_{z=0}^{z=\sqrt{1-y^2}} dy\,dx = \int_0^\pi\int_0^1 y\sqrt{1-y^2}\sin x\,dy\,dx$
$= \int_0^\pi \left[-\frac{1}{3}(1-y^2)^{3/2}\sin x\right]_{y=0}^{y=1} dx = \int_0^\pi \frac{1}{3}\sin x\,dx = -\frac{1}{3}\cos x\Big]_0^\pi = \frac{2}{3}$

8. $\int_0^1\int_0^y\int_x^1 6xyz\,dz\,dx\,dy = \int_0^1\int_0^y \left[3xyz^2\right]_{z=x}^{z=1} dx\,dy = \int_0^1\int_0^y (3xy - 3x^3y)\,dx\,dy$
$= \int_0^1 \left[\frac{3}{2}x^2y - \frac{3}{4}x^4y\right]_{x=0}^{x=y} dy = \int_0^1 \left(\frac{3}{2}y^3 - \frac{3}{4}y^5\right) dy = \left[\frac{3}{8}y^4 - \frac{1}{8}y^6\right]_0^1 = \frac{1}{4}$

9. The region R is more easily described by polar coordinates: $R = \{(r,\theta) \mid 2 \le r \le 4, 0 \le \theta \le \pi\}$. Thus $\iint_R f(x,y)\,dA = \int_0^\pi\int_2^4 f(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$.

10. The region R is a type II region that can be described as the region enclosed by the lines $y = 4 - x$, $y = 4 + x$, and the x-axis. So using rectangular coordinates, we can say $R = \{(x,y) \mid y - 4 \le x \le 4 - y, 0 \le y \le 4\}$ and $\iint_R f(x,y)\,dA = \int_0^4\int_{y-4}^{4-y} f(x,y)\,dx\,dy$.

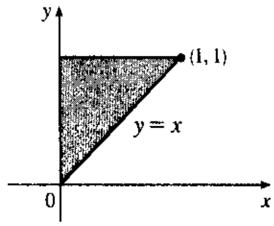
11.



The region whose area is given by $\int_0^{\pi/2}\int_0^{\sin 2\theta} r\,dr\,d\theta$ is $\left\{(r,\theta) \mid 0 \le \theta \le \frac{\pi}{2}, 0 \le r \le \sin 2\theta\right\}$, which is the region contained in the loop in the first quadrant of the four-leaved rose $r = \sin 2\theta$.

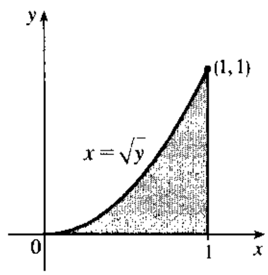
12. The solid is $\left\{(\rho,\theta,\phi) \mid 1 \le \rho \le 2, 0 \le \theta \le \frac{\pi}{2}, 0 \le \phi \le \frac{\pi}{2}\right\}$ which is the region in the first octant on or between the two spheres $\rho = 1$ and $\rho = 2$.

13.



$$\int_0^1 \int_x^1 \cos(y^2)\,dy\,dx = \int_0^1 \int_0^y \cos(y^2)\,dx\,dy$$
$$= \int_0^1 \cos(y^2)\left[\,x\,\right]_{x=0}^{x=y} dy = \int_0^1 y\cos(y^2)\,dy$$
$$= \left[\tfrac{1}{2}\sin(y^2)\right]_0^1 = \tfrac{1}{2}\sin 1$$

14.

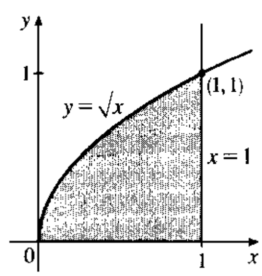


$$\int_0^1 \int_{\sqrt{y}}^1 \frac{ye^{x^2}}{x^3}\,dx\,dy = \int_0^1 \int_0^{x^2} \frac{ye^{x^2}}{x^3}\,dy\,dx$$
$$= \int_0^1 \frac{e^{x^2}}{x^3}\left[\tfrac{1}{2}y^2\right]_{y=0}^{y=x^2} dx = \int_0^1 \tfrac{1}{2}xe^{x^2}\,dx$$
$$= \tfrac{1}{4}e^{x^2}\Big]_0^1 = \tfrac{1}{4}(e-1)$$

15. $\iint_R ye^{xy}\,dA = \int_0^3 \int_0^2 ye^{xy}\,dx\,dy = \int_0^3 \left[e^{xy}\right]_{x=0}^{x=2} dy = \int_0^3 (e^{2y}-1)\,dy = \left[\tfrac{1}{2}e^{2y}-y\right]_0^3$
$= \tfrac{1}{2}e^6 - 3 - \tfrac{1}{2} = \tfrac{1}{2}e^6 - \tfrac{7}{2}$

16. $\iint_D xy\,dA = \int_0^1 \int_{y^2}^{y+2} xy\,dx\,dy = \int_0^1 y\left[\tfrac{1}{2}x^2\right]_{x=y^2}^{x=y+2} dy = \tfrac{1}{2}\int_0^1 y((y+2)^2 - y^4)\,dy$
$= \tfrac{1}{2}\int_0^1 (y^3 + 4y^2 + 4y - y^5)\,dy = \tfrac{1}{2}\left[\tfrac{1}{4}y^4 + \tfrac{4}{3}y^3 + 2y^2 - \tfrac{1}{6}y^6\right]_0^1 = \tfrac{41}{24}$

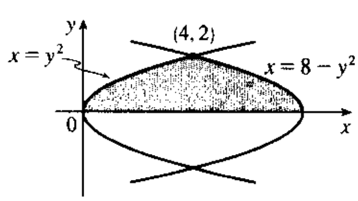
17.



$$\iint_D \frac{y}{1+x^2}\,dA = \int_0^1 \int_0^{\sqrt{x}} \frac{y}{1+x^2}\,dy\,dx = \int_0^1 \frac{1}{1+x^2}\left[\tfrac{1}{2}y^2\right]_{y=0}^{y=\sqrt{x}} dx$$
$$= \tfrac{1}{2}\int_0^1 \frac{x}{1+x^2}\,dx = \left[\tfrac{1}{4}\ln(1+x^2)\right]_0^1 = \tfrac{1}{4}\ln 2$$

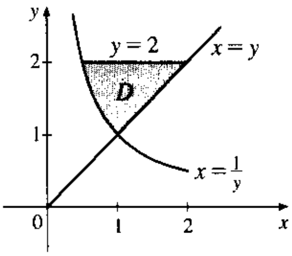
18. $$\iint_D \frac{1}{1+x^2}\,dA = \int_0^1 \int_x^1 \frac{1}{1+x^2}\,dy\,dx = \int_0^1 \frac{1}{1+x^2}\left[\,y\,\right]_{y=x}^{y=1} dx = \int_0^1 \frac{1-x}{1+x^2}\,dx$$
$$= \int_0^1 \left(\frac{1}{1+x^2} - \frac{x}{1+x^2}\right) dx = \left[\tan^{-1}x - \tfrac{1}{2}\ln(1+x^2)\right]_0^1$$
$$= \tan^{-1}1 - \tfrac{1}{2}\ln 2 - \left(\tan^{-1}0 - \tfrac{1}{2}\ln 1\right) = \tfrac{\pi}{4} - \tfrac{1}{2}\ln 2$$

19.



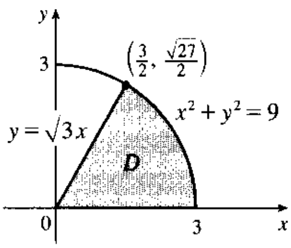
$$\iint_D y\,dA = \int_0^2 \int_{y^2}^{8-y^2} y\,dx\,dy$$
$$= \int_0^2 y\left[x\right]_{x=y^2}^{x=8-y^2} dy = \int_0^2 y(8-y^2-y^2)\,dy$$
$$= \int_0^2 (8y - 2y^3)\,dy = \left[4y^2 - \tfrac{1}{2}y^4\right]_0^2 = 8$$

20.



$$\iint_D y\,dA = \int_1^2 \int_{1/y}^{y} y\,dx\,dy = \int_1^2 y\left(y - \frac{1}{y}\right) dy$$
$$= \int_1^2 (y^2 - 1)\,dy = \left[\tfrac{1}{3}y^3 - y\right]_1^2$$
$$= \left(\tfrac{8}{3} - 2\right) - \left(\tfrac{1}{3} - 1\right) = \tfrac{4}{3}$$

21.

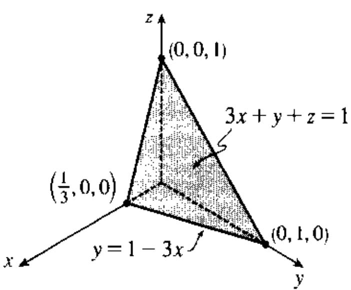


$$\iint_D (x^2 + y^2)^{3/2}\,dA = \int_0^{\pi/3} \int_0^3 (r^2)^{3/2} r\,dr\,d\theta$$
$$= \int_0^{\pi/3} d\theta \int_0^3 r^4\,dr = \left[\theta\right]_0^{\pi/3} \left[\frac{1}{5}r^5\right]_0^3$$
$$= \frac{\pi}{3}\frac{3^5}{5} = \frac{81\pi}{5}$$

22. $\iint_D x\,dA = \int_0^{\pi/2} \int_1^{\sqrt{2}} (r\cos\theta)\,r\,dr\,d\theta = \int_0^{\pi/2} \cos\theta\,d\theta \int_1^{\sqrt{2}} r^2\,dr = \left[\sin\theta\right]_0^{\pi/2} \left[\frac{1}{3}r^3\right]_1^{\sqrt{2}}$
$= 1 \cdot \frac{1}{3}(2^{3/2} - 1) = \frac{1}{3}(2^{3/2} - 1)$

23. $\iiint_E xy\,dV = \int_0^3 \int_0^x \int_0^{x+y} xy\,dz\,dy\,dx = \int_0^3 \int_0^x xy\left[z\right]_{z=0}^{z=x+y} dy\,dx = \int_0^3 \int_0^x xy(x+y)\,dy\,dx$
$= \int_0^3 \int_0^x (x^2y + xy^2)\,dy\,dx = \int_0^3 \left[\frac{1}{2}x^2y^2 + \frac{1}{3}xy^3\right]_{y=0}^{y=x} dx = \int_0^3 \left(\frac{1}{2}x^4 + \frac{1}{3}x^4\right) dx$
$= \frac{5}{6}\int_0^3 x^4\,dx = \left[\frac{1}{6}x^5\right]_0^3 = \frac{81}{2} = 40.5$

24.



$$\iiint_T xy\,dV = \int_0^{1/3} \int_0^{1-3x} \int_0^{1-3x-y} xy\,dz\,dy\,dx$$
$$= \int_0^{1/3} \int_0^{1-3x} xy(1 - 3x - y)\,dy\,dx$$
$$= \int_0^{1/3} \int_0^{1-3x} (xy - 3x^2y - xy^2)\,dy\,dx$$
$$= \int_0^{1/3} \left[\tfrac{1}{2}xy^2 - \tfrac{3}{2}x^2y^2 - \tfrac{1}{3}xy^3\right]_{y=0}^{y=1-3x} dx$$
$$= \int_0^{1/3} \left[\tfrac{1}{2}x(1-3x)^2 - \tfrac{3}{2}x^2(1-3x)^2 - \tfrac{1}{3}x(1-3x)^3\right] dx$$
$$= \int_0^{1/3} \left(\tfrac{1}{6}x - \tfrac{3}{2}x^2 + \tfrac{9}{2}x^3 - \tfrac{9}{2}x^4\right) dx$$
$$= \left[\tfrac{1}{12}x^2 - \tfrac{1}{2}x^3 + \tfrac{9}{8}x^4 - \tfrac{9}{10}x^5\right]_0^{1/3} = \tfrac{1}{1080}$$

25. $\iiint_E y^2z^2\,dV = \int_{-1}^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} \int_0^{1-y^2-z^2} y^2z^2\,dx\,dz\,dy = \int_{-1}^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} y^2z^2(1 - y^2 - z^2)\,dz\,dy$
$= \int_0^{2\pi} \int_0^1 (r^2\cos^2\theta)(r^2\sin^2\theta)(1 - r^2)\,r\,dr\,d\theta = \int_0^{2\pi} \int_0^1 \frac{1}{4}\sin^2 2\theta(r^5 - r^7)\,dr\,d\theta$
$= \int_0^{2\pi} \frac{1}{8}(1 - \cos 4\theta)\left[\frac{1}{6}r^6 - \frac{1}{8}r^8\right]_{r=0}^{r=1} d\theta = \frac{1}{192}\left[\theta - \frac{1}{4}\sin 4\theta\right]_0^{2\pi} = \frac{2\pi}{192} = \frac{\pi}{96}$

26. $\iiint_E z\,dV = \int_0^1 \int_0^{\sqrt{1-y^2}} \int_0^{2-y} z\,dx\,dz\,dy = \int_0^1 \int_0^{\sqrt{1-y^2}} (2-y)z\,dz\,dy = \int_0^1 \frac{1}{2}(2-y)(1-y^2)\,dy$
$= \int_0^1 \frac{1}{2}(2 - y - 2y^2 + y^3)\,dy = \frac{13}{24}$

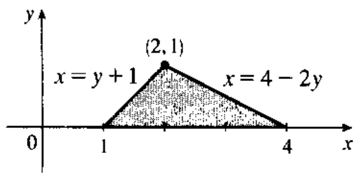
27. $\iiint_E yz\,dV = \int_{-2}^2 \int_0^{\sqrt{4-x^2}} \int_0^y yz\,dz\,dy\,dx = \int_{-2}^2 \int_0^{\sqrt{4-x^2}} \frac{1}{2}y^3\,dy\,dx = \int_0^{\pi} \int_0^2 \frac{1}{2}r^3(\sin^3\theta)\,r\,dr\,d\theta$
$= \frac{16}{5}\int_0^{\pi} \sin^3\theta\,d\theta = \frac{16}{5}\left[-\cos\theta + \frac{1}{3}\cos^3\theta\right]_0^{\pi} = \frac{64}{15}$

28. $\iiint_H z^3\sqrt{x^2+y^2+z^2}\,dV = \int_0^{2\pi}\int_0^{\pi/2}\int_0^1 (\rho^3\cos^3\phi)\rho(\rho^2\sin\phi)\,d\rho\,d\phi\,d\theta$

$= \int_0^{2\pi} d\theta \int_0^{\pi/2}\cos^3\phi\sin\phi\,d\phi \int_0^1 \rho^6\,d\rho = 2\pi\left[-\frac14\cos^4\phi\right]_0^{\pi/2}\left(\frac17\right) = \frac{\pi}{14}$

29. $V = \int_0^2\int_1^4 (x^2+4y^2)\,dy\,dx = \int_0^2\left[x^2y + \frac43 y^3\right]_{y=1}^{y=4} dx = \int_0^2 (3x^2+84)\,dx = 176$

30.



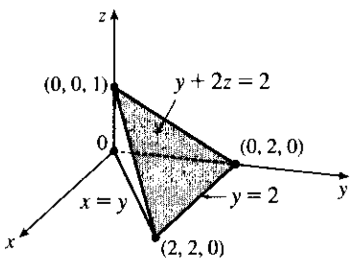
$V = \int_0^1\int_{y+1}^{4-2y}\int_0^{x^2y} dz\,dx\,dy = \int_0^1\int_{y+1}^{4-2y} x^2y\,dx\,dy$

$= \int_0^1 \frac13\left[(4-2y)^3y - (y+1)^3y\right] dy$

$= \int_0^1 3(-y^4+5y^3-11y^2+7y)\,dy$

$= 3\left(-\frac15+\frac54-\frac{11}{3}+\frac72\right) = \frac{53}{20}$

31.



$V = \int_0^2\int_0^y\int_0^{(2-y)/2} dz\,dx\,dy$

$= \int_0^2\int_0^y \left(1-\frac12 y\right) dx\,dy$

$= \int_0^2 \left(y - \frac12 y^2\right) dy = \frac23$

32. $V = \int_0^{2\pi}\int_0^2\int_0^{3-r\sin\theta} r\,dz\,dr\,d\theta = \int_0^{2\pi}\int_0^2 (3r - r^2\sin\theta)\,dr\,d\theta = \int_0^{2\pi}\left[6-\frac83\sin\theta\right] d\theta$

$= 6\theta\big]_0^{2\pi} + 0 = 12\pi$

33. Using the wedge above the plane $z=0$ and below the plane $z=mx$ and noting that we have the same volume for $m<0$ as for $m>0$ (so use $m>0$), we have

$$V = 2\int_0^{a/3}\int_0^{\sqrt{a^2-9y^2}} mx\,dx\,dy = 2\int_0^{a/3}\frac12 m(a^2-9y^2)\,dy = m\left[a^2y-3y^3\right]_0^{a/3}$$

$$= m\left(\frac13 a^3 - \frac19 a^3\right) = \frac29 ma^3$$

34. The paraboloid and the half-cone intersect when $x^2+y^2 = \sqrt{x^2+y^2}$, that is when $x^2+y^2=1$ or 0. So

$$V = \iint_{x^2+y^2\le 1}\int_{x^2+y^2}^{\sqrt{x^2+y^2}} dz\,dA = \int_0^{2\pi}\int_0^1\int_{r^2}^{r} r\,dz\,dr\,d\theta$$

$$= \int_0^{2\pi}\int_0^1 (r^2-r^3)\,dr\,d\theta = \int_0^{2\pi}\left(\frac13-\frac14\right)d\theta = \frac{1}{12}(2\pi) = \frac{\pi}{6}$$

35. (a) $m = \int_0^1\int_0^{1-y^2} y\,dx\,dy = \int_0^1 (y-y^3)\,dy = \frac12-\frac14 = \frac14$

(b) $M_y = \int_0^1\int_0^{1-y^2} xy\,dx\,dy = \int_0^1 \frac12 y(1-y^2)^2\,dy = -\frac{1}{12}(1-y^2)^3\big]_0^1 = \frac{1}{12}$,

$M_x = \int_0^1\int_0^{1-y^2} y^2\,dx\,dy = \int_0^1 (y^2-y^4)\,dy = \frac{2}{15}$. Hence $(\overline{x},\overline{y}) = \left(\frac13, \frac{8}{15}\right)$.

(c) $I_x = \int_0^1\int_0^{1-y^2} y^3\,dx\,dy = \int_0^1 (y^3-y^5)\,dy = \frac{1}{12}$,

$I_y = \int_0^1\int_0^{1-y^2} yx^2\,dx\,dy = \int_0^1 \frac13 y(1-y^2)^3\,dy = -\frac{1}{24}(1-y^2)^4\big]_0^1 = \frac{1}{24}$,

$I_0 = I_x + I_y = \frac18$, $\overline{\overline{y}}^2 = \frac{1/12}{1/4} = \frac13 \quad\Rightarrow\quad \overline{\overline{y}} = \frac{1}{\sqrt3}$, and $\overline{\overline{x}}^2 = \frac{1/24}{1/4} = \frac16 \quad\Rightarrow\quad \overline{\overline{x}} = \frac{1}{\sqrt6}$.

36. (a) $m = \frac{1}{4}\pi K a^2$ where K is constant,

$M_y = \iint_{x^2+y^2\le a^2} Kx\,dA = K\int_0^{\pi/2}\int_0^a r^2\cos\theta\,dr\,d\theta = \frac{1}{3}Ka^3\int_0^{\pi/2}\cos\theta\,d\theta = \frac{1}{3}a^3K$, and

$M_x = K\int_0^{\pi/2}\int_0^a r^2\sin\theta\,dr\,d\theta = \frac{1}{3}a^3K$ [by symmetry $M_y = M_x$].

Hence the centroid is $(\overline{x},\overline{y}) = \left(\frac{4}{3\pi}a, \frac{4}{3\pi}a\right)$.

(b) $m = \int_0^{\pi/2}\int_0^a r^4\cos\theta\sin^2\theta\,dr\,d\theta = \left[\frac{1}{3}\sin^3\theta\right]_0^{\pi/2}\left(\frac{1}{5}a^5\right) = \frac{1}{15}a^5$,

$M_y = \int_0^{\pi/2}\int_0^a r^5\cos^2\theta\sin^2\theta\,dr\,d\theta = \frac{1}{8}\left[\theta - \frac{1}{4}\sin 4\theta\right]_0^{\pi/2}\left(\frac{1}{6}a^6\right) = \frac{1}{96}\pi a^6$, and

$M_x = \int_0^{\pi/2}\int_0^a r^5\cos\theta\sin^3\theta\,dr\,d\theta = \left[\frac{1}{4}\sin^4\theta\right]_0^{\pi/2}\left(\frac{1}{6}a^6\right) = \frac{1}{24}a^6$. Hence $(\overline{x},\overline{y}) = \left(\frac{5}{32}\pi a, \frac{5}{8}a\right)$.

37. (a) The equation of the cone with the suggested orientation is $(h-z) = \frac{h}{a}\sqrt{x^2+y^2}$, $0\le z\le h$. Then $V = \frac{1}{3}\pi a^2 h$ is the volume of one frustum of a cone; by symmetry $M_{yz} = M_{xz} = 0$; and

$$M_{xy} = \iint_{x^2+y^2\le a^2}\int_0^{h-(h/a)\sqrt{x^2+y^2}} z\,dz\,dA = \int_0^{2\pi}\int_0^a\int_0^{(h/a)(a-r)} rz\,dz\,dr\,d\theta$$

$$= \pi\int_0^a r\frac{h^2}{a^2}(a-r)^2\,dr = \frac{\pi h^2}{a^2}\int_0^a (a^2r - 2ar^2 + r^3)\,dr = \frac{\pi h^2}{a^2}\left(\frac{a^4}{2} - \frac{2a^4}{3} + \frac{a^4}{4}\right) = \frac{\pi h^2 a^2}{12}$$

Hence the centroid is $(\overline{x},\overline{y},\overline{z}) = \left(0, 0, \frac{1}{4}h\right)$.

(b) $$I_z = \int_0^{2\pi}\int_0^a\int_0^{(h/a)(a-r)} r^3\,dz\,dr\,d\theta = 2\pi\int_0^a \frac{h}{a}(ar^3 - r^4)\,dr = \frac{2\pi h}{a}\left(\frac{a^5}{4} - \frac{a^5}{5}\right) = \frac{\pi a^4 h}{10}$$

38. $1\le z^2\le 4 \;\Rightarrow\; 1/a^2 \le x^2+y^2 \le 4/a^2$. Let $D = \left\{(x,y) \mid 1/a^2 \le x^2+y^2 \le 4/a^2\right\}$.

$z = f(x,y) = a\sqrt{x^2+y^2}$, so $f_x(x,y) = ax(x^2+y^2)^{-1/2}$, $f_y(x,y) = ay(x^2+y^2)^{-1/2}$, and

$$A(S) = \iint_D \sqrt{\frac{a^2x^2+a^2y^2}{x^2+y^2} + 1}\,dA = \iint_D \sqrt{a^2+1}\,dA = \sqrt{a^2+1}\,A(D)$$

$$= \sqrt{a^2+1}\left[\pi\left(\frac{2}{a}\right)^2 - \pi\left(\frac{1}{a}\right)^2\right] = \frac{3\pi}{a^2}\sqrt{a^2+1}$$

39. Let D represent the given triangle; then D can be described as the area enclosed by the x- and y-axes and the line $y = 2-2x$, or equivalently $D = \{(x,y) \mid 0\le x\le 1, 0\le y\le 2-2x\}$. We want to find the surface area of the part of the graph of $z = x^2 + y$ that lies over D, so using Equation 16.6.3 [ET 15.6.3] we have

$$A(S) = \iint_D \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial y}\right)^2}\,dA = \iint_D \sqrt{1 + (2x)^2 + (1)^2}\,dA$$

$$= \int_0^1\int_0^{2-2x}\sqrt{2+4x^2}\,dy\,dx = \int_0^1\sqrt{2+4x^2}\,\left[y\right]_{y=0}^{y=2-2x}\,dx = \int_0^1 (2-2x)\sqrt{2+4x^2}\,dx$$

$$= \int_0^1 2\sqrt{2+4x^2}\,dx - \int_0^1 2x\sqrt{2+4x^2}\,dx$$

Using Formula 21 in the Table of Integrals with $a = \sqrt{2}$, $u = 2x$, and $du = 2\,dx$, we have

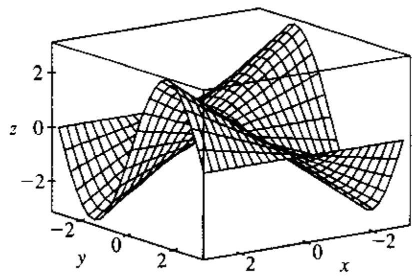
$\int 2\sqrt{2+4x^2}\,dx = x\sqrt{2+4x^2} + \ln\left(2x + \sqrt{2+4x^2}\right)$. If we substitute $u = 2+4x^2$ in the second integral, then $du = 8x\,dx$ and $\int 2x\sqrt{2+4x^2}\,dx = \frac{1}{4}\int \sqrt{u}\,du = \frac{1}{4}\cdot\frac{2}{3}u^{3/2} = \frac{1}{6}(2+4x^2)^{3/2}$. Thus

$$A(S) = \left[x\sqrt{2+4x^2} + \ln\left(2x+\sqrt{2+4x^2}\right) - \tfrac{1}{6}(2+4x^2)^{3/2}\right]_0^1$$

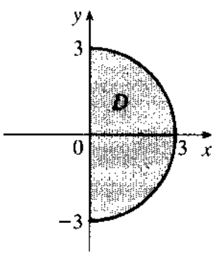
$$= \sqrt{6} + \ln\left(2+\sqrt{6}\right) - \tfrac{1}{6}(6)^{3/2} - \ln\sqrt{2} + \tfrac{\sqrt{2}}{3} = \ln\tfrac{2+\sqrt{6}}{\sqrt{2}} + \tfrac{\sqrt{2}}{3}$$

$$= \ln\left(\sqrt{2}+\sqrt{3}\right) + \tfrac{\sqrt{2}}{3} \approx 1.6176$$

40. Using Formula 16.6.3 [ET 15.6.3] with $\partial z/\partial x = \sin y$, $\partial z/\partial y = x\cos y$, we get $S = \int_{-\pi}^{\pi}\int_{-3}^{3}\sqrt{\sin^2 y + x^2\cos^2 y + 1}\,dx\,dy \approx 62.9714$.



41.



$$\int_0^3\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}(x^3+xy^2)\,dy\,dx = \int_0^3\int_{-\sqrt{9-x^2}}^{\sqrt{9-x^2}}x(x^2+y^2)\,dy\,dx$$

$$= \int_{-\pi/2}^{\pi/2}\int_0^3 (r\cos\theta)(r^2)\,r\,dr\,d\theta$$

$$= \int_{-\pi/2}^{\pi/2}\cos\theta\,d\theta\int_0^3 r^4\,dr$$

$$= \left[\sin\theta\right]_{-\pi/2}^{\pi/2}\left[\tfrac{1}{5}r^5\right]_0^3 = 2\cdot\tfrac{1}{5}(243) = \tfrac{486}{5} = 97.2$$

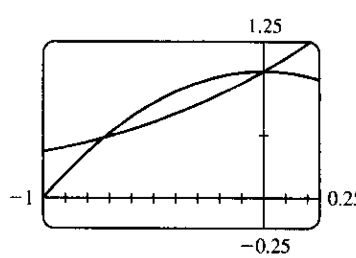
42. The region of integration is the solid hemisphere $x^2+y^2+z^2 \le 4$, $x \ge 0$.

$$\int_{-2}^{2}\int_0^{\sqrt{4-y^2}}\int_{-\sqrt{4-x^2-y^2}}^{\sqrt{4-x^2-y^2}} y^2\sqrt{x^2+y^2+z^2}\,dz\,dx\,dy$$

$$= \int_{-\pi/2}^{\pi/2}\int_0^{\pi}\int_0^2(\rho\sin\phi\sin\theta)^2\left(\sqrt{\rho^2}\right)\rho^2\sin\phi\,d\rho\,d\phi\,d\theta = \int_{-\pi/2}^{\pi/2}\sin^2\theta\,d\theta\int_0^{\pi}\sin^3\phi\,d\phi\int_0^2\rho^5\,d\rho$$

$$= \left[\tfrac{1}{2}\theta - \tfrac{1}{4}\sin 2\theta\right]_{-\pi/2}^{\pi/2}\left[-\tfrac{1}{3}(2+\sin^2\phi)\cos\phi\right]_0^{\pi}\left[\tfrac{1}{6}\rho^6\right]_0^2 = \left(\tfrac{\pi}{2}\right)\left(\tfrac{2}{3}+\tfrac{2}{3}\right)\left(\tfrac{32}{3}\right) = \tfrac{64}{9}\pi$$

43. From the graph, it appears that $1-x^2 = e^x$ at $x \approx -0.71$ and at $x = 0$, with $1-x^2 > e^x$ on $(-0.71, 0)$. So the desired integral is



$$\iint_D y^2\,dA \approx \int_{-0.71}^{0}\int_{e^x}^{1-x^2} y^2\,dy\,dx$$

$$= \tfrac{1}{3}\int_{-0.71}^{0}\left[(1-x^2)^3 - e^{3x}\right]dx$$

$$= \tfrac{1}{3}\left[x - x^3 + \tfrac{3}{5}x^5 - \tfrac{1}{7}x^7 - \tfrac{1}{3}e^{3x}\right]_{-0.71}^{0} \approx 0.0512$$

44. Let the tetrahedron be called T. The front face of T is given by the plane $x + \frac{1}{2}y + \frac{1}{3}z = 1$, or $z = 3 - 3x - \frac{3}{2}y$, which intersects the xy-plane in the line $y = 2 - 2x$. So the total mass is $m = \iiint_T \rho(x,y,z)\,dV = \int_0^1 \int_0^{2-2x} \int_0^{3-3x-3y/2} (x^2 + y^2 + z^2)\,dz\,dy\,dx = \frac{7}{5}$. The center of mass is

$$(\overline{x},\overline{y},\overline{z}) = \left(m^{-1} \iiint_T x\rho(x,y,z)\,dV, m^{-1} \iiint_T y\rho(x,y,z)\,dV, m^{-1} \iiint_T z\rho(x,y,z)\,dV\right)$$
$$= \left(\tfrac{4}{21}, \tfrac{11}{21}, \tfrac{8}{7}\right)$$

45. (a) $f(x,y)$ is a joint density function, so we know that $\iint_{\mathbb{R}^2} f(x,y)\,dA = 1$. Since $f(x,y) = 0$ outside the rectangle $[0,3] \times [0,2]$, we can say

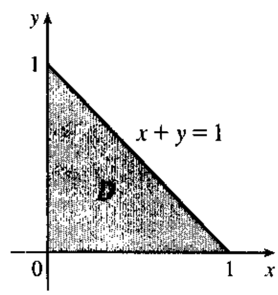
$$\iint_{\mathbb{R}^2} f(x,y)\,dA = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y)\,dy\,dx = \int_0^3 \int_0^2 C(x+y)\,dy\,dx$$
$$= C \int_0^3 \left[xy + \tfrac{1}{2}y^2\right]_{y=0}^{y=2} dx = C \int_0^3 (2x+2)\,dx = C\left[x^2 + 2x\right]_0^3 = 15C$$

Then $15C = 1 \Rightarrow C = \frac{1}{15}$.

(b) $P(X \le 2, Y \ge 1) = \int_{-\infty}^{2} \int_1^{\infty} f(x,y)\,dy\,dx = \int_0^2 \int_1^2 \frac{1}{15}(x,y)\,dy\,dx = \frac{1}{15} \int_0^2 \left[xy + \frac{1}{2}y^2\right]_{y=1}^{y=2} dx$

$$= \tfrac{1}{15} \int_0^2 \left(x + \tfrac{3}{2}\right) dx = \tfrac{1}{15}\left[\tfrac{1}{2}x^2 + \tfrac{3}{2}x\right]_0^2 = \tfrac{1}{3}$$

(c) $P(X + Y \le 1) = P((X,Y) \in D)$ where D is the triangular region shown in the figure. Thus



$$P(X + Y \le 1) = \iint_D f(x,y)\,dA = \int_0^1 \int_0^{1-x} \tfrac{1}{15}(x+y)\,dy\,dx$$
$$= \tfrac{1}{15} \int_0^1 \left[xy + \tfrac{1}{2}y^2\right]_{y=0}^{y=1-x} dx$$
$$= \tfrac{1}{15} \int_0^1 \left[x(1-x) + \tfrac{1}{2}(1-x)^2\right] dx$$
$$= \tfrac{1}{30} \int_0^1 (1 - x^2)\,dx = \tfrac{1}{30}\left[x - \tfrac{1}{3}x^3\right]_0^1 = \tfrac{1}{45}$$

46. Each lamp has exponential density function

$$f(t) = \begin{cases} 0 & \text{if } t < 0 \\ \frac{1}{800}e^{-t/800} & \text{if } t \ge 0 \end{cases}$$

If X, Y, and Z are the lifetimes of the individual bulbs, then X, Y, and Z are independent, so the joint density function is the product of the individual density functions:

$$f(x,y,z) = \begin{cases} \frac{1}{800^3}e^{-(x+y+z)/800} & \text{if } x \ge 0, y \ge 0, z \ge 0 \\ 0 & \text{otherwise} \end{cases}$$

The probability that all three bulbs fail within a total of 1000 hours is $P(X + Y + Z \le 1000)$, or equivalently $P((X,Y,Z) \in E)$ where E is the solid region in the first octant bounded by the coordinate planes and the plane

$x + y + z = 1000$. The plane $x + y + z = 1000$ meets the xy-plane in the line $x + y = 1000$, so we have

$$P(X + Y + Z \le 1000) = \iiint_E f(x, y, z)\, dV = \int_0^{1000} \int_0^{1000-x} \int_0^{1000-x-y} \tfrac{1}{800^3} e^{-(x+y+z)/800}\, dz\, dy\, dx$$

$$= \tfrac{1}{800^3} \int_0^{1000} \int_0^{1000-x} -800 \left[e^{-(x+y+z)/800} \right]_{z=0}^{z=1000-x-y} dy\, dx$$

$$= \tfrac{-1}{800^2} \int_0^{1000} \int_0^{1000-x} \left[e^{-5/4} - e^{-(x+y)/800} \right] dy\, dx$$

$$= \tfrac{-1}{800^2} \int_0^{1000} \left[e^{-5/4} y + 800 e^{-(x+y)/800} \right]_{y=0}^{y=1000-x} dx$$

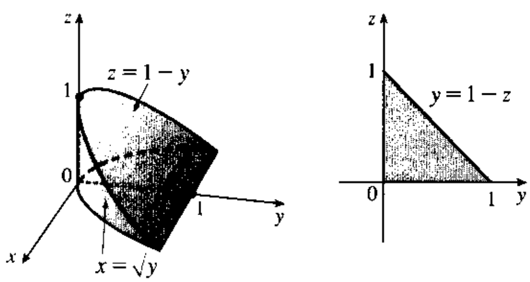
$$= \tfrac{-1}{800^2} \int_0^{1000} \left[e^{-5/4}(1800 - x) - 800 e^{-x/800} \right] dx$$

$$= \tfrac{-1}{800^2} \left[-\tfrac{1}{2} e^{-5/4}(1800 - x)^2 + 800^2 e^{-x/800} \right]_0^{1000}$$

$$= \tfrac{-1}{800^2} \left[-\tfrac{1}{2} e^{-5/4}(800)^2 + 800^2 e^{-5/4} + \tfrac{1}{2} e^{-5/4}(1800)^2 - 800^2 \right]$$

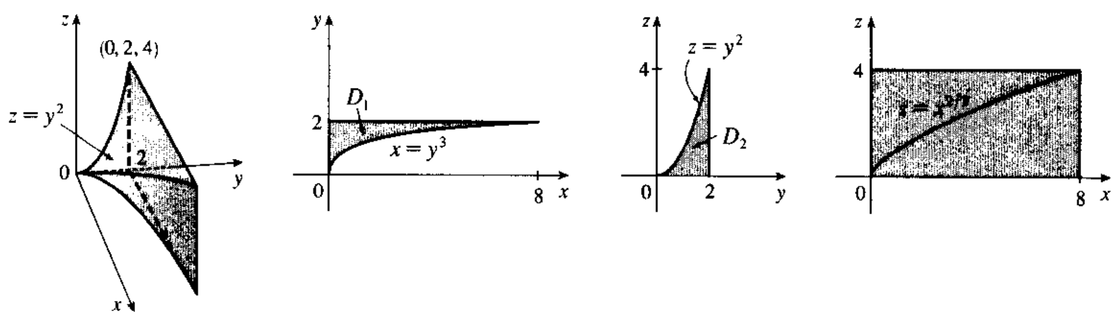
$$= 1 - \tfrac{97}{32} e^{-5/4} \approx 0.1315$$

47.



$$\int_{-1}^{1} \int_{x^2}^{1} \int_0^{1-y} f(x, y, z)\, dz\, dy\, dx = \int_0^1 \int_0^{1-z} \int_{-\sqrt{y}}^{\sqrt{y}} f(x, y, z)\, dx\, dy\, dz$$

48.



$\int_0^2 \int_0^{y^3} \int_0^{y^2} f(x, y, z)\, dz\, dx\, dy = \iiint_E f(x, y, z)\, dV$ where

$E = \left\{(x, y, z) \mid 0 \le y \le 2, 0 \le x \le y^3, 0 \le z \le y^2\right\}$. If D_1, D_2, and D_3 are the projections of E on the xy-, yz-, and xz-planes, then $D_1 = \left\{(x, y) \mid 0 \le y \le 2, 0 \le x \le y^3\right\} = \{(x, y) \mid 0 \le x \le 8, \sqrt[3]{x} \le y \le 2\}$,

$D_2 = \{(y, z) \mid 0 \le z \le 4, \sqrt{z} \le y \le 2\} = \left\{(y, z) \mid 0 \le y \le 2, 0 \le z \le y^2\right\}$,

$D_3 = \{(x, z) \mid 0 \le x \le 8, 0 \le z \le 4\}$.

Therefore we have

$$\begin{aligned}\int_0^2\int_0^{y^3}\int_0^{y^2} f(x,y,z)\,dz\,dx\,dy &= \int_0^8\int_{\sqrt[3]{x}}^2\int_0^{y^2} f(x,y,z)\,dz\,dy\,dx\\ &= \int_0^4\int_{\sqrt{z}}^2\int_0^{y^3} f(x,y,z)\,dx\,dy\,dz = \int_0^2\int_0^{y^2}\int_0^{y^3} f(x,y,z)\,dx\,dz\,dy\\ &= \int_0^8\int_0^{x^{2/3}}\int_{\sqrt[3]{x}}^2 f(x,y,z)\,dy\,dz\,dx + \int_0^8\int_{x^{2/3}}^4\int_{\sqrt{z}}^2 f(x,y,z)\,dy\,dz\,dx\\ &= \int_0^4\int_0^{z^{3/2}}\int_{\sqrt{z}}^2 f(x,y,z)\,dy\,dx\,dz + \int_0^4\int_{z^{3/2}}^8\int_{\sqrt[3]{x}}^2 f(x,y,z)\,dy\,dx\,dz\end{aligned}$$

49. Since $u = x - y$ and $v = x + y$, $x = \frac{1}{2}(u + v)$ and $y = \frac{1}{2}(v - u)$.

Thus $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{vmatrix} = \dfrac{1}{2}$ and $\displaystyle\iint_R \frac{x-y}{x+y}\,dA = \int_2^4\int_{-2}^0 \frac{u}{v}\left(\frac{1}{2}\right)du\,dv = -\int_2^4 \frac{dv}{v} = -\ln 2.$

50. $\dfrac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} 2u & 0 & 0 \\ 0 & 2v & 0 \\ 0 & 0 & 2w \end{vmatrix} = 8uvw$, so

$$\begin{aligned}V = \iiint_E dV &= \int_0^1\int_0^{1-u}\int_0^{1-u-v} 8uvw\,dw\,dv\,du = \int_0^1\int_0^{1-u} 4uv(1-u-v)^2\,du\\ &= \int_0^1\int_0^{1-u}\left[4u(1-u)^2v - 8u(1-u)v^2 + 4uv^3\right]dv\,du\\ &= \int_0^1\left[2u(1-u)^4 - \tfrac{8}{3}u(1-u)^4 + u(1-u)^4\right]du = \int_0^1 \tfrac{1}{3}u(1-u)^4du\\ &= \int_0^1 \tfrac{1}{3}\left[(1-u)^4 - (1-u)^5\right]du = \tfrac{1}{3}\left[-\tfrac{1}{5}(1-u)^5 + \tfrac{1}{6}(1-u)^6\right]_0^1 = \tfrac{1}{3}\left(-\tfrac{1}{6}+\tfrac{1}{5}\right) = \tfrac{1}{90}\end{aligned}$$

51. Let $u = y - x$ and $v = y + x$ so $x = y - u = (v - x) - u \;\Rightarrow\; x = \frac{1}{2}(v - u)$ and

$y = v - \frac{1}{2}(v-u) = \frac{1}{2}(v+u)$. $\left|\dfrac{\partial(x,y)}{\partial(u,v)}\right| = \left|\dfrac{\partial x}{\partial u}\dfrac{\partial y}{\partial v} - \dfrac{\partial x}{\partial v}\dfrac{\partial y}{\partial u}\right| = \left|-\frac{1}{2}\left(\frac{1}{2}\right) - \frac{1}{2}\left(\frac{1}{2}\right)\right| = \left|-\frac{1}{2}\right| = \frac{1}{2}$. R is the image under this transformation of the square with vertices $(u,v) = (0,0)$, $(-2,0)$, $(0,2)$, and $(-2,2)$. So

$$\begin{aligned}\iint_R xy\,dA &= \int_0^2\int_{-2}^0 \frac{v^2-u^2}{4}\left(\frac{1}{2}\right)du\,dv = \tfrac{1}{8}\int_0^2\left[v^2u - \tfrac{1}{3}u^3\right]_{u=-2}^{u=0}dv = \tfrac{1}{8}\int_0^2\left(2v^2 - \tfrac{8}{3}\right)dv\\ &= \tfrac{1}{8}\left[\tfrac{2}{3}v^3 - \tfrac{8}{3}v\right]_0^2 = 0\end{aligned}$$

This result could have been anticipated by symmetry, since the integrand is an odd function of y and R is symmetric about the x-axis.

52. By the Extreme Value Theorem (15.7.8 [ET 14.7.8]), f has an absolute minimum value m and an absolute maximum value M in D. Then by Property 16.3.11 [ET 15.3.11], $mA(D) \le \iint_D f(x,y)\,dA \le MA(D)$.

Dividing through by the positive number $A(D)$, we get $m \leq \dfrac{1}{A(D)} \iint_D f(x,y)\,dA \leq M$. This says that the average value of f over D lies between m and M. But f is continuous on D and takes on the values m and M, and so by the Intermediate Value Theorem must take on all values between m and M. Specifically, there exists a point (x_0, y_0) in D such that $f(x_0, y_0) = \dfrac{1}{A(D)} \iint_D f(x,y)\,dA$ or equivalently $\iint_D f(x,y)\,dA = f(x_0,y_0)\,A(D)$.

53. For each r such that D_r lies within the domain, $A(D_r) = \pi r^2$, and by the Mean Value Theorem for Double Integrals there exists (x_r, y_r) in D_r such that $f(x_r, y_r) = \dfrac{1}{\pi r^2} \iint_{D_r} f(x,y)\,dA$. But $\lim\limits_{r\to 0^+} (x_r, y_r) = (a,b)$, so $\lim\limits_{r\to 0^+} \dfrac{1}{\pi r^2} \iint_{D_r} f(x,y)\,dA = \lim\limits_{r\to 0^+} f(x_r, y_r) = f(a,b)$ by the continuity of f.

54. (a) $$\iint_D \frac{1}{(x^2+y^2)^{n/2}}\,dA = \int_0^{2\pi}\int_r^R \frac{1}{(t^2)^{n/2}}\,t\,dt\,d\theta = 2\pi \int_r^R t^{1-n}\,dt$$

$$= \begin{cases} \left.\dfrac{2\pi}{2-n}t^{2-n}\right]_r^R = \dfrac{2\pi}{2-n}\left(R^{2-n} - r^{2-n}\right) & \text{if } n \neq 2 \\ 2\pi \ln(R/r) & \text{if } n = 2 \end{cases}$$

(b) The integral in part (a) has a limit as $r \to 0^+$ for all values of n such that $2 - n > 0 \;\Leftrightarrow\; n < 2$.

(c) $$\iiint_E \frac{1}{(x^2+y^2+z^2)^{n/2}}\,dV = \int_r^R \int_0^\pi \int_0^{2\pi} \frac{1}{(\rho^2)^{n/2}}\rho^2 \sin\phi\,d\theta\,d\phi\,d\rho$$

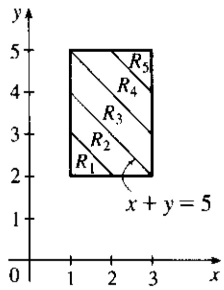
$$= 2\pi \int_r^R \int_0^\pi \rho^{2-n} \sin\phi\,d\phi\,d\rho$$

$$= \begin{cases} \left.\dfrac{4\pi}{3-n}\rho^{3-n}\right]_r^R = \dfrac{4\pi}{3-n}\left(R^{3-n} - r^{3-n}\right) & \text{if } n \neq 3 \\ 4\pi \ln(R/r) & \text{if } n = 3 \end{cases}$$

(d) As $r \to 0^+$, the above integral has a limit, provided that $3 - n > 0 \;\Leftrightarrow\; n < 3$.

☐ PROBLEMS PLUS

1.



Let $R = \bigcup_{i=1}^{5} R_i$, where

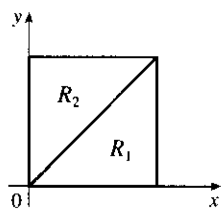
$R_i = \{(x, y) \mid x + y \geq i + 2, x + y < i + 3, 1 \leq x \leq 3, 2 \leq y \leq 5\}$.

$\iint_R [\![x + y]\!]\, dA = \sum_{i=1}^{5} \iint_{R_i} [\![x + y]\!]\, dA = \sum_{i=1}^{5} [\![x + y]\!] \iint_{R_i} dA$, since

$[\![x + y]\!] = \text{constant} = i + 2$ for $(x, y) \in R_i$. Therefore

$$\begin{aligned}\iint_R [\![x + y]\!]\, dA &= \textstyle\sum_{i=1}^{5} (i + 2)\,[A(R_i)] \\ &= 3A(R_1) + 4A(R_2) + 5A(R_3) + 6A(R_4) + 7A(R_5) \\ &= 3\left(\tfrac{1}{2}\right) + 4\left(\tfrac{3}{2}\right) + 5(2) + 6\left(\tfrac{3}{2}\right) + 7\left(\tfrac{1}{2}\right) = 30\end{aligned}$$

2.



Let $R = \{(x, y) \mid 0 \leq x, y \leq 1\}$. For $x, y \in R$, $\max\{x^2, y^2\} = x^2$ if $x \geq y$, and $\max\{x^2, y^2\} = y^2$ if $x \leq y$. Therefore we divide R into two regions: $R = R_1 \cup R_2$, where $R_1 = \{(x, y) \mid 0 \leq x \leq 1, 0 \leq y \leq x\}$ and $R_2 = \{(x, y) \mid 0 \leq y \leq 1, 0 \leq x \leq y\}$. Now $\max\{x^2, y^2\} = x^2$ for $(x, y) \in R_1$, and $\max\{x^2, y^2\} = y^2$ for $(x, y) \in R_2$ $\Rightarrow$

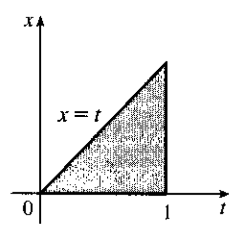
$$\begin{aligned}\int_0^1\!\int_0^1 e^{\max\{x^2,y^2\}}\, dy\, dx &= \iint_R e^{\max\{x^2,y^2\}}\, dA = \iint_{R_1} e^{\max\{x^2,y^2\}}\, dA + \iint_{R_2} e^{\max\{x^2,y^2\}}\, dA \\ &= \int_0^1\!\int_0^x e^{x^2}\, dy\, dx + \int_0^1\!\int_0^y e^{y^2}\, dx\, dy = \int_0^1 x e^{x^2}\, dx + \int_0^1 y e^{y^2}\, dy \\ &= e^{x^2}\Big]_0^1 = e - 1\end{aligned}$$

3. $f_{\text{ave}} = \dfrac{1}{b - a} \displaystyle\int_a^b f(x)\, dx = \dfrac{1}{1 - 0} \int_0^1 \left[\int_x^1 \cos(t^2)\, dt\right] dx$

$= \int_0^1 \int_x^1 \cos(t^2)\, dt\, dx$

$= \int_0^1 \int_0^t \cos(t^2)\, dx\, dt$ [changing the order of integration]

$= \int_0^1 t \cos(t^2)\, dt = \frac{1}{2} \sin(t^2)\big]_0^1 = \frac{1}{2} \sin 1$



4. Let $u = \mathbf{a} \cdot \mathbf{r}$, $v = \mathbf{b} \cdot \mathbf{r}$, $w = \mathbf{c} \cdot \mathbf{r}$, where $\mathbf{a} = \langle a_1, a_2, a_3 \rangle$, $\mathbf{b} = \langle b_1, b_2, b_3 \rangle$, $\mathbf{c} = \langle c_1, c_2, c_3 \rangle$. Under this change of variables, E corresponds to the rectangular box $0 \leq u \leq \alpha$, $0 \leq v \leq \beta$, $0 \leq w \leq \gamma$. So, by Formula 16.9.13 [ET 15.9.13],

$\displaystyle\int_0^\gamma \int_0^\beta \int_0^\alpha uvw\, du\, dv\, dw = \iiint_E (\mathbf{a} \cdot \mathbf{r})(\mathbf{b} \cdot \mathbf{r})(\mathbf{c} \cdot \mathbf{r}) \left|\frac{\partial(u, v, w)}{\partial(x, y, z)}\right| dV$. But

$$\left|\frac{\partial(u,v,w)}{\partial(x,y,z)}\right| = \left\|\begin{matrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{matrix}\right\| = |\mathbf{a}\cdot\mathbf{b}\times\mathbf{c}| \quad\Rightarrow$$

$$\iiint_E (\mathbf{a}\cdot\mathbf{r})(\mathbf{b}\cdot\mathbf{r})(\mathbf{c}\cdot\mathbf{r})\,dV = \frac{1}{|\mathbf{a}\cdot\mathbf{b}\times\mathbf{c}|}\int_0^\gamma\int_0^\beta\int_0^\alpha uvw\,du\,dv\,dw$$

$$= \frac{1}{|\mathbf{a}\cdot\mathbf{b}\times\mathbf{c}|}\left(\frac{\alpha^2}{2}\right)\left(\frac{\beta^2}{2}\right)\left(\frac{\gamma^2}{2}\right) = \frac{(\alpha\beta\gamma)^2}{8\,|\mathbf{a}\cdot\mathbf{b}\times\mathbf{c}|}$$

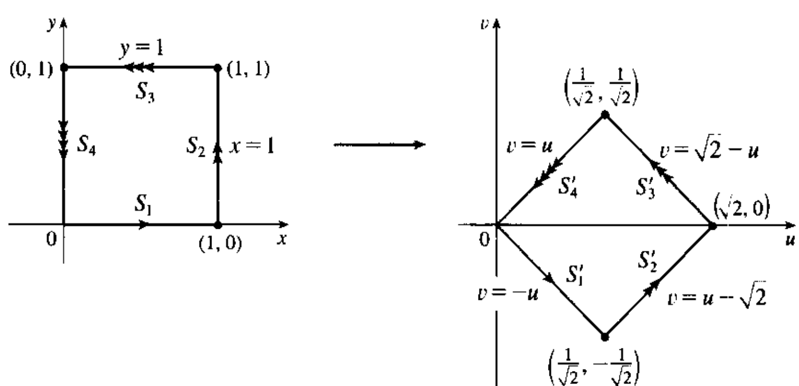
5. Since $|xy| < 1$, except at $(1,1)$, the formula for the sum of a geometric series gives $\dfrac{1}{1-xy} = \sum\limits_{n=0}^{\infty}(xy)^n$, so

$$\int_0^1\int_0^1 \frac{1}{1-xy}\,dx\,dy = \int_0^1\int_0^1 \sum_{n=0}^{\infty}(xy)^n\,dx\,dy = \sum_{n=0}^{\infty}\int_0^1\int_0^1 (xy)^n\,dx\,dy = \sum_{n=0}^{\infty}\left[\int_0^1 x^n\,dx\right]\left[\int_0^1 y^n\,dy\right]$$

$$= \sum_{n=0}^{\infty}\frac{1}{n+1}\cdot\frac{1}{n+1} = \sum_{n=0}^{\infty}\frac{1}{(n+1)^2} = \frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots = \sum_{n=1}^{\infty}\frac{1}{n^2}$$

6. Let $x = \dfrac{u-v}{\sqrt{2}}$ and $y = \dfrac{u+v}{\sqrt{2}}$. We know the region of integration in the xy-plane, so to find its image in the uv-plane we get u and v in terms of x and y, and then use the methods of Section 16.9 [ET 15.9].

$x + y = \dfrac{u-v}{\sqrt{2}} + \dfrac{u+v}{\sqrt{2}} = \sqrt{2}\,u$, so $u = \dfrac{x+y}{\sqrt{2}}$, and similarly $v = \dfrac{y-x}{\sqrt{2}}$. S_1 is given by $y = 0$, $0 \le x \le 1$, so from the equations derived above, the image of S_1 is S_1': $u = \frac{1}{\sqrt{2}}x$, $v = -\frac{1}{\sqrt{2}}x$, $0 \le x \le 1$, that is, $v = -u$, $0 \le u \le \frac{1}{\sqrt{2}}$. Similarly, the image of S_2 is S_2': $v = u - \sqrt{2}$, $\frac{1}{\sqrt{2}} \le u \le \sqrt{2}$, the image of S_3 is S_3': $v = \sqrt{2} - u$, $\frac{1}{\sqrt{2}} \le u \le \sqrt{2}$, and the image of S_4 is S_4': $v - u$, $0 \le u \le \frac{1}{\sqrt{2}}$.



The Jacobian of the transformation is $\dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \partial x/\partial u & \partial x/\partial v \\ \partial y/\partial u & \partial y/\partial v \end{vmatrix} = \begin{vmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{vmatrix} = 1$. From the diagram, we see that we must evaluate two integrals: one over the region $\left\{(u,v) \mid 0 \le u \le \frac{1}{\sqrt{2}},\ -u \le v \le u\right\}$ and the other over $\left\{(u,v) \mid \frac{1}{\sqrt{2}} \le u \le \sqrt{2},\ -\sqrt{2}+u \le v \le \sqrt{2}-u\right\}$. So

$$\int_0^1\int_0^1 \frac{dx\,dy}{1-xy} = \int_0^{\sqrt{2}/2}\int_{-u}^{u} \frac{dv\,du}{1-\left[\frac{1}{\sqrt{2}}(u+v)\right]\left[\frac{1}{\sqrt{2}}(u-v)\right]} + \int_{\sqrt{2}/2}^{\sqrt{2}}\int_{-\sqrt{2}+u}^{\sqrt{2}-u} \frac{dv\,du}{1-\left[\frac{1}{\sqrt{2}}(u+v)\right]\left[\frac{1}{\sqrt{2}}(u-v)\right]}$$

$$= \int_0^{\sqrt{2}/2}\int_{-u}^{u} \frac{2\,dv\,du}{2-u^2+v^2} + \int_{\sqrt{2}/2}^{\sqrt{2}}\int_{-\sqrt{2}+u}^{\sqrt{2}-u} \frac{2\,dv\,du}{2-u^2+v^2}$$

$$= 2\left[\int_0^{\sqrt{2}/2} \frac{1}{\sqrt{2-u^2}}\left[\arctan\frac{v}{\sqrt{2-u^2}}\right]_{-u}^{u} du + \int_{\sqrt{2}/2}^{\sqrt{2}} \frac{1}{\sqrt{2-u^2}}\left[\arctan\frac{v}{\sqrt{2-u^2}}\right]_{-\sqrt{2}+u}^{\sqrt{2}-u} du\right]$$

$$= 4\left[\int_0^{\sqrt{2}/2} \frac{1}{\sqrt{2-u^2}}\arctan\frac{u}{\sqrt{2-u^2}}\,du + \int_{\sqrt{2}/2}^{\sqrt{2}} \frac{1}{\sqrt{2-u^2}}\arctan\frac{\sqrt{2}-u}{\sqrt{2-u^2}}\,du\right]$$

Now let $u = \sqrt{2}\sin\theta$, so $du = \sqrt{2}\cos\theta\,d\theta$ and the limits change to 0 and $\frac{\pi}{6}$ (in the first integral) and $\frac{\pi}{6}$ and $\frac{\pi}{2}$ (in the second integral). Continuing:

$$\int_0^1\int_0^1 \frac{dx\,dy}{1-xy} = 4\left[\int_0^{\pi/6} \frac{1}{\sqrt{2-2\sin^2\theta}}\arctan\left(\frac{\sqrt{2}\sin\theta}{\sqrt{2-2\sin^2\theta}}\right)\left(\sqrt{2}\cos\theta\,d\theta\right) + \int_{\pi/6}^{\pi/2} \frac{1}{\sqrt{2-2\sin^2\theta}}\arctan\left(\frac{\sqrt{2}-\sqrt{2}\sin\theta}{\sqrt{2-2\sin^2\theta}}\right)\left(\sqrt{2}\cos\theta\,d\theta\right)\right]$$

$$= 4\left[\int_0^{\pi/6} \frac{\sqrt{2}\cos\theta}{\sqrt{2}\cos\theta}\arctan\left(\frac{\sqrt{2}\sin\theta}{\sqrt{2}\cos\theta}\right)d\theta + \int_{\pi/6}^{\pi/2} \frac{\sqrt{2}\cos\theta}{\sqrt{2}\cos\theta}\arctan\left(\frac{\sqrt{2}\,(1-\sin\theta)}{\sqrt{2}\cos\theta}\right)d\theta\right]$$

$$= 4\left[\int_0^{\pi/6} \arctan(\tan\theta)\,d\theta + \int_{\pi/6}^{\pi/2} \arctan\left(\frac{1-\sin\theta}{\cos\theta}\right)d\theta\right]$$

But (following the hint)

$$\frac{1-\sin\theta}{\cos\theta} = \frac{1-\cos\left(\frac{\pi}{2}-\theta\right)}{\sin\left(\frac{\pi}{2}-\theta\right)} = \frac{1-\left[1-2\sin^2\left(\frac{1}{2}\left(\frac{\pi}{2}-\theta\right)\right)\right]}{2\sin\left(\frac{1}{2}\left(\frac{\pi}{2}-\theta\right)\right)\cos\left(\frac{1}{2}\left(\frac{\pi}{2}-\theta\right)\right)} \quad \text{[half-angle formulas]}$$

$$= \frac{2\sin^2\left(\frac{1}{2}\left(\frac{\pi}{2}-\theta\right)\right)}{2\sin\left(\frac{1}{2}\left(\frac{\pi}{2}-\theta\right)\right)\cos\left(\frac{1}{2}\left(\frac{\pi}{2}-\theta\right)\right)} = \tan\left(\tfrac{1}{2}\left(\tfrac{\pi}{2}-\theta\right)\right)$$

Continuing:

$$\int_0^1\int_0^1 \frac{dx\,dy}{1-xy} = 4\left[\int_0^{\pi/6} \arctan(\tan\theta)\,d\theta + \int_{\pi/6}^{\pi/2} \arctan\left(\tan\left(\tfrac{1}{2}\left(\tfrac{\pi}{2}-\theta\right)\right)\right)d\theta\right]$$

$$= 4\left[\int_0^{\pi/6}\theta\,d\theta + \int_{\pi/6}^{\pi/2}\left[\tfrac{1}{2}\left(\tfrac{\pi}{2}-\theta\right)\right]d\theta\right]$$

$$= 4\left(\left[\frac{\theta^2}{2}\right]_0^{\pi/6} + \left[\frac{\pi\theta}{4} - \frac{\theta^2}{4}\right]_{\pi/6}^{\pi/2}\right) = 4\left(\frac{3\pi^2}{72}\right) = \frac{\pi^2}{6}$$

7. (a) Since $|xyz| < 1$ except at $(1,1,1)$, the formula for the sum of a geometric series gives $\dfrac{1}{1-xyz} = \sum\limits_{n=0}^{\infty}(xyz)^n$, so

$$\int_0^1\int_0^1\int_0^1 \frac{1}{1-xyz}\,dx\,dy\,dz = \int_0^1\int_0^1\int_0^1 \sum_{n=0}^{\infty}(xyz)^n\,dx\,dy\,dz = \sum_{n=0}^{\infty}\int_0^1\int_0^1\int_0^1 (xyz)^n\,dx\,dy\,dz$$

$$= \sum_{n=0}^{\infty}\left[\int_0^1 x^n\,dx\right]\left[\int_0^1 y^n\,dy\right]\left[\int_0^1 z^n\,dz\right]$$

$$= \sum_{n=0}^{\infty}\frac{1}{n+1}\cdot\frac{1}{n+1}\cdot\frac{1}{n+1}$$

$$= \sum_{n=0}^{\infty}\frac{1}{(n+1)^3} = \frac{1}{1^3}+\frac{1}{2^3}+\frac{1}{3^3}+\cdots = \sum_{n=1}^{\infty}\frac{1}{n^3}$$

(b) Since $|-xyz| < 1$, except at $(1,1,1)$, the formula for the sum of a geometric series gives $\dfrac{1}{1+xyz} = \sum\limits_{n=0}^{\infty}(-xyz)^n$, so

$$\int_0^1\int_0^1\int_0^1 \frac{1}{1+xyz}\,dx\,dy\,dz = \int_0^1\int_0^1\int_0^1 \sum_{n=0}^{\infty}(-xyz)^n\,dx\,dy\,dz$$

$$= \sum_{n=0}^{\infty}\int_0^1\int_0^1\int_0^1 (-xyz)^n\,dx\,dy\,dz$$

$$= \sum_{n=0}^{\infty}(-1)^n\left[\int_0^1 x^n\,dx\right]\left[\int_0^1 y^n\,dy\right]\left[\int_0^1 z^n\,dz\right]$$

$$= \sum_{n=0}^{\infty}(-1)^n\frac{1}{n+1}\cdot\frac{1}{n+1}\cdot\frac{1}{n+1}$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n}{(n+1)^3} = \frac{1}{1^3}-\frac{1}{2^3}+\frac{1}{3^3}-\cdots = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^3}$$

To evaluate this sum, we first write out a few terms: $s = 1 - \frac{1}{2^3} + \frac{1}{3^3} - \frac{1}{4^3} + \frac{1}{5^3} - \frac{1}{6^3} \approx 0.8998$. Notice that $a_7 = \frac{1}{7^3} < 0.003$. By the Alternating Series Estimation Theorem from Section 12.5 [ET 11.5], we have $|s - s_6| \le a_7 < 0.003$. This error of 0.003 will not affect the second decimal place, so we have $s \approx 0.90$.

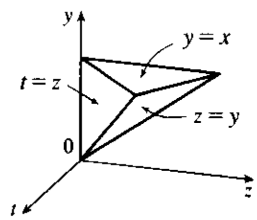
8.

$$\int_0^{\infty}\frac{\arctan \pi x - \arctan x}{x}\,dx = \int_0^{\infty}\left[\frac{\arctan yx}{x}\right]_{y=1}^{y=\pi}dx = \int_0^{\infty}\int_1^{\pi}\frac{1}{1+y^2x^2}\,dy\,dx$$

$$= \int_1^{\pi}\int_0^{\infty}\frac{1}{1+y^2x^2}\,dx\,dy = \int_1^{\pi}\lim_{t\to\infty}\left[\frac{\arctan yx}{y}\right]_{x=0}^{x=t}dy$$

$$= \int_1^{\pi}\frac{\pi}{2y}\,dy = \tfrac{\pi}{2}\left[\ln y\right]_1^{\pi} = \tfrac{\pi}{2}\ln \pi$$

9. $\int_0^x \int_0^y \int_0^z f(t)\,dt\,dz\,dy = \iiint_E f(t)\,dV$, where



$E = \{(t,z,y) \mid 0 \le t \le z, 0 \le z \le y, 0 \le y \le x\}$.

If we let D be the projection of E on the yt-plane then

$D = \{(y,t) \mid 0 \le t \le x, t \le y \le x\}$. And we see from the diagram

that $E = \{(t,z,y) \mid t \le z \le y, t \le y \le x, 0 \le t \le x\}$. So

$$\begin{aligned}\int_0^x \int_0^y \int_0^z f(t)\,dt\,dz\,dy &= \int_0^x \int_t^x \int_t^y f(t)\,dz\,dy\,dt = \int_0^x \left[\int_t^x (y-t)\,f(t)\,dy\right] dt \\ &= \int_0^x \left[\left(\tfrac{1}{2}y^2 - ty\right) f(t)\right]_{y=t}^{y=x} dt = \int_0^x \left[\tfrac{1}{2}x^2 - tx - \tfrac{1}{2}t^2 + t^2\right] f(t)\,dt \\ &= \int_0^x \left[\tfrac{1}{2}x^2 - tx + \tfrac{1}{2}t^2\right] f(t)\,dt = \int_0^x \left(\tfrac{1}{2}x^2 - 2tx + t^2\right) f(t)\,dt \\ &= \tfrac{1}{2}\int_0^x (x-t)^2\,f(t)\,dt\end{aligned}$$

10. (a) Consider a polar division of the disk, similar to that in Figure 16.4.4 [ET 15.4.4], where $0 = \theta_0 < \theta_1 < \theta_2 < \cdots < \theta_n = 2\pi$, $0 = r_1 < r_2 < \cdots < r_m = R$, and where the polar subrectangle R_{ij}, as well as r_i^*, θ_j^*, Δr and $\Delta\theta$ are the same as in that figure. Thus $\Delta A_i = r_i^*\,\Delta r\,\Delta\theta$. The mass of R_{ij} is $\rho\,\Delta A_i$, and its distance from m is $s_{ij} \approx \sqrt{(r_i^*)^2 + d^2}$. According to Newton's Law of Gravitation, the force of attraction experienced by m due to this polar subrectangle is in the direction from m towards R_{ij} and has magnitude $\dfrac{Gm\rho\,\Delta A_i}{s_{ij}^2}$. The symmetry of the lamina with respect to the x- and y-axes and the position of m are such that all horizontal components of the gravitational force cancel, so that the total force is simply in the z-direction. Thus, we need only be concerned with the components of this vertical force; that is, $\dfrac{Gm\rho\,\Delta A_i}{s_{ij}^2}\sin\alpha$, where α is the angle between the origin, r_i^* and the mass m. Thus $\sin\alpha = \dfrac{d}{s_{ij}}$ and the previous result becomes $\dfrac{Gm\rho d\,\Delta A_i}{s_{ij}^3}$. The total attractive force is just the Riemann sum

$$\sum_{i=1}^{m}\sum_{j=1}^{n} \frac{Gm\rho d\,\Delta A_i}{s_{ij}^3} = \sum_{i=1}^{m}\sum_{j=1}^{n} \frac{Gm\rho d(r_i^*)\,\Delta r\,\Delta\theta}{[(r_i^*)^2 + d^2]^{3/2}} \text{ which becomes } \int_0^R \int_0^{2\pi} \frac{Gm\rho d}{(r^2+d^2)^{3/2}}\,r\,d\theta\,dr$$

as $m \to \infty$ and $n \to \infty$. Therefore,

$$F = 2\pi Gm\rho d \int_0^R \frac{r}{(r^2+d^2)^{3/2}}\,dr = 2\pi Gm\rho d\left[-\frac{1}{\sqrt{r^2+d^2}}\right]_0^R = 2\pi Gm\rho d\left(\frac{1}{d} - \frac{1}{\sqrt{R^2+d^2}}\right)$$

(b) This is just the result of part (a) in the limit as $R \to \infty$. In this case $\dfrac{1}{\sqrt{R^2+d^2}} \to 0$, and we are left with

$$F = 2\pi Gm\rho d\left(\frac{1}{d} - 0\right) = 2\pi Gm\rho.$$