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Chapter 7

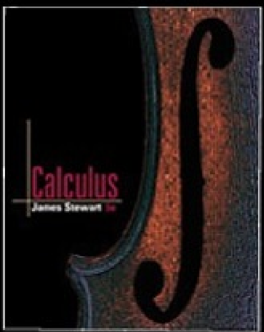
Adapted from the

Complete Solutions Manual

for

James Stewart's

Calculus – 5th Edition



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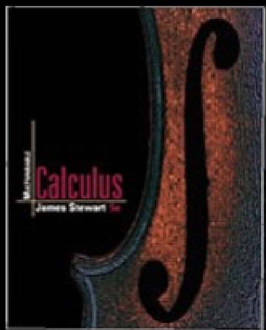

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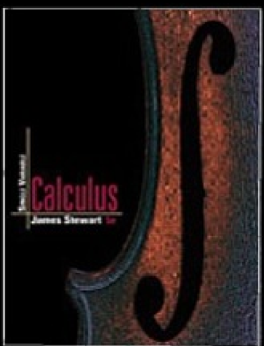

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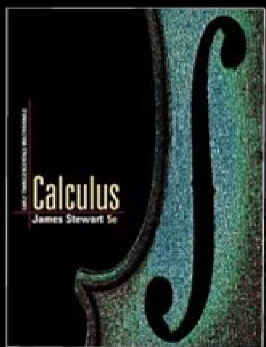

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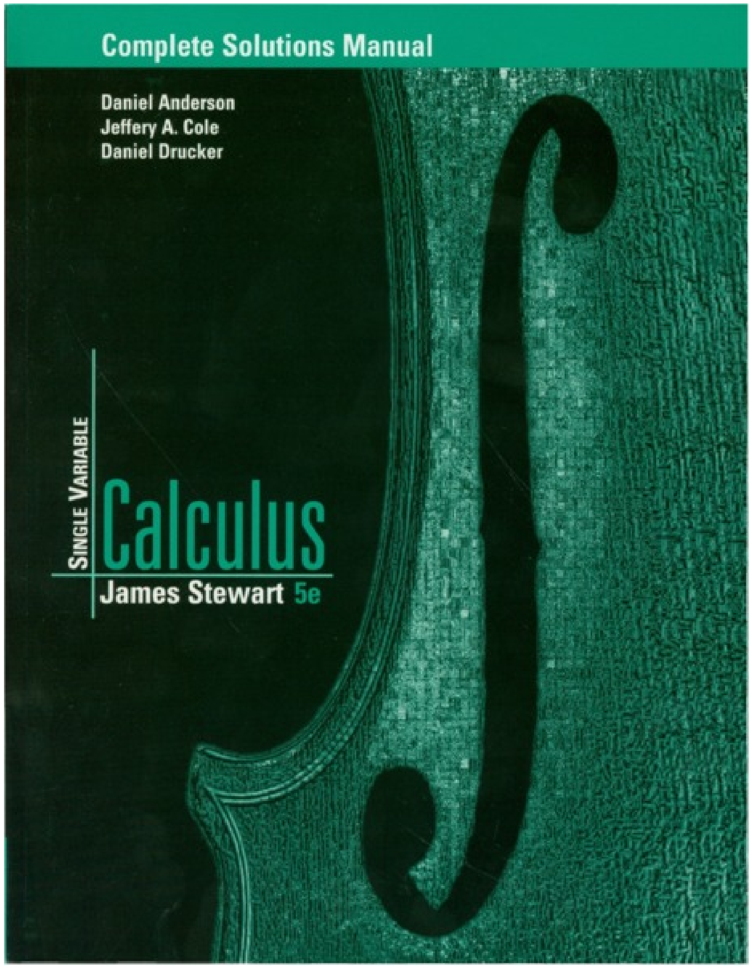


7 ☐ INVERSE FUNCTIONS:
Exponential, Logarithmic, and Inverse Trigonometric Functions

7.1 Inverse Functions

1. (a) See Definition 1.

(b) It must pass the Horizontal Line Test.

2. (a) $f^{-1}(y) = x \quad \Leftrightarrow \quad f(x) = y$ for any y in B. The domain of f^{-1} is B and the range of f^{-1} is A.

(b) See the steps in (5).

(c) Reflect the graph of f about the line $y = x$.

3. f is not one-to-one because $2 \neq 6$, but $f(2) = 2.0 = f(6)$.

4. f is one-to-one since for any two different domain values, there are different range values.

5. No horizontal line intersects the graph of f more than once. Thus, by the Horizontal Line Test, f is one-to-one.

6. The horizontal line $y = 0$ (the x-axis) intersects the graph of f in more than one point. Thus, by the Horizontal Line Test, f is not one-to-one.

7. The horizontal line $y = 0$ (the x-axis) intersects the graph of f in more than one point. Thus, by the Horizontal Line Test, f is not one-to-one.

8. No horizontal line intersects the graph of f more than once. Thus, by the Horizontal Line Test, f is one-to-one.

9. The graph of $f(x) = \frac{1}{2}(x + 5)$ is a line with slope $\frac{1}{2}$. It passes the Horizontal Line Test, so f is one-to-one.
Algebraic solution: If $x_1 \neq x_2$, then $x_1 + 5 \neq x_2 + 5 \quad \Rightarrow \quad \frac{1}{2}(x_1 + 5) \neq \frac{1}{2}(x_2 + 5) \quad \Rightarrow \quad f(x_1) \neq f(x_2)$, so f is one-to-one.

10. The graph of $f(x) = 1 + 4x - x^2$ is a parabola with axis of symmetry $x = -\frac{b}{2a} = -\frac{4}{2(-1)} = 2$. Pick any x-values equidistant from 2 to find two equal function values. For example, $f(1) = 4$ and $f(3) = 4$, so f is not 1-1.

11. $x_1 \neq x_2 \quad \Rightarrow \quad \sqrt{x_1} \neq \sqrt{x_2} \quad \Rightarrow \quad g(x_1) \neq g(x_2)$, so g is 1-1.

12. $g(x) = |x| \quad \Rightarrow \quad g(-1) = 1 = g(1)$, so g is not one-to-one.

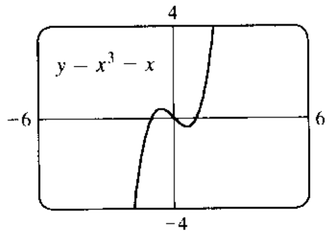
13. $h(x) = x^4 + 5 \quad \Rightarrow \quad h(1) = 6 = h(-1)$, so h is not 1-1.

14. $x_1 \neq x_2 \quad \Rightarrow \quad x_1^4 \neq x_2^4 \quad [\text{since } x \geq 0] \quad \Rightarrow \quad x_1^4 + 5 \neq x_2^4 + 5 \quad \Rightarrow \quad h(x_1) \neq h(x_2)$, so h is 1-1.

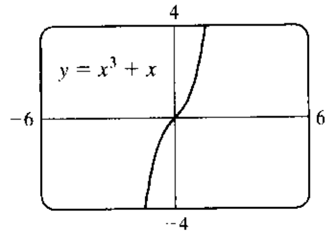
15. A football will attain every height h up to its maximum height twice: once on the way up, and again on the way down. Thus, even if t_1 does not equal t_2, $f(t_1)$ may equal $f(t_2)$, so f is not 1-1.

16. f is not 1-1 because eventually we all stop growing and therefore, there are two times at which we have the same height.

17. f does not pass the Horizontal Line Test, so f is not 1-1.



18. f passes the Horizontal Line Test, so f is 1-1.



19. Since $f(2) = 9$ and f is 1-1, we know that $f^{-1}(9) = 2$. Remember, if the point $(2, 9)$ is on the graph of f, then the point $(9, 2)$ is on the graph of f^{-1}.

20. $f(x) = x + \cos x \quad \Rightarrow \quad f'(x) = 1 - \sin x \geq 0$, with equality only if $x = \frac{\pi}{2} + 2n\pi$. So f is increasing on $\mathbb{R}$, and hence, 1-1. By inspection, $f(0) = 0 + \cos 0 = 1$, so $f^{-1}(1) = 0$.

21. $h(x) = x + \sqrt{x} \quad \Rightarrow \quad h'(x) = 1 + 1/(2\sqrt{x}) > 0$ on $(0, \infty)$. So h is increasing and hence, 1-1. By inspection, $h(4) = 4 + \sqrt{4} = 6$, so $h^{-1}(6) = 4$.

22. (a) f is 1-1 because it passes the Horizontal Line Test.

(b) Domain of $f = [-3, 3] =$ Range of f^{-1}. Range of $f = [-2, 2] =$ Domain of f^{-1}.

(c) Since $f(-2) = 1$, $f^{-1}(1) = -2$.

23. We solve $C = \frac{5}{9}(F - 32)$ for F: $\frac{9}{5}C = F - 32 \quad \Rightarrow \quad F = \frac{9}{5}C + 32$. This gives us a formula for the inverse function, that is, the Fahrenheit temperature F as a function of the Celsius temperature C. $F \geq -459.67 \quad \Rightarrow$ $\frac{9}{5}C + 32 \geq -459.67 \quad \Rightarrow \quad \frac{9}{5}C \geq -491.67 \quad \Rightarrow \quad C \geq -273.15$, the domain of the inverse function.

24. $m = \dfrac{m_0}{\sqrt{1 - v^2/c^2}} \quad \Rightarrow \quad 1 - \dfrac{v^2}{c^2} = \dfrac{m_0^2}{m^2} \quad \Rightarrow \quad \dfrac{v^2}{c^2} = 1 - \dfrac{m_0^2}{m^2} \quad \Rightarrow \quad v^2 = c^2\left(1 - \dfrac{m_0^2}{m^2}\right) \quad \Rightarrow$

$v = c\sqrt{1 - \dfrac{m_0^2}{m^2}}$. This formula gives us the speed v of the particle in terms of its mass m, that is, $v = f^{-1}(m)$.

25. $y = f(x) = 3 - 2x \quad \Rightarrow \quad 2x = 3 - y \quad \Rightarrow \quad x = \dfrac{3 - y}{2}$. Interchange x and y: $y = \dfrac{3 - x}{2}$. So $f^{-1}(x) = \dfrac{3 - x}{2}$.

26. $f(x) = \dfrac{4x - 1}{2x + 3} \quad \Rightarrow \quad y = \dfrac{4x - 1}{2x + 3} \quad \Rightarrow \quad y(2x + 3) = 4x - 1 \quad \Rightarrow \quad 2xy + 3y = 4x - 1 \quad \Rightarrow$

$3y + 1 = 4x - 2xy \quad \Rightarrow \quad 3y + 1 = (4 - 2y)x \quad \Rightarrow \quad x = \dfrac{3y + 1}{4 - 2y}$. Interchange x and y: $y = \dfrac{3x + 1}{4 - 2x}$.

So $f^{-1}(x) = \dfrac{3x + 1}{4 - 2x}$.

27. $f(x) = \sqrt{10 - 3x} \quad \Rightarrow \quad y = \sqrt{10 - 3x} \quad (y \geq 0) \quad \Rightarrow \quad y^2 = 10 - 3x \quad \Rightarrow \quad 3x = 10 - y^2 \quad \Rightarrow$ $x = -\frac{1}{3}y^2 + \frac{10}{3}$. Interchange x and y: $y = -\frac{1}{3}x^2 + \frac{10}{3}$. So $f^{-1}(x) = -\frac{1}{3}x^2 + \frac{10}{3}$. Note that the domain of f^{-1} is $x \geq 0$.

28. $y = f(x) = 2x^3 + 3 \quad \Rightarrow \quad y - 3 = 2x^3 \quad \Rightarrow \quad \dfrac{y - 3}{2} = x^3 \quad \Rightarrow \quad x = \sqrt[3]{\dfrac{y - 3}{2}}$.

Interchange x and y: $y = \sqrt[3]{\dfrac{x - 3}{2}}$. So $f^{-1}(x) = \sqrt[3]{\dfrac{x - 3}{2}}$.

29. For $f(x) = \dfrac{1-\sqrt{x}}{1+\sqrt{x}}$, the domain is $x \geq 0$. $f(0) = 1$ and as x increases, y decreases. As $x \to \infty$,

$\dfrac{1-\sqrt{x}}{1+\sqrt{x}} \cdot \dfrac{1/\sqrt{x}}{1/\sqrt{x}} = \dfrac{1/\sqrt{x}-1}{1/\sqrt{x}+1} \to \dfrac{-1}{1} = -1$, so the range of f is $-1 < y \leq 1$. Thus, the domain of f^{-1} is $-1 < x \leq 1$.

$$y = \frac{1-\sqrt{x}}{1+\sqrt{x}} \;\Rightarrow\; y(1+\sqrt{x}) = 1-\sqrt{x} \;\Rightarrow\; y + y\sqrt{x} = 1-\sqrt{x} \;\Rightarrow\; \sqrt{x} + y\sqrt{x} = 1-y \;\Rightarrow$$

$\sqrt{x}(1+y) = 1-y \;\Rightarrow\; \sqrt{x} = \dfrac{1-y}{1+y} \;\Rightarrow\; x = \left(\dfrac{1-y}{1+y}\right)^2$. Interchange x and y: $y = \left(\dfrac{1-x}{1+x}\right)^2$. So

$f^{-1}(x) = \left(\dfrac{1-x}{1+x}\right)^2$ with $-1 < x \leq 1$.

30. $y = f(x) = 2x^2 - 8x,\, x \geq 2 \;\Rightarrow\; 2x^2 - 8x - y = 0,\, x \geq 2 \;\Rightarrow$

$x = \dfrac{8+\sqrt{64+8y}}{4}$ $\left[\begin{array}{c}\text{quadratic formula with}\\ a = 2,\, b = -8, \text{ and } c = -y\end{array}\right]$ $= \dfrac{8+2\sqrt{16+2y}}{4} = 2 + \frac{1}{2}\sqrt{16+2y}$. Interchange x and y: $y = 2 + \frac{1}{2}\sqrt{16+2x}$. So $f^{-1}(x) = 2 + \frac{1}{2}\sqrt{16+2x}$.

Alternate solution (by completing the square): $y = 2x^2 - 8x,\, x \geq 2 \;\Rightarrow\; x^2 - 4x = y/2,\, x \geq 2 \;\Rightarrow$

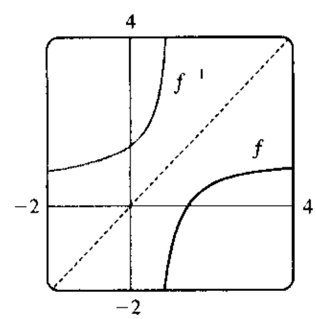
$(x-2)^2 = x^2 - 4x + 4 = \dfrac{y}{2} + 4 = \dfrac{y+8}{2} = \dfrac{2y+16}{4},\, x \geq 2 \;\Rightarrow\; x - 2 = +\sqrt{\dfrac{2y+16}{4}} \;\Rightarrow$

$x = 2 + \frac{1}{2}\sqrt{2y+16}$. Interchange x and y: $y = 2 + \frac{1}{2}\sqrt{2x+16}$. So $f^{-1}(x) = 2 + \frac{1}{2}\sqrt{2x+16}$.

31. $y = f(x) = 1 - \dfrac{2}{x^2} \;\Rightarrow\; 1 - y = \dfrac{2}{x^2} \;\Rightarrow\; x^2 = \dfrac{2}{1-y} \;\Rightarrow$

$x = \sqrt{\dfrac{2}{1-y}}$, since $x > 0$. Interchange x and y: $y = \sqrt{\dfrac{2}{1-x}}$.

So $f^{-1}(x) = \sqrt{\dfrac{2}{1-x}}$.

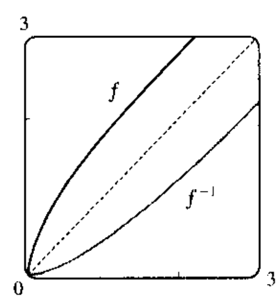


32. $y = f(x) = \sqrt{x^2+2x},\, x > 0 \;\Rightarrow\; y > 0$ and $y^2 = x^2 + 2x \;\Rightarrow$

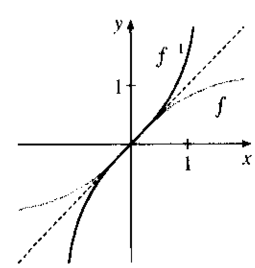
$x^2 + 2x - y^2 = 0$. Now we use the quadratic formula:

$x = \dfrac{-2 \pm \sqrt{2^2 - 4\cdot 1\cdot(-y^2)}}{2\cdot 1} = -1 \pm \sqrt{1+y^2}$. But $x > 0$, so the negative root is inadmissible. Interchange x and y: $y = -1 + \sqrt{1+x^2}$.

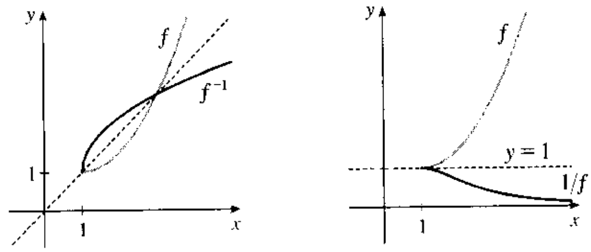
So $f^{-1}(x) = -1 + \sqrt{1+x^2},\, x > 0$.



33. The function f is one-to-one, so its inverse exists and the graph of its inverse can be obtained by reflecting the graph of f about the line $y = x$.



34. The function f is one-to-one, so its inverse exists and the graph of its inverse can be obtained by reflecting the graph of f about the line $y = x$. For the graph of $1/f$, the y-coordinates are simply the reciprocals of f. For example, if $f(5) = 9$, then $1/f(5) = \frac{1}{9}$. If we draw the horizontal line $y = 1$, we see that the only place where the graphs intersect is on that line.



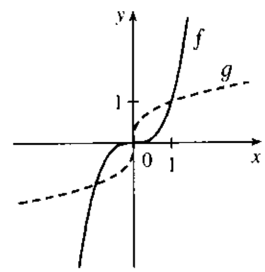
35. (a) $x_1 \neq x_2 \Rightarrow x_1^3 \neq x_2^3 \Rightarrow f(x_1) \neq f(x_2)$, so f is one-to-one.

(b) $f'(x) = 3x^2$ and $f(2) = 8 \Rightarrow g(8) = 2$, so $g'(8) = 1/f'(g(8)) = 1/f'(2) = \frac{1}{12}$.

(c) $y = x^3 \Rightarrow x = y^{1/3}$. Interchanging x and y gives $y = x^{1/3}$, so $f^{-1}(x) = x^{1/3}$. Domain(g) = range$(f) = \mathbb{R}$. Range(g) = domain$(f) = \mathbb{R}$.

(d) $g(x) = x^{1/3} \Rightarrow g'(x) = \frac{1}{3}x^{-2/3} \Rightarrow g'(8) = \frac{1}{3}\left(\frac{1}{4}\right) = \frac{1}{12}$ as in part (b).

(e)



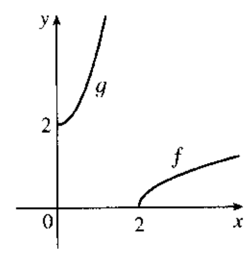
36. (a) $x_1 \neq x_2 \Rightarrow x_1 - 2 \neq x_2 - 2 \Rightarrow \sqrt{x_1 - 2} \neq \sqrt{x_2 - 2} \Rightarrow f(x_1) \neq f(x_2)$, so f is 1-1.

(b) $f(6) = 2$, so $g(2) = 6$. Also $f'(x) = \dfrac{1}{2\sqrt{x-2}}$, so $g'(2) = \dfrac{1}{f'(g(2))} = \dfrac{1}{f'(6)} = \dfrac{1}{1/4} = 4$.

(c) $y = \sqrt{x-2} \Rightarrow y^2 = x - 2 \Rightarrow x = y^2 + 2$. Interchange x and y: $y = x^2 + 2$. So $g(x) = x^2 + 2$. Domain $= [0, \infty)$, range $= [2, \infty)$.

(d) $g(x) = x^2 + 2 \Rightarrow g'(x) = 2x \Rightarrow g'(2) = 4$.

(e)



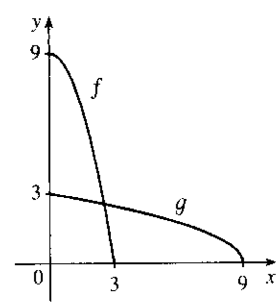
37. (a) Since $x \geq 0$, $x_1 \neq x_2 \Rightarrow x_1^2 \neq x_2^2 \Rightarrow 9 - x_1^2 \neq 9 - x_2^2 \Rightarrow f(x_1) \neq f(x_2)$, so f is 1-1.

(b) $f'(x) = -2x$ and $f(1) = 8 \Rightarrow g(8) = 1$, so $g'(8) = \dfrac{1}{f'(g(8))} = \dfrac{1}{f'(1)} = \dfrac{1}{(-2)} = -\dfrac{1}{2}$.

(c) $y = 9 - x^2 \Rightarrow x^2 = 9 - y \Rightarrow x = \sqrt{9-y}$. Interchange x and y: $y = \sqrt{9-x}$, so $f^{-1}(x) = \sqrt{9-x}$. Domain (g) = range $(f) = [0, 9]$. Range (g) = domain $(f) = [0, 3]$.

(d) $g'(x) = -1/\left(2\sqrt{9-x}\right) \Rightarrow g'(8) = -\frac{1}{2}$ as in part (b).

(e)



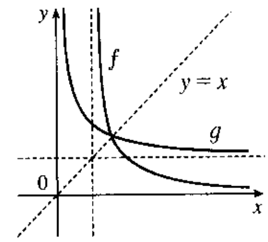
38. (a) $x_1 \neq x_2 \Rightarrow x_1 - 1 \neq x_2 - 1 \Rightarrow \dfrac{1}{x_1 - 1} \neq \dfrac{1}{x_2 - 1} \Rightarrow f(x_1) \neq f(x_2)$, so f is 1-1.

(b) $g(2) = \frac{3}{2}$ since $f\left(\frac{3}{2}\right) = 2$. Also $f'(x) = -1/(x-1)^2$, so $g'(2) = 1/f'\left(\frac{3}{2}\right) = \frac{1}{-4} = -\frac{1}{4}$.

(c) $y = 1/(x-1) \Rightarrow x - 1 = 1/y \Rightarrow x = 1 + 1/y$. Interchange x and y: $y = 1 + 1/x$. So $g(x) = 1 + 1/x$, $x > 0$ (since $y > 1$).
Domain $= (0, \infty)$, range $= (1, \infty)$.

(d) $g'(x) = -1/x^2$, so $g'(2) = -\frac{1}{4}$.

(e)



39. $f(0) = 1 \Rightarrow f^{-1}(1) = 0$, and $f(x) = x^3 + x + 1 \Rightarrow f'(x) = 3x^2 + 1$ and $f'(0) = 1$. Thus,
$(f^{-1})'(1) = \dfrac{1}{f'(f^{-1}(1))} = \dfrac{1}{f'(0)} = \dfrac{1}{1} = 1.$

40. $f(1) = 2 \Rightarrow f^{-1}(2) = 1$, and $f(x) = x^5 - x^3 + 2x \Rightarrow f'(x) = 5x^4 - 3x^2 + 2$ and $f'(1) = 4$. Thus,
$(f^{-1})'(2) = \dfrac{1}{f'(f^{-1}(2))} = \dfrac{1}{f'(1)} = \dfrac{1}{4}.$

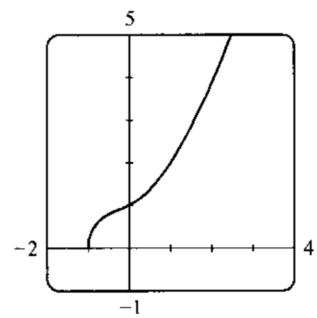
41. $f(0) = 3 \Rightarrow f^{-1}(3) = 0$, and $f(x) = 3 + x^2 + \tan(\pi x/2) \Rightarrow f'(x) = 2x + \frac{\pi}{2}\sec^2(\pi x/2)$ and $f'(0) = \frac{\pi}{2} \cdot 1 = \frac{\pi}{2}$. Thus, $(f^{-1})'(3) = 1/f'(f^{-1}(3)) = 1/f'(0) = 2/\pi$.

42. $f(1) = 2 \Rightarrow f^{-1}(2) = 1$, and $f(x) = \sqrt{x^3 + x^2 + x + 1} \Rightarrow f'(x) = \dfrac{3x^2 + 2x + 1}{2\sqrt{x^3 + x^2 + x + 1}}$ and
$f'(1) = \dfrac{3 + 2 + 1}{2\sqrt{1 + 1 + 1 + 1}} = \dfrac{3}{2}$. Thus, $(f^{-1})'(2) = 1/f'(f^{-1}(2)) = 1/f'(1) = \frac{2}{3}$.

43. $f(4) = 5 \Rightarrow g(5) = 4$. Thus, $g'(5) = \dfrac{1}{f'(g(5))} = \dfrac{1}{f'(4)} = \dfrac{1}{2/3} = \dfrac{3}{2}$.

44. $f(3) = 2 \Rightarrow g(2) = 3$. Thus, $g'(2) = \dfrac{1}{f'(g(2))} = \dfrac{1}{f'(3)} = 9$. Hence, $G(x) = \dfrac{1}{g(x)} \Rightarrow$
$G'(x) = -\dfrac{g'(x)}{[g(x)]^2} \Rightarrow G'(2) = -\dfrac{g'(2)}{[g(2)]^2} = -\dfrac{9}{(3)^2} = -1.$

45. We see that the graph of $y = f(x) = \sqrt{x^3 + x^2 + x + 1}$ is increasing, so f is 1-1. Enter $x = \sqrt{y^3 + y^2 + y + 1}$ and use your CAS to solve the equation for y. Using Derive, we get two (irrelevant) solutions involving imaginary expressions, as well as one which can be simplified to the following:



$$y = f^{-1}(x) = -\frac{\sqrt[3]{4}}{6}\left(\sqrt[3]{D - 27x^2 + 20} - \sqrt[3]{D + 27x^2 - 20} + \sqrt[3]{2}\right)$$

where $D = 3\sqrt{3}\sqrt{27x^4 - 40x^2 + 16}$. Maple and Mathematica each give two complex expressions and one real expression, and the real expression is equivalent to that given by Derive. For example, Maple's expression simplifies to $\dfrac{1}{6}\dfrac{M^{2/3} - 8 - 2M^{1/3}}{2M^{1/3}}$, where $M = 108x^2 + 12\sqrt{48 - 120x^2 + 81x^4} - 80$.

46. Since $\sin(2n\pi) = 0$, $h(x) = \sin x$ is not one-to-one. $h'(x) = \cos x > 0$ on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, so h is increasing and hence 1-1 on $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Let $y = f^{-1}(x) = \sin^{-1} x$ so that $\sin y = x$. Differentiating $\sin y = x$ implicitly with respect to x gives us $\cos y \dfrac{dy}{dx} = 1 \Rightarrow \dfrac{dy}{dx} = \dfrac{1}{\cos y}$. Now $\cos^2 y + \sin^2 y = 1 \Rightarrow \cos y = \pm\sqrt{1 - \sin^2 y}$, but since $\cos y > 0$ on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, we have $\dfrac{dy}{dx} = \dfrac{1}{\sqrt{1 - \sin^2 y}} = \dfrac{1}{\sqrt{1 - x^2}}$.

47. (a) If the point (x, y) is on the graph of $y = f(x)$, then the point $(x - c, y)$ is that point shifted c units to the left. Since f is 1-1, the point (y, x) is on the graph of $y = f^{-1}(x)$ and the point corresponding to $(x - c, y)$ on the graph of f is $(y, x - c)$ on the graph of f^{-1}. Thus, the curve's reflection is shifted *down* the same number of units as the curve itself is shifted to the left. So an expression for the inverse function is $g^{-1}(x) = f^{-1}(x) - c$.

(b) If we compress (or stretch) a curve horizontally, the curve's reflection in the line $y = x$ is compressed (or stretched) *vertically* by the same factor. Using this geometric principle, we see that the inverse of $h(x) = f(cx)$ can be expressed as $h^{-1}(x) = (1/c)\, f^{-1}(x)$.

48. (a) We know that $g'(x) = \dfrac{1}{f'(g(x))}$. Thus,

$$g''(x) = -\frac{f''(g(x)) \cdot g'(x)}{[f'(g(x))]^2} = -\frac{f''(g(x)) \cdot [1/f'(g'(x))]}{[f'(g(x))]^2} = -\frac{f''(g(x))}{f'(g(x))[f'(g(x))]^2} = -\frac{f''(g(x))}{[f'(g(x))]^3}.$$

(b) f is increasing $\Rightarrow$ $f'(g(x)) > 0$ $\Rightarrow$ $[f'(g(x))]^3 > 0$. f is concave upward $\Rightarrow$ $f''(g(x)) > 0$. So $g''(x) = -\dfrac{f''(g(x))}{[f'(g(x))]^3} < 0$, which implies that g (f's inverse) is concave downward.

7.2 Exponential Functions and Their Derivatives

1. (a) $f(x) = a^x$, $a > 0$

(b) $\mathbb{R}$

(c) $(0, \infty)$

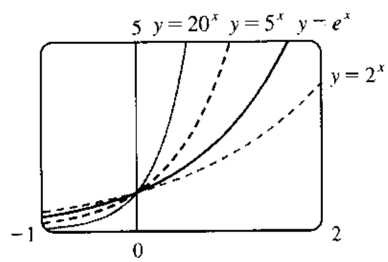
(d) See Figures 6(c), 6(b), and 6(a), respectively.

2. (a) The number e is the value of a such that the slope of the tangent line at $x = 0$ on the graph of $y = a^x$ is exactly 1.

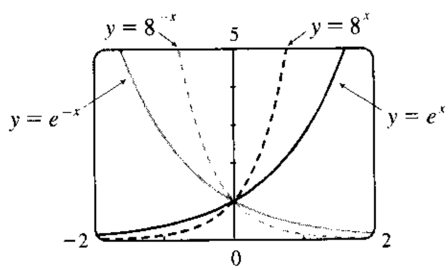
(b) $e \approx 2.71828$

(c) $f(x) = e^x$

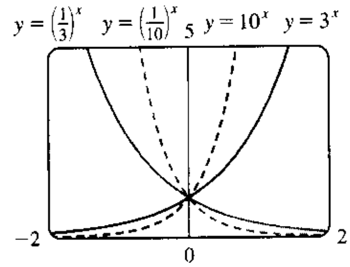
3. All of these graphs approach 0 as $x \to -\infty$, all of them pass through the point $(0, 1)$, and all of them are increasing and approach ∞ as $x \to \infty$. The larger the base, the faster the function increases for $x > 0$, and the faster it approaches 0 as $x \to -\infty$.



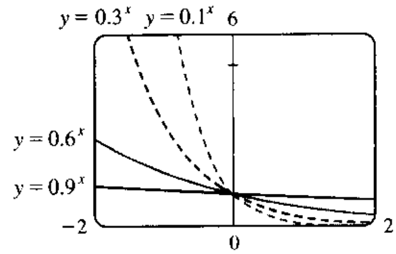
4. The graph of e^{-x} is the reflection of the graph of e^x about the y-axis, and the graph of 8^{-x} is the reflection of that of 8^x about the y-axis. The graph of 8^x increases more quickly than that of e^x for $x > 0$, and approaches 0 faster as $x \to -\infty$.



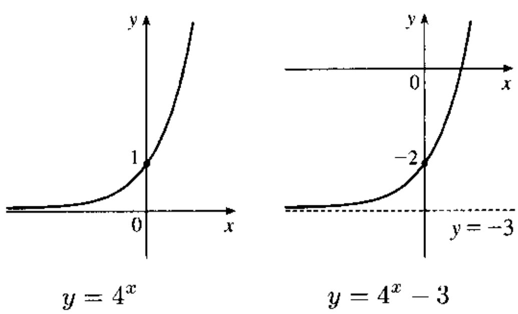
5. The functions with bases greater than 1 (3^x and 10^x) are increasing, while those with bases less than 1 $[(\frac{1}{3})^x$ and $(\frac{1}{10})^x]$ are decreasing. The graph of $(\frac{1}{3})^x$ is the reflection of that of 3^x about the y-axis, and the graph of $(\frac{1}{10})^x$ is the reflection of that of 10^x about the y-axis. The graph of 10^x increases more quickly than that of 3^x for $x > 0$, and approaches 0 faster as $x \to -\infty$.



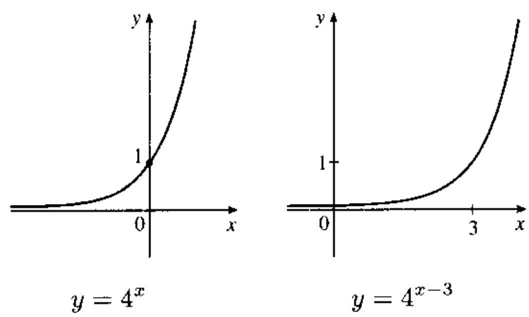
6. Each of the graphs approaches ∞ as $x \to -\infty$, and each approaches 0 as $x \to \infty$. The smaller the base, the faster the function grows as $x \to -\infty$, and the faster it approaches 0 as $x \to \infty$.



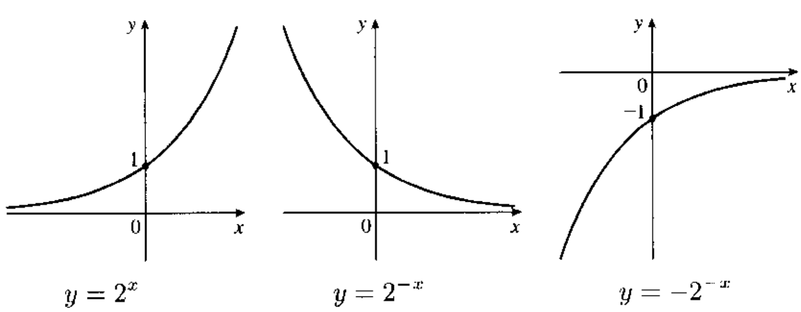
7. We start with the graph of $y = 4^x$ (Figure 3) and then shift 3 units downward. This shift doesn't affect the domain, but the range of $y = 4^x - 3$ is $(-3, \infty)$. There is a horizontal asymptote of $y = -3$.



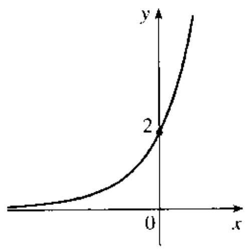
8. We start with the graph of $y = 4^x$ (Figure 3) and then shift 3 units to the right. There is a horizontal asymptote of $y = 0$.



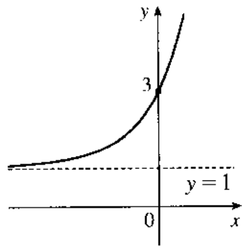
9. We start with the graph of $y = 2^x$ (Figure 3), reflect it about the y-axis, and then about the x-axis (or just rotate $180°$ to handle both reflections) to obtain the graph of $y = -2^{-x}$. In each graph, $y = 0$ is the horizontal asymptote.



10. We start with the graph of $y = e^x$ (Figure 12), vertically stretch by a factor of 2, and then shift 1 unit upward. There is a horizontal asymptote of $y = 1$.

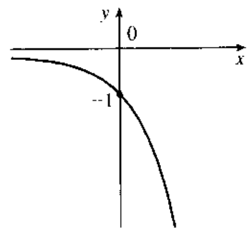


$y = 2e^x$

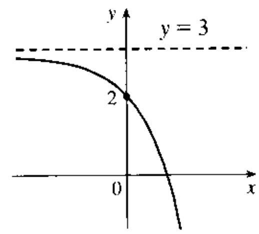


$y = 1 + 2e^x$

11. We start with the graph of $y = e^x$ (Figure 12), reflect it about the x-axis, and then shift 3 units upward. Note the horizontal asymptote of $y = 3$.

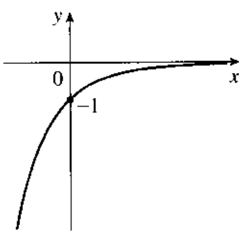


$y = -e^x$

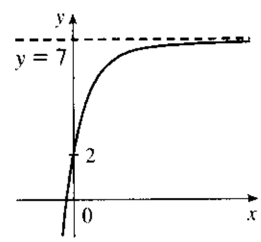


$y = 3 - e^x$

12. We start with the graph of $y = e^x$ (Figure 12), reflect it about the y-axis, and then about the x-axis (or just rotate 180° to handle both reflections) to obtain the graph of $y = -e^{-x}$. Now shift this graph 1 unit upward, vertically stretch by a factor of 5, and then shift 2 units upward.



$y = -e^{-x}$



$y = 2 + 5(1 - e^{-x})$

13. (a) To find the equation of the graph that results from shifting the graph of $y = e^x$ 2 units downward, we subtract 2 from the original function to get $y = e^x - 2$.

(b) To find the equation of the graph that results from shifting the graph of $y = e^x$ 2 units to the right, we replace x with $x - 2$ in the original function to get $y = e^{(x-2)}$.

(c) To find the equation of the graph that results from reflecting the graph of $y = e^x$ about the x-axis, we multiply the original function by -1 to get $y = -e^x$.

(d) To find the equation of the graph that results from reflecting the graph of $y = e^x$ about the y-axis, we replace x with $-x$ in the original function to get $y = e^{-x}$.

(e) To find the equation of the graph that results from reflecting the graph of $y = e^x$ about the x-axis and then about the y-axis, we first multiply the original function by -1 (to get $y = -e^x$) and then replace x with $-x$ in this equation to get $y = -e^{-x}$.

14. (a) This reflection consists of first reflecting the graph about the x-axis (giving the graph with equation $y = -e^x$) and then shifting this graph $2 \cdot 4 = 8$ units upward. So the equation is $y = -e^x + 8$.

(b) This reflection consists of first reflecting the graph about the y-axis (giving the graph with equation $y = e^{-x}$) and then shifting this graph $2 \cdot 2 = 4$ units to the right. So the equation is $y = e^{-(x-4)}$.

15. (a) The denominator $1 + e^x$ is never equal to zero because $e^x > 0$, so the domain of $f(x) = 1/(1 + e^x)$ is $\mathbb{R}$.

(b) $1 - e^x = 0 \quad \Leftrightarrow \quad e^x = 1 \quad \Leftrightarrow \quad x = 0$, so the domain of $f(x) = 1/(1 - e^x)$ is $(-\infty, 0) \cup (0, \infty)$.

16. (a) The sine and exponential functions have domain $\mathbb{R}$, so $g(t) = \sin(e^{-t})$ also has domain $\mathbb{R}$.

(b) The function $g(t) = \sqrt{1 - 2^t}$ has domain $\{t \mid 1 - 2^t \geq 0\} = \{t \mid 2^t \leq 1\} = \{t \mid t \leq 0\} = (-\infty, 0]$.

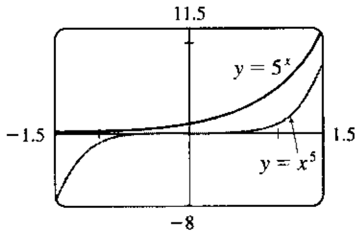
17. Use $y = Ca^x$ with the points $(1, 6)$ and $(3, 24)$. $6 = Ca^1 \quad \left[C = \frac{6}{a}\right]$ and $24 = Ca^3 \quad \Rightarrow \quad 24 = \left(\frac{6}{a}\right)a^3 \quad \Rightarrow$

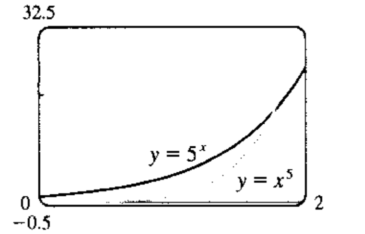
$4 = a^2 \quad \Rightarrow \quad a = 2$ [since $a > 0$] and $C = \frac{6}{2} = 3$. The function is $f(x) = 3 \cdot 2^x$.

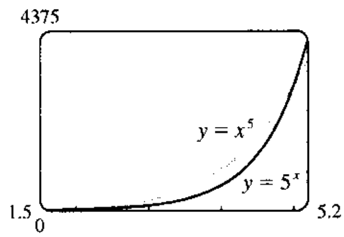
18. Given the y-intercept $(0, 2)$, we have $y = Ca^x = 2a^x$. Using the point $\left(2, \frac{2}{9}\right)$ gives us $\frac{2}{9} = 2a^2 \Rightarrow \frac{1}{9} = a^2$ $\Rightarrow a = \frac{1}{3}$ [since $a > 0$]. The function is $f(x) = 2\left(\frac{1}{3}\right)^x$ or $f(x) = 2(3)^{-x}$.

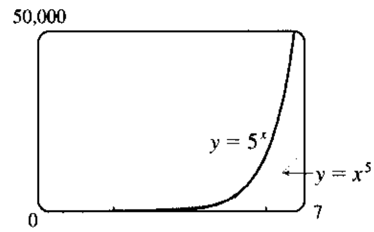
19. $2 \text{ ft} = 24 \text{ in}$, $f(24) = 24^2 \text{ in} = 576 \text{ in} = 48 \text{ ft}$. $g(24) = 2^{24} \text{ in} = 2^{24}/(12 \cdot 5280) \text{ mi} \approx 265 \text{ mi}$

20. We see from the graphs that for x less than about 1.8, $g(x) = 5^x > f(x) = x^5$, and then near the point $(1.8, 17.1)$ the curves intersect. Then $f(x) > g(x)$ from $x \approx 1.8$ until $x = 5$. At $(5, 3125)$ there is another point of intersection, and for $x > 5$ we see that $g(x) > f(x)$. In fact, g increases much more rapidly than f beyond that point.

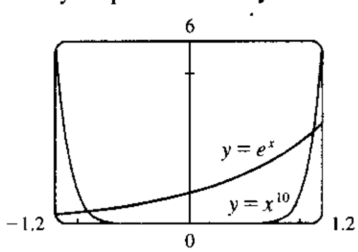


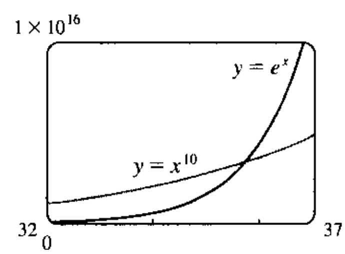




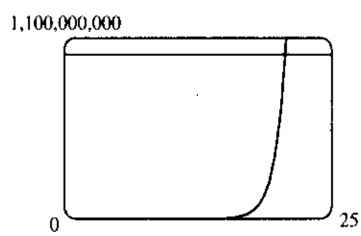


21. The graph of g finally surpasses that of f at $x \approx 35.8$.





22. We graph $y = e^x$ and $y = 1{,}000{,}000{,}000$ and determine where $e^x = 1 \times 10^9$. This seems to be true at $x \approx 20.723$, so $e^x > 1 \times 10^9$ for $x > 20.723$.



23. $\lim\limits_{x\to\infty} (1.001)^x = \infty$ by (3), since $1.001 > 1$.

24. Let $t = -x^2$. As $x \to \infty$, $t \to -\infty$. So $\lim\limits_{x\to\infty} e^{-x^2} = \lim\limits_{t\to-\infty} e^t = 0$ by (11).

25. Divide numerator and denominator by e^{3x}: $\lim\limits_{x\to\infty} \dfrac{e^{3x} - e^{-3x}}{e^{3x} + e^{-3x}} = \lim\limits_{x\to\infty} \dfrac{1 - e^{-6x}}{1 + e^{-6x}} = \dfrac{1-0}{1+0} = 1$

26. If we let $t = \tan x$, then as $x \to (\pi/2)^+$, $t \to -\infty$. Thus, $\lim\limits_{x\to(\pi/2)^+} e^{\tan x} = \lim\limits_{t\to-\infty} e^t = 0$.

27. Let $t = 3/(2 - x)$. As $x \to 2^+$, $t \to -\infty$. So $\lim\limits_{x\to2^+} e^{3/(2-x)} = \lim\limits_{t\to-\infty} e^t = 0$ by (11).

28. Let $t = 3/(2-x)$. As $x \to 2^-$, $t \to \infty$. So $\lim\limits_{x\to 2^-} e^{3/(2-x)} = \lim\limits_{t\to\infty} e^t = \infty$ by (11).

29. By the Product Rule, $f(x) = x^2 e^x \Rightarrow f'(x) = x^2 \dfrac{d}{dx}(e^x) + e^x \dfrac{d}{dx}(x^2) = x^2 e^x + e^x(2x) = xe^x(x+2)$.

30. By the Quotient Rule, $y = \dfrac{e^x}{1+x} \Rightarrow y' = \dfrac{(1+x)e^x - e^x(1)}{(1+x)^2} = \dfrac{e^x + xe^x - e^x}{(x+1)^2} = \dfrac{xe^x}{(x+1)^2}$.

31. By (10), $y = e^{ax^3} \Rightarrow y' = e^{ax^3} \dfrac{d}{dx}(ax^3) = 3ax^2 e^{ax^3}$.

32. $y = e^u(\cos u + cu) \Rightarrow y' = e^u(-\sin u + c) + (\cos u + cu)e^u = e^u(\cos u - \sin u + cu + c)$

33. $f(u) = e^{1/u} \Rightarrow f'(u) = e^{1/u} \cdot \dfrac{d}{du}\left(\dfrac{1}{u}\right) = e^{1/u}\left(\dfrac{-1}{u^2}\right) = \left(\dfrac{-1}{u^2}\right)e^{1/u}$

34. By the Product Rule, $g(x) = \sqrt{x}\,e^x = x^{1/2}e^x \Rightarrow g'(x) = x^{1/2}(e^x) + e^x\left(\frac{1}{2}x^{-1/2}\right) = \frac{1}{2}x^{-1/2}e^x(2x+1)$.

35. By (10), $F(t) = e^{t\sin 2t} \Rightarrow$
$F'(t) = e^{t\sin 2t}(t\sin 2t)' = e^{t\sin 2t}(t\cdot 2\cos 2t + \sin 2t \cdot 1) = e^{t\sin 2t}(2t\cos 2t + \sin 2t)$

36. $y = e^{k\tan\sqrt{x}} \Rightarrow y' = e^{k\tan\sqrt{x}} \cdot \dfrac{d}{dx}(k\tan\sqrt{x}) = e^{k\tan\sqrt{x}}\left(k\sec^2\sqrt{x}\cdot\frac{1}{2}x^{-1/2}\right) = \dfrac{k\sec^2\sqrt{x}}{2\sqrt{x}}\,e^{k\tan\sqrt{x}}$

37. $y = \sqrt{1+2e^{3x}} \Rightarrow y' = \dfrac{1}{2}(1+2e^{3x})^{-1/2}\dfrac{d}{dx}(1+2e^{3x}) = \dfrac{1}{2\sqrt{1+2e^{3x}}}(2e^{3x}\cdot 3) = \dfrac{3e^{3x}}{\sqrt{1+2e^{3x}}}$

38. $y = \cos(e^{\pi x}) \Rightarrow y' = -\sin(e^{\pi x})\cdot e^{\pi x}\cdot \pi = -\pi e^{\pi x}\sin(e^{\pi x})$

39. $y = e^{e^x} \Rightarrow y' = e^{e^x}\cdot\dfrac{d}{dx}(e^x) = e^{e^x}\cdot e^x$ or $e^{e^x + x}$

40. $y = \sqrt{1+xe^{-2x}} \Rightarrow y' = \frac{1}{2}(1+xe^{-2x})^{-1/2}\left[x(-2e^{-2x}) + e^{-2x}\right] = \dfrac{e^{-2x}(-2x+1)}{2\sqrt{1+xe^{-2x}}}$

41. By the Quotient Rule, $y = \dfrac{ae^x + b}{ce^x + d} \Rightarrow$

$$y' = \frac{(ce^x+d)(ae^x) - (ae^x+b)(ce^x)}{(ce^x+d)^2} = \frac{(ace^x + ad - ace^x - bc)e^x}{(ce^x+d)^2} = \frac{(ad-bc)e^x}{(ce^x+d)^2}.$$

42. $y = \dfrac{e^x + e^{-x}}{e^x - e^{-x}} \Rightarrow y' = \dfrac{(e^x - e^{-x})(e^x - e^{-x}) - (e^x + e^{-x})(e^x + e^{-x})}{(e^x - e^{-x})^2}$

$$= \frac{(e^{2x} - 2 + e^{-2x}) - (e^{2x} + 2 + e^{-2x})}{(e^x - e^{-x})^2} = -\frac{4}{(e^x - e^{-x})^2}$$

43. $y = e^{2x}\cos\pi x \Rightarrow y' = e^{2x}(-\pi\sin\pi x) + (\cos\pi x)(2e^{2x}) = e^{2x}(2\cos\pi x - \pi\sin\pi x)$.
At $(0,1)$, $y' = 1(2-0) = 2$, so an equation of the tangent line is $y - 1 = 2(x-0)$, or $y = 2x+1$.

44. $y = \dfrac{e^x}{x} \Rightarrow y' = \dfrac{x\cdot e^x - e^x\cdot 1}{x^2} = \dfrac{e^x(x-1)}{x^2}$. At $(1,e)$, $y' = 0$, and an equation of the tangent line is $y - e = 0(x-1)$, or $y = e$.

45. $\dfrac{d}{dx}\left(e^{x^2y}\right) = \dfrac{d}{dx}(x+y) \Rightarrow e^{x^2y}(x^2y' + y\cdot 2x) = 1 + y' \Rightarrow x^2e^{x^2y}y' + 2xye^{x^2y} = 1 + y' \Rightarrow$

$x^2e^{x^2y}y' - y' = 1 - 2xye^{x^2y} \Rightarrow y'(x^2e^{x^2y} - 1) = 1 - 2xye^{x^2y} \Rightarrow y' = \dfrac{1 - 2xye^{x^2y}}{x^2e^{x^2y} - 1}$

46. $xe^y + ye^x = 1 \;\Rightarrow\; xe^y y' + e^y \cdot 1 + ye^x + e^x y' = 0 \;\Rightarrow\; y'(xe^y + e^x) = -e^y - ye^x \;\Rightarrow$

$y' = -\dfrac{e^y + ye^x}{xe^y + e^x}$. At $(0,1)$, $y' = -\dfrac{e + 1 \cdot 1}{0 + 1} = -(e+1)$, so an equation for the tangent line is

$y - 1 = -(e+1)(x - 0)$, or $y = -(e+1)x + 1$.

47. $y = e^x + e^{-x/2} \;\Rightarrow\; y' = e^x - \frac{1}{2}e^{-x/2} \;\Rightarrow\; y'' = e^x + \frac{1}{4}e^{-x/2}$, so

$2y'' - y' - y = 2\left(e^x + \frac{1}{4}e^{-x/2}\right) - \left(e^x - \frac{1}{2}e^{-x/2}\right) - \left(e^x + e^{-x/2}\right) = 0.$

48. $y = Ae^{-x} + Bxe^{-x} \;\Rightarrow\; y' = -Ae^{-x} + Be^{-x} - Bxe^{-x} = (B - A)e^{-x} - Bxe^{-x} \;\Rightarrow$

$y'' = (A - B)e^{-x} - Be^{-x} + Bxe^{-x} = (A - 2B)e^{-x} + Bxe^{-x}$, so

$y'' + 2y' + y = (A - 2B)e^{-x} + Bxe^{-x} + 2\left[(B - A)e^{-x} - Bxe^{-x}\right] + Ae^{-x} + Bxe^{-x} = 0.$

49. $y = e^{rx} \;\Rightarrow\; y' = re^{rx} \;\Rightarrow\; y'' = r^2e^{rx}$, so if $y = e^{rx}$ satisfies the differential equation $y'' + 6y' + 8y = 0$, then $r^2e^{rx} + 6re^{rx} + 8e^{rx} = 0$; that is, $e^{rx}(r^2 + 6r + 8) = 0$. Since $e^{rx} > 0$ for all x, we must have $r^2 + 6r + 8 = 0$, or $(r+2)(r+4) = 0$, so $r = -2$ or -4.

50. $y = e^{\lambda x} \;\Rightarrow\; y' = \lambda e^{\lambda x} \;\Rightarrow\; y'' = \lambda^2 e^{\lambda x}$. Thus, $y + y' = y'' \;\Leftrightarrow\; e^{\lambda x} + \lambda e^{\lambda x} = \lambda^2 e^{\lambda x} \;\Leftrightarrow$

$e^{\lambda x}(\lambda^2 - \lambda - 1) = 0 \;\Leftrightarrow\; \lambda = \frac{1 \pm \sqrt{5}}{2}$, since $e^{\lambda x} \neq 0$.

51. $f(x) = e^{2x} \;\Rightarrow\; f'(x) = 2e^{2x} \;\Rightarrow\; f''(x) = 2 \cdot 2e^{2x} = 2^2e^{2x} \;\Rightarrow$

$f'''(x) = 2^2 \cdot 2e^{2x} = 2^3e^{2x} \;\Rightarrow\; \cdots \;\Rightarrow\; f^{(n)}(x) = 2^n e^{2x}$

52. $f(x) = xe^{-x} \;\Rightarrow\; f'(x) = x(-e^{-x}) + e^{-x} = (1 - x)e^{-x} \;\Rightarrow$

$f''(x) = (1 - x)(-e^{-x}) + e^{-x}(-1) = (x - 2)e^{-x} \;\Rightarrow\; f'''(x) = (x - 2)(-e^{-x}) + e^{-x} = (3 - x)e^{-x} \;\Rightarrow$

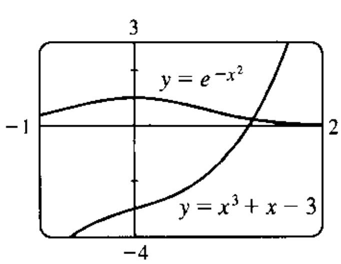
$f^{(4)}(x) = (3 - x)(-e^{-x}) + e^{-x}(-1) = (x - 4)e^{-x} \;\Rightarrow\; \cdots \;\Rightarrow\; f^{(n)}(x) = (-1)^n(x - n)e^{-x}$.

So $D^{1000}xe^{-x} = (x - 1000)e^{-x}$.

53. (a) $f(x) = e^x + x$ is continuous on $\mathbb{R}$ and $f(-1) = e^{-1} - 1 < 0 < 1 = f(0)$, so by the Intermediate Value Theorem, $e^x + x = 0$ has a root in $(-1, 0)$.

(b) $f(x) = e^x + x \;\Rightarrow\; f'(x) = e^x + 1$, so $x_{n+1} = x_n - \dfrac{e^{x_n} + x_n}{e^{x_n} + 1}$. Using $x_1 = -0.5$, we get $x_2 \approx -0.566311$, $x_3 \approx -0.567143 \approx x_4$, so the root is -0.567143 to six decimal places.

54.



From the graph, it appears that the curves intersect at about $x \approx 1.2$ or 1.3. We use Newton's Method with $f(x) = x^3 + x - 3 - e^{-x^2}$, so $f'(x) = 3x^2 + 1 + 2xe^{-x^2}$, and the formula is $x_{n+1} = x_n - f(x_n)/f'(x_n)$. We take $x_1 = 1.2$, and the formula gives $x_2 \approx 1.252462$, $x_3 \approx 1.251045$, and $x_4 \approx x_5 \approx 1.251044$. So the root of the equation, correct to six decimal places, is $x = 1.251044$.

55. (a) $m(t) = 24 \cdot 2^{-t/25} \Rightarrow$

$m(40) = 24 \cdot 2^{-40/25} \approx 7.92$ mg

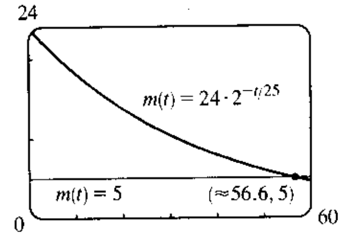
(b) $m'(t) = 24\,\dfrac{d}{dt}\left(2^{-t/25}\right)$

$\approx 24(0.69)2^{-t/25}\,\dfrac{d}{dt}\left(-\dfrac{t}{25}\right)$ [(7) and (10)]

$= 24(0.69)\left(-\frac{1}{25}\right)\,2^{-t/25}$

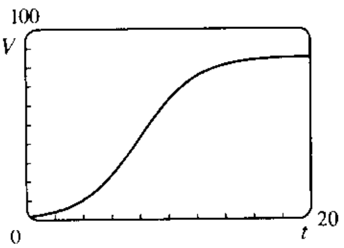
so $m'(40) \approx -\frac{24}{25}(0.69)2^{-40/25} \approx -0.22$ mg/yr.

(c)



From the graph, we can determine that $m(t) = 5 \Rightarrow t \approx 56.6$ yr.

56.



From the graph, we estimate that the most rapid increase in the percentage of households in the United States with at least one VCR occurs at about $t = 8$. To maximize the first derivative, we need to determine the values for which the second derivative is 0. We'll use $V(t) = \dfrac{a}{1 + be^{ct}}$, and substitute $a = 85$, $b = 53$, and $c = -0.5$ later.

$V'(t) = -\dfrac{a(bce^{ct})}{(1 + be^{ct})^2}$ [by the Reciprocal Rule] and

$$V''(t) = -abc \cdot \frac{(1 + be^{ct})^2 \cdot ce^{ct} - e^{ct} \cdot 2(1 + be^{ct}) \cdot bce^{ct}}{[(1 + be^{ct})^2]^2}$$

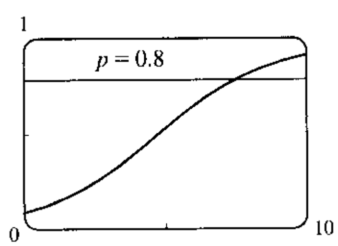
$$= \frac{-abc \cdot ce^{ct}(1 + be^{ct})[(1 + be^{ct}) - 2be^{ct}]}{(1 + be^{ct})^4} = \frac{-abc^2e^{ct}(1 - be^{ct})}{(1 + be^{ct})^3}$$

So $V''(t) = 0 \Leftrightarrow 1 = be^{ct} \Leftrightarrow e^{ct} = 1/b$. Now graph $y = e^{-0.5t}$ and $y = \frac{1}{53}$. These graphs intersect at $t \approx 7.94$ years, which corresponds to roughly midyear 1988. [Alternatively, we could use the rootfinder on a calculator to solve $e^{-0.5t} = \frac{1}{53}$. Or, if you have already studied logarithms, you can solve $e^{ct} = 1/b$ as follows: $ct = \ln(1/b) \Leftrightarrow t = (1/c)\ln(1/b) = -2\ln\frac{1}{53} \approx 7.94$ years.

57. (a) $\displaystyle\lim_{t\to\infty} p(t) = \lim_{t\to\infty} \frac{1}{1 + ae^{-kt}} = \frac{1}{1 + a \cdot 0} = 1$, since $k > 0 \Rightarrow -kt \to -\infty \Rightarrow e^{-kt} \to 0$.

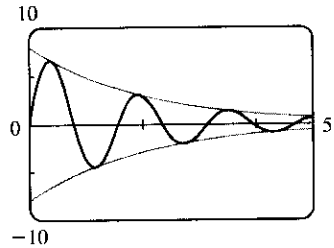
(b) $p(t) = \left(1 + ae^{-kt}\right)^{-1} \Rightarrow \dfrac{dp}{dt} = -\left(1 + ae^{-kt}\right)^{-2}\left(-kae^{-kt}\right) = \dfrac{kae^{-kt}}{(1 + ae^{-kt})^2}$

(c)



From the graph of $p(t) = \left(1 + 10e^{-0.5t}\right)^{-1}$, it seems that $p(t) = 0.8$ (indicating that 80% of the population has heard the rumor) when $t \approx 7.4$ hours.

58. (a)

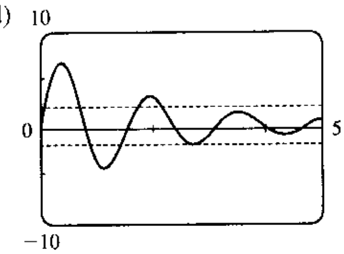


The displacement function is squeezed between the other two functions. This is because $-1 \le \sin 4t \le 1 \;\Rightarrow$ $-8e^{-t/2} \le 8e^{-t/2}\sin 4t \le 8e^{-t/2}$.

(b) The maximum value of the displacement is about 6.6 cm, occurring at $t \approx 0.36$ s. It occurs just before the graph of the displacement function touches the graph of $8e^{-t/2}$ (when $t = \frac{\pi}{8} \approx 0.39$).

(c) The velocity of the object is the derivative of its displacement function, that is,
$\frac{d}{dt}\left(8e^{-t/2}\sin 4t\right) = 8\left[e^{-t/2}\cos 4t(4) + \sin 4t\left(-\frac{1}{2}\right)e^{-t/2}\right]$.
If the displacement is zero, then we must have $\sin 4t = 0$ (since the exponential term in the displacement function is always positive). The first time that $\sin 4t = 0$ after $t = 0$ occurs at $t = \frac{\pi}{4}$. Substituting this into our expression for the velocity, and noting that the second term vanishes, we get $v\left(\frac{\pi}{4}\right) = 8e^{-\pi/8}\cos\left(4 \cdot \frac{\pi}{4}\right) \cdot 4 = -32e^{-\pi/8} \approx -21.6$ cm/s.

(d)

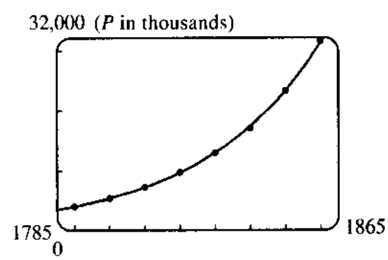


The graph indicates that the displacement is less than 2 cm from equilibrium whenever t is larger than about 2.8.

59. (a) Using a calculator or CAS, we obtain the model $Q = ab^t$ with $a = 100.0124369$ and $b = 0.000045145933$. We can change this model to one with base e and exponent $\ln b$ [$b^t = e^{t \ln b}$ from precalculus mathematics or from Section 7.3]: $Q = ae^{t \ln b} = 100.012437e^{-10.005531t}$.

(b) Use $Q'(t) = ab^t \ln b$ or the calculator command `nDeriv(Y₁,X,.04)` with $Y_1 = ab^x$ to get $Q'(0.04) \approx -670.63\ \mu$A. The result of Example 2 in Section 2.1 was $-670\ \mu$A.

60. (a) $P = ab^t$ with $a = 4.502714 \times 10^{-20}$ and $b = 1.029953851$, where P is measured in thousands of people. The fit appears to be very good.



(b) **For 1800:** $m_1 = \dfrac{5308 - 3929}{1800 - 1790} = 137.9$, $m_2 = \dfrac{7240 - 5308}{1810 - 1800} = 193.2$.

So $P'(1800) \approx (m_1 + m_2)/2 = 165.55$ thousand people/year.

For 1850: $m_1 = \dfrac{23{,}192 - 17{,}063}{1850 - 1840} = 612.9$, $m_2 = \dfrac{31{,}443 - 23{,}192}{1860 - 1850} = 825.1$.

So $P'(1850) \approx (m_1 + m_2)/2 = 719$ thousand people/year.

(c) Use the calculator command `nDeriv(Y₁,X,year)` with $Y_1 = ab^x$ to get $P'(1800) \approx 156.85$ and $P'(1850) \approx 686.07$. These estimates are somewhat less than the ones in part (b).

(d) $P(1870) \approx 41{,}946.56$. The difference of 3.4 million people is most likely due to the Civil War (1861–1865).

61. $f(x) = x - e^x \Rightarrow f'(x) = 1 - e^x = 0 \Leftrightarrow e^x = 1 \Leftrightarrow x = 0$. Now $f'(x) > 0$ for all $x < 0$ and $f'(x) < 0$ for all $x > 0$, so the absolute maximum value is $f(0) = 0 - 1 = -1$.

62. $g(x) = \dfrac{e^x}{x} \Rightarrow g'(x) = \dfrac{xe^x - e^x}{x^2} = 0 \Leftrightarrow e^x(x-1) = 0 \Rightarrow x = 1$. Now $g'(x) > 0 \Leftrightarrow \dfrac{xe^x - e^x}{x^2} > 0 \Leftrightarrow x - 1 > 0 \Leftrightarrow x > 1$ and $g'(x) < 0 \Leftrightarrow \dfrac{xe^x - e^x}{x^2} < 0 \Leftrightarrow x - 1 < 0 \Leftrightarrow x < 1$. Thus there is an absolute minimum value of $g(1) = e$ at $x = 1$.

63. (a) $f(x) = xe^x \Rightarrow f'(x) = e^x + xe^x = e^x(1+x) > 0 \Leftrightarrow 1 + x > 0 \Leftrightarrow x > -1$, so f is increasing on $(-1, \infty)$ and decreasing on $(-\infty, -1)$.

(b) $f''(x) = e^x(1+x) + e^x = e^x(2+x) > 0 \Leftrightarrow 2 + x > 0 \Leftrightarrow x > -2$, so f is CU on $(-2, \infty)$ and CD on $(-\infty, -2)$.

(c) f has an inflection point at $\left(-2, -2e^{-2}\right)$.

64. (a) $f(x) = x^2e^x \Rightarrow f'(x) = 2xe^x + x^2e^x = (x^2 + 2x)e^x$. $f'(x) > 0 \Leftrightarrow x(x+2) > 0 \Leftrightarrow x < -2$ or $x > 0$, $f'(x) < 0 \Leftrightarrow -2 < x < 0$, so f is increasing on $(-\infty, -2)$ and $(0, \infty)$ and decreasing on $(-2, 0)$.

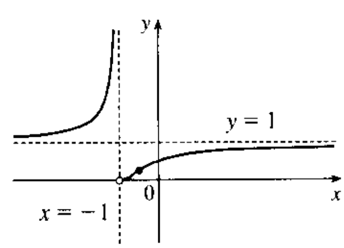
(b) $f''(x) = (2x+2)e^x + (x^2+2x)e^x = (x^2 + 4x + 2)e^x = 0 \Leftrightarrow x^2 + 4x + 2 = 0 \Leftrightarrow x = -2 \pm \sqrt{2}$. $f''(x) > 0$ when $x > -2 + \sqrt{2}$ or $x < -2 - \sqrt{2}$, so f is CU on $\left(-\infty, -2 - \sqrt{2}\right)$ and $\left(-2 + \sqrt{2}, \infty\right)$ and CD on $\left(-2 - \sqrt{2}, -2 + \sqrt{2}\right)$.

(c) f has inflection points at $\left(-2 + \sqrt{2}, \left(6 - 4\sqrt{2}\right)e^{-2+\sqrt{2}}\right)$ and $\left(-2 - \sqrt{2}, \left(6 + 4\sqrt{2}\right)e^{-2-\sqrt{2}}\right)$.

65. $y = f(x) = e^{-1/(x+1)}$ **A.** $D = \{x \mid x \neq -1\} = (-\infty, -1) \cup (-1, \infty)$ **B.** No x-intercept; y-intercept $= f(0) = e^{-1}$ **C.** No symmetry **D.** $\lim\limits_{x \to \pm\infty} e^{-1/(x+1)} = 1$ since $-1/(x+1) \to 0$, so $y = 1$ is a HA. $\lim\limits_{x \to -1^+} e^{-1/(x+1)} = 0$ since $-1/(x+1) \to -\infty$, $\lim\limits_{x \to -1^-} e^{-1/(x+1)} = \infty$ since $-1/(x+1) \to \infty$, so $x = -1$ is a VA. **E.** $f'(x) = e^{-1/(x+1)}/(x+1)^2 \Rightarrow f'(x) > 0$ for all x except 1, so f is increasing on $(-\infty, -1)$ and $(-1, \infty)$. **F.** No extreme values

G. $f''(x) = \dfrac{e^{-1/(x+1)}}{(x+1)^4} + \dfrac{e^{-1/(x+1)}(-2)}{(x+1)^3} = -\dfrac{e^{-1/(x+1)}(2x+1)}{(x+1)^4}$ $\Rightarrow f''(x) > 0 \Leftrightarrow 2x + 1 < 0 \Leftrightarrow x < -\frac{1}{2}$, so f is CU on $(-\infty, -1)$ and $\left(-1, -\frac{1}{2}\right)$, and CD on $\left(-\frac{1}{2}, \infty\right)$. f has an IP at $\left(-\frac{1}{2}, e^{-2}\right)$.

H.



66. $y = f(x) = e^{2x} - e^x$ **A.** $D = \mathbb{R}$ **B.** y-intercept: $f(0) = 0$;

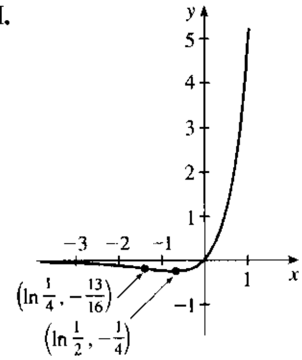
x-intercepts: $f(x) = 0 \Rightarrow e^{2x} = e^x \Rightarrow e^x = 1 \Rightarrow x = 0$.

C. No symmetry **D.** $\lim\limits_{x \to -\infty} e^{2x} - e^x = 0$, so $y = 0$ is a HA. No VA.

E. $f'(x) = 2e^{2x} - e^x = e^x(2e^x - 1)$, so $f'(x) > 0 \Leftrightarrow e^x > \frac{1}{2}$ (*) $\Leftrightarrow x > \ln\frac{1}{2} = -\ln 2$ and $f'(x) < 0 \Leftrightarrow e^x < \frac{1}{2} \Leftrightarrow x < \ln\frac{1}{2}$, so f is decreasing on $\left(-\infty, \ln\frac{1}{2}\right)$ and increasing on $\left(\ln\frac{1}{2}, \infty\right)$. **F.** Local minimum value $f\left(\ln\frac{1}{2}\right) = e^{2\ln(1/2)} - e^{\ln(1/2)} = \left(\frac{1}{2}\right)^2 - \frac{1}{2} = -\frac{1}{4}$

G. $f''(x) = 4e^{2x} - e^x = e^x(4e^x - 1)$, so $f''(x) > 0 \Leftrightarrow e^x > \frac{1}{4} \Leftrightarrow x > \ln\frac{1}{4}$ and $f''(x) < 0 \Leftrightarrow x < \ln\frac{1}{4}$.

H.



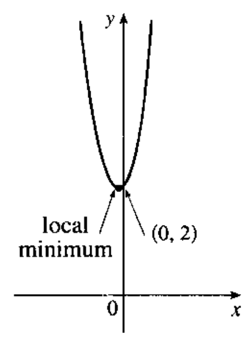
Thus, f is CD on $\left(-\infty, \ln\frac{1}{4}\right)$ and CU on $\left(\ln\frac{1}{4}, \infty\right)$. f has an IP at $\left(\ln\frac{1}{4}, \left(\frac{1}{4}\right)^2 - \frac{1}{4}\right) = \left(\ln\frac{1}{4}, -\frac{3}{16}\right)$.

(*) If you have not yet learned about logarithms, graph $y = e^x$ and $y = \frac{1}{2}$, find the point of intersection, and use decimal approximations for the rest of the solution.

67. $y = f(x) = e^{3x} + e^{-2x}$ **A.** $D = \mathbb{R}$ **B.** y-intercept $= f(0) = 2$; no x-intercept **C.** No symmetry **D.** No asymptotes

E. $f'(x) = 3e^{3x} - 2e^{-2x}$, so $f'(x) > 0 \Leftrightarrow 3e^{3x} > 2e^{-2x}$ [multiply by e^{2x}] $\Leftrightarrow e^{5x} > \frac{2}{3}$ (*) $\Leftrightarrow 5x > \ln\frac{2}{3} \Leftrightarrow x > \frac{1}{5}\ln\frac{2}{3} \approx -0.081$. Similarly, $f'(x) < 0 \Leftrightarrow x < \frac{1}{5}\ln\frac{2}{3}$. f is decreasing on $\left(-\infty, \frac{1}{5}\ln\frac{2}{3}\right)$ and increasing on $\left(\frac{1}{5}\ln\frac{2}{3}, \infty\right)$.

H.

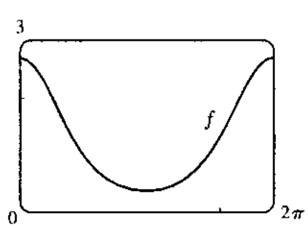


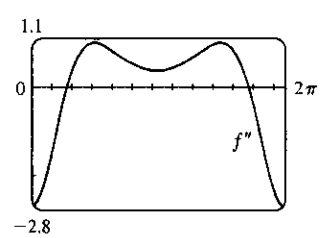
F. Local minimum value $f\left(\frac{1}{5}\ln\frac{2}{3}\right) = \left(\frac{2}{3}\right)^{3/5} + \left(\frac{2}{3}\right)^{-2/5} \approx 1.96$; no local maximum.

G. $f''(x) = 9e^{3x} + 4e^{-2x}$, so $f''(x) > 0$ for all x, and f is CU on $(-\infty, \infty)$. No IP

(*) If you have not yet learned about logarithms, graph $y = e^{5x}$ and $y = \frac{2}{3}$, find the point of intersection, and use decimal approximations for the rest of the solution.

68. The function $f(x) = e^{\cos x}$ is periodic with period 2π, so we consider it only on the interval $[0, 2\pi]$. We see that it has local maxima of about $f(0) \approx 2.72$ and $f(2\pi) \approx 2.72$, and a local minimum of about $f(3.14) \approx 0.37$. To find the exact

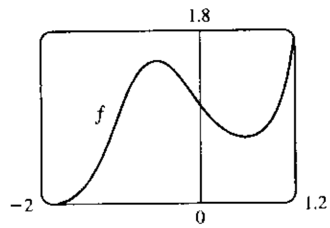


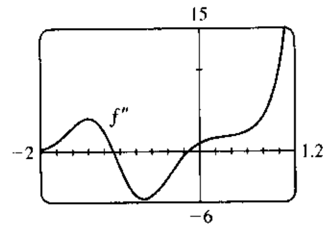


values, we calculate $f'(x) = -\sin x e^{\cos x}$. This is 0 when $-\sin x = 0 \Leftrightarrow x = 0, \pi$ or 2π (since we are only considering $x \in [0, 2\pi]$). Also $f'(x) > 0 \Leftrightarrow \sin x < 0 \Leftrightarrow 0 < x < \pi$. So $f(0) = f(2\pi) = e$

(both maxima) and $f(\pi) = e^{\cos\pi} = 1/e$ (minimum). To find the inflection points, we calculate and graph $f''(x) = \dfrac{d}{dx}\left(-\sin x\, e^{\cos x}\right) = -\cos x\, e^{\cos x} - \sin x\left(e^{\cos x}\right)(-\sin x) = e^{\cos x}\left(\sin^2 x - \cos x\right)$. From the graph of $f''(x)$, we see that f has inflection points at $x \approx 0.90$ and at $x \approx 5.38$. These x-coordinates correspond to inflection points $(0.90, 1.86)$ and $(5.38, 1.86)$.

69. $f(x) = e^{x^3 - x} \to 0$ as $x \to -\infty$, and $f(x) \to \infty$ as $x \to \infty$. From the graph, it appears that f has a local minimum of about $f(0.58) = 0.68$, and a local maximum of about $f(-0.58) = 1.47$. To find the exact values, we calculate





$f'(x) = (3x^2 - 1)e^{x^3 - x}$, which is 0 when $3x^2 - 1 = 0 \iff x = \pm\frac{1}{\sqrt{3}}$. The negative root corresponds to the local maximum $f\left(-\frac{1}{\sqrt{3}}\right) = e^{(-1/\sqrt{3})^3 - (-1/\sqrt{3})} = e^{2\sqrt{3}/9}$, and the positive root corresponds to the local minimum $f\left(\frac{1}{\sqrt{3}}\right) = e^{(1/\sqrt{3})^3 - (1/\sqrt{3})} = e^{-2\sqrt{3}/9}$. To estimate the inflection points, we calculate and graph

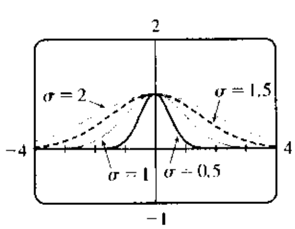
$f''(x) = \dfrac{d}{dx}\left[(3x^2 - 1)e^{x^3 - x}\right] = (3x^2 - 1)e^{x^3 - x}(3x^2 - 1) + e^{x^3 - x}(6x) = e^{x^3 - x}\left(9x^4 - 6x^2 + 6x + 1\right)$.

From the graph, it appears that $f''(x)$ changes sign (and thus f has inflection points) at $x \approx -0.15$ and $x \approx -1.09$. From the graph of f, we see that these x-values correspond to inflection points at about $(-0.15, 1.15)$ and $(-1.09, 0.82)$.

70. (a) As $|x| \to \infty$, $t = -x^2/(2\sigma^2) \to -\infty$, and $e^t \to 0$. The HA is $y = 0$. Since t takes on its maximum value at $x = 0$, so does e^t. Showing this result using derivatives, we have $f(x) = e^{-x^2/(2\sigma^2)} \Rightarrow$
$f'(x) = e^{-x^2/(2\sigma^2)}(-x/\sigma^2)$. $f'(x) = 0 \iff x = 0$. Because f' changes from positive to negative at $x = 0$, $f(0) = 1$ is a local maximum. For inflection points, we find
$f''(x) = -\dfrac{1}{\sigma^2}\left[e^{-x^2/(2\sigma^2)} \cdot 1 + xe^{-x^2/(2\sigma^2)}(-x/\sigma^2)\right] = \dfrac{-1}{\sigma^2}e^{-x^2/(2\sigma^2)}(1 - x^2/\sigma^2)$.
$f''(x) = 0 \iff x^2 = \sigma^2 \iff x = \pm\sigma$. $f''(x) < 0 \iff x^2 < \sigma^2 \iff -\sigma < x < \sigma$. So f is CD on $(-\sigma, \sigma)$ and CU on $(-\infty, -\sigma)$ and (σ, ∞). IP at $\left(\pm\sigma, e^{-1/2}\right)$.

(b) Since we have IP at $x = \pm\sigma$, the inflection points move away from the y-axis as σ increases.

(c)



From the graph, we see that as σ increases, the graph tends to spread out and there is more area between the curve and the x-axis.

71. Let $u = -3x$. Then $du = -3\,dx$, so
$\int_0^5 e^{-3x}\,dx = -\frac{1}{3}\int_0^{-15} e^u\,du = -\frac{1}{3}\left[e^u\right]_0^{-15} = -\frac{1}{3}\left(e^{-15} - e^0\right) = \frac{1}{3}\left(1 - e^{-15}\right)$.

72. Let $u = -x^2$, so $du = -2x\,dx$. When $x = 0$, $u = 0$; when $x = 1$, $u = -1$. Thus,

$\int_0^1 xe^{-x^2}\,dx = \int_0^{-1} e^u\left(-\frac{1}{2}\,du\right) = -\frac{1}{2}\left[e^u\right]_0^{-1} = -\frac{1}{2}\left(e^{-1} - e^0\right) = \frac{1}{2}(1 - 1/e)$.

73. Let $u = 1 + e^x$. Then $du = e^x\,dx$, so $\int e^x\sqrt{1 + e^x}\,dx = \int \sqrt{u}\,du = \frac{2}{3}u^{3/2} + C = \frac{2}{3}(1 + e^x)^{3/2} + C$.

74. Let $u = \tan x$. Then $du = \sec^2 x\,dx$, so $\int \sec^2 x\, e^{\tan x}\,dx = \int e^u\,du = e^u + C = e^{\tan x} + C$.

75. $\displaystyle\int \frac{e^x + 1}{e^x}\,dx = \int (1 + e^{-x})\,dx = x - e^{-x} + C$

76. Let $u = \dfrac{1}{x}$. Then $du = -\dfrac{1}{x^2}\,dx$, so $\displaystyle\int \frac{e^{1/x}}{x^2}\,dx = -\int e^u\,du = -e^u + C = -e^{1/x} + C$.

77. Let $u = \sqrt{x}$. Then $du = \dfrac{1}{2\sqrt{x}}\,dx$, so $\displaystyle\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx = 2\int e^u\,du = 2e^u + C = 2e^{\sqrt{x}} + C$.

78. Let $u = e^x$. Then $du = e^x\,dx$, so $\int e^x \sin(e^x)\,dx = \int \sin u\,du = -\cos u + C = -\cos(e^x) + C$.

79. Area $= \int_0^1 \left(e^{3x} - e^x\right)dx = \left[\frac{1}{3}e^{3x} - e^x\right]_0^1 = \left(\frac{1}{3}e^3 - e\right) - \left(\frac{1}{3} - 1\right) = \frac{1}{3}e^3 - e + \frac{2}{3} \approx 4.644$

80. $f''(x) = 3e^x + 5\sin x \;\Rightarrow\; f'(x) = 3e^x - 5\cos x + C \;\Rightarrow\; 2 = f'(0) = 3 - 5 + C \;\Rightarrow\; C = 4$, so

$f'(x) = 3e^x - 5\cos x + 4 \;\Rightarrow\; f(x) = 3e^x - 5\sin x + 4x + D \;\Rightarrow\; 1 = f(0) = 3 + D \;\Rightarrow\; D = -2$,

so $f(x) = 3e^x - 5\sin x + 4x - 2$.

81. $V = \int_0^1 \pi(e^x)^2\,dx = \pi\int_0^1 e^{2x}\,dx = \frac{1}{2}\pi\left[e^{2x}\right]_0^1 = \frac{\pi}{2}\left(e^2 - 1\right)$

82. $V = \int_0^1 2\pi x e^{-x^2}\,dx$. Let $u = x^2$. Thus $du = 2x\,dx$, so $V = \pi\int_0^1 e^{-u}\,du = \pi\left[-e^{-u}\right]_0^1 = \pi(1 - 1/e)$.

83. We use Theorem 7.1.7. Note that $f(0) = 3 + 0 + e^0 = 4$, so $f^{-1}(4) = 0$. Also $f'(x) = 1 + e^x$. Therefore,

$$\left(f^{-1}\right)'(4) = \frac{1}{f'(f^{-1}(4))} = \frac{1}{f'(0)} = \frac{1}{1 + e^0} = \frac{1}{2}.$$

84. We recognize this limit as the definition of the derivative of the function $f(x) = e^{\sin x}$ at $x = \pi$, since it is of the form $\displaystyle\lim_{x\to\pi} \frac{f(x) - f(\pi)}{x - \pi}$. Therefore, the limit is equal to $f'(\pi) = (\cos\pi)e^{\sin\pi} = -1\cdot e^0 = -1$.

85. (a) Let $f(x) = e^x - 1 - x$. Now $f(0) = e^0 - 1 = 0$, and for $x \ge 0$, we have $f'(x) = e^x - 1 \ge 0$. Now, since $f(0) = 0$ and f is increasing on $[0, \infty)$, $f(x) \ge 0$ for $x \ge 0 \;\Rightarrow\; e^x - 1 - x \ge 0 \;\Rightarrow\; e^x \ge 1 + x$.

(b) For $0 \le x \le 1$, $x^2 \le x$, so $e^{x^2} \le e^x$ [since e^x is increasing]. Hence [from (a)] $1 + x^2 \le e^{x^2} \le e^x$.
So $\frac{4}{3} = \int_0^1 \left(1 + x^2\right)dx \le \int_0^1 e^{x^2}\,dx \le \int_0^1 e^x\,dx = e - 1 < e \;\Rightarrow\; \frac{4}{3} \le \int_0^1 e^{x^2}\,dx \le e$.

86. (a) Let $f(x) = e^x - 1 - x - \frac{1}{2}x^2$. Thus, $f'(x) = e^x - 1 - x$, which is positive for $x \ge 0$ by Exercise 85(a). Thus $f(x)$ is increasing on $(0, \infty)$, so on that interval, $0 = f(0) \le f(x) = e^x - 1 - x - \frac{1}{2}x^2 \;\Rightarrow$ $e^x \ge 1 + x + \frac{1}{2}x^2$.

(b) Using the same argument as in Exercise 85(b), from part (a) we have $1 + x^2 + \frac{1}{2}x^4 \le e^{x^2} \le e^x$ [for $0 \le x \le 1$] $\;\Rightarrow\; \int_0^1 \left(1 + x^2 + \frac{1}{2}x^4\right)dx \le \int_0^1 e^{x^2}\,dx \le \int_0^1 e^x\,dx \;\Rightarrow\; \frac{43}{30} \le \int_0^1 e^{x^2}\,dx \le e - 1$.

87. (a) By Exercise 85(a), the result holds for $n = 1$. Suppose that $e^x \geq 1 + x + \dfrac{x^2}{2!} + \cdots + \dfrac{x^k}{k!}$ for $x \geq 0$.

Let $f(x) = e^x - 1 - x - \dfrac{x^2}{2!} - \cdots - \dfrac{x^k}{k!} - \dfrac{x^{k+1}}{(k+1)!}$. Then $f'(x) = e^x - 1 - x - \cdots - \dfrac{x^k}{k!} \geq 0$ by assumption. Hence $f(x)$ is increasing on $(0, \infty)$. So $0 \leq x$ implies that

$0 = f(0) \leq f(x) = e^x - 1 - x - \cdots - \dfrac{x^k}{k!} - \dfrac{x^{k+1}}{(k+1)!}$, and hence $e^x \geq 1 + x + \cdots + \dfrac{x^k}{k!} + \dfrac{x^{k+1}}{(k+1)!}$

for $x \geq 0$. Therefore, for $x \geq 0$, $e^x \geq 1 + x + \dfrac{x^2}{2!} + \cdots + \dfrac{x^n}{n!}$ for every positive integer n, by mathematical induction.

(b) Taking $n = 4$ and $x = 1$ in (a), we have $e = e^1 \geq 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} = 2.708\overline{3} > 2.7$.

(c) $e^x \geq 1 + x + \cdots + \dfrac{x^k}{k!} + \dfrac{x^{k+1}}{(k+1)!} \quad \Rightarrow \quad \dfrac{e^x}{x^k} \geq \dfrac{1}{x^k} + \dfrac{1}{x^{k-1}} + \cdots + \dfrac{1}{k!} + \dfrac{x}{(k+1)!} \geq \dfrac{x}{(k+1)!}$.

But $\displaystyle\lim_{x\to\infty} \frac{x}{(k+1)!} = \infty$, so $\displaystyle\lim_{x\to\infty} \frac{e^x}{x^k} = \infty$.

7.3 Logarithmic Functions

1. (a) It is defined as the inverse of the exponential function with base a, that is, $\log_a x = y \quad \Leftrightarrow \quad a^y = x$.

(b) $(0, \infty)$ (c) $\mathbb{R}$ (d) See Figure 1.

2. (a) The natural logarithm is the logarithm with base e, denoted $\ln x$.

(b) The common logarithm is the logarithm with base 10, denoted $\log x$.

(c) See Figure 3.

3. (a) $\log_{10} 1000 = 3$ because $10^3 = 1000$. *Or:* $\log_{10} 1000 = \log_{10} 10^3 = 3$ by (2).

(b) $\log_{16} 4 = \frac{1}{2}$ because $16^{1/2} = 4$. *Or:* $\log_{16} 4 = \log_{16} 16^{1/2} = \frac{1}{2}$ by (2).

4. (a) By (6), $\ln e^{-100} = -100$. (b) $\log_3 81 = 4$ since $3^4 = 81$.

5. (a) $\log_5 \frac{1}{25} = \log_5 5^{-2} = -2$ by (2). (b) $e^{\ln 15} = 15$ by (6).

6. (a) $\log_{10} 0.1 = -1$ since $10^{-1} = 0.1$.

(b) $\log_8 320 - \log_8 5 = \log_8 \frac{320}{5} = \log_8 64 = 2$ since $8^2 = 64$.

7. (a) $\log_{12} 3 + \log_{12} 48 = \log_{12}(3 \cdot 48) = \log_{12} 144 = 2$ since $12^2 = 144$.

(b) $\log_2 5 - \log_2 90 + 2\log_2 3 = \log_2 5 + \log_2 3^2 - \log_2 90 = \log_2(5 \cdot 9) - \log_2 90$
$= \log_2\left(\frac{45}{90}\right) = \log_2\left(\frac{1}{2}\right) = -1$ since $2^{-1} = \frac{1}{2}$.

8. (a) $2^{(\log_2 3 + \log_2 5)} = 2^{\log_2 15} = 15$ [*Or:* $2^{(\log_2 3 + \log_2 5)} = 2^{\log_2 3} \cdot 2^{\log_2 5} = 3 \cdot 5 = 15$]

(b) $e^{3\ln 2} = e^{\ln(2^3)} = e^{\ln 8} = 8$ [*Or:* $e^{3\ln 2} = \left(e^{\ln 2}\right)^3 = 2^3 = 8$]

9. $\log_2\left(\dfrac{x^3 y}{z^2}\right) = \log_2(x^3 y) - \log_2 z^2 = \log_2 x^3 + \log_2 y - \log_2 z^2 = 3\log_2 x + \log_2 y - 2\log_2 z$

(assuming that the variables are positive)

10. $\ln\sqrt{a(b^2+c^2)} = \ln(a(b^2+c^2))^{1/2} = \frac{1}{2}\ln(a(b^2+c^2)) = \frac{1}{2}\left[\ln a + \ln(b^2+c^2)\right]$
$= \frac{1}{2}\ln a + \frac{1}{2}\ln(b^2+c^2)$

11. $\ln(uv)^{10} = 10\ln(uv) = 10(\ln u + \ln v) = 10\ln u + 10\ln v$

12. $\ln\dfrac{3x^2}{(x+1)^5} = \ln 3x^2 - \ln(x+1)^5 = \ln 3 + \ln x^2 - 5\ln(x+1) = \ln 3 + 2\ln x - 5\ln(x+1)$

13. $\log_{10} a - \log_{10} b + \log_{10} c = \log_{10}\dfrac{a}{b} + \log_{10} c = \log_{10}\left(\dfrac{a}{b}\cdot c\right) = \log_{10}\dfrac{ac}{b}$

14. $\ln(x+y) + \ln(x-y) - 2\ln z = \ln((x+y)(x-y)) - \ln z^2 = \ln(x^2-y^2) - \ln z^2 = \ln\dfrac{x^2-y^2}{z^2}$

15. $2\ln 4 - \ln 2 = \ln 4^2 - \ln 2 = \ln 16 - \ln 2 = \ln\frac{16}{2} = \ln 8$

16. $\ln 3 + \frac{1}{3}\ln 8 = \ln 3 + \ln 8^{1/3} = \ln 3 + \ln 2 = \ln(3\cdot 2) = \ln 6$

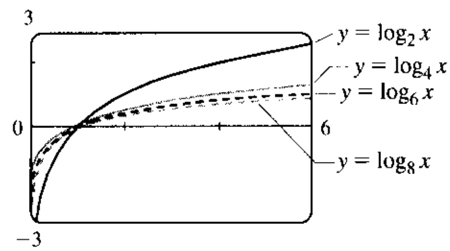
17. $\frac{1}{2}\ln x - 5\ln(x^2+1) = \ln x^{1/2} - \ln(x^2+1)^5 = \ln\dfrac{\sqrt{x}}{(x^2+1)^5}$

18. $\ln x + a\ln y - b\ln z = \ln x + \ln y^a - \ln z^b = \ln(x\cdot y^a) - \ln z^b = \ln\left(xy^a/z^b\right)$

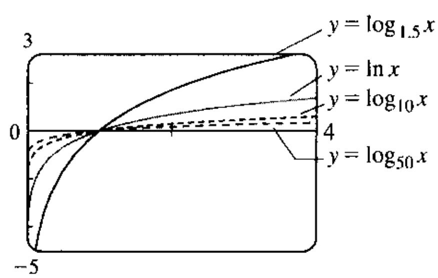
19. (a) $\log_{12} e = \dfrac{\ln e}{\ln 12} = \dfrac{1}{\ln 12} \approx 0.402430$ (b) $\log_6 13.54 = \dfrac{\ln 13.54}{\ln 6} \approx 1.454240$

(c) $\log_2 \pi = \dfrac{\ln \pi}{\ln 2} \approx 1.651496$

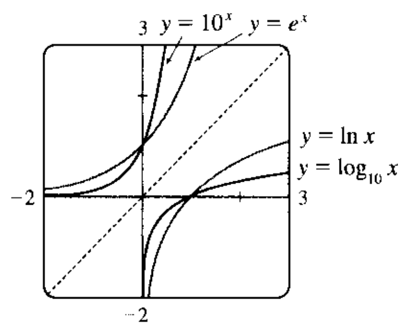
20. To graph the functions, we use $\log_2 x = \dfrac{\ln x}{\ln 2}$, $\log_4 x = \dfrac{\ln x}{\ln 4}$, etc. These graphs all approach $-\infty$ as $x \to 0^+$, and they all pass through the point $(1,0)$. Also, they are all increasing, and all approach ∞ as $x \to \infty$. The smaller the base, the larger the rate of increase of the function (for $x > 1$) and the closer the approach to the y-axis (as $x \to 0^+$).



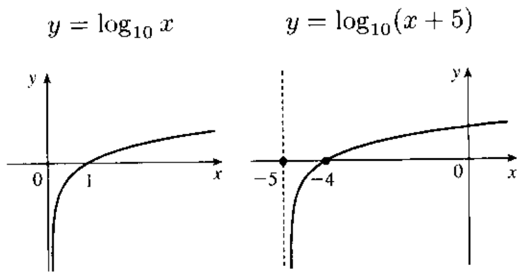
21. To graph these functions, we use $\log_{1.5} x = \dfrac{\ln x}{\ln 1.5}$ and $\log_{50} x = \dfrac{\ln x}{\ln 50}$. These graphs all approach $-\infty$ as $x \to 0^+$, and they all pass through the point $(1,0)$. Also, they are all increasing, and all approach ∞ as $x \to \infty$. The functions with larger bases increase extremely slowly, and the ones with smaller bases do so somewhat more quickly. The functions with large bases approach the y-axis more closely as $x \to 0^+$.



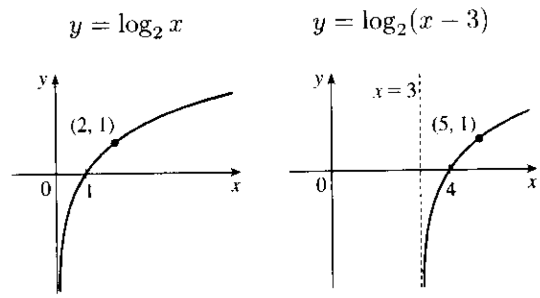
22. We see that the graph of $\ln x$ is the reflection of the graph of e^x about the line $y = x$, and that the graph of $\log_{10} x$ is the reflection of the graph of 10^x about the same line. The graph of 10^x increases more quickly than that of e^x. Also note that $\log_{10} x \to \infty$ as $x \to \infty$ more slowly than $\ln x$.



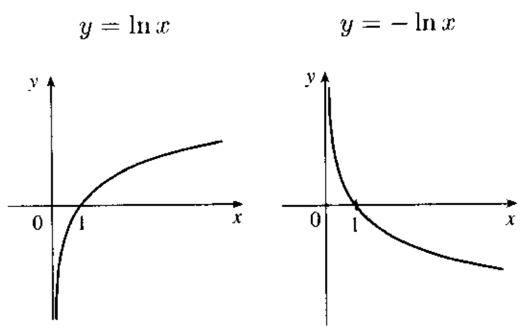
23. Shift the graph of $y = \log_{10} x$ five units to the left to obtain the graph of $y = \log_{10}(x + 5)$. Note the vertical asymptote of $x = -5$.



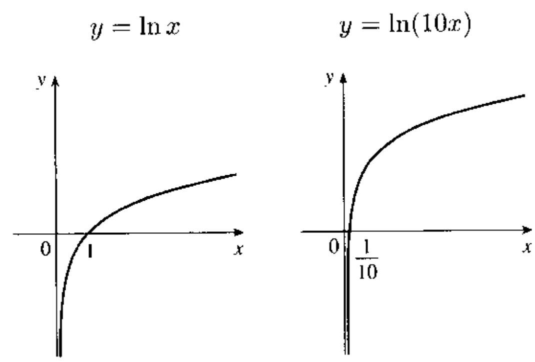
24. $\log_2(x - 3)$: Start with the graph of $y = \log_2 x$ and shift 3 units to the right.



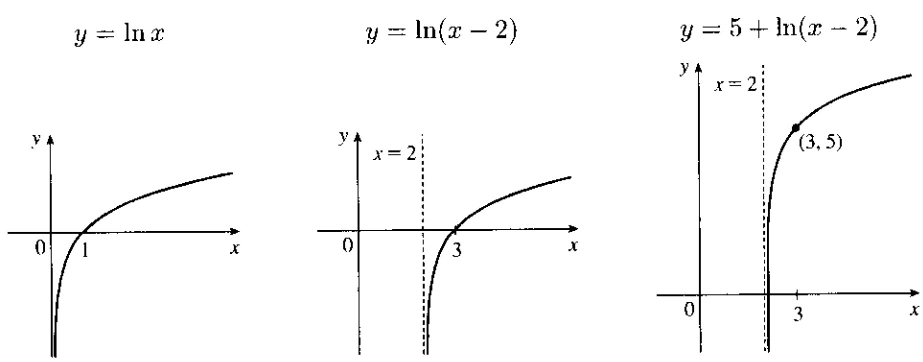
25. Reflect the graph of $y = \ln x$ about the x-axis to obtain the graph of $y = -\ln x$.



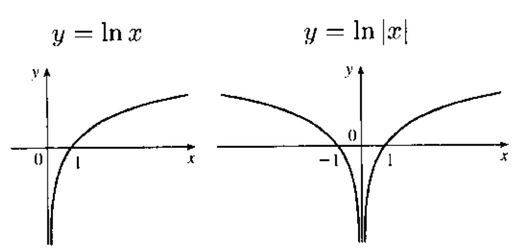
26. $y = \ln(10x)$: Start with the graph of $y = \ln x$ and compress horizontally by a factor of 10. *Or*: $y = \ln(10x) = \ln 10 + \ln x$, so we could start with $y = \ln x$ and shift $\ln 10$ units upward .



27. $y = 5 + \ln(x - 2)$: Start with the graph of $y = \ln x$, shift 2 units to the right and then shift 5 units upward.



28. Reflect the portion of the graph of $y = \ln x$ to the right of the y-axis about the y-axis. The graph of $y = \ln |x|$ is that reflection in addition to the original portion.



29. (a) $2\ln x = 1 \quad \Rightarrow \quad \ln x = \frac{1}{2} \quad \Rightarrow \quad x = e^{1/2} = \sqrt{e}$

(b) $e^{-x} = 5 \quad \Rightarrow \quad -x = \ln 5 \quad \Rightarrow \quad x = -\ln 5$

30. (a) $e^{2x+3} - 7 = 0 \quad \Rightarrow \quad e^{2x+3} = 7 \quad \Rightarrow \quad 2x + 3 = \ln 7 \quad \Rightarrow \quad 2x = \ln 7 - 3 \quad \Rightarrow \quad x = \frac{1}{2}(\ln 7 - 3)$

(b) $\ln(5 - 2x) = -3 \quad \Rightarrow \quad 5 - 2x = e^{-3} \quad \Rightarrow \quad 2x = 5 - e^{-3} \quad \Rightarrow \quad x = \frac{1}{2}(5 - e^{-3})$

31. (a) $5^{x-3} = 10 \quad \Leftrightarrow \quad \log_{10} 5^{x-3} = \log_{10} 10 \quad \Leftrightarrow \quad (x-3)\log_{10} 5 = 1 \quad \Leftrightarrow \quad x - 3 = 1/\log_{10} 5 \quad \Leftrightarrow$
$x = 3 + 1/\log_{10} 5$

(b) $\log_{10}(x+1) = 4 \quad \Leftrightarrow \quad x + 1 = 10^4 \quad \Leftrightarrow \quad x = 10{,}000 - 1 = 9999$

32. (a) $e^{3x+1} = k \quad \Leftrightarrow \quad 3x + 1 = \ln k \quad \Leftrightarrow \quad x = \frac{1}{3}(\ln k - 1)$

(b) $\log_2(mx) = c \quad \Leftrightarrow \quad mx = 2^c \quad \Leftrightarrow \quad x = 2^c/m$

33. $\ln(\ln x) = 1 \quad \Leftrightarrow \quad e^{\ln(\ln x)} = e^1 \quad \Leftrightarrow \quad \ln x = e^1 = e \quad \Leftrightarrow \quad e^{\ln x} = e^e \quad \Leftrightarrow \quad x = e^e$

34. $e^{e^x} = 10 \quad \Leftrightarrow \quad \ln\left(e^{e^x}\right) = \ln 10 \quad \Leftrightarrow \quad e^x \ln e = e^x = \ln 10 \quad \Leftrightarrow \quad \ln e^x = \ln(\ln 10) \quad \Leftrightarrow \quad x = \ln(\ln 10)$

35. $2\ln x = \ln 2 + \ln(3x - 4) \quad \Rightarrow \quad \ln x^2 = \ln[2(3x-4)] \quad \Rightarrow \quad \ln x^2 = \ln(6x - 8) \quad \Rightarrow \quad x^2 = 6x - 8 \quad \Rightarrow$
$x^2 - 6x + 8 = 0 \quad \Rightarrow \quad (x-2)(x-4) = 0 \quad \Rightarrow \quad x = 2$ or $x = 4$, both are valid solutions.

36. $\ln(2x+1) = 2 - \ln x \quad \Rightarrow \quad \ln x + \ln(2x+1) = \ln e^2 \quad \Rightarrow \quad \ln[x(2x+1)] = \ln e^2 \quad \Rightarrow \quad 2x^2 + x = e^2 \quad \Rightarrow$

$2x^2 + x - e^2 = 0 \quad \Rightarrow \quad x = \dfrac{-1 + \sqrt{1 + 8e^2}}{4}$ [since $x > 0$].

37. $e^{ax} = Ce^{bx} \quad \Leftrightarrow \quad \ln e^{ax} = \ln[C(e^{bx})] \quad \Leftrightarrow \quad ax = \ln C + bx + \ln e^{bx} \quad \Leftrightarrow \quad ax = \ln C + bx \quad \Leftrightarrow$

$ax - bx = \ln C \quad \Leftrightarrow \quad (a-b)x = \ln C \quad \Leftrightarrow \quad x = \dfrac{\ln C}{a - b}$

38. $7e^x - e^{2x} = 12 \quad \Leftrightarrow \quad (e^x)^2 - 7e^x + 12 = 0 \quad \Leftrightarrow \quad (e^x - 3)(e^x - 4) = 0$, so we have either $e^x = 3 \quad \Leftrightarrow$
$x = \ln 3$, or $e^x = 4 \quad \Leftrightarrow \quad x = \ln 4$.

39. $e^{2+5x} = 100 \quad \Rightarrow \quad \ln\left(e^{2+5x}\right) = \ln 100 \quad \Rightarrow \quad 2 + 5x = \ln 100 \quad \Rightarrow \quad 5x = \ln 100 - 2 \quad \Rightarrow$
$x = \frac{1}{5}(\ln 100 - 2) \approx 0.5210$

40. $\ln(1 + \sqrt{x}) = 2 \quad \Rightarrow \quad 1 + \sqrt{x} = e^2 \quad \Rightarrow \quad \sqrt{x} = e^2 - 1 \quad \Rightarrow \quad x = (e^2 - 1)^2 \approx 40.8200$

41. $\ln(e^x - 2) = 3 \quad \Rightarrow \quad e^x - 2 = e^3 \quad \Rightarrow \quad e^x = e^3 + 2 \quad \Rightarrow \quad x = \ln(e^3 + 2) \approx 3.0949$

42. $3^{1/(x-4)} = 7 \quad \Rightarrow \quad \ln 3^{1/(x-4)} = \ln 7 \quad \Rightarrow \quad \dfrac{1}{x-4}\ln 3 = \ln 7 \quad \Rightarrow \quad x - 4 = \dfrac{\ln 3}{\ln 7} \quad \Rightarrow$

$x = 4 + \dfrac{\ln 3}{\ln 7} \approx 4.5646$

43. (a) $e^x < 10 \quad \Rightarrow \quad \ln e^x < \ln 10 \quad \Rightarrow \quad x < \ln 10 \quad \Rightarrow \quad x \in (-\infty, \ln 10)$

(b) $\ln x > -1 \quad \Rightarrow \quad e^{\ln x} > e^{-1} \quad \Rightarrow \quad x > e^{-1} \quad \Rightarrow \quad x \in (1/e, \infty)$

44. (a) $2 < \ln x < 9 \quad \Rightarrow \quad e^2 < e^{\ln x} < e^9 \quad \Rightarrow \quad e^2 < x < e^9 \quad \Rightarrow \quad x \in (e^2, e^9)$

(b) $e^{2-3x} > 4 \quad \Rightarrow \quad \ln e^{2-3x} > \ln 4 \quad \Rightarrow \quad 2 - 3x > \ln 4 \quad \Rightarrow \quad -3x > \ln 4 - 2 \quad \Rightarrow$
$x < -\frac{1}{3}(\ln 4 - 2) \quad \Rightarrow \quad x \in \left(-\infty, \frac{1}{3}(2 - \ln 4)\right)$

45. 3 ft = 36 in, so we need x such that $\log_2 x = 36 \quad \Leftrightarrow \quad x = 2^{36} = 68{,}719{,}476{,}736$. In miles, this is

$68{,}719{,}476{,}736 \text{ in} \cdot \dfrac{1 \text{ ft}}{12 \text{ in}} \cdot \dfrac{1 \text{ mi}}{5280 \text{ ft}} \approx 1{,}084{,}587.7$ mi.

46. (a) $v(t) = ce^{-kt} \;\Rightarrow\; a(t) = v'(t) = -kce^{-kt} = -kv(t)$

(b) $v(0) = ce^0 = c$, so c is the initial velocity.

(c) $v(t) = ce^{-kt} = c/2 \;\Rightarrow\; e^{-kt} = \frac{1}{2} \;\Rightarrow\; -kt = \ln\frac{1}{2} = -\ln 2 \;\Rightarrow\; t = (\ln 2)/k$

47. If I is the intensity of the 1989 San Francisco earthquake, then $\log_{10}(I/S) = 7.1 \;\Rightarrow$ $\log_{10}(16I/S) = \log_{10} 16 + \log_{10}(I/S) = \log_{10} 16 + 7.1 \approx 8.3$.

48. Let I_1 and I_2 be the intensities of the music and the mower. Then $10\log_{10}\left(\frac{I_1}{I_0}\right) = 120$ and $10\log_{10}\left(\frac{I_2}{I_0}\right) = 106$, so $\log_{10}\left(\frac{I_1}{I_2}\right) = \log_{10}\left(\frac{I_1/I_0}{I_2/I_0}\right) = \log_{10}\left(\frac{I_1}{I_0}\right) - \log_{10}\left(\frac{I_2}{I_0}\right) = 12 - 10.6 = 1.4 \;\Rightarrow\; \frac{I_1}{I_2} = 10^{1.4} \approx 25$.

49. (a) $n = 100 \cdot 2^{t/3} \;\Rightarrow\; \frac{n}{100} = 2^{t/3} \;\Rightarrow\; \log_2\left(\frac{n}{100}\right) = \frac{t}{3} \;\Rightarrow\; t = 3\log_2\left(\frac{n}{100}\right)$. Using formula (7), we can write this as $t = 3 \cdot \frac{\ln(n/100)}{\ln 2}$. This function tells us how long it will take to obtain n bacteria (given the number n).

(b) $n = 50{,}000 \;\Rightarrow\; t = 3\log_2 \frac{50{,}000}{100} = 3\log_2 500 = 3\left(\frac{\ln 500}{\ln 2}\right) \approx 26.9$ hours

50. (a) $Q = Q_0\left(1 - e^{-t/a}\right) \;\Rightarrow\; \frac{Q}{Q_0} = 1 - e^{-t/a} \;\Rightarrow\; e^{-t/a} = 1 - \frac{Q}{Q_0} \;\Rightarrow\; -\frac{t}{a} = \ln\left(1 - \frac{Q}{Q_0}\right) \;\Rightarrow$ $t = -a\ln(1 - Q/Q_0)$. This gives us the time t necessary to obtain a given charge Q.

(b) $Q = 0.9Q_0$ and $a = 2 \;\Rightarrow\; t = -2\ln(1 - 0.9(Q_0/Q_0)) = -2\ln 0.1 \approx 4.6$ seconds.

51. Let $t = 2 - x$. As $x \to 2^-$, $t \to 0^+$. $\lim\limits_{x\to 2^-} \ln(2 - x) = \lim\limits_{t\to 0^+} \ln t = -\infty$ by (8).

52. Let $t = x^2 - 5x + 6$. As $x \to 3^+$, $t = (x-2)(x-3) \to 0^+$. $\lim\limits_{x\to 3^+} \log_{10}(x^2 - 5x + 6) = \lim\limits_{t\to 0^+} \log_{10} t = -\infty$ by (4).

53. $\lim\limits_{x\to 0} \ln(\cos x) = \ln 1 = 0$. [$\ln(\cos x)$ is continuous at $x = 0$ since it is the composite of two continuous functions.]

54. $\lim\limits_{x\to 0^+} \ln(\sin x) = -\infty$ since $\sin x \to 0^+$ as $x \to 0^+$.

55. $\lim\limits_{x\to\infty} [\ln(1 + x^2) - \ln(1 + x)] = \lim\limits_{x\to\infty} \ln\frac{1 + x^2}{1 + x} = \ln\left(\lim\limits_{x\to\infty} \frac{1 + x^2}{1 + x}\right) = \ln\left(\lim\limits_{x\to\infty} \frac{\frac{1}{x} + x}{\frac{1}{x} + 1}\right) = \infty$, since the limit in parentheses is ∞.

56. $\lim\limits_{x\to\infty} [\ln(2 + x) - \ln(1 + x)] = \lim\limits_{x\to\infty} \ln\left(\frac{2 + x}{1 + x}\right) = \lim\limits_{x\to\infty} \ln\left(\frac{2/x + 1}{1/x + 1}\right) = \ln\frac{1}{1} = \ln 1 = 0$

57. The domain of $f(x) = \log_2(5x - 3)$ is $\{x \mid 5x - 3 > 0\} = \left\{x \mid x > \frac{3}{5}\right\} = \left(\frac{3}{5}, \infty\right)$. Since $5x - 3$ takes on all positive values for x in $\left(\frac{3}{5}, \infty\right)$, the range of f is $\mathbb{R}$.

58. The domain of $G(t) = \ln(e^t - 2)$ is $\{t \mid e^t - 2 > 0\} = \{t \mid e^t > 2\} = \{t \mid t > \ln 2\} = (\ln 2, \infty)$. Since $e^t - 2$ takes on all positive values for t in $(\ln 2, \infty)$, the range of G is $\mathbb{R}$.

59. (a) For $f(x) = \sqrt{3 - e^{2x}}$, we must have $3 - e^{2x} \geq 0 \;\Rightarrow\; e^{2x} \leq 3 \;\Rightarrow\; 2x \leq \ln 3 \;\Rightarrow\; x \leq \frac{1}{2}\ln 3$. Thus, the domain of f is $\left(-\infty, \frac{1}{2}\ln 3\right]$.

(b) $y = f(x) = \sqrt{3 - e^{2x}}$ [note that $y \geq 0$] $\Rightarrow$ $y^2 = 3 - e^{2x}$ $\Rightarrow$ $e^{2x} = 3 - y^2$ $\Rightarrow$ $2x = \ln(3 - y^2)$ $\Rightarrow$ $x = \frac{1}{2}\ln(3 - y^2)$. Interchange x and y: $y = \frac{1}{2}\ln(3 - x^2)$. So $f^{-1}(x) = \frac{1}{2}\ln(3 - x^2)$. For the domain of f^{-1}, we must have $3 - x^2 > 0$ $\Rightarrow$ $x^2 < 3$ $\Rightarrow$ $|x| < \sqrt{3}$ $\Rightarrow$ $-\sqrt{3} < x < \sqrt{3}$ $\Rightarrow$ $0 \leq x < \sqrt{3}$ since $x \geq 0$. Note that the domain of f^{-1}, $[0, \sqrt{3})$, equals the range of f.

60. (a) For $f(x) = \ln(2 + \ln x)$, we must have $2 + \ln x > 0$ $\Rightarrow$ $\ln x > -2$ $\Rightarrow$ $x > e^{-2}$. Thus, the domain of f is (e^{-2}, ∞).

(b) $y = f(x) = \ln(2 + \ln x)$ $\Rightarrow$ $e^y = 2 + \ln x$ $\Rightarrow$ $\ln x = e^y - 2$ $\Rightarrow$ $x = e^{e^y - 2}$. Interchange x and y: $y = e^{e^x - 2}$. So $f^{-1}(x) = e^{e^x - 2}$. The domain of f^{-1}, as well as the range of f, is $\mathbb{R}$.

61. $y = \ln(x + 3)$ $\Rightarrow$ $e^y = e^{\ln(x+3)} = x + 3$ $\Rightarrow$ $x = e^y - 3$.
Interchange x and y: the inverse function is $y = e^x - 3$.

62. $y = 2^{10^x}$ $\Rightarrow$ $\log_2 y = 10^x$ $\Rightarrow$ $\log_{10}(\log_2 y) = x$.
Interchange x and y: $y = \log_{10}(\log_2 x)$ is the inverse function.

63. $f(x) = e^{x^3}$ $\Rightarrow$ $y = e^{x^3}$ $\Rightarrow$ $\ln y = x^3$ $\Rightarrow$ $x = \sqrt[3]{\ln y}$. Interchange x and y: $y = \sqrt[3]{\ln x}$.
So $f^{-1}(x) = \sqrt[3]{\ln x}$.

64. $y = (\ln x)^2$, $x \geq 1$, $\ln x = \sqrt{y}$ $\Rightarrow$ $x = e^{\sqrt{y}}$. Interchange x and y: $y = e^{\sqrt{x}}$ is the inverse function.

65. $y = \dfrac{10^x}{10^x + 1}$ $\Rightarrow$ $10^x y + y = 10^x$ $\Rightarrow$ $10^x(1 - y) = y$ $\Rightarrow$ $10^x = \dfrac{y}{1 - y}$ $\Rightarrow$ $x = \log_{10}\left(\dfrac{y}{1 - y}\right)$.

Interchange x and y: $y = \log_{10}\left(\dfrac{x}{1 - x}\right)$ is the inverse function.

66. $y = \dfrac{1 + e^x}{1 - e^x}$ $\Rightarrow$ $y - ye^x = 1 + e^x$ $\Rightarrow$ $e^x(y + 1) = y - 1$ $\Rightarrow$ $e^x = \dfrac{y - 1}{y + 1}$ $\Rightarrow$ $x = \ln\left(\dfrac{y - 1}{y + 1}\right)$.

Interchange x and y: $y = \ln\left(\dfrac{x - 1}{x + 1}\right)$ is the inverse function.

67. $f(x) = e^{3x} - e^x$ $\Rightarrow$ $f'(x) = 3e^{3x} - e^x$. Thus, $f'(x) > 0$ $\Leftrightarrow$ $3e^{3x} > e^x$ $\Leftrightarrow$ $\dfrac{3e^{3x}}{e^x} > \dfrac{e^x}{e^x}$ $\Leftrightarrow$ $3e^{2x} > 1$ $\Leftrightarrow$ $e^{2x} > \frac{1}{3}$ $\Leftrightarrow$ $2x > \ln\left(\frac{1}{3}\right) = -\ln 3$ $\Leftrightarrow$ $x > -\frac{1}{2}\ln 3$, so f is increasing on $\left(-\frac{1}{2}\ln 3, \infty\right)$.

68. $y = 2e^x - e^{-3x}$ $\Rightarrow$ $y' = 2e^x + 3e^{-3x}$ $\Rightarrow$ $y'' = 2e^x - 9e^{-3x}$. Thus, $y'' < 0$ $\Leftrightarrow$ $2e^x < 9e^{-3x}$ $\Leftrightarrow$ $e^{4x} < \frac{9}{2}$ $\Leftrightarrow$ $4x < \ln\frac{9}{2}$ $\Leftrightarrow$ $x < \frac{1}{4}\ln\frac{9}{2}$, so f is concave downward on $\left(-\infty, \frac{1}{4}\ln\frac{9}{2}\right)$.

69. (a) We have to show that $-f(x) = f(-x)$.

$$\begin{aligned} -f(x) &= -\ln\left(x + \sqrt{x^2+1}\right) = \ln\left(\left(x + \sqrt{x^2+1}\right)^{-1}\right) = \ln\frac{1}{x + \sqrt{x^2+1}} \\ &= \ln\left(\frac{1}{x + \sqrt{x^2+1}} \cdot \frac{x - \sqrt{x^2+1}}{x - \sqrt{x^2+1}}\right) = \ln\frac{x - \sqrt{x^2+1}}{x^2 - x^2 - 1} \\ &= \ln\left(\sqrt{x^2+1} - x\right) = f(-x) \end{aligned}$$

Thus, f is an odd function.

(b) Let $y = \ln\left(x + \sqrt{x^2+1}\right)$. Then $e^y = x + \sqrt{x^2+1}$ $\Leftrightarrow$ $(e^y - x)^2 = x^2 + 1$ $\Leftrightarrow$ $e^{2y} - 2xe^y + x^2 = x^2 + 1$ $\Leftrightarrow$ $2xe^y = e^{2y} - 1$ $\Leftrightarrow$ $x = \dfrac{e^{2y} - 1}{2e^y} = \frac{1}{2}(e^y - e^{-y})$. Thus, the inverse function is $f^{-1}(x) = \frac{1}{2}(e^x - e^{-x})$.

70. Let (a, e^{-a}) be the point where the tangent meets the curve. The tangent has slope $-e^{-a}$ and is perpendicular to the line $2x - y = 8$, which has slope 2. So $-e^{-a} = -\frac{1}{2} \Rightarrow e^{-a} = \frac{1}{2} \Rightarrow e^a = 2 \Rightarrow a = \ln(e^a) = \ln 2$. Thus, the point on the curve is $(\ln 2, \frac{1}{2})$ and the equation of the tangent is $y - \frac{1}{2} = -\frac{1}{2}(x - \ln 2)$ or $x + 2y = 1 + \ln 2$.

71. $x^{1/\ln x} = 2 \Rightarrow \ln(x^{1/\ln x}) = \ln(2) \Rightarrow \dfrac{1}{\ln x} \cdot \ln x = \ln 2 \Rightarrow 1 = \ln 2$, a contradiction, so the given equation has no solution. The function $f(x) = x^{1/\ln x} = (e^{\ln x})^{1/\ln x} = e^1 = e$ for all $x > 0$, so the function $f(x) = x^{1/\ln x}$ is the constant function $f(x) = e$.

72. (a) $\lim\limits_{x\to\infty} x^{\ln x} = \lim\limits_{x\to\infty} \left(e^{\ln x}\right)^{\ln x} = \lim\limits_{x\to\infty} e^{(\ln x)^2} = \infty$ since $(\ln x)^2 \to \infty$ as $x \to \infty$.

(b) $\lim\limits_{x\to 0^+} x^{-\ln x} = \lim\limits_{x\to 0^+} \left(e^{\ln x}\right)^{-\ln x} = \lim\limits_{x\to 0^+} e^{-(\ln x)^2} = 0$ since $-(\ln x)^2 \to -\infty$ as $x \to 0^+$.

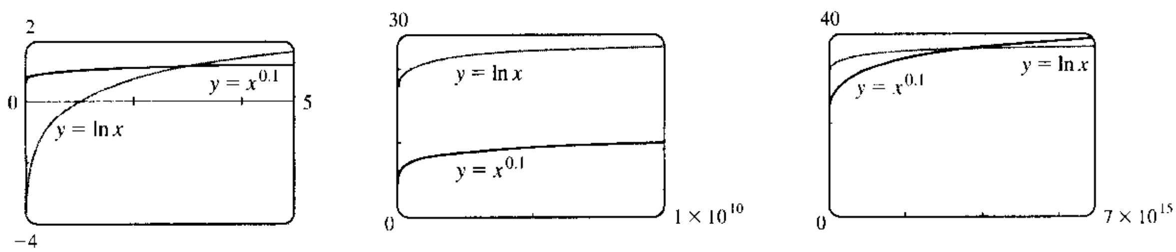
(c) $\lim\limits_{x\to 0^+} x^{1/x} = \lim\limits_{x\to 0^+} \left(e^{\ln x}\right)^{1/x} = \lim\limits_{x\to 0^+} e^{(\ln x/x)} = 0$ since $\dfrac{\ln x}{x} \to -\infty$ as $x \to 0^+$. Note that as $x \to 0^+$, $\ln x$ is a large negative number and x is a small positive number, so $(\ln x)/x \to -\infty$.

(d) $\lim\limits_{x\to\infty} (\ln 2x)^{-\ln x} = \lim\limits_{x\to\infty} \left[e^{\ln(\ln 2x)}\right]^{-\ln x} = \lim\limits_{x\to\infty} e^{-\ln x \ln(\ln 2x)} = 0$ since $-\ln x \ln(\ln 2x) \to -\infty$ as $x \to \infty$.

73. (a) Let $\varepsilon > 0$ be given. We need N such that $|a^x - 0| < \varepsilon$ when $x < N$. But $a^x < \varepsilon \Leftrightarrow x < \log_a \varepsilon$. Let $N = \log_a \varepsilon$. Then $x < N \Rightarrow x < \log_a \varepsilon \Rightarrow |a^x - 0| = a^x < \varepsilon$, so $\lim\limits_{x\to -\infty} a^x = 0$.

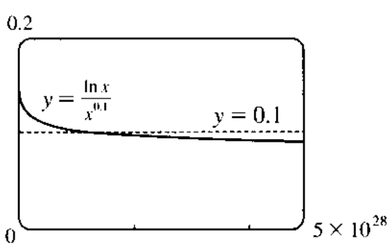
(b) Let $M > 0$ be given. We need N such that $a^x > M$ when $x > N$. But $a^x > M \Leftrightarrow x > \log_a M$. Let $N = \log_a M$. Then $x > N \Rightarrow x > \log_a M \Rightarrow a^x > M$, so $\lim\limits_{x\to\infty} a^x = \infty$.

74. (a)



From the graphs, we see that $f(x) = x^{0.1} > g(x) = \ln x$ for approximately $0 < x < 3.06$, and then $g(x) > f(x)$ for $3.06 < x < 3.43 \times 10^{15}$ (approximately). At that point, the graph of f finally surpasses the graph of g for good.

(b)



(c) From the graph at left, it seems that $\dfrac{\ln x}{x^{0.1}} < 0.1$ whenever $x > 1.3 \times 10^{28}$ (approximately). So we can take $N = 1.3 \times 10^{28}$, or any larger number.

75. $\ln(x^2 - 2x - 2) \le 0 \Rightarrow 0 < x^2 - 2x - 2 \le 1$. Now $x^2 - 2x - 2 \le 1$ gives $x^2 - 2x - 3 \le 0$ and hence $(x-3)(x+1) \le 0$. So $-1 \le x \le 3$. Now $0 < x^2 - 2x - 2 \Rightarrow x < 1 - \sqrt{3}$ or $x > 1 + \sqrt{3}$. Therefore, $\ln(x^2 - 2x - 2) \le 0 \Leftrightarrow -1 \le x < 1 - \sqrt{3}$ or $1 + \sqrt{3} < x \le 3$.

76. (a) The primes less than 25 are 2, 3, 5, 7, 11, 13, 17, 19, and 23. There are 9 of them, so $\pi(25) = 9$. We use the sieve of Eratosthenes, and arrive at the figure at right. There are 25 numbers left over, so $\pi(100) = 25$.

	2	3	~~4~~	5	~~6~~	7	~~8~~	~~9~~	~~10~~
11	~~12~~	13	~~14~~	~~15~~	~~16~~	17	~~18~~	19	~~20~~
~~21~~	~~22~~	23	~~24~~	~~25~~	~~26~~	~~27~~	~~28~~	29	~~30~~
31	~~32~~	~~33~~	~~34~~	~~35~~	~~36~~	37	~~38~~	~~39~~	~~40~~
41	~~42~~	43	~~44~~	~~45~~	~~46~~	47	~~48~~	~~49~~	~~50~~
~~51~~	~~52~~	53	~~54~~	~~55~~	~~56~~	57	~~58~~	59	~~60~~
61	~~62~~	~~63~~	~~64~~	~~65~~	~~66~~	67	~~68~~	~~69~~	~~70~~
71	~~72~~	73	~~74~~	~~75~~	~~76~~	~~77~~	~~78~~	~~79~~	~~80~~
~~81~~	~~82~~	83	~~84~~	~~35~~	~~86~~	~~87~~	~~88~~	89	~~90~~
91	~~92~~	~~93~~	~~94~~	~~95~~	~~96~~	97	~~98~~	~~99~~	~~100~~

(b) Let $f(n) = \dfrac{\pi(n)}{n/\ln n}$. We compute $f(100) = \dfrac{25}{100/\ln 100} \approx 1.15$, $f(1000) \approx 1.16$, $f(10^4) \approx 1.13$, $f(10^5) \approx 1.10$, $f(10^6) \approx 1.08$, and $f(10^7) \approx 1.07$.

(c) By the Prime Number Theorem, the number of primes less than a billion, that is, $\pi(10^9)$, should be close to $10^9/\ln 10^9 \approx 48{,}254{,}942$. In fact, $\pi(10^9) = 50{,}847{,}543$, so our estimate is off by about 5.1%. Do not attempt this calculation at home.

7.4 Derivatives of Logarithmic Functions

1. The differentiation formula for logarithmic functions, $\dfrac{d}{dx}(\log_a x) = \dfrac{1}{x \ln a}$, is simplest when $a = e$ because $\ln e = 1$.

2. $f(x) = \ln(x^2 + 10) \quad \Rightarrow \quad f'(x) = \dfrac{1}{x^2 + 10} \dfrac{d}{dx}(x^2 + 10) = \dfrac{2x}{x^2 + 10}$

3. $f(\theta) = \ln(\cos\theta) \quad \Rightarrow \quad f'(\theta) = \dfrac{1}{\cos\theta} \dfrac{d}{d\theta}(\cos\theta) = \dfrac{-\sin\theta}{\cos\theta} = -\tan\theta$

4. $f(x) = \cos(\ln x) \quad \Rightarrow \quad f'(x) = -\sin(\ln x) \cdot \dfrac{1}{x} = \dfrac{-\sin(\ln x)}{x}$

5. $f(x) = \log_2(1 - 3x) \quad \Rightarrow \quad f'(x) = \dfrac{1}{(1 - 3x)\ln 2} \dfrac{d}{dx}(1 - 3x) = \dfrac{-3}{(1 - 3x)\ln 2}$ or $\dfrac{3}{(3x - 1)\ln 2}$

6. $f(x) = \log_{10}\left(\dfrac{x}{x - 1}\right) = \log_{10} x - \log_{10}(x - 1) \quad \Rightarrow \quad f'(x) = \dfrac{1}{x \ln 10} - \dfrac{1}{(x - 1)\ln 10}$ or $-\dfrac{1}{x(x - 1)\ln 10}$

7. $f(x) = \sqrt[5]{\ln x} = (\ln x)^{1/5} \quad \Rightarrow \quad f'(x) = \frac{1}{5}(\ln x)^{-4/5} \dfrac{d}{dx}(\ln x) = \dfrac{1}{5(\ln x)^{4/5}} \cdot \dfrac{1}{x} = \dfrac{1}{5x\sqrt[5]{(\ln x)^4}}$

8. $f(x) = \ln \sqrt[5]{x} = \ln x^{1/5} = \frac{1}{5}\ln x \quad \Rightarrow \quad f'(x) = \dfrac{1}{5} \cdot \dfrac{1}{x} = \dfrac{1}{5x}$

9. $f(x) = \sqrt{x}\,\ln x \quad \Rightarrow \quad f'(x) = \sqrt{x}\left(\dfrac{1}{x}\right) + (\ln x) \cdot \dfrac{1}{2\sqrt{x}} = \dfrac{1}{\sqrt{x}} + \dfrac{\ln x}{2\sqrt{x}} = \dfrac{2 + \ln x}{2\sqrt{x}}$

10. $f(t) = \dfrac{1 + \ln t}{1 - \ln t} \quad \Rightarrow$

$$f'(t) = \frac{(1 - \ln t)(1/t) - (1 + \ln t)(-1/t)}{(1 - \ln t)^2} = \frac{(1/t)[(1 - \ln t) + (1 + \ln t)]}{(1 - \ln t)^2} = \frac{2}{t(1 - \ln t)^2}$$

11. $F(t) = \ln \dfrac{(2t + 1)^3}{(3t - 1)^4} = \ln(2t + 1)^3 - \ln(3t - 1)^4 = 3\ln(2t + 1) - 4\ln(3t - 1) \quad \Rightarrow$

$$F'(t) = 3 \cdot \frac{1}{2t + 1} \cdot 2 - 4 \cdot \frac{1}{3t - 1} \cdot 3 = \frac{6}{2t + 1} - \frac{12}{3t - 1}, \text{ or combined, } \frac{-6(t + 3)}{(2t + 1)(3t - 1)}.$$

12. $h(x) = \ln(x + \sqrt{x^2 - 1}) \Rightarrow$

$$h'(x) = \frac{1}{x + \sqrt{x^2-1}}\left(1 + \frac{x}{\sqrt{x^2-1}}\right) = \frac{1}{x+\sqrt{x^2-1}} \cdot \frac{\sqrt{x^2-1}+x}{\sqrt{x^2-1}} = \frac{1}{\sqrt{x^2-1}}$$

13. $g(x) = \ln \dfrac{a-x}{a+x} = \ln(a-x) - \ln(a+x) \Rightarrow$

$$g'(x) = \frac{1}{a-x}(-1) - \frac{1}{a+x} = \frac{-(a+x)-(a-x)}{(a-x)(a+x)} = \frac{-2a}{a^2-x^2}$$

14. $F(y) = y\ln(1+e^y) \Rightarrow F'(y) = y \cdot \dfrac{1}{1+e^y} \cdot e^y + \ln(1+e^y) \cdot 1 = \dfrac{ye^y}{1+e^y} + \ln(1+e^y)$

15. $f(u) = \dfrac{\ln u}{1+\ln(2u)} \Rightarrow$

$$f'(u) = \frac{[1+\ln(2u)]\cdot \frac{1}{u} - \ln u \cdot \frac{1}{2u}\cdot 2}{[1+\ln(2u)]^2} = \frac{\frac{1}{u}[1+\ln(2u) - \ln u]}{[1+\ln(2u)]^2}$$

$$= \frac{1 + (\ln 2 + \ln u) - \ln u}{u[1+\ln(2u)]^2} = \frac{1+\ln 2}{u[1+\ln(2u)]^2}$$

16. $y = \ln(x^4 \sin^2 x) = \ln x^4 + \ln(\sin x)^2 = 4\ln x + 2\ln\sin x \Rightarrow y' = 4\cdot\dfrac{1}{x} + 2\cdot\dfrac{1}{\sin x}\cdot\cos x = \dfrac{4}{x} + 2\cot x$

17. $h(t) = t^3 - 3^t \Rightarrow h'(t) = 3t^2 - 3^t \ln 3$

18. $y = 10^{\tan\theta} \Rightarrow y' = 10^{\tan\theta}(\ln 10)(\sec^2\theta)$

19. $y = \ln|2 - x - 5x^2| \Rightarrow y' = \dfrac{1}{2-x-5x^2}\cdot(-1-10x) = \dfrac{-10x-1}{2-x-5x^2}$ or $\dfrac{10x+1}{5x^2+x-2}$

20. $G(u) = \ln\sqrt{\dfrac{3u+2}{3u-2}} = \frac{1}{2}[\ln(3u+2) - \ln(3u-2)] \Rightarrow G'(u) = \dfrac{1}{2}\left(\dfrac{3}{3u+2} - \dfrac{3}{3u-2}\right) = \dfrac{-6}{9u^2-4}$

21. $y = \ln(e^{-x} + xe^{-x}) = \ln(e^{-x}(1+x)) = \ln(e^{-x}) + \ln(1+x) = -x + \ln(1+x) \Rightarrow$

$$y' = -1 + \frac{1}{1+x} = \frac{-1-x+1}{1+x} = -\frac{x}{1+x}$$

22. $y = [\ln(1+e^x)]^2 \Rightarrow y' = 2[\ln(1+e^x)]\cdot\dfrac{1}{1+e^x}\cdot e^x = \dfrac{2e^x\ln(1+e^x)}{1+e^x}$

23. Using Formula 7 and the Chain Rule, $y = 5^{-1/x} \Rightarrow y' = 5^{-1/x}(\ln 5)[-1\cdot(-x^{-2})] = 5^{-1/x}(\ln 5)/x^2$

24. $y = 2^{3^{x^2}} \Rightarrow y' = 2^{3^{x^2}}(\ln 2)\dfrac{d}{dx}\left(3^{x^2}\right) = 2^{3^{x^2}}(\ln 2)3^{x^2}(\ln 3)(2x)$

25. $y = x\ln x \Rightarrow y' = x(1/x) + (\ln x)\cdot 1 = 1 + \ln x \Rightarrow y'' = 1/x$

26. $y = \dfrac{\ln x}{x^2} \Rightarrow y' = \dfrac{x^2(1/x) - (\ln x)(2x)}{(x^2)^2} = \dfrac{x(1-2\ln x)}{x^4} = \dfrac{1-2\ln x}{x^3} \Rightarrow$

$$y'' = \frac{x^3(-2/x) - (1-2\ln x)(3x^2)}{(x^3)^2} = \frac{x^2(-2-3+6\ln x)}{x^6} = \frac{6\ln x - 5}{x^4}$$

27. $y = \log_{10} x \Rightarrow y' = \dfrac{1}{x\ln 10} = \dfrac{1}{\ln 10}\left(\dfrac{1}{x}\right) \Rightarrow y'' = \dfrac{1}{\ln 10}\left(-\dfrac{1}{x^2}\right) = -\dfrac{1}{x^2\ln 10}$

28. $y = \ln(\sec x + \tan x) \Rightarrow y' = \dfrac{\sec x\,\tan x + \sec^2 x}{\sec x + \tan x} = \sec x \Rightarrow y'' = \sec x\,\tan x$

29. $f(x) = \dfrac{x}{1 - \ln(x-1)} \quad \Rightarrow$

$$f'(x) = \frac{[1-\ln(x-1)]\cdot 1 - x\cdot\dfrac{-1}{x-1}}{[1-\ln(x-1)]^2} = \frac{\dfrac{(x-1)[1-\ln(x-1)]+x}{x-1}}{[1-\ln(x-1)]^2} = \frac{x-1-(x-1)\ln(x-1)+x}{(x-1)[1-\ln(x-1)]^2}$$

$$= \frac{2x-1-(x-1)\ln(x-1)}{(x-1)[1-\ln(x-1)]^2}$$

$$\text{Dom}(f) = \{x \mid x-1>0 \text{ and } 1-\ln(x-1)\neq 0\} = \{x \mid x>1 \text{ and } \ln(x-1)\neq 1\}$$
$$= \{x \mid x>1 \text{ and } x-1\neq e^1\} = \{x \mid x>1 \text{ and } x\neq 1+e\} = (1,1+e)\cup(1+e,\infty)$$

30. $f(x) = \dfrac{1}{1+\ln x} \quad \Rightarrow \quad f'(x) = -\dfrac{1/x}{(1+\ln x)^2}$ [Reciprocal Rule] $= -\dfrac{1}{x(1+\ln x)^2}$.
$\text{Dom}(f) = \{x \mid x>0 \text{ and } \ln x \neq -1\} = \{x \mid x>0 \text{ and } x\neq 1/e\} = (0,1/e)\cup(1/e,\infty)$.

31. $f(x) = x^2\ln(1-x^2) \quad \Rightarrow \quad f'(x) = 2x\ln(1-x^2) + \dfrac{x^2(-2x)}{1-x^2} = 2x\ln(1-x^2) - \dfrac{2x^3}{1-x^2}$.
$\text{Dom}(f) = \{x \mid 1-x^2>0\} = \{x \mid |x|<1\} = (-1,1)$.

32. $f(x) = \ln\ln\ln x \quad \Rightarrow \quad f'(x) = \dfrac{1}{\ln\ln x}\cdot\dfrac{1}{\ln x}\cdot\dfrac{1}{x}$.
$\text{Dom}(f) = \{x \mid \ln\ln x>0\} = \{x \mid \ln x>1\} = \{x \mid x>e\} = (e,\infty)$.

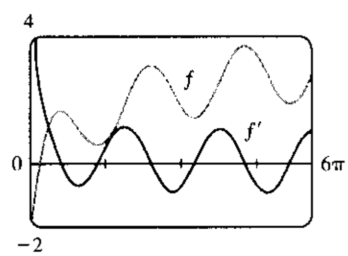
33. $f(x) = \dfrac{x}{\ln x} \quad \Rightarrow \quad f'(x) = \dfrac{\ln x - x(1/x)}{(\ln x)^2} = \dfrac{\ln x - 1}{(\ln x)^2} \quad \Rightarrow \quad f'(e) = \dfrac{1-1}{1^2} = 0$

34. $f(x) = x^2\ln x \quad \Rightarrow \quad f'(x) = 2x\ln x + x^2\left(\dfrac{1}{x}\right) = 2x\ln x + x \quad \Rightarrow \quad f'(1) = 2\ln 1 + 1 = 1$

35. $y = f(x) = \ln\ln x \quad \Rightarrow \quad f'(x) = \dfrac{1}{\ln x}\left(\dfrac{1}{x}\right) \quad \Rightarrow \quad f'(e) = \dfrac{1}{e}$, so an equation of the tangent line at $(e,0)$ is
$y - 0 = \dfrac{1}{e}(x-e)$, or $y = \dfrac{1}{e}x - 1$, or $x - ey = e$.

36. $y = \ln(x^3-7) \quad \Rightarrow \quad y' = \dfrac{1}{x^3-7}\cdot 3x^2 \quad \Rightarrow \quad y'(2) = \dfrac{12}{8-7} = 12$, so an equation of a tangent line at $(2,0)$ is
$y - 0 = 12(x-2)$ or $y = 12x - 24$.

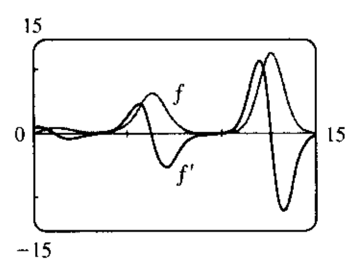
37. $f(x) = \sin x + \ln x \quad \Rightarrow \quad f'(x) = \cos x + 1/x$. This is reasonable, because the graph shows that f increases when f' is positive, and $f'(x) = 0$ when f has a horizontal tangent.



38. $f(x) = x^{\cos x} = e^{\ln x\cos x} \quad \Rightarrow$

$$f'(x) = e^{\ln x\cos x}\left[\ln x(-\sin x) + \cos x\left(\frac{1}{x}\right)\right]$$
$$= x^{\cos x}\left[\frac{\cos x}{x} - \sin x\ln x\right]$$

This is reasonable, because the graph shows that f increases when $f'(x)$ is positive.



39. $y = (2x+1)^5(x^4-3)^6 \quad \Rightarrow \quad \ln y = \ln\left((2x+1)^5(x^4-3)^6\right) \quad \Rightarrow$

$\ln y = 5\ln(2x+1) + 6\ln(x^4-3) \quad \Rightarrow \quad \dfrac{1}{y}y' = 5\cdot\dfrac{1}{2x+1}\cdot 2 + 6\cdot\dfrac{1}{x^4-3}\cdot 4x^3 \quad \Rightarrow$

$y' = y\left(\dfrac{10}{2x+1} + \dfrac{24x^3}{x^4-3}\right) = (2x+1)^5(x^4-3)^6\left(\dfrac{10}{2x+1} + \dfrac{24x^3}{x^4-3}\right).$

[The answer could be simplified to $y' = 2(2x+1)^4(x^4-3)^5(29x^4+12x^3-15)$, but this is unnecessary.]

40. $y = \sqrt{x}\,e^{x^2}(x^2+1)^{10} \quad \Rightarrow \quad \ln y = \ln\sqrt{x} + \ln e^{x^2} + \ln(x^2+1)^{10} \quad \Rightarrow \quad \ln y = \frac{1}{2}\ln x + x^2 + 10\ln(x^2+1)$

$\Rightarrow \quad \dfrac{1}{y}y' = \dfrac{1}{2}\cdot\dfrac{1}{x} + 2x + 10\cdot\dfrac{1}{x^2+1}\cdot 2x \quad \Rightarrow \quad y' = \sqrt{x}\,e^{x^2}(x^2+1)^{10}\left(\dfrac{1}{2x} + 2x + \dfrac{20x}{x^2+1}\right)$

41. $y = \dfrac{\sin^2 x\,\tan^4 x}{(x^2+1)^2} \quad \Rightarrow \quad \ln y = \ln(\sin^2 x\,\tan^4 x) - \ln(x^2+1)^2 \quad \Rightarrow$

$\ln y = \ln(\sin x)^2 + \ln(\tan x)^4 - \ln(x^2+1)^2 \quad \Rightarrow \quad \ln y = 2\,\ln|\sin x| + 4\,\ln|\tan x| - 2\,\ln(x^2+1) \quad \Rightarrow$

$\dfrac{1}{y}y' = 2\cdot\dfrac{1}{\sin x}\cdot\cos x + 4\cdot\dfrac{1}{\tan x}\cdot\sec^2 x - 2\cdot\dfrac{1}{x^2+1}\cdot 2x \quad \Rightarrow$

$y' = \dfrac{\sin^2 x\,\tan^4 x}{(x^2+1)^2}\left(2\cot x + \dfrac{4\sec^2 x}{\tan x} - \dfrac{4x}{x^2+1}\right)$

42. $y = \sqrt[4]{\dfrac{x^2+1}{x^2-1}} \quad \Rightarrow \quad \ln y = \frac{1}{4}\ln(x^2+1) - \frac{1}{4}\ln(x^2-1) \quad \Rightarrow \quad \dfrac{1}{y}y' = \dfrac{1}{4}\cdot\dfrac{1}{x^2+1}\cdot 2x - \dfrac{1}{4}\cdot\dfrac{1}{x^2-1}\cdot 2x \quad \Rightarrow$

$y' = \sqrt[4]{\dfrac{x^2+1}{x^2-1}}\cdot\dfrac{1}{2}\left(\dfrac{x}{x^2+1} - \dfrac{x}{x^2-1}\right) = \dfrac{1}{2}\sqrt[4]{\dfrac{x^2+1}{x^2-1}}\left(\dfrac{-2x}{x^4-1}\right) = \dfrac{x}{1-x^4}\sqrt[4]{\dfrac{x^2+1}{x^2-1}}$

43. $y = x^x \quad \Rightarrow \quad \ln y = \ln x^x \quad \Rightarrow \quad \ln y = x\ln x \quad \Rightarrow \quad y'/y = x(1/x) + (\ln x)\cdot 1 \quad \Rightarrow$

$y' = y(1+\ln x) \quad \Rightarrow \quad y' = x^x(1+\ln x)$

44. $y = x^{1/x} \quad \Rightarrow \quad \ln y = \dfrac{1}{x}\ln x \quad \Rightarrow \quad \dfrac{y'}{y} = \dfrac{1}{x}\left(\dfrac{1}{x}\right) + (\ln x)\left(-\dfrac{1}{x^2}\right) \quad \Rightarrow \quad y' = x^{1/x}\dfrac{1-\ln x}{x^2}$

45. $y = x^{\sin x} \quad \Rightarrow \quad \ln y = \ln x^{\sin x} \quad \Rightarrow \quad \ln y = \sin x\ln x \quad \Rightarrow \quad \dfrac{y'}{y} = (\sin x)\cdot\dfrac{1}{x} + (\ln x)(\cos x) \quad \Rightarrow$

$y' = y\left(\dfrac{\sin x}{x} + \ln x\cos x\right) \quad \Rightarrow \quad y' = x^{\sin x}\left(\dfrac{\sin x}{x} + \ln x\,\cos x\right)$

46. $y = (\sin x)^x \quad \Rightarrow \quad \ln y = x\ln(\sin x) \quad \Rightarrow \quad \dfrac{y'}{y} = x\cdot\dfrac{1}{\sin x}\cdot\cos x + [\ln(\sin x)]\cdot 1 \quad \Rightarrow$

$y' = (\sin x)^x\,[x\cot x + \ln(\sin x)]$

47. $y = (\ln x)^x \quad \Rightarrow \quad \ln y = \ln(\ln x)^x \quad \Rightarrow \quad \ln y = x\ln\ln x \quad \Rightarrow \quad \dfrac{y'}{y} = x\cdot\dfrac{1}{\ln x}\cdot\dfrac{1}{x} + (\ln\ln x)\cdot 1 \quad \Rightarrow$

$y' = y\left(\dfrac{x}{x\ln x} + \ln\ln x\right) \quad \Rightarrow \quad y' = (\ln x)^x\left(\dfrac{1}{\ln x} + \ln\ln x\right)$

48. $y = x^{\ln x} \quad \Rightarrow \quad \ln y = \ln x\ln x = (\ln x)^2 \quad \Rightarrow \quad \dfrac{y'}{y} = 2\ln x\left(\dfrac{1}{x}\right) \quad \Rightarrow \quad y' = x^{\ln x}\left(\dfrac{2\ln x}{x}\right)$

49. $y = x^{e^x} \quad \Rightarrow \quad \ln y = e^x\ln x \quad \Rightarrow \quad \dfrac{y'}{y} = e^x\cdot\dfrac{1}{x} + (\ln x)\cdot e^x \quad \Rightarrow \quad y' = x^{e^x}e^x\left(\ln x + \dfrac{1}{x}\right)$

50. $y = (\ln x)^{\cos x} \quad \Rightarrow \quad \ln y = \cos x \ln(\ln x) \quad \Rightarrow \quad \dfrac{y'}{y} = \cos x \cdot \dfrac{1}{\ln x} \cdot \dfrac{1}{x} + (\ln \ln x)(-\sin x) \quad \Rightarrow$

$y' = (\ln x)^{\cos x}\left(\dfrac{\cos x}{x \ln x} - \sin x \ln \ln x\right)$

51. $y = \ln(x^2 + y^2) \quad \Rightarrow \quad y' = \dfrac{1}{x^2 + y^2}\dfrac{d}{dx}(x^2 + y^2) \quad \Rightarrow \quad y' = \dfrac{2x + 2yy'}{x^2 + y^2} \quad \Rightarrow \quad x^2y' + y^2y' = 2x + 2yy'$

$\Rightarrow \quad x^2y' + y^2y' - 2yy' = 2x \quad \Rightarrow \quad (x^2 + y^2 - 2y)y' = 2x \quad \Rightarrow \quad y' = \dfrac{2x}{x^2 + y^2 - 2y}$

52. $x^y = y^x \quad \Rightarrow \quad y \ln x = x \ln y \quad \Rightarrow \quad y \cdot \dfrac{1}{x} + (\ln x) \cdot y' = x \cdot \dfrac{1}{y} \cdot y' + \ln y \quad \Rightarrow \quad y' \ln x - \dfrac{x}{y}y' = \ln y - \dfrac{y}{x} \quad \Rightarrow$

$y' = \dfrac{\ln y - y/x}{\ln x - x/y}$

53. $f(x) = \ln(x - 1) \quad \Rightarrow \quad f'(x) = 1/(x-1) = (x-1)^{-1} \quad \Rightarrow \quad f''(x) = -(x-1)^{-2} \quad \Rightarrow$

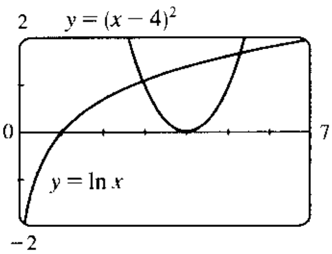
$f'''(x) = 2(x-1)^{-3} \quad \Rightarrow \quad f^{(4)}(x) = -2 \cdot 3(x-1)^{-4} \quad \Rightarrow \quad \cdots \quad \Rightarrow$

$f^{(n)}(x) = (-1)^{n-1} \cdot 2 \cdot 3 \cdot 4 \cdot \cdots \cdot (n-1)(x-1)^{-n} = (-1)^{n-1}\dfrac{(n-1)!}{(x-1)^n}$

54. $y = x^8 \ln x$, so $D^9 y = D^8 y' = D^8\left(8x^7 \ln x + x^7\right)$. But the eighth derivative of x^7 is 0, so we now have

$$D^8\left(8x^7 \ln x\right) = D^7\left(8 \cdot 7x^6 \ln x + 8x^6\right) = D^7\left(8 \cdot 7x^6 \ln x\right)$$
$$= D^6\left(8 \cdot 7 \cdot 6x^5 \ln x\right) = \cdots = D\left(8!\,x^0 \ln x\right) = 8!/x.$$

55.

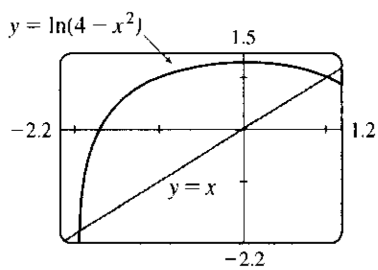


From the graph, it appears that the curves $y = (x-4)^2$ and $y = \ln x$ intersect just to the left of $x = 3$ and to the right of $x = 5$, at about $x = 5.3$. Let $f(x) = \ln x - (x-4)^2$. Then $f'(x) = 1/x - 2(x-4)$, so Newton's Method says that

$x_{n+1} = x_n - f(x_n)/f'(x_n) = x_n - \dfrac{\ln x_n - (x_n - 4)^2}{1/x_n - 2(x_n - 4)}$. Taking

$x_0 = 3$, we get $x_1 \approx 2.957738$, $x_2 \approx 2.958516 \approx x_3$, so the first root is 2.958516, to six decimal places. Taking $x_0 = 5$, we get $x_1 \approx 5.290755$, $x_2 \approx 5.290718 \approx x_3$, so the second (and final) root is 5.290718, to six decimal places.

56.



We use Newton's Method with $f(x) = \ln(4 - x^2) - x$ and

$f'(x) = \dfrac{1}{4 - x^2}(-2x) - 1 = -1 - \dfrac{2x}{4 - x^2}$. The formula is

$x_{n+1} = x_n - f(x_n)/f'(x_n)$. From the graphs it seems that the roots occur at approximately $x = -1.9$ and $x = 1.1$. However, if we use $x_1 = -1.9$ as an initial approximation to the first root, we get $x_2 \approx -2.009611$, and $f(x) = \ln(x-2)^2 - x$ is undefined at this point, making it impossible to calculate x_3. We must use a more accurate first estimate, such as $x_1 = -1.95$. With this approximation, we get $x_1 = -1.95$, $x_2 \approx -1.1967495$, $x_3 \approx -1.964760$, $x_4 \approx x_5 \approx -1.964636$. Calculating the second root gives $x_1 = 1.1$, $x_2 \approx 1.058649$, $x_3 \approx 1.058007$, $x_4 \approx x_5 \approx 1.058006$. So, correct to six decimal places, the two roots of the equation $\ln(4 - x^2) = x$ are $x = -1.964636$ and $x = 1.058006$.

57. $f(x) = \dfrac{\ln x}{\sqrt{x}} \Rightarrow f'(x) = \dfrac{\sqrt{x}\,(1/x) - (\ln x)[1/(2\sqrt{x}\,)]}{x} = \dfrac{2 - \ln x}{2x^{3/2}} \Rightarrow$

$f''(x) = \dfrac{2x^{3/2}(-1/x) - (2 - \ln x)(3x^{1/2})}{4x^3} = \dfrac{3\ln x - 8}{4x^{5/2}} > 0 \Leftrightarrow \ln x > \frac{8}{3} \Leftrightarrow x > e^{8/3}$, so f is CU on $\left(e^{8/3}, \infty\right)$ and CD on $\left(0, e^{8/3}\right)$. The inflection point is $\left(e^{8/3}, \frac{8}{3}e^{-4/3}\right)$.

58. $f(x) = x\ln x$, $f'(x) = \ln x + 1 = 0$ when $\ln x = -1 \Leftrightarrow x = e^{-1}$. $f'(x) > 0 \Leftrightarrow \ln x + 1 > 0 \Leftrightarrow \ln x > -1 \Leftrightarrow x > 1/e$. $f'(x) < 0 \Leftrightarrow \ln x + 1 < 0 \Leftrightarrow x < 1/e$. Therefore, there is an absolute minimum value of $f(1/e) = (1/e)\ln(1/e) = -1/e$.

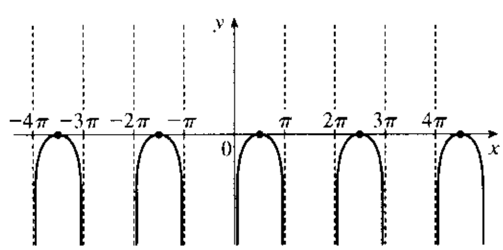
59. $y = f(x) = \ln(\sin x)$

A. $D = \{x \text{ in } \mathbb{R} \mid \sin x > 0\} = \bigcup_{n=-\infty}^{\infty} (2n\pi, (2n+1)\pi)$

$= \cdots \cup (-4\pi, -3\pi) \cup (-2\pi, -\pi) \cup (0, \pi) \cup (2\pi, 3\pi) \cup \cdots$

B. No y-intercept; x-intercepts: $f(x) = 0 \Leftrightarrow \ln(\sin x) = 0 \Leftrightarrow \sin x = e^0 = 1 \Leftrightarrow x = 2n\pi + \frac{\pi}{2}$ for each integer n. **C.** f is periodic with period 2π. **D.** $\lim_{x\to(2n\pi)^+} f(x) = -\infty$ and $\lim_{x\to[(2n+1)\pi]^-} f(x) = -\infty$, so the lines $x = n\pi$ are VAs for all integers n. **E.** $f'(x) = \frac{\cos x}{\sin x} = \cot x$, so $f'(x) > 0$ when $2n\pi < x < 2n\pi + \frac{\pi}{2}$ for each integer n, and $f'(x) < 0$ when $2n\pi + \frac{\pi}{2} < x < (2n+1)\pi$. Thus, f is increasing on $\left(2n\pi, 2n\pi + \frac{\pi}{2}\right)$ and decreasing on $\left(2n\pi + \frac{\pi}{2}, (2n+1)\pi\right)$ for each integer n. **F.** Local maximum values $f\left(2n\pi + \frac{\pi}{2}\right) = 0$, no local minimum. **G.** $f''(x) = -\csc^2 x < 0$, so f is CD on $(2n\pi, (2n+1)\pi)$ for each integer n. No IP

H.



60. $y = \ln(\tan^2 x)$ **A.** $D = \{x \mid x \neq n\pi/2\}$ **B.** x-intercepts $n\pi + \frac{\pi}{4}$, no y-intercept. **C.** $f(-x) = f(x)$, so the curve is symmetric about the y-axis. Also $f(x + \pi) = f(x)$, so f is periodic with period π, and we consider parts D–G only for $-\frac{\pi}{2} < x < \frac{\pi}{2}$. **D.** $\lim_{x\to 0} \ln(\tan^2 x) = -\infty$ and $\lim_{x\to(\pi/2)^-} \ln(\tan^2 x) = \infty$,

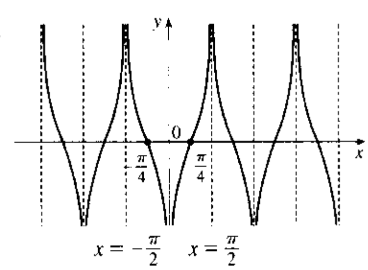
$\lim_{x\to-(-\pi/2)^+} \ln(\tan^2 x) = \infty$, so $x = 0$, $x = \pm\frac{\pi}{2}$ are VA. **E.** $f'(x) = \dfrac{2\tan x\sec^2 x}{\tan^2 x} = 2\dfrac{\sec^2 x}{\tan x} > 0 \Leftrightarrow$

$\tan x > 0 \Leftrightarrow 0 < x < \frac{\pi}{2}$, so f is increasing on $\left(0, \frac{\pi}{2}\right)$ and decreasing on $\left(-\frac{\pi}{2}, 0\right)$. **F.** No maximum or minimum

G. $f'(x) = \dfrac{2}{\sin x\cos x} = \dfrac{4}{\sin 2x} \Rightarrow f''(x) = \dfrac{-8\cos 2x}{\sin^2 2x} < 0$

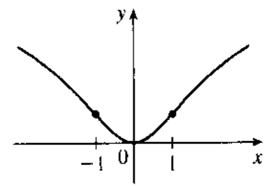
$\Leftrightarrow \cos 2x > 0 \Leftrightarrow -\frac{\pi}{4} < x < \frac{\pi}{4}$, so f is CD on $\left(-\frac{\pi}{4}, 0\right)$ and $\left(0, \frac{\pi}{4}\right)$ and CU on $\left(-\frac{\pi}{2}, -\frac{\pi}{4}\right)$ and $\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$. IP are $\left(\pm\frac{\pi}{4}, 0\right)$.

H.



61. $y = f(x) = \ln(1+x^2)$ **A.** $D = \mathbb{R}$ **B.** Both intercepts are 0. **C.** $f(-x) = f(x)$, so the curve is symmetric about the y-axis. **D.** $\lim\limits_{x\to\pm\infty} \ln(1+x^2) = \infty$, no asymptotes. **E.** $f'(x) = \dfrac{2x}{1+x^2} > 0 \;\Leftrightarrow\; x > 0$, so f is increasing on $(0,\infty)$ and decreasing on $(-\infty, 0)$. **F.** $f(0) = 0$ is a local and absolute minimum. **G.** $f''(x) = \dfrac{2(1+x^2) - 2x(2x)}{(1+x^2)^2} = \dfrac{2(1-x^2)}{(1+x^2)^2} > 0 \;\Leftrightarrow\; |x| < 1$, so f is CU on $(-1,1)$, CD on $(-\infty,-1)$ and $(1,\infty)$. IP $(1, \ln 2)$ and $(-1, \ln 2)$.

H.



62. $y = f(x) = \ln(x^2 - 3x + 2) = \ln[(x-1)(x-2)]$

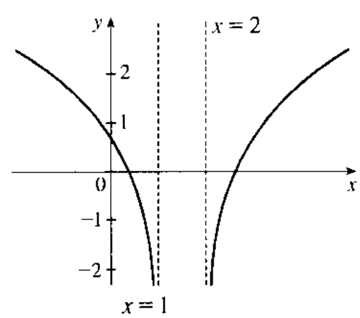
A. $D = \{x \text{ in } \mathbb{R}\colon x^2 - 3x + 2 > 0\} = (-\infty, 1) \cup (2, \infty)$.

B. y-intercept: $f(0) = \ln 2$; x-intercepts: $f(x) = 0 \;\Leftrightarrow\; x^2 - 3x + 2 = e^0 \;\Leftrightarrow\; x^2 - 3x + 1 = 0 \;\Leftrightarrow\; x = \dfrac{3 \pm \sqrt{5}}{2} \;\Rightarrow\; x \approx 0.38, 2.62$ **C.** No symmetry **D.** $\lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 2^+} f(x) = -\infty$, so $x = 1$ and $x = 2$ are VAs. No HA. **E.** $f'(x) = \dfrac{2x-3}{x^2-3x+2} = \dfrac{2(x-3/2)}{(x-1)(x-2)}$, so $f'(x) < 0$ for $x < 1$ and $f'(x) > 0$ for $x > 2$. Thus, f is decreasing on $(-\infty, 1)$ and increasing on $(2, \infty)$. **F.** No extreme values

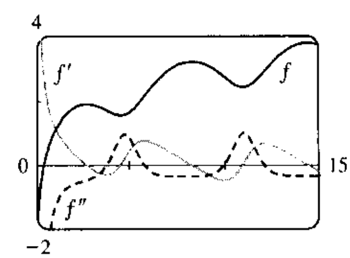
G.
$$f''(x) = \frac{(x^2-3x+2)\cdot 2 - (2x-3)^2}{(x^2-3x+2)^2} = \frac{2x^2 - 6x + 4 - 4x^2 + 12x - 9}{(x^2-3x+2)^2} = \frac{-2x^2 + 6x - 5}{(x^2-3x+2)^2}$$

The numerator is negative for all x and the denominator is positive, so $f''(x) < 0$ for all x in the domain of f. Thus, f is CD on $(-\infty, 1)$ and $(2, \infty)$. No IP

H.



63. We use the CAS to calculate $f'(x) = \dfrac{2 + \sin x + x\cos x}{2x + x\sin x}$ and $f''(x) = \dfrac{2x^2 \sin x + 4\sin x - \cos^2 x + x^2 + 5}{x^2(\cos^2 x - 4\sin x - 5)}$. From the graphs, it seems that $f' > 0$ (and so f is increasing) on approximately the intervals $(0, 2.7)$, $(4.5, 8.2)$ and $(10.9, 14.3)$. It seems that f'' changes sign (indicating inflection points) at $x \approx 3.8$, 5.7, 10.0 and 12.0.

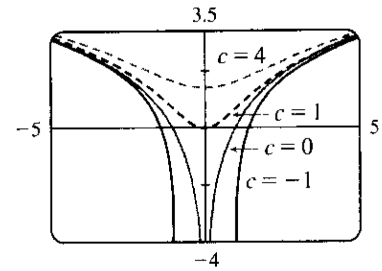


Looking back at the graph of $f(x) = \ln(2x + x\sin x)$, this implies that the inflection points have approximate coordinates $(3.8, 1.7)$, $(5.7, 2.1)$, $(10.0, 2.7)$, and $(12.0, 2.9)$.

64. We see that if $c \le 0$, $f(x) = \ln(x^2 + c)$ is only defined for $x^2 > -c \;\Rightarrow\; |x| > \sqrt{-c}$, and $\lim\limits_{x\to\sqrt{-c}^+} f(x) = \lim\limits_{x\to-\sqrt{-c}^-} f(x) = -\infty$, since $\ln y \to -\infty$ as $y \to 0$. Thus, for $c < 0$, there are vertical asymptotes at $x = \pm\sqrt{c}$, and as c decreases (that is, $|c|$ increases), the asymptotes get further apart. For $c = 0$,

$\lim_{x\to 0} f(x) = -\infty$, so there is a vertical asymptote at $x = 0$. If $c > 0$, there is no asymptote. To find the maxima, minima, and inflection points, we differentiate: $f(x) = \ln(x^2+c) \Rightarrow f'(x) = \frac{1}{x^2+c}(2x)$, so by the First Derivative Test there is a local and absolute minimum at $x = 0$. Differentiating again, we get

$f''(x) = \frac{1}{x^2+c}(2) + 2x\left[-(x^2+c)^{-2}(2x)\right] = \frac{2(c-x^2)}{(x^2+c)^2}$. Now if $c \le 0$, this is always negative, so f is concave down on both of the intervals on which it is defined. If $c > 0$, then f'' changes sign when $c = x^2 \Leftrightarrow x = \pm\sqrt{c}$. So for $c > 0$ there are inflection points at $\pm\sqrt{c}$, and as c increases, the inflection points get further apart.



65. $\int_2^4 \frac{3}{x}\,dx = 3\int_2^4 \frac{1}{x}\,dx = 3\Big[\ln|x|\Big]_2^4 = 3(\ln 4 - \ln 2) = 3\ln\frac{4}{2} = 3\ln 2$

66. $\int_1^2 \frac{4+u^2}{u^3}\,du = \int_1^2 (4u^{-3} + u^{-1})\,du = \left[\frac{4}{-2}u^{-2} + \ln|u|\right]_1^2 = \left[\frac{-2}{u^2} + \ln u\right]_1^2$
$= \left(-\frac{1}{2} + \ln 2\right) - (-2 + \ln 1) = \frac{3}{2} + \ln 2$

67. $\int_1^2 \frac{dt}{8-3t} = \left[-\frac{1}{3}\ln|8-3t|\right]_1^2 = -\frac{1}{3}\ln 2 - \left(-\frac{1}{3}\ln 5\right) = \frac{1}{3}(\ln 5 - \ln 2) = \frac{1}{3}\ln\frac{5}{2}$

Or: Let $u = 8 - 3t$. Then $du = -3\,dt$, so

$\int_1^2 \frac{dt}{8-3t} = \int_5^2 \frac{-\frac{1}{3}\,du}{u} = \left[-\frac{1}{3}\ln|u|\right]_5^2 = -\frac{1}{3}\ln 2 - \left(-\frac{1}{3}\ln 5\right) = \frac{1}{3}(\ln 5 - \ln 2) = \frac{1}{3}\ln\frac{5}{2}$.

68. $\int_4^9 \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^2 dx = \int_4^9 \left(x + 2 + \frac{1}{x}\right) dx = \left[\frac{1}{2}x^2 + 2x + \ln x\right]_4^9 = \frac{81}{2} + 18 + \ln 9 - (8 + 8 + \ln 4)$
$= \frac{85}{2} + \ln\frac{9}{4}$

69. $\int_1^e \frac{x^2+x+1}{x}\,dx = \int_1^e \left(x + 1 + \frac{1}{x}\right) dx = \left[\frac{1}{2}x^2 + x + \ln x\right]_1^e = \left(\frac{1}{2}e^2 + e + 1\right) - \left(\frac{1}{2} + 1 + 0\right)$
$= \frac{1}{2}e^2 + e - \frac{1}{2}$

70. Let $u = \ln x$. Then $du = \frac{1}{x}\,dx$, so $\int_e^6 \frac{dx}{x\ln x} = \int_1^{\ln 6} \frac{1}{u}\,du = \Big[\ln|u|\Big]_1^{\ln 6} = \ln\ln 6 - \ln 1 = \ln\ln 6$

71. Let $u = 6x - x^3$. Then $du = (6 - 3x^2)\,dx = 3(2 - x^2)\,dx$, so

$\int \frac{2-x^2}{6x-x^3}\,dx = \int \frac{\frac{1}{3}\,du}{u} = \frac{1}{3}\ln|u| + C = \frac{1}{3}\ln|6x - x^3| + C$.

72. Let $u = 2 + \sin x$. Then $du = \cos x\,dx$, so

$\int \frac{\cos x}{2+\sin x}\,dx = \int \frac{1}{u}\,du = \ln|u| + C = \ln|2+\sin x| + C = \ln(2+\sin x) + C$ [since $2 + \sin x > 0$].

73. Let $u = \ln x$. Then $du = \dfrac{dx}{x} \quad\Rightarrow\quad \displaystyle\int \frac{(\ln x)^2}{x}\,dx = \int u^2\,du = \frac{1}{3}u^3 + C = \frac{1}{3}(\ln x)^3 + C.$

74. Let $u = e^x + 1$. Then $du = e^x\,dx$, so $\displaystyle\int \frac{e^x}{e^x+1}\,dx = \int \frac{du}{u} = \ln|u| + C = \ln(e^x+1) + C.$

75. $\displaystyle\int_1^2 10^t\,dt = \left[\frac{10^t}{\ln 10}\right]_1^2 = \frac{10^2}{\ln 10} - \frac{10^1}{\ln 10} = \frac{100-10}{\ln 10} = \frac{90}{\ln 10}$

76. Let $u = x^2$. Then $du = 2x\,dx$, so $\displaystyle\int x2^{x^2}\,dx = \tfrac{1}{2}\int 2^u\,du = \frac{1}{2}\frac{2^u}{\ln 2} + C = \frac{1}{2\ln 2}2^{x^2} + C.$

77. (a) $\dfrac{d}{dx}(\ln|\sin x| + C) = \dfrac{1}{\sin x}\cos x = \cot x$

(b) Let $u = \sin x$. Then $du = \cos x\,dx$, so $\displaystyle\int \cot x\,dx = \int \frac{\cos x}{\sin x}\,dx = \int \frac{du}{u} = \ln|u| + C = \ln|\sin x| + C.$

78. Let $u = x - 2$. Then the area is

$$A = -\int_{-4}^{-1} \frac{2}{x-2}\,dx = -2\int_{-6}^{-3} \frac{du}{u} = \Big[-2\ln|u|\Big]_{-6}^{-3} = -2\ln 3 + 2\ln 6 = 2\ln 2 \approx 1.386.$$

79. The cross-sectional area is $\pi\big(1/\sqrt{x+1}\big)^2 = \pi/(x+1)$. Therefore, the volume is

$$\int_0^1 \frac{\pi}{x+1}\,dx = \pi[\ln(x+1)]_0^1 = \pi(\ln 2 - \ln 1) = \pi\ln 2.$$

80. Using cylindrical shells, we get $V = \displaystyle\int_0^3 \frac{2\pi x}{x^2+1}\,dx = \pi\big[\ln(1+x^2)\big]_0^3 = \pi\ln 10.$

81. $W = \int_{V_1}^{V_2} P\,dV = \displaystyle\int_{600}^{1000} \frac{C}{V}\,dV = C\int_{600}^{1000} \frac{1}{V}\,dV = C\Big[\ln|V|\Big]_{600}^{1000}$

$= C(\ln 1000 - \ln 600) = C\ln\frac{1000}{600} = C\ln\frac{5}{3}$

Initially, $PV = C$, where $P = 150$ kPa and $V = 600\text{ cm}^3$, so $C = (150)(600) = 90{,}000$. Thus,

$W = 90{,}000\ln\frac{5}{3} \approx 45{,}974\text{ kPa}\cdot\text{cm}^3 = 45{,}974(10^3\text{ Pa})(10^{-6}\text{ m}^3) = 45{,}974\text{ Pa}\cdot\text{m}^3 = 45{,}974\text{ N}\cdot\text{m}$

$[\text{Pa} = \text{N/m}^2] \quad = 45{,}974\text{ J}$

82. $f''(x) = x^{-2}, x > 0 \quad\Rightarrow\quad f'(x) = -1/x + C \quad\Rightarrow\quad f(x) = -\ln x + Cx + D$. $0 = f(1) = C + D$ and

$0 = f(2) = -\ln 2 + 2C + D = -\ln 2 + 2C - C = -\ln 2 + C \quad\Rightarrow\quad C = \ln 2$ and $D = -\ln 2$. So

$f(x) = -\ln x + (\ln 2)x - \ln 2.$

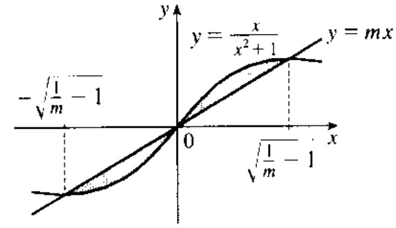
83. $f(x) = 2x + \ln x \quad\Rightarrow\quad f'(x) = 2 + 1/x$. If $g = f^{-1}$, then $f(1) = 2 \quad\Rightarrow\quad g(2) = 1$, so

$g'(2) = 1/f'(g(2)) = 1/f'(1) = \frac{1}{3}.$

84. $f(x) = e^x + \ln x \quad\Rightarrow\quad f'(x) = e^x + 1/x$. $h = f^{-1}$ and $f(1) = e \quad\Rightarrow\quad h(e) = 1$, so

$h'(e) = 1/f'(1) = 1/(e+1).$

85. The curve and the line will determine a region when they intersect at two or more points. So we solve the equation $x/(x^2+1) = mx \Rightarrow x = 0$ or $mx^2 + m - 1 = 0 \Rightarrow x = 0$ or $x = \dfrac{\pm\sqrt{-4(m)(m-1)}}{2m} = \pm\sqrt{\dfrac{1}{m} - 1}$. Note that if $m = 1$, this has only the solution $x = 0$, and no region is determined. But if $1/m - 1 > 0 \Leftrightarrow 1/m > 1 \Leftrightarrow 0 < m < 1$, then there are two solutions. [Another way of seeing this is to observe that the slope of the tangent to $y = x/(x^2+1)$ at the origin is $y' = 1$ and therefore we must have $0 < m < 1$.] Note that we cannot just integrate between the positive and negative roots, since the curve and the line cross at the origin. Since mx and $x/(x^2+1)$ are both odd functions, the total area is twice the area between the curves on the interval $\left[0, \sqrt{1/m - 1}\right]$. So the total area enclosed is



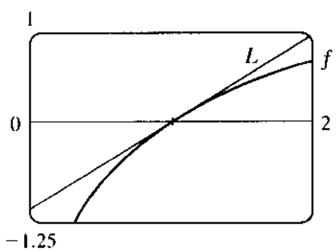
$$2\int_0^{\sqrt{1/m-1}} \left[\frac{x}{x^2+1} - mx\right] dx = 2\left[\tfrac{1}{2}\ln(x^2+1) - \tfrac{1}{2}mx^2\right]_0^{\sqrt{1/m-1}}$$

$$= \left[\ln\left(\frac{1}{m} - 1 + 1\right) - m\left(\frac{1}{m} - 1\right)\right] - (\ln 1 - 0)$$

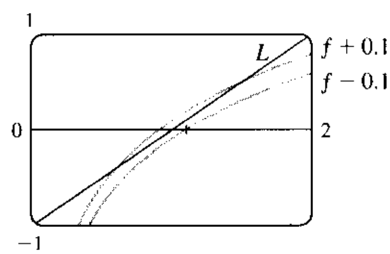
$$= \ln\left(\frac{1}{m}\right) + m - 1 = m - \ln m - 1$$

86. (a) Let $f(x) = \ln x \Rightarrow f'(x) = 1/x \Rightarrow f''(x) = -1/x^2$. The linear approximation to $\ln x$ near 1 is $\ln x \approx f(1) + f'(1)(x-1) = \ln 1 + \frac{1}{1}(x-1) = x - 1$.

(b)



(c)



From the graph, it appears that the linear approximation is accurate to within 0.1 for x between about 0.62 and 1.51.

87. If $f(x) = \ln(1+x)$, then $f'(x) = \dfrac{1}{1+x}$, so $f'(0) = 1$.

Thus, $\displaystyle\lim_{x\to 0}\frac{\ln(1+x)}{x} = \lim_{x\to 0}\frac{f(x)}{x} = \lim_{x\to 0}\frac{f(x) - f(0)}{x - 0} = f'(0) = 1$.

88. Let $m = n/x$. Then $n = xm$, and as $n \to \infty$, $m \to \infty$.

Therefore, $\displaystyle\lim_{n\to\infty}\left(1 + \frac{x}{n}\right)^n = \lim_{m\to\infty}\left(1 + \frac{1}{m}\right)^{mx} = \left[\lim_{m\to\infty}\left(1 + \frac{1}{m}\right)^m\right]^x = e^x$ by Equation 9.

7.2* The Natural Logarithmic Function

1. $\ln \dfrac{x^3 y}{z^2} = \ln x^3 y - \ln z^2 = \ln x^3 + \ln y - \ln z^2 = 3\ln x + \ln y - 2\ln z$

2. $\ln \sqrt{a(b^2+c^2)} = \ln(a(b^2+c^2))^{1/2} = \frac{1}{2}\ln(a(b^2+c^2)) = \frac{1}{2}\left[\ln a + \ln(b^2+c^2)\right]$
$= \frac{1}{2}\ln a + \frac{1}{2}\ln(b^2+c^2)$

3. $\ln(uv)^{10} = 10\ln(uv) = 10(\ln u + \ln v) = 10\ln u + 10\ln v$

4. $\ln \dfrac{3x^2}{(x+1)^5} = \ln 3x^2 - \ln(x+1)^5 = \ln 3 + \ln x^2 - 5\ln(x+1) = \ln 3 + 2\ln x - 5\ln(x+1)$

5. $2\ln 4 - \ln 2 = \ln 4^2 - \ln 2 = \ln 16 - \ln 2 = \ln \frac{16}{2} = \ln 8$

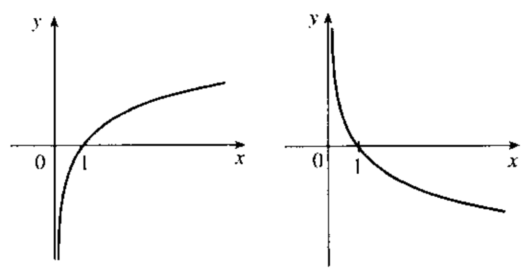
6. $\ln 3 + \frac{1}{3}\ln 8 = \ln 3 + \ln 8^{1/3} = \ln 3 + \ln 2 = \ln(3 \cdot 2) = \ln 6$

7. $\frac{1}{2}\ln x - 5\ln(x^2+1) = \ln x^{1/2} - \ln(x^2+1)^5 = \ln \dfrac{\sqrt{x}}{(x^2+1)^5}$

8. $\ln x + a\ln y - b\ln z = \ln x + \ln y^a - \ln z^b = \ln(x \cdot y^a) - \ln z^b = \ln(xy^a/z^b)$

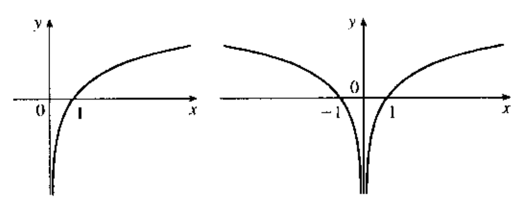
9. Reflect the graph of $y = \ln x$ about the x-axis to obtain the graph of $y = -\ln x$.

$y = \ln x$ $\qquad$ $y = -\ln x$

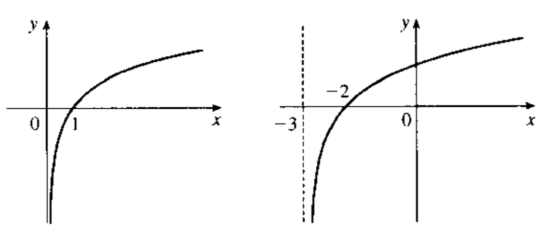


10. Reflect the portion of the graph of $y = \ln x$ to the right of the y-axis about the y-axis. The graph of $y = \ln|x|$ is that reflection in addition to the original portion.

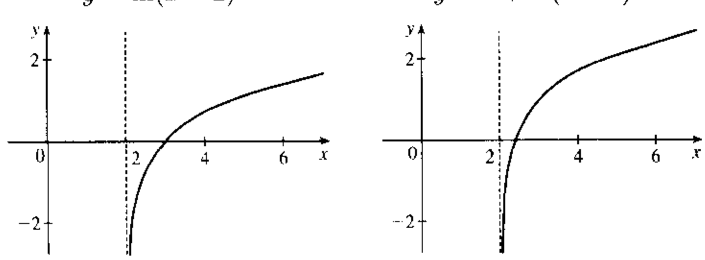
$y = \ln x$ $\qquad$ $y = \ln|x|$



11. $y = \ln x$ $\qquad$ $y = \ln(x+3)$



12. $y = \ln(x-2)$ $\qquad$ $y = 1 + \ln(x-2)$



13. $f(x) = \sqrt{x}\ln x \quad\Rightarrow\quad f'(x) = \dfrac{1}{2\sqrt{x}}\ln x + \sqrt{x}\left(\dfrac{1}{x}\right) = \dfrac{\ln x + 2}{2\sqrt{x}}$

14. $f(x) = \ln(x^2 + 10) \quad\Rightarrow\quad f'(x) = \dfrac{1}{x^2 + 10}\,\dfrac{d}{dx}\,(x^2 + 10) = \dfrac{2x}{x^2 + 10}$

15. $f(\theta) = \ln(\cos\theta) \quad\Rightarrow\quad f'(\theta) = \dfrac{1}{\cos\theta}\,\dfrac{d}{d\theta}\,(\cos\theta) = \dfrac{-\sin\theta}{\cos\theta} = -\tan\theta$

16. $f(x) = \cos(\ln x) \quad\Rightarrow\quad f'(x) = -\sin(\ln x)\cdot\dfrac{1}{x} = \dfrac{-\sin(\ln x)}{x}$

17. $f(x) = \sqrt[5]{\ln x} = (\ln x)^{1/5} \quad\Rightarrow\quad f'(x) = \frac{1}{5}(\ln x)^{-4/5}\,\dfrac{d}{dx}\,(\ln x) = \dfrac{1}{5(\ln x)^{4/5}}\cdot\dfrac{1}{x} = \dfrac{1}{5x\,\sqrt[5]{(\ln x)^4}}$

18. $f(x) = \ln\sqrt[5]{x} = \ln x^{1/5} = \frac{1}{5}\ln x \quad\Rightarrow\quad f'(x) = \dfrac{1}{5}\cdot\dfrac{1}{x} = \dfrac{1}{5x}$

19. $g(x) = \ln\dfrac{a - x}{a + x} = \ln(a - x) - \ln(a + x) \quad\Rightarrow$

$$g'(x) = \frac{1}{a - x}(-1) - \frac{1}{a + x} = \frac{-(a + x) - (a - x)}{(a - x)(a + x)} = \frac{-2a}{a^2 - x^2}$$

20. $h(x) = \ln\left(x + \sqrt{x^2 - 1}\right) \quad\Rightarrow$

$$h'(x) = \frac{1}{x + \sqrt{x^2 - 1}}\left(1 + \frac{x}{\sqrt{x^2 - 1}}\right) = \frac{1}{x + \sqrt{x^2 - 1}}\cdot\frac{\sqrt{x^2 - 1} + x}{\sqrt{x^2 - 1}} = \frac{1}{\sqrt{x^2 - 1}}$$

21. $f(u) = \dfrac{\ln u}{1 + \ln(2u)} \quad\Rightarrow$

$$f'(u) = \frac{[1 + \ln(2u)]\cdot\frac{1}{u} - \ln u\cdot\frac{1}{2u}\cdot 2}{[1 + \ln(2u)]^2} = \frac{\frac{1}{u}\,[1 + \ln(2u) - \ln u]}{[1 + \ln(2u)]^2}$$

$$= \frac{1 + (\ln 2 + \ln u) - \ln u}{u\,[1 + \ln(2u)]^2} = \frac{1 + \ln 2}{u\,[1 + \ln(2u)]^2}$$

22. $f(t) = \dfrac{1 + \ln t}{1 - \ln t} \quad\Rightarrow$

$$f'(t) = \frac{(1 - \ln t)(1/t) - (1 + \ln t)(-1/t)}{(1 - \ln t)^2} = \frac{(1/t)[(1 - \ln t) + (1 + \ln t)]}{(1 - \ln t)^2} = \frac{2}{t(1 - \ln t)^2}$$

23. $F(t) = \ln\dfrac{(2t + 1)^3}{(3t - 1)^4} = \ln(2t + 1)^3 - \ln(3t - 1)^4 = 3\ln(2t + 1) - 4\ln(3t - 1) \quad\Rightarrow$

$F'(t) = 3\cdot\dfrac{1}{2t + 1}\cdot 2 - 4\cdot\dfrac{1}{3t - 1}\cdot 3 = \dfrac{6}{2t + 1} - \dfrac{12}{3t - 1}$, or combined, $\dfrac{-6(t + 3)}{(2t + 1)(3t - 1)}$.

24. $y = \ln(x^4\sin^2 x) = \ln x^4 + \ln(\sin x)^2 = 4\,\ln x + 2\,\ln\sin x \quad\Rightarrow\quad y' = 4\cdot\dfrac{1}{x} + 2\cdot\dfrac{1}{\sin x}\cdot\cos x = \dfrac{4}{x} + 2\cot x$

25. $y = \ln|2 - x - 5x^2| \quad\Rightarrow\quad y' = \dfrac{1}{2 - x - 5x^2}\cdot(-1 - 10x) = \dfrac{-10x - 1}{2 - x - 5x^2}$ or $\dfrac{10x + 1}{5x^2 + x - 2}$

26. $G(u) = \ln\sqrt{\dfrac{3u + 2}{3u - 2}} = \frac{1}{2}[\ln(3u + 2) - \ln(3u - 2)] \quad\Rightarrow\quad G'(u) = \dfrac{1}{2}\left(\dfrac{3}{3u + 2} - \dfrac{3}{3u - 2}\right) = \dfrac{-6}{9u^2 - 4}$

27. $y = \ln\left(\dfrac{x + 1}{x - 1}\right)^{3/5} = \frac{3}{5}[\ln(x + 1) - \ln(x - 1)] \quad\Rightarrow\quad y' = \dfrac{3}{5}\left(\dfrac{1}{x + 1} - \dfrac{1}{x - 1}\right) = \dfrac{-6}{5(x^2 - 1)}$

28. $y = (\ln\tan x)^2 \quad\Rightarrow\quad y' = 2(\ln\tan x)\cdot\dfrac{1}{\tan x}\cdot\sec^2 x = \dfrac{2(\ln\tan x)\,\sec^2 x}{\tan x}$

29. $y = \tan[\ln(ax+b)] \quad \Rightarrow \quad y' = \sec^2[\ln(ax+b)] \cdot \dfrac{1}{ax+b} \cdot a = \sec^2[\ln(ax+b)]\, \dfrac{a}{ax+b}$

30. $y = \ln|\tan 2x| \quad \Rightarrow \quad y' = \dfrac{2\sec^2 2x}{\tan 2x}$

31. $y = \ln\ln x \quad \Rightarrow \quad y' = \dfrac{1}{\ln x}\dfrac{d}{dx}(\ln x) = \dfrac{1}{\ln x}\cdot\dfrac{1}{x} = \dfrac{1}{x\ln x} \quad \Rightarrow$

$$y'' = -\frac{\frac{d}{dx}(x\ln x)}{(x\ln x)^2} \quad \text{[Reciprocal Rule]} \quad = -\frac{x\cdot\frac{1}{x} + \ln x \cdot 1}{(x\ln x)^2} = -\frac{1+\ln x}{(x\ln x)^2}$$

32. $y = \dfrac{\ln x}{x^2} \quad \Rightarrow \quad y' = \dfrac{x^2(1/x) - (\ln x)(2x)}{(x^2)^2} = \dfrac{x(1-2\ln x)}{x^4} = \dfrac{1-2\ln x}{x^3} \quad \Rightarrow$

$$y'' = \frac{x^3(-2/x) - (1-2\ln x)(3x^2)}{(x^3)^2} = \frac{x^2(-2-3+6\ln x)}{x^6} = \frac{6\ln x - 5}{x^4}$$

33. $f(x) = \dfrac{x}{1-\ln(x-1)} \quad \Rightarrow$

$$f'(x) = \frac{[1-\ln(x-1)]\cdot 1 - x\cdot\frac{-1}{x-1}}{[1-\ln(x-1)]^2} = \frac{\frac{(x-1)[1-\ln(x-1)]+x}{x-1}}{[1-\ln(x-1)]^2} = \frac{x-1-(x-1)\ln(x-1)+x}{(x-1)[1-\ln(x-1)]^2}$$

$$= \frac{2x-1-(x-1)\ln(x-1)}{(x-1)[1-\ln(x-1)]^2}$$

$\text{Dom}(f) = \{x \mid x-1>0 \text{ and } 1-\ln(x-1)\neq 0\} = \{x \mid x>1 \text{ and } \ln(x-1)\neq 1\}$

$= \{x \mid x>1 \text{ and } x-1\neq e^1\} = \{x \mid x>1 \text{ and } x\neq 1+e\} = (1, 1+e)\cup(1+e,\infty)$

34. $f(x) = \dfrac{1}{1+\ln x} \quad \Rightarrow \quad f'(x) = -\dfrac{1/x}{(1+\ln x)^2}$ [Reciprocal Rule] $= -\dfrac{1}{x(1+\ln x)^2}$.

$\text{Dom}(f) = \{x \mid x>0 \text{ and } \ln x \neq -1\} = \{x \mid x>0 \text{ and } x\neq 1/e\} = (0, 1/e)\cup(1/e,\infty)$.

35. $f(x) = \sqrt{1-\ln x}$ is defined $\Leftrightarrow$ $x>0$ [so that $\ln x$ is defined] and $1-\ln x \geq 0$ $\Leftrightarrow$ $x>0$ and $\ln x \leq 1$ $\Leftrightarrow$ $0<x\leq e$, so the domain of f is $(0,e]$. Now

$$f'(x) = \frac{1}{2}(1-\ln x)^{-1/2}\cdot\frac{d}{dx}(1-\ln x) = \frac{1}{2\sqrt{1-\ln x}}\cdot\left(-\frac{1}{x}\right) = \frac{-1}{2x\sqrt{1-\ln x}}.$$

36. $f(x) = \ln\ln\ln x \quad \Rightarrow \quad f'(x) = \dfrac{1}{\ln\ln x}\cdot\dfrac{1}{\ln x}\cdot\dfrac{1}{x}$.

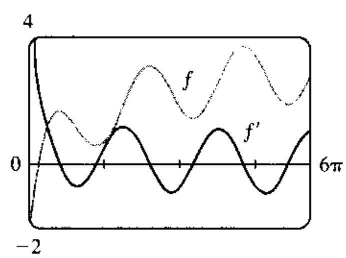
$\text{Dom}(f) = \{x \mid \ln\ln x > 0\} = \{x \mid \ln x > 1\} = \{x \mid x>e\} = (e,\infty)$.

37. $f(x) = \dfrac{x}{\ln x} \quad \Rightarrow \quad f'(x) = \dfrac{\ln x - x(1/x)}{(\ln x)^2} = \dfrac{\ln x - 1}{(\ln x)^2} \quad \Rightarrow \quad f'(e) = \dfrac{1-1}{1^2} = 0$

38. $f(t) = t\ln(4+3t) \quad \Rightarrow \quad f'(t) = t\cdot\dfrac{1}{4+3t}\cdot 3 + \ln(4+3t) = \dfrac{3t}{4+3t} + \ln(4+3t)$,

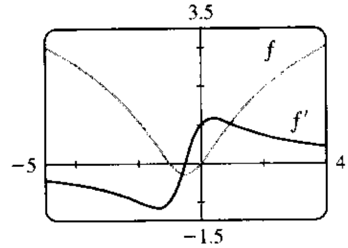
so $f'(-1) = \frac{-3}{1} + \ln 1 = -3 + 0 = -3$.

39. $f(x) = \sin x + \ln x \quad \Rightarrow \quad f'(x) = \cos x + 1/x$. This is reasonable, because the graph shows that f increases when f' is positive, and $f'(x) = 0$ when f has a horizontal tangent.



40. $f(x) = \ln(x^2 + x + 1) \;\Rightarrow\; f'(x) = \dfrac{1}{x^2 + x + 1}\,(2x + 1)$.

Notice from the graph that f is increasing when $f'(x)$ is positive.



41. $y = \sin(2\ln x) \;\Rightarrow\; y' = \cos(2\ln x)\cdot\dfrac{2}{x}$. At $(1, 0)$, $y' = \cos 0 \cdot \dfrac{2}{1} = 2$, so an equation of the tangent line is $y - 0 = 2\cdot(x - 1)$, or $y = 2x - 2$.

42. $y = \ln(x^3 - 7) \;\Rightarrow\; y' = \dfrac{1}{x^3 - 7}\cdot 3x^2 \;\Rightarrow\; y'(2) = \dfrac{12}{8 - 7} = 12$, so an equation of a tangent line at $(2, 0)$ is $y - 0 = 12(x - 2)$ or $y = 12x - 24$.

43. $y = \ln(x^2 + y^2) \;\Rightarrow\; y' = \dfrac{1}{x^2 + y^2}\dfrac{d}{dx}(x^2 + y^2) \;\Rightarrow\; y' = \dfrac{2x + 2yy'}{x^2 + y^2} \;\Rightarrow\; x^2y' + y^2y' = 2x + 2yy'$

$\Rightarrow\; x^2y' + y^2y' - 2yy' = 2x \;\Rightarrow\; (x^2 + y^2 - 2y)y' = 2x \;\Rightarrow\; y' = \dfrac{2x}{x^2 + y^2 - 2y}$

44. $\ln xy = \ln x + \ln y = y\sin x \;\Rightarrow\; 1/x + y'/y = y\cos x + y'\sin x \;\Rightarrow\; y'(1/y - \sin x) = y\cos x - 1/x \;\Rightarrow$

$y' = \dfrac{y\cos x - 1/x}{1/y - \sin x} = \left(\dfrac{y}{x}\right)\dfrac{xy\cos x - 1}{1 - y\sin x}$

45. $f(x) = \ln(x - 1) \;\Rightarrow\; f'(x) = 1/(x - 1) = (x - 1)^{-1} \;\Rightarrow\; f''(x) = -(x - 1)^{-2} \;\Rightarrow$

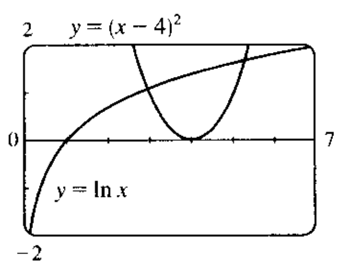
$f'''(x) = 2(x - 1)^{-3} \;\Rightarrow\; f^{(4)}(x) = -2\cdot 3(x - 1)^{-4} \;\Rightarrow\; \cdots \;\Rightarrow$

$f^{(n)}(x) = (-1)^{n-1}\cdot 2\cdot 3\cdot 4\cdot\;\cdots\;\cdot(n - 1)(x - 1)^{-n} = (-1)^{n-1}\dfrac{(n - 1)!}{(x - 1)^n}$

46. $y = x^8\ln x$, so $D^9y = D^8y' = D^8\left(8x^7\ln x + x^7\right)$. But the eighth derivative of x^7 is 0, so we now have

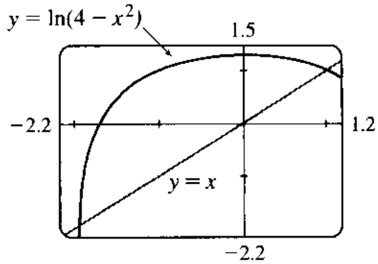
$$\begin{aligned} D^8\left(8x^7\ln x\right) &= D^7\left(8\cdot 7x^6\ln x + 8x^6\right) = D^7\left(8\cdot 7x^6\ln x\right) \\ &= D^6\left(8\cdot 7\cdot 6x^5\ln x\right) = \cdots = D\left(8!\,x^0\ln x\right) = 8!/x. \end{aligned}$$

47.



From the graph, it appears that the curves $y = (x - 4)^2$ and $y = \ln x$ intersect just to the left of $x = 3$ and to the right of $x = 5$, at about $x = 5.3$. Let $f(x) = \ln x - (x - 4)^2$. Then $f'(x) = 1/x - 2(x - 4)$, so Newton's Method says that

$x_{n+1} = x_n - f(x_n)/f'(x_n) = x_n - \dfrac{\ln x_n - (x_n - 4)^2}{1/x_n - 2(x_n - 4)}$. Taking $x_0 = 3$, we get $x_1 \approx 2.957738$, $x_2 \approx 2.958516 \approx x_3$, so the first root is 2.958516, to six decimal places. Taking $x_0 = 5$, we get $x_1 \approx 5.290755$, $x_2 \approx 5.290718 \approx x_3$, so the second (and final) root is 5.290718, to six decimal places.

48. 

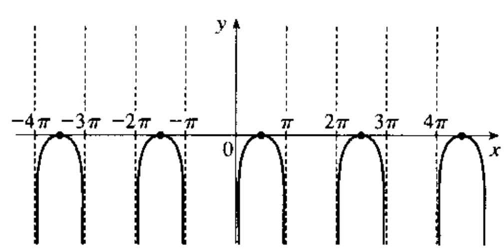
We use Newton's Method with $f(x) = \ln(4 - x^2) - x$ and $f'(x) = \dfrac{1}{4 - x^2}(-2x) - 1 = -1 - \dfrac{2x}{4 - x^2}$. The formula is $x_{n+1} = x_n - f(x_n)/f'(x_n)$. From the graphs it seems that the roots occur at approximately $x = -1.9$ and $x = 1.1$. However, if we use $x_1 = -1.9$ as an initial approximation to the first root, we get $x_2 \approx -2.009611$, and $f(x) = \ln(x - 2)^2 - x$ is undefined at this point, making it impossible to calculate x_3. We must use a more accurate first estimate, such as $x_1 = -1.95$. With this approximation, we get $x_1 = -1.95$, $x_2 \approx -1.1967495$, $x_3 \approx -1.964760$, $x_4 \approx x_5 \approx -1.964636$. Calculating the second root gives $x_1 = 1.1$, $x_2 \approx 1.058649$, $x_3 \approx 1.058007$, $x_4 \approx x_5 \approx 1.058006$. So, correct to six decimal places, the two roots of the equation $\ln(4 - x^2) = x$ are $x = -1.964636$ and $x = 1.058006$.

49. $y = f(x) = \ln(\sin x)$

A. $D = \{x \text{ in } \mathbb{R} \mid \sin x > 0\} = \displaystyle\bigcup_{n=-\infty}^{\infty} (2n\pi, (2n+1)\pi)$

$$= \cdots \cup (-4\pi, -3\pi) \cup (-2\pi, -\pi) \cup (0, \pi) \cup (2\pi, 3\pi) \cup \cdots$$

B. No y-intercept; x-intercepts: $f(x) = 0 \iff \ln(\sin x) = 0 \iff \sin x = e^0 = 1 \iff x = 2n\pi + \frac{\pi}{2}$ for each integer n. **C.** f is periodic with period 2π. **D.** $\lim\limits_{x \to (2n\pi)^+} f(x) = -\infty$ and $\lim\limits_{x \to [(2n+1)\pi]^-} f(x) = -\infty$, so the lines $x = n\pi$ are VAs for all integers n. **E.** $f'(x) = \frac{\cos x}{\sin x} = \cot x$, so $f'(x) > 0$ when $2n\pi < x < 2n\pi + \frac{\pi}{2}$ for each integer n, and $f'(x) < 0$ when $2n\pi + \frac{\pi}{2} < x < (2n+1)\pi$. Thus, f is increasing on $\left(2n\pi, 2n\pi + \frac{\pi}{2}\right)$ and decreasing on $\left(2n\pi + \frac{\pi}{2}, (2n+1)\pi\right)$ for each integer n. **F.** Local maximum values $f\left(2n\pi + \frac{\pi}{2}\right) = 0$, no local minimum. **G.** $f''(x) = -\csc^2 x < 0$, so f is CD on $(2n\pi, (2n+1)\pi)$ for each integer n. No IP

H. 

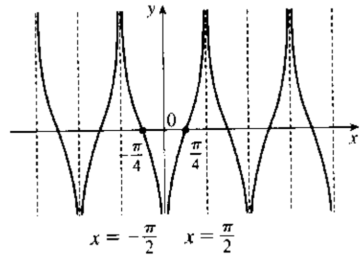
50. $y = \ln(\tan^2 x)$ **A.** $D = \{x \mid x \neq n\pi/2\}$ **B.** x-intercepts $n\pi + \frac{\pi}{4}$, no y-intercept. **C.** $f(-x) = f(x)$, so the curve is symmetric about the y-axis. Also $f(x + \pi) = f(x)$, so f is periodic with period π, and we consider parts D–G only for $-\frac{\pi}{2} < x < \frac{\pi}{2}$. **D.** $\lim\limits_{x \to 0} \ln(\tan^2 x) = -\infty$ and $\lim\limits_{x \to (\pi/2)^-} \ln(\tan^2 x) = \infty$, $\lim\limits_{x \to (-\pi/2)^+} \ln(\tan^2 x) = \infty$, so $x = 0$, $x = \pm\frac{\pi}{2}$ are VA. **E.** $f'(x) = \dfrac{2\tan x \sec^2 x}{\tan^2 x} = 2\dfrac{\sec^2 x}{\tan x} > 0 \iff$

$\tan x > 0 \quad \Leftrightarrow \quad 0 < x < \frac{\pi}{2}$, so f is increasing on $\left(0, \frac{\pi}{2}\right)$ and decreasing on $\left(-\frac{\pi}{2}, 0\right)$. **F.** No maximum or minimum

G. $f'(x) = \dfrac{2}{\sin x \cos x} = \dfrac{4}{\sin 2x} \quad \Rightarrow \quad f''(x) = \dfrac{-8\cos 2x}{\sin^2 2x} < 0$

$\Leftrightarrow \quad \cos 2x > 0 \Leftrightarrow \quad -\frac{\pi}{4} < x < \frac{\pi}{4}$, so f is CD on $\left(-\frac{\pi}{4}, 0\right)$ and $\left(0, \frac{\pi}{4}\right)$ and CU on $\left(-\frac{\pi}{2}, -\frac{\pi}{4}\right)$ and $\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$. IP are $\left(\pm\frac{\pi}{4}, 0\right)$.

H.



51. $y = f(x) = \ln(1 + x^2)$ **A.** $D = \mathbb{R}$ **B.** Both intercepts are 0. **C.** $f(-x) = f(x)$, so the curve is symmetric about the y-axis. **D.** $\lim\limits_{x \to \pm\infty} \ln(1 + x^2) = \infty$, no asymptotes. **E.** $f'(x) = \dfrac{2x}{1 + x^2} > 0 \quad \Leftrightarrow$

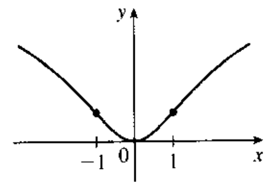
$x > 0$, so f is increasing on $(0, \infty)$ and decreasing on $(-\infty, 0)$.

F. $f(0) = 0$ is a local and absolute minimum.

G. $f''(x) = \dfrac{2(1 + x^2) - 2x(2x)}{(1 + x^2)^2} = \dfrac{2(1 - x^2)}{(1 + x^2)^2} > 0 \quad \Leftrightarrow$

$|x| < 1$, so f is CU on $(-1, 1)$, CD on $(-\infty, -1)$ and $(1, \infty)$. IP $(1, \ln 2)$ and $(-1, \ln 2)$.

H.



52. $y = f(x) = \ln(x^2 - 3x + 2) = \ln\left[(x - 1)(x - 2)\right]$

A. $D = \left\{x \text{ in } \mathbb{R}\text{: } x^2 - 3x + 2 > 0\right\} = (-\infty, 1) \cup (2, \infty)$.

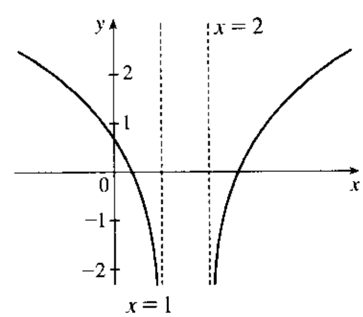
B. y-intercept: $f(0) = \ln 2$; x-intercepts: $f(x) = 0 \quad \Leftrightarrow \quad x^2 - 3x + 2 = e^0 \quad \Leftrightarrow \quad x^2 - 3x + 1 = 0 \quad \Leftrightarrow$

$x = \dfrac{3 \pm \sqrt{5}}{2} \quad \Rightarrow \quad x \approx 0.38, 2.62$ **C.** No symmetry **D.** $\lim\limits_{x \to 1^-} f(x) = \lim\limits_{x \to 2^+} f(x) = -\infty$, so $x = 1$ and $x = 2$ are VAs. No HA. **E.** $f'(x) = \dfrac{2x - 3}{x^2 - 3x + 2} = \dfrac{2(x - 3/2)}{(x - 1)(x - 2)}$, so $f'(x) < 0$ for $x < 1$ and $f'(x) > 0$ for $x > 2$. Thus, f is decreasing on $(-\infty, 1)$ and increasing on $(2, \infty)$. **F.** No extreme values

G.
$$f''(x) = \frac{(x^2 - 3x + 2) \cdot 2 - (2x - 3)^2}{(x^2 - 3x + 2)^2} = \frac{2x^2 - 6x + 4 - 4x^2 + 12x - 9}{(x^2 - 3x + 2)^2} = \frac{-2x^2 + 6x - 5}{(x^2 - 3x + 2)^2}$$

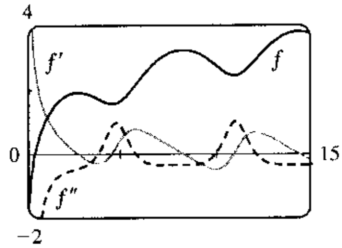
The numerator is negative for all x and the denominator is positive, so $f''(x) < 0$ for all x in the domain of f. Thus, f is CD on $(-\infty, 1)$ and $(2, \infty)$. No IP

H.



53. We use the CAS to calculate $f'(x) = \dfrac{2+\sin x + x\cos x}{2x + x\sin x}$ and

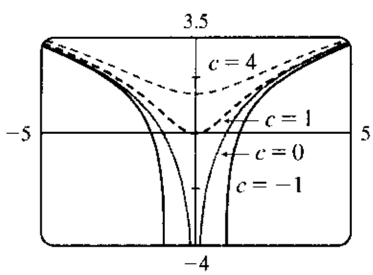
$f''(x) = \dfrac{2x^2 \sin x + 4\sin x - \cos^2 x + x^2 + 5}{x^2(\cos^2 x - 4\sin x - 5)}$. From the graphs, it seems that $f' > 0$ (and so f is increasing) on approximately the intervals $(0, 2.7)$, $(4.5, 8.2)$ and $(10.9, 14.3)$. It seems that f'' changes sign (indicating inflection points) at $x \approx 3.8$, 5.7, 10.0 and 12.0.



Looking back at the graph of $f(x) = \ln(2x + x\sin x)$, this implies that the inflection points have approximate coordinates $(3.8, 1.7)$, $(5.7, 2.1)$, $(10.0, 2.7)$, and $(12.0, 2.9)$.

54. We see that if $c \le 0$, $f(x) = \ln(x^2 + c)$ is only defined for $x^2 > -c \;\Rightarrow\; |x| > \sqrt{-c}$, and $\lim\limits_{x \to \sqrt{-c}^+} f(x) = \lim\limits_{x \to -\sqrt{-c}^-} f(x) = -\infty$, since $\ln y \to -\infty$ as $y \to 0$. Thus, for $c < 0$, there are vertical asymptotes at $x = \pm\sqrt{c}$, and as c decreases (that is, $|c|$ increases), the asymptotes get further apart. For $c = 0$, $\lim\limits_{x \to 0} f(x) = -\infty$, so there is a vertical asymptote at $x = 0$. If $c > 0$, there is no asymptote. To find the maxima, minima, and inflection points, we differentiate: $f(x) = \ln(x^2 + c) \;\Rightarrow\; f'(x) = \dfrac{1}{x^2 + c}(2x)$, so by the First Derivative Test there is a local and absolute minimum at $x = 0$. Differentiating again, we get

$f''(x) = \dfrac{1}{x^2 + c}(2) + 2x\left[-(x^2 + c)^{-2}(2x)\right] = \dfrac{2(c - x^2)}{(x^2 + c)^2}$. Now if $c \le 0$, this is always negative, so f is concave down on both of the intervals on which it is defined. If $c > 0$, then f'' changes sign when $c = x^2 \;\Leftrightarrow\; x = \pm\sqrt{c}$. So for $c > 0$ there are inflection points at $\pm\sqrt{c}$, and as c increases, the inflection points get further apart.



55. $y = (2x+1)^5(x^4 - 3)^6 \;\Rightarrow\; \ln y = \ln\left((2x+1)^5(x^4-3)^6\right) \;\Rightarrow$

$\ln y = 5\ln(2x+1) + 6\ln(x^4 - 3) \;\Rightarrow\; \dfrac{1}{y}y' = 5 \cdot \dfrac{1}{2x+1} \cdot 2 + 6 \cdot \dfrac{1}{x^4 - 3} \cdot 4x^3 \;\Rightarrow$

$y' = y\left(\dfrac{10}{2x+1} + \dfrac{24x^3}{x^4 - 3}\right) = (2x+1)^5(x^4-3)^6\left(\dfrac{10}{2x+1} + \dfrac{24x^3}{x^4-3}\right)$.

[The answer could be simplified to $y' = 2(2x+1)^4(x^4-3)^5(29x^4 + 12x^3 - 15)$, but this is unnecessary.]

56. $y = \dfrac{(x^3+1)^4 \sin^2 x}{x^{1/3}} \;\Rightarrow\; \ln|y| = 4\ln|x^3 + 1| + 2\ln|\sin x| - \frac{1}{3}\ln|x|$.

So $\dfrac{y'}{y} = 4\dfrac{3x^2}{x^3+1} + 2\dfrac{\cos x}{\sin x} - \dfrac{1}{3x} \;\Rightarrow\; y' = \dfrac{(x^3+1)^4\sin^2 x}{x^{1/3}}\left(\dfrac{12x^2}{x^3+1} + 2\cot x - \dfrac{1}{3x}\right)$.

57. $y = \dfrac{\sin^2 x \tan^4 x}{(x^2+1)^2} \quad \Rightarrow \quad \ln y = \ln(\sin^2 x \tan^4 x) - \ln(x^2+1)^2 \quad \Rightarrow$

$\ln y = \ln(\sin x)^2 + \ln(\tan x)^4 - \ln(x^2+1)^2 \quad \Rightarrow \quad \ln y = 2\ln|\sin x| + 4\ln|\tan x| - 2\ln(x^2+1) \quad \Rightarrow$

$\dfrac{1}{y}y' = 2 \cdot \dfrac{1}{\sin x} \cdot \cos x + 4 \cdot \dfrac{1}{\tan x} \cdot \sec^2 x - 2 \cdot \dfrac{1}{x^2+1} \cdot 2x \quad \Rightarrow$

$y' = \dfrac{\sin^2 x \tan^4 x}{(x^2+1)^2}\left(2\cot x + \dfrac{4\sec^2 x}{\tan x} - \dfrac{4x}{x^2+1}\right)$

58. $y = \sqrt[4]{\dfrac{x^2+1}{x^2-1}} \quad \Rightarrow \quad \ln y = \frac{1}{4}\ln(x^2+1) - \frac{1}{4}\ln(x^2-1) \quad \Rightarrow \quad \dfrac{1}{y}y' = \dfrac{1}{4} \cdot \dfrac{1}{x^2+1} \cdot 2x - \dfrac{1}{4} \cdot \dfrac{1}{x^2-1} \cdot 2x \quad \Rightarrow$

$y' = \sqrt[4]{\dfrac{x^2+1}{x^2-1}} \cdot \dfrac{1}{2}\left(\dfrac{x}{x^2+1} - \dfrac{x}{x^2-1}\right) = \dfrac{1}{2}\sqrt[4]{\dfrac{x^2+1}{x^2-1}}\left(\dfrac{-2x}{x^4-1}\right) = \dfrac{x}{1-x^4}\sqrt[4]{\dfrac{x^2+1}{x^2-1}}$

59. $\displaystyle\int_2^4 \frac{3}{x}\,dx = 3\int_2^4 \frac{1}{x}\,dx = 3\Big[\ln|x|\Big]_2^4 = 3(\ln 4 - \ln 2) = 3\ln\frac{4}{2} = 3\ln 2$

60. $\displaystyle\int_1^2 \frac{4+u^2}{u^3}\,du = \int_1^2 (4u^{-3} + u^{-1})\,du = \left[\frac{4}{-2}u^{-2} + \ln|u|\right]_1^2 = \left[\frac{-2}{u^2} + \ln u\right]_1^2$

$= (-\frac{1}{2} + \ln 2) - (-2 + \ln 1) = \frac{3}{2} + \ln 2$

61. $\displaystyle\int_1^2 \frac{dt}{8-3t} = \left[-\frac{1}{3}\ln|8-3t|\right]_1^2 = -\frac{1}{3}\ln 2 - \left(-\frac{1}{3}\ln 5\right) = \frac{1}{3}(\ln 5 - \ln 2) = \frac{1}{3}\ln\frac{5}{2}$

Or: Let $u = 8 - 3t$. Then $du = -3\,dt$, so

$\displaystyle\int_1^2 \frac{dt}{8-3t} = \int_5^2 \frac{-\frac{1}{3}\,du}{u} = \left[-\frac{1}{3}\ln|u|\right]_5^2 = -\frac{1}{3}\ln 2 - \left(-\frac{1}{3}\ln 5\right) = \frac{1}{3}(\ln 5 - \ln 2) = \frac{1}{3}\ln\frac{5}{2}.$

62. $\displaystyle\int_4^9 \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^2 dx = \int_4^9 \left(x + 2 + \frac{1}{x}\right) dx = \left[\tfrac{1}{2}x^2 + 2x + \ln x\right]_4^9 = \frac{81}{2} + 18 + \ln 9 - (8 + 8 + \ln 4)$

$= \frac{85}{2} + \ln\frac{9}{4}$

63. $\displaystyle\int_1^e \frac{x^2+x+1}{x}\,dx = \int_1^e \left(x + 1 + \frac{1}{x}\right) dx = \left[\tfrac{1}{2}x^2 + x + \ln x\right]_1^e = \left(\tfrac{1}{2}e^2 + e + 1\right) - \left(\tfrac{1}{2} + 1 + 0\right)$

$= \frac{1}{2}e^2 + e - \frac{1}{2}$

64. Let $u = \ln x$. Then $du = \dfrac{1}{x}\,dx$, so $\displaystyle\int_e^6 \frac{dx}{x\ln x} = \int_1^{\ln 6} \frac{1}{u}\,du = \Big[\ln|u|\Big]_1^{\ln 6} = \ln\ln 6 - \ln 1 = \ln\ln 6$

65. Let $u = 6x - x^3$. Then $du = (6 - 3x^2)\,dx = 3(2 - x^2)\,dx$, so

$\displaystyle\int \frac{2-x^2}{6x-x^3}\,dx = \int \frac{\frac{1}{3}\,du}{u} = \frac{1}{3}\ln|u| + C = \frac{1}{3}\ln\left|6x - x^3\right| + C.$

66. Let $u = 2 + \sin x$. Then $du = \cos x\,dx$, so

$$\int \frac{\cos x}{2+\sin x}\,dx = \int \frac{1}{u}\,du = \ln|u| + C = \ln|2+\sin x| + C = \ln(2+\sin x) + C \quad [\text{since } 2+\sin x > 0].$$

67. Let $u = \ln x$. Then $du = \dfrac{dx}{x} \;\Rightarrow\; \displaystyle\int \frac{(\ln x)^2}{x}\,dx = \int u^2\,du = \frac{1}{3}u^3 + C = \frac{1}{3}(\ln x)^3 + C.$

68. Let $u = 1 + \sqrt{x}$. Then $du = \dfrac{1}{2\sqrt{x}}\,dx$, so $\displaystyle\int \frac{(1+\sqrt{x})^4}{\sqrt{x}}\,dx = 2\int u^4\,du = \tfrac{2}{5}u^5 + C = \tfrac{2}{5}(1+\sqrt{x})^5 + C.$

69. (a) $\dfrac{d}{dx}(\ln|\sin x| + C) = \dfrac{1}{\sin x}\cos x = \cot x$

(b) Let $u = \sin x$. Then $du = \cos x\,dx$, so $\displaystyle\int \cot x\,dx = \int \frac{\cos x}{\sin x}\,dx = \int \frac{du}{u} = \ln|u| + C = \ln|\sin x| + C.$

70. Let $u = x - 2$. Then the area is

$$A = -\int_{-4}^{-1} \frac{2}{x-2}\,dx = -2\int_{-6}^{-3} \frac{du}{u} = \Big[-2\ln|u|\Big]_{-6}^{-3} = -2\ln 3 + 2\ln 6 = 2\ln 2 \approx 1.386.$$

71. The cross-sectional area is $\pi\left(1/\sqrt{x+1}\right)^2 = \pi/(x+1)$. Therefore, the volume is

$$\int_0^1 \frac{\pi}{x+1}\,dx = \pi[\ln(x+1)]_0^1 = \pi(\ln 2 - \ln 1) = \pi \ln 2.$$

72. Using cylindrical shells, we get $V = \displaystyle\int_0^3 \frac{2\pi x}{x^2+1}\,dx = \pi\left[\ln(1+x^2)\right]_0^3 = \pi \ln 10.$

73. $W = \int_{V_1}^{V_2} P\,dV = \displaystyle\int_{600}^{1000} \frac{C}{V}\,dV = C\int_{600}^{1000} \frac{1}{V}\,dV = C\Big[\ln|V|\Big]_{600}^{1000}$

$= C(\ln 1000 - \ln 600) = C \ln \frac{1000}{600} = C \ln \frac{5}{3}$

Initially, $PV = C$, where $P = 150$ kPa and $V = 600\text{ cm}^3$, so $C = (150)(600) = 90{,}000$. Thus,

$W = 90{,}000 \ln \frac{5}{3} \approx 45{,}974\text{ kPa}\cdot\text{cm}^3 = 45{,}974(10^3\text{ Pa})(10^{-6}\text{ m}^3) = 45{,}974\text{ Pa}\cdot\text{m}^3 = 45{,}974\text{ N}\cdot\text{m}$

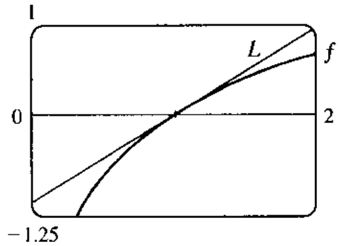
$[\text{Pa} = \text{N/m}^2] \quad = 45{,}974$ J

74. $f''(x) = x^{-2}, x > 0 \;\Rightarrow\; f'(x) = -1/x + C \;\Rightarrow\; f(x) = -\ln x + Cx + D.$ $0 = f(1) = C + D$ and $0 = f(2) = -\ln 2 + 2C + D = -\ln 2 + 2C - C = -\ln 2 + C \;\Rightarrow\; C = \ln 2$ and $D = -\ln 2$. So $f(x) = -\ln x + (\ln 2)x - \ln 2$.

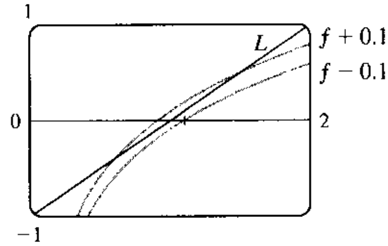
75. $f(x) = 2x + \ln x \;\Rightarrow\; f'(x) = 2 + 1/x$. If $g = f^{-1}$, then $f(1) = 2 \;\Rightarrow\; g(2) = 1$, so $g'(2) = 1/f'(g(2)) = 1/f'(1) = \frac{1}{3}$.

76. (a) Let $f(x) = \ln x \Rightarrow f'(x) = 1/x \Rightarrow f''(x) = -1/x^2$. The linear approximation to $\ln x$ near 1 is $\ln x \approx f(1) + f'(1)(x-1) = \ln 1 + \frac{1}{1}(x-1) = x - 1$.

(b)

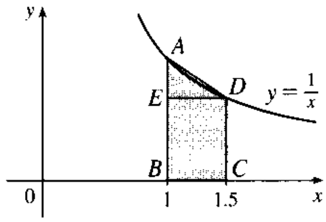


(c)



From the graph, it appears that the linear approximation is accurate to within 0.1 for x between about 0.62 and 1.51.

77. (a)



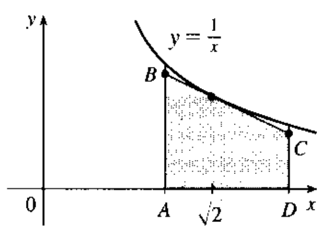
We interpret $\ln 1.5$ as the area under the curve $y = 1/x$ from $x = 1$ to $x = 1.5$. The area of the rectangle $BCDE$ is $\frac{1}{2} \cdot \frac{2}{3} = \frac{1}{3}$. The area of the trapezoid $ABCD$ is $\frac{1}{2} \cdot \frac{1}{2}\left(1 + \frac{2}{3}\right) = \frac{5}{12}$. Thus, by comparing areas, we observe that $\frac{1}{3} < \ln 1.5 < \frac{5}{12}$.

(b) With $f(t) = 1/t$, $n = 10$, and $\Delta t = 0.05$, we have

$$\ln 1.5 = \int_1^{1.5} (1/t)\,dt \approx (0.05)[f(1.025) + f(1.075) + \cdots + f(1.475)]$$
$$= (0.05)\left[\tfrac{1}{1.025} + \tfrac{1}{1.075} + \cdots + \tfrac{1}{1.475}\right] \approx 0.4054$$

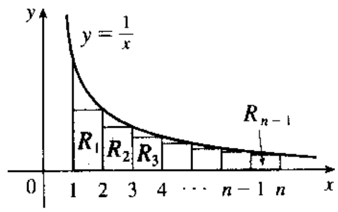
78. (a) $y = \dfrac{1}{t}$, $y' = -\dfrac{1}{t^2}$. The slope of AD is $\dfrac{1/2 - 1}{2 - 1} = -\dfrac{1}{2}$. Let c be the t-coordinate of the point on $y = \dfrac{1}{t}$ with slope $-\dfrac{1}{2}$. Then $-\dfrac{1}{c^2} = -\dfrac{1}{2} \Rightarrow c^2 = 2 \Rightarrow c = \sqrt{2}$ since $c > 0$. Therefore the tangent line is given by $y - \frac{1}{\sqrt{2}} = -\frac{1}{2}\left(t - \sqrt{2}\right) \Rightarrow y = -\frac{1}{2}t + \sqrt{2}$.

(b)

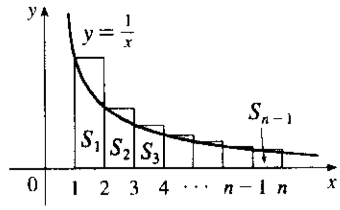


Since the graph of $y = 1/t$ is concave upward, the graph lies above the tangent line, that is, above the line segment BC. Now $|AB| = -\frac{1}{2} + \sqrt{2}$ and $|CD| = -1 + \sqrt{2}$. So the area of the trapezoid $ABCD$ is

$\frac{1}{2}\left[\left(-\frac{1}{2} + \sqrt{2}\right) + \left(-1 + \sqrt{2}\right)1\right] = -\frac{3}{4} + \sqrt{2} \approx 0.6642$. So $\ln 2 >$ area of trapezoid $ABCD > 0.66$.

79. 

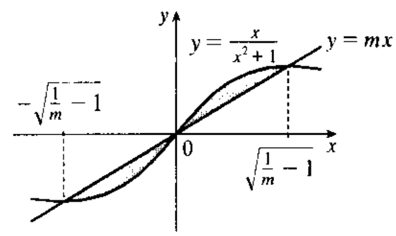
The area of R_i is $\dfrac{1}{i+1}$ and so $\dfrac{1}{2} + \dfrac{1}{3} + \cdots + \dfrac{1}{n} < \displaystyle\int_1^n \frac{1}{t}\,dt = \ln n$.



The area of S_i is $\dfrac{1}{i}$ and so $1 + \dfrac{1}{2} + \cdots + \dfrac{1}{n-1} > \displaystyle\int_1^n \frac{1}{t}\,dt = \ln n$.

80. If $f(x) = \ln(x^r)$, then $f'(x) = (1/x^r)(rx^{r-1}) = r/x$. But if $g(x) = r\ln x$, then $g'(x) = r/x$. So f and g must differ by a constant: $\ln(x^r) = r\ln x + C$. Put $x = 1$: $\ln(1^r) = r\ln 1 + C \Rightarrow C = 0$, so $\ln(x^r) = r\ln x$.

81. The curve and the line will determine a region when they intersect at two or more points. So we solve the equation $x/(x^2+1) = mx \Rightarrow x = 0$ or $mx^2 + m - 1 = 0 \Rightarrow x = 0$ or $x = \dfrac{\pm\sqrt{-4(m)(m-1)}}{2m} = \pm\sqrt{\dfrac{1}{m} - 1}$. Note that if $m = 1$, this has only the solution $x = 0$, and no region is determined. But if $1/m - 1 > 0 \Leftrightarrow 1/m > 1 \Leftrightarrow 0 < m < 1$, then there are two solutions. [Another way of seeing this is to observe that the slope of the tangent to $y = x/(x^2+1)$ at the origin is $y' = 1$ and therefore we must have $0 < m < 1$.] Note that we cannot just integrate between the positive and negative roots, since the curve and the line cross at the origin. Since mx and $x/(x^2+1)$ are both odd functions, the total area is twice the area between the curves on the interval $\left[0, \sqrt{1/m - 1}\right]$. So the total area enclosed is



$$2\int_0^{\sqrt{1/m-1}} \left[\frac{x}{x^2+1} - mx\right] dx = 2\left[\tfrac{1}{2}\ln(x^2+1) - \tfrac{1}{2}mx^2\right]_0^{\sqrt{1/m-1}}$$

$$= \left[\ln\left(\frac{1}{m} - 1 + 1\right) - m\left(\frac{1}{m} - 1\right)\right] - (\ln 1 - 0)$$

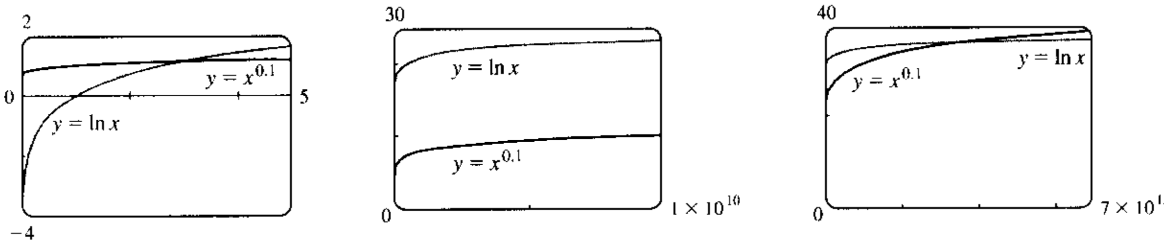
$$= \ln\left(\frac{1}{m}\right) + m - 1 = m - \ln m - 1$$

82. $\displaystyle\lim_{x\to\infty} [\ln(2+x) - \ln(1+x)] = \lim_{x\to\infty} \ln\left(\frac{2+x}{1+x}\right) = \lim_{x\to\infty} \ln\left(\frac{2/x+1}{1/x+1}\right) = \ln\frac{1}{1} = \ln 1 = 0$

83. If $f(x) = \ln(1+x)$, then $f'(x) = \dfrac{1}{1+x}$, so $f'(0) = 1$.

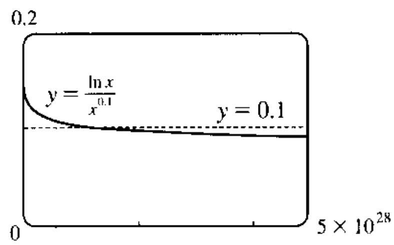
Thus, $\displaystyle\lim_{x\to 0} \frac{\ln(1+x)}{x} = \lim_{x\to 0} \frac{f(x)}{x} = \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = f'(0) = 1.$

84. (a)



From the graphs, we see that $f(x) = x^{0.1} > g(x) = \ln x$ for approximately $0 < x < 3.06$, and then $g(x) > f(x)$ for $3.06 < x < 3.43 \times 10^{15}$ (approximately). At that point, the graph of f finally surpasses the graph of g for good.

(b)



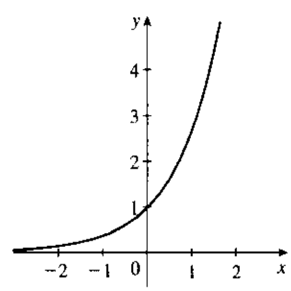
(c) From the graph at left, it seems that $\dfrac{\ln x}{x^{0.1}} < 0.1$ whenever $x > 1.3 \times 10^{28}$ (approximately). So we can take $N = 1.3 \times 10^{28}$, or any larger number.

7.3* The Natural Exponential Function

1. (a) e is the number such that $\ln e = 1$.

(b) $e \approx 2.71828$

(c)



The function value at $x = 0$ is 1 and the slope at $x = 0$ is 1.

2. (a) $e^{\ln 6} = 6$ (b) $\ln\sqrt{e} = \ln(e^{1/2}) = \frac{1}{2}$

3. (a) $\ln e^{\sqrt{2}} = \sqrt{2}$ (b) $e^{3\ln 2} = \left(e^{\ln 2}\right)^3 = 2^3 = 8$

4. (a) $\ln e^{\sin x} = \sin x$ (b) $e^{x + \ln x} = e^x e^{\ln x} = xe^x$

5. (a) $2\ln x = 1 \quad\Rightarrow\quad \ln x = \frac{1}{2} \quad\Rightarrow\quad x = e^{1/2} = \sqrt{e}$

(b) $e^{-x} = 5 \quad\Rightarrow\quad -x = \ln 5 \quad\Rightarrow\quad x = -\ln 5$

6. (a) $e^{2x+3} - 7 = 0 \quad \Rightarrow \quad e^{2x+3} = 7 \quad \Rightarrow \quad 2x + 3 = \ln 7 \quad \Rightarrow \quad 2x = \ln 7 - 3 \quad \Rightarrow \quad x = \frac{1}{2}(\ln 7 - 3)$

(b) $\ln(5 - 2x) = -3 \quad \Rightarrow \quad 5 - 2x = e^{-3} \quad \Rightarrow \quad 2x = 5 - e^{-3} \quad \Rightarrow \quad x = \frac{1}{2}(5 - e^{-3})$

7. $\ln(\ln x) = 1 \quad \Leftrightarrow \quad e^{\ln(\ln x)} = e^1 \quad \Leftrightarrow \quad \ln x = e^1 = e \quad \Leftrightarrow \quad e^{\ln x} = e^e \quad \Leftrightarrow \quad x = e^e$

8. $e^{e^x} = 10 \quad \Leftrightarrow \quad \ln\left(e^{e^x}\right) = \ln 10 \quad \Leftrightarrow \quad e^x \ln e = e^x = \ln 10 \quad \Leftrightarrow \quad \ln e^x = \ln(\ln 10) \quad \Leftrightarrow \quad x = \ln(\ln 10)$

9. $2\ln x = \ln 2 + \ln(3x - 4) \quad \Rightarrow \quad \ln x^2 = \ln[2(3x - 4)] \quad \Rightarrow \quad \ln x^2 = \ln(6x - 8) \quad \Rightarrow \quad x^2 = 6x - 8 \quad \Rightarrow$
$x^2 - 6x + 8 = 0 \quad \Rightarrow \quad (x - 2)(x - 4) = 0 \quad \Rightarrow \quad x = 2$ or $x = 4$, both are valid solutions.

10. $\ln(2x + 1) = 2 - \ln x \quad \Rightarrow \quad \ln x + \ln(2x + 1) = \ln e^2 \quad \Rightarrow \quad \ln[x(2x + 1)] = \ln e^2 \quad \Rightarrow \quad 2x^2 + x = e^2 \quad \Rightarrow$
$2x^2 + x - e^2 = 0 \quad \Rightarrow \quad x = \dfrac{-1 + \sqrt{1 + 8e^2}}{4}$ [since $x > 0$].

11. $e^{ax} = Ce^{bx} \quad \Leftrightarrow \quad \ln e^{ax} = \ln[C(e^{bx})] \quad \Leftrightarrow \quad ax = \ln C + bx + \ln e^{bx} \quad \Leftrightarrow \quad ax = \ln C + bx \quad \Leftrightarrow$
$ax - bx = \ln C \quad \Leftrightarrow \quad (a - b)x = \ln C \quad \Leftrightarrow \quad x = \dfrac{\ln C}{a - b}$

12. $7e^x - e^{2x} = 12 \quad \Leftrightarrow \quad (e^x)^2 - 7e^x + 12 = 0 \quad \Leftrightarrow \quad (e^x - 3)(e^x - 4) = 0$, so we have either $e^x = 3 \quad \Leftrightarrow$
$x = \ln 3$, or $e^x = 4 \quad \Leftrightarrow \quad x = \ln 4$.

13. $e^{2+5x} = 100 \quad \Rightarrow \quad \ln\left(e^{2+5x}\right) = \ln 100 \quad \Rightarrow \quad 2 + 5x = \ln 100 \quad \Rightarrow \quad 5x = \ln 100 - 2 \quad \Rightarrow$
$x = \frac{1}{5}(\ln 100 - 2) \approx 0.5210$

14. $\ln(1 + \sqrt{x}) = 2 \quad \Rightarrow \quad 1 + \sqrt{x} = e^2 \quad \Rightarrow \quad \sqrt{x} = e^2 - 1 \quad \Rightarrow \quad x = (e^2 - 1)^2 \approx 40.8200$

15. $\ln(e^x - 2) = 3 \quad \Rightarrow \quad e^x - 2 = e^3 \quad \Rightarrow \quad e^x = e^3 + 2 \quad \Rightarrow \quad x = \ln(e^3 + 2) \approx 3.0949$

16. $e^{1/(x-4)} = 7 \quad \Rightarrow \quad \ln e^{1/(x-4)} = \ln 7 \quad \Rightarrow \quad \dfrac{1}{x - 4} = \ln 7 \quad \Rightarrow \quad \dfrac{1}{\ln 7} = x - 4 \quad \Rightarrow \quad x = 4 + \dfrac{1}{\ln 7} \approx 4.5139$

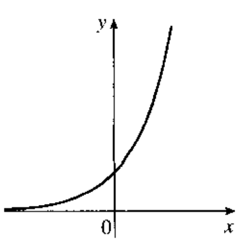
17. (a) $e^x < 10 \quad \Rightarrow \quad \ln e^x < \ln 10 \quad \Rightarrow \quad x < \ln 10 \quad \Rightarrow \quad x \in (-\infty, \ln 10)$

(b) $\ln x > -1 \quad \Rightarrow \quad e^{\ln x} > e^{-1} \quad \Rightarrow \quad x > e^{-1} \quad \Rightarrow \quad x \in (1/e, \infty)$

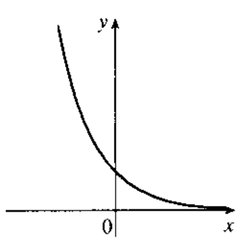
18. (a) $2 < \ln x < 9 \quad \Rightarrow \quad e^2 < e^{\ln x} < e^9 \quad \Rightarrow \quad e^2 < x < e^9 \quad \Rightarrow \quad x \in (e^2, e^9)$

(b) $e^{2-3x} > 4 \quad \Rightarrow \quad \ln e^{2-3x} > \ln 4 \quad \Rightarrow \quad 2 - 3x > \ln 4 \quad \Rightarrow \quad -3x > \ln 4 - 2 \quad \Rightarrow$
$x < -\frac{1}{3}(\ln 4 - 2) \quad \Rightarrow \quad x \in \left(-\infty, \frac{1}{3}(2 - \ln 4)\right)$

19.

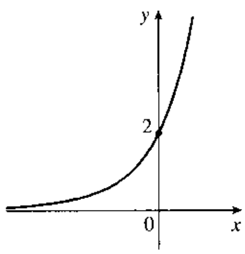


$y = e^x$

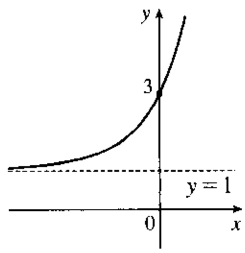


$y = e^{-x}$

20. We start with the graph of $y = e^x$ (Figure 2), vertically stretch by a factor of 2, and then shift 1 unit upward. There is a horizontal asymptote of $y = 1$.

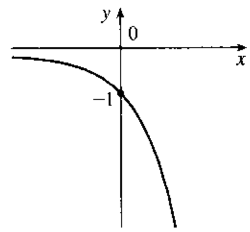


$y = 2e^x$

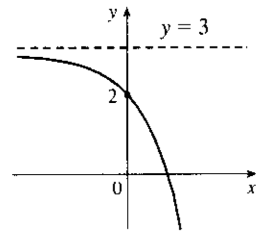


$y = 1 + 2e^x$

21. We start with the graph of $y = e^x$ (Figure 2), reflect it about the x-axis, and then shift 3 units upward. Note the horizontal asymptote of $y = 3$.

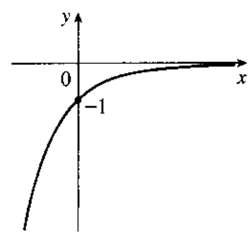


$y = -e^x$

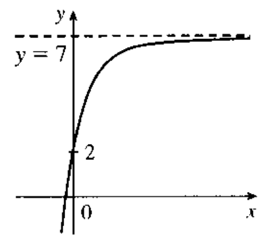


$y = 3 - e^x$

22. We start with the graph of $y = e^x$ (Figure 2), reflect it about the y-axis, and then about the x-axis (or just rotate 180° to handle both reflections) to obtain the graph of $y = -e^{-x}$. Now shift this graph 1 unit upward, vertically stretch by a factor of 5, and then shift 2 units upward.



$y = -e^{-x}$



$y = 2 + 5(1 - e^{-x})$

23. (a) To find the equation of the graph that results from shifting the graph of $y = e^x$ 2 units downward, we subtract 2 from the original function to get $y = e^x - 2$.

(b) To find the equation of the graph that results from shifting the graph of $y = e^x$ 2 units to the right, we replace x with $x - 2$ in the original function to get $y = e^{(x-2)}$.

(c) To find the equation of the graph that results from reflecting the graph of $y = e^x$ about the x-axis, we multiply the original function by -1 to get $y = -e^x$.

(d) To find the equation of the graph that results from reflecting the graph of $y = e^x$ about the y-axis, we replace x with $-x$ in the original function to get $y = e^{-x}$.

(e) To find the equation of the graph that results from reflecting the graph of $y = e^x$ about the x-axis and then about the y-axis, we first multiply the original function by -1 (to get $y = -e^x$) and then replace x with $-x$ in this equation to get $y = -e^{-x}$.

24. (a) This reflection consists of first reflecting the graph about the x-axis (giving the graph with equation $y = -e^x$) and then shifting this graph $2 \cdot 4 = 8$ units upward. So the equation is $y = -e^x + 8$.

(b) This reflection consists of first reflecting the graph about the y-axis (giving the graph with equation $y = e^{-x}$) and then shifting this graph $2 \cdot 2 = 4$ units to the right. So the equation is $y = e^{-(x-4)}$.

25. $\lim\limits_{x\to\infty} e^{1-x^3} = \lim\limits_{x\to\infty} (e^1 \cdot e^{-x^3}) = e \lim\limits_{x\to\infty} \dfrac{1}{e^{x^3}} = e \cdot 0 = 0$

26. If we let $t = \tan x$, then as $x \to (\pi/2)^+$, $t \to -\infty$. Thus, $\lim\limits_{x\to(\pi/2)^+} e^{\tan x} = \lim\limits_{t\to-\infty} e^t = 0$.

27. Divide numerator and denominator by e^{3x}: $\lim\limits_{x\to\infty} \dfrac{e^{3x} - e^{-3x}}{e^{3x} + e^{-3x}} = \lim\limits_{x\to\infty} \dfrac{1 - e^{-6x}}{1 + e^{-6x}} = \dfrac{1-0}{1+0} = 1$

28. Divide numerator and denominator by e^{-3x}: $\lim\limits_{x\to-\infty} \dfrac{e^{3x} - e^{-3x}}{e^{3x} + e^{-3x}} = \lim\limits_{x\to-\infty} \dfrac{e^{6x} - 1}{e^{6x} + 1} = \dfrac{0-1}{0+1} = -1$

29. Let $t = 3/(2-x)$. As $x \to 2^+$, $t \to -\infty$. So $\lim\limits_{x\to 2^+} e^{3/(2-x)} = \lim\limits_{t\to-\infty} e^t = 0$ by (6).

30. Let $t = 3/(2-x)$. As $x \to 2^-$, $t \to \infty$. So $\lim\limits_{x\to 2^-} e^{3/(2-x)} = \lim\limits_{t\to\infty} e^t = \infty$ by (6).

31. By the Product Rule, $f(x) = x^2 e^x \Rightarrow f'(x) = x^2 \dfrac{d}{dx}(e^x) + e^x \dfrac{d}{dx}(x^2) = x^2 e^x + e^x(2x) = xe^x(x+2)$.

32. By the Quotient Rule, $y = \dfrac{e^x}{1+x} \Rightarrow y' = \dfrac{(1+x)\,e^x - e^x(1)}{(1+x)^2} = \dfrac{e^x + xe^x - e^x}{(x+1)^2} = \dfrac{xe^x}{(x+1)^2}$.

33. By (9), $y = e^{ax^3} \Rightarrow y' = e^{ax^3} \dfrac{d}{dx}(ax^3) = 3ax^2 e^{ax^3}$.

34. $y = e^u(\cos u + cu) \Rightarrow y' = e^u(-\sin u + c) + (\cos u + cu)e^u = e^u(\cos u - \sin u + cu + c)$

35. $f(u) = e^{1/u} \Rightarrow f'(u) = e^{1/u} \cdot \dfrac{d}{du}\left(\dfrac{1}{u}\right) = e^{1/u}\left(\dfrac{-1}{u^2}\right) = \left(\dfrac{-1}{u^2}\right) e^{1/u}$

36. $y = e^x \ln x \Rightarrow y' = e^x\left(\dfrac{1}{x}\right) + (\ln x)(e^x) = e^x\left(\ln x + \dfrac{1}{x}\right)$

37. By (9), $F(t) = e^{t\sin 2t} \Rightarrow$
$F'(t) = e^{t\sin 2t}(t\sin 2t)' = e^{t\sin 2t}(t \cdot 2\cos 2t + \sin 2t \cdot 1) = e^{t\sin 2t}(2t\cos 2t + \sin 2t)$

38. $y = e^{k\tan\sqrt{x}} \Rightarrow y' = e^{k\tan\sqrt{x}} \cdot \dfrac{d}{dx}(k\tan\sqrt{x}) = e^{k\tan\sqrt{x}}\left(k\sec^2\sqrt{x} \cdot \tfrac{1}{2}x^{-1/2}\right) = \dfrac{k\sec^2\sqrt{x}}{2\sqrt{x}}\, e^{k\tan\sqrt{x}}$

39. $y = \sqrt{1+2e^{3x}} \Rightarrow y' = \dfrac{1}{2}(1+2e^{3x})^{-1/2} \dfrac{d}{dx}(1+2e^{3x}) = \dfrac{1}{2\sqrt{1+2e^{3x}}}(2e^{3x} \cdot 3) = \dfrac{3e^{3x}}{\sqrt{1+2e^{3x}}}$

40. $y = \cos(e^{\pi x}) \Rightarrow y' = -\sin(e^{\pi x}) \cdot e^{\pi x} \cdot \pi = -\pi e^{\pi x}\sin(e^{\pi x})$

41. $y = e^{e^x} \Rightarrow y' = e^{e^x} \cdot \dfrac{d}{dx}(e^x) = e^{e^x} \cdot e^x$ or $e^{e^x + x}$

42. $y = \sqrt{1 + xe^{-2x}} \Rightarrow y' = \frac{1}{2}\left(1 + xe^{-2x}\right)^{-1/2}\left[x(-2e^{-2x}) + e^{-2x}\right] = \dfrac{e^{-2x}(-2x+1)}{2\sqrt{1+xe^{-2x}}}$

43. By the Quotient Rule, $y = \dfrac{ae^x + b}{ce^x + d} \Rightarrow$
$y' = \dfrac{(ce^x + d)(ae^x) - (ae^x + b)(ce^x)}{(ce^x + d)^2} = \dfrac{(ace^x + ad - ace^x - bc)e^x}{(ce^x + d)^2} = \dfrac{(ad - bc)e^x}{(ce^x + d)^2}$.

44. $y = \dfrac{e^x + e^{-x}}{e^x - e^{-x}} \Rightarrow y' = \dfrac{(e^x - e^{-x})(e^x - e^{-x}) - (e^x + e^{-x})(e^x + e^{-x})}{(e^x - e^{-x})^2}$

$$= \frac{(e^{2x} - 2 + e^{-2x}) - (e^{2x} + 2 + e^{-2x})}{(e^x - e^{-x})^2} = -\frac{4}{(e^x - e^{-x})^2}$$

45. $y = e^{2x}\cos \pi x \;\Rightarrow\; y' = e^{2x}(-\pi \sin \pi x) + (\cos \pi x)(2e^{2x}) = e^{2x}(2\cos \pi x - \pi \sin \pi x)$.
At $(0, 1)$, $y' = 1(2 - 0) = 2$, so an equation of the tangent line is $y - 1 = 2(x - 0)$, or $y = 2x + 1$.

46. $y = \dfrac{e^x}{x} \;\Rightarrow\; y' = \dfrac{x \cdot e^x - e^x \cdot 1}{x^2} = \dfrac{e^x(x-1)}{x^2}$. At $(1, e)$, $y' = 0$, and an equation of the tangent line is $y - e = 0(x - 1)$, or $y = e$.

47. $\dfrac{d}{dx}\left(e^{x^2 y}\right) = \dfrac{d}{dx}(x + y) \;\Rightarrow\; e^{x^2 y}(x^2 y' + y \cdot 2x) = 1 + y' \;\Rightarrow\; x^2 e^{x^2 y} y' + 2xye^{x^2 y} = 1 + y' \;\Rightarrow$
$x^2 e^{x^2 y} y' - y' = 1 - 2xye^{x^2 y} \;\Rightarrow\; y'(x^2 e^{x^2 y} - 1) = 1 - 2xye^{x^2 y} \;\Rightarrow\; y' = \dfrac{1 - 2xye^{x^2 y}}{x^2 e^{x^2 y} - 1}$

48. $y = Ae^{-x} + Bxe^{-x} \;\Rightarrow\; y' = -Ae^{-x} + Be^{-x} - Bxe^{-x} = (B - A)e^{-x} - Bxe^{-x} \;\Rightarrow$
$y'' = (A - B)e^{-x} - Be^{-x} + Bxe^{-x} = (A - 2B)e^{-x} + Bxe^{-x}$, so
$y'' + 2y' + y = (A - 2B)e^{-x} + Bxe^{-x} + 2\left[(B - A)e^{-x} - Bxe^{-x}\right] + Ae^{-x} + Bxe^{-x} = 0$.

49. $y = e^{rx} \;\Rightarrow\; y' = re^{rx} \;\Rightarrow\; y'' = r^2 e^{rx}$, so if $y = e^{rx}$ satisfies the differential equation $y'' + 6y' + 8y = 0$, then $r^2 e^{rx} + 6re^{rx} + 8e^{rx} = 0$; that is, $e^{rx}(r^2 + 6r + 8) = 0$. Since $e^{rx} > 0$ for all x, we must have $r^2 + 6r + 8 = 0$, or $(r + 2)(r + 4) = 0$, so $r = -2$ or -4.

50. $y = e^{\lambda x} \;\Rightarrow\; y' = \lambda e^{\lambda x} \;\Rightarrow\; y'' = \lambda^2 e^{\lambda x}$. Thus, $y + y' = y'' \;\Leftrightarrow\; e^{\lambda x} + \lambda e^{\lambda x} = \lambda^2 e^{\lambda x} \;\Leftrightarrow$
$e^{\lambda x}(\lambda^2 - \lambda - 1) = 0 \;\Leftrightarrow\; \lambda = \frac{1 \pm \sqrt{5}}{2}$, since $e^{\lambda x} \neq 0$.

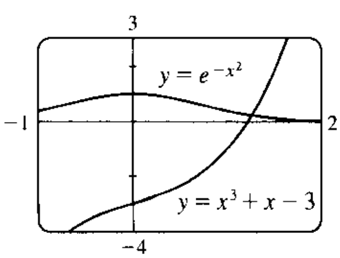
51. $f(x) = e^{2x} \;\Rightarrow\; f'(x) = 2e^{2x} \;\Rightarrow\; f''(x) = 2 \cdot 2e^{2x} = 2^2 e^{2x} \;\Rightarrow$
$f'''(x) = 2^2 \cdot 2e^{2x} = 2^3 e^{2x} \;\Rightarrow\; \cdots \;\Rightarrow\; f^{(n)}(x) = 2^n e^{2x}$

52. $f(x) = xe^{-x} \;\Rightarrow\; f'(x) = x(-e^{-x}) + e^{-x} = (1 - x)e^{-x} \;\Rightarrow$
$f''(x) = (1 - x)(-e^{-x}) + e^{-x}(-1) = (x - 2)e^{-x} \;\Rightarrow\; f'''(x) = (x - 2)(-e^{-x}) + e^{-x} = (3 - x)e^{-x} \;\Rightarrow$
$f^{(4)}(x) = (3 - x)(-e^{-x}) + e^{-x}(-1) = (x - 4)e^{-x} \;\Rightarrow\; \cdots \;\Rightarrow\; f^{(n)}(x) = (-1)^n(x - n)e^{-x}$.
So $D^{1000}xe^{-x} = (x - 1000)e^{-x}$.

53. (a) $f(x) = e^x + x$ is continuous on $\mathbb{R}$ and $f(-1) = e^{-1} - 1 < 0 < 1 = f(0)$, so by the Intermediate Value Theorem, $e^x + x = 0$ has a root in $(-1, 0)$.

(b) $f(x) = e^x + x \;\Rightarrow\; f'(x) = e^x + 1$, so $x_{n+1} = x_n - \dfrac{e^{x_n} + x_n}{e^{x_n} + 1}$. Using $x_1 = -0.5$, we get $x_2 \approx -0.566311$, $x_3 \approx -0.567143 \approx x_4$, so the root is -0.567143 to six decimal places.

54.



From the graph, it appears that the curves intersect at about $x \approx 1.2$ or 1.3. We use Newton's Method with $f(x) = x^3 + x - 3 - e^{-x^2}$, so $f'(x) = 3x^2 + 1 + 2xe^{-x^2}$, and the formula is $x_{n+1} = x_n - f(x_n)/f'(x_n)$. We take $x_1 = 1.2$, and the formula gives $x_2 \approx 1.252462$, $x_3 \approx 1.251045$, and $x_4 \approx x_5 \approx 1.251044$. So the root of the equation, correct to six decimal places, is $x = 1.251044$.

55. (a) $m(t) = 24 \cdot e^{-(\ln 2)t/25} = 24 \cdot 2^{-t/25} \quad \Rightarrow \quad m(40) = 24 \cdot 2^{-40/25} \approx 7.92$ mg.

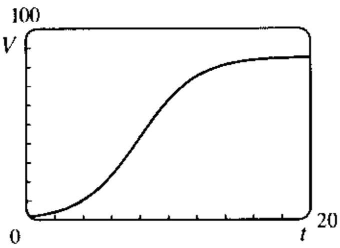
(b) $m'(t) = 24 \frac{d}{dt}\left[e^{-(\ln 2)t/25}\right] = 24 \cdot e^{-(\ln 2)t/25}\left(-\frac{\ln 2}{25}\right)$, so

$$m'(40) = 24e^{-(\ln 2)(40)/25}\left(-\tfrac{\ln 2}{25}\right) \approx -0.22 \text{ mg/yr}$$

(c) $m(t) = 5 \quad \Rightarrow \quad 24e^{-(\ln 2)t/25} = 5 \quad \Rightarrow \quad e^{-(\ln 2)t/25} = \frac{5}{24} \quad \Rightarrow \quad -(\ln 2)t/25 = \ln \frac{5}{24} \quad \Rightarrow$

$$t = -25\,\frac{\ln \frac{5}{24}}{\ln 2} \approx 56.6 \text{ yr}$$

56.



From the graph, we estimate that the most rapid increase in the percentage of households in the United States with at least one VCR occurs at about $t = 8$. To maximize the first derivative, we need to determine the values for which the second derivative is 0. We'll use $V(t) = \dfrac{a}{1 + be^{ct}}$, and substitute $a = 85$, $b = 53$, and $c = -0.5$ later.

$$V'(t) = -\frac{a(bce^{ct})}{(1 + be^{ct})^2} \quad \text{[by the Reciprocal Rule] and}$$

$$V''(t) = -abc \cdot \frac{(1 + be^{ct})^2 \cdot ce^{ct} - e^{ct} \cdot 2(1 + be^{ct}) \cdot bce^{ct}}{\left[(1 + be^{ct})^2\right]^2}$$

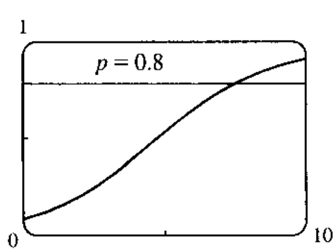
$$= \frac{-abc \cdot ce^{ct}(1 + be^{ct})[(1 + be^{ct}) - 2be^{ct}]}{(1 + be^{ct})^4} = \frac{-abc^2e^{ct}(1 - be^{ct})}{(1 + be^{ct})^3}$$

So $V''(t) = 0 \quad \Leftrightarrow \quad 1 = be^{ct} \quad \Leftrightarrow \quad e^{ct} = 1/b$. Now graph $y = e^{-0.5t}$ and $y = \frac{1}{53}$. These graphs intersect at $t \approx 7.94$ years, which corresponds to roughly midyear 1988. [Alternatively, we could use the rootfinder on a calculator to solve $e^{-0.5t} = \frac{1}{53}$. Or, if you have already studied logarithms, you can solve $e^{ct} = 1/b$ as follows: $ct = \ln(1/b) \quad \Leftrightarrow \quad t = (1/c)\ln(1/b) = -2\ln \frac{1}{53} \approx 7.94$ years.

57. (a) $\lim\limits_{t\to\infty} p(t) = \lim\limits_{t\to\infty} \dfrac{1}{1 + ae^{-kt}} = \dfrac{1}{1 + a \cdot 0} = 1$, since $k > 0 \quad \Rightarrow \quad -kt \to -\infty \quad \Rightarrow \quad e^{-kt} \to 0$.

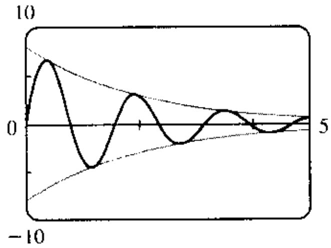
(b) $p(t) = \left(1 + ae^{-kt}\right)^{-1} \quad \Rightarrow \quad \dfrac{dp}{dt} = -\left(1 + ae^{-kt}\right)^{-2}\left(-kae^{-kt}\right) = \dfrac{kae^{-kt}}{(1 + ae^{-kt})^2}$

(c)



From the graph of $p(t) = \left(1 + 10e^{-0.5t}\right)^{-1}$, it seems that $p(t) = 0.8$ (indicating that 80% of the population has heard the rumor) when $t \approx 7.4$ hours.

58. (a)



The displacement function is squeezed between the other two functions. This is because $-1 \le \sin 4t \le 1 \quad \Rightarrow$ $-8e^{-t/2} \le 8e^{-t/2} \sin 4t \le 8e^{-t/2}$.

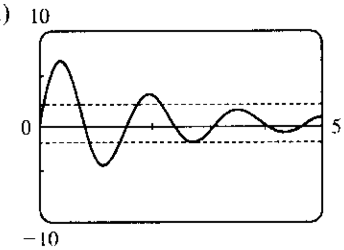
(b) The maximum value of the displacement is about 6.6 cm, occurring at $t \approx 0.36$ s. It occurs just before the graph of the displacement function touches the graph of $8e^{-t/2}$ (when $t = \frac{\pi}{8} \approx 0.39$).

(c) The velocity of the object is the derivative of its displacement function, that is,

$$\frac{d}{dt}\left(8e^{-t/2}\sin 4t\right) = 8\left[e^{-t/2}\cos 4t(4) + \sin 4t\left(-\tfrac{1}{2}\right)e^{-t/2}\right].$$

If the displacement is zero, then we must have $\sin 4t = 0$ (since the exponential term in the displacement function is always positive). The first time that $\sin 4t = 0$ after $t = 0$ occurs at $t = \frac{\pi}{4}$. Substituting this into our expression for the velocity, and noting that the second term vanishes, we get $v\left(\frac{\pi}{4}\right) = 8e^{-\pi/8}\cos\left(4 \cdot \frac{\pi}{4}\right) \cdot 4 = -32e^{-\pi/8} \approx -21.6$ cm/s.

(d)



The graph indicates that the displacement is less than 2 cm from equilibrium whenever t is larger than about 2.8.

59. $f(x) = x - e^x \quad \Rightarrow \quad f'(x) = 1 - e^x = 0 \quad \Leftrightarrow \quad e^x = 1 \quad \Leftrightarrow \quad x = 0$. Now $f'(x) > 0$ for all $x < 0$ and $f'(x) < 0$ for all $x > 0$, so the absolute maximum value is $f(0) = 0 - 1 = -1$.

60. $g(x) = \dfrac{e^x}{x} \quad \Rightarrow \quad g'(x) = \dfrac{xe^x - e^x}{x^2} = 0 \quad \Leftrightarrow \quad e^x(x-1) = 0 \quad \Rightarrow \quad x = 1$. Now $g'(x) > 0 \quad \Leftrightarrow$ $\dfrac{xe^x - e^x}{x^2} > 0 \quad \Leftrightarrow \quad x - 1 > 0 \quad \Leftrightarrow \quad x > 1$ and $g'(x) < 0 \quad \Leftrightarrow \quad \dfrac{xe^x - e^x}{x^2} < 0 \quad \Leftrightarrow \quad x - 1 < 0 \quad \Leftrightarrow$ $x < 1$. Thus there is an absolute minimum value of $g(1) = e$ at $x = 1$.

61. $y = xe^{3x} \quad \Rightarrow \quad y' = xe^{3x} \cdot 3 + e^{3x} \cdot 1 = (3x+1)e^{3x} \quad \Rightarrow \quad y'' = (3x+1)e^{3x} \cdot 3 + e^{3x} \cdot 3 = (9x+6)e^{3x}$. The curve is concave upward at $x \quad \Leftrightarrow \quad y'' > 0$ at $x \quad \Leftrightarrow \quad 9x + 6 > 0 \quad \Leftrightarrow \quad x > -\frac{2}{3}$. Thus, the curve is concave upward on $\left(-\frac{2}{3}, \infty\right)$.

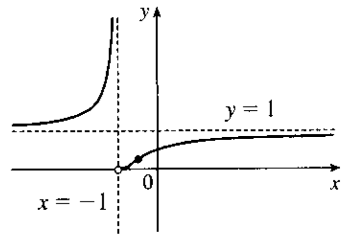
62. $f(x) = x^2e^{-x} \quad \Rightarrow \quad f'(x) = x^2(-e^{-x}) + e^{-x} \cdot 2x = (2x - x^2)e^{-x}$, so $f'(x) > 0 \quad \Leftrightarrow \quad 2x - x^2 > 0 \quad \Leftrightarrow$ $x(2-x) > 0 \quad \Leftrightarrow \quad 0 < x < 2$, so f is increasing on $(0, 2)$.

63. $y = f(x) = e^{-1/(x+1)}$ **A.** $D = \{x \mid x \neq -1\} = (-\infty, -1) \cup (-1, \infty)$ **B.** No x-intercept; y-intercept $= f(0) = e^{-1}$ **C.** No symmetry **D.** $\lim\limits_{x \to \pm\infty} e^{-1/(x+1)} = 1$ since $-1/(x+1) \to 0$, so $y = 1$ is a HA. $\lim\limits_{x \to -1^+} e^{-1/(x+1)} = 0$ since $-1/(x+1) \to -\infty$, $\lim\limits_{x \to -1^-} e^{-1/(x+1)} = \infty$ since $-1/(x+1) \to \infty$, so $x = -1$ is a VA. **E.** $f'(x) = e^{-1/(x+1)}/(x+1)^2 \Rightarrow f'(x) > 0$ for all x except 1, so f is increasing on $(-\infty, -1)$ and $(-1, \infty)$. **F.** No extreme values

G. $f''(x) = \dfrac{e^{-1/(x+1)}}{(x+1)^4} + \dfrac{e^{-1/(x+1)}(-2)}{(x+1)^3} = -\dfrac{e^{-1/(x+1)}(2x+1)}{(x+1)^4}$

$\Rightarrow f''(x) > 0 \Leftrightarrow 2x + 1 < 0 \Leftrightarrow x < -\frac{1}{2}$, so f is CU on $(-\infty, -1)$ and $\left(-1, -\frac{1}{2}\right)$, and CD on $\left(-\frac{1}{2}, \infty\right)$. f has an IP at $\left(-\frac{1}{2}, e^{-2}\right)$.

H.



64. $y = f(x) = e^{2x} - e^x$ **A.** $D = \mathbb{R}$ **B.** y-intercept: $f(0) = 0$; x-intercepts: $f(x) = 0 \Rightarrow e^{2x} = e^x \Rightarrow e^x = 1 \Rightarrow x = 0$.

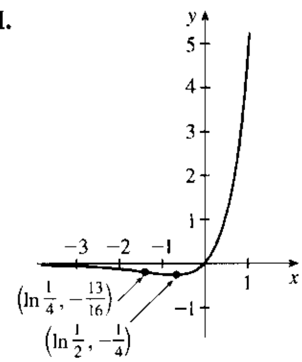
C. No symmetry **D.** $\lim\limits_{x \to -\infty} e^{2x} - e^x = 0$, so $y = 0$ is a HA. No VA.

E. $f'(x) = 2e^{2x} - e^x = e^x(2e^x - 1)$, so $f'(x) > 0 \Leftrightarrow e^x > \frac{1}{2} \Leftrightarrow x > \ln\frac{1}{2} = -\ln 2$ and $f'(x) < 0 \Leftrightarrow e^x < \frac{1}{2} \Leftrightarrow x < \ln\frac{1}{2}$, so f is decreasing on $\left(-\infty, \ln\frac{1}{2}\right)$ and increasing on $\left(\ln\frac{1}{2}, \infty\right)$. **F.** Local minimum value $f\left(\ln\frac{1}{2}\right) = e^{2\ln(1/2)} - e^{\ln(1/2)} = \left(\frac{1}{2}\right)^2 - \frac{1}{2} = -\frac{1}{4}$

G. $f''(x) = 4e^{2x} - e^x = e^x(4e^x - 1)$, so $f''(x) > 0 \Leftrightarrow e^x > \frac{1}{4} \Leftrightarrow x > \ln\frac{1}{4}$ and $f''(x) < 0 \Leftrightarrow x < \ln\frac{1}{4}$.

Thus, f is CD on $\left(-\infty, \ln\frac{1}{4}\right)$ and CU on $\left(\ln\frac{1}{4}, \infty\right)$. f has an IP at $\left(\ln\frac{1}{4}, \left(\frac{1}{4}\right)^2 - \frac{1}{4}\right) = \left(\ln\frac{1}{4}, -\frac{3}{16}\right)$.

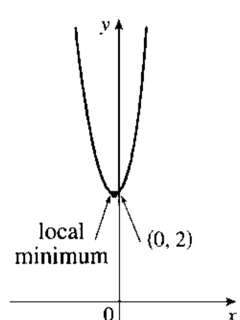
H.



65. $y = f(x) = e^{3x} + e^{-2x}$ **A.** $D = \mathbb{R}$ **B.** y-intercept $= f(0) = 2$; no x-intercept **C.** No symmetry **D.** No asymptotes

E. $f'(x) = 3e^{3x} - 2e^{-2x}$, so $f'(x) > 0 \Leftrightarrow 3e^{3x} > 2e^{-2x}$ [multiply by e^{2x}] $\Leftrightarrow e^{5x} > \frac{2}{3} \Leftrightarrow 5x > \ln\frac{2}{3} \Leftrightarrow x > \frac{1}{5}\ln\frac{2}{3} \approx -0.081$. Similarly, $f'(x) < 0 \Leftrightarrow x < \frac{1}{5}\ln\frac{2}{3}$. f is decreasing on $\left(-\infty, \frac{1}{5}\ln\frac{2}{3}\right)$ and increasing on $\left(\frac{1}{5}\ln\frac{2}{3}, \infty\right)$.

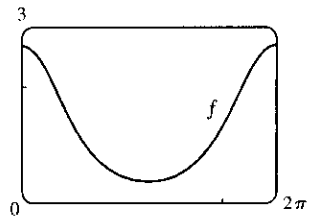
H.

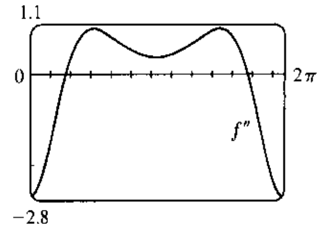


F. Local minimum value $f\left(\frac{1}{5}\ln\frac{2}{3}\right) = \left(\frac{2}{3}\right)^{3/5} + \left(\frac{2}{3}\right)^{-2/5} \approx 1.96$; no local maximum.

G. $f''(x) = 9e^{3x} + 4e^{-2x}$, so $f''(x) > 0$ for all x, and f is CU on $(-\infty, \infty)$. No IP

66. The function $f(x) = e^{\cos x}$ is periodic with period 2π, so we consider it only on the interval $[0, 2\pi]$. We see that it has local maxima of about $f(0) \approx 2.72$ and $f(2\pi) \approx 2.72$, and a local minimum of about $f(3.14) \approx 0.37$. To find the exact values, we calculate $f'(x) = -\sin x e^{\cos x}$. This is 0 when $-\sin x = 0 \iff x = 0, \pi$ or 2π (since we are only considering $x \in [0, 2\pi]$). Also $f'(x) > 0 \iff \sin x < 0 \iff 0 < x < \pi$. So $f(0) = f(2\pi) = e$ (both maxima) and $f(\pi) = e^{\cos \pi} = 1/e$ (minimum). To find the inflection points, we calculate and graph $f''(x) = \dfrac{d}{dx}\left(-\sin x\, e^{\cos x}\right) = -\cos x\, e^{\cos x} - \sin x\left(e^{\cos x}\right)(-\sin x) = e^{\cos x}\left(\sin^2 x - \cos x\right)$. From the graph of $f''(x)$, we see that f has inflection points at $x \approx 0.90$ and at $x \approx 5.38$. These x-coordinates correspond to inflection points $(0.90, 1.86)$ and $(5.38, 1.86)$.

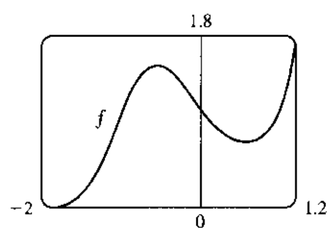


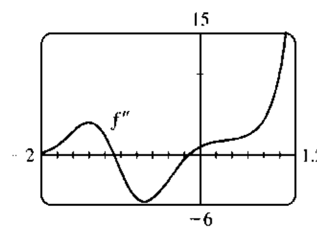


67. $f(x) = e^{x^3 - x} \to 0$ as $x \to -\infty$, and $f(x) \to \infty$ as $x \to \infty$. From the graph, it appears that f has a local minimum of about $f(0.58) = 0.68$, and a local maximum of about $f(-0.58) = 1.47$. To find the exact values, we calculate $f'(x) = (3x^2 - 1)e^{x^3 - x}$, which is 0 when $3x^2 - 1 = 0 \iff x = \pm\frac{1}{\sqrt{3}}$. The negative root corresponds to the local maximum $f\left(-\frac{1}{\sqrt{3}}\right) = e^{(-1/\sqrt{3})^3 - (-1/\sqrt{3})} = e^{2\sqrt{3}/9}$, and the positive root corresponds to the local minimum $f\left(\frac{1}{\sqrt{3}}\right) = e^{(1/\sqrt{3})^3 - (1/\sqrt{3})} = e^{-2\sqrt{3}/9}$. To estimate the inflection points, we calculate and graph

$$f''(x) = \frac{d}{dx}\left[(3x^2 - 1)e^{x^3 - x}\right] = (3x^2 - 1)e^{x^3 - x}(3x^2 - 1) + e^{x^3 - x}(6x) = e^{x^3 - x}\left(9x^4 - 6x^2 + 6x + 1\right).$$

From the graph, it appears that $f''(x)$ changes sign (and thus f has inflection points) at $x \approx -0.15$ and $x \approx -1.09$. From the graph of f, we see that these x-values correspond to inflection points at about $(-0.15, 1.15)$ and $(-1.09, 0.82)$.





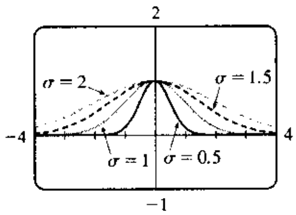
68. (a) As $|x| \to \infty$, $t = -x^2/(2\sigma^2) \to -\infty$, and $e^t \to 0$. The HA is $y = 0$. Since t takes on its maximum value at $x = 0$, so does e^t. Showing this result using derivatives, we have $f(x) = e^{-x^2/(2\sigma^2)} \Rightarrow$ $f'(x) = e^{-x^2/(2\sigma^2)}(-x/\sigma^2)$. $f'(x) = 0 \iff x = 0$. Because f' changes from positive to negative at $x = 0$, $f(0) = 1$ is a local maximum. For inflection points, we find

$f''(x) = -\frac{1}{\sigma^2}\left[e^{-x^2/(2\sigma^2)} \cdot 1 + xe^{-x^2/(2\sigma^2)}(-x/\sigma^2)\right] = \frac{-1}{\sigma^2}e^{-x^2/(2\sigma^2)}(1 - x^2/\sigma^2).$

$f''(x) = 0 \Leftrightarrow x^2 = \sigma^2 \Leftrightarrow x = \pm\sigma$. $f''(x) < 0 \Leftrightarrow x^2 < \sigma^2 \Leftrightarrow -\sigma < x < \sigma$. So f is CD on $(-\sigma, \sigma)$ and CU on $(-\infty, -\sigma)$ and (σ, ∞). IP at $\left(\pm\sigma, e^{-1/2}\right)$.

(b) Since we have IP at $x = \pm\sigma$, the inflection points move away from the y-axis as σ increases.

(c)



From the graph, we see that as σ increases, the graph tends to spread out and there is more area between the curve and the x-axis.

69. Let $u = -3x$. Then $du = -3\,dx$, so

$\int_0^5 e^{-3x}\,dx = -\frac{1}{3}\int_0^{-15} e^u\,du = -\frac{1}{3}[e^u]_0^{-15} = -\frac{1}{3}(e^{-15} - e^0) = \frac{1}{3}(1 - e^{-15})$.

70. Let $u = -x^2$, so $du = -2x\,dx$. When $x = 0$, $u = 0$; when $x = 1$, $u = -1$. Thus,

$\int_0^1 xe^{-x^2}\,dx = \int_0^{-1} e^u\left(-\frac{1}{2}\,du\right) = -\frac{1}{2}\left[e^u\right]_0^{-1} = -\frac{1}{2}(e^{-1} - e^0) = \frac{1}{2}(1 - 1/e)$.

71. Let $u = 1 + e^x$. Then $du = e^x\,dx$, so $\int e^x\sqrt{1 + e^x}\,dx = \int \sqrt{u}\,du = \frac{2}{3}u^{3/2} + C = \frac{2}{3}(1 + e^x)^{3/2} + C$.

72. Let $u = \tan x$. Then $du = \sec^2 x\,dx$, so $\int \sec^2 x\, e^{\tan x}\,dx = \int e^u\,du = e^u + C = e^{\tan x} + C$.

73. $\displaystyle\int \frac{e^x + 1}{e^x}\,dx = \int (1 + e^{-x})\,dx = x - e^{-x} + C$

74. Let $u = \dfrac{1}{x}$. Then $du = -\dfrac{1}{x^2}\,dx$, so $\displaystyle\int \frac{e^{1/x}}{x^2}\,dx = -\int e^u\,du = -e^u + C = -e^{1/x} + C$.

75. Let $u = \sqrt{x}$. Then $du = \dfrac{1}{2\sqrt{x}}\,dx$, so $\displaystyle\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx = 2\int e^u\,du = 2e^u + C = 2e^{\sqrt{x}} + C$.

76. Let $u = e^x$. Then $du = e^x\,dx$, so $\int e^x \sin(e^x)\,dx = \int \sin u\,du = -\cos u + C = -\cos(e^x) + C$.

77. Area $= \int_0^1 (e^{3x} - e^x)\,dx = \left[\frac{1}{3}e^{3x} - e^x\right]_0^1 = \left(\frac{1}{3}e^3 - e\right) - \left(\frac{1}{3} - 1\right) = \frac{1}{3}e^3 - e + \frac{2}{3} \approx 4.644$

78. $f''(x) = 3e^x + 5\sin x \Rightarrow f'(x) = 3e^x - 5\cos x + C \Rightarrow 2 = f'(0) = 3 - 5 + C \Rightarrow C = 4$, so $f'(x) = 3e^x - 5\cos x + 4 \Rightarrow f(x) = 3e^x - 5\sin x + 4x + D \Rightarrow 1 = f(0) = 3 + D \Rightarrow D = -2$, so $f(x) = 3e^x - 5\sin x + 4x - 2$.

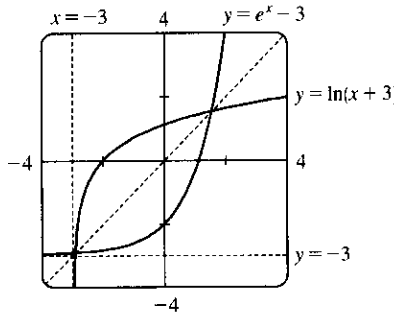
79. $V = \int_0^1 \pi(e^x)^2\,dx = \pi\int_0^1 e^{2x}\,dx = \frac{1}{2}\pi\left[e^{2x}\right]_0^1 = \frac{\pi}{2}(e^2 - 1)$

80. $V = \int_0^1 2\pi xe^{-x^2}\,dx$. Let $u = x^2$. Thus $du = 2x\,dx$, so $V = \pi\int_0^1 e^{-u}\,du = \pi\left[-e^{-u}\right]_0^1 = \pi(1 - 1/e)$.

81. $y = \ln(x+3) \Rightarrow e^y = x+3 \Rightarrow$

$x = e^y - 3$. Interchanging x and y, we get

$y = e^x - 3$, so $f^{-1}(x) = e^x - 3$.

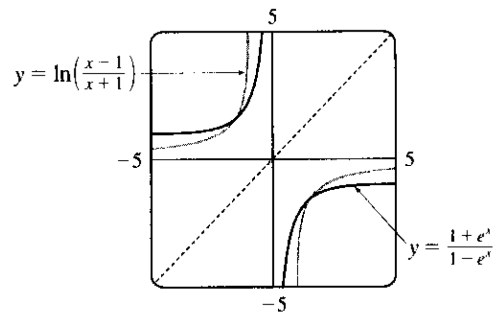


82. $y = \dfrac{1+e^x}{1-e^x} \Rightarrow y - ye^x = 1 + e^x \Rightarrow$

$e^x(y+1) = y - 1 \Rightarrow e^x = \dfrac{y-1}{y+1} \Rightarrow$

$x = \ln\left(\dfrac{y-1}{y+1}\right)$. Interchange x and y:

$y = \ln\left(\dfrac{x-1}{x+1}\right)$ is the inverse function.



83. We use Theorem 7.1.7. Note that $f(0) = 3 + 0 + e^0 = 4$, so $f^{-1}(4) = 0$. Also $f'(x) = 1 + e^x$. Therefore,

$$(f^{-1})'(4) = \frac{1}{f'(f^{-1}(4))} = \frac{1}{f'(0)} = \frac{1}{1+e^0} = \frac{1}{2}.$$

84. We recognize this limit as the definition of the derivative of the function $f(x) = e^{\sin x}$ at $x = \pi$, since it is of the form $\lim\limits_{x\to\pi} \dfrac{f(x) - f(\pi)}{x - \pi}$. Therefore, the limit is equal to $f'(\pi) = (\cos\pi)e^{\sin\pi} = -1 \cdot e^0 = -1$.

85. Using the second law of logarithms and Equation 5, we have $\ln(e^x/e^y) = \ln e^x - \ln e^y = x - y = \ln(e^{x-y})$. Since ln is a one-to-one function, it follows that $e^x/e^y = e^{x-y}$.

86. Using the third law of logarithms and Equation 5, we have $\ln e^{rx} = rx = r\ln e^x = \ln(e^x)^r$. Since ln is a one-to-one function, it follows that $e^{rx} = (e^x)^r$.

87. (a) Let $f(x) = e^x - 1 - x$. Now $f(0) = e^0 - 1 = 0$, and for $x \ge 0$, we have $f'(x) = e^x - 1 \ge 0$. Now, since $f(0) = 0$ and f is increasing on $[0, \infty)$, $f(x) \ge 0$ for $x \ge 0 \Rightarrow e^x - 1 - x \ge 0 \Rightarrow e^x \ge 1 + x$.

(b) For $0 \le x \le 1$, $x^2 \le x$, so $e^{x^2} \le e^x$ [since e^x is increasing]. Hence [from (a)] $1 + x^2 \le e^{x^2} \le e^x$. So $\frac{4}{3} = \int_0^1 (1 + x^2)\,dx \le \int_0^1 e^{x^2}\,dx \le \int_0^1 e^x\,dx = e - 1 < e \Rightarrow \frac{4}{3} \le \int_0^1 e^{x^2}\,dx \le e$.

88. (a) Let $f(x) = e^x - 1 - x - \frac{1}{2}x^2$. Thus, $f'(x) = e^x - 1 - x$, which is positive for $x \ge 0$ by Exercise 87(a). Thus $f(x)$ is increasing on $(0, \infty)$, so on that interval, $0 = f(0) \le f(x) = e^x - 1 - x - \frac{1}{2}x^2 \Rightarrow$ $e^x \ge 1 + x + \frac{1}{2}x^2$.

(b) Using the same argument as in Exercise 87(b), from part (a) we have $1 + x^2 + \frac{1}{2}x^4 \le e^{x^2} \le e^x$ [for $0 \le x \le 1$] $\Rightarrow \int_0^1 \left(1 + x^2 + \frac{1}{2}x^4\right) dx \le \int_0^1 e^{x^2}\,dx \le \int_0^1 e^x\,dx \Rightarrow \frac{43}{30} \le \int_0^1 e^{x^2}\,dx \le e - 1$.

89. (a) By Exercise 87(a), the result holds for $n = 1$. Suppose that $e^x \geq 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^k}{k!}$ for $x \geq 0$.

Let $f(x) = e^x - 1 - x - \frac{x^2}{2!} - \cdots - \frac{x^k}{k!} - \frac{x^{k+1}}{(k+1)!}$. Then $f'(x) = e^x - 1 - x - \cdots - \frac{x^k}{k!} \geq 0$

by assumption. Hence $f(x)$ is increasing on $(0, \infty)$. So $0 \leq x$ implies that

$0 = f(0) \leq f(x) = e^x - 1 - x - \cdots - \frac{x^k}{k!} - \frac{x^{k+1}}{(k+1)!}$, and hence $e^x \geq 1 + x + \cdots + \frac{x^k}{k!} + \frac{x^{k+1}}{(k+1)!}$

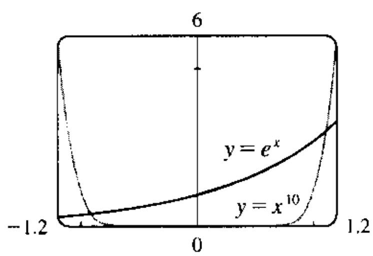
for $x \geq 0$. Therefore, for $x \geq 0$, $e^x \geq 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^n}{n!}$ for every positive integer n, by mathematical induction.

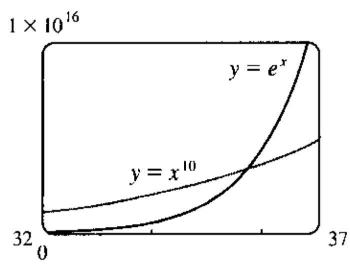
(b) Taking $n = 4$ and $x = 1$ in (a), we have $e = e^1 \geq 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} = 2.708\overline{3} > 2.7$.

(c) $e^x \geq 1 + x + \cdots + \frac{x^k}{k!} + \frac{x^{k+1}}{(k+1)!} \quad \Rightarrow \quad \frac{e^x}{x^k} \geq \frac{1}{x^k} + \frac{1}{x^{k-1}} + \cdots + \frac{1}{k!} + \frac{x}{(k+1)!} \geq \frac{x}{(k+1)!}$.

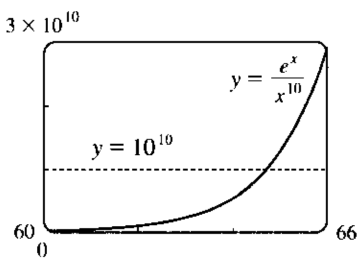
But $\lim\limits_{x \to \infty} \frac{x}{(k+1)!} = \infty$, so $\lim\limits_{x \to \infty} \frac{e^x}{x^k} = \infty$.

90. (a) The graph of g finally surpasses that of f at $x \approx 35.8$.





(b)



(c) From the graph in part (b), it seems that $e^x / x^{10} > 10^{10}$ whenever $x > 65$, approximately. So we can take $N \geq 65$.

7.4* General Logarithmic and Exponential Functions

1. (a) $a^x = e^{x \ln a}$

(b) The domain of $f(x) = a^x$ is $\mathbb{R}$.

(c) The range of $f(x) = a^x$ $(a \neq 1)$ is $(0, \infty)$.

(d) (i) See Figure 1. (ii) See Figure 3. (iii) See Figure 2.

2. (a) $\log_a x$ is the number y such that $a^y = x$.

(b) The domain of $f(x) = \log_a x$ is $(0, \infty)$.

(c) The range of $f(x) = \log_a x$ is $\mathbb{R}$.

(d) See Figure 9.

3. $5^{\sqrt{7}} = (e^{\ln 5})^{\sqrt{7}} = e^{\sqrt{7} \ln 5}$

4. $10^{x^2} = (e^{\ln 10})^{x^2} = e^{x^2 \ln 10}$

5. $(\cos x)^x = (e^{\ln \cos x})^x = e^{x \ln(\cos x)}$.

6. $x^{\cos x} = (e^{\ln x})^{\cos x} = e^{(\cos x)(\ln x)}$

7. (a) $\log_{10} 1000 = 3$ because $10^3 = 1000$.

(b) $\log_2 \frac{1}{16} = -4$ since $2^{-4} = \frac{1}{16}$. [*Or:* $\log_2 \frac{1}{16} = \log_2 2^{-4} = -4$]

8. (a) $\log_{10} 0.1 = -1$ since $10^{-1} = 0.1$.

(b) $\log_8 320 - \log_8 5 = \log_8 \frac{320}{5} = \log_8 64 = 2$ since $8^2 = 64$.

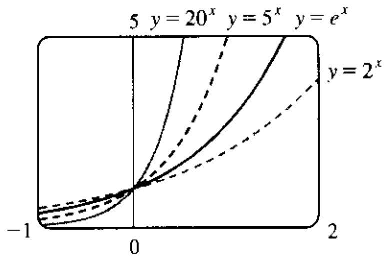
9. (a) $\log_{12} 3 + \log_{12} 48 = \log_{12}(3 \cdot 48) = \log_{12} 144 = 2$ since $12^2 = 144$.

(b) $\log_5 5^{\sqrt{2}} = \sqrt{2}$ by the cancellation property $\log_a a^x = x$.
[*Or:* $\log_5 5^{\sqrt{2}} = \sqrt{2} \log_5 5 = \sqrt{2} \cdot 1 = \sqrt{2}$]

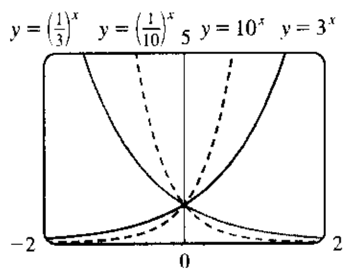
10. (a) $\log_a \dfrac{1}{a} = -1$ since $a^{-1} = \dfrac{1}{a}$. [*Or:* $\log_a \dfrac{1}{a} = \log_a a^{-1} = -1$]

(b) $10^{(\log_{10} 4 + \log_{10} 7)} = 10^{\log_{10} 4} \cdot 10^{\log_{10} 7} = 4 \cdot 7 = 28$
[*Or:* $10^{(\log_{10} 4 + \log_{10} 7)} = 10^{\log_{10}(4 \cdot 7)} = 10^{\log_{10} 28} = 28$]

11. All of these graphs approach 0 as $x \to -\infty$, all of them pass through the point $(0, 1)$, and all of them are increasing and approach ∞ as $x \to \infty$. The larger the base, the faster the function increases for $x > 0$, and the faster it approaches 0 as $x \to -\infty$.



12. The functions with bases greater than 1 (3^x and 10^x) are increasing, while those with bases less than 1 $\left[\left(\frac{1}{3}\right)^x \text{ and } \left(\frac{1}{10}\right)^x\right]$ are decreasing. The graph of $\left(\frac{1}{3}\right)^x$ is the reflection of that of 3^x about the y-axis, and the graph of $\left(\frac{1}{10}\right)^x$ is the reflection of that of 10^x about the y-axis. The graph of 10^x increases more quickly than that of 3^x for $x > 0$, and approaches 0 faster as $x \to -\infty$.

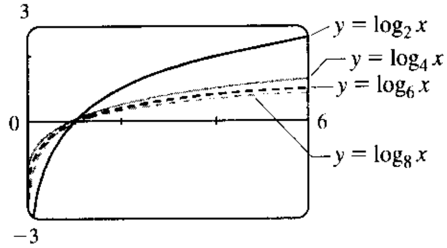


13. (a) $\log_{12} e = \dfrac{\ln e}{\ln 12} = \dfrac{1}{\ln 12} \approx 0.402430$

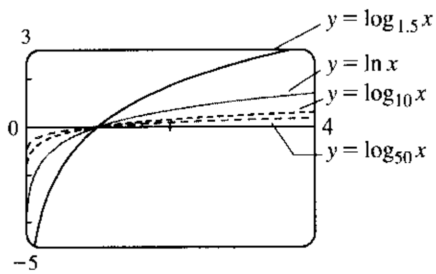
(b) $\log_6 13.54 = \dfrac{\ln 13.54}{\ln 6} \approx 1.454240$

(c) $\log_2 \pi = \dfrac{\ln \pi}{\ln 2} \approx 1.651496$

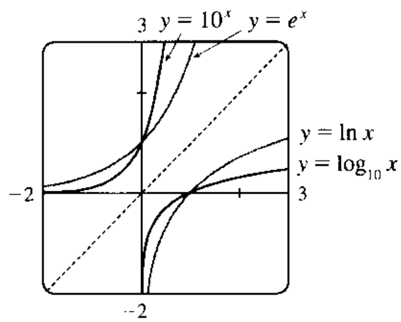
14. To graph the functions, we use $\log_2 x = \dfrac{\ln x}{\ln 2}$, $\log_4 x = \dfrac{\ln x}{\ln 4}$, etc. These graphs all approach $-\infty$ as $x \to 0^+$, and they all pass through the point $(1, 0)$. Also, they are all increasing, and all approach ∞ as $x \to \infty$. The smaller the base, the larger the rate of increase of the function (for $x > 1$) and the closer the approach to the y-axis (as $x \to 0^+$).



15. To graph these functions, we use $\log_{1.5} x = \dfrac{\ln x}{\ln 1.5}$ and $\log_{50} x = \dfrac{\ln x}{\ln 50}$. These graphs all approach $-\infty$ as $x \to 0^+$, and they all pass through the point $(1, 0)$. Also, they are all increasing, and all approach ∞ as $x \to \infty$. The functions with larger bases increase extremely slowly, and the ones with smaller bases do so somewhat more quickly. The functions with large bases approach the y-axis more closely as $x \to 0^+$.



16. We see that the graph of $\ln x$ is the reflection of the graph of e^x about the line $y = x$, and that the graph of $\log_{10} x$ is the reflection of the graph of 10^x about the same line. The graph of 10^x increases more quickly than that of e^x. Also note that $\log_{10} x \to \infty$ as $x \to \infty$ more slowly than $\ln x$.



17. Use $y = Ca^x$ with the points $(1, 6)$ and $(3, 24)$. $6 = Ca^1$ $\left[C = \frac{6}{a}\right]$ and $24 = Ca^3 \Rightarrow 24 = \left(\dfrac{6}{a}\right)a^3 \Rightarrow$ $4 = a^2 \Rightarrow a = 2$ [since $a > 0$] and $C = \frac{6}{2} = 3$. The function is $f(x) = 3 \cdot 2^x$.

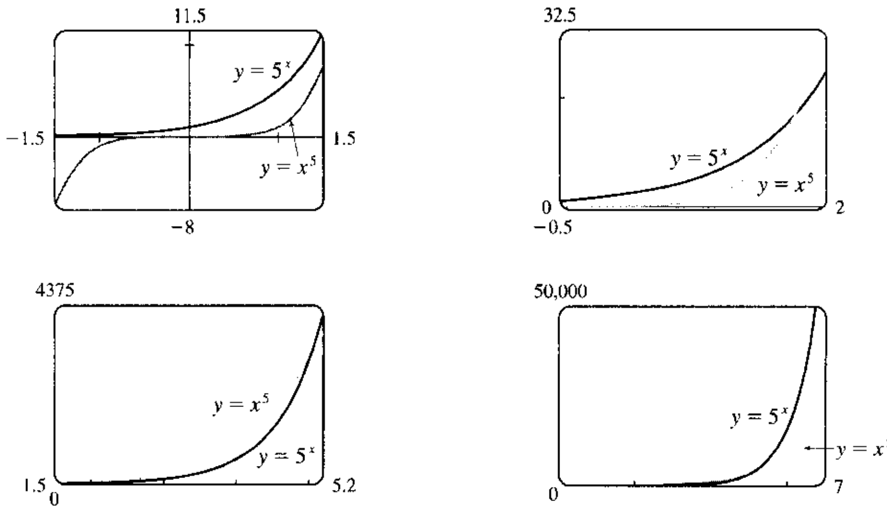
18. Given the y-intercept $(0, 2)$, we have $y = Ca^x = 2a^x$. Using the point $\left(2, \frac{2}{9}\right)$ gives us $\frac{2}{9} = 2a^2 \Rightarrow \frac{1}{9} = a^2$ $\Rightarrow a = \frac{1}{3}$ [since $a > 0$]. The function is $f(x) = 2\left(\frac{1}{3}\right)^x$ or $f(x) = 2(3)^{-x}$.

19. (a) 2 ft = 24 in, $f(24) = 24^2$ in $= 576$ in $= 48$ ft. $g(24) = 2^{24}$ in $= 2^{24}/(12 \cdot 5280)$ mi ≈ 265 mi

(b) 3 ft = 36 in, so we need x such that $\log_2 x = 36 \Leftrightarrow x = 2^{36} = 68{,}719{,}476{,}736$. In miles, this is

$$68{,}719{,}476{,}736 \text{ in} \cdot \frac{1 \text{ ft}}{12 \text{ in}} \cdot \frac{1 \text{ mi}}{5280 \text{ ft}} \approx 1{,}084{,}587.7 \text{ mi}.$$

20. We see from the graphs that for x less than about 1.8, $g(x) = 5^x > f(x) = x^5$, and then near the point $(1.8, 17.1)$ the curves intersect. Then $f(x) > g(x)$ from $x \approx 1.8$ until $x = 5$. At $(5, 3125)$ there is another point of intersection, and for $x > 5$ we see that $g(x) > f(x)$. In fact, g increases much more rapidly than f beyond that point.



21. $\lim\limits_{t \to \infty} 2^{-t^2} = \lim\limits_{u \to -\infty} 2^u \quad [\text{where } u = -t^2] \quad = 0$

22. Let $t = x^2 - 5x + 6$. As $x \to 3^+$, $t = (x-2)(x-3) \to 0^+$. $\lim\limits_{x \to 3^+} \log_{10}(x^2 - 5x + 6) = \lim\limits_{t \to 0^+} \log_{10} t = -\infty$ by (4) in Section 7.3 .

23. $h(t) = t^3 - 3^t \quad \Rightarrow \quad h'(t) = 3t^2 - 3^t \ln 3$

24. $g(x) = x^4 4^x \quad \Rightarrow \quad g'(x) = x^4 4^x \ln 4 + 4^x \cdot 4x^3 = x^3 4^x (x \ln 4 + 4)$

25. Using Formula 4 and the Chain Rule, $y = 5^{-1/x} \quad \Rightarrow \quad y' = 5^{-1/x}(\ln 5)\left[-1 \cdot (-x^{-2})\right] = 5^{-1/x}(\ln 5)/x^2$

26. $y = 10^{\tan \theta} \quad \Rightarrow \quad y' = 10^{\tan \theta}(\ln 10)(\sec^2 \theta)$

27. $f(u) = (2^u + 2^{-u})^{10} \quad \Rightarrow$

$$f'(u) = 10(2^u + 2^{-u})^9 \frac{d}{du}(2^u + 2^{-u}) = 10(2^u + 2^{-u})^9 \left[2^u \ln 2 + 2^{-u} \ln 2 \cdot (-1)\right]$$
$$= 10 \ln 2 (2^u + 2^{-u})^9 (2^u - 2^{-u})$$

28. $y = 2^{3^{x^2}} \quad \Rightarrow \quad y' = 2^{3^{x^2}}(\ln 2)\frac{d}{dx}\left(3^{x^2}\right) = 2^{3^{x^2}}(\ln 2) 3^{x^2}(\ln 3)(2x)$

29. $f(x) = \log_3(x^2 - 4) \quad \Rightarrow \quad f'(x) = \dfrac{1}{(x^2 - 4)\ln 3}(2x) = \dfrac{2x}{(x^2 - 4)\ln 3}$

30. $f(x) = \log_{10}\left(\dfrac{x}{x-1}\right) = \log_{10} x - \log_{10}(x-1) \quad \Rightarrow \quad f'(x) = \dfrac{1}{x \ln 10} - \dfrac{1}{(x-1)\ln 10}$ or $-\dfrac{1}{x(x-1)\ln 10}$

31. $y = x^x \quad \Rightarrow \quad \ln y = x \ln x \quad \Rightarrow \quad y'/y = \ln x + x(1/x) \quad \Rightarrow \quad y' = x^x(\ln x + 1)$

32. $y = x^{1/x} \quad \Rightarrow \quad \ln y = \dfrac{1}{x} \ln x \quad \Rightarrow \quad \dfrac{y'}{y} = -\dfrac{1}{x^2} \ln x + \dfrac{1}{x}\left(\dfrac{1}{x}\right) \quad \Rightarrow \quad y' = x^{1/x} \dfrac{1 - \ln x}{x^2}$

33. $y = x^{\sin x} \quad \Rightarrow \quad \ln y = \sin x \ln x \quad \Rightarrow \quad \dfrac{y'}{y} = \cos x \ln x + \dfrac{\sin x}{x} \quad \Rightarrow \quad y' = x^{\sin x}\left(\cos x \ln x + \dfrac{\sin x}{x}\right)$

34. $y = (\sin x)^x \quad \Rightarrow \quad \ln y = x \ln(\sin x) \quad \Rightarrow \quad y'/y = \ln(\sin x) + x(\cos x)/(\sin x) \quad \Rightarrow$
$y' = (\sin x)^x [\ln(\sin x) + x \cot x]$

35. $y = (\ln x)^x \quad \Rightarrow \quad \ln y = x \ln \ln x \quad \Rightarrow \quad \dfrac{y'}{y} = \ln \ln x + x \cdot \dfrac{1}{\ln x} \cdot \dfrac{1}{x} \quad \Rightarrow \quad y' = (\ln x)^x \left(\ln \ln x + \dfrac{1}{\ln x} \right)$

36. $y = x^{\ln x} \quad \Rightarrow \quad \ln y = \ln x \ln x = (\ln x)^2 \quad \Rightarrow \quad \dfrac{y'}{y} = 2 \ln x \left(\dfrac{1}{x} \right) \quad \Rightarrow \quad y' = x^{\ln x} \left(\dfrac{2 \ln x}{x} \right)$

37. $y = x^{e^x} \quad \Rightarrow \quad \ln y = e^x \ln x \quad \Rightarrow \quad \dfrac{y'}{y} = e^x \ln x + \dfrac{e^x}{x} \quad \Rightarrow \quad y' = x^{e^x} e^x \left(\ln x + \dfrac{1}{x} \right)$

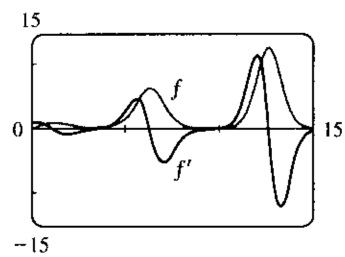
38. $y = (\ln x)^{\cos x} \quad \Rightarrow \quad \ln y = \cos x \ln(\ln x) \quad \Rightarrow \quad \dfrac{y'}{y} = \cos x \cdot \dfrac{1}{\ln x} \cdot \dfrac{1}{x} + (\ln \ln x)(-\sin x) \quad \Rightarrow$
$y' = (\ln x)^{\cos x} \left(\dfrac{\cos x}{x \ln x} - \sin x \ln \ln x \right)$

39. $y = 10^x \quad \Rightarrow \quad y' = 10^x \ln 10$, so at $(1, 10)$, the slope of the tangent line is $10^1 \ln 10 = 10 \ln 10$, and its equation is $y - 10 = 10 \ln 10(x - 1)$, or $y = (10 \ln 10)x + 10(1 - \ln 10)$.

40. $f(x) = x^{\cos x} = e^{\ln x \cos x} \quad \Rightarrow$

$$f'(x) = e^{\ln x \cos x} \left[\ln x(-\sin x) + \cos x \left(\frac{1}{x} \right) \right] = x^{\cos x} \left[\frac{\cos x}{x} - \sin x \ln x \right]$$

This is reasonable, because the graph shows that f increases when $f'(x)$ is positive.



41. $\displaystyle\int_1^2 10^t \, dt = \left[\frac{10^t}{\ln 10} \right]_1^2 = \frac{10^2}{\ln 10} - \frac{10^1}{\ln 10} = \frac{100 - 10}{\ln 10} = \frac{90}{\ln 10}$

42. Let $v = -2u$. Then $dv = -2\,du$ and

$$\int_0^1 4^{-2u} \, du = \int_0^{-2} 4^v \left(-\frac{1}{2} \right) dv = -\frac{1}{2} \left[\frac{4^v}{\ln 4} \right]_0^{-2} = -\frac{1}{2 \ln 4} \left(4^{-2} - 4^0 \right) = -\frac{1}{2 \ln 2^2} \left(\frac{1}{16} - 1 \right)$$
$$= -\frac{1}{4 \ln 2} \left(-\frac{15}{16} \right) = \frac{15}{64 \ln 2}$$

43. $\displaystyle\int \frac{\log_{10} x}{x} \, dx = \int \frac{(\ln x)/(\ln 10)}{x} \, dx = \frac{1}{\ln 10} \int \frac{\ln x}{x} \, dx$. Now put $u = \ln x$, so $du = \dfrac{1}{x} \, dx$, and the expression becomes $\displaystyle\frac{1}{\ln 10} \int u \, du = \frac{1}{\ln 10} \left(\tfrac{1}{2} u^2 + C_1 \right) = \frac{1}{2 \ln 10} (\ln x)^2 + C$.

Or: The substitution $u = \log_{10} x$ gives $du = \dfrac{dx}{x \ln 10}$ and we get $\displaystyle\int \frac{\log_{10} x}{x} \, dx = \tfrac{1}{2} \ln 10 (\log_{10} x)^2 + C$.

44. $\int (x^5 + 5^x) \, dx = \frac{1}{6} x^6 + \frac{1}{\ln 5} 5^x + C$

45. Let $u = \sin \theta$. Then $du = \cos \theta \, d\theta$ and $\displaystyle\int 3^{\sin \theta} \cos \theta \, d\theta = \int 3^u du = \frac{3^u}{\ln 3} + C = \frac{1}{\ln 3} 3^{\sin \theta} + C$.

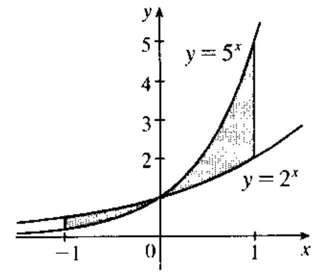
46. Let $u = 2^x + 1$. Then $du = 2^x \ln 2 \, dx$, so $\displaystyle\int \frac{2^x}{2^x + 1} \, dx = \int \frac{1}{u} \frac{du}{\ln 2} = \frac{1}{\ln 2} \ln |u| + C = \frac{1}{\ln 2} \ln(2^x + 1) + C$.

47. $A = \int_{-1}^{0} (2^x - 5^x)\,dx + \int_0^1 (5^x - 2^x)\,dx$

$= \left[\dfrac{2^x}{\ln 2} - \dfrac{5^x}{\ln 5}\right]_{-1}^{0} + \left[\dfrac{5^x}{\ln 5} - \dfrac{2^x}{\ln 2}\right]_0^1$

$= \left(\dfrac{1}{\ln 2} - \dfrac{1}{\ln 5}\right) - \left(\dfrac{1/2}{\ln 2} - \dfrac{1/5}{\ln 5}\right) + \left(\dfrac{5}{\ln 5} - \dfrac{2}{\ln 2}\right) - \left(\dfrac{1}{\ln 5} - \dfrac{1}{\ln 2}\right)$

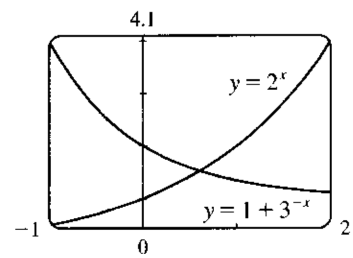
$= \dfrac{16}{5 \ln 5} - \dfrac{1}{2 \ln 2}$



48. Using disks, the volume is $V = \int_0^1 \pi \left[10^{-x}\right]^2 dx = \pi \int_0^1 10^{-2x}\,dx$. To evaluate the integral, we let $u = -2x \;\Rightarrow\; du = -2\,dx$, $x = 0 \;\Rightarrow\; u = 0$, and $x = 1 \;\Rightarrow\; u = -2$, so we have

$$V = -\frac{\pi}{2}\int_0^{-2} 10^u\,du = -\frac{\pi}{2}\left[\frac{1}{\ln 10}10^u\right]_0^{-2} = -\frac{\pi}{2\ln 10}\left(10^{-2} - 1\right) = \frac{99\pi}{200 \ln 10}$$

49. We see that the graphs of $y = 2^x$ and $y = 1 + 3^{-x}$ intersect at $x \approx 0.6$. We let $f(x) = 2^x - 1 - 3^{-x}$ and calculate $f'(x) = 2^x \ln 2 + 3^{-x} \ln 3$, and using the formula $x_{n+1} = x_n - f(x_n)/f'(x_n)$ (Newton's Method), we get $x_1 = 0.6$, $x_2 \approx x_3 \approx 0.600967$. So, correct to six decimal places, the root occurs at $x = 0.600967$.



50. $x^y = y^x \;\Rightarrow\; y \ln x = x \ln y \;\Rightarrow\; y \cdot \dfrac{1}{x} + (\ln x) \cdot y' = x \cdot \dfrac{1}{y} \cdot y' + \ln y \;\Rightarrow\; y' \ln x - \dfrac{x}{y} y' = \ln y - \dfrac{y}{x} \;\Rightarrow$

$y' = \dfrac{\ln y - y/x}{\ln x - x/y}$

51. $y = \dfrac{10^x}{10^x + 1} \;\Leftrightarrow\; (10^x + 1)y = 10^x \;\Leftrightarrow\; 10^x \cdot y + y = 10^x \;\Leftrightarrow\; y = 10^x - 10^x y \;\Leftrightarrow$

$y = 10^x(1 - y) \;\Leftrightarrow\; 10^x = \dfrac{y}{1-y} \;\Leftrightarrow\; \log_{10} 10^x = \log_{10}\left(\dfrac{y}{1-y}\right) \;\Leftrightarrow\; x = \log_{10} y - \log_{10}(1 - y)$.

Interchange x and y: $y = \log_{10} x - \log_{10}(1 - x)$ is the inverse function.

52. $\lim\limits_{x\to 0^+} x^{-\ln x} = \lim\limits_{x\to 0^+} \left(e^{\ln x}\right)^{-\ln x} = \lim\limits_{x\to 0^+} e^{-(\ln x)^2} = 0$ since $-(\ln x)^2 \to -\infty$ as $x \to 0^+$.

53. If I is the intensity of the 1989 San Francisco earthquake, then $\log_{10}(I/S) = 7.1 \;\Rightarrow$ $\log_{10}(16I/S) = \log_{10} 16 + \log_{10}(I/S) = \log_{10} 16 + 7.1 \approx 8.3$.

54. Let I_1 and I_2 be the intensities of the music and the mower. Then $10\log_{10}\left(\dfrac{I_1}{I_0}\right) = 120$ and $10\log_{10}\left(\dfrac{I_2}{I_0}\right) = 106$, so $\log_{10}\left(\dfrac{I_1}{I_2}\right) = \log_{10}\left(\dfrac{I_1/I_0}{I_2/I_0}\right) = \log_{10}\left(\dfrac{I_1}{I_0}\right) - \log_{10}\left(\dfrac{I_2}{I_0}\right) = 12 - 10.6 = 1.4 \;\Rightarrow\; \dfrac{I_1}{I_2} = 10^{1.4} \approx 25$.

55. We find I with the loudness formula from Exercise 55, substituting $I_0 = 10^{-12}$ and $L = 50$:

$50 = 10\log_{10}\dfrac{I}{10^{-12}} \;\Leftrightarrow\; 5 = \log_{10}\dfrac{I}{10^{-12}} \;\Leftrightarrow\; 10^5 = \dfrac{I}{10^{-12}} \;\Leftrightarrow\; I = 10^{-7}$ watt/m^2. Now we differentiate L with respect to I: $L = 10\log_{10}\dfrac{I}{I_0} \;\Rightarrow\; \dfrac{dL}{dI} = 10\dfrac{1}{(I/I_0)\ln 10}\left(\dfrac{1}{I_0}\right) = \dfrac{10}{\ln 10}\left(\dfrac{1}{I}\right)$.

Substituting $I = 10^{-7}$, we get $L'(50) = \dfrac{10}{\ln 10}\left(\dfrac{1}{10^{-7}}\right) = \dfrac{10^8}{\ln 10} \approx 4.34 \times 10^7 \dfrac{\text{dB}}{\text{watt/m}^2}$.

56. (a) $I(x) = I_0 a^x \quad\Rightarrow\quad I'(x) = I_0(\ln a)a^x = (I_0 a^x)\ln a = I(x)\ln a$

(b) We substitute $I_0 = 8$, $a = 0.38$ and $x = 20$ into the first expression for $I'(x)$ above: $I'(20) = 8(\ln 0.38)(0.38)^{20} \approx -3.05 \times 10^{-8}$.

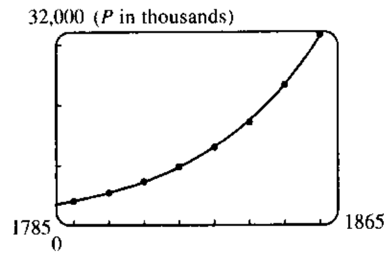
(c) The average value of the function $I(x)$ between $x = 0$ and $x = 20$ is

$$\frac{\int_0^{20} I(x)\,dx}{20-0} = \frac{1}{20}\int_0^{20} 8(0.38)^x\,dx = \frac{2}{5}\left[\frac{(0.38)^x}{\ln 0.38}\right]_0^{20} = \frac{2(0.38^{20}-1)}{5\ln 0.38} \approx 0.41.$$

57. (a) Using a calculator or CAS, we obtain the model $Q = ab^t$ with $a = 100.0124369$ and $b = 0.000045145933$. We can change this model to one with base e and exponent $\ln b$ [$b^t = e^{t\ln b}$ from precalculus mathematics or from Section 7.3]: $Q = ae^{t\ln b} = 100.012437e^{-10.005531t}$.

(b) Use $Q'(t) = ab^t \ln b$ or the calculator command `nDeriv(Y₁,X,.04)` with `Y₁=ab^x` to get $Q'(0.04) \approx -670.63\ \mu\text{A}$. The result of Example 2 in Section 2.1 was $-670\ \mu\text{A}$.

58. (a) $P = ab^t$ with $a = 4.502714 \times 10^{-20}$ and $b = 1.029953851$, where P is measured in thousands of people. The fit appears to be very good.



(b) **For 1800:** $m_1 = \dfrac{5308-3929}{1800-1790} = 137.9$, $m_2 = \dfrac{7240-5308}{1810-1800} = 193.2$.

So $P'(1800) \approx (m_1+m_2)/2 = 165.55$ thousand people/year.

For 1850: $m_1 = \dfrac{23{,}192-17{,}063}{1850-1840} = 612.9$, $m_2 = \dfrac{31{,}443-23{,}192}{1860-1850} = 825.1$.

So $P'(1850) \approx (m_1+m_2)/2 = 719$ thousand people/year.

(c) Use the calculator command `nDeriv(Y₁,X,year)` with `Y₁=ab^x` to get $P'(1800) \approx 156.85$ and $P'(1850) \approx 686.07$. These estimates are somewhat less than the ones in part (b).

(d) $P(1870) \approx 41{,}946.56$. The difference of 3.4 million people is most likely due to the Civil War (1861–1865).

59. Using Definition 1 and the second law of exponents for e^x, we have

$$a^{x-y} = e^{(x-y)\ln a} = e^{x\ln a - y\ln a} = \frac{e^{x\ln a}}{e^{y\ln a}} = \frac{a^x}{a^y}.$$

60. Using Definition 1, the first law of logarithms, and the first law of exponents for e^x, we have $(ab)^x = e^{x\ln(ab)} = e^{x(\ln a + \ln b)} = e^{x\ln a + x\ln b} = e^{x\ln a}e^{x\ln b} = a^x b^x$.

61. Let $\log_a x = r$ and $\log_a y = s$. Then $a^r = x$ and $a^s = y$.

(a) $xy = a^r a^s = a^{r+s} \quad\Rightarrow\quad \log_a(xy) = r + s = \log_a x + \log_a y$

(b) $\dfrac{x}{y} = \dfrac{a^r}{a^s} = a^{r-s} \quad\Rightarrow\quad \log_a \dfrac{x}{y} = r - s = \log_a x - \log_a y$

(c) $x^y = (a^r)^y = a^{ry} \quad\Rightarrow\quad \log_a(x^y) = ry = y\log_a x$

62. Let $m = n/x$. Then $n = xm$, and as $n \to \infty$, $m \to \infty$.

Therefore, $\displaystyle\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n = \lim_{m\to\infty}\left(1+\frac{1}{m}\right)^{mx} = \left[\lim_{m\to\infty}\left(1+\frac{1}{m}\right)^m\right]^x = e^x$ by Equation 9.

7.5 Inverse Trigonometric Functions

1. (a) $\sin^{-1}\left(\frac{\sqrt{3}}{2}\right) = \frac{\pi}{3}$ since $\sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}$ and $\frac{\pi}{3}$ is in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

(b) $\cos^{-1}(-1) = \pi$ since $\cos\pi = -1$ and π is in $[0, \pi]$.

2. (a) $\arctan(-1) = -\frac{\pi}{4}$ since $\tan\left(-\frac{\pi}{4}\right) = -1$ and $-\frac{\pi}{4}$ is in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

(b) $\csc^{-1} 2 = \frac{\pi}{6}$ since $\csc\frac{\pi}{6} = 2$ and $\frac{\pi}{6}$ is in $\left(0, \frac{\pi}{2}\right] \cup \left(\pi, \frac{3\pi}{2}\right]$.

3. (a) $\tan^{-1}\sqrt{3} = \frac{\pi}{3}$ since $\tan\frac{\pi}{3} = \sqrt{3}$ and $\frac{\pi}{3}$ is in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$.

(b) $\arcsin\left(-\frac{1}{\sqrt{2}}\right) = -\frac{\pi}{4}$ since $\sin\left(-\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}$ and $-\frac{\pi}{4}$ is in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

4. (a) $\sec^{-1}\sqrt{2} = \frac{\pi}{4}$ since $\sec\frac{\pi}{4} = \sqrt{2}$ and $\frac{\pi}{4}$ is in $\left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right)$.

(b) $\arcsin 1 = \frac{\pi}{2}$ since $\sin\frac{\pi}{2} = 1$ and $\frac{\pi}{2}$ is in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

5. (a) $\arccos(\cos 2\pi) = \arccos(1) = 0$

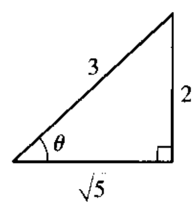
(b) $\tan(\tan^{-1} 5) = 5$

6. (a) $\tan^{-1}\left(\tan\dfrac{3\pi}{4}\right) = \tan^{-1}(-1) = -\dfrac{\pi}{4}$

(b) $\cos\left(\arcsin\dfrac{1}{2}\right) = \cos\left(\dfrac{\pi}{6}\right) = \dfrac{\sqrt{3}}{2}$

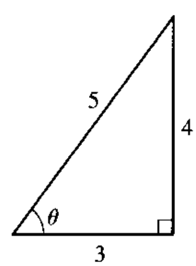
7. Let $\theta = \sin^{-1}\left(\frac{2}{3}\right)$.

Then $\tan\left(\sin^{-1}\left(\frac{2}{3}\right)\right) = \tan\theta = \dfrac{2}{\sqrt{5}}$.



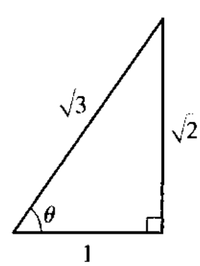
8. Let $\theta = \arccos\frac{3}{5}$.

Then $\csc\left(\arccos\left(\frac{3}{5}\right)\right) = \csc\theta = \frac{5}{4}$.



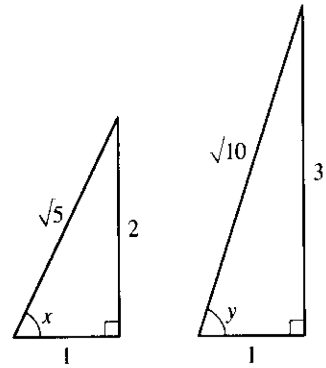
9. Let $\theta = \tan^{-1}\sqrt{2}$. Then

$$\sin\left(2\tan^{-1}\sqrt{2}\right) = \sin(2\theta) = 2\sin\theta\cos\theta = 2\left(\frac{\sqrt{2}}{\sqrt{3}}\right)\left(\frac{1}{\sqrt{3}}\right) = \frac{2\sqrt{2}}{3}$$



10. Let $x = \tan^{-1} 2$ and $y = \tan^{-1} 3$. Then

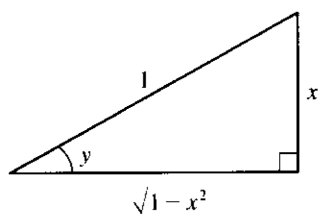
$$\cos\left(\tan^{-1} 2 + \tan^{-1} 3\right) = \cos(x+y) = \cos x \cos y - \sin x \sin y$$
$$= \frac{1}{\sqrt{5}}\frac{1}{\sqrt{10}} - \frac{2}{\sqrt{5}}\frac{3}{\sqrt{10}}$$
$$= \frac{-5}{\sqrt{50}} = \frac{-5}{5\sqrt{2}} = \frac{-1}{\sqrt{2}}$$



11. Let $y = \sin^{-1} x$. Then $-\frac{\pi}{2} \le y \le \frac{\pi}{2} \Rightarrow \cos y \ge 0$, so $\cos(\sin^{-1} x) = \cos y = \sqrt{1 - \sin^2 y} = \sqrt{1 - x^2}$

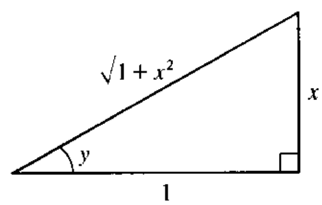
12. Let $y = \sin^{-1} x$. Then $\sin y = x$, so from the triangle we see that

$$\tan(\sin^{-1} x) = \tan y = \frac{x}{\sqrt{1 - x^2}}.$$



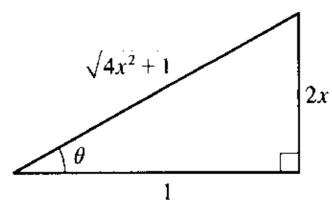
13. Let $y = \tan^{-1} x$. Then $\tan y = x$, so from the triangle we see that

$$\sin(\tan^{-1} x) = \sin y = \frac{x}{\sqrt{1 + x^2}}.$$

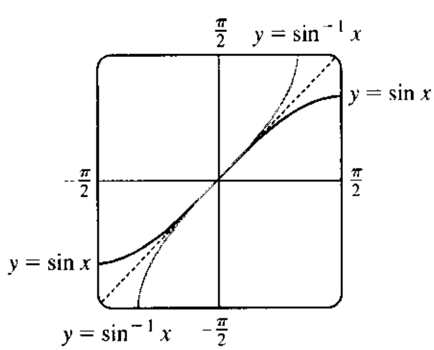


14. Let $\theta = \arctan 2x$. Then $\tan\theta = 2x$, so from the diagram we see that

$$\csc(\arctan 2x) = \csc\theta = \frac{\sqrt{4x^2 + 1}}{2x}.$$

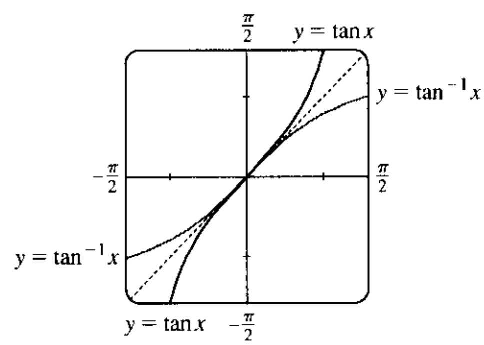


15.



The graph of $\sin^{-1} x$ is the reflection of the graph of $\sin x$ about the line $y = x$.

16.



The graph of $\tan^{-1} x$ is the reflection of the graph of $\tan x$ about the line $y = x$.

17. Let $y = \cos^{-1} x$. Then $\cos y = x$ and $0 \le y \le \pi \Rightarrow -\sin y \frac{dy}{dx} = 1 \Rightarrow$

$$\frac{dy}{dx} = -\frac{1}{\sin y} = -\frac{1}{\sqrt{1 - \cos^2 y}} = -\frac{1}{\sqrt{1 - x^2}}.$$ [Note that $\sin y \ge 0$ for $0 \le y \le \pi$.]

18. (a) Let $a = \sin^{-1} x$ and $b = \cos^{-1} x$. Then $\cos a = \sqrt{1 - \sin^2 a} = \sqrt{1 - x^2}$ since $\cos a \geq 0$ for $-\frac{\pi}{2} \leq a \leq \frac{\pi}{2}$. Similarly, $\sin b = \sqrt{1 - x^2}$. So

$$\sin\left(\sin^{-1} x + \cos^{-1} x\right) = \sin(a + b) = \sin a \cos b + \cos a \sin b = x \cdot x + \sqrt{1 - x^2}\,\sqrt{1 - x^2}$$
$$= x^2 + \left(1 - x^2\right) = 1$$

But $-\frac{\pi}{2} \leq \sin^{-1} x + \cos^{-1} x \leq \frac{3\pi}{2}$, and so $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$.

(b) We differentiate $\sin^{-1} x + \cos^{-1} x = \frac{\pi}{2}$ with respect to x, and get

$$\frac{1}{\sqrt{1 - x^2}} + \frac{d}{dx} \cos^{-1} x = 0 \quad \Rightarrow \quad \frac{d}{dx} \cos^{-1} x = -\frac{1}{\sqrt{1 - x^2}}.$$

19. Let $y = \cot^{-1} x$. Then $\cot y = x \quad \Rightarrow \quad -\csc^2 y \dfrac{dy}{dx} = 1 \quad \Rightarrow \quad \dfrac{dy}{dx} = -\dfrac{1}{\csc^2 y} = -\dfrac{1}{1 + \cot^2 y} = -\dfrac{1}{1 + x^2}$.

20. Let $y = \sec^{-1} x$. Then $\sec y = x$ and $y \in \left(0, \frac{\pi}{2}\right] \cup \left[\pi, \frac{3\pi}{2}\right)$. Differentiate with respect to x:

$\sec y \, \tan y \left(\dfrac{dy}{dx}\right) = 1 \quad \Rightarrow \quad \dfrac{dy}{dx} = \dfrac{1}{\sec y \, \tan y} = \dfrac{1}{\sec y \sqrt{\sec^2 y - 1}} = \dfrac{1}{x\sqrt{x^2 - 1}}$. Note that

$\tan^2 y = \sec^2 y - 1 \quad \Rightarrow \quad \tan y = \sqrt{\sec^2 y - 1}$ since $\tan y > 0$ when $0 < y < \frac{\pi}{2}$ or $\pi < y < \frac{3\pi}{2}$.

21. Let $y = \csc^{-1} x$. Then $\csc y = x \quad \Rightarrow \quad -\csc y \cot y \dfrac{dy}{dx} = 1 \quad \Rightarrow$

$\dfrac{dy}{dx} = -\dfrac{1}{\csc y \cot y} = -\dfrac{1}{\csc y \sqrt{\csc^2 y - 1}} = -\dfrac{1}{x\sqrt{x^2 - 1}}$. Note that $\cot y \geq 0$ on the domain of $\csc^{-1} x$.

22. $y = \sqrt{\tan^{-1} x} = \left(\tan^{-1} x\right)^{1/2} \quad \Rightarrow$

$$y' = \tfrac{1}{2}\left(\tan^{-1} x\right)^{-1/2} \cdot \frac{d}{dx}\left(\tan^{-1} x\right) = \frac{1}{2\sqrt{\tan^{-1} x}} \cdot \frac{1}{1 + x^2} = \frac{1}{2\sqrt{\tan^{-1} x}\left(1 + x^2\right)}$$

23. $y = \tan^{-1} \sqrt{x} \quad \Rightarrow \quad y' = \dfrac{1}{1 + (\sqrt{x})^2} \cdot \dfrac{d}{dx}(\sqrt{x}) = \dfrac{1}{1 + x}\left(\tfrac{1}{2} x^{-1/2}\right) = \dfrac{1}{2\sqrt{x}\,(1 + x)}$

24. $h(x) = \sqrt{1 - x^2} \arcsin x \quad \Rightarrow$

$$h'(x) = \sqrt{1 - x^2} \cdot \frac{1}{\sqrt{1 - x^2}} + \arcsin x \left[\tfrac{1}{2}\left(1 - x^2\right)^{-1/2}(-2x)\right] = 1 - \frac{x \arcsin x}{\sqrt{1 - x^2}}$$

25. $y = \sin^{-1}(2x + 1) \quad \Rightarrow$

$$y' = \frac{1}{\sqrt{1 - (2x + 1)^2}} \cdot \frac{d}{dx}(2x + 1) = \frac{1}{\sqrt{1 - \left(4x^2 + 4x + 1\right)}} \cdot 2 = \frac{2}{\sqrt{-4x^2 - 4x}} = \frac{1}{\sqrt{-x^2 - x}}$$

26. $f(x) = x \ln(\arctan x) \quad \Rightarrow$

$$f'(x) = x \cdot \frac{1}{\arctan x} \cdot \frac{1}{1 + x^2} + \ln(\arctan x) \cdot 1 = \frac{x}{\left(1 + x^2\right) \arctan x} + \ln(\arctan x)$$

27. $H(x) = \left(1 + x^2\right) \arctan x \quad \Rightarrow \quad H'(x) = \left(1 + x^2\right)\dfrac{1}{1 + x^2} + (\arctan x)(2x) = 1 + 2x \arctan x$

28. $h(t) = e^{\sec^{-1} t} \quad \Rightarrow \quad h'(t) = e^{\sec^{-1} t} \dfrac{d}{dt}\left(\sec^{-1} t\right) = \dfrac{e^{\sec^{-1} t}}{t\sqrt{t^2 - 1}}$

29. $y = \cos^{-1}\left(e^{2x}\right) \quad \Rightarrow \quad y' = -\dfrac{1}{\sqrt{1 - \left(e^{2x}\right)^2}} \cdot \dfrac{d}{dx}\left(e^{2x}\right) = -\dfrac{2e^{2x}}{\sqrt{1 - e^{4x}}}$

30. $y = x \cos^{-1} x - \sqrt{1 - x^2} \quad \Rightarrow \quad y' = \cos^{-1} x - \dfrac{x}{\sqrt{1 - x^2}} + \dfrac{x}{\sqrt{1 - x^2}} = \cos^{-1} x$

31. $y = \arctan(\cos\theta) \;\Rightarrow\; y' = \dfrac{1}{1+(\cos\theta)^2}(-\sin\theta) = -\dfrac{\sin\theta}{1+\cos^2\theta}$

32. $y = \tan^{-1}\left(x - \sqrt{x^2+1}\right) \;\Rightarrow$

$$y' = \frac{1}{1+\left(x-\sqrt{x^2+1}\right)^2}\left(1 - \frac{x}{\sqrt{x^2+1}}\right) = \frac{1}{1+x^2-2x\sqrt{x^2+1}+x^2+1}\left(\frac{\sqrt{x^2+1}-x}{\sqrt{x^2+1}}\right)$$

$$= \frac{\sqrt{x^2+1}-x}{2\left(1+x^2-x\sqrt{x^2+1}\right)\sqrt{x^2+1}} = \frac{\sqrt{x^2+1}-x}{2\left[\sqrt{x^2+1}\,(1+x^2) - x(x^2+1)\right]}$$

$$= \frac{\sqrt{x^2+1}-x}{2\left[(1+x^2)\left(\sqrt{x^2+1}-x\right)\right]} = \frac{1}{2(1+x^2)}$$

33. $h(t) = \cot^{-1}(t) + \cot^{-1}(1/t) \;\Rightarrow$

$$h'(t) = -\frac{1}{1+t^2} - \frac{1}{1+(1/t)^2}\cdot\frac{d}{dt}\frac{1}{t} = -\frac{1}{1+t^2} - \frac{t^2}{t^2+1}\cdot\left(-\frac{1}{t^2}\right) = -\frac{1}{1+t^2} + \frac{1}{t^2+1} = 0.$$

Note that this makes sense because $h(t) = \dfrac{\pi}{2}$ for $t > 0$ and $h(t) = -\dfrac{\pi}{2}$ for $t < 0$.

34. $y = \tan^{-1}\left(\dfrac{x}{a}\right) + \ln\sqrt{\dfrac{x-a}{x+a}} = \tan^{-1}\left(\dfrac{x}{a}\right) + \frac{1}{2}\ln(x-a) - \frac{1}{2}\ln(x+a) \;\Rightarrow$

$$y' = \frac{a}{x^2+a^2} + \frac{1/2}{x-a} - \frac{1/2}{x+a} = \frac{a}{x^2+a^2} + \frac{a}{x^2-a^2} = \frac{2ax^2}{x^4-a^4}$$

35. $y = \arccos\left(\dfrac{b+a\cos x}{a+b\cos x}\right) \;\Rightarrow$

$$y' = -\frac{1}{\sqrt{1-\left(\dfrac{b+a\cos x}{a+b\cos x}\right)^2}}\,\frac{(a+b\cos x)(-a\sin x)-(b+a\cos x)(-b\sin x)}{(a+b\cos x)^2}$$

$$= \frac{1}{\sqrt{a^2+b^2\cos^2 x - b^2 - a^2\cos^2 x}}\,\frac{(a^2-b^2)\sin x}{|a+b\cos x|}$$

$$= \frac{1}{\sqrt{a^2-b^2}\,\sqrt{1-\cos^2 x}}\,\frac{(a^2-b^2)\sin x}{|a+b\cos x|} = \frac{\sqrt{a^2-b^2}}{|a+b\cos x|}\,\frac{\sin x}{|\sin x|}$$

But $0 \le x \le \pi$, so $|\sin x| = \sin x$. Also $a > b > 0 \;\Rightarrow\; b\cos x \ge -b > -a$, so $a + b\cos x > 0$.

Thus $y' = \dfrac{\sqrt{a^2-b^2}}{a+b\cos x}$.

36. $f(x) = \arcsin(e^x) \;\Rightarrow\; f'(x) = \dfrac{1}{\sqrt{1-(e^x)^2}}\cdot e^x = \dfrac{e^x}{\sqrt{1-e^{2x}}}$.

Domain$(f) = \{x \mid -1 \le e^x \le 1\} = \{x \mid 0 < e^x \le 1\} = (-\infty, 0]$.

Domain$(f') = \{x \mid 1 - e^{2x} > 0\} = \{x \mid e^{2x} < 1\} = \{x \mid 2x < 0\} = (-\infty, 0)$.

37. $g(x) = \cos^{-1}(3-2x) \;\Rightarrow\; g'(x) = -\dfrac{1}{\sqrt{1-(3-2x)^2}}(-2) = \dfrac{2}{\sqrt{1-(3-2x)^2}}$.

Domain$(g) = \{x \mid -1 \le 3-2x \le 1\} = \{x \mid -4 \le -2x \le -2\} = \{x \mid 2 \ge x \ge 1\} = [1, 2]$.

Domain$(g') = \{x \mid 1-(3-2x)^2 > 0\} = \{x \mid (3-2x)^2 < 1\} = \{x \mid |3-2x| < 1\}$

$= \{x \mid -1 < 3-2x < 1\} = \{x \mid -4 < -2x < -2\} = \{x \mid 2 > x > 1\} = (1, 2)$

38. $\tan^{-1}(xy) = 1 + x^2y \;\Rightarrow\; \dfrac{1}{1+x^2y^2}(xy' + y \cdot 1) = 0 + x^2y' + 2xy \;\Rightarrow$

$$y'\left(\frac{x}{1+x^2y^2} - x^2\right) = 2xy - \frac{y}{1+x^2y^2} \;\Rightarrow$$

$$y' = \frac{2xy - \dfrac{y}{1+x^2y^2}}{\dfrac{x}{1+x^2y^2} - x^2} = \frac{2xy(1+x^2y^2) - y}{x - x^2(1+x^2y^2)} = \frac{y(-1-2x-2x^3y^2)}{x(1-x-x^3y^2)}.$$

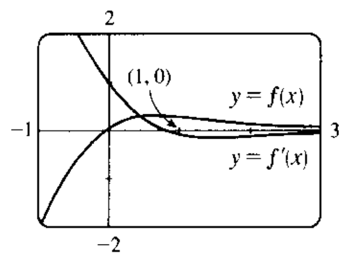
39. $g(x) = x\sin^{-1}\left(\dfrac{x}{4}\right) + \sqrt{16-x^2} \;\Rightarrow\; g'(x) = \sin^{-1}\left(\dfrac{x}{4}\right) + \dfrac{x}{4\sqrt{1-(x/4)^2}} - \dfrac{x}{\sqrt{16-x^2}} = \sin^{-1}\left(\dfrac{x}{4}\right) \;\Rightarrow$

$g'(2) = \sin^{-1}\frac{1}{2} = \frac{\pi}{6}$

40. $y = 3\arccos\dfrac{x}{2} \;\Rightarrow\; y' = 3\left[-\dfrac{1}{\sqrt{1-(x/2)^2}}\right]\left(\dfrac{1}{2}\right)$, so at $(1, \pi)$, $y' = -\dfrac{3}{2\sqrt{1-\frac{1}{4}}} = -\sqrt{3}$. An equation of the tangent line is $y - \pi = -\sqrt{3}\,(x-1)$, or $y = -\sqrt{3}\,x + \pi + \sqrt{3}$.

41. $f(x) = e^{-x}\arctan x \;\Rightarrow$

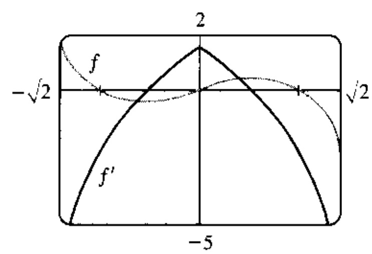
$f'(x) = \dfrac{e^{-x}}{1+x^2} - e^{-x}\arctan x$. The answer is reasonable since f' is positive where f is increasing and f' is negative where f is decreasing.



42. $f(x) = x\arcsin(1-x^2) \;\Rightarrow$

$$f'(x) = x\left[\frac{-2x}{\sqrt{1-(1-x^2)^2}}\right] + \arcsin(1-x^2)\cdot 1$$

$$= \arcsin(1-x^2) - \frac{2x^2}{\sqrt{2x^2-x^4}}$$



This is reasonable because the graphs show that f is increasing when f' is positive, and that f has an inflection point when f' changes from increasing to decreasing .

43. $\displaystyle\lim_{x\to-1^+}\sin^{-1}x = \sin^{-1}(-1) = -\frac{\pi}{2}$

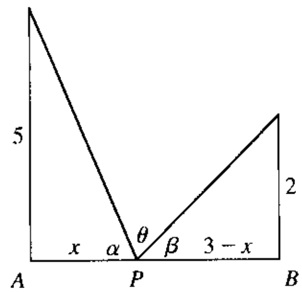
44. Let $t = \dfrac{1+x^2}{1+2x^2}$. As $x \to \infty$, $t = \dfrac{1+x^2}{1+2x^2} = \dfrac{1/x^2+1}{1/x^2+2} \to \dfrac{1}{2}$.

$\displaystyle\lim_{x\to\infty}\arccos\left(\frac{1+x^2}{1+2x^2}\right) = \lim_{t\to1/2}\arccos t = \arccos\tfrac{1}{2} = \tfrac{\pi}{3}.$

45. Let $t = e^x$. As $x \to \infty$, $t \to \infty$. $\displaystyle\lim_{x\to\infty}\arctan(e^x) = \lim_{t\to\infty}\arctan t = \tfrac{\pi}{2}$ by (8).

46. Let $t = \ln x$. As $x \to 0^+$, $t \to -\infty$. $\displaystyle\lim_{x\to0^+}\tan^{-1}(\ln x) = \lim_{t\to-\infty}\tan^{-1}t = -\tfrac{\pi}{2}$ by (8).

47.



From the figure, $\tan\alpha = \dfrac{5}{x}$ and $\tan\beta = \dfrac{2}{3-x}$. Since

$\alpha + \beta + \theta = 180^\circ = \pi$, $\theta = \pi - \tan^{-1}\left(\dfrac{5}{x}\right) - \tan^{-1}\left(\dfrac{2}{3-x}\right) \Rightarrow$

$$\frac{d\theta}{dx} = -\frac{1}{1+\left(\frac{5}{x}\right)^2}\left(-\frac{5}{x^2}\right) - \frac{1}{1+\left(\frac{2}{3-x}\right)^2}\left[\frac{2}{(3-x)^2}\right]$$

$$= \frac{x^2}{x^2+25}\cdot\frac{5}{x^2} - \frac{(3-x)^2}{(3-x)^2+4}\cdot\frac{2}{(3-x)^2}.$$

Now $\dfrac{d\theta}{dx} = 0 \Rightarrow \dfrac{5}{x^2+25} = \dfrac{2}{x^2-6x+13} \Rightarrow 2x^2+50 = 5x^2-30x+65 \Rightarrow$

$3x^2 - 30x + 15 = 0 \Rightarrow x^2 - 10x + 5 = 0 \Rightarrow x = 5 \pm 2\sqrt{5}$. We reject the root with the + sign, since it is larger than 3. $d\theta/dx > 0$ for $x < 5 - 2\sqrt{5}$ and $d\theta/dx < 0$ for $x > 5 - 2\sqrt{5}$, so θ is maximized when $|AP| = x = 5 - 2\sqrt{5} \approx 0.53$.

48. Let x be the distance from the observer to the wall. Then, from the given figure,

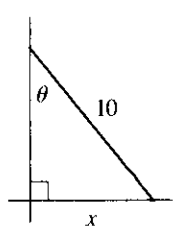
$\theta = \tan^{-1}\left(\dfrac{h+d}{x}\right) - \tan^{-1}\left(\dfrac{d}{x}\right), x > 0 \Rightarrow$

$$\frac{d\theta}{dx} = \frac{1}{1+[(h+d)/x]^2}\left[-\frac{h+d}{x^2}\right] - \frac{1}{1+(d/x)^2}\left[-\frac{d}{x^2}\right] = -\frac{h+d}{x^2+(h+d)^2} + \frac{d}{x^2+d^2}$$

$$= \frac{d[x^2+(h+d)^2] - (h+d)(x^2+d^2)}{[x^2+(h+d)^2](x^2+d^2)} = \frac{h^2d + hd^2 - hx^2}{[x^2+(h+d)^2](x^2+d^2)} = 0 \Leftrightarrow$$

$hx^2 = h^2d + hd^2 \Leftrightarrow x^2 = hd + d^2 \Leftrightarrow x = \sqrt{d(h+d)}$. Since $d\theta/dx > 0$ for all $x < \sqrt{d(h+d)}$ and $d\theta/dx < 0$ for all $x > \sqrt{d(h+d)}$, the absolute maximum occurs when $x = \sqrt{d(h+d)}$.

49.

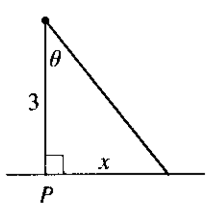


$\dfrac{dx}{dt} = 2$ ft/s, $\sin\theta = \dfrac{x}{10} \Rightarrow \theta = \sin^{-1}\left(\dfrac{x}{10}\right)$, $\dfrac{d\theta}{dx} = \dfrac{1/10}{\sqrt{1-(x/10)^2}}$,

$$\frac{d\theta}{dt} = \frac{d\theta}{dx}\frac{dx}{dt} = \frac{1/10}{\sqrt{1-(x/10)^2}}(2) \text{ rad/s},$$

$$\left.\frac{d\theta}{dt}\right]_{x=6} = \frac{2/10}{\sqrt{1-(6/10)^2}} \text{ rad/s} = \tfrac{1}{4} \text{ rad/s}$$

50.



$\dfrac{d\theta}{dt} = 4$ rev/min $= 8\pi\cdot 60$ rad/h. From the diagram, we see that $\tan\theta = \dfrac{x}{3}$

$\Rightarrow \theta = \tan^{-1}\left(\dfrac{x}{3}\right)$. Thus, $8\pi\cdot 60 = \dfrac{d\theta}{dt} = \dfrac{d\theta}{dx}\dfrac{dx}{dt} = \dfrac{1/3}{1+(x/3)^2}\dfrac{dx}{dt}$.

So $\dfrac{dx}{dt} = 8\pi\cdot 60\cdot 3\left[1+\left(\dfrac{x}{3}\right)^2\right]$ km/h, and at $x = 1$,

$$\frac{dx}{dt} = 8\pi\cdot 60\cdot 3\left[1+\tfrac{1}{9}\right] \text{ km/h} = 1600\pi \text{ km/h}.$$

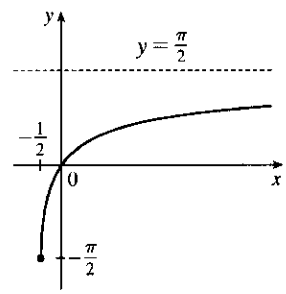
51. $y = f(x) = \sin^{-1}(x/(x+1))$ **A.** $D = \{x \mid -1 \le x/(x+1) \le 1\}$. For $x > -1$ we have $-x - 1 \le x \le x + 1 \quad \Leftrightarrow \quad 2x \ge -1 \quad \Leftrightarrow \quad x \ge -\frac{1}{2}$, so $D = \left[-\frac{1}{2}, \infty\right)$. **B.** Intercepts are 0 **C.** No symmetry **D.** $\lim\limits_{x\to\infty} \sin^{-1}\left(\dfrac{x}{x+1}\right) = \lim\limits_{x\to\infty} \sin^{-1}\left(\dfrac{1}{1+1/x}\right) = \sin^{-1} 1 = \frac{\pi}{2}$, so $y = \frac{\pi}{2}$ is a HA.

E. $f'(x) = \dfrac{1}{\sqrt{1 - [x/(x+1)]^2}} \dfrac{(x+1) - x}{(x+1)^2} = \dfrac{1}{(x+1)\sqrt{2x+1}} > 0,$

so f is increasing on $\left(-\frac{1}{2}, \infty\right)$. **F.** No local maximum or minimum, $f\left(-\frac{1}{2}\right) = \sin^{-1}(-1) = -\frac{\pi}{2}$ is an absolute minimum

G. $$f''(x) = -\frac{\sqrt{2x+1} + (x+1)/\sqrt{2x+1}}{(x+1)^2(2x+1)} = -\frac{3x+2}{(x+1)^2(2x+1)^{3/2}} < 0 \text{ on } D, \text{ so } f \text{ is CD on } \left(-\tfrac{1}{2}, \infty\right).$$

H.



52. $y = f(x) = \tan^{-1}\left(\dfrac{x-1}{x+1}\right)$ **A.** $D = \{x \mid x \ne -1\}$

B. x-intercept $= 1$, y-intercept $= f(0) = \tan^{-1}(-1) = -\frac{\pi}{4}$ **C.** No symmetry

D. $\lim\limits_{x\to\pm\infty} \tan^{-1}\left(\dfrac{x-1}{x+1}\right) = \lim\limits_{x\to\pm\infty} \tan^{-1}\left(\dfrac{1-1/x}{1+1/x}\right) = \tan^{-1} 1 = \frac{\pi}{4}$, so $y = \frac{\pi}{4}$ is a HA. Also

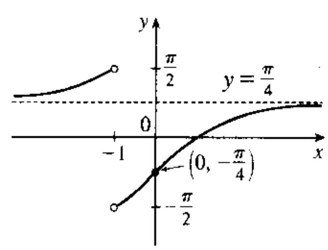
$\lim\limits_{x\to-1^+} \tan^{-1}\left(\dfrac{x-1}{x+1}\right) = -\dfrac{\pi}{2}$ and $\lim\limits_{x\to-1^-} \tan^{-1}\left(\dfrac{x-1}{x+1}\right) = \dfrac{\pi}{2}$.

E. $$f'(x) = \frac{1}{1 + [(x-1)/(x+1)]^2} \frac{(x+1) - (x-1)}{(x+1)^2} = \frac{2}{(x+1)^2 + (x-1)^2} = \frac{1}{x^2+1} > 0$$

so f is increasing on $(-\infty, -1)$ and $(-1, \infty)$. **F.** No extreme values

G. $f''(x) = -2x/(x^2+1)^2 > 0 \quad \Leftrightarrow \quad x < 0$, so f is CU on $(-\infty, -1)$ and $(-1, 0)$, and CD on $(0, \infty)$. IP at $\left(0, -\frac{\pi}{4}\right)$

H.

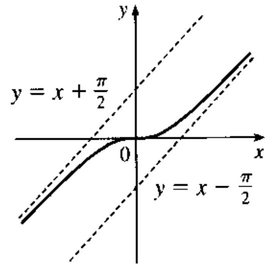


53. $y = f(x) = x - \tan^{-1} x$ **A.** $D = \mathbb{R}$ **B.** Intercepts are 0 **C.** $f(-x) = -f(x)$, so the curve is symmetric about the origin. **D.** $\lim\limits_{x\to\infty}(x - \tan^{-1} x) = \infty$ and $\lim\limits_{x\to-\infty}(x - \tan^{-1} x) = -\infty$, no HA.

But $f(x) - \left(x - \frac{\pi}{2}\right) = -\tan^{-1} x + \frac{\pi}{2} \to 0$ as $x \to \infty$, and $f(x) - \left(x + \frac{\pi}{2}\right) = -\tan^{-1} x - \frac{\pi}{2} \to 0$ as $x \to -\infty$, so $y = x \pm \frac{\pi}{2}$ are slant asymptotes. **E.** $f'(x) = 1 - \dfrac{1}{x^2+1} = \dfrac{x^2}{x^2+1} > 0$, so f is increasing on $\mathbb{R}$. **F.** No extrema

G. $f''(x) = \dfrac{(1+x^2)(2x) - x^2(2x)}{(1+x^2)^2} = \dfrac{2x}{(1+x^2)^2} > 0 \Leftrightarrow x > 0$, so f is CU on $(0, \infty)$, CD on $(-\infty, 0)$. IP at $(0, 0)$.

H.



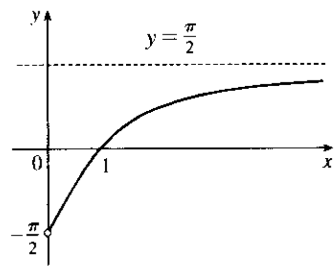
54. $y = \tan^{-1}(\ln x)$ **A.** $D = (0, \infty)$ **B.** No y-intercept, x-intercept when $\tan^{-1}(\ln x) = 0 \Leftrightarrow \ln x = 0 \Leftrightarrow x = 1$. **C.** No symmetry **D.** $\lim\limits_{x\to\infty} \tan^{-1}(\ln x) = \frac{\pi}{2}$, so $y = \frac{\pi}{2}$ is a HA. Also $\lim\limits_{x\to 0^+} \tan^{-1}(\ln x) = -\frac{\pi}{2}$.

E. $f'(x) = \dfrac{1}{x\left[1 + (\ln x)^2\right]} > 0$, so f is increasing on $(0, \infty)$.

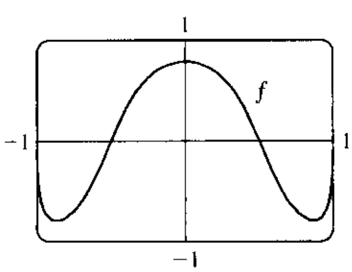
F. No maximum or minimum

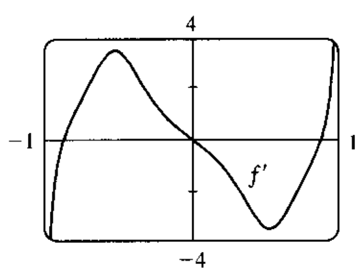
G. $f''(x) = \dfrac{-\left[1 + (\ln x)^2 + x(2\ln x/x)\right]}{x^2\left[1 + (\ln x)^2\right]^2} = -\dfrac{(1 + \ln x)^2}{x^2[1 + (\ln x)^2]^2} < 0$, so f is CD on $(0, \infty)$.

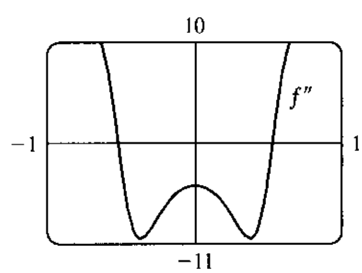
H.



55. $f(x) = \arctan(\cos(3\arcsin x))$. We use a CAS to compute f' and f'', and to graph f, f', and f'':

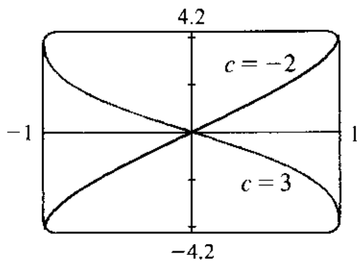


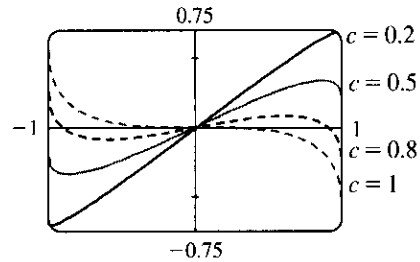




From the graph of f', it appears that the only maximum occurs at $x = 0$ and there are minima at $x = \pm 0.87$. From the graph of f'', it appears that there are inflection points at $x = \pm 0.52$.

56. First note that the function $f(x) = x - c\sin^{-1} x$ is only defined on the interval $[-1, 1]$, since $\sin^{-1}$ is only defined on that interval. We differentiate to get $f'(x) = 1 - c/\sqrt{1 - x^2}$. Now if $c \le 0$, then $f'(x) \ge 1$, so there is no extremum and f is increasing on its domain. If $c > 1$, then $f'(x) < 0$, so there is no local extremum and f is decreasing on its domain, and if $c = 1$, then there is still no extremum, since $f'(x)$ does not change sign at $x = 0$. So we can only have local extrema if $0 < c < 1$. In this case, f is increasing where $f'(x) > 0 \quad\Leftrightarrow\quad \sqrt{1-x^2} > c$ $\Leftrightarrow\quad |x| < \sqrt{1-c^2}$, and decreasing where $\sqrt{1-c^2} < |x| \le 1$. f has a maximum at $x = \sqrt{1-c^2}$ and a minimum at $x = -\sqrt{1-c^2}$.





57. $f(x) = 2x + 5(1-x^2)^{-1/2} = 2x + 5/\sqrt{1-x^2} \quad\Rightarrow\quad F(x) = x^2 + 5\sin^{-1}x + C$

58. $f'(x) = 4 - 3(1+x^2)^{-1} \quad\Rightarrow\quad f(x) = 4x - 3\tan^{-1}x + C \quad\Rightarrow\quad f\left(\frac{\pi}{4}\right) = \pi - 3 + C = 0 \quad\Rightarrow\quad C = 3 - \pi$, so $f(x) = 4x - 3\tan^{-1}x + 3 - \pi$.

59. $$\int_{1/2}^{\sqrt{3}/2} \frac{6}{\sqrt{1-t^2}}\,dt = 6\int_{1/2}^{\sqrt{3}/2} \frac{1}{\sqrt{1-t^2}}\,dt = 6\left[\sin^{-1}t\right]_{1/2}^{\sqrt{3}/2} = 6\left[\sin^{-1}\left(\tfrac{\sqrt{3}}{2}\right) - \sin^{-1}\left(\tfrac{1}{2}\right)\right]$$
$$= 6\left(\tfrac{\pi}{3} - \tfrac{\pi}{6}\right) = 6\left(\tfrac{\pi}{6}\right) = \pi$$

60. $$\int_0^1 \frac{4}{t^2+1}\,dt = 4\int_0^1 \frac{1}{1+t^2}\,dt = 4\left[\tan^{-1}t\right]_0^1 = 4\left(\tan^{-1}1 - \tan^{-1}0\right) = 4\left(\tfrac{\pi}{4} - 0\right) = \pi$$

61. Let $u = 4x$. Then $du = 4\,dx$, so

$$\int_0^{\sqrt{3}/4} \frac{dx}{1+16x^2} = \frac{1}{4}\int_0^{\sqrt{3}} \frac{1}{1+u^2}\,du = \tfrac{1}{4}\left[\tan^{-1}u\right]_0^{\sqrt{3}} = \tfrac{1}{4}\left(\tan^{-1}\sqrt{3} - \tan^{-1}0\right) = \tfrac{1}{4}\left(\tfrac{\pi}{3} - 0\right) = \tfrac{\pi}{12}.$$

62. Let $u = 2t$. Then $\sqrt{1-4t^2} = \sqrt{1-u^2}$ and $du = 2\,dt$, so

$$\int \frac{dt}{\sqrt{1-4t^2}} = \int \frac{\frac{1}{2}\,du}{\sqrt{1-u^2}} = \frac{1}{2}\sin^{-1}u + C = \frac{1}{2}\sin^{-1}(2t) + C.$$

63. Let $u = \sin^{-1}x$. Then $du = \dfrac{1}{\sqrt{1-x^2}}\,dx$, so $\displaystyle\int_0^{1/2} \frac{\sin^{-1}x}{\sqrt{1-x^2}}\,dx = \int_0^{\pi/6} u\,du = \left.\frac{u^2}{2}\right]_0^{\pi/6} = \frac{1}{2}\left(\frac{\pi}{6}\right)^2 = \frac{\pi^2}{72}.$

64. Let $u = -\cos x$. Then $du = \sin x\,dx$, so

$$\int_0^{\pi/2} \frac{\sin x}{1+\cos^2 x}\,dx = \int_{-1}^{0} \frac{1}{1+u^2}\,du = \left[\tan^{-1}u\right]_{-1}^{0} = \tan^{-1}0 - \tan^{-1}(-1) = 0 - \left(-\tfrac{\pi}{4}\right) = \tfrac{\pi}{4}.$$

65. $$\int \frac{x+9}{x^2+9}\,dx = \int \frac{x}{x^2+9}\,dx + 9\int \frac{1}{x^2+9}\,dx = \tfrac{1}{2}\ln(x^2+9) + 3\tan^{-1}\frac{x}{3} + C$$

(Let $u = x^2 + 9$ in the first integral; use Equation 14 in the second.)

66. Let $u = \tan^{-1} x$. Then $du = dx/(1 + x^2)$, so $\displaystyle\int \frac{\tan^{-1} x}{1 + x^2}\,dx = \int u\,du = \tfrac{1}{2}u^2 + C = \tfrac{1}{2}(\tan^{-1} x)^2 + C$.

67. Let $u = t^3$. Then $du = 3t^2\,dt$ and $\displaystyle\int \frac{t^2}{\sqrt{1 - t^6}}\,dt = \int \frac{\frac{1}{3}\,du}{\sqrt{1 - u^2}} = \frac{1}{3}\sin^{-1} u + C = \frac{1}{3}\sin^{-1}(t^3) + C$.

68. Let $u = \frac{1}{2}x$. Then $du = \frac{1}{2}\,dx \quad\Rightarrow$

$$\int \frac{dx}{x\sqrt{x^2 - 4}} = \int \frac{dx}{2x\sqrt{(x/2)^2 - 1}} = \int \frac{2\,du}{4u\sqrt{u^2 - 1}} = \frac{1}{2}\int \frac{du}{u\sqrt{u^2 - 1}}$$
$$= \tfrac{1}{2}\sec^{-1} u + C = \tfrac{1}{2}\sec^{-1}\left(\tfrac{1}{2}x\right) + C$$

69. Let $u = \sqrt{x}$. Then $du = \dfrac{dx}{2\sqrt{x}}$ and $\displaystyle\int \frac{dx}{\sqrt{x}(1 + x)} = \int \frac{2\,du}{1 + u^2} = 2\tan^{-1} u + C = 2\tan^{-1}\sqrt{x} + C$.

70. Let $u = e^{2x}$. Then $du = 2e^{2x}\,dx \quad\Rightarrow\quad \displaystyle\int \frac{e^{2x}\,dx}{\sqrt{1 - e^{4x}}} = \frac{1}{2}\int \frac{du}{\sqrt{1 - u^2}} = \tfrac{1}{2}\sin^{-1} u + C = \tfrac{1}{2}\sin^{-1}(e^{2x}) + C$

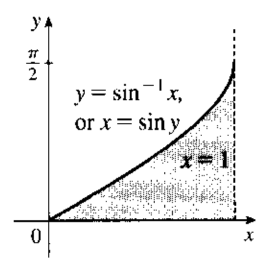
71. Let $u = x/a$. Then $du = dx/a$, so

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \int \frac{dx}{a\sqrt{1 - (x/a)^2}} = \int \frac{du}{\sqrt{1 - u^2}} = \sin^{-1} u + C = \sin^{-1}\frac{x}{a} + C.$$

72. We use the disk method: $A = \displaystyle\int_0^2 \pi\left[\frac{1}{\sqrt{x^2 + 4}}\right]^2 dx = \pi\int_0^2 \frac{1}{x^2 + 4}\,dx$. By Formula 14, this is equal to

$\pi\left[\tfrac{1}{2}\tan^{-1}(x/2)\right]_0^2 = \tfrac{\pi}{2}\left(\tfrac{\pi}{4} - 0\right) = \tfrac{\pi^2}{8}$.

73.



The integral represents the area below the curve $y = \sin^{-1} x$ on the interval $x \in [0, 1]$. The bounding curves are $y = \sin^{-1} x \quad\Leftrightarrow\quad x = \sin y$, $y = 0$ and $x = 1$. We see that y ranges between $\sin^{-1} 0 = 0$ and $\sin^{-1} 1 = \frac{\pi}{2}$. So we have to integrate the function $x = 1 - \sin y$ between $y = 0$ and $y = \frac{\pi}{2}$:

$\int_0^1 \sin^{-1} x\,dx = \int_0^{\pi/2}(1 - \sin y)\,dy = \left(\frac{\pi}{2} + \cos\frac{\pi}{2}\right) - (0 + \cos 0) = \frac{\pi}{2} - 1$.

74. Let $a = \arctan x$ and $b = \arctan y$. Then by the addition formula for the tangent (see Reference Page 2 in the textbook), $\tan(a + b) = \dfrac{\tan a + \tan b}{1 - (\tan a)(\tan b)} = \dfrac{\tan(\arctan x) + \tan(\arctan y)}{1 - \tan(\arctan x)\,\tan(\arctan y)} \quad\Rightarrow\quad \tan(a + b) = \dfrac{x + y}{1 - xy}$

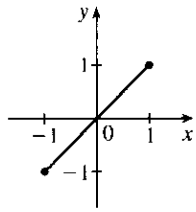
$\Rightarrow\quad \arctan x + \arctan y = a + b = \arctan\left(\dfrac{x + y}{1 - xy}\right)$, since $-\frac{\pi}{2} < \arctan x + \arctan y < \frac{\pi}{2}$.

75. (a) $\arctan\frac{1}{2} + \arctan\frac{1}{3} = \arctan\left(\dfrac{\frac{1}{2} + \frac{1}{3}}{1 - \frac{1}{2}\cdot\frac{1}{3}}\right) = \arctan 1 = \dfrac{\pi}{4}$

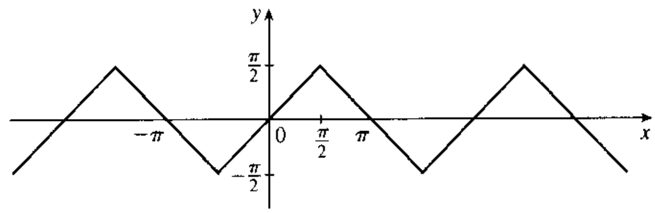
(b) $2\arctan\frac{1}{3} + \arctan\frac{1}{7} = \left(\arctan\frac{1}{3} + \arctan\frac{1}{3}\right) + \arctan\frac{1}{7} = \arctan\left(\dfrac{\frac{1}{3} + \frac{1}{3}}{1 - \frac{1}{3}\cdot\frac{1}{3}}\right) + \arctan\frac{1}{7}$

$= \arctan\frac{3}{4} + \arctan\frac{1}{7} = \arctan\left(\dfrac{\frac{3}{4} + \frac{1}{7}}{1 - \frac{3}{4}\cdot\frac{1}{7}}\right) = \arctan 1 = \dfrac{\pi}{4}$

76. (a) $f(x) = \sin(\sin^{-1} x)$

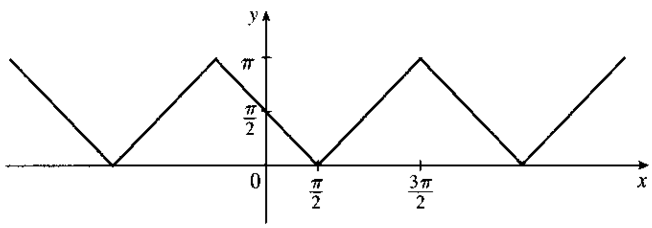


(b) $g(x) = \sin^{-1}(\sin x)$



(c) $g'(x) = \dfrac{d}{dx}\sin^{-1}(\sin x) = \dfrac{1}{\sqrt{1-\sin^2 x}}\cos x = \dfrac{\cos x}{\sqrt{\cos^2 x}} = \dfrac{\cos x}{|\cos x|}$

(d) $h(x) = \cos^{-1}(\sin x)$, so $h'(x) = -\dfrac{\cos x}{\sqrt{1-\sin^2 x}} = \dfrac{\cos x}{|\cos x|}$



77. Let $f(x) = 2\sin^{-1} x - \cos^{-1}(1-2x^2)$. Then

$$f'(x) = \frac{2}{\sqrt{1-x^2}} - \frac{4x}{\sqrt{1-(1-2x^2)^2}} = \frac{2}{\sqrt{1-x^2}} - \frac{4x}{2x\sqrt{1-x^2}} = 0 \qquad [\text{since } x \geq 0]$$

Thus $f'(x) = 0$ for all $x \in [0,1)$. Thus $f(x) = C$. To find C let $x = 0$. Thus $2\sin^{-1}(0) - \cos^{-1}(1) = 0 = C$. Therefore we see that $f(x) = 2\sin^{-1} x - \cos^{-1}(1-2x^2) = 0 \;\Rightarrow\; 2\sin^{-1} x = \cos^{-1}(1-2x^2)$.

78. Let $f(x) = \sin^{-1}\left(\dfrac{x-1}{x+1}\right) - 2\tan^{-1}\sqrt{x} + \frac{\pi}{2}$. Note that the domain of f is $[0,\infty)$. Thus

$$f'(x) = \frac{1}{\sqrt{1-\left(\dfrac{x-1}{x+1}\right)^2}}\,\frac{(x+1)-(x-1)}{(x+1)^2} - \frac{2}{1+x}\cdot\frac{1}{2\sqrt{x}} = \frac{1}{\sqrt{x}\,(x+1)} - \frac{1}{\sqrt{x}\,(x+1)} = 0$$

Then $f(x) = C$. To find C, we let $x = 0 \;\Rightarrow\; \sin^{-1}(-1) - 2\tan^{-1}(0) + \frac{\pi}{2} = C \;\Rightarrow\; -\frac{\pi}{2} - 0 + \frac{\pi}{2} = 0 = C$.
Thus, $f(x) = 0 \;\Rightarrow\; \sin^{-1}\left(\dfrac{x-1}{x+1}\right) = 2\tan^{-1}\sqrt{x} - \frac{\pi}{2}$.

79. $y = \sec^{-1} x \;\Rightarrow\; \sec y = x \;\Rightarrow\; \sec y \tan y \dfrac{dy}{dx} = 1 \;\Rightarrow\; \dfrac{dy}{dx} = \dfrac{1}{\sec y \tan y}$. Now $\tan^2 y = \sec^2 y - 1 = x^2 - 1$, so $\tan y = \pm\sqrt{x^2-1}$. For $y \in \left[0, \frac{\pi}{2}\right)$, $x \geq 1$, so $\sec y = x = |x|$ and $\tan y \geq 0$ $\Rightarrow\; \dfrac{dy}{dx} = \dfrac{1}{x\sqrt{x^2-1}} = \dfrac{1}{|x|\sqrt{x^2-1}}$. For $y \in \left(\frac{\pi}{2}, \pi\right]$, $x \leq -1$, so $|x| = -x$ and $\tan y = -\sqrt{x^2-1} \;\Rightarrow$

$$\frac{dy}{dx} = \frac{1}{\sec y\,\tan y} = \frac{1}{x\left(-\sqrt{x^2-1}\right)} = \frac{1}{(-x)\sqrt{x^2-1}} = \frac{1}{|x|\sqrt{x^2-1}}$$

80. (a) Since $|\arctan(1/x)| < \frac{\pi}{2}$, we have $0 \le |x \arctan(1/x)| \le \frac{\pi}{2}|x| \to 0$ as $x \to 0$. So, by the Squeeze Theorem, $\lim\limits_{x \to 0} f(x) = 0 = f(0)$, so f is continuous at 0.

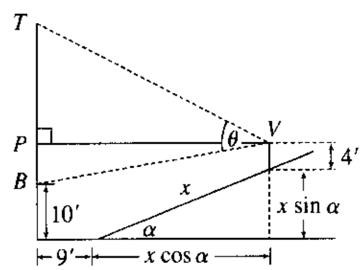
(b) Here $\dfrac{f(x) - f(0)}{x - 0} = \dfrac{x \arctan(1/x) - 0}{x} = \arctan\left(\dfrac{1}{x}\right)$. So (see Exercise 44 in Section 3.2 for a discussion of left- and right-hand derivatives)

$$f'_-(0) = \lim_{x \to 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^-} \arctan\left(\frac{1}{x}\right) = \lim_{y \to -\infty} \arctan y = -\frac{\pi}{2}, \text{ while}$$

$$f'_+(0) = \lim_{x \to 0^+} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0^+} \arctan\left(\frac{1}{x}\right) = \lim_{y \to \infty} \arctan y = \frac{\pi}{2}. \text{ So } f'(0) \text{ does not exist.}$$

APPLIED PROJECT Where to Sit at the Movies

1.



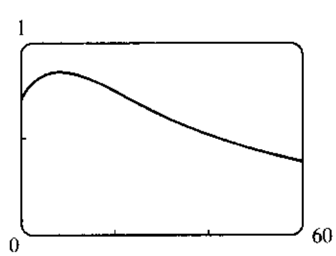
$|VP| = 9 + x\cos\alpha$, $|PT| = 35 - (4 + x\sin\alpha) = 31 - x\sin\alpha$, and

$|PB| = (4 + x\sin\alpha) - 10 = x\sin\alpha - 6$. So using the Pythagorean Theorem, we have

$|VT| = \sqrt{|VP|^2 + |PT|^2} = \sqrt{(9 + x\cos\alpha)^2 + (31 - x\sin\alpha)^2} = a$, and

$|VB| = \sqrt{|VP|^2 + |PB|^2} = \sqrt{(9 + x\cos\alpha)^2 + (x\sin\alpha - 6)^2} = b$. Using the Law of Cosines on $\triangle VBT$, we get $25^2 = a^2 + b^2 - 2ab\cos\theta \iff \cos\theta = \dfrac{a^2 + b^2 - 625}{2ab} \iff \theta = \arccos\left(\dfrac{a^2 + b^2 - 625}{2ab}\right)$, as required.

2. From the graph of θ, it appears that the value of x which maximizes θ is $x \approx 8.25$ ft. Assuming that the first row is at $x = 0$, the row closest to this value of x is the fourth row, at $x = 9$ ft, and from the graph, the viewing angle in this row seems to be about 0.85 radians, or about 49°.



3. With a CAS, we type in the definition of θ, substitute in the proper values of a and b in terms of x and $\alpha = 20^\circ = \frac{\pi}{9}$ radians, and then use the differentiation command to find the derivative. We use a numerical root finder and find that the root of the equation $d\theta/dx = 0$ is $x \approx 8.253062$, as approximated in Problem 2.

4. From the graph in Problem 2, it seems that the average value of the function on the interval $[0, 60]$ is about 0.6. We can use a CAS to approximate $\frac{1}{60}\int_0^{60} \theta(x)\,dx \approx 0.625 \approx 36^\circ$. (The calculation is much faster if we reduce the number of digits of accuracy required.) The minimum value is $\theta(60) \approx 0.38$ and, from Problem 2, the maximum value is about 0.85.

7.6 Hyperbolic Functions

1. (a) $\sinh 0 = \frac{1}{2}(e^0 - e^0) = 0$ (b) $\cosh 0 = \frac{1}{2}(e^0 + e^0) = \frac{1}{2}(1+1) = 1$

2. (a) $\tanh 0 = \dfrac{(e^0 - e^{-0})/2}{(e^0 + e^{-0})/2} = 0$ (b) $\tanh 1 = \dfrac{e^1 - e^{-1}}{e^1 + e^{-1}} = \dfrac{e^2 - 1}{e^2 + 1} \approx 0.76159$

3. (a) $\sinh(\ln 2) = \dfrac{e^{\ln 2} - e^{-\ln 2}}{2} = \dfrac{e^{\ln 2} - (e^{\ln 2})^{-1}}{2} = \dfrac{2 - 2^{-1}}{2} = \dfrac{2 - \frac{1}{2}}{2} = \dfrac{3}{4}$

(b) $\sinh 2 = \frac{1}{2}(e^2 - e^{-2}) \approx 3.62686$

4. (a) $\cosh 3 = \frac{1}{2}(e^3 + e^{-3}) \approx 10.06766$ (b) $\cosh(\ln 3) = \dfrac{e^{\ln 3} + e^{-\ln 3}}{2} = \dfrac{3 + \frac{1}{3}}{2} = \dfrac{5}{3}$

5. (a) $\operatorname{sech} 0 = \dfrac{1}{\cosh 0} = \dfrac{1}{1} = 1$ (b) $\cosh^{-1} 1 = 0$ because $\cosh 0 = 1$.

6. (a) $\sinh 1 = \frac{1}{2}(e^1 - e^{-1}) \approx 1.17520$

(b) Using Equation 3, we have $\sinh^{-1} 1 = \ln\left(1 + \sqrt{1^2 + 1}\right) = \ln\left(1 + \sqrt{2}\right) \approx 0.88137$.

7. $\sinh(-x) = \frac{1}{2}\left[e^{-x} - e^{-(-x)}\right] = \frac{1}{2}(e^{-x} - e^x) = -\frac{1}{2}(e^x - e^{-x}) = -\sinh x$

8. $\cosh(-x) = \frac{1}{2}\left[e^{-x} + e^{-(-x)}\right] = \frac{1}{2}(e^{-x} + e^x) = \frac{1}{2}(e^x + e^{-x}) = \cosh x$

9. $\cosh x + \sinh x = \frac{1}{2}(e^x + e^{-x}) + \frac{1}{2}(e^x - e^{-x}) = \frac{1}{2}(2e^x) = e^x$

10. $\cosh x - \sinh x = \frac{1}{2}(e^x + e^{-x}) - \frac{1}{2}(e^x - e^{-x}) = \frac{1}{2}(2e^{-x}) = e^{-x}$

11. $\sinh x \cosh y + \cosh x \sinh y = \left[\frac{1}{2}(e^x - e^{-x})\right]\left[\frac{1}{2}(e^y + e^{-y})\right] + \left[\frac{1}{2}(e^x + e^{-x})\right]\left[\frac{1}{2}(e^y - e^{-y})\right]$

$$= \tfrac{1}{4}\left[(e^{x+y} + e^{x-y} - e^{-x+y} - e^{-x-y}) + (e^{x+y} - e^{x-y} + e^{-x+y} - e^{-x-y})\right]$$

$$= \tfrac{1}{4}(2e^{x+y} - 2e^{-x-y}) = \tfrac{1}{2}\left[e^{x+y} - e^{-(x+y)}\right] = \sinh(x+y)$$

12. $\cosh x \cosh y + \sinh x \sinh y = \left[\frac{1}{2}(e^x + e^{-x})\right]\left[\frac{1}{2}(e^y + e^{-y})\right] + \left[\frac{1}{2}(e^x - e^{-x})\right]\left[\frac{1}{2}(e^y - e^{-y})\right]$

$$= \tfrac{1}{4}\left[\left(e^{x+y} + e^{x-y} + e^{-x+y} + e^{-x-y}\right) + \left(e^{x+y} - e^{x-y} - e^{-x+y} + e^{-x-y}\right)\right]$$

$$= \tfrac{1}{4}\left(2e^{x+y} + 2e^{-x-y}\right) = \tfrac{1}{2}\left[e^{x+y} + e^{-(x+y)}\right] = \cosh(x+y)$$

13. Divide both sides of the identity $\cosh^2 x - \sinh^2 x = 1$ by $\sinh^2 x$:

$$\frac{\cosh^2 x}{\sinh^2 x} - \frac{\sinh^2 x}{\sinh^2 x} = \frac{1}{\sinh^2 x} \quad \Leftrightarrow \quad \coth^2 x - 1 = \operatorname{csch}^2 x.$$

14. $\tanh(x+y) = \dfrac{\sinh(x+y)}{\cosh(x+y)} = \dfrac{\sinh x \cosh y + \cosh x \sinh y}{\cosh x \cosh y + \sinh x \sinh y} = \dfrac{\dfrac{\sinh x \cosh y}{\cosh x \cosh y} + \dfrac{\cosh x \sinh y}{\cosh x \cosh y}}{\dfrac{\cosh x \cosh y}{\cosh x \cosh y} + \dfrac{\sinh x \sinh y}{\cosh x \cosh y}}$

$$= \frac{\tanh x + \tanh y}{1 + \tanh x \tanh y}$$

15. Putting $y = x$ in the result from Exercise 11, we have

$\sinh 2x = \sinh(x+x) = \sinh x \cosh x + \cosh x \sinh x = 2 \sinh x \cosh x.$

16. Putting $y = x$ in the result from Exercise 12, we have

$\cosh 2x = \cosh(x+x) = \cosh x \cosh x + \sinh x \sinh x = \cosh^2 x + \sinh^2 x.$

17. $\tanh(\ln x) = \dfrac{\sinh(\ln x)}{\cosh(\ln x)} = \dfrac{(e^{\ln x} - e^{-\ln x})/2}{(e^{\ln x} + e^{-\ln x})/2} = \dfrac{x - (e^{\ln x})^{-1}}{x + (e^{\ln x})^{-1}} = \dfrac{x - x^{-1}}{x + x^{-1}}$

$$= \frac{x - 1/x}{x + 1/x} = \frac{(x^2-1)/x}{(x^2+1)/x} = \frac{x^2-1}{x^2+1}$$

18. $\dfrac{1 + \tanh x}{1 - \tanh x} = \dfrac{1 + (\sinh x)/\cosh x}{1 - (\sinh x)/\cosh x} = \dfrac{\cosh x + \sinh x}{\cosh x - \sinh x} = \dfrac{\frac{1}{2}(e^x + e^{-x}) + \frac{1}{2}(e^x - e^{-x})}{\frac{1}{2}(e^x + e^{-x}) - \frac{1}{2}(e^x - e^{-x})}$

$$= \frac{e^x + e^{-x} + e^x - e^{-x}}{e^x + e^{-x} - e^x + e^{-x}} = \frac{2e^x}{2e^{-x}} = e^{2x}$$

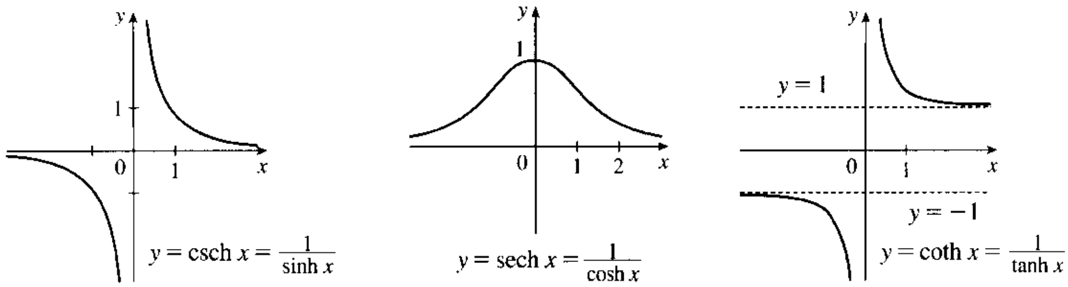
Or: Using the results of Exercises 9 and 10, $\dfrac{\cosh x + \sinh x}{\cosh x - \sinh x} = \dfrac{e^x}{e^{-x}} = e^{2x}$

19. By Exercise 9, $(\cosh x + \sinh x)^n = (e^x)^n = e^{nx} = \cosh nx + \sinh nx$.

20. $\sinh x = \frac{3}{4} \Rightarrow \operatorname{csch} x = 1/\sinh x = \frac{4}{3}$. $\cosh^2 x = \sinh^2 x + 1 = \frac{9}{16} + 1 = \frac{25}{16} \Rightarrow \cosh x = \frac{5}{4}$ (since $\cosh x > 0$). $\operatorname{sech} x = 1/\cosh x = \frac{4}{5}$, $\tanh x = \sinh x/\cosh x = \frac{3/4}{5/4} = \frac{3}{5}$, and $\coth x = 1/\tanh x = \frac{5}{3}$.

21. $\tanh x = \frac{4}{5} > 0$, so $x > 0$. $\coth x = 1/\tanh x = \frac{5}{4}$, $\operatorname{sech}^2 x = 1 - \tanh^2 x = 1 - \left(\frac{4}{5}\right)^2 = \frac{9}{25} \Rightarrow$ $\operatorname{sech} x = \frac{3}{5}$ (since $\operatorname{sech} x > 0$), $\cosh x = 1/\operatorname{sech} x = \frac{5}{3}$, $\sinh x = \tanh x \cosh x = \frac{4}{5} \cdot \frac{5}{3} = \frac{4}{3}$, and $\operatorname{csch} x = 1/\sinh x = \frac{3}{4}$.

22.



23. (a) $\displaystyle\lim_{x\to\infty} \tanh x = \lim_{x\to\infty} \frac{e^x - e^{-x}}{e^x + e^{-x}} \cdot \frac{e^{-x}}{e^{-x}} = \lim_{x\to\infty} \frac{1 - e^{-2x}}{1 + e^{-2x}} = \frac{1-0}{1+0} = 1$

(b) $\displaystyle\lim_{x\to-\infty} \tanh x = \lim_{x\to-\infty} \frac{e^x - e^{-x}}{e^x + e^{-x}} \cdot \frac{e^{x}}{e^{x}} = \lim_{x\to-\infty} \frac{e^{2x} - 1}{e^{2x} + 1} = \frac{0-1}{0+1} = -1$

(c) $\displaystyle\lim_{x\to\infty} \sinh x = \lim_{x\to\infty} \frac{e^x - e^{-x}}{2} = \infty$

(d) $\displaystyle\lim_{x\to-\infty} \sinh x = \lim_{x\to-\infty} \frac{e^x - e^{-x}}{2} = -\infty$

(e) $\displaystyle\lim_{x\to\infty} \operatorname{sech} x = \lim_{x\to\infty} \frac{2}{e^x + e^{-x}} = 0$

(f) $\displaystyle\lim_{x\to\infty} \coth x = \lim_{x\to\infty} \frac{e^x + e^{-x}}{e^x - e^{-x}} \cdot \frac{e^{-x}}{e^{-x}} = \lim_{x\to\infty} \frac{1 + e^{-2x}}{1 - e^{-2x}} = \frac{1+0}{1-0} = 1$ [*Or:* Use part (a)]

(g) $\displaystyle\lim_{x\to0^+} \coth x = \lim_{x\to0^+} \frac{\cosh x}{\sinh x} = \infty$, since $\sinh x \to 0$ through positive values and $\cosh x \to 1$.

(h) $\displaystyle\lim_{x\to0^-} \coth x = \lim_{x\to0^-} \frac{\cosh x}{\sinh x} = -\infty$, since $\sinh x \to 0$ through negative values and $\cosh x \to 1$.

(i) $\displaystyle\lim_{x\to-\infty} \operatorname{csch} x = \lim_{x\to-\infty} \frac{2}{e^x - e^{-x}} = 0$

24. (a) $\displaystyle\frac{d}{dx} \cosh x = \frac{d}{dx}\left[\tfrac{1}{2}\left(e^x + e^{-x}\right)\right] = \tfrac{1}{2}\left(e^x - e^{-x}\right) = \sinh x$

(b) $\displaystyle\frac{d}{dx} \tanh x = \frac{d}{dx}\left[\frac{\sinh x}{\cosh x}\right] = \frac{\cosh x \cosh x - \sinh x \sinh x}{\cosh^2 x} = \frac{\cosh^2 x - \sinh^2 x}{\cosh^2 x} = \frac{1}{\cosh^2 x} = \operatorname{sech}^2 x$

(c) $\displaystyle\frac{d}{dx} \operatorname{csch} x = \frac{d}{dx}\left[\frac{1}{\sinh x}\right] = -\frac{\cosh x}{\sinh^2 x} = -\frac{1}{\sinh x} \cdot \frac{\cosh x}{\sinh x} = -\operatorname{csch} x \coth x$

(d) $\displaystyle\frac{d}{dx} \operatorname{sech} x = \frac{d}{dx}\left[\frac{1}{\cosh x}\right] = -\frac{\sinh x}{\cosh^2 x} = -\frac{1}{\cosh x} \cdot \frac{\sinh x}{\cosh x} = -\operatorname{sech} x \tanh x$

(e) $\displaystyle\frac{d}{dx} \coth x = \frac{d}{dx}\left[\frac{\cosh x}{\sinh x}\right] = \frac{\sinh x \sinh x - \cosh x \cosh x}{\sinh^2 x} = \frac{\sinh^2 x - \cosh^2 x}{\sinh^2 x} = -\frac{1}{\sinh^2 x}$
$= -\operatorname{csch}^2 x$

25. Let $y = \sinh^{-1} x$. Then $\sinh y = x$ and, by Example 1(a), $\cosh^2 y - \sinh^2 y = 1 \Rightarrow$ [with $\cosh y > 0$] $\cosh y = \sqrt{1 + \sinh^2 y} = \sqrt{1 + x^2}$. So by Exercise 9, $e^y = \sinh y + \cosh y = x + \sqrt{1 + x^2} \Rightarrow$ $y = \ln\left(x + \sqrt{1 + x^2}\,\right)$.

26. Let $y = \cosh^{-1} x$. Then $\cosh y = x$ and $y \geq 0$, so $\sinh y = \sqrt{\cosh^2 y - 1} = \sqrt{x^2 - 1}$. So, by Exercise 9, $e^y = \cosh y + \sinh y = x + \sqrt{x^2 - 1} \quad \Rightarrow \quad y = \ln\left(x + \sqrt{x^2 - 1}\right)$.

Another method: Write $x = \cosh y = \frac{1}{2}\left(e^y + e^{-y}\right)$ and solve a quadratic, as in Example 3.

27. (a) Let $y = \tanh^{-1} x$. Then $x = \tanh y = \dfrac{\sinh y}{\cosh y} = \dfrac{(e^y - e^{-y})/2}{(e^y + e^{-y})/2} \cdot \dfrac{e^y}{e^y} = \dfrac{e^{2y} - 1}{e^{2y} + 1} \quad \Rightarrow$

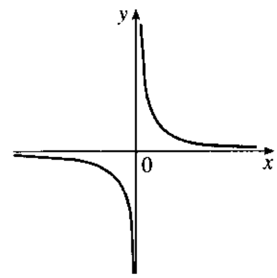
$xe^{2y} + x = e^{2y} - 1 \quad \Rightarrow \quad 1 + x = e^{2y} - xe^{2y} \quad \Rightarrow \quad 1 + x = e^{2y}(1 - x) \quad \Rightarrow$

$e^{2y} = \dfrac{1 + x}{1 - x} \quad \Rightarrow \quad 2y = \ln\left(\dfrac{1 + x}{1 - x}\right) \quad \Rightarrow \quad y = \frac{1}{2}\ln\left(\dfrac{1 + x}{1 - x}\right)$.

(b) Let $y = \tanh^{-1} x$. Then $x = \tanh y$, so from Exercise 18 we have

$e^{2y} = \dfrac{1 + \tanh y}{1 - \tanh y} = \dfrac{1 + x}{1 - x} \quad \Rightarrow \quad 2y = \ln\left(\dfrac{1 + x}{1 - x}\right) \quad \Rightarrow \quad y = \frac{1}{2}\ln\left(\dfrac{1 + x}{1 - x}\right)$.

28. (a) (i) $y = \operatorname{csch}^{-1} x \quad \Leftrightarrow \quad \operatorname{csch} y = x \quad (x \neq 0)$



(ii) We sketch the graph of csch^{-1} by reflecting the graph of csch (see Exercise 22) about the line $y = x$.

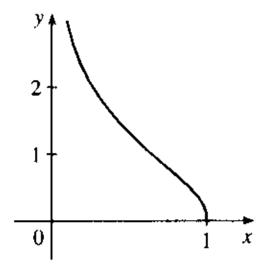
(iii) Let $y = \operatorname{csch}^{-1} x$. Then $x = \operatorname{csch} y = \dfrac{2}{e^y - e^{-y}} \quad \Rightarrow \quad xe^y - xe^{-y} = 2$

$\Rightarrow \quad x(e^y)^2 - 2e^y - x = 0 \quad \Rightarrow \quad e^y = \dfrac{1 \pm \sqrt{x^2 + 1}}{x}$.

But $e^y > 0$, so for $x > 0$, $e^y = \dfrac{1 + \sqrt{x^2 + 1}}{x}$ and for $x < 0$, $e^y = \dfrac{1 - \sqrt{x^2 + 1}}{x}$.

Thus, $\operatorname{csch}^{-1} x = \ln\left(\dfrac{1}{x} + \dfrac{\sqrt{x^2 + 1}}{|x|}\right)$.

(b) (i) $y = \operatorname{sech}^{-1} x \quad \Leftrightarrow \quad \operatorname{sech} y = x$ and $y > 0$.



(ii) We sketch the graph of sech^{-1} by reflecting the graph of sech (see Exercise 22) about the line $y = x$.

(iii) Let $y = \operatorname{sech}^{-1} x$, so $x = \operatorname{sech} y = \dfrac{2}{e^y + e^{-y}} \quad \Rightarrow \quad xe^y + xe^{-y} = 2$

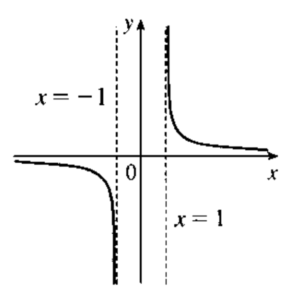
$\Rightarrow \quad x(e^y)^2 - 2e^y + x = 0 \quad \Leftrightarrow \quad e^y = \dfrac{1 \pm \sqrt{1 - x^2}}{x}$.

But $y > 0 \quad \Rightarrow \quad e^y > 1$. This rules out the minus sign because $\dfrac{1 - \sqrt{1 - x^2}}{x} > 1 \quad \Leftrightarrow \quad 1 - \sqrt{1 - x^2} > x$

$\Leftrightarrow \quad 1 - x > \sqrt{1 - x^2} \quad \Leftrightarrow \quad 1 - 2x + x^2 > 1 - x^2 \quad \Leftrightarrow \quad x^2 > x \quad \Leftrightarrow \quad x > 1$, but $x = \operatorname{sech} y \leq 1$. Thus,

$e^y = \dfrac{1 + \sqrt{1 - x^2}}{x} \quad \Rightarrow \quad \operatorname{sech}^{-1} x = \ln\left(\dfrac{1 + \sqrt{1 - x^2}}{x}\right)$.

(c) (i) $y = \coth^{-1} x \quad \Leftrightarrow \quad \coth y = x$



(ii) We sketch the graph of $\coth^{-1}$ by reflecting the graph of coth (see Exercise 22) about the line $y = x$.

(iii) Let $y = \coth^{-1} x$. Then $x = \coth y = \dfrac{e^y + e^{-y}}{e^y - e^{-y}} \quad \Rightarrow$

$xe^y - xe^{-y} = e^y + e^{-y} \quad \Rightarrow \quad (x - 1)\,e^y = (x + 1)\,e^{-y} \quad \Rightarrow$

$e^{2y} = \dfrac{x + 1}{x - 1} \quad \Rightarrow \quad 2y = \ln\dfrac{x + 1}{x - 1} \quad \Rightarrow \quad \coth^{-1} x = \frac{1}{2}\ln\dfrac{x + 1}{x - 1}$

29. (a) Let $y = \cosh^{-1} x$. Then $\cosh y = x$ and $y \geq 0 \;\Rightarrow\; \sinh y \dfrac{dy}{dx} = 1 \;\Rightarrow$

$\dfrac{dy}{dx} = \dfrac{1}{\sinh y} = \dfrac{1}{\sqrt{\cosh^2 y - 1}} = \dfrac{1}{\sqrt{x^2 - 1}}$ (since $\sinh y \geq 0$ for $y \geq 0$). *Or:* Use Formula 4.

(b) Let $y = \tanh^{-1} x$. Then $\tanh y = x \;\Rightarrow\; \operatorname{sech}^2 y \dfrac{dy}{dx} = 1 \;\Rightarrow\; \dfrac{dy}{dx} = \dfrac{1}{\operatorname{sech}^2 y} = \dfrac{1}{1 - \tanh^2 y} = \dfrac{1}{1 - x^2}$.

Or: Use Formula 5.

(c) Let $y = \operatorname{csch}^{-1} x$. Then $\operatorname{csch} y = x \;\Rightarrow\; -\operatorname{csch} y \coth y \dfrac{dy}{dx} = 1 \;\Rightarrow\; \dfrac{dy}{dx} = -\dfrac{1}{\operatorname{csch} y \coth y}$.

By Exercise 13, $\coth y = \pm\sqrt{\operatorname{csch}^2 y + 1} = \pm\sqrt{x^2 + 1}$. If $x > 0$, then $\coth y > 0$, so $\coth y = \sqrt{x^2 + 1}$. If $x < 0$, then $\coth y < 0$, so $\coth y = -\sqrt{x^2 + 1}$. In either case we have

$\dfrac{dy}{dx} = -\dfrac{1}{\operatorname{csch} y \coth y} = -\dfrac{1}{|x| \sqrt{x^2 + 1}}$.

(d) Let $y = \operatorname{sech}^{-1} x$. Then $\operatorname{sech} y = x \;\Rightarrow\; -\operatorname{sech} y \tanh y \dfrac{dy}{dx} = 1 \;\Rightarrow$

$\dfrac{dy}{dx} = -\dfrac{1}{\operatorname{sech} y \tanh y} = -\dfrac{1}{\operatorname{sech} y \sqrt{1 - \operatorname{sech}^2 y}} = -\dfrac{1}{x\sqrt{1 - x^2}}$. (Note that $y > 0$ and so $\tanh y > 0$.)

(e) Let $y = \coth^{-1} x$. Then $\coth y = x \;\Rightarrow\; -\operatorname{csch}^2 y \dfrac{dy}{dx} = 1 \;\Rightarrow$

$\dfrac{dy}{dx} = -\dfrac{1}{\operatorname{csch}^2 y} = \dfrac{1}{1 - \coth^2 y} = \dfrac{1}{1 - x^2}$ by Exercise 13.

30. $f(x) = \tanh 4x \;\Rightarrow\; f'(x) = 4 \operatorname{sech}^2 4x$

31. $f(x) = x \cosh x \;\Rightarrow\; f'(x) = x\,(\cosh x)' + (\cosh x)(x)' = x \sinh x + \cosh x$

32. $g(x) = \sinh^2 x \;\Rightarrow\; g'(x) = 2 \sinh x \cosh x$

33. $h(x) = \sinh(x^2) \;\Rightarrow\; h'(x) = \cosh(x^2) \cdot 2x = 2x \cosh(x^2)$

34. $F(x) = \sinh x \tanh x \;\Rightarrow\; F'(x) = \sinh x \operatorname{sech}^2 x + \tanh x \cosh x$

35. $G(x) = \dfrac{1 - \cosh x}{1 + \cosh x} \;\Rightarrow$

$$G'(x) = \frac{(1 + \cosh x)(-\sinh x) - (1 - \cosh x)(\sinh x)}{(1 + \cosh x)^2} = \frac{-\sinh x - \sinh x \cosh x - \sinh x + \sinh x \cosh x}{(1 + \cosh x)^2} = \frac{-2 \sinh x}{(1 + \cosh x)^2}$$

36. $f(t) = e^t \operatorname{sech} t \;\Rightarrow\; f'(t) = e^t(-\operatorname{sech} t \tanh t) + (\operatorname{sech} t)\, e^t = e^t \operatorname{sech} t\,(1 - \tanh t)$

37. $h(t) = \coth \sqrt{1 + t^2} \;\Rightarrow\; h'(t) = -\operatorname{csch}^2 \sqrt{1 + t^2} \cdot \frac{1}{2}(1 + t^2)^{-1/2}\,(2t) = -\dfrac{t \operatorname{csch}^2 \sqrt{1 + t^2}}{\sqrt{1 + t^2}}$

38. $f(t) = \ln(\sinh t) \;\Rightarrow\; f'(t) = \dfrac{1}{\sinh t} \cosh t = \coth t$

39. $H(t) = \tanh(e^t) \;\Rightarrow\; H'(t) = \operatorname{sech}^2(e^t) \cdot e^t = e^t \operatorname{sech}^2(e^t)$

40. $y = \sinh(\cosh x) \;\Rightarrow\; y' = \cosh(\cosh x) \cdot \sinh x$

41. $y = e^{\cosh 3x} \;\Rightarrow\; y' = e^{\cosh 3x} \cdot \sinh 3x \cdot 3 = 3e^{\cosh 3x} \sinh 3x$

42. $y = x^2 \sinh^{-1}(2x) \;\Rightarrow\; y' = x^2 \cdot \dfrac{1}{\sqrt{1+(2x)^2}} \cdot 2 + \sinh^{-1}(2x) \cdot 2x = 2x\left[\dfrac{x}{\sqrt{1+4x^2}} + \sinh^{-1}(2x)\right]$

43. $y = \tanh^{-1}\sqrt{x} \;\Rightarrow\; y' = \dfrac{1}{1-(\sqrt{x})^2} \cdot \frac{1}{2}x^{-1/2} = \dfrac{1}{2\sqrt{x}(1-x)}$

44. $y = x\tanh^{-1}x + \ln\sqrt{1-x^2} = x\tanh^{-1}x + \frac{1}{2}\ln(1-x^2) \;\Rightarrow$

$y' = \tanh^{-1}x + \dfrac{x}{1-x^2} + \dfrac{1}{2}\left(\dfrac{1}{1-x^2}\right)(-2x) = \tanh^{-1}x$

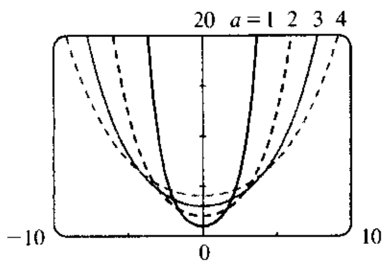
45. $y = x\sinh^{-1}(x/3) - \sqrt{9+x^2} \;\Rightarrow$

$y' = \sinh^{-1}\left(\dfrac{x}{3}\right) + x\dfrac{1/3}{\sqrt{1+(x/3)^2}} - \dfrac{2x}{2\sqrt{9+x^2}} = \sinh^{-1}\left(\dfrac{x}{3}\right) + \dfrac{x}{\sqrt{9+x^2}} - \dfrac{x}{\sqrt{9+x^2}} = \sinh^{-1}\left(\dfrac{x}{3}\right)$

46. $y = \operatorname{sech}^{-1}\sqrt{1-x^2} \;\Rightarrow\; y' = -\dfrac{1}{\sqrt{1-x^2}\sqrt{1-(1-x^2)}}\dfrac{-2x}{2\sqrt{1-x^2}} = \dfrac{x}{(1-x^2)\,|x|}$

47. $y = \coth^{-1}\sqrt{x^2+1} \;\Rightarrow\; y' = \dfrac{1}{1-(x^2+1)}\dfrac{2x}{2\sqrt{x^2+1}} = -\dfrac{1}{x\sqrt{x^2+1}}$

48.



For $y = a\cosh(x/a)$ with $a > 0$, we have the y-intercept equal to a. As a increases, the graph flattens.

49. (a) $y = 20\cosh(x/20) - 15 \;\Rightarrow\; y' = 20\sinh(x/20)\cdot\frac{1}{20} = \sinh(x/20)$. Since the right pole is positioned at $x = 7$, we have $y'(7) = \sinh\frac{7}{20} \approx 0.3572$.

(b) If α is the angle between the tangent line and the x-axis, then $\tan\alpha =$ slope of the line $= \sinh\frac{7}{20}$, so $\alpha = \tan^{-1}\left(\sinh\frac{7}{20}\right) \approx 0.343$ rad $\approx 19.66^\circ$. Thus, the angle between the line and the pole is $\theta = 90^\circ - \alpha \approx 70.34^\circ$.

50. We differentiate the function twice, then substitute into the differential equation: $y = \dfrac{T}{\rho g}\cosh\dfrac{\rho g x}{T} \;\Rightarrow$

$\dfrac{dy}{dx} = \dfrac{T}{\rho g}\sinh\left(\dfrac{\rho g x}{T}\right)\dfrac{\rho g}{T} = \sinh\dfrac{\rho g x}{T} \;\Rightarrow\; \dfrac{d^2y}{dx^2} = \cosh\left(\dfrac{\rho g x}{T}\right)\dfrac{\rho g}{T} = \dfrac{\rho g}{T}\cosh\dfrac{\rho g x}{T}$.

We evaluate the two sides separately: LHS $= \dfrac{d^2y}{dx^2} = \dfrac{\rho g}{T}\cosh\dfrac{\rho g x}{T}$,

RHS $= \dfrac{\rho g}{T}\sqrt{1+\left(\dfrac{dy}{dx}\right)^2} = \dfrac{\rho g}{T}\sqrt{1+\sinh^2\dfrac{\rho g x}{T}} = \dfrac{\rho g}{T}\cosh\dfrac{\rho g x}{T}$, by the identity proved in Example 1(a).

51. (a) $y = A\sinh mx + B\cosh mx \;\Rightarrow\; y' = mA\cosh mx + mB\sinh mx \;\Rightarrow$
$y'' = m^2A\sinh mx + m^2B\cosh mx = m^2(A\sinh mx + B\cosh mx) = m^2y$

(b) From part (a), a solution of $y'' = 9y$ is $y(x) = A\sinh 3x + B\cosh 3x$. So $-4 = y(0) = A\sinh 0 + B\cosh 0 = B$, so $B = -4$. Now $y'(x) = 3A\cosh 3x - 12\sinh 3x \;\Rightarrow$ $6 = y'(0) = 3A \;\Rightarrow\; A = 2$, so $y = 2\sinh 3x - 4\cosh 3x$.

52. $\displaystyle\lim_{x\to\infty}\frac{\sinh x}{e^x}=\lim_{x\to\infty}\frac{e^x-e^{-x}}{2e^x}=\lim_{x\to\infty}\frac{1-e^{-2x}}{2}=\frac{1-0}{2}=\frac{1}{2}$

53. The tangent to $y=\cosh x$ has slope 1 when $y'=\sinh x=1 \quad\Rightarrow\quad x=\sinh^{-1}1=\ln\left(1+\sqrt{2}\right)$, by Equation 3. Since $\sinh x=1$ and $y=\cosh x=\sqrt{1+\sinh^2 x}$, we have $\cosh x=\sqrt{2}$. The point is $\left(\ln\left(1+\sqrt{2}\right),\sqrt{2}\right)$.

54. $\cosh x=\cosh[\ln(\sec\theta+\tan\theta)]=\frac{1}{2}\left[e^{\ln(\sec\theta+\tan\theta)}+e^{-\ln(\sec\theta+\tan\theta)}\right]$

$$=\frac{1}{2}\left[\sec\theta+\tan\theta+\frac{1}{\sec\theta+\tan\theta}\right]=\frac{1}{2}\left[\sec\theta+\tan\theta+\frac{\sec\theta-\tan\theta}{(\sec\theta+\tan\theta)(\sec\theta-\tan\theta)}\right]$$

$$=\frac{1}{2}\left[\sec\theta+\tan\theta+\frac{\sec\theta-\tan\theta}{\sec^2\theta-\tan^2\theta}\right]=\tfrac{1}{2}(\sec\theta+\tan\theta+\sec\theta-\tan\theta)=\sec\theta$$

55. Let $u=\cosh x$. Then $du=\sinh x\,dx$, so $\int\sinh x\cosh^2 x\,dx=\int u^2\,du=\frac{1}{3}u^3+C=\frac{1}{3}\cosh^3 x+C$.

56. Let $u=1+4x$. Then $du=4\,dx$, so $\int\sinh(1+4x)\,dx=\frac{1}{4}\int\sinh u\,du=\frac{1}{4}\cosh u+C=\frac{1}{4}\cosh(1+4x)+C$.

57. Let $u=\sqrt{x}$. Then $du=\dfrac{dx}{2\sqrt{x}}$ and $\displaystyle\int\frac{\sinh\sqrt{x}}{\sqrt{x}}\,dx=\int\sinh u\cdot 2\,du=2\cosh u+C=2\cosh\sqrt{x}=C$.

58. Let $u=\cosh x$. Then $du=\sinh x\,dx$, and

$$\int\tanh x\,dx=\int\frac{\sinh x}{\cosh x}\,dx=\int\frac{du}{u}=\ln|u|+C=\ln(\cosh x)+C.$$

59. $\displaystyle\int\frac{\cosh x}{\cosh^2 x-1}\,dx=\int\frac{\cosh x}{\sinh^2 x}\,dx=\int\frac{\cosh x}{\sinh x}\cdot\frac{1}{\sinh x}\,dx=\int\coth x\operatorname{csch} x\,dx=-\operatorname{csch} x+C$

60. Let $u=2+\tanh x$. Then $du=\operatorname{sech}^2 x\,dx$, so

$$\int\frac{\operatorname{sech}^2 x}{2+\tanh x}\,dx=\int\frac{1}{u}\,du=\ln|u|+C=\ln|2+\tanh x|+C=\ln(2+\tanh x)+C$$

(since $2+\tanh x>1$).

61. Let $t=3u$. Then $dt=3\,du$ and

$$\int_4^6\frac{1}{\sqrt{t^2-9}}\,dt=\int_{4/3}^2\frac{1}{\sqrt{9u^2-9}}3\,du=\int_{4/3}^2\frac{du}{\sqrt{u^2-1}}=\left[\cosh^{-1}u\right]_{4/3}^2=\cosh^{-1}2-\cosh^{-1}\left(\tfrac{4}{3}\right)\quad\text{or}$$

$$=\left[\cosh^{-1}u\right]_{4/3}^2=\left[\ln\left(u+\sqrt{u^2-1}\right)\right]_{4/3}^2$$

$$=\ln\left(2+\sqrt{3}\right)-\ln\left(\frac{4+\sqrt{7}}{3}\right)=\ln\left(\frac{6+3\sqrt{3}}{4+\sqrt{7}}\right)$$

62. Let $u=4t$. Then $du=4\,dt$ and

$$\int_0^1\frac{dt}{\sqrt{16t^2+1}}=\int_0^4\frac{\frac{1}{4}\,du}{\sqrt{u^2+1}}=\frac{1}{4}\left[\sinh^{-1}u\right]_0^4=\frac{1}{4}\left[\ln\left(u+\sqrt{u^2+1}\right)\right]_0^4$$

$$=\tfrac{1}{4}\left[\ln\left(4+\sqrt{17}\right)-\ln 1\right]=\tfrac{1}{4}\ln\left(4+\sqrt{17}\right)$$

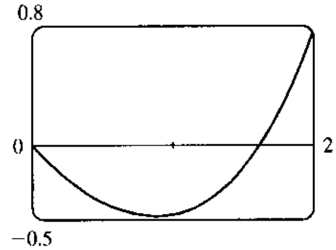
63. Let $u=e^x$. Then $du=e^x\,dx$ and $\displaystyle\int\frac{e^x}{1-e^{2x}}\,dx=\int\frac{du}{1-u^2}=\tanh^{-1}u+C=\tanh^{-1}(e^x)+C$

$\left[\text{or }\dfrac{1}{2}\ln\left(\dfrac{1+e^x}{1-e^x}\right)+C\right]$.

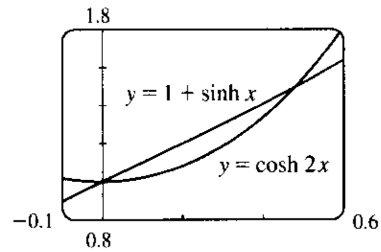
64. We want $\int_0^1 \sinh cx\,dx = 1$. To calculate the integral, we put $u = cx$, so $du = c\,dx$, the upper limit becomes c, and the equation becomes

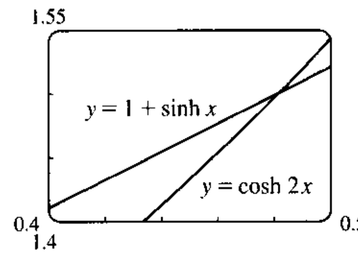
$$\frac{1}{c}\int_0^c \sinh u\,du = 1 \quad\Leftrightarrow\quad \frac{1}{c}[\cosh c - 1] = 1 \quad\Leftrightarrow\quad \cosh c - 1 = c.$$

We plot the function $f(c) = \cosh c - c - 1$, and see that its positive root lies at approximately $c = 1.62$. So the equation $\int_0^1 \sinh cx\,dx = 1$ holds for $c \approx 1.62$.



65. (a) From the graphs, we estimate that the two curves $y = \cosh 2x$ and $y = 1 + \sinh x$ intersect at $x = 0$ and at $x = a \approx 0.481$.





(b) We have found the two roots of the equation $\cosh 2x = 1 + \sinh x$ to be $x = 0$ and $x = a \approx 0.481$. Note from the first graph that $1 + \sinh x > \cosh 2x$ on the interval $(0, a)$, so the area between the two curves is

$$\begin{aligned} A &= \textstyle\int_0^a (1 + \sinh x - \cosh 2x)\,dx = \left[x + \cosh x - \frac{1}{2}\sinh 2x\right]_0^a \\ &= \left[a + \cosh a - \tfrac{1}{2}\sinh 2a\right] - \left[0 + \cosh 0 - \tfrac{1}{2}\sinh 0\right] \approx 0.0402 \end{aligned}$$

66. The area of the triangle with vertices O, P, and $(\cosh t, 0)$ is $\frac{1}{2}\sinh t\cosh t$, and the area under the curve $x^2 - y^2 = 1$, from $x = 1$ to $x = \cosh t$, is $\int_1^{\cosh t}\sqrt{x^2-1}\,dx$. Therefore, the area of the shaded region is $A(t) = \frac{1}{2}\sinh t\cosh t - \int_1^{\cosh t}\sqrt{x^2-1}\,dx$. So, by FTC1,

$$\begin{aligned} A'(t) &= \tfrac{1}{2}(\cosh^2 t + \sinh^2 t) - \sqrt{\cosh^2 t - 1}\,\sinh t = \tfrac{1}{2}(\cosh^2 t + \sinh^2 t) - \sqrt{\sinh^2 t}\,\sinh t \\ &= \tfrac{1}{2}(\cosh^2 t + \sinh^2 t) - \sinh^2 t = \tfrac{1}{2}(\cosh^2 t - \sinh^2 t) = \tfrac{1}{2}(1) = \tfrac{1}{2} \end{aligned}$$

Thus $A(t) = \frac{1}{2}t + C$, since $A'(t) = \frac{1}{2}$. To calculate C, we let $t = 0$. Thus, $A(0) = \frac{1}{2}\sinh 0\cosh 0 - \int_1^{\cosh 0}\sqrt{x^2-1}\,dx = \frac{1}{2}(0) + C \quad\Rightarrow\quad C = 0$. Thus $A(t) = \frac{1}{2}t$.

67. If $ae^x + be^{-x} = \alpha\cosh(x+\beta)$ [or $\alpha\sinh(x+\beta)$], then $ae^x + be^{-x} = \frac{\alpha}{2}\left(e^{x+\beta} \pm e^{-x-\beta}\right) = \frac{\alpha}{2}\left(e^x e^\beta \pm e^{-x}e^{-\beta}\right) = \left(\frac{\alpha}{2}e^\beta\right)e^x \pm \left(\frac{\alpha}{2}e^{-\beta}\right)e^{-x}$. Comparing coefficients of e^x and e^{-x}, we have $a = \frac{\alpha}{2}e^\beta$ **(1)** and $b = \pm\frac{\alpha}{2}e^{-\beta}$ **(2)**. We need to find α and β. Dividing equation **(1)** by equation **(2)** gives us $\dfrac{a}{b} = \pm e^{2\beta} \quad\Rightarrow\quad (*) \quad 2\beta = \ln\left(\pm\frac{a}{b}\right) \quad\Rightarrow\quad \beta = \frac{1}{2}\ln\left(\pm\frac{a}{b}\right)$. Solving equations **(1)** and **(2)** for e^β gives us $e^\beta = \dfrac{2a}{\alpha}$ and $e^\beta = \pm\dfrac{\alpha}{2b}$, so $\dfrac{2a}{\alpha} = \pm\dfrac{\alpha}{2b} \quad\Rightarrow\quad \alpha^2 = \pm 4ab \quad\Rightarrow\quad \alpha = 2\sqrt{\pm ab}$.

(*) If $\dfrac{a}{b} > 0$, we use the $+$ sign and obtain a cosh function, whereas if $\dfrac{a}{b} < 0$, we use the $-$ sign and obtain a sinh function.

In summary, if a and b have the same sign, we have $ae^x + be^{-x} = 2\sqrt{ab}\cosh\left(x + \frac{1}{2}\ln\frac{a}{b}\right)$, whereas, if a and b have the opposite sign, then $ae^x + be^{-x} = 2\sqrt{-ab}\sinh\left(x + \frac{1}{2}\ln\left(-\frac{a}{b}\right)\right)$.

7.7 Indeterminate Forms and L'Hospital's Rule

The use of l'Hospital's Rule is indicated by an H above the equal sign: $\stackrel{H}{=}$.

1. (a) $\lim_{x\to a} \frac{f(x)}{g(x)}$ is an indeterminate form of type $\frac{0}{0}$.

(b) $\lim_{x\to a} \frac{f(x)}{p(x)} = 0$ because the numerator approaches 0 while the denominator becomes large.

(c) $\lim_{x\to a} \frac{h(x)}{p(x)} = 0$ because the numerator approaches a finite number while the denominator becomes large.

(d) If $\lim_{x\to a} p(x) = \infty$ and $f(x) \to 0$ through positive values, then $\lim_{x\to a} \frac{p(x)}{f(x)} = \infty$. [For example, take $a = 0$, $p(x) = 1/x^2$, and $f(x) = x^2$.] If $f(x) \to 0$ through negative values, then $\lim_{x\to a} \frac{p(x)}{f(x)} = -\infty$. [For example, take $a = 0$, $p(x) = 1/x^2$, and $f(x) = -x^2$.] If $f(x) \to 0$ through both positive and negative values, then the limit might not exist. [For example, take $a = 0$, $p(x) = 1/x^2$, and $f(x) = x$.]

(e) $\lim_{x\to a} \frac{p(x)}{q(x)}$ is an indeterminate form of type $\frac{\infty}{\infty}$.

2. (a) $\lim_{x\to a} [f(x)p(x)]$ is an indeterminate form of type $0 \cdot \infty$.

(b) When x is near a, $p(x)$ is large and $h(x)$ is near 1, so $h(x)p(x)$ is large. Thus, $\lim_{x\to a} [h(x)p(x)] = \infty$.

(c) When x is near a, $p(x)$ and $q(x)$ are both large, so $p(x)q(x)$ is large. Thus, $\lim_{x\to a} [p(x)q(x)] = \infty$.

3. (a) When x is near a, $f(x)$ is near 0 and $p(x)$ is large, so $f(x) - p(x)$ is large negative. Thus, $\lim_{x\to a} [f(x) - p(x)] = -\infty$.

(b) $\lim_{x\to a} [p(x) - q(x)]$ is an indeterminate form of type $\infty - \infty$.

(c) When x is near a, $p(x)$ and $q(x)$ are both large, so $p(x) + q(x)$ is large. Thus, $\lim_{x\to a} [p(x) + q(x)] = \infty$.

4. (a) $\lim_{x\to a} [f(x)]^{g(x)}$ is an indeterminate form of type 0^0.

(b) If $y = [f(x)]^{p(x)}$, then $\ln y = p(x) \ln f(x)$. When x is near a, $p(x) \to \infty$ and $\ln f(x) \to -\infty$, so $\ln y \to -\infty$. Therefore, $\lim_{x\to a} [f(x)]^{p(x)} = \lim_{x\to a} y = \lim_{x\to a} e^{\ln y} = 0$, provided f^p is defined.

(c) $\lim_{x\to a} [h(x)]^{p(x)}$ is an indeterminate form of type 1^∞.

(d) $\lim_{x\to a} [p(x)]^{f(x)}$ is an indeterminate form of type ∞^0.

(e) If $y = [p(x)]^{q(x)}$, then $\ln y = q(x) \ln p(x)$. When x is near a, $q(x) \to \infty$ and $\ln p(x) \to \infty$, so $\ln y \to \infty$. Therefore, $\lim_{x\to a} [p(x)]^{q(x)} = \lim_{x\to a} y = \lim_{x\to a} e^{\ln y} = \infty$.

(f) $\lim_{x\to a} \sqrt[q(x)]{p(x)} = \lim_{x\to a} [p(x)]^{1/q(x)}$ is an indeterminate form of type ∞^0.

5. This limit has the form $\frac{0}{0}$. We can simply factor the numerator to evaluate this limit.

$$\lim_{x\to -1} \frac{x^2 - 1}{x + 1} = \lim_{x\to -1} \frac{(x+1)(x-1)}{x+1} = \lim_{x\to -1} (x - 1) = -2$$

6. $\lim\limits_{x\to -2}\dfrac{x+2}{x^2+3x+2}=\lim\limits_{x\to -2}\dfrac{x+2}{(x+1)(x+2)}=\lim\limits_{x\to -2}\dfrac{1}{x+1}=-1$

7. This limit has the form $\frac{0}{0}$. $\lim\limits_{x\to 1}\dfrac{x^9-1}{x^5-1}\overset{\text{H}}{=}\lim\limits_{x\to 1}\dfrac{9x^8}{5x^4}=\dfrac{9}{5}\lim\limits_{x\to 1}x^4=\dfrac{9}{5}(1)=\dfrac{9}{5}$

8. $\lim\limits_{x\to 1}\dfrac{x^a-1}{x^b-1}\overset{\text{H}}{=}\lim\limits_{x\to 1}\dfrac{ax^{a-1}}{bx^{b-1}}=\dfrac{a}{b}$

9. This limit has the form $\frac{0}{0}$. $\lim\limits_{x\to(\pi/2)^+}\dfrac{\cos x}{1-\sin x}\overset{\text{H}}{=}\lim\limits_{x\to(\pi/2)^+}\dfrac{-\sin x}{-\cos x}=\lim\limits_{x\to(\pi/2)^+}\tan x=-\infty.$

10. $\lim\limits_{x\to 0}\dfrac{x+\tan x}{\sin x}\overset{\text{H}}{=}\lim\limits_{x\to 0}\dfrac{1+\sec^2 x}{\cos x}=\dfrac{1+1^2}{1}=2$

11. This limit has the form $\frac{0}{0}$. $\lim\limits_{t\to 0}\dfrac{e^t-1}{t^3}\overset{\text{H}}{=}\lim\limits_{t\to 0}\dfrac{e^t}{3t^2}=\infty$ since $e^t\to 1$ and $3t^2\to 0^+$ as $t\to 0$.

12. $\lim\limits_{t\to 0}\dfrac{e^{3t}-1}{t}\overset{\text{H}}{=}\lim\limits_{t\to 0}\dfrac{3e^{3t}}{1}=3$

13. This limit has the form $\frac{0}{0}$. $\lim\limits_{x\to 0}\dfrac{\tan px}{\tan qx}\overset{\text{H}}{=}\lim\limits_{x\to 0}\dfrac{p\sec^2 px}{q\sec^2 qx}=\dfrac{p(1)^2}{q(1)^2}=\dfrac{p}{q}$

14. $\lim\limits_{\theta\to\pi/2}\dfrac{1-\sin\theta}{\csc\theta}=\dfrac{0}{1}=0$. L'Hospital's Rule does not apply.

15. This limit has the form $\frac{\infty}{\infty}$. $\lim\limits_{x\to\infty}\dfrac{\ln x}{x}\overset{\text{H}}{=}\lim\limits_{x\to\infty}\dfrac{1/x}{1}=0$

16. $\lim\limits_{x\to\infty}\dfrac{e^x}{x}\overset{\text{H}}{=}\lim\limits_{x\to\infty}\dfrac{e^x}{1}=\lim\limits_{x\to\infty}e^x=\infty$

17. $\lim\limits_{x\to 0^+}[(\ln x)/x]=-\infty$ since $\ln x\to-\infty$ as $x\to 0^+$ and dividing by small values of x just increases the magnitude of the quotient $(\ln x)/x$. L'Hospital's Rule does not apply.

18. $\lim\limits_{x\to\infty}\dfrac{\ln\ln x}{x}\overset{\text{H}}{=}\lim\limits_{x\to\infty}\dfrac{\frac{1}{\ln x}\cdot\frac{1}{x}}{1}=\lim\limits_{x\to\infty}\dfrac{1}{x\ln x}=0$

19. This limit has the form $\frac{0}{0}$. $\lim\limits_{t\to 0}\dfrac{5^t-3^t}{t}\overset{\text{H}}{=}\lim\limits_{t\to 0}\dfrac{5^t\ln 5-3^t\ln 3}{1}=\ln 5-\ln 3=\ln\frac{5}{3}$

20. $\lim\limits_{x\to 1}\dfrac{\ln x}{\sin\pi x}\overset{\text{H}}{=}\lim\limits_{x\to 1}\dfrac{1/x}{\pi\cos\pi x}=\dfrac{1}{\pi(-1)}=-\dfrac{1}{\pi}$

21. This limit has the form $\frac{0}{0}$. $\lim\limits_{x\to 0}\dfrac{e^x-1-x}{x^2}\overset{\text{H}}{=}\lim\limits_{x\to 0}\dfrac{e^x-1}{2x}\overset{\text{H}}{=}\lim\limits_{x\to 0}\dfrac{e^x}{2}=\dfrac{1}{2}$

22. $\lim\limits_{x\to 0}\dfrac{e^x-1-x-x^2/2}{x^3}\overset{\text{H}}{=}\lim\limits_{x\to 0}\dfrac{e^x-1-x}{3x^2}\overset{\text{H}}{=}\lim\limits_{x\to 0}\dfrac{e^x-1}{6x}\overset{\text{H}}{=}\lim\limits_{x\to 0}\dfrac{e^x}{6}=\dfrac{1}{6}$

23. This limit has the form $\frac{\infty}{\infty}$. $\lim\limits_{x\to\infty}\dfrac{e^x}{x^3}\overset{\text{H}}{=}\lim\limits_{x\to\infty}\dfrac{e^x}{3x^2}\overset{\text{H}}{=}\lim\limits_{x\to\infty}\dfrac{e^x}{6x}\overset{\text{H}}{=}\lim\limits_{x\to\infty}\dfrac{e^x}{6}=\infty$

24. $\lim\limits_{x\to 0}\dfrac{\sin x}{\sinh x}\overset{\text{H}}{=}\lim\limits_{x\to 0}\dfrac{\cos x}{\cosh x}=\dfrac{1}{1}=1$

25. This limit has the form $\frac{0}{0}$. $\lim\limits_{x\to 0}\dfrac{\sin^{-1}x}{x}\overset{\text{H}}{=}\lim\limits_{x\to 0}\dfrac{1/\sqrt{1-x^2}}{1}=\lim\limits_{x\to 0}\dfrac{1}{\sqrt{1-x^2}}=\dfrac{1}{1}=1$

26. $\lim\limits_{x\to 0}\dfrac{\sin x-x}{x^3}\overset{\text{H}}{=}\lim\limits_{x\to 0}\dfrac{\cos x-1}{3x^2}\overset{\text{H}}{=}\lim\limits_{x\to 0}\dfrac{-\sin x}{6x}\overset{\text{H}}{=}\lim\limits_{x\to 0}\dfrac{-\cos x}{6}=-\dfrac{1}{6}$

27. This limit has the form $\frac{0}{0}$. $\lim\limits_{x\to 0}\dfrac{1-\cos x}{x^2}\overset{\text{H}}{=}\lim\limits_{x\to 0}\dfrac{\sin x}{2x}\overset{\text{H}}{=}\lim\limits_{x\to 0}\dfrac{\cos x}{2}=\dfrac{1}{2}$

28. $\lim\limits_{x\to\infty}\dfrac{(\ln x)^2}{x}\overset{\text{H}}{=}\lim\limits_{x\to\infty}\dfrac{2(\ln x)(1/x)}{1}=2\lim\limits_{x\to\infty}\dfrac{\ln x}{x}\overset{\text{H}}{=}2\lim\limits_{x\to\infty}\dfrac{1/x}{1}=2(0)=0$

29. $\lim\limits_{x\to0}\dfrac{x+\sin x}{x+\cos x}=\dfrac{0+0}{0+1}=\dfrac{0}{1}=0$. L'Hospital's Rule does not apply.

30. $\lim\limits_{x\to0}\dfrac{\cos mx-\cos nx}{x^2}\overset{\text{H}}{=}\lim\limits_{x\to0}\dfrac{-m\sin mx+n\sin nx}{2x}\overset{\text{H}}{=}\lim\limits_{x\to0}\dfrac{-m^2\cos mx+n^2\cos nx}{2}=\frac{1}{2}(n^2-m^2)$

31. This limit has the form $\frac{\infty}{\infty}$. $\lim\limits_{x\to\infty}\dfrac{x}{\ln(1+2e^x)}\overset{\text{H}}{=}\lim\limits_{x\to\infty}\dfrac{1}{\dfrac{1}{1+2e^x}\cdot 2e^x}=\lim\limits_{x\to\infty}\dfrac{1+2e^x}{2e^x}\overset{\text{H}}{=}\lim\limits_{x\to\infty}\dfrac{2e^x}{2e^x}=1$

32. $\lim\limits_{x\to0}\dfrac{x}{\tan^{-1}(4x)}\overset{\text{H}}{=}\lim\limits_{x\to0}\dfrac{1}{\dfrac{1}{1+(4x)^2}\cdot 4}=\lim\limits_{x\to0}\dfrac{1+16x^2}{4}=\dfrac{1}{4}$

33. This limit has the form $\frac{0}{0}$. $\lim\limits_{x\to1}\dfrac{1-x+\ln x}{1+\cos\pi x}\overset{\text{H}}{=}\lim\limits_{x\to1}\dfrac{-1+1/x}{-\pi\sin\pi x}\overset{\text{H}}{=}\lim\limits_{x\to1}\dfrac{-1/x^2}{-\pi^2\cos\pi x}=\dfrac{-1}{-\pi^2(-1)}=-\dfrac{1}{\pi^2}$

34. $\lim\limits_{x\to\infty}\dfrac{\sqrt{x^2+2}}{\sqrt{2x^2+1}}=\lim\limits_{x\to\infty}\sqrt{\dfrac{x^2+2}{2x^2+1}}=\sqrt{\lim\limits_{x\to\infty}\dfrac{x^2+2}{2x^2+1}}=\sqrt{\lim\limits_{x\to\infty}\dfrac{1+2/x^2}{2+1/x^2}}=\sqrt{\dfrac{1}{2}}$

35. This limit has the form $\frac{0}{0}$. $\lim\limits_{x\to1}\dfrac{x^a-ax+a-1}{(x-1)^2}\overset{\text{H}}{=}\lim\limits_{x\to1}\dfrac{ax^{a-1}-a}{2(x-1)}\overset{\text{H}}{=}\lim\limits_{x\to1}\dfrac{a(a-1)x^{a-2}}{2}=\dfrac{a(a-1)}{2}$

36. $\lim\limits_{x\to0}\dfrac{1-e^{-2x}}{\sec x}=\dfrac{1-1}{1}=0$. L'Hospital's Rule does not apply.

37. This limit has the form $0\cdot(-\infty)$. We need to write this product as a quotient, but keep in mind that we will have to differentiate both the numerator and the denominator. If we differentiate $\dfrac{1}{\ln x}$, we get a complicated expression that results in a more difficult limit. Instead we write the quotient as $\dfrac{\ln x}{x^{-1/2}}$.

$$\lim_{x\to0^+}\sqrt{x}\ln x=\lim_{x\to0^+}\frac{\ln x}{x^{-1/2}}\overset{\text{H}}{=}\lim_{x\to0^+}\frac{1/x}{-\frac{1}{2}x^{-3/2}}\cdot\frac{-2x^{3/2}}{-2x^{3/2}}=\lim_{x\to0^+}(-2\sqrt{x})=0$$

38. $\lim\limits_{x\to-\infty}x^2e^x=\lim\limits_{x\to-\infty}\dfrac{x^2}{e^{-x}}\overset{\text{H}}{=}\lim\limits_{x\to-\infty}\dfrac{2x}{-e^{-x}}\overset{\text{H}}{=}\lim\limits_{x\to-\infty}\dfrac{2}{e^{-x}}=\lim\limits_{x\to-\infty}2e^x=0$

39. This limit has the form $\infty\cdot0$. We'll change it to the form $\frac{0}{0}$.

$$\lim_{x\to0}\cot 2x\sin 6x=\lim_{x\to0}\frac{\sin 6x}{\tan 2x}\overset{\text{H}}{=}\lim_{x\to0}\frac{6\cos 6x}{2\sec^2 2x}=\frac{6(1)}{2(1)^2}=3$$

40. $\lim\limits_{x\to0^+}\sin x\ln x=\lim\limits_{x\to0^+}\dfrac{\ln x}{\csc x}\overset{\text{H}}{=}\lim\limits_{x\to0^+}\dfrac{1/x}{-\csc x\cot x}=-\lim\limits_{x\to0^+}\left(\dfrac{\sin x}{x}\cdot\tan x\right)$

$$=-\left(\lim_{x\to0^+}\frac{\sin x}{x}\right)\left(\lim_{x\to0^+}\tan x\right)=-1\cdot0=0$$

41. This limit has the form $\infty\cdot0$. $\lim\limits_{x\to\infty}x^3e^{-x^2}=\lim\limits_{x\to\infty}\dfrac{x^3}{e^{x^2}}\overset{\text{H}}{=}\lim\limits_{x\to\infty}\dfrac{3x^2}{2xe^{x^2}}=\lim\limits_{x\to\infty}\dfrac{3x}{2e^{x^2}}\overset{\text{H}}{=}\lim\limits_{x\to\infty}\dfrac{3}{4xe^{x^2}}=0$

42. $\lim\limits_{x\to\pi/4}(1-\tan x)\sec x=(1-1)\sqrt{2}=0$. L'Hospital's Rule does not apply.

43. This limit has the form $0\cdot(-\infty)$.

$$\lim_{x\to1^+}\ln x\tan(\pi x/2)=\lim_{x\to1^+}\frac{\ln x}{\cot(\pi x/2)}\overset{\text{H}}{=}\lim_{x\to1^+}\frac{1/x}{(-\pi/2)\csc^2(\pi x/2)}=\frac{1}{(-\pi/2)(1)^2}=-\frac{2}{\pi}$$

44. $\lim\limits_{x\to\infty} x\tan(1/x) = \lim\limits_{x\to\infty} \dfrac{\tan(1/x)}{1/x} \overset{\text{H}}{=} \lim\limits_{x\to\infty} \dfrac{\sec^2(1/x)(-1/x^2)}{-1/x^2} = \lim\limits_{x\to\infty} \sec^2(1/x) = 1^2 = 1$

45. $\lim\limits_{x\to 0}\left(\dfrac{1}{x} - \csc x\right) = \lim\limits_{x\to 0}\left(\dfrac{1}{x} - \dfrac{1}{\sin x}\right) = \lim\limits_{x\to 0} \dfrac{\sin x - x}{x\sin x}$

$$\overset{\text{H}}{=} \lim_{x\to 0} \frac{\cos x - 1}{x\cos x + \sin x} \overset{\text{H}}{=} \lim_{x\to 0} \frac{-\sin x}{2\cos x - x\sin x} = \frac{0}{2} = 0$$

46. $\lim\limits_{x\to 0}(\csc x - \cot x) = \lim\limits_{x\to 0}\left(\dfrac{1}{\sin x} - \dfrac{\cos x}{\sin x}\right) = \lim\limits_{x\to 0} \dfrac{1-\cos x}{\sin x} \overset{\text{H}}{=} \lim\limits_{x\to 0} \dfrac{\sin x}{\cos x} = 0$

47. We will multiply and divide by the conjugate of the expression to change the form of the expression.

$$\lim_{x\to\infty}\left(\sqrt{x^2+x} - x\right) = \lim_{x\to\infty}\left(\frac{\sqrt{x^2+x}-x}{1}\cdot\frac{\sqrt{x^2+x}+x}{\sqrt{x^2+x}+x}\right) = \lim_{x\to\infty} \frac{(x^2+x)-x^2}{\sqrt{x^2+x}+x}$$

$$= \lim_{x\to\infty} \frac{x}{\sqrt{x^2+x}+x} = \lim_{x\to\infty} \frac{1}{\sqrt{1+1/x}+1} = \frac{1}{\sqrt{1}+1} = \frac{1}{2}.$$

As an alternate solution, write $\sqrt{x^2+x} - x$ as $\sqrt{x^2+x} - \sqrt{x^2}$, factor out $\sqrt{x^2}$, rewrite as $\left(\sqrt{1+1/x} - 1\right)/(1/x)$, and apply l'Hospital's Rule.

48. $\lim\limits_{x\to 1}\left(\dfrac{1}{\ln x} - \dfrac{1}{x-1}\right) = \lim\limits_{x\to 1} \dfrac{x-1-\ln x}{(x-1)\ln x} \overset{\text{H}}{=} \lim\limits_{x\to 1} \dfrac{1-1/x}{(x-1)(1/x)+\ln x}\cdot\dfrac{x}{x}$

$$= \lim_{x\to 1} \frac{x-1}{x-1+x\ln x} \overset{\text{H}}{=} \lim_{x\to 1} \frac{1}{1+1+\ln x} = \frac{1}{2+0} = \frac{1}{2}$$

49. The limit has the form $\infty - \infty$ and we will change the form to a product by factoring out x.

$\lim\limits_{x\to\infty}(x - \ln x) = \lim\limits_{x\to\infty} x\left(1 - \dfrac{\ln x}{x}\right) = \infty$ since $\lim\limits_{x\to\infty} \dfrac{\ln x}{x} \overset{\text{H}}{=} \lim\limits_{x\to\infty} \dfrac{1/x}{1} = 0.$

50. As $x \to \infty$, $1/x \to 0$, and $e^{1/x} \to 1$. So the limit has the form $\infty - \infty$ and we will change the form to a product by factoring out x.

$$\lim_{x\to\infty}\left(xe^{1/x} - x\right) = \lim_{x\to\infty} x\left(e^{1/x} - 1\right) = \lim_{x\to\infty} \frac{e^{1/x}-1}{1/x} \overset{\text{H}}{=} \lim_{x\to\infty} \frac{e^{1/x}(-1/x^2)}{-1/x^2} = \lim_{x\to\infty} e^{1/x} = e^0 = 1$$

51. $y = x^{x^2} \quad\Rightarrow\quad \ln y = x^2\ln x$, so

$\lim\limits_{x\to 0^+} \ln y = \lim\limits_{x\to 0^+} x^2\ln x = \lim\limits_{x\to 0^+} \dfrac{\ln x}{1/x^2} \overset{\text{H}}{=} \lim\limits_{x\to 0^+} \dfrac{1/x}{-2/x^3} = \lim\limits_{x\to 0^+}\left(-\dfrac{1}{2}x^2\right) = 0 \quad\Rightarrow$

$\lim\limits_{x\to 0^+} x^{x^2} = \lim\limits_{x\to 0^+} e^{\ln y} = e^0 = 1.$

52. $y = (\tan 2x)^x \quad\Rightarrow\quad \ln y = x\cdot\ln\tan 2x$, so

$\lim\limits_{x\to 0^+} \ln y = \lim\limits_{x\to 0^+} x\cdot\ln\tan 2x = \lim\limits_{x\to 0^+} \dfrac{\ln\tan 2x}{1/x}$

$$\overset{\text{H}}{=} \lim_{x\to 0^+} \frac{(1/\tan 2x)(2\sec^2 2x)}{-1/x^2} = \lim_{x\to 0^+} \frac{-2x^2\cos 2x}{\sin 2x\cos^2 2x} = \lim_{x\to 0^+} \frac{2x}{\sin 2x}\cdot\lim_{x\to 0^+} \frac{-x}{\cos 2x} = 1\cdot 0 = 0 \quad\Rightarrow$$

$\lim\limits_{x\to 0^+} (\tan 2x)^x = \lim\limits_{x\to 0^+} e^{\ln y} = e^0 = 1.$

53. $y = (1-2x)^{1/x} \quad\Rightarrow\quad \ln y = \dfrac{1}{x}\ln(1-2x)$, so $\lim\limits_{x\to 0} \ln y = \lim\limits_{x\to 0} \dfrac{\ln(1-2x)}{x} \overset{\text{H}}{=} \lim\limits_{x\to 0} \dfrac{-2/(1-2x)}{1} = -2 \quad\Rightarrow$

$\lim\limits_{x\to 0}(1-2x)^{1/x} = \lim\limits_{x\to 0} e^{\ln y} = e^{-2}.$

54. $y = \left(1 + \dfrac{a}{x}\right)^{bx} \Rightarrow \ln y = bx\ln\left(1 + \dfrac{a}{x}\right)$, so

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{b\ln(1+a/x)}{1/x} \overset{\text{H}}{=} \lim_{x\to\infty} \frac{b\left(\dfrac{1}{1+a/x}\right)\left(-\dfrac{a}{x^2}\right)}{-1/x^2} = \lim_{x\to\infty} \frac{ab}{1+a/x} = ab \Rightarrow$$

$$\lim_{x\to\infty}\left(1+\frac{a}{x}\right)^{bx} = \lim_{x\to\infty} e^{\ln y} = e^{ab}.$$

55. $y = \left(1 + \dfrac{3}{x} + \dfrac{5}{x^2}\right)^x \Rightarrow \ln y = x\ln\left(1 + \dfrac{3}{x} + \dfrac{5}{x^2}\right) \Rightarrow$

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{\ln\left(1+\dfrac{3}{x}+\dfrac{5}{x^2}\right)}{1/x} \overset{\text{H}}{=} \lim_{x\to\infty} \frac{\left(-\dfrac{3}{x^2}-\dfrac{10}{x^3}\right)\Big/\left(1+\dfrac{3}{x}+\dfrac{5}{x^2}\right)}{-1/x^2} = \lim_{x\to\infty} \frac{3+\dfrac{10}{x}}{1+\dfrac{3}{x}+\dfrac{5}{x^2}} = 3,$$

so $\displaystyle\lim_{x\to\infty}\left(1+\frac{3}{x}+\frac{5}{x^2}\right)^x = \lim_{x\to\infty} e^{\ln y} = e^3.$

56. $y = x^{(\ln 2)/(1+\ln x)} \Rightarrow \ln y = \dfrac{\ln 2}{1+\ln x}\ln x \Rightarrow$

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{(\ln 2)(\ln x)}{1+\ln x} \overset{\text{H}}{=} \lim_{x\to\infty} \frac{(\ln 2)(1/x)}{1/x} = \lim_{x\to\infty} \ln 2 = \ln 2,$$

so $\displaystyle\lim_{x\to\infty} x^{(\ln 2)/(1+\ln x)} = \lim_{x\to\infty} e^{\ln y} = e^{\ln 2} = 2.$

57. $y = x^{1/x} \Rightarrow \ln y = (1/x)\ln x \Rightarrow \displaystyle\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{\ln x}{x} \overset{\text{H}}{=} \lim_{x\to\infty} \frac{1/x}{1} = 0 \Rightarrow$

$$\lim_{x\to\infty} x^{1/x} = \lim_{x\to\infty} e^{\ln y} = e^0 = 1$$

58. $y = (e^x + x)^{1/x} \Rightarrow \ln y = \dfrac{1}{x}\ln(e^x + x)$, so

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{\ln(e^x+x)}{x} \overset{\text{H}}{=} \lim_{x\to\infty} \frac{e^x+1}{e^x+x} \overset{\text{H}}{=} \lim_{x\to\infty} \frac{e^x}{e^x+1} \overset{\text{H}}{=} \lim_{x\to\infty} \frac{e^x}{e^x} = 1 \Rightarrow$$

$$\lim_{x\to\infty} (e^x+x)^{1/x} = \lim_{x\to\infty} e^{\ln y} = e^1 = e.$$

59. $y = \left(\dfrac{x}{x+1}\right)^x \Rightarrow \ln y = x\ln\left(\dfrac{x}{x+1}\right) \Rightarrow$

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} x\ln\left(\frac{x}{x+1}\right) = \lim_{x\to\infty} \frac{\ln x - \ln(x+1)}{1/x} \overset{\text{H}}{=} \lim_{x\to\infty} \frac{1/x - 1/(x+1)}{-1/x^2}$$

$$= \lim_{x\to\infty}\left(-x + \frac{x^2}{x+1}\right) = \lim_{x\to\infty} \frac{-x}{x+1} = -1$$

so $\displaystyle\lim_{x\to\infty}\left(\frac{x}{x+1}\right)^x = \lim_{x\to\infty} e^{\ln y} = e^{-1}$

Or: $\displaystyle\lim_{x\to\infty}\left(\frac{x}{x+1}\right)^x = \lim_{x\to\infty}\left[\left(\frac{x+1}{x}\right)^{-1}\right]^x = \left[\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x\right]^{-1} = e^{-1}$

60. $y = (\cos 3x)^{5/x} \Rightarrow \ln y = \dfrac{5}{x}\ln(\cos 3x) \Rightarrow \displaystyle\lim_{x\to 0} \ln y = 5\lim_{x\to 0} \frac{\ln(\cos 3x)}{x} \overset{\text{H}}{=} 5\lim_{x\to 0} \frac{-3\tan 3x}{1} = 0,$

so $\displaystyle\lim_{x\to 0}(\cos 3x)^{5/x} = e^0 = 1.$

61. $y=(\cos x)^{1/x^2} \Rightarrow \ln y=\dfrac{1}{x^2}\ln\cos x \Rightarrow$

$$\lim_{x\to0^+}\ln y=\lim_{x\to0^+}\frac{\ln\cos x}{x^2}\overset{\text{H}}{=}\lim_{x\to0^+}\frac{-\tan x}{2x}\overset{\text{H}}{=}\lim_{x\to0^+}\frac{-\sec^2 x}{2}=-\frac{1}{2}\Rightarrow$$

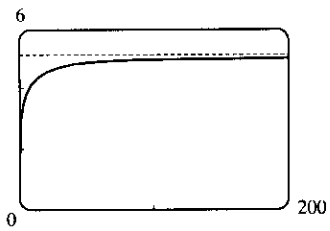
$$\lim_{x\to0^+}(\cos x)^{1/x^2}=\lim_{x\to0^+}e^{\ln y}=e^{-1/2}=1/\sqrt{e}$$

62. $y=\left(\dfrac{2x-3}{2x+5}\right)^{2x+1} \Rightarrow \ln y=(2x+1)\ln\left(\dfrac{2x-3}{2x+5}\right) \Rightarrow$

$$\lim_{x\to\infty}\ln y=\lim_{x\to\infty}\frac{\ln(2x-3)-\ln(2x+5)}{1/(2x+1)}\overset{\text{H}}{=}\lim_{x\to\infty}\frac{2/(2x-3)-2/(2x+5)}{-2/(2x+1)^2}=\lim_{x\to\infty}\frac{-8(2x+1)^2}{(2x-3)(2x+5)}$$

$$=\lim_{x\to\infty}\frac{-8(2+1/x)^2}{(2-3/x)(2+5/x)}=-8 \Rightarrow \lim_{x\to\infty}\left(\frac{2x-3}{2x+5}\right)^{2x+1}=e^{-8}$$

63.



From the graph, it appears that $\lim_{x\to\infty} x\,[\ln(x+5)-\ln x]=5$.

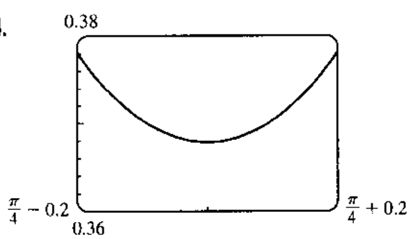
To prove this, we first note that

$\ln(x+5)-\ln x=\ln\dfrac{x+5}{x}=\ln\left(1+\dfrac{5}{x}\right)\to\ln 1=0$ as $x\to\infty$. Thus,

$$\lim_{x\to\infty}x\,[\ln(x+5)-\ln x]=\lim_{x\to\infty}\frac{\ln(x+5)-\ln x}{1/x}\overset{\text{H}}{=}\lim_{x\to\infty}\frac{\dfrac{1}{x+5}-\dfrac{1}{x}}{-1/x^2}$$

$$=\lim_{x\to\infty}\left[\frac{x-(x+5)}{x(x+5)}\cdot\frac{-x^2}{1}\right]=\lim_{x\to\infty}\frac{5x^2}{x^2+5x}=5$$

64.



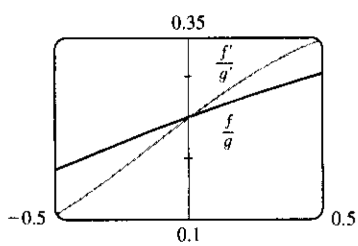
From the graph, it appears that $\lim_{x\to\pi/4}(\tan x)^{\tan 2x}\approx 0.368$.

The limit has the form 1^∞. Now $y=(\tan x)^{\tan 2x} \Rightarrow$ $\ln y=\tan 2x\,\ln(\tan x)$, so

$$\lim_{x\to\pi/4}\ln y=\lim_{x\to\pi/4}\frac{\ln(\tan x)}{\cot 2x}\overset{\text{H}}{=}\lim_{x\to\pi/4}\frac{\sec^2 x/\tan x}{-2\csc^2 2x}=\frac{2/1}{-2(1)}=-1$$

$$\Rightarrow \lim_{x\to\pi/4}(\tan x)^{\tan 2x}=\lim_{x\to\pi/4}e^{\ln y}=e^{-1}=1/e\approx 0.3679.$$

65.

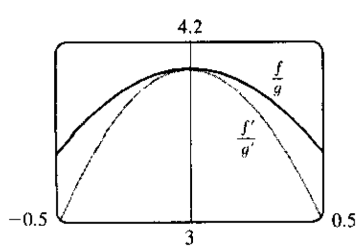


From the graph, it appears that

$\lim_{x\to0}\dfrac{f(x)}{g(x)}=\lim_{x\to0}\dfrac{f'(x)}{g'(x)}=0.25$. We calculate

$$\lim_{x\to0}\frac{f(x)}{g(x)}=\lim_{x\to0}\frac{e^x-1}{x^3+4x}\overset{\text{H}}{=}\lim_{x\to0}\frac{e^x}{3x^2+4}=\frac{1}{4}.$$

66.



From the graph, it appears that $\lim_{x\to0}\dfrac{f(x)}{g(x)}=\lim_{x\to0}\dfrac{f'(x)}{g'(x)}=4$.

We calculate

$$\lim_{x\to0}\frac{f(x)}{g(x)}=\lim_{x\to0}\frac{2x\sin x}{\sec x-1}\overset{\text{H}}{=}\lim_{x\to0}\frac{2(x\cos x+\sin x)}{\sec x\,\tan x}$$

$$\overset{\text{H}}{=}\lim_{x\to0}\frac{2(-x\sin x+\cos x+\cos x)}{\sec x(\sec^2 x)+\tan x(\sec x\,\tan x)}=\frac{4}{1}=4$$

67. $y = f(x) = xe^{-x}$ **A.** $D = \mathbb{R}$ **B.** Intercepts are 0 **C.** No symmetry

D. $\lim_{x\to\infty} xe^{-x} = \lim_{x\to\infty} \frac{x}{e^x} \overset{H}{=} \lim_{x\to\infty} \frac{1}{e^x} = 0$, so $y = 0$ is a HA.

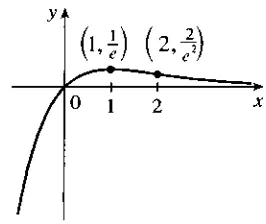
$\lim_{x\to-\infty} xe^{-x} = -\infty$ **E.** $f'(x) = e^{-x} - xe^{-x} = e^{-x}(1-x) > 0 \Leftrightarrow$

$x < 1$, so f is increasing on $(-\infty, 1)$ and decreasing on $(1, \infty)$.

F. Absolute and local maximum value $f(1) = 1/e$.

G. $f''(x) = e^{-x}(x-2) > 0 \Leftrightarrow x > 2$, so f is CU on $(2, \infty)$ and CD on $(-\infty, 2)$. IP at $(2, 2/e^2)$

H.



68. $y = f(x) = x(\ln x)^2$ **A.** $D = (0, \infty)$ **B.** x-intercept $= 1$, no y-intercept **C.** No symmetry

D. $\lim_{x\to\infty} x(\ln x)^2 = \infty$, $\lim_{x\to0^+} x(\ln x)^2 = \lim_{x\to0^+} \frac{(\ln x)^2}{1/x} \overset{H}{=} \lim_{x\to0^+} \frac{2(\ln x)(1/x)}{-1/x^2} = \lim_{x\to0^+} \frac{2\ln x}{-1/x} \overset{H}{=}$

$\lim_{x\to0^+} \frac{2/x}{1/x^2} = \lim_{x\to0^+} 2x = 0$, no asymptote **E.** $f'(x) = (\ln x)^2 + 2\ln x = (\ln x)(\ln x + 2) = 0$ when $\ln x = 0$

$\Leftrightarrow x = 1$ and when $\ln x = -2 \Leftrightarrow x = e^{-2}$. $f'(x) > 0$ when $0 < x < e^{-2}$ and when $x > 1$, so

f is increasing on $(0, e^{-2})$ and $(1, \infty)$ and decreasing on $(e^{-2}, 1)$.

F. Local maximum value $f(e^{-2}) = 4e^{-2}$,

local minimum value $f(1) = 0$

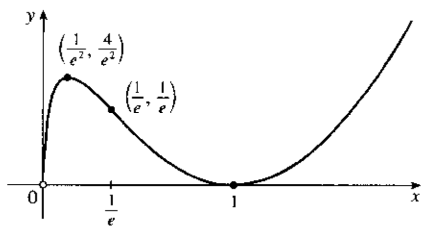
G. $f''(x) = 2(\ln x)(1/x) + 2/x = (2/x)(\ln x + 1) = 0$

when $\ln x = -1 \Leftrightarrow x = e^{-1}$. $f''(x) > 0 \Leftrightarrow$

$x > 1/e$, so f is CU on $(1/e, \infty)$, CD on $(0, 1/e)$.

IP at $(1/e, 1/e)$

H.



69. $y = f(x) = xe^{-x^2}$ **A.** $D = \mathbb{R}$ **B.** Intercepts are 0 **C.** $f(-x) = -f(x)$, so the curve is symmetric about the origin. **D.** $\lim_{x\to\pm\infty} xe^{-x^2} = \lim_{x\to\pm\infty} \frac{x}{e^{x^2}} \overset{H}{=} \lim_{x\to\pm\infty} \frac{1}{2xe^{x^2}} = 0$, so $y = 0$ is a HA.

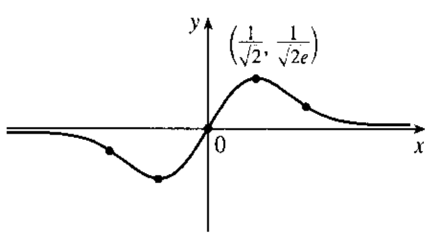
E. $f'(x) = e^{-x^2} - 2x^2e^{-x^2} = e^{-x^2}(1 - 2x^2) > 0 \Leftrightarrow x^2 < \frac{1}{2} \Leftrightarrow |x| < \frac{1}{\sqrt{2}}$, so f is increasing on $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ and decreasing on $\left(-\infty, -\frac{1}{\sqrt{2}}\right)$ and $\left(\frac{1}{\sqrt{2}}, \infty\right)$. **F.** Local maximum value $f\left(\frac{1}{\sqrt{2}}\right) = 1/\sqrt{2e}$, local minimum value $f\left(-\frac{1}{\sqrt{2}}\right) = -1/\sqrt{2e}$ **G.** $f''(x) = -2xe^{-x^2}(1 - 2x^2) - 4xe^{-x^2} = 2xe^{-x^2}(2x^2 - 3) > 0$

$\Leftrightarrow x > \sqrt{\frac{3}{2}}$ or $-\sqrt{\frac{3}{2}} < x < 0$, so f is CU on $\left(\sqrt{\frac{3}{2}}, \infty\right)$

and $\left(-\sqrt{\frac{3}{2}}, 0\right)$ and CD on $\left(-\infty, -\sqrt{\frac{3}{2}}\right)$ and $\left(0, \sqrt{\frac{3}{2}}\right)$.

IP are $(0, 0)$ and $\left(\pm\sqrt{\frac{3}{2}}, \pm\sqrt{\frac{3}{2}}\, e^{-3/2}\right)$.

H.



70. $y = f(x) = e^x/x$ **A.** $D = \{x \mid x \neq 0\}$ **B.** No intercept **C.** No symmetry **D.** $\lim_{x\to\infty} \frac{e^x}{x} \overset{H}{=} \lim_{x\to\infty} \frac{e^x}{1} = \infty$,

$\lim_{x\to-\infty} \frac{e^x}{x} = 0$, so $y = 0$ is a HA. $\lim_{x\to 0^+} \frac{e^x}{x} = \infty$, $\lim_{x\to 0^-} \frac{e^x}{x} = -\infty$, so $x = 0$ is a VA.

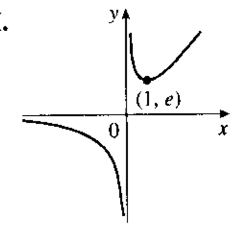
E. $f'(x) = \frac{xe^x - e^x}{x^2} > 0 \Leftrightarrow (x-1)e^x > 0 \Leftrightarrow x > 1$, so f is increasing on $(1,\infty)$, and decreasing on $(-\infty, 0)$ and $(0,1)$.

F. $f(1) = e$ is a local minimum value.

G. $f''(x) = \frac{x^2(xe^x) - 2x(xe^x - e^x)}{x^4} = \frac{e^x(x^2 - 2x + 2)}{x^3} > 0$

$\Leftrightarrow x > 0$ since $x^2 - 2x + 2 > 0$ for all x. So f is CU on $(0,\infty)$ and CD on $(-\infty, 0)$. No IP

H.



71. $y = f(x) = x - \ln(1+x)$ **A.** $D = \{x \mid x > -1\} = (-1,\infty)$ **B.** Intercepts are 0 **C.** No symmetry

D. $\lim_{x\to-1^+} [x - \ln(1+x)] = \infty$, so $x = -1$ is a VA. $\lim_{x\to\infty} [x - \ln(1+x)] = \lim_{x\to\infty} x\left[1 - \frac{\ln(1+x)}{x}\right] = \infty$,

since $\lim_{x\to\infty} \frac{\ln(1+x)}{x} \overset{H}{=} \lim_{x\to\infty} \frac{1/(1+x)}{1} = 0$.

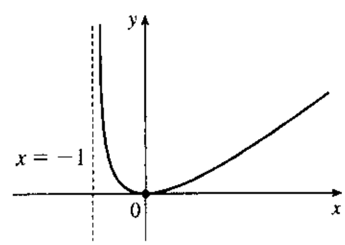
E. $f'(x) = 1 - \frac{1}{1+x} = \frac{x}{1+x} > 0 \Leftrightarrow x > 0$ since $x + 1 > 0$.

So f is increasing on $(0,\infty)$ and decreasing on $(-1,0)$.

F. $f(0) = 0$ is an absolute minimum.

G. $f''(x) = 1/(1+x)^2 > 0$, so f is CU on $(-1,\infty)$.

H.



72. $y = f(x) = e^x - 3e^{-x} - 4x$ **A.** $D = \mathbb{R}$ **B.** y-intercept $= -2$; x-intercept ≈ 2.22 **C.** No symmetry

D. $\lim_{x\to\infty} (e^x - 3e^{-x} - 4x) = \lim_{x\to\infty} x\left(\frac{e^x}{x} - 3\frac{e^{-x}}{x} - 4\right) = \infty$, since $\lim_{x\to\infty} \frac{e^x}{x} \overset{H}{=} \lim_{x\to\infty} \frac{e^x}{1} = \infty$.

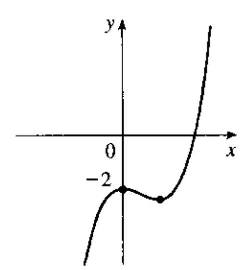
Similarly, $\lim_{x\to-\infty} (e^x - 3e^{-x} - 4x) = -\infty$. No HA; no VA

E. $f'(x) = e^x + 3e^{-x} - 4 = e^{-x}(e^{2x} - 4e^x + 3) = e^{-x}(e^x - 3)(e^x - 1) > 0 \Leftrightarrow e^x > 3$ or $e^x < 1 \Leftrightarrow$ $x > \ln 3$ or $x < 0$. So f is increasing on $(-\infty, 0)$ and $(\ln 3, \infty)$ and decreasing on $(0, \ln 3)$. **F.** Local maximum value $f(0) = -2$, local minimum value $f(\ln 3) = 2 - 4\ln 3$

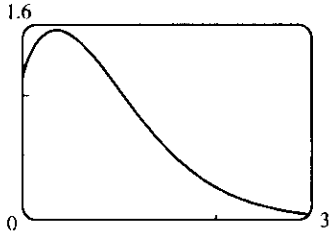
G. $f''(x) = e^x - 3e^{-x} = e^{-x}(e^{2x} - 3) > 0 \Leftrightarrow e^{2x} > 3 \Leftrightarrow$ $x > \frac{1}{2}\ln 3$, so f is CU on $\left(\frac{1}{2}\ln 3, \infty\right)$ and CD on $\left(-\infty, \frac{1}{2}\ln 3\right)$.

IP at $\left(\frac{1}{2}\ln 3, -2\ln 3\right)$.

H.



73. (a)



(b) $y = f(x) = x^{-x}$. We note that

$$\ln f(x) = \ln x^{-x} = -x\ln x = -\frac{\ln x}{1/x}, \text{ so}$$

$$\lim_{x\to 0^+} \ln f(x) \overset{\text{H}}{=} \lim_{x\to 0^+} -\frac{1/x}{-x^{-2}} = \lim_{x\to 0^+} x = 0. \text{ Thus}$$

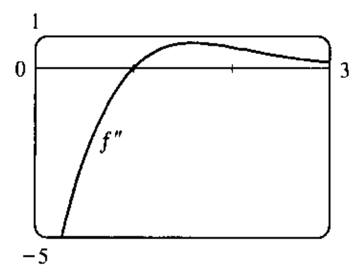
$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} e^{\ln f(x)} = e^0 = 1.$$

(c) From the graph, it appears that there is a local and absolute maximum of about $f(0.37) \approx 1.44$. To find the exact value, we differentiate: $f(x) = x^{-x} = e^{-x\ln x} \Rightarrow$

$f'(x) = e^{-x\ln x}\left[-x\left(\frac{1}{x}\right) + \ln x(-1)\right] = -x^{-x}(1 + \ln x)$. This is 0 only when $1 + \ln x = 0 \Leftrightarrow$ $x = e^{-1}$. Also $f'(x)$ changes from positive to negative at e^{-1}. So the maximum value is $f(1/e) = (1/e)^{-1/e} = e^{1/e}$.

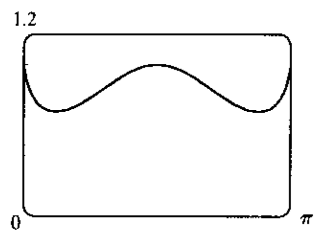
(d) We differentiate again to get

$$f''(x) = -x^{-x}(1/x) + (1 + \ln x)^2(x^{-x})$$
$$= x^{-x}\left[(1 + \ln x)^2 - 1/x\right]$$



From the graph of $f''(x)$, it seems that $f''(x)$ changes from negative to positive at $x = 1$, so we estimate that f has an IP at $x = 1$.

74. (a) $f(x) = (\sin x)^{\sin x}$ is continuous where $\sin x > 0$, that is, on intervals of the form $(2n\pi, (2n+1)\pi)$, so we have graphed f on $(0, \pi)$.



(b) $y = (\sin x)^{\sin x} \Rightarrow \ln y = \sin x \ln\sin x$, so

$$\lim_{x\to 0^+} \ln y = \lim_{x\to 0^+} \sin x\ln\sin x = \lim_{x\to 0^+} \frac{\ln\sin x}{\csc x} \overset{\text{H}}{=} \lim_{x\to 0^+} \frac{\cot x}{-\csc x\cot x} = \lim_{x\to 0^+}(-\sin x) = 0$$

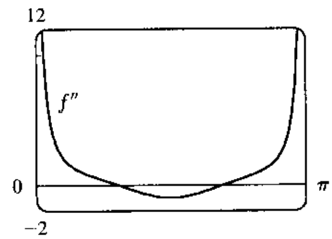
$$\Rightarrow \lim_{x\to 0^+} y = e^0 = 1.$$

(c) It appears that we have a local maximum at $(1.57, 1)$ and local minima at $(0.38, 0.69)$ and $(2.76, 0.69)$. $y = (\sin x)^{\sin x} \Rightarrow \ln y = \sin x\ln\sin x \Rightarrow$

$\frac{y'}{y} = (\sin x)\left(\frac{\cos x}{\sin x}\right) + (\ln\sin x)\cos x = \cos x(1 + \ln\sin x) \Rightarrow y' = (\sin x)^{\sin x}(\cos x)(1 + \ln\sin x)$.

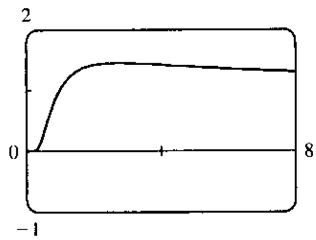
$y' = 0 \Rightarrow \cos x = 0$ or $\ln\sin x = -1 \Rightarrow x_2 = \frac{\pi}{2}$ or $\sin x = e^{-1}$. On $(0, \pi)$, $\sin x = e^{-1} \Rightarrow$ $x_1 = \sin^{-1}(e^{-1})$ and $x_3 = \pi - \sin^{-1}(e^{-1})$. Approximating these points gives us $(x_1, f(x_1)) \approx (0.3767, 0.6922)$, $(x_2, f(x_2)) \approx (1.5708, 1)$, and $(x_3, f(x_3)) \approx (2.7649, 0.6922)$. The approximations confirm our estimates.

(d)



From the graph, we see that $f''(x) = 0$ at $x \approx 0.94$ and $x \approx 2.20$. Since f'' changes sign at these values, they are x-coordinates of inflection points.

75. (a) $f(x) = x^{1/x}$

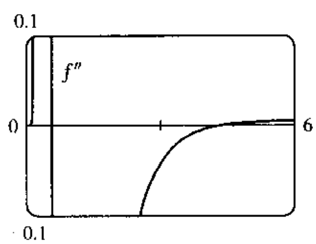


(b) Recall that $a^b = e^{b \ln a}$. $\lim\limits_{x \to 0^+} x^{1/x} = \lim\limits_{x \to 0^+} e^{(1/x)\ln x}$. As $x \to 0^+$, $\dfrac{\ln x}{x} \to -\infty$, so $x^{1/x} = e^{(1/x)\ln x} \to 0$. This indicates that there is a hole at $(0, 0)$. As $x \to \infty$, we have the indeterminate form ∞^0. $\lim\limits_{x \to \infty} x^{1/x} = \lim\limits_{x \to \infty} e^{(1/x)\ln x}$, but $\lim\limits_{x \to \infty} \dfrac{\ln x}{x} \overset{\text{H}}{=} \lim\limits_{x \to \infty} \dfrac{1/x}{1} = 0$, so $\lim\limits_{x \to \infty} x^{1/x} = e^0 = 1$. This indicates that $y = 1$ is a HA.

(c) Estimated maximum: $(2.72, 1.45)$. No estimated minimum. We use logarithmic differentiation to find any critical numbers. $y = x^{1/x} \quad \Rightarrow \quad \ln y = \dfrac{1}{x} \ln x \quad \Rightarrow \quad \dfrac{y'}{y} = \dfrac{1}{x} \cdot \dfrac{1}{x} + (\ln x)\left(-\dfrac{1}{x^2}\right) \quad \Rightarrow$

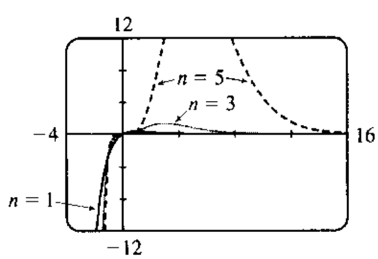
$y' = x^{1/x}\left(\dfrac{1 - \ln x}{x^2}\right) = 0 \quad \Rightarrow \quad \ln x = 1 \quad \Rightarrow \quad x = e$. For $0 < x < e$, $y' > 0$ and for $x > e$, $y' < 0$, so $f(e) = e^{1/e}$ is a local maximum value. This point is approximately $(2.7183, 1.4447)$, which agrees with our estimate.

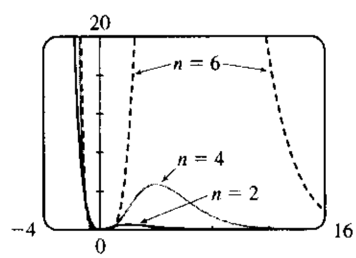
(d)



From the graph, we see that $f''(x) = 0$ at $x \approx 0.58$ and $x \approx 4.37$. Since f'' changes sign at these values, they are x-coordinates of inflection points.

76.





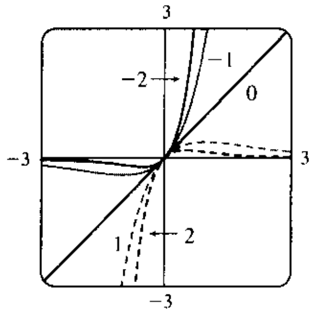
The first figure shows representative examples of $f(x) = x^n e^{-x}$ with n odd. n is even in the second figure. All curves pass through the origin and approach $y = 0$ as $x \to \infty$. $f'(x) = \dfrac{x^n(n - x)}{xe^x} = 0 \quad \Leftrightarrow \quad x = n$ or $x = 0$ (the latter for $n > 1$). At $x = 0$, we have a local minimum for n even. At $x = n$, we have a local maximum for all n. As n increases, $(n, f(n))$ gets farther away from the origin. $f''(x) = \dfrac{x^n(x^2 - 2nx + n^2 - n)}{x^2 e^x} = 0 \quad \Leftrightarrow$ $x = n \pm \sqrt{n}$ or $x = 0$ (the latter for $n > 2$). As n increases, the IP move farther away from the origin—they are symmetric about the line $x = n$.

77. If $c < 0$, then $\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty} \frac{x}{e^{cx}} \overset{\text{H}}{=} \lim_{x\to-\infty} \frac{1}{ce^{cx}} = 0$, and $\lim_{x\to\infty} f(x) = \infty$.

If $c > 0$, then $\lim_{x\to-\infty} f(x) = -\infty$, and $\lim_{x\to\infty} f(x) \overset{\text{H}}{=} \lim_{x\to\infty} \frac{1}{ce^{cx}} = 0$.

If $c = 0$, then $f(x) = x$, so $\lim_{x\to\pm\infty} f(x) = \pm\infty$ respectively.

So we see that $c = 0$ is a transitional value. We now exclude the case $c = 0$, since we know how the function behaves in that case. To find the maxima and minima of f, we differentiate: $f(x) = xe^{-cx} \Rightarrow$ $f'(x) = x(-ce^{-cx}) + e^{-cx} = (1 - cx)e^{-cx}$. This is 0 when $1 - cx = 0 \Leftrightarrow x = 1/c$. If $c < 0$ then this represents a minimum value of $f(1/c) = 1/(ce)$, since $f'(x)$ changes from negative to positive at $x = 1/c$; and if $c > 0$, it represents a maximum value. As $|c|$ increases, the maximum or minimum point gets closer to the origin. To find the inflection points, we differentiate again: $f'(x) = e^{-cx}(1 - cx) \Rightarrow$ $f''(x) = e^{-cx}(-c) + (1 - cx)(-ce^{-cx}) = (cx - 2)ce^{-cx}$. This changes sign when $cx - 2 = 0 \Leftrightarrow x = 2/c$. So as $|c|$ increases, the points of inflection get closer to the origin.



78. We see that both numerator and denominator approach 0, so we can use l'Hospital's Rule:

$$\lim_{x\to a} \frac{\sqrt{2a^3x - x^4} - a\sqrt[3]{aax}}{a - \sqrt[4]{ax^3}} \overset{\text{H}}{=} \lim_{x\to a} \frac{\frac{1}{2}(2a^3x - x^4)^{-1/2}(2a^3 - 4x^3) - a(\frac{1}{3})(aax)^{-2/3}a^2}{-\frac{1}{4}(ax^3)^{-3/4}(3ax^2)}$$

$$= \frac{\frac{1}{2}(2a^3a - a^4)^{-1/2}(2a^3 - 4a^3) - \frac{1}{3}a^3(a^2a)^{-2/3}}{-\frac{1}{4}(aa^3)^{-3/4}(3aa^2)}$$

$$= \frac{(a^4)^{-1/2}(-a^3) - \frac{1}{3}a^3(a^3)^{-2/3}}{-\frac{3}{4}a^3(a^4)^{-3/4}} = \frac{-a - \frac{1}{3}a}{-\frac{3}{4}} = \frac{4}{3}\left(\frac{4}{3}a\right) = \frac{16}{9}a$$

79. First we will find $\lim_{n\to\infty}\left(1 + \frac{i}{n}\right)^{nt}$, which is of the form 1^∞. $y = \left(1 + \frac{i}{n}\right)^{nt} \Rightarrow \ln y = nt\ln\left(1 + \frac{i}{n}\right)$, so

$$\lim_{n\to\infty} \ln y = \lim_{n\to\infty} nt\ln\left(1 + \frac{i}{n}\right) = t\lim_{n\to\infty} \frac{\ln(1 + i/n)}{1/n} \overset{\text{H}}{=} t\lim_{n\to\infty} \frac{(-i/n^2)}{(1 + i/n)(-1/n^2)} = t\lim_{n\to\infty} \frac{i}{1 + i/n} = ti$$

$\Rightarrow \lim_{n\to\infty} y = e^{it}$. Thus, as $n \to \infty$, $A = A_0\left(1 + \frac{i}{n}\right)^{nt} \to A_0e^{it}$.

80. (a) $\lim_{t\to\infty} v = \lim_{t\to\infty} \frac{mg}{c}\left(1 - e^{-ct/m}\right) = \frac{mg}{c}\lim_{t\to\infty}\left(1 - e^{-ct/m}\right)$

$= \frac{mg}{c}(1 - 0)$ [because $-ct/m \to -\infty$ as $t \to \infty$] $= \frac{mg}{c}$,

which is the speed the object approaches as time goes on, the so-called limiting velocity.

(b) $\lim_{m\to\infty} v = \lim_{m\to\infty} \frac{mg}{c}\left(1 - e^{-ct/m}\right) = \frac{g}{c}\lim_{m\to\infty} \frac{1 - e^{-ct/m}}{1/m} \overset{\text{H}}{=} \frac{g}{c}\lim_{m\to\infty} \frac{-e^{-ct/m}(ct/m^2)}{-1/m^2}$

$= \frac{g}{c}(ct)\lim_{m\to\infty} e^{-ct/m} = gt(1)$ [because $-ct/m \to 0$ as $m \to \infty$] $= gt$.

The speed of a very heavy falling object is approximately proportional to the elapsed time t, provided it can fall for time t in an environment where the given model continues to hold. [If t is too large, the object may hit the ground in less than time t, or it may have to start falling too high above the earth, where there is almost no air.].

81. Both numerator and denominator approach 0 as $x \to 0$, so we use l'Hospital's Rule (and FTC1):

$$\lim_{x \to 0} \frac{S(x)}{x^3} = \lim_{x \to 0} \frac{\int_0^x \sin(\pi t^2/2)dt}{x^3} \overset{\text{H}}{=} \lim_{x \to 0} \frac{\sin(\pi x^2/2)}{3x^2} \overset{\text{H}}{=} \lim_{x \to 0} \frac{\pi x \cos(\pi x^2/2)}{6x} = \frac{\pi}{6} \cdot \cos 0 = \frac{\pi}{6}$$

82. Both numerator and denominator approach 0 as $a \to 0$, so we use l'Hospital's Rule. (Note that we are differentiating with respect to a, since that is the quantity which is changing.) We also use the Fundamental Theorem of Calculus, Part 1:

$$\lim_{a \to 0} T(x,t) = \lim_{a \to 0} \frac{C \int_0^a e^{-(x-u)^2/(4kt)}\,du}{a\sqrt{4\pi kt}} \overset{\text{H}}{=} \lim_{a \to 0} \frac{Ce^{-(x-a)^2/(4kt)}}{\sqrt{4\pi kt}} = \frac{Ce^{-x^2/(4kt)}}{\sqrt{4\pi kt}}$$

83. Since $f(2) = 0$, the given limit has the form $\frac{0}{0}$.

$$\lim_{x \to 0} \frac{f(2+3x) + f(2+5x)}{x} \overset{\text{H}}{=} \lim_{x \to 0} \frac{f'(2+3x) \cdot 3 + f'(2+5x) \cdot 5}{1} = f'(2) \cdot 3 + f'(2) \cdot 5 = 8f'(2) = 8 \cdot 7 = 56$$

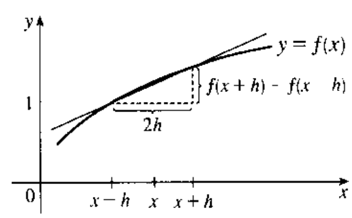
84. $L = \lim_{x \to 0} \left(\frac{\sin 2x}{x^3} + a + \frac{b}{x^2} \right) = \lim_{x \to 0} \frac{\sin 2x + ax^3 + bx}{x^3} \overset{\text{H}}{=} \lim_{x \to 0} \frac{2\cos 2x + 3ax^2 + b}{3x^2}$. As $x \to 0$, $3x^2 \to 0$, and $(2\cos 2x + 3ax^2 + b) \to b + 2$, so the last limit exists only if $b + 2 = 0$, that is, $b = -2$. Thus,

$\lim_{x \to 0} \frac{2\cos 2x + 3ax^2 - 2}{3x^2} \overset{\text{H}}{=} \lim_{x \to 0} \frac{-4\sin 2x + 6ax}{6x} \overset{\text{H}}{=} \lim_{x \to 0} \frac{-8\cos 2x + 6a}{6} = \frac{6a - 8}{6}$, which is equal to 0 if and only if $a = \frac{4}{3}$. Hence, $L = 0$ if and only if $b = -2$ and $a = \frac{4}{3}$.

85. Since $\lim_{h \to 0} [f(x+h) - f(x-h)] = f(x) - f(x) = 0$ (f is differentiable and hence continuous) and $\lim_{h \to 0} 2h = 0$, we use l'Hospital's Rule:

$$\lim_{h \to 0} \frac{f(x+h) - f(x-h)}{2h} \overset{\text{H}}{=} \lim_{h \to 0} \frac{f'(x+h)(1) - f'(x-h)(-1)}{2} = \frac{f'(x) + f'(x)}{2} = \frac{2f'(x)}{2} = f'(x)$$

$\frac{f(x+h) - f(x-h)}{2h}$ is the slope of the secant line between $(x-h, f(x-h))$ and $(x+h, f(x+h))$. As $h \to 0$, this line gets closer to the tangent line and its slope approaches $f'(x)$.



86. Since $\lim_{h \to 0} [f(x+h) - 2f(x) + f(x-h)] = f(x) - 2f(x) + f(x) = 0$ (f is differentiable and hence continuous) and $\lim_{h \to 0} h^2 = 0$, we can apply l'Hospital's Rule:

$$\lim_{h \to 0} \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} \overset{\text{H}}{=} \lim_{h \to 0} \frac{f'(x+h) - f'(x-h)}{2h} = f''(x)$$

At the last step, we have applied the result of Exercise 85 to $f'(x)$.

87. $\lim\limits_{x\to\infty}\dfrac{e^x}{x^n}\overset{\text{H}}{=}\lim\limits_{x\to\infty}\dfrac{e^x}{nx^{n-1}}\overset{\text{H}}{=}\lim\limits_{x\to\infty}\dfrac{e^x}{n(n-1)x^{n-2}}\overset{\text{H}}{=}\cdots\overset{\text{H}}{=}\lim\limits_{x\to\infty}\dfrac{e^x}{n!}=\infty$

88. $\lim\limits_{x\to\infty}\dfrac{\ln x}{x^p}\overset{\text{H}}{=}\lim\limits_{x\to\infty}\dfrac{1/x}{px^{p-1}}=\lim\limits_{x\to\infty}\dfrac{1}{px^p}=0$ since $p>0$.

89. $\lim\limits_{x\to0^+}x^\alpha\ln x=\lim\limits_{x\to0^+}\dfrac{\ln x}{x^{-\alpha}}\overset{\text{H}}{=}\lim\limits_{x\to0^+}\dfrac{1/x}{-\alpha x^{-\alpha-1}}=\lim\limits_{x\to0^+}\dfrac{x^\alpha}{-\alpha}=0$ since $\alpha>0$.

90. Using l'Hospital's Rule and FTC1, we have

$$\lim_{x\to0}\frac{\int_0^x\sin(t^2)\,dt}{x^3}=\lim_{x\to0}\frac{\frac{d}{dx}\int_0^x\sin(t^2)\,dt}{3x^2}=\lim_{x\to0}\frac{\sin(x^2)}{3x^2}=\frac{1}{3}\lim_{x\to0}\frac{\sin(x^2)}{x^2}=\frac{1}{3}$$

91. Let the radius of the circle be r. We see that $A(\theta)$ is the area of the whole figure (a sector of the circle with radius 1), minus the area of $\triangle OPR$. But the area of the sector of the circle is $\frac{1}{2}r^2\theta$ (see Reference Page 1), and the area of the triangle is $\frac{1}{2}r\,|PQ|=\frac{1}{2}r(r\sin\theta)=\frac{1}{2}r^2\sin\theta$. So we have $A(\theta)=\frac{1}{2}r^2\theta-\frac{1}{2}r^2\sin\theta=\frac{1}{2}r^2(\theta-\sin\theta)$. Now by elementary trigonometry, $B(\theta)=\frac{1}{2}\,|QR|\,|PQ|=\frac{1}{2}(r-|OQ|)\,|PQ|=\frac{1}{2}(r-r\cos\theta)(r\sin\theta)=\frac{1}{2}r^2(1-\cos\theta)\sin\theta$.
So the limit we want is

$$\begin{aligned}\lim_{\theta\to0^+}\frac{A(\theta)}{B(\theta)}&=\lim_{\theta\to0^+}\frac{\frac{1}{2}r^2(\theta-\sin\theta)}{\frac{1}{2}r^2(1-\cos\theta)\sin\theta}\overset{\text{H}}{=}\lim_{\theta\to0^+}\frac{1-\cos\theta}{(1-\cos\theta)\cos\theta+\sin\theta\,(\sin\theta)}\\&=\lim_{\theta\to0^+}\frac{1-\cos\theta}{\cos\theta-\cos^2\theta+\sin^2\theta}\overset{\text{H}}{=}\lim_{\theta\to0^+}\frac{\sin\theta}{-\sin\theta-2\cos\theta\,(-\sin\theta)+2\sin\theta\,(\cos\theta)}\\&=\lim_{\theta\to0^+}\frac{\sin\theta}{-\sin\theta+4\sin\theta\,\cos\theta}=\lim_{\theta\to0^+}\frac{1}{-1+4\cos\theta}=\frac{1}{-1+4\cos0}=\frac{1}{3}\end{aligned}$$

92. The area $A(t)=\int_0^t\sin(x^2)\,dx$, and the area $B(t)=\frac{1}{2}t\sin(t^2)$. Since $\lim\limits_{t\to0^+}A(t)=0=\lim\limits_{t\to0^+}B(t)$, we can use l'Hospital's Rule:

$$\lim_{t\to0^+}\frac{A(t)}{B(t)}\overset{\text{H}}{=}\lim_{t\to0^+}\frac{\sin(t^2)}{\frac{1}{2}\sin(t^2)+\frac{1}{2}t[2t\cos(t^2)]}\qquad\text{[by FTC1 and the Product Rule]}$$

$$\overset{\text{H}}{=}\lim_{t\to0^+}\frac{2t\cos(t^2)}{t\cos(t^2)-2t^3\sin(t^2)+2t\cos(t^2)}=\lim_{t\to0^+}\frac{2\cos(t^2)}{3\cos(t^2)-2t^2\sin(t^2)}$$

$$=\frac{2}{3-0}=\frac{2}{3}$$

93. (a) We show that $\lim\limits_{x\to0}\dfrac{f(x)}{x^n}=0$ for every integer $n\geq0$. Let $y=\dfrac{1}{x^2}$. Then

$$\lim_{x\to0}\frac{f(x)}{x^{2n}}=\lim_{x\to0}\frac{e^{-1/x^2}}{(x^2)^n}=\lim_{y\to\infty}\frac{y^n}{e^y}\overset{\text{H}}{=}\lim_{y\to\infty}\frac{ny^{n-1}}{e^y}\overset{\text{H}}{=}\cdots\overset{\text{H}}{=}\lim_{y\to\infty}\frac{n!}{e^y}=0\;\Rightarrow$$

$\lim\limits_{x\to0}\dfrac{f(x)}{x^n}=\lim\limits_{x\to0}x^n\dfrac{f(x)}{x^{2n}}=\lim\limits_{x\to0}x^n\lim\limits_{x\to0}\dfrac{f(x)}{x^{2n}}=0$. Thus, $f'(0)=\lim\limits_{x\to0}\dfrac{f(x)-f(0)}{x-0}=\lim\limits_{x\to0}\dfrac{f(x)}{x}=0.$

(b) Using the Chain Rule and the Quotient Rule we see that $f^{(n)}(x)$ exists for $x \neq 0$. In fact, we prove by induction that for each $n \geq 0$, there is a polynomial p_n and a non-negative integer k_n with $f^{(n)}(x) = p_n(x)f(x)/x^{k_n}$ for $x \neq 0$. This is true for $n = 0$; suppose it is true for the nth derivative. Then $f'(x) = f(x)(2/x^3)$, so

$$f^{(n+1)}(x) = \left[x^{k_n}\left[p_n'(x)f(x) + p_n(x)f'(x)\right] - k_n x^{k_n - 1} p_n(x)f(x)\right]x^{-2k_n}$$

$$= \left[x^{k_n} p_n'(x) + p_n(x)(2/x^3) - k_n x^{k_n - 1} p_n(x)\right] f(x)x^{-2k_n}$$

$$= \left[x^{k_n+3} p_n'(x) + 2p_n(x) - k_n x^{k_n+2} p_n(x)\right] f(x)x^{-(2k_n+3)}$$

which has the desired form.

Now we show by induction that $f^{(n)}(0) = 0$ for all n. By part (a), $f'(0) = 0$. Suppose that $f^{(n)}(0) = 0$. Then

$$f^{(n+1)}(0) = \lim_{x\to 0} \frac{f^{(n)}(x) - f^{(n)}(0)}{x - 0} = \lim_{x\to 0} \frac{f^{(n)}(x)}{x} = \lim_{x\to 0} \frac{p_n(x)f(x)/x^{k_n}}{x} = \lim_{x\to 0} \frac{p_n(x)f(x)}{x^{k_n+1}}$$

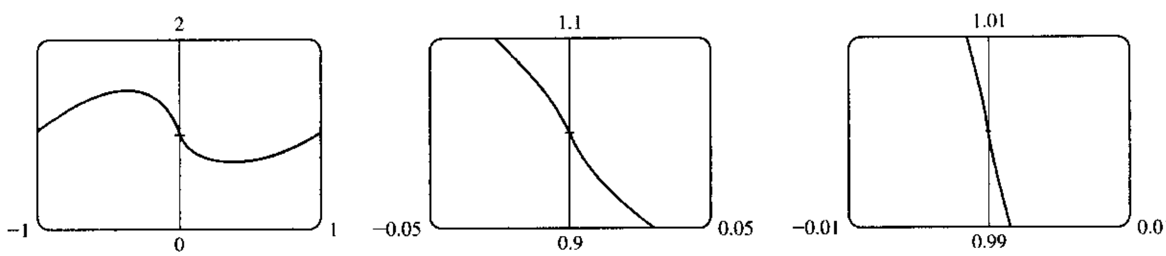
$$= \lim_{x\to 0} p_n(x) \lim_{x\to 0} \frac{f(x)}{x^{k_n+1}} = p_n(0)\cdot 0 = 0$$

94. (a) For f to be continuous, we need $\lim_{x\to 0} f(x) = f(0) = 1$. We note that for $x \neq 0$, $\ln f(x) = \ln |x|^x = x\ln|x|$.

So $\lim_{x\to 0} \ln f(x) = \lim_{x\to 0} x\ln|x| = \lim_{x\to 0} \frac{\ln|x|}{1/x} \overset{H}{=} \lim_{x\to 0} \frac{1/x}{-1/x^2} = 0$. Therefore,

$\lim_{x\to 0} f(x) = \lim_{x\to 0} e^{\ln f(x)} = e^0 = 1$. So f is continuous at 0.

(b) From the graphs, it appears that f is differentiable at 0.



(c) To find f', we use logarithmic differentiation: $\ln f(x) = x\ln|x| \Rightarrow \frac{f'(x)}{f(x)} = x\left(\frac{1}{x}\right) + \ln|x| \Rightarrow$

$f'(x) = f(x)(1 + \ln|x|) = |x|^x(1 + \ln|x|)$, $x \neq 0$. Now $f'(x) \to -\infty$ as $x \to 0$ [since $|x|^x \to 1$ and $(1 + \ln|x|) \to -\infty$], so the curve has a vertical tangent at $(0, 1)$ and is therefore not differentiable there. The fact cannot be seen in the graphs in part (b) because $\ln|x| \to -\infty$ very slowly as $x \to 0$.

7 Review

CONCEPT CHECK

1. (a) See Definition 1 in Section 7.1. It must pass the Horizontal Line Test.

(b) See Definition 2 in Section 7.1. The graph of f^{-1} is obtained by reflecting the graph of f about the line $y = x$.

(c) $g'(a) = \left(f^{-1}\right)'(a) = \dfrac{1}{f'(g(a))}$

2. (a) The function $f(x) = e^x$ has domain $\mathbb{R}$ and range $(0, \infty)$.

(b) The function $f(x) = \ln x$ has domain $(0, \infty)$ and range $\mathbb{R}$.

(c) The graphs are reflections of one another about the line $y = x$. See Figure 7.3.3 or Figure 7.3*.1.

(d) $\log_a x = \dfrac{\ln x}{\ln a}$

3. (a) See Definition 7.5.1. Domain $= [-1, 1]$, Range $= \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

(b) See Definition 7.5.4. Domain $= [-1, 1]$, Range $= [0, \pi]$

(c) See Definition 7.5.7. Domain $= \mathbb{R}$, Range $= \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. See Figure 10 in Section 7.5.

4. $\sinh x = \dfrac{e^x - e^{-x}}{2}$, $\cosh x = \dfrac{e^x + e^{-x}}{2}$, $\tanh x = \dfrac{\sinh x}{\cosh x} = \dfrac{e^x - e^{-x}}{e^x + e^{-x}}$

5. (a) $y = e^x \;\Rightarrow\; y' = e^x$

(b) $y = a^x \;\Rightarrow\; y' = a^x \ln a$

(c) $y = \ln x \;\Rightarrow\; y' = 1/x$

(d) $y = \log_a x \;\Rightarrow\; y' = 1/(x \ln a)$

(e) $y = \sin^{-1} x \;\Rightarrow\; y' = 1/\sqrt{1 - x^2}$

(f) $y = \cos^{-1} x \;\Rightarrow\; y' = -1/\sqrt{1 - x^2}$

(g) $y = \tan^{-1} x \;\Rightarrow\; y' = 1/(1 + x^2)$

(h) $y = \sinh x \;\Rightarrow\; y' = \cosh x$

(i) $y = \cosh x \;\Rightarrow\; y' = \sinh x$

(j) $y = \tanh x \;\Rightarrow\; y' = \operatorname{sech}^2 x$

(k) $y = \sinh^{-1} x \;\Rightarrow\; y' = 1/\sqrt{1 + x^2}$

(l) $y = \cosh^{-1} x \;\Rightarrow\; y' = 1/\sqrt{x^2 - 1}$

(m) $y = \tanh^{-1} x \;\Rightarrow\; y' = 1/(1 - x^2)$

6. (a) e is the number such that $\displaystyle\lim_{h \to 0} \frac{e^h - 1}{h} = 1$.

(b) $e = \displaystyle\lim_{x \to 0} (1 + x)^{1/x}$

(c) The differentiation formula for $y = a^x$ $[y' = a^x \ln a]$ is simplest when $a = e$ because $\ln e = 1$.

(d) The differentiation formula for $y = \log_a x$ $[y' = 1/(x \ln a)]$ is simplest when $a = e$ because $\ln e = 1$.

7. (a) See l'Hospital's Rule and the three notes that follow it in Section 7.7.

(b) Write fg as $\dfrac{f}{1/g}$ or $\dfrac{g}{1/f}$.

(c) Convert the difference into a quotient using a common denominator, rationalizing, factoring, or some other method.

(d) Convert the power to a product by taking the natural logarithm of both sides of $y = f^g$ or by writing f^g as $e^{g \ln f}$.

TRUE-FALSE QUIZ

1. True. If f is one-to-one, with domain $\mathbb{R}$, then $f^{-1}(f(6)) = 6$ by the first cancellation equation [see (4) in Section 7.1].

2. False. By Theorem 7 in Section 7.1, $(f^{-1})'(6) = \dfrac{1}{f'(f^{-1}(6))}$, *not* $\dfrac{1}{f'(6)}$ unless $f^{-1}(6) = 6$.

3. False. For example, $\cos\frac{\pi}{2} = \cos\left(-\frac{\pi}{2}\right)$, so $\cos x$ is not 1-1.

4. False. It is true that $\tan\frac{3\pi}{4} = -1$, but since the range of $\tan^{-1}$ is $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, we must have $\tan^{-1}(-1) = -\frac{\pi}{4}$.

5. True, since $\ln x$ is an increasing function on $(0, \infty)$.

6. True, by Equation 7.4*.1.

7. True. We can divide by e^x since $e^x \neq 0$ for every x.

8. False. For example, $\ln(1+1) = \ln 2$, but $\ln 1 + \ln 1 = 0$. In fact $\ln a + \ln b = \ln(ab)$.

9. False. Let $x = e$. Then $(\ln x)^6 = (\ln e)^6 = 1^6 = 1$, but $6\ln x = 6\ln e = 6 \cdot 1 = 6 \neq 1 = (\ln x)^6$.

10. False. $\dfrac{d}{dx}10^x = 10^x \ln 10$

11. False. $\ln 10$ is a constant, so its derivative is 0.

12. True. $y = e^{3x} \Rightarrow \ln y = 3x \Rightarrow x = \frac{1}{3}\ln y \Rightarrow$ the inverse function is $y = \frac{1}{3}\ln x$.

13. False. The "-1" is not an exponent; it is an indication of an inverse function.

14. False. For example, $\tan^{-1} 20$ is defined; $\sin^{-1} 20$ and $\cos^{-1} 20$ are not.

15. True. See Figure 2 in Section 7.6.

16. True. $\ln\frac{1}{10} = -\ln 10 = -\int_1^{10}(1/x)\,dx$, by Equation 7.4.4 or by Definition 7.2*.1.

17. True. $\int_2^{16}(1/x)\,dx = \ln x\big]_2^{16} = \ln 16 - \ln 2 = \ln\frac{16}{2} = \ln 8 = \ln 2^3 = 3\ln 2$

18. False. L'Hospital's Rule does not apply since $\displaystyle\lim_{x\to\pi^-}\frac{\tan x}{1-\cos x} = \frac{0}{2} = 0$.

EXERCISES

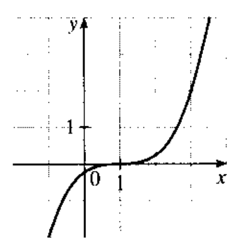
1. No. f is not 1-1 because the graph of f fails the Horizontal Line Test.

2. (a) g is one-to-one because it passes the Horizontal Line Test.

(b) When $y = 2$, $x \approx 0.2$. So $g^{-1}(2) \approx 0.2$.

(c) The range of g is $[-1, 3.5]$, which is the same as the domain of g^{-1}.

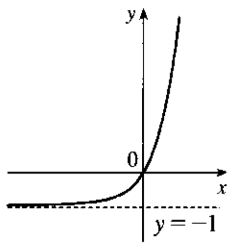
(d) We reflect the graph of g through the line $y = x$ to obtain the graph of g^{-1}.



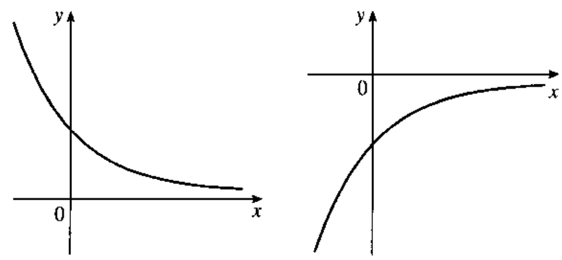
3. (a) $f^{-1}(3) = 7$ since $f(7) = 3$. (b) $(f^{-1})'(3) = \dfrac{1}{f'(f^{-1}(3))} = \dfrac{1}{f'(7)} = \dfrac{1}{8}$

4. $y = \dfrac{x+1}{2x+1}$. Interchanging x and y gives us $x = \dfrac{y+1}{2y+1} \Rightarrow 2xy + x = y + 1 \Rightarrow 2xy - y = 1 - x \Rightarrow$ $y(2x-1) = 1 - x \Rightarrow y = \dfrac{1-x}{2x-1} = f^{-1}(x)$.

5. $y = 5^x - 1$

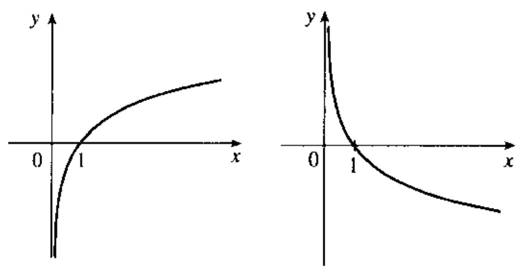


6. $y = e^{-x}$ $y = -e^{-x}$

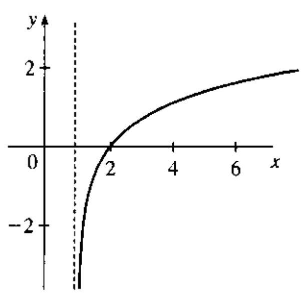


7. Reflect the graph of $y = \ln x$ about the x-axis to obtain the graph of $y = -\ln x$.

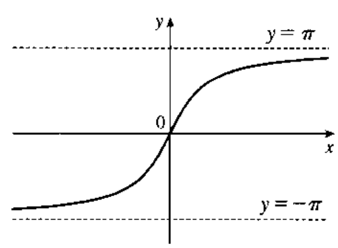
$y = \ln x$ $y = -\ln x$



8. $y = \ln(x-1)$



9. $y = 2\arctan x$



10. We have seen that if $a > 1$, then $a^x > x^a$ for sufficiently large x. (See Exercise 7.2.20.) In general, we could show that $\lim\limits_{x\to\infty}(a^x/x^a) = \infty$ by using l'Hospital's Rule repeatedly. Also, $\log_a x$ increases much more slowly than either x^a or a^x. [Compare the graph of $\log_a x$ with those of x^a and a^x, or use l'Hospital's Rule to show that $\lim\limits_{x\to\infty}[(\log_a x)/x^a] = 0$.] So for large x, $\log_a x < x^a < a^x$.

11. (a) $e^{2\ln 3} = \left(e^{\ln 3}\right)^2 = 3^2 = 9$

(b) $\log_{10} 25 + \log_{10} 4 = \log_{10}(25 \cdot 4) = \log_{10} 100 = \log_{10} 10^2 = 2$

12. (a) $\ln e^{\pi} = \pi$

(b) $\tan\left(\arcsin \frac{1}{2}\right) = \tan \frac{\pi}{6} = \frac{1}{\sqrt{3}}$

13. $\ln x = \frac{1}{3} \quad \Leftrightarrow \quad \log_e x = \frac{1}{3} \quad \Rightarrow \quad x = e^{1/3}$

14. $e^x = \frac{1}{3} \quad \Rightarrow \quad x = \ln \frac{1}{3} = \ln 1 - \ln 3 = -\ln 3$

15. $e^{e^x} = 17 \quad \Rightarrow \quad \ln e^{e^x} = \ln 17 \quad \Rightarrow \quad e^x = \ln 17 \quad \Rightarrow \quad \ln e^x = \ln(\ln 17) \quad \Rightarrow \quad x = \ln \ln 17$

16. $\ln(1 + e^{-x}) = 3 \quad \Rightarrow \quad 1 + e^{-x} = e^3 \quad \Rightarrow \quad e^{-x} = e^3 - 1 \quad \Rightarrow \quad \ln e^{-x} = \ln(e^3 - 1) \quad \Rightarrow \quad -x = \ln(e^3 - 1)$
$\Rightarrow \quad x = -\ln(e^3 - 1)$

17. $\ln(x + 1) + \ln(x - 1) = 1 \quad \Rightarrow \quad \ln[(x + 1)(x - 1)] = 1 \quad \Rightarrow \quad \ln(x^2 - 1) = \ln e \quad \Rightarrow \quad x^2 - 1 = e \quad \Rightarrow$
$x^2 = e + 1 \quad \Rightarrow \quad x = \sqrt{e + 1}$ since $\ln(x - 1)$ is defined only when $x > 1$.

18. $\log_5(c^x) = d \quad \Rightarrow \quad x \log_5 c = d \quad \Rightarrow \quad x = \dfrac{d}{\log_5 c}$.

Or: $\log_5(c^x) = d \quad \Rightarrow \quad 5^d = c^x \quad \Rightarrow \quad \ln 5^d = \ln c^x \quad \Rightarrow \quad d \ln 5 = x \ln c \quad \Rightarrow \quad x = \dfrac{d \ln 5}{\ln c}$.

19. $\tan^{-1} x = 1 \quad \Rightarrow \quad \tan \tan^{-1} x = \tan 1 \quad \Rightarrow \quad x = \tan 1 \;\; (\approx 1.5574)$

20. $\sin x = 0.3 \quad \Rightarrow \quad x = \sin^{-1} 0.3 = \alpha$ for $-\frac{\pi}{2} \leq x \leq \frac{\pi}{2}$. The reference angle for α is $\pi - \alpha$, so all solutions are $x = \alpha + 2n\pi$ and $x = \pi - \alpha + 2n\pi$ [or $(2n + 1)\pi - \alpha$]

21. $f(t) = t^2 \ln t \quad \Rightarrow \quad f'(t) = t^2 \cdot \dfrac{1}{t} + (\ln t)(2t) = t + 2t \ln t$ or $t(1 + 2\ln t)$

22. $g(t) = \dfrac{e^t}{1 + e^t} \quad \Rightarrow \quad g'(t) = \dfrac{(1 + e^t)e^t - e^t(e^t)}{(1 + e^t)^2} = \dfrac{e^t}{(1 + e^t)^2}$

23. $h(\theta) = e^{\tan 2\theta} \quad \Rightarrow \quad h'(\theta) = e^{\tan 2\theta} \cdot \sec^2 2\theta \cdot 2 = 2 \sec^2 2\theta \, e^{\tan 2\theta}$

24. $h(u) = 10^{\sqrt{u}} \quad \Rightarrow \quad h'(u) = 10^{\sqrt{u}} \cdot \ln 10 \cdot \dfrac{1}{2\sqrt{u}} = \dfrac{(\ln 10)10^{\sqrt{u}}}{2\sqrt{u}}$

25. $y = \ln|\sec 5x + \tan 5x| \quad \Rightarrow$
$y' = \dfrac{1}{\sec 5x + \tan 5x}(\sec 5x \, \tan 5x \cdot 5 + \sec^2 5x \cdot 5) = \dfrac{5 \sec 5x \, (\tan 5x + \sec 5x)}{\sec 5x + \tan 5x} = 5 \sec 5x$

26. $y = e^{-t}(t^2 - 2t + 2) \quad \Rightarrow$
$y' = e^{-t}(2t - 2) + (t^2 - 2t + 2)(-e^{-t}) = e^{-t}(2t - 2 - t^2 + 2t - 2) = e^{-t}(-t^2 + 4t - 4)$

27. $y = e^{cx}(c \sin x - \cos x) \quad \Rightarrow \quad y' = ce^{cx}(c \sin x - \cos x) + e^{cx}(c \cos x + \sin x) = (c^2 + 1)e^{cx} \sin x$

28. $y = \sin^{-1}(e^x) \quad \Rightarrow \quad y' = 1/\sqrt{1 - (e^x)^2} \cdot e^x = e^x/\sqrt{1 - e^{2x}}$

29. $y = \ln(\sec^2 x) = 2 \ln|\sec x| \quad \Rightarrow \quad y' = (2/\sec x)(\sec x \tan x) = 2 \tan x$

30. $y = \ln(x^2 e^x) = 2 \ln|x| + x \quad \Rightarrow \quad y' = 2/x + 1$

31. $y = xe^{-1/x} \quad \Rightarrow \quad y' = e^{-1/x} + xe^{-1/x}(1/x^2) = e^{-1/x}(1 + 1/x)$

32. $y = x^r e^{sx} \quad \Rightarrow \quad y' = rx^{r-1}e^{sx} + sx^r e^{sx}$

33. $y = 2^{-t^2} \quad \Rightarrow \quad y' = 2^{-t^2}(\ln 2)(-2t) = (-2\ln 2)t\,2^{-t^2}$

34. $y = e^{\cos x} + \cos(e^x) \quad \Rightarrow \quad y' = -\sin x e^{\cos x} - e^x \sin(e^x)$

35. $H(v) = v \tan^{-1} v \quad \Rightarrow \quad H'(v) = v \cdot \dfrac{1}{1 + v^2} + \tan^{-1} v \cdot 1 = \dfrac{v}{1 + v^2} + \tan^{-1} v$

36. $F(z) = \log_{10}(1 + z^2) \quad \Rightarrow \quad F'(z) = \dfrac{1}{(\ln 10)(1 + z^2)} \cdot 2z = \dfrac{2z}{(\ln 10)(1 + z^2)}$

37. $y = x \sinh(x^2) \quad \Rightarrow \quad y' = x \cosh(x^2) \cdot 2x + \sinh(x^2) \cdot 1 = 2x^2 \cosh(x^2) + \sinh(x^2)$

38. $y = (\cos x)^x \quad \Rightarrow \quad \ln y = \ln(\cos x)^x = x \ln \cos x \quad \Rightarrow \quad \dfrac{y'}{y} = x \cdot \dfrac{1}{\cos x} \cdot (-\sin x) + \ln \cos x \cdot 1 \quad \Rightarrow$

$y' = (\cos x)^x (\ln \cos x - x \tan x)$

39. $y = \ln \sin x - \frac{1}{2} \sin^2 x \quad \Rightarrow \quad y' = \dfrac{1}{\sin x} \cdot \cos x - \frac{1}{2} \cdot 2 \sin x \cdot \cos x = \cot x - \sin x \cos x$

40. $y = \arctan(\arcsin \sqrt{x}\,) \quad \Rightarrow \quad y' = \dfrac{1}{1 + (\arcsin \sqrt{x}\,)^2} \cdot \dfrac{1}{\sqrt{1-x}} \cdot \dfrac{1}{2\sqrt{x}}$

41. $y = \ln \dfrac{1}{x} + \dfrac{1}{\ln x} = \ln x^{-1} + (\ln x)^{-1} = -\ln x + (\ln x)^{-1} \quad \Rightarrow$

$y' = -1 \cdot \dfrac{1}{x} + (-1)(\ln x)^{-2} \cdot \dfrac{1}{x} = -\dfrac{1}{x} - \dfrac{1}{x\,(\ln x)^2}$

42. $xe^y = y - 1 \quad \Rightarrow \quad e^y + xe^y y' = y' \quad \Rightarrow \quad y' = e^y/(1 - xe^y)$

43. $y = \ln(\cosh 3x) \quad \Rightarrow \quad y' = (1/\cosh 3x)(\sinh 3x)(3) = 3 \tanh 3x$

44. $y = \dfrac{(x^2+1)^4}{(2x+1)^3(3x-1)^5} \quad \Rightarrow$

$\ln y = \ln \dfrac{(x^2+1)^4}{(2x+1)^3(3x-1)^5} = \ln(x^2+1)^4 - \ln[(2x+1)^3(3x-1)^5]$

$= 4\ln(x^2+1) - [\ln(2x+1)^3 + \ln(3x-1)^5] = 4\ln(x^2+1) - 3\ln(2x+1) - 5\ln(3x-1) \quad \Rightarrow$

$\dfrac{y'}{y} = 4 \cdot \dfrac{1}{x^2+1} \cdot 2x - 3 \cdot \dfrac{1}{2x+1} \cdot 2 - 5 \cdot \dfrac{1}{3x-1} \cdot 3 \quad \Rightarrow$

$y' = \dfrac{(x^2+1)^4}{(2x+1)^3(3x-1)^5}\left(\dfrac{8x}{x^2+1} - \dfrac{6}{2x+1} - \dfrac{15}{3x-1}\right).$

[The answer could be simplified to $y' = -\dfrac{(x^2+56x+9)(x^2+1)^3}{(2x+1)^4(3x-1)^6}$, but this is unnecessary.]

45. $y = \cosh^{-1}(\sinh x) \quad \Rightarrow \quad y' = (\cosh x)/\sqrt{\sinh^2 x - 1}$

46. $y = x \tanh^{-1}\sqrt{x} \quad \Rightarrow \quad y' = \tanh^{-1}\sqrt{x} + x\dfrac{1}{1-(\sqrt{x})^2}\dfrac{1}{2\sqrt{x}} = \tanh^{-1}\sqrt{x} + \dfrac{\sqrt{x}}{2(1-x)}$

47. $f(x) = e^{\sin^3(\ln(x^2+1))} \quad \Rightarrow$

$f'(x) = e^{\sin^3(\ln(x^2+1))} \cdot 3\sin^2(\ln(x^2+1)) \cdot \cos(\ln(x^2+1)) \cdot \dfrac{1}{x^2+1} \cdot 2x$

$= \dfrac{6x}{x^2+1} \sin^2(\ln(x^2+1)) \cdot \cos(\ln(x^2+1)) \cdot e^{\sin^3(\ln(x^2+1))}$

48. $\dfrac{d}{dx}\left(\dfrac{1}{2}\tan^{-1} x + \dfrac{1}{4}\ln\dfrac{(x+1)^2}{x^2+1}\right) = \dfrac{d}{dx}\left(\dfrac{1}{2}\tan^{-1} x + \dfrac{1}{2}\ln|x+1| - \dfrac{1}{4}\ln(x^2+1)\right)$

$= \dfrac{1}{2}\dfrac{1}{x^2+1} + \dfrac{1}{2}\dfrac{1}{x+1} - \dfrac{1}{4}\dfrac{2x}{x^2+1} = \dfrac{1}{2}\left(\dfrac{1}{x^2+1} - \dfrac{x}{x^2+1} + \dfrac{1}{x+1}\right)$

$= \dfrac{1}{2}\left(\dfrac{1-x}{x^2+1} + \dfrac{1}{x+1}\right) = \dfrac{1}{2}\left(\dfrac{1-x^2}{(x^2+1)(1+x)} + \dfrac{x^2+1}{(x^2+1)(1+x)}\right)$

$= \dfrac{1}{2}\dfrac{2}{(x^2+1)(1+x)} = \dfrac{1}{(1+x)(x^2+1)}$

49. $f(x) = e^{g(x)} \quad \Rightarrow \quad f'(x) = e^{g(x)} g'(x)$

50. $f(x) = g(e^x) \quad \Rightarrow \quad f'(x) = g'(e^x)e^x$

51. $f(x) = \ln|g(x)| \quad \Rightarrow \quad f'(x) = \dfrac{1}{g(x)}g'(x) = \dfrac{g'(x)}{g(x)}$

52. $f(x) = g(\ln x) \quad \Rightarrow \quad f'(x) = g'(\ln x) \cdot \dfrac{1}{x} = \dfrac{g'(\ln x)}{x}$

53. $f(x) = 2^x \quad \Rightarrow \quad f'(x) = 2^x \ln 2 \quad \Rightarrow \quad f''(x) = 2^x (\ln 2)^2 \quad \Rightarrow \quad \cdots \quad \Rightarrow \quad f^{(n)}(x) = 2^x (\ln 2)^n$

54. $f(x) = \ln(2x) = \ln 2 + \ln x \quad \Rightarrow \quad f'(x) = x^{-1}, f''(x) = -x^{-2}, f'''(x) = 2x^{-3}, f^{(4)}(x) = -2 \cdot 3x^{-4}, \ldots,$
$f^{(n)}(x) = (-1)^{n-1}(n-1)!\, x^{-n}$

55. We first show it is true for $n = 1$: $f'(x) = e^x + xe^x = (x+1)e^x$. We now assume it is true for $n = k$:

$f^{(k)}(x) = (x+k)e^x$. With this assumption, we must show it is true for $n = k + 1$:

$$f^{(k+1)}(x) = \frac{d}{dx}\left[f^{(k)}(x)\right] = \frac{d}{dx}\left[(x+k)e^x\right] = e^x + (x+k)e^x = \left[x + (k+1)\right]e^x.$$

Therefore, $f^{(n)}(x) = (x+n)e^x$ by mathematical induction.

56. Using implicit differentiation, $y = x + \arctan y \quad \Rightarrow \quad y' = 1 + \dfrac{1}{1+y^2}y' \quad \Rightarrow \quad y'\left(1 - \dfrac{1}{1+y^2}\right) = 1 \quad \Rightarrow$

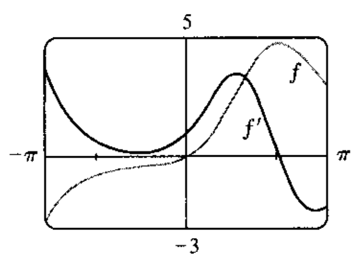
$$y'\left(\frac{y^2}{1+y^2}\right) = 1 \quad \Rightarrow \quad y' = \frac{1+y^2}{y^2} = \frac{1}{y^2} + 1.$$

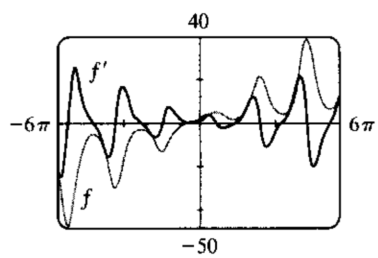
57. $y = (2+x)e^{-x} \quad \Rightarrow \quad y' = (2+x)(-e^{-x}) + e^{-x} \cdot 1 = e^{-x}[-(2+x)+1] = e^{-x}(-x-1)$. At $(0, 2)$, $y' = 1(-1) = -1$, so an equation of the tangent line is $y - 2 = -1(x - 0)$, or $y = -x + 2$.

58. $y = f(x) = x \ln x \quad \Rightarrow \quad f'(x) = \ln x + 1$, so the slope of the tangent at (e, e) is $f'(e) = 2$ and an equation is $y - e = 2(x - e)$ or $y = 2x - e$.

59. $y = [\ln(x+4)]^2 \quad \Rightarrow \quad y' = 2\dfrac{\ln(x+4)}{x+4} = 0 \quad \Leftrightarrow \quad \ln(x+4) = 0 \quad \Leftrightarrow \quad x + 4 = 1 \quad \Leftrightarrow \quad x = -3$, so the tangent is horizontal at $(-3, 0)$.

60. $f(x) = xe^{\sin x} \quad \Rightarrow \quad f'(x) = x\left[e^{\sin x}(\cos x)\right] + e^{\sin x}(1) = e^{\sin x}(x \cos x + 1)$. As a check on our work, we notice from the graphs that $f'(x) > 0$ when f is increasing. Also, we see in the larger viewing rectangle a certain similarity in the graphs of f and f': the sizes of the oscillations of f and f' are linked.





61. (a) The line $x - 4y = 1$ has slope $\frac{1}{4}$. The tangent to $y = e^x$ has slope $\frac{1}{4}$ when $y' = e^x = \frac{1}{4} \quad \Rightarrow$
$x = \ln \frac{1}{4} = -\ln 4$, so an equation is $y - \frac{1}{4} = \frac{1}{4}(x + \ln 4)$ or $y = \frac{1}{4}x + \frac{1}{4}(\ln 4 + 1)$.

(b) The slope of the tangent at the point (a, e^a) is $\left.\dfrac{d}{dx}e^x\right]_{x=a} = e^a$. An equation of the tangent line is thus
$y - e^a = e^a(x - a)$. We substitute $x = 0$, $y = 0$ into this equation, since we want the line to pass through the origin: $0 - e^a = e^a(0 - a) \quad \Leftrightarrow \quad -e^a = e^a(-a) \quad \Leftrightarrow \quad a = 1$. So an equation of the tangent is
$y - e = e(x - 1)$, or $y = ex$.

62. (a) $\lim\limits_{t\to\infty} C(t) = \lim\limits_{t\to\infty}\left[K\left(e^{-at}-e^{-bt}\right)\right] = K\lim\limits_{t\to\infty}\left(e^{-at}-e^{-bt}\right) = K(0-0) = 0$ because $-at\to-\infty$ and $-bt\to-\infty$ as $t\to\infty$.

(b) $C(t) = K\left(e^{-at}-e^{-bt}\right) \quad\Rightarrow\quad C'(t) = K\left(e^{-at}(-a) - e^{-bt}(-b)\right) = K\left(-ae^{-at}+be^{-bt}\right)$

(c) $C'(t) = 0 \quad\Rightarrow\quad be^{-bt} = ae^{-at} \quad\Rightarrow\quad \dfrac{b}{a} = e^{(-a+b)t} \quad\Rightarrow\quad \ln\dfrac{b}{a} = (b-a)t \quad\Rightarrow\quad t = \dfrac{\ln(b/a)}{b-a}$

63. $\lim\limits_{x\to\infty} e^{-3x} = 0$ since $-3x\to-\infty$ as $x\to\infty$ and $\lim\limits_{t\to-\infty} e^t = 0$.

64. $\lim\limits_{x\to 10^-} \ln\left(100-x^2\right) = -\infty$ since as $x\to 10^-$, $\left(100-x^2\right)\to 0^+$.

65. Let $t = 2/(x-3)$. As $x\to 3^-$, $t\to-\infty$. $\lim\limits_{x\to 3^-} e^{2/(x-3)} = \lim\limits_{t\to-\infty} e^t = 0$

66. If $y = x^3 - x = x(x^2-1)$, then as $x\to\infty$, $y\to\infty$. $\lim\limits_{x\to\infty}\arctan(x^3-x) = \lim\limits_{y\to\infty}\arctan y = \frac{\pi}{2}$ by (7.5.8).

67. Let $t = \sinh x$. As $x\to 0^+$, $t\to 0^+$. $\lim\limits_{x\to 0^+}\ln(\sinh x) = \lim\limits_{t\to 0^+}\ln t = -\infty$

68. $-1\le\sin x\le 1 \quad\Rightarrow\quad -e^{-x}\le e^{-x}\sin x\le e^{-x}$. Now $\lim\limits_{x\to\infty}\left(\pm e^{-x}\right) = 0$, so by the Squeeze Theorem,

$\lim\limits_{x\to\infty} e^{-x}\sin x = 0$.

69. $\lim\limits_{x\to\infty}\dfrac{(1+2^x)/2^x}{(1-2^x)/2^x} = \lim\limits_{x\to\infty}\dfrac{1/2^x+1}{1/2^x-1} = \dfrac{0+1}{0-1} = -1$

70. Let $t = x/4$, so $x = 4t$. As $x\to\infty$, $t\to\infty$. $\lim\limits_{x\to\infty}\left(1+\dfrac{4}{x}\right)^x = \lim\limits_{t\to\infty}\left(1+\dfrac{1}{t}\right)^{4t} = \left[\lim\limits_{t\to\infty}\left(1+\dfrac{1}{t}\right)^t\right]^4 = e^4$

71. $\lim\limits_{x\to 0}\dfrac{\tan\pi x}{\ln(1+x)} \overset{\text{H}}{=} \lim\limits_{x\to 0}\dfrac{\pi\sec^2\pi x}{1/(1+x)} = \dfrac{\pi\cdot 1^2}{1/1} = \pi$

72. $\lim\limits_{x\to 0}\dfrac{1-\cos x}{x^2+x} \overset{\text{H}}{=} \lim\limits_{x\to 0}\dfrac{\sin x}{2x+1} = \dfrac{0}{1} = 0$

73. $\lim\limits_{x\to 0}\dfrac{e^{4x}-1-4x}{x^2} \overset{\text{H}}{=} \lim\limits_{x\to 0}\dfrac{4e^{4x}-4}{2x} \overset{\text{H}}{=} \lim\limits_{x\to 0}\dfrac{16e^{4x}}{2} = \lim\limits_{x\to 0} 8e^{4x} = 8\cdot 1 = 8$

74. $\lim\limits_{x\to\infty}\dfrac{e^{4x}-1-4x}{x^2} \overset{\text{H}}{=} \lim\limits_{x\to\infty}\dfrac{4e^{4x}-4}{2x} \overset{\text{H}}{=} \lim\limits_{x\to\infty}\dfrac{16e^{4x}}{2} = \lim\limits_{x\to\infty} 8e^{4x} = \infty$

75. $\lim\limits_{x\to\infty} x^3e^{-x} = \lim\limits_{x\to\infty}\dfrac{x^3}{e^x} \overset{\text{H}}{=} \lim\limits_{x\to\infty}\dfrac{3x^2}{e^x} \overset{\text{H}}{=} \lim\limits_{x\to\infty}\dfrac{6x}{e^x} \overset{\text{H}}{=} \lim\limits_{x\to\infty}\dfrac{6}{e^x} = 0$

76. $\lim\limits_{x\to 0^+} x^2\ln x = \lim\limits_{x\to 0^+}\dfrac{\ln x}{1/x^2} \overset{\text{H}}{=} \lim\limits_{x\to 0^+}\dfrac{1/x}{-2/x^3} = \lim\limits_{x\to 0^+}\left(-\frac{1}{2}x^2\right) = 0$

77. $\lim\limits_{x\to 1^+}\left(\dfrac{x}{x-1}-\dfrac{1}{\ln x}\right) = \lim\limits_{x\to 1^+}\left(\dfrac{x\ln x - x + 1}{(x-1)\ln x}\right) \overset{\text{H}}{=} \lim\limits_{x\to 1^+}\dfrac{x\cdot(1/x)+\ln x - 1}{(x-1)\cdot(1/x)+\ln x}$

$$= \lim_{x\to 1^+}\frac{\ln x}{1-1/x+\ln x} \overset{\text{H}}{=} \lim_{x\to 1^+}\frac{1/x}{1/x^2+1/x} = \frac{1}{1+1} = \frac{1}{2}$$

78. $y = (\tan x)^{\cos x} \;\Rightarrow\; \ln y = \cos x \ln \tan x$, so

$$\lim_{x\to(\pi/2)^-} \ln y = \lim_{x\to(\pi/2)^-} \frac{\ln \tan x}{\sec x} \overset{\text{H}}{=} \lim_{x\to(\pi/2)^-} \frac{(1/\tan x)\sec^2 x}{\sec x \tan x} = \lim_{x\to(\pi/2)^-} \frac{\sec x}{\tan^2 x}$$

$$= \lim_{x\to(\pi/2)^-} \frac{\cos x}{\sin^2 x} = \frac{0}{1^2} = 0, \text{ so}$$

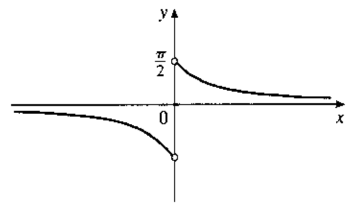
$$\lim_{x\to(\pi/2)^-} (\tan x)^{\cos x} = \lim_{x\to(\pi/2)^-} e^{\ln y} = e^0 = 1.$$

79. $y = f(x) = \tan^{-1}(1/x)$ **A.** $D = \{x \mid x \neq 0\}$ **B.** No intercept **C.** $f(-x) = -f(x)$, so the curve is symmetric about the origin. **D.** $\lim_{x\to\pm\infty} \tan^{-1}(1/x) = \tan^{-1} 0 = 0$, so $y = 0$ is a HA. $\lim_{x\to 0^+} \tan^{-1}(1/x) = \frac{\pi}{2}$ and $\lim_{x\to 0^-} \tan^{-1}(1/x) = -\frac{\pi}{2}$ since $\dfrac{1}{x} \to \pm\infty$ as $x \to 0^\pm$.

E. $f'(x) = \dfrac{1}{1+(1/x)^2}\left(-1/x^2\right) = \dfrac{-1}{x^2+1} \;\Rightarrow\; f'(x) < 0$, so f is decreasing on $(-\infty, 0)$ and $(0, \infty)$. **F.** No maximum nor minimum

G. $f''(x) = \dfrac{2x}{(x^2+1)^2} > 0 \;\Leftrightarrow\; x > 0$, so f is CU on $(0,\infty)$ and CD on $(-\infty, 0)$.

H.



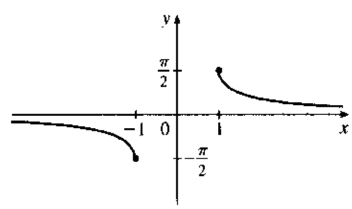
80. $y = f(x) = \sin^{-1}(1/x)$ **A.** $D = \{x \mid -1 \le 1/x \le 1\} = (-\infty, -1] \cup [1, \infty)$. **B.** No intercept **C.** $f(-x) = -f(x)$, symmetric about the origin **D.** $\lim_{x\to\pm\infty} \sin^{-1}(1/x) = \sin^{-1}(0) = 0$, so $y = 0$ is a HA.

E. $f'(x) = \dfrac{1}{\sqrt{1-(1/x)^2}}\left(-\dfrac{1}{x^2}\right) = \dfrac{-1}{\sqrt{x^4 - x^2}} < 0$, so f is decreasing on $(-\infty, -1)$ and $(1, \infty)$.

F. No local extreme value, but $f(1) = \frac{\pi}{2}$ is the absolute maximum value and $f(-1) = -\frac{\pi}{2}$ is the absolute minimum value.

G. $f''(x) = \dfrac{4x^3 - 2x}{2(x^4 - x^2)^{3/2}} = \dfrac{x(2x^2-1)}{(x^4-x^2)^{3/2}} > 0$ for $x > 1$ and $f''(x) < 0$ for $x < -1$, so f is CU on $(1, \infty)$ and CD on $(-\infty, -1)$. No IP

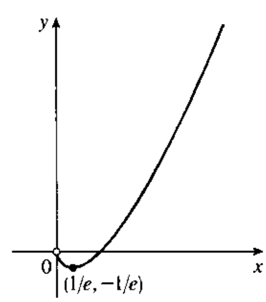
H.



81. $y = f(x) = x \ln x$ **A.** $D = (0, \infty)$ **B.** No y-intercept; x-intercept 1. **C.** No symmetry **D.** No asymptote [Note that the graph approaches the point $(0, 0)$ as $x \to 0^+$.]

E. $f'(x) = x(1/x) + (\ln x)(1) = 1 + \ln x$, so $f'(x) \to -\infty$ as $x \to 0^+$ and $f'(x) \to \infty$ as $x \to \infty$. $f'(x) = 0 \;\Leftrightarrow\; \ln x = -1 \;\Leftrightarrow\; x = e^{-1} = 1/e$. $f'(x) > 0$ for $x > 1/e$, so f is decreasing on $(0, 1/e)$ and increasing on $(1/e, \infty)$. **F.** Local minimum: $f(1/e) = -1/e$. No local maximum. **G.** $f''(x) = 1/x$, so $f''(x) > 0$ for $x > 0$. The graph is CU on $(0, \infty)$ and there is no IP.

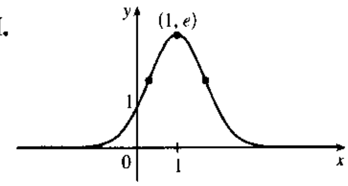
H.



82. $y = f(x) = e^{2x-x^2}$ **A.** $D = \mathbb{R}$ **B.** y-intercept 1; no x-intercept **C.** No symmetry **D.** $\lim\limits_{x\to\pm\infty} e^{2x-x^2} = 0$, so $y = 0$ is a HA. **E.** $y = f(x) = e^{2x-x^2} \Rightarrow f'(x) = 2(1-x)e^{2x-x^2} > 0 \Leftrightarrow x < 1$, so f is increasing on $(-\infty, 1)$ and decreasing on $(1, \infty)$. **F.** $f(1) = e$ is a local and absolute maximum value.

G. $f''(x) = 2\left(2x^2 - 4x + 1\right)e^{2x-x^2} = 0 \Leftrightarrow x = 1 \pm \frac{\sqrt{2}}{2}$. $f''(x) > 0 \Leftrightarrow x < 1 - \frac{\sqrt{2}}{2}$ or $x > 1 + \frac{\sqrt{2}}{2}$, so f is CU on $\left(-\infty, 1 - \frac{\sqrt{2}}{2}\right)$ and $\left(1 + \frac{\sqrt{2}}{2}, \infty\right)$, and CD on $\left(1 - \frac{\sqrt{2}}{2}, 1 + \frac{\sqrt{2}}{2}\right)$. IP at $\left(1 \pm \frac{\sqrt{2}}{2}, \sqrt{e}\right)$

H.



83. $y = f(x) = e^x + e^{-3x}$ **A.** $D = \mathbb{R}$ **B.** y-intercept 2; no x-intercept **C.** No symmetry

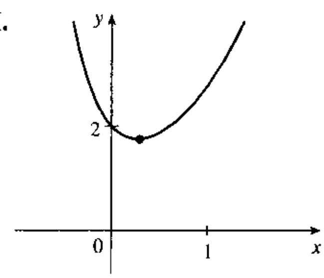
D. $\lim\limits_{x\to\pm\infty} \left(e^x + e^{-3x}\right) = \infty$, no asymptote **E.** $y = f(x) = e^x + e^{-3x} \Rightarrow$

$f'(x) = e^x - 3e^{-3x} = e^{-3x}\left(e^{4x} - 3\right) > 0 \Leftrightarrow e^{4x} > 3 \Leftrightarrow$ $4x > \ln 3 \Leftrightarrow x > \frac{1}{4}\ln 3 \approx 0.27$, so f is increasing on $\left(\frac{1}{4}\ln 3, \infty\right)$ and decreasing on $\left(-\infty, \frac{1}{4}\ln 3\right)$.

F. Absolute minimum value $f\left(\frac{1}{4}\ln 3\right) = 3^{1/4} + 3^{-3/4} \approx 1.75$.

G. $f''(x) = e^x + 9e^{-3x} > 0$, so f is CU on $(-\infty, \infty)$. No IP

H.

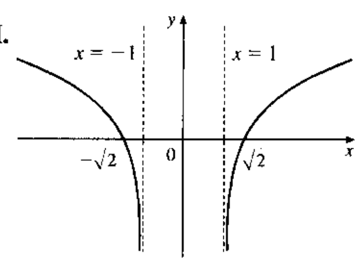


84. $y = f(x) = \ln\left(x^2 - 1\right)$ **A.** $D = (-\infty, -1) \cup (1, \infty)$ **B.** No y-intercept; x-intercepts $\pm\sqrt{2}$ **C.** Symmetric about the y-axis **D.** $\lim\limits_{x\to\pm\infty} \ln\left(x^2 - 1\right) = \infty$, $\lim\limits_{x\to 1^+} \ln\left(x^2 - 1\right) = -\infty$, $\lim\limits_{x\to -1^-} \ln\left(x^2 - 1\right) = -\infty$, so $x = 1$ and $x = -1$ are VA. **E.** $y = f(x) = \ln\left(x^2 - 1\right) \Rightarrow f'(x) = \dfrac{2x}{x^2 - 1} > 0$ for $x > 1$ and $f'(x) < 0$ for $x < -1$, so f is increasing on $(1, \infty)$ and decreasing on $(-\infty, -1)$. Note that the domain of f is $|x| > 1$. **F.** No extreme value

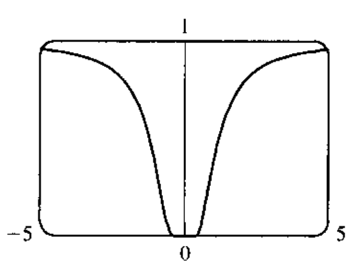
G. $f''(x) = -2\dfrac{x^2 + 1}{\left(x^2 - 1\right)^2} < 0$, so f is CD on $(-\infty, -1)$ and $(1, \infty)$.

No IP

H.



85.



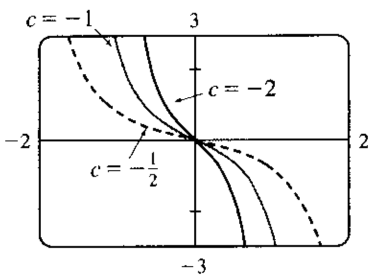
From the graph, we estimate the points of inflection to be about $(\pm 0.82, 0.22)$. $f(x) = e^{-1/x^2} \Rightarrow f'(x) = 2x^{-3}e^{-1/x^2} \Rightarrow$

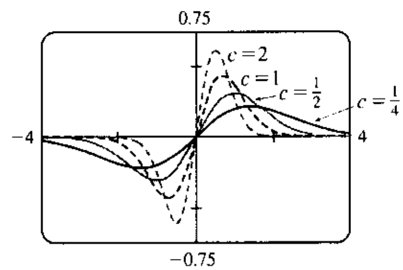
$$f''(x) = 2\left[x^{-3}\left(2x^{-3}\right)e^{-1/x^2} + e^{-1/x^2}\left(-3x^{-4}\right)\right] = 2x^{-6}e^{-1/x^2}\left(2 - 3x^2\right).$$

This is 0 when $2 - 3x^2 = 0 \Leftrightarrow x = \pm\sqrt{\frac{2}{3}}$, so the inflection points are $\left(\pm\sqrt{\frac{2}{3}}, e^{-3/2}\right)$.

86. We exclude the case $c = 0$, since in that case $f(x) = 0$ for all x. To find the maxima and minima, we differentiate: $f(x) = cxe^{-cx^2} \Rightarrow f'(x) = c\left[xe^{-cx^2}(-2cx) + e^{-cx^2}(1)\right] = ce^{-cx^2}(-2cx^2+1)$. This is 0 where $-2cx^2 + 1 = 0 \Leftrightarrow x = \pm 1/\sqrt{2c}$. So if $c > 0$, there are two maxima or minima, whose x-coordinates approach 0 as c increases. The negative root gives a minimum and the positive root gives a maximum, by the First Derivative Test. By substituting back into the equation, we see that $f(\pm 1/\sqrt{2c}) = c(\pm 1/\sqrt{2c})\, e^{-c(\pm 1/\sqrt{2c})^2} = \pm\sqrt{c/2e}$. So as c increases, the extreme points become more pronounced. Note that if $c > 0$, then $\lim_{x\to\pm\infty} f(x) = 0$. If $c < 0$, then there are no extreme values, and $\lim_{x\to\pm\infty} f(x) = \mp\infty$.

To find the points of inflection, we differentiate again: $f'(x) = ce^{-cx^2}(-2cx^2+1) \Rightarrow$ $f''(x) = c\left[e^{-cx^2}(-4cx) + (-2cx^2+1)\left(-2cxe^{-cx^2}\right)\right] = -2c^2xe^{-cx^2}(3-2cx^2)$. This is 0 at $x = 0$ and where $3 - 2cx^2 = 0 \Leftrightarrow x = \pm\sqrt{3/(2c)} \Rightarrow$ IP at $\left(\pm\sqrt{3/(2c)}, \pm\sqrt{3c/2}\, e^{-3/2}\right)$. If $c > 0$ there are three inflection points, and as c increases, the x-coordinates of the nonzero inflection points approach 0. If $c < 0$, there is only one inflection point, the origin.





87. $s(t) = Ae^{-ct}\cos(\omega t + \delta) \Rightarrow$

$$v(t) = s'(t) = A\{e^{-ct}[-\omega\sin(\omega t+\delta)] + \cos(\omega t+\delta)(-ce^{-ct})\}$$
$$= -Ae^{-ct}[\omega\sin(\omega t+\delta) + c\cos(\omega t+\delta)] \Rightarrow$$

$$a(t) = v'(t) = -A\{e^{-ct}[\omega^2\cos(\omega t+\delta) - c\omega\sin(\omega t+\delta)] + [\omega\sin(\omega t+\delta) + c\cos(\omega t+\delta)](-ce^{-ct})\}$$
$$= -Ae^{-ct}[\omega^2\cos(\omega t+\delta) - c\omega\sin(\omega t+\delta) - c\omega\sin(\omega t+\delta) - c^2\cos(\omega t+\delta)]$$
$$= -Ae^{-ct}[(\omega^2 - c^2)\cos(\omega t+\delta) - 2c\omega\sin(\omega t+\delta)]$$
$$= Ae^{-ct}[(c^2-\omega^2)\cos(\omega t+\delta) + 2c\omega\sin(\omega t+\delta)]$$

88. (a) Let $f(x) = \ln x + x - 3$. Then $f'(x) = 1/x + 1 > 0$ (for $x > 0$) and $f(2) \approx -0.307$ and $f(e) \approx 0.718$. f is differentiable on $(2, e)$, continuous on $[2, e]$ and $f(2) < 0$, $f(e) > 0$. Therefore, by the Intermediate Value Theorem there exists a number c in $(2, e)$ such that $f(c) = 0$. Thus, there is one root. But $f'(x) > 0$ for $x \in (2, e)$, so f is increasing on $(2, e)$, which means that there is exactly one root.

(b) We use Newton's Method with $f(x) = \ln x + x - 3$, $f'(x) = 1/x + 1$, and $x_1 = 2$.
$x_2 = x_1 - \dfrac{\ln x_1 + x_1 - 3}{1/x_1 + 1} = 2 - \dfrac{\ln 2 + 2 - 3}{1/2 + 1} \approx 2.20457$. Similarly, $x_3 \approx 2.20794$, $x_4 = 2.20794$. Thus, the root of the equation, correct to four decimal places, is 2.2079.

89. Let $P(t) = \dfrac{64}{1 + 31e^{-0.7944t}} = \dfrac{A}{1 + Be^{ct}} = A(1 + Be^{ct})^{-1}$, where $A = 64$, $B = 31$, and $c = -0.7944$.

$$P'(t) = -A(1 + Be^{ct})^{-2}(Bce^{ct}) = -ABce^{ct}(1 + Be^{ct})^{-2}$$

$$P''(t) = -ABce^{ct}\left[-2(1 + Be^{ct})^{-3}(Bce^{ct})\right] + (1 + Be^{ct})^{-2}(-ABc^2e^{ct})$$

$$= -ABc^2e^{ct}(1 + Be^{ct})^{-3}\left[-2Be^{ct} + (1 + Be^{ct})\right] = -\frac{ABc^2e^{ct}(1 - Be^{ct})}{(1 + Be^{ct})^3}$$

The population is increasing most rapidly when its graph changes from CU to CD; that is, when $P''(t) = 0$ in this case. $P''(t) = 0 \;\Rightarrow\; Be^{ct} = 1 \;\Rightarrow\; e^{ct} = \dfrac{1}{B} \;\Rightarrow$ $ct = \ln\dfrac{1}{B} \;\Rightarrow\; t = \dfrac{\ln(1/B)}{c} = \dfrac{\ln(1/31)}{-0.7944} \approx 4.32$ days. Note that

$$P\left(\frac{1}{c}\ln\frac{1}{B}\right) = \frac{A}{1 + Be^{c(1/c)\ln(1/B)}} = \frac{A}{1 + Be^{\ln(1/B)}} = \frac{A}{1 + B(1/B)} = \frac{A}{1+1} = \frac{A}{2},$$ one-half the limit of P as $t \to \infty$.

90. Let $t = 4u$. Then $dt = 4\,du$ and

$$\int_0^4 \frac{1}{16 + t^2}\,dt = \int_0^1 \frac{1}{16 + 16u^2}\cdot 4\,du = \frac{1}{4}\int_0^1 \frac{du}{1 + u^2} = \frac{1}{4}\left[\tan^{-1} u\right]_0^1$$
$$= \tfrac{1}{4}(\tan^{-1} 1 - \tan^{-1} 0) = \tfrac{1}{4}\left(\tfrac{\pi}{4} - 0\right) = \tfrac{\pi}{16}$$

91. Let $u = -2y^2$. Then $du = -4y\,dy$ and

$$\int_0^1 ye^{-2y^2}\,dy = \int_0^{-2} e^u\left(-\tfrac{1}{4}\,du\right) = -\tfrac{1}{4}\left[e^u\right]_0^{-2} = -\tfrac{1}{4}(e^{-2} - 1) = \tfrac{1}{4}(1 - e^{-2}).$$

92. $\displaystyle\int_2^5 \frac{dr}{1 + 2r} = \tfrac{1}{2}[\ln|1 + 2r|]_2^5 = \tfrac{1}{2}(\ln 11 - \ln 5) = \tfrac{1}{2}\ln\tfrac{11}{5}$

93. $\displaystyle\int_2^4 \frac{1 + x - x^2}{x^2}\,dx = \int_2^4 \left(x^{-2} + \frac{1}{x} - 1\right)dx = \left[-\frac{1}{x} + \ln x - x\right]_2^4$
$$= \left(-\tfrac{1}{4} + \ln 4 - 4\right) - \left(-\tfrac{1}{2} + \ln 2 - 2\right) = \ln 2 - \tfrac{7}{4}$$

94. Let $u = \sin x$. Then $du = \cos x\,dx$, so

$$\int_0^{\pi/2} \frac{\cos x}{1 + \sin^2 x}\,dx = \int_0^1 \frac{1}{1 + u^2}\,du = \left[\tan^{-1} u\right]_0^1 = \tan^{-1} 1 - \tan^{-1} 0 = \tfrac{\pi}{4} - 0 = \tfrac{\pi}{4}$$

95. Let $u = \sqrt{x}$. Then $du = \dfrac{dx}{2\sqrt{x}} \;\Rightarrow\; \displaystyle\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\,dx = 2\int e^u\,du = 2e^u + C = 2e^{\sqrt{x}} + C.$

96. Let $u = \ln x$. Then $du = \dfrac{dx}{x} \;\Rightarrow\; \displaystyle\int \frac{\cos(\ln x)}{x}\,dx = \int \cos u\,du = \sin u + C = \sin(\ln x) + C.$

97. Let $u = x^2 + 2x$. Then $du = (2x + 2)\,dx = 2(x + 1)\,dx$ and

$$\int \frac{x + 1}{x^2 + 2x}\,dx = \int \frac{\frac{1}{2}\,du}{u} = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln\left|x^2 + 2x\right| + C.$$

98. Let $u = e^{-x}$. Then $du = -e^{-x}\,dx$ and

$$\int \frac{e^{-x}}{1+e^{-2x}}\,dx = \int \frac{-du}{1+u^2} = -\tan^{-1} u + C = -\tan^{-1}(e^{-x}) + C.$$

99. Let $u = \ln(\cos x)$. Then $du = \dfrac{-\sin x}{\cos x}\,dx = -\tan x\,dx \quad\Rightarrow$

$\int \tan x \ln(\cos x)\,dx = -\int u\,du = -\frac{1}{2}u^2 + C = -\frac{1}{2}[\ln(\cos x)]^2 + C.$

100. Let $u = x^2$. Then $du = 2x\,dx \quad\Rightarrow\quad \displaystyle\int \frac{x}{\sqrt{1-x^4}}\,dx = \frac{1}{2}\int \frac{du}{\sqrt{1-u^2}} = \frac{1}{2}\sin^{-1} u + C = \frac{1}{2}\sin^{-1}(x^2) + C.$

101. Let $u = \tan\theta$. Then $du = \sec^2\theta\,d\theta$ and $\displaystyle\int 2^{\tan\theta}\sec^2\theta\,d\theta = \int 2^u\,du = \frac{2^u}{\ln 2} + C = \frac{2^{\tan\theta}}{\ln 2} + C.$

102. $\displaystyle\int \sinh au\,du = \frac{1}{a}\cosh au + C$

103. Let $u = 1 + \sec\theta$, so $du = \sec\theta\tan\theta\,d\theta \quad\Rightarrow\quad \displaystyle\int \frac{\sec\theta\tan\theta}{1+\sec\theta}\,d\theta = \int \frac{1}{u}\,du = \ln|u| + C = \ln|1+\sec\theta| + C.$

104. $1 + e^{2x} > e^{2x} \quad\Rightarrow\quad \sqrt{1+e^{2x}} > \sqrt{e^{2x}} = e^x \quad\Rightarrow\quad \int_0^1 \sqrt{1+e^{2x}}\,dx \ge \int_0^1 e^x\,dx = e^x\big]_0^1 = e - 1$

105. $\cos x \le 1 \quad\Rightarrow\quad e^x\cos x \le e^x \quad\Rightarrow\quad \int_0^1 e^x\cos x\,dx \le \int_0^1 e^x\,dx = e^x\big]_0^1 = e - 1$

106. For $0 \le x \le 1$, $0 \le \sin^{-1} x \le \frac{\pi}{2}$, so $\int_0^1 x\sin^{-1} x\,dx \le \int_0^1 x\left(\frac{\pi}{2}\right)dx = \frac{\pi}{4}x^2\big]_0^1 = \frac{\pi}{4}.$

107. $\displaystyle f(x) = \int_1^{\sqrt{x}} \frac{e^s}{s}\,ds \quad\Rightarrow\quad f'(x) = \frac{d}{dx}\int_1^{\sqrt{x}} \frac{e^s}{s}\,ds = \frac{e^{\sqrt{x}}}{\sqrt{x}}\frac{d}{dx}\sqrt{x} = \frac{e^{\sqrt{x}}}{\sqrt{x}}\frac{1}{2\sqrt{x}} = \frac{e^{\sqrt{x}}}{2x}$

108. $\displaystyle f(x) = \int_{\ln x}^{2x} e^{-t^2}\,dt \quad\Rightarrow\quad f'(x) = \frac{d}{dx}\int_{\ln x}^{2x} e^{-t^2}\,dt = -\frac{d}{dx}\int_0^{\ln x} e^{-t^2}\,dt + \frac{d}{dx}\int_0^{2x} e^{-t^2}\,dt$

$$= -e^{-(\ln x)^2}\left(\frac{1}{x}\right) + e^{-(2x)^2}(2) = -\frac{e^{-(\ln x)^2}}{x} + 2e^{-4x^2}$$

109. $\displaystyle f_{\text{ave}} = \frac{1}{4-1}\int_1^4 \frac{1}{x}\,dx = \frac{1}{3}\Big[\ln|x|\Big]_1^4 = \frac{1}{3}[\ln 4 - \ln 1] = \frac{1}{3}\ln 4$

110. $A = \int_{-2}^0 (e^{-x} - e^x)\,dx + \int_0^1 (e^x - e^{-x})\,dx = \left[-e^{-x} - e^x\right]_{-2}^0 + \left[e^x + e^{-x}\right]_0^1$

$= [(-1-1) - (-e^2 - e^{-2})] + [(e + e^{-1}) - (1+1)] = e^2 + e + e^{-1} + e^{-2} - 4$

111. $\displaystyle V = \int_0^1 \frac{2\pi x}{1+x^4}\,dx$ by cylindrical shells. Let $u = x^2 \quad\Rightarrow\quad du = 2x\,dx$. Then

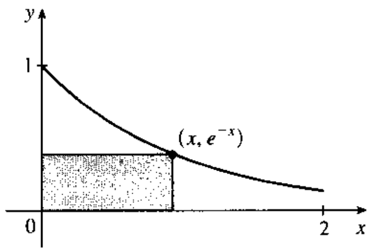
$$V = \int_0^1 \frac{\pi}{1+u^2}\,du = \pi\left[\tan^{-1} u\right]_0^1 = \pi\left(\tan^{-1} 1 - \tan^{-1} 0\right) = \pi\left(\frac{\pi}{4}\right) = \frac{\pi^2}{4}.$$

112. $f(x) = x + x^2 + e^x \quad\Rightarrow\quad f'(x) = 1 + 2x + e^x$ and $f(0) = 1 \quad\Rightarrow\quad g(1) = 0$, so

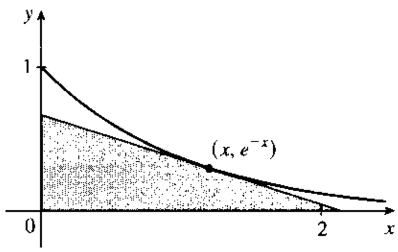
$$g'(1) = \frac{1}{f'(g(1))} = \frac{1}{f'(0)} = \frac{1}{2}.$$

113. $f(x) = \ln x + \tan^{-1} x \quad\Rightarrow\quad f(1) = \ln 1 + \tan^{-1} 1 = \frac{\pi}{4} \quad\Rightarrow\quad g\left(\frac{\pi}{4}\right) = 1.$

$$f'(x) = \frac{1}{x} + \frac{1}{1+x^2}, \text{ so } g'\left(\frac{\pi}{4}\right) = \frac{1}{f'(1)} = \frac{1}{3/2} = \frac{2}{3}.$$

114. 

The area of such a rectangle is just the product of its sides, that is, $A(x) = x \cdot e^{-x}$. We want to find the maximum of this function, so we differentiate: $A'(x) = x(-e^{-x}) + e^{-x}(1) = e^{-x}(1-x)$. This is 0 only at $x = 1$, and changes from positive to negative there, so by the First Derivative Test this gives a local maximum. So the largest area is $A(1) = 1/e$.

115. 

We find the equation of a tangent to the curve $y = e^{-x}$, so that we can find the x- and y-intercepts of this tangent, and then we can find the area of the triangle. The slope of the tangent at the point (a, e^{-a}) is given by $\left.\dfrac{d}{dx}e^{-x}\right]_{x=a} = -e^{-a}$, and so the equation of the tangent is $y - e^{-a} = -e^{-a}(x-a) \quad\Leftrightarrow\quad y = e^{-a}(a - x + 1)$.

The y-intercept of this line is $y = e^{-a}(a - 0 + 1) = e^{-a}(a+1)$. To find the x-intercept we set $y = 0 \quad\Rightarrow\quad e^{-a}(a - x + 1) = 0 \quad\Rightarrow\quad x = a + 1$. So the area of the triangle is $A(a) = \frac{1}{2}\left[e^{-a}(a+1)\right](a+1) = \frac{1}{2}e^{-a}(a+1)^2$. We differentiate this with respect to a: $A'(a) = \frac{1}{2}\left[e^{-a}(2)(a+1) + (a+1)^2 e^{-a}(-1)\right] = \frac{1}{2}e^{-a}(1 - a^2)$. This is 0 at $a = \pm 1$, and the root $a = 1$ gives a maximum, by the First Derivative Test. So the maximum area of the triangle is $A(1) = \frac{1}{2}e^{-1}(1+1)^2 = 2e^{-1} = 2/e$.

116. Using Formula 5.2.3 with $a = 0$ and $b = 1$, we have $\displaystyle\int_0^1 e^x\,dx = \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n} e^{i/n}$. This series is a geometric series with $a = r = e^{1/n}$, so $\displaystyle\sum_{i=1}^{n} e^{i/n} = e^{1/n}\frac{e^{n/n} - 1}{e^{1/n} - 1} = e^{1/n}\frac{e - 1}{e^{1/n} - 1} \quad\Rightarrow$

$$\int_0^1 e^x\,dx = \lim_{n\to\infty}\frac{1}{n}\sum_{i=1}^{n} e^{i/n} = \lim_{n\to\infty}(e-1)e^{1/n}\frac{1/n}{e^{1/n} - 1}.$$ As $n \to \infty$, $1/n \to 0^+$, so $e^{1/n} \to e^0 = 1$.

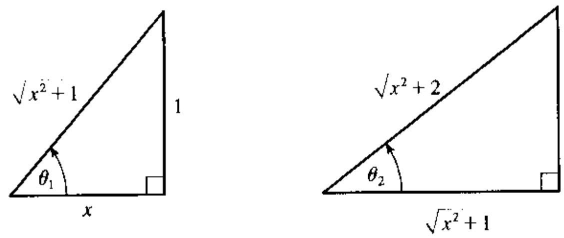
Let $t = 1/n$. Then $e^{1/n} - 1 = e^t - 1 \to 0^+$, so l'Hospital's Rule gives $\displaystyle\lim_{t\to 0}\frac{t}{e^t - 1} = \lim_{t\to 0}\frac{1}{e^t} = 1$ and we have $\int_0^1 e^x\,dx = \left[\lim_{t\to 0^+}(e-1)e^t\right]\left[\lim_{t\to 0^+}\dfrac{t}{e^t - 1}\right] = e - 1$.

117. $\displaystyle\lim_{x\to -1}F(x) = \lim_{x\to -1}\frac{b^{x+1} - a^{x+1}}{x+1} \stackrel{H}{=} \lim_{x\to -1}\frac{b^{x+1}\ln b - a^{x+1}\ln a}{1} = \ln b - \ln a = F(-1)$, so F is continuous at -1.

118. Let $\theta_1 = \operatorname{arccot} x$, so $\cot\theta_1 = x = x/1$. So $\sin(\operatorname{arccot} x) = \sin\theta_1 = \dfrac{1}{\sqrt{x^2+1}}$.

Let $\theta_2 = \arctan\left[\dfrac{1}{\sqrt{x^2+1}}\right]$, so $\tan\theta_2 = \dfrac{1}{\sqrt{x^2+1}}$.

Hence, $\cos(\arctan(\sin(\operatorname{arccot} x))) = \cos\theta_2 = \dfrac{\sqrt{x^2+1}}{\sqrt{x^2+2}} = \sqrt{\dfrac{x^2+1}{x^2+2}}$.



119. Differentiating both sides of the given equation, using the Fundamental Theorem for each side, gives

$f(x) = e^{2x} + 2xe^{2x} + e^{-x}f(x)$. So $f(x)(1-e^{-x}) = e^{2x} + 2xe^{2x}$. Hence $f(x) = \dfrac{e^{2x}(1+2x)}{1-e^{-x}}$.

120. (a) Let $f(x) = x - \ln x - 1$, so $f'(x) = 1 - \dfrac{1}{x} = \dfrac{x-1}{x}$. Since $x > 0$, $f'(x) < 0$ for $0 < x < 1$ and $f'(x) > 0$ for $x > 1$. So there is an absolute minimum at $x = 1$ with $f(1) = 0$.
So for $x > 0$, $x \neq 1$, $x - \ln x - 1 = f(x) > f(1) = 0$, and hence $\ln x < x - 1$.

(b) Here let $f(x) = \ln x - \dfrac{x-1}{x} = \ln x - 1 + \dfrac{1}{x}$. So $f'(x) = \dfrac{1}{x} - \dfrac{1}{x^2} = \dfrac{x-1}{x^2}$. As in (a), we see that there is an absolute minimum value at $x = 1$ and that $f(1) = 0$. So for $x > 0$, $x \neq 1$, $\ln x - \dfrac{x-1}{x} = f(x) > f(1) = 0$ and hence $\dfrac{x-1}{x} < \ln x$.

(c) Let $b > a > 0$, so $b/a > 1$. Letting $x = b/a$ in the inequalities in (a) and (b) gives
$\dfrac{b-a}{b} = \dfrac{b/a - 1}{b/a} < \ln\dfrac{b}{a} < \dfrac{b}{a} - 1 = \dfrac{b-a}{a}$. Noting that $\ln\dfrac{b}{a} = \ln b - \ln a$, the result follows after dividing through by $b - a$.

(d) Let $f(x) = \ln x$. From the given diagram, we see that
(slope of tangent at $x = b$) < (slope of secant line) < (slope of tangent at $x = a$). Since $f'(x) = \dfrac{1}{x}$, we therefore have $\dfrac{1}{b} < \dfrac{\ln b - \ln a}{b-a} < \dfrac{1}{a}$. To make this geometric argument more rigorous, we could use the Mean Value Theorem: For any a and b with $0 < a < b$, there exists some $c \in (a, b)$ for which
$f'(c) = \dfrac{1}{c} = \dfrac{\ln b - \ln a}{b-a}$. But $\dfrac{1}{x}$ is a decreasing function on $(0,\infty)$, so $\dfrac{1}{b} < \dfrac{1}{c} = \dfrac{\ln b - \ln a}{b-a} < \ln\dfrac{1}{a}$.

(e) Since $\dfrac{1}{b} < \dfrac{1}{x} < \dfrac{1}{a}$ for $a < x < b$, Property 8 says that $\dfrac{1}{b}(b-a) < \displaystyle\int_a^b \frac{1}{x}\,dx < \frac{1}{a}(b-a) \quad\Rightarrow$
$\dfrac{1}{b}(b-a) < \ln b - \ln a < \dfrac{1}{a}(b-a) \quad\Rightarrow\quad \dfrac{1}{b} < \dfrac{\ln b - \ln a}{b-a} < \dfrac{1}{a}$. (Note from the proof of Property 8 that we are justified in making all of the inequalities strict.)

PROBLEMS PLUS

1. Let $y = f(x) = e^{-x^2}$. The area of the rectangle under the curve from $-x$ to x is $A(x) = 2xe^{-x^2}$ where $x \geq 0$. We maximize $A(x)$: $A'(x) = 2e^{-x^2} - 4x^2e^{-x^2} = 2e^{-x^2}(1 - 2x^2) = 0 \quad \Rightarrow \quad x = \frac{1}{\sqrt{2}}$. This gives a maximum since $A'(x) > 0$ for $0 \leq x < \frac{1}{\sqrt{2}}$ and $A'(x) < 0$ for $x > \frac{1}{\sqrt{2}}$. We next determine the points of inflection of $f(x)$. Notice that $f'(x) = -2xe^{-x^2} = -A(x)$. So $f''(x) = -A'(x)$ and hence, $f''(x) < 0$ for $-\frac{1}{\sqrt{2}} < x < \frac{1}{\sqrt{2}}$ and $f''(x) > 0$ for $x < -\frac{1}{\sqrt{2}}$ and $x > \frac{1}{\sqrt{2}}$. So $f(x)$ changes concavity at $x = \pm\frac{1}{\sqrt{2}}$, and the two vertices of the rectangle of largest area are at the inflection points.

2. We use proof by contradiction. Suppose that $\log_2 5$ is a rational number. Then $\log_2 5 = m/n$ where m and n are positive integers $\Rightarrow \quad 2^{m/n} = 5 \quad \Rightarrow \quad 2^m = 5^n$. But this is impossible since 2^m is even and 5^n is odd. So $\log_2 5$ is irrational.

3. Consider the statement that $\dfrac{d^n}{dx^n}(e^{ax}\sin bx) = r^n e^{ax}\sin(bx + n\theta)$. For $n = 1$,

$\dfrac{d}{dx}(e^{ax}\sin bx) = ae^{ax}\sin bx + be^{ax}\cos bx$, and

$$re^{ax}\sin(bx + \theta) = re^{ax}[\sin bx\cos\theta + \cos bx\sin\theta] = re^{ax}\left(\frac{a}{r}\sin bx + \frac{b}{r}\cos bx\right)$$
$$= ae^{ax}\sin bx + be^{ax}\cos bx$$

since $\tan\theta = \dfrac{b}{a} \quad \Rightarrow \quad \sin\theta = \dfrac{b}{r}$ and $\cos\theta = \dfrac{a}{r}$.

So the statement is true for $n = 1$. Assume it is true for $n = k$. Then

$$\frac{d^{k+1}}{dx^{k+1}}(e^{ax}\sin bx) = \frac{d}{dx}\left[r^k e^{ax}\sin(bx + k\theta)\right] = r^k ae^{ax}\sin(bx + k\theta) + r^k e^{ax} b\cos(bx + k\theta)$$
$$= r^k e^{ax}[a\sin(bx + k\theta) + b\cos(bx + k\theta)]$$

But

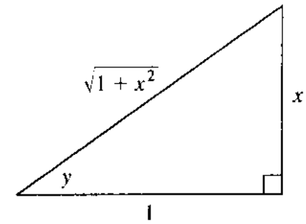
$$\sin[bx + (k+1)\theta] = \sin[(bx + k\theta) + \theta] = \sin(bx + k\theta)\cos\theta + \sin\theta\cos(bx + k\theta)$$
$$= \frac{a}{r}\sin(bx + k\theta) + \frac{b}{r}\cos(bx + k\theta)$$

Hence, $a\sin(bx + k\theta) + b\cos(bx + k\theta) = r\sin[bx + (k+1)\theta]$. So

$$\frac{d^{k+1}}{dx^{k+1}}(e^{ax}\sin bx) = r^k e^{ax}[a\sin(bx + k\theta) + b\cos(bx + k\theta)] = r^k e^{ax}[r\sin(bx + (k+1)\theta)]$$
$$= r^{k+1}e^{ax}[\sin(bx + (k+1)\theta)]$$

Therefore, the statement is true for all n by mathematical induction.

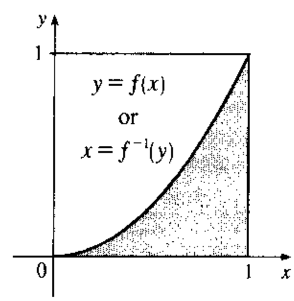
4. Let $y = \tan^{-1} x$. Then $\tan y = x$, so from the triangle we see that $\sin(\tan^{-1} x) = \sin y = \dfrac{x}{\sqrt{1+x^2}}$. Using this fact we have that $\sin(\tan^{-1}(\sinh x)) = \dfrac{\sinh x}{\sqrt{1+\sinh^2 x}} = \dfrac{\sinh x}{\cosh x} = \tanh x$. Hence, $\sin^{-1}(\tanh x) = \sin^{-1}\left(\sin(\tan^{-1}(\sinh x))\right) = \tan^{-1}(\sinh x)$.



5. We first show that $\dfrac{x}{1+x^2} < \tan^{-1} x$ for $x > 0$. Let $f(x) = \tan^{-1} x - \dfrac{x}{1+x^2}$. Then

$$f'(x) = \frac{1}{1+x^2} - \frac{1(1+x^2) - x(2x)}{(1+x^2)^2} = \frac{(1+x^2)-(1-x^2)}{(1+x^2)^2} = \frac{2x^2}{(1+x^2)^2} > 0 \text{ for } x > 0.$$ So $f(x)$ is increasing on $(0,\infty)$. Hence, $0 < x \;\Rightarrow\; 0 = f(0) < f(x) = \tan^{-1} x - \dfrac{x}{1+x^2}$. So $\dfrac{x}{1+x^2} < \tan^{-1} x$ for $0 < x$. We next show that $\tan^{-1} x < x$ for $x > 0$. Let $h(x) = x - \tan^{-1} x$. Then $h'(x) = 1 - \dfrac{1}{1+x^2} = \dfrac{x^2}{1+x^2} > 0$. Hence, $h(x)$ is increasing on $(0,\infty)$. So for $0 < x$, $0 = h(0) < h(x) = x - \tan^{-1} x$. Hence, $\tan^{-1} x < x$ for $x > 0$, and we conclude that $\dfrac{x}{1+x^2} < \tan^{-1} x < x$ for $x > 0$.

6. The shaded region has area $\int_0^1 f(x)\,dx = \frac{1}{3}$. The integral $\int_0^1 f^{-1}(y)\,dy$ gives the area of the unshaded region, which we know to be $1 - \frac{1}{3} = \frac{2}{3}$. So $\int_0^1 f^{-1}(y)\,dy = \frac{2}{3}$.



7. By the Fundamental Theorem of Calculus, $f(x) = \int_1^x \sqrt{1+t^3}\,dt \;\Rightarrow\; f'(x) = \sqrt{1+x^3} > 0$ for $x > -1$. So f is increasing on $(-1,\infty)$ and hence is one-to-one. Note that $f(1) = 0$, so $f^{-1}(1) = 0 \;\Rightarrow$ $(f^{-1})'(0) = 1/f'(1) = \frac{1}{\sqrt{2}}$.

8. $y = \dfrac{x}{\sqrt{a^2-1}} - \dfrac{2}{\sqrt{a^2-1}} \arctan \dfrac{\sin x}{a + \sqrt{a^2-1} + \cos x}$. Let $k = a + \sqrt{a^2-1}$. Then

$$y' = \frac{1}{\sqrt{a^2-1}} - \frac{2}{\sqrt{a^2-1}} \cdot \frac{1}{1 + \sin^2 x/(k+\cos x)^2} \cdot \frac{\cos x(k+\cos x) + \sin^2 x}{(k+\cos x)^2}$$

$$= \frac{1}{\sqrt{a^2-1}} - \frac{2}{\sqrt{a^2-1}} \cdot \frac{k\cos x + \cos^2 x + \sin^2 x}{(k+\cos x)^2 + \sin^2 x} = \frac{1}{\sqrt{a^2-1}} - \frac{2}{\sqrt{a^2-1}} \cdot \frac{k\cos x + 1}{k^2 + 2k\cos x + 1}$$

$$= \frac{k^2 + 2k\cos x + 1 - 2k\cos x - 2}{\sqrt{a^2-1}\,(k^2+2k\cos x+1)} = \frac{k^2-1}{\sqrt{a^2-1}\,(k^2+2k\cos x+1)}$$

But $k^2 = 2a^2 + 2a\sqrt{a^2-1} - 1 = 2a\left(a + \sqrt{a^2-1}\right) - 1 = 2ak - 1$, so $k^2 + 1 = 2ak$, and

$k^2 - 1 = 2(ak-1)$. So $y' = \dfrac{2(ak-1)}{\sqrt{a^2-1}\,(2ak + 2k\cos x)} = \dfrac{ak-1}{\sqrt{a^2-1}k\,(a+\cos x)}$. But

$ak - 1 = a^2 + a\sqrt{a^2-1} - 1 = k\sqrt{a^2-1}$, so $y' = 1/(a+\cos x)$.

9. If $L = \lim\limits_{x\to\infty}\left(\dfrac{x+a}{x-a}\right)^x$, then L has the indeterminate form 1^∞, so

$$\ln L = \lim_{x\to\infty}\ln\left(\frac{x+a}{x-a}\right)^x = \lim_{x\to\infty} x\ln\left(\frac{x+a}{x-a}\right) = \lim_{x\to\infty}\frac{\ln(x+a)-\ln(x-a)}{1/x}$$

$$\overset{\text{H}}{=} \lim_{x\to\infty}\frac{\dfrac{1}{x+a}-\dfrac{1}{x-a}}{-1/x^2} = \lim_{x\to\infty}\left[\frac{(x-a)-(x+a)}{(x+a)(x-a)}\cdot\frac{-x^2}{1}\right]$$

$$= \lim_{x\to\infty}\frac{2ax^2}{x^2-a^2} = \lim_{x\to\infty}\frac{2a}{1-a^2/x^2} = 2a.$$

Hence, $\ln L = 2a$, so $L = e^{2a}$. From the original equation, we want $L = e^1 \;\Rightarrow\; 2a = 1 \;\Rightarrow\; a = \frac{1}{2}$.

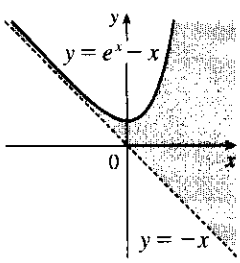
10. *Case (i) (first graph):* For $x + y \geq 0$, that is, $y \geq -x$, $|x+y| = x + y \leq e^x \;\Rightarrow\; y \leq e^x - x$.

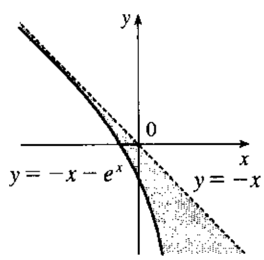
Note that $y = e^x - x$ is always above the line $y = -x$ and that $y = -x$ is a slant asymptote.

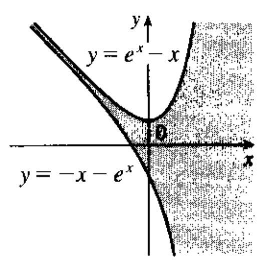
Case (ii) (second graph): For $x + y < 0$, that is, $y < -x$, $|x+y| = -x - y \leq e^x \;\Rightarrow\; y \geq -x - e^x$.

Note that $-x - e^x$ is always below the line $y = -x$ and $y = -x$ is a slant asymptote.

Putting the two pieces together gives the third graph.







11. Both sides of the inequality are positive, so $\cosh(\sinh x) < \sinh(\cosh x) \;\Leftrightarrow\; \cosh^2(\sinh x) < \sinh^2(\cosh x)$

$\Leftrightarrow\; \sinh^2(\sinh x) + 1 < \sinh^2(\cosh x) \;\Leftrightarrow\; 1 < [\sinh(\cosh x) - \sinh(\sinh x)][\sinh(\cosh x) + \sinh(\sinh x)]$

$\Leftrightarrow\; 1 < \left[\sinh\left(\dfrac{e^x+e^{-x}}{2}\right) - \sinh\left(\dfrac{e^x-e^{-x}}{2}\right)\right]\left[\sinh\left(\dfrac{e^x+e^{-x}}{2}\right) + \sinh\left(\dfrac{e^x-e^{-x}}{2}\right)\right]$

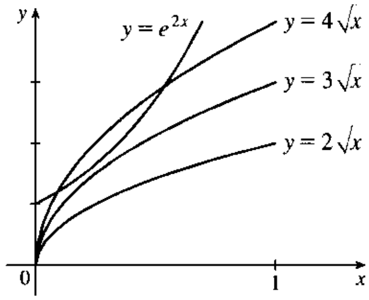
$\Leftrightarrow\; 1 < [2\cosh(e^x/2)\sinh(e^x/2)][2\sinh(e^x/2)\cosh(e^x/2)]$ [use the addition formulas and cancel]

$\Leftrightarrow\; 1 < [2\sinh(e^x/2)\cosh(e^x/2)][2\sinh(e^x/2)\cosh(e^x/2)] \;\Leftrightarrow\; 1 < \sinh e^x \sinh e^{-x}$,

by the half-angle formula. Now both e^x and e^{-x} are positive, and $\sinh y > y$ for $y > 0$, since $\sinh 0 = 0$ and $(\sinh y - y)' = \cosh y - 1 > 0$ for $x > 0$, so $1 = e^x e^{-x} < \sinh e^x \sinh e^{-x}$. So, following this chain of reasoning backward, we arrive at the desired result.

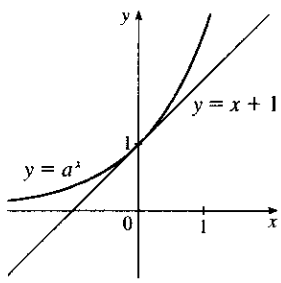
12. First, we recognize some symmetry in the inequality: $\dfrac{e^{x+y}}{xy} \geq e^2 \quad \Leftrightarrow \quad \dfrac{e^x}{x} \cdot \dfrac{e^y}{y} \geq e \cdot e$. This suggests that we need to show that $\dfrac{e^x}{x} \geq e$ for $x > 0$. If we can do this, then the inequality $\dfrac{e^y}{y} \geq e$ is true, and the given inequality follows. $f(x) = \dfrac{e^x}{x} \quad \Rightarrow \quad f'(x) = \dfrac{xe^x - e^x}{x^2} = \dfrac{e^x(x-1)}{x^2} = 0 \quad \Rightarrow \quad x = 1$. By the First Derivative Test, we have a minimum of $f(1) = e$, so $e^x/x \geq e$ for all x.

13.



Let $f(x) = e^{2x}$ and $g(x) = k\sqrt{x}$ $(k > 0)$. From the graphs of f and g, we see that f will intersect g exactly once when f and g share a tangent line. Thus, we must have $f = g$ and $f' = g'$ at $x = a$. $f(a) = g(a) \quad \Rightarrow$ $e^{2a} = k\sqrt{a}$ **(1)** and $f'(a) = g'(a) \quad \Rightarrow \quad 2e^{2a} = \dfrac{k}{2\sqrt{a}} \quad \Rightarrow$ $e^{2a} = \dfrac{k}{4\sqrt{a}}$. So we must have $k\sqrt{a} = \dfrac{k}{4\sqrt{a}} \quad \Rightarrow \quad (\sqrt{a})^2 = \dfrac{k}{4k} \quad \Rightarrow$ $a = \frac{1}{4}$. From **(1)**, $e^{2(1/4)} = k\sqrt{1/4} \quad \Rightarrow \quad k = 2e^{1/2} = 2\sqrt{e} \approx 3.297$.

14.



We see that at $x = 0$, $f(x) = a^x = 1 + x = 1$, so if $y = a^x$ is to lie above $y = 1 + x$, the two curves must just touch at $(0, 1)$, that is, we must have $f'(0) = 1$.

[To see this analytically, note that $a^x \geq 1 + x \quad \Rightarrow \quad a^x - 1 \geq x \quad \Rightarrow$ $\dfrac{a^x - 1}{x} \geq 1$ for $x > 0$, so $f'(0) = \lim\limits_{x \to 0^+} \dfrac{a^x - 1}{x} \geq 1$. Similarly, for $x < 0$, $a^x - 1 \geq x \quad \Rightarrow \quad \dfrac{a^x - 1}{x} \leq 1$, so $f'(0) = \lim\limits_{x \to 0^-} \dfrac{a^x - 1}{x} \leq 1$. Since $1 \leq f'(0) \leq 1$, we must have $f'(0) = 1$.] But $f'(x) = a^x \ln a \quad \Rightarrow$ $f'(0) = \ln a$, so we have $\ln a = 1 \quad \Leftrightarrow \quad a = e$.

Another method: The inequality certainly holds for $x \leq -1$, so consider $x > -1$, $x \neq 0$. Then $a^x \geq 1 + x \quad \Rightarrow$ $a \geq (1 + x)^{1/x}$ for $x > 0 \quad \Rightarrow \quad a \geq \lim\limits_{x \to 0^+} (1 + x)^{1/x} = e$, by Equation 7.4.8. Also, $a^x \geq 1 + x \quad \Rightarrow$ $a \leq (1 + x)^{1/x}$ for $x < 0 \quad \Rightarrow \quad a \leq \lim\limits_{x \to 0^-} (1 + x)^{1/x} = e$. So since $e \leq a \leq e$, we must have $a = e$.

15. Suppose that the curve $y = a^x$ intersects the line $y = x$. Then $a^{x_0} = x_0$ for some $x_0 > 0$, and hence $a = x_0^{1/x_0}$. We find the maximum value of $g(x) = x^{1/x}$, > 0, because if a is larger than the maximum value of this function, then the curve $y = a^x$ does not intersect the line $y = x$.

$g'(x) = e^{(1/x)\ln x}\left(-\dfrac{1}{x^2}\ln x + \dfrac{1}{x} \cdot \dfrac{1}{x}\right) = x^{1/x}\left(\dfrac{1}{x^2}\right)(1 - \ln x)$. This is 0 only where $x = e$, and for $0 < x < e$, $f'(x) > 0$, while for $x > e$, $f'(x) < 0$, so g has an absolute maximum of $g(e) = e^{1/e}$. So if $y = a^x$ intersects $y = x$, we must have $0 < a \leq e^{1/e}$. Conversely, suppose that $0 < a \leq e^{1/e}$. Then $a^e \leq e$, so the graph of $y = a^x$ lies below or touches the graph of $y = x$ at $x = e$. Also $a^0 = 1 > 0$, so the graph of $y = a^x$ lies above that of $y = x$ at $x = 0$. Therefore, by the Intermediate Value Theorem, the graphs of $y = a^x$ and $y = x$ must intersect somewhere between $x = 0$ and $x = e$.