Chapter 8

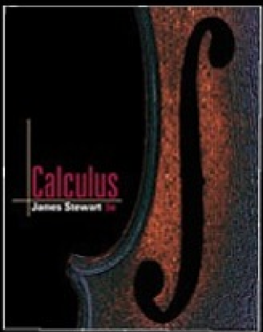
Adapted from the

Complete Solutions Manual

for

James Stewart's

Calculus – 5th Edition



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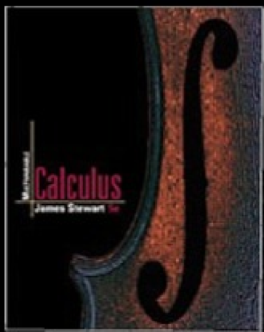

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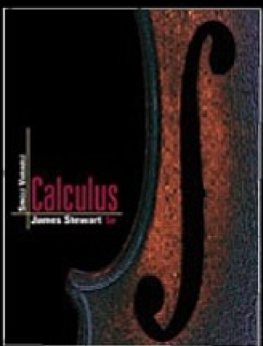

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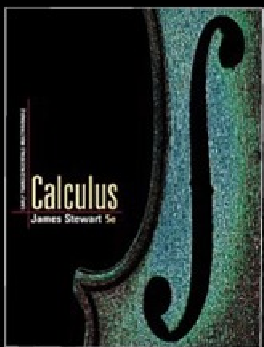

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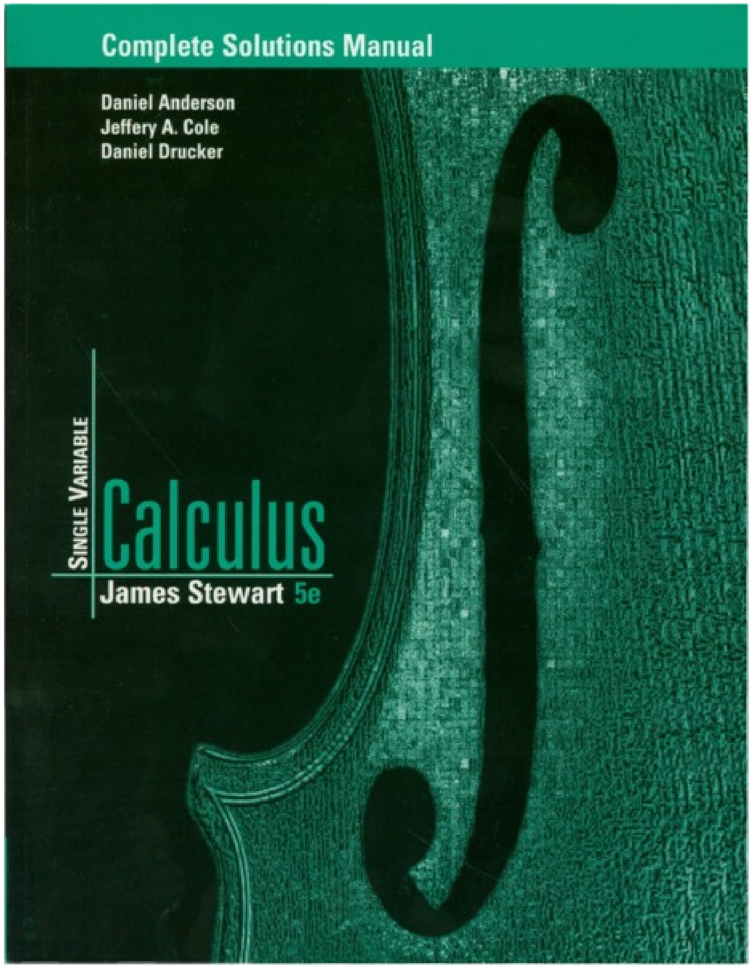


8 ☐ TECHNIQUES OF INTEGRATION

8.1 Integration by Parts

1. Let $u = \ln x$, $dv = x\,dx \quad\Rightarrow\quad du = dx/x$, $v = \frac{1}{2}x^2$. Then by Equation 2, $\int u\,dv = uv - \int v\,du$,

$$\int x\ln x\,dx = \tfrac{1}{2}x^2\ln x - \int \tfrac{1}{2}x^2(dx/x) = \tfrac{1}{2}x^2\ln x - \tfrac{1}{2}\int x\,dx = \tfrac{1}{2}x^2\ln x - \tfrac{1}{2}\cdot\tfrac{1}{2}x^2 + C$$

$$= \tfrac{1}{2}x^2\ln x - \tfrac{1}{4}x^2 + C$$

2. Let $u = \theta$, $dv = \sec^2\theta\,d\theta \quad\Rightarrow\quad du = d\theta$, $v = \tan\theta$. Then
$\int \theta\sec^2\theta\,d\theta = \theta\tan\theta - \int\tan\theta\,d\theta = \theta\tan\theta - \ln|\sec\theta| + C$.

3. Let $u = x$, $dv = \cos 5x\,dx \quad\Rightarrow\quad du = dx$, $v = \frac{1}{5}\sin 5x$. Then by Equation 2,
$\int x\cos 5x\,dx = \frac{1}{5}x\sin 5x - \int \frac{1}{5}\sin 5x\,dx = \frac{1}{5}x\sin 5x + \frac{1}{25}\cos 5x + C$.

4. Let $u = x$, $dv = e^{-x}\,dx \quad\Rightarrow\quad du = dx$, $v = -e^{-x}$. Then
$\int xe^{-x}\,dx = -xe^{-x} + \int e^{-x}\,dx = -xe^{-x} - e^{-x} + C$.

5. Let $u = r$, $dv = e^{r/2}\,dr \quad\Rightarrow\quad du = dr$, $v = 2e^{r/2}$. Then
$\int re^{r/2}\,dr = 2re^{r/2} - \int 2e^{r/2}\,dr = 2re^{r/2} - 4e^{r/2} + C$.

6. Let $u = t$, $dv = \sin 2t\,dt \quad\Rightarrow\quad du = dt$, $v = -\frac{1}{2}\cos 2t$. Then
$\int t\sin 2t\,dt = -\frac{1}{2}t\cos 2t + \frac{1}{2}\int\cos 2t\,dt = -\frac{1}{2}t\cos 2t + \frac{1}{4}\sin 2t + C$.

7. Let $u = x^2$, $dv = \sin\pi x\,dx \quad\Rightarrow\quad du = 2x\,dx$ and $v = -\frac{1}{\pi}\cos\pi x$. Then
$I = \int x^2\sin\pi x\,dx = -\frac{1}{\pi}x^2\cos\pi x + \frac{2}{\pi}\int x\cos\pi x\,dx$ (∗).
Next let $U = x$, $dV = \cos\pi x\,dx \quad\Rightarrow\quad dU = dx$, $V = \frac{1}{\pi}\sin\pi x$, so
$\int x\cos\pi x\,dx = \frac{1}{\pi}x\sin\pi x - \frac{1}{\pi}\int\sin\pi x\,dx = \frac{1}{\pi}x\sin\pi x + \frac{1}{\pi^2}\cos\pi x + C_1$. Substituting for $\int x\cos\pi x\,dx$ in (∗),
we get $I = -\frac{1}{\pi}x^2\cos\pi x + \frac{2}{\pi}\left(\frac{1}{\pi}x\sin\pi x + \frac{1}{\pi^2}\cos\pi x + C_1\right) = -\frac{1}{\pi}x^2\cos\pi x + \frac{2}{\pi^2}x\sin\pi x + \frac{2}{\pi^3}\cos\pi x + C$,
where $C = \frac{2}{\pi}C_1$.

8. Let $u = x^2$, $dv = \cos mx\,dx \quad\Rightarrow\quad du = 2x\,dx$, $v = \frac{1}{m}\sin mx$.
Then $I = \displaystyle\int x^2\cos mx\,dx = \frac{1}{m}x^2\sin mx - \frac{2}{m}\int x\sin mx\,dx$ (∗). Next let
$U = x$, $dV = \sin mx\,dx \quad\Rightarrow\quad dU = dx$, $V = -\frac{1}{m}\cos mx$, so
$\int x\sin mx\,dx = -\frac{1}{m}x\cos mx + \frac{1}{m}\int\cos mx\,dx = -\frac{1}{m}x\cos mx + \frac{1}{m^2}\sin mx + C_1$.
Substituting for $\int x\sin mx\,dx$ in (∗), we get
$I = \frac{1}{m}x^2\sin mx - \frac{2}{m}\left(-\frac{1}{m}x\cos mx + \frac{1}{m^2}\sin mx + C_1\right) = \frac{1}{m}x^2\sin mx + \frac{2}{m^2}x\cos mx - \frac{2}{m^3}\sin mx + C$,
where $C = -\frac{2}{m}C_1$.

9. Let $u = \ln(2x+1)$, $dv = dx \Rightarrow du = \dfrac{2}{2x+1}\,dx$, $v = x$. Then

$$\begin{aligned}\int \ln(2x+1)\,dx &= x\ln(2x+1) - \int \frac{2x}{2x+1}\,dx = x\ln(2x+1) - \int \frac{(2x+1)-1}{2x+1}\,dx \\ &= x\ln(2x+1) - \int\left(1 - \frac{1}{2x+1}\right)dx = x\ln(2x+1) - x + \tfrac{1}{2}\ln(2x+1) + C \\ &= \tfrac{1}{2}(2x+1)\,\ln(2x+1) - x + C\end{aligned}$$

10. Let $u = \sin^{-1}x$, $dv = dx \Rightarrow du = \dfrac{dx}{\sqrt{1-x^2}}$, $v = x$. Then

$\int \sin^{-1}x\,dx = x\sin^{-1}x - \displaystyle\int \frac{x}{\sqrt{1-x^2}}\,dx$. Setting $t = 1 - x^2$, we get $dt = -2x\,dx$, so

$-\displaystyle\int \frac{x\,dx}{\sqrt{1-x^2}} = -\int t^{-1/2}\left(-\tfrac{1}{2}\,dt\right) = \tfrac{1}{2}\left(2t^{1/2}\right) + C = t^{1/2} + C = \sqrt{1-x^2} + C$. Hence,

$\int \sin^{-1}x\,dx = x\sin^{-1}x + \sqrt{1-x^2} + C$.

11. Let $u = \arctan 4t$, $dv = dt \Rightarrow du = \dfrac{4}{1+(4t)^2}\,dt = \dfrac{4}{1+16t^2}\,dt$, $v = t$. Then

$$\begin{aligned}\int \arctan 4t\,dt &= t\arctan 4t - \int \frac{4t}{1+16t^2}\,dt = t\arctan 4t - \frac{1}{8}\int \frac{32t}{1+16t^2}\,dt \\ &= t\arctan 4t - \tfrac{1}{8}\ln(1+16t^2) + C\end{aligned}$$

12. Let $u = \ln p$, $dv = p^5\,dp \Rightarrow du = \dfrac{1}{p}\,dp$, $v = \frac{1}{6}p^6$. Then

$\int p^5 \ln p\,dp = \frac{1}{6}p^6\ln p - \frac{1}{6}\int p^5\,dp = \frac{1}{6}p^6\ln p - \frac{1}{36}p^6 + C$.

13. First let $u = (\ln x)^2$, $dv = dx \Rightarrow du = 2\ln x \cdot \frac{1}{x}\,dx$, $v = x$. Then by Equation 2,

$I = \int (\ln x)^2\,dx = x(\ln x)^2 - 2\int x\ln x \cdot \frac{1}{x}\,dx = x(\ln x)^2 - 2\int \ln x\,dx$. Next let $U = \ln x$, $dV = dx \Rightarrow$

$dU = 1/x\,dx$, $V = x$ to get $\int \ln x\,dx = x\ln x - \int x\cdot(1/x)\,dx = x\ln x - \int dx = x\ln x - x + C_1$. Thus,

$I = x(\ln x)^2 - 2(x\ln x - x + C_1) = x(\ln x)^2 - 2x\ln x + 2x + C$, where $C = -2C_1$.

14. Let $u = t^3$, $dv = e^t\,dt \Rightarrow du = 3t^2\,dt$, $v = e^t$. Then $I = \int t^3 e^t\,dt = t^3e^t - \int 3t^2e^t\,dt$. Integrate by parts twice more with $dv = e^t\,dt$.

$$\begin{aligned}I &= t^3e^t - \left(3t^2e^t - \textstyle\int 6te^t\,dt\right) = t^3e^t - 3t^2e^t + 6te^t - \textstyle\int 6e^t\,dt \\ &= t^3e^t - 3t^2e^t + 6te^t - 6e^t + C = \left(t^3 - 3t^2 + 6t - 6\right)e^t + C\end{aligned}$$

More generally, if $p(t)$ is a polynomial of degree n in t, then repeated integration by parts shows that

$\int p(t)\,e^t\,dt = \left[p(t) - p'(t) + p''(t) - p'''(t) + \cdots + (-1)^n\,p^{(n)}(t)\right]e^t + C$.

15. First let $u = \sin 3\theta$, $dv = e^{2\theta}\,d\theta \quad \Rightarrow \quad du = 3\cos 3\theta\,d\theta$, $v = \frac{1}{2}e^{2\theta}$. Then
$I = \int e^{2\theta}\sin 3\theta\,d\theta = \frac{1}{2}e^{2\theta}\sin 3\theta - \frac{3}{2}\int e^{2\theta}\cos 3\theta\,d\theta$. Next let $U = \cos 3\theta$,
$dV = e^{2\theta}\,d\theta \quad \Rightarrow \quad dU = -3\sin 3\theta\,d\theta$, $V = \frac{1}{2}e^{2\theta}$ to get
$\int e^{2\theta}\cos 3\theta\,d\theta = \frac{1}{2}e^{2\theta}\cos 3\theta + \frac{3}{2}\int e^{2\theta}\sin 3\theta\,d\theta$. Substituting in the previous formula gives
$I = \frac{1}{2}e^{2\theta}\sin 3\theta - \frac{3}{4}e^{2\theta}\cos 3\theta - \frac{9}{4}\int e^{2\theta}\sin 3\theta\,d\theta = \frac{1}{2}e^{2\theta}\sin 3\theta - \frac{3}{4}e^{2\theta}\cos 3\theta - \frac{9}{4}I \quad \Rightarrow$
$\frac{13}{4}I = \frac{1}{2}e^{2\theta}\sin 3\theta - \frac{3}{4}e^{2\theta}\cos 3\theta + C_1$. Hence, $I = \frac{1}{13}e^{2\theta}(2\sin 3\theta - 3\cos 3\theta) + C$, where $C = \frac{4}{13}C_1$.

16. First let $u = e^{-\theta}$, $dv = \cos 2\theta\,d\theta \quad \Rightarrow \quad du = -e^{-\theta}\,d\theta$, $v = \frac{1}{2}\sin 2\theta$. Then
$I = \int e^{-\theta}\cos 2\theta\,d\theta = \frac{1}{2}e^{-\theta}\sin 2\theta - \int \frac{1}{2}\sin 2\theta\left(-e^{-\theta}\,d\theta\right) = \frac{1}{2}e^{-\theta}\sin 2\theta + \frac{1}{2}\int e^{-\theta}\sin 2\theta\,d\theta$.
Next let $U = e^{-\theta}$, $dV = \sin 2\theta\,d\theta \quad \Rightarrow \quad dU = -e^{-\theta}\,d\theta$, $V = -\frac{1}{2}\cos 2\theta$, so
$\int e^{-\theta}\sin 2\theta\,d\theta = -\frac{1}{2}e^{-\theta}\cos 2\theta - \int\left(-\frac{1}{2}\right)\cos 2\theta\left(-e^{-\theta}\,d\theta\right) = -\frac{1}{2}e^{-\theta}\cos 2\theta - \frac{1}{2}\int e^{-\theta}\cos 2\theta\,d\theta$.
So $I = \frac{1}{2}e^{-\theta}\sin 2\theta + \frac{1}{2}\left[\left(-\frac{1}{2}e^{-\theta}\cos 2\theta\right) - \frac{1}{2}I\right] = \frac{1}{2}e^{-\theta}\sin 2\theta - \frac{1}{4}e^{-\theta}\cos 2\theta - \frac{1}{4}I \quad \Rightarrow$
$\frac{5}{4}I = \frac{1}{2}e^{-\theta}\sin 2\theta - \frac{1}{4}e^{-\theta}\cos 2\theta + C_1 \quad \Rightarrow$
$I = \frac{4}{5}\left(\frac{1}{2}e^{-\theta}\sin 2\theta - \frac{1}{4}e^{-\theta}\cos 2\theta + C_1\right) = \frac{2}{5}e^{-\theta}\sin 2\theta - \frac{1}{5}e^{-\theta}\cos 2\theta + C$.

17. Let $u = y$, $dv = \sinh y\,dy \quad \Rightarrow \quad du = dy$, $v = \cosh y$. Then
$\int y\sinh y\,dy = y\cosh y - \int \cosh y\,dy = y\cosh y - \sinh y + C$.

18. Let $u = y$, $dv = \cosh ay\,dy \quad \Rightarrow \quad du = dy$, $v = \dfrac{\sinh ay}{a}$. Then
$$\int y\cosh ay\,dy = \frac{y\sinh ay}{a} - \frac{1}{a}\int \sinh ay\,dy = \frac{y\sinh ay}{a} - \frac{\cosh ay}{a^2} + C.$$

19. Let $u = t$, $dv = \sin 3t\,dt \quad \Rightarrow \quad du = dt$, $v = -\frac{1}{3}\cos 3t$. Then
$\int_0^\pi t\sin 3t\,dt = \left[-\frac{1}{3}t\cos 3t\right]_0^\pi + \frac{1}{3}\int_0^\pi \cos 3t\,dt = \left(\frac{1}{3}\pi - 0\right) + \frac{1}{9}\left[\sin 3t\right]_0^\pi = \frac{\pi}{3}$.

20. First let $u = x^2 + 1$, $dv = e^{-x}\,dx \quad \Rightarrow \quad du = 2x\,dx$, $v = -e^{-x}$. By (6),
$\int_0^1 (x^2+1)e^{-x}\,dx = \left[-(x^2+1)e^{-x}\right]_0^1 + \int_0^1 2xe^{-x}\,dx = -2e^{-1} + 1 + 2\int_0^1 xe^{-x}\,dx$. Next let
$U = x$, $dV = e^{-x}\,dx \quad \Rightarrow \quad dU = dx$, $V = -e^{-x}$. By (6) again,
$\int_0^1 xe^{-x}\,dx = \left[-xe^{-x}\right]_0^1 + \int_0^1 e^{-x}\,dx = -e^{-1} + \left[-e^{-x}\right]_0^1 = -e^{-1} - e^{-1} + 1 = -2e^{-1} + 1$. So
$\int_0^1 (x^2+1)e^{-x}\,dx = -2e^{-1} + 1 + 2\left(-2e^{-1} + 1\right) = -2e^{-1} + 1 - 4e^{-1} + 2 = -6e^{-1} + 3$.

21. Let $u = \ln x$, $dv = x^{-2}\,dx \quad \Rightarrow \quad du = \dfrac{1}{x}\,dx$, $v = -x^{-1}$. By (6),
$$\int_1^2 \frac{\ln x}{x^2}\,dx = \left[-\frac{\ln x}{x}\right]_1^2 + \int_1^2 x^{-2}\,dx = -\tfrac{1}{2}\ln 2 + \ln 1 + \left[-\frac{1}{x}\right]_1^2 = -\tfrac{1}{2}\ln 2 + 0 - \tfrac{1}{2} + 1 = \tfrac{1}{2} - \tfrac{1}{2}\ln 2.$$

22. Let $u = \ln t$, $dv = \sqrt{t}\,dt \quad \Rightarrow \quad du = dt/t$, $v = \frac{2}{3}t^{3/2}$. By Formula 6,
$$\int_1^4 \sqrt{t}\ln t\,dt = \left[\tfrac{2}{3}t^{3/2}\ln t\right]_1^4 - \tfrac{2}{3}\int_1^4 \sqrt{t}\,dt = \tfrac{2}{3}\cdot 8\cdot\ln 4 - 0 - \left[\tfrac{2}{3}\cdot\tfrac{2}{3}t^{3/2}\right]_1^4 = \tfrac{16}{3}\ln 4 - \tfrac{4}{9}(8-1) = \tfrac{16}{3}\ln 4 - \tfrac{28}{9}.$$

23. Let $u = y$, $dv = \dfrac{dy}{e^{2y}} = e^{-2y}dy \Rightarrow du = dy$, $v = -\dfrac{1}{2}e^{-2y}$. Then

$$\int_0^1 \frac{y}{e^{2y}}\,dy = \left[-\tfrac{1}{2}ye^{-2y}\right]_0^1 + \tfrac{1}{2}\int_0^1 e^{-2y}dy = \left(-\tfrac{1}{2}e^{-2} + 0\right) - \tfrac{1}{4}\left[e^{-2y}\right]_0^1 = -\tfrac{1}{2}e^{-2} - \tfrac{1}{4}e^{-2} + \tfrac{1}{4} = \tfrac{1}{4} - \tfrac{3}{4}e^{-2}.$$

24. Let $u = x$, $dv = \csc^2 x\,dx \Rightarrow du = dx$, $v = -\cot x$. Then

$$\int_{\pi/4}^{\pi/2} x\csc^2 x\,dx = \left[-x\cot x\right]_{\pi/4}^{\pi/2} + \int_{\pi/4}^{\pi/2}\cot x\,dx = -\tfrac{\pi}{2}\cdot 0 + \tfrac{\pi}{4}\cdot 1 + \Big[\ln|\sin x|\Big]_{\pi/4}^{\pi/2} \quad \text{[see Exercise 5.5.75]}$$

$$= \tfrac{\pi}{4} + \ln 1 - \ln\tfrac{1}{\sqrt{2}} = \tfrac{\pi}{4} + 0 - \ln 2^{-1/2} = \tfrac{\pi}{4} + \tfrac{1}{2}\ln 2$$

25. Let $u = \cos^{-1}x$, $dv = dx \Rightarrow du = -\dfrac{dx}{\sqrt{1-x^2}}$, $v = x$. Then

$$I = \int_0^{1/2}\cos^{-1}x\,dx = \left[x\cos^{-1}x\right]_0^{1/2} + \int_0^{1/2}\frac{x\,dx}{\sqrt{1-x^2}} = \tfrac{1}{2}\cdot\tfrac{\pi}{3} + \int_1^{3/4} t^{-1/2}\left[-\frac{1}{2}\,dt\right], \text{ where } t = 1 - x^2$$

$\Rightarrow dt = -2x\,dx$. Thus, $I = \frac{\pi}{6} + \frac{1}{2}\int_{3/4}^1 t^{-1/2}\,dt = \frac{\pi}{6} + \left[\sqrt{t}\right]_{3/4}^1 = \frac{\pi}{6} + 1 - \frac{\sqrt{3}}{2} = \frac{1}{6}\left(\pi + 6 - 3\sqrt{3}\right)$.

26. Let $u = x$, $dv = 5^x\,dx \Rightarrow du = dx$, $v = (5^x/\ln 5)$. Then

$$\int_0^1 x5^x\,dx = \left[\frac{x5^x}{\ln 5}\right]_0^1 - \int_0^1\frac{5^x}{\ln 5}\,dx = \frac{5}{\ln 5} - 0 - \frac{1}{\ln 5}\left[\frac{5^x}{\ln 5}\right]_0^1 = \frac{5}{\ln 5} - \frac{5}{(\ln 5)^2} + \frac{1}{(\ln 5)^2}$$

$$= \frac{5}{\ln 5} - \frac{4}{(\ln 5)^2}$$

27. Let $u = \ln(\sin x)$, $dv = \cos x\,dx \Rightarrow du = \dfrac{\cos x}{\sin x}\,dx$, $v = \sin x$. Then

$I = \int \cos x\ln(\sin x)\,dx = \sin x\ln(\sin x) - \int\cos x\,dx = \sin x\ln(\sin x) - \sin x + C$.

Another method: Substitute $t = \sin x$, so $dt = \cos x\,dx$. Then $I = \int\ln t\,dt = t\ln t - t + C$ (see Example 2) and so $I = \sin x\,(\ln\sin x - 1) + C$.

28. Let $u = \arctan(1/x)$, $dv = dx \Rightarrow du = \dfrac{1}{1+(1/x)^2}\cdot\dfrac{-1}{x^2}\,dx = \dfrac{-dx}{x^2+1}$, $v = x$. Then

$$\int_1^{\sqrt{3}}\arctan(1/x)\,dx = \left[x\arctan\left(\frac{1}{x}\right)\right]_1^{\sqrt{3}} + \int_1^{\sqrt{3}}\frac{x\,dx}{x^2+1} = \sqrt{3}\,\frac{\pi}{6} - 1\cdot\frac{\pi}{4} + \frac{1}{2}\left[\ln(x^2+1)\right]_1^{\sqrt{3}}$$

$$= \frac{\pi\sqrt{3}}{6} - \frac{\pi}{4} + \frac{1}{2}(\ln 4 - \ln 2) = \frac{\pi\sqrt{3}}{6} - \frac{\pi}{2} + \frac{1}{2}\ln\frac{4}{2} = \frac{\pi\sqrt{3}}{6} - \frac{\pi}{2} + \frac{1}{2}\ln 2$$

29. Let $w = \ln x \Rightarrow dw = dx/x$. Then $x = e^w$ and $dx = e^w\,dw$, so

$$\int\cos(\ln x)\,dx = \int e^w\cos w\,dw = \tfrac{1}{2}e^w(\sin w + \cos w) + C \quad \text{[by the method of Example 4]}$$

$$= \tfrac{1}{2}x\left[\sin(\ln x) + \cos(\ln x)\right] + C$$

30. Let $u = r^2$, $dv = \dfrac{r}{\sqrt{4+r^2}}\,dr \Rightarrow du = 2r\,dr$, $v = \sqrt{4+r^2}$. By (6),

$$\int_0^1\frac{r^3}{\sqrt{4+r^2}}\,dr = \left[r^2\sqrt{4+r^2}\right]_0^1 - 2\int_0^1 r\sqrt{4+r^2}\,dr = \sqrt{5} - \tfrac{2}{3}\left[(4+r^2)^{3/2}\right]_0^1$$

$$= \sqrt{5} - \tfrac{2}{3}(5)^{3/2} + \tfrac{2}{3}(8) = \sqrt{5}\left(1 - \tfrac{10}{3}\right) + \tfrac{16}{3} = \tfrac{16}{3} - \tfrac{7}{3}\sqrt{5}$$

31. Let $u = (\ln x)^2$, $dv = x^4\,dx$ $\Rightarrow$ $du = 2\dfrac{\ln x}{x}\,dx$, $v = \dfrac{x^5}{5}$. By (6),

$$\int_1^2 x^4(\ln x)^2\,dx = \left[\frac{x^5}{5}(\ln x)^2\right]_1^2 - 2\int_1^2 \frac{x^4}{5}\ln x\,dx = \tfrac{32}{5}(\ln 2)^2 - 0 - 2\int_1^2 \frac{x^4}{5}\ln x\,dx.$$

Let $U = \ln x$, $dV = \dfrac{x^4}{5}\,dx$ $\Rightarrow$ $dU = \dfrac{1}{x}\,dx$, $V = \dfrac{x^5}{25}$.

Then $\displaystyle\int_1^2 \frac{x^4}{5}\ln x\,dx = \left[\frac{x^5}{25}\ln x\right]_1^2 - \int_1^2 \frac{x^4}{25}\,dx = \tfrac{32}{25}\ln 2 - 0 - \left[\frac{x^5}{125}\right]_1^2 = \tfrac{32}{25}\ln 2 - \left(\tfrac{32}{125} - \tfrac{1}{125}\right).$

So $\int_1^2 x^4(\ln x)^2\,dx = \frac{32}{5}(\ln 2)^2 - 2\left(\frac{32}{25}\ln 2 - \frac{31}{125}\right) = \frac{32}{5}(\ln 2)^2 - \frac{64}{25}\ln 2 + \frac{62}{125}$.

32. Let $u = \sin(t-s)$, $dv = e^s\,ds$ $\Rightarrow$ $du = -\cos(t-s)\,ds$, $v = e^s$. Then
$I = \int_0^t e^s\sin(t-s)\,ds = \left[e^s\sin(t-s)\right]_0^t + \int_0^t e^s\cos(t-s)\,ds = e^t\sin 0 - e^0\sin t + I_1$. For
I_1, let $U = \cos(t-s)$, $dV = e^s\,ds$ $\Rightarrow$ $dU = \sin(t-s)\,ds$, $V = e^s$. So
$I_1 = \left[e^s\cos(t-s)\right]_0^t - \int_0^t e^s\sin(t-s)ds = e^t\cos 0 - e^0\cos t - I$. Thus, $I = -\sin t + e^t - \cos t - I$ $\Rightarrow$
$2I = e^t - \cos t - \sin t$ $\Rightarrow$ $I = \frac{1}{2}\left(e^t - \cos t - \sin t\right)$.

33. Let $w = \sqrt{x}$, so that $x = w^2$ and $dx = 2w\,dw$. Thus, $\int\sin\sqrt{x}\,dx = \int 2w\sin w\,dw$. Now use parts with $u = 2w$, $dv = \sin w\,dw$, $du = 2\,dw$, $v = -\cos w$ to get

$$\begin{aligned}\int 2w\sin w\,dw &= -2w\cos w + \int 2\cos w\,dw = -2w\cos w + 2\sin w + C\\ &= -2\sqrt{x}\cos\sqrt{x} + 2\sin\sqrt{x} + C = 2\left(\sin\sqrt{x} - \sqrt{x}\cos\sqrt{x}\right) + C\end{aligned}$$

34. Let $w = \sqrt{x}$, so that $x = w^2$ and $dx = 2w\,dw$. Thus, $\int_1^4 e^{\sqrt{x}}\,dx = \int_1^2 e^w 2w\,dw$. Now use parts with $u = 2w$, $dv = e^w\,dw$, $du = 2\,dw$, $v = e^w$ to get $\int_1^2 e^w 2w\,dw = \left[2we^w\right]_1^2 - 2\int_1^2 e^w\,dw = 4e^2 - 2e - 2(e^2 - e) = 2e^2$.

35. Let $x = \theta^2$, so that $dx = 2\theta\,d\theta$. Thus, $\int_{\sqrt{\pi/2}}^{\sqrt{\pi}}\theta^3\cos(\theta^2)\,d\theta = \int_{\sqrt{\pi/2}}^{\sqrt{\pi}}\theta^2\cos(\theta^2)\cdot\frac{1}{2}(2\theta\,d\theta) = \frac{1}{2}\int_{\pi/2}^{\pi} x\cos x\,dx$.
Now use parts with $u = x$, $dv = \cos x\,dx$, $du = dx$, $v = \sin x$ to get

$$\begin{aligned}\tfrac{1}{2}\int_{\pi/2}^{\pi} x\cos x\,dx &= \tfrac{1}{2}\left(\left[x\sin x\right]_{\pi/2}^{\pi} - \int_{\pi/2}^{\pi}\sin x\,dx\right) = \tfrac{1}{2}\left[x\sin x + \cos x\right]_{\pi/2}^{\pi}\\ &= \tfrac{1}{2}(\pi\sin\pi + \cos\pi) - \tfrac{1}{2}\left(\tfrac{\pi}{2}\sin\tfrac{\pi}{2} + \cos\tfrac{\pi}{2}\right) = \tfrac{1}{2}(\pi\cdot 0 - 1) - \tfrac{1}{2}\left(\tfrac{\pi}{2}\cdot 1 + 0\right) = -\tfrac{1}{2} - \tfrac{\pi}{4}\end{aligned}$$

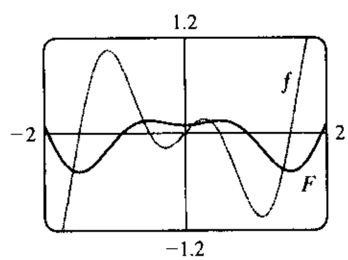
36. $\int x^5e^{x^2}\,dx = \int\left(x^2\right)^2e^{x^2}x\,dx = \int t^2e^t\frac{1}{2}\,dt$ [where $t = x^2$ $\Rightarrow$ $\frac{1}{2}\,dt = x\,dx$]
$= \frac{1}{2}\left(t^2 - 2t + 2\right)e^t + C$ [by Example 3] $= \frac{1}{2}\left(x^4 - 2x^2 + 2\right)e^{x^2} + C$

In Exercises 37–40, let $f(x)$ denote the integrand and $F(x)$ its antiderivative (with $C = 0$).

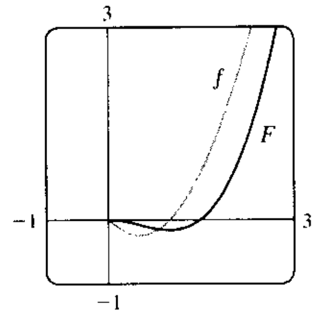
37. Let $u = x$, $dv = \cos\pi x\,dx$ $\Rightarrow$ $du = dx$, $v = (\sin\pi x)/\pi$. Then

$$\int x\cos\pi x\,dx = x\cdot\frac{\sin\pi x}{\pi} - \int\frac{\sin\pi x}{\pi}\,dx = \frac{x\sin\pi x}{\pi} + \frac{\cos\pi x}{\pi^2} + C.$$

We see from the graph that this is reasonable, since F has extreme values where f is 0.



38. Let $u = \ln x$, $dv = x^{3/2}\,dx \quad\Rightarrow\quad du = \dfrac{1}{x}\,dx$, $v = \frac{2}{5}x^{5/2}$. Then

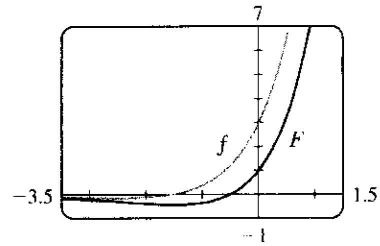


$$\int x^{3/2}\ln x\,dx = \tfrac{2}{5}x^{5/2}\ln x - \tfrac{2}{5}\int x^{3/2}\,dx = \tfrac{2}{5}x^{5/2}\ln x - \left(\tfrac{2}{5}\right)^2 x^{5/2} + C$$

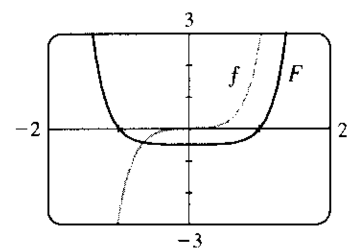
$$= \tfrac{2}{5}x^{5/2}\ln x - \tfrac{4}{25}x^{5/2} + C.$$

We see from the graph that this is reasonable, since F has a minimum where f changes from negative to positive.

39. Let $u = 2x + 3$, $dv = e^x\,dx \quad\Rightarrow\quad du = 2\,dx$, $v = e^x$. Then $\int(2x+3)e^x\,dx = (2x+3)e^x - 2\int e^x\,dx = (2x+3)e^x - 2e^x + C = (2x+1)\,e^x + C$. We see from the graph that this is reasonable, since F has a minimum where f changes from negative to positive.



40. $\int x^3 e^{x^2}\,dx = \int x^2 \cdot x e^{x^2}\,dx = I$.



Let $u = x^2$, $dv = xe^{x^2}\,dx \quad\Rightarrow\quad du = 2x\,dx$, $v = \frac{1}{2}e^{x^2}$. Then $I = \frac{1}{2}x^2e^{x^2} - \int xe^{x^2}\,dx = \frac{1}{2}x^2e^{x^2} - \frac{1}{2}e^{x^2} + C = \frac{1}{2}e^{x^2}\left(x^2 - 1\right) + C$. We see from the graph that this is reasonable, since F has a minimum where f changes from negative to positive.

41. (a) Take $n = 2$ in Example 6 to get $\int \sin^2 x\,dx = -\frac{1}{2}\cos x\sin x + \frac{1}{2}\int 1\,dx = \dfrac{x}{2} - \dfrac{\sin 2x}{4} + C$.

(b) $\int \sin^4 x\,dx = -\frac{1}{4}\cos x\sin^3 x + \frac{3}{4}\int \sin^2 x\,dx = -\frac{1}{4}\cos x\sin^3 x + \frac{3}{8}x - \frac{3}{16}\sin 2x + C$.

42. (a) Let $u = \cos^{n-1} x$, $dv = \cos x\,dx \quad\Rightarrow\quad du = -(n-1)\cos^{n-2} x\sin x\,dx$, $v = \sin x$ in (2):

$$\int \cos^n x\,dx = \cos^{n-1} x\sin x + (n-1)\int \cos^{n-2} x\sin^2 x\,dx$$
$$= \cos^{n-1} x\sin x + (n-1)\int \cos^{n-2} x\left(1 - \cos^2 x\right)dx$$
$$= \cos^{n-1} x\sin x + (n-1)\int \cos^{n-2} x\,dx - (n-1)\int \cos^n x\,dx$$

Rearranging terms gives $n\int \cos^n x\,dx = \cos^{n-1} x\sin x + (n-1)\int \cos^{n-2} x\,dx$ or

$$\int \cos^n x\,dx = \frac{1}{n}\cos^{n-1} x\sin x + \frac{n-1}{n}\int \cos^{n-2} x\,dx$$

(b) Take $n = 2$ in part (a) to get $\int \cos^2 x\,dx = \frac{1}{2}\cos x\sin x + \frac{1}{2}\int 1\,dx = \dfrac{x}{2} + \dfrac{\sin 2x}{4} + C$.

(c) $\int \cos^4 x\,dx = \frac{1}{4}\cos^3 x\sin x + \frac{3}{4}\int \cos^2 x\,dx = \frac{1}{4}\cos^3 x\sin x + \frac{3}{8}x + \frac{3}{16}\sin 2x + C$

43. (a) From Example 6, $\displaystyle\int \sin^n x\,dx = -\frac{1}{n}\cos x\sin^{n-1} x + \frac{n-1}{n}\int \sin^{n-2} x\,dx$. Using (6),

$$\int_0^{\pi/2} \sin^n x\,dx = \left[-\frac{\cos x\sin^{n-1} x}{n}\right]_0^{\pi/2} + \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2} x\,dx$$

$$= (0-0) + \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2} x\,dx = \frac{n-1}{n}\int_0^{\pi/2} \sin^{n-2} x\,dx$$

(b) Using $n = 3$ in part (a), we have $\int_0^{\pi/2} \sin^3 x\,dx = \frac{2}{3}\int_0^{\pi/2} \sin x\,dx = \left[-\frac{2}{3}\cos x\right]_0^{\pi/2} = \frac{2}{3}$.

Using $n = 5$ in part (a), we have $\int_0^{\pi/2} \sin^5 x\,dx = \frac{4}{5}\int_0^{\pi/2} \sin^3 x\,dx = \frac{4}{5}\cdot\frac{2}{3} = \frac{8}{15}$.

(c) The formula holds for $n = 1$ (that is, $2n + 1 = 3$) by (b). Assume it holds for some $k \geq 1$. Then $\displaystyle\int_0^{\pi/2} \sin^{2k+1} x\,dx = \frac{2\cdot 4\cdot 6\cdot\cdots\cdot(2k)}{3\cdot 5\cdot 7\cdot\cdots\cdot(2k+1)}$. By Example 6,

$$\int_0^{\pi/2} \sin^{2k+3} x\,dx = \frac{2k+2}{2k+3}\int_0^{\pi/2} \sin^{2k+1} x\,dx = \frac{2k+2}{2k+3}\cdot\frac{2\cdot 4\cdot 6\cdot\cdots\cdot(2k)}{3\cdot 5\cdot 7\cdot\cdots\cdot(2k+1)}$$
$$= \frac{2\cdot 4\cdot 6\cdot\cdots\cdot(2k)[2(k+1)]}{3\cdot 5\cdot 7\cdot\cdots\cdot(2k+1)[2(k+1)+1]},$$

so the formula holds for $n = k + 1$. By induction, the formula holds for all $n \geq 1$.

44. Using Exercise 43(a), we see that the formula holds for $n = 1$, because

$\int_0^{\pi/2} \sin^2 x\,dx = \frac{1}{2}\int_0^{\pi/2} 1\,dx = \frac{1}{2}\left[x\right]_0^{\pi/2} = \frac{1}{2}\cdot\frac{\pi}{2}$.

Now assume it holds for some $k \geq 1$. Then $\displaystyle\int_0^{\pi/2} \sin^{2k} x\,dx = \frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2k-1)}{2\cdot 4\cdot 6\cdot\cdots\cdot(2k)}\,\frac{\pi}{2}$. By Exercise 43(a),

$$\int_0^{\pi/2} \sin^{2(k+1)} x\,dx = \frac{2k+1}{2k+2}\int_0^{\pi/2} \sin^{2k} x\,dx = \frac{2k+1}{2k+2}\cdot\frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2k-1)}{2\cdot 4\cdot 6\cdot\cdots\cdot(2k)}\,\frac{\pi}{2}$$
$$= \frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2k-1)(2k+1)}{2\cdot 4\cdot 6\cdot\cdots\cdot(2k)(2k+2)}\cdot\frac{\pi}{2},$$

so the formula holds for $n = k + 1$. By induction, the formula holds for all $n \geq 1$.

45. Let $u = (\ln x)^n$, $dv = dx \;\Rightarrow\; du = n(\ln x)^{n-1}(dx/x)$, $v = x$. By Equation 2,
$\int (\ln x)^n\,dx = x(\ln x)^n - \int nx(\ln x)^{n-1}(dx/x) = x(\ln x)^n - n\int(\ln x)^{n-1}\,dx$.

46. Let $u = x^n$, $dv = e^x\,dx \;\Rightarrow\; du = nx^{n-1}\,dx$, $v = e^x$. By Equation 2, $\int x^n e^x\,dx = x^n e^x - n\int x^{n-1}e^x\,dx$.

47. Let $u = \left(x^2+a^2\right)^n$, $dv = dx \;\Rightarrow\; du = n\left(x^2+a^2\right)^{n-1}2x\,dx$, $v = x$. Then

$$\int \left(x^2+a^2\right)^n dx = x\left(x^2+a^2\right)^n - 2n\int x^2\left(x^2+a^2\right)^{n-1} dx$$
$$= x\left(x^2+a^2\right)^n - 2n\left[\int\left(x^2+a^2\right)^n dx - a^2\int\left(x^2+a^2\right)^{n-1} dx\right] \quad \left[\text{since } x^2 = \left(x^2+a^2\right) - a^2\right]$$

$\Rightarrow\; (2n+1)\int\left(x^2+a^2\right)^n dx = x\left(x^2+a^2\right)^n + 2na^2\int\left(x^2+a^2\right)^{n-1} dx$, and

$$\int\left(x^2+a^2\right)^n dx = \frac{x\left(x^2+a^2\right)^n}{2n+1} + \frac{2na^2}{2n+1}\int\left(x^2+a^2\right)^{n-1} dx \quad [\text{provided } 2n+1 \neq 0].$$

48. Let $u = \sec^{n-2} x$, $dv = \sec^2 x\,dx \;\Rightarrow\; du = (n-2)\sec^{n-3} x\sec x\tan x\,dx$, $v = \tan x$. Then by Equation 2,

$$\int \sec^n x\,dx = \tan x\sec^{n-2} x - (n-2)\int\sec^{n-2} x\tan^2 x\,dx$$
$$= \tan x\sec^{n-2} x - (n-2)\int\sec^{n-2} x\left(\sec^2 x - 1\right) dx$$
$$= \tan x\sec^{n-2} x - (n-2)\int\sec^n x\,dx + (n-2)\int\sec^{n-2} x\,dx$$

so $(n-1)\int\sec^n x\,dx = \tan x\sec^{n-2} x + (n-2)\int\sec^{n-2} x\,dx$. If $n - 1 \neq 0$, then

$$\int\sec^n x\,dx = \frac{\tan x\sec^{n-2} x}{n-1} + \frac{n-2}{n-1}\int\sec^{n-2} x\,dx.$$

49. Take $n = 3$ in Exercise 45 to get

$\int (\ln x)^3\,dx = x\,(\ln x)^3 - 3\int (\ln x)^2\,dx = x(\ln x)^3 - 3x(\ln x)^2 + 6x\ln x - 6x + C$ [by Exercise 13].

Or: Instead of using Exercise 13, apply Exercise 45 again with $n = 2$.

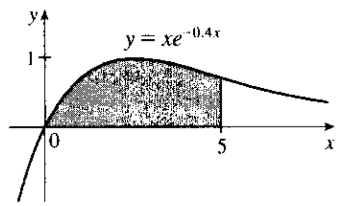
50. Take $n = 4$ in Exercise 46 to get

$$\begin{aligned}\int x^4e^x\,dx &= x^4e^x - 4\int x^3e^x\,dx = x^4e^x - 4\left(x^3 - 3x^2 + 6x - 6\right)e^x + C \quad \text{[by Exercise 14]}\\ &= e^x\left(x^4 - 4x^3 + 12x^2 - 24x + 24\right) + C\end{aligned}$$

Or: Instead of using Exercise 14, apply Exercise 46 with $n = 3$, then $n = 2$, then $n = 1$.

51. Area $= \int_0^5 xe^{-0.4x}dx$. Let $u = x$, $dv = e^{-0.4x}dx \Rightarrow$ $du = dx$, $v = -2.5e^{-0.4x}$. Then

$$\begin{aligned}\text{area} &= \left[-2.5xe^{-0.4x}\right]_0^5 + 2.5\int_0^5 e^{-0.4x}dx\\ &= -12.5e^{-2} + 0 + 2.5\left[-2.5e^{-0.4x}\right]_0^5\\ &= -12.5e^{-2} - 6.25(e^{-2} - 1) = 6.25 - 18.75e^{-2} \quad\text{or}\quad \tfrac{25}{4} - \tfrac{75}{4}e^{-2}\end{aligned}$$

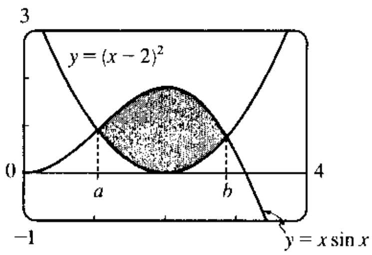


52. The curves $y = x\ln x$ and $y = 5\ln x$ intersect when $x\ln x = 5\ln x \Leftrightarrow x\ln x - 5\ln x = 0 \Leftrightarrow$ $(x-5)\ln x = 0$; that is, when $x = 1$ or $x = 5$. For $1 < x < 5$, we have $5\ln x > x\ln x$ since $\ln x > 0$. Thus, area $= \int_1^5 (5\ln x - x\ln x)\,dx = \int_1^5 [(5-x)\ln x]\,dx$. Let $u = \ln x$, $dv = (5-x)\,dx \Rightarrow du = dx/x$, $v = 5x - \frac{1}{2}x^2$. Then

$$\begin{aligned}\text{area} &= \left[(\ln x)\left(5x - \tfrac12 x^2\right)\right]_1^5 - \int_1^5 \left[\left(5x - \tfrac12 x^2\right)\tfrac1x\right]dx = (\ln 5)\left(\tfrac{25}{2}\right) - 0 - \int_1^5 \left(5 - \tfrac12 x\right)dx\\ &= \tfrac{25}{2}\ln 5 - \left[5x - \tfrac14 x^2\right]_1^5 = \tfrac{25}{2}\ln 5 - \left[\left(25 - \tfrac{25}{4}\right) - \left(5 - \tfrac14\right)\right] = \tfrac{25}{2}\ln 5 - 14\end{aligned}$$

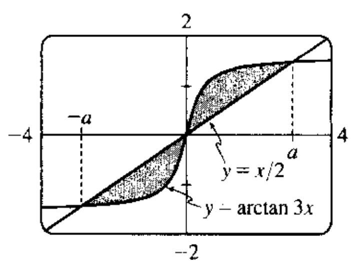
53. The curves $y = x\sin x$ and $y = (x-2)^2$ intersect at $a \approx 1.04748$ and $b \approx 2.87307$, so

$$\begin{aligned}\text{area} &= \int_a^b \left[x\sin x - (x-2)^2\right]dx\\ &= \left[-x\cos x + \sin x - \tfrac13(x-2)^3\right]_a^b \quad\text{[by Example 1]}\\ &\approx 2.81358 - 0.63075 = 2.18283\end{aligned}$$



54. The curves $y = \arctan 3x$ and $y = x/2$ intersect at $x = \pm a \approx \pm 2.91379$, so

$$\begin{aligned}\text{area} &= \int_{-a}^{a}\left|\arctan 3x - \tfrac12 x\right|dx = 2\int_0^a\left(\arctan 3x - \tfrac12 x\right)dx\\ &= 2\left[x\arctan 3x - \tfrac16\ln(1+9x^2) - \tfrac14 x^2\right]_0^a \quad\text{[see Example 5]}\\ &\approx 2(1.39768) = 2.79536.\end{aligned}$$



55. $V = \int_0^1 2\pi x\cos(\pi x/2)dx$. Let $u = x$, $dv = \cos(\pi x/2)dx \Rightarrow du = dx$, $v = \frac{2}{\pi}\sin(\pi x/2)$.

$$\begin{aligned}V &= 2\pi\left[\frac{2}{\pi}x\sin\left(\frac{\pi x}{2}\right)\right]_0^1 - 2\pi\cdot\frac{2}{\pi}\int_0^1 \sin\left(\frac{\pi x}{2}\right)dx = 2\pi\left(\frac{2}{\pi} - 0\right) - 4\left[-\frac{2}{\pi}\cos\left(\frac{\pi x}{2}\right)\right]_0^1\\ &= 4 + \tfrac{8}{\pi}(0-1) = 4 - \tfrac{8}{\pi}.\end{aligned}$$

56. Volume $= \int_0^1 2\pi x(e^x - e^{-x})\,dx = 2\pi \int_0^1 (xe^x - xe^{-x})\,dx$

$$= 2\pi\left[\int_0^1 xe^x\,dx - \int_0^1 xe^{-x}\,dx\right] \quad \text{[both integrals by parts]}$$

$$= 2\pi\left[(xe^x - e^x) - (-xe^{-x} - e^{-x})\right]_0^1 = 2\pi[2/e - 0] = 4\pi/e$$

57. Volume $= \int_{-1}^0 2\pi(1-x)e^{-x}\,dx$. Let $u = 1 - x$, $dv = e^{-x}\,dx \Rightarrow du = -\,dx$, $v = -e^{-x}$.

$$V = 2\pi\left[(1-x)(-e^{-x})\right]_{-1}^0 - 2\pi\int_{-1}^0 e^{-x}\,dx = 2\pi\left[(x-1)(e^{-x}) + e^{-x}\right]_{-1}^0$$

$$= 2\pi\left[xe^{-x}\right]_{-1}^0 = 2\pi(0+e) = 2\pi e$$

58. Volume $= \int_1^\pi 2\pi y \cdot \ln y\,dy = 2\pi\left[\tfrac{1}{2}y^2 \ln y - \tfrac{1}{4}y^2\right]_1^\pi$ [by parts with $u = \ln y$ and $dv = y\,dy$]

$$= 2\pi\left[\tfrac{1}{4}y^2(2\ln y - 1)\right]_1^\pi = 2\pi\left[\frac{\pi^2(2\ln\pi - 1)}{4} - \frac{(0-1)}{4}\right] = \pi^3\ln\pi - \frac{\pi^3}{2} + \frac{\pi}{2}$$

59. The average value of $f(x) = x^2 \ln x$ on the interval $[1,3]$ is $f_{\text{ave}} = \dfrac{1}{3-1}\displaystyle\int_1^3 x^2 \ln x\,dx = \tfrac{1}{2}I$.

Let $u = \ln x$, $dv = x^2\,dx \Rightarrow du = (1/x)\,dx$, $v = \frac{1}{3}x^3$. So

$$I = \left[\tfrac{1}{3}x^3 \ln x\right]_1^3 - \int_1^3 \tfrac{1}{3}x^2 dx = (9\ln 3 - 0) - \left[\tfrac{1}{9}x^3\right]_1^3 = 9\ln 3 - \left(3 - \tfrac{1}{9}\right) = 9\ln 3 - \tfrac{26}{9}.$$

Thus, $f_{\text{ave}} = \frac{1}{2}I = \frac{1}{2}\left(9\ln 3 - \frac{26}{9}\right) = \frac{9}{2}\ln 3 - \frac{13}{9}$.

60. The rocket will have height $H = \int_0^{60} v(t)\,dt$ after 60 seconds.

$$H = \int_0^{60}\left[-gt - v_e \ln\left(\frac{m - rt}{m}\right)\right]dt = -g\left[\tfrac{1}{2}t^2\right]_0^{60} - v_e\left[\int_0^{60}\ln(m - rt)\,dt - \int_0^{60}\ln m\,dt\right]$$

$$= -g(1800) + v_e(\ln m)(60) - v_e\int_0^{60}\ln(m - rt)\,dt$$

Let $u = \ln(m - rt)$, $dv = dt \Rightarrow du = \dfrac{1}{m - rt}(-r)\,dt$, $v = t$. Then

$$\int_0^{60}\ln(m - rt)\,dt = [t\ln(m - rt)]_0^{60} + \int_0^{60}\frac{rt}{m - rt}\,dt = 60\ln(m - 60r) + \int_0^{60}\left(-1 + \frac{m}{m - rt}\right)dt$$

$$= 60\ln(m - 60r) + \left[-t - \frac{m}{r}\ln(m - rt)\right]_0^{60}$$

$$= 60\ln(m - 60r) - 60 - \frac{m}{r}\ln(m - 60r) + \frac{m}{r}\ln m$$

So $H = -1800g + 60v_e \ln m - 60v_e \ln(m - 60r) + 60v_e + \dfrac{m}{r}v_e \ln(m - 60r) - \dfrac{m}{r}v_e \ln m$. Substituting $g = 9.8$, $m = 30{,}000$, $r = 160$, and $v_e = 3000$ gives us $H \approx 14{,}844$ m.

61. Since $v(t) > 0$ for all t, the desired distance is $s(t) = \int_0^t v(w)dw = \int_0^t w^2 e^{-w}\,dw$.

First let $u = w^2$, $dv = e^{-w}\,dw \Rightarrow du = 2w\,dw$, $v = -e^{-w}$. Then $s(t) = \left[-w^2 e^{-w}\right]_0^t + 2\int_0^t we^{-w}\,dw$.

Next let $U = w$, $dV = e^{-w}\,dw \Rightarrow dU = dw$, $V = -e^{-w}$. Then

$$s(t) = -t^2 e^{-t} + 2\left(\left[-we^{-w}\right]_0^t + \int_0^t e^{-w}\,dw\right) = -t^2 e^{-t} + 2\left(-te^{-t} + 0 + \left[-e^{-w}\right]_0^t\right)$$

$$= -t^2 e^{-t} + 2(-te^{-t} - e^{-t} + 1) = -t^2 e^{-t} - 2te^{-t} - 2e^{-t} + 2$$

$$= 2 - e^{-t}(t^2 + 2t + 2) \text{ meters}$$

62. Suppose $f(0) = g(0) = 0$ and let $u = f(x)$, $dv = g''(x)\,dx \quad \Rightarrow \quad du = f'(x)\,dx$, $v = g'(x)$.
Then $\int_0^a f(x)g''(x)\,dx = [f(x)g'(x)]_0^a - \int_0^a f'(x)g'(x)\,dx = f(a)g'(a) - \int_0^a f'(x)g'(x)\,dx$.
Now let $U = f'(x)$, $dV = g'(x)\,dx \quad \Rightarrow \quad dU = f''(x)\,dx$ and $V = g(x)$, so
$\int_0^a f'(x)g'(x)\,dx = [f'(x)g(x)]_0^a - \int_0^a f''(x)g(x)\,dx = f'(a)g(a) - \int_0^a f''(x)g(x)\,dx$.
Combining the two results, we get $\int_0^a f(x)g''(x)\,dx = f(a)g'(a) - f'(a)g(a) + \int_0^a f''(x)g(x)\,dx$.

63. For $I = \int_1^4 xf''(x)\,dx$, let $u = x$, $dv = f''(x)\,dx \quad \Rightarrow \quad du = dx$, $v = f'(x)$. Then
$I = [xf'(x)]_1^4 - \int_1^4 f'(x)\,dx = 4f'(4) - 1 \cdot f'(1) - [f(4) - f(1)] = 4 \cdot 3 - 1 \cdot 5 - (7 - 2) = 12 - 5 - 5 = 2$.
We used the fact that f'' is continuous to guarantee that I exists.

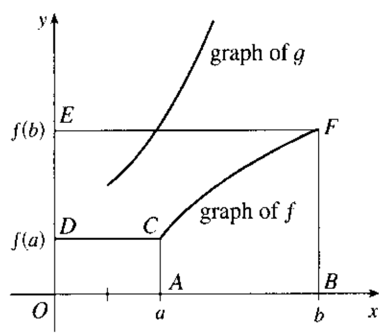
64. (a) Take $g(x) = x$ and $g'(x) = 1$ in Equation 1.

(b) By part (a), $\int_a^b f(x)\,dx = bf(b) - a\,f(a) - \int_a^b x\,f'(x)\,dx$. Now let $y = f(x)$, so that $x = g(y)$ and $dy = f'(x)\,dx$. Then $\int_a^b x\,f'(x)\,dx = \int_{f(a)}^{f(b)} g(y)\,dy$. The result follows.

(c) Part (b) says that the area of region $ABFC$ is

$$= \quad bf(b) \quad - \quad af(a) \quad - \quad \int_{f(a)}^{f(b)} g(y)\,dy$$

= (area of rectangle $OBFE$) − (area of rectangle $OACD$) − (area of region $DCFE$)



(d) We have $f(x) = \ln x$, so $f^{-1}(x) = e^x$, and since $g = f^{-1}$, we have $g(y) = e^y$. By part (b),
$\int_1^e \ln x\,dx = e\ln e - 1\ln 1 - \int_{\ln 1}^{\ln e} e^y\,dy = e - \int_0^1 e^y\,dy = e - \left[e^y\right]_0^1 = e - (e-1) = 1$.

65. Using the formula for volumes of rotation and the figure, we see that
Volume $= \int_0^d \pi b^2\,dy - \int_0^c \pi a^2\,dy - \int_c^d \pi[g(y)]^2\,dy = \pi b^2 d - \pi a^2 c - \int_c^d \pi[g(y)]^2\,dy$. Let $y = f(x)$, which gives $dy = f'(x)\,dx$ and $g(y) = x$, so that $V = \pi b^2 d - \pi a^2 c - \pi \int_a^b x^2 f'(x)\,dx$. Now integrate by parts with $u = x^2$, and $dv = f'(x)\,dx \quad \Rightarrow \quad du = 2x\,dx$, $v = f(x)$, and
$\int_a^b x^2\,f'(x)\,dx = \left[x^2\,f(x)\right]_a^b - \int_a^b 2x\,f(x)\,dx = b^2\,f(b) - a^2\,f(a) - \int_a^b 2x\,f(x)\,dx$, but $f(a) = c$ and $f(b) = d$
$\Rightarrow \quad V = \pi b^2 d - \pi a^2 c - \pi\left[b^2 d - a^2 c - \int_a^b 2xf(x)\,dx\right] = \int_a^b 2\pi x f(x)\,dx$.

66. (a) We note that for $0 \le x \le \frac{\pi}{2}$, $0 \le \sin x \le 1$, so $\sin^{2n+2} x \le \sin^{2n+1} x \le \sin^{2n} x$. So by the second Comparison Property of the Integral, $I_{2n+2} \le I_{2n+1} \le I_{2n}$.

(b) Substituting directly into the result from Exercise 44, we get

$$\frac{I_{2n+2}}{I_{2n}} = \frac{\dfrac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot [2(n+1)-1]}{2 \cdot 4 \cdot 6 \cdot \cdots \cdot [2(n+1)]}\dfrac{\pi}{2}}{\dfrac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{2 \cdot 4 \cdot 6 \cdot \cdots \cdot (2n)}\dfrac{\pi}{2}} = \frac{2(n+1)-1}{2(n+1)} = \frac{2n+1}{2n+2}$$

(c) We divide the result from part (a) by I_{2n}. The inequalities are preserved since I_{2n} is positive: $\frac{I_{2n+2}}{I_{2n}} \le \frac{I_{2n+1}}{I_{2n}} \le \frac{I_{2n}}{I_{2n}}$. Now from part (b), the left term is equal to $\frac{2n+1}{2n+2}$, so the expression becomes $\frac{2n+1}{2n+2} \le \frac{I_{2n+1}}{I_{2n}} \le 1$. Now $\lim_{n\to\infty} \frac{2n+1}{2n+2} = \lim_{n\to\infty} 1 = 1$, so by the Squeeze Theorem, $\lim_{n\to\infty} \frac{I_{2n+1}}{I_{2n}} = 1$.

(d) We substitute the results from Exercises 43 and 44 into the result from part (c):

$$1 = \lim_{n\to\infty} \frac{I_{2n+1}}{I_{2n}} = \lim_{n\to\infty} \frac{\dfrac{2\cdot 4\cdot 6\cdot\cdots\cdot(2n)}{3\cdot 5\cdot 7\cdot\cdots\cdot(2n+1)}}{\dfrac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}{2\cdot 4\cdot 6\cdot\cdots\cdot(2n)}\dfrac{\pi}{2}}$$

$$= \lim_{n\to\infty} \left[\frac{2\cdot 4\cdot 6\cdot\cdots\cdot(2n)}{3\cdot 5\cdot 7\cdot\cdots\cdot(2n+1)}\right]\left[\frac{2\cdot 4\cdot 6\cdot\cdots\cdot(2n)}{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}\left(\frac{2}{\pi}\right)\right]$$

$$= \lim_{n\to\infty} \frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdot\cdots\cdot\frac{2n}{2n-1}\cdot\frac{2n}{2n+1}\cdot\frac{2}{\pi} \qquad \text{[rearrange terms]}$$

Multiplying both sides by $\frac{\pi}{2}$ gives us the *Wallis product*:

$$\frac{\pi}{2} = \frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdot\cdots$$

(e) The area of the kth rectangle is k. At the $2n$th step, the area is increased from $2n-1$ to $2n$ by multiplying the width by $\frac{2n}{2n-1}$, and at the $(2n+1)$th step, the area is increased from $2n$ to $2n+1$ by multiplying the height by $\frac{2n+1}{2n}$. These two steps multiply the ratio of width to height by $\frac{2n}{2n-1}$ and $\frac{1}{(2n+1)/(2n)} = \frac{2n}{2n+1}$ respectively. So, by part (d), the limiting ratio is $\frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdot\cdots = \frac{\pi}{2}$.

8.2 Trigonometric Integrals

The symbols $\overset{s}{=}$ and $\overset{c}{=}$ indicate the use of the substitutions $\{u = \sin x, du = \cos x\,dx\}$ and $\{u = \cos x, du = -\sin x\,dx\}$, respectively.

1. $\int \sin^3 x \cos^2 x\,dx = \int \sin^2 x \cos^2 x \sin x\,dx = \int (1-\cos^2 x)\cos^2 x \sin x\,dx \overset{c}{=} \int (1-u^2)u^2(-du)$
$= \int (u^2-1)u^2\,du = \int (u^4-u^2)du = \frac{1}{5}u^5 - \frac{1}{3}u^3 + C = \frac{1}{5}\cos^5 x - \frac{1}{3}\cos^3 x + C$

2. $\int \sin^6 x \cos^3 x\,dx = \int \sin^6 x \cos^2 x \cos x\,dx = \int \sin^6 x\,(1-\sin^2 x)\cos x\,dx \overset{s}{=} \int u^6\,(1-u^2)\,du$
$= \int (u^6 - u^8)\,du = \frac{1}{7}u^7 - \frac{1}{9}u^9 + C = \frac{1}{7}\sin^7 x - \frac{1}{9}\sin^9 x + C$

3. $\int_{\pi/2}^{3\pi/4} \sin^5 x \cos^3 x\,dx = \int_{\pi/2}^{3\pi/4} \sin^5 x \cos^2 x \cos x\,dx = \int_{\pi/2}^{3\pi/4} \sin^5 x\,(1-\sin^2 x)\cos x\,dx$
$\overset{s}{=} \int_1^{\sqrt{2}/2} u^5(1-u^2)du = \int_1^{\sqrt{2}/2} (u^5-u^7)du = \left[\frac{1}{6}u^6 - \frac{1}{8}u^8\right]_1^{\sqrt{2}/2}$
$= \left(\frac{1/8}{6} - \frac{1/16}{8}\right) - \left(\frac{1}{6} - \frac{1}{8}\right) = -\frac{11}{384}$

4. $\int_0^{\pi/2} \cos^5 x\,dx = \int_0^{\pi/2} (\cos^2 x)^2 \cos x\,dx = \int_0^{\pi/2} (1-\sin^2 x)^2 \cos x\,dx \overset{s}{=} \int_0^1 (1-u^2)^2 du$
$= \int_0^1 (1-2u^2+u^4)du = \left[u - \frac{2}{3}u^3 + \frac{1}{5}u^5\right]_0^1 = \left(1 - \frac{2}{3} + \frac{1}{5}\right) - 0 = \frac{8}{15}$

5. $\int \cos^5 x \sin^4 x\,dx = \int \cos^4 x \sin^4 x \cos x\,dx = \int \left(1-\sin^2 x\right)^2 \sin^4 x \cos x\,dx \overset{s}{=} \int \left(1-u^2\right)^2 u^4\,du$
$= \int \left(1-2u^2+u^4\right) u^4\,du = \int \left(u^4-2u^6+u^8\right) du = \frac{1}{5}u^5 - \frac{2}{7}u^7 + \frac{1}{9}u^9 + C$
$= \frac{1}{5}\sin^5 x - \frac{2}{7}\sin^7 x + \frac{1}{9}\sin^9 x + C$

6. $\int \sin^3(mx)\,dx = \int \left(1-\cos^2 mx\right)\sin mx\,dx = -\frac{1}{m}\int \left(1-u^2\right)du \quad [u=\cos mx,\ du=-m\sin mx\,dx]$
$= -\frac{1}{m}\left(u - \frac{1}{3}u^3\right) + C = -\frac{1}{m}\left(\cos mx - \frac{1}{3}\cos^3 mx\right) + C$
$= \frac{1}{3m}\cos^3 mx - \frac{1}{m}\cos mx + C$

7. $\int_0^{\pi/2} \cos^2\theta\,d\theta = \int_0^{\pi/2} \frac{1}{2}(1+\cos 2\theta)\,d\theta$ [half-angle identity]
$= \frac{1}{2}\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{\pi/2} = \frac{1}{2}\left[\left(\frac{\pi}{2}+0\right) - (0+0)\right] = \frac{\pi}{4}$

8. $\int_0^{\pi/2} \sin^2(2\theta)\,d\theta = \int_0^{\pi/2} \frac{1}{2}(1-\cos 4\theta)d\theta = \frac{1}{2}\left[\theta - \frac{1}{4}\sin 4\theta\right]_0^{\pi/2} = \frac{1}{2}\left[\left(\frac{\pi}{2}-0\right) - (0-0)\right] = \frac{\pi}{4}$

9. $\int_0^{\pi} \sin^4(3t)\,dt = \int_0^{\pi} \left[\sin^2(3t)\right]^2 dt = \int_0^{\pi} \left[\frac{1}{2}(1-\cos 6t)\right]^2 dt = \frac{1}{4}\int_0^{\pi}(1-2\cos 6t+\cos^2 6t)\,dt$
$= \frac{1}{4}\int_0^{\pi}\left[1-2\cos 6t + \frac{1}{2}(1+\cos 12t)\right] dt = \frac{1}{4}\int_0^{\pi}\left(\frac{3}{2} - 2\cos 6t + \frac{1}{2}\cos 12t\right) dt$
$= \frac{1}{4}\left[\frac{3}{2}t - \frac{1}{3}\sin 6t + \frac{1}{24}\sin 12t\right]_0^{\pi} = \frac{1}{4}\left[\left(\frac{3\pi}{2} - 0 + 0\right) - (0-0+0)\right] = \frac{3\pi}{8}$

10. $\int_0^{\pi} \cos^6\theta\,d\theta = \int_0^{\pi}(\cos^2\theta)^3\,d\theta = \int_0^{\pi}\left[\frac{1}{2}(1+\cos 2\theta)\right]^3 d\theta = \frac{1}{8}\int_0^{\pi}(1+3\cos 2\theta+3\cos^2 2\theta+\cos^3 2\theta)\,d\theta$
$= \frac{1}{8}\left[\theta + \frac{3}{2}\sin 2\theta\right]_0^{\pi} + \frac{1}{8}\int_0^{\pi}\left[\frac{3}{2}(1+\cos 4\theta)\right] d\theta + \frac{1}{8}\int_0^{\pi}\left[(1-\sin^2 2\theta)\cos 2\theta\right] d\theta$
$= \frac{1}{8}\pi + \frac{3}{16}\left[\theta + \frac{1}{4}\sin 4\theta\right]_0^{\pi} + \frac{1}{8}\int_0^{0}(1-u^2)\left(\frac{1}{2}du\right) \quad [u=\sin 2\theta,\ du = 2\cos 2\theta\,d\theta]$
$= \frac{\pi}{8} + \frac{3\pi}{16} + 0 = \frac{5\pi}{16}$

11. $\int (1+\cos\theta)^2\,d\theta = \int (1+2\cos\theta+\cos^2\theta)\,d\theta = \theta + 2\sin\theta + \frac{1}{2}\int(1+\cos 2\theta)\,d\theta$
$= \theta + 2\sin\theta + \frac{1}{2}\theta + \frac{1}{4}\sin 2\theta + C = \frac{3}{2}\theta + 2\sin\theta + \frac{1}{4}\sin 2\theta + C$

12. Let $u = x$, $dv = \cos^2 x\,dx \Rightarrow du = dx$, $v = \int \cos^2 x\,dx = \int \frac{1}{2}(1+\cos 2x)dx = \frac{1}{2}x + \frac{1}{4}\sin 2x$, so
$\int x\cos^2 x\,dx = x\left(\frac{1}{2}x + \frac{1}{4}\sin 2x\right) - \int\left(\frac{1}{2}x + \frac{1}{4}\sin 2x\right) dx = \frac{1}{2}x^2 + \frac{1}{4}x\sin 2x - \frac{1}{4}x^2 + \frac{1}{8}\cos 2x + C$
$= \frac{1}{4}x^2 + \frac{1}{4}x\sin 2x + \frac{1}{8}\cos 2x + C$

13. $\int_0^{\pi/4} \sin^4 x\cos^2 x\,dx = \int_0^{\pi/4}\sin^2 x\,(\sin x\cos x)^2\,dx = \int_0^{\pi/4}\frac{1}{2}(1-\cos 2x)\left(\frac{1}{2}\sin 2x\right)^2 dx$
$= \frac{1}{8}\int_0^{\pi/4}(1-\cos 2x)\,\sin^2 2x\,dx = \frac{1}{8}\int_0^{\pi/4}\sin^2 2x\,dx - \frac{1}{8}\int_0^{\pi/4}\sin^2 2x\cos 2x\,dx$
$= \frac{1}{16}\int_0^{\pi/4}(1-\cos 4x)\,dx - \frac{1}{16}\left[\frac{1}{3}\sin^3 2x\right]_0^{\pi/4} = \frac{1}{16}\left[x - \frac{1}{4}\sin 4x - \frac{1}{3}\sin^3 2x\right]_0^{\pi/4}$
$= \frac{1}{16}\left(\frac{\pi}{4} - 0 - \frac{1}{3}\right) = \frac{1}{192}(3\pi - 4)$

14. $\int_0^{\pi/2}\sin^2 x\,\cos^2 x\,dx = \int_0^{\pi/2}\frac{1}{4}\left(4\sin^2 x\,\cos^2 x\right) dx = \int_0^{\pi/2}\frac{1}{4}(2\sin x\,\cos x)^2 dx = \frac{1}{4}\int_0^{\pi/2}\sin^2 2x\,dx$
$= \frac{1}{4}\int_0^{\pi/2}\frac{1}{2}(1-\cos 4x)\,dx = \frac{1}{8}\int_0^{\pi/2}(1-\cos 4x)\,dx = \frac{1}{8}\left[x - \frac{1}{4}\sin 4x\right]_0^{\pi/2}$
$= \frac{1}{8}\left(\frac{\pi}{2}\right) = \frac{\pi}{16}$

15. $\int \sin^3 x\,\sqrt{\cos x}\,dx = \int \left(1-\cos^2 x\right)\sqrt{\cos x}\sin x\,dx \overset{c}{=} \int\left(1-u^2\right)u^{1/2}\,(-du) = \int\left(u^{5/2} - u^{1/2}\right) du$
$= \frac{2}{7}u^{7/2} - \frac{2}{3}u^{3/2} + C = \frac{2}{7}(\cos x)^{7/2} - \frac{2}{3}(\cos x)^{3/2} + C$
$= \left(\frac{2}{7}\cos^3 x - \frac{2}{3}\cos x\right)\sqrt{\cos x} + C$

16. Let $u = \sin\theta$. Then $du = \cos\theta\, d\theta$ and

$$\int \cos\theta \cos^5(\sin\theta)\,d\theta = \int \cos^5 u\,du = \int (\cos^2 u)^2 \cos u\,du = \int (1-\sin^2 u)^2 \cos u\,du$$
$$= \int (1 - 2\sin^2 u + \sin^4 u)\cos u\,du = I$$

Now let $x = \sin u$. Then $dx = \cos u\,du$ and

$$I = \int (1 - 2x^2 + x^4)\,dx = x - \tfrac{2}{3}x^3 + \tfrac{1}{5}x^5 + C = \sin u - \tfrac{2}{3}\sin^3 u + \tfrac{1}{5}\sin^5 u + C$$
$$= \sin(\sin\theta) - \tfrac{2}{3}\sin^3(\sin\theta) + \tfrac{1}{5}\sin^5(\sin\theta) + C$$

17. $$\int \cos^2 x \tan^3 x\,dx = \int \frac{\sin^3 x}{\cos x}\,dx \overset{c}{=} \int \frac{(1-u^2)(-du)}{u} = \int \left[\frac{-1}{u} + u\right] du$$
$$= -\ln|u| + \tfrac{1}{2}u^2 + C = \tfrac{1}{2}\cos^2 x - \ln|\cos x| + C$$

18. $$\int \cot^5\theta \sin^4\theta\,d\theta = \int \frac{\cos^5\theta}{\sin^5\theta}\sin^4\theta\,d\theta = \int \frac{\cos^5\theta}{\sin\theta}\,d\theta = \int \frac{\cos^4\theta}{\sin\theta}\cos\theta\,d\theta = \int \frac{(1-\sin^2\theta)^2}{\sin\theta}\cos\theta\,d\theta$$
$$\overset{s}{=} \int \frac{(1-u^2)^2}{u}\,du = \int \frac{1-2u^2+u^4}{u}\,du = \int \left(\frac{1}{u} - 2u + u^3\right) du$$
$$= \ln|u| - u^2 + \tfrac{1}{4}u^4 + C = \ln|\sin\theta| - \sin^2\theta + \tfrac{1}{4}\sin^4\theta + C$$

19. $$\int \frac{1-\sin x}{\cos x}\,dx = \int (\sec x - \tan x)\,dx = \ln|\sec x + \tan x| - \ln|\sec x| + C \quad \left[\begin{array}{c}\text{by (1) and the boxed}\\ \text{formula above it}\end{array}\right]$$
$$= \ln|(\sec x + \tan x)\cos x| + C = \ln|1 + \sin x| + C$$
$$= \ln(1 + \sin x) + C \quad \text{since } 1 + \sin x \geq 0$$

Or: $$\int \frac{1-\sin x}{\cos x}\,dx = \int \frac{1-\sin x}{\cos x}\cdot\frac{1+\sin x}{1+\sin x}\,dx = \int \frac{(1-\sin^2 x)\,dx}{\cos x\,(1+\sin x)} = \int \frac{\cos x\,dx}{1+\sin x}$$
$$= \int \frac{dw}{w} \quad [\text{where } w = 1 + \sin x,\ dw = \cos x\,dx]$$
$$= \ln|w| + C = \ln|1 + \sin x| + C = \ln(1 + \sin x) + C$$

20. $\int \cos^2 x \sin 2x\,dx = 2\int \cos^3 x \sin x\,dx \overset{c}{=} -2\int u^3\,du = -\tfrac{1}{2}u^4 + C = -\tfrac{1}{2}\cos^4 x + C$

21. Let $u = \tan x$, $du = \sec^2 x\,dx$. Then $\int \sec^2 x \tan x\,dx = \int u\,du = \tfrac{1}{2}u^2 + C = \tfrac{1}{2}\tan^2 x + C$.
Or: Let $v = \sec x$, $dv = \sec x \tan x\,dx$. Then $\int \sec^2 x \tan x\,dx = \int v\,dv = \tfrac{1}{2}v^2 + C = \tfrac{1}{2}\sec^2 x + C$.

22. $\int_0^{\pi/2} \sec^4(t/2)\,dt = \int_0^{\pi/4} \sec^4 x\,(2\,dx) \quad [x = t/2,\ dx = \tfrac{1}{2}\,dt] \quad = 2\int_0^{\pi/4} \sec^2 x(1 + \tan^2 x)\,dx$
$$= 2\int_0^1 (1 + u^2)\,du \quad [u = \tan x,\ du = \sec^2 x\,dx] \quad = 2\left[u + \tfrac{1}{3}u^3\right]_0^1 = 2\left(1 + \tfrac{1}{3}\right) = \tfrac{8}{3}$$

23. $\int \tan^2 x\,dx = \int (\sec^2 x - 1)\,dx = \tan x - x + C$

24. $\int \tan^4 x\,dx = \int \tan^2 x\,(\sec^2 x - 1)\,dx = \int \tan^2 x \sec^2 x\,dx - \int \tan^2 x\,dx = \tfrac{1}{3}\tan^3 x - \tan x + x + C$
(Set $u = \tan x$ in the first integral and use Exercise 23 for the second.)

25. $\int \sec^6 t\,dt = \int \sec^4 t \cdot \sec^2 t\,dt = \int (\tan^2 t + 1)^2 \sec^2 t\,dt = \int (u^2 + 1)^2\,du \qquad [u = \tan t,\ du = \sec^2 t\,dt]$
$$= \int (u^4 + 2u^2 + 1)\,du = \tfrac{1}{5}u^5 + \tfrac{2}{3}u^3 + u + C = \tfrac{1}{5}\tan^5 t + \tfrac{2}{3}\tan^3 t + \tan t + C$$

26. $\int_0^{\pi/4} \sec^4\theta \tan^4\theta\,d\theta = \int_0^{\pi/4} (\tan^2\theta + 1)\,\tan^4\theta \sec^2\theta\,d\theta = \int_0^1 (u^2 + 1)u^4\,du \qquad [u = \tan\theta,\ du = \sec^2\theta\,d\theta]$
$$= \int_0^1 (u^6 + u^4)\,du = \left[\tfrac{1}{7}u^7 + \tfrac{1}{5}u^5\right]_0^1 = \tfrac{1}{7} + \tfrac{1}{5} = \tfrac{12}{35}$$

27. $\int_0^{\pi/3} \tan^5 x \sec^4 x\,dx = \int_0^{\pi/3} \tan^5 x\,(\tan^2 x + 1)\sec^2 x\,dx$

$= \int_0^{\sqrt{3}} u^5(u^2+1)\,du \qquad [u = \tan x,\ du = \sec^2 x\,dx]$

$= \int_0^{\sqrt{3}} (u^7 + u^5)\,du = \left[\frac{1}{8}u^8 + \frac{1}{6}u^6\right]_0^{\sqrt{3}} = \frac{81}{8} + \frac{27}{6} = \frac{81}{8} + \frac{9}{2} = \frac{81}{8} + \frac{36}{8} = \frac{117}{8}$

Alternate solution:

$\int_0^{\pi/3} \tan^5 x \sec^4 x\,dx = \int_0^{\pi/3} \tan^4 x \sec^3 x \sec x \tan x\,dx = \int_0^{\pi/3} (\sec^2 x - 1)^2 \sec^3 x \sec x \tan x\,dx$

$= \int_1^2 (u^2-1)^2 u^3\,du \qquad [u = \sec x,\ du = \sec x \tan x\,dx]$

$= \int_1^2 (u^4 - 2u^2 + 1)u^3\,du = \int_1^2 (u^7 - 2u^5 + u^3)\,du$

$= \left[\frac{1}{8}u^8 - \frac{1}{3}u^6 + \frac{1}{4}u^4\right]_1^2 = \left(32 - \frac{64}{3} + 4\right) - \left(\frac{1}{8} - \frac{1}{3} + \frac{1}{4}\right) = \frac{117}{8}$

28. $\int \tan^3(2x)\sec^5(2x)\,dx = \int \tan^2(2x)\sec^4(2x)\cdot\sec(2x)\tan(2x)\,dx$

$= \int (u^2-1)u^4\left(\frac{1}{2}\,du\right) \qquad [u = \sec(2x),\ du = 2\sec(2x)\tan(2x)\,dx]$

$= \frac{1}{2}\int (u^6 - u^4)\,du = \frac{1}{14}u^7 - \frac{1}{10}u^5 + C = \frac{1}{14}\sec^7(2x) - \frac{1}{10}\sec^5(2x) + C$

29. $\int \tan^3 x \sec x\,dx = \int \tan^2 x \sec x \tan x\,dx = \int \left(\sec^2 x - 1\right)\sec x \tan x\,dx$

$= \int (u^2 - 1)\,du \qquad [u = \sec x,\ du = \sec x \tan x\,dx]$

$= \frac{1}{3}u^3 - u + C = \frac{1}{3}\sec^3 x - \sec x + C$

30. $\int_0^{\pi/3} \tan^5 x \sec^6 x\,dx = \int_0^{\pi/3} \tan^5 x \sec^4 x \sec^2 x\,dx = \int_0^{\pi/3} \tan^5 x\left(1 + \tan^2 x\right)^2 \sec^2 x\,dx$

$= \int_0^{\sqrt{3}} u^5\left(1 + u^2\right)^2\,du \quad [u = \tan x,\ du = \sec^2 x\,dx] \quad = \int_0^{\sqrt{3}} u^5\left(1 + 2u^2 + u^4\right)du$

$= \int_0^{\sqrt{3}} \left(u^5 + 2u^7 + u^9\right)du = \left[\frac{1}{6}u^6 + \frac{1}{4}u^8 + \frac{1}{10}u^{10}\right]_0^{\sqrt{3}} = \frac{27}{6} + \frac{81}{4} + \frac{243}{10} = \frac{981}{20}$

Alternate solution:

$\int_0^{\pi/3} \tan^5 x \sec^6 x\,dx = \int_0^{\pi/3} \tan^4 x \sec^5 x \sec x \tan x\,dx = \int_0^{\pi/3} \left(\sec^2 x - 1\right)^2 \sec^5 x \sec x \tan x\,dx$

$= \int_1^2 \left(u^2 - 1\right)^2 u^5\,du \qquad [u = \sec x,\ du = \sec x \tan x\,dx]$

$= \int_1^2 \left(u^4 - 2u^2 + 1\right)u^5\,du = \int_1^2 \left(u^9 - 2u^7 + u^5\right)du$

$= \left[\frac{1}{10}u^{10} - \frac{1}{4}u^8 + \frac{1}{6}u^6\right]_1^2 = \left(\frac{512}{5} - 64 + \frac{32}{3}\right) - \left(\frac{1}{10} - \frac{1}{4} + \frac{1}{6}\right) = \frac{981}{20}$

31. $\int \tan^5 x\,dx = \int \left(\sec^2 x - 1\right)^2 \tan x\,dx = \int \sec^4 x \tan x\,dx - 2\int \sec^2 x \tan x\,dx + \int \tan x\,dx$

$= \int \sec^3 x \sec x \tan x\,dx - 2\int \tan x \sec^2 x\,dx + \int \tan x\,dx$

$= \frac{1}{4}\sec^4 x - \tan^2 x + \ln|\sec x| + C \quad \left[\text{or } \frac{1}{4}\sec^4 x - \sec^2 x + \ln|\sec x| + C\right]$

32. $\int \tan^6 ay\,dy = \int \tan^4 ay\left(\sec^2 ay - 1\right)dy = \int \tan^4 ay \sec^2 ay\,dy - \int \tan^4 ay\,dy$

$= \frac{1}{5a}\tan^5 ay - \int \tan^2 ay\left(\sec^2 ay - 1\right)dy$

$= \frac{1}{5a}\tan^5 ay - \int \tan^2 ay \sec^2 ay\,dy + \int \left(\sec^2 ay - 1\right)dy$

$= \frac{1}{5a}\tan^5 ay - \frac{1}{3a}\tan^3 ay + \frac{1}{a}\tan ay - y + C$

33. $\displaystyle\int \frac{\tan^3\theta}{\cos^4\theta}\,d\theta = \int \tan^3\theta \sec^4\theta\,d\theta = \int \tan^3\theta\cdot(\tan^2\theta + 1)\cdot\sec^2\theta\,d\theta$

$= \int u^3(u^2+1)\,du \qquad [u = \tan\theta,\ du = \sec^2\theta\,d\theta]$

$= \int (u^5 + u^3)\,du = \frac{1}{6}u^6 + \frac{1}{4}u^4 + C = \frac{1}{6}\tan^6\theta + \frac{1}{4}\tan^4\theta + C$

34. $\int \tan^2 x \sec x\,dx = \int (\sec^2 x - 1)\sec x\,dx = \int \sec^3 x\,dx - \int \sec x\,dx$

$= \frac{1}{2}(\sec x \tan x + \ln|\sec x + \tan x|) - \ln|\sec x + \tan x| + C$ [by Example 8 and (1)]

$= \frac{1}{2}(\sec x \tan x - \ln|\sec x + \tan x|) + C$

35. $\int_{\pi/6}^{\pi/2} \cot^2 x\,dx = \int_{\pi/6}^{\pi/2} (\csc^2 x - 1)\,dx = [-\cot x - x]_{\pi/6}^{\pi/2} = \left(0 - \frac{\pi}{2}\right) - \left(-\sqrt{3} - \frac{\pi}{6}\right) = \sqrt{3} - \frac{\pi}{3}$

36. $\int_{\pi/4}^{\pi/2} \cot^3 x\,dx = \int_{\pi/4}^{\pi/2} \cot x\,(\csc^2 x - 1)\,dx = \int_{\pi/4}^{\pi/2} \cot x \csc^2 x\,dx - \int_{\pi/4}^{\pi/2} \frac{\cos x}{\sin x}\,dx$

$= \left[-\frac{1}{2}\cot^2 x - \ln|\sin x|\right]_{\pi/4}^{\pi/2} = (0 - \ln 1) - \left[-\frac{1}{2} - \ln\frac{1}{\sqrt{2}}\right] = \frac{1}{2} + \ln\frac{1}{\sqrt{2}} = \frac{1}{2}(1 - \ln 2)$

37. $\int \cot^3 \alpha \csc^3 \alpha\,d\alpha = \int \cot^2 \alpha \csc^2 \alpha \cdot \csc\alpha \cot\alpha\,d\alpha = \int (\csc^2 \alpha - 1)\csc^2 \alpha \cdot \csc\alpha\cot\alpha\,d\alpha$

$= \int (u^2 - 1)u^2 \cdot (-du)$ $[u = \csc\alpha,\ du = -\csc\alpha\cot\alpha\,d\alpha]$

$= \int (u^2 - u^4)\,du = \frac{1}{3}u^3 - \frac{1}{5}u^5 + C = \frac{1}{3}\csc^3\alpha - \frac{1}{5}\csc^5\alpha + C$

38. $\int \csc^4 x \cot^6 x\,dx = \int \cot^6 x\,(\cot^2 x + 1)\csc^2 x\,dx$

$= \int u^6(u^2 + 1) \cdot (-du)$ $[u = \cot x,\ du = -\csc^2 x\,dx]$

$= \int (-u^8 - u^6)\,du = -\frac{1}{9}u^9 - \frac{1}{7}u^7 + C = -\frac{1}{9}\cot^9 x - \frac{1}{7}\cot^7 x + C$

39. $I = \displaystyle\int \csc x\,dx = \int \frac{\csc x\,(\csc x - \cot x)}{\csc x - \cot x}\,dx = \int \frac{-\csc x \cot x + \csc^2 x}{\csc x - \cot x}\,dx$. Let $u = \csc x - \cot x \Rightarrow$ $du = (-\csc x \cot x + \csc^2 x)\,dx$. Then $I = \int du/u = \ln|u| = \ln|\csc x - \cot x| + C$.

40. Let $u = \csc x$, $dv = \csc^2 x\,dx$. Then $du = -\csc x \cot x\,dx$, $v = -\cot x \Rightarrow$

$$\int \csc^3 x\,dx = -\csc x \cot x - \int \csc x \cot^2 x\,dx = -\csc x \cot x - \int \csc x\,(\csc^2 x - 1)\,dx$$

$$= -\csc x \cot x + \int \csc x\,dx - \int \csc^3 x\,dx$$

Solving for $\int \csc^3 x\,dx$ and using Exercise 39, we get $\int \csc^3 x\,dx = -\frac{1}{2}\csc x \cot x + \frac{1}{2}\int \csc x\,dx = -\frac{1}{2}\csc x \cot x + \frac{1}{2}\ln|\csc x - \cot x| + C$. Thus,

$$\int_{\pi/6}^{\pi/3} \csc^3 x\,dx = \left[-\tfrac{1}{2}\csc x \cot x + \tfrac{1}{2}\ln|\csc x - \cot x|\right]_{\pi/6}^{\pi/3}$$

$$= -\tfrac{1}{2} \cdot \tfrac{2}{\sqrt{3}} \cdot \tfrac{1}{\sqrt{3}} + \tfrac{1}{2}\ln\left|\tfrac{2}{\sqrt{3}} - \tfrac{1}{\sqrt{3}}\right| + \tfrac{1}{2} \cdot 2 \cdot \sqrt{3} - \tfrac{1}{2}\ln\left|2 - \sqrt{3}\right|$$

$$= -\tfrac{1}{3} + \sqrt{3} + \tfrac{1}{2}\ln\tfrac{1}{\sqrt{3}} - \tfrac{1}{2}\ln\left(2 - \sqrt{3}\right) \approx 1.7825$$

41. Use Equation 2(b):

$\int \sin 5x \sin 2x\,dx = \int \frac{1}{2}[\cos(5x - 2x) - \cos(5x + 2x)]\,dx = \frac{1}{2}\int (\cos 3x - \cos 7x)\,dx$

$= \frac{1}{6}\sin 3x - \frac{1}{14}\sin 7x + C$

42. Use Equation 2(a):

$\int \sin 3x \cos x\,dx = \int \frac{1}{2}[\sin(3x + x) + \sin(3x - x)]\,dx = \frac{1}{2}\int (\sin 4x + \sin 2x)\,dx$

$= -\frac{1}{8}\cos 4x - \frac{1}{4}\cos 2x + C$

43. Use Equation 2(c):

$\int \cos 7\theta \cos 5\theta\,d\theta = \int \frac{1}{2}[\cos(7\theta - 5\theta) + \cos(7\theta + 5\theta)]\,d\theta = \frac{1}{2}\int (\cos 2\theta + \cos 12\theta)\,d\theta$

$= \frac{1}{2}\left(\frac{1}{2}\sin 2\theta + \frac{1}{12}\sin 12\theta\right) + C = \frac{1}{4}\sin 2\theta + \frac{1}{24}\sin 12\theta + C$

44. $\displaystyle\int \frac{\cos x + \sin x}{\sin 2x}\,dx = \frac{1}{2}\int \frac{\cos x + \sin x}{\sin x \cos x}\,dx = \frac{1}{2}\int (\csc x + \sec x)\,dx$

$= \frac{1}{2}\left(\ln|\csc x - \cot x| + \ln|\sec x + \tan x|\right) + C$ [by Exercise 39 and (1)]

45. $\displaystyle\int \frac{1 - \tan^2 x}{\sec^2 x}\,dx = \int \left(\cos^2 x - \sin^2 x\right)dx = \int \cos 2x\,dx = \frac{1}{2}\sin 2x + C$

46. $\displaystyle\int \frac{dx}{\cos x - 1} = \int \frac{1}{\cos x - 1}\cdot\frac{\cos x + 1}{\cos x + 1}\,dx = \int \frac{\cos x + 1}{\cos^2 x - 1}\,dx = \int \frac{\cos x + 1}{-\sin^2 x}\,dx$

$= \int \left(-\cot x \csc x - \csc^2 x\right)dx = \csc x + \cot x + C$

47. Let $u = \tan(t^2) \;\Rightarrow\; du = 2t\sec^2(t^2)\,dt$. Then
$\int t\sec^2(t^2)\tan^4(t^2)\;dt = \int u^4\left(\frac{1}{2}\,du\right) = \frac{1}{10}u^5 + C = \frac{1}{10}\tan^5(t^2) + C.$

48. Let $u = \tan^7 x$, $dv = \sec x\,\tan x\,dx \;\Rightarrow\; du = 7\tan^6 x\,\sec^2 x\,dx$, $v = \sec x$. Then
$\int \tan^8 x\,\sec x\,dx = \int \tan^7 x\cdot\sec x\,\tan x\,dx = \tan^7 x\,\sec x - \int 7\tan^6 x\sec^2 x\sec x\,dx$

$= \tan^7 x\,\sec x - 7\int \tan^6 x\left(\tan^2 x + 1\right)\sec x\,dx$

$= \tan^7 x\,\sec x - 7\int \tan^8 x\sec x\,dx - 7\int \tan^6 x\sec x\,dx.$

Thus, $8\int \tan^8 x\sec x\,dx = \tan^7 x\,\sec x - 7\int \tan^6 x\sec x\,dx$ and

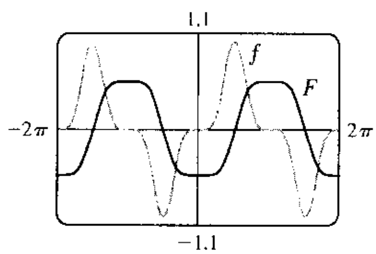
$\displaystyle\int_0^{\pi/4} \tan^8 x\sec x\,dx = \frac{1}{8}\left[\tan^7 x\,\sec x\right]_0^{\pi/4} - \frac{7}{8}\int_0^{\pi/4}\tan^6 x\sec x\,dx = \frac{\sqrt{2}}{8} - \frac{7}{8}I.$

49. Let $u = \cos x \;\Rightarrow\; du = -\sin x\,dx$. Then

$\int \sin^5 x\,dx = \int \left(1 - \cos^2 x\right)^2 \sin x\,dx = \int \left(1 - u^2\right)^2(-du)$

$= \int \left(-1 + 2u^2 - u^4\right)du = -\frac{1}{5}u^5 + \frac{2}{3}u^3 - u + C$

$= -\frac{1}{5}\cos^5 x + \frac{2}{3}\cos^3 x - \cos x + C$



Notice that F is increasing when $f(x) > 0$, so the graphs serve as a check on our work.

50. $\int \sin^4 x\cos^4 x\,dx = \int \left(\frac{1}{2}\sin 2x\right)^4 dx = \frac{1}{16}\int \sin^4 2x\,dx = \frac{1}{16}\int \left[\frac{1}{2}(1 - \cos 4x)\right]^2 dx$

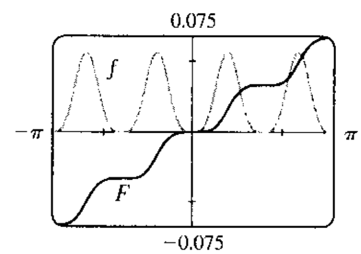
$= \frac{1}{64}\int \left(1 - 2\cos 4x + \cos^2 4x\right)dx$

$= \frac{1}{64}\left(x - \frac{1}{2}\sin 4x\right) + \frac{1}{128}\int (1 + \cos 8x)\,dx$

$= \frac{1}{64}\left(x - \frac{1}{2}\sin 4x\right) + \frac{1}{128}\left(x + \frac{1}{8}\sin 8x\right) + C$

$= \frac{3}{128}x - \frac{1}{128}\sin 4x + \frac{1}{1024}\sin 8x + C$

Notice that $f(x) = 0$ whenever F has a horizontal tangent.

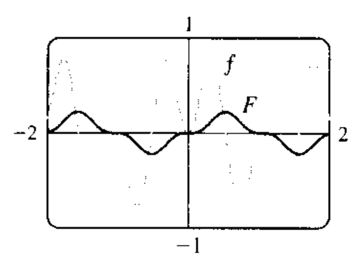


51. $\int \sin 3x\sin 6x\,dx = \int \frac{1}{2}[\cos(3x - 6x) - \cos(3x + 6x)]\,dx$

$= \frac{1}{2}\int (\cos 3x - \cos 9x)\,dx$

$= \frac{1}{6}\sin 3x - \frac{1}{18}\sin 9x + C$

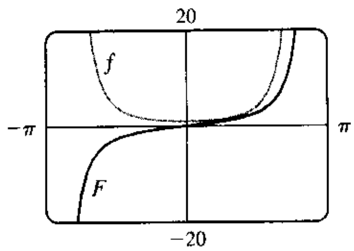
Notice that $f(x) = 0$ whenever F has a horizontal tangent.



52. $\int \sec^4 \frac{x}{2}\,dx = \int \left(\tan^2 \frac{x}{2} + 1\right) \sec^2 \frac{x}{2}\,dx$

$= \int \left(u^2 + 1\right) 2\,du \quad \left[u = \tan \frac{x}{2},\ du = \frac{1}{2}\sec^2 \frac{x}{2}\,dx\right]$

$= \frac{2}{3}u^3 + 2u + C = \frac{2}{3}\tan^3 \frac{x}{2} + 2\tan \frac{x}{2} + C$



Notice that F is increasing and f is positive on the intervals on which they are defined. Also, F has no horizontal tangent and f is never zero.

53. $f_{\text{ave}} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \sin^2 x \cos^3 x\,dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} \sin^2 x \left(1 - \sin^2 x\right) \cos x\,dx$

$= \frac{1}{2\pi} \int_0^0 u^2 \left(1 - u^2\right) du \qquad [\text{where } u = \sin x]$

$= 0$

54. (a) Let $u = \cos x$. Then $du = -\sin x\,dx \quad \Rightarrow \quad \int \sin x \cos x\,dx = \int u(-du) = -\frac{1}{2}u^2 + C = -\frac{1}{2}\cos^2 x + C_1$.

(b) Let $u = \sin x$. Then $du = \cos x\,dx \quad \Rightarrow \quad \int \sin x \cos x\,dx = \int u\,du = \frac{1}{2}u^2 + C = \frac{1}{2}\sin^2 x + C_2$.

(c) $\int \sin x \cos x\,dx = \int \frac{1}{2}\sin 2x\,dx = -\frac{1}{4}\cos 2x + C_3$

(d) Let $u = \sin x$, $dv = \cos x\,dx$. Then $du = \cos x\,dx$, $v = \sin x$, so

$\int \sin x \cos x\,dx = \sin^2 x - \int \sin x \cos x\,dx$, by Equation 8.1.2, so $\int \sin x \cos x\,dx = \frac{1}{2}\sin^2 x + C_4$.

The answers differ from one another by constants. Since $\cos 2x = 1 - 2\sin^2 x = 2\cos^2 x - 1$, we find that

$-\frac{1}{4}\cos 2x = \frac{1}{2}\sin^2 x - \frac{1}{4} = -\frac{1}{2}\cos^2 x + \frac{1}{4}$.

55. For $0 < x < \frac{\pi}{2}$, we have $0 < \sin x < 1$, so $\sin^3 x < \sin x$. Hence the area is

$\int_0^{\pi/2} \left(\sin x - \sin^3 x\right) dx = \int_0^{\pi/2} \sin x \left(1 - \sin^2 x\right) dx = \int_0^{\pi/2} \cos^2 x \sin x\,dx$. Now let $u = \cos x \quad \Rightarrow$

$du = -\sin x\,dx$. Then area $= \int_1^0 u^2\,(-du) = \int_0^1 u^2 du = \left[\frac{1}{3}u^3\right]_0^1 = \frac{1}{3}$.

56. $\sin x > 0$ for $0 < x < \frac{\pi}{2}$, so the sign of $2\sin^2 x - \sin x$ $\left[\text{which equals } 2\sin x \left(\sin x - \frac{1}{2}\right)\right]$ is the same as that of $\sin x - \frac{1}{2}$. Thus $2\sin^2 x - \sin x$ is positive on $\left(\frac{\pi}{6}, \frac{\pi}{2}\right)$ and negative on $\left(0, \frac{\pi}{6}\right)$. The desired area is

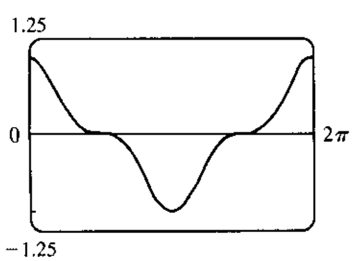
$$\int_0^{\pi/6} \left(\sin x - 2\sin^2 x\right) dx + \int_{\pi/6}^{\pi/2} \left(2\sin^2 x - \sin x\right) dx$$

$$= \int_0^{\pi/6} (\sin x - 1 + \cos 2x)\,dx + \int_{\pi/6}^{\pi/2} (1 - \cos 2x - \sin x)\,dx$$

$$= \left[-\cos x - x + \tfrac{1}{2}\sin 2x\right]_0^{\pi/6} + \left[x - \tfrac{1}{2}\sin 2x + \cos x\right]_{\pi/6}^{\pi/2}$$

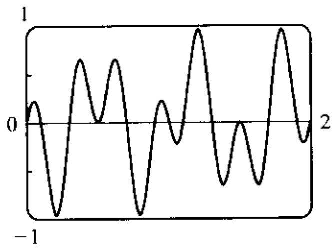
$$= -\frac{\sqrt{3}}{2} - \frac{\pi}{6} + \frac{\sqrt{3}}{4} - (-1) + \frac{\pi}{2} - \left(\frac{\pi}{6} - \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{2}\right) = 1 + \frac{\pi}{6} - \frac{\sqrt{3}}{2}$$

57.



It seems from the graph that $\int_0^{2\pi} \cos^3 x\,dx = 0$, since the area below the x-axis and above the graph looks about equal to the area above the axis and below the graph. By Example 1, the integral is $\left[\sin x - \frac{1}{3}\sin^3 x\right]_0^{2\pi} = 0$. Note that due to symmetry, the integral of any odd power of $\sin x$ or $\cos x$ between limits which differ by $2n\pi$ (n any integer) is 0.

58.



It seems from the graph that $\int_0^2 \sin 2\pi x \cos 5\pi x\,dx = 0$, since each bulge above the x-axis seems to have a corresponding depression below the x-axis. To evaluate the integral, we use a trigonometric identity:

$$\begin{aligned}\int_0^1 \sin 2\pi x \cos 5\pi x\,dx &= \tfrac{1}{2}\int_0^2 [\sin(2\pi x - 5\pi x) + \sin(2\pi x + 5\pi x)]\,dx \\ &= \tfrac{1}{2}\int_0^2 [\sin(-3\pi x) + \sin 7\pi x]\,dx \\ &= \tfrac{1}{2}\left[\tfrac{1}{3\pi}\cos(-3\pi x) - \tfrac{1}{7\pi}\cos 7\pi x\right]_0^2 \\ &= \tfrac{1}{2}\left[\tfrac{1}{3\pi}(1-1) - \tfrac{1}{7\pi}(1-1)\right] = 0\end{aligned}$$

59. $V = \int_{\pi/2}^{\pi} \pi \sin^2 x\,dx = \pi\int_{\pi/2}^{\pi} \tfrac{1}{2}(1-\cos 2x)\,dx = \pi\left[\tfrac{1}{2}x - \tfrac{1}{4}\sin 2x\right]_{\pi/2}^{\pi} = \pi\left(\tfrac{\pi}{2} - 0 - \tfrac{\pi}{4} + 0\right) = \tfrac{\pi^2}{4}$

60. Volume $= \int_0^{\pi/4} \pi(\tan^2 x)^2\,dx = \pi\int_0^{\pi/4} \tan^2 x\,(\sec^2 x - 1)\,dx = \pi\int_0^{\pi/4}\tan^2 x\sec^2 x\,dx - \pi\int_0^{\pi/4}\tan^2 x\,dx$

$= \pi\int_0^{\pi/4} u^2 du - \pi\int_0^{\pi/4}(\sec^2 x - 1)\,dx$ [where $u = \tan x$ and $du = \sec^2 x\,dx$]

$= \pi\left[\tfrac{1}{3}u^3\right]_{x=0}^{\pi/4} - \pi[\tan x - x]_0^{\pi/4} = \pi\left[\tfrac{1}{3}\tan^3 x - \tan x + x\right]_0^{\pi/4} = \pi\left[\tfrac{1}{3} - 1 + \tfrac{\pi}{4}\right] = \pi\left(\tfrac{\pi}{4} - \tfrac{2}{3}\right)$

61. Volume $= \pi\int_0^{\pi/2}\left[(1+\cos x)^2 - 1^2\right]dx = \pi\int_0^{\pi/2}\left(2\cos x + \cos^2 x\right)dx$

$= \pi\left[2\sin x + \tfrac{1}{2}x + \tfrac{1}{4}\sin 2x\right]_0^{\pi/2} = \pi\left(2 + \tfrac{\pi}{4}\right) = 2\pi + \tfrac{\pi^2}{4}$

62. Volume $= \pi\int_0^{\pi/2}\left[1^2 - (1-\cos x)^2\right]dx = \pi\int_0^{\pi/2}\left(2\cos x - \cos^2 x\right)dx$

$= \pi\left[2\sin x - \tfrac{1}{2}x - \tfrac{1}{4}\sin 2x\right]_0^{\pi/2} = \pi\left[\left(2 - \tfrac{\pi}{4} - 0\right) - 0\right] = 2\pi - \tfrac{\pi^2}{4}$

63. $s = f(t) = \int_0^t \sin\omega u\cos^2\omega u\,du$. Let $y = \cos\omega u \Rightarrow dy = -\omega\sin\omega u\,du$. Then

$s = -\tfrac{1}{\omega}\int_1^{\cos\omega t} y^2 dy = -\tfrac{1}{\omega}\left[\tfrac{1}{3}y^3\right]_1^{\cos\omega t} = \tfrac{1}{3\omega}\left(1 - \cos^3\omega t\right)$.

64. (a) We want to calculate the square root of the average value of $[E(t)]^2 = [155\sin(120\pi t)]^2 = 155^2\sin^2(120\pi t)$.

First, we calculate the average value itself, by integrating $[E(t)]^2$ over one cycle (between $t = 0$ and $t = \tfrac{1}{60}$, since there are 60 cycles per second) and dividing by $\left(\tfrac{1}{60} - 0\right)$:

$$\begin{aligned}[E(t)]^2_{\text{ave}} &= \tfrac{1}{1/60}\int_0^{1/60}\left[155^2\sin^2(120\pi t)\right]dt = 60\cdot 155^2\int_0^{1/60}\tfrac{1}{2}[1-\cos(240\pi t)]\,dt \\ &= 60\cdot 155^2\left(\tfrac{1}{2}\right)\left[t - \tfrac{1}{240\pi}\sin(240\pi t)\right]_0^{1/60} = 60\cdot 155^2\left(\tfrac{1}{2}\right)\left[\left(\tfrac{1}{60} - 0\right) - (0-0)\right] = \tfrac{155^2}{2}\end{aligned}$$

The RMS value is just the square root of this quantity, which is $\tfrac{155}{\sqrt{2}} \approx 110$ V.

(b) $220 = \sqrt{[E(t)]^2_{\text{ave}}} \Rightarrow$

$$\begin{aligned}220^2 = [E(t)]^2_{\text{ave}} &= \tfrac{1}{1/60}\int_0^{1/60} A^2\sin^2(120\pi t)\,dt = 60A^2\int_0^{1/60}\tfrac{1}{2}[1-\cos(240\pi t)]\,dt \\ &= 30A^2\left[t - \tfrac{1}{240\pi}\sin(240\pi t)\right]_0^{1/60} = 30A^2\left[\left(\tfrac{1}{60} - 0\right) - (0-0)\right] = \tfrac{1}{2}A^2\end{aligned}$$

Thus, $220^2 = \tfrac{1}{2}A^2 \Rightarrow A = 220\sqrt{2} \approx 311$ V.

65. Just note that the integrand is odd $[f(-x) = -f(x)]$.

Or: If $m \neq n$, calculate

$$\int_{-\pi}^{\pi} \sin mx \cos nx\, dx = \int_{-\pi}^{\pi} \tfrac{1}{2}[\sin(m-n)x + \sin(m+n)x]\, dx$$
$$= \frac{1}{2}\left[-\frac{\cos(m-n)x}{m-n} - \frac{\cos(m+n)x}{m+n}\right]_{-\pi}^{\pi} = 0$$

If $m = n$, then the first term in each set of brackets is zero.

66. $\int_{-\pi}^{\pi} \sin mx \sin nx\, dx = \int_{-\pi}^{\pi} \frac{1}{2}[\cos(m-n)x - \cos(m+n)x]\, dx$. If $m \neq n$, this is equal to $\frac{1}{2}\left[\frac{\sin(m-n)x}{m-n} - \frac{\sin(m+n)x}{m+n}\right]_{-\pi}^{\pi} = 0$. If $m = n$, we get $\int_{-\pi}^{\pi} \frac{1}{2}[1 - \cos(m+n)x]\, dx = \left[\frac{1}{2}x\right]_{-\pi}^{\pi} - \left[\frac{\sin(m+n)x}{2(m+n)}\right]_{-\pi}^{\pi} = \pi - 0 = \pi$.

67. $\int_{-\pi}^{\pi} \cos mx \cos nx\, dx = \int_{-\pi}^{\pi} \frac{1}{2}[\cos(m-n)x + \cos(m+n)x]\, dx$. If $m \neq n$, this is equal to $\frac{1}{2}\left[\frac{\sin(m-n)x}{m-n} + \frac{\sin(m+n)x}{m+n}\right]_{-\pi}^{\pi} = 0$. If $m = n$, we get $\int_{-\pi}^{\pi} \frac{1}{2}[1 + \cos(m+n)x]\, dx = \left[\frac{1}{2}x\right]_{-\pi}^{\pi} + \left[\frac{\sin(m+n)x}{2(m+n)}\right]_{-\pi}^{\pi} = \pi + 0 = \pi$.

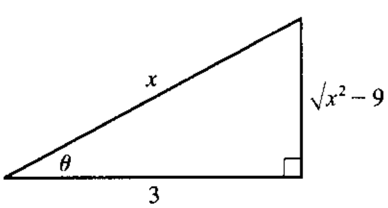
68. $\displaystyle \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin mx\, dx = \frac{1}{\pi}\int_{-\pi}^{\pi}\left[\left(\sum_{n=1}^{m} a_n \sin nx\right)\sin mx\right] dx = \sum_{n=1}^{m} \frac{a_n}{\pi}\int_{-\pi}^{\pi} \sin mx \sin nx\, dx.$

By Exercise 66, every term is zero except the mth one, and that term is $\frac{a_m}{\pi} \cdot \pi = a_m$.

8.3 Trigonometric Substitution

1. Let $x = 3\sec\theta$, where $0 \le \theta < \frac{\pi}{2}$ or $\pi \le \theta < \frac{3\pi}{2}$. Then $dx = 3\sec\theta\tan\theta\, d\theta$ and

$$\sqrt{x^2 - 9} = \sqrt{9\sec^2\theta - 9} = \sqrt{9(\sec^2\theta - 1)} = \sqrt{9\tan^2\theta}$$
$$= 3\,|\tan\theta| = 3\tan\theta \text{ for the relevant values of } \theta.$$



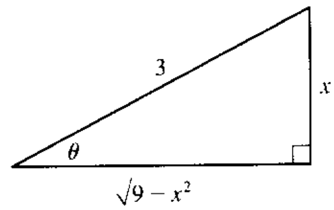
$$\int \frac{1}{x^2\sqrt{x^2-9}}\, dx = \int \frac{1}{9\sec^2\theta \cdot 3\tan\theta} 3\sec\theta\tan\theta\, d\theta = \tfrac{1}{9}\int \cos\theta\, d\theta = \tfrac{1}{9}\sin\theta + C = \frac{1}{9}\frac{\sqrt{x^2-9}}{x} + C$$

Note that $-\sec(\theta + \pi) = \sec\theta$, so the figure is sufficient for the case $\pi \le \theta < \frac{3\pi}{2}$.

2. Let $x = 3\sin\theta$, where $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. Then $dx = 3\cos\theta\, d\theta$ and

$$\sqrt{9-x^2} = \sqrt{9 - 9\sin^2\theta} = \sqrt{9(1-\sin^2\theta)} = \sqrt{9\cos^2\theta}$$

$$= 3\,|\cos\theta| = 3\cos\theta \text{ for the relevant values of } \theta.$$



$$\int x^3\sqrt{9-x^2}\,dx = \int 3^3\sin^3\theta \cdot 3\cos\theta\cdot 3\cos\theta\,d\theta = 3^5\int \sin^3\theta\cos^2\theta\,d\theta$$

$$= 3^5\int \sin^2\theta\cos^2\theta\sin\theta\,d\theta = 3^5\int\left(1-\cos^2\theta\right)\cos^2\theta\sin\theta\,d\theta$$

$$= 3^5\int\left(1-u^2\right)u^2\,(-du) \qquad [u = \cos\theta,\ du = -\sin\theta\,d\theta]$$

$$= 3^5\int\left(u^4-u^2\right)du = 3^5\left(\tfrac{1}{5}u^5 - \tfrac{1}{3}u^3\right) + C = 3^5\left(\tfrac{1}{5}\cos^5\theta - \tfrac{1}{3}\cos^3\theta\right) + C$$

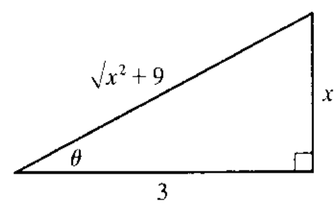
$$= 3^5\left[\frac{1}{5}\frac{\left(9-x^2\right)^{5/2}}{3^5} - \frac{1}{3}\frac{\left(9-x^2\right)^{3/2}}{3^3}\right] + C$$

$$= \tfrac{1}{5}\left(9-x^2\right)^{5/2} - 3\left(9-x^2\right)^{3/2} + C \text{ or } -\tfrac{1}{5}\left(x^2+6\right)(9-x)^{3/2} + C$$

3. Let $x = 3\tan\theta$, where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. Then $dx = 3\sec^2\theta\,d\theta$ and

$$\sqrt{x^2+9} = \sqrt{9\tan^2\theta + 9} = \sqrt{9(\tan^2\theta+1)} = \sqrt{9\sec^2\theta}$$

$$= 3\,|\sec\theta| = 3\sec\theta \text{ for the relevant values of } \theta.$$



$$\int\frac{x^3}{\sqrt{x^2+9}}\,dx = \int\frac{3^3\tan^3\theta}{3\sec\theta}3\sec^2\theta\,d\theta = 3^3\int\tan^3\theta\sec\theta d\theta = 3^3\int\tan^2\theta\tan\theta\sec\theta\,d\theta$$

$$= 3^3\int\left(\sec^2\theta - 1\right)\tan\theta\sec\theta\,d\theta = 3^3\int\left(u^2-1\right)du \qquad [u = \sec\theta,\ du = \sec\theta\tan\theta\,d\theta]$$

$$= 3^3\left(\tfrac{1}{3}u^3 - u\right) + C = 3^3\left(\tfrac{1}{3}\sec^3\theta - \sec\theta\right) + C = 3^3\left[\frac{1}{3}\frac{\left(x^2+9\right)^{3/2}}{3^3} - \frac{\sqrt{x^2+9}}{3}\right] + C$$

$$= \tfrac{1}{3}\left(x^2+9\right)^{3/2} - 9\sqrt{x^2+9} + C \quad\text{or}\quad \tfrac{1}{3}\left(x^2-18\right)\sqrt{x^2+9} + C$$

4. Let $x = 4\sin\theta$, where $-\pi/2 \le \theta \le \pi/2$. Then $dx = 4\cos\theta\,d\theta$ and $\sqrt{16-x^2} = \sqrt{16-16\sin^2\theta} = \sqrt{16\cos^2\theta} = 4\,|\cos\theta| = 4\cos\theta$. When $x = 0$, $4\sin\theta = 0 \;\Rightarrow\; \theta = 0$, and when $x = 2\sqrt{3}$, $4\sin\theta = 2\sqrt{3} \;\Rightarrow\; \sin\theta = \frac{\sqrt{3}}{2} \;\Rightarrow\; \theta = \frac{\pi}{3}$. Thus, substitution gives

$$\int_0^{2\sqrt{3}}\frac{x^3}{\sqrt{16-x^2}}\,dx = \int_0^{\pi/3}\frac{4^3\sin^3\theta}{4\cos\theta}4\cos\theta\,d\theta = 4^3\int_0^{\pi/3}\sin^3\theta\,d\theta$$

$$= 4^3\int_0^{\pi/3}\left(1-\cos^2\theta\right)\sin\theta\,d\theta$$

$$\overset{c}{=} -4^3\int_1^{1/2}\left(1-u^2\right)du = -64\left[u - \tfrac{1}{3}u^3\right]_1^{1/2}$$

$$= -64\left[\left(\tfrac{1}{2} - \tfrac{1}{24}\right) - \left(1 - \tfrac{1}{3}\right)\right] = -64\left(-\tfrac{5}{24}\right) = \tfrac{40}{3}$$

Or: Let $u = 16 - x^2$, $x^2 = 16 - u$, $du = -2x\,dx$.

5. Let $t = \sec\theta$, so $dt = \sec\theta\tan\theta\,d\theta$, $t = \sqrt{2} \;\Rightarrow\; \theta = \frac{\pi}{4}$, and $t = 2 \;\Rightarrow\; \theta = \frac{\pi}{3}$. Then

$$\int_{\sqrt{2}}^{2} \frac{1}{t^3\sqrt{t^2-1}}\,dt = \int_{\pi/4}^{\pi/3} \frac{1}{\sec^3\theta\tan\theta}\sec\theta\tan\theta\,d\theta = \int_{\pi/4}^{\pi/3}\frac{1}{\sec^2\theta}\,d\theta = \int_{\pi/4}^{\pi/3}\cos^2\theta\,d\theta$$

$$= \int_{\pi/4}^{\pi/3}\tfrac{1}{2}(1+\cos 2\theta)\,d\theta = \tfrac{1}{2}\left[\theta + \tfrac{1}{2}\sin 2\theta\right]_{\pi/4}^{\pi/3}$$

$$= \tfrac{1}{2}\left[\left(\tfrac{\pi}{3} + \tfrac{1}{2}\tfrac{\sqrt{3}}{2}\right) - \left(\tfrac{\pi}{4} + \tfrac{1}{2}\cdot 1\right)\right] = \tfrac{1}{2}\left(\tfrac{\pi}{12} + \tfrac{\sqrt{3}}{4} - \tfrac{1}{2}\right) = \tfrac{\pi}{24} + \tfrac{\sqrt{3}}{8} - \tfrac{1}{4}$$

6. Let $x = 2\tan\theta$, so $dx = 2\sec^2\theta\,d\theta$, $x = 0 \;\Rightarrow\; \theta = 0$, and $x = 2 \;\Rightarrow\; \theta = \frac{\pi}{4}$. Then

$$\int_0^2 x^3\sqrt{x^2+4}\,dx = \int_0^{\pi/4} 2^3\tan^3\theta\cdot 2\sec\theta\cdot 2\sec^2\theta\,d\theta = 2^5\int_0^{\pi/4}\tan^2\theta\sec^2\theta\sec\theta\tan\theta\,d\theta$$

$$= 2^5\int_0^{\pi/4}(\sec^2\theta - 1)\,\sec^2\theta\sec\theta\tan\theta\,d\theta$$

$$= 2^5\int_1^{\sqrt{2}}(u^2-1)u^2\,du \quad [u = \sec\theta,\, du = \sec\theta\tan\theta\,d\theta]$$

$$= 2^5\int_1^{\sqrt{2}}(u^4-u^2)\,du = 2^5\left[\tfrac{1}{5}u^5 - \tfrac{1}{3}u^3\right]_1^{\sqrt{2}} = 2^5\left[\left(\tfrac{1}{5}\cdot 4\sqrt{2} - \tfrac{1}{3}\cdot 2\sqrt{2}\right) - \left(\tfrac{1}{5} - \tfrac{1}{3}\right)\right]$$

$$= 32\left(\tfrac{2}{15}\sqrt{2} + \tfrac{2}{15}\right) = \tfrac{64}{15}\left(\sqrt{2}+1\right)$$

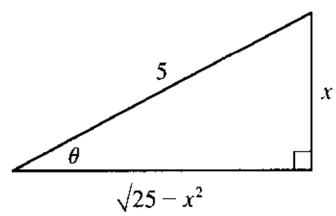
Or: Let $u = x^2 + 4$, $x^2 = u - 4$, $du = 2x\,dx$.

7. Let $x = 5\sin\theta$, so $dx = 5\cos\theta\,d\theta$. Then

$$\int\frac{1}{x^2\sqrt{25-x^2}}\,dx = \int\frac{1}{5^2\sin^2\theta\cdot 5\cos\theta}5\cos\theta\,d\theta$$

$$= \tfrac{1}{25}\int\csc^2\theta\,d\theta = -\tfrac{1}{25}\cot\theta + C$$

$$= -\frac{1}{25}\frac{\sqrt{25-x^2}}{x} + C$$



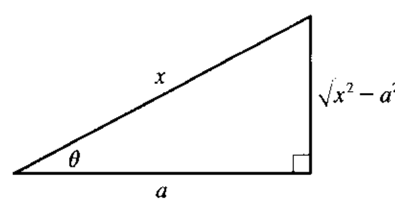
8. Let $x = a\sec\theta$, where $0 \le \theta < \frac{\pi}{2}$ or $\pi \le \theta < \frac{3\pi}{2}$. Then

$dx = a\sec\theta\tan\theta\,d\theta$ and $\sqrt{x^2-a^2} = a\tan\theta$, so

$$\int\frac{\sqrt{x^2-a^2}}{x^4}dx = \int\frac{a\tan\theta}{a^4\sec^4\theta}a\sec\theta\tan\theta\,d\theta$$

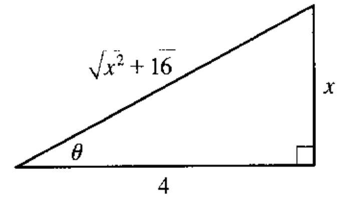
$$= \frac{1}{a^2}\int\sin^2\theta\cos\theta\,d\theta$$

$$= \frac{1}{3a^2}\sin^3\theta + C = \frac{\left(x^2-a^2\right)^{3/2}}{3a^2x^3} + C$$



9. Let $x = 4\tan\theta$, where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. Then $dx = 4\sec^2\theta\,d\theta$ and

$$\begin{aligned}\sqrt{x^2+16} &= \sqrt{16\tan^2\theta + 16} = \sqrt{16(\tan^2\theta+1)}\\ &= \sqrt{16\sec^2\theta} = 4\,|\sec\theta|\\ &= 4\sec\theta \text{ for the relevant values of } \theta.\end{aligned}$$

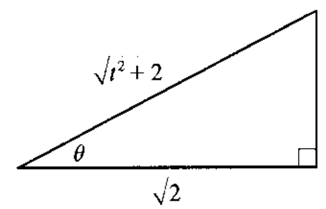


$$\begin{aligned}\int \frac{dx}{\sqrt{x^2+16}} &= \int \frac{4\sec^2\theta\,d\theta}{4\sec\theta} = \int \sec\theta\,d\theta = \ln|\sec\theta + \tan\theta| + C_1\\ &= \ln\left|\frac{\sqrt{x^2+16}}{4} + \frac{x}{4}\right| + C_1 = \ln\left|\sqrt{x^2+16}+x\right| - \ln|4| + C_1\\ &= \ln\left(\sqrt{x^2+16}+x\right) + C, \text{ where } C = C_1 - \ln 4.\end{aligned}$$

(Since $\sqrt{x^2+16}+x > 0$, we don't need the absolute value.)

10. Let $t = \sqrt{2}\tan\theta$, where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. Then $dt = \sqrt{2}\sec^2\theta\,d\theta$ and

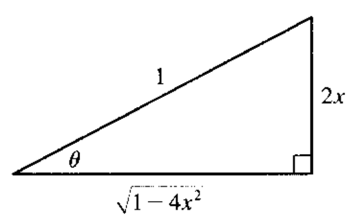
$$\begin{aligned}\sqrt{t^2+2} &= \sqrt{2\tan^2\theta+2} = \sqrt{2(\tan^2\theta+1)} = \sqrt{2\sec^2\theta}\\ &= \sqrt{2}\,|\sec\theta| = \sqrt{2}\sec\theta \text{ for the relevant values of } \theta.\end{aligned}$$



$$\begin{aligned}\int \frac{t^5}{\sqrt{t^2+2}}\,dt &= \int \frac{4\sqrt{2}\tan^5\theta}{\sqrt{2}\sec\theta}\sqrt{2}\sec^2\theta\,d\theta = 4\sqrt{2}\int \tan^5\theta\sec\theta\,d\theta = 4\sqrt{2}\int\left(\sec^2\theta-1\right)^2\sec\theta\tan\theta\,d\theta\\ &= 4\sqrt{2}\int\left(u^2-1\right)^2du \quad [u=\sec\theta,\ du=\sec\theta\tan\theta\,d\theta] \quad = 4\sqrt{2}\int\left(u^4-2u^2+1\right)du\\ &= 4\sqrt{2}\left(\frac{1}{5}u^5 - \frac{2}{3}u^3 + u\right) + C = \frac{4\sqrt{2}}{15}u(3u^4-10u^2+15) + C\\ &= \frac{4\sqrt{2}}{15}\cdot\frac{\sqrt{t^2+2}}{\sqrt{2}}\left[3\cdot\frac{(t^2+2)^2}{2^2} - 10\,\frac{t^2+2}{2} + 15\right] + C\\ &= \tfrac{4}{15}\sqrt{t^2+2}\cdot\tfrac{1}{4}\left[3(t^4+4t^2+4) - 20(t^2+2) + 60\right] + C\\ &= \tfrac{1}{15}\sqrt{t^2+2}\,(3t^4-8t^2+32) + C\end{aligned}$$

11. Let $2x = \sin\theta$, where $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. Then $x = \frac{1}{2}\sin\theta$,

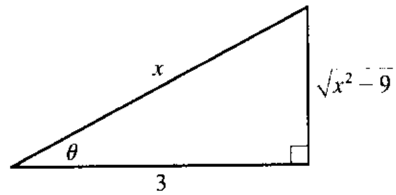
$dx = \frac{1}{2}\cos\theta\,d\theta$, and $\sqrt{1-4x^2} = \sqrt{1-(2x)^2} = \cos\theta$.



$$\begin{aligned}\int\sqrt{1-4x^2}\,dx &= \int\cos\theta\left(\tfrac{1}{2}\cos\theta\right)d\theta = \tfrac{1}{4}\int(1+\cos 2\theta)\,d\theta\\ &= \tfrac{1}{4}\left(\theta + \tfrac{1}{2}\sin 2\theta\right) + C = \tfrac{1}{4}(\theta + \sin\theta\cos\theta) + C\\ &= \tfrac{1}{4}\left[\sin^{-1}(2x) + 2x\sqrt{1-4x^2}\right] + C\end{aligned}$$

12. $\int_0^1 x\sqrt{x^2+4}\,dx = \int_4^5 \sqrt{u}\left(\frac{1}{2}\,du\right) \quad [u = x^2+4,\ du = 2x\,dx] \quad = \frac{1}{2}\cdot\frac{2}{3}\left[u^{3/2}\right]_4^5 = \frac{1}{3}\left(5\sqrt{5}-8\right)$

13. Let $x = 3\sec\theta$, where $0 \le \theta < \frac{\pi}{2}$ or $\pi \le \theta < \frac{3\pi}{2}$. Then $dx = 3\sec\theta\tan\theta\,d\theta$ and $\sqrt{x^2-9} = 3\tan\theta$, so

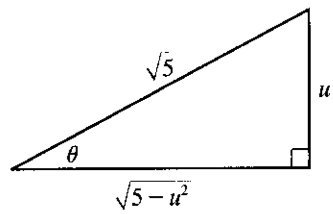


$$\int \frac{\sqrt{x^2-9}}{x^3}\,dx = \int \frac{3\tan\theta}{27\sec^3\theta}\,3\sec\theta\tan\theta\,d\theta = \frac{1}{3}\int\frac{\tan^2\theta}{\sec^2\theta}d\theta$$

$$= \tfrac{1}{3}\int \sin^2\theta\,d\theta = \tfrac{1}{3}\int\tfrac{1}{2}(1-\cos 2\theta)\,d\theta = \tfrac{1}{6}\theta - \tfrac{1}{12}\sin 2\theta + C = \tfrac{1}{6}\theta - \tfrac{1}{6}\sin\theta\cos\theta + C$$

$$= \frac{1}{6}\sec^{-1}\left(\frac{x}{3}\right) - \frac{1}{6}\frac{\sqrt{x^2-9}}{x}\frac{3}{x} + C = \frac{1}{6}\sec^{-1}\left(\frac{x}{3}\right) - \frac{\sqrt{x^2-9}}{2x^2} + C$$

14. Let $u = \sqrt{5}\sin\theta$, so $du = \sqrt{5}\cos\theta\,d\theta$. Then



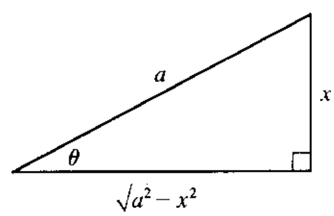
$$\int \frac{du}{u\sqrt{5-u^2}} = \int \frac{1}{\sqrt{5}\sin\theta\cdot\sqrt{5}\cos\theta}\sqrt{5}\cos\theta\,d\theta = \frac{1}{\sqrt{5}}\int\csc\theta\,d\theta$$

$$= \tfrac{1}{\sqrt{5}}\ln|\csc\theta - \cot\theta| + C \quad \text{[by Exercise 8.2.39]}$$

$$= \frac{1}{\sqrt{5}}\ln\left|\frac{\sqrt{5}}{u} - \frac{\sqrt{5-u^2}}{u}\right| + C$$

$$= \frac{1}{\sqrt{5}}\ln\left|\frac{\sqrt{5}-\sqrt{5-u^2}}{u}\right| + C$$

15. Let $x = a\sin\theta$, where $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. Then $dx = a\cos\theta\,d\theta$ and

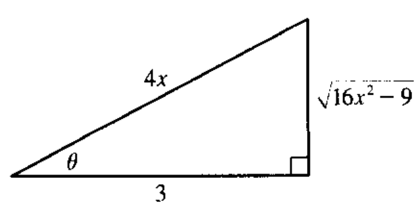


$$\int \frac{x^2\,dx}{(a^2-x^2)^{3/2}} = \int \frac{a^2\sin^2\theta\,a\cos\theta\,d\theta}{a^3\cos^3\theta} = \int\tan^2\theta\,d\theta$$

$$= \int\left(\sec^2\theta - 1\right)d\theta = \tan\theta - \theta + C$$

$$= \frac{x}{\sqrt{a^2-x^2}} - \sin^{-1}\frac{x}{a} + C$$

16. Let $4x = 3\sec\theta$, where $0 \le \theta < \frac{\pi}{2}$ or $\pi \le \theta < \frac{3\pi}{2}$. Then $dx = \frac{3}{4}\sec\theta\tan\theta\,d\theta$ and $\sqrt{16x^2-9} = 3\tan\theta$, so



$$\int \frac{dx}{x^2\sqrt{16x^2-9}} = \int \frac{\frac{3}{4}\sec\theta\tan\theta d\theta}{\left(\frac{3}{4}\right)^2\sec^2\theta\,3\tan\theta}$$

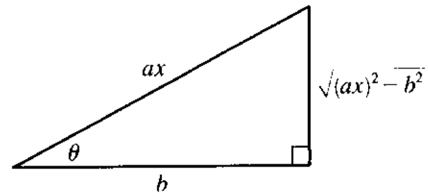
$$= \frac{4}{9}\int\cos\theta\,d\theta = \frac{4}{9}\sin\theta + C = \frac{4}{9}\frac{\sqrt{16x^2-9}}{4x} + C = \frac{\sqrt{16x^2-9}}{9x} + C$$

17. Let $u = x^2 - 7$, so $du = 2x\,dx$. Then $\displaystyle\int \frac{x}{\sqrt{x^2-7}}\,dx = \frac{1}{2}\int\frac{1}{\sqrt{u}}\,du = \tfrac{1}{2}\cdot 2\sqrt{u} + C = \sqrt{x^2-7} + C.$

18. Let $ax = b\sec\theta$, so $(ax)^2 = b^2\sec^2\theta \quad\Rightarrow$

$(ax)^2 - b^2 = b^2\sec^2\theta - b^2 = b^2(\sec^2\theta - 1) = b^2\tan^2\theta$. So

$\sqrt{(ax^2) - b^2} = b\tan\theta$, $dx = \dfrac{b}{a}\sec\theta\tan\theta\,d\theta$, and

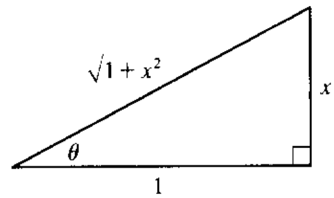


$$\int\frac{dx}{\left[(ax)^2 - b^2\right]^{3/2}} = \int\frac{\frac{b}{a}\sec\theta\tan\theta}{b^3\tan^3\theta}\,d\theta = \frac{1}{ab^2}\int\frac{\sec\theta}{\tan^2\theta}\,d\theta = \frac{1}{ab^2}\int\frac{\cos\theta}{\sin^2\theta}\,d\theta = \frac{1}{ab^2}\int\csc\theta\cot\theta\,d\theta$$

$$= -\frac{1}{ab^2}\csc\theta + C = -\frac{1}{ab^2}\frac{ax}{\sqrt{(ax)^2 - b^2}} + C = -\frac{x}{b^2\sqrt{(ax)^2 - b^2}} + C$$

19. Let $x = \tan\theta$, where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. Then $dx = \sec^2\theta\,d\theta$

and $\sqrt{1 + x^2} = \sec\theta$, so



$$\int\frac{\sqrt{1+x^2}}{x}\,dx = \int\frac{\sec\theta}{\tan\theta}\sec^2\theta\,d\theta = \int\frac{\sec\theta}{\tan\theta}(1 + \tan^2\theta)\,d\theta$$

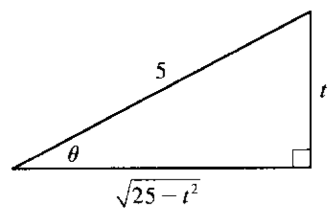
$$= \int(\csc\theta + \sec\theta\tan\theta)\,d\theta$$

$$= \ln|\csc\theta - \cot\theta| + \sec\theta + C \quad \text{[by Exercise 8.2.39]}$$

$$= \ln\left|\frac{\sqrt{1+x^2}}{x} - \frac{1}{x}\right| + \frac{\sqrt{1+x^2}}{1} + C = \ln\left|\frac{\sqrt{1+x^2} - 1}{x}\right| + \sqrt{1+x^2} + C$$

20. Let $t = 5\sin\theta$, where $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. Then $dt = 5\cos\theta\,d\theta$

and $\sqrt{25 - t^2} = 5\cos\theta$, so



$$\int\frac{t}{\sqrt{25 - t^2}}\,dt = \int\frac{5\sin\theta}{5\cos\theta}5\cos\theta\,d\theta = 5\int\sin\theta\,d\theta$$

$$= -5\cos\theta + C = -5\cdot\frac{\sqrt{25 - t^2}}{5} + C = -\sqrt{25 - t^2} + C$$

Or: Let $u = 25 - t^2$, so $du = -2t\,dt$.

21. Let $u = 4 - 9x^2 \quad\Rightarrow\quad du = -18x\,dx$. Then $x^2 = \frac{1}{9}(4 - u)$ and

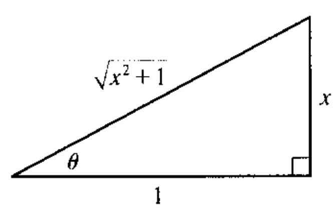
$$\int_0^{2/3} x^3\sqrt{4 - 9x^2}\,dx = \int_4^0 \tfrac{1}{9}(4 - u)u^{1/2}\left(-\tfrac{1}{18}\right)du = \tfrac{1}{162}\int_0^4\left(4u^{1/2} - u^{3/2}\right)du$$

$$= \tfrac{1}{162}\left[\tfrac{8}{3}u^{3/2} - \tfrac{2}{5}u^{5/2}\right]_0^4 = \tfrac{1}{162}\left[\tfrac{64}{3} - \tfrac{64}{5}\right] = \tfrac{64}{1215}$$

Or: Let $3x = 2\sin\theta$, where $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$.

22. Let $x = \tan\theta$, where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. Then $dx = \sec^2\theta\,d\theta$,

$\sqrt{x^2 + 1} = \sec\theta$ and $x = 0 \quad\Rightarrow\quad \theta = 0$, $x = 1 \quad\Rightarrow\quad \theta = \frac{\pi}{4}$, so

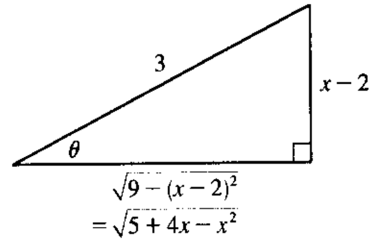


$$\int_0^1\sqrt{x^2 + 1}\,dx = \int_0^{\pi/4}\sec\theta\sec^2\theta\,d\theta = \int_0^{\pi/4}\sec^3\theta\,d\theta$$

$$= \tfrac{1}{2}\Big[\sec\theta\tan\theta + \ln|\sec\theta + \tan\theta|\Big]_0^{\pi/4} \quad \text{[by Example 8.2.8]}$$

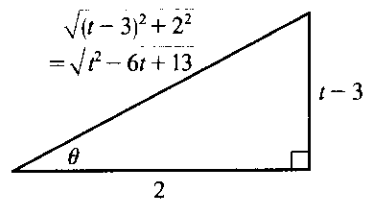
$$= \tfrac{1}{2}\left[\sqrt{2}\cdot 1 + \ln\left(1 + \sqrt{2}\right) - 0 - \ln(1 + 0)\right] = \tfrac{1}{2}\left[\sqrt{2} + \ln\left(1 + \sqrt{2}\right)\right]$$

23. $5+4x-x^2=-(x^2-4x+4)+9=-(x-2)^2+9$. Let $x-2=3\sin\theta$, $-\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$, so $dx=3\cos\theta\,d\theta$. Then



$$\begin{aligned}\int\sqrt{5+4x-x^2}\,dx&=\int\sqrt{9-(x-2)^2}\,dx=\int\sqrt{9-9\sin^2\theta}\,3\cos\theta\,d\theta\\&=\int\sqrt{9\cos^2\theta}\,3\cos\theta\,d\theta=\int 9\cos^2\theta\,d\theta\\&=\tfrac{9}{2}\int(1+\cos 2\theta)\,d\theta=\tfrac{9}{2}\left(\theta+\tfrac{1}{2}\sin 2\theta\right)+C\\&=\tfrac{9}{2}\theta+\tfrac{9}{4}\sin 2\theta+C=\tfrac{9}{2}\theta+\tfrac{9}{4}(2\sin\theta\cos\theta)+C\\&=\frac{9}{2}\sin^{-1}\left(\frac{x-2}{3}\right)+\frac{9}{2}\cdot\frac{x-2}{3}\cdot\frac{\sqrt{5+4x-x^2}}{3}+C\\&=\frac{9}{2}\sin^{-1}\left(\frac{x-2}{3}\right)+\frac{1}{2}(x-2)\sqrt{5+4x-x^2}+C\end{aligned}$$

24. $t^2-6t+13=\left(t^2-6t+9\right)+4=(t-3)^2+2^2$. Let $t-3=2\tan\theta$, so $dt=2\sec^2\theta\,d\theta$. Then



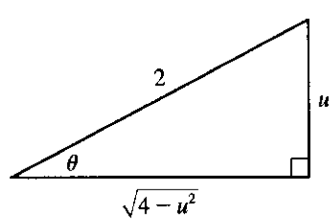
$$\begin{aligned}\int\frac{dt}{\sqrt{t^2-6t+13}}&=\int\frac{1}{\sqrt{(2\tan\theta)^2+2^2}}\,2\sec^2\theta\,d\theta=\int\frac{2\sec^2\theta}{2\sec\theta}\,d\theta\\&=\int\sec\theta\,d\theta=\ln|\sec\theta+\tan\theta|+C_1\quad\text{[by Formula 8.2.1]}\\&=\ln\left|\frac{\sqrt{t^2-6t+13}}{2}+\frac{t-3}{2}\right|+C_1\\&=\ln\left|\sqrt{t^2-6t+13}+t-3\right|+C\qquad\text{where } C=C_1-\ln 2\end{aligned}$$

25. $9x^2+6x-8=(3x+1)^2-9$, so let $u=3x+1$, $du=3dx$. Then $\displaystyle\int\frac{dx}{\sqrt{9x^2+6x-8}}=\int\frac{\frac{1}{3}\,du}{\sqrt{u^2-9}}$. Now let $u=3\sec\theta$, where $0\le\theta<\frac{\pi}{2}$ or $\pi\le\theta<\frac{3\pi}{2}$. Then $du=3\sec\theta\tan\theta\,d\theta$ and $\sqrt{u^2-9}=3\tan\theta$, so

$$\begin{aligned}\int\frac{\frac{1}{3}\,du}{\sqrt{u^2-9}}&=\int\frac{\sec\theta\tan\theta\,d\theta}{3\tan\theta}=\tfrac{1}{3}\int\sec\theta d\theta=\tfrac{1}{3}\ln|\sec\theta+\tan\theta|+C_1=\tfrac{1}{3}\ln\left|\frac{u+\sqrt{u^2-9}}{3}\right|+C_1\\&=\tfrac{1}{3}\ln\left|u+\sqrt{u^2-9}\right|+C=\tfrac{1}{3}\ln\left|3x+1+\sqrt{9x^2+6x-8}\right|+C\end{aligned}$$

26. $4x-x^2=-\left(x^2-4x+4\right)+4=4-(x-2)^2$, so let $u=x-2$. Then $x=u+2$ and $dx=du$, so

$$\begin{aligned}\int\frac{x^2\,dx}{\sqrt{4x-x^2}}&=\int\frac{(u+2)^2\,du}{\sqrt{4-u^2}}=\int\frac{(2\sin\theta+2)^2}{2\cos\theta}2\cos\theta\,d\theta\quad[\text{Put } u=2\sin\theta]\\&=4\int\left(\sin^2\theta+2\sin\theta+1\right)d\theta\\&=2\int(1-\cos 2\theta)\,d\theta+8\int\sin\theta d\theta+4\int d\theta\\&=2\theta-\sin 2\theta-8\cos\theta+4\theta+C\\&=6\theta-8\cos\theta-2\sin\theta\cos\theta+C\\&=6\sin^{-1}\left(\tfrac{1}{2}u\right)-4\sqrt{4-u^2}-\tfrac{1}{2}u\sqrt{4-u^2}+C\\&=6\sin^{-1}\left(\frac{x-2}{2}\right)-4\sqrt{4x-x^2}-\left(\frac{x-2}{2}\right)\sqrt{4x-x^2}+C\end{aligned}$$



27. $x^2 + 2x + 2 = (x+1)^2 + 1$. Let $u = x + 1$, $du = dx$. Then

$$\int \frac{dx}{(x^2+2x+2)^2} = \int \frac{du}{(u^2+1)^2} = \int \frac{\sec^2\theta\,d\theta}{\sec^4\theta} \qquad \left[\begin{array}{c}\text{where } u = \tan\theta,\ du = \sec^2\theta\,d\theta,\\ \text{and } u^2+1 = \sec^2\theta\end{array}\right]$$

$$= \int \cos^2\theta\,d\theta = \tfrac{1}{2}\int(1+\cos 2\theta)\,d\theta = \tfrac{1}{2}(\theta + \sin\theta\cos\theta) + C$$

$$= \frac{1}{2}\left[\tan^{-1}u + \frac{u}{1+u^2}\right] + C = \frac{1}{2}\left[\tan^{-1}(x+1) + \frac{x+1}{x^2+2x+2}\right] + C$$

28. $5 - 4x - x^2 = -(x^2+4x+4) + 9 = 9 - (x+2)^2$. Let $u = x + 2 \;\Rightarrow\; du = dx$. Then

$$\int \frac{dx}{(5-4x-x^2)^{5/2}} = \int \frac{du}{(9-u^2)^{5/2}} = \int \frac{3\cos\theta\,d\theta}{(3\cos\theta)^5} \qquad \left[\begin{array}{c}\text{where } u = 3\sin\theta,\ du = 3\cos\theta\,d\theta,\\ \text{and } \sqrt{9-u^2} = 3\cos\theta\end{array}\right]$$

$$= \tfrac{1}{81}\int \sec^4\theta\,d\theta = \tfrac{1}{81}\int(\tan^2\theta+1)\sec^2\theta\,d\theta = \tfrac{1}{81}\left[\tfrac{1}{3}\tan^3\theta + \tan\theta\right] + C$$

$$= \frac{1}{243}\left[\frac{u^3}{(9-u^2)^{3/2}} + \frac{3u}{\sqrt{9-u^2}}\right] + C = \frac{1}{243}\left[\frac{(x+2)^3}{(5-4x-x^2)^{3/2}} + \frac{3(x+2)}{\sqrt{5-4x-x^2}}\right] + C$$

29. Let $u = x^2$, $du = 2x\,dx$. Then

$$\int x\sqrt{1-x^4}\,dx = \int\sqrt{1-u^2}\left(\tfrac{1}{2}\,du\right) = \tfrac{1}{2}\int\cos\theta\cdot\cos\theta\,d\theta \qquad \left[\begin{array}{c}\text{where } u = \sin\theta,\ du = \cos\theta\,d\theta,\\ \text{and } \sqrt{1-u^2} = \cos\theta\end{array}\right]$$

$$= \tfrac{1}{2}\int\tfrac{1}{2}(1+\cos 2\theta)d\theta = \tfrac{1}{4}\theta + \tfrac{1}{8}\sin 2\theta + C = \tfrac{1}{4}\theta + \tfrac{1}{4}\sin\theta\cos\theta + C$$

$$= \tfrac{1}{4}\sin^{-1}u + \tfrac{1}{4}u\sqrt{1-u^2} + C = \tfrac{1}{4}\sin^{-1}(x^2) + \tfrac{1}{4}x^2\sqrt{1-x^4} + C$$

30. Let $u = \sin t$, $du = \cos t\,dt$. Then

$$\int_0^{\pi/2}\frac{\cos t}{\sqrt{1+\sin^2 t}}\,dt = \int_0^1\frac{1}{\sqrt{1+u^2}}\,du = \int_0^{\pi/4}\frac{1}{\sec\theta}\sec^2\theta\,d\theta \qquad \left[\begin{array}{c}\text{where } u = \tan\theta,\ du = \sec^2\theta\,d\theta,\\ \text{and } \sqrt{1+u^2} = \sec\theta\end{array}\right]$$

$$= \int_0^{\pi/4}\sec\theta\,d\theta = \Big[\ln|\sec\theta+\tan\theta|\Big]_0^{\pi/4} \qquad \text{[by (1) in Section 8.2]}$$

$$= \ln(\sqrt{2}+1) - \ln(1+0) = \ln(\sqrt{2}+1)$$

31. (a) Let $x = a\tan\theta$, where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. Then $\sqrt{x^2+a^2} = a\sec\theta$ and

$$\int\frac{dx}{\sqrt{x^2+a^2}} = \int\frac{a\sec^2\theta\,d\theta}{a\sec\theta} = \int\sec\theta\,d\theta = \ln|\sec\theta+\tan\theta| + C_1 = \ln\left|\frac{\sqrt{x^2+a^2}}{a} + \frac{x}{a}\right| + C_1$$

$$= \ln\left(x + \sqrt{x^2+a^2}\right) + C \quad \text{where } C = C_1 - \ln|a|$$

(b) Let $x = a\sinh t$, so that $dx = a\cosh t\,dt$ and $\sqrt{x^2+a^2} = a\cosh t$. Then

$$\int\frac{dx}{\sqrt{x^2+a^2}} = \int\frac{a\cosh t\,dt}{a\cosh t} = t + C = \sinh^{-1}\frac{x}{a} + C.$$

32. (a) Let $x = a\tan\theta$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. Then

$$I = \int\frac{x^2}{(x^2+a^2)^{3/2}}\,dx = \int\frac{a^2\tan^2\theta}{a^3\sec^3\theta}a\sec^2\theta\,d\theta = \int\frac{\tan^2\theta}{\sec\theta}\,d\theta = \int\frac{\sec^2\theta-1}{\sec\theta}\,d\theta$$

$$= \int(\sec\theta - \cos\theta)\,d\theta = \ln|\sec\theta+\tan\theta| - \sin\theta + C$$

$$= \ln\left|\frac{\sqrt{x^2+a^2}}{a} + \frac{x}{a}\right| - \frac{x}{\sqrt{x^2+a^2}} + C = \ln\left(x + \sqrt{x^2+a^2}\right) - \frac{x}{\sqrt{x^2+a^2}} + C_1$$

(b) Let $x = a\sinh t$. Then

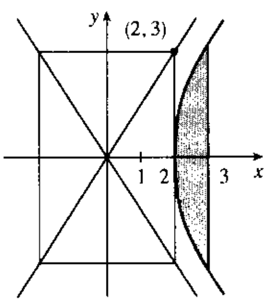
$$I = \int \frac{a^2\sinh^2 t}{a^3\cosh^3 t} a\cosh t\,dt = \int \tanh^2 t\,dt = \int \left(1 - \operatorname{sech}^2 t\right) dt = t - \tanh t + C$$
$$= \sinh^{-1}\frac{x}{a} - \frac{x}{\sqrt{a^2+x^2}} + C$$

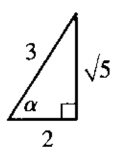
33. The average value of $f(x) = \sqrt{x^2-1}/x$ on the interval $[1,7]$ is

$$\frac{1}{7-1}\int_1^7 \frac{\sqrt{x^2-1}}{x}\,dx = \frac{1}{6}\int_0^\alpha \frac{\tan\theta}{\sec\theta}\cdot\sec\theta\tan\theta\,d\theta \quad \left[\begin{array}{c}\text{where } x=\sec\theta,\ dx=\sec\theta\tan\theta\,d\theta,\\ \sqrt{x^2-1}=\tan\theta, \text{ and } \alpha=\sec^{-1}7\end{array}\right]$$
$$= \tfrac{1}{6}\int_0^\alpha \tan^2\theta\,d\theta = \tfrac{1}{6}\int_0^\alpha (\sec^2\theta - 1)\,d\theta$$
$$= \tfrac{1}{6}\Big[\tan\theta - \theta\Big]_0^\alpha = \tfrac{1}{6}(\tan\alpha - \alpha)$$
$$= \tfrac{1}{6}\left(\sqrt{48} - \sec^{-1}7\right)$$

34. $9x^2 - 4y^2 = 36 \quad\Rightarrow\quad y = \pm\frac{3}{2}\sqrt{x^2-4} \quad\Rightarrow$

area $= 2\int_2^3 \frac{3}{2}\sqrt{x^2-4}\,dx = 3\int_2^3 \sqrt{x^2-4}\,dx$





$$= 3\int_0^\alpha 2\tan\theta\, 2\sec\theta\tan\theta\,d\theta \quad \left[\begin{array}{c}\text{where } x = 2\sec\theta,\\ dx = 2\sec\theta\tan\theta\,d\theta,\\ \alpha = \sec^{-1}\frac{3}{2}\end{array}\right]$$
$$= 12\int_0^\alpha \left(\sec^2\theta - 1\right)\sec\theta d\theta = 12\int_0^\alpha \left(\sec^3\theta - \sec\theta\right) d\theta$$
$$= 12\left[\tfrac{1}{2}(\sec\theta\tan\theta + \ln|\sec\theta+\tan\theta|) - \ln|\sec\theta+\tan\theta|\right]_0^\alpha$$
$$= 6\Big[\sec\theta\tan\theta - \ln|\sec\theta+\tan\theta|\Big]_0^\alpha$$
$$= 6\left[\tfrac{3\sqrt{5}}{4} - \ln\left(\tfrac{3}{2} + \tfrac{\sqrt{5}}{2}\right)\right] = \tfrac{9\sqrt{5}}{2} - 6\ln\left(\tfrac{3+\sqrt{5}}{2}\right)$$

35. Area of $\triangle POQ = \frac{1}{2}(r\cos\theta)(r\sin\theta) = \frac{1}{2}r^2\sin\theta\cos\theta$. Area of region $PQR = \int_{r\cos\theta}^{r}\sqrt{r^2-x^2}\,dx$. Let $x = r\cos u \quad\Rightarrow\quad dx = -r\sin u\,du$ for $\theta \le u \le \frac{\pi}{2}$. Then we obtain

$$\int \sqrt{r^2-x^2}\,dx = \int r\sin u\,(-r\sin u)\,du = -r^2\int \sin^2 u\,du = -\tfrac{1}{2}r^2(u - \sin u\cos u) + C$$
$$= -\tfrac{1}{2}r^2\cos^{-1}(x/r) + \tfrac{1}{2}x\sqrt{r^2-x^2} + C$$

so

$$\text{area of region } PQR = \tfrac{1}{2}\left[-r^2\cos^{-1}(x/r) + x\sqrt{r^2-x^2}\right]_{r\cos\theta}^{r}$$
$$= \tfrac{1}{2}\left[0 - \left(-r^2\theta + r\cos\theta\, r\sin\theta\right)\right]$$
$$= \tfrac{1}{2}r^2\theta - \tfrac{1}{2}r^2\sin\theta\cos\theta$$

and thus, (area of sector POR) = (area of $\triangle POQ$) + (area of region PQR) $= \frac{1}{2}r^2\theta$.

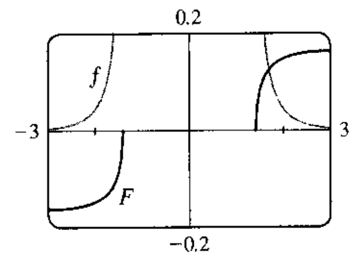
36. Let $x = \sqrt{2}\sec\theta$, where $0 \le \theta < \frac{\pi}{2}$ or $\pi \le \theta < \frac{3\pi}{2}$, so $dx = \sqrt{2}\sec\theta\tan\theta d\theta$. Then

$$\int \frac{dx}{x^4\sqrt{x^2-2}} = \int \frac{\sqrt{2}\sec\theta\tan\theta\,d\theta}{4\sec^4\theta\,\sqrt{2}\tan\theta}$$

$$= \tfrac{1}{4}\int \cos^3\theta\,d\theta = \tfrac{1}{4}\int\left(1-\sin^2\theta\right)\cos\theta\,d\theta$$

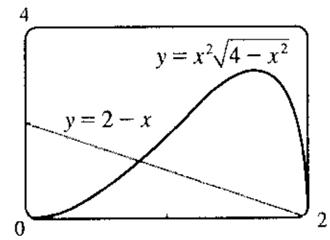
$$= \tfrac{1}{4}\left[\sin\theta - \tfrac{1}{3}\sin^3\theta\right] + C \qquad [\text{substitute } u = \sin\theta]$$

$$= \frac{1}{4}\left[\frac{\sqrt{x^2-2}}{x} - \frac{\left(x^2-2\right)^{3/2}}{3x^3}\right] + C$$



From the graph, it appears that our answer is reasonable. [Notice that $f(x)$ is large when F increases rapidly and small when F levels out.]

37. From the graph, it appears that the curve $y = x^2\sqrt{4-x^2}$ and the line $y = 2 - x$ intersect at about $x = 0.81$ and $x = 2$, with $x^2\sqrt{4-x^2} > 2 - x$ on $(0.81, 2)$. So the area bounded by the curve and the line is $A \approx \int_{0.81}^{2}\left[x^2\sqrt{4-x^2} - (2-x)\right]dx = \int_{0.81}^{2} x^2\sqrt{4-x^2}\,dx - \left[2x - \frac{1}{2}x^2\right]_{0.81}^{2}$.



To evaluate the integral, we put $x = 2\sin\theta$, where $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. Then $dx = 2\cos\theta\,d\theta$, $x = 2 \;\Rightarrow\; \theta = \sin^{-1}1 = \frac{\pi}{2}$, and $x = 0.81 \;\Rightarrow\; \theta = \sin^{-1}0.405 \approx 0.417$. So

$$\int_{0.81}^{2} x^2\sqrt{4-x^2}\,dx \approx \int_{0.417}^{\pi/2} 4\sin^2\theta\,(2\cos\theta)(2\cos\theta\,d\theta) = 4\int_{0.417}^{\pi/2}\sin^2 2\theta\,d\theta = 4\int_{0.417}^{\pi/2}\tfrac{1}{2}(1-\cos 4\theta)\,d\theta$$

$$= 2\left[\theta - \tfrac{1}{4}\sin 4\theta\right]_{0.417}^{\pi/2} = 2\left[\left(\tfrac{\pi}{2} - 0\right) - \left(0.417 - \tfrac{1}{4}(0.995)\right)\right] \approx 2.81$$

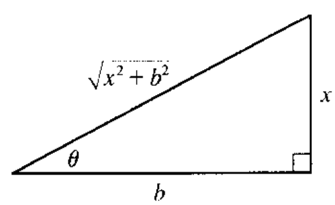
Thus, $A \approx 2.81 - \left[\left(2\cdot 2 - \frac{1}{2}\cdot 2^2\right) - \left(2\cdot 0.81 - \frac{1}{2}\cdot 0.81^2\right)\right] \approx 2.10$.

38. Let $x = b\tan\theta$, so that $dx = b\sec^2\theta\,d\theta$ and $\sqrt{x^2+b^2} = b\sec\theta$.

$$E(P) = \int_{-a}^{L-a}\frac{\lambda b}{4\pi\varepsilon_0(x^2+b^2)^{3/2}}\,dx = \frac{\lambda b}{4\pi\varepsilon_0}\int_{\theta_1}^{\theta_2}\frac{1}{(b\sec\theta)^3}\,b\sec^2\theta\,d\theta$$

$$= \frac{\lambda}{4\pi\varepsilon_0 b}\int_{\theta_1}^{\theta_2}\frac{1}{\sec\theta}\,d\theta = \frac{\lambda}{4\pi\varepsilon_0 b}\int_{\theta_1}^{\theta_2}\cos\theta\,d\theta = \frac{\lambda}{4\pi\varepsilon_0 b}\Big[\sin\theta\Big]_{\theta_1}^{\theta_2}$$

$$= \frac{\lambda}{4\pi\varepsilon_0 b}\left[\frac{x}{\sqrt{x^2+b^2}}\right]_{-a}^{L-a} = \frac{\lambda}{4\pi\varepsilon_0 b}\left(\frac{L-a}{\sqrt{(L-a)^2+b^2}} + \frac{a}{\sqrt{a^2+b^2}}\right)$$



39. Let the equation of the large circle be $x^2 + y^2 = R^2$. Then the equation of the small circle is $x^2 + (y-b)^2 = r^2$, where $b = \sqrt{R^2 - r^2}$ is the distance between the centers of the circles. The desired area is

$$A = \int_{-r}^{r}\left[\left(b + \sqrt{r^2-x^2}\right) - \sqrt{R^2-x^2}\right]dx = 2\int_0^r\left(b + \sqrt{r^2-x^2} - \sqrt{R^2-x^2}\right)dx$$

$$= 2\int_0^r b\,dx + 2\int_0^r\sqrt{r^2-x^2}\,dx - 2\int_0^r\sqrt{R^2-x^2}\,dx$$

The first integral is just $2br = 2r\sqrt{R^2-r^2}$. To evaluate the other two integrals, note that

$$\int\sqrt{a^2-x^2}\,dx = \int a^2\cos^2\theta\,d\theta \quad [x = a\sin\theta,\ dx = a\cos\theta\,d\theta] \quad = \tfrac{1}{2}a^2\int(1+\cos 2\theta)\,d\theta$$

$$= \tfrac{1}{2}a^2\left(\theta + \tfrac{1}{2}\sin 2\theta\right) + C = \tfrac{1}{2}a^2(\theta + \sin\theta\cos\theta) + C$$

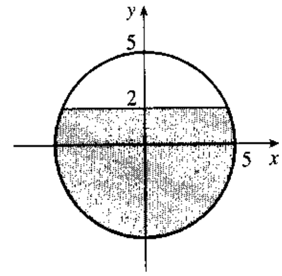
$$= \frac{a^2}{2}\arcsin\left(\frac{x}{a}\right) + \frac{a^2}{2}\left(\frac{x}{a}\right)\frac{\sqrt{a^2-x^2}}{a} + C = \frac{a^2}{2}\arcsin\left(\frac{x}{a}\right) + \frac{x}{2}\sqrt{a^2-x^2} + C$$

so the desired area is

$$A = 2r\sqrt{R^2-r^2} + \left[r^2\arcsin(x/r) + x\sqrt{r^2-x^2}\right]_0^r - \left[R^2\arcsin(x/R) + x\sqrt{R^2-x^2}\right]_0^r$$

$$= 2r\sqrt{R^2-r^2} + r^2\left(\tfrac{\pi}{2}\right) - \left[R^2\arcsin(r/R) + r\sqrt{R^2-r^2}\right] = r\sqrt{R^2-r^2} + \tfrac{\pi}{2}r^2 - R^2\arcsin(r/R)$$

40. Note that the circular cross-sections of the tank are the same everywhere, so the percentage of the total capacity that is being used is equal to the percentage of any cross-section that is under water. The underwater area is



$$A = 2\int_{-5}^{2}\sqrt{25-y^2}\,dy$$

$$= \left[25\arcsin(y/5) + y\sqrt{25-y^2}\right]_{-5}^{2} \quad [\text{substitute } y = 5\sin\theta]$$

$$= 25\arcsin\tfrac{2}{5} + 2\sqrt{21} + \tfrac{25}{2}\pi \approx 58.72 \text{ ft}^2$$

so the fraction of the total capacity in use is $\dfrac{A}{\pi(5)^2} \approx \dfrac{58.72}{25\pi} \approx 0.748$ or 74.8%.

41. We use cylindrical shells and assume that $R > r$. $x^2 = r^2 - (y-R)^2 \Rightarrow x = \pm\sqrt{r^2-(y-R)^2}$, so $g(y) = 2\sqrt{r^2-(y-R)^2}$ and

$$V = \int_{R-r}^{R+r} 2\pi y \cdot 2\sqrt{r^2-(y-R)^2}\,dy = \int_{-r}^{r} 4\pi(u+R)\sqrt{r^2-u^2}\,du \quad [\text{where } u = y - R]$$

$$= 4\pi\int_{-r}^{r} u\sqrt{r^2-u^2}\,du + 4\pi R\int_{-r}^{r}\sqrt{r^2-u^2}\,du \quad \left[\begin{matrix}\text{where } u = r\sin\theta,\ du = r\cos\theta\,d\theta \\ \text{in the second integral}\end{matrix}\right]$$

$$= 4\pi\left[-\tfrac{1}{3}\left(r^2-u^2\right)^{3/2}\right]_{-r}^{r} + 4\pi R\int_{-\pi/2}^{\pi/2} r^2\cos^2\theta\,d\theta = -\tfrac{4\pi}{3}(0-0) + 4\pi Rr^2\int_{-\pi/2}^{\pi/2}\cos^2\theta\,d\theta$$

$$= 2\pi Rr^2\int_{-\pi/2}^{\pi/2}(1+\cos 2\theta)\,d\theta = 2\pi Rr^2\left[\theta + \tfrac{1}{2}\sin 2\theta\right]_{-\pi/2}^{\pi/2} = 2\pi^2 Rr^2$$

Another method: Use washers instead of shells, so $V = 8\pi R\int_0^r\sqrt{r^2-y^2}\,dy$ as in Exercise 6.2.61(a), but evaluate the integral using $y = r\sin\theta$.

8.4 Integration of Rational Functions by Partial Fractions

1. (a) $\dfrac{2x}{(x+3)(3x+1)} = \dfrac{A}{x+3} + \dfrac{B}{3x+1}$

(b) $\dfrac{1}{x^3+2x^2+x} = \dfrac{1}{x(x^2+2x+1)} = \dfrac{1}{x(x+1)^2} = \dfrac{A}{x} + \dfrac{B}{x+1} + \dfrac{C}{(x+1)^2}$

2. (a) $\dfrac{x-1}{x^3+x^2} = \dfrac{x-1}{x^2(x+1)} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x+1}$

(b) $\dfrac{x-1}{x^3+x} = \dfrac{x-1}{x(x^2+1)} = \dfrac{A}{x} + \dfrac{Bx+C}{x^2+1}$

3. (a) $\dfrac{2}{x^2+3x-4} = \dfrac{2}{(x+4)(x-1)} = \dfrac{A}{x+4} + \dfrac{B}{x-1}$

(b) x^2+x+1 is irreducible, so $\dfrac{x^2}{(x-1)(x^2+x+1)} = \dfrac{A}{x-1} + \dfrac{Bx+C}{x^2+x+1}$.

4. (a) $\dfrac{x^3}{x^2+4x+3} = x-4+\dfrac{13x+12}{x^2+4x+3} = x-4+\dfrac{13x+12}{(x+1)(x+3)} = x-4+\dfrac{A}{x+1}+\dfrac{B}{x+3}$

(b) $\dfrac{2x+1}{(x+1)^3(x^2+4)^2} = \dfrac{A}{x+1}+\dfrac{B}{(x+1)^2}+\dfrac{C}{(x+1)^3}+\dfrac{Dx+E}{x^2+4}+\dfrac{Fx+G}{(x^2+4)^2}$

5. (a) $\dfrac{x^4}{x^4-1} = \dfrac{(x^4-1)+1}{x^4-1} = 1+\dfrac{1}{x^4-1}$ [or use long division] $= 1+\dfrac{1}{(x^2-1)(x^2+1)}$

$$= 1+\frac{1}{(x-1)(x+1)(x^2+1)} = 1+\frac{A}{x-1}+\frac{B}{x+1}+\frac{Cx+D}{x^2+1}$$

(b) $\dfrac{t^4+t^2+1}{(t^2+1)(t^2+4)^2} = \dfrac{At+B}{t^2+1}+\dfrac{Ct+D}{t^2+4}+\dfrac{Et+F}{(t^2+4)^2}$

6. (a) $\dfrac{x^4}{(x^3+x)(x^2-x+3)} = \dfrac{x^4}{x(x^2+1)(x^2-x+3)} = \dfrac{x^3}{(x^2+1)(x^2-x+3)} = \dfrac{Ax+B}{x^2+1}+\dfrac{Cx+D}{x^2-x+3}$

(b) $\dfrac{1}{x^6-x^3} = \dfrac{1}{x^3(x^3-1)} = \dfrac{1}{x^3(x-1)(x^2+x+1)} = \dfrac{A}{x}+\dfrac{B}{x^2}+\dfrac{C}{x^3}+\dfrac{D}{x-1}+\dfrac{Ex+F}{x^2+x+1}$

7. $\displaystyle\int \frac{x}{x-6}\,dx = \int \frac{(x-6)+6}{x-6}\,dx = \int\left(1+\frac{6}{x-6}\right)dx = x+6\ln|x-6|+C$

8. $\displaystyle\int \frac{r^2}{r+4}\,dr = \int\left(\frac{r^2-16}{r+4}+\frac{16}{r+4}\right)dr = \int\left(r-4+\frac{16}{r+4}\right)dr$ [or use long division]

$$= \tfrac{1}{2}r^2-4r+16\ln|r+4|+C$$

9. $\dfrac{x-9}{(x+5)(x-2)} = \dfrac{A}{x+5}+\dfrac{B}{x-2}$. Multiply both sides by $(x+5)(x-2)$ to get $x-9 = A(x-2)+B(x+5)$. Substituting 2 for x gives $-7 = 7B \Leftrightarrow B=-1$. Substituting -5 for x gives $-14=-7A \Leftrightarrow A=2$. Thus,

$$\int \frac{x-9}{(x+5)(x-2)}\,dx = \int\left(\frac{2}{x+5}+\frac{-1}{x-2}\right)dx = 2\ln|x+5|-\ln|x-2|+C$$

10. $\dfrac{1}{(t+4)(t-1)} = \dfrac{A}{t+4}+\dfrac{B}{t-1} \Rightarrow 1 = A(t-1)+B(t+4)$.

$t=1 \Rightarrow 1=5B \Rightarrow B=\frac{1}{5}$. $t=-4 \Rightarrow 1=-5A \Rightarrow A=-\frac{1}{5}$. Thus,

$$\int \frac{1}{(t+4)(t-1)}\,dt = \int\left(\frac{-1/5}{t+4}+\frac{1/5}{t-1}\right)dt = -\tfrac{1}{5}\ln|t+4|+\tfrac{1}{5}\ln|t-1|+C \quad \text{or} \quad \frac{1}{5}\ln\left|\frac{t-1}{t+4}\right|+C$$

11. $\dfrac{1}{x^2-1} = \dfrac{1}{(x+1)(x-1)} = \dfrac{A}{x+1}+\dfrac{B}{x-1}$. Multiply both sides by $(x+1)(x-1)$ to get $1 = A(x-1)+B(x+1)$. Substituting 1 for x gives $1=2B \Leftrightarrow B=\frac{1}{2}$. Substituting -1 for x gives $1=-2A \Leftrightarrow A=-\frac{1}{2}$. Thus,

$$\int_2^3 \frac{1}{x^2-1}\,dx = \int_2^3\left(\frac{-1/2}{x+1}+\frac{1/2}{x-1}\right)dx = \left[-\tfrac{1}{2}\ln|x+1|+\tfrac{1}{2}\ln|x-1|\right]_2^3$$

$$= \left(-\tfrac{1}{2}\ln 4+\tfrac{1}{2}\ln 2\right)-\left(-\tfrac{1}{2}\ln 3+\tfrac{1}{2}\ln 1\right) = \tfrac{1}{2}(\ln 2+\ln 3-\ln 4) \quad \left[\text{or } \tfrac{1}{2}\ln\tfrac{3}{2}\right]$$

12. $\dfrac{x-1}{x^2+3x+2} = \dfrac{A}{x+1}+\dfrac{B}{x+2}$. Multiply both sides by $(x+1)(x+2)$ to get $x-1 = A(x+2)+B(x+1)$. Substituting -2 for x gives $-3=-B \Leftrightarrow B=3$. Substituting -1 for x gives $-2=A$. Thus,

$$\int_0^1 \frac{x-1}{x^2+3x+2}\,dx = \int_0^1\left(\frac{-2}{x+1}+\frac{3}{x+2}\right)dx = \left[-2\ln|x+1|+3\ln|x+2|\right]_0^1$$

$$= (-2\ln 2+3\ln 3)-(-2\ln 1+3\ln 2) = 3\ln 3-5\ln 2 \quad \left[\text{or } \ln\tfrac{27}{32}\right]$$

13. $\displaystyle\int \frac{ax}{x^2 - bx}\,dx = \int \frac{ax}{x(x-b)}\,dx = \int \frac{a}{x-b}\,dx = a\ln|x-b| + C$

14. If $a \neq b$, $\dfrac{1}{(x+a)(x+b)} = \dfrac{1}{b-a}\left(\dfrac{1}{x+a} - \dfrac{1}{x+b}\right)$, so if $a \neq b$, then

$$\int \frac{dx}{(x+a)(x+b)} = \frac{1}{b-a}\left(\ln|x+a| - \ln|x+b|\right) + C = \frac{1}{b-a}\ln\left|\frac{x+a}{x+b}\right| + C$$

If $a = b$, then $\displaystyle\int \frac{dx}{(x+a)^2} = -\frac{1}{x+a} + C$.

15. $\dfrac{2x+3}{(x+1)^2} = \dfrac{A}{x+1} + \dfrac{B}{(x+1)^2} \;\Rightarrow\; 2x+3 = A(x+1) + B$. Take $x = -1$ to get $B = 1$, and equate coefficients of x to get $A = 2$. Now

$$\int_0^1 \frac{2x+3}{(x+1)^2}dx = \int_0^1 \left[\frac{2}{x+1} + \frac{1}{(x+1)^2}\right]dx = \left[2\ln(x+1) - \frac{1}{x+1}\right]_0^1$$

$$= 2\ln 2 - \tfrac{1}{2} - (2\ln 1 - 1) = 2\ln 2 + \tfrac{1}{2}$$

16. $\dfrac{x^3 - 4x - 10}{x^2 - x - 6} = x + 1 + \dfrac{3x-4}{(x-3)(x+2)}$. Write $\dfrac{3x-4}{(x-3)(x+2)} = \dfrac{A}{x-3} + \dfrac{B}{x+2}$. Then $3x - 4 = A(x+2) + B(x-3)$. Taking $x = 3$ and $x = -2$, we get $5 = 5A \;\Leftrightarrow\; A = 1$ and $-10 = -5B \;\Leftrightarrow\; B = 2$, so

$$\int_0^1 \frac{x^3 - 4x - 10}{x^2 - x - 6}\,dx = \int_0^1 \left(x + 1 + \frac{1}{x-3} + \frac{2}{x+2}\right)dx = \left[\frac{1}{2}x^2 + x + \ln|x-3| + 2\ln(x+2)\right]_0^1$$

$$= \left(\tfrac{1}{2} + 1 + \ln 2 + 2\ln 3\right) - \left(0 + 0 + \ln 3 + 2\ln 2\right) = \tfrac{3}{2} + \ln 3 - \ln 2 = \tfrac{3}{2} + \ln\tfrac{3}{2}$$

17. $\dfrac{4y^2 - 7y - 12}{y(y+2)(y-3)} = \dfrac{A}{y} + \dfrac{B}{y+2} + \dfrac{C}{y-3} \;\Rightarrow\; 4y^2 - 7y - 12 = A(y+2)(y-3) + By(y-3) + Cy(y+2)$.

Setting $y = 0$ gives $-12 = -6A$, so $A = 2$. Setting $y = -2$ gives $18 = 10B$, so $B = \frac{9}{5}$. Setting $y = 3$ gives $3 = 15C$, so $C = \frac{1}{5}$. Now

$$\int_1^2 \frac{4y^2 - 7y - 12}{y(y+2)(y-3)}\,dy = \int_1^2 \left(\frac{2}{y} + \frac{9/5}{y+2} + \frac{1/5}{y-3}\right)dy = \left[2\ln|y| + \tfrac{9}{5}\ln|y+2| + \tfrac{1}{5}\ln|y-3|\right]_1^2$$

$$= 2\ln 2 + \tfrac{9}{5}\ln 4 + \tfrac{1}{5}\ln 1 - 2\ln 1 - \tfrac{9}{5}\ln 3 - \tfrac{1}{5}\ln 2$$

$$= 2\ln 2 + \tfrac{18}{5}\ln 2 - \tfrac{1}{5}\ln 2 - \tfrac{9}{5}\ln 3 = \tfrac{27}{5}\ln 2 - \tfrac{9}{5}\ln 3 = \tfrac{9}{5}(3\ln 2 - \ln 3) = \tfrac{9}{5}\ln\tfrac{8}{3}$$

18. $\dfrac{x^2 + 2x - 1}{x^3 - x} = \dfrac{x^2 + 2x - 1}{x(x+1)(x-1)} = \dfrac{A}{x} + \dfrac{B}{x+1} + \dfrac{C}{x-1}$. Multiply both sides by $x(x+1)(x-1)$ to get $x^2 + 2x - 1 = A(x+1)(x-1) + Bx(x-1) + Cx(x+1)$. Substituting 0 for x gives $-1 = -A \;\Leftrightarrow\; A = 1$. Substituting -1 for x gives $-2 = 2B \;\Leftrightarrow\; B = -1$. Substituting 1 for x gives $2 = 2C \;\Leftrightarrow\; C = 1$. Thus,

$$\int \frac{x^2 + 2x - 1}{x^3 - x}\,dx = \int\left(\frac{1}{x} - \frac{1}{x+1} + \frac{1}{x-1}\right)dx = \ln|x| - \ln|x+1| + \ln|x-1| + C = \ln\left|\frac{x(x-1)}{x+1}\right| + C.$$

19. $\dfrac{1}{(x+5)^2(x-1)} = \dfrac{A}{x+5} + \dfrac{B}{(x+5)^2} + \dfrac{C}{x-1} \Rightarrow 1 = A(x+5)(x-1) + B(x-1) + C(x+5)^2$. Setting $x=-5$ gives $1=-6B$, so $B=-\frac{1}{6}$. Setting $x=1$ gives $1=36C$, so $C=\frac{1}{36}$. Setting $x=-2$ gives $1 = A(3)(-3) + B(-3) + C(3^2) = -9A - 3B + 9C = -9A + \frac{1}{2} + \frac{1}{4} = -9A + \frac{3}{4}$, so $9A = -\frac{1}{4}$ and $A = -\frac{1}{36}$. Now

$$\int \frac{1}{(x+5)^2(x-1)}\,dx = \int \left[\frac{-1/36}{x+5} - \frac{1/6}{(x+5)^2} + \frac{1/36}{x-1}\right] dx$$
$$= -\frac{1}{36}\ln|x+5| + \frac{1}{6(x+5)} + \frac{1}{36}\ln|x-1| + C$$

20. $\dfrac{x^2}{(x-3)(x+2)^2} = \dfrac{A}{x-3} + \dfrac{B}{x+2} + \dfrac{C}{(x+2)^2} \Rightarrow x^2 = A(x+2)^2 + B(x-3)(x+2) + C(x-3)$. Setting $x=3$ gives $A=\frac{9}{25}$. Take $x=-2$ to get $C=-\frac{4}{5}$, and equate the coefficients of x^2 to get $1 = A + B \Rightarrow B = \frac{16}{25}$. Then

$$\int \frac{x^2}{(x-3)(x+2)^2}\,dx = \int \left[\frac{9/25}{x-3} + \frac{16/25}{x+2} - \frac{4/5}{(x+2)^2}\right] dx$$
$$= \frac{9}{25}\ln|x-3| + \frac{16}{25}\ln|x+2| + \frac{4}{5(x+2)} + C$$

21. $\dfrac{5x^2+3x-2}{x^3+2x^2} = \dfrac{5x^2+3x-2}{x^2(x+2)} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x+2}$. Multiply by $x^2(x+2)$ to get $5x^2+3x-2 = Ax(x+2) + B(x+2) + Cx^2$. Set $x=-2$ to get $C=3$, and take $x=0$ to get $B=-1$. Equating the coefficients of x^2 gives $5 = A + C \Rightarrow A = 2$. So

$$\int \frac{5x^2+3x-2}{x^3+2x^2}\,dx = \int \left(\frac{2}{x} - \frac{1}{x^2} + \frac{3}{x+2}\right) dx = 2\ln|x| + \frac{1}{x} + 3\ln|x+2| + C.$$

22. $\dfrac{1}{s^2(s-1)^2} = \dfrac{A}{s} + \dfrac{B}{s^2} + \dfrac{C}{s-1} + \dfrac{D}{(s-1)^2} \Rightarrow 1 = As(s-1)^2 + B(s-1)^2 + Cs^2(s-1) + Ds^2$. Set $s=0$, giving $B=1$. Then set $s=1$ to get $D=1$. Equate the coefficients of s^3 to get $0 = A + C$ or $A = -C$, and finally set $s=2$ to get $1 = 2A + 1 - 4A + 4$ or $A = 2$. Now

$$\int \frac{ds}{s^2(s-1)^2} = \int \left[\frac{2}{s} + \frac{1}{s^2} - \frac{2}{s-1} + \frac{1}{(s-1)^2}\right] ds = 2\ln|s| - \frac{1}{s} - 2\ln|s-1| - \frac{1}{s-1} + C.$$

23. $\dfrac{x^2}{(x+1)^3} = \dfrac{A}{x+1} + \dfrac{B}{(x+1)^2} + \dfrac{C}{(x+1)^3}$. Multiply by $(x+1)^3$ to get $x^2 = A(x+1)^2 + B(x+1) + C$. Setting $x=-1$ gives $C=1$. Equating the coefficients of x^2 gives $A=1$, and setting $x=0$ gives $B=-2$. Now $\displaystyle\int \frac{x^2\,dx}{(x+1)^3} = \int \left[\frac{1}{x+1} - \frac{2}{(x+1)^2} + \frac{1}{(x+1)^3}\right] dx = \ln|x+1| + \frac{2}{x+1} - \frac{1}{2(x+1)^2} + C.$

24. $\dfrac{x}{x+1} = \dfrac{(x+1)-1}{x+1} = 1 - \dfrac{1}{x+1}$, so $\dfrac{x^3}{(x+1)^3} = \left[1 - \dfrac{1}{x+1}\right]^3 = 1 - \dfrac{3}{x+1} + \dfrac{3}{(x+1)^2} - \dfrac{1}{(x+1)^3}$. Thus,

$$\int \frac{x^3}{(x+1)^3}\,dx = \int \left[1 - \frac{3}{x+1} + \frac{3}{(x+1)^2} - \frac{1}{(x+1)^3}\right] dx = x - 3\ln|x+1| - \frac{3}{x+1} + \frac{1}{2(x+1)^2} + C.$$

25. $\dfrac{10}{(x-1)(x^2+9)} = \dfrac{A}{x-1} + \dfrac{Bx+C}{x^2+9}$. Multiply both sides by $(x-1)(x^2+9)$ to get $10 = A(x^2+9) + (Bx+C)(x-1)$ $(*)$. Substituting 1 for x gives $10 = 10A \Leftrightarrow A = 1$. Substituting 0 for x gives $10 = 9A - C \Rightarrow C = 9(1) - 10 = -1$. The coefficients of the x^2-terms in $(*)$ must be equal, so $0 = A + B \Rightarrow B = -1$. Thus,

$$\int \frac{10}{(x-1)(x^2+9)}\,dx = \int \left(\frac{1}{x-1} + \frac{-x-1}{x^2+9}\right) dx = \int \left(\frac{1}{x-1} - \frac{x}{x^2+9} - \frac{1}{x^2+9}\right) dx$$

$$= \ln|x-1| - \tfrac{1}{2}\ln(x^2+9) \;\;[\text{let } u = x^2+9] \; - \tfrac{1}{3}\tan^{-1}\left(\tfrac{x}{3}\right) \;\;[\text{Formula 10}] + C$$

26. $\dfrac{x^2-x+6}{x^3+3x} = \dfrac{x^2-x+6}{x(x^2+3)} = \dfrac{A}{x} + \dfrac{Bx+C}{x^2+3}$. Multiply by $x(x^2+3)$ to get $x^2 - x + 6 = A(x^2+3) + (Bx+C)x$. Substituting 0 for x gives $6 = 3A \Leftrightarrow A = 2$. The coefficients of the x^2-terms must be equal, so $1 = A + B \Rightarrow B = 1 - 2 = -1$. The coefficients of the x-terms must be equal, so $-1 = C$. Thus,

$$\int \frac{x^2-x+6}{x^3+3x}\,dx = \int \left(\frac{2}{x} + \frac{-x-1}{x^2+3}\right) dx = \int \left(\frac{2}{x} - \frac{x}{x^2+3} - \frac{1}{x^2+3}\right) dx$$

$$= 2\ln|x| - \frac{1}{2}\ln(x^2+3) - \frac{1}{\sqrt{3}}\tan^{-1}\frac{x}{\sqrt{3}} + C$$

27. $\dfrac{x^3+x^2+2x+1}{(x^2+1)(x^2+2)} = \dfrac{Ax+B}{x^2+1} + \dfrac{Cx+D}{x^2+2}$. Multiply both sides by $(x^2+1)(x^2+2)$ to get

$x^3 + x^2 + 2x + 1 = (Ax+B)(x^2+2) + (Cx+D)(x^2+1) \Leftrightarrow$

$x^3 + x^2 + 2x + 1 = (Ax^3 + Bx^2 + 2Ax + 2B) + (Cx^3 + Dx^2 + Cx + D) \Leftrightarrow$

$x^3 + x^2 + 2x + 1 = (A+C)x^3 + (B+D)x^2 + (2A+C)x + (2B+D)$. Comparing coefficients gives us the following system of equations:

$$A + C = 1 \quad \textbf{(1)} \qquad B + D = 1 \quad \textbf{(2)}$$
$$2A + C = 2 \quad \textbf{(3)} \qquad 2B + D = 1 \quad \textbf{(4)}$$

Subtracting equation **(1)** from equation **(3)** gives us $A = 1$, so $C = 0$. Subtracting equation **(2)** from equation **(4)** gives us $B = 0$, so $D = 1$. Thus, $I = \displaystyle\int \frac{x^3+x^2+2x+1}{(x^2+1)(x^2+2)}\,dx = \int \left(\frac{x}{x^2+1} + \frac{1}{x^2+2}\right) dx$. For $\displaystyle\int \frac{x}{x^2+1}\,dx$, let $u = x^2+1$ so $du = 2x\,dx$ and then $\displaystyle\int \frac{x}{x^2+1}\,dx = \frac{1}{2}\int \frac{1}{u}\,du = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln(x^2+1) + C$. For $\displaystyle\int \frac{1}{x^2+2}\,dx$, use Formula 10 with $a = \sqrt{2}$. So $\displaystyle\int \frac{1}{x^2+2}\,dx = \int \frac{1}{x^2+(\sqrt{2})^2}\,dx = \frac{1}{\sqrt{2}}\tan^{-1}\frac{x}{\sqrt{2}} + C$.

Thus, $I = \dfrac{1}{2}\ln(x^2+1) + \dfrac{1}{\sqrt{2}}\tan^{-1}\dfrac{x}{\sqrt{2}} + C$.

28. $\dfrac{x^2-2x-1}{(x-1)^2(x^2+1)} = \dfrac{A}{x-1} + \dfrac{B}{(x-1)^2} + \dfrac{Cx+D}{x^2+1} \Rightarrow$

$x^2 - 2x - 1 = A(x-1)(x^2+1) + B(x^2+1) + (Cx+D)(x-1)^2$. Setting $x = 1$ gives $B = -1$.

Equating the coefficients of x^3 gives $A = -C$. Equating the constant terms gives $-1 = -A - 1 + D$, so $D = A$, and setting $x = 2$ gives $-1 = 5A - 5 - 2A + A$ or $A = 1$. We have

$$\int \frac{x^2 - 2x - 1}{(x-1)^2(x^2+1)}\,dx = \int \left[\frac{1}{x-1} - \frac{1}{(x-1)^2} - \frac{x-1}{x^2+1}\right] dx$$
$$= \ln|x-1| + \frac{1}{x-1} - \tfrac{1}{2}\ln(x^2+1) + \tan^{-1}x + C$$

29. $\displaystyle\int \frac{x+4}{x^2+2x+5}\,dx = \int \frac{x+1}{x^2+2x+5}\,dx + \int \frac{3}{x^2+2x+5}\,dx = \frac{1}{2}\int \frac{(2x+2)\,dx}{x^2+2x+5} + \int \frac{3\,dx}{(x+1)^2+4}$

$$= \frac{1}{2}\ln\left|x^2+2x+5\right| + 3\int \frac{2\,du}{4(u^2+1)} \qquad \begin{bmatrix}\text{where } x+1 = 2u,\\ \text{and } dx = 2\,du\end{bmatrix}$$
$$= \frac{1}{2}\ln(x^2+2x+5) + \frac{3}{2}\tan^{-1}u + C = \frac{1}{2}\ln(x^2+2x+5) + \frac{3}{2}\tan^{-1}\left(\frac{x+1}{2}\right) + C$$

30. $\displaystyle\frac{x^3-2x^2+x+1}{x^4+5x^2+4} = \frac{x^3-2x^2+x+1}{(x^2+1)(x^2+4)} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{x^2+4} \quad\Rightarrow$

$x^3 - 2x^2 + x + 1 = (Ax+B)(x^2+4) + (Cx+D)(x^2+1)$. Equating coefficients gives $A + C = 1$, $B + D = -2$, $4A + C = 1$, $4B + D = 1 \quad\Rightarrow\quad A = 0$, $C = 1$, $B = 1$, $D = -3$. Now

$$\int \frac{x^3-2x^2+x+1}{x^4+5x^2+4}\,dx = \int \frac{dx}{x^2+1} + \int \frac{x-3}{x^2+4}\,dx = \tan^{-1}x + \tfrac{1}{2}\ln(x^2+4) - \tfrac{3}{2}\tan^{-1}(x/2) + C.$$

31. $\displaystyle\frac{1}{x^3-1} = \frac{1}{(x-1)(x^2+x+1)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+x+1} \quad\Rightarrow\quad 1 = A(x^2+x+1) + (Bx+C)(x-1).$

Take $x = 1$ to get $A = \frac{1}{3}$. Equating coefficients of x^2 and then comparing the constant terms, we get $0 = \frac{1}{3} + B$, $1 = \frac{1}{3} - C$, so $B = -\frac{1}{3}$, $C = -\frac{2}{3} \quad\Rightarrow$

$$\int \frac{1}{x^3-1}\,dx = \int \frac{\frac{1}{3}}{x-1}\,dx + \int \frac{-\frac{1}{3}x - \frac{2}{3}}{x^2+x+1}\,dx = \tfrac{1}{3}\ln|x-1| - \frac{1}{3}\int \frac{x+2}{x^2+x+1}\,dx$$
$$= \tfrac{1}{3}\ln|x-1| - \frac{1}{3}\int \frac{x+1/2}{x^2+x+1}\,dx - \frac{1}{3}\int \frac{(3/2)\,dx}{(x+1/2)^2+3/4}$$
$$= \tfrac{1}{3}\ln|x-1| - \tfrac{1}{6}\ln(x^2+x+1) - \tfrac{1}{2}\left(\tfrac{2}{\sqrt{3}}\right)\tan^{-1}\left(\frac{x+\frac{1}{2}}{\sqrt{3}/2}\right) + K$$
$$= \tfrac{1}{3}\ln|x-1| - \tfrac{1}{6}\ln(x^2+x+1) - \tfrac{1}{\sqrt{3}}\tan^{-1}\left(\tfrac{1}{\sqrt{3}}(2x+1)\right) + K$$

32. $\displaystyle\int_0^1 \frac{x}{x^2+4x+13}\,dx = \int_0^1 \frac{\frac{1}{2}(2x+4)}{x^2+4x+13}\,dx - 2\int_0^1 \frac{dx}{(x+2)^2+9}$

$$= \frac{1}{2}\int_{13}^{18}\frac{dy}{y} - 2\int_{2/3}^{1}\frac{3\,du}{9u^2+9} \qquad \begin{bmatrix}\text{where } y = x^2+4x+13,\ dy = (2x+4)\,dx,\\ x+2 = 3u, \text{ and } dx = 3\,du\end{bmatrix}$$
$$= \tfrac{1}{2}\left[\ln y\right]_{13}^{18} - \tfrac{2}{3}\left[\tan^{-1}u\right]_{2/3}^{1} = \tfrac{1}{2}\ln\tfrac{18}{13} - \tfrac{2}{3}\left(\tfrac{\pi}{4} - \tan^{-1}\left(\tfrac{2}{3}\right)\right)$$
$$= \tfrac{1}{2}\ln\tfrac{18}{13} - \tfrac{\pi}{6} + \tfrac{2}{3}\tan^{-1}\left(\tfrac{2}{3}\right)$$

33. Let $u = x^3 + 3x^2 + 4$. Then $du = 3(x^2 + 2x)\,dx \quad \Rightarrow$

$$\int_2^5 \frac{x^2 + 2x}{x^3 + 3x^2 + 4}\,dx = \frac{1}{3}\int_{24}^{204} \frac{du}{u} = \tfrac{1}{3}[\ln u]_{24}^{204} = \tfrac{1}{3}(\ln 204 - \ln 24) = \tfrac{1}{3}\ln\tfrac{204}{24} = \tfrac{1}{3}\ln\tfrac{17}{2}.$$

34. $\dfrac{x^3}{x^3+1} = \dfrac{(x^3+1)-1}{x^3+1} = 1 - \dfrac{1}{x^3+1} = 1 - \left(\dfrac{A}{x+1} + \dfrac{Bx+C}{x^2-x+1}\right) \quad \Rightarrow$

$1 = A(x^2 - x + 1) + (Bx + C)(x + 1)$. Equate the terms of degree 2, 1 and 0 to get $0 = A + B$, $0 = -A + B + C$, $1 = A + C$. Solve the three equations to get $A = \frac{1}{3}$, $B = -\frac{1}{3}$, and $C = \frac{2}{3}$. So

$$\begin{aligned}\int \frac{x^3}{x^3+1}\,dx &= \int \left[1 - \frac{\frac{1}{3}}{x+1} + \frac{\frac{1}{3}x - \frac{2}{3}}{x^2 - x + 1}\right] dx \\ &= x - \tfrac{1}{3}\ln|x+1| + \frac{1}{6}\int \frac{2x-1}{x^2-x+1}\,dx - \frac{1}{2}\int \frac{dx}{\left(x-\frac{1}{2}\right)^2 + \frac{3}{4}} \\ &= x - \tfrac{1}{3}\ln|x+1| + \tfrac{1}{6}\ln(x^2 - x + 1) - \tfrac{1}{\sqrt{3}}\tan^{-1}\!\left(\tfrac{1}{\sqrt{3}}(2x-1)\right) + K\end{aligned}$$

35. $\dfrac{1}{x^4 - x^2} = \dfrac{1}{x^2(x-1)(x+1)} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x-1} + \dfrac{D}{x+1}$. Multiply by $x^2(x-1)(x+1)$ to get $1 = Ax(x-1)(x+1) + B(x-1)(x+1) + Cx^2(x+1) + Dx^2(x-1)$. Setting $x = 1$ gives $C = \frac{1}{2}$, taking $x = -1$ gives $D = -\frac{1}{2}$. Equating the coefficients of x^3 gives $0 = A + C + D = A$. Finally, setting $x = 0$ yields $B = -1$. Now $\displaystyle\int \frac{dx}{x^4 - x^2} = \int \left[\frac{-1}{x^2} + \frac{1/2}{x-1} - \frac{1/2}{x+1}\right] dx = \frac{1}{x} + \tfrac{1}{2}\ln\left|\frac{x-1}{x+1}\right| + C.$

36. Let $u = x^4 + 5x^2 + 4 \quad \Rightarrow \quad du = (4x^3 + 10x)\,dx = 2(2x^3 + 5x)\,dx$, so

$$\int_0^1 \frac{2x^3 + 5x}{x^4 + 5x^2 + 4}\,dx = \frac{1}{2}\int_4^{10} \frac{du}{u} = \tfrac{1}{2}\Big[\ln|u|\Big]_4^{10} = \tfrac{1}{2}(\ln 10 - \ln 4) = \tfrac{1}{2}\ln\tfrac{5}{2}.$$

37. $\displaystyle\int \frac{x-3}{(x^2+2x+4)^2}\,dx = \int \frac{x-3}{\left[(x+1)^2+3\right]^2}\,dx = \int \frac{u-4}{(u^2+3)^2}\,du \qquad$ [with $u = x + 1$]

$$\begin{aligned}&= \int \frac{u\,du}{(u^2+3)^2} - 4\int \frac{du}{(u^2+3)^2} = \frac{1}{2}\int \frac{dv}{v^2} - 4\int \frac{\sqrt{3}\sec^2\theta\,d\theta}{9\sec^4\theta} \qquad \left[\begin{array}{c} v = u^2 + 3 \text{ in the first integral;} \\ u = \sqrt{3}\tan\theta \text{ in the second}\end{array}\right] \\ &= \frac{-1}{(2v)} - \frac{4\sqrt{3}}{9}\int \cos^2\theta\,d\theta = \frac{-1}{2(u^2+3)} - \frac{2\sqrt{3}}{9}(\theta + \sin\theta\cos\theta) + C \\ &= \frac{-1}{2(x^2+2x+4)} - \frac{2\sqrt{3}}{9}\left[\tan^{-1}\!\left(\frac{x+1}{\sqrt{3}}\right) + \frac{\sqrt{3}(x+1)}{x^2+2x+4}\right] + C \\ &= \frac{-1}{2(x^2+2x+4)} - \frac{2\sqrt{3}}{9}\tan^{-1}\!\left(\frac{x+1}{\sqrt{3}}\right) - \frac{2(x+1)}{3(x^2+2x+4)} + C\end{aligned}$$

38. $\dfrac{x^4+1}{x(x^2+1)^2} = \dfrac{A}{x} + \dfrac{Bx+C}{x^2+1} + \dfrac{Dx+E}{(x^2+1)^2} \quad \Rightarrow \quad x^4 + 1 = A(x^2+1)^2 + (Bx+C)x(x^2+1) + (Dx+E)x.$

Setting $x = 0$ gives $A = 1$, and equating the coefficients of x^4 gives $1 = A + B$, so $B = 0$. Now

$\dfrac{C}{x^2+1} + \dfrac{Dx+E}{(x^2+1)^2} = \dfrac{x^4+1}{x(x^2+1)^2} - \dfrac{1}{x} = \dfrac{1}{x}\left[\dfrac{x^4+1-(x^4+2x^2+1)}{(x^2+1)^2}\right] = \dfrac{-2x}{(x^2+1)^2}$, so we can take $C=0$,

$D=-2$, and $E=0$. Hence, $\displaystyle\int \frac{x^4+1}{x(x^2+1)^2}dx = \int\left[\frac{1}{x} - \frac{2x}{(x^2+1)^2}\right]dx = \ln|x| + \frac{1}{x^2+1} + C.$

39. Let $u = \sqrt{x+1}$. Then $x = u^2 - 1$, $dx = 2u\,du \quad\Rightarrow$

$$\int \frac{dx}{x\sqrt{x+1}} = \int \frac{2u\,du}{(u^2-1)\,u} = 2\int \frac{du}{u^2-1} = \ln\left|\frac{u-1}{u+1}\right| + C = \ln\left|\frac{\sqrt{x+1}-1}{\sqrt{x+1}+1}\right| + C.$$

40. Let $u = \sqrt{x+2}$. Then $x = u^2 - 2$, $dx = 2u\,du \quad\Rightarrow$

$I = \displaystyle\int \frac{dx}{x - \sqrt{x+2}} = \int \frac{2u\,du}{u^2-2-u} = 2\int \frac{u\,du}{u^2-u-2}$ and $\dfrac{u}{u^2-u-2} = \dfrac{A}{u-2} + \dfrac{B}{u+1} \quad\Rightarrow$

$u = A(u+1) + B(u-2)$. Substituting -1 for u gives $-1 = -3B \quad\Leftrightarrow\quad B = \frac{1}{3}$ and substituting 2 for u gives $2 = 3A \quad\Leftrightarrow\quad A = \frac{2}{3}$. Thus,

$$\begin{aligned} I &= \frac{2}{3}\int\left[\frac{2}{u-2} + \frac{1}{u+1}\right]du = \tfrac{2}{3}(2\ln|u-2| + \ln|u+1|) + C \\ &= \tfrac{2}{3}\left[2\ln\left|\sqrt{x+2}-2\right| + \ln\left(\sqrt{x+2}+1\right)\right] + C \end{aligned}$$

41. Let $u = \sqrt{x}$, so $u^2 = x$ and $dx = 2u\,du$. Thus,

$$\begin{aligned} \int_9^{16} \frac{\sqrt{x}}{x-4}\,dx &= \int_3^4 \frac{u}{u^2-4}2u\,du = 2\int_3^4 \frac{u^2}{u^2-4}\,du = 2\int_3^4\left(1 + \frac{4}{u^2-4}\right)du \quad \text{[by long division]} \\ &= 2 + 8\int_3^4 \frac{du}{(u+2)(u-2)}. \quad (*) \end{aligned}$$

Multiply $\dfrac{1}{(u+2)(u-2)} = \dfrac{A}{u+2} + \dfrac{B}{u-2}$ by $(u+2)(u-2)$ to get $1 = A(u-2) + B(u+2)$. Equating coefficients we get $A + B = 0$ and $-2A + 2B = 1$. Solving gives us $B = \frac{1}{4}$ and $A = -\frac{1}{4}$, so

$\dfrac{1}{(u+2)(u-2)} = \dfrac{-1/4}{u+2} + \dfrac{1/4}{u-2}$ and $(*)$ is

$$\begin{aligned} 2 + 8\int_3^4\left(\frac{-1/4}{u+2} + \frac{1/4}{u-2}\right)du &= 2 + 8\left[-\tfrac{1}{4}\ln|u+2| + \tfrac{1}{4}\ln|u-2|\right]_3^4 \\ &= 2 + \Big[2\ln|u-2| - 2\ln|u+2|\Big]_3^4 = 2 + 2\left[\ln\left|\frac{u-2}{u+2}\right|\right]_3^4 \\ &= 2 + 2\left(\ln\tfrac{2}{6} - \ln\tfrac{1}{5}\right) = 2 + 2\ln\tfrac{2/6}{1/5} \\ &= 2 + 2\ln\tfrac{5}{3} \ \text{ or } \ 2 + \ln\left(\tfrac{5}{3}\right)^2 = 2 + \ln\tfrac{25}{9} \end{aligned}$$

42. Let $u = \sqrt[3]{x}$. Then $x = u^3$, $dx = 3u^2\,du \quad\Rightarrow$

$$\begin{aligned} \int_0^1 \frac{1}{1+\sqrt[3]{x}}\,dx &= \int_0^1 \frac{3u^2\,du}{1+u} = \int_0^1\left(3u - 3 + \frac{3}{1+u}\right)du = \left[\tfrac{3}{2}u^2 - 3u + 3\ln(1+u)\right]_0^1 \\ &= 3\left(\ln 2 - \tfrac{1}{2}\right) \end{aligned}$$

43. Let $u = \sqrt[3]{x^2+1}$. Then $x^2 = u^3 - 1$, $2x\,dx = 3u^2\,du \Rightarrow$

$$\int \frac{x^3\,dx}{\sqrt[3]{x^2+1}} = \int \frac{(u^3-1)\frac{3}{2}u^2\,du}{u} = \tfrac{3}{2}\int (u^4 - u)\,du = \tfrac{3}{10}u^5 - \tfrac{3}{4}u^2 + C$$
$$= \tfrac{3}{10}(x^2+1)^{5/3} - \tfrac{3}{4}(x^2+1)^{2/3} + C$$

44. Let $u = \sqrt{x}$. Then $x = u^2$, $dx = 2u\,du \Rightarrow$

$$\int_{1/3}^{3} \frac{\sqrt{x}}{x^2+x}\,dx = \int_{1/\sqrt{3}}^{\sqrt{3}} \frac{u \cdot 2u\,du}{u^4+u^2} = 2\int_{1/\sqrt{3}}^{\sqrt{3}} \frac{du}{u^2+1} = 2\big[\tan^{-1} u\big]_{1/\sqrt{3}}^{\sqrt{3}} = 2\left(\tfrac{\pi}{3} - \tfrac{\pi}{6}\right) = \tfrac{\pi}{3}$$

45. If we were to substitute $u = \sqrt{x}$, then the square root would disappear but a cube root would remain. On the other hand, the substitution $u = \sqrt[3]{x}$ would eliminate the cube root but leave a square root. We can eliminate both roots by means of the substitution $u = \sqrt[6]{x}$. (Note that 6 is the least common multiple of 2 and 3.)

Let $u = \sqrt[6]{x}$. Then $x = u^6$, so $dx = 6u^5\,du$ and $\sqrt{x} = u^3$, $\sqrt[3]{x} = u^2$. Thus,

$$\int \frac{dx}{\sqrt{x} - \sqrt[3]{x}} = \int \frac{6u^5\,du}{u^3 - u^2} = 6\int \frac{u^5}{u^2(u-1)}\,du = 6\int \frac{u^3}{u-1}\,du$$
$$= 6\int \left(u^2 + u + 1 + \frac{1}{u-1}\right) du \qquad \text{[by long division]}$$
$$= 6\left(\tfrac{1}{3}u^3 + \tfrac{1}{2}u^2 + u + \ln|u-1|\right) + C = 2\sqrt{x} + 3\sqrt[3]{x} + 6\sqrt[6]{x} + 6\ln\left|\sqrt[6]{x} - 1\right| + C$$

46. Let $u = \sqrt[12]{x}$. Then $x = u^{12}$, $dx = 12u^{11}\,du \Rightarrow$

$$\int \frac{dx}{\sqrt[3]{x} + \sqrt[4]{x}} = \int \frac{12u^{11}du}{u^4+u^3} = 12\int \frac{u^8du}{u+1} = 12\int \left(u^7 - u^6 + u^5 - u^4 + u^3 - u^2 + u - 1 + \frac{1}{u+1}\right) du$$
$$= \tfrac{3}{2}u^8 - \tfrac{12}{7}u^7 + 2u^6 - \tfrac{12}{5}u^5 + 3u^4 - 4u^3 + 6u^2 - 12u + 12\ln|u+1| + C$$
$$= \tfrac{3}{2}x^{2/3} - \tfrac{12}{7}x^{7/12} + 2\sqrt{x} - \tfrac{12}{5}x^{5/12} + 3\sqrt[3]{x} - 4\sqrt[4]{x} + 6\sqrt[6]{x} - 12\sqrt[12]{x} + 12\ln\left(\sqrt[12]{x} + 1\right) + C$$

47. Let $u = e^x$. Then $x = \ln u$, $dx = \dfrac{du}{u} \Rightarrow$

$$\int \frac{e^{2x}\,dx}{e^{2x} + 3e^x + 2} = \int \frac{u^2\,(du/u)}{u^2 + 3u + 2} = \int \frac{u\,du}{(u+1)(u+2)} = \int \left[\frac{-1}{u+1} + \frac{2}{u+2}\right] du$$
$$= 2\ln|u+2| - \ln|u+1| + C = \ln\left[(e^x+2)^2/(e^x+1)\right] + C$$

48. Let $u = \sin x$. Then $du = \cos x\,dx \Rightarrow$

$$\int \frac{\cos x\,dx}{\sin^2 x + \sin x} = \int \frac{du}{u^2+u} = \int \frac{du}{u(u+1)} = \int \left[\frac{1}{u} - \frac{1}{u+1}\right] du$$
$$= \ln\left|\frac{u}{u+1}\right| + C = \ln\left|\frac{\sin x}{1+\sin x}\right| + C$$

49. Let $u = \ln(x^2 - x + 2)$, $dv = dx$. Then $du = \dfrac{2x-1}{x^2-x+2}\,dx$, $v = x$, and (by integration by parts)

$$\int \ln(x^2-x+2)\,dx = x\ln(x^2-x+2) - \int \frac{2x^2-x}{x^2-x+2}\,dx = x\ln(x^2-x+2) - \int\left(2 + \frac{x-4}{x^2-x+2}\right)dx$$

$$= x\ln(x^2-x+2) - 2x - \int \frac{\frac{1}{2}(2x-1)}{x^2-x+2}\,dx + \frac{7}{2}\int \frac{dx}{(x-\frac{1}{2})^2+\frac{7}{4}}$$

$$= x\ln(x^2-x+2) - 2x - \frac{1}{2}\ln(x^2-x+2) + \frac{7}{2}\int \frac{\frac{\sqrt{7}}{2}\,du}{\frac{7}{4}(u^2+1)} \quad \left[\begin{array}{c}\text{where } x-\frac{1}{2} = \frac{\sqrt{7}}{2}u,\\ dx = \frac{\sqrt{7}}{2}\,du,\\ (x-\frac{1}{2})^2+\frac{7}{4} = \frac{7}{4}(u^2+1)\end{array}\right.$$

$$= (x-\tfrac{1}{2})\ln(x^2-x+2) - 2x + \sqrt{7}\tan^{-1}u + C$$

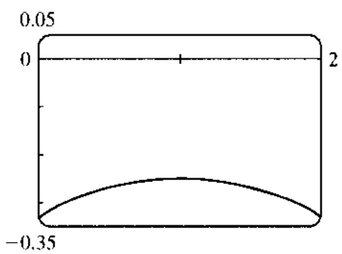
$$= (x-\tfrac{1}{2})\ln(x^2-x+2) - 2x + \sqrt{7}\tan^{-1}\frac{2x-1}{\sqrt{7}} + C$$

50. Let $u = \tan^{-1}x$, $dv = x\,dx \quad \Rightarrow \quad du = dx/(1+x^2)$, $v = \frac{1}{2}x^2$.

Then $\int x\tan^{-1}x\,dx = \frac{1}{2}x^2\tan^{-1}x - \dfrac{1}{2}\displaystyle\int \frac{x^2}{1+x^2}\,dx$. To evaluate the last integral, use long division or observe that $\displaystyle\int \frac{x^2}{1+x^2}\,dx = \int \frac{(1+x^2)-1}{1+x^2}\,dx = \int 1\,dx - \int \frac{1}{1+x^2}\,dx = x - \tan^{-1}x + C_1$. So

$\int x\tan^{-1}x\,dx = \frac{1}{2}x^2\tan^{-1}x - \frac{1}{2}(x - \tan^{-1}x + C_1) = \frac{1}{2}(x^2\tan^{-1}x + \tan^{-1}x - x) + C$.

51.



From the graph, we see that the integral will be negative, and we guess that the area is about the same as that of a rectangle with width 2 and height 0.3, so we estimate the integral to be $-(2\cdot 0.3) = -0.6$. Now

$$\frac{1}{x^2-2x-3} = \frac{1}{(x-3)(x+1)} = \frac{A}{x-3} + \frac{B}{x+1} \quad \Leftrightarrow$$

$1 = (A+B)x + A - 3B$, so $A = -B$ and $A - 3B = 1 \quad \Leftrightarrow \quad A = \frac{1}{4}$ and $B = -\frac{1}{4}$, so the integral becomes

$$\int_0^2 \frac{dx}{x^2-2x-3} = \frac{1}{4}\int_0^2 \frac{dx}{x-3} - \frac{1}{4}\int_0^2 \frac{dx}{x+1} = \tfrac{1}{4}\Big[\ln|x-3| - \ln|x+1|\Big]_0^2$$

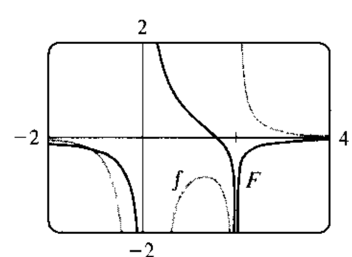
$$= \frac{1}{4}\left[\ln\left|\frac{x-3}{x+1}\right|\right]_0^2 = \tfrac{1}{4}(\ln\tfrac{1}{3} - \ln 3) = -\tfrac{1}{2}\ln 3 \approx -0.55$$

52. $\dfrac{1}{x^3-2x^2} = \dfrac{1}{x^2(x-2)} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x-2} \Rightarrow 1 = (A+C)x^2 + (B-2A)x - 2B$, so $A + C = B - 2A = 0$ and $-2B = 1 \quad \Rightarrow \quad B = -\frac{1}{2}$, $A = -\frac{1}{4}$, and $C = \frac{1}{4}$. So the general antiderivative of $\dfrac{1}{x^3-2x^2}$ is

$$\int \frac{dx}{x^3-2x^2} = -\frac{1}{4}\int\frac{dx}{x} - \frac{1}{2}\int\frac{dx}{x^2} + \frac{1}{4}\int\frac{dx}{x-2}$$

$$= -\tfrac{1}{4}\ln|x| - \tfrac{1}{2}(-1/x) + \tfrac{1}{4}\ln|x-2| + C$$

$$= \frac{1}{4}\ln\left|\frac{x-2}{x}\right| + \frac{1}{2x} + C$$



We plot this function with $C = 0$ on the same screen as $y = \dfrac{1}{x^3-2x^2}$.

53. $\displaystyle\int \frac{dx}{x^2-2x} = \int \frac{dx}{(x-1)^2-1} = \int \frac{du}{u^2-1}$ [put $u = x - 1$]

$\displaystyle = \frac{1}{2}\ln\left|\frac{u-1}{u+1}\right| + C$ [by Equation 6] $\displaystyle = \frac{1}{2}\ln\left|\frac{x-2}{x}\right| + C$

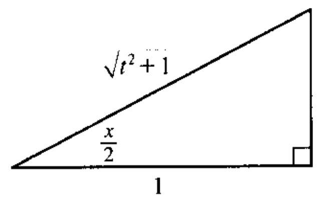
54. $\displaystyle\int \frac{(2x+1)\,dx}{4x^2+12x-7} = \frac{1}{4}\int \frac{(8x+12)\,dx}{4x^2+12x-7} - \int \frac{2\,dx}{(2x+3)^2-16}$

$\displaystyle = \tfrac{1}{4}\ln\left|4x^2+12x-7\right| - \int \frac{du}{u^2-16}$ [put $u = 2x+3$]

$= \tfrac{1}{4}\ln\left|4x^2+12x-7\right| - \tfrac{1}{8}\ln\left|(u-4)/(u+4)\right| + C$ [by Equation 6]

$= \tfrac{1}{4}\ln\left|4x^2+12x-7\right| - \tfrac{1}{8}\ln\left|(2x-1)/(2x+7)\right| + C$

55. (a) If $t = \tan\left(\frac{x}{2}\right)$, then $\frac{x}{2} = \tan^{-1} t$. The figure gives



$\displaystyle\cos\left(\frac{x}{2}\right) = \frac{1}{\sqrt{1+t^2}}$ and $\displaystyle\sin\left(\frac{x}{2}\right) = \frac{t}{\sqrt{1+t^2}}$.

(b) $\displaystyle\cos x = \cos\left(2\cdot\frac{x}{2}\right) = 2\cos^2\left(\frac{x}{2}\right) - 1$

$\displaystyle = 2\left(\frac{1}{\sqrt{1+t^2}}\right)^2 - 1 = \frac{2}{1+t^2} - 1 = \frac{1-t^2}{1+t^2}$

$\displaystyle\sin x = \sin\left(2\cdot\frac{x}{2}\right) = 2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right) = 2\,\frac{t}{\sqrt{1+t^2}}\,\frac{1}{\sqrt{1+t^2}} = \frac{2t}{1+t^2}$

(c) $\displaystyle\frac{x}{2} = \arctan t \;\Rightarrow\; x = 2\arctan t \;\Rightarrow\; dx = \frac{2}{1+t^2}\,dt$

56. Let $t = \tan(x/2)$. Then, using Exercise 55, $\displaystyle dx = \frac{2}{1+t^2}\,dt$, $\displaystyle\sin x = \frac{2t}{1+t^2}$ $\Rightarrow$

$$\int \frac{dx}{3-5\sin x} = \int \frac{2\,dt/(1+t^2)}{3-10t/(1+t^2)} = \int \frac{2\,dt}{3(1+t^2)-10t} = 2\int \frac{dt}{3t^2-10t+3}$$

$$= \frac{1}{4}\int\left[\frac{1}{t-3} - \frac{3}{3t-1}\right]dt = \tfrac{1}{4}\left(\ln|t-3| - \ln|3t-1|\right) + C = \frac{1}{4}\ln\left|\frac{\tan(x/2)-3}{3\tan(x/2)-1}\right| + C$$

57. Let $t = \tan(x/2)$. Then, using the expressions in Exercise 55, we have

$$\int \frac{1}{3\sin x - 4\cos x}\,dx = \int \frac{1}{3\left(\frac{2t}{1+t^2}\right) - 4\left(\frac{1-t^2}{1+t^2}\right)}\,\frac{2\,dt}{1+t^2} = 2\int \frac{dt}{3(2t)-4(1-t^2)} = \int \frac{dt}{2t^2+3t-2}$$

$$= \int \frac{dt}{(2t-1)(t+2)} = \int\left[\frac{2}{5}\,\frac{1}{2t-1} - \frac{1}{5}\,\frac{1}{t+2}\right]dt \quad \text{[using partial fractions]}$$

$$= \tfrac{1}{5}\Big[\ln|2t-1| - \ln|t+2|\Big] + C = \frac{1}{5}\ln\left|\frac{2t-1}{t+2}\right| + C = \frac{1}{5}\ln\left|\frac{2\tan(x/2)-1}{\tan(x/2)+2}\right| + C$$

58. Let $t = \tan(x/2)$. Then, by Exercise 55,

$$\int_{\pi/3}^{\pi/2} \frac{dx}{1+\sin x - \cos x} = \int_{1/\sqrt{3}}^{1} \frac{2dt/(1+t^2)}{1+2t/(1+t^2)-(1-t^2)/(1+t^2)} = \int_{1/\sqrt{3}}^{1} \frac{2dt}{1+t^2+2t-1+t^2}$$

$$= \int_{1/\sqrt{3}}^{1}\left[\frac{1}{t} - \frac{1}{t+1}\right]dt = \Big[\ln t - \ln(t+1)\Big]_{1/\sqrt{3}}^{1} = \ln\frac{1}{2} - \ln\frac{1}{\sqrt{3}+1} = \ln\frac{\sqrt{3}+1}{2}$$

59. Let $t = \tan(x/2)$. Then, by Exercise 55,

$$\int \frac{dx}{2\sin x + \sin 2x} = \frac{1}{2}\int \frac{dx}{\sin x + \sin x \cos x} = \frac{1}{2}\int \frac{2\,dt/(1+t^2)}{2t/(1+t^2) + 2t(1-t^2)/(1+t^2)^2}$$

$$= \frac{1}{2}\int \frac{(1+t^2)\,dt}{t(1+t^2)+t(1-t^2)} = \frac{1}{4}\int \frac{(1+t^2)\,dt}{t} = \frac{1}{4}\int\left(\frac{1}{t}+t\right)dt$$

$$= \tfrac{1}{4}\ln|t| + \tfrac{1}{8}t^2 + C = \tfrac{1}{4}\ln\left|\tan\left(\tfrac{1}{2}x\right)\right| + \tfrac{1}{8}\tan^2\left(\tfrac{1}{2}x\right) + C$$

60. $x^2 - 6x + 8 = (x-3)^2 - 1$ is positive for $5 \le x \le 10$, so

$$\text{area} = \int_5^{10} \frac{dx}{(x-3)^2 - 1} = \int_2^7 \frac{du}{u^2-1} \quad [\text{put } u = x-3] \quad = \left[\frac{1}{2}\ln\left|\frac{u-1}{u+1}\right|\right]_2^7$$

$$= \tfrac{1}{2}\ln\tfrac{3}{4} - \tfrac{1}{2}\ln\tfrac{1}{3} = \tfrac{1}{2}(\ln 3 - 2\ln 2 + \ln 3) = \ln 3 - \ln 2 = \ln\tfrac{3}{2}$$

61. $\dfrac{x+1}{x-1} = 1 + \dfrac{2}{x-1} > 0$ for $2 \le x \le 3$, so

$$\text{area} = \int_2^3 \left[1 + \frac{2}{x-1}\right]dx = \Big[x + 2\ln|x-1|\Big]_2^3 = (3 + 2\ln 2) - (2 + 2\ln 1) = 1 + 2\ln 2.$$

62. (a) We use disks, so the volume is $V = \pi\int_0^1 \left[\dfrac{1}{x^2+3x+2}\right]^2 dx = \pi\int_0^1 \dfrac{dx}{(x+1)^2(x+2)^2}$. To evaluate the integral, we use partial fractions: $\dfrac{1}{(x+1)^2(x+2)^2} = \dfrac{A}{x+1} + \dfrac{B}{(x+1)^2} + \dfrac{C}{x+2} + \dfrac{D}{(x+2)^2} \Rightarrow$ $1 = A(x+1)(x+2)^2 + B(x+2)^2 + C(x+1)^2(x+2) + D(x+1)^2$. We set $x = -1$, giving $B = 1$, then set $x = -2$, giving $D = 1$. Now equating coefficients of x^3 gives $A = -C$, and then equating constants gives $1 = 4A + 4 + 2(-A) + 1 \Rightarrow A = -2 \Rightarrow C = 2$. So the expression becomes

$$V = \pi\int_0^1 \left[\frac{-2}{x+1} + \frac{1}{(x+1)^2} + \frac{2}{(x+2)} + \frac{1}{(x+2)^2}\right]dx = \pi\left[2\ln\left|\frac{x+2}{x+1}\right| - \frac{1}{x+1} - \frac{1}{x+2}\right]_0^1$$

$$= \pi\left[\left(2\ln\tfrac{3}{2} - \tfrac{1}{2} - \tfrac{1}{3}\right) - \left(2\ln 2 - 1 - \tfrac{1}{2}\right)\right] = \pi\left(2\ln\tfrac{3/2}{2} + \tfrac{2}{3}\right) = \pi\left(\tfrac{2}{3} + \ln\tfrac{9}{16}\right)$$

(b) In this case, we use cylindrical shells, so the volume is $V = 2\pi\displaystyle\int_0^1 \frac{x\,dx}{x^2+3x+2} = 2\pi\int_0^1 \frac{x\,dx}{(x+1)(x+2)}$.

We use partial fractions to simplify the integrand: $\dfrac{x}{(x+1)(x+2)} = \dfrac{A}{x+1} + \dfrac{B}{x+2} \Rightarrow$ $x = (A+B)x + 2A + B$. So $A + B = 1$ and $2A + B = 0 \Rightarrow A = -1$ and $B = 2$. So the volume is

$$2\pi\int_0^1 \left[\frac{-1}{x+1} + \frac{2}{x+2}\right]dx = 2\pi\Big[-\ln|x+1| + 2\ln|x+2|\Big]_0^1$$

$$= 2\pi(-\ln 2 + 2\ln 3 + \ln 1 - 2\ln 2) = 2\pi(2\ln 3 - 3\ln 2) = 2\pi\ln\tfrac{9}{8}$$

63. $\dfrac{P+S}{P[(r-1)P-S]} = \dfrac{A}{P} + \dfrac{B}{(r-1)P-S} \Rightarrow P + S = A[(r-1)P - S] + BP = [(r-1)A + B]P - AS$ $\Rightarrow (r-1)A + B = 1, -A = 1 \Rightarrow A = -1, B = r$. Now

$$t = \int \frac{P+S}{P[(r-1)P-S]}\,dP = \int\left[\frac{-1}{P} + \frac{r}{(r-1)P-S}\right]dP = -\int\frac{dP}{P} + \frac{r}{r-1}\int\frac{r-1}{(r-1)P-S}\,dP$$

so $t = -\ln P + \dfrac{r}{r-1}\ln|(r-1)P - S| + C$. Here $r = 0.10$ and $S = 900$, so

$$t = -\ln P + \tfrac{0.1}{-0.9}\ln|-0.9P - 900| + C = -\ln P - \tfrac{1}{9}\ln(|-1|\,|0.9P + 900|)$$
$$= -\ln P - \tfrac{1}{9}\ln(0.9P + 900) + C$$

When $t = 0$, $P = 10{,}000$, so $0 = -\ln 10{,}000 - \frac{1}{9}\ln(9900) + C$. Thus, $C = \ln 10{,}000 + \frac{1}{9}\ln 9900$ $[\approx 10.2326]$, so our equation becomes

$$t = \ln 10,000 - \ln P + \tfrac{1}{9}\ln 9900 - \tfrac{1}{9}\ln(0.9P + 900) = \ln\frac{10{,}000}{P} + \frac{1}{9}\ln\frac{9900}{0.9P + 900}$$
$$= \ln\frac{10{,}000}{P} + \frac{1}{9}\ln\frac{1100}{0.1P + 100} = \ln\frac{10{,}000}{P} + \frac{1}{9}\ln\frac{11{,}000}{P + 1000}$$

64. If we subtract and add $2x^2$, we get

$$x^4 + 1 = x^4 + 2x^2 + 1 - 2x^2 = (x^2+1)^2 - 2x^2 = (x^2+1)^2 - \left(\sqrt{2}\,x\right)^2$$
$$= \left[(x^2+1) - \sqrt{2}\,x\right]\left[(x^2+1) + \sqrt{2}\,x\right] = \left(x^2 - \sqrt{2}\,x + 1\right)\left(x^2 + \sqrt{2}\,x + 1\right)$$

So we can decompose $\dfrac{1}{x^4+1} = \dfrac{Ax+B}{x^2+\sqrt{2}x+1} + \dfrac{Cx+D}{x^2-\sqrt{2}x+1} \Rightarrow$
$1 = (Ax+B)\left(x^2 - \sqrt{2}x + 1\right) + (Cx+D)\left(x^2 + \sqrt{2}x + 1\right)$. Setting the constant terms equal gives $B + D = 1$, then from the coefficients of x^3 we get $A + C = 0$. Now from the coefficients of x we get
$A + C + (B - D)\sqrt{2} = 0 \;\Leftrightarrow\; [(1-D) - D]\sqrt{2} = 0 \;\Rightarrow\; D = \frac{1}{2} \;\Rightarrow\; B = \frac{1}{2}$, and finally, from the coefficients of x^2 we get $\sqrt{2}\,(C - A) + B + D = 0 \;\Rightarrow\; C - A = -\frac{1}{\sqrt{2}} \;\Rightarrow\; C = -\frac{\sqrt{2}}{4}$ and $A = \frac{\sqrt{2}}{4}$.
So we rewrite the integrand, splitting the terms into forms which we know how to integrate:

$$\frac{1}{x^4+1} = \frac{\frac{\sqrt{2}}{4}x + \frac{1}{2}}{x^2 + \sqrt{2}\,x + 1} + \frac{-\frac{\sqrt{2}}{4}x + \frac{1}{2}}{x^2 - \sqrt{2}\,x + 1} = \frac{1}{4\sqrt{2}}\left[\frac{2x + 2\sqrt{2}}{x^2 + \sqrt{2}\,x + 1} - \frac{2x - 2\sqrt{2}}{x^2 - \sqrt{2}\,x + 1}\right]$$
$$= \frac{\sqrt{2}}{8}\left[\frac{2x + \sqrt{2}}{x^2 + \sqrt{2}\,x + 1} - \frac{2x - \sqrt{2}}{x^2 - \sqrt{2}\,x + 1}\right] + \frac{1}{4}\left[\frac{1}{\left(x + \frac{1}{\sqrt{2}}\right)^2 + \frac{1}{2}} + \frac{1}{\left(x - \frac{1}{\sqrt{2}}\right)^2 + \frac{1}{2}}\right]$$

Now we integrate:
$$\int \frac{dx}{x^4+1} = \frac{\sqrt{2}}{8}\ln\left(\frac{x^2 + \sqrt{2}\,x + 1}{x^2 - \sqrt{2}\,x + 1}\right) + \frac{\sqrt{2}}{4}\left[\tan^{-1}\left(\sqrt{2}\,x + 1\right) + \tan^{-1}\left(\sqrt{2}\,x - 1\right)\right] + C.$$

65. (a) In Maple, we define $f(x)$, and then use `convert(f,parfrac,x);` to obtain
$$f(x) = \frac{24{,}110/4879}{5x+2} - \frac{668/323}{2x+1} - \frac{9438/80{,}155}{3x-7} + \frac{(22{,}098x + 48{,}935)/260{,}015}{x^2 + x + 5}.$$
In Mathematica, we use the command `Apart`, and in Derive, we use `Expand`.

(b) $$\int f(x)\,dx = \tfrac{24{,}110}{4879}\cdot\tfrac{1}{5}\ln|5x+2| - \tfrac{668}{323}\cdot\tfrac{1}{2}\ln|2x+1| - \tfrac{9438}{80{,}155}\cdot\tfrac{1}{3}\ln|3x-7| + \frac{1}{260{,}015}\int\frac{22{,}098\left(x+\frac{1}{2}\right)+37{,}886}{\left(x+\frac{1}{2}\right)^2+\frac{19}{4}}\,dx + C$$

$$= \tfrac{24{,}110}{4879}\cdot\tfrac{1}{5}\ln|5x+2| - \tfrac{668}{323}\cdot\tfrac{1}{2}\ln|2x+1| - \tfrac{9438}{80{,}155}\cdot\tfrac{1}{3}\ln|3x-7| + \tfrac{1}{260{,}015}\left[22{,}098\cdot\tfrac{1}{2}\ln(x^2+x+5) + 37{,}886\cdot\sqrt{\tfrac{4}{19}}\tan^{-1}\left(\tfrac{1}{\sqrt{19/4}}\left(x+\tfrac{1}{2}\right)\right)\right] + C$$

$$= \tfrac{4822}{4879}\ln|5x+2| - \tfrac{334}{323}\ln|2x+1| - \tfrac{3146}{80{,}155}\ln|3x-7| + \tfrac{11{,}049}{260{,}015}\ln(x^2+x+5) + \tfrac{75{,}772}{260{,}015\sqrt{19}}\tan^{-1}\left[\tfrac{1}{\sqrt{19}}(2x+1)\right] + C$$

Using a CAS, we get

$$\frac{4822\ln(5x+2)}{4879} - \frac{334\ln(2x+1)}{323} - \frac{3146\ln(3x-7)}{80{,}155} + \frac{11{,}049\ln(x^2+x+5)}{260{,}015} + \frac{3988\sqrt{19}}{260{,}015}\tan^{-1}\left[\frac{\sqrt{19}}{19}(2x+1)\right]$$

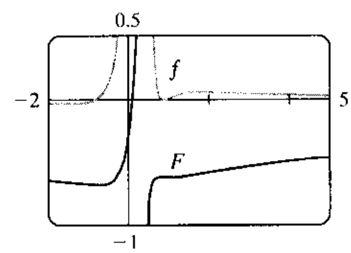
The main difference in this answer is that the absolute value signs and the constant of integration have been omitted. Also, the fractions have been reduced and the denominators rationalized.

66. (a) In Maple, we define $f(x)$, and then use `convert(f,parfrac,x);` to get

$$f(x) = \frac{5828/1815}{(5x-2)^2} - \frac{59{,}096/19{,}965}{5x-2} + \frac{2(2843x+816)/3993}{2x^2+1} + \frac{(313x-251)/363}{(2x^2+1)^2}.$$

In Mathematica, we use the command `Apart`, and in Derive, we use `Expand`.

(b) As we saw in Exercise 65, computer algebra systems omit the absolute value signs in $\int (1/y)\,dy = \ln|y|$. So we use the CAS to integrate the expression in part (a) and add the necessary absolute value signs and constant of integration to get



$$\int f(x)\,dx = -\frac{5828}{9075(5x-2)} - \frac{59{,}096\ln|5x-2|}{99{,}825} + \frac{2843\ln(2x^2+1)}{7986} + \frac{503}{15{,}972}\sqrt{2}\tan^{-1}\left(\sqrt{2}\,x\right) - \frac{1}{2904}\frac{1004x+626}{2x^2+1} + C$$

(c) From the graph, we see that f goes from negative to positive at $x \approx -0.78$, then back to negative at $x \approx 0.8$, and finally back to positive at $x = 1$. Also, $\lim_{x\to 0.4} f(x) = \infty$. So we see (by the First Derivative Test) that $\int f(x)\,dx$ has minima at $x \approx -0.78$ and $x = 1$, and a maximum at $x \approx 0.80$, and that $\int f(x)\,dx$ is unbounded as $x \to 0.4$. Note also that just to the right of $x = 0.4$, f has large values, so $\int f(x)\,dx$ increases rapidly, but slows down as f drops toward 0. $\int f(x)\,dx$ decreases from about 0.8 to 1, then increases slowly since f stays small and positive.

67. There are only finitely many values of x where $Q(x) = 0$ (assuming that Q is not the zero polynomial). At all other values of x, $F(x)/Q(x) = G(x)/Q(x)$, so $F(x) = G(x)$. In other words, the values of F and G agree at all except perhaps finitely many values of x. By continuity of F and G, the polynomials F and G must agree at those values of x too.

More explicitly: if a is a value of x such that $Q(a) = 0$, then $Q(x) \neq 0$ for all x sufficiently close to a. Thus,

$$F(a) = \lim_{x \to a} F(x) \text{ [by continuity of } F] = \lim_{x \to a} G(x) \quad [\text{whenever } Q(x) \neq 0]$$
$$= G(a) \quad [\text{by continuity of } G]$$

68. Let $f(x) = ax^2 + bx + c$. We calculate the partial fraction decomposition of $\dfrac{f(x)}{x^2(x+1)^3}$. Since $f(0) = 1$, we must have $c = 1$, so $\dfrac{f(x)}{x^2(x+1)^3} = \dfrac{ax^2 + bx + 1}{x^2(x+1)^3} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x+1} + \dfrac{D}{(x+1)^2} + \dfrac{E}{(x+1)^3}$. Now in order for the integral not to contain any logarithms (that is, in order for it to be a rational function), we must have $A = C = 0$, so $ax^2 + bx + 1 = B(x+1)^3 + Dx^2(x+1) + Ex^2$. Equating constant terms gives $B = 1$, then equating coefficients of x gives $3B = b \quad \Rightarrow \quad b = 3$. This is the quantity we are looking for, since $f'(0) = b$.

8.5 Strategy for Integration

1. $\displaystyle\int \frac{\sin x + \sec x}{\tan x}\,dx = \int \left(\frac{\sin x}{\tan x} + \frac{\sec x}{\tan x}\right) dx = \int (\cos x + \csc x)\,dx = \sin x + \ln|\csc x - \cot x| + C$

2. $\displaystyle\int \tan^3\theta\,d\theta = \int (\sec^2\theta - 1)\tan\theta\,d\theta = \int \tan\theta\sec^2\theta\,d\theta - \int \frac{\sin\theta}{\cos\theta}\,d\theta$

$$= \int u\,du + \int \frac{dv}{v} \quad \begin{bmatrix} u = \tan\theta, & v = \cos\theta, \\ du = \sec^2\theta\,d\theta & dv = -\sin\theta\,d\theta \end{bmatrix}$$

$$= \tfrac{1}{2}u^2 + \ln|v| + C = \tfrac{1}{2}\tan^2\theta + \ln|\cos\theta| + C$$

3. $\displaystyle\int_0^2 \frac{2t}{(t-3)^2}\,dt = \int_{-3}^{-1} \frac{2(u+3)}{u^2}\,du \ [u = t-3,\ du = dt] = \int_{-3}^{-1} \left(\frac{2}{u} + \frac{6}{u^2}\right) du = \left[2\ln|u| - \frac{6}{u}\right]_{-3}^{-1}$

$$= (2\ln 1 + 6) - (2\ln 3 + 2) = 4 - 2\ln 3 \text{ or } 4 - \ln 9$$

4. Let $u = x^2$. Then $du = 2x\,dx \quad \Rightarrow \quad \displaystyle\int \frac{x\,dx}{\sqrt{3 - x^4}} = \frac{1}{2}\int \frac{du}{\sqrt{3 - u^2}} = \frac{1}{2}\sin^{-1}\frac{u}{\sqrt{3}} + C = \frac{1}{2}\sin^{-1}\frac{x^2}{\sqrt{3}} + C.$

5. Let $u = \arctan y$. Then $du = \dfrac{dy}{1 + y^2} \quad \Rightarrow \quad \displaystyle\int_{-1}^{1} \frac{e^{\arctan y}}{1 + y^2}\,dy = \int_{-\pi/4}^{\pi/4} e^u\,du = [e^u]_{-\pi/4}^{\pi/4} = e^{\pi/4} - e^{-\pi/4}.$

6. $\displaystyle\int x\csc x\cot x\,dx \quad \begin{bmatrix} u = x, & dv = \csc x\cot x\,dx, \\ du = dx & v = -\csc x \end{bmatrix} = -x\csc x - \int (-\csc x)\,dx$

$$= -x\csc x + \ln|\csc x - \cot x| + C$$

7. $\displaystyle\int_1^3 r^4 \ln r\,dr \quad \begin{bmatrix} u = \ln r, & dv = r^4\,dr, \\ du = \dfrac{dr}{r} & v = \dfrac{1}{5}r^5 \end{bmatrix} = \left[\frac{1}{5}r^5 \ln r\right]_1^3 - \int_1^3 \frac{1}{5}r^4\,dr = \frac{243}{5}\ln 3 - 0 - \left[\frac{1}{25}r^5\right]_1^3$

$$= \tfrac{243}{5}\ln 3 - \left(\tfrac{243}{25} - \tfrac{1}{25}\right) = \tfrac{243}{5}\ln 3 - \tfrac{242}{25}$$

8. $\dfrac{x-1}{x^2-4x-5} = \dfrac{x-1}{(x-5)(x+1)} = \dfrac{A}{x-5} + \dfrac{B}{x+1} \Rightarrow x - 1 = A(x+1) + B(x-5)$. Setting $x = -1$ gives $-2 = -6B$, so $B = \frac{1}{3}$. Setting $x = 5$ gives $4 = 6A$, so $A = \frac{2}{3}$. Now

$$\int_0^4 \frac{x-1}{x^2-4x-5}\,dx = \int_0^4 \left(\frac{2/3}{x-5} + \frac{1/3}{x+1}\right) dx = \left[\tfrac{2}{3}\ln|x-5| + \tfrac{1}{3}\ln|x+1|\right]_0^4$$
$$= \tfrac{2}{3}\ln 1 + \tfrac{1}{3}\ln 5 - \tfrac{2}{3}\ln 5 - \tfrac{1}{3}\ln 1 = -\tfrac{1}{3}\ln 5$$

9. $\displaystyle\int \frac{x-1}{x^2-4x+5}\,dx = \int \frac{(x-2)+1}{(x-2)^2+1}\,dx = \int \left(\frac{u}{u^2+1} + \frac{1}{u^2+1}\right) du \quad [u = x-2,\ du = dx]$

$$= \tfrac{1}{2}\ln(u^2+1) + \tan^{-1}u + C = \tfrac{1}{2}\ln(x^2-4x+5) + \tan^{-1}(x-2) + C$$

10. $\displaystyle\int \frac{x}{x^4+x^2+1}\,dx = \int \frac{\frac{1}{2}\,du}{u^2+u+1} \quad [u = x^2,\ du = 2x\,dx] \quad = \frac{1}{2}\int \frac{du}{\left(u+\frac{1}{2}\right)^2 + \frac{3}{4}}$

$$= \frac{1}{2}\int \frac{\frac{\sqrt{3}}{2}\,dv}{\frac{3}{4}(v^2+1)} \quad \left[u + \tfrac{1}{2} = \tfrac{\sqrt{3}}{2}v,\ du = \tfrac{\sqrt{3}}{2}\,dv\right] \quad = \frac{\sqrt{3}}{4}\cdot\frac{4}{3}\int \frac{dv}{v^2+1}$$
$$= \tfrac{1}{\sqrt{3}}\tan^{-1}v + C = \tfrac{1}{\sqrt{3}}\tan^{-1}\left(\tfrac{2}{\sqrt{3}}\left(x^2+\tfrac{1}{2}\right)\right) + C$$

11. $\int \sin^3\theta\,\cos^5\theta\,d\theta = \int \cos^5\theta\sin^2\theta\sin\theta\,d\theta = -\int \cos^5\theta\,(1-\cos^2\theta)(-\sin\theta)\,d\theta$

$$= -\int u^5(1-u^2)\,du \quad \begin{bmatrix} u = \cos\theta, \\ du = -\sin\theta\,d\theta \end{bmatrix}$$
$$= \int (u^7 - u^5)\,du = \tfrac{1}{8}u^8 - \tfrac{1}{6}u^6 + C = \tfrac{1}{8}\cos^8\theta - \tfrac{1}{6}\cos^6\theta + C$$

Another solution:

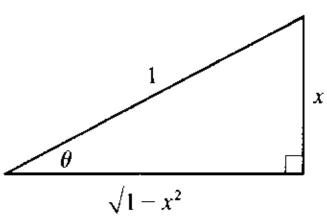
$\int \sin^3\theta\,\cos^5\theta\,d\theta = \int \sin^3\theta\,(\cos^2\theta)^2\cos\theta\,d\theta = \int \sin^3\theta\,(1-\sin^2\theta)^2\cos\theta\,d\theta$

$$= \int u^3(1-u^2)^2\,du \quad \begin{bmatrix} u = \sin\theta, \\ du = \cos\theta\,d\theta \end{bmatrix} \quad = \int u^3(1-2u^2+u^4)\,du$$
$$= \int (u^3 - 2u^5 + u^7)\,du = \tfrac{1}{4}u^4 - \tfrac{1}{3}u^6 + \tfrac{1}{8}u^8 + C = \tfrac{1}{4}\sin^4\theta - \tfrac{1}{3}\sin^6\theta + \tfrac{1}{8}\sin^8\theta + C$$

12. Let $u = \cos x$. Then $du = -\sin x\,dx \Rightarrow$
$\int \sin x\cos(\cos x)\,dx = -\int \cos u\,du = -\sin u + C = -\sin(\cos x) + C.$

13. Let $x = \sin\theta$, where $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. Then $dx = \cos\theta\,d\theta$ and $(1-x^2)^{1/2} = \cos\theta$, so

$$\int \frac{dx}{(1-x^2)^{3/2}} = \int \frac{\cos\theta\,d\theta}{(\cos\theta)^3} = \int \sec^2\theta\,d\theta$$
$$= \tan\theta + C = \frac{x}{\sqrt{1-x^2}} + C$$



14. Let $u = \ln x$. Then $du = dx/x \Rightarrow$

$$\int \frac{\sqrt{1+\ln x}}{x\ln x}\,dx = \int \frac{\sqrt{1+u}}{u}\,du = \int \frac{v}{v^2-1}\,2v\,dv \quad \left[\text{put } v = \sqrt{1+u},\ u = v^2-1,\ du = 2v\,dv\right]$$
$$= 2\int \left(1 + \frac{1}{v^2-1}\right) dv = 2v + \ln\left|\frac{v-1}{v+1}\right| + C = 2\sqrt{1+\ln x} + \ln\left(\frac{\sqrt{1+\ln x}-1}{\sqrt{1+\ln x}+1}\right) + C$$

15. Let $u = 1 - x^2 \Rightarrow du = -2x\,dx$. Then

$$\int_0^{1/2} \frac{x}{\sqrt{1-x^2}}\,dx = -\frac{1}{2}\int_1^{3/4} \frac{1}{\sqrt{u}}\,du = \frac{1}{2}\int_{3/4}^1 u^{-1/2}\,du = \tfrac{1}{2}\left[2u^{1/2}\right]_{3/4}^1 = \left[\sqrt{u}\right]_{3/4}^1 = 1 - \tfrac{\sqrt{3}}{2}$$

16. $\displaystyle\int_0^{\sqrt{2}/2} \frac{x^2}{\sqrt{1-x^2}}\,dx = \int_0^{\pi/4} \frac{\sin^2\theta}{\cos\theta}\cos\theta\,d\theta \qquad [x = \sin\theta,\ dx = \cos\theta\,d\theta]$

$= \int_0^{\pi/4} \frac{1}{2}(1-\cos 2\theta)\,d\theta = \frac{1}{2}\left[\theta - \frac{1}{2}\sin 2\theta\right]_0^{\pi/4} = \frac{1}{2}\left[\left(\frac{\pi}{4} - \frac{1}{2}\right) - (0-0)\right] = \frac{\pi}{8} - \frac{1}{4}$

17. $\int x\sin^2 x\,dx \qquad \begin{bmatrix} u = x, & dv = \sin^2 x\,dx, \\ du = dx & v = \int \sin^2 x\,dx = \int \frac{1}{2}(1-\cos 2x)\,dx = \frac{1}{2}x - \frac{1}{2}\sin x\cos x \end{bmatrix}$

$= \frac{1}{2}x^2 - \frac{1}{2}x\sin x\cos x - \int\left(\frac{1}{2}x - \frac{1}{2}\sin x\cos x\right)dx$

$= \frac{1}{2}x^2 - \frac{1}{2}x\sin x\cos x - \frac{1}{4}x^2 + \frac{1}{4}\sin^2 x + C = \frac{1}{4}x^2 - \frac{1}{2}x\sin x\cos x + \frac{1}{4}\sin^2 x + C$

Note: $\int \sin x\cos x\,dx = \int s\,ds = \frac{1}{2}s^2 + C \quad [\text{where } s = \sin x,\ ds = \cos x\,dx]$.

A slightly different method is to write $\int x\sin^2 x\,dx = \int x\cdot\frac{1}{2}(1-\cos 2x)\,dx = \frac{1}{2}\int x\,dx - \frac{1}{2}\int x\cos 2x\,dx$. If we evaluate the second integral by parts, we arrive at the equivalent answer $\frac{1}{4}x^2 - \frac{1}{4}x\sin 2x - \frac{1}{8}\cos 2x + C$.

18. Let $u = e^{2t}$, $du = 2e^{2t}\,dt$. Then

$$\int \frac{e^{2t}}{1+e^{4t}}\,dt = \int \frac{\frac{1}{2}(2e^{2t})\,dt}{1+(e^{2t})^2} = \int \frac{\frac{1}{2}du}{1+u^2} = \frac{1}{2}\tan^{-1}u + C = \frac{1}{2}\tan^{-1}(e^{2t}) + C.$$

19. Let $u = e^x$. Then $\int e^{x+e^x}\,dx = \int e^{e^x}e^x\,dx = \int e^u\,du = e^u + C = e^{e^x} + C$.

20. Let $u = \sqrt[3]{x}$. Then $x = u^3 \quad\Rightarrow\quad \int e^{\sqrt[3]{x}}\,dx = \int e^u\cdot 3u^2\,du$. Now use parts: let $w = u^2$, $dv = e^u\,du \quad\Rightarrow$ $dw = 2u\,du$, $v = e^u \quad\Rightarrow\quad 3\int e^u u^2\,du = 3\left(u^2e^u - 2\int ue^u\,du\right)$. Now use parts again with $W = u$, $dV = e^u\,du$ to get $\int e^u\,3u^2\,du = e^u\left(3u^2 - 6u + 6\right) + C = 3e^{\sqrt[3]{x}}\left(x^{2/3} - 2\sqrt[3]{x} + 2\right) + C$.

21. Integrate by parts three times, first with $u = t^3$, $dv = e^{-2t}\,dt$:

$$\int t^3e^{-2t}\,dt = -\frac{1}{2}t^3e^{-2t} + \frac{1}{2}\int 3t^2e^{-2t}\,dt = -\frac{1}{2}t^3e^{-2t} - \frac{3}{4}t^2e^{-2t} + \frac{1}{2}\int 3te^{-2t}\,dt$$

$$= -e^{-2t}\left[\frac{1}{2}t^3 + \frac{3}{4}t^2\right] - \frac{3}{4}te^{-2t} + \frac{3}{4}\int e^{-2t}\,dt = -e^{-2t}\left[\frac{1}{2}t^3 + \frac{3}{4}t^2 + \frac{3}{4}t + \frac{3}{8}\right] + C$$

$$= -\frac{1}{8}e^{-2t}\left(4t^3 + 6t^2 + 6t + 3\right) + C$$

22. Integrate by parts: $u = \sin^{-1}x$, $dv = x\,dx \quad\Rightarrow\quad du = \left(1/\sqrt{1-x^2}\right)dx$, $v = \frac{1}{2}x^2$, so

$$\int x\sin^{-1}x\,dx = \frac{1}{2}x^2\sin^{-1}x - \frac{1}{2}\int\frac{x^2\,dx}{\sqrt{1-x^2}} = \frac{1}{2}x^2\sin^{-1}x - \frac{1}{2}\int\frac{\sin^2\theta\cos\theta\,d\theta}{\cos\theta} \quad \begin{bmatrix}\text{where } x = \sin\theta \\ \text{for } -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}\end{bmatrix}$$

$$= \frac{1}{2}x^2\sin^{-1}x - \frac{1}{4}\int(1-\cos 2\theta)\,d\theta = \frac{1}{2}x^2\sin^{-1}x - \frac{1}{4}(\theta - \sin\theta\cos\theta) + C$$

$$= \frac{1}{2}x^2\sin^{-1}x - \frac{1}{4}\left[\sin^{-1}x - x\sqrt{1-x^2}\right] + C = \frac{1}{4}\left[\left(2x^2-1\right)\sin^{-1}x + x\sqrt{1-x^2}\right] + C$$

23. Let $u = 1 + \sqrt{x}$. Then $x = (u-1)^2$, $dx = 2(u-1)\,du \quad\Rightarrow$

$$\int_0^1\left(1+\sqrt{x}\right)^8 dx = \int_1^2 u^8\cdot 2(u-1)\,du = 2\int_1^2\left(u^9 - u^8\right)du = \left[\frac{1}{5}u^{10} - 2\cdot\frac{1}{9}u^9\right]_1^2$$

$$= \frac{1024}{5} - \frac{1024}{9} - \frac{1}{5} + \frac{2}{9} = \frac{4097}{45}$$

24. Let $u = \ln(x^2-1)$, $dv = dx \;\Leftrightarrow\; du = \dfrac{2x}{x^2-1}$, $v = x$. Then

$$\int \ln(x^2-1)\,dx = x\ln(x^2-1) - \int \frac{2x^2}{x^2-1}\,dx = x\ln(x^2-1) - \int \left[2 + \frac{2}{(x-1)(x+1)}\right] dx$$
$$= x\ln(x^2-1) - \int \left[2 + \frac{1}{x-1} - \frac{1}{x+1}\right] dx$$
$$= x\ln(x^2-1) - 2x - \ln|x-1| + \ln|x+1| + C$$

25. $\dfrac{3x^2-2}{x^2-2x-8} = 3 + \dfrac{6x+22}{(x-4)(x+2)} = 3 + \dfrac{A}{x-4} + \dfrac{B}{x+2} \;\Rightarrow\; 6x+22 = A(x+2)+B(x-4)$. Setting $x=4$ gives $46 = 6A$, so $A = \frac{23}{3}$. Setting $x=-2$ gives $10=-6B$, so $B = -\frac{5}{3}$. Now

$$\int \frac{3x^2-2}{x^2-2x-8}\,dx = \int \left(3 + \frac{23/3}{x-4} - \frac{5/3}{x+2}\right) dx = 3x + \tfrac{23}{3}\ln|x-4| - \tfrac{5}{3}\ln|x+2| + C.$$

26. $\displaystyle\int \frac{3x^2-2}{x^3-2x-8}\,dx = \int \frac{du}{u} \quad \begin{bmatrix} u = x^3-2x-8, \\ du = (3x^2-2)\,dx \end{bmatrix} \quad = \ln|u| + C = \ln\left|x^3-2x-8\right| + C$

27. Let $u = \ln(\sin x)$. Then $du = \cot x\,dx \;\Rightarrow\; \int \cot x \ln(\sin x)\,dx = \int u\,du = \frac{1}{2}u^2 + C = \frac{1}{2}[\ln(\sin x)]^2 + C.$

28. $\int \sin\sqrt{at}\,dt = \int \sin u \cdot \frac{2}{a}u\,du \quad [u=\sqrt{at},\, u^2 = at,\, 2u\,du = a\,dt] \quad = \frac{2}{a}\int u\sin u\,du$

$$= \tfrac{2}{a}[-u\cos u + \sin u] + C \quad \text{[integration by parts]} \quad = -\tfrac{2}{a}\sqrt{at}\,\cos\sqrt{at} + \tfrac{2}{a}\sin\sqrt{at} + C$$
$$= -2\sqrt{\tfrac{t}{a}}\,\cos\sqrt{at} + \tfrac{2}{a}\sin\sqrt{at} + C$$

29. $\displaystyle\int_0^5 \frac{3w-1}{w+2}\,dw = \int_0^5 \left(3 - \frac{7}{w+2}\right) dw = \Big[3w - 7\ln|w+2|\Big]_0^5$

$$= 15 - 7\ln 7 + 7\ln 2 = 15 + 7(\ln 2 - \ln 7) = 15 + 7\ln\tfrac{2}{7}$$

30. $x^2 - 4x < 0$ on $[0,4]$, so

$$\int_{-2}^{2} \left|x^2-4x\right|\,dx = \int_{-2}^{0}(x^2-4x)\,dx + \int_0^2 (4x-x^2)\,dx = \left[\tfrac{1}{3}x^3 - 2x^2\right]_{-2}^{0} + \left[2x^2 - \tfrac{1}{3}x^3\right]_0^2$$
$$= 0 - \left(-\tfrac{8}{3} - 8\right) + \left(8 - \tfrac{8}{3}\right) - 0 = 16$$

31. As in Example 5,

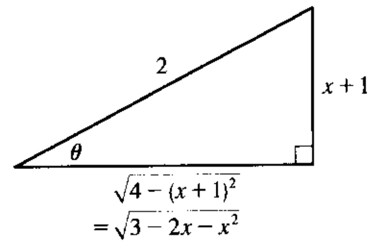
$$\int \sqrt{\frac{1+x}{1-x}}\,dx = \int \frac{\sqrt{1+x}}{\sqrt{1-x}} \cdot \frac{\sqrt{1+x}}{\sqrt{1+x}}\,dx = \int \frac{1+x}{\sqrt{1-x^2}}\,dx = \int \frac{dx}{\sqrt{1-x^2}} + \int \frac{x\,dx}{\sqrt{1-x^2}}$$
$$= \sin^{-1}x - \sqrt{1-x^2} + C$$

Another method: Substitute $u = \sqrt{(1+x)/(1-x)}$.

32. $\displaystyle\int \frac{\sqrt{2x-1}}{2x+3}\,dx = \int \frac{u\cdot u\,du}{u^2+4} \quad \begin{bmatrix} u = \sqrt{2x-1},\, 2x+3 = u^2+4, \\ u^2 = 2x-1,\, u\,du = dx \end{bmatrix} \quad = \int \left(1 - \frac{4}{u^2+4}\right) du$

$$= u - 4\cdot\tfrac{1}{2}\tan^{-1}\left(\tfrac{1}{2}u\right) + C = \sqrt{2x-1} - 2\tan^{-1}\left(\tfrac{1}{2}\sqrt{2x-1}\right) + C$$

33. $3 - 2x - x^2 = -(x^2 + 2x + 1) + 4 = 4 - (x+1)^2$. Let $x + 1 = 2\sin\theta$, where $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$. Then $dx = 2\cos\theta\, d\theta$ and



$$\begin{aligned}\int \sqrt{3 - 2x - x^2}\, dx &= \int \sqrt{4 - (x+1)^2}\, dx = \int \sqrt{4 - 4\sin^2\theta}\, 2\cos\theta\, d\theta \\ &= 4\int \cos^2\theta\, d\theta = 2\int (1 + \cos 2\theta)\, d\theta \\ &= 2\theta + \sin 2\theta + C = 2\theta + 2\sin\theta\cos\theta + C \\ &= 2\sin^{-1}\left(\frac{x+1}{2}\right) + 2\cdot\frac{x+1}{2}\cdot\frac{\sqrt{3-2x-x^2}}{2} + C \\ &= 2\sin^{-1}\left(\frac{x+1}{2}\right) + \frac{x+1}{2}\sqrt{3-2x-x^2} + C\end{aligned}$$

34. $$\int_{\pi/4}^{\pi/2} \frac{1 + 4\cot x}{4 - \cot x}\, dx = \int_{\pi/4}^{\pi/2} \left[\frac{(1 + 4\cos x/\sin x)}{(4 - \cos x/\sin x)}\cdot\frac{\sin x}{\sin x}\right] dx = \int_{\pi/4}^{\pi/2} \frac{\sin x + 4\cos x}{4\sin x - \cos x}\, dx$$

$$= \int_{3/\sqrt{2}}^{4} \frac{1}{u}\, du \quad \left[\begin{aligned} u &= 4\sin x - \cos x, \\ du &= (4\cos x + \sin x)\, dx\end{aligned}\right] \quad = \Big[\ln|u|\Big]_{3/\sqrt{2}}^{4}$$

$$= \ln 4 - \ln\frac{3}{\sqrt{2}} = \ln\frac{4}{3/\sqrt{2}} = \ln\left(\frac{4}{3}\sqrt{2}\right)$$

35. Because $f(x) = x^8 \sin x$ is the product of an even function and an odd function, it is odd. Therefore, $\int_{-1}^{1} x^8 \sin x\, dx = 0$ [by (5.5.6)(b)].

36. $\sin 4x \cos 3x = \frac{1}{2}(\sin x + \sin 7x)$ by Formula 8.2.2(a), so

$$\int \sin 4x \cos 3x\, dx = \tfrac{1}{2}\int (\sin x + \sin 7x)\, dx = \tfrac{1}{2}\left[-\cos x - \tfrac{1}{7}\cos 7x\right] + C = -\tfrac{1}{2}\cos x - \tfrac{1}{14}\cos 7x + C.$$

37. $$\int_0^{\pi/4} \cos^2\theta \tan^2\theta\, d\theta = \int_0^{\pi/4} \sin^2\theta\, d\theta = \int_0^{\pi/4} \tfrac{1}{2}(1 - \cos 2\theta)\, d\theta = \left[\tfrac{1}{2}\theta - \tfrac{1}{4}\sin 2\theta\right]_0^{\pi/4}$$
$$= \left(\tfrac{\pi}{8} - \tfrac{1}{4}\right) - (0 - 0) = \tfrac{\pi}{8} - \tfrac{1}{4}$$

38. $$\int_0^{\pi/4} \tan^5\theta \sec^3\theta\, d\theta = \int_0^{\pi/4} (\tan^2\theta)^2 \sec^2\theta \cdot \sec\theta\tan\theta\, d\theta = \int_1^{\sqrt{2}} (u^2 - 1)^2 u^2\, du \quad \left[\begin{aligned} u &= \sec\theta \\ du &= \sec\theta\tan\theta\, d\theta\end{aligned}\right]$$
$$= \int_1^{\sqrt{2}} (u^6 - 2u^4 + u^2)\, du = \left[\tfrac{1}{7}u^7 - \tfrac{2}{5}u^5 + \tfrac{1}{3}u^3\right]_1^{\sqrt{2}}$$
$$= \left(\tfrac{8}{7}\sqrt{2} - \tfrac{8}{5}\sqrt{2} + \tfrac{2}{3}\sqrt{2}\right) - \left(\tfrac{1}{7} - \tfrac{2}{5} + \tfrac{1}{3}\right) = \tfrac{22}{105}\sqrt{2} - \tfrac{8}{105} = \tfrac{2}{105}(11\sqrt{2} - 4)$$

39. Let $u = 1 - x^2$. Then $du = -2x\, dx \Rightarrow$

$$\int \frac{x\, dx}{1 - x^2 + \sqrt{1 - x^2}} = -\frac{1}{2}\int \frac{du}{u + \sqrt{u}} = -\int \frac{v\, dv}{v^2 + v} \quad [v = \sqrt{u},\ u = v^2,\ du = 2v\, dv]$$
$$= -\int \frac{dv}{v + 1} = -\ln|v + 1| + C = -\ln\left(\sqrt{1 - x^2} + 1\right) + C$$

40. $4y^2 - 4y - 3 = (2y - 1)^2 - 2^2$, so let $u = 2y - 1 \Rightarrow du = 2\, dy$. Thus,

$$\int \frac{dy}{\sqrt{4y^2 - 4y - 3}} = \int \frac{dy}{\sqrt{(2y-1)^2 - 2^2}} = \frac{1}{2}\int \frac{du}{\sqrt{u^2 - 2^2}}$$
$$= \tfrac{1}{2}\ln\left|u + \sqrt{u^2 - 2^2}\right| \quad \text{[by Formula 20 in the table in this section]}$$
$$= \tfrac{1}{2}\ln\left|2y - 1 + \sqrt{4y^2 - 4y - 3}\right| + C$$

41. Let $u = \theta$, $dv = \tan^2\theta\, d\theta = (\sec^2\theta - 1)\, d\theta \Rightarrow du = d\theta$ and $v = \tan\theta - \theta$. So

$$\int \theta\tan^2\theta\, d\theta = \theta(\tan\theta - \theta) - \int(\tan\theta - \theta)\, d\theta = \theta\tan\theta - \theta^2 - \ln|\sec\theta| + \tfrac{1}{2}\theta^2 + C$$
$$= \theta\tan\theta - \tfrac{1}{2}\theta^2 - \ln|\sec\theta| + C$$

42. Integrate by parts with $u = \tan^{-1}x$, $dv = x^2\, dx \Rightarrow du = dx/(1+x^2)$, $v = \frac{1}{3}x^3$:

$$\int x^2\tan^{-1}x\, dx = \tfrac{1}{3}x^3\tan^{-1}x - \int \frac{x^3}{3}\frac{dx}{1+x^2} = \tfrac{1}{3}x^3\tan^{-1}x - \frac{1}{3}\int\left[x - \frac{x}{x^2+1}\right]dx$$
$$= \tfrac{1}{3}x^3\tan^{-1}x - \tfrac{1}{6}x^2 + \tfrac{1}{6}\ln(x^2+1) + C$$

43. Let $u = 1 + e^x$, so that $du = e^x\, dx$. Then
$\int e^x\sqrt{1+e^x}\, dx = \int u^{1/2}\, du = \frac{2}{3}u^{3/2} + C = \frac{2}{3}(1+e^x)^{3/2} + C$.
Or: Let $u = \sqrt{1+e^x}$, so that $u^2 = 1 + e^x$ and $2u\, du = e^x\, dx$. Then
$\int e^x\sqrt{1+e^x}\, dx = \int u\cdot 2u\, du = \int 2u^2 du = \frac{2}{3}u^3 + C = \frac{2}{3}(1+e^x)^{3/2} + C$.

44. Let $u = \sqrt{1+e^x}$. Then $u^2 = 1 + e^x$, $2u\, du = e^x\, dx = (u^2-1)\, dx$, and $dx = \dfrac{2u}{u^2-1}du$, so

$$\int\sqrt{1+e^x}\, dx = \int u\cdot\frac{2u}{u^2-1}\, du = \int\frac{2u^2}{u^2-1}\, du = \int\left(2 + \frac{2}{u^2-1}\right)du = \int\left(2 + \frac{1}{u-1} - \frac{1}{u+1}\right)du$$
$$= 2u + \ln|u-1| - \ln|u+1| + C = 2\sqrt{1+e^x} + \ln\left(\sqrt{1+e^x}-1\right) - \ln\left(\sqrt{1+e^x}+1\right) + C$$

45. Let $t = x^3$. Then $dt = 3x^2\, dx \Rightarrow I = \int x^5e^{-x^3}\, dx = \frac{1}{3}\int te^{-t}\, dt$. Now integrate by parts with $u = t$, $dv = e^{-t}\, dt$: $I = -\frac{1}{3}te^{-t} + \frac{1}{3}\int e^{-t}\, dt = -\frac{1}{3}te^{-t} - \frac{1}{3}e^{-t} + C = -\frac{1}{3}e^{-x^3}(x^3+1) + C$.

46. Let $u = e^x$. Then $x = \ln u$, $dx = du/u \Rightarrow$

$$\int\frac{1+e^x}{1-e^x}\, dx = \int\frac{(1+u)\, du}{(1-u)u} = -\int\frac{(u+1)\, du}{(u-1)u} = -\int\left(\frac{2}{u-1} - \frac{1}{u}\right)du$$
$$= \ln|u| - 2\ln|u-1| + C = \ln e^x - 2\ln|e^x-1| + C = x - 2\ln|e^x-1| + C$$

47. $$\int\frac{x+a}{x^2+a^2}\, dx = \frac{1}{2}\int\frac{2x\, dx}{x^2+a^2} + a\int\frac{dx}{x^2+a^2} = \tfrac{1}{2}\ln(x^2+a^2) + a\cdot\frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + C$$
$$= \ln\sqrt{x^2+a^2} + \tan^{-1}(x/a) + C$$

48. Let $u = x^2$. Then $du = 2x\, dx \Rightarrow$
$$\int\frac{x\, dx}{x^4-a^4} = \int\frac{\frac{1}{2}\, du}{u^2-(a^2)^2} = \frac{1}{4a^2}\ln\left|\frac{u-a^2}{u+a^2}\right| + C = \frac{1}{4a^2}\ln\left|\frac{x^2-a^2}{x^2+a^2}\right| + C.$$

49. Let $u = \sqrt{4x+1} \Rightarrow u^2 = 4x+1 \Rightarrow 2u\, du = 4\, dx \Rightarrow dx = \frac{1}{2}u\, du$. So

$$\int\frac{1}{x\sqrt{4x+1}}\, dx = \int\frac{\frac{1}{2}u\, du}{\frac{1}{4}(u^2-1)\, u} = 2\int\frac{du}{u^2-1} = 2\left(\tfrac{1}{2}\right)\ln\left|\frac{u-1}{u+1}\right| + C \quad \text{[by Formula 19]}$$
$$= \ln\left|\frac{\sqrt{4x+1}-1}{\sqrt{4x+1}+1}\right| + C$$

50. As in Exercise 49, let $u = \sqrt{4x+1}$. Then $\int \frac{dx}{x^2\sqrt{4x+1}} = \int \frac{\frac{1}{2}u\,du}{\left[\frac{1}{4}(u^2-1)\right]^2 u} = 8\int \frac{du}{(u^2-1)^2}$. Now

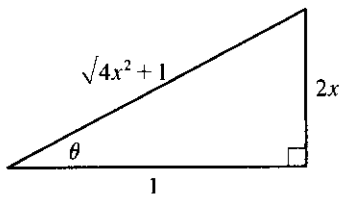
$\frac{1}{(u^2-1)^2} = \frac{1}{(u+1)^2(u-1)^2} = \frac{A}{u+1} + \frac{B}{(u+1)^2} + \frac{C}{u-1} + \frac{D}{(u-1)^2} \Rightarrow$

$1 = A(u+1)(u-1)^2 + B(u-1)^2 + C(u-1)(u+1)^2 + D(u+1)^2$. $u = 1 \Rightarrow D = \frac{1}{4}$, $u = -1 \Rightarrow$ $B = \frac{1}{4}$. Equating coefficients of u^3 gives $A + C = 0$, and equating coefficients of 1 gives $1 = A + B - C + D$ $\Rightarrow 1 = A + \frac{1}{4} - C + \frac{1}{4} \Rightarrow \frac{1}{2} = A - C$. So $A = \frac{1}{4}$ and $C = -\frac{1}{4}$. Therefore,

$$\begin{aligned}\int \frac{dx}{x^2\sqrt{4x+1}} &= 8\int\left[\frac{1/4}{u+1} + \frac{1/4}{(u+1)^2} + \frac{-1/4}{u-1} + \frac{1/4}{(u-1)^2}\right]du \\ &= \int\left[\frac{2}{u+1} + 2(u+1)^{-2} - \frac{2}{u-1} + 2(u-1)^{-2}\right]du \\ &= 2\ln|u+1| - \frac{2}{u+1} - 2\ln|u-1| - \frac{2}{u-1} + C \\ &= 2\ln\left(\sqrt{4x+1}+1\right) - \frac{2}{\sqrt{4x+1}+1} - 2\ln\left|\sqrt{4x+1}-1\right| - \frac{2}{\sqrt{4x+1}-1} + C\end{aligned}$$

51. Let $2x = \tan\theta \Rightarrow x = \frac{1}{2}\tan\theta$, $dx = \frac{1}{2}\sec^2\theta\,d\theta$, $\sqrt{4x^2+1} = \sec\theta$, so

$$\begin{aligned}\int \frac{dx}{x\sqrt{4x^2+1}} &= \int \frac{\frac{1}{2}\sec^2\theta\,d\theta}{\frac{1}{2}\tan\theta\sec\theta} = \int \frac{\sec\theta}{\tan\theta}\,d\theta = \int \csc\theta\,d\theta \\ &= -\ln|\csc\theta + \cot\theta| + C \qquad [\text{or } \ln|\csc\theta - \cot\theta| + C] \\ &= -\ln\left|\frac{\sqrt{4x^2+1}}{2x} + \frac{1}{2x}\right| + C \qquad \left[\text{or } \ln\left|\frac{\sqrt{4x^2+1}}{2x} - \frac{1}{2x}\right| + C\right]\end{aligned}$$



52. Let $u = x^2$. Then $du = 2x\,dx \Rightarrow$

$$\begin{aligned}\int \frac{dx}{x(x^4+1)} &= \int \frac{x\,dx}{x^2(x^4+1)} = \frac{1}{2}\int \frac{du}{u(u^2+1)} = \frac{1}{2}\int\left[\frac{1}{u} - \frac{u}{u^2+1}\right]du = \tfrac{1}{2}\ln|u| - \tfrac{1}{4}\ln(u^2+1) + C \\ &= \tfrac{1}{2}\ln(x^2) - \tfrac{1}{4}\ln(x^4+1) + C = \tfrac{1}{4}\left[\ln(x^4) - \ln(x^4+1)\right] + C = \frac{1}{4}\ln\left(\frac{x^4}{x^4+1}\right) + C\end{aligned}$$

Or: Write $I = \int \frac{x^3\,dx}{x^4(x^4+1)}$ and let $u = x^4$.

53. $\int x^2\sinh(mx)dx = \frac{1}{m}x^2\cosh(mx) - \frac{2}{m}\int x\cosh(mx)\,dx$ $\quad\left[\begin{matrix} u = x^2, & dv = \sinh(mx)\,dx, \\ du = 2x\,dx & v = \frac{1}{m}\cosh(mx)\end{matrix}\right]$

$= \frac{1}{m}x^2\cosh(mx) - \frac{2}{m}\left(\frac{1}{m}x\sinh(mx) - \frac{1}{m}\int\sinh(mx)\,dx\right)$ $\quad\left[\begin{matrix} U = x, & dV = \cosh(mx)\,dx, \\ dU = dx & V = \frac{1}{m}\sinh(mx)\end{matrix}\right]$

$= \frac{1}{m}x^2\cosh(mx) - \frac{2}{m^2}x\sinh(mx) + \frac{2}{m^3}\cosh(mx) + C$

54. $\int(x+\sin x)^2\,dx = \int\left(x^2 + 2x\sin x + \sin^2 x\right)dx = \frac{1}{3}x^3 + 2(\sin x - x\cos x) + \frac{1}{2}(x - \sin x\cos x) + C$

$= \frac{1}{3}x^3 + \frac{1}{2}x + 2\sin x - \frac{1}{2}\sin x\cos x - 2x\cos x + C$

55. Let $u = \sqrt{x+1}$. Then $x = u^2 - 1 \quad \Rightarrow$

$$\int \frac{dx}{x+4+4\sqrt{x+1}} = \int \frac{2u\,du}{u^2+3+4u} = \int \left[\frac{-1}{u+1} + \frac{3}{u+3}\right] du$$

$$= 3\ln|u+3| - \ln|u+1| + C = 3\ln\left(\sqrt{x+1}+3\right) - \ln\left(\sqrt{x+1}+1\right) + C$$

56. Let $t = \sqrt{x^2-1}$. Then $dt = \left(x/\sqrt{x^2-1}\right)dx$, $x^2 - 1 = t^2$, $x = \sqrt{t^2+1}$, so

$I = \int \frac{x\ln x}{\sqrt{x^2-1}}\,dx = \int \ln\sqrt{t^2+1}\,dt = \frac{1}{2}\int \ln\left(t^2+1\right)dt$. Now use parts with $u = \ln\left(t^2+1\right)$, $dv = dt$:

$$I = \tfrac{1}{2}t\ln\left(t^2+1\right) - \int \frac{t^2}{t^2+1}\,dt = \tfrac{1}{2}t\ln\left(t^2+1\right) - \int\left[1 - \frac{1}{t^2+1}\right]dt$$

$$= \tfrac{1}{2}t\ln\left(t^2+1\right) - t + \tan^{-1}t + C = \sqrt{x^2-1}\,\ln x - \sqrt{x^2-1} + \tan^{-1}\sqrt{x^2-1} + C$$

Another method: First integrate by parts with $u = \ln x$, $dv = \left(x/\sqrt{x^2-1}\right)dx$ and then use substitution $\left(x = \sec\theta \text{ or } u = \sqrt{x^2-1}\right)$.

57. Let $u = \sqrt[3]{x+c}$. Then $x = u^3 - c \quad \Rightarrow$

$$\int x\sqrt[3]{x+c}\,dx = \int \left(u^3 - c\right)u \cdot 3u^2\,du = 3\int\left(u^6 - cu^3\right)du = \tfrac{3}{7}u^7 - \tfrac{3}{4}cu^4 + C$$

$$= \tfrac{3}{7}(x+c)^{7/3} - \tfrac{3}{4}c(x+c)^{4/3} + C$$

58. Integrate by parts with $u = \ln(1+x)$, $dv = x^2\,dx \quad \Rightarrow \quad du = dx/(1+x)$, $v = \frac{1}{3}x^3$:

$$\int x^2\ln(1+x)\,dx = \tfrac{1}{3}x^3\ln(1+x) - \int \frac{x^3\,dx}{3(1+x)} = \tfrac{1}{3}x^3\ln(1+x) - \frac{1}{3}\int\left(x^2 - x + 1 - \frac{1}{x+1}\right)dx$$

$$= \tfrac{1}{3}x^3\ln(1+x) - \tfrac{1}{9}x^3 + \tfrac{1}{6}x^2 - \tfrac{1}{3}x + \tfrac{1}{3}\ln(1+x) + C$$

59. Let $u = e^x$. Then $x = \ln u$, $dx = du/u \quad \Rightarrow$

$$\int \frac{dx}{e^{3x} - e^x} = \int \frac{du/u}{u^3 - u} = \int \frac{du}{(u-1)u^2(u+1)} = \int \left[\frac{1/2}{u-1} - \frac{1}{u^2} - \frac{1/2}{u+1}\right] du$$

$$= \frac{1}{u} + \frac{1}{2}\ln\left|\frac{u-1}{u+1}\right| + C = e^{-x} + \frac{1}{2}\ln\left|\frac{e^x-1}{e^x+1}\right| + C$$

60. Let $u = \sqrt[3]{x}$. Then $x = u^3$, $dx = 3u^2\,du \quad \Rightarrow$

$$\int \frac{dx}{x + \sqrt[3]{x}} = \int \frac{3u^2\,du}{u^3 + u} = \frac{3}{2}\int \frac{2u\,du}{u^2+1} = \tfrac{3}{2}\ln\left(u^2+1\right) + C = \tfrac{3}{2}\ln\left(x^{2/3}+1\right) + C.$$

61. Let $u = x^5$. Then $du = 5x^4\,dx \quad \Rightarrow$

$$\int \frac{x^4\,dx}{x^{10}+16} = \int \frac{\frac{1}{5}\,du}{u^2+16} = \tfrac{1}{5} \cdot \tfrac{1}{4}\tan^{-1}\left(\tfrac{1}{4}u\right) + C = \tfrac{1}{20}\tan^{-1}\left(\tfrac{1}{4}x^5\right) + C.$$

62. Let $u = x + 1$. Then $du = dx \quad \Rightarrow$

$$\int \frac{x^3}{(x+1)^{10}}\,dx = \int \frac{(u-1)^3}{u^{10}}\,du = \int\left(u^{-7} - 3u^{-8} + 3u^{-9} - u^{-10}\right)du$$

$$= -\tfrac{1}{6}u^{-6} + \tfrac{3}{7}u^{-7} - \tfrac{3}{8}u^{-8} + \tfrac{1}{9}u^{-9} + C$$

$$= (x+1)^{-9}\left[-\tfrac{1}{6}(x+1)^3 + \tfrac{3}{7}(x+1)^2 - \tfrac{3}{8}(x+1) + \tfrac{1}{9}\right] + C$$

63. Let $y = \sqrt{x}$ so that $dy = \dfrac{1}{2\sqrt{x}}\,dx \Rightarrow dx = 2\sqrt{x}\,dy = 2y\,dy$. Then

$$\int \sqrt{x}\,e^{\sqrt{x}}\,dx = \int ye^y(2y\,dy) = \int 2y^2e^y\,dy \qquad \begin{bmatrix} u = 2y^2, & dv = e^y\,dy, \\ du = 4y\,dy & v = e^y \end{bmatrix}$$
$$= 2y^2e^y - \int 4ye^y\,dy \qquad \begin{bmatrix} U = 4y, & dV = e^y\,dy, \\ dU = 4\,dy & V = e^y \end{bmatrix}$$
$$= 2y^2e^y - \left(4ye^y - \textstyle\int 4e^y\,dy\right) = 2y^2e^y - 4ye^y + 4e^y + C$$
$$= 2(y^2 - 2y + 2)e^y + C = 2(x - 2\sqrt{x} + 2)\,e^{\sqrt{x}} + C$$

64. Let $u = \tan x$. Then

$$\int_{\pi/4}^{\pi/3} \frac{\ln(\tan x)\,dx}{\sin x\cos x} = \int_{\pi/4}^{\pi/3} \frac{\ln(\tan x)}{\tan x}\sec^2 x\,dx = \int_1^{\sqrt{3}} \frac{\ln u}{u}\,du$$
$$= \left[\tfrac{1}{2}(\ln u)^2\right]_1^{\sqrt{3}} = \tfrac{1}{2}\left(\ln\sqrt{3}\right)^2 = \tfrac{1}{8}(\ln 3)^2$$

65. $$\int \frac{dx}{\sqrt{x+1}+\sqrt{x}} = \int \left(\frac{1}{\sqrt{x+1}+\sqrt{x}} \cdot \frac{\sqrt{x+1}-\sqrt{x}}{\sqrt{x+1}-\sqrt{x}}\right) dx = \int \left(\sqrt{x+1}-\sqrt{x}\,\right) dx$$
$$= \tfrac{2}{3}\left[(x+1)^{3/2} - x^{3/2}\right] + C$$

66. $$\int \frac{u^3+1}{u^3-u^2}\,du = \int \left[1 + \frac{u^2+1}{(u-1)u^2}\right] du = u + \int \left[\frac{2}{u-1} - \frac{1}{u} - \frac{1}{u^2}\right] du = u + 2\ln|u-1| - \ln|u| + \frac{1}{u} + C.$$

Thus,

$$\int_2^3 \frac{u^3+1}{u^3-u^2}\,du = \left[u + 2\ln(u-1) - \ln u + \frac{1}{u}\right]_2^3 = \left(3 + 2\ln 2 - \ln 3 + \tfrac{1}{3}\right) - \left(2 + 2\ln 1 - \ln 2 + \tfrac{1}{2}\right)$$
$$= 1 + 3\ln 2 - \ln 3 - \tfrac{1}{6} = \tfrac{5}{6} + \ln\tfrac{8}{3}$$

67. Let $u = \sqrt{t}$. Then $du = dt/\left(2\sqrt{t}\,\right) \Rightarrow$

$$\int_1^3 \frac{\arctan\sqrt{t}}{\sqrt{t}}\,dt = \int_1^{\sqrt{3}} \tan^{-1}u\,(2\,du) = 2\left[u\tan^{-1}u - \tfrac{1}{2}\ln(1+u^2)\right]_1^{\sqrt{3}} \qquad \text{[Example 5 in Section 8.1]}$$
$$= 2\left[\left(\sqrt{3}\tan^{-1}\sqrt{3} - \tfrac{1}{2}\ln 4\right) - \left(\tan^{-1}1 - \tfrac{1}{2}\ln 2\right)\right]$$
$$= 2\left[\left(\sqrt{3}\cdot\tfrac{\pi}{3} - \ln 2\right) - \left(\tfrac{\pi}{4} - \tfrac{1}{2}\ln 2\right)\right] = \tfrac{2}{3}\sqrt{3}\,\pi - \tfrac{1}{2}\pi - \ln 2$$

68. Let $u = e^x$. Then $x = \ln u$, $dx = du/u \Rightarrow$

$$\int \frac{dx}{1+2e^x-e^{-x}} = \int \frac{du/u}{1+2u-1/u} = \int \frac{du}{2u^2+u-1} = \int \left[\frac{2/3}{2u-1} - \frac{1/3}{u+1}\right] du$$
$$= \tfrac{1}{3}\ln|2u-1| - \tfrac{1}{3}\ln|u+1| + C = \tfrac{1}{3}\ln|(2e^x-1)/(e^x+1)| + C$$

69. Let $u = e^x$. Then $x = \ln u$, $dx = du/u \Rightarrow$

$$\int \frac{e^{2x}}{1+e^x}\,dx = \int \frac{u^2}{1+u}\frac{du}{u} = \int \frac{u}{1+u}\,du = \int \left(1 - \frac{1}{1+u}\right) du$$
$$= u - \ln|1+u| + C = e^x - \ln(1+e^x) + C$$

70. Use parts with $u = \ln(x+1)$, $dv = dx/x^2$:

$$\int \frac{\ln(x+1)}{x^2}\,dx = -\frac{1}{x}\ln(x+1) + \int \frac{dx}{x(x+1)} = -\frac{1}{x}\ln(x+1) + \int \left[\frac{1}{x} - \frac{1}{x+1}\right] dx$$
$$= -\tfrac{1}{x}\ln(x+1) + \ln|x| - \ln(x+1) + C = -\left(1 + \tfrac{1}{x}\right)\ln(x+1) + \ln|x| + C$$

71. $\dfrac{x}{x^4+4x^2+3} = \dfrac{x}{(x^2+3)(x^2+1)} = \dfrac{Ax+B}{x^2+3} + \dfrac{Cx+D}{x^2+1} \quad\Rightarrow$

$$x = (Ax+B)(x^2+1) + (Cx+D)(x^2+3) = (Ax^3+Bx^2+Ax+B) + (Cx^3+Dx^2+3Cx+3D)$$
$$= (A+C)x^3 + (B+D)x^2 + (A+3C)x + (B+3D) \quad\Rightarrow$$

$A+C=0$, $B+D=0$, $A+3C=1$, $B+3D=0 \;\Rightarrow\; A=-\frac{1}{2}$, $C=\frac{1}{2}$, $B=0$, $D=0$. Thus,

$$\int \frac{x}{x^4+4x^2+3}\,dx = \int \left(\frac{-\frac{1}{2}x}{x^2+3} + \frac{\frac{1}{2}x}{x^2+1}\right)dx$$
$$= -\tfrac{1}{4}\ln(x^2+3) + \tfrac{1}{4}\ln(x^2+1) + C \quad \text{or} \quad \frac{1}{4}\ln\left(\frac{x^2+1}{x^2+3}\right) + C$$

72. Let $u = \sqrt[6]{t}$. Then $t = u^6$, $dt = 6u^5\,du \quad\Rightarrow$

$$\int \frac{\sqrt{t}\,dt}{1+\sqrt[3]{t}} = \int \frac{u^3\cdot 6u^5\,du}{1+u^2} = 6\int \frac{u^8}{u^2+1}\,du = 6\int \left(u^6 - u^4 + u^2 - 1 + \frac{1}{u^2+1}\right)du$$
$$= 6\left(\tfrac{1}{7}u^7 - \tfrac{1}{5}u^5 + \tfrac{1}{3}u^3 - u + \tan^{-1}u\right) + C$$
$$= 6\left(\tfrac{1}{7}t^{7/6} - \tfrac{1}{5}t^{5/6} + \tfrac{1}{3}t^{1/2} - t^{1/6} + \tan^{-1}t^{1/6}\right) + C$$

73. $\dfrac{1}{(x-2)(x^2+4)} = \dfrac{A}{x-2} + \dfrac{Bx+C}{x^2+4} \quad\Rightarrow$

$1 = A(x^2+4) + (Bx+C)(x-2) = (A+B)x^2 + (C-2B)x + (4A-2C)$. So $0 = A+B = C-2B$, $1 = 4A-2C$. Setting $x=2$ gives $A = \frac{1}{8} \;\Rightarrow\; B = -\frac{1}{8}$ and $C = -\frac{1}{4}$. So

$$\int \frac{1}{(x-2)(x^2+4)}\,dx = \int \left(\frac{\frac{1}{8}}{x-2} + \frac{-\frac{1}{8}x - \frac{1}{4}}{x^2+4}\right)dx = \frac{1}{8}\int \frac{dx}{x-2} - \frac{1}{16}\int \frac{2x\,dx}{x^2+4} - \frac{1}{4}\int \frac{dx}{x^2+4}$$
$$= \tfrac{1}{8}\ln|x-2| - \tfrac{1}{16}\ln(x^2+4) - \tfrac{1}{8}\tan^{-1}(x/2) + C$$

74. Let $u = e^x$. Then $x = \ln u$, $dx = du/u \quad\Rightarrow$

$$\int \frac{dx}{e^x - e^{-x}} = \int \frac{e^x\,dx}{e^{2x}-1} = \int \frac{u}{u^2-1}\,\frac{du}{u} = \int \frac{du}{u^2-1} = \frac{1}{2}\ln\left|\frac{u-1}{u+1}\right| + C = \frac{1}{2}\ln\left(\frac{|e^x-1|}{e^x+1}\right) + C.$$

75. $\int \sin x \sin 2x \sin 3x\,dx = \int \sin x \cdot \frac{1}{2}[\cos(2x-3x) - \cos(2x+3x)]\,dx = \frac{1}{2}\int (\sin x\cos x - \sin x \cos 5x)\,dx$

$$= \tfrac{1}{4}\int \sin 2x\,dx - \tfrac{1}{2}\int \tfrac{1}{2}[\sin(x+5x) + \sin(x-5x)]\,dx$$
$$= -\tfrac{1}{8}\cos 2x - \tfrac{1}{4}\int (\sin 6x - \sin 4x)\,dx = -\tfrac{1}{8}\cos 2x + \tfrac{1}{24}\cos 6x - \tfrac{1}{16}\cos 4x + C$$

76. $\int (x^2 - bx)\sin 2x\,dx = -\frac{1}{2}(x^2-bx)\cos 2x + \frac{1}{2}\int (2x-b)\cos 2x\,dx$

$[u = x^2 - bx,\ dv = \sin 2x\,dx,\ du = (2x-b)\,dx,\ v = -\frac{1}{2}\cos 2x]$

$$= -\tfrac{1}{2}(x^2-bx)\cos 2x + \tfrac{1}{2}\left[\tfrac{1}{2}(2x-b)\sin 2x - \int \sin 2x\,dx\right]$$

$[U = 2x-b,\ dV = \cos 2x\,dx,\ dU = 2\,dx,\ V = \frac{1}{2}\sin 2x]$

$$= -\tfrac{1}{2}(x^2-bx)\cos 2x + \tfrac{1}{4}(2x-b)\sin 2x + \tfrac{1}{4}\cos 2x + C$$

77. Let $u = x^{3/2}$ so that $u^2 = x^3$ and $du = \frac{3}{2}x^{1/2}\,dx \;\Rightarrow\; \sqrt{x}\,dx = \frac{2}{3}\,du$. Then

$$\int \frac{\sqrt{x}}{1+x^3}\,dx = \int \frac{\frac{2}{3}}{1+u^2}\,du = \frac{2}{3}\tan^{-1}u + C = \frac{2}{3}\tan^{-1}(x^{3/2}) + C.$$

78. $$\int \frac{\sec x\,\cos 2x}{\sin x + \sec x}\,dx = \int \frac{\sec x\,\cos 2x}{\sin x + \sec x}\cdot\frac{2\cos x}{2\cos x}\,dx = \int \frac{2\cos 2x}{2\sin x\cos x + 2}\,dx$$

$$= \int \frac{2\cos 2x}{\sin 2x + 2}\,dx = \int \frac{1}{u}\,du \qquad \begin{bmatrix} u = \sin 2x + 2, \\ du = 2\cos 2x\,dx \end{bmatrix}$$

$$= \ln|u| + C = \ln|\sin 2x + 2| + C = \ln(\sin 2x + 2) + C$$

79. Let $u = x$, $dv = \sin^2 x\cos x\,dx \;\Rightarrow\; du = dx$, $v = \frac{1}{3}\sin^3 x$. Then

$$\int x\sin^2 x\,\cos x\,dx = \tfrac{1}{3}x\sin^3 x - \int \tfrac{1}{3}\sin^3 x\,dx = \tfrac{1}{3}x\sin^3 x - \tfrac{1}{3}\int(1-\cos^2 x)\sin x\,dx$$

$$= \frac{1}{3}x\sin^3 x + \frac{1}{3}\int(1-y^2)\,dy \qquad \begin{bmatrix} y = \cos x, \\ dy = -\sin x\,dx \end{bmatrix}$$

$$= \tfrac{1}{3}x\sin^3 x + \tfrac{1}{3}y - \tfrac{1}{9}y^3 + C = \tfrac{1}{3}x\sin^3 x + \tfrac{1}{3}\cos x - \tfrac{1}{9}\cos^3 x + C$$

80. $$\int \frac{\sin x\,\cos x}{\sin^4 x + \cos^4 x}\,dx = \int \frac{\sin x\,\cos x}{(\sin^2 x)^2 + (\cos^2 x)^2}\,dx = \int \frac{\sin x\,\cos x}{(\sin^2 x)^2 + (1-\sin^2 x)^2}\,dx$$

$$= \int \frac{1}{u^2 + (1-u)^2}\left(\frac{1}{2}\,du\right) \qquad \begin{bmatrix} u = \sin^2 x, \\ du = 2\sin x\cos x\,dx \end{bmatrix}$$

$$= \int \frac{1}{4u^2 - 4u + 2}\,du = \int \frac{1}{(4u^2 - 4u + 1) + 1}\,du$$

$$= \int \frac{1}{(2u-1)^2 + 1}\,du = \frac{1}{2}\int \frac{1}{y^2+1}\,dy \qquad \begin{bmatrix} y = 2u - 1, \\ dy = 2\,du \end{bmatrix}$$

$$= \tfrac{1}{2}\tan^{-1}y + C = \tfrac{1}{2}\tan^{-1}(2u-1) + C = \tfrac{1}{2}\tan^{-1}(2\sin^2 x - 1) + C$$

Another solution:

$$\int \frac{\sin x\,\cos x}{\sin^4 x + \cos^4 x}\,dx = \int \frac{(\sin x\,\cos x)/\cos^4 x}{(\sin^4 x + \cos^4 x)/\cos^4 x}\,dx = \int \frac{\tan x\sec^2 x}{\tan^4 x + 1}\,dx$$

$$= \int \frac{1}{u^2+1}\left(\frac{1}{2}\,du\right) \qquad \begin{bmatrix} u = \tan^2 x, \\ du = 2\tan x\sec^2 x\,dx \end{bmatrix}$$

$$= \tfrac{1}{2}\tan^{-1}u + C = \tfrac{1}{2}\tan^{-1}(\tan^2 x) + C$$

81. The function $y = 2xe^{x^2}$ *does* have an elementary antiderivative, so we'll use this fact to help evaluate the integral.

$$\int(2x^2+1)e^{x^2}\,dx = \int 2x^2 e^{x^2}\,dx + \int e^{x^2}\,dx = \int x(2xe^{x^2})\,dx + \int e^{x^2}\,dx$$

$$= xe^{x^2} - \int e^{x^2}\,dx + \int e^{x^2}\,dx \qquad \begin{bmatrix} u = x, & dv = 2xe^{x^2}\,dx, \\ du = dx & v = e^{x^2} \end{bmatrix} \qquad = xe^{x^2} + C$$

8.6 Integration Using Tables and Computer Algebra Systems

Keep in mind that there are several ways to approach many of these exercises, and different methods can lead to different forms of the answer.

1. We could make the substitution $u = \sqrt{2}\,x$ to obtain the radical $\sqrt{7-u^2}$ and then use Formula 33 with $a = \sqrt{7}$. Alternatively, we will factor $\sqrt{2}$ out of the radical and use $a = \sqrt{\frac{7}{2}}$.

$$\int \frac{\sqrt{7-2x^2}}{x^2}\,dx = \sqrt{2}\int \frac{\sqrt{\frac{7}{2}-x^2}}{x^2}\,dx \overset{33}{=} \sqrt{2}\left[-\frac{1}{x}\sqrt{\tfrac{7}{2}-x^2} - \sin^{-1}\frac{x}{\sqrt{\frac{7}{2}}}\right] + C$$

$$= -\frac{1}{x}\sqrt{7-2x^2} - \sqrt{2}\,\sin^{-1}\left(\sqrt{\tfrac{2}{7}}\,x\right) + C$$

2. $$\int \frac{3x}{\sqrt{3-2x}}\,dx = 3\int \frac{x}{\sqrt{3+(-2)x}}\,dx \overset{55}{=} 3\left[\frac{2}{3(-2)^2}(-2x-2\cdot 3)\sqrt{3+(-2)x}\right] + C$$
$$= \tfrac{1}{2}(-2x-6)\sqrt{3-2x} + C = -(x+3)\sqrt{3-2x} + C$$

3. Let $u = \pi x \;\Rightarrow\; du = \pi\,dx$, so

$$\int \sec^3(\pi x)\,dx = \tfrac{1}{\pi}\int \sec^3 u\,du \overset{71}{=} \tfrac{1}{\pi}\left(\tfrac{1}{2}\sec u\tan u + \tfrac{1}{2}\ln|\sec u + \tan u|\right) + C$$
$$= \tfrac{1}{2\pi}\sec \pi x\tan \pi x + \tfrac{1}{2\pi}\ln|\sec \pi x + \tan \pi x| + C$$

4. $$\int e^{2\theta}\sin 3\theta\,d\theta \overset{98}{=} \frac{e^{2\theta}}{2^2+3^2}(2\sin 3\theta - 3\cos 3\theta) + C = \tfrac{2}{13}e^{2\theta}\sin 3\theta - \tfrac{3}{13}e^{2\theta}\cos 3\theta + C$$

5. $$\int_0^1 2x\cos^{-1}x\,dx \overset{91}{=} 2\left[\frac{2x^2-1}{4}\cos^{-1}x - \frac{x\sqrt{1-x^2}}{4}\right]_0^1 = 2\left[\left(\tfrac{1}{4}\cdot 0 - 0\right) - \left(-\tfrac{1}{4}\cdot\tfrac{\pi}{2} - 0\right)\right] = 2\left(\tfrac{\pi}{8}\right) = \tfrac{\pi}{4}$$

6. $$\int_2^3 \frac{1}{x^2\sqrt{4x^2-7}}\,dx = \int_4^6 \frac{1}{\left(\frac{1}{2}u\right)^2\sqrt{u^2-7}}\left(\frac{1}{2}\,du\right) \qquad [u = 2x,\ du = 2\,dx]$$
$$= 2\int_4^6 \frac{du}{u^2\sqrt{u^2-7}} \overset{45}{=} 2\left[\frac{\sqrt{u^2-7}}{7u}\right]_4^6$$
$$= 2\left(\tfrac{\sqrt{29}}{42} - \tfrac{3}{28}\right) = \tfrac{\sqrt{29}}{21} - \tfrac{3}{14}$$

7. By Formula 99 with $a = -3$ and $b = 4$,

$$\int e^{-3x}\cos 4x\,dx = \frac{e^{-3x}}{(-3)^2+4^2}(-3\cos 4x + 4\sin 4x) + C = \frac{e^{-3x}}{25}(-3\cos 4x + 4\sin 4x) + C.$$

8. Let $u = x/2$, so $dx = 2\,du$, and we use Formula 72:

$$\int \csc^3(x/2)\,dx = 2\int \csc^3 u\,du = -\csc u\cot u + \ln|\csc u - \cot u| + C$$
$$= -\csc(x/2)\cot(x/2) + \ln|\csc(x/2) - \cot(x/2)| + C$$

9. Let $u = 2x$ and $a = 3$. Then $du = 2\,dx$ and

$$\int \frac{dx}{x^2\sqrt{4x^2+9}} = \int \frac{\frac{1}{2}\,du}{\frac{u^2}{4}\sqrt{u^2+a^2}} = 2\int \frac{du}{u^2\sqrt{a^2+u^2}} \overset{28}{=} -2\,\frac{\sqrt{a^2+u^2}}{a^2u} + C$$
$$= -2\,\frac{\sqrt{4x^2+9}}{9\cdot 2x} + C = -\frac{\sqrt{4x^2+9}}{9x} + C$$

10. Let $u = \sqrt{2}\,y$ and $a = \sqrt{3}$. Then $du = \sqrt{2}\,dy$ and

$$\begin{aligned}\int \frac{\sqrt{2y^2-3}}{y^2}\,dy &= \int \frac{\sqrt{u^2-a^2}}{\frac{1}{2}u^2}\,\frac{du}{\sqrt{2}} = \sqrt{2}\int \frac{\sqrt{u^2-a^2}}{u^2}\,du \\ &\overset{42}{=} \sqrt{2}\left(-\frac{\sqrt{u^2-a^2}}{u} + \ln\left|u+\sqrt{u^2-a^2}\right|\right) + C \\ &= \sqrt{2}\left(-\frac{\sqrt{2y^2-3}}{\sqrt{2}\,y} + \ln\left|\sqrt{2}\,y+\sqrt{2y^2-3}\right|\right) + C \\ &= -\frac{\sqrt{2y^2-3}}{y} + \sqrt{2}\ln\left|\sqrt{2}\,y+\sqrt{2y^2-3}\right| + C\end{aligned}$$

11. $$\begin{aligned}\int_{-1}^{0} t^2e^{-t}\,dt &\overset{97}{=} \left[\frac{1}{-1}t^2e^{-t}\right]_{-1}^{0} - \frac{2}{-1}\int_{-1}^{0} te^{-t}\,dt = e + 2\int_{-1}^{0} te^{-t}\,dt \overset{96}{=} e + 2\left[\frac{1}{(-1)^2}\,(-t-1)\,e^{-t}\right]_{-1}^{0} \\ &= e + 2\left[-e^0+0\right] = e-2\end{aligned}$$

12. Let $u = 3x$. Then $du = 3\,dx$, so

$$\begin{aligned}\textstyle\int x^2\cos 3x\,dx &= \tfrac{1}{27}\textstyle\int u^2\cos u\,du \overset{85}{=} \tfrac{1}{27}\left(u^2\sin u - 2\textstyle\int u\sin u\,du\right) \\ &\overset{82}{=} \tfrac{1}{3}x^2\sin 3x - \tfrac{2}{27}(\sin 3x - 3x\cos 3x) + C \\ &= \tfrac{1}{27}\left[(9x^2-2)\sin 3x + 6x\cos 3x\right] + C\end{aligned}$$

Thus, $\int_0^\pi x^2\cos 3x\,dx = \frac{1}{27}\left[(9x^2-2)\sin 3x + 6x\cos 3x\right]_0^\pi = \frac{1}{27}\left[(0+6\pi(-1)) - (0+0)\right] = -\frac{6\pi}{27} = -\frac{2\pi}{9}$.

13. $$\begin{aligned}\int \frac{\tan^3(1/z)}{z^2}\,dz \quad \left[\begin{array}{l} u = 1/z, \\ du = -dz/z^2 \end{array}\right] \quad &= -\int \tan^3 u\,du \overset{69}{=} -\frac{1}{2}\tan^2 u - \ln|\cos u| + C \\ &= -\frac{1}{2}\tan^2\left(\frac{1}{z}\right) - \ln\left|\cos\left(\frac{1}{z}\right)\right| + C\end{aligned}$$

14. Let $u = \sqrt{x}$. Then $u^2 = x$ and $2u\,du = dx$, so

$$\begin{aligned}\int \sin^{-1}\sqrt{x}\,dx = 2\int u\sin^{-1}u\,du &\overset{90}{=} \frac{2u^2-1}{2}\sin^{-1}u + \frac{u\sqrt{1-u^2}}{2} + C \\ &= \frac{2x-1}{2}\sin^{-1}\sqrt{x} + \frac{\sqrt{x(1-x)}}{2} + C\end{aligned}$$

15. Let $u = e^x$. Then $du = e^x\,dx$, so $\int e^x\text{sech}(e^x)\,dx = \int \text{sech}\,u\,du \overset{107}{=} \tan^{-1}|\sinh u| + C = \tan^{-1}[\sinh(e^x)] + C$

16. Let $u = x^2$, so that $du = 2x\,dx$. Then

$$\begin{aligned}\int x\sin(x^2)\cos(3x^2)\,dx = \frac{1}{2}\int \sin u\cos 3u\,du &\overset{81}{=} -\frac{1}{2}\,\frac{\cos(1-3)u}{2(1-3)} - \frac{1}{2}\,\frac{\cos(1+3)u}{2(1+3)} + C \\ &= \tfrac{1}{8}\cos 2u - \tfrac{1}{16}\cos 4u + C = \tfrac{1}{8}\cos(2x^2) - \tfrac{1}{16}\cos(4x^2) + C\end{aligned}$$

17. Let $z = 6 + 4y - 4y^2 = 6 - (4y^2 - 4y + 1) + 1 = 7 - (2y - 1)^2$, $u = 2y - 1$, and $a = \sqrt{7}$. Then $z = a^2 - u^2$, $du = 2\,dy$, and

$$\begin{aligned}
\int y\sqrt{6+4y-4y^2}\,dy &= \int y\sqrt{z}\,dy = \int \tfrac{1}{2}(u+1)\sqrt{a^2-u^2}\,\tfrac{1}{2}\,du \\
&= \tfrac{1}{4}\int u\sqrt{a^2-u^2}\,du + \tfrac{1}{4}\int \sqrt{a^2-u^2}\,du \\
&= \tfrac{1}{4}\int \sqrt{a^2-u^2}\,du - \tfrac{1}{8}\int(-2u)\sqrt{a^2-u^2}\,du \\
&\overset{30}{=} \frac{u}{8}\sqrt{a^2-u^2} + \frac{a^2}{8}\sin^{-1}\!\left(\frac{u}{a}\right) - \frac{1}{8}\int\sqrt{w}\,dw \qquad \begin{bmatrix} w = a^2 - u^2, \\ dw = -2u\,du \end{bmatrix} \\
&= \frac{2y-1}{8}\sqrt{6+4y-4y^2} + \frac{7}{8}\sin^{-1}\frac{2y-1}{\sqrt{7}} - \frac{1}{8}\cdot\frac{2}{3}w^{3/2} + C \\
&= \frac{2y-1}{8}\sqrt{6+4y-4y^2} + \frac{7}{8}\sin^{-1}\frac{2y-1}{\sqrt{7}} - \frac{1}{12}(6+4y-4y^2)^{3/2} + C.
\end{aligned}$$

This can be rewritten as

$$\begin{aligned}
\sqrt{6+4y-4y^2}\left[\frac{1}{8}(2y-1) - \frac{1}{12}(6+4y-4y^2)\right] &+ \frac{7}{8}\sin^{-1}\frac{2y-1}{\sqrt{7}} + C \\
&= \left(\frac{1}{3}y^2 - \frac{1}{12}y - \frac{5}{8}\right)\sqrt{6+4y-4y^2} + \frac{7}{8}\sin^{-1}\left(\frac{2y-1}{\sqrt{7}}\right) + C \\
&= \frac{1}{24}(8y^2 - 2y - 15)\sqrt{6+4y-4y^2} + \frac{7}{8}\sin^{-1}\left(\frac{2y-1}{\sqrt{7}}\right) + C
\end{aligned}$$

18. Let $u = x^2$. Then $du = 2x\,dx$, so by Formula 48,

$$\begin{aligned}
\int \frac{x^5\,dx}{x^2+\sqrt{2}} &= \frac{1}{2}\int\frac{u^2}{u+\sqrt{2}}\,du = \tfrac{1}{2}\cdot\tfrac{1}{2}\left[\left(u+\sqrt{2}\right)^2 - 4\sqrt{2}\left(u+\sqrt{2}\right) + 4\ln\left|u+\sqrt{2}\right|\right] + C \\
&= \tfrac{1}{4}\left[\left(x^2+\sqrt{2}\right)^2 - 4\sqrt{2}\left(x^2+\sqrt{2}\right) + 4\ln\left(x^2+\sqrt{2}\right)\right] + C \\
&= \tfrac{1}{4}x^4 - \tfrac{1}{\sqrt{2}}x^2 + \ln\left(x^2+\sqrt{2}\right) + K
\end{aligned}$$

Or: Let $u = x^2 + \sqrt{2}$.

19. Let $u = \sin x$. Then $du = \cos x\,dx$, so

$$\begin{aligned}
\int \sin^2 x\cos x\,\ln(\sin x)\,dx &= \int u^2\ln u\,du \overset{101}{=} \frac{u^{2+1}}{(2+1)^2}\left[(2+1)\ln u - 1\right] + C = \tfrac{1}{9}u^3(3\ln u - 1) + C \\
&= \tfrac{1}{9}\sin^3 x\,[3\ln(\sin x) - 1] + C
\end{aligned}$$

20. Let $u = e^x$. Then $x = \ln u$, $dx = du/u$, so

$$\begin{aligned}
\int\frac{dx}{e^x(1+2e^x)} &= \int\frac{du/u}{u(1+2u)} = \int\frac{du}{u^2(1+2u)} \overset{50}{=} -\frac{1}{u} + 2\ln\left|\frac{1+2u}{u}\right| + C \\
&= -e^{-x} + 2\ln(e^{-x}+2) + C
\end{aligned}$$

21. Let $u = e^x$ and $a = \sqrt{3}$. Then $du = e^x\,dx$ and

$$\int\frac{e^x}{3-e^{2x}}\,dx = \int\frac{du}{a^2-u^2} \overset{19}{=} \frac{1}{2a}\ln\left|\frac{u+a}{u-a}\right| + C = \frac{1}{2\sqrt{3}}\ln\left|\frac{e^x+\sqrt{3}}{e^x-\sqrt{3}}\right| + C.$$

22. Let $u = x^2$ and $a = 2$. Then $du = 2x\,dx$ and

$$\int_0^2 x^3\sqrt{4x^2 - x^4}\,dx = \frac{1}{2}\int_0^2 x^2\sqrt{2\cdot 2\cdot x^2 - (x^2)^2}\cdot 2x\,dx = \frac{1}{2}\int_0^4 u\sqrt{2au - u^2}\,du$$

$$\overset{114}{=} \left[\frac{2u^2 - au - 3a^2}{12}\sqrt{2au - u^2} + \frac{a^3}{4}\cos^{-1}\left(\frac{a-u}{a}\right)\right]_0^4$$

$$= \left[\frac{2u^2 - 2u - 12}{12}\sqrt{4u - u^2} + \frac{8}{4}\cos^{-1}\left(\frac{2-u}{2}\right)\right]_0^4$$

$$= \left[\frac{u^2 - u - 6}{6}\sqrt{4u - u^2} + 2\cos^{-1}\left(\frac{2-u}{2}\right)\right]_0^4$$

$$= [0 + 2\cos^{-1}(-1)] - (0 + 2\cos^{-1}1) = 2\cdot\pi - 2\cdot 0 = 2\pi$$

23. $\int \sec^5 x\,dx \overset{77}{=} \frac{1}{4}\tan x\sec^3 x + \frac{3}{4}\int \sec^3 x\,dx \overset{77}{=} \frac{1}{4}\tan x\sec^3 x + \frac{3}{4}\left(\frac{1}{2}\tan x\sec x + \frac{1}{2}\int \sec x\,dx\right)$

$$\overset{14}{=} \frac{1}{4}\tan x\sec^3 x + \frac{3}{8}\tan x\sec x + \frac{3}{8}\ln|\sec x + \tan x| + C$$

24. Let $u = 2x$. Then $du = 2\,dx$, so

$$\int \sin^6 2x\,dx = \frac{1}{2}\int \sin^6 u\,du \overset{73}{=} \frac{1}{2}\left(-\frac{1}{6}\sin^5 u\cos u + \frac{5}{6}\int \sin^4 u\,du\right)$$

$$\overset{73}{=} -\frac{1}{12}\sin^5 u\cos u + \frac{5}{12}\left(-\frac{1}{4}\sin^3 u\cos u + \frac{3}{4}\int \sin^2 u\,du\right)$$

$$\overset{63}{=} -\frac{1}{12}\sin^5 u\cos u - \frac{5}{48}\sin^3 u\cos u + \frac{5}{16}\left(\frac{1}{2}u - \frac{1}{4}\sin 2u\right) + C$$

$$= -\frac{1}{12}\sin^5 2x\cos 2x - \frac{5}{48}\sin^3 2x\cos 2x - \frac{5}{64}\sin 4x + \frac{5}{16}x + C$$

25. Let $u = \ln x$ and $a = 2$. Then $du = \dfrac{dx}{x}$ and

$$\int \frac{\sqrt{4 + (\ln x)^2}}{x}\,dx = \int \sqrt{a^2 + u^2}\,du \overset{21}{=} \frac{u}{2}\sqrt{a^2 + u^2} + \frac{a^2}{2}\ln\left(u + \sqrt{a^2 + u^2}\right) + C$$

$$= \frac{1}{2}(\ln x)\sqrt{4 + (\ln x)^2} + 2\ln\left[\ln x + \sqrt{4 + (\ln x)^2}\right] + C$$

26. $\int x^4 e^{-x}\,dx \overset{97}{=} -x^4 e^{-x} + 4\int x^3 e^{-x}\,dx \overset{97}{=} -x^4 e^{-x} + 4\left(-x^3 e^{-x} + 3\int x^2 e^{-x}\,dx\right)$

$$\overset{97}{=} -(x^4 + 4x^3)e^{-x} + 12\left(-x^2 e^{-x} + 2\int x e^{-x}\,dx\right)$$

$$\overset{96}{=} -(x^4 + 4x^3 + 12x^2)e^{-x} + 24\left[(-x - 1)e^{-x}\right] + C$$

$$= -(x^4 + 4x^3 + 12x^2 + 24x + 24)e^{-x} + C$$

So $\int_0^1 x^4 e^{-x}\,dx = \left[-(x^4 + 4x^3 + 12x^2 + 24x + 24)e^{-x}\right]_0^1$

$$= -(1 + 4 + 12 + 24 + 24)e^{-1} + 24e^0 = 24 - 65e^{-1}.$$

27. Let $u = e^x$. Then $x = \ln u$, $dx = du/u$, so

$$\int \sqrt{e^{2x} - 1}\,dx = \int \frac{\sqrt{u^2 - 1}}{u}\,du \overset{41}{=} \sqrt{u^2 - 1} - \cos^{-1}(1/u) + C = \sqrt{e^{2x} - 1} - \cos^{-1}(e^{-x}) + C.$$

28. Let $u = \alpha t - 3$ and assume that $\alpha \neq 0$. Then $du = \alpha\, dt$ and

$$\begin{aligned}
\int e^t \sin(\alpha t - 3)\, dt &= \tfrac{1}{\alpha}\int e^{(u+3)/\alpha} \sin u\, du = \tfrac{1}{\alpha} e^{3/\alpha} \int e^{(1/\alpha)u} \sin u\, du \\
&\overset{98}{=} \frac{1}{\alpha} e^{3/\alpha} \frac{e^{(1/\alpha)u}}{(1/\alpha)^2 + 1^2}\left(\frac{1}{\alpha}\sin u - \cos u\right) + C \\
&= \frac{1}{\alpha} e^{3/\alpha} e^{(1/\alpha)u} \frac{\alpha^2}{1+\alpha^2}\left(\frac{1}{\alpha}\sin u - \cos u\right) + C \\
&= \frac{1}{1+\alpha^2} e^{(u+3)/\alpha}(\sin u - \alpha \cos u) + C \\
&= \frac{1}{1+\alpha^2} e^t \left[\sin(\alpha t - 3) - \alpha \cos(\alpha t - 3)\right] + C
\end{aligned}$$

29. $$\int \frac{x^4\, dx}{\sqrt{x^{10} - 2}} = \int \frac{x^4\, dx}{\sqrt{(x^5)^2 - 2}} = \frac{1}{5}\int \frac{du}{\sqrt{u^2 - 2}} \qquad [u = x^5,\ du = 5x^4\, dx]$$

$$\overset{43}{=} \tfrac{1}{5} \ln\left|u + \sqrt{u^2 - 2}\right| + C = \tfrac{1}{5} \ln\left|x^5 + \sqrt{x^{10} - 2}\right| + C$$

30. Let $u = \tan\theta$ and $a = 3$. Then $du = \sec^2\theta\, d\theta$ and

$$\begin{aligned}
\int \frac{\sec^2\theta \tan^2\theta}{\sqrt{9 - \tan^2\theta}}\, d\theta &= \int \frac{u^2}{\sqrt{a^2 - u^2}}\, du \overset{34}{=} -\frac{u}{2}\sqrt{a^2 - u^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{u}{a}\right) + C \\
&= -\frac{1}{2}\tan\theta \sqrt{9 - \tan^2\theta} + \frac{9}{2}\sin^{-1}\left(\frac{\tan\theta}{3}\right) + C
\end{aligned}$$

31. Using cylindrical shells, we get

$$V = 2\pi \int_0^2 x \cdot x\sqrt{4 - x^2}\, dx = 2\pi \int_0^2 x^2\sqrt{4 - x^2}\, dx \overset{31}{=} 2\pi\left[\frac{x}{8}(2x^2 - 4)\sqrt{4 - x^2} + \frac{16}{8}\sin^{-1}\frac{x}{2}\right]_0^2$$

$$= 2\pi\left[(0 + 2\sin^{-1} 1) - (0 + 2\sin^{-1} 0\right] = 2\pi\left(2 \cdot \frac{\pi}{2}\right) = 2\pi^2$$

32. Using disks, we get

$$\text{Volume} = \int_0^{\pi/4} \pi \tan^4 x\, dx \overset{75}{=} \pi\left(\left[\tfrac{1}{3}\tan^3 x\right]_0^{\pi/4} - \int_0^{\pi/4} \tan^2 x\, dx\right) \overset{65}{=} \pi\left[\tfrac{1}{3}\tan^3 x - \tan x + x\right]_0^{\pi/4}$$

$$= \pi\left(\tfrac{1}{3} - 1 + \tfrac{\pi}{4}\right) = \pi\left(\tfrac{\pi}{4} - \tfrac{2}{3}\right)$$

33. (a) $$\frac{d}{du}\left[\frac{1}{b^3}\left(a + bu - \frac{a^2}{a + bu} - 2a \ln|a + bu|\right) + C\right] = \frac{1}{b^3}\left[b + \frac{ba^2}{(a + bu)^2} - \frac{2ab}{(a + bu)}\right]$$

$$= \frac{1}{b^3}\left[\frac{b(a + bu)^2 + ba^2 - (a + bu)2ab}{(a + bu)^2}\right] = \frac{1}{b^3}\left[\frac{b^3 u^2}{(a + bu)^2}\right] = \frac{u^2}{(a + bu)^2}$$

(b) Let $t = a + bu \Rightarrow dt = b\, du$. Note that $u = \dfrac{t - a}{b}$ and $du = \dfrac{1}{b}\, dt$.

$$\begin{aligned}
\int \frac{u^2\, du}{(a + bu)^2} &= \frac{1}{b^3}\int \frac{(t - a)^2}{t^2}\, dt = \frac{1}{b^3}\int \frac{t^2 - 2at + a^2}{t^2}\, dt \\
&= \frac{1}{b^3}\int\left(1 - \frac{2a}{t} + \frac{a^2}{t^2}\right) dt = \frac{1}{b^3}\left(t - 2a \ln|t| - \frac{a^2}{t}\right) + C \\
&= \frac{1}{b^3}\left(a + bu - \frac{a^2}{a + bu} - 2a \ln|a + bu|\right) + C
\end{aligned}$$

34. (a) $\frac{d}{du}\left[\frac{u}{8}(2u^2-a^2)\sqrt{a^2-u^2}+\frac{a^4}{8}\sin^{-1}\frac{u}{a}+C\right]$

$$=\frac{u}{8}(2u^2-a^2)\frac{-u}{\sqrt{a^2-u^2}}+\sqrt{a^2-u^2}\left[\frac{u}{8}(4u)+(2u^2-a^2)\tfrac{1}{8}\right]+\frac{a^4}{8}\frac{1/a}{\sqrt{1-u^2/a^2}}$$

$$=-\frac{u^2(2u^2-a^2)}{8\sqrt{a^2-u^2}}+\sqrt{a^2-u^2}\left[\frac{u^2}{2}+\frac{2u^2-a^2}{8}\right]+\frac{a^4}{8\sqrt{a^2-u^2}}$$

$$=\tfrac{1}{2}(a^2-u^2)^{-1/2}\left[-\frac{u^2}{4}(2u^2-a^2)+u^2(a^2-u^2)+\tfrac{1}{4}(a^2-u^2)(2u^2-a^2)+\frac{a^4}{4}\right]$$

$$=\tfrac{1}{2}(a^2-u^2)^{-1/2}\left[2u^2a^2-2u^4\right]=\frac{u^2(a^2-u^2)}{\sqrt{a^2-u^2}}=u^2\sqrt{a^2-u^2}$$

(b) Let $u=a\sin\theta \Rightarrow du=a\cos\theta\,d\theta$. Then

$$\int u^2\sqrt{a^2-u^2}\,du=\int a^2\sin^2\theta\,a\sqrt{1-\sin^2\theta}\,a\cos\theta\,d\theta=a^4\int\sin^2\theta\cos^2\theta\,d\theta$$

$$=a^4\int\tfrac{1}{2}(1+\cos2\theta)\tfrac{1}{2}(1-\cos2\theta)d\theta=\tfrac{1}{4}a^4\int(1-\cos^2 2\theta)d\theta$$

$$=\tfrac{1}{4}a^4\int\left[1-\tfrac{1}{2}(1+\cos4\theta)\right]d\theta=\tfrac{1}{4}a^4\left(\tfrac{1}{2}\theta-\tfrac{1}{8}\sin4\theta\right)+C$$

$$=\tfrac{1}{4}a^4\left(\tfrac{1}{2}\theta-\tfrac{1}{8}2\sin2\theta\cos2\theta\right)+C=\tfrac{1}{4}a^4\left[\tfrac{1}{2}\theta-\tfrac{1}{2}\sin\theta\cos\theta(1-2\sin^2\theta)\right]+C$$

$$=\frac{a^4}{8}\left[\sin^{-1}\frac{u}{a}-\frac{u}{a}\frac{\sqrt{a^2-u^2}}{a}\left(1-\frac{2u^2}{a^2}\right)\right]+C$$

$$=\frac{a^4}{8}\left[\sin^{-1}\frac{u}{a}-\frac{u}{a}\frac{\sqrt{a^2-u^2}}{a}\frac{a^2-2u^2}{a^2}\right]+C$$

$$=\frac{u}{8}(2u^2-a^2)\sqrt{a^2-u^2}+\frac{a^4}{8}\sin^{-1}\frac{u}{a}+C$$

35. Maple, Mathematica and Derive all give $\int x^2\sqrt{5-x^2}\,dx=-\frac{1}{4}x(5-x^2)^{3/2}+\frac{5}{8}x\sqrt{5-x^2}+\frac{25}{8}\sin^{-1}\left(\frac{1}{\sqrt{5}}x\right)$. Using Formula 31, we get $\int x^2\sqrt{5-x^2}\,dx=\frac{1}{8}x(2x^2-5)\sqrt{5-x^2}+\frac{1}{8}(5^2)\sin^{-1}\left(\frac{1}{\sqrt{5}}x\right)+C$. But $-\frac{1}{4}x(5-x^2)^{3/2}+\frac{5}{8}x\sqrt{5-x^2}=\frac{1}{8}x\sqrt{5-x^2}\left[5-2(5-x^2)\right]=\frac{1}{8}x(2x^2-5)\sqrt{5-x^2}$, and the $\sin^{-1}$ terms are the same in each expression, so the answers are equivalent.

36. Maple and Mathematica both give $\int x^2(1+x^3)^4\,dx=\frac{1}{15}x^{15}+\frac{1}{3}x^{12}+\frac{2}{3}x^9+\frac{2}{3}x^6+\frac{1}{3}x^3$, while Derive gives $\int x^2(1+x^3)^4\,dx=\frac{1}{15}(x^3+1)^5$. Using the substitution $u=1+x^3 \Rightarrow du=3x^2\,dx$, we get $\int x^2(1+x^3)^4\,dx=\int u^4\left(\frac{1}{3}\,du\right)=\frac{1}{15}u^5+C=\frac{1}{15}(1+x^3)^5+C$. We can use the Binomial Theorem or a CAS to expand this expression, and we get $\frac{1}{15}(1+x^3)^5+C=\frac{1}{15}+\frac{1}{3}x^3+\frac{2}{3}x^6+\frac{2}{3}x^9+\frac{1}{3}x^{12}+\frac{1}{15}x^{15}+C$.

37. Maple and Derive both give $\int\sin^3x\cos^2x\,dx=-\frac{1}{5}\sin^2x\cos^3x-\frac{2}{15}\cos^3x$ (although Derive factors the expression), and Mathematica gives $\int\sin^3x\cos^2x\,dx=-\frac{1}{8}\cos x-\frac{1}{48}\cos3x+\frac{1}{80}\cos5x$. We can use a CAS to show that both of these expressions are equal to $-\frac{1}{3}\cos^3x+\frac{1}{5}\cos^5x$. Using Formula 86, we write

$$\int\sin^3x\cos^2x\,dx=-\tfrac{1}{5}\sin^2x\cos^3x+\tfrac{2}{5}\int\sin x\cos^2x\,dx=-\tfrac{1}{5}\sin^2x\cos^3x+\tfrac{2}{5}\left(-\tfrac{1}{3}\cos^3x\right)+C$$

$$=-\tfrac{1}{5}\sin^2x\cos^3x-\tfrac{2}{15}\cos^3x+C$$

38. Maple gives $\int \tan^2 x \sec^4 dx = \dfrac{1}{5}\dfrac{\sin^3 x}{\cos^5 x} + \dfrac{2}{15}\dfrac{\sin^3 x}{\cos^3 x}$,

Mathematica gives $\int \tan^2 x \sec^4 dx = -\frac{1}{120}\sec^5 x\,(-20\sin x + 5\sin 3x + \sin 5x)$, and

Derive gives $\int \tan^2 x \sec^4 dx = -\frac{2}{15}\tan x - \dfrac{\sin x}{15\cos^3 x} + \dfrac{\sin x}{5\cos^5 x}$. All of these expressions can be "simplified"

to $-\dfrac{1}{15}\dfrac{\sin x\,(\cos^2 x - 2\cos^4 x - 3)}{\cos^5 x}$ using Maple. Using the identity $1 + \tan^2 x = \sec^2 x$, we write

$\int \tan^2 x \sec^4 x\,dx = \int \tan^2 x\,(1 + \tan^2 x)\sec^2 x\,dx = \int (\tan^2 x + \tan^4 x)\sec^2 x\,dx$.

Now we substitute $u = \tan x \;\Rightarrow\; du = \sec^2 x\,dx$, and the integral becomes

$\int (u^2 + u^4)\,du = \frac{1}{3}u^3 + \frac{1}{5}u^5 + C = \frac{1}{3}\tan^3 x + \frac{1}{5}\tan^5 x + C$. If we write

$\sin^5 x = \sin^3 x\,(1 - \cos^2 x)$ and substitute into the numerator of the $\tan^5 x$ term, this becomes

$$\frac{1}{3}\frac{\sin^3 x}{\cos^3 x} + \frac{1}{5}\frac{\sin^3 x\,(1-\cos^2 x)}{\cos^5 x} + C = \frac{1}{5}\frac{\sin^3 x}{\cos^5 x} + \left(\frac{1}{3} - \frac{1}{5}\right)\frac{\sin^3 x}{\cos^3 x} + C = \frac{1}{5}\frac{\sin^3 x}{\cos^5 x} + \frac{2}{15}\frac{\sin^3 x}{\cos^3 x} + C,$$

which is the same as Maple's expression.

39. Maple gives $\int x\sqrt{1+2x}\,dx = \frac{1}{10}(1+2x)^{5/2} - \frac{1}{6}(1+2x)^{3/2}$, Mathematica gives $\sqrt{1+2x}\left(\frac{2}{5}x^2 + \frac{1}{15}x - \frac{1}{15}\right)$,

and Derive gives $\frac{1}{15}(1+2x)^{3/2}(3x-1)$. The first two expressions can be simplified to Derive's result. If we use Formula 54, we get

$$\int x\sqrt{1+2x}\,dx = \frac{2}{15(2)^2}(3\cdot 2x - 2\cdot 1)(1+2x)^{3/2} + C = \frac{1}{30}(6x-2)(1+2x)^{3/2} + C$$
$$= \frac{1}{15}(3x-1)(1+2x)^{3/2}$$

40. Maple and Derive both give $\int \sin^4 x\,dx = -\frac{1}{4}\sin^3 x\cos x - \frac{3}{8}\cos x\sin x + \frac{3}{8}x$, while Mathematica gives

$\frac{1}{32}(12x - 8\sin 2x + \sin 4x)$, which can be expanded and simplified to give the other expression. Now

$$\int \sin^4 x\,dx \overset{73}{=} -\frac{1}{4}\sin^3 x\cos x + \frac{3}{4}\int \sin^2 x\,dx \overset{63}{=} -\frac{1}{4}\sin^3 x\cos x + \frac{3}{4}\left(\frac{1}{2}x - \frac{1}{4}\sin 2x\right) + C$$
$$= -\frac{1}{4}\sin^3 x\cos x - \frac{3}{8}\sin x\cos x + \frac{3}{8}x + C \text{ since } \sin 2x = 2\sin x\cos x$$

41. Maple gives $\int \tan^5 x\,dx = \frac{1}{4}\tan^4 x - \frac{1}{2}\tan^2 x + \frac{1}{2}\ln(1+\tan^2 x)$, Mathematica

gives $\int \tan^5 x\,dx = \frac{1}{4}[-1 - 2\cos(2x)]\sec^4 x - \ln(\cos x)$, and Derive gives

$\int \tan^5 x\,dx = \frac{1}{4}\tan^4 x - \frac{1}{2}\tan^2 x - \ln(\cos x)$. These expressions are equivalent, and none includes absolute value bars or a constant of integration. Note that Mathematica's and Derive's expressions suggest that the integral is undefined where $\cos x < 0$, which is not the case.

Using Formula 75, $\int \tan^5 x\,dx = \frac{1}{5-1}\tan^{5-1} x - \int \tan^{5-2} x\,dx = \frac{1}{4}\tan^4 x - \int \tan^3 x\,dx$. Using Formula 69,

$\int \tan^3 x\,dx = \frac{1}{2}\tan^2 x + \ln|\cos x| + C$, so $\int \tan^5 x\,dx = \frac{1}{4}\tan^4 x - \frac{1}{2}\tan^2 x - \ln|\cos x| + C$.

42. Maple gives $\int x^5\sqrt{x^2+1}\,dx = \frac{1}{35}x^4\sqrt{1+x^2} - \frac{4}{105}x^2\sqrt{1+x^2} + \frac{8}{105}\sqrt{1+x^2} + \frac{1}{7}x^6\sqrt{1+x^2}$. When we use

the `factor` command on this expression, it becomes $\frac{1}{105}(1+x^2)^{3/2}(15x^4 - 12x^2 + 8)$. Mathematica gives

$\sqrt{1+x^2}\left(\frac{8}{105} - \frac{4}{105}x^2 + \frac{1}{35}x^4 + \frac{1}{7}x^6\right)$, which again factors to give the above expression, and Derive gives the

factored form immediately. If we substitute $u = \sqrt{x^2+1} \;\Rightarrow\; x^4 = (u^2-1)^2$, $x\,dx = u\,du$, then the integral becomes

$$\begin{aligned}\int (u^2-1)^2 u(u\,du) &= \int (u^4-2u^2+1)u^2\,du = \tfrac{1}{7}u^7 - \tfrac{2}{5}u^5 + \tfrac{1}{3}u^3 + C \\ &= (x^2+1)^{3/2}\left[\tfrac{1}{7}(x^2+1)^2 - \tfrac{2}{5}(x^2+1) + \tfrac{1}{3}\right] + C \\ &= \tfrac{1}{105}(x^2+1)^{3/2}\left[15(x^2+1)^2 - 42(x^2+1) + 35\right] + C \\ &= \tfrac{1}{105}(x^2+1)^{3/2}(15x^4 - 12x^2 + 8) + C\end{aligned}$$

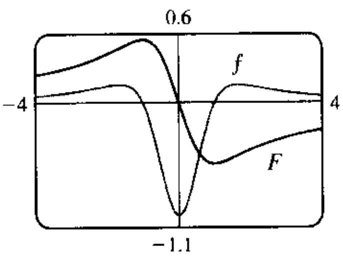
43. Derive gives $I = \int 2^x\sqrt{4^x-1}\,dx = \dfrac{2^{x-1}\sqrt{2^{2x}-1}}{\ln 2} - \dfrac{\ln\left(\sqrt{2^{2x}-1}+2^x\right)}{2\ln 2}$ immediately. Neither Maple nor Mathematica is able to evaluate I in its given form. However, if we instead write I as $\int 2^x\sqrt{(2^x)^2-1}\,dx$, both systems give the same answer as Derive (after minor simplification). Our trick works because the CAS now recognizes 2^x as a promising substitution.

44. None of Maple, Mathematica and Derive is able to evaluate $\int (1+\ln x)\sqrt{1+(x\ln x)^2}\,dx$. However, if we let $u = x\ln x$, then $du = (1+\ln x)dx$ and the integral is simply $\int \sqrt{1+u^2}\,du$, which any CAS can evaluate. The antiderivative is $\frac{1}{2}\ln\left(x\ln x + \sqrt{1+(x\ln x)^2}\right) + \frac{1}{2}x\ln x\sqrt{1+(x\ln x)^2} + C$.

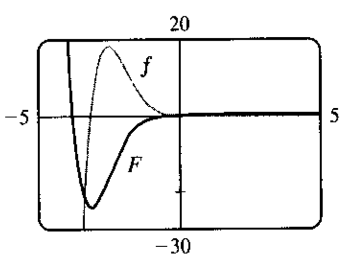
45. Maple gives the antiderivative

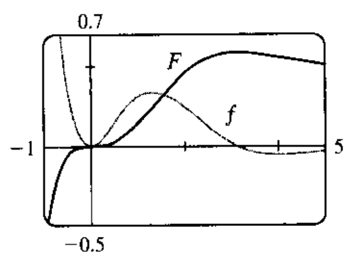
$$F(x) = \int \frac{x^2-1}{x^4+x^2+1}\,dx = -\tfrac{1}{2}\ln(x^2+x+1) + \tfrac{1}{2}\ln(x^2-x+1).$$

We can see that at 0, this antiderivative is 0. From the graphs, it appears that F has a maximum at $x=-1$ and a minimum at $x=1$ [since $F'(x) = f(x)$ changes sign at these x-values], and that F has inflection points at $x \approx -1.7$, $x = 0$, and $x \approx 1.7$ [since $f(x)$ has extrema at these x-values].



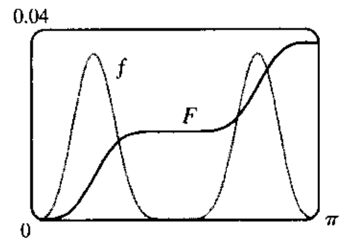
46. Maple gives the antiderivative which, after we use the `simplify` command, becomes $\int xe^{-x}\sin x\,dx = -\frac{1}{2}e^{-x}(\cos x + x\cos x + x\sin x)$. At $x=0$, this antiderivative has the value $-\frac{1}{2}$, so we use $F(x) = -\frac{1}{2}e^{-x}(\cos x + x\cos x + x\sin x) + \frac{1}{2}$ to make $F(0)=0$.





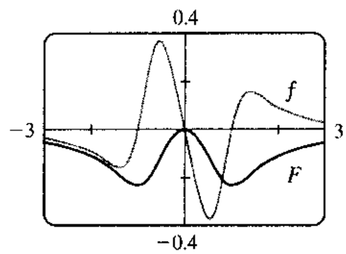
From the graphs, it appears that F has a minimum at $x \approx -3.1$ and a maximum at $x \approx 3.1$ [note that $f(x) = 0$ at $x = \pm\pi$], and that F has inflection points where f' changes sign, at $x \approx -2.5$, $x = 0$, $x \approx 1.3$ and $x \approx 4.1$.

47. Since $f(x) = \sin^4 x \cos^6 x$ is everywhere positive, we know that its antiderivative F is increasing. Maple gives $\int f(x)\,dx = -\frac{1}{10}\sin^3 x\cos^7 x - \frac{3}{80}\sin x\cos^7 x + \frac{1}{160}\cos^5 x\sin x + \frac{1}{128}\cos^3 x\sin x + \frac{3}{256}\cos x\sin x + \frac{3}{256}x$ and this expression is 0 at $x = 0$.



F has a minimum at $x = 0$ and a maximum at $x = \pi$. F has inflection points where f' changes sign, that is, at $x \approx 0.7$, $x = \pi/2$, and $x \approx 2.5$.

48. From the graph of $f(x) = \dfrac{x^3 - x}{x^6 + 1}$, we can see that F has a maximum at $x = 0$, and minima at $x \approx \pm 1$. The antiderivative given by Maple is $F(x) = -\frac{1}{3}\ln(x^2+1) + \frac{1}{6}\ln(x^4 - x^2 + 1)$, and $F(0) = 0$. Note that f is odd, and its antiderivative F is even.



F has inflection points where f' changes sign, that is, at $x \approx \pm 0.5$ and $x \approx \pm 1.4$.

DISCOVERY PROJECT Patterns in Integrals

1. (a) The CAS results are listed. Note that the absolute value symbols are missing, as is the familiar "$+C$".

(i) $\displaystyle\int \frac{1}{(x+2)(x+3)}\,dx = \ln(x+2) - \ln(x+3)$

(ii) $\displaystyle\int \frac{1}{(x+1)(x+5)}\,dx = \frac{\ln(x+1)}{4} - \frac{\ln(x+5)}{4}$

(iii) $\displaystyle\int \frac{1}{(x+2)(x-5)}\,dx = \frac{\ln(x-5)}{7} - \frac{\ln(x+2)}{7}$

(iv) $\displaystyle\int \frac{1}{(x+2)^2}\,dx = -\frac{1}{x+2}$

(b) If $a \neq b$, it appears that $\ln(x+a)$ is divided by $b-a$ and $\ln(x+b)$ is divided by $a-b$, so we guess that

$$\int \frac{1}{(x+a)(x+b)}\,dx = \frac{\ln(x+a)}{b-a} + \frac{\ln(x+b)}{a-b} + C$$

If $a = b$, as in part (a)(iv), it appears that

$$\int \frac{1}{(x+a)^2}\,dx = -\frac{1}{x+a} + C$$

(c) The CAS verifies our guesses. Now

$$\frac{1}{(x+a)(x+b)} = \frac{A}{x+a} + \frac{B}{x+b} \quad \Rightarrow \quad 1 = A(x+b) + B(x+a)$$

Setting $x = -b$ gives $B = 1/(a-b)$ and setting $x = -a$ gives $A = 1/(b-a)$. So

$$\int \frac{1}{(x+a)(x+b)}\,dx = \int \left[\frac{1/(b-a)}{x+a} + \frac{1/(a-b)}{x+b}\right] dx = \frac{\ln|x+a|}{b-a} + \frac{\ln|x+b|}{a-b} + C$$

and our guess for $a \neq b$ is correct.

If $a = b$, then $\dfrac{1}{(x+a)(x+b)} = \dfrac{1}{(x+a)^2} = (x+a)^{-2}$. Letting $u = x + a \Rightarrow du = dx$, we have $\displaystyle\int (x+a)^{-2}\,dx = \int u^{-2}\,du = -\frac{1}{u} + C = -\frac{1}{x+a} + C$, and our guess for $a = b$ is also correct.

2. (a) (i) $\int \sin x \, \cos 2x\,dx = \dfrac{\cos x}{2} - \dfrac{\cos 3x}{6}$

(ii) $\int \sin 3x \, \cos 7x\,dx = \dfrac{\cos 4x}{8} - \dfrac{\cos 10x}{20}$

(iii) $\int \sin 8x \, \cos 3x\,dx = -\dfrac{\cos 11x}{22} - \dfrac{\cos 5x}{10}$

(b) Looking at the sums and differences of a and b in part (a), we guess that

$$\int \sin ax \, \cos bx\,dx = \frac{\cos((a-b)x)}{2(b-a)} - \frac{\cos((a+b)x)}{2(a+b)} + C$$

Note that $\cos((a-b)x) = \cos((b-a)x)$.

(c) The CAS verifies our guess. To integrate directly, we can use Formula 2(a) from Section 8.2.

$$\int \sin ax \cos bx\,dx = \int \left\{\tfrac{1}{2}\left[\sin(ax - bx) + \sin(ax + bx)\right]\right\} dx = \tfrac{1}{2}\int \left[\sin((a-b)x) + \sin((a+b)x)\right] dx$$

$$= \frac{1}{2}\left[-\frac{\cos((a-b)x)}{b-a} - \frac{\cos((a+b)x)}{a+b}\right] + C = \frac{\cos((a-b)x)}{2(b-a)} - \frac{\cos((a+b)x)}{2(a+b)} + C$$

Our formula is valid for $a \neq b$.

3. (a) (i) $\int \ln x\,dx = x\ln x - x$

(ii) $\int x \ln x\,dx = \frac{1}{2}x^2 \ln x - \frac{1}{4}x^2$

(iii) $\int x^2 \ln x\,dx = \frac{1}{3}x^3 \ln x - \frac{1}{9}x^3$

(iv) $\int x^3 \ln x\,dx = \frac{1}{4}x^4 \ln x - \frac{1}{16}x^4$

(v) $\int x^7 \ln x\,dx = \frac{1}{8}x^8 \ln x - \frac{1}{64}x^8$

(b) We guess that $\int x^n \ln x\,dx = \dfrac{1}{n+1}x^{n+1}\ln x - \dfrac{1}{(n+1)^2}x^{n+1}$.

(c) Let $u = \ln x$, $dv = x^n\,dx \Rightarrow du = \dfrac{dx}{x}$, $v = \dfrac{1}{n+1}x^{n+1}$.

Then
$$\int x^n \ln x\,dx = \frac{1}{n+1}x^{n+1}\ln x - \frac{1}{n+1}\int x^n\,dx$$
$$= \frac{1}{n+1}x^{n+1}\ln x - \frac{1}{n+1}\cdot\frac{1}{n+1}x^{n+1},$$

which verifies our guess. We must have $n + 1 \neq 0 \Leftrightarrow n \neq -1$.

4. (a) (i) $\int xe^x\,dx = e^x(x-1)$

(ii) $\int x^2e^x\,dx = e^x(x^2-2x+2)$

(iii) $\int x^3e^x\,dx = e^x(x^3-3x^2+6x-6)$

(iv) $\int x^4e^x\,dx = e^x(x^4-4x^3+12x^2-24x+24)$

(v) $\int x^5e^x\,dx = e^x(x^5-5x^4+20x^3-60x^2+120x-120)$

(b) Notice from part (a) that we can write

$$\int x^4e^x\,dx = e^x(x^4-4x^3+4\cdot 3x^2-4\cdot 3\cdot 2x+4\cdot 3\cdot 2\cdot 1)$$

and

$$\int x^5e^x\,dx = e^x(x^5-5x^4+5\cdot 4x^3-5\cdot 4\cdot 3x^2+5\cdot 4\cdot 3\cdot 2x-5\cdot 4\cdot 3\cdot 2\cdot 1)$$

So we guess that

$$\begin{aligned}\int x^6e^x\,dx &= e^x(x^6-6x^5+6\cdot 5x^4-6\cdot 5\cdot 4x^3+6\cdot 5\cdot 4\cdot 3x^2-6\cdot 5\cdot 4\cdot 3\cdot 2x+6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1)\\ &= e^x(x^6-6x^5+30x^4-120x^3+360x^2-720x+720)\end{aligned}$$

The CAS verifies our guess.

(c) From the results in part (a), as well as our prediction in part (b), we speculate that

$$\begin{aligned}\int x^ne^x\,dx &= e^x\left[x^n-nx^{n-1}+n(n-1)x^{n-2}-n(n-1)(n-2)x^{n-3}+\cdots\pm n!x\mp n!\right]\\ &= e^x\sum_{i=0}^{n}(-1)^{n-i}\frac{n!}{i!}x^i\end{aligned}$$

(We have reversed the order of the polynomial's terms.)

(d) Let S_n be the statement that $\int x^ne^x\,dx = e^x\sum_{i=0}^{n}(-1)^{n-i}\frac{n!}{i!}x^i$.

S_1 is true by part (a)(i). Suppose S_k is true for some k, and consider S_{k+1}. Integrating by parts with $u=x^{k+1}$, $dv=e^x\,dx \;\Rightarrow\; du=(k+1)x^k\,dx$, $v=e^x$, we get

$$\begin{aligned}\int x^{k+1}e^x\,dx &= x^{k+1}e^x-(k+1)\int x^ke^x\,dx\\ &= x^{k+1}e^x-(k+1)\left[e^x\sum_{i=0}^{k}(-1)^{k-i}\frac{k!}{i!}x^i\right]\\ &= e^x\left[x^{k+1}-(k+1)\sum_{i=0}^{k}(-1)^{k-i}\frac{k!}{i!}x^i\right]\\ &= e^x\left[x^{k+1}+\sum_{i=0}^{k}(-1)^{k-i+1}\frac{(k+1)k!}{i!}x^i\right]\\ &= e^x\sum_{i=0}^{k+1}(-1)^{(k+1)-i}\frac{(k+1)!}{i!}x^i\end{aligned}$$

This verifies S_n for $n=k+1$. Thus, by mathematical induction, S_n is true for all n, where n is a positive integer.

8.7 Approximate Integration

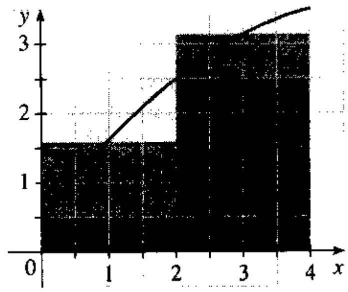
1. (a) $\Delta x = (b-a)/n = (4-0)/2 = 2$

$$L_2 = \sum_{i=1}^{2} f(x_{i-1})\,\Delta x = f(x_0)\cdot 2 + f(x_1)\cdot 2 = 2\,[f(0)+f(2)] = 2(0.5+2.5) = 6$$

$$R_2 = \sum_{i=1}^{2} f(x_i)\,\Delta x = f(x_1)\cdot 2 + f(x_2)\cdot 2 = 2\,[f(2)+f(4)] = 2(2.5+3.5) = 12$$

$$M_2 = \sum_{i=1}^{2} f(\overline{x}_i)\Delta x = f(\overline{x}_1)\cdot 2 + f(\overline{x}_2)\cdot 2 = 2\,[f(1)+f(3)] \approx 2(1.6+3.2) = 9.6$$

(b)

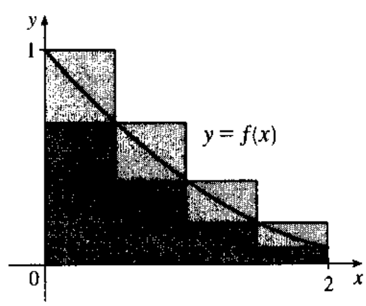


L_2 is an underestimate, since the area under the small rectangles is less than the area under the curve, and R_2 is an overestimate, since the area under the large rectangles is greater than the area under the curve. It appears that M_2 is an overestimate, though it is fairly close to I. See the solution to Exercise 45 for a proof of the fact that if f is concave down on $[a,b]$, then the Midpoint Rule is an overestimate of $\int_a^b f(x)\,dx$.

(c) $T_2 = \left(\frac{1}{2}\,\Delta x\right)[f(x_0) + 2f(x_1) + f(x_2)] = \frac{2}{2}[f(0) + 2f(2) + f(4)] = 0.5 + 2(2.5) + 3.5 = 9.$
This approximation is an underestimate, since the graph is concave down. Thus, $T_2 = 9 < I$. See the solution to Exercise 45 for a general proof of this conclusion.

(d) For any n, we will have $L_n < T_n < I < M_n < R_n$.

2.



The diagram shows that $L_4 > T_4 > \int_0^2 f(x)\,dx > R_4$, and it appears that M_4 is a bit less than $\int_0^2 f(x)\,dx$. In fact, for any function that is concave upward, it can be shown that

$L_n > T_n > \int_0^2 f(x)\,dx > M_n > R_n$.

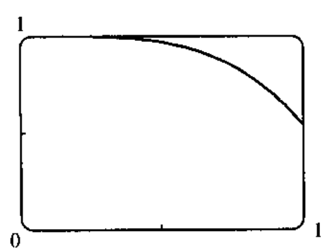
(a) Since $0.9540 > 0.8675 > 0.8632 > 0.7811$, it follows that
$L_n = 0.9540$, $T_n = 0.8675$, $M_n = 0.8632$, and $R_n = 0.7811$.

(b) Since $M_n < \int_0^2 f(x)\,dx < T_n$, we have
$0.8632 < \int_0^2 f(x)\,dx < 0.8675$.

3. $f(x) = \cos(x^2)$, $\Delta x = \frac{1-0}{4} = \frac{1}{4}$

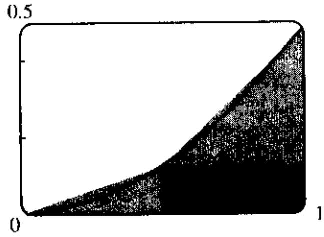
(a) $T_4 = \frac{1}{4\cdot 2}\left[f(0) + 2f\left(\frac{1}{4}\right) + 2f\left(\frac{2}{4}\right) + 2f\left(\frac{3}{4}\right) + f(1)\right] \approx 0.895759$

(b) $M_4 = \frac{1}{4}\left[f\left(\frac{1}{8}\right) + f\left(\frac{3}{8}\right) + f\left(\frac{5}{8}\right) + f\left(\frac{7}{8}\right)\right] \approx 0.908907$



The graph shows that f is concave down on $[0,1]$. So T_4 is an underestimate and M_4 is an overestimate. We can conclude that $0.895759 < \int_0^1 \cos(x^2)\,dx < 0.908907$.

4.



(a) Since f is increasing on $[0,1]$, L_2 will underestimate I (since the area of the darkest rectangle is less than the area under the curve), and R_2 will overestimate I. Since f is concave upward on $[0,1]$, M_2 will underestimate I and T_2 will overestimate I (the area under the straight line segments is greater than the area under the curve).

(b) For any n, we will have $L_n < M_n < I < T_n < R_n$.

(c) $L_5 = \sum_{i=1}^{5} f(x_{i-1})\,\Delta x = \frac{1}{5}[f(0.0) + f(0.2) + f(0.4) + f(0.6) + f(0.8)] \approx 0.1187$

$R_5 = \sum_{i=1}^{5} f(x_i)\,\Delta x = \frac{1}{5}[f(0.2) + f(0.4) + f(0.6) + f(0.8) + f(1)] \approx 0.2146$

$M_5 = \sum_{i=1}^{5} f(\overline{x}_i)\,\Delta x = \frac{1}{5}[f(0.1) + f(0.3) + f(0.5) + f(0.7) + f(0.9)] \approx 0.1622$

$T_5 = \left(\frac{1}{2}\,\Delta x\right)[f(0) + 2f(0.2) + 2f(0.4) + 2f(0.6) + 2f(0.8) + f(1)] \approx 0.1666$

From the graph, it appears that the Midpoint Rule gives the best approximation. (This is in fact the case, since $I \approx 0.16371405$.)

5. $f(x) = x^2 \sin x$, $\Delta x = \dfrac{b-a}{n} = \dfrac{\pi - 0}{8} = \dfrac{\pi}{8}$

(a) $M_8 = \frac{\pi}{8}\left[f\left(\frac{\pi}{16}\right) + f\left(\frac{3\pi}{16}\right) + f\left(\frac{5\pi}{16}\right) + \cdots + f\left(\frac{15\pi}{16}\right)\right] \approx 5.932957$

(b) $S_8 = \frac{\pi}{8\cdot 3}\left[f(0) + 4f\left(\frac{\pi}{8}\right) + 2f\left(\frac{2\pi}{8}\right) + 4f\left(\frac{3\pi}{8}\right) + 2f\left(\frac{4\pi}{8}\right) + 4f\left(\frac{5\pi}{8}\right) + 2f\left(\frac{6\pi}{8}\right) + 4f\left(\frac{7\pi}{8}\right) + f(\pi)\right]$
≈ 5.869247

Actual: $\int_0^\pi x^2 \sin x\,dx \overset{84}{=} \left[-x^2\cos x\right]_0^\pi + 2\int_0^\pi x\cos x\,dx \overset{83}{=} \left[-\pi^2(-1) - 0\right] + 2[\cos x + x\sin x]_0^\pi$
$= \pi^2 + 2[(-1+0) - (1+0)] = \pi^2 - 4 \approx 5.869604$

Errors: $E_M = \text{actual} - M_8 = \int_0^\pi x^2\sin x\,dx - M_8 \approx -0.063353$

$E_S = \text{actual} - S_8 = \int_0^\pi x^2 \sin x\,dx - S_8 \approx 0.000357$

6. $f(x) = e^{-\sqrt{x}}$, $\Delta x = \dfrac{b-a}{n} = \dfrac{1-0}{6} = \dfrac{1}{6}$

(a) $M_6 = \frac{1}{6}\left[f\left(\frac{1}{12}\right) + f\left(\frac{3}{12}\right) + f\left(\frac{5}{12}\right) + f\left(\frac{7}{12}\right) + f\left(\frac{9}{12}\right) + f\left(\frac{11}{12}\right)\right] \approx 0.525100$

(b) $S_6 = \frac{1}{6\cdot 3}\left[f(0) + 4f\left(\frac{1}{6}\right) + 2f\left(\frac{2}{6}\right) + 4f\left(\frac{3}{6}\right) + 2f\left(\frac{4}{6}\right) + 4f\left(\frac{5}{6}\right) + f(1)\right] \approx 0.533979$

Actual: $\int_0^1 e^{-\sqrt{x}}\,dx = \int_0^{-1} e^u 2u\,du \quad [u = -\sqrt{x},\, u^2 = x,\, 2u\,du = dx]$
$\overset{96}{=} 2[(u-1)e^u]_0^{-1} = 2\left[-2e^{-1} - (-1e^0)\right] = 2 - 4e^{-1} \approx 0.528482$

Errors: $E_M = \text{actual} - M_6 = \int_0^1 e^{-\sqrt{x}}\,dx - M_6 \approx 0.003382$

$E_S = \text{actual} - S_6 = \int_0^1 e^{-\sqrt{x}}\,dx - S_6 \approx -0.005497$

7. $f(x) = \sqrt[4]{1+x^2}$, $\Delta x = \dfrac{2-0}{8} = \dfrac{1}{4}$

(a) $T_8 = \frac{1}{4\cdot 2}\left[f(0) + 2f\left(\frac{1}{4}\right) + 2f\left(\frac{1}{2}\right) + \cdots + 2f\left(\frac{3}{2}\right) + 2f\left(\frac{7}{4}\right) + f(2)\right] \approx 2.413790$

(b) $M_8 = \frac{1}{4}\left[f\left(\frac{1}{8}\right) + f\left(\frac{3}{8}\right) + \cdots + f\left(\frac{13}{8}\right) + f\left(\frac{15}{8}\right)\right] \approx 2.411453$

(c) $S_8 = \frac{1}{4\cdot 3}\left[f(0) + 4f\left(\frac{1}{4}\right) + 2f\left(\frac{1}{2}\right) + 4f\left(\frac{3}{4}\right) + 2f(1) + 4f\left(\frac{5}{4}\right) + 2f\left(\frac{3}{2}\right) + 4f\left(\frac{7}{4}\right) + f(2)\right] \approx 2.412232$

8. $f(x)=\sin(x^2)$, $\Delta x=\dfrac{\frac{1}{2}-0}{4}=\dfrac{1}{8}$

(a) $T_4=\frac{1}{8\cdot 2}\left[f(0)+2f\left(\frac{1}{8}\right)+2f\left(\frac{2}{8}\right)+2f\left(\frac{3}{8}\right)+f\left(\frac{1}{2}\right)\right]\approx 0.042743$

(b) $M_4=\frac{1}{8}\left[f\left(\frac{1}{16}\right)+f\left(\frac{3}{16}\right)+f\left(\frac{5}{16}\right)+f\left(\frac{7}{16}\right)\right]\approx 0.040850$

(c) $S_4=\frac{1}{8\cdot 3}\left[f(0)+4f\left(\frac{1}{8}\right)+2f\left(\frac{2}{8}\right)+4f\left(\frac{3}{8}\right)+f\left(\frac{1}{2}\right)\right]\approx 0.041478$

9. $f(x)=\dfrac{\ln x}{1+x}$, $\Delta x=\dfrac{2-1}{10}=\dfrac{1}{10}$

(a) $T_{10}=\frac{1}{10\cdot 2}[f(1)+2f(1.1)+2f(1.2)+\cdots+2f(1.8)+2f(1.9)+f(2)]\approx 0.146879$

(b) $M_{10}=\frac{1}{10}[f(1.05)+f(1.15)+\cdots+f(1.85)+f(1.95)]\approx 0.147391$

(c) $S_{10}=\frac{1}{10\cdot 3}[f(1)+4f(1.1)+2f(1.2)+4f(1.3)+2f(1.4)+4f(1.5)+2f(1.6)+4f(1.7)$
$+2f(1.8)+4f(1.9)+f(2)]$
≈ 0.147219

10. $f(t)=\dfrac{1}{1+t^2+t^4}$, $\Delta t=\dfrac{3-0}{6}=\dfrac{1}{2}$

(a) $T_6=\frac{1}{2\cdot 2}\left[f(0)+2f\left(\frac{1}{2}\right)+2f(1)+2f\left(\frac{3}{2}\right)+2f(2)+2f\left(\frac{5}{2}\right)+f(3)\right]\approx 0.895122$

(b) $M_6=\frac{1}{2}\left[f\left(\frac{1}{4}\right)+f\left(\frac{3}{4}\right)+f\left(\frac{5}{4}\right)+f\left(\frac{7}{4}\right)+f\left(\frac{9}{4}\right)+f\left(\frac{11}{4}\right)\right]\approx 0.895478$

(c) $S_6=\frac{1}{2\cdot 3}\left[f(0)+4f\left(\frac{1}{2}\right)+2f(1)+4f\left(\frac{3}{2}\right)+2f(2)+4f\left(\frac{5}{2}\right)+f(3)\right]\approx 0.898014$

11. $f(t)=\sin(e^{t/2})$, $\Delta t=\dfrac{\frac{1}{2}-0}{8}=\dfrac{1}{16}$

(a) $T_8=\frac{1}{16\cdot 2}\left[f(0)+2f\left(\frac{1}{16}\right)+2f\left(\frac{2}{16}\right)+\cdots+2f\left(\frac{7}{16}\right)+f\left(\frac{1}{2}\right)\right]\approx 0.451948$

(b) $M_8=\frac{1}{16}\left[f\left(\frac{1}{32}\right)+f\left(\frac{3}{32}\right)+f\left(\frac{5}{32}\right)+\cdots+f\left(\frac{13}{32}\right)+f\left(\frac{15}{32}\right)\right]\approx 0.451991$

(c) $S_8=\frac{1}{16\cdot 3}\left[f(0)+4f\left(\frac{1}{16}\right)+2f\left(\frac{2}{16}\right)+\cdots+4f\left(\frac{7}{16}\right)+f\left(\frac{1}{2}\right)\right]\approx 0.451976$

12. $f(x)=\sqrt{1+\sqrt{x}}$, $\Delta x=\dfrac{4-0}{8}=\dfrac{1}{2}$

(a) $T_8=\frac{1}{2\cdot 2}\left[f(0)+2f\left(\frac{1}{2}\right)+2f(1)+\cdots+2f(3)+2f\left(\frac{7}{2}\right)+f(4)\right]\approx 6.042985$

(b) $M_8=\frac{1}{2}\left[f\left(\frac{1}{4}\right)+f\left(\frac{3}{4}\right)+\cdots+f\left(\frac{13}{4}\right)+f\left(\frac{15}{4}\right)\right]\approx 6.084778$

(c) $S_8=\frac{1}{2\cdot 3}\left[f(0)+4f\left(\frac{1}{2}\right)+2f(1)+4f\left(\frac{3}{2}\right)+2f(2)+4f\left(\frac{5}{2}\right)+2f(3)+4f\left(\frac{7}{2}\right)+f(4)\right]\approx 6.061678$

13. $f(x)=e^{1/x}$, $\Delta x=\dfrac{2-1}{4}=\dfrac{1}{4}$

(a) $T_4=\frac{1}{4\cdot 2}[f(1)+2f(1.25)+2f(1.5)+2f(1.75)+f(2)]\approx 2.031893$

(b) $M_4=\frac{1}{4}[f(1.125)+f(1.375)+f(1.625)+f(1.875)]\approx 2.014207$

(c) $S_4=\frac{1}{4\cdot 3}[f(1)+4f(1.25)+2f(1.5)+4f(1.75)+f(2)]\approx 2.020651$

14. $f(x)=\sqrt{x}\sin x$, $\Delta x=\dfrac{4-0}{8}=\dfrac{1}{2}$

(a) $T_8=\frac{1}{2\cdot 2}\left\{f(0)+2\left[f\left(\frac{1}{2}\right)+f(1)+f\left(\frac{3}{2}\right)+f(2)+f\left(\frac{5}{2}\right)+f(3)+f\left(\frac{7}{2}\right)\right]+f(4)\right\}\approx 1.732865$

(b) $M_8=\frac{1}{2}\left[f\left(\frac{1}{4}\right)+f\left(\frac{3}{4}\right)+f\left(\frac{5}{4}\right)+f\left(\frac{7}{4}\right)+\cdots+f\left(\frac{13}{4}\right)+f\left(\frac{15}{4}\right)\right]\approx 1.787427$

(c) $S_8=\frac{1}{2\cdot 3}\left[f(0)+4f\left(\frac{1}{2}\right)+2f(1)+4f\left(\frac{3}{2}\right)+2f(2)+4f\left(\frac{5}{2}\right)+2f(3)+4f\left(\frac{7}{2}\right)+f(4)\right]\approx 1.772142$

15. $f(x) = \dfrac{\cos x}{x}$, $\Delta x = \dfrac{5-1}{8} = \dfrac{1}{2}$

(a) $T_8 = \frac{1}{2\cdot 2}\left[f(1) + 2f\left(\frac{3}{2}\right) + 2f(2) + \cdots + 2f(4) + 2f\left(\frac{9}{2}\right) + f(5)\right] \approx -0.495333$

(b) $M_8 = \frac{1}{2}\left[f\left(\frac{5}{4}\right) + f\left(\frac{7}{4}\right) + f\left(\frac{9}{4}\right) + f\left(\frac{11}{4}\right) + f\left(\frac{13}{4}\right) + f\left(\frac{15}{4}\right) + f\left(\frac{17}{4}\right) + f\left(\frac{19}{4}\right)\right] \approx -0.543321$

(c) $S_8 = \frac{1}{2\cdot 3}\left[f(1) + 4f\left(\frac{3}{2}\right) + 2f(2) + 4f\left(\frac{5}{2}\right) + 2f(3) + 4f\left(\frac{7}{2}\right) + 2f(4) + 4f\left(\frac{9}{2}\right) + f(5)\right]$
≈ -0.526123

16. $f(x) = \ln(x^3 + 2)$, $\Delta x = \dfrac{6-4}{10} = \dfrac{1}{5}$

(a) $T_{10} = \frac{1}{5\cdot 2}\left[f(4) + 2f(4.2) + 2f(4.4) + \cdots + 2f(5.6) + 2f(5.8) + f(6)\right] \approx 9.649753$

(b) $M_{10} = \frac{1}{5}[f(4.1) + f(4.3) + \cdots + f(5.7) + f(5.9)] \approx 9.650912$

(c) $S_{10} = \frac{1}{5\cdot 3}[f(4) + 4f(4.2) + 2f(4.4) + 4f(4.6) + 2f(4.8) + 4f(5)$
$+ 2f(5.2) + 4f(5.4) + 2f(5.6) + 4f(5.8) + f(6)]$
≈ 9.650526

17. $f(y) = \dfrac{1}{1+y^5}$, $\Delta y = \dfrac{3-0}{6} = \dfrac{1}{2}$

(a) $T_6 = \frac{1}{2\cdot 2}\left[f(0) + 2f\left(\frac{1}{2}\right) + 2f\left(\frac{2}{2}\right) + 2f\left(\frac{3}{2}\right) + 2f\left(\frac{4}{2}\right) + 2f\left(\frac{5}{2}\right) + f(3)\right] \approx 1.064275$

(b) $M_6 = \frac{1}{2}\left[f\left(\frac{1}{4}\right) + f\left(\frac{3}{4}\right) + f\left(\frac{5}{4}\right) + f\left(\frac{7}{4}\right) + f\left(\frac{9}{4}\right) + f\left(\frac{11}{4}\right)\right] \approx 1.067416$

(c) $S_6 = \frac{1}{2\cdot 3}\left[f(0) + 4f\left(\frac{1}{2}\right) + 2f\left(\frac{2}{2}\right) + 4f\left(\frac{3}{2}\right) + 2f\left(\frac{4}{2}\right) + 4f\left(\frac{5}{2}\right) + f(3)\right] \approx 1.074915$

18. $f(x) = \dfrac{e^x}{x}$, $\Delta x = \dfrac{4-2}{10} = \dfrac{1}{5}$

(a) $T_{10} = \frac{1}{5\cdot 2}\{f(2) + 2[f(2.2) + f(2.4) + f(2.6) + \cdots + f(3.8)] + f(4)\} \approx 14.704592$

(b) $M_{10} = \frac{1}{5}[f(2.1) + f(2.3) + f(2.5) + f(2.7) + \cdots + f(3.7) + f(3.9)] \approx 14.662669$

(c) $S_{10} = \frac{1}{5\cdot 3}[f(2) + 4f(2.2) + 2f(2.4) + 4f(2.6) + \cdots + 2f(3.6) + 4f(3.8) + f(4)] \approx 14.676696$

19. $f(x) = e^{-x^2}$, $\Delta x = \dfrac{2-0}{10} = \dfrac{1}{5}$

(a) $T_{10} = \frac{1}{5\cdot 2}\{f(0) + 2[f(0.2) + f(0.4) + \cdots + f(1.8)] + f(2)\} \approx 0.881839$
$M_{10} = \frac{1}{5}[f(0.1) + f(0.3) + f(0.5) + \cdots + f(1.7) + f(1.9)] \approx 0.882202$

(b) $f(x) = e^{-x^2}$, $f'(x) = -2xe^{-x^2}$, $f''(x) = (4x^2 - 2)e^{-x^2}$, $f'''(x) = 4x(3 - 2x^2)e^{-x^2}$.
$f'''(x) = 0 \;\Leftrightarrow\; x = 0$ or $x = \pm\sqrt{\frac{3}{2}}$. So to find the maximum value of $|f''(x)|$ on $[0, 2]$, we need only consider its values at $x = 0$, $x = 2$, and $x = \sqrt{\frac{3}{2}}$. $|f''(0)| = 2$, $|f''(2)| \approx 0.2564$ and $\left|f''\left(\sqrt{\frac{3}{2}}\right)\right| \approx 0.8925$.
Thus, taking $K = 2$, $a = 0$, $b = 2$, and $n = 10$ in Theorem 3, we get $|E_T| \le 2 \cdot 2^3/(12 \cdot 10^2) = \frac{1}{75} = 0.01\overline{3}$, and $|E_M| \le |E_T|/2 \le 0.00\overline{6}$.

(c) Take $K = 2$ [as in part (b)] in Theorem 3. $|E_T| \le \dfrac{K(b-a)^3}{12n^2} \le 10^{-5} \;\Leftrightarrow\; \dfrac{2(2-0)^3}{12n^2} \le 10^{-5} \;\Leftrightarrow$
$\frac{3}{4}n^2 \ge 10^5 \;\Leftrightarrow\; n \ge 365.1\ldots \;\Leftrightarrow\; n \ge 366$. Take $n = 366$ for T_n. For E_M, again take $K = 2$ in Theorem 3 to get $|E_M| \le 10^{-5} \;\Leftrightarrow\; \frac{3}{2}n^2 \ge 10^5 \;\Leftrightarrow\; n \ge 258.2 \;\Rightarrow\; n \ge 259$. Take $n = 259$ for M_n.

20. (a) $T_8 = \frac{1}{8\cdot 2}\left\{f(0) + 2\left[f\left(\frac{1}{8}\right) + f\left(\frac{2}{8}\right) + \cdots + f\left(\frac{7}{8}\right)\right] + f(1)\right\} \approx 0.902333$
$M_8 = \frac{1}{8}\left[f\left(\frac{1}{16}\right) + f\left(\frac{3}{16}\right) + f\left(\frac{5}{16}\right) + \cdots + f\left(\frac{15}{16}\right)\right] = 0.905620$

(b) $f(x) = \cos(x^2)$, $f'(x) = -2x\sin(x^2)$, $f''(x) = -2\sin(x^2) - 4x^2\cos(x^2)$. For $0 \le x \le 1$, sin and cos are positive, so $|f''(x)| = 2\sin(x^2) + 4x^2\cos(x^2) \le 2\cdot 1 + 4\cdot 1\cdot 1 = 6$ since $\sin(x^2) \le 1$ and $\cos(x^2) \le 1$ for all x, and $x^2 \le 1$ for $0 \le x \le 1$. So for $n = 8$, we take $K = 6$, $a = 0$, and $b = 1$ in Theorem 3, to get $|E_T| \le 6\cdot 1^3/(12\cdot 8^2) = \frac{1}{128} = 0.0078125$ and $|E_M| \le \frac{1}{256} = 0.00390625$. [A better estimate is obtained by noting from a graph of f'' that $|f''(x)| \le 4$ for $0 \le x \le 1$.]

(c) Using $K = 6$ as in part (b), we have $|E_T| \le 6\cdot 1^3/(12n^2) = 1/(2n^2) \le 10^{-5} \Rightarrow 2n^2 \ge 10^5 \Rightarrow n \ge \sqrt{\frac{1}{2}\cdot 10^5}$ or $n \ge 224$. To guarantee that $|E_M| \le 0.00001$, we need $6\cdot 1^3/(24n^2) \le 10^{-5} \Rightarrow 4n^2 \ge 10^5 \Rightarrow n \ge \sqrt{\frac{1}{4}\cdot 10^5}$ or $n \ge 159$.

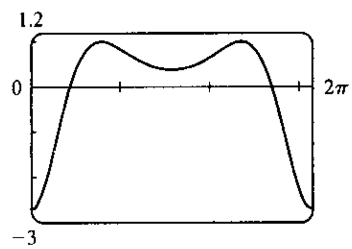
21. (a) $T_{10} = \frac{1}{10\cdot 2}\{f(0) + 2[f(0.1) + f(0.2) + \cdots + f(0.9)] + f(1)\} \approx 1.71971349$
$S_{10} = \frac{1}{10\cdot 3}[f(0) + 4f(0.1) + 2f(0.2) + 4f(0.3) + \cdots + 4f(0.9) + f(1)] \approx 1.71828278$
Since $I = \int_0^1 e^x\,dx = [e^x]_0^1 = e - 1 \approx 1.71828183$, $E_T = I - T_{10} \approx -0.00143166$ and $E_S = I - S_{10} \approx -0.00000095$.

(b) $f(x) = e^x \Rightarrow f''(x) = e^x \le e$ for $0 \le x \le 1$. Taking $K = e$, $a = 0$, $b = 1$, and $n = 10$ in Theorem 3, we get $|E_T| \le e(1)^3/(12\cdot 10^2) \approx 0.002265 > 0.00143166$ [actual $|E_T|$ from (a)]. $f^{(4)}(x) = e^x < e$ for $0 \le x \le 1$. Using Theorem 4, we have $|E_S| \le e(1)^5/(180\cdot 10^4) \approx 0.0000015 > 0.00000095$ [actual $|E_S|$ from (a)]. We see that the actual errors are about two-thirds the size of the error estimates.

(c) From part (b), we take $K = e$ to get $|E_T| \le \dfrac{K(b-a)^3}{12n^2} \le 0.00001 \Rightarrow n^2 \ge \dfrac{e(1^3)}{12(0.00001)} \Rightarrow n \ge 150.5$. Take $n = 151$ for T_n. Now $|E_M| \le \dfrac{K(b-a)^3}{24n^2} \le 0.00001 \Rightarrow n \ge 106.4$. Take $n = 107$ for M_n. Finally, $|E_S| \le \dfrac{K(b-a)^5}{180n^4} \le 0.00001 \Rightarrow n^4 \ge \dfrac{e(1^5)}{180(0.00001)} \Rightarrow n \ge 6.23$. Take $n = 8$ for S_n (since n has to be even for Simpson's Rule).

22. From Example 7(b), we take $K = 76e$ to get $|E_S| \le 76e(1)^5/(180n^4) \le 0.00001 \Rightarrow n^4 \ge 76e/[180(0.00001)] \Rightarrow n \ge 18.4$. Take $n = 20$ (since n must be even).

23. (a) Using a CAS, we differentiate $f(x) = e^{\cos x}$ twice, and find that $f''(x) = e^{\cos x}(\sin^2 x - \cos x)$. From the graph, we see that the maximum value of $|f''(x)|$ occurs at the endpoints of the interval $[0, 2\pi]$. Since $f''(0) = -e$, we can use $K = e$ or $K = 2.8$.



(b) A CAS gives $M_{10} \approx 7.954926518$. (In Maple, use `student[middlesum]`.)

(c) Using Theorem 3 for the Midpoint Rule, with $K = e$, we get $|E_M| \le \dfrac{e(2\pi - 0)^3}{24\cdot 10^2} \approx 0.280945995$. With $K = 2.8$, we get $|E_M| \le \dfrac{2.8(2\pi - 0)^3}{24\cdot 10^2} = 0.289391916$.

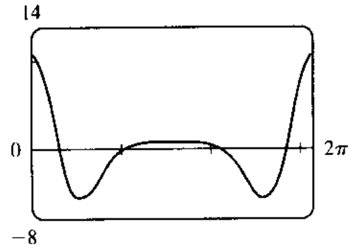
(d) A CAS gives $I \approx 7.954926521$.

(e) The actual error is only about 3×10^{-9}, much less than the estimate in part (c).

(f) We use the CAS to differentiate twice more, and then graph

$f^{(4)}(x) = e^{\cos x}\left(\sin^4 x - 6\sin^2 x\cos x + 3 - 7\sin^2 x + \cos x\right)$.

From the graph, we see that the maximum value of $\left|f^{(4)}(x)\right|$ occurs at the endpoints of the interval $[0, 2\pi]$. Since $f^{(4)}(0) = 4e$, we can use $K = 4e$ or $K = 10.9$.



(g) A CAS gives $S_{10} \approx 7.953789422$. (In Maple, use `student[simpson]`.)

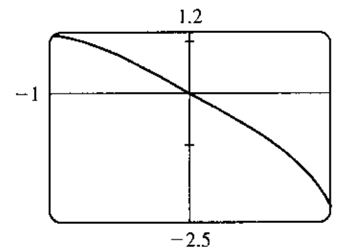
(h) Using Theorem 4 with $K = 4e$, we get $|E_S| \le \dfrac{4e(2\pi - 0)^5}{180 \cdot 10^4} \approx 0.059153618$. With $K = 10.9$, we get $|E_S| \le \dfrac{10.9(2\pi - 0)^5}{180 \cdot 10^4} \approx 0.059299814$.

(i) The actual error is about $7.954926521 - 7.953789422 \approx 0.00114$. This is quite a bit smaller than the estimate in part (h), though the difference is not nearly as great as it was in the case of the Midpoint Rule.

(j) To ensure that $|E_S| \le 0.0001$, we use Theorem 4: $|E_S| \le \dfrac{4e(2\pi)^5}{180 \cdot n^4} \le 0.0001 \Rightarrow \dfrac{4e(2\pi)^5}{180 \cdot 0.0001} \le n^4 \Rightarrow n^4 \ge 5{,}915{,}362 \Leftrightarrow n \ge 49.3$. So we must take $n \ge 50$ to ensure that $|I - S_n| \le 0.0001$. ($K = 10.9$ leads to the same value of n.)

24. (a) Using the CAS, we differentiate $f(x) = \sqrt{4 - x^3}$ twice, and find that $f''(x) = -\dfrac{9x^4}{4(4 - x^3)^{3/2}} - \dfrac{3x}{(4 - x^3)^{1/2}}$.

From the graph, we see that $|f''(x)| < 2.2$ on $[-1, 1]$.



(b) A CAS gives $M_{10} \approx 3.995804152$. (In Maple, use `student[middlesum]`.)

(c) Using Theorem 3 for the Midpoint Rule, with $K = 2.2$, we get $|E_M| \le \dfrac{2.2\,[1 - (-1)]^3}{24 \cdot 10^2} \approx 0.00733$.

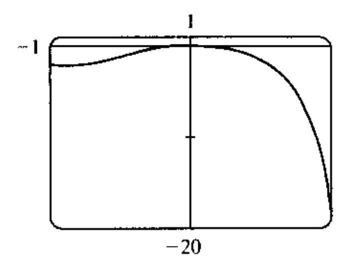
(d) A CAS gives $I \approx 3.995487677$.

(e) The actual error is about -0.0003165, much less than the estimate in part (c).

(f) We use the CAS to differentiate twice more, and then graph

$f^{(4)}(x) = \dfrac{9}{16}\dfrac{x^2(x^6 - 224x^3 - 1280)}{(4 - x^3)^{7/2}}$.

From the graph, we see that $\left|f^{(4)}(x)\right| < 18.1$ on $[-1, 1]$.



(g) A CAS gives $S_{10} \approx 3.995449790$. (In Maple, use `student[simpson]`.)

(h) Using Theorem 4 with $K = 18.1$, we get $|E_S| \le \dfrac{18.1\,[1 - (-1)]^5}{180 \cdot 10^4} \approx 0.000322$.

(i) The actual error is about $3.995487677 - 3.995449790 \approx 0.0000379$. This is quite a bit smaller than the estimate in part (h).

(j) To ensure that $|E_S| \le 0.0001$,we use Theorem 4: $|E_S| \le \dfrac{18.1(2)^5}{180 \cdot n^4} \le 0.0001 \;\Rightarrow\; \dfrac{18.1(2)^5}{180 \cdot 0.0001} \le n^4 \;\Rightarrow$ $n^4 \ge 32{,}178 \;\Rightarrow n \ge 13.4$. So we must take $n \ge 14$ to ensure that $|I - S_n| \le 0.0001$.

25. $I = \int_0^1 x^3 dx = \left[\frac{1}{4}x^4\right]_0^1 = 0.25$. $f(x) = x^3$.

$n = 4$:

$$L_4 = \tfrac{1}{4}\left[0^3 + \left(\tfrac{1}{4}\right)^3 + \left(\tfrac{2}{4}\right)^3 + \left(\tfrac{3}{4}\right)^3\right] = 0.140625$$

$$R_4 = \tfrac{1}{4}\left[\left(\tfrac{1}{4}\right)^3 + \left(\tfrac{2}{4}\right)^3 + \left(\tfrac{3}{4}\right)^3 + 1^3\right] = 0.390625$$

$$T_4 = \tfrac{1}{4 \cdot 2}\left[0^3 + 2\left(\tfrac{1}{4}\right)^3 + 2\left(\tfrac{2}{4}\right)^3 + 2\left(\tfrac{3}{4}\right)^3 + 1^3\right] = 0.265625,$$

$$M_4 = \tfrac{1}{4}\left[\left(\tfrac{1}{8}\right)^3 + \left(\tfrac{3}{8}\right)^3 + \left(\tfrac{5}{8}\right)^3 + \left(\tfrac{7}{8}\right)^3\right] = 0.2421875,$$

$$E_L = I - L_4 = \tfrac{1}{4} - 0.140625 = 0.109375,\; E_R = \tfrac{1}{4} - 0.390625 = -0.140625,$$

$$E_T = \tfrac{1}{4} - 0.265625 = -0.015625,\; E_M = \tfrac{1}{4} - 0.2421875 = 0.0078125$$

$n = 8$:

$$L_8 = \tfrac{1}{8}\left[f(0) + f\left(\tfrac{1}{8}\right) + f\left(\tfrac{2}{8}\right) + \cdots + f\left(\tfrac{7}{8}\right)\right] \approx 0.191406$$

$$R_8 = \tfrac{1}{8}\left[f\left(\tfrac{1}{8}\right) + f\left(\tfrac{2}{8}\right) + \cdots + f\left(\tfrac{7}{8}\right) + f(1)\right] \approx 0.316406$$

$$T_8 = \tfrac{1}{8 \cdot 2}\left\{f(0) + 2\left[f\left(\tfrac{1}{8}\right) + f\left(\tfrac{2}{8}\right) + \cdots + f\left(\tfrac{7}{8}\right)\right] + f(1)\right\} \approx 0.253906$$

$$M_8 = \tfrac{1}{8}\left[f\left(\tfrac{1}{16}\right) + f\left(\tfrac{3}{16}\right) + \cdots + f\left(\tfrac{13}{16}\right) + f\left(\tfrac{15}{16}\right)\right] = 0.248047$$

$$E_L \approx \tfrac{1}{4} - 0.191406 \approx 0.058594,\; E_R \approx \tfrac{1}{4} - 0.316406 \approx -0.066406,$$

$$E_T \approx \tfrac{1}{4} - 0.253906 \approx -0.003906,\; E_M \approx \tfrac{1}{4} - 0.248047 \approx 0.001953.$$

$n = 16$:

$$L_{16} = \tfrac{1}{16}\left[f(0) + f\left(\tfrac{1}{16}\right) + f\left(\tfrac{2}{16}\right) + \cdots + f\left(\tfrac{15}{16}\right)\right] \approx 0.219727$$

$$R_{16} = \tfrac{1}{16}\left[f\left(\tfrac{1}{16}\right) + f\left(\tfrac{2}{16}\right) + \cdots + f\left(\tfrac{15}{16}\right) + f(1)\right] \approx 0.282227$$

$$T_{16} = \tfrac{1}{16 \cdot 2}\left\{f(0) + 2\left[f\left(\tfrac{1}{16}\right) + f\left(\tfrac{2}{16}\right) + \cdots + f\left(\tfrac{15}{16}\right)\right] + f(1)\right\} \approx 0.250977$$

$$M_{16} = \tfrac{1}{16}\left[f\left(\tfrac{1}{32}\right) + f\left(\tfrac{3}{32}\right) + \cdots + f\left(\tfrac{31}{32}\right)\right] \approx 0.249512$$

$$E_L \approx \tfrac{1}{4} - 0.219727 \approx 0.030273,\; E_R \approx \tfrac{1}{4} - 0.282227 \approx -0.032227,$$

$$E_T \approx \tfrac{1}{4} - 0.250977 \approx -0.000977,\; E_M \approx \tfrac{1}{4} - 0.249512 \approx 0.000488.$$

n	L_n	R_n	T_n	M_n
4	0.140625	0.390625	0.265625	0.242188
8	0.191406	0.316406	0.253906	0.248047
16	0.219727	0.282227	0.250977	0.249512

n	E_L	E_R	E_T	E_M
4	0.109375	−0.140625	−0.015625	0.007813
8	0.058594	−0.066406	−0.003906	0.001953
16	0.030273	−0.032227	−0.000977	0.000488

Observations:

1. E_L and E_R are always opposite in sign, as are E_T and E_M.
2. As n is doubled, E_L and E_R are decreased by about a factor of 2, and E_T and E_M are decreased by a factor of about 4.
3. The Midpoint approximation is about twice as accurate as the Trapezoidal approximation.
4. All the approximations become more accurate as the value of n increases.
5. The Midpoint and Trapezoidal approximations are much more accurate than the endpoint approximations.

26. $\int_0^2 e^x\,dx = [e^x]_0^2 = e^2 - 1 \approx 6.389056$. $f(x) = e^x$

$n = 4$: $\Delta x = (2-0)/4 = \frac{1}{2}$

$$L_4 = \tfrac{1}{2}\left[e^0 + e^{1/2} + e^1 + e^{3/2}\right] \approx 4.924346$$

$$R_4 = \tfrac{1}{2}\left[e^{1/2} + e^1 + e^{3/2} + e^2\right] \approx 8.118874$$

$$T_4 = \tfrac{1}{2\cdot 2}\left[e^0 + 2e^{1/2} + 2e^1 + 2e^{3/2} + e^2\right] \approx 6.521610$$

$$M_4 = \tfrac{1}{2}\left[e^{1/4} + e^{3/4} + e^{5/4} + e^{7/4}\right] \approx 6.322986.$$

$E_L \approx 6.389056 - 4.924346 \approx 1.464710$, $E_R \approx 6.389056 - 8.118874 = -1.729818$,

$E_T \approx 6.389056 - 6.521610 \approx -0.132554$, $E_M \approx 6.389056 - 6.322986 = 0.0660706$.

$n = 8$: $\Delta x = (2-0)/8 = \frac{1}{4}$

$$L_8 = \tfrac{1}{4}\left[e^0 + e^{1/4} + e^{1/2} + e^{3/4} + e^1 + e^{5/4} + e^{3/2} + e^{7/4}\right] \approx 5.623666$$

$$R_8 = \tfrac{1}{4}\left[e^{1/4} + e^{1/2} + e^{3/4} + e^1 + e^{5/4} + e^{3/2} + e^{7/4} + e^2\right] \approx 7.220930$$

$$T_8 = \tfrac{1}{4\cdot 2}\left[e^0 + 2e^{1/4} + 2e^{1/2} + 2e^{3/4} + 2e^1 + 2e^{5/4} + 2e^{3/2} + 2e^{7/4} + e^2\right] \approx 6.422298$$

$$M_8 = \tfrac{1}{4}\left[e^{1/8} + e^{3/8} + e^{5/8} + e^{7/8} + e^{9/8} + e^{11/8} + e^{13/8} + e^{15/8}\right] \approx 6.372448$$

$E_L \approx 6.389056 - 5.623666 \approx 0.765390$, $E_R \approx 6.389056 - 7.220930 \approx -0.831874$,

$E_T \approx 6.389056 - 6.422298 \approx -0.033242$, $E_M \approx 6.389056 - 6.372448 \approx 0.016608$.

$n = 16$: $\Delta x = (2-0)/16 = \frac{1}{8}$

$$L_{16} = \tfrac{1}{8}\left[f(0) + f\left(\tfrac{1}{8}\right) + f\left(\tfrac{2}{8}\right) + \cdots + f\left(\tfrac{14}{8}\right) + f\left(\tfrac{15}{8}\right)\right] \approx 5.998057$$

$$R_{16} = \tfrac{1}{8}\left[f\left(\tfrac{1}{8}\right) + f\left(\tfrac{2}{8}\right) + f\left(\tfrac{3}{8}\right) + \cdots + f\left(\tfrac{15}{8}\right) + f(2)\right] \approx 6.796689$$

$$T_{16} = \tfrac{1}{8\cdot 2}\left(f(0) + 2\left[f\left(\tfrac{1}{8}\right) + f\left(\tfrac{2}{8}\right) + f\left(\tfrac{3}{8}\right) + \cdots + f\left(\tfrac{15}{8}\right)\right] + f(2)\right) \approx 6.397373$$

$$M_{16} = \tfrac{1}{8}\left[f\left(\tfrac{1}{16}\right) + f\left(\tfrac{3}{16}\right) + f\left(\tfrac{5}{16}\right) + \cdots + f\left(\tfrac{29}{16}\right) + f\left(\tfrac{31}{16}\right)\right] \approx 6.384899$$

$E_L \approx 6.389056 - 5.998057 \approx 0.390999$, $E_R \approx 6.389056 - 6.796689 \approx -0.407633$,

$E_T \approx 6.389056 - 6.397373 \approx -0.008317$, $E_M \approx 6.389056 - 6.384899 \approx 0.004158$.

n	E_L	E_R	E_T	E_M
4	1.464710	−1.729818	−0.132554	0.066071
8	0.765390	−0.831874	−0.033242	0.016608
16	0.390999	−0.407633	−0.008317	0.004158

Observations:

1. E_L and E_R are always opposite in sign, as are E_T and E_M.
2. As n is doubled, E_L and E_R are decreased by a factor of about 2, and E_T and E_M are decreased by a factor of about 4.
3. The Midpoint approximation is about twice as accurate as the Trapezoidal approximation.
4. All the approximations become more accurate as the value of n increases.
5. The Midpoint and Trapezoidal approximations are much more accurate than the endpoint approximations.

27. $\int_1^4 \sqrt{x}\,dx = \left[\frac{2}{3}x^{3/2}\right]_1^4 = \frac{2}{3}(8-1) = \frac{14}{3} \approx 4.666667$

$n = 6$: $\Delta x = (4-1)/6 = \frac{1}{2}$

$T_6 = \frac{1}{2\cdot 2}\left[\sqrt{1} + 2\sqrt{1.5} + 2\sqrt{2} + 2\sqrt{2.5} + 2\sqrt{3} + 2\sqrt{3.5} + \sqrt{4}\right] \approx 4.661488$

$M_6 = \frac{1}{2}\left[\sqrt{1.25} + \sqrt{1.75} + \sqrt{2.25} + \sqrt{2.75} + \sqrt{3.25} + \sqrt{3.75}\right] \approx 4.669245$

$S_6 = \frac{1}{2\cdot 3}\left[\sqrt{1} + 4\sqrt{1.5} + 2\sqrt{2} + 4\sqrt{2.5} + 2\sqrt{3} + 4\sqrt{3.5} + \sqrt{4}\right] \approx 4.666563$

$E_T \approx \frac{14}{3} - 4.661488 \approx 0.005178$, $E_M \approx \frac{14}{3} - 4.669245 \approx -0.002578$,

$E_S \approx \frac{14}{3} - 4.666563 \approx 0.000104$.

$n = 12$: $\Delta x = (4-1)/12 = \frac{1}{4}$

$T_{12} = \frac{1}{4\cdot 2}(f(1) + 2\left[f(1.25) + f(1.5) + \cdots + f(3.5) + f(3.75)\right] + f(4)) \approx 4.665367$

$M_{12} = \frac{1}{4}[f(1.125) + f(1.375) + f(1.625) + \cdots + f(3.875)] \approx 4.667316$

$S_{12} = \frac{1}{4\cdot 3}[f(1) + 4f(1.25) + 2f(1.5) + 4f(1.75) + \cdots + 4f(3.75) + f(4)] \approx 4.666659$

$E_T \approx \frac{14}{3} - 4.665367 \approx 0.001300$, $E_M \approx \frac{14}{3} - 4.667316 \approx -0.000649$,

$E_S \approx \frac{14}{3} - 4.666659 \approx 0.000007$.

Note: These errors were computed more precisely and then rounded to six places. That is, they were not computed by comparing the rounded values of T_n, M_n, and S_n with the rounded value of the actual integral.

n	T_n	M_n	S_n
6	4.661488	4.669245	4.666563
12	4.665367	4.667316	4.666659

n	E_T	E_M	E_S
6	0.005178	−0.002578	0.000104
12	0.001300	−0.000649	0.000007

Observations:

1. E_T and E_M are opposite in sign and decrease by a factor of about 4 as n is doubled.
2. The Simpson's approximation is much more accurate than the Midpoint and Trapezoidal approximations, and seems to decrease by a factor of about 16 as n is doubled.

28. $I = \int_{-1}^{2} xe^x\,dx = [xe^x - e^x]_{-1}^{2} = e^2 + 2/e \approx 8.124815$. $f(x) = xe^x$.

$n = 6$: $\Delta x = [2-(-1)]/6 = \frac{1}{2}$

$T_6 = \frac{1}{2\cdot 2}\{f(-1) + 2[f(-0.5) + f(0) + \cdots + f(1.5)] + f(2)\} \approx 8.583514$

$M_6 = \frac{1}{2}[f(-0.75) + f(-0.25) + \cdots + f(1.75)] \approx 7.896632$

$S_6 = \frac{1}{2\cdot 3}[f(-1) + 4f(-0.5) + 2f(0) + 4f(0.5) + 2f(1) + 4f(1.5) + f(2)] \approx 8.136885$

$E_T \approx I - 8.583514 \approx -0.458699$, $E_M \approx I - 7.896632 \approx 0.228183$,

$E_S \approx I - 8.136885 \approx -0.012070$.

$n = 12$: $\Delta x = [2-(-1)]/12 = \frac{1}{4}$

$T_{12} = \frac{1}{4\cdot 2}\{f(-1) + 2[f(-0.75) + f(-0.5) + \cdots + f(1.75)] + f(2)\} \approx 8.240073$

$M_{12} = \frac{1}{4}\left[f\left(-\frac{7}{8}\right) + f\left(-\frac{5}{8}\right) + \cdots + f\left(\frac{13}{8}\right) + f\left(\frac{15}{8}\right)\right] \approx 8.067259$

$S_{12} = \frac{1}{4\cdot 3}[f(-1) + 4f(-0.75) + 2f(-0.5) + \cdots + 2f(1.5) + 4f(1.75) + f(2)] \approx 8.125593$

$E_T \approx I - 8.240073 \approx -0.115258$, $E_M \approx I - 8.067259 \approx 0.057556$,

$E_S \approx I - 8.125593 \approx -0.000778$

n	T_n	M_n	S_n
6	8.583514	7.896632	8.136885
12	8.240073	8.067259	8.125593

n	E_T	E_M	E_S
6	−0.458699	0.228183	−0.012070
12	−0.115258	0.057556	−0.000778

Observations:

1. E_T and E_M are opposite in sign and decrease by a factor of about 4 as n is doubled.
2. The Simpson's approximation is much more accurate than the Midpoint and Trapezoidal approximations, and seems to decrease by a factor of about 16 as n is doubled.

29. $\Delta x = (4-0)/4 = 1$

(a) $T_4 = \frac{1}{2}[f(0) + 2f(1) + 2f(2) + 2f(3) + f(4)] \approx \frac{1}{2}[0 + 2(3) + 2(5) + 2(3) + 1] = 11.5$

(b) $M_4 = 1 \cdot [f(0.5) + f(1.5) + f(2.5) + f(3.5)] \approx 1 + 4.5 + 4.5 + 2 = 12$

(c) $S_4 = \frac{1}{3}[f(0) + 4f(1) + 2f(2) + 4f(3) + f(4)] \approx \frac{1}{3}[0 + 4(3) + 2(5) + 4(3) + 1] = 11.\overline{6}$

30. If x = distance from left end of pool and $w = w(x)$ = width at x, then Simpson's Rule with $n = 8$ and $\Delta x = 2$ gives Area $= \int_0^{16} w\,dx \approx \frac{2}{3}[0 + 4(6.2) + 2(7.2) + 4(6.8) + 2(5.6) + 4(5.0) + 2(4.8) + 4(4.8) + 0] \approx 84\text{ m}^2$.

31. (a) We are given the function values at the endpoints of 8 intervals of length 0.4, so we'll use the Midpoint Rule with $n = 8/2 = 4$ and $\Delta x = (3.2 - 0)/4 = 0.8$.

$$\begin{aligned}\int_0^{3.2} f(x)\,dx \approx M_4 &= 0.8[f(0.4) + f(1.2) + f(2.0) + f(2.8)] \\ &= 0.8[6.5 + 6.4 + 7.6 + 8.8] \\ &= 0.8(29.3) = 23.44\end{aligned}$$

(b) $-4 \le f''(x) \le 1 \quad \Rightarrow \quad |f''(x)| \le 4$, so use $K = 4$, $a = 0$, $b = 3.2$, and $n = 4$ in Theorem 3. So $|E_M| \le \dfrac{4(3.2-0)^3}{24(4)^2} = \dfrac{128}{375} = 0.341\overline{3}$.

32. We use Simpson's Rule with $n = 10$ and $\Delta x = \frac{1}{2}$:

$$\begin{aligned}\text{distance} = \int_0^5 v(t)\,dt \approx S_{10} &= \tfrac{1}{2\cdot 3}[f(0) + 4f(0.5) + 2f(1) + \cdots + 4f(4.5) + f(5)] \\ &= \tfrac{1}{6}[0 + 4(4.67) + 2(7.34) + 4(8.86) + 2(9.73) + 4(10.22) \\ &\qquad + 2(10.51) + 4(10.67) + 2(10.76) + 4(10.81) + 10.81] \\ &= \tfrac{1}{6}(268.41) = 44.735\text{ m}\end{aligned}$$

33. By the Net Change Theorem, the increase in velocity is equal to $\int_0^6 a(t)\,dt$. We use Simpson's Rule with $n = 6$ and $\Delta t = (6-0)/6 = 1$ to estimate this integral:

$$\begin{aligned}\int_0^6 a(t)\,dt \approx S_6 &= \tfrac{1}{3}[a(0) + 4a(1) + 2a(2) + 4a(3) + 2a(4) + 4a(5) + a(6)] \\ &\approx \tfrac{1}{3}[0 + 4(0.5) + 2(4.1) + 4(9.8) + 2(12.9) + 4(9.5) + 0] = \tfrac{1}{3}(113.2) = 37.7\overline{3}\text{ ft/s}\end{aligned}$$

34. By the Net Change Theorem, the total amount of water that leaked out during the first six hours is equal to $\int_0^6 r(t)\,dt$. We use Simpson's Rule with $n = 6$ and $\Delta t = \frac{6-0}{6} = 1$ to estimate this integral:

$$\begin{aligned}\int_0^6 r(t)\,dt \approx S_6 &= \tfrac{1}{3}[r(0) + 4r(1) + 2r(2) + 4r(3) + 2r(4) + 4r(5) + r(6)]\\ &\approx \tfrac{1}{3}[4 + 4(3) + 2(2.4) + 4(1.9) + 2(1.4) + 4(1.1) + 1]\\ &= \tfrac{1}{3}(36.6) = 12.2 \text{ liters}\end{aligned}$$

The function values were obtained from a high-resolution graph.

35. By the Net Change Theorem, the energy used is equal to $\int_0^6 P(t)\,dt$. We use Simpson's Rule with $n = 12$ and $\Delta t = (6-0)/12 = \frac{1}{2}$ to estimate this integral:

$$\begin{aligned}\int_0^6 P(t)\,dt \approx S_{12} &= \tfrac{1/2}{3}[P(0) + 4P(0.5) + 2P(1) + 4P(1.5) + 2P(2) + 4P(2.5)\\ &\qquad + 2P(3) + 4P(3.5) + 2P(4) + 4P(4.5) + 2P(5) + 4P(5.5) + P(6)]\\ &= \tfrac{1}{6}[1814 + 4(1735) + 2(1686) + 4(1646) + 2(1637) + 4(1609) + 2(1604)\\ &\qquad + 4(1611) + 2(1621) + 4(1666) + 2(1745) + 4(1886) + 2052]\\ &= \tfrac{1}{6}(61{,}064) = 10{,}177.\overline{3} \text{ megawatt-hours.}\end{aligned}$$

36. By the Net Change Theorem, the total amount of data transmitted is equal to $\int_0^8 D(t)\,dt \times 3600$ [since $D(t)$ is measured in megabits per second and t is in hours]. We use Simpson's Rule with $n = 8$ and $\Delta t = (8-0)/8 = 1$ to estimate this integral:

$$\begin{aligned}\int_0^8 D(t)\,dt \approx S_8 &= \tfrac{1}{3}[D(0) + 4D(1) + 2D(2) + 4D(3) + 2D(4) + 4D(5) + 2D(6) + 4D(7) + D(8)]\\ &\approx \tfrac{1}{3}[0.35 + 4(0.32) + 2(0.41) + 4(0.50) + 2(0.51) + 4(0.56) + 2(0.56) + 4(0.83) + 0.88]\\ &= \tfrac{1}{3}(13.03) = 4.34\overline{3}\end{aligned}$$

Now multiply by 3600 to obtain 15,636 megabits.

37. Let $y = f(x)$ denote the curve. Using cylindrical shells, $V = \int_2^{10} 2\pi x f(x)\,dx = 2\pi \int_2^{10} x f(x)\,dx = 2\pi I$. Now use Simpson's Rule to approximate I:

$$\begin{aligned}I \approx S_8 &= \tfrac{10-2}{3(8)}[2f(2) + 4\cdot 3f(3) + 2\cdot 4f(4) + 4\cdot 5f(5) + 2\cdot 6f(6)\\ &\qquad + 4\cdot 7f(7) + 2\cdot 8f(8) + 4\cdot 9f(9) + 10f(10)]\\ &\approx \tfrac{1}{3}[2(0) + 12(1.5) + 8(1.9) + 20(2.2) + 12(3.0) + 28(3.8) + 16(4.0) + 36(3.1) + 10(0)]\\ &= \tfrac{1}{3}(395.2)\end{aligned}$$

Thus, $V \approx 2\pi \cdot \frac{1}{3}(395.2) \approx 827.7$ or 828 cubic units.

38. Work $= \displaystyle\int_0^{18} f(x)\,dx \approx S_6 = \frac{18-0}{6\cdot 3}[f(0) + 4f(3) + 2f(6) + 4f(9) + 2f(12) + 4f(15) + f(18)]$

$= 1\cdot[9.8 + 4(9.1) + 2(8.5) + 4(8.0) + 2(7.7) + 4(7.5) + 7.4] = 148$ joules

39. Volume $= \pi \int_0^2 \left(\sqrt[3]{1+x^3}\right)^2 dx = \pi \int_0^2 \left(1+x^3\right)^{2/3} dx$. $V \approx \pi \cdot S_{10}$ where $f(x) = \left(1+x^3\right)^{2/3}$ and $\Delta x = (2-0)/10 = \frac{1}{5}$. Therefore,

$$\begin{aligned}V \approx \pi \cdot S_{10} &= \pi \tfrac{1}{5\cdot 3}[f(0) + 4f(0.2) + 2f(0.4) + 4f(0.6) + 2f(0.8) + 4f(1)\\ &\qquad + 2f(1.2) + 4f(1.4) + 2f(1.6) + 4f(1.8) + f(2)] \approx 12.325078\end{aligned}$$

40. Using Simpson's Rule with $n = 10$, $\Delta x = \frac{\pi/2}{10}$, $L = 1$, $\theta_0 = \frac{42\pi}{180}$ radians, $g = 9.8\text{ m/s}^2$, $k^2 = \sin^2(\frac{1}{2}\theta_0)$, and $f(x) = 1/\sqrt{1 - k^2 \sin^2 x}$, we get

$$T = 4\sqrt{\frac{L}{g}} \int_0^{\pi/2} \frac{dx}{\sqrt{1 - k^2 \sin^2 x}} \approx 4\sqrt{\frac{L}{g}}\, S_{10}$$

$$= 4\sqrt{\tfrac{1}{9.8}}\left(\tfrac{\pi/2}{10 \cdot 3}\right)\left[f(0) + 4f\left(\tfrac{\pi}{20}\right) + 2f\left(\tfrac{2\pi}{20}\right) + \cdots + 4f\left(\tfrac{9\pi}{20}\right) + f\left(\tfrac{\pi}{2}\right)\right] \approx 2.07665$$

41. $I(\theta) = \dfrac{N^2 \sin^2 k}{k^2}$, where $k = \dfrac{\pi N d \sin\theta}{\lambda}$, $N = 10{,}000$, $d = 10^{-4}$, and $\lambda = 632.8 \times 10^{-9}$. So

$I(\theta) = \dfrac{(10^4)^2 \sin^2 k}{k^2}$, where $k = \dfrac{\pi(10^4)(10^{-4})\sin\theta}{632.8 \times 10^{-9}}$. Now $n = 10$ and $\Delta\theta = \dfrac{10^{-6} - (-10^{-6})}{10} = 2 \times 10^{-7}$,

so $M_{10} = 2 \times 10^{-7}[I(-0.0000009) + I(-0.0000007) + \cdots + I(0.0000009)] \approx 59.4$.

42. $f(x) = \cos(\pi x)$, $\Delta x = \frac{20 - 0}{10} = 2 \quad \Rightarrow$

$$\begin{aligned} T_{10} &= \tfrac{2}{2}\{f(0) + 2[f(2) + f(4) + \cdots + f(18)] + f(20)\} \\ &= 1[\cos 0 + 2(\cos 2\pi + \cos 4\pi + \cdots + \cos 18\pi) + \cos 20\pi] \\ &= 1 + 2(1 + 1 + 1 + 1 + 1 + 1 + 1 + 1 + 1) + 1 = 20 \end{aligned}$$

The actual value is $\int_0^{20} \cos(\pi x)\,dx = \frac{1}{\pi}[\sin \pi x]_0^{20} = \frac{1}{\pi}(\sin 20\pi - \sin 0) = 0$. The discrepancy is due to the fact that the function is sampled only at points of the form $2n$, where its value is $f(2n) = \cos(2n\pi) = 1$.

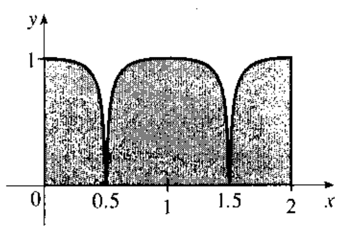
43. Consider the function f whose graph is shown. The area $\int_0^2 f(x)\,dx$ is close to 2. The Trapezoidal Rule gives

$T_2 = \frac{2-0}{2 \cdot 2}[f(0) + 2f(1) + f(2)] = \frac{1}{2}[1 + 2 \cdot 1 + 1] = 2$.

The Midpoint Rule gives

$M_2 = \frac{2-0}{2}[f(0.5) + f(1.5)] = 1[0 + 0] = 0$,

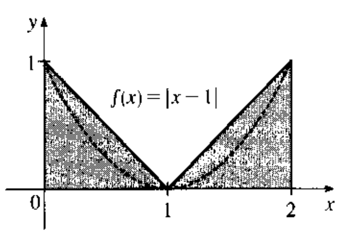
so the Trapezoidal Rule is more accurate.



44. Consider the function $f(x) = |x - 1|$, $0 \le x \le 2$. The area $\int_0^2 f(x)\,dx$ is exactly 1. So is the right endpoint approximation:

$R_2 = f(1)\,\Delta x + f(2)\,\Delta x = 0 \cdot 1 + 1 \cdot 1 = 1$. But Simpson's Rule approximates f with the parabola $y = (x - 1)^2$, shown dashed, and

$S_2 = \dfrac{\Delta x}{3}[f(0) + 4f(1) + f(2)] = \dfrac{1}{3}[1 + 4 \cdot 0 + 1] = \dfrac{2}{3}$.



45. Since the Trapezoidal and Midpoint approximations on the interval $[a, b]$ are the sums of the Trapezoidal and Midpoint approximations on the subintervals $[x_{i-1}, x_i]$, $i = 1, 2, \ldots, n$, we can focus our attention on one such interval. The condition $f''(x) < 0$ for $a \le x \le b$ means that the graph of f is concave down as in Figure 5. In that figure, T_n is the area of the trapezoid $AQRD$, $\int_a^b f(x)\,dx$ is the area of the region $AQPRD$, and M_n is the area of the trapezoid $ABCD$, so $T_n < \int_a^b f(x)\,dx < M_n$. In general, the condition $f'' < 0$ implies that the graph of f on $[a, b]$ lies above the chord joining the points $(a, f(a))$ and $(b, f(b))$. Thus, $\int_a^b f(x)\,dx > T_n$. Since M_n is the area under a tangent to the graph, and since $f'' < 0$ implies that the tangent lies above the graph, we also have $M_n > \int_a^b f(x)\,dx$. Thus, $T_n < \int_a^b f(x)\,dx < M_n$.

46. Let f be a polynomial of degree ≤ 3; say $f(x) = Ax^3 + Bx^2 + Cx + D$. It will suffice to show that Simpson's estimate is exact when there are two subintervals ($n = 2$), because for a larger even number of subintervals the sum of exact estimates is exact. As in the derivation of Simpson's Rule, we can assume that $x_0 = -h$, $x_1 = 0$, and $x_2 = h$. Then Simpson's approximation is

$$\begin{aligned}\int_{-h}^{h} f(x)\,dx &\approx \tfrac{1}{3}h[f(-h) + 4f(0) + f(h)] \\ &= \tfrac{1}{3}h\left[\left(-Ah^3 + Bh^2 - Ch + D\right) + 4D + \left(Ah^3 + Bh^2 + Ch + D\right)\right] \\ &= \tfrac{1}{3}h\left[2Bh^2 + 6D\right] = \tfrac{2}{3}Bh^3 + 2Dh\end{aligned}$$

The exact value of the integral is

$$\begin{aligned}\int_{-h}^{h}\left(Ax^3 + Bx^2 + Cx + D\right)dx &= 2\int_0^h \left(Bx^2 + D\right)dx \quad \text{[by Theorem 5.5.6(a) and (b)]} \\ &= 2\left[\tfrac{1}{3}Bx^3 + Dx\right]_0^h = \tfrac{2}{3}Bh^3 + 2Dh\end{aligned}$$

Thus, Simpson's Rule is exact.

47. $T_n = \frac{1}{2}\,\Delta x\,[f(x_0) + 2f(x_1) + \cdots + 2f(x_{n-1}) + f(x_n)]$ and $M_n = \Delta x\,[f(\overline{x}_1) + f(\overline{x}_2) + \cdots + f(\overline{x}_{n-1}) + f(\overline{x}_n)]$, where $\overline{x}_i = \frac{1}{2}(x_{i-1} + x_i)$. Now

$$\begin{aligned}T_{2n} = \tfrac{1}{2}\left(\tfrac{1}{2}\Delta x\right)[f(x_0) + 2f(\overline{x}_1) + 2f(x_1) + 2f(\overline{x}_2) + 2f(x_2) + \cdots \\ + 2f(\overline{x}_{n-1}) + 2f(x_{n-1}) + 2f(\overline{x}_n) + f(x_n)]\end{aligned}$$

so

$$\begin{aligned}\tfrac{1}{2}(T_n + M_n) &= \tfrac{1}{2}T_n + \tfrac{1}{2}M_n \\ &= \tfrac{1}{4}\,\Delta x\,[f(x_0) + 2f(x_1) + \cdots + 2f(x_{n-1}) + f(x_n)] \\ &\qquad + \tfrac{1}{4}\Delta x\,[2f(\overline{x}_1) + 2f(\overline{x}_2) + \cdots + 2f(\overline{x}_{n-1}) + 2f(\overline{x}_n)] \\ &= T_{2n}\end{aligned}$$

48. $T_n = \dfrac{\Delta x}{2}\left[f(x_0) + 2\displaystyle\sum_{i=1}^{n-1} f(x_i) + f(x_n)\right]$ and $M_n = \Delta x \displaystyle\sum_{i=1}^{n} f\left(x_i - \frac{\Delta x}{2}\right)$, so

$$\tfrac{1}{3}T_n + \tfrac{2}{3}M_n = \tfrac{1}{3}(T_n + 2M_n) = \frac{\Delta x}{3 \cdot 2}\left[f(x_0) + 2\sum_{i=1}^{n-1} f(x_i) + f(x_n) + 4\sum_{i=1}^{n} f\left(x_i - \frac{\Delta x}{2}\right)\right]$$

where $\Delta x = \dfrac{b-a}{n}$. Let $\delta x = \dfrac{b-a}{2n}$. Then $\Delta x = 2\delta x$, so

$$\begin{aligned}\tfrac{1}{3}T_n + \tfrac{2}{3}M_n &= \frac{\delta x}{3}\left[f(x_0) + 2\sum_{i=1}^{n-1} f(x_i) + f(x_n) + 4\sum_{i=1}^{n} f(x_i - \delta x)\right] \\ &= \tfrac{1}{3}\delta x[f(x_0) + 4f(x_1 - \delta x) + 2f(x_1) + 4f(x_2 - \delta x) \\ &\qquad + 2f(x_2) + \cdots + 2f(x_{n-1}) + 4f(x_n - \delta x) + f(x_n)]\end{aligned}$$

Since $x_0, x_1 - \delta x, x_1, x_2 - \delta x, x_2, \ldots, x_{n-1}, x_n - \delta x, x_n$ are the subinterval endpoints for S_{2n}, and since $\delta x = \dfrac{b-a}{2n}$ is the width of the subintervals for S_{2n}, the last expression for $\frac{1}{3}T_n + \frac{2}{3}M_n$ is the usual expression for S_{2n}. Therefore, $\frac{1}{3}T_n + \frac{2}{3}M_n = S_{2n}$.

8.8 Improper Integrals

1. (a) Since $\int_1^\infty x^4 e^{-x^4}\,dx$ has an infinite interval of integration, it is an improper integral of Type I.

(b) Since $y = \sec x$ has an infinite discontinuity at $x = \frac{\pi}{2}$, $\int_0^{\pi/2} \sec x\,dx$ is a Type II improper integral.

(c) Since $y = \dfrac{x}{(x-2)(x-3)}$ has an infinite discontinuity at $x = 2$, $\displaystyle\int_0^2 \frac{x}{x^2-5x+6}\,dx$ is a Type II improper integral.

(d) Since $\displaystyle\int_{-\infty}^0 \frac{1}{x^2+5}\,dx$ has an infinite interval of integration, it is an improper integral of Type I.

2. (a) Since $y = 1/(2x-1)$ is defined and continuous on $[1,2]$, the integral is proper.

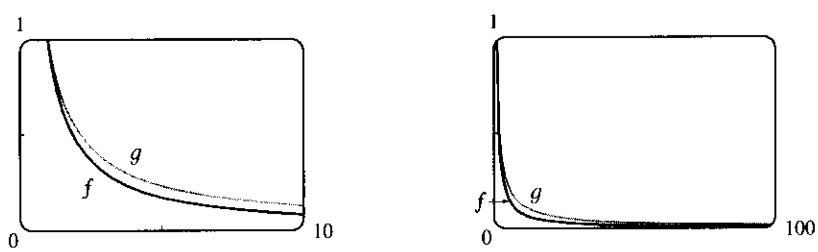
(b) Since $y = \dfrac{1}{2x-1}$ has an infinite discontinuity at $x = \frac{1}{2}$, $\displaystyle\int_0^1 \frac{1}{2x-1}\,dx$ is a Type II improper integral.

(c) Since $\displaystyle\int_{-\infty}^\infty \frac{\sin x}{1+x^2}\,dx$ has an infinite interval of integration, it is an improper integral of Type I.

(d) Since $y = \ln(x-1)$ has an infinite discontinuity at $x = 1$, $\int_1^2 \ln(x-1)dx$ is a Type II improper integral.

3. The area under the graph of $y = 1/x^3 = x^{-3}$ between $x = 1$ and $x = t$ is $A(t) = \int_1^t x^{-3}\,dx = \left[-\frac{1}{2}x^{-2}\right]_1^t = -\frac{1}{2}t^{-2} - \left(-\frac{1}{2}\right) = \frac{1}{2} - 1/(2t^2)$. So the area for $1 \le x \le 10$ is $A(10) = 0.5 - 0.005 = 0.495$, the area for $1 \le x \le 100$ is $A(100) = 0.5 - 0.00005 = 0.49995$, and the area for $1 \le x \le 1000$ is $A(1000) = 0.5 - 0.0000005 = 0.4999995$. The total area under the curve for $x \ge 1$ is $\lim\limits_{t\to\infty} A(t) = \lim\limits_{t\to\infty}\left[\frac{1}{2} - 1/(2t^2)\right] = \frac{1}{2}$.

4. (a)



(b) The area under the graph of f from $x = 1$ to $x = t$ is

$$F(t) = \int_1^t f(x)\,dx = \int_1^t x^{-1.1}dx = \left[-\tfrac{1}{0.1}x^{-0.1}\right]_1^t = -10(t^{-0.1} - 1) = 10(1 - t^{-0.1})$$

and the area under the graph of g is

$$G(t) = \int_1^t g(x)dx = \int_1^t x^{-0.9}dx = \left[\tfrac{1}{0.1}x^{0.1}\right]_1^t = 10(t^{0.1} - 1)$$

t	$F(t)$	$G(t)$
10	2.06	2.59
100	3.69	5.85
10^4	6.02	15.12
10^6	7.49	29.81
10^{10}	9	90
10^{20}	9.9	990

(c) The total area under the graph of f is $\lim\limits_{t\to\infty} F(t) = \lim\limits_{t\to\infty} 10(1 - t^{-0.1}) = 10$.

The total area under the graph of g does not exist, since $\lim\limits_{t\to\infty} G(t) = \lim\limits_{t\to\infty} 10(t^{0.1} - 1) = \infty$.

5. $I = \int_1^\infty \frac{1}{(3x+1)^2}\,dx = \lim_{t\to\infty}\int_1^t \frac{1}{(3x+1)^2}\,dx$. Now

$$\int \frac{1}{(3x+1)^2}\,dx = \frac{1}{3}\int \frac{1}{u^2}\,du \quad [u = 3x+1,\, du = 3\,dx]$$
$$= -\frac{1}{3u} + C = -\frac{1}{3(3x+1)} + C,$$

so $I = \lim_{t\to\infty}\left[-\frac{1}{3(3x+1)}\right]_1^t = \lim_{t\to\infty}\left[-\frac{1}{3(3t+1)} + \frac{1}{12}\right] = 0 + \frac{1}{12} = \frac{1}{12}$. Convergent

6. $\int_{-\infty}^0 \frac{1}{2x-5}\,dx = \lim_{t\to-\infty}\int_t^0 \frac{1}{2x-5}\,dx = \lim_{t\to-\infty}\left[\frac{1}{2}\ln|2x-5|\right]_t^0 = \lim_{t\to\infty}\left[\frac{1}{2}\ln 5 - \frac{1}{2}\ln|2t-5|\right] = -\infty$. Divergent

7. $\int_{-\infty}^{-1} \frac{1}{\sqrt{2-w}}\,dw = \lim_{t\to-\infty}\int_t^{-1} \frac{1}{\sqrt{2-w}}\,dw = \lim_{t\to-\infty}\left[-2\sqrt{2-w}\right]_t^{-1} \quad [u = 2-w,\, du = -dw]$

$$= \lim_{t\to-\infty}\left[-2\sqrt{3} + 2\sqrt{2-t}\right] = \infty.$$ Divergent

8. $\int_0^\infty \frac{x}{(x^2+2)^2}\,dx = \lim_{t\to\infty}\int_0^t \frac{x}{(x^2+2)^2}\,dx = \lim_{t\to\infty}\frac{1}{2}\left[\frac{-1}{x^2+2}\right]_0^t = \frac{1}{2}\lim_{t\to\infty}\left(\frac{-1}{t^2+2} + \frac{1}{2}\right)$

$$= \tfrac{1}{2}\left(0 + \tfrac{1}{2}\right) = \tfrac{1}{4}.$$ Convergent

9. $\int_4^\infty e^{-y/2}\,dy = \lim_{t\to\infty}\int_4^t e^{-y/2}\,dy = \lim_{t\to\infty}\left[-2e^{-y/2}\right]_4^t = \lim_{t\to\infty}(-2e^{-t/2} + 2e^{-2}) = 0 + 2e^{-2} = 2e^{-2}$.

Convergent

10. $\int_{-\infty}^{-1} e^{-2t}\,dt = \lim_{x\to-\infty}\int_x^{-1} e^{-2t}\,dt = \lim_{x\to-\infty}\left[-\frac{1}{2}e^{-2t}\right]_x^{-1} = \lim_{x\to-\infty}\left[-\frac{1}{2}e^2 + \frac{1}{2}e^{-2x}\right] = \infty$. Divergent

11. $\int_{-\infty}^\infty \frac{x\,dx}{1+x^2} = \int_{-\infty}^0 \frac{x\,dx}{1+x^2} + \int_0^\infty \frac{x\,dx}{1+x^2}$ and

$\int_{-\infty}^0 \frac{x\,dx}{1+x^2} = \lim_{t\to-\infty}\left[\frac{1}{2}\ln(1+x^2)\right]_t^0 = \lim_{t\to-\infty}\left[0 - \frac{1}{2}\ln(1+t^2)\right] = -\infty$. Divergent

12. $I = \int_{-\infty}^\infty (2-v^4)\,dv = I_1 + I_2 = \int_{-\infty}^0 (2-v^4)\,dv + \int_0^\infty (2-v^4)\,dv$, but

$I_1 = \lim_{t\to-\infty}\left[2v - \frac{1}{5}v^5\right]_t^0 = \lim_{t\to-\infty}\left(-2t + \frac{1}{5}t^5\right) = -\infty$. Since I_1 is divergent, I is divergent, and there is no need to evaluate I_2. Divergent

13. $\int_{-\infty}^\infty xe^{-x^2}\,dx = \int_{-\infty}^0 xe^{-x^2}\,dx + \int_0^\infty xe^{-x^2}\,dx$.

$\int_{-\infty}^0 xe^{-x^2}\,dx = \lim_{t\to-\infty}\left(-\frac{1}{2}\right)\left[e^{-x^2}\right]_t^0 = \lim_{t\to-\infty}\left(-\frac{1}{2}\right)\left(1 - e^{-t^2}\right) = -\frac{1}{2}\cdot 1 = -\frac{1}{2}$, and

$\int_0^\infty xe^{-x^2}\,dx = \lim_{t\to\infty}\left(-\frac{1}{2}\right)\left[e^{-x^2}\right]_0^t = \lim_{t\to\infty}\left(-\frac{1}{2}\right)\left(e^{-t^2} - 1\right) = -\frac{1}{2}\cdot(-1) = \frac{1}{2}$.

Therefore, $\int_{-\infty}^\infty xe^{-x^2}\,dx = -\frac{1}{2} + \frac{1}{2} = 0$. Convergent

14. $\int_{-\infty}^\infty x^2e^{-x^3}\,dx = \int_{-\infty}^0 x^2e^{-x^3}\,dx + \int_0^\infty x^2e^{-x^3}\,dx$, and

$\int_{-\infty}^0 x^2e^{-x^3}\,dx = \lim_{t\to-\infty}\left[-\frac{1}{3}e^{-x^3}\right]_t^0 = -\frac{1}{3} + \frac{1}{3}\left(\lim_{t\to-\infty} e^{-t^3}\right) = \infty$. Divergent

15. $\int_{2\pi}^\infty \sin\theta\,d\theta = \lim_{t\to\infty}\int_{2\pi}^t \sin\theta\,d\theta = \lim_{t\to\infty}\left[-\cos\theta\right]_{2\pi}^t = \lim_{t\to\infty}(-\cos t + 1)$. This limit does not exist, so the integral is divergent. Divergent

16. $\int_0^\infty \cos^2\alpha\,d\alpha = \lim_{t\to\infty}\int_0^t \frac{1}{2}(1+\cos 2\alpha)\,d\alpha = \lim_{t\to\infty}\left[\frac{1}{2}\alpha + \frac{1}{4}\sin 2\alpha\right]_0^t = \lim_{t\to\infty}\left[\frac{1}{2}t + \frac{1}{4}\sin 2t\right] = \infty$ since $\left|\frac{1}{4}\sin 2t\right| \le \frac{1}{4}$ for all t, but $\frac{1}{2}t \to \infty$ as $t \to \infty$. Divergent

17. $$\int_1^\infty \frac{x+1}{x^2+2x}\,dx = \lim_{t\to\infty}\int_1^t \frac{\frac{1}{2}(2x+2)}{x^2+2x}\,dx = \frac{1}{2}\lim_{t\to\infty}\left[\ln(x^2+2x)\right]_1^t = \frac{1}{2}\lim_{t\to\infty}\left[\ln(t^2+2t) - \ln 3\right]$$
$= \infty$. Divergent

18. $$\int_0^\infty \frac{dz}{z^2+3z+2} = \lim_{t\to\infty}\int_0^t \left[\frac{1}{z+1} - \frac{1}{z+2}\right]dz = \lim_{t\to\infty}\left[\ln\left(\frac{z+1}{z+2}\right)\right]_0^t$$
$$= \lim_{t\to\infty}\left[\ln\left(\frac{t+1}{t+2}\right) - \ln\left(\frac{1}{2}\right)\right] = \ln 1 + \ln 2 = \ln 2.$$ Convergent

19. $$\int_0^\infty se^{-5s}\,ds = \lim_{t\to\infty}\int_0^t se^{-5s}\,ds = \lim_{t\to\infty}\left[-\frac{1}{5}se^{-5s} - \frac{1}{25}e^{-5s}\right]_0^t \quad \left[\begin{array}{l}\text{by integration by}\\ \text{parts with } u = s\end{array}\right]$$
$$= \lim_{t\to\infty}\left(-\tfrac{1}{5}te^{-5t} - \tfrac{1}{25}e^{-5t} + \tfrac{1}{25}\right) = 0 - 0 + \tfrac{1}{25} \quad \text{[by l'Hospital's Rule]}$$
$= \frac{1}{25}$. Convergent

20. $$\int_{-\infty}^6 re^{r/3}\,dr = \lim_{t\to-\infty}\int_t^6 re^{r/3}\,dr = \lim_{t\to-\infty}\left[3re^{r/3} - 9e^{r/3}\right]_t^6 \quad \left[\begin{array}{l}\text{by integration by}\\ \text{parts with } u = r\end{array}\right]$$
$$= \lim_{t\to-\infty}\left(18e^2 - 9e^2 - 3te^{t/3} + 9e^{t/3}\right) = 9e^2 - 0 + 0 \quad \text{[by l'Hospital's Rule]}$$
$= 9e^2$. Convergent

21. $\displaystyle\int_1^\infty \frac{\ln x}{x}\,dx = \lim_{t\to\infty}\left[\frac{(\ln x)^2}{2}\right]_1^t$ (by substitution with $u = \ln x$, $du = dx/x$) $= \displaystyle\lim_{t\to\infty}\frac{(\ln t)^2}{2} = \infty$. Divergent

22. $\int_{-\infty}^\infty e^{-|x|}\,dx = \int_{-\infty}^0 e^x\,dx + \int_0^\infty e^{-x}\,dx$, $\int_{-\infty}^0 e^x\,dx = \lim_{t\to-\infty}\left[e^x\right]_t^0 = \lim_{t\to-\infty}\left(1 - e^t\right) = 1$, and $\int_0^\infty e^{-x}\,dx = \lim_{t\to\infty}\left[-e^{-x}\right]_0^t = \lim_{t\to\infty}\left(1 - e^{-t}\right) = 1$. Therefore, $\int_{-\infty}^\infty e^{-|x|}\,dx = 1 + 1 = 2$. Convergent

23. $$\int_{-\infty}^\infty \frac{x^2}{9+x^6}\,dx = \int_{-\infty}^0 \frac{x^2}{9+x^6}\,dx + \int_0^\infty \frac{x^2}{9+x^6}\,dx = 2\int_0^\infty \frac{x^2}{9+x^6}\,dx \quad \text{[since the integrand is even].}$$
Now $$\int \frac{x^2\,dx}{9+x^6} \quad \left[\begin{array}{l} u = x^3 \\ du = 3x^2dx\end{array}\right] = \int \frac{\frac{1}{3}\,du}{9+u^2} \quad \left[\begin{array}{l} u = 3v \\ du = 3\,dv\end{array}\right] = \int \frac{\frac{1}{3}(3\,dv)}{9+9v^2} = \frac{1}{9}\int \frac{dv}{1+v^2}$$
$$= \frac{1}{9}\tan^{-1} v + C = \frac{1}{9}\tan^{-1}\left(\frac{u}{3}\right) + C = \frac{1}{9}\tan^{-1}\left(\frac{x^3}{3}\right) + C,$$
so $$2\int_0^\infty \frac{x^2}{9+x^6}\,dx = 2\lim_{t\to\infty}\int_0^t \frac{x^2}{9+x^6}\,dx = 2\lim_{t\to\infty}\left[\frac{1}{9}\tan^{-1}\left(\frac{x^3}{3}\right)\right]_0^t$$
$$= 2\lim_{t\to\infty}\frac{1}{9}\tan^{-1}\left(\frac{t^3}{3}\right) = \frac{2}{9}\cdot\frac{\pi}{2} = \frac{\pi}{9}.$$ Convergent

24. Integrate by parts with $u = \ln x$, $dv = dx/x^3$ $\Rightarrow$ $du = dx/x$, $v = -1/(2x^2)$.
$$\int_1^\infty \frac{\ln x}{x^3}\,dx = \lim_{t\to\infty}\int_1^t \frac{\ln x}{x^3}\,dx = \lim_{t\to\infty}\left(\left[-\frac{1}{2x^2}\ln x\right]_1^t + \frac{1}{2}\int_1^t \frac{1}{x^3}\,dx\right)$$
$$= \lim_{t\to\infty}\left(-\frac{1}{2}\frac{\ln t}{t^2} + 0 - \frac{1}{4t^2} + \frac{1}{4}\right) = \frac{1}{4}$$
since $\displaystyle\lim_{t\to\infty}\frac{\ln t}{t^2} \overset{\text{H}}{=} \lim_{t\to\infty}\frac{1/t}{2t} = \lim_{t\to\infty}\frac{1}{2t^2} = 0$. Convergent

25. Integrate by parts with $u = \ln x$, $dv = dx/x^2 \Rightarrow du = dx/x$, $v = -1/x$.

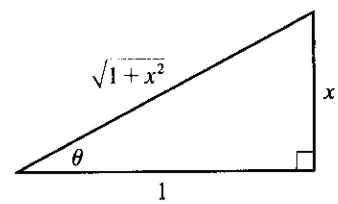
$$\int_1^\infty \frac{\ln x}{x^2}\,dx = \lim_{t\to\infty}\int_1^t \frac{\ln x}{x^2}\,dx = \lim_{t\to\infty}\left[-\frac{\ln x}{x} - \frac{1}{x}\right]_1^t = \lim_{t\to\infty}\left(-\frac{\ln t}{t} - \frac{1}{t} + 0 + 1\right)$$
$$= -0 - 0 + 0 + 1 = 1$$

since $\lim\limits_{t\to\infty}\frac{\ln t}{t} \overset{\text{H}}{=} \lim\limits_{t\to\infty}\frac{1/t}{1} = 0$. Convergent

26. $\displaystyle\int_0^\infty \frac{x\arctan x}{(1+x^2)^2}\,dx = \lim_{t\to\infty}\int_0^t \frac{x\arctan x}{(1+x^2)^2}\,dx$. Let $u = \arctan x$, $dv = \dfrac{x\,dx}{(1+x^2)^2}$. Then $du = \dfrac{dx}{1+x^2}$,

$v = \dfrac{1}{2}\displaystyle\int \frac{2x\,dx}{(1+x^2)^2} = \frac{-1/2}{1+x^2}$, and

$$\int \frac{x\arctan x}{(1+x^2)^2}\,dx = -\frac{1}{2}\frac{\arctan x}{1+x^2} + \frac{1}{2}\int \frac{dx}{(1+x^2)^2} \quad \begin{bmatrix} x = \tan\theta, \\ dx = \sec^2\theta\,d\theta \end{bmatrix}$$
$$= -\frac{1}{2}\frac{\arctan x}{1+x^2} + \frac{1}{2}\int \frac{\sec^2\theta\,d\theta}{(\sec^2\theta)^2}$$
$$= -\frac{1}{2}\frac{\arctan x}{1+x^2} + \frac{1}{2}\int \cos^2\theta\,d\theta$$
$$= -\frac{1}{2}\frac{\arctan x}{1+x^2} + \frac{\theta}{4} + \frac{\sin\theta\cos\theta}{4} + C$$
$$= -\frac{1}{2}\frac{\arctan x}{1+x^2} + \frac{1}{4}\arctan x + \frac{1}{4}\frac{x}{1+x^2} + C$$



It follows that

$$\int_0^\infty \frac{x\arctan x}{(1+x^2)^2}\,dx = \lim_{t\to\infty}\left[-\frac{1}{2}\frac{\arctan x}{1+x^2} + \frac{1}{4}\arctan x + \frac{1}{4}\frac{x}{1+x^2}\right]_0^t$$
$$= \lim_{t\to\infty}\left(-\frac{1}{2}\frac{\arctan t}{1+t^2} + \frac{1}{4}\arctan t + \frac{1}{4}\frac{t}{1+t^2}\right) = 0 + \frac{1}{4}\cdot\frac{\pi}{2} + 0 = \frac{\pi}{8}.$$

Convergent.

27. There is an infinite discontinuity at the left endpoint of $[0, 3]$.

$$\int_0^3 \frac{dx}{\sqrt{x}} = \lim_{t\to 0^+}\int_t^3 \frac{dx}{\sqrt{x}} = \lim_{t\to 0^+}\left[2\sqrt{x}\right]_t^3 = \lim_{t\to 0^+}\left(2\sqrt{3} - 2\sqrt{t}\right) = 2\sqrt{3}. \quad \text{Convergent}$$

28. There is an infinite discontinuity at the left endpoint of $[0, 3]$.

$$\int_0^3 \frac{dx}{x\sqrt{x}} = \lim_{t\to 0^+}\int_t^3 \frac{dx}{x^{3/2}} = \lim_{t\to 0^+}\left[\frac{-2}{\sqrt{x}}\right]_t^3 = \frac{-2}{\sqrt{3}} + \lim_{t\to 0^+}\frac{2}{\sqrt{t}} = \infty. \quad \text{Divergent}$$

29. There is an infinite discontinuity at the right endpoint of $[-1, 0]$.

$$\int_{-1}^0 \frac{dx}{x^2} = \lim_{t\to 0^-}\int_{-1}^t \frac{dx}{x^2} = \lim_{t\to 0^-}\left[\frac{-1}{x}\right]_{-1}^t = \lim_{t\to 0^-}\left[-\frac{1}{t} + \frac{1}{-1}\right] = \infty. \quad \text{Divergent}$$

30. $\displaystyle\int_1^9 \frac{dx}{\sqrt[3]{x-9}} = \lim_{t\to 9^-}\int_1^t \frac{dx}{\sqrt[3]{x-9}} = \lim_{t\to 9^-}\left[\tfrac{3}{2}(x-9)^{2/3}\right]_1^t = \lim_{t\to 9^-}\left[\tfrac{3}{2}(t-9)^{2/3} - \tfrac{3}{2}(4)\right] = 0 - 6 = -6.$
Convergent

31. $\displaystyle\int_{-2}^3 \frac{dx}{x^4} = \int_{-2}^0 \frac{dx}{x^4} + \int_0^3 \frac{dx}{x^4}$, but $\displaystyle\int_{-2}^0 \frac{dx}{x^4} = \lim_{t\to 0^-}\left[-\frac{x^{-3}}{3}\right]_{-2}^t = \lim_{t\to 0^-}\left[-\frac{1}{3t^3} - \frac{1}{24}\right] = \infty.$ Divergent

32. $\displaystyle\int_0^1 \frac{dx}{\sqrt{1-x^2}} = \lim_{t\to 1^-}\int_0^t \frac{dx}{\sqrt{1-x^2}} = \lim_{t\to 1^-}\left[\sin^{-1}x\right]_0^t = \lim_{t\to 1^-}\sin^{-1}t = \frac{\pi}{2}.$ Convergent

33. There is an infinite discontinuity at $x = 1$. $\int_0^{33}(x-1)^{-1/5}\,dx = \int_0^1(x-1)^{-1/5}\,dx + \int_1^{33}(x-1)^{-1/5}\,dx$. Here $\int_0^1(x-1)^{-1/5}\,dx = \lim\limits_{t\to1^-}\int_0^t(x-1)^{-1/5}\,dx = \lim\limits_{t\to1^-}\left[\frac{5}{4}(x-1)^{4/5}\right]_0^t = \lim\limits_{t\to1^-}\left[\frac{5}{4}(t-1)^{4/5} - \frac{5}{4}\right] = -\frac{5}{4}$ and $\int_1^{33}(x-1)^{-1/5}\,dx = \lim\limits_{t\to1^+}\int_t^{33}(x-1)^{-1/5}\,dx = \lim\limits_{t\to1^+}\left[\frac{5}{4}(x-1)^{4/5}\right]_t^{33} = \lim\limits_{t\to1^+}\left[\frac{5}{4}\cdot16 - \frac{5}{4}(t-1)^{4/5}\right] = 20$.
Thus, $\int_0^{33}(x-1)^{-1/5}\,dx = -\frac{5}{4} + 20 = \frac{75}{4}$. Convergent

34. $f(y) = 1/(4y-1)$ has an infinite discontinuity at $y = \frac{1}{4}$.

$$\begin{aligned}\int_{1/4}^1 \frac{1}{4y-1}\,dy &= \lim_{t\to(1/4)^+}\int_t^1 \frac{1}{4y-1}\,dy = \lim_{t\to(1/4)^+}\left[\tfrac{1}{4}\ln|4y-1|\right]_t^1 \\ &= \lim_{t\to(1/4)^+}\left[\tfrac{1}{4}\ln3 - \tfrac{1}{4}\ln(4t-1)\right] = \infty\end{aligned}$$

so $\displaystyle\int_{1/4}^1 \frac{1}{4y-1}\,dy$ diverges, and hence, $\displaystyle\int_0^1 \frac{1}{4y-1}\,dy$ diverges. Divergent

35. $\int_0^\pi \sec x\,dx = \int_0^{\pi/2}\sec x\,dx + \int_{\pi/2}^\pi \sec x\,dx$. $\int_0^{\pi/2}\sec x\,dx = \lim\limits_{t\to\pi/2^-}\int_0^t \sec x\,dx$

$$= \lim_{t\to\pi/2^-}\Big[\ln|\sec x + \tan x|\Big]_0^t = \lim_{t\to\pi/2^-}\ln|\sec t + \tan t| = \infty. \text{ Divergent}$$

36. $\displaystyle\int_0^4 \frac{dx}{x^2+x-6} = \int_0^4 \frac{dx}{(x+3)(x-2)} = \int_0^2 \frac{dx}{(x-2)(x+3)} + \int_2^4 \frac{dx}{(x-2)(x+3)}$, and

$$\begin{aligned}\int_0^2 \frac{dx}{(x-2)(x+3)} &= \lim_{t\to2^-}\int_0^t\left[\frac{1/5}{x-2} - \frac{1/5}{x+3}\right]dx \quad \text{[partial fractions]} \quad = \lim_{t\to2^-}\left[\frac{1}{5}\ln\left|\frac{x-2}{x+3}\right|\right]_0^t \\ &= \lim_{t\to2^-}\frac{1}{5}\left[\ln\left|\frac{t-2}{t+3}\right| - \ln\frac{2}{3}\right] = -\infty. \text{ Divergent}\end{aligned}$$

37. There is an infinite discontinuity at $x = 0$. $\displaystyle\int_{-1}^1 \frac{e^x}{e^x-1}\,dx = \int_{-1}^0 \frac{e^x}{e^x-1}\,dx + \int_0^1 \frac{e^x}{e^x-1}\,dx$.

$$\int_{-1}^0 \frac{e^x}{e^x-1}\,dx = \lim_{t\to0^-}\int_{-1}^t \frac{e^x}{e^x-1}\,dx = \lim_{t\to0^-}\Big[\ln|e^x-1|\Big]_{-1}^t = \lim_{t\to0^-}\Big[\ln|e^t-1| - \ln|e^{-1}-1|\Big] = -\infty,$$

so $\displaystyle\int_{-1}^1 \frac{e^x}{e^x-1}\,dx$ is divergent. The integral $\displaystyle\int_0^1 \frac{e^x}{e^x-1}\,dx$ also diverges since

$$\int_0^1 \frac{e^x}{e^x-1}\,dx = \lim_{t\to0^+}\int_t^1 \frac{e^x}{e^x-1}\,dx = \lim_{t\to0^+}\Big[\ln|e^x-1|\Big]_t^1 = \lim_{t\to0^+}\Big[\ln|e-1| - \ln|e^t-1|\Big] = \infty.$$

Divergent

38. $\displaystyle\int_0^2 \frac{x-3}{2x-3}\,dx = \int_0^{3/2}\frac{x-3}{2x-3}\,dx + \int_{3/2}^2 \frac{x-3}{2x-3}\,dx$ and

$$\int \frac{x-3}{2x-3}\,dx = \frac{1}{2}\int \frac{2x-6}{2x-3}\,dx = \frac{1}{2}\int\left[1 - \frac{3}{2x-3}\right]dx = \tfrac{1}{2}x - \tfrac{3}{4}\ln|2x-3| + C, \text{ so}$$

$$\int_0^{3/2}\frac{x-3}{2x-3}\,dx = \lim_{t\to3/2^-}\tfrac{1}{4}\Big[2x - 3\ln|2x-3|\Big]_0^t = \infty. \quad \text{Divergent}$$

39. $I = \int_0^2 z^2 \ln z\, dz = \lim\limits_{t\to 0^+} \int_t^2 z^2 \ln z\, dz \overset{101}{=} \lim\limits_{t\to 0^+} \left[\frac{z^3}{3^2}(3\ln z - 1)\right]_t^2$

$= \lim\limits_{t\to 0^+} \left[\frac{8}{9}(3\ln 2 - 1) - \frac{1}{9}t^3(3\ln t - 1)\right] = \frac{8}{3}\ln 2 - \frac{8}{9} - \frac{1}{9}\lim\limits_{t\to 0^+}\left[t^3(3\ln t - 1)\right] = \frac{8}{3}\ln 2 - \frac{8}{9} - \frac{1}{9}L.$

Now $L = \lim\limits_{t\to 0^+}\left[t^3(3\ln t - 1)\right] = \lim\limits_{t\to 0^+} \dfrac{3\ln t - 1}{t^{-3}} \overset{H}{=} \lim\limits_{t\to 0^+} \dfrac{3/t}{-3/t^4} = \lim\limits_{t\to 0^+}(-t^3) = 0$. Thus, $L = 0$ and

$I = \frac{8}{3}\ln 2 - \frac{8}{9}$. Convergent

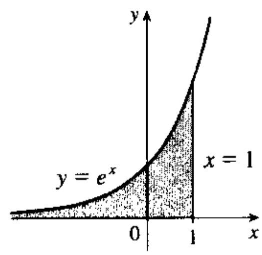
40. Integrate by parts with $u = \ln x$, $dv = dx/\sqrt{x} \quad \Rightarrow \quad du = dx/x$, $v = 2\sqrt{x}$.

$$\int_0^1 \frac{\ln x}{\sqrt{x}}\, dx = \lim_{t\to 0^+} \int_t^1 \frac{\ln x}{\sqrt{x}}\, dx = \lim_{t\to 0^+}\left(\left[2\sqrt{x}\ln x\right]_t^1 - 2\int_t^1 \frac{dx}{\sqrt{x}}\right) = \lim_{t\to 0^+}\left(-2\sqrt{t}\ln t - 4\left[\sqrt{x}\right]_t^1\right)$$

$$= \lim_{t\to 0^+}\left(-2\sqrt{t}\ln t - 4 + 4\sqrt{t}\right) = -4$$

since $\lim\limits_{t\to 0^+} \sqrt{t}\ln t = \lim\limits_{t\to 0^+} \dfrac{\ln t}{t^{-1/2}} \overset{H}{=} \lim\limits_{t\to 0^+} \dfrac{1/t}{-t^{-3/2}/2} = \lim\limits_{t\to 0^+}\left(-2\sqrt{t}\right) = 0$. Convergent

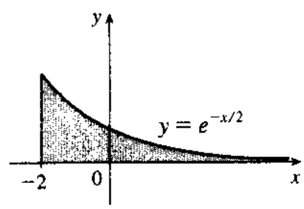
41.



Area $= \int_{-\infty}^1 e^x\, dx = \lim\limits_{t\to -\infty}\left[e^x\right]_t^1$

$= e - \lim\limits_{t\to -\infty} e^t = e$

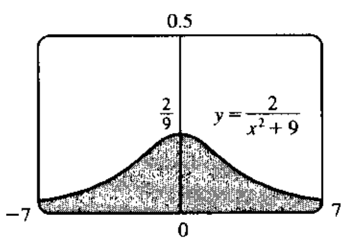
42.



Area $= \int_{-2}^{\infty} e^{-x/2}\, dx = -2\lim\limits_{t\to\infty}\left[e^{-x/2}\right]_{-2}^t$

$= -2\lim\limits_{t\to\infty} e^{-t/2} + 2e = 2e$

43.

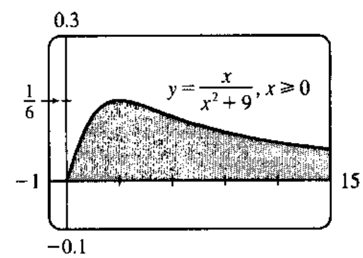


$$\text{Area} = \int_{-\infty}^{\infty} \frac{2}{x^2+9}\, dx = 2\cdot 2\int_0^{\infty} \frac{1}{x^2+9}\, dx$$

$$= 4\lim_{t\to\infty}\int_0^t \frac{1}{x^2+9}\, dx = 4\lim_{t\to\infty}\left[\frac{1}{3}\tan^{-1}\frac{x}{3}\right]_0^t$$

$$= \frac{4}{3}\lim_{t\to\infty}\left[\tan^{-1}\frac{t}{3} - 0\right] = \frac{4}{3}\cdot\frac{\pi}{2} = \frac{2\pi}{3}$$

44.



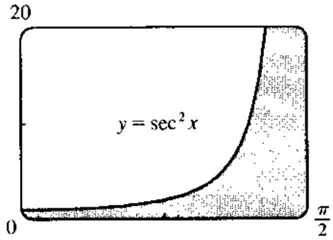
$$\text{Area} = \int_0^{\infty} \frac{x}{x^2+9}\, dx = \lim_{t\to\infty}\int_0^t \frac{x}{x^2+9}\, dx$$

$$= \lim_{t\to\infty}\left[\tfrac{1}{2}\ln(x^2+9)\right]_0^t$$

$$= \tfrac{1}{2}\lim_{t\to\infty}\left[\ln(t^2+9) - \ln 9\right] = \infty$$

Infinite area

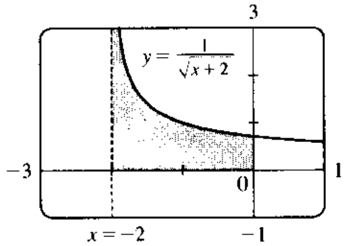
45.



$$\text{Area} = \int_0^{\pi/2} \sec^2 x\,dx = \lim_{t\to(\pi/2)^-} \int_0^t \sec^2 x\,dx = \lim_{t\to(\pi/2)^-} [\tan x]_0^t = \lim_{t\to(\pi/2)^-} (\tan t - 0) = \infty$$

Infinite area

46.



$$\text{Area} = \int_{-2}^{0} \frac{1}{\sqrt{x+2}}\,dx = \lim_{t\to -2^+} \int_t^0 \frac{1}{\sqrt{x+2}}\,dx = \lim_{t\to -2^+} \left[2\sqrt{x+2}\,\right]_t^0 = \lim_{t\to -2^+} \left(2\sqrt{2} - 2\sqrt{t+2}\,\right) = 2\sqrt{2} - 0 = 2\sqrt{2}$$

47. (a)

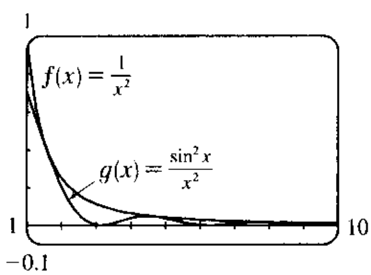
t	$\int_1^t g(x)\,dx$
2	0.447453
5	0.577101
10	0.621306
100	0.668479
1000	0.672957
10,000	0.673407

$g(x) = \dfrac{\sin^2 x}{x^2}$.

It appears that the integral is convergent.

(b) $-1 \le \sin x \le 1 \;\Rightarrow\; 0 \le \sin^2 x \le 1 \;\Rightarrow\; 0 \le \dfrac{\sin^2 x}{x^2} \le \dfrac{1}{x^2}$. Since $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$ is convergent (Equation 2 with $p = 2 > 1$), $\displaystyle\int_1^\infty \frac{\sin^2 x}{x^2}\,dx$ is convergent by the Comparison Theorem.

(c)



Since $\int_1^\infty f(x)\,dx$ is finite and the area under $g(x)$ is less than the area under $f(x)$ on any interval $[1, t]$, $\int_1^\infty g(x)\,dx$ must be finite; that is, the integral is convergent.

48. (a)

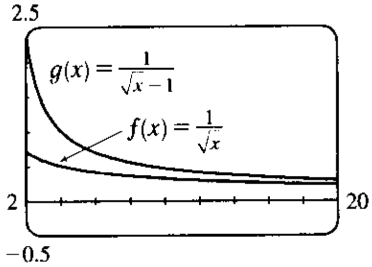
t	$\int_2^t g(x)dx$
5	3.830327
10	6.801200
100	23.328769
1000	69.023361
10,000	208.124560

$g(x) = \dfrac{1}{\sqrt{x} - 1}$.

It appears that the integral is divergent.

(b) For $x \ge 2$, $\sqrt{x} > \sqrt{x} - 1 \;\Rightarrow\; \dfrac{1}{\sqrt{x}} < \dfrac{1}{\sqrt{x}-1}$. Since $\displaystyle\int_2^\infty \frac{1}{\sqrt{x}}\,dx$ is divergent (Equation 2 with $p = \frac{1}{2} \le 1$), $\displaystyle\int_2^\infty \frac{1}{\sqrt{x}-1}\,dx$ is divergent by the Comparison Theorem.

(c)



Since $\int_2^\infty f(x)\,dx$ is infinite and the area under $g(x)$ is greater than the area under $f(x)$ on any interval $[2, t]$, $\int_2^\infty g(x)\,dx$ must be infinite; that is, the integral is divergent.

49. For $x \ge 1$, $\dfrac{\cos^2 x}{1+x^2} \le \dfrac{1}{1+x^2} < \dfrac{1}{x^2}$. $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$ is convergent by Equation 2 with $p = 2 > 1$, so $\displaystyle\int_1^\infty \frac{\cos^2 x}{1+x^2}\,dx$ is convergent by the Comparison Theorem.

50. For $x \ge 1$, $\dfrac{2+e^{-x}}{x} > \dfrac{2}{x}$ [since $e^{-x} > 0$] $> \dfrac{1}{x}$. $\displaystyle\int_1^\infty \frac{1}{x}\,dx$ is divergent by Equation 2 with $p = 1 \le 1$, so $\displaystyle\int_1^\infty \frac{2+e^{-x}}{x}\,dx$ is divergent by the Comparison Theorem.

51. For $x \ge 1$, $x + e^{2x} > e^{2x} > 0 \;\Rightarrow\; \dfrac{1}{x+e^{2x}} \le \dfrac{1}{e^{2x}} = e^{-2x}$ on $[1, \infty)$.

$\displaystyle\int_1^\infty e^{-2x}\,dx = \lim_{t\to\infty}\left[-\frac{1}{2}e^{-2x}\right]_1^t = \lim_{t\to\infty}\left[-\frac{1}{2}e^{-2t} + \frac{1}{2}e^{-2}\right] = \frac{1}{2}e^{-2}$. Therefore, $\displaystyle\int_1^\infty e^{-2x}\,dx$ is convergent, and by the Comparison Theorem, $\displaystyle\int_1^\infty \frac{dx}{x+e^{2x}}$ is also convergent.

52. For $x \ge 1$, $0 < \dfrac{x}{\sqrt{1+x^6}} < \dfrac{x}{\sqrt{x^6}} = \dfrac{x}{x^3} = \dfrac{1}{x^2}$. $\displaystyle\int_1^\infty \frac{1}{x^2}\,dx$ is convergent by Equation 2 with $p = 2 > 1$, so $\displaystyle\int_1^\infty \frac{x}{\sqrt{1+x^6}}\,dx$ is convergent by the Comparison Theorem.

53. $\dfrac{1}{x\sin x} \ge \dfrac{1}{x}$ on $(0, \frac{\pi}{2}]$ since $0 \le \sin x \le 1$. $\displaystyle\int_0^{\pi/2} \frac{dx}{x} = \lim_{t\to 0^+}\int_t^{\pi/2} \frac{dx}{x} = \lim_{t\to 0^+}\left[\ln x\right]_t^{\pi/2}$.

But $\ln t \to -\infty$ as $t \to 0^+$, so $\displaystyle\int_0^{\pi/2} \frac{dx}{x}$ is divergent, and by the Comparison Theorem, $\displaystyle\int_0^{\pi/2} \frac{dx}{x\sin x}$ is also divergent.

54. For $0 \le x \le 1$, $e^{-x} \le 1 \;\Rightarrow\; \dfrac{e^{-x}}{\sqrt{x}} \le \dfrac{1}{\sqrt{x}}$.

$\displaystyle\int_0^1 \frac{1}{\sqrt{x}}\,dx = \lim_{t\to 0^+}\int_t^1 \frac{1}{\sqrt{x}}\,dx = \lim_{t\to 0^+}\left[2\sqrt{x}\right]_t^1 = \lim_{t\to 0^+}\left(2 - 2\sqrt{t}\right) = 2$ is convergent. Therefore, $\displaystyle\int_0^1 \frac{e^{-x}}{\sqrt{x}}\,dx$ is convergent by the Comparison Theorem.

55. $\displaystyle\int_0^\infty \frac{dx}{\sqrt{x}\,(1+x)} = \int_0^1 \frac{dx}{\sqrt{x}\,(1+x)} + \int_1^\infty \frac{dx}{\sqrt{x}\,(1+x)} = \lim_{t\to 0^+}\int_t^1 \frac{dx}{\sqrt{x}\,(1+x)} + \lim_{t\to\infty}\int_1^t \frac{dx}{\sqrt{x}\,(1+x)}$. Now

$$\int \frac{dx}{\sqrt{x}\,(1+x)} = \int \frac{2u\,du}{u(1+u^2)} \quad [u=\sqrt{x}, x=u^2, dx = 2u\,du]$$

$$= 2\int \frac{du}{1+u^2} = 2\tan^{-1}u + C = 2\tan^{-1}\sqrt{x} + C,$$

so $\displaystyle\int_0^\infty \frac{dx}{\sqrt{x}\,(1+x)} = \lim_{t\to 0^+}\left[2\tan^{-1}\sqrt{x}\right]_t^1 + \lim_{t\to\infty}\left[2\tan^{-1}\sqrt{x}\right]_1^t$

$$= \lim_{t\to 0^+}\left[2\left(\tfrac{\pi}{4}\right) - 2\tan^{-1}\sqrt{t}\right] + \lim_{t\to\infty}\left[2\tan^{-1}\sqrt{t} - 2\left(\tfrac{\pi}{4}\right)\right] = \tfrac{\pi}{2} - 0 + 2\left(\tfrac{\pi}{2}\right) - \tfrac{\pi}{2} = \pi.$$

56. $\displaystyle\int_2^\infty \frac{dx}{x\sqrt{x^2-4}} = \int_2^3 \frac{dx}{x\sqrt{x^2-4}} + \int_3^\infty \frac{dx}{x\sqrt{x^2-4}} = \lim_{t\to 2^+}\int_t^3 \frac{dx}{x\sqrt{x^2-4}} + \lim_{t\to\infty}\int_3^t \frac{dx}{x\sqrt{x^2-4}}$. Now

$$\int \frac{dx}{x\sqrt{x^2-4}} = \int \frac{2\sec\theta\,\tan\theta\,d\theta}{2\sec\theta\,2\tan\theta} \quad [x = 2\sec\theta, \text{ where } 0 \le \theta < \pi/2 \text{ or } \pi \le \theta < 3\pi/2]$$

$$= \tfrac{1}{2}\theta + C = \tfrac{1}{2}\sec^{-1}\left(\tfrac{1}{2}x\right) + C, \text{ so}$$

$$\int_2^\infty \frac{dx}{x\sqrt{x^2-4}} = \lim_{t\to 2^+}\left[\tfrac{1}{2}\sec^{-1}\left(\tfrac{1}{2}x\right)\right]_t^3 + \lim_{t\to\infty}\left[\tfrac{1}{2}\sec^{-1}\left(\tfrac{1}{2}x\right)\right]_3^t$$

$$= \tfrac{1}{2}\sec^{-1}\left(\tfrac{3}{2}\right) - 0 + \tfrac{1}{2}\left(\tfrac{\pi}{2}\right) - \tfrac{1}{2}\sec^{-1}\left(\tfrac{3}{2}\right) = \tfrac{\pi}{4}$$

57. If $p=1$, then $\displaystyle\int_0^1 \frac{dx}{x^p} = \lim_{t\to 0^+}\int_t^1 \frac{dx}{x} = \lim_{t\to 0^+}\left[\ln x\right]_t^1 = \infty$. Divergent.

If $p \neq 1$, then $\displaystyle\int_0^1 \frac{dx}{x^p} = \lim_{t\to 0^+}\int_t^1 \frac{dx}{x^p}$ (note that the integral is not improper if $p<0$)

$$= \lim_{t\to 0^+}\left[\frac{x^{-p+1}}{-p+1}\right]_t^1 = \lim_{t\to 0^+}\frac{1}{1-p}\left[1 - \frac{1}{t^{p-1}}\right]$$

If $p>1$, then $p-1>0$, so $\dfrac{1}{t^{p-1}} \to \infty$ as $t\to 0^+$, and the integral diverges.

If $p<1$, then $p-1<0$, so $\dfrac{1}{t^{p-1}} \to 0$ as $t\to 0^+$ and $\displaystyle\int_0^1 \frac{dx}{x^p} = \frac{1}{1-p}\left[\lim_{t\to 0^+}\left(1-t^{1-p}\right)\right] = \frac{1}{1-p}$.

Thus, the integral converges if and only if $p<1$, and in that case its value is $\dfrac{1}{1-p}$.

58. Let $u = \ln x$. Then $du = dx/x \Rightarrow \displaystyle\int_e^\infty \frac{dx}{x(\ln x)^p} = \int_1^\infty \frac{du}{u^p}$. By Example 4, this converges to $\dfrac{1}{p-1}$ if $p>1$ and diverges otherwise.

59. First suppose $p=-1$. Then

$$\int_0^1 x^p \ln x\,dx = \int_0^1 \frac{\ln x}{x}\,dx = \lim_{t\to 0^+}\int_t^1 \frac{\ln x}{x}\,dx = \lim_{t\to 0^+}\left[\tfrac{1}{2}(\ln x)^2\right]_t^1 = -\tfrac{1}{2}\lim_{t\to 0^+}(\ln t)^2 = -\infty,$$

so the integral diverges. Now suppose $p\neq -1$. Then integration by parts gives

$\displaystyle\int x^p \ln x\,dx = \frac{x^{p+1}}{p+1}\ln x - \int \frac{x^p}{p+1}\,dx = \frac{x^{p+1}}{p+1}\ln x - \frac{x^{p+1}}{(p+1)^2} + C$. If $p<-1$, then $p+1<0$, so

$$\int_0^1 x^p \ln x\,dx = \lim_{t\to 0^+}\left[\frac{x^{p+1}}{p+1}\ln x - \frac{x^{p+1}}{(p+1)^2}\right]_t^1 = \frac{-1}{(p+1)^2} - \left(\frac{1}{p+1}\right)\lim_{t\to 0^+}\left[t^{p+1}\left(\ln t - \frac{1}{p+1}\right)\right] = \infty.$$

If $p > -1$, then $p+1 > 0$ and

$$\int_0^1 x^p \ln x\,dx = \frac{-1}{(p+1)^2} - \left(\frac{1}{p+1}\right)\lim_{t\to 0^+}\frac{\ln t - 1/(p+1)}{t^{-(p+1)}} \overset{\text{H}}{=} \frac{-1}{(p+1)^2} - \left(\frac{1}{p+1}\right)\lim_{t\to 0^+}\frac{1/t}{-(p+1)t^{-(p+2)}}$$

$$= \frac{-1}{(p+1)^2} + \frac{1}{(p+1)^2}\lim_{t\to 0^+} t^{p+1} = \frac{-1}{(p+1)^2}$$

Thus, the integral converges to $-\dfrac{1}{(p+1)^2}$ if $p > -1$ and diverges otherwise.

60. (a) $\boldsymbol{n = 0}$: $\displaystyle\int_0^\infty x^n e^{-x}dx = \lim_{t\to\infty}\int_0^t e^{-x}dx = \lim_{t\to\infty}\left[-e^{-x}\right]_0^t$

$$= \lim_{t\to\infty}\left[-e^{-t}+1\right] = 0 + 1 = 1$$

$\boldsymbol{n = 1}$: $\displaystyle\int_0^\infty x^n e^{-x}dx = \lim_{t\to\infty}\int_0^t xe^{-x}dx$. To evaluate $\displaystyle\int xe^{-x}dx$, we'll use integration by parts with $u = x$, $dv = e^{-x}dx \;\Rightarrow\; du = dx$, $v = -e^{-x}$.

So $\displaystyle\int xe^{-x}dx = -xe^{-x} - \int -e^{-x}dx = -xe^{-x} - e^{-x} + C = (-x-1)e^{-x} + C$ and

$$\lim_{t\to\infty}\int_0^t xe^{-x}dx = \lim_{t\to\infty}\left[(-x-1)e^{-x}\right]_0^t$$

$$= \lim_{t\to\infty}\left[(-t-1)e^{-t}+1\right] = \lim_{t\to\infty}\left[-te^{-t}-e^{-t}+1\right]$$

$$= 0 - 0 + 1 \quad \text{[use l'Hospital's Rule]} \quad = 1$$

$\boldsymbol{n = 2}$: $\displaystyle\int_0^\infty x^n e^{-x}dx = \lim_{t\to\infty}\int_0^t x^2e^{-x}dx$. To evaluate $\displaystyle\int x^2e^{-x}dx$, we could use integration by parts again or Formula 97. Thus,

$$\lim_{t\to\infty}\int_0^t x^2e^{-x}dx = \lim_{t\to\infty}\left[-x^2e^{-x}\right]_0^t + 2\lim_{t\to\infty}\int_0^t xe^{-x}dx$$

$$= 0 + 0 + 2(1) \quad \text{[use l'Hospital's Rule and the result for } n = 1\text{]} \quad = 2$$

$\boldsymbol{n = 3}$: $\displaystyle\int_0^\infty x^n e^{-x}dx = \lim_{t\to\infty}\int_0^t x^3e^{-x}dx \overset{97}{=} \lim_{t\to\infty}\left[-x^3e^{-x}\right]_0^t + 3\lim_{t\to\infty}\int_0^t x^2e^{-x}dx$

$$= 0 + 0 + 3(2) \quad \text{[use l'Hospital's Rule and the result for } n = 2\text{]} \quad = 6$$

(b) For $n = 1, 2$, and 3, we have $\displaystyle\int_0^\infty x^ne^{-x}dx = 1, 2$, and 6. The values for the integral are equal to the factorials for n, so we guess $\displaystyle\int_0^\infty x^ne^{-x}dx = n!$.

(c) Suppose that $\displaystyle\int_0^\infty x^ke^{-x}dx = k!$ for some positive integer k. Then $\displaystyle\int_0^\infty x^{k+1}e^{-x}dx = \lim_{t\to\infty}\int_0^t x^{k+1}e^{-x}dx$.

To evaluate $\displaystyle\int x^{k+1}e^{-x}dx$, we use parts with $u = x^{k+1}$, $dv = e^{-x}dx \;\Rightarrow\; du = (k+1)x^kdx$, $v = -e^{-x}$.

So $\displaystyle\int x^{k+1}e^{-x}dx = -x^{k+1}e^{-x} - \int -(k+1)x^ke^{-x}dx = -x^{k+1}e^{-x} + (k+1)\int x^ke^{-x}dx$ and

$$\lim_{t\to\infty}\int_0^t x^{k+1}e^{-x}dx = \lim_{t\to\infty}\left[-x^{k+1}e^{-x}\right]_0^t + (k+1)\lim_{t\to\infty}\int_0^t x^ke^{-x}dx$$

$$= \lim_{t\to\infty}\left[-t^{k+1}e^{-t}+0\right] + (k+1)k! = 0 + 0 + (k+1)! = (k+1)!,$$

so the formula holds for $k+1$. By induction, the formula holds for all positive integers. (Since $0! = 1$, the formula holds for $n = 0$, too.)

61. (a) $I = \int_{-\infty}^{\infty} x\,dx = \int_{-\infty}^{0} x\,dx + \int_{0}^{\infty} x\,dx$, and

$\int_0^\infty x\,dx = \lim\limits_{t\to\infty} \int_0^t x\,dx = \lim\limits_{t\to\infty} \left[\frac{1}{2}x^2\right]_0^t = \lim\limits_{t\to\infty} \left[\frac{1}{2}t^2 - 0\right] = \infty$, so I is divergent.

(b) $\int_{-t}^{t} x\,dx = \left[\frac{1}{2}x^2\right]_{-t}^{t} = \frac{1}{2}t^2 - \frac{1}{2}t^2 = 0$, so $\lim\limits_{t\to\infty}\int_{-t}^{t} x\,dx = 0$. Therefore, $\int_{-\infty}^{\infty} x\,dx \neq \lim\limits_{t\to\infty}\int_{-t}^{t} x\,dx$.

62. Let $k = \dfrac{M}{2RT}$ so that $\overline{v} = \dfrac{4}{\sqrt{\pi}} k^{3/2} \displaystyle\int_0^\infty v^3 e^{-kv^2}\,dv$. Let I denote the integral and use parts to integrate I.

Let $\alpha = v^2$, $d\beta = ve^{-kv^2}\,dv \quad \Rightarrow \quad d\alpha = 2v\,dv$, $\beta = -\dfrac{1}{2k}e^{-kv^2}$:

$$I = \lim_{t\to\infty}\left[-\frac{1}{2k}v^2 e^{-kv^2}\right]_0^t + \frac{1}{k}\int_0^\infty ve^{-kv^2}\,dv = -\frac{1}{2k}\lim_{t\to\infty}\left(t^2e^{-kt^2}\right) + \frac{1}{k}\lim_{t\to\infty}\left[-\frac{1}{2k}e^{-kv^2}\right]_0^t$$

$$\overset{\text{H}}{=} -\frac{1}{2k}\cdot 0 - \frac{1}{2k^2}(0-1) = \frac{1}{2k^2}$$

Thus, $\overline{v} = \dfrac{4}{\sqrt{\pi}}k^{3/2}\cdot\dfrac{1}{2k^2} = \dfrac{2}{(k\pi)^{1/2}} = \dfrac{2}{[\pi M/(2RT)]^{1/2}} = \dfrac{2\sqrt{2}\sqrt{RT}}{\sqrt{\pi M}} = \sqrt{\dfrac{8RT}{\pi M}}$.

63. Volume $= \displaystyle\int_1^\infty \pi\left(\frac{1}{x}\right)^2 dx = \pi\lim_{t\to\infty}\int_1^t \frac{dx}{x^2} = \pi\lim_{t\to\infty}\left[-\frac{1}{x}\right]_1^t = \pi\lim_{t\to\infty}\left(1-\frac{1}{t}\right) = \pi < \infty$.

64. Work $= \displaystyle\int_R^\infty \frac{GMm}{r^2}\,dr = \lim_{t\to\infty}\int_R^t \frac{GMm}{r^2}\,dr = \lim_{t\to\infty} GMm\left[\frac{-1}{r}\right]_R^t = GMm\lim_{t\to\infty}\left(\frac{-1}{t}+\frac{1}{R}\right) = \frac{GMm}{R}$,

where M = mass of Earth $= 5.98\times 10^{24}$ kg, m = mass of satellite $= 10^3$ kg,

R = radius of Earth $= 6.37\times 10^6$ m, and G = gravitational constant $= 6.67\times 10^{-11}$ N·m²/kg.

Therefore, Work $= \dfrac{6.67\times 10^{-11}\cdot 5.98\times 10^{24}\cdot 10^3}{6.37\times 10^6} \approx 6.26\times 10^{10}$ J.

65. Work $= \displaystyle\int_R^\infty F\,dr = \lim_{t\to\infty}\int_R^t \frac{GmM}{r^2}\,dr = \lim_{t\to\infty} GmM\left(\frac{1}{R}-\frac{1}{t}\right) = \frac{GmM}{R}$. The initial kinetic energy provides the work, so $\frac{1}{2}mv_0^2 = \dfrac{GmM}{R} \quad \Rightarrow \quad v_0 = \sqrt{\dfrac{2GM}{R}}$.

66. $y(s) = \displaystyle\int_s^R \frac{2r}{\sqrt{r^2-s^2}}\,x(r)\,dr$ and $x(r) = \frac{1}{2}(R-r)^2 \quad \Rightarrow$

$$y(s) = \lim_{t\to s^+}\int_t^R \frac{r(R-r)^2}{\sqrt{r^2-s^2}}\,dr = \lim_{t\to s^+}\int_t^R \frac{r^3-2Rr^2+R^2r}{\sqrt{r^2-s^2}}\,dr$$

$$= \lim_{t\to s^+}\left[\int_t^R \frac{r^3\,dr}{\sqrt{r^2-s^2}} - 2R\int_t^R \frac{r^2\,dr}{\sqrt{r^2-s^2}} + R^2\int_t^R \frac{r\,dr}{\sqrt{r^2-s^2}}\right]$$

$$= \lim_{t\to s^+}\left(I_1 - 2RI_2 + R^2I_3\right) = L$$

For I_1: Let $u = \sqrt{r^2-s^2} \quad \Rightarrow \quad u^2 = r^2-s^2$, $r^2 = u^2+s^2$, $2r\,dr = 2u\,du$, so, omitting limits and constant of integration,

$$I_1 = \int \frac{(u^2+s^2)u}{u}\,du = \int (u^2+s^2)\,du = \tfrac{1}{3}u^3 + s^2u = \tfrac{1}{3}u(u^2+3s^2)$$

$$= \tfrac{1}{3}\sqrt{r^2-s^2}\left(r^2-s^2+3s^2\right) = \tfrac{1}{3}\sqrt{r^2-s^2}\left(r^2+2s^2\right)$$

For I_2: Using Formula 44, $I_2 = \frac{r}{2}\sqrt{r^2 - s^2} + \frac{s^2}{2}\ln\left|r + \sqrt{r^2 - s^2}\right|$.

For I_3: Let $u = r^2 - s^2 \Rightarrow du = 2r\,dr$. Then $I_3 = \frac{1}{2}\int \frac{du}{\sqrt{u}} = \frac{1}{2}\cdot 2\sqrt{u} = \sqrt{r^2 - s^2}$.

Thus,

$$L = \lim_{t\to s^+}\left[\tfrac{1}{3}\sqrt{r^2 - s^2}\,(r^2 + 2s^2) - 2R\left(\frac{r}{2}\sqrt{r^2 - s^2} + \frac{s^2}{2}\ln\left|r + \sqrt{r^2 - s^2}\right|\right) + R^2\sqrt{r^2 - s^2}\right]_t^R$$

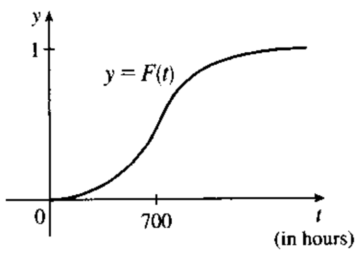
$$= \lim_{t\to s^+}\left[\tfrac{1}{3}\sqrt{R^2 - s^2}\,(R^2 + 2s^2) - 2R\left(\frac{R}{2}\sqrt{R^2 - s^2} + \frac{s^2}{2}\ln\left|R + \sqrt{R^2 - s^2}\right|\right) + R^2\sqrt{R^2 - s^2}\right]$$

$$- \lim_{t\to s^+}\left[\tfrac{1}{3}\sqrt{t^2 - s^2}\,(t^2 + 2s^2) - 2R\left(\frac{t}{2}\sqrt{t^2 - s^2} + \frac{s^2}{2}\ln\left|t + \sqrt{t^2 - s^2}\right|\right) + R^2\sqrt{t^2 - s^2}\right]$$

$$= \left[\tfrac{1}{3}\sqrt{R^2 - s^2}\,(R^2 + 2s^2) - Rs^2\ln\left|R + \sqrt{R^2 - s^2}\right|\right] - \left[-Rs^2\ln|s|\right]$$

$$= \tfrac{1}{3}\sqrt{R^2 - s^2}\,(R^2 + 2s^2) - Rs^2\ln\left(\frac{R + \sqrt{R^2 - s^2}}{s}\right)$$

67. (a) We would expect a small percentage of bulbs to burn out in the first few hundred hours, most of the bulbs to burn out after close to 700 hours, and a few overachievers to burn on and on.



(b) $r(t) = F'(t)$ is the rate at which the fraction $F(t)$ of burnt-out bulbs increases as t increases. This could be interpreted as a fractional burnout rate.

(c) $\int_0^\infty r(t)\,dt = \lim_{x\to\infty} F(x) = 1$, since all of the bulbs will eventually burn out.

68. $I = \int_0^\infty te^{kt}\,dt = \lim_{s\to\infty}\left[\frac{1}{k^2}(kt - 1)\,e^{kt}\right]_0^s$ [Formula 96, or parts]

$= \lim_{s\to\infty}\left[\left(\frac{1}{k}se^{ks} - \frac{1}{k^2}e^{ks}\right) - \left(-\frac{1}{k^2}\right)\right]$.

Since $k < 0$ the first two terms approach 0 (you can verify that the first term does so with l'Hospital's Rule), so the limit is equal to $1/k^2$. Thus, $M = -kI = -k(1/k^2) = -1/k = -1/(-0.000121) \approx 8264.5$ years.

69. $I = \int_a^\infty \frac{1}{x^2 + 1}\,dx = \lim_{t\to\infty}\int_a^t \frac{1}{x^2 + 1}\,dx = \lim_{t\to\infty}\left[\tan^{-1}x\right]_a^t = \lim_{t\to\infty}\left(\tan^{-1}t - \tan^{-1}a\right) = \frac{\pi}{2} - \tan^{-1}a$.

$I < 0.001 \Rightarrow \frac{\pi}{2} - \tan^{-1}a < 0.001 \Rightarrow \tan^{-1}a > \frac{\pi}{2} - 0.001 \Rightarrow a > \tan\left(\frac{\pi}{2} - 0.001\right) \approx 1000$.

70. $f(x) = e^{-x^2}$ and $\Delta x = \frac{4 - 0}{8} = \frac{1}{2}$.

$$\int_0^4 f(x)\,dx \approx S_8 = \frac{1}{2\cdot 3}[f(0) + 4f(0.5) + 2f(1) + \cdots + 2f(3) + 4f(3.5) + f(4)]$$

$$\approx \tfrac{1}{6}(5.31717808) \approx 0.8862$$

Now $x > 4 \Rightarrow -x\cdot x < -x\cdot 4 \Rightarrow e^{-x^2} < e^{-4x} \Rightarrow \int_4^\infty e^{-x^2}\,dx < \int_4^\infty e^{-4x}\,dx$.

$\int_4^\infty e^{-4x}\,dx = \lim_{t\to\infty}\left[-\frac{1}{4}e^{-4x}\right]_4^t = -\frac{1}{4}\left(0 - e^{-16}\right) = 1/(4e^{16}) \approx 0.0000000281 < 0.0000001$, as desired.

71. (a) $F(s) = \int_0^\infty f(t)e^{-st}\,dt = \int_0^\infty e^{-st}\,dt = \lim_{n\to\infty}\left[-\frac{e^{-st}}{s}\right]_0^n = \lim_{n\to\infty}\left(\frac{e^{-sn}}{-s} + \frac{1}{s}\right)$. This converges to $\frac{1}{s}$ only if $s > 0$. Therefore $F(s) = \frac{1}{s}$ with domain $\{s \mid s > 0\}$.

(b) $$F(s) = \int_0^\infty f(t)e^{-st}\,dt = \int_0^\infty e^t e^{-st}\,dt = \lim_{n\to\infty}\int_0^n e^{t(1-s)}\,dt = \lim_{n\to\infty}\left[\frac{1}{1-s}e^{t(1-s)}\right]_0^n$$
$$= \lim_{n\to\infty}\left(\frac{e^{(1-s)n}}{1-s} - \frac{1}{1-s}\right)$$

This converges only if $1 - s < 0 \;\Rightarrow\; s > 1$, in which case $F(s) = \frac{1}{s-1}$ with domain $\{s \mid s > 1\}$.

(c) $F(s) = \int_0^\infty f(t)e^{-st}\,dt = \lim_{n\to\infty}\int_0^n te^{-st}\,dt$. Use integration by parts: let $u = t$, $dv = e^{-st}\,dt \;\Rightarrow$ $du = dt$, $v = -\frac{e^{-st}}{s}$. Then $F(s) = \lim_{n\to\infty}\left[-\frac{t}{s}e^{-st} - \frac{1}{s^2}e^{-st}\right]_0^n = \lim_{n\to\infty}\left(\frac{-n}{se^{sn}} - \frac{1}{s^2e^{sn}} + 0 + \frac{1}{s^2}\right) = \frac{1}{s^2}$ only if $s > 0$. Therefore, $F(s) = \frac{1}{s^2}$ and the domain of F is $\{s \mid s > 0\}$.

72. $0 \le f(t) \le Me^{at} \;\Rightarrow\; 0 \le f(t)e^{-st} \le Me^{at}e^{-st}$ for $t \ge 0$. Now use the Comparison Theorem:

$$\int_0^\infty Me^{at}e^{-st}\,dt = \lim_{n\to\infty} M\int_0^n e^{t(a-s)}\,dt = M\cdot\lim_{n\to\infty}\left[\frac{1}{a-s}e^{t(a-s)}\right]_0^n = M\cdot\lim_{n\to\infty}\frac{1}{a-s}\left[e^{n(a-s)} - 1\right]$$

This is convergent only when $a - s < 0 \;\Rightarrow\; s > a$. Therefore, by the Comparison Theorem, $F(s) = \int_0^\infty f(t)\,e^{-st}\,dt$ is also convergent for $s > a$.

73. $G(s) = \int_0^\infty f'(t)e^{-st}\,dt$. Integrate by parts with $u = e^{-st}$, $dv = f'(t)\,dt \;\Rightarrow\; du = -se^{-st}$, $v = f(t)$:

$$G(s) = \lim_{n\to\infty}\left[f(t)e^{-st}\right]_0^n + s\int_0^\infty f(t)e^{-st}\,dt = \lim_{n\to\infty} f(n)e^{-sn} - f(0) + sF(s)$$

But $0 \le f(t) \le Me^{at} \;\Rightarrow\; 0 \le f(t)e^{-st} \le Me^{at}e^{-st}$ and $\lim_{t\to\infty} Me^{t(a-s)} = 0$ for $s > a$. So by the Squeeze Theorem, $\lim_{t\to\infty} f(t)e^{-st} = 0$ for $s > a \;\Rightarrow\; G(s) = 0 - f(0) + sF(s) = sF(s) - f(0)$ for $s > a$.

74. Assume without loss of generality that $a < b$. Then

$$\begin{aligned}
\int_{-\infty}^a f(x)\,dx + \int_a^\infty f(x)\,dx &= \lim_{t\to-\infty}\int_t^a f(x)\,dx + \lim_{u\to\infty}\int_a^u f(x)\,dx \\
&= \lim_{t\to-\infty}\int_t^a f(x)\,dx + \lim_{u\to\infty}\left[\int_a^b f(x)\,dx + \int_b^u f(x)\,dx\right] \\
&= \lim_{t\to-\infty}\int_t^a f(x)\,dx + \int_a^b f(x)\,dx + \lim_{u\to\infty}\int_b^u f(x)\,dx \\
&= \lim_{t\to-\infty}\left[\int_t^a f(x)\,dx + \int_a^b f(x)\,dx\right] + \int_b^\infty f(x)\,dx \\
&= \lim_{t\to-\infty}\int_t^b f(x)\,dx + \int_b^\infty f(x)\,dx \\
&= \int_{-\infty}^b f(x)\,dx + \int_b^\infty f(x)\,dx
\end{aligned}$$

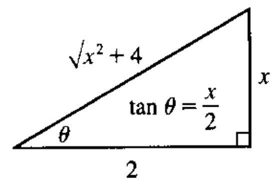
75. We use integration by parts: let $u = x$, $dv = xe^{-x^2}\,dx \quad\Rightarrow\quad du = dx$, $v = -\frac{1}{2}e^{-x^2}$. So

$$\int_0^\infty x^2e^{-x^2}\,dx = \lim_{t\to\infty}\left[-\tfrac{1}{2}xe^{-x^2}\right]_0^t + \tfrac{1}{2}\int_0^\infty e^{-x^2}\,dx$$
$$= \lim_{t\to\infty}\left[-t\big/\!\left(2e^{t^2}\right)\right] + \tfrac{1}{2}\int_0^\infty e^{-x^2}\,dx = \tfrac{1}{2}\int_0^\infty e^{-x^2}\,dx$$

(The limit is 0 by l'Hospital's Rule.)

76. $\int_0^\infty e^{-x^2}\,dx$ is the area under the curve $y = e^{-x^2}$ for $0 \le x < \infty$ and $0 < y \le 1$. Solving $y = e^{-x^2}$ for x, we get $y = e^{-x^2} \quad\Rightarrow\quad \ln y = -x^2 \quad\Rightarrow\quad -\ln y = x^2 \quad\Rightarrow\quad x = \pm\sqrt{-\ln y}$. Since x is positive, choose $x = \sqrt{-\ln y}$, and the area is represented by $\int_0^1 \sqrt{-\ln y}\,dy$. Therefore, each integral represents the same area, so the integrals are equal.

77. For the first part of the integral, let $x = 2\tan\theta \quad\Rightarrow\quad dx = 2\sec^2\theta\,d\theta$.



$\displaystyle\int \frac{1}{\sqrt{x^2+4}}\,dx = \int \frac{2\sec^2\theta}{2\sec\theta}\,d\theta = \int \sec\theta\,d\theta = \ln|\sec\theta + \tan\theta|$. From the figure, $\tan\theta = \dfrac{x}{2}$, and $\sec\theta = \dfrac{\sqrt{x^2+4}}{2}$. So

$$I = \int_0^\infty\left(\frac{1}{\sqrt{x^2+4}} - \frac{C}{x+2}\right)dx = \lim_{t\to\infty}\left[\ln\left|\frac{\sqrt{x^2+4}}{2} + \frac{x}{2}\right| - C\ln|x+2|\right]_0^t$$
$$= \lim_{t\to\infty}\left[\ln\frac{\sqrt{t^2+4}+t}{2} - C\ln(t+2) - (\ln 1 - C\ln 2)\right]$$
$$= \lim_{t\to\infty}\left[\ln\left(\frac{\sqrt{t^2+4}+t}{2\,(t+2)^C}\right) + \ln 2^C\right]$$
$$= \ln\left(\lim_{t\to\infty}\frac{t+\sqrt{t^2+4}}{(t+2)^C}\right) + \ln 2^{C-1}$$

Now $L = \displaystyle\lim_{t\to\infty}\frac{t+\sqrt{t^2+4}}{(t+2)^C} \overset{\text{H}}{=} \lim_{t\to\infty}\frac{1+t/\sqrt{t^2+4}}{C\,(t+2)^{C-1}} = \frac{2}{C\lim\limits_{t\to\infty}(t+2)^{C-1}}$.

If $C < 1$, $L = \infty$ and I diverges. If $C = 1$, $L = 2$ and I converges to $\ln 2 + \ln 2^0 = \ln 2$. If $C > 1$, $L = 0$ and I diverges to $-\infty$.

78. $$I = \int_0^\infty\left(\frac{x}{x^2+1} - \frac{C}{3x+1}\right)dx = \lim_{t\to\infty}\left[\tfrac{1}{2}\ln(x^2+1) - \tfrac{1}{3}C\ln(3x+1)\right]_0^t$$
$$= \lim_{t\to\infty}\left[\ln(t^2+1)^{1/2} - \ln(3t+1)^{C/3}\right]$$
$$= \lim_{t\to\infty}\left(\ln\frac{(t^2+1)^{1/2}}{(3t+1)^{C/3}}\right) = \ln\left(\lim_{t\to\infty}\frac{\sqrt{t^2+1}}{(3t+1)^{C/3}}\right)$$

For $C \le 0$, the integral diverges. For $C > 0$, we have

$$L = \lim_{t\to\infty}\frac{\sqrt{t^2+1}}{(3t+1)^{C/3}} \overset{\text{H}}{=} \lim_{t\to\infty}\frac{t\big/\sqrt{t^2+1}}{C\,(3t+1)^{(C/3)-1}} = \frac{1}{C}\lim_{t\to\infty}\frac{1}{(3t+1)^{(C/3)-1}}.$$

For $C/3 < 1 \quad\Leftrightarrow\quad C < 3$, $L = \infty$ and I diverges. For $C = 3$, $L = \frac{1}{3}$ and $I = \ln\frac{1}{3}$. For $C > 3$, $L = 0$ and I diverges to $-\infty$.

8 Review

CONCEPT CHECK

1. See Formula 8.1.1 or 8.1.2. We try to choose $u = f(x)$ to be a function that becomes simpler when differentiated (or at least not more complicated) as long as $dv = g'(x)\,dx$ can be readily integrated to give v.
2. See the Strategy for Evaluating $\int \sin^m x \cos^n x\,dx$ on page 520.
3. If $\sqrt{a^2 - x^2}$ occurs, try $x = a\sin\theta$; if $\sqrt{a^2 + x^2}$ occurs, try $x = a\tan\theta$, and if $\sqrt{x^2 - a^2}$ occurs, try $x = a\sec\theta$. See the Table of Trigonometric Substitutions on page 526.
4. See Equation 2 and Expressions 7, 9, and 11 in Section 8.4.
5. See the Midpoint Rule, the Trapezoidal Rule, and Simpson's Rule, as well as their associated error bounds, all in Section 8.7. We would expect the best estimate to be given by Simpson's Rule.
6. See Definitions 1(a), (b), and (c) in Section 8.8.
7. See Definitions 3(b), (a), and (c) in Section 8.8.
8. See the Comparison Theorem after Example 8 in Section 8.8.

TRUE-FALSE QUIZ

1. False. Since the numerator has a higher degree than the denominator,

$$\frac{x(x^2+4)}{x^2-4} = x + \frac{8x}{x^2-4} = x + \frac{A}{x+2} + \frac{B}{x-2}.$$

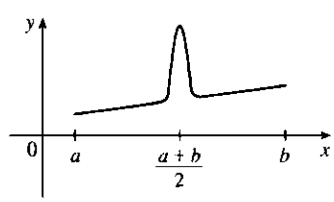
2. True. In fact, $A = -1$, $B = C = 1$.
3. False. It can be put in the form $\dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x-4}$.
4. False. The form is $\dfrac{A}{x} + \dfrac{Bx+C}{x^2+4}$.
5. False. This is an improper integral, since the denominator vanishes at $x = 1$.

$$\int_0^4 \frac{x}{x^2-1}\,dx = \int_0^1 \frac{x}{x^2-1}\,dx + \int_1^4 \frac{x}{x^2-1}\,dx \text{ and}$$

$$\int_0^1 \frac{x}{x^2-1}\,dx = \lim_{t\to 1^-}\int_0^t \frac{x}{x^2-1}\,dx = \lim_{t\to 1^-}\left[\tfrac{1}{2}\ln\left|x^2-1\right|\right]_0^t = \lim_{t\to 1^-}\tfrac{1}{2}\ln\left|t^2-1\right| = \infty$$

So the integral diverges.

6. True by Theorem 8.8.2 with $p = \sqrt{2} > 1$.
7. False. See Exercise 61 in Section 8.8.
8. False. For example, with $n = 1$ the Trapezoidal Rule is much more accurate than the Midpoint Rule for the function in the diagram.



9. (a) True. See the end of Section 8.5.

(b) False. Examples include the functions $f(x) = e^{x^2}$, $g(x) = \sin(x^2)$, and $h(x) = \dfrac{\sin x}{x}$.

10. True. If f is continuous on $[0, \infty)$, then $\int_0^1 f(x)\,dx$ is finite. Since $\int_1^\infty f(x)\,dx$ is finite, so is $\int_0^\infty f(x)\,dx = \int_0^1 f(x)\,dx + \int_1^\infty f(x)\,dx$.

11. False. If $f(x) = 1/x$, then f is continuous and decreasing on $[1, \infty)$ with $\lim\limits_{x\to\infty} f(x) = 0$, but $\int_1^\infty f(x)\,dx$ is divergent.

12. True.

$$\int_a^\infty [f(x) + g(x)]\,dx = \lim_{t\to\infty} \int_a^t [f(x) + g(x)]\,dx = \lim_{t\to\infty} \left(\int_a^t f(x)\,dx + \int_a^t g(x)\,dx\right)$$
$$= \lim_{t\to\infty} \int_a^t f(x)\,dx + \lim_{t\to\infty} \int_a^t g(x)\,dx \quad \left[\begin{matrix}\text{since both limits}\\ \text{in the sum exist}\end{matrix}\right]$$
$$= \int_a^\infty f(x)\,dx + \int_a^\infty g(x)\,dx$$

Since the two integrals are finite, so is their sum.

13. False. Take $f(x) = 1$ for all x and $g(x) = -1$ for all x. Then $\int_a^\infty f(x)\,dx = \infty$ [divergent] and $\int_a^\infty g(x)\,dx = -\infty$ [divergent], but $\int_a^\infty [f(x) + g(x)]\,dx = 0$ [convergent].

14. False. $\int_0^\infty f(x)\,dx$ could converge or diverge. For example, if $g(x) = 1$, then $\int_0^\infty f(x)\,dx$ diverges if $f(x) = 1$ and converges if $f(x) = 0$.

EXERCISES

1. $$\int_0^5 \frac{x}{x+10}\,dx = \int_0^5 \left(1 - \frac{10}{x+10}\right) dx = \Big[x - 10\ln(x+10)\Big]_0^5$$
$$= 5 - 10\ln 15 + 10\ln 10 = 5 + 10\ln\tfrac{10}{15} = 5 + 10\ln\tfrac{2}{3}$$

2. $$\int_0^5 ye^{-0.6y}\,dy \quad \left[\begin{matrix} u = y, & dv = e^{-0.6y}\,dy, \\ du = dy & v = -\frac{5}{3}e^{-0.6y}\end{matrix}\right] = \left[-\tfrac{5}{3}ye^{-0.6y}\right]_0^5 - \int_0^5 \left(-\tfrac{5}{3}e^{-0.6y}\right) dy$$
$$= -\tfrac{25}{3}e^{-3} - \tfrac{25}{9}\left[e^{-0.6y}\right]_0^5 = -\tfrac{25}{3}e^{-3} - \tfrac{25}{9}(e^{-3} - 1)$$
$$= -\tfrac{25}{3}e^{-3} - \tfrac{25}{9}e^{-3} + \tfrac{25}{9} = \tfrac{25}{9} - \tfrac{100}{9}e^{-3}$$

3. $$\int_0^{\pi/2} \frac{\cos\theta}{1+\sin\theta}\,d\theta = \Big[\ln(1+\sin\theta)\Big]_0^{\pi/2} = \ln 2 - \ln 1 = \ln 2$$

4. $$\int_1^4 \frac{dt}{(2t+1)^3} \quad \left[\begin{matrix} u = 2t+1, \\ du = 2\,dt\end{matrix}\right] = \int_3^9 \frac{\frac{1}{2}\,du}{u^3} = \frac{-1}{4}\left[\frac{1}{u^2}\right]_3^9 = -\frac{1}{4}\left(\frac{1}{81} - \frac{1}{9}\right) = -\frac{1}{4}\left(-\frac{8}{81}\right) = \frac{2}{81}$$

5. Let $u = \sec x$. Then $du = \sec x\tan x\,dx$, so

$$\int \tan^7 x\sec^3 x\,dx = \int \tan^6 x\sec^2 x\sec x\tan x\,dx = \int \left(u^2 - 1\right)^3 u^2\,du = \int \left(u^8 - 3u^6 + 3u^4 - u^2\right) du$$
$$= \tfrac{1}{9}u^9 - \tfrac{3}{7}u^7 + \tfrac{3}{5}u^5 - \tfrac{1}{3}u^3 + C = \tfrac{1}{9}\sec^9 x - \tfrac{3}{7}\sec^7 x + \tfrac{3}{5}\sec^5 x - \tfrac{1}{3}\sec^3 x + C$$

6. $\dfrac{1}{y^2 - 4y - 12} = \dfrac{1}{(y-6)(y+2)} = \dfrac{A}{y-6} + \dfrac{B}{y+2} \;\Rightarrow\; 1 = A(y+2) + B(y-6)$. Letting $y = -2 \;\Rightarrow\; B = -\frac{1}{8}$ and letting $y = 6 \;\Rightarrow\; A = \frac{1}{8}$. So

$$\int \frac{1}{y^2 - 4y - 12}\,dy = \int \left(\frac{1/8}{y-6} + \frac{-1/8}{y+2}\right) dy = \tfrac{1}{8}\ln|y-6| - \tfrac{1}{8}\ln|y+2| + C$$

7. Let $u = \ln t$, $du = dt/t$. Then $\int \frac{\sin(\ln t)}{t}\,dt = \int \sin u\,du = -\cos u + C = -\cos(\ln t) + C$.

8. Let $x = \tan\theta$, $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. Then

$$\int \frac{dx}{x^2\sqrt{1+x^2}} = \int \frac{\sec^2\theta\,d\theta}{\tan^2\theta\sec\theta} = \int \frac{\sec\theta\,d\theta}{\tan^2\theta} = \int \frac{\cos\theta\,d\theta}{\sin^2\theta} = \int \frac{du}{u^2} \quad [\text{put } u = \sin\theta]$$

$$= -\frac{1}{u} + C = -\frac{1}{\sin\theta} + C = -\frac{\sqrt{1+x^2}}{x} + C$$

9. $$\int_1^4 x^{3/2}\ln x\,dx \quad \begin{bmatrix} u = \ln x, & dv = x^{3/2}\,dx, \\ du = dx/x & v = \frac{2}{5}x^{5/2} \end{bmatrix} = \frac{2}{5}\left[x^{5/2}\ln x\right]_1^4 - \frac{2}{5}\int_1^4 x^{3/2}\,dx$$

$$= \tfrac{2}{5}(32\ln 4 - \ln 1) - \tfrac{2}{5}\left[\tfrac{2}{5}x^{5/2}\right]_1^4$$

$$= \tfrac{2}{5}(64\ln 2) - \tfrac{4}{25}(32-1)$$

$$= \tfrac{128}{5}\ln 2 - \tfrac{124}{25} \quad \left(\text{or } \tfrac{64}{5}\ln 4 - \tfrac{124}{25}\right)$$

10. Let $u = \arctan x$, $du = dx/(1+x^2)$. Then

$$\int_0^1 \frac{\sqrt{\arctan x}}{1+x^2}\,dx = \int_0^{\pi/4}\sqrt{u}\,du = \frac{2}{3}\left[u^{3/2}\right]_0^{\pi/4} = \frac{2}{3}\left[\frac{\pi^{3/2}}{4^{3/2}} - 0\right] = \frac{2}{3}\cdot\frac{1}{8}\pi^{3/2} = \frac{1}{12}\pi^{3/2}.$$

11. Let $x = \sec\theta$. Then

$$\int_1^2 \frac{\sqrt{x^2-1}}{x}\,dx = \int_0^{\pi/3}\frac{\tan\theta}{\sec\theta}\sec\theta\tan\theta\,d\theta = \int_0^{\pi/3}\tan^2\theta\,d\theta = \int_0^{\pi/3}(\sec^2\theta - 1)\,d\theta$$

$$= \left[\tan\theta - \theta\right]_0^{\pi/3} = \sqrt{3} - \tfrac{\pi}{3}$$

12. $\int_{-1}^1 \frac{\sin x}{1+x^2}\,dx = 0$ by Theorem 5.5.6(b), since $f(x) = \frac{\sin x}{1+x^2}$ is an odd function.

13. $\int \frac{dx}{x^3+x} = \int\left(\frac{1}{x} - \frac{x}{x^2+1}\right)dx = \ln|x| - \frac{1}{2}\ln(x^2+1) + C$

14. $\int \frac{x^2+2}{x+2}\,dx = \int\left(x - 2 + \frac{6}{x+2}\right)dx = \frac{1}{2}x^2 - 2x + 6\ln|x+2| + C$

15. $$\int \sin^2\theta\cos^5\theta\,d\theta = \int \sin^2\theta\left(\cos^2\theta\right)^2\cos\theta\,d\theta = \int \sin^2\theta\left(1-\sin^2\theta\right)^2\cos\theta\,d\theta$$

$$= \int u^2\left(1-u^2\right)^2du \quad [u = \sin\theta,\ du = \cos\theta\,d\theta] \quad = \int u^2\left(1 - 2u^2 + u^4\right)du$$

$$= \int\left(u^2 - 2u^4 + u^6\right)du = \tfrac{1}{3}u^3 - \tfrac{2}{5}u^5 + \tfrac{1}{7}u^7 + C = \tfrac{1}{3}\sin^3\theta - \tfrac{2}{5}\sin^5\theta + \tfrac{1}{7}\sin^7\theta + C$$

16. $$\int \frac{\sec^6\theta}{\tan^2\theta}\,d\theta = \int \frac{(\tan^2\theta+1)^2\sec^2\theta}{\tan^2\theta}\,d\theta \quad \begin{bmatrix} u = \tan\theta, \\ du = \sec^2\theta\,d\theta \end{bmatrix}$$

$$= \int \frac{(u^2+1)^2}{u^2}\,du = \int \frac{u^4+2u^2+1}{u^2}\,du = \int\left(u^2 + 2 + \frac{1}{u^2}\right)du$$

$$= \frac{u^3}{3} + 2u - \frac{1}{u} + C = \frac{1}{3}\tan^3\theta + 2\tan\theta - \cot\theta + C$$

17. Integrate by parts with $u = x$, $dv = \sec x \tan x\, dx \quad \Rightarrow \quad du = dx$, $v = \sec x$:

$\int x \sec x \, \tan x\, dx = x \sec x - \int \sec x\, dx \overset{14}{=} x \sec x - \ln|\sec x + \tan x| + C.$

18. $\dfrac{x^2+8x-3}{x^3+3x^2} = \dfrac{x^2+8x-3}{x^2(x+3)} = \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x+3} \quad \Rightarrow \quad x^2+8x-3 = Ax(x+3) + B(x+3) + Cx^2.$

Taking $x = 0$, we get $-3 = 3B$, so $B = -1$. Taking $x = -3$, we get $-18 = 9C$, so $C = -2$. Taking $x = 1$, we get $6 = 4A + 4B + C = 4A - 4 - 2$, so $4A = 12$ and $A = 3$. Now

$$\int \frac{x^2+8x-3}{x^3+3x^2}\,dx = \int \left(\frac{3}{x} - \frac{1}{x^2} - \frac{2}{x+3}\right) dx = 3\ln|x| + \frac{1}{x} - 2\ln|x+3| + C$$

19. $$\int \frac{x+1}{9x^2+6x+5}\,dx = \int \frac{x+1}{(9x^2+6x+1)+4}\,dx = \int \frac{x+1}{(3x+1)^2+4}\,dx \quad \begin{bmatrix} u = 3x+1, \\ du = 3\,dx \end{bmatrix}$$

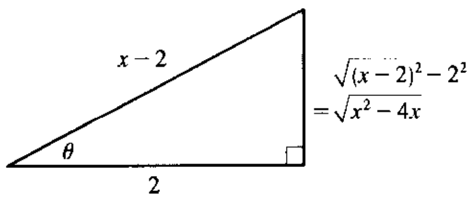
$$= \int \frac{\left[\frac{1}{3}(u-1)\right]+1}{u^2+4}\left(\frac{1}{3}\,du\right) = \frac{1}{3}\cdot\frac{1}{3}\int \frac{(u-1)+3}{u^2+4}\,du$$

$$= \frac{1}{9}\int \frac{u}{u^2+4}\,du + \frac{1}{9}\int \frac{2}{u^2+2^2}\,du = \frac{1}{9}\cdot\frac{1}{2}\ln(u^2+4) + \frac{2}{9}\cdot\frac{1}{2}\tan^{-1}\left(\frac{1}{2}u\right) + C$$

$$= \tfrac{1}{18}\ln(9x^2+6x+5) + \tfrac{1}{9}\tan^{-1}\left[\tfrac{1}{2}(3x+1)\right] + C$$

20. $$\int \frac{dt}{\sin^2 t + \cos 2t} = \int \frac{dt}{\sin^2 t + (\cos^2 t - \sin^2 t)} = \int \frac{dt}{\cos^2 t} = \int \sec^2 t\,dt = \tan t + C$$

21.



$$\int \frac{dx}{\sqrt{x^2-4x}} = \int \frac{dx}{\sqrt{(x^2-4x+4)-4}} = \int \frac{dx}{\sqrt{(x-2)^2-2^2}}$$

$$= \int \frac{2\sec\theta\tan\theta\,d\theta}{2\tan\theta} \quad \begin{bmatrix} x-2 = 2\sec\theta, \\ dx = 2\sec\theta\tan\theta\,d\theta \end{bmatrix} = \int \sec\theta\,d\theta = \ln|\sec\theta + \tan\theta| + C_1$$

$$= \ln\left|\frac{x-2}{2} + \frac{\sqrt{x^2-4x}}{2}\right| + C_1 = \ln\left|x - 2 + \sqrt{x^2-4x}\right| + C, \text{ where } C = C_1 - \ln 2$$

22. Let $u = x + 1$. Then

$$\int \frac{x^3}{(x+1)^{10}}\,dx = \int \frac{(u-1)^3}{u^{10}}\,dx = \int \frac{u^3-3u^2+3u-1}{u^{10}}\,du$$

$$= \int \left(u^{-7} - 3u^{-8} + 3u^{-9} - u^{-10}\right) du = -\tfrac{1}{6}u^{-6} + \tfrac{3}{7}u^{-7} - \tfrac{3}{8}u^{-8} + \tfrac{1}{9}u^{-9} + C$$

$$= \frac{-1}{6(x+1)^6} + \frac{3}{7(x+1)^7} - \frac{3}{8(x+1)^8} + \frac{1}{9(x+1)^9} + C$$

23. Let $u = \cot 4x$. Then $du = -4\csc^2 4x\,dx \Rightarrow$

$$\int \csc^4 4x\,dx = \int (\cot^2 4x + 1)\csc^2 4x\,dx = \int (u^2+1)\left(-\tfrac{1}{4}\,du\right)$$
$$= -\tfrac{1}{4}\left(\tfrac{1}{3}u^3 + u\right) + C = -\tfrac{1}{12}\left(\cot^3 4x + 3\cot 4x\right) + C$$

24. Let $u = \cos x$, $dv = e^x\,dx \Rightarrow du = -\sin x\,dx$, $v = e^x$: $(*)\ I = \int e^x \cos x\,dx = e^x\cos x + \int e^x \sin x\,dx$.

To integrate $\int e^x \sin x\,dx$, let $U = \sin x$, $dV = e^x\,dx \Rightarrow dU = \cos x\,dx$, $V = e^x$. Then

$\int e^x \sin x\,dx = e^x \sin x - \int e^x \cos x\,dx = e^x \sin x - I$. By substitution in $(*)$, $I = e^x\cos x + e^x \sin x - I \Rightarrow$

$2I = e^x(\cos x + \sin x) \Rightarrow I = \tfrac{1}{2}e^x(\cos x + \sin x) + C$.

25. $\dfrac{3x^3 - x^2 + 6x - 4}{(x^2+1)(x^2+2)} = \dfrac{Ax+B}{x^2+1} + \dfrac{Cx+D}{x^2+2} \Rightarrow$

$3x^3 - x^2 + 6x - 4 = (Ax+B)(x^2+2) + (Cx+D)(x^2+1)$. Equating the coefficients gives $A + C = 3$,

$B + D = -1$, $2A + C = 6$, and $2B + D = -4 \Rightarrow A = 3$, $C = 0$, $B = -3$, and $D = 2$. Now

$$\int \frac{3x^3 - x^2 + 6x - 4}{(x^2+1)(x^2+2)}\,dx = 3\int \frac{x-1}{x^2+1}\,dx + 2\int \frac{dx}{x^2+2}$$
$$= \tfrac{3}{2}\ln(x^2+1) - 3\tan^{-1}x + \sqrt{2}\tan^{-1}\left(\tfrac{1}{\sqrt{2}}x\right) + C$$

26. Let $u = e^x$. Then $x = \ln u$, $dx = \dfrac{du}{u}$, so

$$\int \frac{dx}{1+e^x} = \int \frac{du/u}{1+u} = \int \left[\frac{1}{u} - \frac{1}{u+1}\right] du = \ln u - \ln(u+1) + C = \ln e^x - \ln(1+e^x) + C$$
$$= x - \ln(1+e^x) + C$$

27. $\int_0^{\pi/2} \cos^3 x \sin 2x\,dx = \int_0^{\pi/2} \cos^3 x\,(2\sin x\cos x)\,dx = \int_0^{\pi/2} 2\cos^4 x \sin x\,dx = \left[-\tfrac{2}{5}\cos^5 x\right]_0^{\pi/2} = \tfrac{2}{5}$

28. Let $u = \sqrt[3]{x}$. Then $x = u^3$, $dx = 3u^2 du \Rightarrow$

$$\int \frac{\sqrt[3]{x}+1}{\sqrt[3]{x}-1}\,dx = \int \frac{u+1}{u-1}\,3u^2 du = 3\int \left(u^2 + 2u + 2 + \frac{2}{u-1}\right) du$$
$$= u^3 + 3u^2 + 6u + 6\ln|u-1| + C = x + 3x^{2/3} + 6\sqrt[3]{x} + 6\ln\left|\sqrt[3]{x} - 1\right| + C$$

29. The product of an odd function and an even function is an odd function, so $f(x) = x^5 \sec x$ is an odd function.

By Theorem 5.5.6(b), $\int_{-1}^{1} x^5 \sec x\,dx = 0$.

30. Let $u = e^{-x}$, $du = -e^{-x}\,dx$. Then

$$\int \frac{dx}{e^x\sqrt{1-e^{-2x}}} = \int \frac{e^{-x}\,dx}{\sqrt{1-(e^{-x})^2}} = \int \frac{-du}{\sqrt{1-u^2}} = -\sin^{-1}u + C = -\sin^{-1}(e^{-x}) + C.$$

31. Let $u = \sqrt{e^x - 1}$. Then $u^2 = e^x - 1$ and $2u\,du = e^x\,dx$. Also, $e^x + 8 = u^2 + 9$. Thus,

$$\int_0^{\ln 10} \frac{e^x\sqrt{e^x - 1}}{e^x + 8}\,dx = \int_0^3 \frac{u \cdot 2u\,du}{u^2 + 9} = 2\int_0^3 \frac{u^2}{u^2 + 9}\,du = 2\int_0^3 \left(1 - \frac{9}{u^2 + 9}\right) du$$

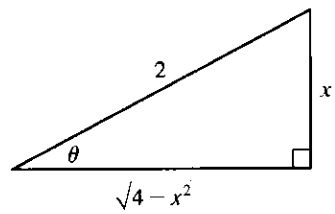
$$= 2\left[u - \frac{9}{3}\tan^{-1}\left(\frac{u}{3}\right)\right]_0^3 = 2\left[(3 - 3\tan^{-1} 1) - 0\right] = 2\left(3 - 3 \cdot \frac{\pi}{4}\right) = 6 - \frac{3\pi}{2}$$

32. $$\int_0^{\pi/4} \frac{x \sin x}{\cos^3 x}\,dx = \int_0^{\pi/4} x \tan x \sec^2 x\,dx \quad \begin{bmatrix} u = x, & dv = \tan x \sec^2 x\,dx, \\ du = dx & v = \frac{1}{2}\tan^2 x \end{bmatrix}$$

$$= \left[\frac{x}{2}\tan^2 x\right]_0^{\pi/4} - \frac{1}{2}\int_0^{\pi/4} \tan^2 x\,dx = \frac{\pi}{8} \cdot 1^2 - 0 - \frac{1}{2}\int_0^{\pi/4} (\sec^2 x - 1)\,dx$$

$$= \frac{\pi}{8} - \frac{1}{2}\Big[\tan x - x\Big]_0^{\pi/4} = \frac{\pi}{8} - \frac{1}{2}\left(1 - \frac{\pi}{4}\right) = \frac{\pi}{4} - \frac{1}{2}$$

33. Let $x = 2\sin\theta \quad \Rightarrow \quad (4 - x^2)^{3/2} = (2\cos\theta)^3$, $dx = 2\cos\theta\,d\theta$, so



$$\int \frac{x^2}{(4 - x^2)^{3/2}}\,dx = \int \frac{4\sin^2\theta}{8\cos^3\theta}\,2\cos\theta\,d\theta$$

$$= \int \tan^2\theta\,d\theta = \int (\sec^2\theta - 1)\,d\theta$$

$$= \tan\theta - \theta + C = \frac{x}{\sqrt{4 - x^2}} - \sin^{-1}\left(\frac{x}{2}\right) + C$$

34. Integrate by parts twice, first with $u = (\arcsin x)^2$, $dv = dx$:

$$I = \int (\arcsin x)^2\,dx = x(\arcsin x)^2 - \int 2x \arcsin x \left(\frac{dx}{\sqrt{1 - x^2}}\right)$$

Now let $U = \arcsin x$, $dV = \dfrac{x}{\sqrt{1 - x^2}}\,dx \quad \Rightarrow \quad dU = \dfrac{1}{\sqrt{1 - x^2}}\,dx$, $V = -\sqrt{1 - x^2}$. So

$$I = x(\arcsin x)^2 - 2\left[\arcsin x\left(-\sqrt{1 - x^2}\right) + \textstyle\int dx\right] = x(\arcsin x)^2 + 2\sqrt{1 - x^2}\arcsin x - 2x + C$$

35. $$\int \frac{1}{\sqrt{x + x^{3/2}}}\,dx = \int \frac{dx}{\sqrt{x\,(1 + \sqrt{x})}} = \int \frac{dx}{\sqrt{x}\sqrt{1 + \sqrt{x}}} \quad \begin{bmatrix} u = 1 + \sqrt{x}, \\ du = \frac{dx}{2\sqrt{x}} \end{bmatrix} \quad = \int \frac{2\,du}{\sqrt{u}} = \int 2u^{-1/2}\,du$$

$$= 4\sqrt{u} + C = 4\sqrt{1 + \sqrt{x}} + C$$

36. $$\int \frac{1 - \tan\theta}{1 + \tan\theta}\,d\theta = \int \frac{\frac{\cos\theta}{\cos\theta} - \frac{\sin\theta}{\cos\theta}}{\frac{\cos\theta}{\cos\theta} + \frac{\sin\theta}{\cos\theta}}\,d\theta = \int \frac{\cos\theta - \sin\theta}{\cos\theta + \sin\theta}\,d\theta = \ln|\cos\theta + \sin\theta| + C$$

37. $\int (\cos x + \sin x)^2 \cos 2x\,dx = \int \left(\cos^2 x + 2\sin x\cos x + \sin^2 x\right)\cos 2x\,dx$

$= \int (1 + \sin 2x)\cos 2x\,dx = \int \cos 2x\,dx + \frac{1}{2}\int \sin 4x\,dx = \frac{1}{2}\sin 2x - \frac{1}{8}\cos 4x + C$

Or: $\int (\cos x + \sin x)^2 \cos 2x\,dx = \int (\cos x + \sin x)^2\left(\cos^2 x - \sin^2 x\right)dx$

$= \int (\cos x + \sin x)^3(\cos x - \sin x)\,dx = \frac{1}{4}(\cos x + \sin x)^4 + C_1$

38. Let $u = (\tan^{-1} x)^2$, $dv = x\,dx \Rightarrow du = 2(\tan^{-1} x)/(1 + x^2)\,dx$, $v = \frac{1}{2}x^2$. Then

$$I = \int x(\tan^{-1} x)^2\,dx = \tfrac{1}{2}x^2(\tan^{-1} x)^2 - \int \frac{x^2 \tan^{-1} x}{1 + x^2}\,dx$$

Now let $w = \tan^{-1} x$, $dw = 1/(1 + x^2)\,dx$, and $x^2 = \tan^2 w$. So

$$\begin{aligned} I &= \tfrac{1}{2}x^2(\tan^{-1} x)^2 - \int w \tan^2 w\,dw = \tfrac{1}{2}x^2(\tan^{-1} x)^2 - \int w \sec^2 w\,dw + \int w\,dw \\ &= \tfrac{1}{2}x^2(\tan^{-1} x)^2 - \left(x \tan^{-1} x - \ln\sqrt{x^2+1}\right) + \tfrac{1}{2}\left(\tan^{-1} x\right)^2 \quad [\text{parts with } u = w,\ dv = \sec^2 w\,dw] \\ &= \tfrac{1}{2}(x^2+1)(\tan^{-1} x)^2 - x\tan^{-1} x + \ln\sqrt{x^2+1} + C \\ &\quad \text{or } \tfrac{1}{2}(x^2+1)(\tan^{-1} x)^2 - x\tan^{-1} x + \tfrac{1}{2}\ln(x^2+1) + C \end{aligned}$$

39. We'll integrate $I = \int \frac{xe^{2x}}{(1+2x)^2}\,dx$ by parts with $u = xe^{2x}$ and $dv = \frac{dx}{(1+2x)^2}$. Then

$du = (x \cdot 2e^{2x} + e^{2x} \cdot 1)\,dx$ and $v = -\frac{1}{2} \cdot \frac{1}{1+2x}$, so

$$\begin{aligned} I &= -\frac{1}{2} \cdot \frac{xe^{2x}}{1+2x} - \int \left[-\frac{1}{2} \cdot \frac{e^{2x}(2x+1)}{1+2x}\right] dx = -\frac{xe^{2x}}{4x+2} + \frac{1}{2} \cdot \frac{1}{2}e^{2x} + C \\ &= e^{2x}\left(\frac{1}{4} - \frac{x}{4x+2}\right) + C. \end{aligned}$$

Thus, $\int_0^{1/2} \frac{xe^{2x}}{(1+2x)^2}\,dx = \left[e^{2x}\left(\frac{1}{4} - \frac{x}{4x+2}\right)\right]_0^{1/2} = e\left(\frac{1}{4} - \frac{1}{8}\right) - 1\left(\frac{1}{4} - 0\right) = \frac{1}{8}e - \frac{1}{4}$.

40. $$\begin{aligned} \int_{\pi/4}^{\pi/3} \frac{\sqrt{\tan\theta}}{\sin 2\theta}\,d\theta &= \int_{\pi/4}^{\pi/3} \frac{\sqrt{\frac{\sin\theta}{\cos\theta}}}{2\sin\theta\cos\theta}\,d\theta = \int_{\pi/4}^{\pi/3} \frac{1}{2}(\sin\theta)^{-1/2}(\cos\theta)^{-3/2}d\theta \\ &= \int_{\pi/4}^{\pi/3} \frac{1}{2}\left(\frac{\sin\theta}{\cos\theta}\right)^{-1/2}(\cos\theta)^{-2}d\theta = \int_{\pi/4}^{\pi/3} \frac{1}{2}(\tan\theta)^{-1/2}\sec^2\theta\,d\theta \\ &= \left[\sqrt{\tan\theta}\right]_{\pi/4}^{\pi/3} = \sqrt{\sqrt{3}} - \sqrt{1} = \sqrt[4]{3} - 1 \end{aligned}$$

41. $$\begin{aligned} \int_1^\infty \frac{1}{(2x+1)^3}\,dx &= \lim_{t\to\infty} \int_1^t \frac{1}{(2x+1)^3}\,dx = \lim_{t\to\infty} \int_1^t \tfrac{1}{2}(2x+1)^{-3}\,2\,dx \\ &= \lim_{t\to\infty}\left[-\frac{1}{4(2x+1)^2}\right]_1^t = -\frac{1}{4}\lim_{t\to\infty}\left[\frac{1}{(2t+1)^2} - \frac{1}{9}\right] = -\frac{1}{4}\left(0 - \frac{1}{9}\right) = \frac{1}{36} \end{aligned}$$

42. $\frac{t^2+1}{t^2-1} = \frac{(t^2-1)+2}{t^2-1} = 1 + \frac{2}{(t+1)(t-1)}$. Now $\frac{2}{(t+1)(t-1)} = \frac{A}{t+1} + \frac{B}{t-1} \Rightarrow$
$2 = A(t-1) + B(t+1)$. Letting $t = 1 \Rightarrow B = 1$ and letting $t = -1 \Rightarrow A = -1$. So

$$\begin{aligned} \int_0^1 \frac{t^2+1}{t^2-1}\,dt &= \lim_{b\to 1^-} \int_0^b \left(1 + \frac{-1}{t+1} + \frac{1}{t-1}\right) dt = \lim_{b\to 1^-}\left[t - \ln|t+1| + \ln|t-1|\right]_0^b \\ &= \lim_{b\to 1^-}\left[t + \ln\left|\frac{t-1}{t+1}\right|\right]_0^b = \lim_{b\to 1^-}\left[\left(b + \ln\left|\frac{b-1}{b+1}\right|\right) - (0+0)\right] = -\infty. \quad \text{Divergent} \end{aligned}$$

43. $\displaystyle\int \frac{dx}{x\ln x} \quad \left[\begin{matrix} u = \ln x, \\ du = \dfrac{dx}{x}\end{matrix}\right] = \int \frac{du}{u} = \ln|u| + C = \ln|\ln x| + C$, so

$\displaystyle\int_2^\infty \frac{dx}{x\ln x} = \lim_{t\to\infty}\int_2^t \frac{dx}{x\ln x} = \lim_{t\to\infty}\Big[\ln|\ln x|\Big]_2^t = \lim_{t\to\infty}[\ln(\ln t) - \ln(\ln 2)] = \infty$, so the integral is divergent.

44. Let $u = \sqrt{y-2}$. Then $y = u^2 + 2$ and $dy = 2u\,du$, so

$$\int \frac{y\,dy}{\sqrt{y-2}} = \int \frac{(u^2+2)2u\,du}{u} = 2\int (u^2+2)\,du = 2\left[\tfrac{1}{3}u^3 + 2u\right] + C$$

Thus, $\displaystyle\int_2^6 \frac{y\,dy}{\sqrt{y-2}} = \lim_{t\to 2^+}\int_t^6 \frac{y\,dy}{\sqrt{y-2}} = \lim_{t\to 2^+}\left[\tfrac{2}{3}(y-2)^{3/2} + 4\sqrt{y-2}\right]_t^6$

$$= \lim_{t\to 2^+}\left[\tfrac{16}{3} + 8 - \tfrac{2}{3}(t-2)^{3/2} - 4\sqrt{t-2}\right] = \tfrac{40}{3}.$$

45. $\displaystyle\int_0^4 \frac{\ln x}{\sqrt{x}}\,dx = \lim_{t\to 0^+}\int_t^4 \frac{\ln x}{\sqrt{x}}\,dx \overset{*}{=} \lim_{t\to 0^+}\left[2\sqrt{x}\,\ln x - 4\sqrt{x}\right]_t^4$

$$= \lim_{t\to 0^+}\left[(2\cdot 2\ln 4 - 4\cdot 2) - \left(2\sqrt{t}\,\ln t - 4\sqrt{t}\right)\right] \overset{**}{=} (4\ln 4 - 8) - (0 - 0) = 4\ln 4 - 8$$

(*) Let $u = \ln x$, $dv = \dfrac{1}{\sqrt{x}}\,dx \;\Rightarrow\; du = \dfrac{1}{x}\,dx$, $v = 2\sqrt{x}$. Then

$$\int \frac{\ln x}{\sqrt{x}}\,dx = 2\sqrt{x}\,\ln x - 2\int \frac{dx}{\sqrt{x}} = 2\sqrt{x}\,\ln x - 4\sqrt{x} + C$$

(**) $\displaystyle\lim_{t\to 0^+}\left(2\sqrt{t}\,\ln t\right) = \lim_{t\to 0^+}\frac{2\ln t}{t^{-1/2}} \overset{H}{=} \lim_{t\to 0^+}\frac{2/t}{-\frac{1}{2}t^{-3/2}} = \lim_{t\to 0^+}\left(-4\sqrt{t}\right) = 0$

46. Note that $f(x) = 1/(2-3x)$ has an infinite discontinuity at $x = \frac{2}{3}$. Now

$$\int_0^{2/3} \frac{1}{2-3x}\,dx = \lim_{t\to(2/3)^-}\int_0^t \frac{1}{2-3x}\,dx = \lim_{t\to(2/3)^-}\left[-\tfrac{1}{3}\ln|2-3x|\right]_0^t$$

$$= -\tfrac{1}{3}\lim_{t\to(2/3)^-}\Big[\ln|2-3t| - \ln 2\Big] = \infty$$

Since $\displaystyle\int_0^{2/3} \frac{1}{2-3x}\,dx$ diverges, so does $\displaystyle\int_0^1 \frac{1}{2-3x}\,dx$.

47. $\displaystyle\int_0^3 \frac{dx}{x^2-x-2} = \int_0^3 \frac{dx}{(x+1)(x-2)} = \int_0^2 \frac{dx}{(x+1)(x-2)} + \int_2^3 \frac{dx}{(x+1)(x-2)}$, and

$$\int_2^3 \frac{dx}{x^2-x-2} = \lim_{t\to 2^+}\int_t^3 \left[\frac{-1/3}{x+1} + \frac{1/3}{x-2}\right]dx = \lim_{t\to 2^+}\left[\frac{1}{3}\ln\left|\frac{x-2}{x+1}\right|\right]_t^3$$

$$= \lim_{t\to 2^+}\left[\tfrac{1}{3}\ln\tfrac{1}{4} - \frac{1}{3}\ln\left|\frac{t-2}{t+1}\right|\right] = \infty$$

so $\displaystyle\int_0^3 \frac{dx}{x^2-x-2}$ diverges.

48. $\int_{-1}^{1} \frac{x+1}{\sqrt[3]{x^4}}\,dx = \int_{-1}^{1} \left(x^{-1/3} + x^{-4/3}\right) dx = \int_{-1}^{0} \left(x^{-1/3} + x^{-4/3}\right) dx + \int_{0}^{1} \left(x^{-1/3} + x^{-4/3}\right) dx$. But

$$\int_0^1 \left(x^{-1/3} + x^{-4/3}\right) dx = \lim_{t\to 0^+} \int_t^1 \left(x^{-1/3} + x^{-4/3}\right) dx = \lim_{t\to 0^+} \left[\tfrac{3}{2}x^{2/3} - 3x^{-1/3}\right]_t^1$$
$$= \lim_{t\to 0^+} \left[\tfrac{3}{2} - 3 - \tfrac{3}{2}t^{2/3} + 3t^{-1/3}\right] = \infty. \quad \text{Divergent}$$

49. Let $u = 2x + 1$. Then

$$\int_{-\infty}^{\infty} \frac{dx}{4x^2 + 4x + 5} = \int_{-\infty}^{\infty} \frac{\frac{1}{2}\,du}{u^2 + 4} = \frac{1}{2}\int_{-\infty}^{0} \frac{du}{u^2+4} + \frac{1}{2}\int_0^{\infty} \frac{du}{u^2+4}$$
$$= \tfrac{1}{2}\lim_{t\to -\infty} \left[\tfrac{1}{2}\tan^{-1}\left(\tfrac{1}{2}u\right)\right]_t^0 + \tfrac{1}{2}\lim_{t\to\infty} \left[\tfrac{1}{2}\tan^{-1}\left(\tfrac{1}{2}u\right)\right]_0^t$$
$$= \tfrac{1}{4}\left[0 - \left(-\tfrac{\pi}{2}\right)\right] + \tfrac{1}{4}\left[\tfrac{\pi}{2} - 0\right] = \tfrac{\pi}{4}$$

50. $\int_1^{\infty} \frac{\tan^{-1} x}{x^2}\,dx = \lim_{t\to\infty} \int_1^t \frac{\tan^{-1} x}{x^2}\,dx$. Integrate by parts:

$$\int \frac{\tan^{-1} x}{x^2}\,dx = \frac{-\tan^{-1} x}{x} + \int \frac{1}{x}\frac{dx}{1+x^2} = \frac{-\tan^{-1} x}{x} + \int \left[\frac{1}{x} - \frac{x}{x^2+1}\right] dx$$
$$= \frac{-\tan^{-1} x}{x} + \ln|x| - \tfrac{1}{2}\ln\left(x^2+1\right) + C = \frac{-\tan^{-1} x}{x} + \frac{1}{2}\ln\frac{x^2}{x^2+1} + C$$

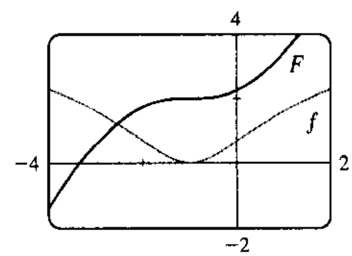
Thus,

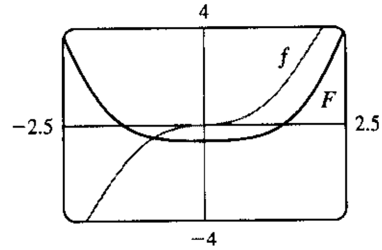
$$\int_1^{\infty} \frac{\tan^{-1} x}{x^2}\,dx = \lim_{t\to\infty} \left[-\frac{\tan^{-1} x}{x} + \frac{1}{2}\ln\frac{x^2}{x^2+1}\right]_1^t = \lim_{t\to\infty} \left[-\frac{\tan^{-1} t}{t} + \frac{1}{2}\ln\frac{t^2}{t^2+1} + \tfrac{\pi}{4} - \tfrac{1}{2}\ln\tfrac{1}{2}\right]$$
$$= 0 + \tfrac{1}{2}\ln 1 + \tfrac{\pi}{4} + \tfrac{1}{2}\ln 2 = \tfrac{\pi}{4} + \tfrac{1}{2}\ln 2$$

51. We first make the substitution $t = x + 1$, so $\ln\left(x^2 + 2x + 2\right) = \ln\left[(x+1)^2 + 1\right] = \ln\left(t^2 + 1\right)$. Then we use parts with $u = \ln\left(t^2+1\right)$, $dv = dt$:

$$\int \ln\left(t^2+1\right) dt = t\,\ln\left(t^2+1\right) - \int \frac{t(2t)\,dt}{t^2+1} = t\,\ln\left(t^2+1\right) - 2\int \frac{t^2\,dt}{t^2+1}$$
$$= t\,\ln\left(t^2+1\right) - 2\int \left(1 - \frac{1}{t^2+1}\right) dt = t\,\ln\left(t^2+1\right) - 2t + 2\arctan t + C$$
$$= (x+1)\,\ln\left(x^2+2x+2\right) - 2x + 2\arctan(x+1) + K, \quad \text{where } K = C - 2$$

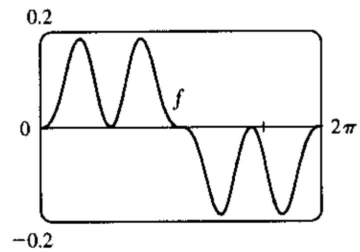
[Alternatively, we could have integrated by parts immediately with $u = \ln\left(x^2 + 2x + 2\right)$.] Notice from the graph that $f = 0$ where F has a horizontal tangent. Also, F is always increasing, and $f \geq 0$.





52. Let $u = x^2 + 1$. Then $x^2 = u - 1$ and $x\,dx = \frac{1}{2}\,du$, so

$$\begin{aligned}\int \frac{x^3}{\sqrt{x^2+1}}\,dx &= \int \frac{(u-1)}{\sqrt{u}}\left(\tfrac{1}{2}\,du\right) = \frac{1}{2}\int\left(u^{1/2} - u^{-1/2}\right)du \\ &= \tfrac{1}{2}\left(\tfrac{2}{3}u^{3/2} - 2u^{1/2}\right) + C \\ &= \tfrac{1}{3}\left(x^2+1\right)^{3/2} - \left(x^2+1\right)^{1/2} + C \\ &= \tfrac{1}{3}\left(x^2+1\right)^{1/2}\left[\left(x^2+1\right) - 3\right] + C \\ &= \tfrac{1}{3}\sqrt{x^2+1}\left(x^2-2\right) + C.\end{aligned}$$



53. From the graph, it seems as though $\int_0^{2\pi} \cos^2 x \sin^3 x\,dx$ is equal to 0. To evaluate the integral, we write the integral as $I = \int_0^{2\pi} \cos^2 x\,(1 - \cos^2 x)\sin x\,dx$ and let $u = \cos x \Rightarrow du = -\sin x\,dx$. Thus, $I = \int_1^1 u^2(1-u^2)(-du) = 0$.

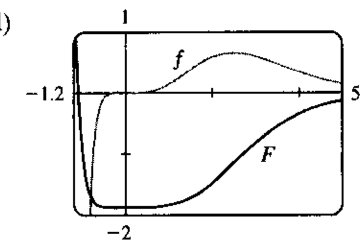
54. (a) To evaluate $\int x^5 e^{-2x}\,dx$ by hand, we would integrate by parts repeatedly, always taking $dv = e^{-2x}$ and starting with $u = x^5$. Each time we would reduce the degree of the x-factor by 1.

(b) To evaluate the integral using tables, we would use Formula 97 (which is proved using integration by parts) until the exponent of x was reduced to 1, and then we would use Formula 96.

(c) $\int x^5 e^{-2x}\,dx$

$$= -\tfrac{1}{8}e^{-2x}\left(4x^5 + 10x^4 + 20x^3 + 30x^2 + 30x + 15\right) + C$$

(d)



55. $u = e^x \Rightarrow du = e^x\,dx$, so

$$\int e^x\sqrt{1-e^{2x}}\,dx = \int\sqrt{1-u^2}\,du \overset{30}{=} \tfrac{1}{2}u\sqrt{1-u^2} + \tfrac{1}{2}\sin^{-1}u + C = \tfrac{1}{2}\left[e^x\sqrt{1-e^{2x}} + \sin^{-1}(e^x)\right] + C$$

56.
$$\begin{aligned}\int \csc^5 t\,dt &\overset{78}{=} -\tfrac{1}{4}\cot t\,\csc^3 t + \tfrac{3}{4}\int \csc^3 t\,dt \\ &\overset{72}{=} -\tfrac{1}{4}\cot t\,\csc^3 t + \tfrac{3}{4}\left[-\tfrac{1}{2}\csc t\,\cot t + \tfrac{1}{2}\ln|\csc t - \cot t|\right] + C \\ &= -\tfrac{1}{4}\cot t\,\csc^3 t - \tfrac{3}{8}\csc t\,\cot t + \tfrac{3}{8}\ln|\csc t - \cot t| + C\end{aligned}$$

57.
$$\begin{aligned}\int\sqrt{x^2+x+1}\,dx &= \int\sqrt{x^2+x+\tfrac{1}{4}+\tfrac{3}{4}}\,dx = \int\sqrt{\left(x+\tfrac{1}{2}\right)^2+\tfrac{3}{4}}\,dx \\ &= \int\sqrt{u^2+\left(\tfrac{\sqrt{3}}{2}\right)^2}\,du \quad \left[u = x+\tfrac{1}{2},\ du = dx\right] \\ &\overset{21}{=} \tfrac{1}{2}u\sqrt{u^2+\tfrac{3}{4}} + \tfrac{3}{8}\ln\left(u+\sqrt{u^2+\tfrac{3}{4}}\right) + C \\ &= \frac{2x+1}{4}\sqrt{x^2+x+1} + \tfrac{3}{8}\ln\left(x+\tfrac{1}{2}+\sqrt{x^2+x+1}\right) + C\end{aligned}$$

58. Let $u = \sin x$. Then $du = \cos x\,dx$, so

$$\int \frac{\cot x\,dx}{\sqrt{1+2\sin x}} = \int \frac{du}{u\sqrt{1+2u}} \overset{\substack{57\text{ with}\\ a=1,\,b=2}}{=} \ln\left|\frac{\sqrt{1+2u}-1}{\sqrt{1+2u}+1}\right| + C = \ln\left|\frac{\sqrt{1+2\sin x}-1}{\sqrt{1+2\sin x}+1}\right| + C$$

59. (a) $\dfrac{d}{du}\left[-\dfrac{1}{u}\sqrt{a^2-u^2} - \sin^{-1}\left(\dfrac{u}{a}\right) + C\right] = \dfrac{1}{u^2}\sqrt{a^2-u^2} + \dfrac{1}{\sqrt{a^2-u^2}} - \dfrac{1}{\sqrt{1-u^2/a^2}}\cdot\dfrac{1}{a}$

$$= \left(a^2-u^2\right)^{-1/2}\left[\frac{1}{u^2}\left(a^2-u^2\right) + 1 - 1\right] = \frac{\sqrt{a^2-u^2}}{u^2}$$

(b) Let $u = a\sin\theta \;\Rightarrow\; du = a\cos\theta\,d\theta$, $a^2 - u^2 = a^2\left(1-\sin^2\theta\right) = a^2\cos^2\theta$.

$$\int \frac{\sqrt{a^2-u^2}}{u^2}\,du = \int \frac{a^2\cos^2\theta}{a^2\sin^2\theta}\,d\theta = \int \frac{1-\sin^2\theta}{\sin^2\theta}\,d\theta = \int \left(\csc^2\theta - 1\right)d\theta = -\cot\theta - \theta + C$$

$$= -\frac{\sqrt{a^2-u^2}}{u} - \sin^{-1}\left(\frac{u}{a}\right) + C$$

60. Work backward, and use integration by parts with $U = u^{-(n-1)}$ and $dV = (a+bu)^{-1/2}\,du \;\Rightarrow$

$dU = \dfrac{-(n-1)\,du}{u^n}$ and $V = \dfrac{2}{b}\sqrt{a+bu}$, to get

$$\int \frac{du}{u^{n-1}\sqrt{a+bu}} = \int U\,dV = UV - \int V\,dU = \frac{2\sqrt{a+bu}}{bu^{n-1}} + \frac{2(n-1)}{b}\int \frac{\sqrt{a+bu}}{u^n}\,du$$

$$= \frac{2\sqrt{a+bu}}{bu^{n-1}} + \frac{2(n-1)}{b}\int \frac{a+bu}{u^n\sqrt{a+bu}}\,du$$

$$= \frac{2\sqrt{a+bu}}{bu^{n-1}} + 2(n-1)\int \frac{du}{u^{n-1}\sqrt{a+bu}} + \frac{2a(n-1)}{b}\int \frac{du}{u^n\sqrt{a+bu}}$$

Rearranging the equation gives $\dfrac{2a(n-1)}{b}\displaystyle\int \frac{du}{u^n\sqrt{a+bu}} = -\frac{2\sqrt{a+bu}}{bu^{n-1}} - (2n-3)\int \frac{du}{u^{n-1}\sqrt{a+bu}} \;\Rightarrow$

$$\int \frac{du}{u^n\sqrt{a+bu}} = \frac{-\sqrt{a+bu}}{a(n-1)u^{n-1}} - \frac{b(2n-3)}{2a(n-1)}\int \frac{du}{u^{n-1}\sqrt{a+bu}}$$

61. For $n \geq 0$, $\int_0^\infty x^n\,dx = \lim\limits_{t\to\infty}\left[x^{n+1}/(n+1)\right]_0^t = \infty$. For $n < 0$, $\int_0^\infty x^n\,dx = \int_0^1 x^n\,dx + \int_1^\infty x^n\,dx$. Both integrals are improper. By (8.8.2), the second integral diverges if $-1 \leq n < 0$. By Exercise 8.8.57, the first integral diverges if $n \leq -1$. Thus, $\int_0^\infty x^n\,dx$ is divergent for all values of n.

62. $I = \displaystyle\int_0^\infty e^{ax}\cos x\,dx = \lim_{t\to\infty}\int_0^t e^{ax}\cos x\,dx \overset{\substack{99\text{ with}\\ b=1}}{=} \lim_{t\to\infty}\left[\frac{e^{ax}}{a^2+1}(a\cos x + \sin x)\right]_0^t$

$$= \lim_{t\to\infty}\left[\frac{e^{at}}{a^2+1}(a\cos t + \sin t) - \frac{1}{a^2+1}(a)\right] = \frac{1}{a^2+1}\lim_{t\to\infty}\left[e^{at}(a\cos t + \sin t) - a\right].$$

For $a \geq 0$, the limit does not exist due to oscillation. For $a < 0$, $\lim\limits_{t\to\infty}\left[e^{at}(a\cos t + \sin t)\right] = 0$ by the Squeeze Theorem, because $\left|e^{at}(a\cos t + \sin t)\right| \leq e^{at}(|a|+1)$, so $I = \dfrac{1}{a^2+1}(-a) = -\dfrac{a}{a^2+1}$.

63. $f(x) = \sqrt{1+x^4}$, $\Delta x = \dfrac{b-a}{n} = \dfrac{1-0}{10} = \dfrac{1}{10}$.

(a) $T_{10} = \frac{1}{10 \cdot 2}\{f(0) + 2\,[f(0.1) + f(0.2) + \cdots + f(0.9)] + f(1)\} \approx 1.090608$

(b) $M_{10} = \frac{1}{10}\left[f\left(\frac{1}{20}\right) + f\left(\frac{3}{20}\right) + f\left(\frac{5}{20}\right) + \cdots + f\left(\frac{19}{20}\right)\right] \approx 1.088840$

(c) $S_{10} = \frac{1}{10 \cdot 3}[f(0) + 4f(0.1) + 2f(0.2) + \cdots + 4f(0.9) + f(1)] \approx 1.089429$

f is concave upward, so the Trapezoidal Rule gives us an overestimate, the Midpoint Rule gives an underestimate, and we cannot tell whether Simpson's Rule gives us an overestimate or an underestimate.

64. $f(x) = \sqrt{\sin x}$, $\Delta x = \dfrac{\frac{\pi}{2} - 0}{10} = \dfrac{\pi}{20}$.

(a) $T_{10} = \frac{\pi}{20 \cdot 2}\left\{f(0) + 2\left[f\left(\frac{\pi}{20}\right) + f\left(\frac{2\pi}{20}\right) + \cdots + f\left(\frac{9\pi}{20}\right)\right] + f\left(\frac{\pi}{2}\right)\right\} \approx 1.185197$

(b) $M_{10} = \frac{\pi}{20}\left[f\left(\frac{\pi}{40}\right) + f\left(\frac{3\pi}{40}\right) + f\left(\frac{5\pi}{40}\right) + \cdots + f\left(\frac{17\pi}{40}\right) + f\left(\frac{19\pi}{40}\right)\right] \approx 1.201932$

(c) $S_{10} = \frac{\pi}{20 \cdot 3}\left[f(0) + 4f\left(\frac{\pi}{20}\right) + 2f\left(\frac{2\pi}{20}\right) + \cdots + 4f\left(\frac{9\pi}{20}\right) + f\left(\frac{\pi}{2}\right)\right] \approx 1.193089$

f is concave downward, so the Trapezoidal Rule gives us an underestimate, the Midpoint Rule gives an overestimate, and we cannot tell whether Simpson's Rule gives us an overestimate or an underestimate.

65. $f(x) = (1+x^4)^{1/2}$, $f'(x) = \frac{1}{2}(1+x^4)^{-1/2}(4x^3) = 2x^3(1+x^4)^{-1/2}$, $f''(x) = (2x^6 + 6x^2)(1+x^4)^{-3/2}$.

A graph of f'' on $[0,1]$ shows that it has its maximum at $x = 1$, so $|f''(x)| \le f''(1) = \sqrt{8}$ on $[0,1]$. By taking $K = \sqrt{8}$, we find that the error in Exercise 63(a) is bounded by $\dfrac{K(b-a)^3}{12n^2} = \dfrac{\sqrt{8}}{1200} \approx 0.0024$, and in (b) by about $\frac{1}{2}(0.0024) = 0.0012$.

Note: Another way to estimate K is to let $x = 1$ in the factor $2x^6 + 6x^2$ (maximizing the numerator) and let $x = 0$ in the factor $(1+x^4)^{-3/2}$ (minimizing the denominator). Doing so gives us $K = 8$ and errors of $0.00\overline{6}$ and $0.00\overline{3}$.

Using $K = 8$ for the Trapezoidal Rule, we have $|E_T| \le \dfrac{K(b-a)^3}{12n^2} \le 0.00001 \quad \Leftrightarrow \quad \dfrac{8(1-0)^3}{12n^2} \le \dfrac{1}{100{,}000}$

$\Leftrightarrow \quad n^2 \ge \dfrac{800{,}000}{12} \quad \Leftrightarrow \quad n \gtrsim 258.2$, so we should take $n = 259$.

For the Midpoint Rule, $|E_M| \le \dfrac{K(b-a)^3}{24n^2} \le 0.00001 \quad \Leftrightarrow \quad n^2 \ge \dfrac{800{,}000}{24} \quad \Leftrightarrow \quad n \gtrsim 182.6$, so we should take $n = 183$.

66. $\displaystyle\int_1^4 \frac{e^x}{x}\,dx \approx S_6 = \frac{(4-1)/6}{3}\,[f(1) + 4f(1.5) + 2f(2) + 4f(2.5) + 2f(3) + 4f(3.5) + f(4)] \approx 17.739438$

67. $\Delta t = \left(\frac{10}{60} - 0\right)/10 = \frac{1}{60}$.

Distance traveled $= \int_0^{10} v\,dt \approx S_{10}$

$$= \tfrac{1}{60 \cdot 3}[40 + 4(42) + 2(45) + 4(49) + 2(52) + 4(54) + 2(56) + 4(57) + 2(57) + 4(55) + 56]$$

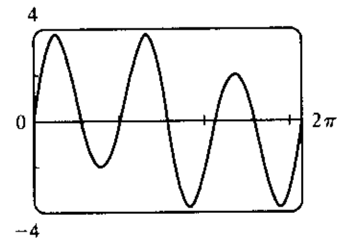
$$= \tfrac{1}{180}(1544) = 8.5\overline{7}\text{ mi}$$

68. We use Simpson's Rule with $n = 6$ and $\Delta t = \frac{24-0}{6} = 4$:

$$\begin{aligned}\text{Increase in bee population} &= \int_0^{24} r(t)\,dt \approx S_6 \\ &= \tfrac{4}{3}[r(0) + 4r(4) + 2r(8) + 4r(12) + 2r(16) + 4r(20) + r(24)] \\ &= \tfrac{4}{3}[0 + 4(300) + 2(3000) + 4(11{,}000) + 2(4000) + 4(400) + 0] \\ &= \tfrac{4}{3}(60{,}800) \approx 81{,}067 \text{ bees}\end{aligned}$$

69. (a) $f(x) = \sin(\sin x)$. A CAS gives

$$f^{(4)}(x) = \sin(\sin x)\left[\cos^4 x + 7\cos^2 x - 3\right] + \cos(\sin x)\left[6\cos^2 x \sin x + \sin x\right]$$



From the graph, we see that $\left|f^{(4)}(x)\right| < 3.8$ for $x \in [0, \pi]$.

(b) We use Simpson's Rule with $f(x) = \sin(\sin x)$ and $\Delta x = \frac{\pi}{10}$:

$$\int_0^{\pi} f(x)\,dx \approx \frac{\pi}{10 \cdot 3}\left[f(0) + 4f\left(\tfrac{\pi}{10}\right) + 2f\left(\tfrac{2\pi}{10}\right) + \cdots + 4f\left(\tfrac{9\pi}{10}\right) + f(\pi)\right] \approx 1.786721$$

From part (a), we know that $\left|f^{(4)}(x)\right| < 3.8$ on $[0, \pi]$, so we use Theorem 8.7.4 with $K = 3.8$, and estimate the error as $|E_S| \le \dfrac{3.8(\pi - 0)^5}{180(10)^4} \approx 0.000646$.

(c) If we want the error to be less than 0.00001, we must have $|E_S| \le \frac{3.8\pi^5}{180n^4} \le 0.00001$, so $n^4 \ge \frac{3.8\pi^5}{180(0.00001)} \approx 646{,}041.6 \Rightarrow n \ge 28.35$. Since n must be even for Simpson's Rule, we must have $n \ge 30$ to ensure the desired accuracy.

70. With an x-axis in the normal position, at $x = 7$ we have $C = 2\pi r = 45 \Rightarrow r(7) = \frac{2\pi}{45}$. Using Simpson's Rule with $n = 4$ and $\Delta x = 7$, we have

$$V = \int_0^{28} \pi[r(x)]^2\,dx \approx S_4 = \tfrac{7}{3}\left[0 + 4\pi\left(\tfrac{45}{2\pi}\right)^2 + 2\pi\left(\tfrac{53}{2\pi}\right)^2 + 4\pi\left(\tfrac{45}{2\pi}\right)^2 + 0\right] = \tfrac{7}{3}\left(\tfrac{21{,}818}{4\pi}\right) \approx 4051 \text{ cm}^3$$

71. $\dfrac{x^3}{x^5+2} \le \dfrac{x^3}{x^5} = \dfrac{1}{x^2}$ for x in $[1, \infty)$. $\displaystyle\int_1^{\infty} \frac{1}{x^2}\,dx$ is convergent by (8.8.2) with $p = 2 > 1$. Therefore, $\displaystyle\int_1^{\infty} \frac{x^3}{x^5+2}\,dx$ is convergent by the Comparison Theorem.

72. The line $y = 3$ intersects the hyperbola $y^2 - x^2 = 1$ at two points on its upper branch, namely $\left(-2\sqrt{2}, 3\right)$ and $\left(2\sqrt{2}, 3\right)$. The desired area is

$$\begin{aligned}A &= \int_{-2\sqrt{2}}^{2\sqrt{2}} \left(3 - \sqrt{x^2+1}\right)dx = 2\int_0^{2\sqrt{2}} \left(3 - \sqrt{x^2+1}\right)dx \\ &\overset{21}{=} 2\left[3x - \tfrac{1}{2}x\sqrt{x^2+1} - \tfrac{1}{2}\ln\left(x + \sqrt{x^2+1}\right)\right]_0^{2\sqrt{2}} \\ &= \left[6x - x\sqrt{x^2+1} - \ln\left(x + \sqrt{x^2+1}\right)\right]_0^{2\sqrt{2}} = 12\sqrt{2} - 2\sqrt{2}\cdot 3 - \ln\left(2\sqrt{2} + 3\right) \\ &= 6\sqrt{2} - \ln\left(3 + 2\sqrt{2}\right)\end{aligned}$$

Another method: $A = 2\int_1^3 \sqrt{y^2 - 1}\,dy$ and use Formula 39.

73. For x in $\left[0, \frac{\pi}{2}\right]$, $0 \le \cos^2 x \le \cos x$. For x in $\left[\frac{\pi}{2}, \pi\right]$, $\cos x \le 0 \le \cos^2 x$. Thus,

$$\begin{aligned}\text{area} &= \int_0^{\pi/2} (\cos x - \cos^2 x)\,dx + \int_{\pi/2}^{\pi} (\cos^2 x - \cos x)\,dx \\ &= \left[\sin x - \tfrac{1}{2}x - \tfrac{1}{4}\sin 2x\right]_0^{\pi/2} + \left[\tfrac{1}{2}x + \tfrac{1}{4}\sin 2x - \sin x\right]_{\pi/2}^{\pi} \\ &= \left[\left(1 - \tfrac{\pi}{4}\right) - 0\right] + \left[\tfrac{\pi}{2} - \left(\tfrac{\pi}{4} - 1\right)\right] = 2\end{aligned}$$

74. The curves $y = \dfrac{1}{2 \pm \sqrt{x}}$ are defined for $x \ge 0$. For $x > 0$, $\dfrac{1}{2 - \sqrt{x}} > \dfrac{1}{2 + \sqrt{x}}$. Thus, the required area is

$$\begin{aligned}\int_0^1 \left(\frac{1}{2 - \sqrt{x}} - \frac{1}{2 + \sqrt{x}}\right) dx &= \int_0^1 \left(\frac{1}{2 - u} - \frac{1}{2 + u}\right) 2u\,du \quad [\text{put } u = \sqrt{x}\,] \\ &= 2\int_0^1 \left(-\frac{u}{u - 2} - \frac{u}{u + 2}\right) du = 2\int_0^1 \left(-1 - \frac{2}{u - 2} - 1 + \frac{2}{u + 2}\right) du \\ &= 2\left[2\ln\left|\frac{u + 2}{u - 2}\right| - 2u\right]_0^1 = 4\ln 3 - 4\end{aligned}$$

75. Using the formula for disks, the volume is

$$\begin{aligned}V &= \int_0^{\pi/2} \pi\,[f(x)]^2\,dx = \pi \int_0^{\pi/2} \left(\cos^2 x\right)^2 dx = \pi \int_0^{\pi/2} \left[\tfrac{1}{2}(1 + \cos 2x)\right]^2 dx \\ &= \tfrac{\pi}{4} \int_0^{\pi/2} (1 + \cos^2 2x + 2\cos 2x)\,dx = \tfrac{\pi}{4} \int_0^{\pi/2} \left[1 + \tfrac{1}{2}(1 + \cos 4x) + 2\cos 2x\right] dx \\ &= \tfrac{\pi}{4}\left[\tfrac{3}{2}x + \tfrac{1}{2}\left(\tfrac{1}{4}\sin 4x\right) + 2\left(\tfrac{1}{2}\sin 2x\right)\right]_0^{\pi/2} = \tfrac{\pi}{4}\left[\left(\tfrac{3\pi}{4} + \tfrac{1}{8}\cdot 0 + 0\right) - 0\right] = \tfrac{3\pi^2}{16}\end{aligned}$$

76. Using the formula for cylindrical shells, the volume is

$$\begin{aligned}V &= \int_0^{\pi/2} 2\pi x f(x)\,dx = 2\pi \int_0^{\pi/2} x\cos^2 x\,dx \\ &= 2\pi \int_0^{\pi/2} x\left[\tfrac{1}{2}(1 + \cos 2x)\right] dx = 2\left(\tfrac{1}{2}\right)\pi \int_0^{\pi/2} (x + x\cos 2x)\,dx \\ &= \pi\left(\left[\tfrac{1}{2}x^2\right]_0^{\pi/2} + \left[x\left(\tfrac{1}{2}\sin 2x\right)\right]_0^{\pi/2} - \int_0^{\pi/2} \tfrac{1}{2}\sin 2x\,dx\right) \quad [\text{parts with } u = x,\ dv = \cos 2x\,dx] \\ &= \pi\left[\tfrac{1}{2}\left(\tfrac{\pi}{2}\right)^2 + 0 - \tfrac{1}{2}\left[-\tfrac{1}{2}\cos 2x\right]_0^{\pi/2}\right] = \tfrac{\pi^3}{8} + \tfrac{\pi}{4}(-1 - 1) = \tfrac{1}{8}\left(\pi^3 - 4\pi\right)\end{aligned}$$

77. By the Fundamental Theorem of Calculus,

$$\int_0^\infty f'(x)\,dx = \lim_{t\to\infty} \int_0^t f'(x)\,dx = \lim_{t\to\infty} [f(t) - f(0)] = \lim_{t\to\infty} f(t) - f(0) = 0 - f(0) = -f(0).$$

78. (a)
$$\begin{aligned}\left(\tan^{-1} x\right)_{\text{ave}} &= \lim_{t\to\infty} \frac{1}{t - 0} \int_0^t \tan^{-1} x\,dx \overset{89}{=} \lim_{t\to\infty} \left\{\frac{1}{t}\left[x\tan^{-1} x - \tfrac{1}{2}\ln\left(1 + x^2\right)\right]_0^t\right\} \\ &= \lim_{t\to\infty} \left[\frac{1}{t}\left(t\tan^{-1} t - \tfrac{1}{2}\ln\left(1 + t^2\right)\right)\right] = \lim_{t\to\infty} \left[\tan^{-1} t - \frac{\ln\left(1 + t^2\right)}{2t}\right] \\ &\overset{H}{=} \frac{\pi}{2} - \lim_{t\to\infty} \frac{2t/\left(1 + t^2\right)}{2} = \frac{\pi}{2} - 0 = \frac{\pi}{2}\end{aligned}$$

(b) $f(x) \geq 0$ and $\int_a^\infty f(x)\,dx$ is divergent $\Rightarrow$ $\lim\limits_{t\to\infty} \int_a^t f(x)\,dx = \infty$.

$$f_{\text{ave}} = \lim_{t\to\infty} \frac{\int_a^t f(x)\,dx}{t-a}\,dx \overset{\text{H}}{=} \lim_{t\to\infty} \frac{f(t)}{1} \quad \text{[by FTC1]} \quad = \lim_{x\to\infty} f(x), \text{ if this limit exists.}$$

(c) Suppose $\int_a^\infty f(x)\,dx$ converges; that is, $\lim\limits_{t\to\infty} \int_a^t f(x)\,dx = L < \infty$. Then

$$f_{\text{ave}} = \lim_{t\to\infty} \left[\frac{1}{t-a}\int_a^t f(x)\,dx\right] = \lim_{t\to\infty} \frac{1}{t-a} \cdot \lim_{t\to\infty} \int_a^t f(x)\,dx = 0 \cdot L = 0$$

(d) $$(\sin x)_{\text{ave}} = \lim_{t\to\infty} \frac{1}{t}\int_0^t \sin x\,dx = \lim_{t\to\infty} \left(\frac{1}{t}\left[-\cos x\right]_0^t\right) = \lim_{t\to\infty}\left(-\frac{\cos t}{t} + \frac{1}{t}\right) = \lim_{t\to\infty} \frac{1-\cos t}{t} = 0$$

9. Let $u = 1/x \quad \Rightarrow \quad x = 1/u \quad \Rightarrow \quad dx = -\left(1/u^2\right) du$.

$$\int_0^\infty \frac{\ln x}{1+x^2}\,dx = \int_\infty^0 \frac{\ln(1/u)}{1+1/u^2}\left(-\frac{du}{u^2}\right) = \int_\infty^0 \frac{-\ln u}{u^2+1}(-du) = \int_\infty^0 \frac{\ln u}{1+u^2}\,du = -\int_0^\infty \frac{\ln u}{1+u^2}\,du$$

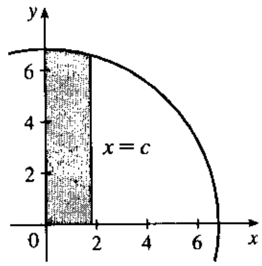
Therefore, $\displaystyle\int_0^\infty \frac{\ln x}{1+x^2}\,dx = -\int_0^\infty \frac{\ln x}{1+x^2}\,dx = 0$.

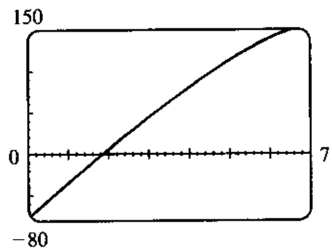
10. If the distance between P and the point charge is d, then the potential V at P is

$$V = W = \int_\infty^d F\,dr = \int_\infty^d \frac{q}{4\pi\varepsilon_0 r^2}\,dr = \lim_{t\to\infty} \frac{q}{4\pi\varepsilon_0}\left[-\frac{1}{r}\right]_t^d = \frac{q}{4\pi\varepsilon_0} \lim_{t\to\infty}\left(-\frac{1}{d} + \frac{1}{t}\right) = -\frac{q}{4\pi\varepsilon_0 d}$$

PROBLEMS PLUS

1.





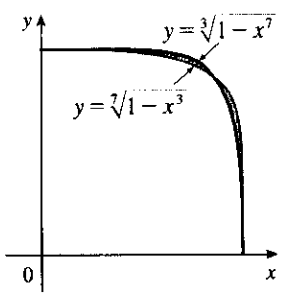
By symmetry, the problem can be reduced to finding the line $x = c$ such that the shaded area is one-third of the area of the quarter-circle. The equation of the circle is $y = \sqrt{49 - x^2}$, so we require that $\int_0^c \sqrt{49 - x^2}\,dx = \frac{1}{3} \cdot \frac{1}{4}\pi(7)^2$ $\Leftrightarrow$ $\left[\frac{1}{2}x\sqrt{49 - x^2} + \frac{49}{2}\sin^{-1}(x/7)\right]_0^c = \frac{49}{12}\pi$ [by Formula 30] $\Leftrightarrow$ $\frac{1}{2}c\sqrt{49 - c^2} + \frac{49}{2}\sin^{-1}(c/7) = \frac{49}{12}\pi$. This equation would be difficult to solve exactly, so we plot the left-hand side as a function of c, and find that the equation holds for $c \approx 1.85$. So the cuts should be made at distances of about 1.85 inches from the center of the pizza.

2. $$\int \frac{1}{x^7 - x}\,dx = \int \frac{dx}{x(x^6 - 1)} = \int \frac{x^5}{x^6(x^6 - 1)}\,dx = \frac{1}{6}\int \frac{1}{u(u-1)}\,du \quad [u = x^6,\, du = 6x^5\,dx]$$
$$= \frac{1}{6}\int\left(\frac{1}{u-1} - \frac{1}{u}\right)du = \frac{1}{6}(\ln|u - 1| - \ln|u|) + C = \frac{1}{6}\ln\left|\frac{u-1}{u}\right| + C$$
$$= \frac{1}{6}\ln\left|\frac{x^6 - 1}{x^6}\right| + C$$

Alternate method: $$\int \frac{1}{x^7 - x}\,dx = \int \frac{x^{-7}}{1 - x^{-6}}\,dx \quad [u = 1 - x^{-6},\, du = 6x^{-7}\,dx]$$
$$= \frac{1}{6}\int \frac{du}{u} = \frac{1}{6}\ln|u| + C = \frac{1}{6}\ln\left|1 - x^{-6}\right| + C$$

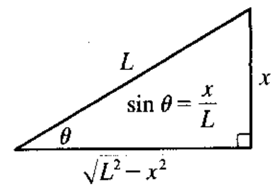
Other methods: Substitute $u = x^3$ or $x^3 = \sec\theta$.

3. The given integral represents the difference of the shaded areas, which appears to be 0. It can be calculated by integrating with respect to either x or y, so we find x in terms of y for each curve: $y = \sqrt[3]{1 - x^7}$ $\Rightarrow$ $x = \sqrt[7]{1 - y^3}$ and $y = \sqrt[7]{1 - x^3}$ $\Rightarrow$ $x = \sqrt[3]{1 - y^7}$, so $\int_0^1\left(\sqrt[3]{1 - y^7} - \sqrt[7]{1 - y^3}\right)dy = \int_0^1\left(\sqrt[7]{1 - x^3} - \sqrt[3]{1 - x^7}\right)dx$. But this equation is of the form $z = -z$. So $\int_0^1\left(\sqrt[3]{1 - x^7} - \sqrt[7]{1 - x^3}\right)dx = 0$.



4. (a) The tangent to the curve $y = f(x)$ at $x = x_0$ has the equation $y - f(x_0) = f'(x_0)(x - x_0)$. The y-intercept of this tangent line is $f(x_0) - f'(x_0)x_0$. Thus, L is the distance from the point $(0, f(x_0) - f'(x_0)x_0)$ to the point $(x_0, f(x_0))$; that is, $L^2 = x_0^2 + [f'(x_0)]^2 x_0^2$, so $[f'(x_0)]^2 = \dfrac{L^2 - x_0^2}{x_0^2}$ and $f'(x_0) = -\dfrac{\sqrt{L^2 - x_0^2}}{x_0}$ for $0 < x_0 < L$.

(b) $\dfrac{dy}{dx} = -\dfrac{\sqrt{L^2 - x^2}}{x} \quad \Rightarrow \quad y = \displaystyle\int \left(-\frac{\sqrt{L^2 - x^2}}{x} \right) dx.$



Let $x = L \sin\theta$. Then $dx = L\cos\theta\, d\theta$ and

$$\begin{aligned} y &= \int \frac{-L\cos\theta\, L\cos\theta\, d\theta}{L\sin\theta} = L \int \frac{\sin^2\theta - 1}{\sin\theta}\, d\theta \\ &= L \textstyle\int (\sin\theta - \csc\theta)\, d\theta \\ &= -L\cos\theta - L\ln|\csc\theta - \cot\theta| + C \\ &= -\sqrt{L^2 - x^2} - L\ln\left(\frac{L}{x} - \frac{\sqrt{L^2 - x^2}}{x} \right) + C. \end{aligned}$$

When $x = L$, $y = 0$, and $0 = -0 - L\ln(1 - 0) + C$, so $C = 0$. Therefore,

$$y = -\sqrt{L^2 - x^2} - L\ln\left(\frac{L - \sqrt{L^2 - x^2}}{x} \right)$$

5. Recall that $\cos A \cos B = \frac{1}{2}[\cos(A + B) + \cos(A - B)]$. So

$$\begin{aligned} f(x) &= \textstyle\int_0^\pi \cos t \cos(x - t)\, dt = \frac{1}{2} \int_0^\pi [\cos(t + x - t) + \cos(t - x + t)]\, dt \\ &= \tfrac{1}{2} \textstyle\int_0^\pi [\cos x + \cos(2t - x)]\, dt = \frac{1}{2} \left[t\cos x + \frac{1}{2}\sin(2t - x) \right]_0^\pi \\ &= \tfrac{\pi}{2}\cos x + \tfrac{1}{4}\sin(2\pi - x) - \tfrac{1}{4}\sin(-x) = \tfrac{\pi}{2}\cos x + \tfrac{1}{4}\sin(-x) - \tfrac{1}{4}\sin(-x) \\ &= \tfrac{\pi}{2}\cos x \end{aligned}$$

The minimum of $\cos x$ on this domain is -1, so the minimum value of $f(x)$ is $f(\pi) = -\frac{\pi}{2}$.

6. n is a positive integer, so

$$\textstyle\int (\ln x)^n\, dx = x(\ln x)^n - \int x \cdot n(\ln x)^{n-1}\, (dx/x) \quad \text{[by parts]} \quad = x(\ln x)^n - n \int (\ln x)^{n-1}\, dx$$

Thus,

$$\begin{aligned} \textstyle\int_0^1 (\ln x)^n\, dx &= \lim_{t\to 0^+} \textstyle\int_t^1 (\ln x)^n\, dx = \lim_{t\to 0^+} [x(\ln x)^n]_t^1 - n \lim_{t\to 0^+} \textstyle\int_t^1 (\ln x)^{n-1}\, dx \\ &= -\lim_{t\to 0^+} \frac{(\ln t)^n}{1/t} - n\int_0^1 (\ln x)^{n-1}\, dx = -n\int_0^1 (\ln x)^{n-1}\, dx \end{aligned}$$

by repeated application of l'Hospital's Rule. We want to prove that $\int_0^1 (\ln x)^n\, dx = (-1)^n n!$ for every positive integer n. For $n = 1$, we have

$$\textstyle\int_0^1 (\ln x)^1\, dx = (-1) \int_0^1 (\ln x)^0\, dx = -\int_0^1 dx = -1 \qquad \left[\text{or } \int_0^1 \ln x\, dx = \lim_{t\to 0^+} [x\ln x - x]_t^1 = -1 \right]$$

Assuming that the formula holds for n, we find that

$$\textstyle\int_0^1 (\ln x)^{n+1}\, dx = -(n + 1) \int_0^1 (\ln x)^n\, dx = -(n + 1)(-1)^n n! = (-1)^{n+1}(n + 1)!$$

This is the formula for $n + 1$. Thus, the formula holds for all positive integers n by induction.

7. In accordance with the hint, we let $I_k = \int_0^1 \left(1 - x^2\right)^k dx$, and we find an expression for I_{k+1} in terms of I_k. We integrate I_{k+1} by parts with $u = \left(1 - x^2\right)^{k+1} \Rightarrow du = (k + 1)\left(1 - x^2\right)^k(-2x)$, $dv = dx \Rightarrow v = x$, and then split the remaining integral into identifiable quantities:

$$\begin{aligned} I_{k+1} &= x\left(1 - x^2\right)^{k+1} \Big|_0^1 + 2(k + 1) \textstyle\int_0^1 x^2\left(1 - x^2\right)^k dx = (2k + 2) \textstyle\int_0^1 \left(1 - x^2\right)^k \left[1 - \left(1 - x^2\right)\right] dx \\ &= (2k + 2)(I_k - I_{k+1}) \end{aligned}$$

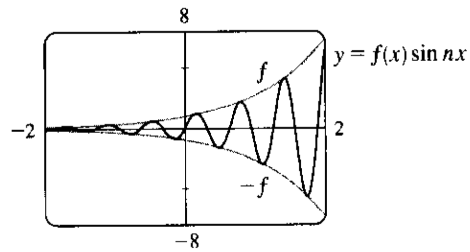
So $I_{k+1}[1+(2k+2)] = (2k+2)I_k \quad\Rightarrow\quad I_{k+1} = \frac{2k+2}{2k+3}I_k$. Now to complete the proof, we use induction: $I_0 = 1 = \dfrac{2^0(0!)^2}{1!}$, so the formula holds for $n = 0$. Now suppose it holds for $n = k$. Then

$$I_{k+1} = \frac{2k+2}{2k+3}I_k = \frac{2k+2}{2k+3}\left[\frac{2^{2k}(k!)^2}{(2k+1)!}\right] = \frac{2(k+1)2^{2k}(k!)^2}{(2k+3)(2k+1)!} = \frac{2(k+1)}{2k+2}\cdot\frac{2(k+1)2^{2k}(k!)^2}{(2k+3)(2k+1)!}$$

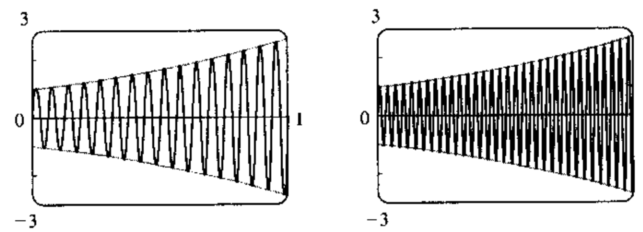
$$= \frac{[2(k+1)]^2\,2^{2k}(k!)^2}{(2k+3)(2k+2)(2k+1)!} = \frac{2^{2(k+1)}\,[(k+1)!]^2}{[2(k+1)+1]!}$$

So by induction, the formula holds for all integers $n \geq 0$.

8. (a) Since $-1 \leq \sin \leq 1$, we have $-f(x) \leq f(x)\sin nx \leq f(x)$, and the graph of $y = f(x)\,\sin nx$ oscillates between $f(x)$ and $-f(x)$. (The diagram shows the case $f(x) = e^x$ and $n = 10$.) As $n \to \infty$, the graph oscillates more and more frequently; see the graphs in part (b).



(b) From the graphs of the integrand, it seems that $\lim\limits_{n\to\infty}\int_0^1 f(x)\,\sin nx\,dx = 0$, since as n increases, the integrand oscillates more and more rapidly, and thus (since f' is continuous) it makes sense that the areas above the x-axis and below it during each oscillation approach equality.



$n = 100$ $\qquad$ $n = 200$

(c) We integrate by parts with $u = f(x) \quad\Rightarrow\quad du = f'(x)\,dx$, $dv = \sin nx\,dx \quad\Rightarrow\quad v = -\dfrac{\cos nx}{n}$:

$$\int_0^1 f(x)\,\sin nx\,dx = \left[-\frac{f(x)\,\cos nx}{n}\right]_0^1 + \int_0^1 \frac{\cos nx}{n}\,f'(x)\,dx$$

$$= \tfrac{1}{n}\left(\int_0^1 \cos nx f'(x)\,dx - [f(x)\,\cos nx]_0^1\right)$$

$$= \tfrac{1}{n}\left[\int_0^1 \cos nx f'(x)\,dx + f(0) - f(1)\,\cos n\right]$$

Taking absolute values of the first and last terms in this equality, and using the facts that $|\alpha \pm \beta| \leq |\alpha| + |\beta|$, $\int_0^1 f(x)\,dx \leq \int_0^1 |f(x)|\,dx$, $|f(0)| = f(0)$ [f is positive], $|f'(x)| \leq M$ for $0 \leq x \leq 1$, and $|\cos nx| \leq 1$,

$$\left|\int_0^1 f(x)\,\sin nx\,dx\right| \leq \tfrac{1}{n}\left[\left|\int_0^1 M\,dx\right| + |f(0)| + |f(1)|\right] = \tfrac{1}{n}\left[M + |f(0)| + |f(1)|\right]$$

which approaches 0 as $n \to \infty$. The result follows by the Squeeze Theorem.

9. $0 < a < b$. Now

$$\int_0^1 [bx + a(1-x)]^t\,dx = \int_a^b \frac{u^t}{(b-a)}\,du \quad [\text{put } u = bx + a(1-x)] \quad = \left[\frac{u^{t+1}}{(t+1)(b-a)}\right]_a^b = \frac{b^{t+1} - a^{t+1}}{(t+1)(b-a)}.$$

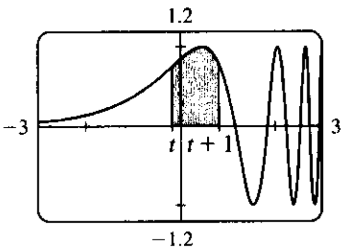
Now let $y = \lim\limits_{t\to 0}\left[\dfrac{b^{t+1} - a^{t+1}}{(t+1)(b-a)}\right]^{1/t}$. Then $\ln y = \lim\limits_{t\to 0}\left[\dfrac{1}{t}\ln\dfrac{b^{t+1} - a^{t+1}}{(t+1)(b-a)}\right]$. This limit is of the form $0/0$,

so we can apply l'Hospital's Rule to get

$$\ln y = \lim_{t\to 0}\left[\frac{b^{t+1}\ln b - a^{t+1}\ln a}{b^{t+1} - a^{t+1}} - \frac{1}{t+1}\right] = \frac{b\ln b - a\ln a}{b-a} - 1 = \frac{b\ln b}{b-a} - \frac{a\ln a}{b-a} - \ln e = \ln\frac{b^{b/(b-a)}}{ea^{a/(b-a)}}.$$

Therefore, $y = e^{-1}\left(\dfrac{b^b}{a^a}\right)^{1/(b-a)}$.

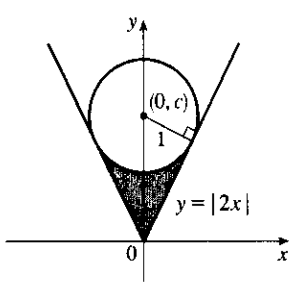
10.



From the graph, it appears that the area under the graph of $f(x) = \sin(e^x)$ on the interval $[t, t+1]$ is greatest when $t \approx -0.2$. To find the exact value, we write the integral as $I = \int_t^{t+1} f(x)\,dx = \int_0^{t+1} f(x)\,dx - \int_0^t f(x)\,dx$, and use FTC1 to find $dI/dt = f(t+1) - f(t) = \sin(e^{t+1}) - \sin(e^t) = 0$ when $\sin(e^{t+1}) = \sin(e^t)$.

Now we have $\sin x = \sin y$ whenever $x - y = 2k\pi$ and also whenever x and y are the same distance from $\left(k + \frac{1}{2}\right)\pi$, k any integer, since $\sin x$ is symmetric about the line $x = \left(k + \frac{1}{2}\right)\pi$. The first possibility is the more obvious one, but if we calculate $e^{t+1} - e^t = 2k\pi$, we get $t = \ln(2k\pi/(e-1))$, which is about 1.3 for $k = 1$ (the least possible value of k). From the graph, this looks unlikely to give the maximum we are looking for. So instead we set $e^{t+1} - \left(k + \frac{1}{2}\right)\pi = \left(k + \frac{1}{2}\right)\pi - e^t \;\Leftrightarrow\; e^{t+1} + e^t = (2k+1)\pi \;\Leftrightarrow\; e^t(e+1) = (2k+1)\pi \;\Leftrightarrow\; t = \ln((2k+1)\pi/(e+1))$. Now $k = 0 \;\Rightarrow\; t = \ln(\pi/(e+1)) \approx -0.16853$, which does give the maximum value, as we have seen from the graph of f.

11.



An equation of the circle with center $(0, c)$ and radius 1 is $x^2 + (y-c)^2 = 1^2$, so an equation of the lower semicircle is $y = c - \sqrt{1 - x^2}$. At the points of tangency, the slopes of the line and semicircle must be equal. For $x \geq 0$, we must have

$$y' = 2 \;\Rightarrow\; \frac{x}{\sqrt{1-x^2}} = 2 \;\Rightarrow\; x = 2\sqrt{1-x^2} \;\Rightarrow$$

$$x^2 = 4(1-x^2) \;\Rightarrow\; 5x^2 = 4 \;\Rightarrow\; x^2 = \tfrac{4}{5} \;\Rightarrow\; x = \tfrac{2}{5}\sqrt{5}$$

and so $y = 2\left(\frac{2}{5}\sqrt{5}\right) = \frac{4}{5}\sqrt{5}$. The slope of the perpendicular line segment is $-\frac{1}{2}$, so an equation of the line segment is $y - \frac{4}{5}\sqrt{5} = -\frac{1}{2}\left(x - \frac{2}{5}\sqrt{5}\right) \;\Leftrightarrow\; y = -\frac{1}{2}x + \frac{1}{5}\sqrt{5} + \frac{4}{5}\sqrt{5} \;\Leftrightarrow\; y = -\frac{1}{2}x + \sqrt{5}$, so $c = \sqrt{5}$ and an equation of the lower semicircle is $y = \sqrt{5} - \sqrt{1-x^2}$. Thus, the shaded area is

$$2\int_0^{(2/5)\sqrt{5}} \left[\left(\sqrt{5}-\sqrt{1-x^2}\right) - 2x\right] dx \overset{30}{=} 2\left[\sqrt{5}\,x - \frac{x}{2}\sqrt{1-x^2} - \frac{1}{2}\sin^{-1}x - x^2\right]_0^{(2/5)\sqrt{5}}$$

$$= 2\left[2 - \frac{\sqrt{5}}{5}\cdot\frac{1}{\sqrt{5}} - \frac{1}{2}\sin^{-1}\left(\frac{2}{\sqrt{5}}\right) - \frac{4}{5}\right] - 2(0)$$

$$= 2\left[1 - \frac{1}{2}\sin^{-1}\left(\frac{2}{\sqrt{5}}\right)\right] = 2 - \sin^{-1}\left(\frac{2}{\sqrt{5}}\right)$$

12. (a) $M\dfrac{dv}{dt} - ub = -Mg \;\Rightarrow\; (M_0 - bt)\dfrac{dv}{dt} = ub - (M_0 - bt)g \;\Rightarrow\; \dfrac{dv}{dt} = \dfrac{ub}{M_0 - bt} - g \;\Rightarrow$ $v(t) = -u\ln(M_0 - bt) - gt + C$. Now $0 = v(0) = -u\ln M_0 + C$, so $C = u\ln M_0$. Thus $v(t) = u\ln M_0 - u\ln(M_0 - bt) - gt = u\ln\dfrac{M_0}{M_0 - bt} - gt$.

(b) Burnout velocity $= v\left(\dfrac{M_2}{b}\right) = u\ln\dfrac{M_0}{M_0 - M_2} - g\dfrac{M_2}{b} = u\ln\dfrac{M_0}{M_1} - g\dfrac{M_2}{b}$.

Note: The reason for the term "burnout velocity" is that M_2 kilograms of fuel is used in M_2/b seconds, so $v(M_2/b)$ is the rocket's velocity when the fuel is used up.

(c) Height at burnout time $= y\left(\dfrac{M_2}{b}\right)$. Now $\dfrac{dy}{dt} = v(t) = u\ln M_0 - gt - u\ln(M_0 - bt)$, so

$y(t) = (u\ln M_0)t - \dfrac{gt^2}{2} - \dfrac{u}{b}(M_0 - bt)\ln(M_0 - bt) + ut + C$. Since $0 = y(0) = \dfrac{u}{b}M_0\ln M_0 + C$, we get $C = -\dfrac{u}{b}M_0\ln M_0$ and $y(t) = u(1+\ln M_0)t - \dfrac{gt^2}{2} + \dfrac{u}{b}(M_0 - bt)\ln(M_0 - bt) - \dfrac{u}{b}M_0\ln M_0$. Therefore, the height at burnout is

$$y\left(\frac{M_2}{b}\right) = u(1+\ln M_0)\frac{M_2}{b} - \frac{g}{2}\left(\frac{M_2}{b}\right)^2 + \frac{u}{b}M_1\ln M_1 - \frac{u}{b}M_0\ln M_0$$

$$= \frac{u}{b}M_2 - \frac{u}{b}M_1\ln M_0 + \frac{u}{b}M_1\ln M_1 - \frac{g}{2}\left(\frac{M_2}{b}\right)^2 = \frac{u}{b}M_2 + \frac{u}{b}M_1\ln\frac{M_1}{M_0} - \frac{g}{2}\left(\frac{M_2}{b}\right)^2$$

[In the calculation of $y(M_2/b)$, repeated use was made of the relation $M_0 = M_1 + M_2$. In particular, $t = M_2/b \;\Rightarrow\; M_0 - bt = M_1$.]

(d) The formula for $y(t)$ in part (c) holds while there is still fuel. Once the fuel is used up, gravity is the only force acting on the rocket. $-M_1 g = M_1\dfrac{dv}{dt} \;\Rightarrow\; \dfrac{dv}{dt} = -g \;\Rightarrow\; v(t) = -gt + c_1$, where $c_1 = v\left(\dfrac{M_2}{b}\right) + \dfrac{gM_2}{b}$

$$\Rightarrow\; v(t) = v\left(\frac{M_2}{b}\right) - g\left(t - \frac{M_2}{b}\right) \;\Rightarrow$$

$y(t) = v\left(\dfrac{M_2}{b}\right)\left(t - \dfrac{M_2}{b}\right) - \dfrac{g}{2}\left(t - \dfrac{M_2}{b}\right)^2 + c_2$, where $c_2 = y\left(\dfrac{M_2}{b}\right)$, so

$$y(t) = y\left(\frac{M_2}{b}\right) + v\left(\frac{M_2}{b}\right)\left(t - \frac{M_2}{b}\right) - \frac{g}{2}\left(t - \frac{M_2}{b}\right)^2,\; t \ge \frac{M_2}{b}.$$

To summarize: For $0 \le t \le \dfrac{M_2}{b}$, $y(t) = u(1+\ln M_0)t - \dfrac{gt^2}{2} + \dfrac{u}{b}(M_0 - bt)\ln(M_0 - bt) - \dfrac{u}{b}M_0\ln M_0$ [from part (c)], and for $t \ge \dfrac{M_2}{b}$, $y(t) = y\left(\dfrac{M_2}{b}\right) + v\left(\dfrac{M_2}{b}\right)\left(t - \dfrac{M_2}{b}\right) - \dfrac{g}{2}\left(t - \dfrac{M_2}{b}\right)^2$ [from above].

$y\left(\dfrac{M_2}{b}\right)$ and $v\left(\dfrac{M_2}{b}\right)$ are given in parts (c) and (b), respectively.

3. We integrate by parts with $u = \dfrac{1}{\ln(1+x+t)}$, $dv = \sin t\,dt$, so $du = \dfrac{-1}{(1+x+t)[\ln(1+x+t)]^2}$ and $v = -\cos t$. The integral becomes

$$I = \int_0^\infty \frac{\sin t\,dt}{\ln(1+x+t)} = \lim_{b\to\infty}\left(\left[\frac{-\cos t}{\ln(1+x+t)}\right]_0^b - \int_0^b \frac{\cos t\,dt}{(1+x+t)[\ln(1+x+t)]^2}\right)$$

$$= \lim_{b\to\infty}\frac{-\cos b}{\ln(1+x+b)} + \frac{1}{\ln(1+x)} + \int_0^\infty \frac{-\cos t\,dt}{(1+x+t)[\ln(1+x+t)]^2} = \frac{1}{\ln(1+x)} + J$$

where $J = \displaystyle\int_0^\infty \frac{-\cos t\,dt}{(1+x+t)[\ln(1+x+t)]^2}$. Now $-1 \le -\cos t \le 1$ for all t; in fact, the inequality is strict except at isolated points. So $-\displaystyle\int_0^\infty \frac{dt}{(1+x+t)[\ln(1+x+t)]^2} < J < \int_0^\infty \frac{dt}{(1+x+t)[\ln(1+x+t)]^2}$ $\Leftrightarrow$ $-\dfrac{1}{\ln(1+x)} < J < \dfrac{1}{\ln(1+x)}$ $\Leftrightarrow$ $0 < I < \dfrac{2}{\ln(1+x)}$.

14. (a) $T_n(x) = \cos(n\arccos x)$. The domain of arccos is $[-1,1]$, and the domain of cos is $\mathbb{R}$, so the domain of $T_n(x)$ is $[-1,1]$. As for the range, $T_0(x) = \cos 0 = 1$, so the range of $T_0(x)$ is $\{1\}$. But since the range of $n\arccos x$ is at least $[0,\pi]$ for $n>0$, and since $\cos y$ takes on all values in $[-1,1]$ for $y\in[0,\pi]$, the range of $T_n(x)$ is $[-1,1]$ for $n>0$.

(b) Using the usual trigonometric identities, $T_2(x) = \cos(2\arccos x) = 2[\cos(\arccos x)]^2 - 1 = 2x^2 - 1$, and

$$\begin{aligned} T_3(x) &= \cos(3\arccos x) = \cos(\arccos x + 2\arccos x) \\ &= \cos(\arccos x)\,\cos(2\arccos x) - \sin(\arccos x)\,\sin(2\arccos x) \\ &= x\,(2x^2-1) - \sin(\arccos x)\,[2\sin(\arccos x)\,\cos(\arccos x)] \\ &= 2x^3 - x - 2[\sin^2(\arccos x)]\,x = 2x^3 - x - 2x[1-\cos^2(\arccos x)] \\ &= 2x^3 - x - 2x(1-x^2) = 4x^3 - 3x \end{aligned}$$

(c) Let $y = \arccos x$. Then

$$\begin{aligned} T_{n+1}(x) &= \cos[(n+1)y] = \cos(y+ny) = \cos y\cos ny - \sin y\sin ny \\ &= 2\cos y\cos ny - (\cos y\cos ny + \sin y\sin ny) = 2xT_n(x) - \cos(ny-y) \\ &= 2xT_n(x) - T_{n-1}(x) \end{aligned}$$

(d) Here we use induction. $T_0(x) = 1$, a polynomial of degree 0. Now assume that $T_k(x)$ is a polynomial of degree k. Then $T_{k+1}(x) = 2xT_k(x) - T_{k-1}(x)$. By assumption, the leading term of T_k is a_kx^k, say, so the leading term of T_{k+1} is $2xa_kx^k = 2a_kx^{k+1}$, and so T_{k+1} has degree $k+1$.

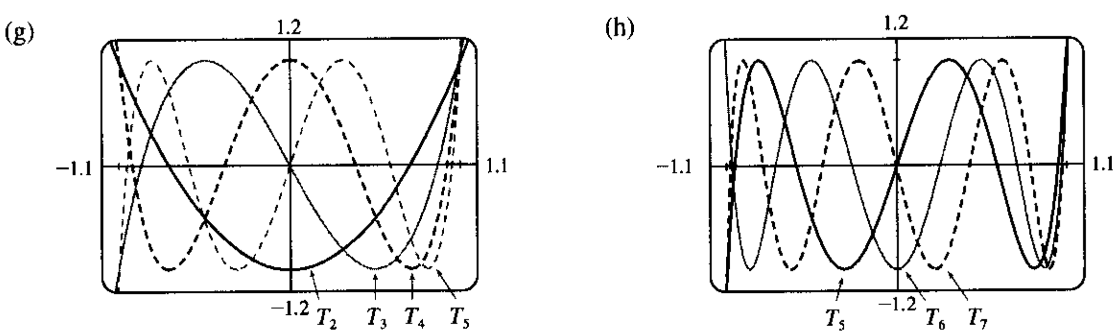
(e) $T_4(x) = 2xT_3(x) - T_2(x) = 2x(4x^3-3x) - (2x^2-1) = 8x^4 - 8x^2 + 1$,

$T_5(x) = 2xT_4(x) - T_3(x) = 2x(8x^4 - 8x^2 + 1) - (4x^3 - 3x) = 16x^5 - 20x^3 + 5x$,

$$T_6(x) = 2xT_5(x) - T_4(x) = 2x(16x^5 - 20x^3 + 5x) - (8x^4 - 8x^2 + 1) = 32x^6 - 48x^4 + 18x^2 - 1,$$

$$T_7(x) = 2xT_6(x) - T_5(x) = 2x(32x^6 - 48x^4 + 18x^2 - 1) - (16x^5 - 20x^3 + 5x)$$
$$= 64x^7 - 112x^5 + 56x^3 - 7x$$

(f) The zeros of $T_n(x) = \cos(n \arccos x)$ occur where $n \arccos x = k\pi + \frac{\pi}{2}$ for some integer k, since then $\cos(n \arccos x) = \cos(k\pi + \frac{\pi}{2}) = 0$. Note that there will be restrictions on k, since $0 \le \arccos x \le \pi$. We continue: $n \arccos x = k\pi + \frac{\pi}{2} \quad \Leftrightarrow \quad \arccos x = \dfrac{k\pi + \frac{\pi}{2}}{n}$. This only has solutions for $0 \le \dfrac{k\pi + \frac{\pi}{2}}{n} \le \pi$ $\Leftrightarrow \quad 0 < k\pi + \frac{\pi}{2} < n\pi \quad \Leftrightarrow \quad 0 \le k < n$. [This makes sense, because then $T_n(x)$ has n zeros, and it is a polynomial of degree n.] So, taking cosines of both sides of the last equation, we find that the zeros of $T_n(x)$ occur at $x = \cos \dfrac{k\pi + \frac{\pi}{2}}{n}$, k an integer with $0 \le k < n$. To find the values of x at which $T_n(x)$ has local extrema, we set $0 = T_n'(x) = -\sin(n \arccos x) \dfrac{-n}{\sqrt{1-x^2}} = \dfrac{n \sin(n \arccos x)}{\sqrt{1-x^2}} \quad \Leftrightarrow \quad \sin(n \arccos x) = 0$ $\Leftrightarrow \quad n \arccos x = k\pi$, k some integer $\quad \Leftrightarrow \quad \arccos x = k\pi/n$. This has solutions for $0 \le k \le n$, but we disallow the cases $k = 0$ and $k = n$, since these give $x = 1$ and $x = -1$ respectively. So the local extrema of $T_n(x)$ occur at $x = \cos(k\pi/n)$, k an integer with $0 < k < n$. [Again, this seems reasonable, since a polynomial of degree n has at most $(n-1)$ extrema.] By the First Derivative Test, the cases where k is even give maxima of $T_n(x)$, since then $n \arccos[\cos(k\pi/n)] = k\pi$ is an even multiple of π, so $\sin(n \arccos x)$ goes from negative to positive at $x = \cos(k\pi/n)$. Similarly, the cases where k is odd represent minima of $T_n(x)$.



(i) From the graphs, it seems that the zeros of T_n and T_{n+1} alternate; that is, between two adjacent zeros of T_n, there is a zero of T_{n+1}, and vice versa. The same is true of the x-coordinates of the extrema of T_n and T_{n+1}: between the x-coordinates of any two adjacent extrema of one, there is the x-coordinate of an extremum of the other.

(j) When n is odd, the function $T_n(x)$ is odd, since all of its terms have odd degree, and so $\int_{-1}^{1} T_n(x)\,dx = 0$. When n is even, $T_n(x)$ is even, and it appears that the integral is negative, but decreases in absolute value as n gets larger.

(k) $\int_{-1}^{1} T_n(x)\,dx = \int_{-1}^{1} \cos(n \arccos x)\,dx$. We substitute $u = \arccos x \Rightarrow x = \cos u \Rightarrow$ $dx = -\sin u\,du$, $x = -1 \Rightarrow u = \pi$, and $x = 1 \Rightarrow u = 0$. So the integral becomes

$$\begin{aligned}\int_0^{\pi} \cos(nu)\,\sin u\,du &= \int_0^{\pi} \tfrac{1}{2}[\sin(u - nu) + \sin(u + nu)]\,du \\ &= \frac{1}{2}\left[\frac{\cos[(1-n)u]}{n-1} - \frac{\cos[(1+n)u]}{n+1}\right]_0^{\pi} \\ &= \begin{cases} \frac{1}{2}\left[\left(\frac{-1}{n-1} - \frac{-1}{n+1}\right) - \left(\frac{1}{n-1} - \frac{1}{n+1}\right)\right] & \text{if } n \text{ is even} \\ \frac{1}{2}\left[\left(\frac{1}{n-1} - \frac{1}{n+1}\right) - \left(\frac{1}{n-1} - \frac{1}{n+1}\right)\right] & \text{if } n \text{ is odd} \end{cases} = \begin{cases} -\frac{2}{n^2-1} & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}\end{aligned}$$

(l) From the graph, we see that as c increases through an integer, the graph of f gains a local extremum, which starts at $x = -1$ and moves rightward, compressing the graph of f as c continues to increase.