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Chapter 10

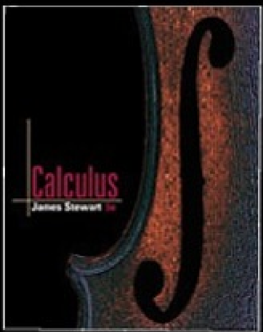
Adapted from the

Complete Solutions Manual

for

James Stewart's

Calculus – 5th Edition



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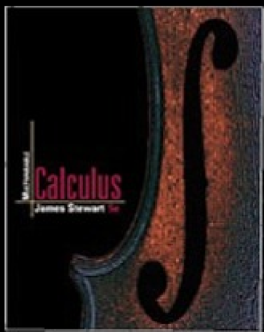

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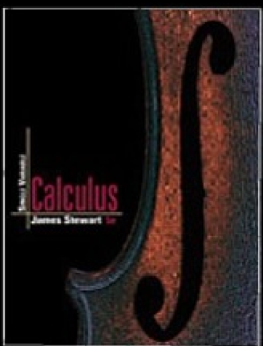

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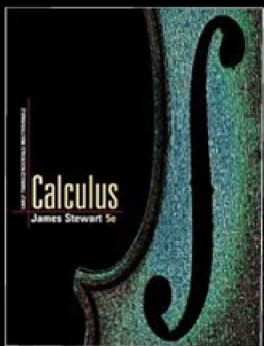

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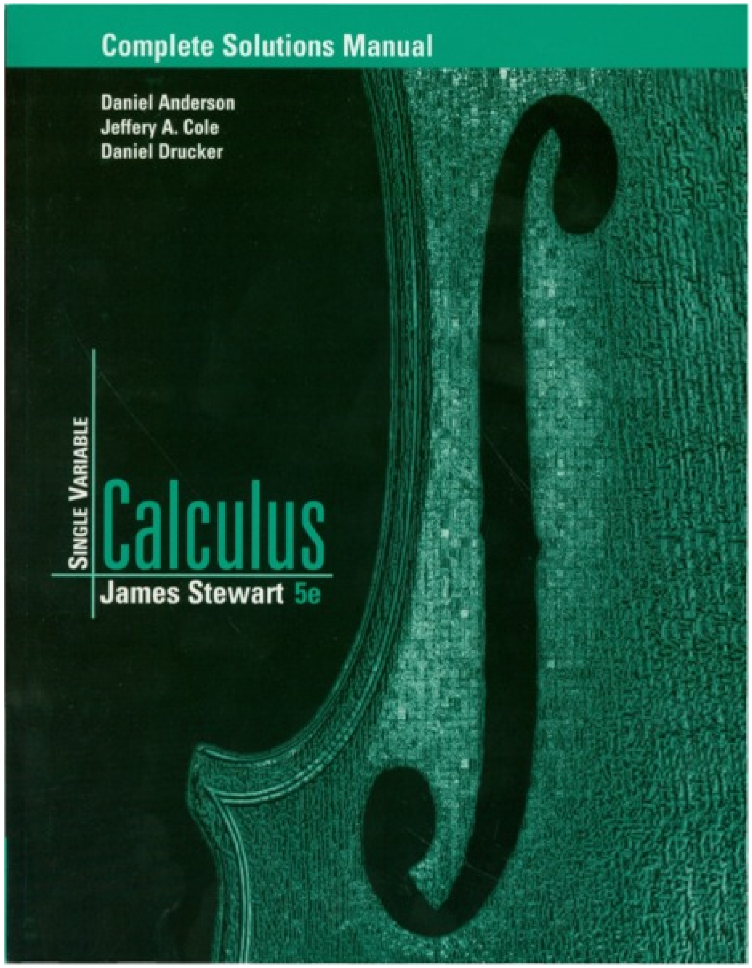


10 □ DIFFERENTIAL EQUATIONS

10.1 Modeling with Differential Equations

1. $y = x - x^{-1} \quad \Rightarrow \quad y' = 1 + x^{-2}$. To show that y is a solution of the differential equation, we will substitute the expressions for y and y' in the left-hand side of the equation and show that the left-hand side is equal to the right-hand side.

$$\text{LHS} = xy' + y = x(1 + x^{-2}) + (x - x^{-1}) = x + x^{-1} + x - x^{-1} = 2x = \text{RHS}$$

2. $y = \sin x \cos x - \cos x \quad \Rightarrow \quad y' = \sin x\,(-\sin x) + \cos x\,(\cos x) - (-\sin x) = \cos^2 x - \sin^2 x + \sin x$.

$$\begin{aligned} \text{LHS} &= y' + (\tan x)y = \cos^2 x - \sin^2 x + \sin x + (\tan x)(\sin x \cos x - \cos x) \\ &= \cos^2 x - \sin^2 x + \sin x + \sin^2 x - \sin x = \cos^2 x = \text{RHS}, \end{aligned}$$

so y is a solution of the differential equation. Also, $y(0) = \sin 0 \cos 0 - \cos 0 = 0 \cdot 1 - 1 = -1$, so the initial condition is satisfied.

3. (a) $y = \sin kt \quad \Rightarrow \quad y' = k \cos kt \quad \Rightarrow \quad y'' = -k^2 \sin kt$. $y'' + 9y = 0 \quad \Rightarrow$
$-k^2 \sin kt + 9 \sin kt = 0$ for all $t \quad \Leftrightarrow \quad (9 - k^2) \sin kt = 0$ for all $t \quad \Leftrightarrow \quad 9 - k^2 = 0 \quad \Leftrightarrow \quad k = \pm 3$

(b) $y = A \sin kt + B \cos kt \quad \Rightarrow \quad y' = Ak \cos kt - Bk \sin kt \quad \Rightarrow \quad y'' = -Ak^2 \sin kt - Bk^2 \cos kt$.
Thus, $y'' + 9y = 0 \quad \Rightarrow \quad -Ak^2 \sin kt - Bk^2 \cos kt + 9(A \sin kt + B \cos kt) = 0 \quad \Rightarrow$
$(9 - k^2) A \sin kt + (9 - k^2) B \cos kt = 0$. The last equation is true for all values of A and B if $k = \pm 3$.

4. $y = e^{rt} \quad \Rightarrow \quad y' = re^{rt} \quad \Rightarrow \quad y'' = r^2 e^{rt}$. $y'' + y' - 6y = 0 \quad \Rightarrow \quad r^2 e^{rt} + re^{rt} - 6e^{rt} = 0 \quad \Rightarrow$
$(r^2 + r - 6)e^{rt} = 0 \quad \Rightarrow \quad (r + 3)(r - 2) = 0 \quad \Rightarrow \quad r = -3$ or 2

5. (a) $y = e^t \quad \Rightarrow \quad y' = e^t \quad \Rightarrow \quad y'' = e^t$. $\text{LHS} = y'' + 2y' + y = e^t + 2e^t + e^t = 4e^t \neq 0$, so $y = e^t$ is not a solution of the differential equation.

(b) $y = e^{-t} \quad \Rightarrow \quad y' = -e^{-t} \quad \Rightarrow \quad y'' = e^{-t}$. $\text{LHS} = y'' + 2y' + y = e^{-t} - 2e^{-t} + e^{-t} = 0 = \text{RHS}$, so $y = e^{-t}$ is a solution.

(c) $y = te^{-t} \quad \Rightarrow \quad y' = t(-e^{-t}) + e^{-t}(1) = e^{-t}(1 - t) \quad \Rightarrow \quad y'' = e^{-t}(t - 2)$.

$$\begin{aligned} \text{LHS} &= y'' + 2y' + y = e^{-t}(t - 2) + 2e^{-t}(1 - t) + te^{-t} \\ &= e^{-t}\left[(t - 2) + 2(1 - t) + t\right] = e^{-t}(0) = 0 = \text{RHS}, \end{aligned}$$

so $y = te^{-t}$ is a solution.

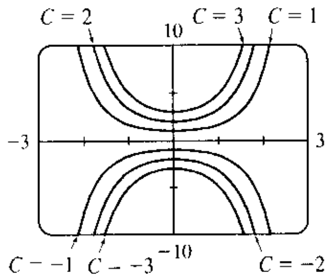
(d) $y = t^2 e^{-t} \quad \Rightarrow \quad y' = te^{-t}(2 - t) \quad \Rightarrow \quad y'' = e^{-t}(t^2 - 4t + 2)$.

$$\begin{aligned} \text{LHS} &= y'' + 2y' + y = e^{-t}(t^2 - 4t + 2) + 2te^{-t}(2 - t) + t^2 e^{-t} \\ &= e^{-t}\left[(t^2 - 4t + 2) + 2t(2 - t) + t^2\right] = e^{-t}(2) \neq 0, \end{aligned}$$

so $y = t^2 e^{-t}$ is not a solution.

6. (a) $y = Ce^{x^2/2} \Rightarrow y' = Ce^{x^2/2}(2x/2) = xCe^{x^2/2} = xy$.

(b)



(c) $y(0) = 5 \Rightarrow Ce^0 = 5 \Rightarrow C = 5$, so the solution is $y = 5e^{x^2/2}$.

(d) $y(1) = 2 \Rightarrow Ce^{1/2} = 2 \Rightarrow C = 2e^{-1/2}$, so the solution is $y = 2e^{-1/2}e^{x^2/2} = 2e^{(x^2-1)/2}$.

7. (a) Since the derivative $y' = -y^2$ is always negative (or 0 if $y = 0$), the function y must be decreasing (or equal to 0) on any interval on which it is defined.

(b) $y = \dfrac{1}{x+C} \Rightarrow y' = -\dfrac{1}{(x+C)^2}$. LHS $= y' = -\dfrac{1}{(x+C)^2} = -\left(\dfrac{1}{x+C}\right)^2 = -y^2 =$ RHS

(c) $y = 0$ is a solution of $y' = -y^2$ that is not a member of the family in part (b).

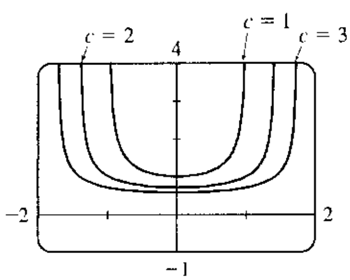
(d) If $y(x) = \dfrac{1}{x+C}$, then $y(0) = \dfrac{1}{0+C} = \dfrac{1}{C}$. Since $y(0) = 0.5$, $\dfrac{1}{C} = \dfrac{1}{2} \Rightarrow C = 2$, so $y = \dfrac{1}{x+2}$.

8. (a) If x is close to 0, then xy^3 is close to 0, and hence, y' is close to 0. Thus, the graph of y must have a tangent line that is nearly horizontal. If x is large, then xy^3 is large, and the graph of y must have a tangent line that is nearly vertical. (In both cases, we assume reasonable values for y.)

(b) $y = (c - x^2)^{-1/2} \Rightarrow y' = x(c - x^2)^{-3/2}$.

RHS $= xy^3 = x\left[(c-x^2)^{-1/2}\right]^3 = x(c-x^2)^{-3/2} = y' =$ LHS

(c)



When x is close to 0, y' is also close to 0. As x gets larger, so does $|y'|$.

(d) $y(0) = (c - 0)^{-1/2} = 1/\sqrt{c}$ and $y(0) = 2 \Rightarrow \sqrt{c} = \frac{1}{2} \Rightarrow c = \frac{1}{4}$, so $y = \left(\frac{1}{4} - x^2\right)^{-1/2}$.

9. (a) $\dfrac{dP}{dt} = 1.2P\left(1 - \dfrac{P}{4200}\right)$. Now $\dfrac{dP}{dt} > 0 \Rightarrow 1 - \dfrac{P}{4200} > 0$ [assuming that $P > 0$] $\Rightarrow \dfrac{P}{4200} < 1 \Rightarrow P < 4200 \Rightarrow$ the population is increasing for $0 < P < 4200$.

(b) $\dfrac{dP}{dt} < 0 \Rightarrow P > 4200$

(c) $\dfrac{dP}{dt} = 0 \Rightarrow P = 4200$ or $P = 0$

10. (a) $y = k \quad \Rightarrow \quad y' = 0$, so $\dfrac{dy}{dt} = y^4 - 6y^3 + 5y^2 \quad \Leftrightarrow \quad 0 = k^4 - 6k^3 + 5k^2 \quad \Leftrightarrow \quad k^2(k^2 - 6k + 5) = 0 \quad \Leftrightarrow$
$k^2(k-1)(k-5) = 0 \quad \Leftrightarrow \quad k = 0, 1,$ or 5

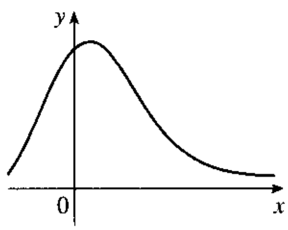
(b) y is increasing $\quad \Leftrightarrow \quad \dfrac{dy}{dt} > 0 \quad \Leftrightarrow \quad y^2(y-1)(y-5) > 0 \quad \Leftrightarrow \quad y \in (-\infty, 0) \cup (0, 1) \cup (5, \infty)$

(c) y is decreasing $\quad \Leftrightarrow \quad \dfrac{dy}{dt} < 0 \quad \Leftrightarrow \quad y \in (1, 5)$

11. (a) This function is increasing *and* also decreasing. But $dy/dt = e^t(y-1)^2 \geq 0$ for all t, implying that the graph of the solution of the differential equation cannot be decreasing on any interval.

(b) When $y = 1$, $dy/dt = 0$, but the graph does not have a horizontal tangent line.

12. The graph for this exercise is shown in the figure at the right.



A. $y' = 1 + xy > 1$ for points in the first quadrant, but we can see that $y' < 0$ for some points in the first quadrant. So equation A is incorrect.

B. $y' = -2xy = 0$ when $x = 0$, but we can see that $y' > 0$ for $x = 0$. So equation B is incorrect.

C. $y' = 1 - 2xy$ seems reasonable since:

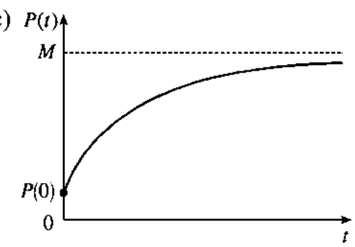
(1) When $x = 0$, y' could be 1.

(2) When $x < 0$, y' could be greater than 1.

(3) Solving $y' = 1 - 2xy$ for y gives us $y = \dfrac{1 - y'}{2x}$. If y' takes on small negative values, then as $x \to \infty$, $y \to 0^+$, as shown in the figure. Thus, the correct equation is C.

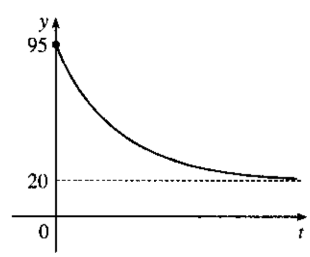
13. (a) P increases most rapidly at the beginning, since there are usually many simple, easily-learned sub-skills associated with learning a skill. As t increases, we would expect dP/dt to remain positive, but decrease. This is because as time progresses, the only points left to learn are the more difficult ones.

(b) $\dfrac{dP}{dt} = k(M - P)$ is always positive, so the level of performance P is increasing. As P gets close to M, dP/dt gets close to 0; that is, the performance levels off, as explained in part (a).

(c)


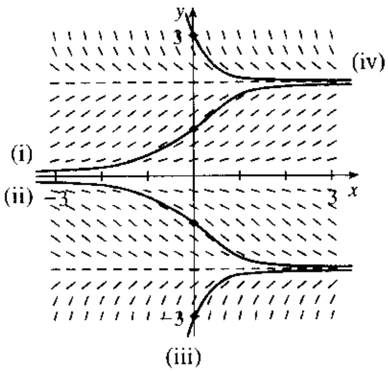
14. (a) The coffee cools most quickly as soon as it is removed from the heat source. The rate of cooling decreases toward 0 since the coffee approaches room temperature.

(b) $\dfrac{dy}{dt} = k(y - R)$, where k is a proportionality constant, y is the temperature of the coffee, and R is the room temperature. The initial condition is $y(0) = 95\,°C$. The answer and the model support each other because as y approaches R, dy/dt approaches 0, so the model seems appropriate.

(c)


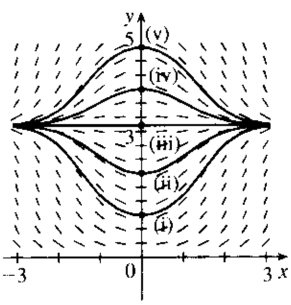
10.2 Direction Fields and Euler's Method

1. (a)



(b) It appears that the constant functions $y = 0$, $y = -2$, and $y = 2$ are equilibrium solutions. Note that these three values of y satisfy the given differential equation $y' = y\left(1 - \frac{1}{4}y^2\right)$.

2. (a)



(b) From the figure, it appears that $y = \pi$ is an equilibrium solution. From the equation $y' = x \sin y$, we see that $y = n\pi$ (n an integer) describes all the equilibrium solutions.

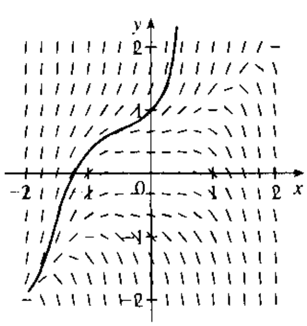
3. $y' = y - 1$. The slopes at each point are independent of x, so the slopes are the same along each line parallel to the x-axis. Thus, IV is the direction field for this equation. Note that for $y = 1$, $y' = 0$.

4. $y' = y - x = 0$ on the line $y = x$, when $x = 0$ the slope is y, and when $y = 0$ the slope is $-x$. Direction field II satisfies these conditions. [Looking at the slope at the point $(0, 2)$, II looks more like it has a slope of 2 than does direction field I.]

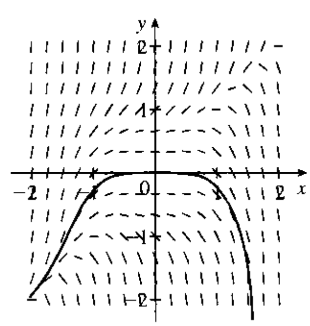
5. $y' = y^2 - x^2 = 0 \;\Rightarrow\; y = \pm x$. There are horizontal tangents on these lines only in graph III, so this equation corresponds to direction field III.

6. $y' = y^3 - x^3 = 0$ on the line $y = x$, when $x = 0$ the slope is y^3, and when $y = 0$ the slope is $-x^3$. The graph is similar to the graph for Exercise 4, but the segments must get steeper very rapidly as they move away from the origin, because x and y are raised to the third power. This is the case in direction field I.

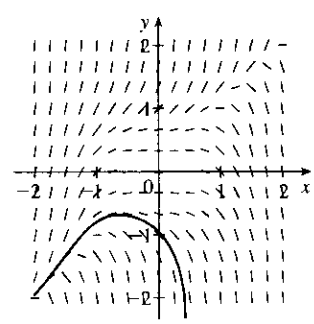
7. (a) $y(0) = 1$



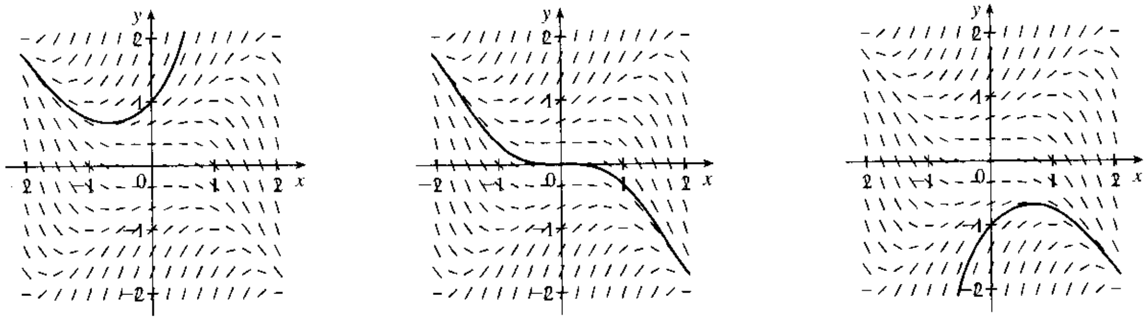
(b) $y(0) = 0$



(c) $y(0) = -1$



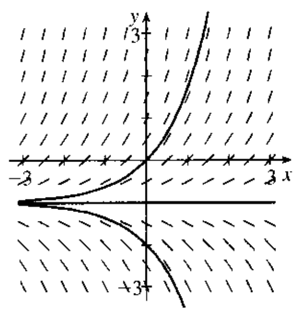
8. (a) $y(0) = 1$ (b) $y(0) = 0$ (c) $y(0) = -1$



9.

x	y	$y' = 1 + y$
0	0	1
0	1	2
0	2	3
0	-3	-2
0	-2	-1

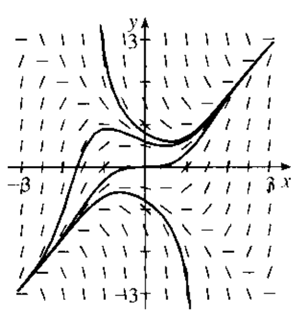
Note that for $y = -1$, $y' = 0$. The three solution curves sketched go through $(0, 0)$, $(0, -1)$, and $(0, -2)$.



10.

x	y	$y' = x^2 - y^2$
± 1	± 3	-8
± 3	± 1	8
± 1	± 0.5	0.75
± 0.5	± 1	-0.75

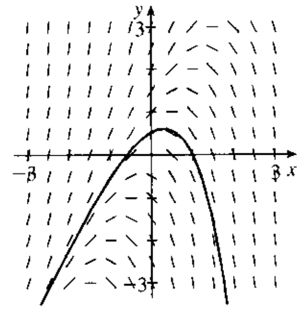
Note that $y' = 0$ for $y = \pm x$. If $|x| < |y|$, then $y' < 0$; that is, the slopes are negative for all points in quadrants I and II above both of the lines $y = x$ and $y = -x$, and all points in quadrants III and IV below both of the lines $y = -x$ and $y = x$. A similar statement holds for positive slopes.



11.

x	y	$y' = y - 2x$
-2	-2	2
-2	2	6
2	2	-2
2	-2	-6

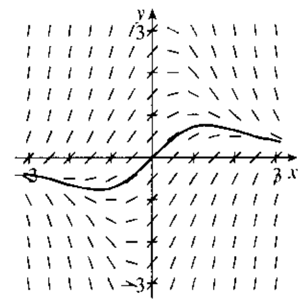
Note that $y' = 0$ for any point on the line $y = 2x$. The slopes are positive to the left of the line and negative to the right of the line. The solution curve in the graph passes through $(1, 0)$.



12.

x	y	$y' = 1 - xy$
± 1	± 1	0
± 2	± 2	-3
± 2	∓ 2	5

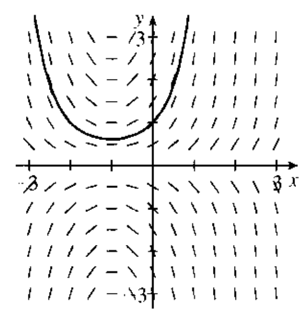
Note that $y' = 0$ for any point on the hyperbola $xy = 1$ (or $y = 1/x$). The slopes are negative at points "inside" the branches and positive at points everywhere else. The solution curve in the graph passes through $(0, 0)$.



13.

x	y	$y' = y + xy$
0	± 2	± 2
1	± 2	± 4
-3	± 2	∓ 4

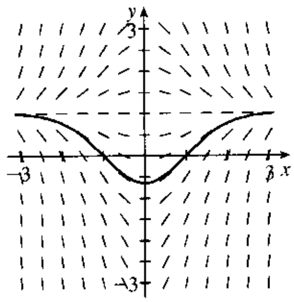
Note that $y' = y(x + 1) = 0$ for any point on $y = 0$ or on $x = -1$. The slopes are positive when the factors y and $x + 1$ have the same sign and negative when they have opposite signs. The solution curve in the graph passes through $(0, 1)$.



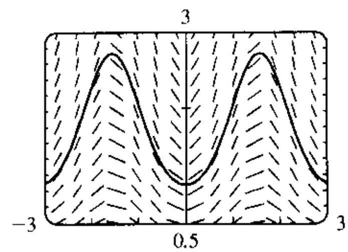
14.

x	y	$y' = x - xy$
± 2	0	± 2
± 2	3	∓ 4
± 2	-1	± 4

Note that $y' = x(1-y) = 0$ for any point on $x = 0$ or on $y = 1$. The slopes are positive when the factors x and $1 - y$ have the same sign and negative when they have opposite signs. The solution curve in the graph passes through $(1, 0)$.

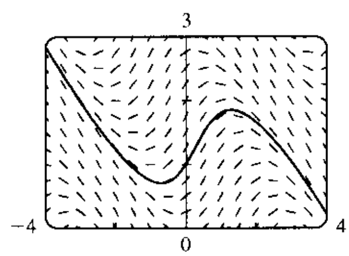


15. In Maple, we can use either `directionfield` (in Maple's share library) or `plots[fieldplot]` to plot the direction field. To plot the solution, we can either use the initial-value option in `directionfield`, or actually solve the equation. In Mathematica, we use `PlotVectorField` for the direction field, and the `Plot[Evaluate[...]]` construction to plot the solution, which is $y = e^{(1-\cos 2x)/2}$. In Derive, use `Direction_Field` (in utility file `ODE_APPR`) to plot the direction field. Then use `DSOLVE1(-y*SIN(2*x),1,x,y,0,1)` (in utility file `ODE1`) to solve the equation. Simplify each result.

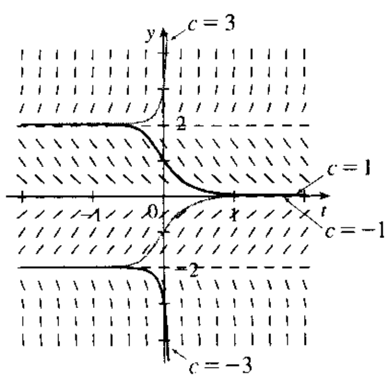


16. See Exercise 15 for specific CAS directions. The exact solution is

$$y = -x - 2\arctan\frac{2 + x - \dfrac{2}{1+\tan(1/2)}}{x - \dfrac{2}{1+\tan(1/2)}}$$

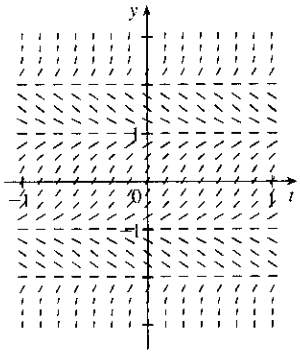


17.



$L = \lim_{t \to \infty} y(t)$ exists for $-2 \le c \le 2$; $L = \pm 2$ for $c = \pm 2$ and $L = 0$ for $-2 < c < 2$. For other values of c, L does not exist.

18.



Note that when $f(y) = 0$ on the graph in the text, we have $y' = f(y) = 0$; so we get horizontal segments at $y = \pm 1, \pm 2$. We get segments with negative slopes only for $1 < |y| < 2$. All other segments have positive slope. For the limiting behavior of solutions:

- If $y(0) > 2$, then $\lim_{t\to\infty} y = \infty$ and $\lim_{t\to-\infty} y = 2$.
- If $1 < y(0) < 2$, then $\lim_{t\to\infty} y = 1$ and $\lim_{t\to-\infty} y = 2$.
- If $-1 < y(0) < 1$, then $\lim_{t\to\infty} y = 1$ and $\lim_{t\to-\infty} y = -1$.
- If $-2 < y(0) < -1$, then $\lim_{t\to\infty} y = -2$ and $\lim_{t\to-\infty} y = -1$.
- If $y < -2$, then $\lim_{t\to\infty} y = -2$ and $\lim_{t\to-\infty} y = -\infty$.

19. (a) $y' = F(x, y) = y$ and $y(0) = 1 \quad\Rightarrow\quad x_0 = 0, y_0 = 1$.

(i) $h = 0.4$ and $y_1 = y_0 + hF(x_0, y_0) \quad\Rightarrow\quad y_1 = 1 + 0.4 \cdot 1 = 1.4$. $x_1 = x_0 + h = 0 + 0.4 = 0.4$, so $y_1 = y\,(0.4) = 1.4$.

(ii) $h = 0.2 \quad\Rightarrow\quad x_1 = 0.2$ and $x_2 = 0.4$, so we need to find y_2.

$y_1 = y_0 + hF(x_0, y_0) = 1 + 0.2y_0 = 1 + 0.2 \cdot 1 = 1.2$,

$y_2 = y_1 + hF(x_1, y_1) = 1.2 + 0.2y_1 = 1.2 + 0.2 \cdot 1.2 = 1.44$.

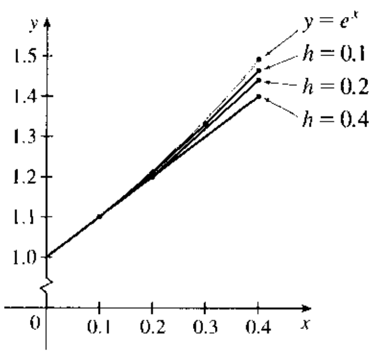
(iii) $h = 0.1 \quad\Rightarrow\quad x_4 = 0.4$, so we need to find y_4. $y_1 = y_0 + hF(x_0, y_0) = 1 + 0.1y_0 = 1 + 0.1 \cdot 1 = 1.1$,

$y_2 = y_1 + hF(x_1, y_1) = 1.1 + 0.1y_1 = 1.1 + 0.1 \cdot 1.1 = 1.21$,

$y_3 = y_2 + hF(x_2, y_2) = 1.21 + 0.1y_2 = 1.21 + 0.1 \cdot 1.21 = 1.331$,

$y_4 = y_3 + hF(x_3, y_3) = 1.331 + 0.1y_3 = 1.331 + 0.1 \cdot 1.331 = 1.4641$.

(b)



We see that the estimates are underestimates since they are all below the graph of $y = e^x$.

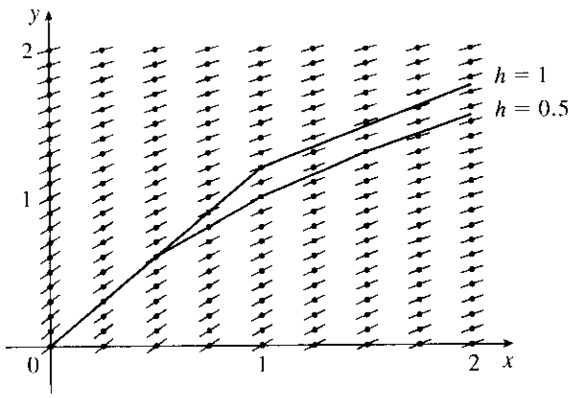
(c) (i) For $h = 0.4$: (exact value) − (approximate value) $= e^{0.4} - 1.4 \approx 0.0918$

(ii) For $h = 0.2$: (exact value) − (approximate value) $= e^{0.4} - 1.44 \approx 0.0518$

(iii) For $h = 0.1$: (exact value) − (approximate value) $= e^{0.4} - 1.4641 \approx 0.0277$

Each time the step size is halved, the error estimate also appears to be halved (approximately).

20.



As x increases, the slopes decrease and all of the estimates are above the true values. Thus, all of the estimates are overestimates.

21. $h = 0.5$, $x_0 = 1$, $y_0 = 0$, and $F(x, y) = y - 2x$.

Note that $x_1 = x_0 + h = 1 + 0.5 = 1.5$, $x_2 = 2$, and $x_3 = 2.5$.

$y_1 = y_0 + hF(x_0, y_0) = 0 + 0.5F(1, 0) = 0.5[0 - 2(1)] = -1.$

$y_2 = y_1 + hF(x_1, y_1) = -1 + 0.5F(1.5, -1) = -1 + 0.5[-1 - 2(1.5)] = -3.$

$y_3 = y_2 + hF(x_2, y_2) = -3 + 0.5F(2, -3) = -3 + 0.5[-3 - 2(2)] = -6.5.$

$y_4 = y_3 + hF(x_3, y_3) = -6.5 + 0.5F(2.5, -6.5) = -6.5 + 0.5[-6.5 - 2(2.5)] = -12.25.$

22. $h = 0.2$, $x_0 = 0$, $y_0 = 0$, and $F(x, y) = 1 - xy$.

Note that $x_1 = x_0 + h = 0 + 0.2 = 0.2$, $x_2 = 0.4$, $x_3 = 0.6$, and $x_4 = 0.8$.

$y_1 = y_0 + hF(x_0, y_0) = 0 + 0.2F(0, 0) = 0.2[1 - (0)(0)] = 0.2.$

$y_2 = y_1 + hF(x_1, y_1) = 0.2 + 0.2F(0.2, 0.2) = 0.2 + 0.2[1 - (0.2)(0.2)] = 0.392.$

$y_3 = y_2 + hF(x_2, y_2) = 0.392 + 0.2F(0.4, 0.392) = 0.392 + 0.2[1 - (0.4)(0.392)] = 0.56064.$

$y_4 = y_3 + hF(x_3, y_3) = 0.56064 + 0.2[1 - (0.6)(0.56064)] = 0.6933632.$

$y_5 = y_4 + hF(x_4, y_4) = 0.6933632 + 0.2[1 - (0.8)(0.6933632)] = 0.782425088.$

Thus, $y(1) \approx 0.7824$.

23. $h = 0.1$, $x_0 = 0$, $y_0 = 1$, and $F(x, y) = y + xy$.

Note that $x_1 = x_0 + h = 0 + 0.1 = 0.1$, $x_2 = 0.2$, $x_3 = 0.3$, and $x_4 = 0.4$.

$y_1 = y_0 + hF(x_0, y_0) = 1 + 0.1F(0, 1) = 1 + 0.1[1 + (0)(1)] = 1.1.$

$y_2 = y_1 + hF(x_1, y_1) = 1.1 + 0.1F(0.1, 1.1) = 1.1 + 0.1[1.1 + (0.1)(1.1)] = 1.221.$

$y_3 = y_2 + hF(x_2, y_2) = 1.221 + 0.1F(0.2, 1.221) = 1.221 + 0.1[1.221 + (0.2)(1.221)] = 1.36752.$

$$y_4 = y_3 + hF(x_3, y_3) = 1.36752 + 0.1F(0.3, 1.36752) = 1.36752 + 0.1[1.36752 + (0.3)(1.36752)] = 1.5452976.$$

$$y_5 = y_4 + hF(x_4, y_4) = 1.5452976 + 0.1F(0.4, 1.5452976) = 1.5452976 + 0.1[1.5452976 + (0.4)(1.5452976)] = 1.761639264.$$

Thus, $y(0.5) \approx 1.7616$.

24. (a) $h = 0.2$, $x_0 = 1$, $y_0 = 0$, and $F(x, y) = x - xy$.

We need to find y_2, because $x_1 = 1.2$ and $x_2 = 1.4$.

$y_1 = y_0 + hF(x_0, y_0) = 0 + 0.2F(1, 0) = 0.2[1 - (1)(0)] = 0.2.$

$y_2 = y_1 + hF(x_1, y_1) = 0.2 + 0.2F(1.2, 0.2) = 0.2 + 0.2[1.2 - (1.2)(0.2)] = 0.392 \approx y(1.4).$

(b) Now $h = 0.1$, so we need to find y_4.

$y_1 = 0 + 0.1[1 - (1)(0)] = 0.1$,

$y_2 = 0.1 + 0.1[1.1 - (1.1)(0.1)] = 0.199$,

$y_3 = 0.199 + 0.1[1.2 - (1.2)(0.199)] = 0.29512$, and

$y_4 = 0.29512 + 0.1[1.3 - (1.3)(0.29512)] = 0.3867544 \approx y(1.4)$.

25. (a) $dy/dx + 3x^2y = 6x^2 \quad\Rightarrow\quad y' = 6x^2 - 3x^2y$. Store this expression in Y_1 and use the following simple program to evaluate $y(1)$ for each part, using $\mathrm{H} = h = 1$ and $\mathrm{N} = 1$ for part (i), $\mathrm{H} = 0.1$ and $\mathrm{N} = 10$ for part (ii), and so forth.

$h \rightarrow$ H: 0 $\rightarrow$ X: 3 $\rightarrow$ Y:

For(I, 1, N): Y + H $\times$ Y_1 $\rightarrow$ Y: X + H $\rightarrow$ X:

End(loop):

Display Y. [To see all iterations, include this statement in the loop.]

(i) $\mathrm{H} = 1, \mathrm{N} = 1 \quad\Rightarrow\quad y(1) = 3$

(ii) $\mathrm{H} = 0.1, \mathrm{N} = 10 \quad\Rightarrow\quad y(1) \approx 2.3928$

(iii) $\mathrm{H} = 0.01, \mathrm{N} = 100 \quad\Rightarrow\quad y(1) \approx 2.3701$

(iv) $\mathrm{H} = 0.001, \mathrm{N} = 1000 \quad\Rightarrow\quad y(1) \approx 2.3681$

(b) $y = 2 + e^{-x^3} \quad\Rightarrow\quad y' = -3x^2e^{-x^3}$

$$\mathrm{LHS} = y' + 3x^2y = -3x^2e^{-x^3} + 3x^2\left(2 + e^{-x^3}\right) = -3x^2e^{-x^3} + 6x^2 + 3x^2e^{-x^3} = 6x^2 = \mathrm{RHS}$$

$y(0) = 2 + e^{-0} = 2 + 1 = 3$

(c) The exact value of $y(1)$ is $2 + e^{-1^3} = 2 + e^{-1}$.

(i) For $h = 1$: (exact value) $-$ (approximate value) $= 2 + e^{-1} - 3 \approx -0.6321$

(ii) For $h = 0.1$: (exact value) $-$ (approximate value) $= 2 + e^{-1} - 2.3928 \approx -0.0249$

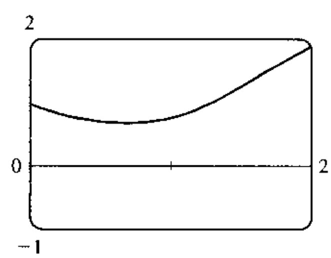
(iii) For $h = 0.01$: (exact value) $-$ (approximate value) $= 2 + e^{-1} - 2.3701 \approx -0.0022$

(iv) For $h = 0.001$: (exact value) $-$ (approximate value) $= 2 + e^{-1} - 2.3681 \approx -0.0002$

In (ii)–(iv), it seems that when the step size is divided by 10, the error estimate is also divided by 10 (approximately).

26. (a) We use the program from the solution to Exercise 25 with $\mathrm{Y}_1 = x^3 - y^3$, $\mathrm{H} = 0.01$, and $\mathrm{N} = \frac{2-0}{0.01} = 200$. With $(x_0, y_0) = (0, 1)$, we get $y(2) \approx 1.9000$.

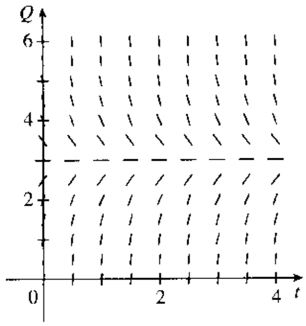
(b)



Notice from the graph that $y(2) \approx 1.9$, which serves as a check on our calculation in part (a).

27. (a) $R\dfrac{dQ}{dt} + \dfrac{1}{C}Q = E(t)$ becomes

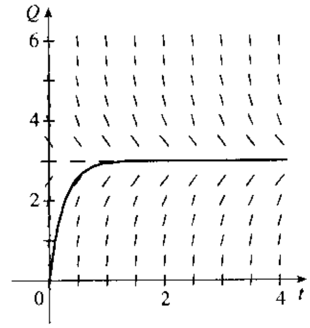
$5Q' + \frac{1}{0.05}Q = 60$ or $Q' + 4Q = 12$.



(b) From the graph, it appears that the limiting value of the charge Q is about 3.

(c) If $Q' = 0$, then $4Q = 12 \;\Rightarrow\; Q = 3$ is an equilibrium solution.

(d)



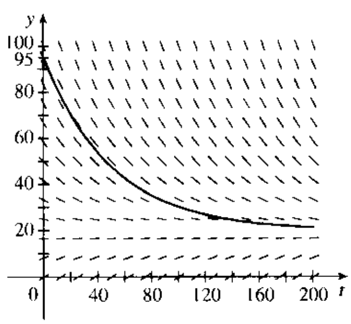
(e) $Q' + 4Q = 12 \;\Rightarrow\; Q' = 12 - 4Q$. Now $Q(0) = 0$, so $t_0 = 0$ and $Q_0 = 0$.

$$Q_1 = Q_0 + hF(t_0, Q_0) = 0 + 0.1(12 - 4 \cdot 0) = 1.2$$
$$Q_2 = Q_1 + hF(t_1, Q_1) = 1.2 + 0.1(12 - 4 \cdot 1.2) = 1.92$$
$$Q_3 = Q_2 + hF(t_2, Q_2) = 1.92 + 0.1(12 - 4 \cdot 1.92) = 2.352$$
$$Q_4 = Q_3 + hF(t_3, Q_3) = 2.352 + 0.1(12 - 4 \cdot 2.352) = 2.6112$$
$$Q_5 = Q_4 + hF(t_4, Q_4) = 2.6112 + 0.1(12 - 4 \cdot 2.6112) = 2.76672$$

Thus, $Q_5 = Q(0.5) \approx 2.77$ C.

28. (a) From Exercise 10.1.14, we have $dy/dt = k(y - R)$. We are given that $R = 20°\text{C}$ and $dy/dt = -1°\text{C/min}$ when $y = 70°\text{C}$. Thus, $-1 = k(70 - 20) \;\Rightarrow\; k = -\frac{1}{50}$ and the differential equation becomes $dy/dt = -\frac{1}{50}(y - 20)$.

(b)



The limiting value of the temperature is 20°C; that is, the temperature of the room.

(c) From part (a), $dy/dt = -\frac{1}{50}(y - 20)$. With $t_0 = 0$, $y_0 = 95$, and $h = 2$ min, we get

$$y_1 = y_0 + hF(t_0, y_0) = 95 + 2\left[-\tfrac{1}{50}(95 - 20)\right] = 92$$
$$y_2 = y_1 + hF(t_1, y_1) = 92 + 2\left[-\tfrac{1}{50}(92 - 20)\right] = 89.12$$
$$y_3 = y_2 + hF(t_2, y_2) = 89.12 + 2\left[-\tfrac{1}{50}(89.12 - 20)\right] = 86.3552$$
$$y_4 = y_3 + hF(t_3, y_3) = 86.3552 + 2\left[-\tfrac{1}{50}(86.3552 - 20)\right] = 83.700992$$
$$y_5 = y_4 + hF(t_4, y_4) = 83.700992 + 2\left[-\tfrac{1}{50}(83.700992 - 20)\right] = 81.15295232$$

Thus, $y(10) \approx 81.15°\text{C}$.

10.3 Separable Equations

1. $\dfrac{dy}{dx} = \dfrac{y}{x} \;\Rightarrow\; \dfrac{dy}{y} = \dfrac{dx}{x}\ [y \neq 0] \;\Rightarrow\; \displaystyle\int \frac{dy}{y} = \int \frac{dx}{x} \;\Rightarrow\; \ln|y| = \ln|x| + C \;\Rightarrow$

$|y| = e^{\ln|x|+C} = e^{\ln|x|}e^C = e^C\,|x| \;\Rightarrow\; y = Kx$, where $K = \pm e^C$ is a constant. (In our derivation, K was nonzero, but we can restore the excluded case $y = 0$ by allowing K to be zero.)

2. $\dfrac{dy}{dx} = \dfrac{e^{2x}}{4y^3} \;\Rightarrow\; 4y^3\,dy = e^{2x}\,dx \;\Rightarrow\; \int 4y^3\,dy = \int e^{2x}\,dx \;\Rightarrow\; y^4 = \frac{1}{2}e^{2x} + C \;\Rightarrow$

$y = \pm\sqrt[4]{\frac{1}{2}e^{2x} + C}$

3. $(x^2+1)y' = xy \;\Rightarrow\; \dfrac{dy}{dx} = \dfrac{xy}{x^2+1} \;\Rightarrow\; \dfrac{dy}{y} = \dfrac{x\,dx}{x^2+1}\ [y \neq 0] \;\Rightarrow\; \displaystyle\int \frac{dy}{y} = \int \frac{x\,dx}{x^2+1} \;\Rightarrow$

$\ln|y| = \frac{1}{2}\ln(x^2+1) + C\quad [u = x^2+1,\ du = 2x\,dx]\quad = \ln(x^2+1)^{1/2} + \ln e^C = \ln\left(e^C\sqrt{x^2+1}\right) \;\Rightarrow$

$|y| = e^C\sqrt{x^2+1} \;\Rightarrow\; y = K\sqrt{x^2+1}$, where $K = \pm e^C$ is a constant. (In our derivation, K was nonzero, but we can restore the excluded case $y = 0$ by allowing K to be zero.)

4. $y' = y^2\sin x \;\Rightarrow\; \dfrac{dy}{dx} = y^2\sin x \;\Rightarrow\; \dfrac{dy}{y^2} = \sin x\,dx\ [y \neq 0] \;\Rightarrow\; \displaystyle\int \frac{dy}{y^2} = \int \sin x\,dx \;\Rightarrow$

$-\dfrac{1}{y} = -\cos x + C \;\Rightarrow\; \dfrac{1}{y} = \cos x - C \;\Rightarrow\; y = \dfrac{1}{\cos x + K}$, where $K = -C$. $y = 0$ is also a solution.

5. $(1+\tan y)\,y' = x^2+1 \;\Rightarrow\; (1+\tan y)\dfrac{dy}{dx} = x^2+1 \;\Rightarrow\; \left(1 + \dfrac{\sin y}{\cos y}\right)dy = (x^2+1)\,dx \;\Rightarrow$

$\displaystyle\int\left(1 - \frac{-\sin y}{\cos y}\right)dy = \int (x^2+1)\,dx \;\Rightarrow\; y - \ln|\cos y| = \tfrac{1}{3}x^3 + x + C$. Note: The left side is equivalent to $y + \ln|\sec y|$.

6. $\dfrac{du}{dr} = \dfrac{1+\sqrt{r}}{1+\sqrt{u}} \;\Rightarrow\; (1+\sqrt{u}\,)\,du = (1+\sqrt{r}\,)\,dr \;\Rightarrow\; \int(1+u^{1/2})\,du = \int(1+r^{1/2})\,dr \;\Rightarrow$

$u + \frac{2}{3}u^{3/2} = r + \frac{2}{3}r^{3/2} + C$

7. $\dfrac{dy}{dt} = \dfrac{te^t}{y\sqrt{1+y^2}} \;\Rightarrow\; y\sqrt{1+y^2}\,dy = te^t\,dt \;\Rightarrow\; \int y\sqrt{1+y^2}\,dy = \int te^t\,dt \;\Rightarrow$

$\frac{1}{3}\left(1+y^2\right)^{3/2} = te^t - e^t + C$ [where the first integral is evaluated by substitution and the second by parts] $\Rightarrow$

$1 + y^2 = [3(te^t - e^t + C)]^{2/3} \;\Rightarrow\; y = \pm\sqrt{[3(te^t - e^t + C)]^{2/3} - 1}$

8. $y' = \dfrac{xy}{2\ln y} \;\Rightarrow\; \dfrac{2\ln y}{y}\,dy = x\,dx \;\Rightarrow\; \displaystyle\int \frac{2\ln y}{y}\,dy = \int x\,dx \;\Rightarrow\; (\ln y)^2 = \frac{x^2}{2} + C \;\Rightarrow$

$\ln y = \pm\sqrt{x^2/2 + C} \;\Rightarrow\; y = e^{\pm\sqrt{x^2/2 + C}}$

9. $\frac{du}{dt} = 2 + 2u + t + tu \Rightarrow \frac{du}{dt} = (1+u)(2+t) \Rightarrow \int \frac{du}{1+u} = \int (2+t)dt \; [u \neq -1] \Rightarrow$

$\ln|1+u| = \frac{1}{2}t^2 + 2t + C \Rightarrow |1+u| = e^{t^2/2+2t+C} = Ke^{t^2/2+2t}$, where $K = e^C \Rightarrow$

$1 + u = \pm Ke^{t^2/2+2t} \Rightarrow u = -1 \pm Ke^{t^2/2+2t}$ where $K > 0$. $u = -1$ is also a solution, so

$u = -1 + Ae^{t^2/2+2t}$, where A is an arbitrary constant.

10. $\frac{dz}{dt} + e^{t+z} = 0 \Rightarrow \frac{dz}{dt} = -e^t e^z \Rightarrow \int e^{-z}\,dz = -\int e^t\,dt \Rightarrow -e^{-z} = -e^t + C \Rightarrow e^{-z} = e^t - C$

$\Rightarrow \frac{1}{e^z} = e^t - C \Rightarrow e^z = \frac{1}{e^t - C} \Rightarrow z = \ln\left(\frac{1}{e^t - C}\right) \Rightarrow z = -\ln(e^t - C)$

11. $\frac{dy}{dx} = y^2 + 1$, $y(1) = 0$. $\int \frac{dy}{y^2+1} = \int dx \Rightarrow \tan^{-1} y = x + C$. $y = 0$ when $x = 1$, so

$1 + C = \tan^{-1} 0 = 0 \Rightarrow C = -1$. Thus, $\tan^{-1} y = x - 1$ and $y = \tan(x-1)$.

12. $\frac{dy}{dx} = \frac{y\cos x}{1+y^2}$, $y(0) = 1$. $(1+y^2)\,dy = y\cos x\,dx \Rightarrow \frac{1+y^2}{y}dy = \cos x\,dx \Rightarrow$

$\int\left(\frac{1}{y} + y\right)dy = \int \cos x\,dx \Rightarrow \ln|y| + \frac{1}{2}y^2 = \sin x + C$. $y(0) = 1 \Rightarrow \ln 1 + \frac{1}{2} = \sin 0 + C \Rightarrow$

$C = \frac{1}{2}$, so $\ln|y| + \frac{1}{2}y^2 = \sin x + \frac{1}{2}$. We cannot solve explicitly for y.

13. $x\cos x = (2y + e^{3y})\,y' \Rightarrow x\cos x\,dx = (2y + e^{3y})\,dy \Rightarrow \int(2y + e^{3y})\,dy = \int x\cos x\,dx \Rightarrow$

$y^2 + \frac{1}{3}e^{3y} = x\sin x + \cos x + C$ [where the second integral is evaluated using integration by parts]. Now

$y(0) = 0 \Rightarrow 0 + \frac{1}{3} = 0 + 1 + C \Rightarrow C = -\frac{2}{3}$. Thus, a solution is $y^2 + \frac{1}{3}e^{3y} = x\sin x + \cos x - \frac{2}{3}$.

We cannot solve explicitly for y.

14. $\frac{dP}{dt} = \sqrt{Pt} \Rightarrow dP/\sqrt{P} = \sqrt{t}\,dt \Rightarrow \int P^{-1/2}\,dP = \int t^{1/2}\,dt \Rightarrow 2P^{1/2} = \frac{2}{3}t^{3/2} + C$.

$P(1) = 2 \Rightarrow 2\sqrt{2} = \frac{2}{3} + C \Rightarrow C = 2\sqrt{2} - \frac{2}{3}$, so $2P^{1/2} = \frac{2}{3}t^{3/2} + 2\sqrt{2} - \frac{2}{3} \Rightarrow$

$\sqrt{P} = \frac{1}{3}t^{3/2} + \sqrt{2} - \frac{1}{3} \Rightarrow P = \left(\frac{1}{3}t^{3/2} + \sqrt{2} - \frac{1}{3}\right)^2$.

15. $\frac{du}{dt} = \frac{2t + \sec^2 t}{2u}$, $u(0) = -5$. $\int 2u\,du = \int(2t + \sec^2 t)\,dt \Rightarrow u^2 = t^2 + \tan t + C$, where

$[u(0)]^2 = 0^2 + \tan 0 + C \Rightarrow C = (-5)^2 = 25$. Therefore, $u^2 = t^2 + \tan t + 25$, so $u = \pm\sqrt{t^2 + \tan t + 25}$.

Since $u(0) = -5$, we must have $u = -\sqrt{t^2 + \tan t + 25}$.

16. $\frac{dy}{dt} = te^y$, $y(1) = 0$. $\int e^{-y}\,dy = \int t\,dt \Rightarrow -e^{-y} = \frac{1}{2}t^2 + C$. Since $y(1) = 0$, $-e^0 = \frac{1}{2} \cdot 1^2 + C$. Therefore,

$C = -1 - \frac{1}{2} = -\frac{3}{2}$ and $-e^{-y} = \frac{1}{2}t^2 - \frac{3}{2}$. So $e^{-y} = \frac{3}{2} - \frac{1}{2}t^2 = \frac{3 - t^2}{2} \Rightarrow e^y = \frac{2}{3 - t^2} \Rightarrow$

$y = \ln 2 - \ln(3 - t^2)$ for $|t| < \sqrt{3}$.

17. $y' \tan x = a + y,\ 0 < x < \pi/2 \quad\Rightarrow\quad \dfrac{dy}{dx} = \dfrac{a+y}{\tan x} \quad\Rightarrow\quad \dfrac{dy}{a+y} = \cot x\,dx \quad [a + y \neq 0] \quad\Rightarrow$

$\displaystyle\int \frac{dy}{a+y} = \int \frac{\cos x}{\sin x}\,dx \quad\Rightarrow\quad \ln|a+y| = \ln|\sin x| + C \quad\Rightarrow$

$|a+y| = e^{\ln|\sin x| + C} = e^{\ln|\sin x|} \cdot e^C = e^C\,|\sin x| \quad\Rightarrow\quad a + y = K\sin x$, where $K = \pm e^C$. (In our derivation, K was nonzero, but we can restore the excluded case $y = -a$ by allowing K to be zero.) $y(\pi/3) = a \quad\Rightarrow$

$a + a = K\sin\left(\frac{\pi}{3}\right) \quad\Rightarrow\quad 2a = K\,\dfrac{\sqrt{3}}{2} \quad\Rightarrow\quad K = \dfrac{4a}{\sqrt{3}}$. Thus, $a + y = \dfrac{4a}{\sqrt{3}}\sin x$ and so $y = \dfrac{4a}{\sqrt{3}}\sin x - a$.

18. $xy' + y = y^2 \quad\Rightarrow\quad x\,\dfrac{dy}{dx} = y^2 - y \quad\Rightarrow\quad x\,dy = (y^2 - y)\,dx \quad\Rightarrow\quad \dfrac{dy}{y^2 - y} = \dfrac{dx}{x} \quad\Rightarrow$

$\displaystyle\int \frac{dy}{y(y-1)} = \int \frac{dx}{x} \quad [y \neq 0, 1] \quad\Rightarrow\quad \int\left(\frac{1}{y-1} - \frac{1}{y}\right)dy = \int\frac{dx}{x} \quad\Rightarrow\quad \ln|y-1| - \ln|y| = \ln|x| + C$

$\Rightarrow\quad \ln\left|\dfrac{y-1}{y}\right| = \ln\left(e^C\,|x|\right) \quad\Rightarrow\quad \left|\dfrac{y-1}{y}\right| = e^C\,|x| \quad\Rightarrow\quad \dfrac{y-1}{y} = Kx$, where $K = \pm e^C \quad\Rightarrow$

$1 - \dfrac{1}{y} = Kx \quad\Rightarrow\quad \dfrac{1}{y} = 1 - Kx \quad\Rightarrow\quad y = \dfrac{1}{1 - Kx}$. [The excluded cases, $y = 0$ and $y = 1$, are ruled out by the initial condition $y(1) = -1$.] Now $y(1) = -1 \quad\Rightarrow\quad -1 = \dfrac{1}{1-K} \quad\Rightarrow\quad 1 - K = -1 \quad\Rightarrow\quad K = 2$,

so $y = \dfrac{1}{1 - 2x}$.

19. $\dfrac{dy}{dx} = 4x^3 y,\ y(0) = 7.\ \dfrac{dy}{y} = 4x^3\,dx$ [if $y \neq 0$] $\quad\Rightarrow\quad \displaystyle\int\frac{dy}{y} = \int 4x^3\,dx \quad\Rightarrow\quad \ln|y| = x^4 + C \quad\Rightarrow$

$e^{\ln|y|} = e^{x^4 + C} \quad\Rightarrow\quad |y| = e^{x^4}e^C \quad\Rightarrow\quad y = Ae^{x^4};\ y(0) = 7 \quad\Rightarrow\quad A = 7 \quad\Rightarrow\quad y = 7e^{x^4}$.

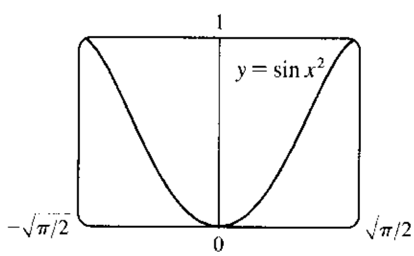
20. $\dfrac{dy}{dx} = \dfrac{y^2}{x^3},\ y(1) = 1.\ \displaystyle\int\frac{dy}{y^2} = \int\frac{dx}{x^3} \quad\Rightarrow\quad -\frac{1}{y} = -\frac{1}{2x^2} + C.\ y(1) = 1 \quad\Rightarrow\quad -1 = -\tfrac{1}{2} + C \quad\Rightarrow$

$C = -\tfrac{1}{2}$. So $\dfrac{1}{y} = \dfrac{1}{2x^2} + \dfrac{1}{2} = \dfrac{2 + 2x^2}{2\cdot 2x^2} \quad\Rightarrow\quad y = \dfrac{2x^2}{x^2 + 1}$.

21. (a) $y' = 2x\sqrt{1 - y^2} \quad\Rightarrow\quad \dfrac{dy}{dx} = 2x\sqrt{1-y^2} \quad\Rightarrow\quad \dfrac{dy}{\sqrt{1-y^2}} = 2x\,dx \quad\Rightarrow\quad \displaystyle\int\frac{dy}{\sqrt{1-y^2}} = \int 2x\,dx \quad\Rightarrow$

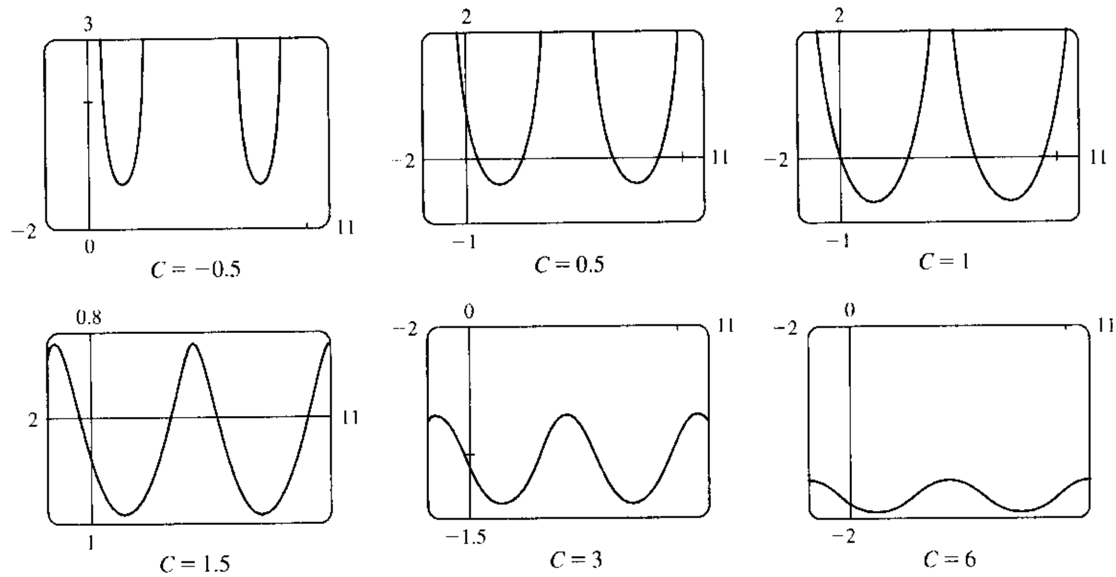
$\sin^{-1} y = x^2 + C$ for $-\frac{\pi}{2} \le x^2 + C \le \frac{\pi}{2}$.

(b) $y(0) = 0 \quad\Rightarrow\quad \sin^{-1} 0 = 0^2 + C \quad\Rightarrow\quad C = 0$, so $\sin^{-1} y = x^2$ and $y = \sin(x^2)$ for $-\sqrt{\pi/2} \le x \le \sqrt{\pi/2}$.



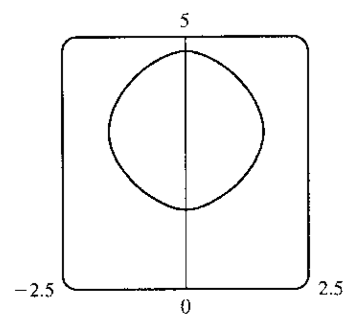
(c) For $\sqrt{1 - y^2}$ to be a real number, we must have $-1 \le y \le 1$; that is, $-1 \le y(0) \le 1$. Thus, the initial-value problem $y' = 2x\sqrt{1 - y^2}$, $y(0) = 2$ does *not* have a solution.

22. $e^{-y}y' + \cos x = 0 \quad \Leftrightarrow \quad \int e^{-y}\,dy = -\int \cos x\,dx \quad \Leftrightarrow \quad -e^{-y} = -\sin x + C_1 \quad \Leftrightarrow \quad y = -\ln(\sin x + C)$.
The solution is periodic, with period 2π. Note that for $C > 1$, the domain of the solution is $\mathbb{R}$, but for $-1 < C \leq 1$ it is only defined on the intervals where $\sin x + C > 0$, and it is meaningless for $C \leq -1$, since then $\sin x + C \leq 0$, and the logarithm is undefined.

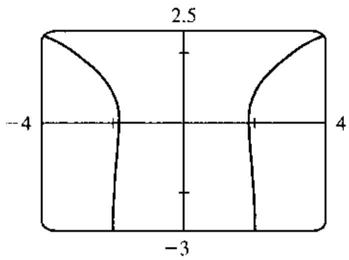


For $-1 < C < 1$, the solution curve consists of concave-up pieces separated by intervals on which the solution is not defined (where $\sin x + C \leq 0$). For $C = 1$, the solution curve consists of concave-up pieces separated by vertical asymptotes at the points where $\sin x + C = 0 \quad \Leftrightarrow \quad \sin x = -1$. For $C > 1$, the curve is continuous, and as C increases, the graph moves downward, and the amplitude of the oscillations decreases.

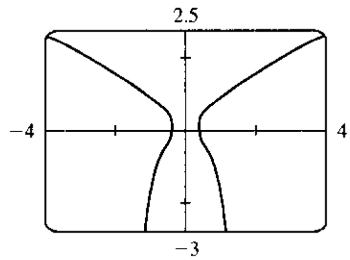
23. $\dfrac{dy}{dx} = \dfrac{\sin x}{\sin y}$, $y(0) = \frac{\pi}{2}$. So $\int \sin y\,dy = \int \sin x\,dx \quad \Leftrightarrow$ $-\cos y = -\cos x + C \quad \Leftrightarrow \quad \cos y = \cos x - C$. From the initial condition, we need $\cos\frac{\pi}{2} = \cos 0 - C \quad \Rightarrow \quad 0 = 1 - C \quad \Rightarrow \quad C = 1$, so the solution is $\cos y = \cos x - 1$. Note that we cannot take $\cos^{-1}$ of both sides, since that would unnecessarily restrict the solution to the case where $-1 \leq \cos x - 1 \quad \Leftrightarrow \quad 0 \leq \cos x$, as $\cos^{-1}$ is defined only on $[-1, 1]$. Instead we plot the graph using Maple's `plots[implicitplot]` or Mathematica's `Plot[Evaluate[···]]`.



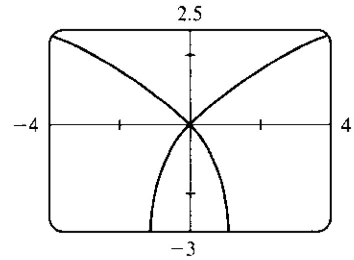
24. $\dfrac{dy}{dx} = \dfrac{x\sqrt{x^2+1}}{ye^y} \quad \Leftrightarrow \quad \int ye^y\,dy = \int x\sqrt{x^2+1}\,dx$. We use parts on the LHS with $u = y$, $dv = e^y\,dy$, and on the RHS we use the substitution $z = x^2 + 1$, so $dz = 2x\,dx$. The equation becomes $ye^y - \int e^y\,dy = \frac{1}{2}\int\sqrt{z}\,dz$ $\Leftrightarrow \quad e^y(y-1) = \frac{1}{3}(x^2+1)^{3/2} + C$, so we see that the curves are symmetric about the y-axis. Every point (x, y) in the plane lies on one of the curves, namely the one for which $C = (y-1)e^y - \frac{1}{3}(x^2+1)^{3/2}$. For example, along the y-axis, $C = (y-1)e^y - \frac{1}{3}$, so the origin lies on the curve with $C = -\frac{4}{3}$. We use Maple's `plots[implicitplot]` command or `Plot[Evaluate[···]]` in Mathematica to plot the solution curves for various values of C.



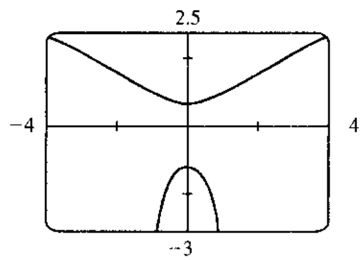
$C = -4$



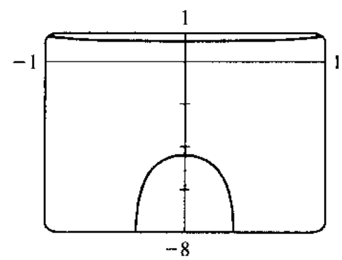
$C = -1.4$



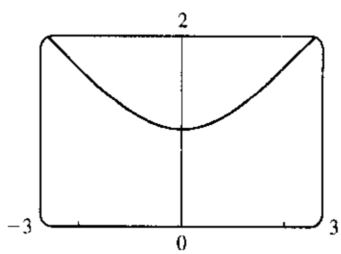
$C = -4/3$



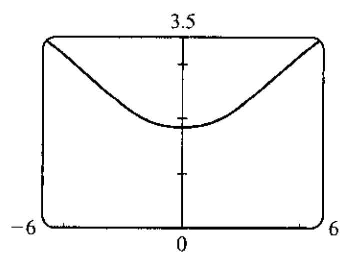
$C = -1$



$C = -0.4$



$C = -1/3$



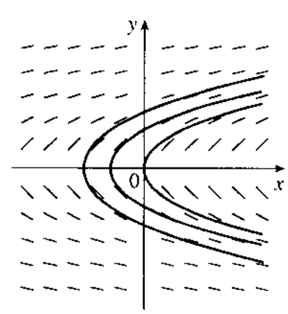
$C = 5$

It seems that the transitional values of C are $-\frac{4}{3}$ and $-\frac{1}{3}$. For $C < -\frac{4}{3}$, the graph consists of left and right branches. At $C = -\frac{4}{3}$, the two branches become connected at the origin, and as C increases, the graph splits into top and bottom branches. At $C = -\frac{1}{3}$, the bottom half disappears. As C increases further, the graph moves upward, but doesn't change shape much.

25. (a)

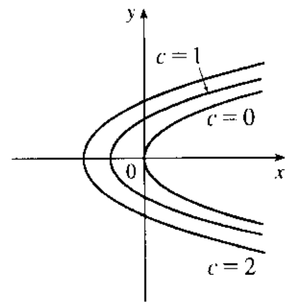
x	y	$y' = 1/y$
0	0.5	2
0	−0.5	−2
0	1	1
0	−1	−1
0	2	0.5

x	y	$y' = 1/y$
0	−2	−0.5
0	4	0.25
0	3	$0.\overline{3}$
0	0.25	4
0	$0.\overline{3}$	3



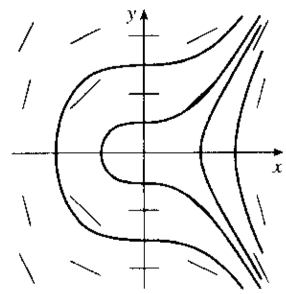
(b) $y' = 1/y \Rightarrow dy/dx = 1/y \Rightarrow$
$y\,dy = dx \Rightarrow \int y\,dy = \int dx \Rightarrow$
$\frac{1}{2}y^2 = x + c \Rightarrow y^2 = 2(x+c)$
or $y = \pm\sqrt{2(x+c)}$.

(c)



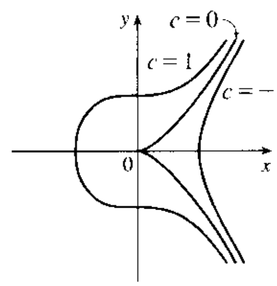
26. (a)

x	y	$y' = x^2/y$
1	1	1
−1	1	1
−1	−1	−1
1	−1	−1
1	2	0.5
2	1	4
2	2	2
1	0.5	2
0.5	1	0.25
2	0.5	8

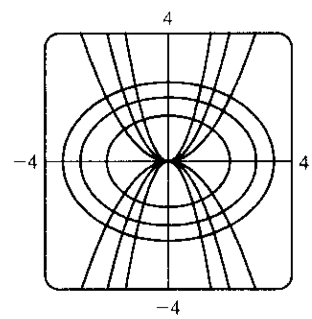


(b) $y' = x^2/y \Rightarrow y\,dy = x^2\,dx$,
so $\frac{1}{2}y^2 = \frac{1}{3}x^3 + c_1$, or
$y = \pm\left(\frac{2}{3}x^3 + c\right)^{1/2}$.

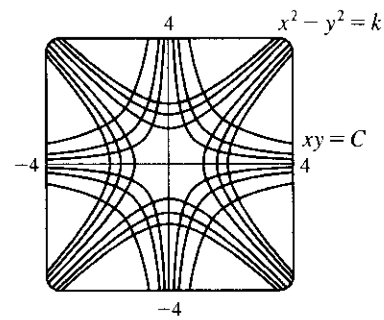
(c)



27. The curves $y = kx^2$ form a family of parabolas with axis the y-axis. Differentiating gives $y' = 2kx$, but $k = y/x^2$, so $y' = 2y/x$. Thus, the slope of the tangent line at any point (x, y) on one of the parabolas is $y' = 2y/x$, so the orthogonal trajectories must satisfy $y' = -x/(2y)$ $\Leftrightarrow$ $2y\,dy = -x\,dx$ $\Leftrightarrow$ $y^2 = -x^2/2 + C_1$ $\Leftrightarrow$ $x^2 + 2y^2 = C$. This is a family of ellipses.



28. The curves $x^2 - y^2 = k$ form a family of hyperbolas. Differentiating gives $2x - 2y\,(dy/dx) = 0$ or $y' = x/y$, the slope of the tangent line at (x, y) on one of the hyperbolas. Thus, the orthogonal trajectories must satisfy $y' = -y/x \Leftrightarrow dy/y = -dx/x \Leftrightarrow \ln|y| = -\ln|x| + C_1 \Leftrightarrow$ $\ln|x| + \ln|y| = C_1 \Leftrightarrow \ln|xy| = C_1 \Leftrightarrow$ $|xy| = e^{C_1} \Leftrightarrow xy = C$. This is a family of hyperbolas.

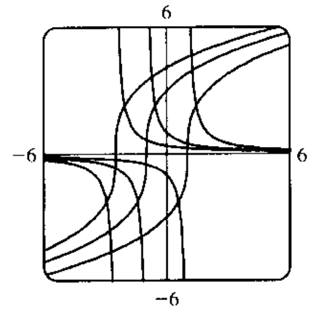


29. Differentiating $y = (x+k)^{-1}$ gives $y' = -\dfrac{1}{(x+k)^2}$, but $k = \dfrac{1}{y} - x$, so

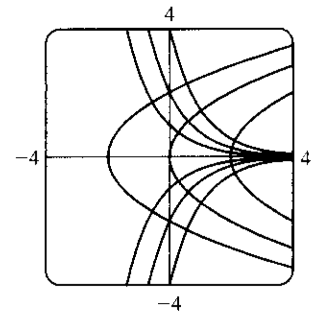
$y' = -\dfrac{1}{(1/y)^2} = -y^2$. Thus, the orthogonal trajectories must satisfy

$y' = -\dfrac{1}{-y^2} = \dfrac{1}{y^2} \quad\Leftrightarrow\quad y^2\,dy = dx \quad\Leftrightarrow\quad \dfrac{y^3}{3} = x + C$ or

$y = [3(x+C)]^{1/3}$



30. Differentiating $y = ke^{-x}$ gives $y' = -ke^{-x}$, but $k = ye^x$, so $y' = -y$. Thus, the orthogonal trajectories must satisfy $y' = -1/(-y) = 1/y \quad\Leftrightarrow\quad y\,dy = dx \quad\Leftrightarrow\quad \frac{1}{2}y^2 = x + C \quad\Leftrightarrow\quad y = \pm[2(C+x)]^{1/2}$. This is a family of parabolas with axis the x-axis.



31. From Exercise 10.2.27, $\dfrac{dQ}{dt} = 12 - 4Q \quad\Leftrightarrow\quad \displaystyle\int \frac{dQ}{12-4Q} = \int dt \quad\Leftrightarrow\quad -\tfrac{1}{4}\ln|12-4Q| = t + C \quad\Leftrightarrow$ $\ln|12-4Q| = -4t - 4C \quad\Leftrightarrow\quad |12-4Q| = e^{-4t-4C} \quad\Leftrightarrow\quad 12 - 4Q = Ke^{-4t}\ [K = \pm e^{-4C}] \quad\Leftrightarrow$ $4Q = 12 - Ke^{-4t} \quad\Leftrightarrow\quad Q = 3 - Ae^{-4t}\ [A = K/4]$. $Q(0) = 0 \quad\Leftrightarrow\quad 0 = 3 - A \quad\Leftrightarrow\quad A = 3 \quad\Leftrightarrow$ $Q(t) = 3 - 3e^{-4t}$. As $t \to \infty$, $Q(t) \to 3 - 0 = 3$ (the limiting value).

32. From Exercise 10.2.28, $\dfrac{dy}{dt} = -\frac{1}{50}(y-20) \quad\Leftrightarrow\quad \displaystyle\int \frac{dy}{y-20} = \int\left(-\tfrac{1}{50}\right)dt \quad\Leftrightarrow\quad \ln|y-20| = -\tfrac{1}{50}t + C \quad\Leftrightarrow$ $y - 20 = Ke^{-t/50} \quad\Leftrightarrow\quad y(t) = Ke^{-t/50} + 20$. $y(0) = 95 \quad\Leftrightarrow\quad 95 = K + 20 \quad\Leftrightarrow\quad K = 75 \quad\Leftrightarrow$ $y(t) = 75e^{-t/50} + 20$.

33. $\dfrac{dP}{dt} = k(M-P) \quad\Leftrightarrow\quad \displaystyle\int \frac{dP}{P-M} = \int(-k)\,dt \quad\Leftrightarrow\quad \ln|P-M| = -kt + C \quad\Leftrightarrow\quad |P-M| = e^{-kt+C}$ $\Leftrightarrow\quad P - M = Ae^{-kt}\ [A = \pm e^C] \quad\Leftrightarrow\quad P = M + Ae^{-kt}$. If we assume that performance is at level 0 when $t = 0$, then $P(0) = 0 \quad\Leftrightarrow\quad 0 = M + A \quad\Leftrightarrow\quad A = -M \quad\Leftrightarrow\quad P(t) = M - Me^{-kt}$. $\lim\limits_{t\to\infty} P(t) = M - M\cdot 0 = M$.

34. (a) $\dfrac{dx}{dt} = k(a-x)(b-x)$, $a \neq b$. Using partial fractions, $\dfrac{1}{(a-x)(b-x)} = \dfrac{1/(b-a)}{a-x} - \dfrac{1/(b-a)}{b-x}$, so $\displaystyle\int \frac{dx}{(a-x)(b-x)} = \int k\,dt \quad\Rightarrow\quad \frac{1}{b-a}\left(-\ln|a-x| + \ln|b-x|\right) = kt + C \quad\Rightarrow$ $\ln\left|\dfrac{b-x}{a-x}\right| = (b-a)(kt+C)$. The concentrations [A] $= a - x$ and [B] $= b - x$ cannot be negative, so $\dfrac{b-x}{a-x} \geq 0$ and $\left|\dfrac{b-x}{a-x}\right| = \dfrac{b-x}{a-x}$. We now have $\ln\left(\dfrac{b-x}{a-x}\right) = (b-a)(kt+C)$. Since $x(0) = 0$, we get

$\ln\left(\dfrac{b}{a}\right) = (b-a)C$. Hence, $\ln\left(\dfrac{b-x}{a-x}\right) = (b-a)kt + \ln\left(\dfrac{b}{a}\right) \Rightarrow \dfrac{b-x}{a-x} = \dfrac{b}{a}e^{(b-a)kt} \Rightarrow$

$x = \dfrac{b\left[e^{(b-a)kt}-1\right]}{be^{(b-a)kt}/a - 1} = \dfrac{ab\left[e^{(b-a)kt}-1\right]}{be^{(b-a)kt}-a}$ moles/L.

(b) If $b = a$, then $\dfrac{dx}{dt} = k(a-x)^2$, so $\displaystyle\int \frac{dx}{(a-x)^2} = \int k\,dt$ and $\dfrac{1}{a-x} = kt + C$. Since $x(0) = 0$, we get

$C = \dfrac{1}{a}$. Thus, $a - x = \dfrac{1}{kt + 1/a}$ and $x = a - \dfrac{a}{akt+1} = \dfrac{a^2kt}{akt+1}\ \dfrac{\text{moles}}{\text{L}}$.

Suppose $x = [\text{C}] = a/2$ when $t = 20$. Then $x(20) = a/2 \Rightarrow \dfrac{a}{2} = \dfrac{20a^2k}{20ak+1} \Rightarrow 40a^2k = 20a^2k + a$

$\Rightarrow 20a^2k = a \Rightarrow k = \dfrac{1}{20a}$, so $x = \dfrac{a^2t/(20a)}{1 + at/(20a)} = \dfrac{at/20}{1+t/20} = \dfrac{at}{t+20}\ \dfrac{\text{moles}}{\text{L}}$.

35. (a) If $a = b$, then $\dfrac{dx}{dt} = k(a-x)(b-x)^{1/2}$ becomes $\dfrac{dx}{dt} = k(a-x)^{3/2} \Rightarrow (a-x)^{-3/2}dx = k\,dt \Rightarrow$

$\int (a-x)^{-3/2}dx = \int k\,dt \Rightarrow 2(a-x)^{-1/2} = kt + C$ [by substitution] $\Rightarrow \dfrac{2}{kt+C} = \sqrt{a-x} \Rightarrow$

$\left(\dfrac{2}{kt+C}\right)^2 = a - x \Rightarrow x(t) = a - \dfrac{4}{(kt+C)^2}$. The initial concentration of HBr is 0, so $x(0) = 0 \Rightarrow$

$0 = a - \dfrac{4}{C^2} \Rightarrow \dfrac{4}{C^2} = a \Rightarrow C^2 = \dfrac{4}{a} \Rightarrow C = 2/\sqrt{a}$ (C is positive since

$kt + C = 2(a-x)^{-1/2} > 0$). Thus, $x(t) = a - \dfrac{4}{(kt + 2/\sqrt{a})^2}$.

(b) $\dfrac{dx}{dt} = k(a-x)(b-x)^{1/2} \Rightarrow \dfrac{dx}{(a-x)\sqrt{b-x}} = k\,dt \Rightarrow \displaystyle\int \frac{dx}{(a-x)\sqrt{b-x}} = \int k\,dt$ $(\star)$. From the

hint, $u = \sqrt{b-x} \Rightarrow u^2 = b - x \Rightarrow 2u\,du = -dx$, so $\displaystyle\int \frac{dx}{(a-x)\sqrt{b-x}} = \int \frac{-2u\,du}{[a-(b-u^2)]u} =$

$\displaystyle -2\int \frac{du}{a-b+u^2} = -2\int \frac{du}{\left(\sqrt{a-b}\right)^2 + u^2} \overset{17}{=} -2\left(\frac{1}{\sqrt{a-b}}\tan^{-1}\frac{u}{\sqrt{a-b}}\right)$. So $(\star)$ becomes

$\dfrac{-2}{\sqrt{a-b}}\tan^{-1}\dfrac{\sqrt{b-x}}{\sqrt{a-b}} = kt + C$. Now $x(0) = 0 \Rightarrow C = \dfrac{-2}{\sqrt{a-b}}\tan^{-1}\dfrac{\sqrt{b}}{\sqrt{a-b}}$

and we have $\dfrac{-2}{\sqrt{a-b}}\tan^{-1}\dfrac{\sqrt{b-x}}{\sqrt{a-b}} = kt - \dfrac{2}{\sqrt{a-b}}\tan^{-1}\dfrac{\sqrt{b}}{\sqrt{a-b}} \Rightarrow$

$\dfrac{2}{\sqrt{a-b}}\left(\tan^{-1}\sqrt{\dfrac{b}{a-b}} - \tan^{-1}\sqrt{\dfrac{b-x}{a-b}}\right) = kt \Rightarrow$

$t(x) = \dfrac{2}{k\sqrt{a-b}}\left(\tan^{-1}\sqrt{\dfrac{b}{a-b}} - \tan^{-1}\sqrt{\dfrac{b-x}{a-b}}\right)$.

36. If $S = \dfrac{dT}{dr}$, then $\dfrac{dS}{dr} = \dfrac{d^2T}{dr^2}$. The differential equation $\dfrac{d^2T}{dr^2} + \dfrac{2}{r}\dfrac{dT}{dr} = 0$ can be written as $\dfrac{dS}{dr} + \dfrac{2}{r}S = 0$. Thus,

$\dfrac{dS}{dr} = \dfrac{-2S}{r} \Rightarrow \dfrac{dS}{S} = -\dfrac{2}{r}dr \Rightarrow \displaystyle\int \frac{1}{S}\,dS = \int -\frac{2}{r}\,dr \Rightarrow \ln|S| = -2\ln|r| + C$. Assuming

$S = dT/dr > 0$ and $r > 0$, we have $S = e^{-2\ln r + C} = e^{\ln r^{-2}}e^C = r^{-2}k$ $[k = e^C]$ $\Rightarrow S = \dfrac{1}{r^2}k \Rightarrow$

$$\frac{dT}{dr} = \frac{1}{r^2}k \quad \Rightarrow \quad dT = \frac{1}{r^2}k\,dr \quad \Rightarrow \quad \int dT = \int \frac{1}{r^2}k\,dr \quad \Rightarrow \quad T(r) = -\frac{k}{r} + A.$$

$T(1) = 15 \;\Rightarrow\; 15 = -k + A$ **(1)** and $T(2) = 25 \;\Rightarrow\; 25 = -\frac{1}{2}k + A$ **(2)**.

Now solve for k and A: -2**(2)** $+$ **(1)** $\Rightarrow$ $-35 = -A$, so $A = 35$ and $k = 20$, and $T(r) = -20/r + 35$.

37. (a) $\frac{dC}{dt} = r - kC \;\Rightarrow\; \frac{dC}{dt} = -(kC - r) \;\Rightarrow\; \int \frac{dC}{kC - r} = \int -dt \;\Rightarrow\; (1/k)\ln|kC - r| = -t + M_1$

$\Rightarrow\; \ln|kC - r| = -kt + M_2 \;\Rightarrow\; |kC - r| = e^{-kt+M_2} \;\Rightarrow\; kC - r = M_3e^{-kt} \;\Rightarrow$

$kC = M_3e^{-kt} + r \;\Rightarrow\; C(t) = M_4e^{-kt} + r/k$. $C(0) = C_0 \;\Rightarrow\; C_0 = M_4 + r/k \;\Rightarrow$

$M_4 = C_0 - r/k \;\Rightarrow\; C(t) = (C_0 - r/k)e^{-kt} + r/k$.

(b) If $C_0 < r/k$, then $C_0 - r/k < 0$ and the formula for $C(t)$ shows that $C(t)$ increases and $\lim_{t\to\infty} C(t) = r/k$. As t increases, the formula for $C(t)$ shows how the role of C_0 steadily diminishes as that of r/k increases.

38. (a) Use 1 billion dollars as the x-unit and 1 day as the t-unit. Initially, there is \$10 billion of old currency in circulation, so all of the \$50 million returned to the banks is old. At time t, the amount of new currency is $x(t)$ billion dollars, so $10 - x(t)$ billion dollars of currency is old. The fraction of circulating money that is old is $[10 - x(t)]/10$, and the amount of old currency being returned to the banks each day is $\frac{10 - x(t)}{10}0.05$ billion dollars. This amount of new currency per day is introduced into circulation, so $\frac{dx}{dt} = \frac{10 - x}{10} \cdot 0.05 = 0.005(10 - x)$ billion dollars per day.

(b) $\frac{dx}{10 - x} = 0.005\,dt \;\Rightarrow\; \frac{-dx}{10 - x} = -0.005\,dt \;\Rightarrow\; \ln(10 - x) = -0.005t + c \;\Rightarrow$

$10 - x = Ce^{-0.005t}$, where $C = e^c \;\Rightarrow\; x(t) = 10 - Ce^{-0.005t}$. From $x(0) = 0$, we get $C = 10$, so $x(t) = 10(1 - e^{-0.005t})$.

(c) The new bills make up 90% of the circulating currency when $x(t) = 0.9 \cdot 10 = 9$ billion dollars.

$9 = 10(1 - e^{-0.005t}) \;\Rightarrow\; 0.9 = 1 - e^{-0.005t} \;\Rightarrow\; e^{-0.005t} = 0.1 \;\Rightarrow\; -0.005t = -\ln 10 \;\Rightarrow$

$t = 200\ln 10 \approx 460.517$ days ≈ 1.26 years.

39. (a) Let $y(t)$ be the amount of salt (in kg) after t minutes. Then $y(0) = 15$. The amount of liquid in the tank is 1000 L at all times, so the concentration at time t (in minutes) is $y(t)/1000$ kg/L and

$\frac{dy}{dt} = -\left[\frac{y(t)}{1000}\,\frac{\text{kg}}{\text{L}}\right]\left(10\,\frac{\text{L}}{\text{min}}\right) = -\frac{y(t)}{100}\,\frac{\text{kg}}{\text{min}}$. $\int \frac{dy}{y} = -\frac{1}{100}\int dt \;\Rightarrow\; \ln y = -\frac{t}{100} + C$, and

$y(0) = 15 \;\Rightarrow\; \ln 15 = C$, so $\ln y = \ln 15 - \frac{t}{100}$. It follows that $\ln\left(\frac{y}{15}\right) = -\frac{t}{100}$ and $\frac{y}{15} = e^{-t/100}$, so $y = 15e^{-t/100}$ kg.

(b) After 20 minutes, $y = 15e^{-20/100} = 15e^{-0.2} \approx 12.3$ kg.

40. (a) If $y(t)$ is the amount of salt (in kg) after t minutes, then $y(0) = 0$ and the total amount of liquid in the tank remains constant at 1000 L.

$$\frac{dy}{dt} = \left(0.05\,\frac{\text{kg}}{\text{L}}\right)\left(5\,\frac{\text{L}}{\text{min}}\right) + \left(0.04\,\frac{\text{kg}}{\text{L}}\right)\left(10\,\frac{\text{L}}{\text{min}}\right) - \left(\frac{y(t)}{1000}\,\frac{\text{kg}}{\text{L}}\right)\left(15\,\frac{\text{L}}{\text{min}}\right)$$

$$= 0.25 + 0.40 - 0.015y = 0.65 - 0.015y = \frac{130 - 3y}{200}\,\frac{\text{kg}}{\text{min}}$$

so $\int \frac{dy}{130 - 3y} = \int \frac{dt}{200}$ and $-\frac{1}{3}\ln|130 - 3y| = \frac{1}{200}t + C$; since $y(0) = 0$, we have $-\frac{1}{3}\ln 130 = C$, so $-\frac{1}{3}\ln|130-3y| = \frac{1}{200}t - \frac{1}{3}\ln 130 \Rightarrow \ln|130-3y| = -\frac{3}{200}t + \ln 130 = \ln\left(130e^{-3t/200}\right)$, and $|130-3y| = 130e^{-3t/200}$. Since y is continuous, $y(0)=0$, and the right-hand side is never zero, we deduce that $130-3y$ is always positive. Thus, $130-3y = 130e^{-3t/200}$ and $y = \frac{130}{3}\left(1-e^{-3t/200}\right)$ kg.

(b) After one hour, $y = \frac{130}{3}\left(1-e^{-3\cdot 60/200}\right) = \frac{130}{3}\left(1-e^{-0.9}\right) \approx 25.7$ kg.

Note: As $t \to \infty$, $y(t) \to \frac{130}{3} = 43\frac{1}{3}$ kg.

41. Assume that the raindrop begins at rest, so that $v(0)=0$. $dm/dt = km$ and $(mv)' = gm \Rightarrow mv' + vm' = gm \Rightarrow mv' + v(km) = gm \Rightarrow v' + vk = g \Rightarrow dv/dt = g - kv \Rightarrow$
$$\int \frac{dv}{g-kv} = \int dt \Rightarrow -(1/k)\ln|g-kv| = t + C \Rightarrow \ln|g-kv| = -kt - kC \Rightarrow g - kv = Ae^{-kt}.$$
$v(0)=0 \Rightarrow A = g$. So $kv = g - ge^{-kt} \Rightarrow v = (g/k)(1-e^{-kt})$. Since $k>0$, as $t\to\infty$, $e^{-kt}\to 0$ and therefore, $\lim_{t\to\infty} v(t) = g/k$.

42. (a) $m\dfrac{dv}{dt} = -kv \Rightarrow \dfrac{dv}{v} = -\dfrac{k}{m}\,dt \Rightarrow \ln|v| = -\dfrac{k}{m}t + C$. Since $v(0)=v_0$, $\ln|v_0| = C$. Therefore,
$\ln\left|\dfrac{v}{v_0}\right| = -\dfrac{k}{m}t \Rightarrow \left|\dfrac{v}{v_0}\right| = e^{-kt/m} \Rightarrow v(t) = \pm v_0 e^{-kt/m}$. The sign is $+$ when $t=0$, and we assume v is continuous, so that the sign is $+$ for all t. Thus, $v(t) = v_0e^{-kt/m}$. $ds/dt = v_0 e^{-kt/m} \Rightarrow$
$s(t) = -\dfrac{mv_0}{k}e^{-kt/m} + C'$. From $s(0)=s_0$, we get $s_0 = -\dfrac{mv_0}{k} + C'$, so $C' = s_0 + \dfrac{mv_0}{k}$ and
$s(t) = s_0 + \dfrac{mv_0}{k}\left(1-e^{-kt/m}\right)$. The distance traveled from time 0 to time t is $s(t)-s_0$, so the total distance traveled is $\lim_{t\to\infty}[s(t)-s_0] = \dfrac{mv_0}{k}$.

Note: In finding the limit, we use the fact that $k>0$ to conclude that $\lim_{t\to\infty} e^{-kt/m} = 0$.

(b) $m\dfrac{dv}{dt} = -kv^2 \Rightarrow \dfrac{dv}{v^2} = -\dfrac{k}{m}\,dt \Rightarrow \dfrac{-1}{v} = -\dfrac{kt}{m} + C \Rightarrow \dfrac{1}{v} = \dfrac{kt}{m} - C$. Since $v(0)=v_0$, $C = -\dfrac{1}{v_0}$ and $\dfrac{1}{v} = \dfrac{kt}{m} + \dfrac{1}{v_0}$. Therefore, $v(t) = \dfrac{1}{kt/m + 1/v_0} = \dfrac{mv_0}{kv_0t + m}$. $\dfrac{ds}{dt} = \dfrac{mv_0}{kv_0t+m} \Rightarrow$

$s(t) = \dfrac{m}{k}\displaystyle\int \dfrac{kv_0\,dt}{kv_0t+m} = \dfrac{m}{k}\ln|kv_0t+m| + C'$. Since $s(0)=s_0$, we get $s_0 = \dfrac{m}{k}\ln m + C' \Rightarrow$

$C' = s_0 - \dfrac{m}{k}\ln m \Rightarrow s(t) = s_0 + \dfrac{m}{k}(\ln|kv_0t+m| - \ln m) = s_0 + \dfrac{m}{k}\ln\left|\dfrac{kv_0t+m}{m}\right|$. We can rewrite the formulas for $v(t)$ and $s(t)$ as $v(t) = \dfrac{v_0}{1+(kv_0/m)t}$ and $s(t) = s_0 + \dfrac{m}{k}\ln\left|1 + \dfrac{kv_0}{m}t\right|$.

Remarks: This model of horizontal motion through a resistive medium was designed to handle the case in which $v_0 > 0$. Then the term $-kv^2$ representing the resisting force causes the object to decelerate. The absolute value in the expression for $s(t)$ is unnecessary (since k, v_0, and m are all positive), and $\lim_{t\to\infty} s(t) = \infty$. In other words, the object travels infinitely far. However, $\lim_{t\to\infty} v(t) = 0$. When $v_0 < 0$, the term $-kv^2$ increases the magnitude of the object's negative velocity. According to the formula for $s(t)$, the position of the object approaches $-\infty$ as t approaches $m/k(-v_0)$: $\lim\limits_{t\to -m/(kv_0)} s(t) = -\infty$. Again the object travels infinitely far, but this time the feat is accomplished in a finite amount of time. Notice also that $\lim\limits_{t\to -m/(kv_0)} v(t) = -\infty$ when $v_0 < 0$, showing that the speed of the object increases without limit.

43. (a) The rate of growth of the area is jointly proportional to $\sqrt{A(t)}$ and $M - A(t)$; that is, the rate is proportional to the product of those two quantities. So for some constant k, $dA/dt = k\sqrt{A}\,(M - A)$. We are interested in the maximum of the function dA/dt (when the tissue grows the fastest), so we differentiate, using the Chain Rule and then substituting for dA/dt from the differential equation:

$$\frac{d}{dt}\left(\frac{dA}{dt}\right) = k\left[\sqrt{A}\,(-1)\frac{dA}{dt} + (M - A)\cdot\tfrac{1}{2}A^{-1/2}\frac{dA}{dt}\right] = \tfrac{1}{2}kA^{-1/2}\frac{dA}{dt}\left[-2A + (M - A)\right]$$

$$= \tfrac{1}{2}kA^{-1/2}\left[k\sqrt{A}(M - A)\right][M - 3A] = \tfrac{1}{2}k^2(M - A)(M - 3A)$$

This is 0 when $M - A = 0$ [this situation never actually occurs, since the graph of $A(t)$ is asymptotic to the line $y = M$, as in the logistic model] and when $M - 3A = 0 \;\Leftrightarrow\; A(t) = M/3$. This represents a maximum by the First Derivative Test, since $\dfrac{d}{dt}\left(\dfrac{dA}{dt}\right)$ goes from positive to negative when $A(t) = M/3$.

(b) From the CAS, we get $A(t) = M\left(\dfrac{Ce^{\sqrt{M}kt} - 1}{Ce^{\sqrt{M}kt} + 1}\right)^2$. To get C in terms of the initial area A_0 and the maximum area M, we substitute $t = 0$ and $A = A_0 = A(0)$: $A_0 = M\left(\dfrac{C-1}{C+1}\right)^2 \;\Leftrightarrow\; (C+1)\sqrt{A_0} = (C-1)\sqrt{M}$

$\Leftrightarrow\; C\sqrt{A_0} + \sqrt{A_0} = C\sqrt{M} - \sqrt{M} \;\Leftrightarrow\; \sqrt{M} + \sqrt{A_0} = C\sqrt{M} - C\sqrt{A_0} \;\Leftrightarrow$

$\sqrt{M} + \sqrt{A_0} = C\left(\sqrt{M} - \sqrt{A_0}\right) \;\Leftrightarrow\; C = \dfrac{\sqrt{M} + \sqrt{A_0}}{\sqrt{M} - \sqrt{A_0}}$. (Notice that if $A_0 = 0$, then $C = 1$.)

44. (a) According to the hint we use the Chain Rule: $m\dfrac{dv}{dt} = m\dfrac{dv}{dx}\cdot\dfrac{dx}{dt} = mv\dfrac{dv}{dx} = -\dfrac{mgR^2}{(x+R)^2} \;\Rightarrow$

$\displaystyle\int v\,dv = \int\frac{-gR^2\,dx}{(x+R)^2} \;\Rightarrow\; \frac{v^2}{2} = \frac{gR^2}{x+R} + C$. When $x = 0$, $v = v_0$, so $\dfrac{v_0^2}{2} = \dfrac{gR^2}{0+R} + C \;\Rightarrow$

$C = \tfrac{1}{2}v_0^2 - gR \;\Rightarrow\; \tfrac{1}{2}v^2 - \tfrac{1}{2}v_0^2 = \dfrac{gR^2}{x+R} - gR$. Now at the top of its flight, the rocket's velocity will be 0, and its height will be $x = h$. Solving for v_0: $-\tfrac{1}{2}v_0^2 = \dfrac{gR^2}{h+R} - gR \;\Rightarrow$

$$\frac{v_0^2}{2} = g\left[-\frac{R^2}{R+h} + \frac{R(R+h)}{R+h}\right] = \frac{gRh}{R+h} \;\Rightarrow\; v_0 = \sqrt{\frac{2gRh}{R+h}}.$$

(b) $v_e = \lim\limits_{h\to\infty} v_0 = \lim\limits_{h\to\infty}\sqrt{\dfrac{2gRh}{R+h}} = \lim\limits_{h\to\infty}\sqrt{\dfrac{2gR}{(R/h)+1}} = \sqrt{2gR}$

(c) $v_e = \sqrt{2\cdot 32\text{ ft/s}^2\cdot 3960\text{ mi}\cdot 5280\text{ ft/mi}} \approx 36{,}581\text{ ft/s} \approx 6.93\text{ mi/s}$

APPLIED PROJECT How Fast Does a Tank Drain?

1. (a) $V = \pi r^2 h \quad \Rightarrow \quad \dfrac{dV}{dt} = \pi r^2 \dfrac{dh}{dt}$ [implicit differentiation] $\Rightarrow$

$$\frac{dh}{dt} = \frac{1}{\pi r^2}\frac{dV}{dt} = \frac{1}{\pi r^2}\left(-a\sqrt{2gh}\right) = \frac{1}{\pi 2^2}\left[-\pi\left(\tfrac{1}{12}\right)^2\sqrt{2\cdot 32}\sqrt{h}\right] = -\tfrac{1}{72}\sqrt{h}$$

(b) $\dfrac{dh}{dt} = -\frac{1}{72}\sqrt{h} \quad \Rightarrow \quad h^{-1/2}\,dh = -\frac{1}{72}\,dt \quad \Rightarrow \quad 2\sqrt{h} = -\frac{1}{72}t + C.$

$h(0) = 6 \quad \Rightarrow \quad 2\sqrt{6} = 0 + C \quad \Rightarrow \quad C = 2\sqrt{6} \quad \Rightarrow \quad h(t) = \left(-\frac{1}{144}t + \sqrt{6}\right)^2.$

(c) We want to find t when $h = 0$, so we set $h = 0 = \left(-\frac{1}{144}t + \sqrt{6}\right)^2 \quad \Rightarrow \quad t = 144\sqrt{6} \approx 5$ min 53 s.

2. (a) $\dfrac{dh}{dt} = k\sqrt{h} \quad \Rightarrow \quad h^{-1/2}\,dh = k\,dt \quad [h \neq 0] \quad \Rightarrow \quad 2\sqrt{h} = kt + C$

$\Rightarrow \quad h(t) = \frac{1}{4}(kt + C)^2$. Since $h(0) = 10$ cm, the relation $2\sqrt{h(t)} = kt + C$ gives us $2\sqrt{10} = C$. Also, $h(68) = 3$ cm, so $2\sqrt{3} = 68k + 2\sqrt{10}$ and $k = -\dfrac{\sqrt{10} - \sqrt{3}}{34}$. Thus,

$$h(t) = \frac{1}{4}\left(2\sqrt{10} - \frac{\sqrt{10} - \sqrt{3}}{34}t\right)^2 \approx 10 - 0.133t + 0.00044t^2.$$

Here is a table of values of $h(t)$ correct to one decimal place.

t (in s)	$h(t)$ (in cm)
10	8.7
20	7.5
30	6.4
40	5.4
50	4.5
60	3.6

(b) The answers to this part are to be obtained experimentally. See the article by Tom Farmer and Fred Gass, *Physical Demonstrations in the Calculus Classroom*, College Mathematics Journal 1992, pp. 146–148.

3. $V(t) = \pi r^2 h(t) = 100\pi h(t) \quad \Rightarrow \quad \dfrac{dV}{dh} = 100\pi$ and $\dfrac{dV}{dt} = \dfrac{dV}{dh}\dfrac{dh}{dt} = 100\pi\dfrac{dh}{dt}.$

Diameter $= 2.5$ inches $\quad \Rightarrow \quad$ radius $= 1.25$ inches $= \frac{5}{4}\cdot\frac{1}{12}$ foot $= \frac{5}{48}$ foot. Thus,

$$\frac{dV}{dt} = -a\sqrt{2gh} \quad \Rightarrow \quad 100\pi\frac{dh}{dt} = -\pi\left(\tfrac{5}{48}\right)^2\sqrt{2\cdot 32h} = -\frac{25\pi}{288}\sqrt{h} \quad \Rightarrow \quad \frac{dh}{dt} = -\frac{\sqrt{h}}{1152} \quad \Rightarrow$$

$$\int h^{-1/2}\,dh = \int -\tfrac{1}{1152}\,dt \quad \Rightarrow \quad 2\sqrt{h} = -\tfrac{1}{1152}t + C \quad \Rightarrow \quad \sqrt{h} = -\tfrac{1}{2304}t + k \quad \Rightarrow \quad h(t) = \left(-\tfrac{1}{2304}t + k\right)^2.$$

The water pressure after t seconds is $62.5h(t)$ lb/ft^2, so the condition that the pressure be at least 2160 lb/ft^2 for 10 minutes (600 seconds) is the condition $62.5\cdot h(600) \geq 2160$; that is, $\left(k - \frac{600}{2304}\right)^2 \geq \frac{2160}{62.5} \quad \Rightarrow$

$\left|k - \frac{25}{96}\right| \geq \sqrt{34.56} \quad \Rightarrow \quad k \geq \frac{25}{96} + \sqrt{34.56}$. Now $h(0) = k^2$, so the height of the tank should be at least $\left(\frac{25}{96} + \sqrt{34.56}\right)^2 \approx 37.69$ ft.

4. (a) If the radius of the circular cross-section at height h is r, then the Pythagorean Theorem gives $r^2 = 2^2 - (2-h)^2$ since the radius of the tank is 2 m. So $A(h) = \pi r^2 = \pi\left[4 - (2-h)^2\right] = \pi(4h - h^2)$.

Thus, $A(h)\dfrac{dh}{dt} = -a\sqrt{2gh} \quad \Rightarrow \quad \pi(4h - h^2)\dfrac{dh}{dt} = -\pi(0.01)^2\sqrt{2\cdot 10h} \quad \Rightarrow$

$(4h - h^2)\dfrac{dh}{dt} = -0.0001\sqrt{20h}.$

(b) From part (a) we have $\left(4h^{1/2} - h^{3/2}\right)dh = \left(-0.0001\sqrt{20}\right)dt \Rightarrow$

$\frac{8}{3}h^{3/2} - \frac{2}{5}h^{5/2} = \left(-0.0001\sqrt{20}\right)t + C$. $h(0) = 2 \Rightarrow \frac{8}{3}(2)^{3/2} - \frac{2}{5}(2)^{5/2} = C \Rightarrow$

$C = \left(\frac{16}{3} - \frac{8}{5}\right)\sqrt{2} = \frac{56}{15}\sqrt{2}$. To find out how long it will take to drain all the water we evaluate t when $h = 0$:

$0 = \left(-0.0001\sqrt{20}\right)t + C \Rightarrow$

$$t = \frac{C}{0.0001\sqrt{20}} = \frac{56\sqrt{2}/15}{0.0001\sqrt{20}} = \frac{11{,}200\sqrt{10}}{3} \approx 11{,}806 \text{ s} \approx 3 \text{ h } 17 \text{ min.}$$

APPLIED PROJECT Which is Faster, Going Up or Coming Down?

1. $mv' = -pv - mg \Rightarrow m\frac{dv}{dt} = -(pv + mg) \Rightarrow \int \frac{dv}{pv + mg} = \int -\frac{1}{m}\,dt \Rightarrow$

$\frac{1}{p}\ln(pv + mg) = -\frac{1}{m}t + C$ $[pv + mg > 0]$. At $t = 0$, $v = v_0$, so $C = \frac{1}{p}\ln(pv_0 + mg)$. Thus,

$\frac{1}{p}\ln(pv + mg) = -\frac{1}{m}t + \frac{1}{p}\ln(pv_0 + mg) \Rightarrow \ln(pv + mg) = -\frac{p}{m}t + \ln(pv_0 + mg) \Rightarrow$

$pv + mg = e^{-pt/m}(pv_0 + mg) \Rightarrow pv = (pv_0 + mg)e^{-pt/m} - mg \Rightarrow$

$v(t) = \left(v_0 + \frac{mg}{p}\right)e^{-pt/m} - \frac{mg}{p}$.

2. $y(t) = \int v(t)\,dt = \int \left[\left(v_0 + \frac{mg}{p}\right)e^{-pt/m} - \frac{mg}{p}\right]dt = \left(v_0 + \frac{mg}{p}\right)e^{-pt/m}\left(-\frac{m}{p}\right) - \frac{mg}{p}t + C$

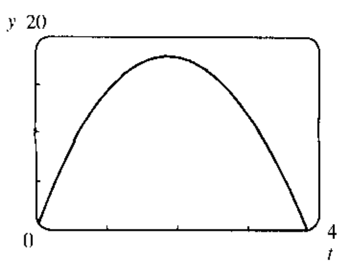
At $t = 0$, $y = 0$, so $C = \left(v_0 + \frac{mg}{p}\right)\frac{m}{p}$. Thus,

$$y(t) = \left(v_0 + \frac{mg}{p}\right)\frac{m}{p} - \left(v_0 + \frac{mg}{p}\right)\frac{m}{p}e^{-pt/m} - \frac{mgt}{p} = \left(v_0 + \frac{mg}{p}\right)\frac{m}{p}\left(1 - e^{-pt/m}\right) - \frac{mgt}{p}$$

3. $v(t) = 0 \Rightarrow \frac{mg}{p} = \left(v_0 + \frac{mg}{p}\right)e^{-pt/m} \Rightarrow e^{pt/m} = \frac{pv_0}{mg} + 1 \Rightarrow \frac{pt}{m} = \ln\left(\frac{pv_0}{mg} + 1\right) \Rightarrow$

$t_1 = \frac{m}{p}\ln\left(\frac{mg + pv_0}{mg}\right)$. With $m = 1$, $v_0 = 20$, $p = \frac{1}{10}$, and $g = 9.8$, we have $t_1 = 10\ln\left(\frac{11.8}{9.8}\right) \approx 1.86$ s.

4.



The figure shows the graph of $y = 1180\left(1 - e^{-0.1t}\right) - 98t$. The zeros are at $t = 0$ and $t_2 \approx 3.84$. Thus, $t_1 - 0 \approx 1.86$ and $t_2 - t_1 \approx 1.98$. So the time it takes to come down is about 0.12 s longer than the time it takes to go up; hence, going up is faster.

5. $y(2t_1) = \left(v_0 + \frac{mg}{p}\right)\frac{m}{p}\left(1 - e^{-2pt_1/m}\right) - \frac{mg}{p}\cdot 2t_1$

$$= \left(\frac{pv_0 + mg}{p}\right)\frac{m}{p}\left[1 - \left(e^{pt_1/m}\right)^{-2}\right] - \frac{mg}{p}\cdot 2\frac{m}{p}\ln\left(\frac{pv_0 + mg}{mg}\right)$$

Substituting $x = e^{pt_1/m} = \frac{pv_0}{mg} + 1 = \frac{pv_0 + mg}{mg}$ (from Problem 3), we get

$y\,(2t_1) = \left(x \cdot \frac{mg}{p}\right)\frac{m}{p}\left(1 - x^{-2}\right) - \frac{m^2 g}{p^2}\cdot 2\ln x = \frac{m^2 g}{p^2}\left(x - \frac{1}{x} - 2\ln x\right)$. Now $p > 0, m > 0, t_1 > 0 \;\Rightarrow$ $x = e^{pt_1/m} > e^0 = 1$. $f(x) = x - \frac{1}{x} - 2\ln x \;\Rightarrow\; f'(x) = 1 + \frac{1}{x^2} - \frac{2}{x} = \frac{x^2 - 2x + 1}{x^2} = \frac{(x-1)^2}{x^2} > 0$ for $x > 1 \;\Rightarrow\; f(x)$ is increasing for $x > 1$. Since $f(1) = 0$, it follows that $f(x) > 0$ for every $x > 1$. Therefore, $y(2t_1) = \frac{m^2 g}{p^2} f(x)$ is positive, which means that the ball has not yet reached the ground at time $2t_1$. This tells us that the time spent going up is always less than the time spent coming down, so *ascent is faster.*

10.4 Exponential Growth and Decay

1. The relative growth rate is $\frac{1}{P}\frac{dP}{dt} = 0.7944$, so $\frac{dP}{dt} = 0.7944P$ and, by Theorem 2, $P(t) = P(0)e^{0.7944t} = 2e^{0.7944t}$. Thus, $P(6) = 2e^{0.7944(6)} \approx 234.99$ or about 235 members.

2. (a) By Theorem 2, $P(t) = P(0)e^{kt} = 60e^{kt}$. In 20 minutes ($\frac{1}{3}$ hour), there are 120 cells, so $P(\frac{1}{3}) = 60e^{k/3} = 120 \;\Rightarrow\; e^{k/3} = 2 \;\Rightarrow\; k/3 = \ln 2 \;\Rightarrow\; k = 3\ln 2 = \ln(2^3) = \ln 8$.

(b) $P(t) = 60e^{(\ln 8)t} = 60 \cdot 8^t$

(c) $P(8) = 60 \cdot 8^8 = 60 \cdot 2^{24} = 1{,}006{,}632{,}960$

(d) $dP/dt = kP \;\Rightarrow\; P'(8) = kP(8) = (\ln 8)P(8) \approx 2.093$ billion cells/h

(e) $P(t) = 20{,}000 \;\Rightarrow\; 60 \cdot 8^t = 20{,}000 \;\Rightarrow\; 8^t = 1000/3 \;\Rightarrow\; t\ln 8 = \ln(1000/3) \;\Rightarrow$ $t = \frac{\ln(1000/3)}{\ln 8} \approx 2.79$ h

3. (a) By Theorem 2, $y(t) = y(0)e^{kt} = 500e^{kt}$. Now $y(3) = 500e^{k(3)} = 8000 \;\Rightarrow\; e^{3k} = \frac{8000}{500} \;\Rightarrow$ $3k = \ln 16 \;\Rightarrow\; k = (\ln 16)/3$. So $y(t) = 500e^{(\ln 16)t/3} = 500 \cdot 16^{t/3}$

(b) $y(4) = 500 \cdot 16^{4/3} \approx 20{,}159$

(c) $dy/dt = ky \;\Rightarrow\; y'(4) = ky(4) = \frac{1}{3}\ln 16\left(500 \cdot 16^{4/3}\right)$ [from part (a)] $\approx 18{,}631$ cells/h

(d) $y(t) = 500 \cdot 16^{t/3} = 30{,}000 \;\Rightarrow\; 16^{t/3} = 60 \;\Rightarrow\; \frac{1}{3}t\ln 16 = \ln 60 \;\Rightarrow\; t = 3(\ln 60)/(\ln 16) \approx 4.4$ h

4. (a) $y(t) = y(0)e^{kt} \;\Rightarrow\; y(2) = y(0)e^{2k} = 600$, $y(8) = y(0)e^{8k} = 75{,}000$. Dividing these equations, we get $e^{8k}/e^{2k} = 75{,}000/600 \;\Rightarrow\; e^{6k} = 125 \;\Rightarrow\; 6k = \ln 125 = \ln 5^3 = 3\ln 5 \;\Rightarrow\; k = \frac{3}{6}\ln 5 = \frac{1}{2}\ln 5$. Thus, $y(0) = 600/e^{2k} = 600/e^{\ln 5} = \frac{600}{5} = 120$.

(b) $y(t) = y(0)e^{kt} = 120e^{(\ln 5)t/2}$ or $y = 120 \cdot 5^{t/2}$

(c) $y(5) = 120 \cdot 5^{5/2} = 120 \cdot 25\sqrt{5} = 3000\sqrt{5} \approx 6708$ bacteria.

(d) $y(t) = 120 \cdot 5^{t/2} \Rightarrow y'(t) = 120 \cdot 5^{t/2} \cdot \ln 5 \cdot \frac{1}{2} = 60 \cdot \ln 5 \cdot 5^{t/2}$.

$y'(5) = 60 \cdot \ln 5 \cdot 5^{5/2} = 60 \cdot \ln 5 \cdot 25\sqrt{5} \approx 5398$ bacteria/hour.

(e) $y(t) = 200{,}000 \Leftrightarrow 120e^{(\ln 5)t/2} = 200{,}000 \Leftrightarrow e^{(\ln 5)t/2} = \frac{5000}{3} \Leftrightarrow (\ln 5)t/2 = \ln \frac{5000}{3} \Leftrightarrow$
$t = \left(2\ln \frac{5000}{3}\right)/\ln 5 \approx 9.2$ h.

5. (a) Let the population (in millions) in the year t be $P(t)$. Since the initial time is the year 1750, we substitute $t - 1750$ for t in Theorem 2, so the exponential model gives $P(t) = P(1750)e^{k(t-1750)}$. Then
$P(1800) = 980 = 790e^{k(1800-1750)} \Rightarrow \frac{980}{790} = e^{k(50)} \Rightarrow \ln \frac{980}{790} = 50k \Rightarrow$
$k = \frac{1}{50}\ln \frac{980}{790} \approx 0.0043104$. So with this model, we have $P(1900) = 790e^{k(1900-1750)} \approx 1508$ million, and $P(1950) = 790e^{k(1950-1750)} \approx 1871$ million. Both of these estimates are much too low.

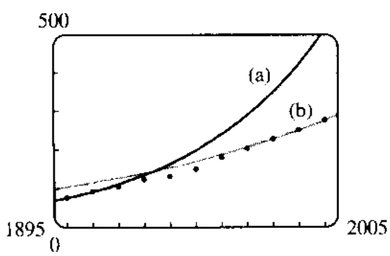
(b) In this case, the exponential model gives $P(t) = P(1850)e^{k(t-1850)} \Rightarrow$
$P(1900) = 1650 = 1260e^{k(1900-1850)} \Rightarrow \ln \frac{1650}{1260} = k(50) \Rightarrow k = \frac{1}{50}\ln \frac{1650}{1260} \approx 0.005393$. So with this model, we estimate $P(1950) = 1260e^{k(1950-1850)} \approx 2161$ million. This is still too low, but closer than the estimate of $P(1950)$ in part (a).

(c) The exponential model gives $P(t) = P(1900)e^{k(t-1900)} \Rightarrow P(1950) = 2560 = 1650e^{k(1950-1900)} \Rightarrow$
$\ln \frac{2560}{1650} = k(50) \Rightarrow k = \frac{1}{50}\ln \frac{2560}{1650} \approx 0.008785$. With this model, we estimate
$P(2000) = 1650e^{k(2000-1900)} \approx 3972$ million. This is much too low. The discrepancy is explained by the fact that the world birth rate (average yearly number of births per person) is about the same as always, whereas the mortality rate (especially the infant mortality rate) is much lower, owing mostly to advances in medical science and to the wars in the first part of the twentieth century. The exponential model assumes, among other things, that the birth and mortality rates will remain constant.

6. (a) Let $P(t)$ be the population (in millions) in the year t. Since the initial time is the year 1900, we substitute $t - 1900$ for t in Theorem 2, and find that the exponential model gives $P(t) = P(1900)e^{k(t-1900)} \Rightarrow$
$P(1910) = 92 = 76e^{k(1910-1900)} \Rightarrow k = \frac{1}{10}\ln \frac{92}{76} \approx 0.0191$. With this model, we estimate
$P(2000) = 76e^{k(2000-1900)} \approx 514$ million. This estimate is much too high. The discrepancy is explained by the fact that, between the years 1900 and 1910, an enormous number of immigrants (compared to the total population) came to the United States. Since that time, immigration (as a proportion of total population) has been much lower. Also, the birth rate in the United States has declined since the turn of the century. So our calculation of the constant k was based partly on factors which no longer exist.

(b) Substituting $t - 1980$ for t in Theorem 2, we find that the exponential model gives $P(t) = P(1980)e^{k(t-1980)}$
$\Rightarrow P(1990) = 250 = 227e^{k(1990-1980)} \Rightarrow k = \frac{1}{10}\ln \frac{250}{227} \approx 0.00965$. With this model, we estimate
$P(2000) = 227e^{k(2000-1980)} \approx 275.3$ million. This is quite accurate. The further estimates are
$P(2010) = 227e^{30k} \approx 303$ million and $P(2020) = 227e^{40k} \approx 334$ million.

(c)



The model in part (a) is quite inaccurate after 1910 (off by 5 million in 1920 and 12 million in 1930). The model in part (b) is more accurate (which is not surprising, since it is based on more recent information).

7. (a) If $y = [N_2O_5]$ then by Theorem 2, $\frac{dy}{dt} = -0.0005y \;\Rightarrow\; y(t) = y(0)e^{-0.0005t} = Ce^{-0.0005t}$.

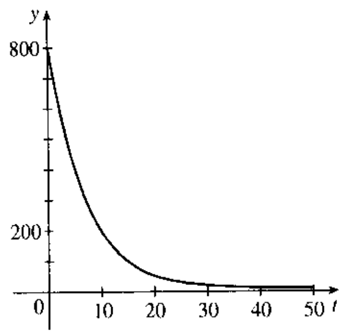
(b) $y(t) = Ce^{-0.0005t} = 0.9C \;\Rightarrow\; e^{-0.0005t} = 0.9 \;\Rightarrow\; -0.0005t = \ln 0.9 \;\Rightarrow$
$t = -2000 \ln 0.9 \approx 211$ s

8. (a) The mass remaining after t days is
$y(t) = y(0)e^{kt} = 800e^{kt}$. Since the half-life is 5.0 days,
$y(5) = 800e^{5k} = 400 \;\Rightarrow\; e^{5k} = \frac{1}{2} \;\Rightarrow$
$5k = \ln\frac{1}{2} \;\Rightarrow\; k = -(\ln 2)/5$, so
$y(t) = 800e^{-(\ln 2)t/5} = 800 \cdot 2^{-t/5}$.

(b) $y(30) = 800 \cdot 2^{-30/5} = 12.5$ mg

(c) $800e^{-(\ln 2)t/5} = 1 \;\Leftrightarrow\; -(\ln 2)\frac{t}{5} = \ln\frac{1}{800} = -\ln 800$
$\Leftrightarrow\; t = 5\,\frac{\ln 800}{\ln 2} \approx 48$ days

(d)



9. (a) If $y(t)$ is the mass (in mg) remaining after t years, then $y(t) = y(0)e^{kt} = 100e^{kt}$. $y(30) = 100e^{30k} = \frac{1}{2}(100)$
$\Rightarrow\; e^{30k} = \frac{1}{2} \;\Rightarrow\; k = -(\ln 2)/30 \;\Rightarrow\; y(t) = 100e^{-(\ln 2)t/30} = 100 \cdot 2^{-t/30}$

(b) $y(100) = 100 \cdot 2^{-100/30} \approx 9.92$ mg

(c) $100e^{-(\ln 2)t/30} = 1 \;\Rightarrow\; -(\ln 2)t/30 = \ln\frac{1}{100} \;\Rightarrow\; t = -30\,\frac{\ln 0.01}{\ln 2} \approx 199.3$ years

10. (a) If $y(t)$ is the mass after t days and $y(0) = A$, then $y(t) = Ae^{kt}$. $y(3) = Ae^{3k} = 0.58A \;\Rightarrow$
$e^{3k} = 0.58 \;\Rightarrow\; 3k = \ln 0.58 \;\Rightarrow\; k = \frac{1}{3}\ln 0.58$. Then $Ae^{(\ln 0.58)t/3} = \frac{1}{2}A \;\Leftrightarrow$
$\ln e^{(\ln 0.58)t/3} = \ln\frac{1}{2} \;\Leftrightarrow\; \frac{(\ln 0.58)t}{3} = \ln\frac{1}{2}$, so the half-life is $t = -\frac{3\ln 2}{\ln 0.58} \approx 3.82$ days.

(b) $Ae^{(\ln 0.58)t/3} = 0.10A \;\Leftrightarrow\; \frac{(\ln 0.58)t}{3} = \ln\frac{1}{10} \;\Leftrightarrow\; t = -\frac{3\ln 10}{\ln 0.58} \approx 12.68$ days

11. Let $y(t)$ be the level of radioactivity. Thus, $y(t) = y(0)e^{-kt}$ and k is determined by using the half-life:
$y(5730) = \frac{1}{2}y(0) \;\Rightarrow\; y(0)e^{-k(5730)} = \frac{1}{2}y(0) \;\Rightarrow\; e^{-5730k} = \frac{1}{2} \;\Rightarrow$
$-5730k = \ln\frac{1}{2} \;\Rightarrow\; k = -\frac{\ln\frac{1}{2}}{5730} = \frac{\ln 2}{5730}$. If 74% of the ^{14}C remains, then we know that $y(t) = 0.74y(0)$
$\Rightarrow\; 0.74 = e^{-t(\ln 2)/5730} \;\Rightarrow\; \ln 0.74 = -\frac{t\ln 2}{5730} \;\Rightarrow\; t = -\frac{5730(\ln 0.74)}{\ln 2} \approx 2489 \approx 2500$ years.

12. From the information given, we know that $\frac{dy}{dx} = 2y \;\Rightarrow\; y = Ce^{2x}$ by Theorem 2. To calculate C we use the point $(0, 5)$: $5 = Ce^{2(0)} \;\Rightarrow\; C = 5$. Thus, the equation of the curve is $y = 5e^{2x}$.

13. (a) Using Newton's Law of Cooling, $\frac{dT}{dt} = k(T - T_s)$, we have $\frac{dT}{dt} = k(T - 75)$.
Now let $y = T - 75$, so $y(0) = T(0) - 75 = 185 - 75 = 110$, so y is a solution of the initial-value problem $dy/dt = ky$ with $y(0) = 110$ and by Theorem 2 we have $y(t) = y(0)e^{kt} = 110e^{kt}$.
$y(30) = 110e^{30k} = 150 - 75 \;\Rightarrow\; e^{30k} = \frac{75}{110} = \frac{15}{22} \;\Rightarrow\; k = \frac{1}{30}\ln\frac{15}{22}$,
so $y(t) = 110e^{\frac{1}{30}t\ln\left(\frac{15}{22}\right)}$ and $y(45) = 110e^{\frac{45}{30}\ln\left(\frac{15}{22}\right)} \approx 62°$F. Thus, $T(45) \approx 62 + 75 = 137°$F.

(b) $T(t) = 100 \;\Rightarrow\; y(t) = 25$. $y(t) = 110e^{\frac{1}{30}t\ln\left(\frac{15}{22}\right)} = 25 \;\Rightarrow\; e^{\frac{1}{30}t\ln\left(\frac{15}{22}\right)} = \frac{25}{110} \;\Rightarrow$
$\frac{1}{30}t\ln\frac{15}{22} = \ln\frac{25}{110} \;\Rightarrow\; t = \frac{30\ln\frac{25}{110}}{\ln\frac{15}{22}} \approx 116$ min.

14. (a) Let $T(t)$ = temperature after t minutes. Newton's Law of Cooling implies that $\frac{dT}{dt} = k(T - 5)$. Let $y(t) = T(t) - 5$. Then $\frac{dy}{dt} = ky$, so $y(t) = y(0)e^{kt} = 15e^{kt} \Rightarrow T(t) = 5 + 15e^{kt} \Rightarrow$ $T(1) = 5 + 15e^k = 12 \Rightarrow e^k = \frac{7}{15} \Rightarrow k = \ln\frac{7}{15}$, so $T(t) = 5 + 15e^{\ln(7/15)t}$ and $T(2) = 5 + 15e^{2\ln(7/15)} \approx 8.3°\text{C}$.

(b) $5 + 15e^{\ln(7/15)t} = 6$ when $e^{\ln(7/15)t} = \frac{1}{15} \Rightarrow \ln\left(\frac{7}{15}\right)t = \ln\frac{1}{15} \Rightarrow t = \dfrac{\ln\frac{1}{15}}{\ln\frac{7}{15}} \approx 3.6$ min.

15. $\frac{dT}{dt} = k(T - 20)$. Letting $y = T - 20$, we get $\frac{dy}{dt} = ky$, so $y(t) = y(0)e^{kt}$.

$y(0) = T(0) - 20 = 5 - 20 = -15$, so $y(25) = y(0)e^{25k} = -15e^{25k}$, and

$y(25) = T(25) - 20 = 10 - 20 = -10$, so $-15e^{25k} = -10 \Rightarrow e^{25k} = \frac{2}{3}$. Thus, $25k = \ln\left(\frac{2}{3}\right)$ and

$k = \frac{1}{25}\ln\left(\frac{2}{3}\right)$, so $y(t) = y(0)e^{kt} = -15e^{(1/25)\ln(2/3)t}$. More simply, $e^{25k} = \frac{2}{3} \Rightarrow e^k = \left(\frac{2}{3}\right)^{1/25} \Rightarrow$

$e^{kt} = \left(\frac{2}{3}\right)^{t/25} \Rightarrow y(t) = -15 \cdot \left(\frac{2}{3}\right)^{t/25}$.

(a) $T(50) = 20 + y(50) = 20 - 15 \cdot \left(\frac{2}{3}\right)^{50/25} = 20 - 15 \cdot \left(\frac{2}{3}\right)^2 = 20 - \frac{20}{3} = 13.\bar{3}\,°\text{C}$

(b) $15 = T(t) = 20 + y(t) = 20 - 15 \cdot \left(\frac{2}{3}\right)^{t/25} \Rightarrow 15 \cdot \left(\frac{2}{3}\right)^{t/25} = 5 \Rightarrow \left(\frac{2}{3}\right)^{t/25} = \frac{1}{3} \Rightarrow$ $(t/25)\ln\left(\frac{2}{3}\right) = \ln\left(\frac{1}{3}\right) \Rightarrow t = 25\ln\left(\frac{1}{3}\right)/\ln\left(\frac{2}{3}\right) \approx 67.74$ min.

16. $\frac{dT}{dt} = k(T - 20)$. Let $y = T - 20$. Then $\frac{dy}{dt} = ky$, so $y(t) = y(0)e^{kt}$. $y(0) = T(0) - 20 = 95 - 20 = 75$, so $y(t) = 75e^{kt}$. When $T(t) = 70$, $\frac{dT}{dt} = -1°\text{C/min}$. Equivalently, $\frac{dy}{dt} = -1$ when $y(t) = 50$. Thus,

$-1 = \frac{dy}{dt} = ky(t) = 50k$ and $50 = y(t) = 75e^{kt}$. The first relation implies $k = -1/50$, so the second relation says $50 = 75e^{-t/50}$. Thus, $e^{-t/50} = \frac{2}{3} \Rightarrow -t/50 = \ln\left(\frac{2}{3}\right) \Rightarrow t = -50\ln\left(\frac{2}{3}\right) \approx 20.27$ min.

17. (a) Let $P(h)$ be the pressure at altitude h. Then $dP/dh = kP \Rightarrow P(h) = P(0)e^{kh} = 101.3e^{kh}$.
$P(1000) = 101.3e^{1000k} = 87.14 \Rightarrow 1000k = \ln\left(\frac{87.14}{101.3}\right) \Rightarrow$
$k = \frac{1}{1000}\ln\left(\frac{87.14}{101.3}\right) \Rightarrow P(h) = 101.3\,e^{\frac{1}{1000}h\ln\left(\frac{87.14}{101.3}\right)}$, so $P(3000) = 101.3e^{3\ln\left(\frac{87.14}{101.3}\right)} \approx 64.5$ kPa.

(b) $P(6187) = 101.3\,e^{\frac{6187}{1000}\ln\left(\frac{87.14}{101.3}\right)} \approx 39.9$ kPa

18. (a) Using $A = A_0\left(1 + \frac{r}{n}\right)^{nt}$ with $A_0 = 500$, $r = 0.14$, and $t = 2$, we have:

(i) Annually: $n = 1$; $A = 500\left(1 + \frac{0.14}{1}\right)^{1\cdot 2} = \$649.80$

(ii) Quarterly: $n = 4$; $A = 500\left(1 + \frac{0.14}{4}\right)^{4\cdot 2} = \$658.40$

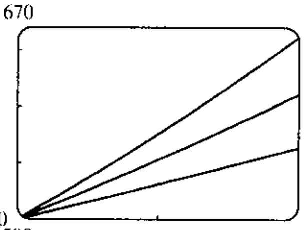
(iii) Monthly: $n = 12$; $A = 500\left(1 + \frac{0.14}{12}\right)^{12\cdot 2} = \$660.49$

(iv) Daily: $n = 365$; $A = 500\left(1 + \frac{0.14}{365}\right)^{365\cdot 2} = \$661.53$

(v) Hourly: $n = 365 \cdot 24$; $A = 500\left(1 + \frac{0.14}{365\cdot 24}\right)^{365\cdot 24\cdot 2} = \$661.56$

(vi) Continuously: $A = 500e^{(0.14)2} = \$661.56$

(b)



$A_{0.14}(2) = \$661.56$,
$A_{0.10}(2) = \$610.70$, and
$A_{0.06}(2) = \$563.75$.

19. (a) Using $A = A_0\left(1 + \frac{r}{n}\right)^{nt}$ with $A_0 = 3000$, $r = 0.05$, and $t = 5$, we have:

(i) Annually: $n = 1$; $A = 3000\left(1 + \frac{0.05}{1}\right)^{1 \cdot 5} = \$3828.84$

(ii) Semiannually: $n = 2$; $A = 3000\left(1 + \frac{0.05}{2}\right)^{2 \cdot 5} = \$3840.25$

(iii) Monthly: $n = 12$; $A = 3000\left(1 + \frac{0.05}{12}\right)^{12 \cdot 5} = \$3850.08$

(iv) Weekly: $n = 52$; $A = 3000\left(1 + \frac{0.05}{52}\right)^{52 \cdot 5} = \$3851.61$

(v) Daily: $n = 365$; $A = 3000\left(1 + \frac{0.05}{365}\right)^{365 \cdot 5} = \$3852.01$

(vi) Continuously: $A = 3000e^{(0.05)5} = \$3852.08$

(b) $dA/dt = 0.05A$ and $A(0) = 3000$.

20. (a) $A_0e^{0.06t} = 2A_0 \quad \Leftrightarrow \quad e^{0.06t} = 2 \quad \Leftrightarrow \quad 0.06t = \ln 2 \quad \Leftrightarrow \quad t = \frac{50}{3}\ln 2 \approx 11.55$, so the investment will double in about 11.55 years.

(b) The annual interest rate in $A = A_0(1 + r)^t$ is r. From part (a), we have $A = A_0e^{0.06t}$. These amounts must be equal, so $(1 + r)^t = e^{0.06t} \quad \Rightarrow \quad 1 + r = e^{0.06} \quad \Rightarrow \quad r = e^{0.06} - 1 \approx 0.0618 = 6.18\%$, which is the equivalent annual interest rate.

21. (a) $\frac{dP}{dt} = kP - m = k\left(P - \frac{m}{k}\right)$. Let $y = P - \frac{m}{k}$, so $\frac{dy}{dt} = \frac{dP}{dt}$ and the differential equation becomes $\frac{dy}{dt} = ky$. The solution is $y = y_0e^{kt} \quad \Rightarrow \quad P - \frac{m}{k} = \left(P_0 - \frac{m}{k}\right)e^{kt} \quad \Rightarrow \quad P(t) = \frac{m}{k} + \left(P_0 - \frac{m}{k}\right)e^{kt}$.

(b) Since $k > 0$, there will be an exponential expansion $\Leftrightarrow \quad P_0 - \frac{m}{k} > 0 \quad \Leftrightarrow \quad m < kP_0$.

(c) The population will be constant if $P_0 - \frac{m}{k} = 0 \quad \Leftrightarrow \quad m = kP_0$. It will decline if $P_0 - \frac{m}{k} < 0 \quad \Leftrightarrow$ $m > kP_0$.

(d) $P_0 = 8{,}000{,}000$, $k = \alpha - \beta = 0.016$, $m = 210{,}000 \quad \Rightarrow \quad m > kP_0 \; (= 128{,}000)$, so by part (c), the population was declining.

22. (a) $\frac{dy}{dt} = ky^{1+c} \quad \Rightarrow \quad y^{-1-c}\,dy = k\,dt \quad \Rightarrow \quad \frac{y^{-c}}{-c} = kt + C$. Since $y(0) = y_0$, we have $C = \frac{y_0^{-c}}{-c}$. Thus, $\frac{y^{-c}}{-c} = kt + \frac{y_0^{-c}}{-c}$, or $y^{-c} = y_0^{-c} - ckt$. So $y^c = \frac{1}{y_0^{-c} - ckt} = \frac{y_0^c}{1 - cy_0^ckt}$ and $y(t) = \frac{y_0}{(1 - cy_0^ckt)^{1/c}}$.

(b) $y(t) \to \infty$ as $1 - cy_0^ckt \to 0$, that is, as $t \to \frac{1}{cy_0^ck}$. Define $T = \frac{1}{cy_0^ck}$. Then $\lim_{t \to T^-} y(t) = \infty$.

(c) According to the data given, we have $c = 0.01$, $y(0) = 2$, and $y(3) = 16$, where the time t is given in months. Thus, $y_0 = 2$ and $16 = y(3) = \frac{y_0}{(1 - cy_0^ck \cdot 3)^{1/c}}$. Since $T = \frac{1}{cy_0^ck}$, we will solve for cy_0^ck.

$16 = \frac{2}{(1 - 3cy_0^ck)^{100}} \quad \Rightarrow \quad 1 - 3cy_0^ck = \left(\frac{1}{8}\right)^{0.01} = 8^{-0.01} \quad \Rightarrow \quad cy_0^ck = \frac{1}{3}\left(1 - 8^{-0.01}\right)$. Thus, doomsday occurs when $t = T = \frac{1}{cy_0^ck} = \frac{3}{1 - 8^{-0.01}} \approx 145.77$ months or 12.15 years.

APPLIED PROJECT Calculus and Baseball

1. (a) $F = ma = m\dfrac{dv}{dt}$, so by the Substitution Rule we have

$$\int_{t_0}^{t_1} F(t)\,dt = \int_{t_0}^{t_1} m\left(\frac{dv}{dt}\right)dt = m\int_{v_0}^{v_1} dv = [mv]_{v_0}^{v_1} = mv_1 - mv_0 = p(t_1) - p(t_0)$$

(b) (i) We have $v_1 = 110$ mi/h $= \frac{110(5280)}{3600}$ ft/s $= 161.\overline{3}$ ft/s, $v_0 = -90$ mi/h $= -132$ ft/s, and the mass of the baseball is $m = \dfrac{w}{g} = \frac{5/16}{32} = \frac{5}{512}$. So the change in momentum is

$p(t_1) - p(t_0) = mv_1 - mv_0 = \frac{5}{512}\left[161.\overline{3} - (-132)\right] \approx 2.86$ slug-ft/s.

(ii) From part (a) and part (b)(i), we have $\int_0^{0.001} F(t)dt = p(0.001) - p(0) \approx 2.86$, so the average force over the interval $[0, 0.001]$ is $\frac{1}{0.001}\int_0^{0.001} F(t)\,dt \approx \frac{1}{0.001}(2.86) = 2860$ lb.

2. (a) $W = \displaystyle\int_{s_0}^{s_1} F(s)\,ds$, where $F(s) = m\dfrac{dv}{dt} = m\dfrac{dv}{ds}\dfrac{ds}{dt} = mv\dfrac{dv}{ds}$ and so, by the Substitution Rule,

$$W = \int_{s_0}^{s_1} F(s)\,ds = \int_{s_0}^{s_1} mv\frac{dv}{ds}\,ds = \int_{v(s_0)}^{v(s_1)} mv\,dv = \left[\tfrac{1}{2}mv^2\right]_{v_0}^{v_1} = \tfrac{1}{2}mv_1^2 - \tfrac{1}{2}mv_0^2$$

(b) From part (b)(i), 90 mi/h = 132 ft/s. Assume $v_0 = v(s_0) = 0$ and $v_1 = v(s_1) = 132$ ft/s (note that s_1 is the point of release of the baseball). $m = \frac{5}{512}$, so the work done is

$$W = \tfrac{1}{2}mv_1^2 - \tfrac{1}{2}mv_0^2 = \tfrac{1}{2}\cdot\tfrac{5}{512}\cdot(132)^2 \approx 85 \text{ ft-lb}$$

3. (a) Here we have a differential equation of the form $dv/dt = kv$, so by Theorem 10.4.2, the solution is $v(t) = v(0)e^{kt}$. In this case $k = -\frac{1}{10}$ and $v(0) = 100$ ft/s, so $v(t) = 100e^{-t/10}$. We are interested in the time t that the ball takes to travel 280 ft, so we find the distance function

$$s(t) = \int_0^t v(x)\,dx = \int_0^t 100e^{-x/10}\,dx = 100\left[-10e^{-x/10}\right]_0^t = -1000\left(e^{-t/10} - 1\right)$$

$$= 1000\left(1 - e^{-t/10}\right)$$

Now we set $s(t) = 280$ and solve for t: $280 = 1000\left(1 - e^{-t/10}\right) \;\Rightarrow\; 1 - e^{-t/10} = \frac{7}{25} \;\Rightarrow$

$-\frac{1}{10}t = \ln\left(1 - \frac{7}{25}\right) \;\Rightarrow\; t \approx 3.285$ seconds.

(b) Let x be the distance of the shortstop from home plate. We calculate the time for the ball to reach home plate as a function of x, then differentiate with respect to x to find the value of x which corresponds to the minimum time. The total time that it takes the ball to reach home is the sum of the times of the two throws, plus the relay time ($\frac{1}{2}$ s). The distance from the fielder to the shortstop is $280 - x$, so to find the time t_1 taken by the first throw, we solve the equation $s_1(t_1) = 280 - x \;\Leftrightarrow\; 1 - e^{-t_1/10} = \dfrac{280 - x}{1000} \;\Leftrightarrow\; t_1 = -10\ln\dfrac{720 + x}{1000}$.

We find the time t_2 taken by the second throw if the shortstop throws with velocity w, since we see that this velocity varies in the rest of the problem. We use $v = we^{-t/10}$ and isolate t_2 in the equation

$s(t_2) = 10w\left(1 - e^{-t_2/10}\right) = x \quad \Leftrightarrow \quad e^{-t_2/10} = 1 - \dfrac{x}{10w} \quad \Leftrightarrow \quad t_2 = -10 \ln \dfrac{10w - x}{10w}$, so the total time is

$t_w(x) = \dfrac{1}{2} - 10\left[\ln \dfrac{720 + x}{1000} + \ln \dfrac{10w - x}{10w}\right]$. To find the minimum, we differentiate:

$\dfrac{dt_w}{dx} = -10\left[\dfrac{1}{720 + x} - \dfrac{1}{10w - x}\right]$, which changes from negative to positive when $720 + x = 10w - x \quad \Leftrightarrow$

$x = 5w - 360$. By the First Derivative Test, t_w has a minimum at this distance from the shortstop to home plate. So if the shortstop throws at $w = 105$ ft/s from a point $x = 5(105) - 360 = 165$ ft from home plate, the minimum time is $t_{105}(165) = \frac{1}{2} - 10\left(\ln \frac{720 + 165}{1000} + \ln \frac{1050 - 165}{1050}\right) \approx 3.431$ seconds. This is longer than the time taken in part (a), so in this case the manager should encourage a direct throw.

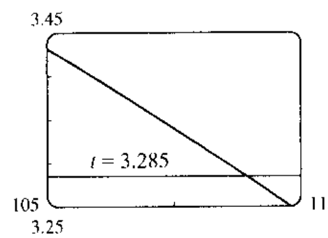
If $w = 115$ ft/s, then $x = 215$ ft from home, and the minimum time is

$t_{115}(215) = \frac{1}{2} - 10\left(\ln \frac{720 + 215}{1000} + \ln \frac{1150 - 215}{1150}\right) \approx 3.242$ seconds. This is less than the time taken in part (a), so in this case, the manager should encourage a relayed throw.

(c) In general, the minimum time is

$$t_w(5w - 360) = \frac{1}{2} - 10\left[\ln \frac{360 + 5w}{1000} + \ln \frac{360 + 5w}{10w}\right]$$

$$= \frac{1}{2} - 10 \ln \frac{(w + 72)^2}{400w}$$



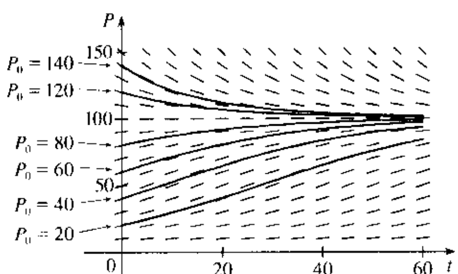
We want to find out when this is about 3.285 seconds, the same time as the direct throw. From the graph, we estimate that this is the case for $w \approx 112.8$ ft/s. So if the shortstop can throw the ball with this velocity, then a relayed throw takes the same time as a direct throw.

10.5 The Logistic Equation

1. (a) $dP/dt = 0.05P - 0.0005P^2 = 0.05P(1 - 0.01P) = 0.05P(1 - P/100)$. Comparing to Equation 1, $dP/dt = kP(1 - P/K)$, we see that the carrying capacity is $K = 100$ and the value of k is 0.05.

(b) The slopes close to 0 occur where P is near 0 or 100. The largest slopes appear to be on the line $P = 50$. The solutions are increasing for $0 < P_0 < 100$ and decreasing for $P_0 > 100$.

(c)

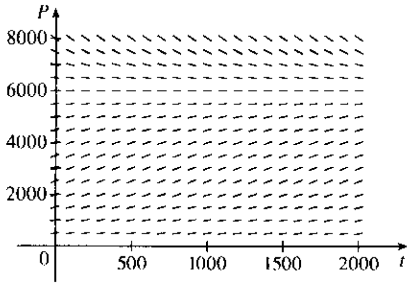


All of the solutions approach $P = 100$ as t increases. As in part (b), the solutions differ since for $0 < P_0 < 100$ they are increasing, and for $P_0 > 100$ they are decreasing. Also, some have an IP and some don't. It appears that the solutions which have $P_0 = 20$ and $P_0 = 40$ have inflection points at $P = 50$.

(d) The equilibrium solutions are $P = 0$ (trivial solution) and $P = 100$. The increasing solutions move away from $P = 0$ and all nonzero solutions approach $P = 100$ as $t \to \infty$.

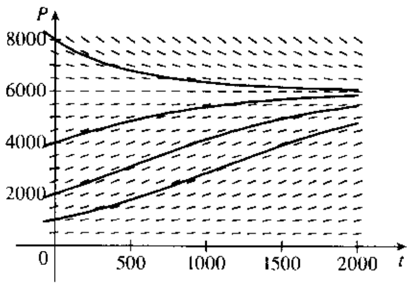
2. (a) $K = 6000$ and $k = 0.0015 \quad \Rightarrow \quad dP/dt = 0.0015P(1 - P/6000)$.

(b)



All of the solution curves approach 6000 as $t \to \infty$.

(c)



The curves with $P_0 = 1000$ and $P_0 = 2000$ appear to be concave upward at first and then concave downward. The curve with $P_0 = 4000$ appears to be concave downward everywhere. The curve with $P_0 = 8000$ appears to be concave upward everywhere. The inflection points are where the population grows the fastest.

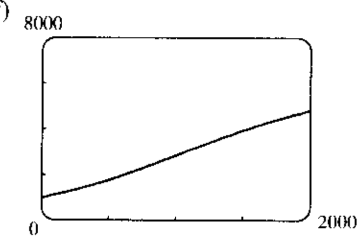
(d) See the solution to Exercise 10.2.25 for a possible program to calculate $P(50)$. [In this case, we use $\text{X} = 0$, $\text{H} = 1$, $\text{N} = 50$, $\text{Y}_1 = 0.0015y(1 - y/6000)$, and $\text{Y} = 1000$.] We find that $P(50) \approx 1064$.

(e) Using Equation 4 with $K = 6000$, $k = 0.0015$, and $P_0 = 1000$, we have $P(t) = \dfrac{K}{1 + Ae^{-kt}} = \dfrac{6000}{1 + Ae^{-0.0015t}}$, where

$$A = \frac{K - P_0}{P_0} = \frac{6000 - 1000}{1000} = 5. \text{ Thus,}$$

$$P(50) = \frac{6000}{1 + 5e^{-0.0015(50)}} \approx 1064.1,$$ which is extremely close to the estimate obtained in part (d).

(f)



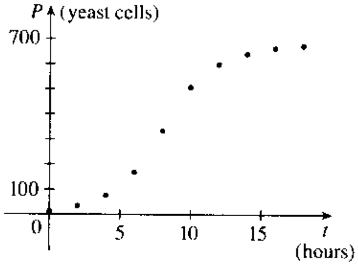
The curves are very similar.

3. (a) $\dfrac{dy}{dt} = ky\left(1 - \dfrac{y}{K}\right) \quad \Rightarrow \quad y(t) = \dfrac{K}{1 + Ae^{-kt}}$ with $A = \dfrac{K - y(0)}{y(0)}$. With $K = 8 \times 10^7$, $k = 0.71$, and $y(0) = 2 \times 10^7$, we get the model $y(t) = \dfrac{8 \times 10^7}{1 + 3e^{-0.71t}}$, so $y(1) = \dfrac{8 \times 10^7}{1 + 3e^{-0.71}} \approx 3.23 \times 10^7$ kg.

(b) $y(t) = 4 \times 10^7 \quad \Rightarrow \quad \dfrac{8 \times 10^7}{1 + 3e^{-0.71t}} = 4 \times 10^7 \quad \Rightarrow \quad 2 = 1 + 3e^{-0.71t} \quad \Rightarrow \quad e^{-0.71t} = \frac{1}{3} \quad \Rightarrow$

$-0.71t = \ln \frac{1}{3} \quad \Rightarrow \quad t = \dfrac{\ln 3}{0.71} \approx 1.55$ years

4. (a)



From the graph, we estimate the carrying capacity K for the yeast population to be 680.

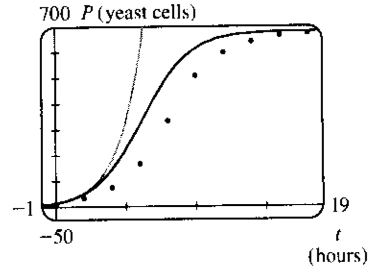
(b) An estimate of the initial relative growth rate is

$$\frac{1}{P_0}\frac{dP}{dt} = \frac{1}{18} \cdot \frac{39 - 18}{2 - 0} = \frac{7}{12} = 0.58\overline{3}.$$

(c) An exponential model is $P(t) = 18e^{7t/12}$. A logistic model is $P(t) = \dfrac{680}{1 + Ae^{-7t/12}}$, where $A = \frac{680 - 18}{18} = \frac{331}{9}$.

(d)

Time in Hours	Observed Values	Exponential Model	Logistic Model
0	18	18	18
2	39	58	55
4	80	186	149
6	171	596	322
8	336	1914	505
10	509	6147	614
12	597	19,739	658
14	640	63,389	673
16	664	203,558	678
18	672	653,679	679



The exponential model is a poor fit for anything beyond the first two observed values.
The logistic model varies more for the middle values than it does for the values at either end, but provides a good general fit, as shown in the figure.

(e) $P(7) = \dfrac{680}{1 + \frac{331}{9} e^{-7(7/12)}} \approx 420$ yeast cells

5. (a) We will assume that the difference in the birth and death rates is 20 million/year. Let $t = 0$ correspond to the year 1990 and use a unit of 1 billion for all calculations. $k \approx \dfrac{1}{P}\dfrac{dP}{dt} = \dfrac{1}{5.3}(0.02) = \dfrac{1}{265}$, so

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{K}\right) = \frac{1}{265}P\left(1 - \frac{P}{100}\right), \ P \text{ in billions}$$

(b) $A = \dfrac{K - P_0}{P_0} = \dfrac{100 - 5.3}{5.3} = \dfrac{947}{53} \approx 17.8679$. $P(t) = \dfrac{K}{1 + Ae^{-kt}} = \dfrac{100}{1 + \frac{947}{53} e^{-(1/265)t}}$, so $P(10) \approx 5.49$ billion.

(c) $P(110) \approx 7.81$, and $P(510) \approx 27.72$. The predictions are 7.81 billion in the year 2100 and 27.72 billion in 2500.

(d) If $K = 50$, then $P(t) = \dfrac{50}{1 + \frac{447}{53} e^{-(1/265)t}}$. So $P(10) \approx 5.48$, $P(110) \approx 7.61$, and $P(510) \approx 22.41$. The predictions become 5.48 billion in the year 2000, 7.61 billion in 2100, and 22.41 billion in the year 2500.

6. (a) If we assume that the carrying capacity for the world population is 100 billion, it would seem reasonable that the carrying capacity for the U.S. is 3–5 billion by using current populations and simple proportions. We will use $K = 4$ billion or 4000 million. With $t = 0$ corresponding to 1980, we have

$$P(t) = \frac{4000}{1 + \left(\frac{4000 - 250}{250}\right) e^{-kt}} = \frac{4000}{1 + 15e^{-kt}}$$

(b) $P(10) = 275 \;\Rightarrow\; \dfrac{4000}{1 + 15e^{-10k}} = 275 \;\Rightarrow\; 1 + 15e^{-10k} = \frac{4000}{275} \;\Rightarrow\; e^{-10k} = \dfrac{\frac{160}{11} - 1}{15} \;\Rightarrow$
$-10k = \ln \frac{149}{165} \;\Rightarrow\; k = -\frac{1}{10} \ln \frac{149}{165} \approx 0.01019992$.

(c) $2100 - 1990 = 110$ and $P(110) \approx 680$ million.
$2200 - 1990 = 210$ and $P(210) \approx 1449$ million, or about 1.4 billion.

(d) $P(t) = 300 \Rightarrow \dfrac{4000}{1 + 15e^{-kt}} = 300 \Rightarrow 1 + 15e^{-kt} = \frac{40}{3} \Rightarrow e^{-kt} = \frac{37}{3} \cdot \frac{1}{15} \Rightarrow -kt = \ln \frac{37}{45}$

$\Rightarrow t = 10 \dfrac{\ln \frac{37}{45}}{\ln \frac{149}{165}} \approx 19.19 \approx 19$. So we predict that the U.S. population will exceed 300 million in the year $1990 + 19 = 2009$.

7. (a) Our assumption is that $\dfrac{dy}{dt} = ky(1-y)$, where y is the fraction of the population that has heard the rumor.

(b) Using the logistic equation (1), $\dfrac{dP}{dt} = kP\left(1 - \dfrac{P}{K}\right)$, we substitute $y = \dfrac{P}{K}$, $P = Ky$, and $\dfrac{dP}{dt} = K\dfrac{dy}{dt}$, to obtain $K\dfrac{dy}{dt} = k(Ky)(1-y) \Leftrightarrow \dfrac{dy}{dt} = ky(1-y)$, our equation in part (a). Now the solution to (1) is $P(t) = \dfrac{K}{1 + Ae^{-kt}}$, where $A = \dfrac{K - P_0}{P_0}$. We use the same substitution to obtain $Ky = \dfrac{K}{1 + \dfrac{K - Ky_0}{Ky_0}e^{-kt}}$

$\Rightarrow y = \dfrac{y_0}{y_0 + (1 - y_0)e^{-kt}}$.

Alternatively, we could use the same steps as outlined in "The Analytic Solution," following Example 2.

(c) Let t be the number of hours since 8 A.M. Then $y_0 = y(0) = \frac{80}{1000} = 0.08$ and $y(4) = \frac{1}{2}$, so $\dfrac{1}{2} = y(4) = \dfrac{0.08}{0.08 + 0.92e^{-4k}}$. Thus, $0.08 + 0.92e^{-4k} = 0.16$, $e^{-4k} = \frac{0.08}{0.92} = \frac{2}{23}$, and $e^{-k} = \left(\frac{2}{23}\right)^{1/4}$, so $y = \dfrac{0.08}{0.08 + 0.92(2/23)^{t/4}} = \dfrac{2}{2 + 23(2/23)^{t/4}}$. Solving this equation for t, we get

$$2y + 23y\left(\frac{2}{23}\right)^{t/4} = 2 \Rightarrow \left(\frac{2}{23}\right)^{t/4} = \frac{2 - 2y}{23y} \Rightarrow \left(\frac{2}{23}\right)^{t/4} = \frac{2}{23} \cdot \frac{1 - y}{y} \Rightarrow$$

$\left(\dfrac{2}{23}\right)^{t/4 - 1} = \dfrac{1 - y}{y}$. It follows that $\dfrac{t}{4} - 1 = \dfrac{\ln[(1-y)/y]}{\ln \frac{2}{23}}$, so $t = 4\left[1 + \dfrac{\ln((1-y)/y)}{\ln \frac{2}{23}}\right]$.

When $y = 0.9$, $\dfrac{1-y}{y} = \frac{1}{9}$, so $t = 4\left(1 - \dfrac{\ln 9}{\ln \frac{2}{23}}\right) \approx 7.6$ h or 7 h 36 min. Thus, 90% of the population will have heard the rumor by 3:36 P.M.

8. (a) $P(0) = P_0 = 400$, $P(1) = 1200$ and $K = 10{,}000$. From the solution to the logistic differential equation $P(t) = \dfrac{P_0 K}{P_0 + (K - P_0)e^{-kt}}$, we get $P = \dfrac{400\,(10{,}000)}{400 + (9600)e^{-kt}} = \dfrac{10{,}000}{1 + 24e^{-kt}}$. $P(1) = 1200 \Rightarrow$ $1 + 24e^{-k} = \frac{100}{12} \Rightarrow e^k = \frac{288}{88} \Rightarrow k = \ln \frac{36}{11}$. So $P = \dfrac{10{,}000}{1 + 24e^{-t\ln(36/11)}} = \dfrac{10{,}000}{1 + 24 \cdot (11/36)^t}$.

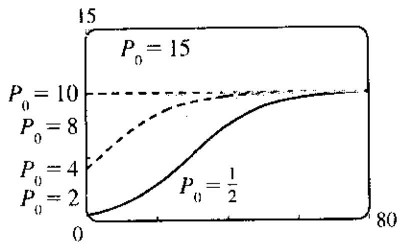
(b) $5000 = \dfrac{10{,}000}{1 + 24(11/36)^t} \Rightarrow 24\left(\frac{11}{36}\right)^t = 1 \Rightarrow t \ln \frac{11}{36} = \ln \frac{1}{24} \Rightarrow t \approx 2.68$ years.

9. (a) $\dfrac{dP}{dt} = kP\left(1 - \dfrac{P}{K}\right) \Rightarrow$

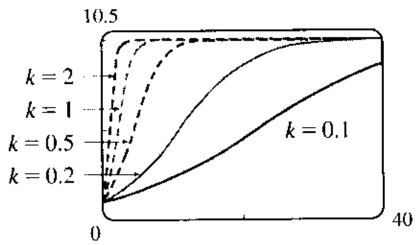
$$\frac{d^2P}{dt^2} = k\left[P\left(-\frac{1}{K}\frac{dP}{dt}\right) + \left(1 - \frac{P}{K}\right)\frac{dP}{dt}\right] = k\frac{dP}{dt}\left(-\frac{P}{K} + 1 - \frac{P}{K}\right)$$
$$= k\left[kP\left(1 - \frac{P}{K}\right)\right]\left(1 - \frac{2P}{K}\right) = k^2P\left(1 - \frac{P}{K}\right)\left(1 - \frac{2P}{K}\right)$$

(b) P grows fastest when P' has a maximum, that is, when $P'' = 0$. From part (a), $P'' = 0 \Leftrightarrow P = 0$, $P = K$, or $P = K/2$. Since $0 < P < K$, we see that $P'' = 0 \Leftrightarrow P = K/2$.

10.



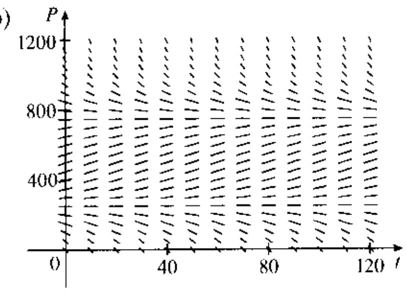
First we keep k constant (at 0.1, say) and change P_0 in the function $P = \dfrac{10P_0}{P_0 + (10 - P_0)e^{-0.1t}}$. (Notice that P_0 is the P-intercept.) If $P_0 = 0$, the function is 0 everywhere. For $0 < P_0 < 5$, the curve has an inflection point, which moves to the right as P_0 decreases. If $5 < P_0 < 10$, the graph is concave down everywhere. (We are considering only $t \geq 0$.) If $P_0 = 10$, the function is the constant function $P = 10$, and if $P_0 > 10$, the function decreases. For all $P_0 \neq 0$, $\lim\limits_{t \to \infty} P = 10$.



Now we instead keep P_0 constant (at $P_0 = 1$) and change k in the function $P = \dfrac{10}{1 + 9e^{-kt}}$. It seems that as k increases, the graph approaches the line $P = 10$ more and more quickly. (Note that the only difference in the shape of the curves is in the horizontal scaling; if we choose suitable x-scales, the graphs all look the same.)

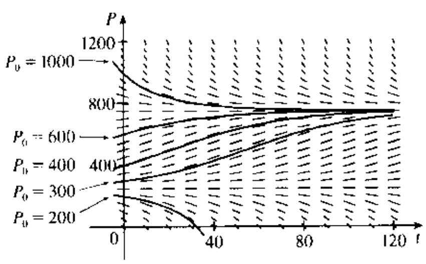
11. (a) The term -15 represents a harvesting of fish at a constant rate—in this case, 15 fish/week. This is the rate at which fish are caught.

(b)



(c) From the graph in part (b), it appears that $P(t) = 250$ and $P(t) = 750$ are the equilibrium solutions. We confirm this analytically by solving the equation $dP/dt = 0$ as follows: $0.08P(1 - P/1000) - 15 = 0 \Rightarrow$ $0.08P - 0.00008P^2 - 15 = 0 \Rightarrow -0.00008(P^2 - 1000P + 187{,}500) = 0 \Rightarrow$ $(P - 250)(P - 750) = 0 \Rightarrow P = 250$ or 750.

(d)



For $0 < P_0 < 250$, $P(t)$ decreases to 0. For $P_0 = 250$, $P(t)$ remains constant. For $250 < P_0 < 750$, $P(t)$ increases and approaches 750. For $P_0 = 750$, $P(t)$ remains constant. For $P_0 > 750$, $P(t)$ decreases and approaches 750.

(e) $\dfrac{dP}{dt} = 0.08P\left(1 - \dfrac{P}{1000}\right) - 15 \;\Leftrightarrow\; -\dfrac{100{,}000}{8} \cdot \dfrac{dP}{dt} = (0.08P - 0.00008P^2 - 15) \cdot \left(-\dfrac{100{,}000}{8}\right) \;\Leftrightarrow$

$-12{,}500\,\dfrac{dP}{dt} = P^2 - 1000P + 187{,}500 \;\Leftrightarrow\; \dfrac{dP}{(P-250)(P-750)} = -\dfrac{1}{12{,}500}\,dt \;\Leftrightarrow$

$\displaystyle\int \left(\frac{-1/500}{P-250} + \frac{1/500}{P-750}\right) dP = -\frac{1}{12{,}500}\,dt \;\Leftrightarrow\; \int \left(\frac{1}{P-250} - \frac{1}{P-750}\right) dP = \tfrac{1}{25}\,dt \;\Leftrightarrow$

$\ln|P-250| - \ln|P-750| = \frac{1}{25}t + C \quad\Leftrightarrow\quad \ln\left|\dfrac{P-250}{P-750}\right| = \frac{1}{25}t + C \quad\Leftrightarrow$

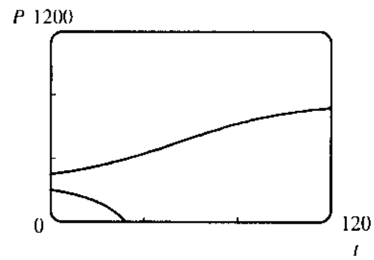
$\left|\dfrac{P-250}{P-750}\right| = e^{t/25+C} = ke^{t/25} \quad\Leftrightarrow\quad \dfrac{P-250}{P-750} = ke^{t/25} \quad\Leftrightarrow\quad P-250 = Pke^{t/25} - 750ke^{t/25} \quad\Leftrightarrow$

$P - Pke^{t/25} = 250 - 750ke^{t/25} \quad\Leftrightarrow\quad P(t) = \dfrac{250-750ke^{t/25}}{1-ke^{t/25}}$. If $t=0$ and $P=200$, then

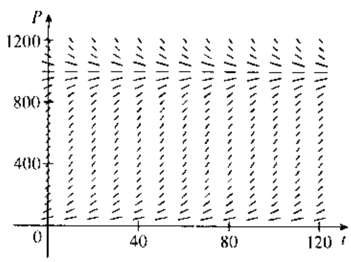
$200 = \dfrac{250-750k}{1-k} \quad\Leftrightarrow\quad 200 - 200k = 250 - 750k \quad\Leftrightarrow$

$550k = 50 \quad\Leftrightarrow\quad k = \frac{1}{11}$. Similarly, if $t=0$ and $P=300$, then $k = -\frac{1}{9}$. Simplifying P with these two values of k gives us

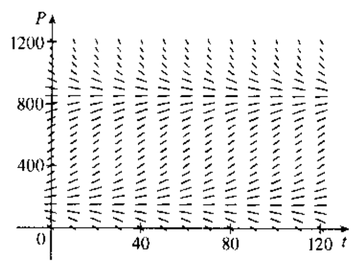
$$P(t) = \frac{250\left(3e^{t/25}-11\right)}{e^{t/25}-11} \text{ and } P(t) = \frac{750\left(e^{t/25}+3\right)}{e^{t/25}+9}.$$



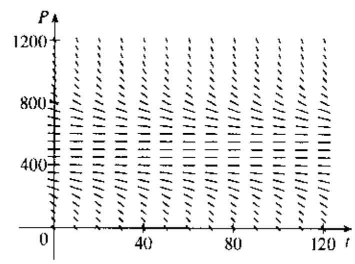
12. (a)



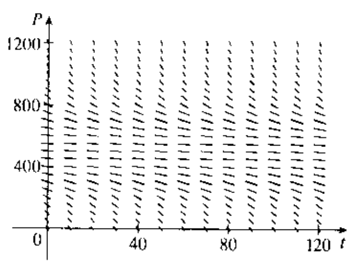
$c = 0$



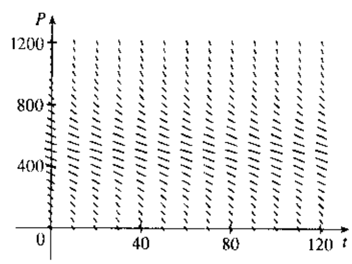
$c = 10$



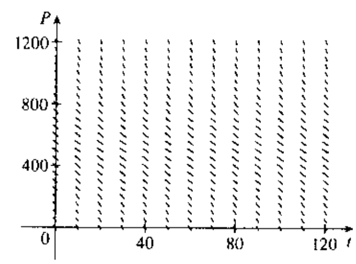
$c = 20$



$c = 21$



$c = 25$



$c = 30$

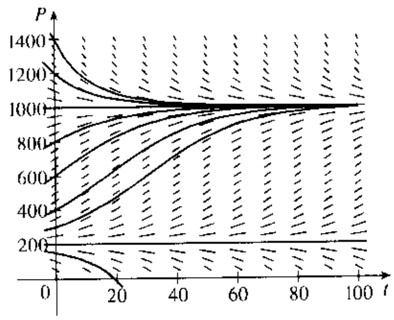
(b) For $0 \le c \le 20$, there is at least one equilibrium solution. For $c > 20$, the population always dies out.

(c) $\dfrac{dP}{dt} = 0.08P - 0.00008P^2 - c$. $\dfrac{dP}{dt} = 0 \quad\Leftrightarrow\quad P = \dfrac{-0.08 \pm \sqrt{(0.08)^2 - 4(-0.00008)(-c)}}{2(-0.00008)}$, which has at least one solution when the discriminant is nonnegative $\Rightarrow$ $0.0064 - 0.00032c \ge 0 \quad\Leftrightarrow\quad c \le 20$. For $0 \le c \le 20$, there is at least one value of P such that $dP/dt = 0$ and hence, at least one equilibrium solution. For $c > 20$, $dP/dt < 0$ and the population always dies out.

(d) The weekly catch should be less than 20 fish per week.

13. (a) $\dfrac{dP}{dt} = (kP)\left(1-\dfrac{P}{K}\right)\left(1-\dfrac{m}{P}\right)$. If $m < P < K$, then $dP/dt = (+)(+)(+) = + \quad\Rightarrow\quad P$ is increasing. If $0 < P < m$, then $dP/dt = (+)(+)(-) = - \quad\Rightarrow\quad P$ is decreasing.

(b)



$k = 0.08$, $K = 1000$, and $m = 200 \Rightarrow$

$$\frac{dP}{dt} = 0.08P\left(1 - \frac{P}{1000}\right)\left(1 - \frac{200}{P}\right)$$

For $0 < P_0 < 200$, the population dies out. For $P_0 = 200$, the population is steady. For $200 < P_0 < 1000$, the population increases and approaches 1000. For $P_0 > 1000$, the population decreases and approaches 1000.

The equilibrium solutions are $P(t) = 200$ and $P(t) = 1000$.

(c) $\dfrac{dP}{dt} = kP\left(1 - \dfrac{P}{K}\right)\left(1 - \dfrac{m}{P}\right) = kP\left(\dfrac{K-P}{K}\right)\left(\dfrac{P-m}{P}\right) = \dfrac{k}{K}(K-P)(P-m) \Leftrightarrow$

$\displaystyle\int \frac{dP}{(K-P)(P-m)} = \int \frac{k}{K}\,dt.$

By partial fractions, $\dfrac{1}{(K-P)(P-m)} = \dfrac{A}{K-P} + \dfrac{B}{P-m}$, so $A(P-m) + B(K-P) = 1$.

If $P = m$, $B = \dfrac{1}{K-m}$; if $P = K$, $A = \dfrac{1}{K-m}$, so $\displaystyle\frac{1}{K-m}\int\left(\frac{1}{K-P} + \frac{1}{P-m}\right)dP = \int \frac{k}{K}\,dt \Rightarrow$

$\dfrac{1}{K-m}\left(-\ln|K-P| + \ln|P-m|\right) = \dfrac{k}{K}t + M \Rightarrow \dfrac{1}{K-m}\ln\left|\dfrac{P-m}{K-P}\right| = \dfrac{k}{K}t + M \Rightarrow$

$\ln\left|\dfrac{P-m}{K-P}\right| = (K-m)\dfrac{k}{K}t + M_1 \Leftrightarrow \dfrac{P-m}{K-P} = De^{(K-m)(k/K)t}$ $[D = \pm e^{M_1}]$.

Let $t = 0$: $\dfrac{P_0 - m}{K - P_0} = D$. So $\dfrac{P-m}{K-P} = \dfrac{P_0-m}{K-P_0}e^{(K-m)(k/K)t}$. Solving for P, we get

$$P(t) = \frac{m(K-P_0) + K(P_0-m)e^{(K-m)(k/K)t}}{K - P_0 + (P_0-m)e^{(K-m)(k/K)t}}.$$

(d) If $P_0 < m$, then $P_0 - m < 0$. Let $N(t)$ be the numerator of the expression for $P(t)$ in part (c). Then $N(0) = P_0(K-m) > 0$, and $P_0 - m < 0 \Leftrightarrow \displaystyle\lim_{t\to\infty} K(P_0-m)\,e^{(K-m)(k/K)t} = -\infty \Rightarrow$ $\displaystyle\lim_{t\to\infty} N(t) = -\infty$. Since N is continuous, there is a number t such that $N(t) = 0$ and thus $P(t) = 0$. So the species will become extinct.

14. (a) $\dfrac{dP}{dt} = c\ln\left(\dfrac{K}{P}\right)P \Rightarrow \displaystyle\int \frac{dP}{P\ln(K/P)} = \int c\,dt$. Let $u = \ln\left(\dfrac{K}{P}\right) = \ln K - \ln P \Rightarrow du = -\dfrac{dP}{P}$

$\Rightarrow \displaystyle\int -\frac{du}{u} = ct + D \Rightarrow \ln|u| = -ct - D \Rightarrow |u| = e^{-(ct+D)} \Rightarrow |\ln(K/P)| = e^{-(ct+D)} \Rightarrow$

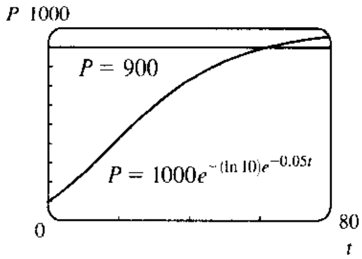
$\ln(K/P) = \pm e^{-(ct+D)}$. Letting $t = 0$, we get $\ln(K/P_0) = \pm e^{-D}$, so

$\ln(K/P) = \pm e^{-ct-D} = \pm e^{-ct}e^{-D} = \ln(K/P_0)\,e^{-ct} \Rightarrow K/P = e^{\ln(K/P_0)e^{-ct}} \Rightarrow$

$P(t) = Ke^{-\ln(K/P_0)e^{-ct}}$, $c \neq 0$.

(b) $\displaystyle\lim_{t\to\infty} P(t) = \lim_{t\to\infty} Ke^{-\ln(K/P_0)e^{-ct}} = Ke^{-\ln(K/P_0)\cdot 0} = Ke^0 = K$

(c)



The graphs look very similar. For the Gompertz function, $P(40) \approx 732$, nearly the same as the logistic function. The Gompertz function reaches $P = 900$ at $t \approx 61.7$ and its value at $t = 80$ is about 959, so it doesn't increase quite as fast as the logistic curve.

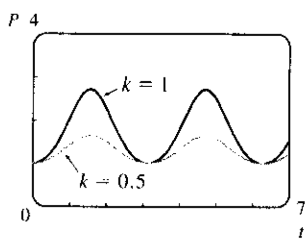
(d) $\dfrac{dP}{dt} = c\ln\left(\dfrac{K}{P}\right)P = cP(\ln K - \ln P) \quad\Rightarrow$

$$\frac{d^2P}{dt^2} = c\left[P\left(-\frac{1}{P}\frac{dP}{dt}\right) + (\ln K - \ln P)\frac{dP}{dt}\right] = c\,\frac{dP}{dt}\left[-1 + \ln\left(\frac{K}{P}\right)\right]$$
$$= c\,[c\ln(K/P)\,P]\,[\ln(K/P) - 1] = c^2P\ln(K/P)\,[\ln(K/P) - 1]$$

Since $0 < P < K$, $P'' = 0 \iff \ln(K/P) = 1 \iff K/P = e \iff P = K/e$. $P'' > 0$ for $0 < P < K/e$ and $P'' < 0$ for $K/e < P < K$, so P' is a maximum (and P grows fastest) when $P = K/e$. *Note:* If $P > K$, then $\ln(K/P) < 0$, so $P''(t) > 0$.

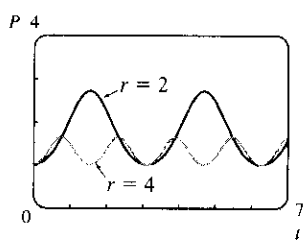
15. (a) $dP/dt = kP\cos(rt - \phi) \;\Rightarrow\; (dP)/P = k\cos(rt - \phi)\,dt \;\Rightarrow\; \int (dP)/P = k\int \cos(rt - \phi)\,dt \;\Rightarrow$ $\ln P = (k/r)\sin(rt - \phi) + C$. (Since this is a growth model, $P > 0$ and we can write $\ln P$ instead of $\ln|P|$.) Since $P(0) = P_0$, we obtain $\ln P_0 = (k/r)\sin(-\phi) + C = -(k/r)\sin\phi + C \;\Rightarrow$ $C = \ln P_0 + (k/r)\sin\phi$. Thus, $\ln P = (k/r)\sin(rt - \phi) + \ln P_0 + (k/r)\sin\phi$, which we can rewrite as $\ln(P/P_0) = (k/r)[\sin(rt - \phi) + \sin\phi]$ or, after exponentiation, $P(t) = P_0e^{(k/r)[\sin(rt-\phi)+\sin\phi]}$.

(b) As k increases, the amplitude increases, but the minimum value stays the same.



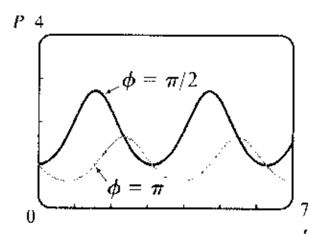
Comparing values of k with $P_0 = 1$, $r = 2$, and $\phi = \pi/2$

As r increases, the amplitude and the period decrease.



Comparing values of r with $P_0 = 1$, $k = 1$, and $\phi = \pi/2$

A change in ϕ produces slight adjustments in the phase shift and amplitude.



Comparing values of ϕ with $P_0 = 1$, $k = 1$, and $r = 2$

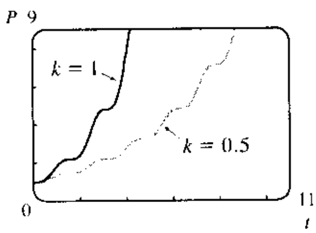
$P(t)$ oscillates between $P_0e^{(k/r)(1+\sin\phi)}$ and $P_0e^{(k/r)(-1+\sin\phi)}$ (the extreme values are attained when $rt - \phi$ is an odd multiple of $\frac{\pi}{2}$), so $\lim\limits_{t\to\infty} P(t)$ does not exist.

16. (a) $dP/dt = kP\cos^2(rt - \phi) \;\Rightarrow\; (dP)/P = k\cos^2(rt - \phi)\,dt \;\Rightarrow\; \int (dP)/P = k\int \cos^2(rt - \phi)\,dt \;\Rightarrow$ $\ln P = k\displaystyle\int \frac{1 + \cos(2(rt - \phi))}{2}\,dt = \frac{k}{2}t + \frac{k}{4r}\sin(2(rt - \phi)) + C$. From $P(0) = P_0$, we get $\ln P_0 = \dfrac{k}{4r}\sin(-2\phi) + C = C - \dfrac{k}{4r}\sin 2\phi$, so $C = \ln P_0 + \dfrac{k}{4r}\sin 2\phi$ and

$\ln P = \frac{k}{2}t + \frac{k}{4r}\sin(2(rt-\phi)) + \ln P_0 + \frac{k}{4r}\sin 2\phi$. Simplifying, we get

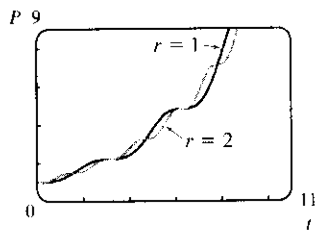
$\ln\frac{P}{P_0} = \frac{k}{2}t + \frac{k}{4r}\left[\sin(2(rt-\phi)) + \sin 2\phi\right] = f(t)$, or $P(t) = P_0 e^{f(t)}$.

(b) An increase in k stretches the graph of P vertically while maintaining $P(0) = P_0$.



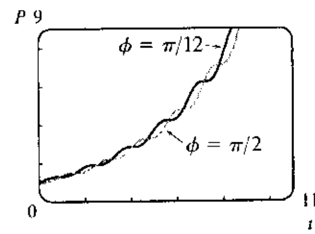
Comparing values of k with $P_0 = 1$, $r = 2$, and $\phi = \pi/2$

An increase in r compresses the graph of P horizontally—similar to changing the period in Exercise 15.



Comparing values of r with $P_0 = 1$, $k = 0.5$, and $\phi = \pi/2$

As in Exercise 15, a change in ϕ only makes slight adjustments in the growth of P, as shown in the figure.



Comparing values of ϕ with $P_0 = 1$, $k = 0.5$, and $r = 2$

$f'(t) = k/2 + [k/(4r)][2r\cos(2(rt-\phi))] = (k/2)[1+\cos(2(rt-\phi))] \geq 0$. Since $P(t) = P_0 e^{f(t)}$, we have $P'(t) = P_0 f'(t) e^{f(t)} \geq 0$, with equality only when $\cos(2(rt-\phi)) = -1$; that is, when $rt - \phi$ is an odd multiple of $\frac{\pi}{2}$. Therefore, $P(t)$ is an increasing function on $(0,\infty)$. P can also be written as $P(t) = P_0 e^{kt/2} e^{(k/4r)[\sin(2(rt-\phi))+\sin 2\phi]}$. The second exponential oscillates between $e^{(k/4r)(1+\sin 2\phi)}$ and $e^{(k/4r)(-1+\sin 2\phi)}$, while the first one, $e^{kt/2}$, grows without bound. So $\lim\limits_{t\to\infty} P(t) = \infty$.

17. By Equation (4), $P(t) = \dfrac{K}{1+Ae^{-kt}}$. By comparison, if $c = (\ln A)/k$ and $u = \frac{1}{2}k(t-c)$, then

$$1 + \tanh u = 1 + \frac{e^u - e^{-u}}{e^u + e^{-u}} = \frac{e^u + e^{-u}}{e^u + e^{-u}} + \frac{e^u - e^{-u}}{e^u + e^{-u}} = \frac{2e^u}{e^u + e^{-u}} \cdot \frac{e^{-u}}{e^{-u}} = \frac{2}{1+e^{-2u}}$$

and $e^{-2u} = e^{-k(t-c)} = e^{kc}e^{-kt} = e^{\ln A}e^{-kt} = Ae^{-kt}$, so

$$\tfrac{1}{2}K\left[1 + \tanh\left(\tfrac{1}{2}k(t-c)\right)\right] = \frac{K}{2}[1+\tanh u] = \frac{K}{2}\cdot\frac{2}{1+e^{-2u}} = \frac{K}{1+e^{-2u}} = \frac{K}{1+Ae^{-kt}} = P(t).$$

10.6 Linear Equations

1. $y' + e^x y = x^2 y^2$ is not linear since it cannot be put into the standard linear form (1), $y' + P(x)\,y = Q(x)$.

2. $y + \sin x = x^3 y' \Rightarrow x^3 y' - y = \sin x \Rightarrow y' + \left(-\dfrac{1}{x^3}\right) y = \dfrac{\sin x}{x^3}$. This equation is in the standard linear form (1), so it is linear.

3. $xy' + \ln x - x^2 y = 0 \Rightarrow xy' - x^2 y = -\ln x \Rightarrow y' + (-x)\,y = -\dfrac{\ln x}{x}$, which is in the standard linear form (1), so this equation is linear.

4. $y' + \cos y = \tan x$ is not linear since it cannot be put into the standard linear form (1), $y' + P(x)\,y = Q(x)$. [$\cos y$ is not of the form $P(x)y$.]

5. Comparing the given equation, $y' + 2y = 2e^x$, with the general form, $y' + P(x)y = Q(x)$, we see that $P(x) = 2$ and the integrating factor is $I(x) = e^{\int P(x)dx} = e^{\int 2\,dx} = e^{2x}$. Multiplying the differential equation by $I(x)$ gives $e^{2x}y' + 2e^{2x}y = 2e^{3x} \Rightarrow (e^{2x}y)' = 2e^{3x} \Rightarrow e^{2x}y = \int 2e^{3x}\,dx \Rightarrow e^{2x}y = \frac{2}{3}e^{3x} + C \Rightarrow$ $y = \frac{2}{3}e^x + Ce^{-2x}$.

6. $y' = x + 5y \Rightarrow y' - 5y = x$. $I(x) = e^{\int P(x)dx} = e^{\int(-5)dx} = e^{-5x}$. Multiplying the differential equation by $I(x)$ gives $e^{-5x}y' - 5e^{-5x}y = xe^{-5x} \Rightarrow (e^{-5x}y)' = xe^{-5x} \Rightarrow$ $e^{-5x}y = \int xe^{-5x}\,dx = -\frac{1}{5}xe^{-5x} - \frac{1}{25}e^{-5x} + C$ [by parts] $\Rightarrow y = -\frac{1}{5}x - \frac{1}{25} + Ce^{5x}$

7. $xy' - 2y = x^2$ [divide by x] $\Rightarrow y' + \left(-\dfrac{2}{x}\right)y = x$ (*).

$I(x) = e^{\int P(x)\,dx} = e^{\int(-2/x)\,dx} = e^{-2\ln|x|} = e^{\ln|x|^{-2}} = e^{\ln(1/x^2)} = 1/x^2$. Multiplying the differential equation (*) by $I(x)$ gives $\dfrac{1}{x^2}y' - \dfrac{2}{x^3}y = \dfrac{1}{x} \Rightarrow \left(\dfrac{1}{x^2}y\right)' = \dfrac{1}{x} \Rightarrow \dfrac{1}{x^2}y = \ln|x| + C \Rightarrow$ $y = x^2(\ln|x| + C) = x^2\ln|x| + Cx^2$.

8. $x^2y' + 2xy = \cos^2 x \Rightarrow y' + \dfrac{2}{x}y = \dfrac{\cos^2 x}{x^2}$. $I(x) = e^{\int P(x)\,dx} = e^{\int 2/x\,dx} = e^{2\ln|x|} = e^{\ln(x^2)} = x^2$. Multiplying by $I(x)$ gives us our original equation back. You may have noticed this immediately, since $P(x)$ is the derivative of the coefficient of y'. We rewrite it as $(x^2y)' = \cos^2 x$. Thus,

$x^2y = \int \cos^2 x\,dx = \int \frac{1}{2}(1 + \cos 2x)\,dx = \frac{1}{2}x + \frac{1}{4}\sin 2x + C \Rightarrow y = \dfrac{1}{2x} + \dfrac{1}{4x^2}\sin 2x + \dfrac{C}{x^2}$ or $y = \dfrac{1}{2x} + \dfrac{1}{2x^2}\sin x\cos x + \dfrac{C}{x^2}$.

9. Since $P(x)$ is the derivative of the coefficient of y' [$P(x) = 1$ and the coefficient is x], we can write the differential equation $xy' + y = \sqrt{x}$ in the easily integrable form $(xy)' = \sqrt{x} \Rightarrow xy = \frac{2}{3}x^{3/2} + C \Rightarrow$ $y = \frac{2}{3}\sqrt{x} + C/x$.

10. $y' - y = 1/x$ [$x \neq 0$], so $I(x) = e^{\int(-1)dx} = e^{-x}$. Multiplying the differential equation by $I(x)$ gives $e^{-x}y' - e^{-x}y = e^{-x}/x \Rightarrow (e^{-x}y)' = e^{-x}/x \Rightarrow y = e^x\left[\int (e^{-x}/x)\,dx + C\right]$.

11. $I(x) = e^{\int 2x\,dx} = e^{x^2}$. Multiplying the differential equation $y' + 2xy = x^2$ by $I(x)$ gives $e^{x^2}y' + 2xe^{x^2}y = x^2e^{x^2} \Rightarrow \left(e^{x^2}y\right)' = x^2e^{x^2}$. Thus

$y = e^{-x^2}\left[\int x^2e^{x^2}\,dx + C\right] = e^{-x^2}\left[\frac{1}{2}xe^{x^2} - \int \frac{1}{2}e^{x^2}\,dx + C\right] = \frac{1}{2}x + Ce^{-x^2} - e^{-x^2}\int \frac{1}{2}e^{x^2}\,dx$.

12. $I(x) = e^{\int -\tan x\,dx} = e^{\ln|\cos x|} = \cos x$ (since $-\frac{\pi}{2} < x < \frac{\pi}{2}$). Multiplying the differential equation by $I(x)$ gives $y'\cos x - y\tan x\cos x = x\cos x\sin 2x \Rightarrow (y\cos x)' = x\cos x\sin 2x$. So

$$y = \frac{1}{\cos x}\left[\int x\cos x\sin 2x\,dx + C\right] = \frac{1}{\cos x}\left[\int 2x\cos^2 x\sin x\,dx + C\right]$$

$$= \frac{1}{\cos x}\left[\frac{-2x\cos^3 x}{3} + \frac{2}{3}\left(\sin x - \frac{\sin^3 x}{3}\right) + C\right] = \frac{-2x\cos^2 x}{3} + \frac{C}{\cos x} + 2\tan x\frac{3 - \sin^2 x}{9}$$

13. $(1 + t)\dfrac{du}{dt} + u = 1 + t,\ t > 0$ [divide by $1 + t$] $\Rightarrow \dfrac{du}{dt} + \dfrac{1}{1 + t}u = 1$ (*), which has the form $u' + P(t)u = Q(t)$. The integrating factor is $I(t) = e^{\int P(t)\,dt} = e^{\int[1/(1+t)]\,dt} = e^{\ln(1+t)} = 1 + t$.

Multiplying (*) by $I(t)$ gives us our original equation back. We rewrite it as $[(1+t)\,u]' = 1+t$. Thus,

$$(1+t)\,u = \int (1+t)\,dt = t + \tfrac{1}{2}t^2 + C \quad\Rightarrow\quad u = \frac{t+\frac{1}{2}t^2+C}{1+t} \text{ or } u = \frac{t^2+2t+2C}{2\,(t+1)}.$$

14. $t\ln t\,\dfrac{dr}{dt} + r = te^t \quad\Rightarrow\quad \dfrac{dr}{dt} + \dfrac{1}{t\ln t}r = \dfrac{e^t}{\ln t}$. $I(t) = e^{\int dt/(t\ln t)} = e^{\ln(\ln t)} = \ln t$. Multiplying by $\ln t$ gives

$$\ln t\,\frac{dr}{dt} + \frac{1}{t}r = e^t \quad\Rightarrow\quad [(\ln t)r]' = e^t \quad\Rightarrow\quad (\ln t)r = e^t + C \quad\Rightarrow\quad r = \frac{e^t+C}{\ln t}.$$

15. $y' = x + y \quad\Rightarrow\quad y' + (-1)y = x$. $I(x) = e^{\int(-1)\,dx} = e^{-x}$. Multiplying by e^{-x} gives $e^{-x}y' - e^{-x}y = xe^{-x}$ $\Rightarrow\quad (e^{-x}y)' = xe^{-x} \quad\Rightarrow\quad e^{-x}y = \int xe^{-x}\,dx = -xe^{-x} - e^{-x} + C$ [integration by parts with $u = x$, $dv = e^{-x}\,dx$] $\Rightarrow\quad y = -x - 1 + Ce^x$. $y(0) = 2 \quad\Rightarrow\quad -1 + C = 2 \quad\Rightarrow\quad C = 3$, so $y = -x - 1 + 3e^x$.

16. $t\dfrac{dy}{dt} + 2y = t^3$, $t > 0$, $y(1) = 0$. Divide by t to get $\dfrac{dy}{dt} + \dfrac{2}{t}y = t^2$, which is linear.

$I(t) = e^{\int(2/t)dt} = e^{2\ln t} = t^2$. Multiplying by t^2 gives $t^2\dfrac{dy}{dt} + 2ty = t^4 \quad\Rightarrow\quad (t^2y)' = t^4 \quad\Rightarrow$

$t^2y = \frac{1}{5}t^5 + C \quad\Rightarrow\quad y = \dfrac{t^3}{5} + \dfrac{C}{t^2}$. Thus, $0 = y(1) = \frac{1}{5} + C \quad\Rightarrow\quad C = -\frac{1}{5}$, so $y = \dfrac{t^3}{5} - \dfrac{1}{5t^2}$.

17. $\dfrac{dv}{dt} - 2tv = 3t^2e^{t^2}$, $v(0) = 5$. $I(t) = e^{\int(-2t)dt} = e^{-t^2}$. Multiply the differential equation by $I(t)$ to get

$$e^{-t^2}\frac{dv}{dt} - 2te^{-t^2}v = 3t^2 \quad\Rightarrow\quad \left(e^{-t^2}v\right)' = 3t^2 \quad\Rightarrow\quad e^{-t^2}v = \int 3t^2\,dt = t^3 + C \quad\Rightarrow\quad v = t^3e^{t^2} + Ce^{t^2}.$$

$5 = v(0) = 0\cdot 1 + C\cdot 1 = C$, so $v = t^3e^{t^2} + 5e^{t^2}$.

18. $2xy' + y = 6x$, $x > 0 \quad\Rightarrow\quad y' + \dfrac{1}{2x}y = 3$. $I(x) = e^{\int 1/(2x)\,dx} = e^{(1/2)\ln x} = e^{\ln x^{1/2}} = \sqrt{x}$. Multiplying by $\sqrt{x}$ gives $\sqrt{x}\,y' + \dfrac{1}{2\sqrt{x}}y = 3\sqrt{x} \quad\Rightarrow\quad (\sqrt{x}\,y)' = 3\sqrt{x} \quad\Rightarrow\quad \sqrt{x}\,y = \int 3\sqrt{x}\,dx = 2x^{3/2} + C \quad\Rightarrow$

$y = 2x + \dfrac{C}{\sqrt{x}}$. $y(4) = 20 \quad\Rightarrow\quad 8 + \dfrac{C}{2} = 20 \quad\Rightarrow\quad C = 24$, so $y = 2x + \dfrac{24}{\sqrt{x}}$.

19. $xy' = y + x^2\sin x \quad\Rightarrow\quad y' - \dfrac{1}{x}y = x\sin x$. $I(x) = e^{\int(-1/x)\,dx} = e^{-\ln x} = e^{\ln x^{-1}} = \dfrac{1}{x}$.

Multiplying by $\dfrac{1}{x}$ gives $\dfrac{1}{x}y' - \dfrac{1}{x^2}y = \sin x \quad\Rightarrow\quad \left(\dfrac{1}{x}y\right)' = \sin x \quad\Rightarrow\quad \dfrac{1}{x}y = -\cos x + C \quad\Rightarrow$

$y = -x\cos x + Cx$. $y(\pi) = 0 \quad\Rightarrow\quad -\pi\cdot(-1) + C\pi = 0 \quad\Rightarrow\quad C = -1$, so $y = -x\cos x - x$.

20. $x\dfrac{dy}{dx} - \dfrac{y}{x+1} = x \quad\Rightarrow\quad y' - \dfrac{y}{x(x+1)} = 1$ $(x > 0)$, so $I(x) = e^{-\int 1/[x(x+1)]dx} = e^{-(\ln|x| - \ln|x+1|)} = \dfrac{x+1}{x}$.

Multiplying the differential equation by $I(x)$ gives $\dfrac{x+1}{x}y' - \dfrac{y}{x(x+1)}\dfrac{x+1}{x} = \dfrac{x+1}{x} \quad\Rightarrow$

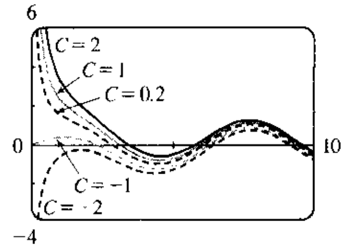
$\left(\dfrac{x+1}{x}y\right)' = \dfrac{x+1}{x}$. Then $y = \dfrac{x}{x+1}\left[\displaystyle\int\left(1+\frac{1}{x}\right)dx + C\right] = \dfrac{x}{x+1}(x + \ln x + C)$. But

$0 = y(1) = \frac{1}{2}[1 + C]$ so $C = -1$ and the solution to the initial-value problem is $y = \dfrac{x}{x+1}(x - 1 + \ln x)$.

21. $y' + \frac{1}{x}y = \cos x$ $(x \neq 0)$, so $I(x) = e^{\int (1/x)\,dx} = e^{\ln|x|} = x$ (for $x > 0$). Multiplying the differential equation by $I(x)$ gives $xy' + y = x\cos x \Rightarrow (xy)' = x\cos x$. Thus,

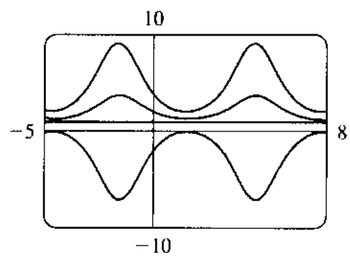
$$y = \frac{1}{x}\left[\int x\cos x\,dx + C\right] = \frac{1}{x}\left[x\sin x + \cos x + C\right]$$

$$= \sin x + \frac{\cos x}{x} + \frac{C}{x}$$



The solutions are asymptotic to the y-axis (except for $C = -1$). In fact, for $C > -1$, $y \to \infty$ as $x \to 0^+$, whereas for $C < -1$, $y \to -\infty$ as $x \to 0^+$. As x gets larger, the solutions approximate $y = \sin x$ more closely. The graphs for larger C lie above those for smaller C. The distance between the graphs lessens as x increases.

22. $I(x) = e^{\int \cos x\,dx} = e^{\sin x}$. Multiplying the differential equation by $I(x)$ gives $e^{\sin x}y' + \cos x \cdot e^{\sin x}y = \cos x \cdot e^{\sin x} \Rightarrow$ $\left(e^{\sin x}y\right)' = \cos x \cdot e^{\sin x} \Rightarrow$ $y = e^{-\sin x}\left[\int \cos x \cdot e^{\sin x}\,dx + C\right] = 1 + Ce^{-\sin x}$. The graphs for $C = -3, 0, 1,$ and 3 are shown. As the values of C get further from zero the graph is stretched away from the line $y = 1$, which is the value for $C = 0$. The graphs are all periodic in x, with a period of 2π.



23. Setting $u = y^{1-n}$, $\frac{du}{dx} = (1-n)\,y^{-n}\frac{dy}{dx}$ or $\frac{dy}{dx} = \frac{y^n}{1-n}\frac{du}{dx} = \frac{u^{n/(1-n)}}{1-n}\frac{du}{dx}$. Then the Bernoulli differential equation becomes $\frac{u^{n/(1-n)}}{1-n}\frac{du}{dx} + P(x)u^{1/(1-n)} = Q(x)u^{n/(1-n)}$ or $\frac{du}{dx} + (1-n)P(x)u = Q(x)(1-n)$.

24. Here $y' + \frac{y}{x} = -y^2$, so $n = 2$, $P(x) = \frac{1}{x}$ and $Q(x) = -1$. Setting $u = y^{-1}$, u satisfies $u' - \frac{1}{x}u = 1$. Then $I(x) = e^{\int(-1/x)\,dx} = \frac{1}{x}$ (for $x > 0$) and $u = x\left(\int \frac{1}{x}\,dx + C\right) = x(\ln|x| + C)$. Thus, $y = \frac{1}{x(C + \ln|x|)}$.

25. $y' + \frac{2}{x}y = \frac{y^3}{x^2}$. Here $n = 3$, $P(x) = \frac{2}{x}$, $Q(x) = \frac{1}{x^2}$ and setting $u = y^{-2}$, u satisfies $u' - \frac{4u}{x} = -\frac{2}{x^2}$.

Then $I(x) = e^{\int(-4/x)\,dx} = x^{-4}$ and $u = x^4\left(\int -\frac{2}{x^6}\,dx + C\right) = x^4\left(\frac{2}{5x^5} + C\right) = Cx^4 + \frac{2}{5x}$.

Thus, $y = \pm\left(Cx^4 + \frac{2}{5x}\right)^{-1/2}$.

26. Here $n = 3$, $P(x) = 1$, $Q(x) = x$ and setting $u = y^{-2}$, u satisfies $u' - 2u = -2x$. Then $I(x) = e^{\int(-2)\,dx} = e^{-2x}$ and $u = e^{2x}\left[\int -2xe^{-2x}\,dx + C\right] = e^{2x}\left(xe^{-2x} + \frac{1}{2}e^{-2x} + C\right) = x + \frac{1}{2} + Ce^{2x}$. So $y^{-2} = x + \frac{1}{2} + Ce^{2x} \Rightarrow y = \pm\left[x + \frac{1}{2} + Ce^{2x}\right]^{-1/2}$.

27. (a) $2\dfrac{dI}{dt} + 10I = 40$ or $\dfrac{dI}{dt} + 5I = 20$. Then the integrating factor is $e^{\int 5\,dt} = e^{5t}$. Multiplying the differential equation by the integrating factor gives $e^{5t}\dfrac{dI}{dt} + 5Ie^{5t} = 20e^{5t} \;\Rightarrow\; \left(e^{5t}I\right)' = 20e^{5t} \;\Rightarrow$ $I(t) = e^{-5t}\left[\int 20e^{5t}\,dt + C\right] = 4 + Ce^{-5t}$. But $0 = I(0) = 4 + C$, so $I(t) = 4 - 4e^{-5t}$.

(b) $I(0.1) = 4 - 4e^{-0.5} \approx 1.57$ A

28. (a) $\dfrac{dI}{dt} + 20I = 40\sin 60t$, so the integrating factor is e^{20t}. Multiplying the differential equation by the integrating factor gives $e^{20t}\dfrac{dI}{dt} + 20Ie^{20t} = 40e^{20t}\sin 60t \;\Rightarrow\; \left(e^{20t}I\right)' = 40e^{20t}\sin 60t \;\Rightarrow$

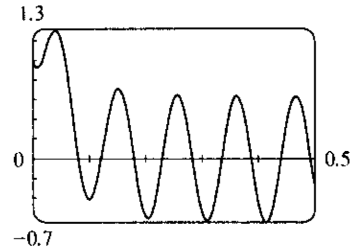
$$I(t) = e^{-20t}\left[\int 40e^{20t}\sin 60t\,dt + C\right]$$
$$= e^{-20t}\left[40e^{20t}\left(\tfrac{1}{4000}\right)(20\sin 60t - 60\cos 60t)\right] + Ce^{-20t}$$
$$= \frac{\sin 60t - 3\cos 60t}{5} + Ce^{-20t}$$

But $1 = I(0) = -\frac{3}{5} + C$, so

$$I(t) = \frac{\sin 60t - 3\cos 60t + 8e^{-20t}}{5}.$$

(b) $I(0.1) = \dfrac{\sin 6 - 3\cos 6 + 8e^{-2}}{5} \approx -0.42$ A

(c)



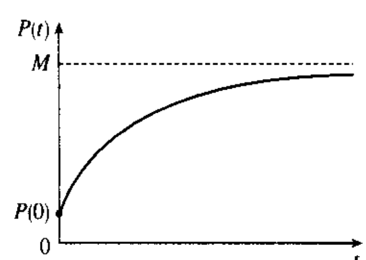
29. $5\dfrac{dQ}{dt} + 20Q = 60$ with $Q(0) = 0$ C. Then the integrating factor is $e^{\int 4\,dt} = e^{4t}$, and multiplying the differential equation by the integrating factor gives $e^{4t}\dfrac{dQ}{dt} + 4e^{4t}Q = 12e^{4t} \;\Rightarrow\; \left(e^{4t}Q\right)' = 12e^{4t} \;\Rightarrow$ $Q(t) = e^{-4t}\left[\int 12e^{4t}\,dt + C\right] = 3 + Ce^{-4t}$. But $0 = Q(0) = 3 + C$ so $Q(t) = 3\left(1 - e^{-4t}\right)$ is the charge at time t and $I = dQ/dt = 12e^{-4t}$ is the current at time t.

30. $2\dfrac{dQ}{dt} + 100Q = 10\sin 60t$ or $\dfrac{dQ}{dt} + 50Q = 5\sin 60t$. Then the integrating factor is $e^{\int 50\,dt} = e^{50t}$, and multiplying the differential equation by the integrating factor gives $e^{50t}\dfrac{dQ}{dt} + 50e^{50t}Q = 5e^{50t}\sin 60t \;\Rightarrow$ $\left(e^{50t}Q\right)' = 5e^{50t}\sin 60t \;\Rightarrow$

$$Q(t) = e^{-50t}\left[\int 5e^{50t}\sin 60t\,dt + C\right] = e^{-50t}\left[5e^{50t}\left(\tfrac{1}{6100}\right)(50\sin 60t - 60\cos 60t)\right] + Ce^{-50t}$$
$$= \tfrac{1}{122}(5\sin 60t - 6\cos 60t) + Ce^{-50t}$$

But $0 = Q(0) = -\frac{6}{122} + C$ so $C = \frac{3}{61}$ and $Q(t) = \dfrac{5\sin 60t - 6\cos 60t}{122} + \dfrac{3e^{-50t}}{61}$ is the charge at time t, while the current is $I(t) = \dfrac{dQ}{dt} = \dfrac{150\cos 60t + 180\sin 60t - 150e^{-50t}}{61}$.

31. $\dfrac{dP}{dt} + kP = kM$, so $I(t) = e^{\int k\,dt} = e^{kt}$. Multiplying the differential equation by $I(t)$ gives $e^{kt}\dfrac{dP}{dt} + kPe^{kt} = kMe^{kt} \;\Rightarrow$ $\left(e^{kt}P\right)' = kMe^{kt} \;\Rightarrow$ $P(t) = e^{-kt}\left(\int kMe^{kt}dt + C\right) = M + Ce^{-kt}$, $k > 0$. Furthermore, it is reasonable to assume that $0 \le P(0) \le M$, so $-M \le C \le 0$.



32. Since $P(0) = 0$, we have $P(t) = M(1 - e^{-kt})$. If $P_1(t)$ is Jim's learning curve, then $P_1(1) = 25$ and $P_1(2) = 45$. Hence, $25 = M_1(1 - e^{-k})$ and $45 = M_1(1 - e^{-2k})$, so $1 - 25/M_1 = e^{-k}$ or

$k = -\ln\left(1 - \dfrac{25}{M_1}\right) = \ln\left(\dfrac{M_1}{M_1 - 25}\right)$. But $45 = M_1(1 - e^{-2k})$ so $45 = M_1\left[1 - \left(\dfrac{M_1 - 25}{M_1}\right)^2\right]$ or

$45 = \dfrac{50M_1 - 625}{M_1}$. Thus, $M_1 = 125$ is the maximum number of units per hour Jim is capable of processing.

Similarly, if $P_2(t)$ is Mark's learning curve, then $P_2(1) = 35$ and $P_2(2) = 50$. So $k = \ln\left(\dfrac{M_2}{M_2 - 35}\right)$ and

$50 = M_2\left[1 - \left(\dfrac{M_2 - 35}{M_2}\right)^2\right]$ or $M_2 = 61.25$. Hence the maximum number of units per hour for Mark is approximately 61. Another approach would be to use the midpoints of the intervals so that $P_1(0.5) = 25$ and $P_1(1.5) = 45$. Doing so gives us $M_1 \approx 52.6$ and $M_2 \approx 51.8$.

33. $y(0) = 0$ kg. Salt is added at a rate of $\left(0.4\ \dfrac{\text{kg}}{\text{L}}\right)\left(5\ \dfrac{\text{L}}{\text{min}}\right) = 2\ \dfrac{\text{kg}}{\text{min}}$. Since solution is drained from the tank at a rate of 3 L/min, but salt solution is added at a rate of 5 L/min, the tank, which starts out with 100 L of water, contains $(100 + 2t)$ L of liquid after t min. Thus, the salt concentration at time t is $\dfrac{y(t)}{100 + 2t}\ \dfrac{\text{kg}}{\text{L}}$. Salt therefore leaves the tank at a rate of $\left(\dfrac{y(t)}{100 + 2t}\ \dfrac{\text{kg}}{\text{L}}\right)\left(3\ \dfrac{\text{L}}{\text{min}}\right) = \dfrac{3y}{100 + 2t}\ \dfrac{\text{kg}}{\text{min}}$. Combining the rates at which salt enters and leaves the tank, we get $\dfrac{dy}{dt} = 2 - \dfrac{3y}{100 + 2t}$. Rewriting this equation as $\dfrac{dy}{dt} + \left(\dfrac{3}{100 + 2t}\right)y = 2$, we see that it is linear. $I(t) = \exp\left(\displaystyle\int \dfrac{3\,dt}{100 + 2t}\right) = \exp\left(\tfrac{3}{2}\ln(100 + 2t)\right) = (100 + 2t)^{3/2}$. Multiplying the differential equation by $I(t)$ gives $(100 + 2t)^{3/2}\dfrac{dy}{dt} + 3(100 + 2t)^{1/2}y = 2(100 + 2t)^{3/2}$ $\Rightarrow$

$\left[(100 + 2t)^{3/2}y\right]' = 2(100 + 2t)^{3/2}$ $\Rightarrow$ $(100 + 2t)^{3/2}y = \tfrac{2}{5}(100 + 2t)^{5/2} + C$ $\Rightarrow$

$y = \tfrac{2}{5}(100 + 2t) + C(100 + 2t)^{-3/2}$. Now $0 = y(0) = \tfrac{2}{5}(100) + C \cdot 100^{-3/2} = 40 + \tfrac{1}{1000}C$ $\Rightarrow$

$C = -40{,}000$, so $y = \left[\tfrac{2}{5}(100 + 2t) - 40{,}000(100 + 2t)^{-3/2}\right]$ kg. From this solution (no pun intended), we calculate the salt concentration at time t to be $C(t) = \dfrac{y(t)}{100 + 2t} = \left[\dfrac{-40{,}000}{(100 + 2t)^{5/2}} + \dfrac{2}{5}\right]\ \dfrac{\text{kg}}{\text{L}}$. In particular,

$C(20) = \dfrac{-40{,}000}{140^{5/2}} + \dfrac{2}{5} \approx 0.2275\ \dfrac{\text{kg}}{\text{L}}$ and $y(20) = \tfrac{2}{5}(140) - 40{,}000(140)^{-3/2} \approx 31.85$ kg.

34. Let $y(t)$ denote the amount of chlorine in the tank at time t (in seconds). $y(0) = (0.05\text{ g/L})\,(400\text{ L}) = 20$ g. The amount of liquid in the tank at time t is $(400 - 6t)$ L since 4 L of water enters the tank each second and 10 L of liquid leaves the tank each second. Thus, the concentration of chlorine at time t is $\dfrac{y(t)}{400 - 6t}\ \dfrac{\text{g}}{\text{L}}$. Chlorine doesn't enter the tank, but it leaves at a rate of $\left[\dfrac{y(t)}{400 - 6t}\ \dfrac{\text{g}}{\text{L}}\right]\left[10\ \dfrac{\text{L}}{\text{s}}\right] = \dfrac{10\,y(t)}{400 - 6t}\ \dfrac{\text{g}}{\text{s}} = \dfrac{5\,y(t)}{200 - 3t}\ \dfrac{\text{g}}{\text{s}}$. Therefore,

$\frac{dy}{dt} = -\frac{5y}{200-3t} \Rightarrow \int \frac{dy}{y} = \int \frac{-5\,dt}{200-3t} \Rightarrow \ln y = \frac{5}{3}\ln(200-3t) + C \Rightarrow$

$y = \exp\left(\frac{5}{3}\ln(200-3t)+C\right) = e^C(200-3t)^{5/3}$. Now $20 = y(0) = e^C \cdot 200^{5/3} \Rightarrow e^C = \frac{20}{200^{5/3}}$, so

$y(t) = 20\frac{(200-3t)^{5/3}}{200^{5/3}} = 20(1-0.015t)^{5/3}$ g for $0 \le t \le 66\frac{2}{3}$ s, at which time the tank is empty.

35. (a) $\frac{dv}{dt} + \frac{c}{m}v = g$ and $I(t) = e^{\int (c/m)dt} = e^{(c/m)t}$, and multiplying the differential equation by $I(t)$ gives

$e^{(c/m)t}\frac{dv}{dt} + \frac{vce^{(c/m)t}}{m} = ge^{(c/m)t} \Rightarrow \left[e^{(c/m)t}v\right]' = ge^{(c/m)t}$. Hence,

$v(t) = e^{-(c/m)t}\left[\int ge^{(c/m)t}\,dt + K\right] = mg/c + Ke^{-(c/m)t}$. But the object is dropped from rest, so $v(0) = 0$

and $K = -mg/c$. Thus, the velocity at time t is $v(t) = (mg/c)\left[1 - e^{-(c/m)t}\right]$.

(b) $\lim\limits_{t\to\infty} v(t) = mg/c$

(c) $s(t) = \int v(t)\,dt = (mg/c)\left[t + (m/c)e^{-(c/m)t}\right] + c_1$ where $c_1 = s(0) - m^2g/c^2$. $s(0)$ is the initial position,

so $s(0) = 0$ and $s(t) = (mg/c)\left[t + (m/c)e^{-(c/m)t}\right] - m^2g/c^2$.

36. $v = (mg/c)(1 - e^{-ct/m}) \Rightarrow \frac{dv}{dm} = \frac{mg}{c}\left(0 - e^{-ct/m}\cdot\frac{ct}{m^2}\right) + \frac{g}{c}\left(1 - e^{-ct/m}\right)\cdot 1 =$

$-\frac{gt}{m}e^{-ct/m} + \frac{g}{c} - \frac{g}{c}e^{-ct/m} = \frac{g}{c}\left(1 - e^{-ct/m} - \frac{ct}{m}e^{-ct/m}\right) \Rightarrow$

$\frac{c}{g}\frac{dv}{dm} = 1 - \left(1 + \frac{ct}{m}\right)e^{-ct/m} = 1 - \frac{1 + ct/m}{e^{ct/m}} = 1 - \frac{1+Q}{e^Q}$, where $Q = \frac{ct}{m} \ge 0$. Since $e^Q > 1 + Q$ for all

$Q > 0$, it follows that $dv/dm > 0$ for $t > 0$. In other words, for all $t > 0$, v increases as m increases.

10.7 Predator-Prey Systems

1. (a) $dx/dt = -0.05x + 0.0001xy$. If $y = 0$, we have $dx/dt = -0.05x$, which indicates that in the absence of y, x declines at a rate proportional to itself. So x represents the predator population and y represents the prey population. The growth of the prey population, $0.1y$ (from $dy/dt = 0.1y - 0.005xy$), is restricted only by encounters with predators (the term $-0.005xy$). The predator population increases only through the term $0.0001xy$; that is, by encounters with the prey and not through additional food sources.

(b) $dy/dt = -0.015y + 0.00008xy$. If $x = 0$, we have $dy/dt = -0.015y$, which indicates that in the absence of x, y would decline at a rate proportional to itself. So y represents the predator population and x represents the prey population. The growth of the prey population, $0.2x$ (from $dx/dt = 0.2x - 0.0002x^2 - 0.006xy = 0.2x(1 - 0.001x) - 0.006xy$), is restricted by a carrying capacity of 1000 [from the term $1 - 0.001x = 1 - x/1000$] and by encounters with predators (the term $-0.006xy$). The predator population increases only through the term $0.00008xy$; that is, by encounters with the prey and not through additional food sources.

2. (a) $dx/dt = 0.12x - 0.0006x^2 + 0.00001xy$. $dy/dt = 0.08y + 0.00004xy$.

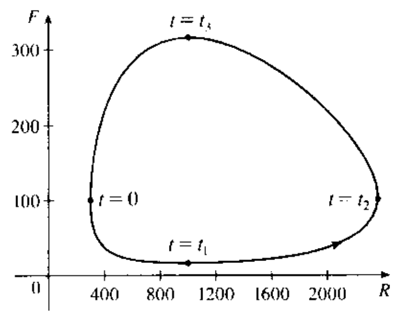
The xy terms represent encounters between the two species x and y. An increase in y makes dx/dt (the growth rate of x) larger due to the positive term $0.00001xy$. An increase in x makes dy/dt (the growth rate of y) larger due to the positive term $0.00004xy$. Hence, the system describes a cooperation model.

(b) $dx/dt = 0.15x - 0.0002x^2 - 0.0006xy = 0.15x(1 - x/750) - 0.0006xy$.

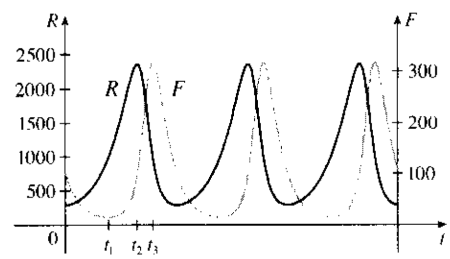
$dy/dt = 0.2y - 0.00008y^2 - 0.0002xy = 0.2y(1 - y/2500) - 0.0002xy$.

The system shows that x and y have carrying capacities of 750 and 2500. An increase in x reduces the growth rate of y due to the negative term $-0.0002xy$. An increase in y reduces the growth rate of x due to the negative term $-0.0006xy$. Hence, the system describes a competition model.

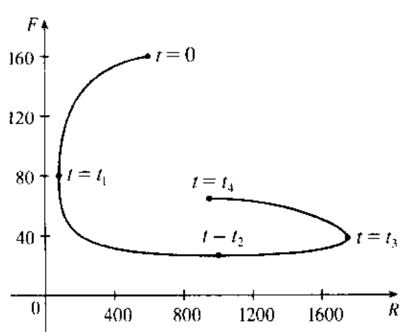
3. (a) At $t = 0$, there are about 300 rabbits and 100 foxes. At $t = t_1$, the number of foxes reaches a minimum of about 20 while the number of rabbits is about 1000. At $t = t_2$, the number of rabbits reaches a maximum of about 2400, while the number of foxes rebounds to 100. At $t = t_3$, the number of rabbits decreases to about 1000 and the number of foxes reaches a maximum of about 315. As t increases, the number of foxes decreases greatly to 100, and the number of rabbits decreases to 300 (the initial populations), and the cycle starts again.



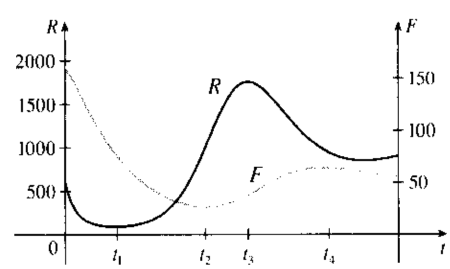
(b)



4. (a) At $t = 0$, there are about 600 rabbits and 160 foxes. At $t = t_1$, the number of rabbits reaches a minimum of about 80 and the number of foxes is also 80. At $t = t_2$, the number of foxes reaches a minimum of about 25 while the number of rabbits rebounds to 1000. At $t = t_3$, the number of foxes has increased to 40 and the rabbit population has reached a maximum of about 1750. The curve ends at $t = t_4$, where the number of foxes has increased to 65 and the number of rabbits has decreased to about 950.



(b)



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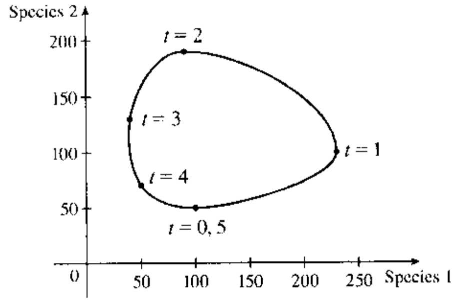

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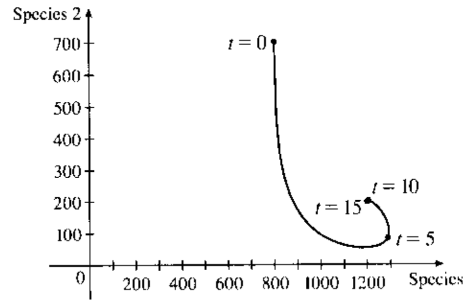


7. $\dfrac{dW}{dR} = \dfrac{-0.02W + 0.00002RW}{0.08R - 0.001RW} \Leftrightarrow (0.08 - 0.001W)R\,dW = (-0.02 + 0.00002R)W\,dR \Leftrightarrow$

$\dfrac{0.08 - 0.001W}{W}\,dW = \dfrac{-0.02 + 0.00002R}{R}\,dR \Leftrightarrow \displaystyle\int\left(\frac{0.08}{W} - 0.001\right)dW = \int\left(-\frac{0.02}{R} + 0.00002\right)dR$

$\Leftrightarrow\ 0.08\ln|W| - 0.001W = -0.02\ln|R| + 0.00002R + K \Leftrightarrow$

$0.08\ln W + 0.02\ln R = 0.001W + 0.00002R + K \Leftrightarrow \ln\left(W^{0.08}R^{0.02}\right) = 0.00002R + 0.001W + K \Leftrightarrow$

$W^{0.08}R^{0.02} = e^{0.00002R + 0.001W + K} \Leftrightarrow R^{0.02}W^{0.08} = Ce^{0.00002R}e^{0.001W} \Leftrightarrow \dfrac{R^{0.02}W^{0.08}}{e^{0.00002R}e^{0.001W}} = C.$

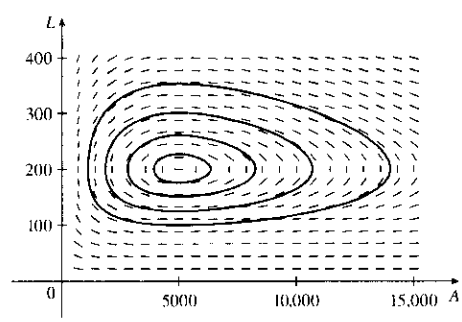
In general, if $\dfrac{dy}{dx} = \dfrac{-ry + bxy}{kx - axy}$, then $C = \dfrac{x^r y^k}{e^{bx}e^{ay}}$.

8. (a) A and L are constant $\Rightarrow A' = 0$ and $L' = 0 \Rightarrow \begin{Bmatrix} 0 = 2A - 0.01AL \\ 0 = -0.5L + 0.0001AL \end{Bmatrix} \Rightarrow \begin{cases} 0 = A(2 - 0.01L) \\ 0 = L(-0.5 + 0.0001A) \end{cases}$

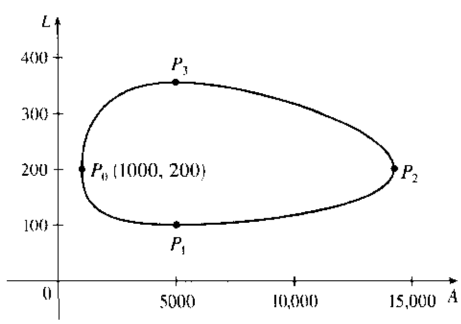
So either $A = L = 0$ or $L = \frac{2}{0.01} = 200$ and $A = \frac{0.5}{0.0001} = 5000$. The trivial solution $A = L = 0$ just says that if there aren't any aphids or ladybugs, then the populations will not change. The non-trivial solution, $L = 200$ and $A = 5000$, indicates the population sizes needed so that there are no changes in either the number of aphids or the number of ladybugs.

(b) $\dfrac{dL}{dA} = \dfrac{dL/dt}{dA/dt} = \dfrac{-0.5L + 0.0001AL}{2A - 0.01AL}$

(c) The solution curves (phase trajectories) are all closed curves that have the equilibrium point $(5000, 200)$ inside them.

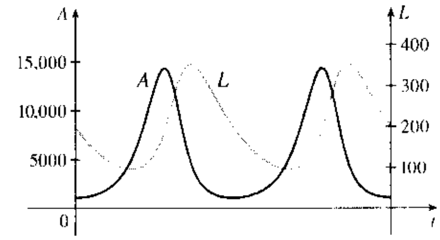


(d)



At $P_0(1000, 200)$, $dA/dt = 0$ and $dL/dt = -80 < 0$, so the number of ladybugs is decreasing and hence, we are proceeding in a counterclockwise direction. At P_0, there aren't enough aphids to support the ladybug population, so the number of ladybugs decreases and the number of aphids begins to increase. The ladybug population reaches a minimum at $P_1(5000, 100)$ while the aphid population increases in a dramatic way, reaching its maximum at $P_2(14{,}250, 200)$. Meanwhile, the ladybug population is increasing from P_1 to $P_3(5000, 355)$, and as we pass through P_2, the increasing number of ladybugs starts to deplete the aphid population. At P_3 the ladybugs reach a maximum population, and start to decrease due to the reduced aphid population. Both populations then decrease until P_0, where the cycle starts over again.

(e) Both graphs have the same period and the graph of L peaks about a quarter of a cycle after the graph of A.



9. (a) Letting $W = 0$ gives us $dR/dt = 0.08R(1 - 0.0002R)$. $dR/dt = 0 \Leftrightarrow R = 0$ or 5000. Since $dR/dt > 0$ for $0 < R < 5000$, we would expect the rabbit population to *increase* to 5000 for these values of R. Since $dR/dt < 0$ for $R > 5000$, we would expect the rabbit population to *decrease* to 5000 for these values of R. Hence, in the absence of wolves, we would expect the rabbit population to stabilize at 5000.

(b) R and W are constant $\Rightarrow$ $R' = 0$ and $W' = 0$ $\Rightarrow$

$$\left\{\begin{array}{l} 0 = 0.08R(1 - 0.0002R) - 0.001RW \\ 0 = -0.02W + 0.00002RW \end{array}\right\} \Rightarrow \left\{\begin{array}{l} 0 = R[0.08(1 - 0.0002R) - 0.001W] \\ 0 = W(-0.02 + 0.00002R) \end{array}\right.$$

The second equation is true if $W = 0$ or $R = \frac{0.02}{0.00002} = 1000$. If $W = 0$ in the first equation, then either $R = 0$ or $R = \frac{1}{0.0002} = 5000$ [as in part (a)]. If $R = 1000$, then $0 = 1000[0.08(1 - 0.0002 \cdot 1000) - 0.001W] \Leftrightarrow 0 = 80(1 - 0.2) - W \Leftrightarrow W = 64$.

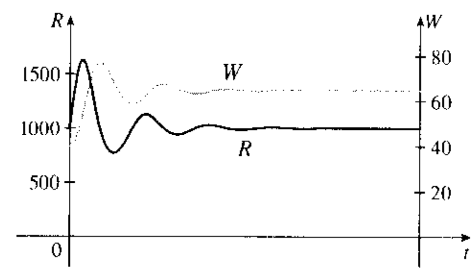
Case (i): $W = 0$, $R = 0$: both populations are zero

Case (ii): $W = 0$, $R = 5000$: see part (a)

Case (iii): $R = 1000$, $W = 64$: the predator/prey interaction balances and the populations are stable.

(c) The populations of wolves and rabbits fluctuate around 64 and 1000, respectively, and eventually stabilize at those values.

(d)



10. (a) If $L = 0$, $dA/dt = 2A(1 - 0.0001A)$, so $dA/dt = 0 \Leftrightarrow A = 0$ or $A = \frac{1}{0.0001} = 10{,}000$. Since $dA/dt > 0$ for $0 < A < 10{,}000$, we expect the aphid population to *increase* to 10,000 for these values of A. Since $dA/dt < 0$ for $A > 10{,}000$, we expect the aphid population to *decrease* to 10,000 for these values of A. Hence, in the absence of ladybugs we expect the aphid population to stabilize at 10,000.

(b) A and L are constant $\Rightarrow$ $A' = 0$ and $L' = 0$ $\Rightarrow$

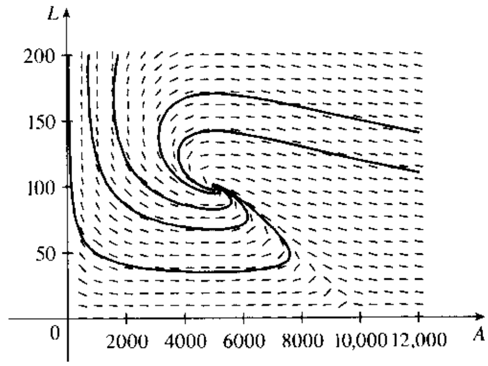
$$\left\{\begin{array}{l} 0 = 2A(1 - 0.0001A) - 0.01AL \\ 0 = -0.5L + 0.0001AL \end{array}\right\} \Rightarrow \left\{\begin{array}{l} 0 = A[2(1 - 0.0001A) - 0.01L] \\ 0 = L(-0.5 + 0.0001A) \end{array}\right.$$

The second equation is true if $L = 0$ or $A = \frac{0.5}{0.0001} = 5000$. If $L = 0$ in the first equation, then either $A = 0$ or $A = \frac{1}{0.0001} = 10{,}000$. If $A = 5000$, then $0 = 5000[2(1 - 0.0001 \cdot 5000) - 0.01L] \Leftrightarrow 0 = 10{,}000(1 - 0.5) - 50L \Leftrightarrow 50L = 5000 \Leftrightarrow L = 100$. The equilibrium solutions are:

(i) $L = 0, A = 0$ (ii) $L = 0, A = 10{,}000$ (iii) $A = 5000, L = 100$

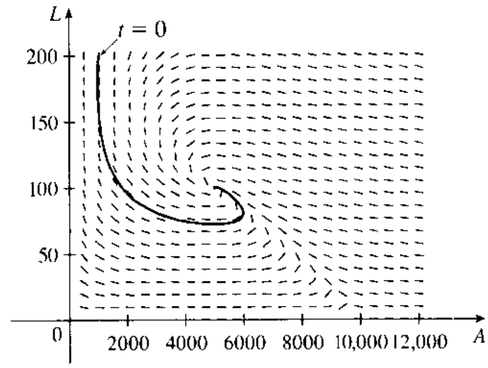
(c) $\dfrac{dL}{dA} = \dfrac{dL/dt}{dA/dt} = \dfrac{-0.5L + 0.0001AL}{2A(1 - 0.0001A) - 0.01AL}$

(d)



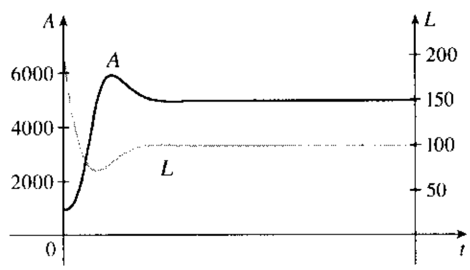
All of the phase trajectories spiral tightly around the equilibrium solution (5000, 100).

(e)



At $t = 0$, the ladybug population decreases rapidly and the aphid population decreases slightly before beginning to increase. As the aphid population continues to increase, the ladybug population reaches a minimum at about (5000, 75). The ladybug population starts to increase and quickly stabilizes at 100, while the aphid population stabilizes at 5000.

(f)



The graph of A peaks just after the graph of L has a minimum.

10 Review

CONCEPT CHECK

1. (a) A differential equation is an equation that contains an unknown function and one or more of its derivatives.

(b) The order of a differential equation is the order of the highest derivative that occurs in the equation.

(c) An initial condition is a condition of the form $y(t_0) = y_0$.

2. $y' = x^2 + y^2 \geq 0$ for all x and y. $y' = 0$ only at the origin, so there is a horizontal tangent at $(0, 0)$, but nowhere else. The graph of the solution is increasing on every interval.

3. See the paragraph preceding Example 1 in Section 10.2.

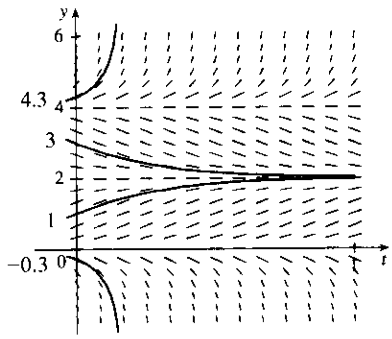
4. See the paragraph after Figure 14 in Section 10.2.
5. A separable equation is a first-order differential equation in which the expression for dy/dx can be factored as a function of x times a function of y, that is, $dy/dx = g(x)f(y)$. We can solve the equation by integrating both sides of the equation $dy/f(y) = g(x)dx$ and solving for y.
6. A first-order linear differential equation is a differential equation that can be put in the form $\frac{dy}{dx} + P(x)\,y = Q(x)$, where P and Q are continuous functions on a given interval. To solve such an equation, multiply it by the integrating factor $I(x) = e^{\int P(x)dx}$ to put it in the form $[I(x)\,y]' = I(x)\,Q(x)$ and then integrate both sides to get $I(x)\,y = \int I(x)\,Q(x)\,dx$, that is, $e^{\int P(x)\,dx}y = \int e^{\int P(x)\,dx}Q(x)\,dx$. Solving for y gives us $y = e^{-\int P(x)\,dx}\int e^{\int P(x)\,dx}Q(x)\,dx$.
7. (a) $\frac{dy}{dt} = ky$; the relative growth rate, $\frac{1}{y}\frac{dy}{dt}$, is constant.
   (b) The equation in part (a) is an appropriate model for population growth, assuming that there is enough room and nutrition to support the growth.
   (c) If $y(0) = y_0$, then the solution is $y(t) = y_0e^{kt}$.
8. (a) $dP/dt = kP(1 - P/K)$, where K is the carrying capacity.
   (b) The equation in part (a) is an appropriate model for population growth, assuming that the population grows at a rate proportional to the size of the population in the beginning, but eventually levels off and approaches its carrying capacity because of limited resources.
9. (a) $dF/dt = kF - aFS$ and $dS/dt = -rS + bFS$.
   (b) In the absence of sharks, an ample food supply would support exponential growth of the fish population, that is, $dF/dt = kF$, where k is a positive constant. In the absence of fish, we assume that the shark population would decline at a rate proportional to itself, that is, $dS/dt = -rS$, where r is a positive constant.

TRUE-FALSE QUIZ

1. True. Since $y^4 \geq 0$, $y' = -1 - y^4 < 0$ and the solutions are decreasing functions.
2. True. $y = \frac{\ln x}{x} \Rightarrow y' = \frac{1 - \ln x}{x^2}$.
   LHS $= x^2y' + xy = x^2 \cdot \frac{1 - \ln x}{x^2} + x \cdot \frac{\ln x}{x} = (1 - \ln x) + \ln x = 1 =$ RHS, so $y = \frac{\ln x}{x}$ is a solution of $x^2y' + xy = 1$.
3. False. $x + y$ cannot be written in the form $g(x)f(y)$.
4. True. $y' = 3y - 2x + 6xy - 1 = 6xy - 2x + 3y - 1 = 2x(3y - 1) + 1(3y - 1) = (2x + 1)(3y - 1)$, so y' can be written in the form $g(x)f(y)$, and hence, is separable.
5. True. $e^xy' = y \Rightarrow y' = e^{-x}y \Rightarrow y' + (-e^{-x})y = 0$, which is of the form $y' + P(x)\,y = Q(x)$, so the equation is linear.
6. False. $y' + xy = e^y$ cannot be put in the form $y' + P(x)\,y = Q(x)$, so it is not linear.
7. True. By comparing $\frac{dy}{dt} = 2y\left(1 - \frac{y}{5}\right)$ with the logistic differential equation (10.5.1), we see that the carrying capacity is 5; that is, $\lim_{t\to\infty} y = 5$.

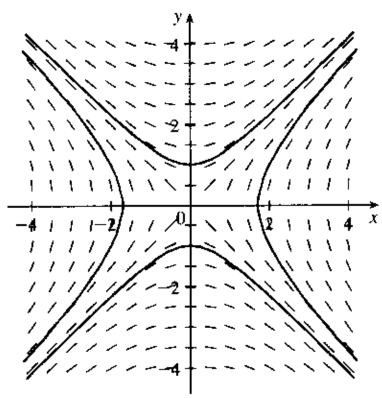
EXERCISES

1. (a)



(b) $\lim_{t\to\infty} y(t)$ appears to be finite for $0 \le c \le 4$. In fact $\lim_{t\to\infty} y(t) = 4$ for $c = 4$, $\lim_{t\to\infty} y(t) = 2$ for $0 < c < 4$, and $\lim_{t\to\infty} y(t) = 0$ for $c = 0$. The equilibrium solutions are $y(t) = 0$, $y(t) = 2$, and $y(t) = 4$.

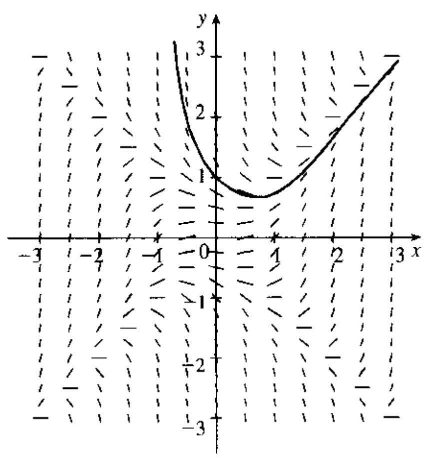
2. (a)



We sketch the direction field and four solution curves, as shown. Note that the slope $y' = x/y$ is not defined on the line $y = 0$.

(b) $y' = x/y \quad \Leftrightarrow \quad y\,dy = x\,dx \quad \Leftrightarrow y^2 = x^2 + C$. For $C = 0$, this is the pair of lines $y = \pm x$. For $C \neq 0$, it is the hyperbola $x^2 - y^2 = -C$.

3. (a)



We estimate that when $x = 0.3$, $y = 0.8$, so $y(0.3) \approx 0.8$.

(b) $h = 0.1$, $x_0 = 0$, $y_0 = 1$ and $F(x, y) = x^2 - y^2$. So $y_n = y_{n-1} + 0.1(x_{n-1}^2 - y_{n-1}^2)$. Thus,

$y_1 = 1 + 0.1(0^2 - 1^2) = 0.9$,

$y_2 = 0.9 + 0.1(0.1^2 - 0.9^2) = 0.82$,

$y_3 = 0.82 + 0.1(0.2^2 - 0.82^2) = 0.75676$. This is close to our graphical estimate of $y(0.3) \approx 0.8$.

(c) The centers of the horizontal line segments of the direction field are located on the lines $y = x$ and $y = -x$. When a solution curve crosses one of these lines, it has a local maximum or minimum.

4. (a) $h = 0.2$, $x_0 = 0$, $y_0 = 1$ and $F(x, y) = 2xy^2$. We need y_2.
$y_1 = 1 + 0.2(2 \cdot 0 \cdot 1^2) = 1$, $y_2 = 1 + 0.2(2 \cdot 0.2 \cdot 1^2) = 1.08 \approx y(0.4)$.

(b) $h = 0.1$ now, so $y_1 = 1 + 0.1(2 \cdot 0 \cdot 1^2) = 1$, $y_2 = 1 + 0.1(2 \cdot 0.1 \cdot 1^2) = 1.02$,
$y_3 = 1.02 + 0.1(2 \cdot 0.2 \cdot 1.02^2) \approx 1.06162$, $y_4 = 1.06162 + 0.1(2 \cdot 0.3 \cdot 1.06162^2) \approx 1.1292 \approx y(0.4)$.

(c) The equation is separable, so we write $\dfrac{dy}{y^2} = 2x\,dx \;\Rightarrow\; \displaystyle\int \frac{dy}{y^2} = \int 2x\,dx \;\Leftrightarrow\; -\frac{1}{y} = x^2 + C$, but $y(0) = 1$, so $C = -1$ and $y(x) = \dfrac{1}{1-x^2} \;\Leftrightarrow\; y(0.4) = \dfrac{1}{1-0.16} \approx 1.1905$. From this we see that the approximation was greatly improved by increasing the number of steps, but the approximations were still far off.

5. $y' = xe^{-\sin x} - y\cos x \;\Rightarrow\; y' + (\cos x)\,y = xe^{-\sin x}$ $(*)$. This is a linear equation and the integrating factor is $I(x) = e^{\int \cos x\,dx} = e^{\sin x}$. Multiplying $(*)$ by $e^{\sin x}$ gives $e^{\sin x}y' + e^{\sin x}(\cos x)\,y = x \;\Rightarrow\; (e^{\sin x}y)' = x$ $\Rightarrow\; e^{\sin x}y = \frac{1}{2}x^2 + C \;\Rightarrow\; y = \left(\frac{1}{2}x^2 + C\right)e^{-\sin x}$.

6. $\dfrac{dx}{dt} = 1 - t + x - tx = 1(1-t) + x(1-t) = (1+x)(1-t) \;\Rightarrow\; \dfrac{dx}{1+x} = (1-t)\,dt \;\Rightarrow$ $\displaystyle\int \frac{dx}{1+x} = \int (1-t)\,dt \;\Rightarrow\; \ln|1+x| = t - \tfrac{1}{2}t^2 + C \;\Rightarrow\; |1+x| = e^{t-t^2/2+C} \;\Rightarrow$ $1 + x = \pm e^{t-t^2/2}\cdot e^C \;\Rightarrow\; x = -1 + Ke^{t-t^2/2}$, where K is any nonzero constant.

7. $(3y^2 + 2y)\,y' = x\cos x \;\Rightarrow\; (3y^2 + 2y)\,dy = (x\cos x)\,dx \;\Rightarrow\; \int(3y^2 + 2y)\,dy = \int(x\cos x)\,dx \;\Rightarrow$ $y^3 + y^2 = \cos x + x\sin x + C$. For the last step, use integration by parts or Formula 83 in the Table of Integrals.

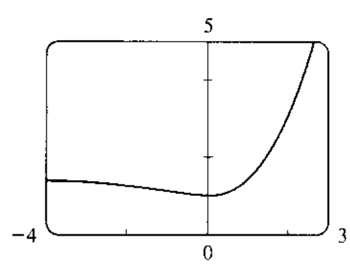
8. $x^2y' - y = 2x^3e^{-1/x} \;\Rightarrow\; y' - \dfrac{1}{x^2}\,y = 2xe^{-1/x}$ $(*)$. This is a linear equation and the integrating factor is $I(x) = e^{\int(-1/x^2)\,dx} = e^{1/x}$. Multiplying $(*)$ by $e^{1/x}$ gives $e^{1/x}y' - e^{1/x}\cdot\dfrac{1}{x^2}\,y = 2x \;\Rightarrow\; (e^{1/x}y)' = 2x \;\Rightarrow$ $e^{1/x}y = x^2 + C \;\Rightarrow\; y = e^{-1/x}(x^2 + C)$.

9. $xyy' = \ln x \;\Rightarrow\; y\,dy = \dfrac{\ln x}{x}\,dx \;\Rightarrow\; \displaystyle\int y\,dy = \int \frac{\ln x}{x}\,dx$ (Make the substitution $u = \ln x$; then $du = dx/x$.) So $\int y\,dy = \int u\,du \;\Rightarrow\; \frac{1}{2}y^2 = \frac{1}{2}u^2 + C \;\Rightarrow \frac{1}{2}y^2 = \frac{1}{2}(\ln x)^2 + C$. $y(1) = 2 \;\Rightarrow$ $\frac{1}{2}2^2 = \frac{1}{2}(\ln 1)^2 + C = C \;\Leftrightarrow\; C = 2$. Therefore, $\frac{1}{2}y^2 = \frac{1}{2}(\ln x)^2 + 2$, or $y = \sqrt{(\ln x)^2 + 4}$. The negative square root is inadmissible, since $y(1) > 0$.

10. $1 + x = 2xyy' \;\Rightarrow\; y' = \dfrac{1+x}{2xy} \;\Leftrightarrow\; y\,dy = \dfrac{1+x}{2x}\,dx \;\Rightarrow\; \dfrac{y^2}{2} = \dfrac{\ln|x|}{2} + \dfrac{x}{2} + c_1$. But $x > 0$, so $y^2 = \ln x + x + c \;\Leftrightarrow\; y(x) = \pm\sqrt{c + x + \ln x}$. But $-2 = y(1)$ so choose the negative square root and $-2 = -\sqrt{c+1}$ so $c = 3$. Thus, the solution is $y(x) = -\sqrt{3 + x + \ln x}$.

11. Since the equation is linear, let $I(x) = e^{\int dx} = e^x$. Then multiplying by $I(x)$ gives $e^xy' + e^xy = \sqrt{x} \;\Rightarrow$ $(e^xy)' = \sqrt{x} \;\Rightarrow\; y(x) = e^{-x}\left(\int\sqrt{x}\,dx + c\right) = e^{-x}\left(\frac{2}{3}x^{3/2} + c\right)$. But $3 = y(0) = c$, so the solution to the initial-value problem is $y(x) = e^{-x}\left(\frac{2}{3}x^{3/2} + 3\right)$.

12. $2yy' = xe^x \;\Rightarrow\; \int 2y\,dy = \int xe^x\,dx \;\Rightarrow$ $y^2 = xe^x - \int e^x\,dx$ (by parts) $= (x-1)\,e^x + C$. We substitute the initial condition: $1^2 = (0-1)\,e^0 + C \;\Rightarrow\; C = 2$. So the solution is $y = \sqrt{(x-1)\,e^x + 2}$. The negative square root is inadmissible due to the initial condition.



13. The curves $kx^2 + y^2 = 1$ form a family of ellipses for $k > 0$, a family of hyperbolas for $k < 0$, and two parallel lines $y = \pm 1$ for $k = 0$. Solving $kx^2 + y^2 = 1$ for k gives $k = \dfrac{1 - y^2}{x^2}$. Differentiating gives $2kx + 2yy' = 0 \Leftrightarrow$ $y' = -\dfrac{kx}{y} = -\left(1 - y^2\right)\dfrac{x}{yx^2} = \dfrac{y^2 - 1}{xy}$. Thus, for $k \neq 0$ the orthogonal trajectories must satisfy $y' = -\dfrac{xy}{y^2 - 1}$ $\Rightarrow \dfrac{y^2 - 1}{y}\,dy = -x\,dx \Rightarrow \dfrac{y^2}{2} - \ln|y| = \dfrac{-x^2}{2} + K \Rightarrow y^2 - 2\ln|y| + x^2 = C$. For $k = 0$, the orthogonal trajectories are given by $x = C_1$ for C_1 an arbitrary constant.

14. Differentiating both sides of $y = \dfrac{k}{1 + x^2}$ gives $y' = -\dfrac{2kx}{(1 + x^2)^2} = -2xy\dfrac{1 + x^2}{(1 + x^2)^2} = -\dfrac{2xy}{1 + x^2}$. Thus, for $k \neq 0$ the orthogonal trajectories must satisfy $y' = \dfrac{1 + x^2}{2xy} \Rightarrow 2y\,dy = \left(\dfrac{1}{x} + x\right)dx \Rightarrow$ $y^2 = \dfrac{x^2}{2} + \ln|x| + C$. For $k = 0$, the orthogonal trajectories are given by $x = C_2$ for C_2 an arbitrary constant.

15. (a) $y(t) = y(0)e^{kt} = 1000e^{kt} \Rightarrow y(2) = 1000e^{2k} = 9000 \Rightarrow e^{2k} = 9 \Rightarrow 2k = \ln 9 \Rightarrow$ $k = \frac{1}{2}\ln 9 = \ln 3 \Rightarrow y(t) = 1000e^{(\ln 3)t} = 1000 \cdot 3^t$

(b) $y(3) = 1000 \cdot 3^3 = 27{,}000$

(c) $y'(t) = 1000 \cdot 3^t \cdot \ln 3$, so $y'(3) = 27{,}000 \ln 3 \approx 29{,}663$ bacteria per hour

(d) $1000 \cdot 3^t = 2 \cdot 1000 \Rightarrow 3^t = 2 \Rightarrow t\ln 3 = \ln 2 \Rightarrow t = (\ln 2)/\ln 3 \approx 0.63$ h

16. (a) If $y(t)$ is the mass remaining after t years, then $y(t) = y(0)e^{kt} = 18e^{kt}$. $y(25) = 18e^{25k} = \frac{1}{2} \cdot 18 \Rightarrow$ $e^{25k} = \frac{1}{2} \Rightarrow 25k = -\ln 2 \Rightarrow k = -\frac{1}{25}\ln 2 \Rightarrow y(t) = 18\,e^{-(\ln 2)t/25} = 18 \cdot 2^{-t/25}$.

(b) $18 \cdot 2^{-t/25} = 2 \Rightarrow 2^{-t/25} = \frac{1}{9} \Rightarrow -\frac{1}{25}t\ln 2 = -\ln 9 \Rightarrow t = 25\,\frac{\ln 9}{\ln 2} \approx 79$ years

17. (a) $C'(t) = -kC(t) \Rightarrow C(t) = C(0)e^{-kt}$ by Theorem 10.4.2. But $C(0) = C_0$, so $C(t) = C_0 e^{-kt}$.

(b) $C(30) = \frac{1}{2}C_0$ since the concentration is reduced by half. Thus, $\frac{1}{2}C_0 = C_0e^{-30k} \Rightarrow \ln\frac{1}{2} = -30k \Rightarrow$ $k = -\frac{1}{30}\ln\frac{1}{2} = \frac{1}{30}\ln 2$. Since 10% of the original concentration remains if 90% is eliminated, we want the value of t such that $C(t) = \frac{1}{10}C_0$. Therefore, $\frac{1}{10}C_0 = C_0e^{-t(\ln 2)/30} \Rightarrow \ln 0.1 = -t(\ln 2)/30 \Rightarrow$ $t = -\frac{30}{\ln 2}\ln 0.1 \approx 100$ h.

18. (a) Let $t = 0$ correspond to 1990 so that $P(t) = 5.28e^{kt}$ is a starting point for the model. When $t = 10$, $P = 6.07$. So $6.07 = 5.28e^{10k} \Rightarrow 10k = \ln\frac{6.07}{5.28} \Rightarrow k = \frac{1}{10}\ln\frac{6.07}{5.28} \approx 0.01394$. For the year 2020, $t = 30$, and $P(30) = 5.28e^{30k} \approx 8.02$ billion.

(b) $P = 10 \Rightarrow 5.28e^{kt} = 10 \Rightarrow \frac{10}{5.28} = e^{kt} \Rightarrow kt = \ln\frac{10}{5.28} \Rightarrow t = 10\dfrac{\ln\frac{10}{5.28}}{\ln\frac{6.07}{5.28}} \approx 45.8$ years; that is, in $1990 + 45 = 2035$.

(c) $P(t) = \dfrac{K}{1 + Ae^{-kt}} = \dfrac{100}{1 + Ae^{-kt}}$, where $A = \dfrac{100 - 5.28}{5.28} \approx 17.94$. Using $k = \dfrac{1}{10}\ln\dfrac{6.07}{5.28}$ from part (a), a model is $P(t) \approx \dfrac{100}{1 + 17.94e^{-0.01394t}}$ and $P(30) \approx 7.81$ billion, slightly lower than our estimate of 8.02 billion in part (a).

(d) $P = 10 \Rightarrow 1 + Ae^{-kt} = \frac{100}{10} \Rightarrow Ae^{-kt} = 9 \Rightarrow e^{-kt} = 9/A \Rightarrow -kt = \ln(9/A) \Rightarrow$ $t = -\dfrac{1}{k}\ln\dfrac{9}{A} \approx 49.47$ years (that is, in 2039), which is later than the prediction of 2035 in part (b).

19. (a) $\dfrac{dL}{dt} \propto L_\infty - L \;\Rightarrow\; \dfrac{dL}{dt} = k(L_\infty - L) \;\Rightarrow\; \displaystyle\int \frac{dL}{L_\infty - L} = \int k\,dt \;\Rightarrow\; -\ln|L_\infty - L| = kt + C \;\Rightarrow$

$\ln|L_\infty - L| = -kt - C \;\Rightarrow\; |L_\infty - L| = e^{-kt-C} \;\Rightarrow\; L_\infty - L = Ae^{-kt} \;\Rightarrow\; L = L_\infty - Ae^{-kt}.$

At $t = 0$, $L = L(0) = L_\infty - A \;\Rightarrow\; A = L_\infty - L(0) \;\Rightarrow\; L(t) = L_\infty - [L_\infty - L(0)]\,e^{-kt}$.

(b) $L_\infty = 53$ cm, $L(0) = 10$ cm, and $k = 0.2 \;\Rightarrow\; L(t) = 53 - (53 - 10)e^{-0.2t} = 53 - 43e^{-0.2t}$.

20. Denote the amount of salt in the tank (in kg) by y. $y(0) = 0$ since initially there is only water in the tank. The rate at which y increases is equal to the rate at which salt flows into the tank minus the rate at which it flows out. That rate is $\dfrac{dy}{dt} = 0.1\dfrac{\text{kg}}{\text{L}} \times 10\dfrac{\text{L}}{\text{min}} - \dfrac{y}{100}\dfrac{\text{kg}}{\text{L}} \times 10\dfrac{\text{L}}{\text{min}} = 1 - \dfrac{y}{10}\dfrac{\text{kg}}{\text{min}} \;\Rightarrow\; \displaystyle\int \frac{dy}{10 - y} = \int \frac{1}{10}\,dt \;\Rightarrow$

$-\ln|10 - y| = \frac{1}{10}t + C \;\Rightarrow\; 10 - y = Ae^{-t/10}$. $y(0) = 0 \;\Rightarrow\; 10 = A \;\Rightarrow\; y = 10(1 - e^{-t/10})$.

At $t = 6$ minutes, $y = 10\left(1 - e^{-6/10}\right) \approx 4.512$ kg.

21. Let P be the population and I be the number of infected people. The rate of spread dI/dt is jointly proportional to I and to $P - I$, so for some constant k, $dI/dt = kI(P - I) \;\Rightarrow\; I = \dfrac{I_0 P}{I_0 + (P - I_0)e^{-kPt}}$ (from the discussion of logistic growth in Section 10.5).

Now, measuring t in days, we substitute $t = 7$, $P = 5000$, $I_0 = 160$ and $I(7) = 1200$ to find k:

$1200 = \dfrac{160 \cdot 5000}{160 + (5000 - 160)e^{-5000 \cdot 7 \cdot k}} \;\Leftrightarrow\; k \approx 0.00006448$. So, putting $I = 5000 \times 80\% = 4000$, we solve

for t: $4000 = \dfrac{160 \cdot 5000}{160 + (5000 - 160)e^{-0.00006448 \cdot 5000 \cdot t}} \;\Leftrightarrow\; 160 + 4840e^{-0.3224t} = 200 \;\Leftrightarrow$

$-0.3224t = \ln\frac{40}{4840} \;\Leftrightarrow\; t \approx 14.9$. So it takes about 15 days for 80% of the population to be infected.

22. $\dfrac{1}{R}\dfrac{dR}{dt} = \dfrac{k}{S}\dfrac{dS}{dt} \;\Rightarrow\; \dfrac{d}{dt}(\ln R) = \dfrac{d}{dt}(k\ln S) \;\Rightarrow\; \ln R = k\ln S + C \;\Rightarrow$

$R = e^{k\ln S + C} = e^C\left(e^{\ln S}\right)^k \;\Rightarrow\; R = AS^k$, where $A = e^C$ is a positive constant.

23. $\dfrac{dh}{dt} = -\dfrac{R}{V}\left(\dfrac{h}{k+h}\right) \;\Rightarrow\; \displaystyle\int \frac{k+h}{h}\,dh = \int\left(-\frac{R}{V}\right)dt \;\Rightarrow\; \int\left(1 + \frac{k}{h}\right)dh = -\frac{R}{V}\int 1\,dt \;\Rightarrow$

$h + k\ln h = -\dfrac{R}{V}t + C$. This equation gives a relationship between h and t, but it is not possible to isolate h and express it in terms of t.

24. $dx/dt = 0.4x - 0.002xy$, $dy/dt = -0.2y + 0.000008xy$

(a) The xy terms represent encounters between the birds and the insects. Since the y-population increases from these terms and the x-population decreases, we expect y to represent the birds and x the insects.

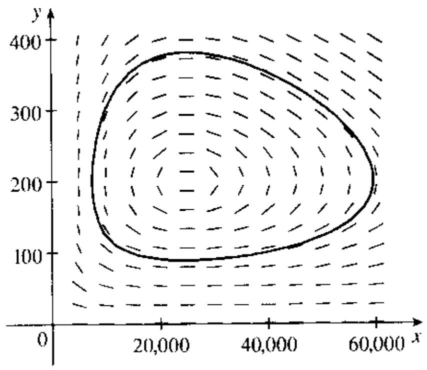
(b) x and y are constant $\;\Rightarrow\; x' = 0$ and $y' = 0 \;\Rightarrow$

$$\begin{Bmatrix} 0 = 0.4x - 0.002xy \\ 0 = -0.2y + 0.000008xy \end{Bmatrix} \;\Rightarrow\; \begin{cases} 0 = 0.4x(1 - 0.005y) \\ 0 = -0.2y(1 - 0.00004x) \end{cases} \;\Rightarrow\; y = 0 \text{ and } x = 0 \text{ (zero populations)}$$

or $y = \frac{1}{0.005} = 200$ and $x = \frac{1}{0.00004} = 25{,}000$. The non-trivial solution represents the population sizes needed so that there are no changes in either the number of birds or the number of insects.

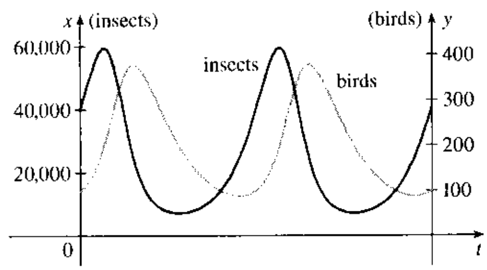
(c) $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{-0.2y + 0.000008xy}{0.4x - 0.002xy}$

(d)



At $(x, y) = (40{,}000, 100)$, $dx/dt = 8000 > 0$, so as t increases we are proceeding in a counterclockwise direction. The populations increase to approximately $(59{,}646, 200)$, at which point the insect population starts to decrease. The birds attain a maximum population of about 380 when the insect population is 25,000. The populations decrease to about $(7370, 200)$, at which point the insect population starts to increase. The birds attain a minimum population of about 88 when the insect population is 25,000, and then the cycle repeats.

(e)



Both graphs have the same period and the bird population peaks about a quarter-cycle after the insect population.

25. (a) $dx/dt = 0.4x(1 - 0.000005x) - 0.002xy$, $dy/dt = -0.2y + 0.000008xy$. If $y = 0$, then $dx/dt = 0.4x(1 - 0.000005x)$, so $dx/dt = 0 \quad\Leftrightarrow\quad x = 0$ or $x = 200{,}000$, which shows that the insect population increases logistically with a carrying capacity of 200,000. Since $dx/dt > 0$ for $0 < x < 200{,}000$ and $dx/dt < 0$ for $x > 200{,}000$, we expect the insect population to stabilize at 200,000.

(b) x and y are constant $\Rightarrow$ $x' = 0$ and $y' = 0$ $\Rightarrow$

$$\left\{\begin{array}{l} 0 = 0.4x(1 - 0.000005x) - 0.002xy \\ 0 = -0.2y + 0.000008xy \end{array}\right\} \Rightarrow \left\{\begin{array}{l} 0 = 0.4x[(1 - 0.000005x) - 0.005y] \\ 0 = y(-0.2 + 0.000008x) \end{array}\right.$$

The second equation is true if $y = 0$ or $x = \frac{0.2}{0.000008} = 25{,}000$. If $y = 0$ in the first equation, then either $x = 0$ or $x = \frac{1}{0.000005} = 200{,}000$. If $x = 25{,}000$, then $0 = 0.4(25{,}000)[(1 - 0.000005 \cdot 25{,}000) - 0.005y] \Rightarrow 0 = 10{,}000[(1 - 0.125) - 0.005y] \Rightarrow 0 = 8750 - 50y \Rightarrow y = 175$.

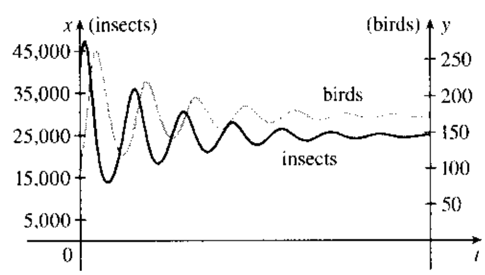
Case (i): $y = 0$, $x = 0$: Zero populations

Case (ii): $y = 0$, $x = 200{,}000$: In the absence of birds, the insect population is always 200,000.

Case (iii): $x = 25{,}000$, $y = 175$: The predator/prey interaction balances and the populations are stable.

(c) The populations of the birds and insects fluctuate around 175 and 25,000, respectively, and eventually stabilize at those values.

(d)



26. First note that, in this question, "weighs" is used in the informal sense, so what we really require is Barbara's mass m in kg as a function of t. Barbara's net intake of calories per day at time t (measured in days) is $c(t) = 1600 - 850 - 15m(t) = 750 - 15m(t)$, where $m(t)$ is her mass at time t. We are given that $m(0) = 60$ kg and $\dfrac{dm}{dt} = \dfrac{c(t)}{10{,}000}$, so $\dfrac{dm}{dt} = \dfrac{750 - 15m}{10{,}000} = \dfrac{150 - 3m}{2000} = \dfrac{-3(m-50)}{2000}$ with $m(0) = 60$. From $\displaystyle\int \frac{dm}{m-50} = \int \frac{-3\,dt}{2000}$, we get $\ln|m - 50| = -\frac{3}{2000}t + C$. Since $m(0) = 60$, $C = \ln 10$. Now $\ln\dfrac{|m-50|}{10} = -\dfrac{3t}{2000}$, so $|m - 50| = 10e^{-3t/2000}$. The quantity $m - 50$ is continuous, initially positive, and the right-hand side is never zero. Thus, $m - 50$ is positive for all t, and $m(t) = 50 + 10e^{-3t/2000}$ kg. As $t \to \infty$, $m(t) \to 50$ kg. Thus, Barbara's mass gradually settles down to 50 kg.

27. (a) $\dfrac{d^2y}{dx^2} = k\sqrt{1 + \left(\dfrac{dy}{dx}\right)^2}$. Setting $z = \dfrac{dy}{dx}$, we get $\dfrac{dz}{dx} = k\sqrt{1+z^2} \;\Rightarrow\; \dfrac{dz}{\sqrt{1+z^2}} = k\,dx$. Using Formula 25 gives $\ln\left(z + \sqrt{1+z^2}\right) = kx + c \;\Rightarrow\; z + \sqrt{1+z^2} = Ce^{kx}$ (where $C = e^c$) $\Rightarrow$ $\sqrt{1+z^2} = Ce^{kx} - z \;\Rightarrow\; 1 + z^2 = C^2e^{2kx} - 2Ce^{kx}z + z^2 \;\Rightarrow\; 2Ce^{kx}z = C^2e^{2kx} - 1 \;\Rightarrow$ $z = \dfrac{C}{2}e^{kx} - \dfrac{1}{2C}e^{-kx}$. Now $\dfrac{dy}{dx} = \dfrac{C}{2}e^{kx} - \dfrac{1}{2C}e^{-kx} \;\Rightarrow\; y = \dfrac{C}{2k}e^{kx} + \dfrac{1}{2Ck}e^{-kx} + C'$. From the diagram in the text, we see that $y(0) = a$ and $y(\pm b) = h$. $a = y(0) = \dfrac{C}{2k} + \dfrac{1}{2Ck} + C' \;\Rightarrow$ $C' = a - \dfrac{C}{2k} - \dfrac{1}{2Ck} \;\Rightarrow\; y = \dfrac{C}{2k}\left(e^{kx} - 1\right) + \dfrac{1}{2Ck}\left(e^{-kx} - 1\right) + a$. From $h = y(\pm b)$, we find $h = \dfrac{C}{2k}\left(e^{kb} - 1\right) + \dfrac{1}{2Ck}\left(e^{-kb} - 1\right) + a$ and $h = \dfrac{C}{2k}\left(e^{-kb} - 1\right) + \dfrac{1}{2Ck}\left(e^{kb} - 1\right) + a$. Subtracting the second equation from the first, we get $0 = \dfrac{C}{k}\dfrac{e^{kb} - e^{-kb}}{2} - \dfrac{1}{Ck}\dfrac{e^{kb} - e^{-kb}}{2} = \dfrac{1}{k}\left(C - \dfrac{1}{C}\right)\sinh kb$. Now $k > 0$ and $b > 0$, so $\sinh kb > 0$ and $C = \pm 1$. If $C = 1$, then $y = \dfrac{1}{2k}\left(e^{kx} - 1\right) + \dfrac{1}{2k}\left(e^{-kx} - 1\right) + a = \dfrac{1}{k}\dfrac{e^{kx} + e^{-kx}}{2} - \dfrac{1}{k} + a = a + \dfrac{1}{k}(\cosh kx - 1)$. If $C = -1$, then $y = -\dfrac{1}{2k}\left(e^{kx} - 1\right) - \dfrac{1}{2k}\left(e^{-kx} - 1\right) + a = \dfrac{-1}{k}\dfrac{e^{kx} + e^{-kx}}{2} + \dfrac{1}{k} + a = a - \dfrac{1}{k}(\cosh kx - 1)$. Since $k > 0$, $\cosh kx \geq 1$, and $y \geq a$, we conclude that $C = 1$ and $y = a + \dfrac{1}{k}(\cosh kx - 1)$, where $h = y(b) = a + \dfrac{1}{k}(\cosh kb - 1)$. Since $\cosh(kb) = \cosh(-kb)$, there is no further information to extract from the condition that $y(b) = y(-b)$. However, we could replace a with the expression $h - \dfrac{1}{k}(\cosh kb - 1)$, obtaining $y = h + \dfrac{1}{k}(\cosh kx - \cosh kb)$. It would be better still to keep a in the expression for y, and use the expression for h to solve for k in terms of a, b, and h. That would enable us to express y in terms of x and the given parameters a, b, and h. Sadly, it is not possible to solve for k in closed form. That would have to be done by numerical methods when specific parameter values are given.

(b) The length of the cable is

$$L = \int_{-b}^{b} \sqrt{1 + (dy/dx)^2}\,dx = \int_{-b}^{b} \sqrt{1 + \sinh^2 kx}\,dx = \int_{-b}^{b} \cosh kx\,dx = 2\int_0^b \cosh kx\,dx$$

$$= 2\Big[(1/k)\sinh kx\Big]_0^b = (2/k)\sinh kb$$

PROBLEMS PLUS

1. We use the Fundamental Theorem of Calculus to differentiate the given equation:

$[f(x)]^2 = 100 + \int_0^x \left\{[f(t)]^2 + [f'(t)]^2\right\} dt \;\Rightarrow\; 2f(x)f'(x) = [f(x)]^2 + [f'(x)]^2 \;\Rightarrow$

$[f(x)]^2 + [f'(x)]^2 - 2f(x)f'(x) = 0 \;\Rightarrow\; [f(x) - f'(x)]^2 = 0 \;\Leftrightarrow\; f(x) = f'(x)$. We can solve this as a separable equation, or else use Theorem 10.4.2 with $k = 1$, which says that the solutions are $f(x) = Ce^x$. Now $[f(0)]^2 = 100$, so $f(0) = C = \pm 10$, and hence $f(x) = \pm 10e^x$ are the only functions satisfying the given equation.

2. $(fg)' = f'g'$, where $f(x) = e^{x^2} \;\Rightarrow\; \left(e^{x^2} g\right)' = 2xe^{x^2} g'$. Since the student's mistake did not affect the answer,

$\left(e^{x^2} g\right)' = e^{x^2} g' + 2xe^{x^2} g = 2xe^{x^2} g'$. So $(2x - 1)g' = 2xg$, or $\dfrac{g'}{g} = \dfrac{2x}{2x - 1} = 1 + \dfrac{1}{2x - 1} \;\Rightarrow$

$\ln|g(x)| = x + \frac{1}{2}\ln(2x - 1) + C \;\Rightarrow\; g(x) = Ae^x\sqrt{2x - 1}$.

3. $$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \to 0} \frac{f(x)\,[f(h) - 1]}{h} \quad [\text{since } f(x + h) = f(x)f(h)]$$

$$= f(x) \lim_{h \to 0} \frac{f(h) - 1}{h} = f(x) \lim_{h \to 0} \frac{f(h) - f(0)}{h - 0} = f(x)f'(0) = f(x)$$

Therefore, $f'(x) = f(x)$ for all x and from Theorem 10.4.2 we get $f(x) = Ae^x$. Now $f(0) = 1 \;\Rightarrow\; A = 1 \;\Rightarrow$ $f(x) = e^x$.

4. $\left(\int f(x)\,dx\right)\left(\int \dfrac{dx}{f(x)}\right) = -1 \Rightarrow \int \dfrac{dx}{f(x)} = \dfrac{-1}{\int f(x)\,dx} \Rightarrow \dfrac{1}{f(x)} = \dfrac{f(x)}{\left[\int f(x)\,dx\right]^2}$ [after differentiating] $\Rightarrow$

$\int f(x)\,dx = \pm f(x)$ [after taking square roots] $\Rightarrow$ $f(x) = \pm f'(x)$ [after differentiating again] $\Rightarrow$

$y = Ae^x$ or $y = Ae^{-x}$ by Theorem 10.4.2. Therefore, $f(x) = Ae^x$ or $f(x) = Ae^{-x}$, for all nonzero constants A, are the functions satisfying the original equation.

5. Let $y(t)$ denote the temperature of the peach pie t minutes after 5:00 P.M. and R the temperature of the room.

Newton's Law of Cooling gives us $dy/dt = k(y - R)$. Solving for y we get $\dfrac{dy}{y - R} = k\,dt \;\Rightarrow$

$\ln|y - R| = kt + C \;\Rightarrow\; |y - R| = e^{kt+C} \;\Rightarrow\; y - R = \pm e^{kt} \cdot e^C \;\Rightarrow\; y = Me^{kt} + R$, where M is a nonzero constant. We are given temperatures at three times.

$$\begin{aligned} y(0) &= 100 \;\Rightarrow\; 100 = M + R \;\Rightarrow\; R = 100 - M \\ y(10) &= 80 \;\Rightarrow\; 80 = Me^{10k} + R \quad \mathbf{(1)} \\ y(20) &= 65 \;\Rightarrow\; 65 = Me^{20k} + R \quad \mathbf{(2)} \end{aligned}$$

Substituting $100 - M$ for R in **(1)** and **(2)** gives us

$$-20 = Me^{10k} - M \;\; \mathbf{(3)} \quad \text{and} \quad -35 = Me^{20k} - M \;\; \mathbf{(4)}$$

Dividing **(3)** by **(4)** gives us $\dfrac{-20}{-35} = \dfrac{M(e^{10k}-1)}{M(e^{20k}-1)} \Rightarrow \dfrac{4}{7} = \dfrac{e^{10k}-1}{e^{20k}-1} \Rightarrow 4e^{20k} - 4 = 7e^{10k} - 7 \Rightarrow$ $4e^{20k} - 7e^{10k} + 3 = 0$. This is a quadratic equation in e^{10k}. $(4e^{10k} - 3)(e^{10k} - 1) = 0 \Rightarrow e^{10k} = \frac{3}{4}$ or 1 $\Rightarrow 10k = \ln\frac{3}{4}$ or $\ln 1 \Rightarrow k = \frac{1}{10}\ln\frac{3}{4}$ since k is a nonzero constant of proportionality. Substituting $\frac{3}{4}$ for e^{10k} in **(3)** gives us $-20 = M \cdot \frac{3}{4} - M \Rightarrow -20 = -\frac{1}{4}M \Rightarrow M = 80$. Now $R = 100 - M$ so $R = 20\ ^\circ\text{C}$.

6. Let b be the number of hours before noon that it began to snow, t the time measured in hours after noon, and $x = x(t)$ = distance traveled by the plow at time t. Then dx/dt = speed of plow. Since the snow falls steadily, the height at time t is $h(t) = k(t+b)$, where k is a constant. We are given that the rate of removal is constant, say R (in m^3/h). If the width of the path is w, then R = height × width × speed $= h(t) \times w \times \dfrac{dx}{dt} = k(t+b)w\,\dfrac{dx}{dt}$.

Thus, $\dfrac{dx}{dt} = \dfrac{C}{t+b}$, where $C = \dfrac{R}{kw}$ is a constant. This is a separable equation. $\displaystyle\int dx = C\int \frac{dt}{t+b} \Rightarrow$ $x(t) = C\ln(t+b) + K$.

Put $t = 0$: $0 = C\ln b + K \Rightarrow K = -C\ln b$, so $x(t) = C\ln(t+b) - C\ln b = C\ln(1 + t/b)$.

Put $t = 1$: $6000 = C\ln(1 + 1/b)$ $[x = 6 \text{ km}]$.

Put $t = 2$: $9000 = C\ln(1 + 2/b)$ $[x = (6+3) \text{ km}]$.

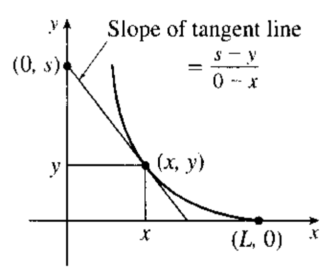
Solve for b: $\dfrac{\ln(1+1/b)}{6000} = \dfrac{\ln(1+2/b)}{9000} \Rightarrow 3\ln\left(1+\dfrac{1}{b}\right) = 2\ln\left(1+\dfrac{2}{b}\right) \Rightarrow \left(1+\dfrac{1}{b}\right)^3 = \left(1+\dfrac{2}{b}\right)^2$

$\Rightarrow 1 + \dfrac{3}{b} + \dfrac{3}{b^2} + \dfrac{1}{b^3} = 1 + \dfrac{4}{b} + \dfrac{4}{b^2} \Rightarrow \dfrac{1}{b} + \dfrac{1}{b^2} - \dfrac{1}{b^3} = 0 \Rightarrow b^2 + b - 1 = 0 \Rightarrow b = \frac{-1 \pm \sqrt{5}}{2}$.

But $b > 0$, so $b = \frac{-1 + \sqrt{5}}{2} \approx 0.618$ h ≈ 37 min. The snow began to fall $\frac{\sqrt{5}-1}{2}$ hours before noon; that is, at about 11:23 A.M.

7. (a) While running from $(L, 0)$ to (x, y), the dog travels a distance

$s = \displaystyle\int_x^L \sqrt{1 + (dy/dx)^2}\,dx = -\int_L^x \sqrt{1 + (dy/dx)^2}\,dx$, so $\dfrac{ds}{dx} = -\sqrt{1 + (dy/dx)^2}$. The dog and rabbit run at the same speed, so the rabbit's position when the dog has traveled a distance s is $(0, s)$. Since the dog runs straight for the rabbit, $\dfrac{dy}{dx} = \dfrac{s - y}{0 - x}$ (see the figure).



Thus, $s = y - x\dfrac{dy}{dx} \Rightarrow \dfrac{ds}{dx} = \dfrac{dy}{dx} - \left(x\dfrac{d^2y}{dx^2} + 1\,\dfrac{dy}{dx}\right) = -x\dfrac{d^2y}{dx^2}$. Equating the two expressions for $\dfrac{ds}{dx}$ gives us $x\dfrac{d^2y}{dx^2} = \sqrt{1 + \left(\dfrac{dy}{dx}\right)^2}$, as claimed.

(b) Letting $z = \frac{dy}{dx}$, we obtain the differential equation $x\frac{dz}{dx} = \sqrt{1+z^2}$, or $\frac{dz}{\sqrt{1+z^2}} = \frac{dx}{x}$. Integrating:

$\ln x = \int \frac{dz}{\sqrt{1+z^2}} \overset{25}{=} \ln\left(z + \sqrt{1+z^2}\right) + C$. When $x = L$, $z = dy/dx = 0$, so $\ln L = \ln 1 + C$.

Therefore, $C = \ln L$, so $\ln x = \ln\left(\sqrt{1+z^2}+z\right) + \ln L = \ln\left[L\left(\sqrt{1+z^2}+z\right)\right] \Rightarrow$

$$x = L\left(\sqrt{1+z^2}+z\right) \Rightarrow \sqrt{1+z^2} = \frac{x}{L} - z \Rightarrow 1 + z^2 = \left(\frac{x}{L}\right)^2 - \frac{2xz}{L} + z^2 \Rightarrow$$

$\left(\frac{x}{L}\right)^2 - 2z\left(\frac{x}{L}\right) - 1 = 0 \Rightarrow z = \frac{(x/L)^2 - 1}{2(x/L)} = \frac{x^2 - L^2}{2Lx} = \frac{x}{2L} - \frac{L}{2}\frac{1}{x}$ [for $x > 0$]. Since $z = \frac{dy}{dx}$,

$y = \frac{x^2}{4L} - \frac{L}{2}\ln x + C_1$. Since $y = 0$ when $x = L$, $0 = \frac{L}{4} - \frac{L}{2}\ln L + C_1 \Rightarrow C_1 = \frac{L}{2}\ln L - \frac{L}{4}$. Thus,

$$y = \frac{x^2}{4L} - \frac{L}{2}\ln x + \frac{L}{2}\ln L - \frac{L}{4} = \frac{x^2 - L^2}{4L} - \frac{L}{2}\ln\left(\frac{x}{L}\right).$$

(c) As $x \to 0^+$, $y \to \infty$, so the dog never catches the rabbit.

8. (a) If the dog runs twice as fast as the rabbit, then the rabbit's position when the dog has traveled a distance s is $(0, s/2)$. Since the dog runs straight toward the rabbit, the tangent line to the dog's path has slope

$\frac{dy}{dx} = \frac{s/2 - y}{0 - x}$. Thus, $s = 2y - 2x\frac{dy}{dx} \Rightarrow \frac{ds}{dx} = 2\frac{dy}{dx} - \left(2x\frac{d^2y}{dx^2} + 2\frac{dy}{dx}\right) = -2x\frac{d^2y}{dx^2}$.

From Problem 7(a), $\frac{ds}{dx} = -\sqrt{1 + \left(\frac{dy}{dx}\right)^2}$, so $2x\frac{d^2y}{dx^2} = \sqrt{1+\left(\frac{dy}{dx}\right)^2}$.

Letting $z = \frac{dy}{dx}$, we obtain the differential equation $2x\frac{dz}{dx} = \sqrt{1+z^2}$, or $\frac{2\,dz}{\sqrt{1+z^2}} = \frac{dx}{x}$.

Integrating, we get $\ln x = \int \frac{2\,dz}{\sqrt{1+z^2}} = 2\ln\left(\sqrt{1+z^2}+z\right) + C$. [See Problem 7(b).]

When $x = L$, $z = dy/dx = 0$, so $\ln L = 2\ln 1 + C = C$. Thus,

$$\ln x = 2\ln\left(\sqrt{1+z^2}+z\right) + \ln L = \ln\left(L\left(\sqrt{1+z^2}+z\right)^2\right) \Rightarrow x = L\left(\sqrt{1+z^2}+z\right)^2 \Rightarrow$$

$$\sqrt{1+z^2} = \sqrt{\frac{x}{L}} - z \Rightarrow 1 + z^2 = \frac{x}{L} - 2\sqrt{\frac{x}{L}}z + z^2 \Rightarrow 2\sqrt{\frac{x}{L}}z = \frac{x}{L} - 1 \Rightarrow$$

$\frac{dy}{dx} = z = \frac{1}{2}\sqrt{\frac{x}{L}} - \frac{1}{2\sqrt{x/L}} = \frac{1}{2\sqrt{L}}x^{1/2} - \frac{\sqrt{L}}{2}x^{-1/2} \Rightarrow y = \frac{1}{3\sqrt{L}}x^{3/2} - \sqrt{L}\,x^{1/2} + C_1$. When

$x = L$, $y = 0$, so $0 = \frac{1}{3\sqrt{L}}L^{3/2} - \sqrt{L}\,L^{1/2} + C_1 = \frac{L}{3} - L + C_1 = C_1 - \frac{2}{3}L$. Therefore, $C_1 = \frac{2}{3}L$ and

$y = \frac{x^{3/2}}{3\sqrt{L}} - \sqrt{L}\,x^{1/2} + \frac{2}{3}L$. As $x \to 0$, $y \to \frac{2}{3}L$, so the dog catches the rabbit when the rabbit is at $\left(0, \frac{2}{3}L\right)$.

(At that point, the dog has traveled a distance of $\frac{4}{3}L$, twice as far as the rabbit has run.)

(b) As in the solutions to part (a) and Problem 7, we get $z = \frac{dy}{dx} = \frac{x^2}{2L^2} - \frac{L^2}{2x^2}$ and hence $y = \frac{x^3}{6L^2} + \frac{L^2}{2x} - \frac{2}{3}L$.

We want to minimize the distance D from the dog at (x, y) to the rabbit at $(0, 2s)$. Now $s = \frac{1}{2}y - \frac{1}{2}x\frac{dy}{dx} \Rightarrow$

$$2s = y - xz \quad \Rightarrow \quad y - 2s = xz = x\left(\frac{x^2}{2L^2} - \frac{L^2}{2x^2}\right) = \frac{x^3}{2L^2} - \frac{L^2}{2x}, \text{ so}$$

$$D = \sqrt{(x-0)^2 + (y-2s)^2} = \sqrt{x^2 + \left(\frac{x^3}{2L^2} - \frac{L^2}{2x}\right)^2}$$

$$= \sqrt{\frac{x^6}{4L^4} + \frac{x^2}{2} + \frac{L^4}{4x^2}} = \sqrt{\left(\frac{x^3}{2L^2} + \frac{L^2}{2x}\right)^2} = \frac{x^3}{2L^2} + \frac{L^2}{2x}$$

$$D' = 0 \quad \Leftrightarrow \quad \frac{3x^2}{2L^2} - \frac{L^2}{2x^2} = 0 \quad \Leftrightarrow \quad \frac{3x^2}{2L^2} = \frac{L^2}{2x^2} \quad \Leftrightarrow \quad x^4 = \frac{L^4}{3} \quad \Leftrightarrow \quad x = \frac{L}{\sqrt[4]{3}},\ x > 0,\ L > 0.$$

Since $D''(x) = \dfrac{3x}{L^2} + \dfrac{L^2}{x^3} > 0$ for all $x > 0$, we know that

$D\left(\dfrac{L}{\sqrt[4]{3}}\right) = \dfrac{\left(L \cdot 3^{-1/4}\right)^3}{2L^2} + \dfrac{L^2}{2(L \cdot 3^{-1/4})} = \dfrac{2L}{3^{3/4}}$ is the minimum value of D, that is, the closest the dog gets to the rabbit. The positions at this distance are

Dog: $(x, y) = \left(\dfrac{L}{\sqrt[4]{3}}, \left(\dfrac{5}{3^{7/4}} - \dfrac{2}{3}\right)L\right) = \left(\dfrac{L}{\sqrt[4]{3}}, \dfrac{5\sqrt[4]{3} - 6}{9}L\right)$

Rabbit: $(0, 2s) = \left(0, \dfrac{8\sqrt[4]{3}L}{9} - \dfrac{2L}{3}\right) = \left(0, \dfrac{8\sqrt[4]{3} - 6}{9}L\right)$

9. (a) We are given that $V = \frac{1}{3}\pi r^2 h$, $dV/dt = 60{,}000\pi$ ft³/h, and $r = 1.5h = \frac{3}{2}h$. So $V = \frac{1}{3}\pi\left(\frac{3}{2}h\right)^2 h = \frac{3}{4}\pi h^3$ $\Rightarrow$ $\dfrac{dV}{dt} = \frac{3}{4}\pi \cdot 3h^2 \dfrac{dh}{dt} = \frac{9}{4}\pi h^2 \dfrac{dh}{dt}$. Therefore, $\dfrac{dh}{dt} = \dfrac{4(dV/dt)}{9\pi h^2} = \dfrac{240{,}000\pi}{9\pi h^2} = \dfrac{80{,}000}{3h^2}$ $(\star)$ $\Rightarrow$ $\int 3h^2\,dh = \int 80{,}000\,dt$ $\Rightarrow$ $h^3 = 80{,}000t + C$. When $t = 0$, $h = 60$. Thus, $C = 60^3 = 216{,}000$, so $h^3 = 80{,}000t + 216{,}000$. Let $h = 100$. Then $100^3 = 1{,}000{,}000 = 80{,}000t + 216{,}000$ $\Rightarrow$ $80{,}000t = 784{,}000$ $\Rightarrow$ $t = 9.8$, so the time required is 9.8 hours.

(b) The floor area of the silo is $F = \pi \cdot 200^2 = 40{,}000\pi$ ft², and the area of the base of the pile is $A = \pi r^2 = \pi\left(\frac{3}{2}h\right)^2 = \frac{9\pi}{4}h^2$. So the area of the floor which is not covered when $h = 60$ is $F - A = 40{,}000\pi - 8100\pi = 31{,}900\pi \approx 100{,}217$ ft². Now $A = \frac{9\pi}{4}h^2$ $\Rightarrow$ $dA/dt = \frac{9\pi}{4} \cdot 2h\,(dh/dt)$, and from $(\star)$ in part (a) we know that when $h = 60$, $dh/dt = \frac{80{,}000}{3(60)^2} = \frac{200}{27}$ ft/h. Therefore, $dA/dt = \frac{9\pi}{4}(2)(60)\left(\frac{200}{27}\right) = 2000\pi \approx 6283$ ft²/h.

(c) At $h = 90$ ft, $dV/dt = 60{,}000\pi - 20{,}000\pi = 40{,}000\pi$ ft³/h. From $(\star)$ in part (a), $\dfrac{dh}{dt} = \dfrac{4(dV/dt)}{9\pi h^2} = \dfrac{4(40{,}000\pi)}{9\pi h^2} = \dfrac{160{,}000}{9h^2}$ $\Rightarrow$ $\int 9h^2\,dh = \int 160{,}000\,dt$ $\Rightarrow$ $3h^3 = 160{,}000t + C$. When $t = 0$, $h = 90$; therefore, $C = 3 \cdot 729{,}000 = 2{,}187{,}000$. So $3h^3 = 160{,}000t + 2{,}187{,}000$. At the top, $h = 100$ $\Rightarrow$ $3(100)^3 = 160{,}000t + 2{,}187{,}000$ $\Rightarrow$ $t = \frac{813{,}000}{160{,}000} \approx 5.1$. The pile reaches the top after about 5.1 h.

10. Let $P(a, b)$ be any first-quadrant point on the curve $y = f(x)$. The tangent line at P has equation $y - b = f'(a)(x - a)$, or equivalently, $y = mx + b - ma$, where $m = f'(a)$. If $Q(0, c)$ is the y-intercept, then $c = b - am$. If $R(k, 0)$ is the x-intercept, then $k = \dfrac{am - b}{m} = a - \dfrac{b}{m}$. Since the tangent line is bisected at P, we know that $|PQ| = |PR|$; that is,

$$\sqrt{(a-0)^2 + [b - (b - am)]^2} = \sqrt{[a - (a - b/m)]^2 + (b-0)^2}$$

Squaring and simplifying gives us $a^2 + a^2m^2 = b^2/m^2 + b^2 \Rightarrow a^2m^2 + a^2m^4 = b^2 + b^2m^2 \Rightarrow$ $a^2m^4 + (a^2 - b^2)m^2 - b^2 = 0 \Rightarrow (a^2m^2 - b^2)(m^2 + 1) = 0 \Rightarrow m^2 = b^2/a^2$. Since m is the slope of the line from a positive y-intercept to a positive x-intercept, m must be negative. Since a and b are positive, we have $m = -b/a$, so we will solve the equivalent differential equation $\dfrac{dy}{dx} = -\dfrac{y}{x} \Rightarrow \dfrac{dy}{y} = -\dfrac{dx}{x} \Rightarrow$ $\displaystyle\int \frac{dy}{y} = -\int \frac{dx}{x} \Rightarrow \ln y = -\ln x + C\ \ [x, y > 0] \Rightarrow y = e^{-\ln x + C} = e^{\ln x^{-1}} \cdot e^C = x^{-1} \cdot A \Rightarrow$ $y = A/x$. Since the point $(3, 2)$ is on the curve, $3 = A/2 \Rightarrow A = 6$ and the curve is $y = 6/x$ with $x > 0$.

11. Let $P(a, b)$ be any point on the curve. If m is the slope of the tangent line at P, then $m = y'(a)$, and an equation of the normal line at P is $y - b = -\dfrac{1}{m}(x - a)$, or equivalently, $y = -\dfrac{1}{m}x + b + \dfrac{a}{m}$. The y-intercept is always 6, so $b + \dfrac{a}{m} = 6 \Rightarrow \dfrac{a}{m} = 6 - b \Rightarrow m = \dfrac{a}{6 - b}$. We will solve the equivalent differential equation $\dfrac{dy}{dx} = \dfrac{x}{6 - y} \Rightarrow (6 - y)\,dy = x\,dx \Rightarrow$ $\displaystyle\int (6 - y)\,dy = \int x\,dx \Rightarrow 6y - \tfrac{1}{2}y^2 = \tfrac{1}{2}x^2 + C \Rightarrow 12y - y^2 = x^2 + K$. Since $(3, 2)$ is on the curve, $12(2) - 2^2 = 3^2 + K \Rightarrow K = 11$. So the curve is given by $12y - y^2 = x^2 + 11 \Rightarrow$ $x^2 + y^2 - 12y + 36 = -11 + 36 \Rightarrow x^2 + (y - 6)^2 = 25$, a circle with center $(0, 6)$ and radius 5.

12. Suppose C is a curve with the required property and let $P = (x_0, y_0)$ be a point on C. The equation of the normal line to C at P is $y - y_0 = -\dfrac{1}{y_0'}(x - x_0)$, where y_0' is the value of $\dfrac{dy}{dx}$ at $x = x_0$. This equation makes sense only if $y_0' \neq 0$. If $y_0' = 0$, then the normal line at P is $x = x_0$, which does not intersect the y-axis at all unless $x_0 = 0$.

So let's assume that $y_0' \neq 0$. Then the normal line to C at P intersects the x-axis at $(x_0 + y_0y_0', 0)$, and it intersects the y-axis at $(0, y_0 + x_0/y_0')$. The condition on C implies that

$$[\text{distance from } P\,(x_0, y_0) \text{ to } (0, y_0 + x_0/y_0')] = [\text{distance from } (0, y_0 + x_0/y_0') \text{ to } (x_0 + y_0y_0', 0)]$$

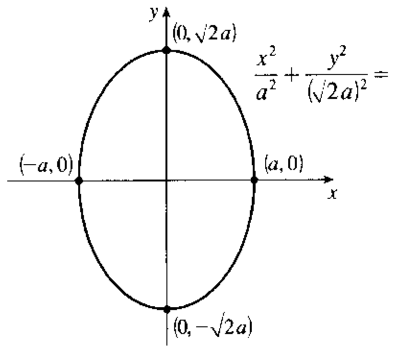
$$\sqrt{(0 - x_0)^2 + (y_0 + x_0/y_0' - y_0)^2} = \sqrt{(x_0 + y_0y_0' - 0)^2 + [0 - (y_0 + x_0/y_0')]^2}$$

Squaring both sides, we get $x_0^2 + x_0^2/(y_0')^2 = (x_0 + y_0y_0')^2 + (y_0 + x_0/y_0')^2$ or $x_0^2 + \dfrac{x_0^2}{(y_0')^2} = x_0^2 + 2x_0y_0y_0' + y_0^2(y_0')^2 + y_0^2 + 2\dfrac{x_0y_0}{y_0'} + \dfrac{x_0^2}{(y_0')^2}$. Subtracting $x_0^2 + \dfrac{x_0^2}{(y_0')^2}$ from both sides and multiplying by y_0', we get

$$0 = y_0^2y_0' + y_0^2(y_0')^3 + 2x_0y_0\left[1 + (y_0')^2\right] = y_0\left\{y_0y_0' + y_0(y_0')^3 + 2x_0\left[1 + (y_0')^2\right]\right\}$$

$$= y_0\left\{y_0y_0'\left[1 + (y_0')^2\right] + 2x_0\left[1 + (y_0')^2\right]\right\} = y_0(y_0y_0' + 2x_0)\left[1 + (y_0')^2\right]$$

Since $1 + (y_0')^2 \geq 1 > 0$, we conclude that $y_0(y_0y_0' + 2x_0) = 0$. Now P is an arbitrary point on C for which $y_0' \neq 0$. Thus, we have shown that $y(yy' + 2x) = 0$ for points (x, y) along C where $y' \neq 0$. One solution of this equation is $y = 0$, but that curve (the x-axis) doesn't satisfy the condition required of C, since its normal lines at points for $x \neq 0$ don't intersect the y-axis. Thus, we can focus our attention on points of C where $y \neq 0$, and conclude that $yy' + 2x = 0$ at points of C where $y \neq 0$ and $y' \neq 0$. Integrating both sides of $yy' + 2x = 0$, we get $\tfrac{1}{2}y^2 + x^2 = c$. Clearly $c > 0$ (since $y \neq 0$), so we can write $c = a^2$, where $a = \sqrt{c} > 0$. Thus, $\tfrac{1}{2}y^2 + x^2 = a^2$ and



$x^2/a^2 + y^2/(\sqrt{2}a)^2 = 1$. This shows that C is (part of) the ellipse centered at $(0,0)$ with semimajor axis $\sqrt{2}\,a$ in the y-direction and semiminor axis a in the x-direction. The points of C where $y = 0$ or $y' = 0$ are the vertices $(0, \pm\sqrt{2}\,a)$ and $(\pm a, 0)$. At these points, the condition on C is satisfied in a degenerate way. [When $P = (\pm a, 0)$, the normal line at P *is* the x-axis, so *all* the points of the normal line can be viewed as points of intersection with the x-axis. The intersection with the y-axis at $(0,0)$ is midway between $(a,0)$ and $(-a,0)$; one of these points is P, and the other can be regarded as an intersection of the normal line with the x-axis. Similarly, when $P = (0, \pm\sqrt{2}\,a)$, the normal line is the y-axis, and the point $(0, \pm\sqrt{2}\,a/2)$, which can be regarded as an intersection of the normal line with the y-axis, is midway between P and $(0,0)$, the intersection with the x-axis.]

Conversely, if C is part of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{2a^2} = 1$ for some $a > 0$, then the normal line at a point (x_0, y_0) of C (other than the four vertices) has equation $y - y_0 = \dfrac{y_0}{2x_0}(x - x_0)$. Its intersections with the coordinate axes are $\left(0, \dfrac{y_0}{2}\right)$ and $(-x_0, 0)$. $\left[\text{distance from } (x_0, y_0) \text{ to } \left(0, \dfrac{y_0}{2}\right)\right]^2 = x_0^2 + \dfrac{y_0^2}{4}$ and $\left[\text{distance from } \left(0, \dfrac{y_0}{2}\right) \text{ to } (-x_0, 0)\right]^2 = x_0^2 + \dfrac{y_0^2}{4}$, so the required condition is met at points other than the four vertices. As we have explained, if we are willing to interpret the condition broadly, then it can be viewed as holding even at the four vertices.

Another method: Let $P(x_0, y_0)$ be a point on the curve. Since the midpoint of the line segment determined by the normal line from (x_0, y_0) to its intersection with the x-axis has x-coordinate 0, the x-coordinate of the point of intersection with the x-axis must be $-x_0$. Hence, the normal line has slope $\dfrac{y_0 - 0}{x_0 - (-x_0)} = \dfrac{y_0}{2x_0}$. So the tangent line has slope $-\dfrac{2x_0}{y_0}$. This gives the differential equation $y' = -\dfrac{2x}{y} \Rightarrow y\,dy = -2x\,dx \Rightarrow$ $\int y\,dy = \int(-2x)\,dx \Rightarrow \frac{1}{2}y^2 = -x^2 + C \Rightarrow x^2 + \frac{1}{2}y^2 = C \ (C > 0)$.