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Chapter 12

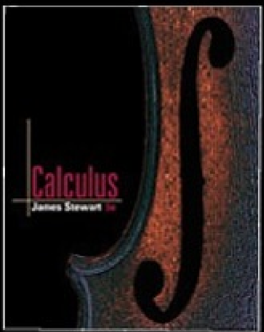
Adapted from the

Complete Solutions Manual

for

James Stewart's

Calculus – 5th Edition



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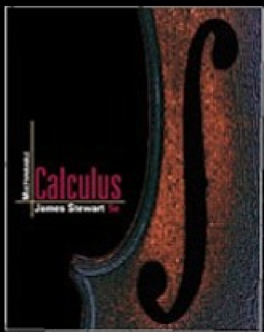

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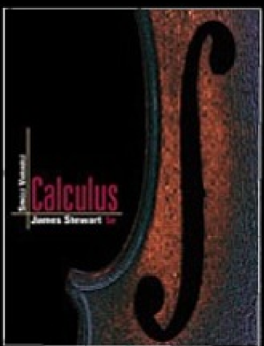

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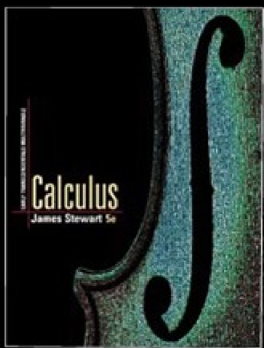

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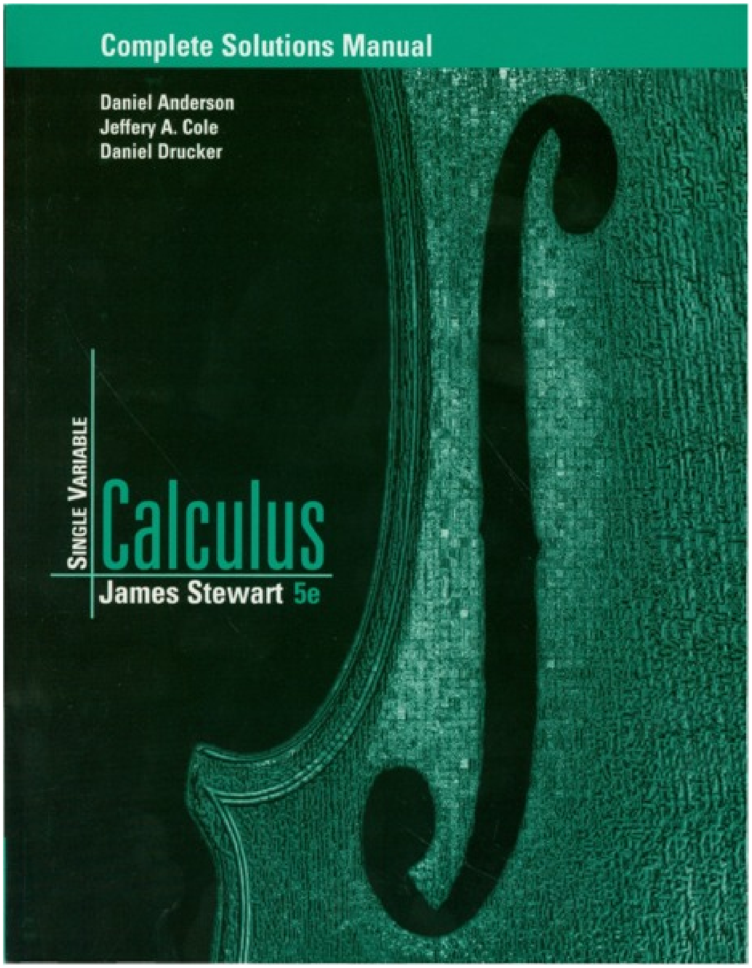


12 ☐ INFINITE SEQUENCES AND SERIES

12.1 Sequences

1. (a) A sequence is an ordered list of numbers. It can also be defined as a function whose domain is the set of positive integers.

(b) The terms a_n approach 8 as n becomes large. In fact, we can make a_n as close to 8 as we like by taking n sufficiently large.

(c) The terms a_n become large as n becomes large. In fact, we can make a_n as large as we like by taking n sufficiently large.

2. (a) From Definition 1, a convergent sequence is a sequence for which $\lim_{n\to\infty} a_n$ exists. Examples: $\{1/n\}$, $\{1/2^n\}$

(b) A divergent sequence is a sequence for which $\lim_{n\to\infty} a_n$ *does not* exist. Examples: $\{n\}$, $\{\sin n\}$

3. $a_n = 1 - (0.2)^n$, so the sequence is $\{0.8, 0.96, 0.992, 0.9984, 0.99968, \ldots\}$.

4. $a_n = \dfrac{n+1}{3n-1}$, so the sequence is $\left\{\dfrac{2}{2}, \dfrac{3}{5}, \dfrac{4}{8}, \dfrac{5}{11}, \dfrac{6}{14}, \ldots\right\} = \left\{1, \dfrac{3}{5}, \dfrac{1}{2}, \dfrac{5}{11}, \dfrac{3}{7}, \ldots\right\}$.

5. $a_n = \dfrac{3(-1)^n}{n!}$, so the sequence is $\left\{\dfrac{-3}{1}, \dfrac{3}{2}, \dfrac{-3}{6}, \dfrac{3}{24}, \dfrac{-3}{120}, \ldots\right\} = \left\{-3, \dfrac{3}{2}, -\dfrac{1}{2}, \dfrac{1}{8}, -\dfrac{1}{40}, \ldots\right\}$.

6. $a_n = 2 \cdot 4 \cdot 6 \cdot \cdots \cdot (2n)$, so the sequence is
$\{2, 2\cdot 4, 2\cdot 4\cdot 6, 2\cdot 4\cdot 6\cdot 8, 2\cdot 4\cdot 6\cdot 8\cdot 10, \ldots\} = \{2, 8, 48, 384, 3840, \ldots\}$.

7. $a_1 = 3$, $a_{n+1} = 2a_n - 1$. Each term is defined in terms of the preceding term.
$a_2 = 2a_1 - 1 = 2(3) - 1 = 5$. $a_3 = 2a_2 - 1 = 2(5) - 1 = 9$. $a_4 = 2a_3 - 1 = 2(9) - 1 = 17$.
$a_5 = 2a_4 - 1 = 2(17) - 1 = 33$. The sequence is $\{3, 5, 9, 17, 33, \ldots\}$.

8. $a_1 = 4$, $a_{n+1} = \dfrac{a_n}{a_n - 1}$. Each term is defined in terms of the preceding term.

$a_2 = \dfrac{a_1}{a_1 - 1} = \dfrac{4}{4-1} = \dfrac{4}{3}$. $a_3 = \dfrac{a_2}{a_2 - 1} = \dfrac{4/3}{\frac{4}{3} - 1} = \dfrac{4/3}{1/3} = 4$. Since $a_3 = a_1$, we can see that the terms of the sequence will alternately equal 4 and 4/3, so the sequence is $\{4, \frac{4}{3}, 4, \frac{4}{3}, 4, \ldots\}$.

9. The numerators are all 1 and the denominators are powers of 2, so $a_n = \dfrac{1}{2^n}$.

10. The numerators are all 1 and the denominators are multiples of 2, so $a_n = \dfrac{1}{2n}$.

11. $\{2, 7, 12, 17, \ldots\}$. Each term is larger than the preceding one by 5, so
$a_n = a_1 + d(n-1) = 2 + 5(n-1) = 5n - 3$.

12. $\{-\frac{1}{4}, \frac{2}{9}, -\frac{3}{16}, \frac{4}{25}, \ldots\}$. The numerator of the nth term is n and its denominator is $(n+1)^2$. Including the alternating signs, we get $a_n = (-1)^n \dfrac{n}{(n+1)^2}$.

13. $\{1, -\frac{2}{3}, \frac{4}{9}, -\frac{8}{27}, \ldots\}$. Each term is $-\frac{2}{3}$ times the preceding one, so $a_n = \left(-\frac{2}{3}\right)^{n-1}$.

14. $\{5, 1, 5, 1, 5, 1, \ldots\}$. The average of 5 and 1 is 3, so we can think of the sequence as alternately adding 2 and -2 to 3. Thus, $a_n = 3 + (-1)^{n+1} \cdot 2$.

15. $a_n = n(n-1)$. $a_n \to \infty$ as $n \to \infty$, so the sequence diverges.

16. $a_n = \dfrac{n+1}{3n-1} = \dfrac{1+1/n}{3-1/n}$, so $a_n \to \dfrac{1+0}{3-0} = \dfrac{1}{3}$ as $n \to \infty$. Converges

17. $a_n = \dfrac{3+5n^2}{n+n^2} = \dfrac{(3+5n^2)/n^2}{(n+n^2)/n^2} = \dfrac{5+3/n^2}{1+1/n}$, so $a_n \to \dfrac{5+0}{1+0} = 5$ as $n \to \infty$. Converges

18. $a_n = \dfrac{\sqrt{n}}{1+\sqrt{n}} = \dfrac{1}{1/\sqrt{n}+1}$, so $a_n \to \dfrac{1}{0+1} = 1$ as $n \to \infty$. Converges

19. $a_n = \dfrac{2^n}{3^{n+1}} = \dfrac{1}{3}\left(\dfrac{2}{3}\right)^n$, so $\lim\limits_{n\to\infty} a_n = \frac{1}{3}\lim\limits_{n\to\infty}\left(\frac{2}{3}\right)^n = \frac{1}{3}\cdot 0 = 0$ by (8) with $r = \frac{2}{3}$. Converges

20. $a_n = \dfrac{n}{1+\sqrt{n}} = \dfrac{\sqrt{n}}{1/\sqrt{n}+1}$. The numerator approaches ∞ and the denominator approaches $0+1=1$ as $n \to \infty$, so $a_n \to \infty$ as $n \to \infty$ and the sequence diverges.

21. $a_n = \dfrac{(-1)^{n-1}n}{n^2+1} = \dfrac{(-1)^{n-1}}{n+1/n}$, so $0 \le |a_n| = \dfrac{1}{n+1/n} \le \dfrac{1}{n} \to 0$ as $n \to \infty$, so $a_n \to 0$ by the Squeeze Theorem and Theorem 6. Converges

22. $a_n = \dfrac{(-1)^n n^3}{n^3+2n^2+1}$. Now $|a_n| = \dfrac{n^3}{n^3+2n^2+1} = \dfrac{1}{1+\frac{2}{n}+\frac{1}{n^3}} \to 1$ as $n \to \infty$, but the terms of the sequence $\{a_n\}$ alternate in sign, so the sequence $a_1, a_3, a_5, \ldots$ converges to -1 and the sequence $a_2, a_4, a_6, \ldots$ converges to $+1$. This shows that the given sequence diverges since its terms don't approach a single real number.

23. $a_n = \cos(n/2)$. This sequence diverges since the terms don't approach any particular real number as $n \to \infty$. The terms take on values between -1 and 1.

24. $a_n = \cos(2/n)$. As $n \to \infty$, $2/n \to 0$, so $\cos(2/n) \to \cos 0 = 1$. Converges

25. $a_n = \dfrac{(2n-1)!}{(2n+1)!} = \dfrac{(2n-1)!}{(2n+1)(2n)(2n-1)!} = \dfrac{1}{(2n+1)(2n)} \to 0$ as $n \to \infty$. Converges

26. $2n \to \infty$ as $n \to \infty$, so since $\lim\limits_{x\to\infty} \arctan x = \frac{\pi}{2}$, we have $\lim\limits_{n\to\infty} \arctan 2n = \frac{\pi}{2}$. Converges

27. $a_n = \dfrac{e^n + e^{-n}}{e^{2n}-1} \cdot \dfrac{e^{-n}}{e^{-n}} = \dfrac{1+e^{-2n}}{e^n - e^{-n}} \to \dfrac{1+0}{e^n - 0} \to 0$ as $n \to \infty$. Converges

28. $a_n = \dfrac{\ln n}{\ln 2n} = \dfrac{\ln n}{\ln 2 + \ln n} = \dfrac{1}{\frac{\ln 2}{\ln n}+1} \to \dfrac{1}{0+1} \to 1$ as $n \to \infty$. Converges

29. $a_n = n^2 e^{-n} = \dfrac{n^2}{e^n}$. Since $\lim\limits_{x\to\infty}\dfrac{x^2}{e^x} \overset{H}{=} \lim\limits_{x\to\infty}\dfrac{2x}{e^x} \overset{H}{=} \lim\limits_{x\to\infty}\dfrac{2}{e^x} = 0$, it follows from Theorem 3 that $\lim\limits_{n\to\infty} a_n = 0$. Converges

30. $a_n = n\cos n\pi = n(-1)^n$. Since $|a_n| = n \to \infty$ as $n \to \infty$, the given sequence diverges.

31. $0 \le \dfrac{\cos^2 n}{2^n} \le \dfrac{1}{2^n}$ [since $0 \le \cos^2 n \le 1$], so since $\lim\limits_{n\to\infty}\dfrac{1}{2^n} = 0$, $\left\{\dfrac{\cos^2 n}{2^n}\right\}$ converges to 0 by the Squeeze Theorem.

32. $a_n = \ln(n+1) - \ln n = \ln\left(\dfrac{n+1}{n}\right) = \ln\left(1+\dfrac{1}{n}\right) \to \ln(1) = 0$ as $n \to \infty$. Converges

33. $a_n = n\sin(1/n) = \dfrac{\sin(1/n)}{1/n}$. Since $\lim\limits_{x\to\infty}\dfrac{\sin(1/x)}{1/x} = \lim\limits_{t\to 0^+}\dfrac{\sin t}{t}$ [where $t = 1/x$] $= 1$, it follows from Theorem 3 that $\{a_n\}$ converges to 1.

34. $a_n = \sqrt{n} - \sqrt{n^2 - 1} = \sqrt{n^2 \cdot \dfrac{1}{n}} - \sqrt{n^2\left(1 - \dfrac{1}{n^2}\right)} = n\left(\dfrac{1}{\sqrt{n}} - \sqrt{1 - \dfrac{1}{n^2}}\right) \to n(0-1) \to -n$ as $n \to \infty$, so $a_n \to -\infty$ as $n \to \infty$. Diverges

35. $a_n = \left(1 + \dfrac{2}{n}\right)^{1/n} \quad \Rightarrow \quad \ln a_n = \dfrac{1}{n}\ln\left(1 + \dfrac{2}{n}\right)$. As $n \to \infty$, $\dfrac{1}{n} \to 0$ and $\ln\left(1 + \dfrac{2}{n}\right) \to 0$, so $\ln a_n \to 0$. Thus, $a_n \to e^0 = 1$ as $n \to \infty$. Converges

36. $a_n = \dfrac{\sin 2n}{1 + \sqrt{n}}$. $|a_n| \le \dfrac{1}{1 + \sqrt{n}}$ and $\lim\limits_{n\to\infty} \dfrac{1}{1+\sqrt{n}} = 0$, so $\dfrac{-1}{1+\sqrt{n}} \le a_n \le \dfrac{1}{1+\sqrt{n}} \quad \Rightarrow \quad \lim\limits_{n\to\infty} a_n = 0$ by the Squeeze Theorem. Converges

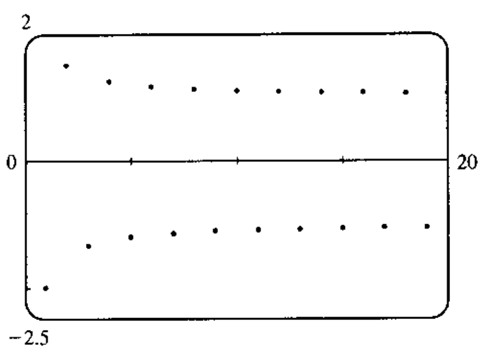
37. $\{0, 1, 0, 0, 1, 0, 0, 0, 1, \ldots\}$ diverges since the sequence takes on only two values, 0 and 1, and never stays arbitrarily close to either one (or any other value) for n sufficiently large.

38. $\left\{\frac{1}{1}, \frac{1}{3}, \frac{1}{2}, \frac{1}{4}, \frac{1}{3}, \frac{1}{5}, \frac{1}{4}, \frac{1}{6}, \ldots\right\}$. $a_{2n-1} = \dfrac{1}{n}$ and $a_{2n} = \dfrac{1}{n+2}$ for all positive integers n. $\lim\limits_{n\to\infty} a_n = 0$ since $\lim\limits_{n\to\infty} a_{2n-1} = \lim\limits_{n\to\infty} \dfrac{1}{n} = 0$ and $\lim\limits_{n\to\infty} a_{2n} = \lim\limits_{n\to\infty} \dfrac{1}{n+2} = 0$. For n sufficiently large, a_n can be made as close to 0 as we like. Converges

39. $a_n = \dfrac{n!}{2^n} = \dfrac{1}{2} \cdot \dfrac{2}{2} \cdot \dfrac{3}{2} \cdot \cdots \cdot \dfrac{(n-1)}{2} \cdot \dfrac{n}{2} \ge \dfrac{1}{2} \cdot \dfrac{n}{2}$ [for $n > 1$] $= \dfrac{n}{4} \to \infty$ as $n \to \infty$, so $\{a_n\}$ diverges.

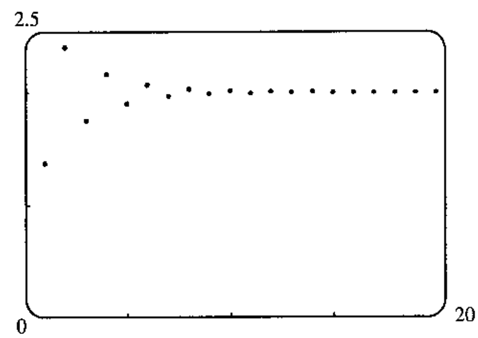
40. $0 < |a_n| = \dfrac{3^n}{n!} = \dfrac{3}{1} \cdot \dfrac{3}{2} \cdot \dfrac{3}{3} \cdot \cdots \cdot \dfrac{3}{(n-1)} \cdot \dfrac{3}{n} \le \dfrac{3}{1} \cdot \dfrac{3}{2} \cdot \dfrac{3}{n}$ [for $n > 2$] $= \dfrac{27}{2n} \to 0$ as $n \to \infty$, so by the Squeeze Theorem and Theorem 6, $\{(-3)^n/n\}$ converges to 0.

41.



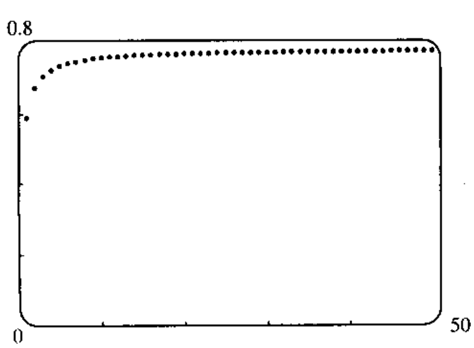
From the graph, we see that the sequence $\left\{(-1)^n \dfrac{n+1}{n}\right\}$ is divergent, since it oscillates between 1 and -1 (approximately).

42.



From the graph, it appears that the sequence converges to 2. $\left\{\left(-\frac{2}{\pi}\right)^n\right\}$ converges to 0 by (6), and hence $\left\{2 + \left(-\frac{2}{\pi}\right)^n\right\}$ converges to $2 + 0 = 2$.

43.

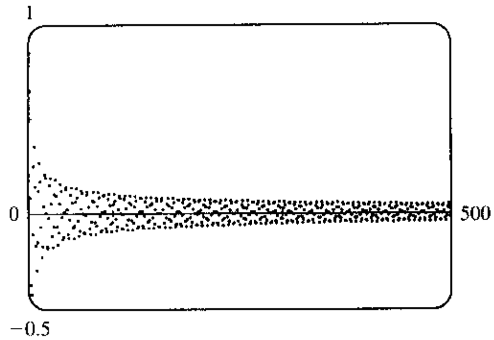


From the graph, it appears that the sequence converges to about 0.78.

$$\lim_{n\to\infty} \frac{2n}{2n+1} = \lim_{n\to\infty} \frac{2}{2 + 1/n} = 1, \text{ so}$$

$$\lim_{n\to\infty} \arctan\left(\frac{2n}{2n+1}\right) = \arctan 1 = \frac{\pi}{4}.$$

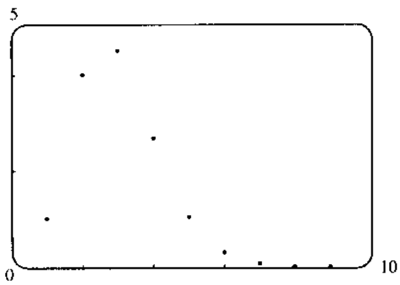
44.



From the graph, it appears that the sequence converges (slowly) to 0.

$0 \le \dfrac{|\sin n|}{\sqrt{n}} \le \dfrac{1}{\sqrt{n}} \to 0$ as $n \to \infty$, so by the Squeeze Theorem and Theorem 6, $\left\{\dfrac{\sin n}{\sqrt{n}}\right\}$ converges to 0.

45.



From the graph, it appears that the sequence converges to 0.

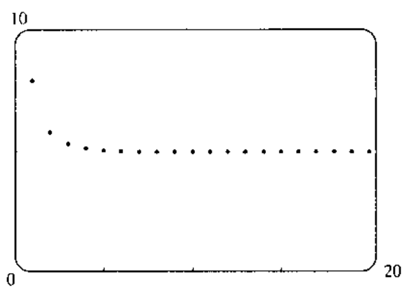
$$0 < a_n = \frac{n^3}{n!} = \frac{n}{n} \cdot \frac{n}{(n-1)} \cdot \frac{n}{(n-2)} \cdot \frac{1}{(n-3)} \cdot \cdots \cdot \frac{1}{3} \cdot \frac{1}{2} \cdot \frac{1}{1}$$

$$\le \frac{n^2}{(n-1)(n-2)(n-3)} \quad [\text{for } n \ge 4]$$

$$= \frac{1/n}{(1-1/n)(1-2/n)(1-3/n)} \to 0 \text{ as } n \to \infty$$

So by the Squeeze Theorem, $\{n^3/n!\}$ converges to 0.

46.



From the graph, it appears that the sequence converges to 5.

$$5 = \sqrt[n]{5^n} \le \sqrt[n]{3^n + 5^n} \le \sqrt[n]{5^n + 5^n} = \sqrt[n]{2}\,\sqrt[n]{5^n}$$

$$= \sqrt[n]{2} \cdot 5 \to 5 \text{ as } n \to \infty \quad \left[\lim_{n\to\infty} 2^{1/n} = 2^0 = 1\right]$$

Hence, $a_n \to 5$ by the Squeeze Theorem.

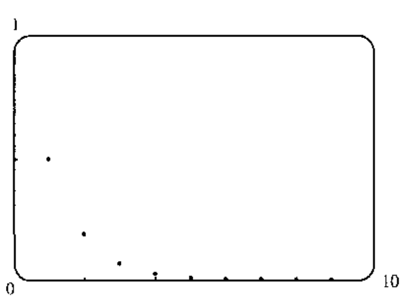
Alternate Solution: Let $y = (3^x + 5^x)^{1/x}$. Then

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{\ln(3^x + 5^x)}{x} \overset{\text{H}}{=} \lim_{x\to\infty} \frac{3^x \ln 3 + 5^x \ln 5}{3^x + 5^x}$$

$$= \lim_{x\to\infty} \frac{\left(\frac{3}{5}\right)^x \ln 3 + \ln 5}{\left(\frac{3}{5}\right)^x + 1} = \ln 5$$

so $\lim\limits_{x\to\infty} y = e^{\ln 5} = 5$, and so $\left\{\sqrt[n]{3^n + 5^n}\right\}$ converges to 5.

47.



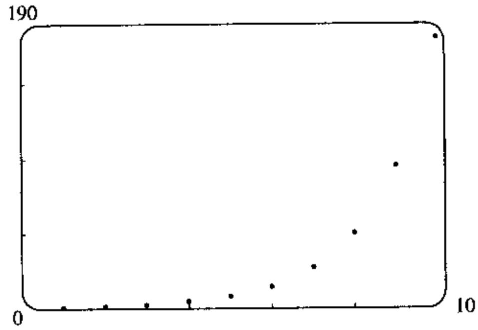
From the graph, it appears that the sequence approaches 0.

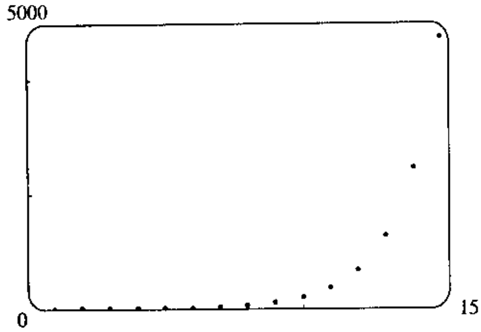
$$0 < a_n = \frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{(2n)^n} = \frac{1}{2n} \cdot \frac{3}{2n} \cdot \frac{5}{2n} \cdot \cdots \cdot \frac{2n-1}{2n}$$

$$\le \frac{1}{2n} \cdot (1) \cdot (1) \cdot \cdots \cdot (1) = \frac{1}{2n} \to 0 \text{ as } n \to \infty$$

So by the Squeeze Theorem, $\left\{\dfrac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{(2n)^n}\right\}$ converges to 0.

48.





From the graphs, it seems that the sequence diverges. $a_n = \dfrac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{n!}$. We first prove by induction that $a_n \geq \left(\dfrac{3}{2}\right)^{n-1}$ for all n. This is clearly true for $n=1$, so let $P(n)$ be the statement that the above is true for n. We must show it is then true for $n+1$. $a_{n+1} = a_n \cdot \dfrac{2n+1}{n+1} \geq \left(\dfrac{3}{2}\right)^{n-1} \cdot \dfrac{2n+1}{n+1}$ (induction hypothesis).

But $\dfrac{2n+1}{n+1} \geq \dfrac{3}{2}$ [since $2(2n+1) \geq 3(n+1) \quad \Leftrightarrow \quad 4n+2 \geq 3n+3 \quad \Leftrightarrow \quad n \geq 1$], and so we get that $a_{n+1} \geq \left(\frac{3}{2}\right)^{n-1} \cdot \frac{3}{2} = \left(\frac{3}{2}\right)^n$ which is $P(n+1)$. Thus, we have proved our first assertion, so since $\left\{\left(\frac{3}{2}\right)^{n-1}\right\}$ diverges (by (8)), so does the given sequence $\{a_n\}$.

49. (a) $a_n = 1000(1.06)^n \quad \Rightarrow \quad a_1 = 1060$, $a_2 = 1123.60$, $a_3 = 1191.02$, $a_4 = 1262.48$, and $a_5 = 1338.23$.

(b) $\lim\limits_{n\to\infty} a_n = 1000 \lim\limits_{n\to\infty} (1.06)^n$, so the sequence diverges by (8) with $r = 1.06 > 1$.

50. $a_{n+1} = \begin{cases} \frac{1}{2}a_n & \text{if } a_n \text{ is an even number} \\ 3a_n + 1 & \text{if } a_n \text{ is an odd number} \end{cases}$ When $a_1 = 11$, the first 40 terms are 11, 34, 17, 52, 26, 13, 40, 20, 10, 5, 16, 8, 4, 2, 1, 4, 2, 1, 4, 2, 1, 4, 2, 1, 4, 2, 1, 4, 2, 1, 4, 2, 1, 4, 2, 1, 4, 2, 1, 4. When $a_1 = 25$, the first 40 terms are 25, 76, 38, 19, 58, 29, 88, 44, 22, 11, 34, 17, 52, 26, 13, 40, 20, 10, 5, 16, 8, 4, 2, 1, 4, 2, 1, 4, 2, 1, 4, 2, 1, 4, 2, 1, 4, 2, 1, 4. The famous Collatz conjecture is that this sequence always reaches 1, regardless of the starting point a_1.

51. If $|r| \geq 1$, then $\{r^n\}$ diverges by (8), so $\{nr^n\}$ diverges also, since $|nr^n| = n\,|r^n| \geq |r^n|$. If $|r| < 1$ then $\lim\limits_{x\to\infty} xr^x = \lim\limits_{x\to\infty} \dfrac{x}{r^{-x}} \overset{\text{H}}{=} \lim\limits_{x\to\infty} \dfrac{1}{(-\ln r)\,r^{-x}} = \lim\limits_{x\to\infty} \dfrac{r^x}{-\ln r} = 0$, so $\lim\limits_{n\to\infty} nr^n = 0$, and hence $\{nr^n\}$ converges whenever $|r| < 1$.

52. (a) Let $\lim\limits_{n\to\infty} a_n = L$. By Definition 1, this means that for every $\varepsilon > 0$ there is an integer N such that $|a_n - L| < \varepsilon$ whenever $n > N$. Thus, $|a_{n+1} - L| < \varepsilon$ whenever $n+1 > N \quad \Leftrightarrow \quad n > N-1$. It follows that $\lim\limits_{n\to\infty} a_{n+1} = L$ and so $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} a_{n+1}$.

(b) If $L = \lim\limits_{n\to\infty} a_n$ then $\lim\limits_{n\to\infty} a_{n+1} = L$ also, so L must satisfy $L = 1/(1+L) \quad \Rightarrow \quad L^2 + L - 1 = 0 \quad \Rightarrow$ $L = \frac{-1+\sqrt{5}}{2}$ (since L has to be non-negative if it exists).

53. Since $\{a_n\}$ is a decreasing sequence, $a_n > a_{n+1}$ for all $n \geq 1$. Because all of its terms lie between 5 and 8, $\{a_n\}$ is a bounded sequence. By the Monotonic Sequence Theorem, $\{a_n\}$ is convergent; that is, $\{a_n\}$ has a limit L. L must be less than 8 since $\{a_n\}$ is decreasing, so $5 \leq L < 8$.

54. $a_n = 1/5^n$ defines a decreasing geometric sequence since $a_{n+1} = \frac{1}{5}a_n < a_n$ for each $n \geq 1$. The sequence is bounded since $0 < a_n \leq \frac{1}{5}$ for all $n \geq 1$.

55. $a_n = \dfrac{1}{2n+3}$ is decreasing since $a_{n+1} = \dfrac{1}{2(n+1)+3} = \dfrac{1}{2n+5} < \dfrac{1}{2n+3} = a_n$ for each $n \geq 1$. The sequence is bounded since $0 < a_n \leq \frac{1}{5}$ for all $n \geq 1$. Note that $a_1 = \frac{1}{5}$.

56. $a_n = \dfrac{2n-3}{3n+4}$ defines an increasing sequence since for $f(x) = \dfrac{2x-3}{3x+4}$,

$f'(x) = \dfrac{(3x+4)(2)-(2x-3)(3)}{(3x+4)^2} = \dfrac{17}{(3x+4)^2} > 0$. The sequence is bounded since $a_n \geq a_1 = -\frac{1}{7}$ for $n \geq 1$, and $a_n < \dfrac{2n-3}{3n} < \dfrac{2n}{3n} = \dfrac{2}{3}$ for $n \geq 1$.

57. $a_n = \cos(n\pi/2)$ is not monotonic. The first few terms are $0, -1, 0, 1, 0, -1, 0, 1, \ldots$. In fact, the sequence consists of the terms $0, -1, 0, 1$ repeated over and over again in that order. The sequence is bounded since $|a_n| \leq 1$ for all $n \geq 1$.

58. $a_n = ne^{-n}$ defines a positive decreasing sequence since the function $f(x) = xe^{-x}$ is decreasing for $x > 1$. $[f'(x) = e^{-x} - xe^{-x} = e^{-x}(1-x) < 0$ for $x > 1.]$ The sequence is bounded above by $a_1 = \frac{1}{e}$ and below by 0.

59. $a_n = \dfrac{n}{n^2+1}$ defines a decreasing sequence since for $f(x) = \dfrac{x}{x^2+1}$,

$f'(x) = \dfrac{(x^2+1)(1)-x(2x)}{(x^2+1)^2} = \dfrac{1-x^2}{(x^2+1)^2} \leq 0$ for $x \geq 1$. The sequence is bounded since $0 < a_n \leq \frac{1}{2}$ for all $n \geq 1$.

60. $a_n = n + \dfrac{1}{n}$ defines an increasing sequence since the function $g(x) = x + \dfrac{1}{x}$ is increasing for $x > 1$. $[g'(x) = 1 - 1/x^2 > 0$ for $x > 1.]$ The sequence is unbounded since $a_n \to \infty$ as $n \to \infty$. (It is, however, bounded below by $a_1 = 2$.)

61. $a_1 = 2^{1/2}, a_2 = 2^{3/4}, a_3 = 2^{7/8}, \ldots$, so $a_n = 2^{(2^n-1)/2^n} = 2^{1-(1/2^n)}$. $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} 2^{1-(1/2^n)} = 2^1 = 2$.

Alternate solution: Let $L = \lim\limits_{n\to\infty} a_n$. (We could show the limit exists by showing that $\{a_n\}$ is bounded and increasing.) Then L must satisfy $L = \sqrt{2 \cdot L} \;\Rightarrow\; L^2 = 2L \;\Rightarrow\; L(L-2) = 0$. $L \neq 0$ since the sequence increases, so $L = 2$.

62. (a) Let P_n be the statement that $a_{n+1} \geq a_n$ and $a_n \leq 3$. P_1 is obviously true. We will assume that P_n is true and then show that as a consequence P_{n+1} must also be true. $a_{n+2} \geq a_{n+1} \quad \Leftrightarrow \quad \sqrt{2 + a_{n+1}} \geq \sqrt{2 + a_n} \quad \Leftrightarrow$ $2 + a_{n+1} \geq 2 + a_n \quad \Leftrightarrow \quad a_{n+1} \geq a_n$, which is the induction hypothesis. $a_{n+1} \leq 3 \quad \Leftrightarrow \quad \sqrt{2 + a_n} \leq 3 \quad \Leftrightarrow$ $2 + a_n \leq 9 \quad \Leftrightarrow \quad a_n \leq 7$, which is certainly true because we are assuming that $a_n \leq 3$. So P_n is true for all n, and so $a_1 \leq a_n \leq 3$ (showing that the sequence is bounded), and hence by the Monotonic Sequence Theorem, $\lim\limits_{n\to\infty} a_n$ exists.

(b) If $L = \lim\limits_{n\to\infty} a_n$, then $\lim\limits_{n\to\infty} a_{n+1} = L$ also, so $L = \sqrt{2 + L} \quad \Rightarrow \quad L^2 = 2 + L \quad \Leftrightarrow \quad L^2 - L - 2 = 0 \quad \Leftrightarrow$ $(L+1)(L-2) = 0 \quad \Leftrightarrow \quad L = 2$ (since L can't be negative).

63. We show by induction that $\{a_n\}$ is increasing and bounded above by 3.

Let P_n be the proposition that $a_{n+1} > a_n$ and $0 < a_n < 3$. Clearly P_1 is true. Assume that P_n is true.

Then $a_{n+1} > a_n \quad \Rightarrow \quad \dfrac{1}{a_{n+1}} < \dfrac{1}{a_n} \quad \Rightarrow \quad -\dfrac{1}{a_{n+1}} > -\dfrac{1}{a_n}$.

Now $a_{n+2} = 3 - \dfrac{1}{a_{n+1}} > 3 - \dfrac{1}{a_n} = a_{n+1} \quad \Leftrightarrow \quad P_{n+1}$. This proves that $\{a_n\}$ is increasing and bounded above by 3, so $1 = a_1 < a_n < 3$, that is, $\{a_n\}$ is bounded, and hence convergent by the Monotonic Sequence Theorem. If $L = \lim\limits_{n\to\infty} a_n$, then $\lim\limits_{n\to\infty} a_{n+1} = L$ also, so L must satisfy $L = 3 - 1/L \quad \Rightarrow \quad L^2 - 3L + 1 = 0 \quad \Rightarrow$ $L = \frac{3 \pm \sqrt{5}}{2}$. But $L > 1$, so $L = \frac{3 + \sqrt{5}}{2}$.

64. We use induction. Let P_n be the statement that $0 < a_{n+1} \leq a_n \leq 2$. Clearly P_1 is true, since $a_2 = 1/(3-2) = 1$. Now assume that P_n is true. Then $a_{n+1} \leq a_n \quad \Rightarrow \quad -a_{n+1} \geq -a_n \quad \Rightarrow \quad 3 - a_{n+1} \geq 3 - a_n \quad \Rightarrow$ $a_{n+2} = \dfrac{1}{3 - a_{n+1}} \leq \dfrac{1}{3 - a_n} = a_{n+1}$. Also $a_{n+2} > 0$ (since $3 - a_{n+1}$ is positive) and $a_{n+1} \leq 2$ by the induction hypothesis, so P_{n+1} is true.

To find the limit, we use the fact that $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} a_{n+1} \quad \Rightarrow \quad L = \frac{1}{3-L} \quad \Rightarrow \quad L^2 - 3L + 1 = 0 \quad \Rightarrow$ $L = \frac{3 \pm \sqrt{5}}{2}$. But $L \leq 2$, so we must have $L = \frac{3 - \sqrt{5}}{2}$.

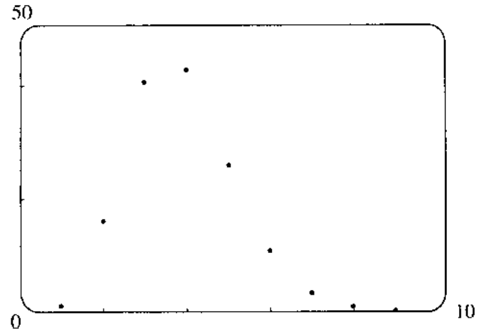
65. (a) Let a_n be the number of rabbit pairs in the nth month. Clearly $a_1 = 1 = a_2$. In the nth month, each pair that is 2 or more months old (that is, a_{n-2} pairs) will produce a new pair to add to the a_{n-1} pairs already present. Thus, $a_n = a_{n-1} + a_{n-2}$, so that $\{a_n\} = \{f_n\}$, the Fibonacci sequence.

(b) $a_n = \dfrac{f_{n+1}}{f_n} \quad \Rightarrow \quad a_{n-1} = \dfrac{f_n}{f_{n-1}} = \dfrac{f_{n-1} + f_{n-2}}{f_{n-1}} = 1 + \dfrac{f_{n-2}}{f_{n-1}} = 1 + \dfrac{1}{f_{n-1}/f_{n-2}} = 1 + \dfrac{1}{a_{n-2}}$. If $L = \lim\limits_{n\to\infty} a_n$, then $L = \lim\limits_{n\to\infty} a_{n-1}$ and $L = \lim\limits_{n\to\infty} a_{n-2}$, so L must satisfy $L = 1 + \dfrac{1}{L} \quad \Rightarrow$ $L^2 - L - 1 = 0 \quad \Rightarrow \quad L = \frac{1 + \sqrt{5}}{2}$ (since L must be positive).

66. (a) If f is continuous, then $f(L) = f\left(\lim\limits_{n\to\infty} a_n\right) = \lim\limits_{n\to\infty} f(a_n) = \lim\limits_{n\to\infty} a_{n+1} = L$ by Exercise 52(a).

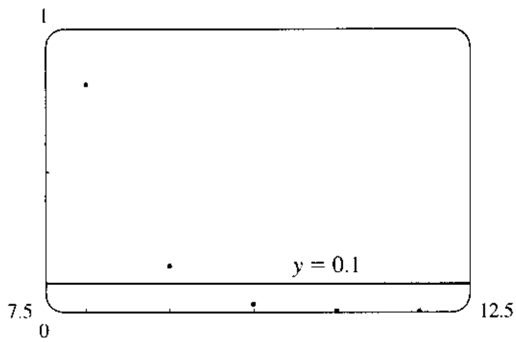
(b) By repeatedly pressing the cosine key on the calculator (that is, taking cosine of the previous answer) until the displayed value stabilizes, we see that $L \approx 0.73909$.

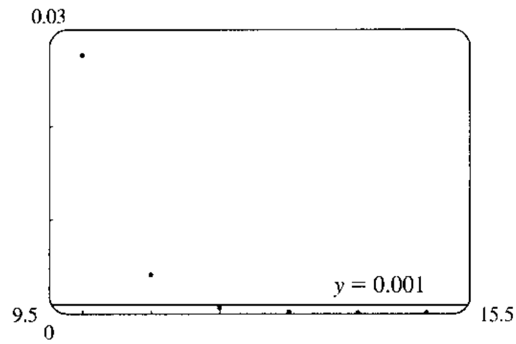
67. (a)



From the graph, it appears that the sequence $\left\{\frac{n^5}{n!}\right\}$ converges to 0, that is, $\lim\limits_{n\to\infty}\frac{n^5}{n!}=0$.

(b)





From the first graph, it seems that the smallest possible value of N corresponding to $\varepsilon = 0.1$ is 9, since $n^5/n! < 0.1$ whenever $n \geq 10$, but $9^5/9! > 0.1$. From the second graph, it seems that for $\varepsilon = 0.001$, the smallest possible value for N is 11.

68. Let $\varepsilon > 0$ and let N be any positive integer larger than $\ln(\varepsilon)/\ln|r|$. If $n > N$ then $n > \ln(\varepsilon)/\ln|r| \Rightarrow n\ln|r| < \ln\varepsilon$ [since $|r| < 1 \Rightarrow \ln|r| < 0$] $\Rightarrow \ln(|r|^n) < \ln\varepsilon \Rightarrow |r|^n < \varepsilon \Rightarrow |r^n - 0| < \varepsilon$, and so by Definition 1, $\lim\limits_{n\to\infty} r^n = 0$.

69. If $\lim\limits_{n\to\infty}|a_n| = 0$ then $\lim\limits_{n\to\infty} -|a_n| = 0$, and since $-|a_n| \leq a_n \leq |a_n|$, we have that $\lim\limits_{n\to\infty} a_n = 0$ by the Squeeze Theorem.

70. (a) $\dfrac{b^{n+1}-a^{n+1}}{b-a} = b^n + b^{n-1}a + b^{n-2}a^2 + b^{n-3}a^3 + \cdots + ba^{n-1} + a^n$

$< b^n + b^{n-1}b + b^{n-2}b^2 + b^{n-3}b^3 + \cdots + bb^{n-1} + b^n = (n+1)b^n$

(b) Since $b - a > 0$, we have $b^{n+1} - a^{n+1} < (n+1)b^n(b-a) \Rightarrow b^{n+1} - (n+1)b^n(b-a) < a^{n+1} \Rightarrow b^n[(n+1)a - nb] < a^{n+1}$.

(c) With this substitution, $(n+1)a - nb = 1$, and so $b^n = \left(1+\dfrac{1}{n}\right)^n < a^{n+1} = \left(1+\dfrac{1}{n+1}\right)^{n+1}$.

(d) With this substitution, we get $\left(1+\dfrac{1}{2n}\right)^n\left(\dfrac{1}{2}\right) < 1 \Rightarrow \left(1+\dfrac{1}{2n}\right)^n < 2 \Rightarrow \left(1+\dfrac{1}{2n}\right)^{2n} < 4$.

(e) $a_n < a_{2n}$ since $\{a_n\}$ is increasing, so $a_n < a_{2n} < 4$.

(f) Since $\{a_n\}$ is increasing and bounded above by 4, $a_1 \leq a_n \leq 4$, and so $\{a_n\}$ is bounded and monotonic, and hence has a limit by Theorem 11.

71. (a) First we show that $a > a_1 > b_1 > b$.

$a_1 - b_1 = \frac{a+b}{2} - \sqrt{ab} = \frac{1}{2}\left(a - 2\sqrt{ab} + b\right) = \frac{1}{2}\left(\sqrt{a} - \sqrt{b}\right)^2 > 0$ (since $a > b$) $\Rightarrow$ $a_1 > b_1$. Also

$a - a_1 = a - \frac{1}{2}(a+b) = \frac{1}{2}(a-b) > 0$ and $b - b_1 = b - \sqrt{ab} = \sqrt{b}\left(\sqrt{b} - \sqrt{a}\right) < 0$, so $a > a_1 > b_1 > b$.

In the same way we can show that $a_1 > a_2 > b_2 > b_1$ and so the given assertion is true for $n = 1$. Suppose it is true for $n = k$, that is, $a_k > a_{k+1} > b_{k+1} > b_k$. Then

$$a_{k+2} - b_{k+2} = \tfrac{1}{2}(a_{k+1} + b_{k+1}) - \sqrt{a_{k+1}b_{k+1}} = \tfrac{1}{2}\left(a_{k+1} - 2\sqrt{a_{k+1}b_{k+1}} + b_{k+1}\right)$$

$$= \tfrac{1}{2}\left(\sqrt{a_{k+1}} - \sqrt{b_{k+1}}\right)^2 > 0$$

$$a_{k+1} - a_{k+2} = a_{k+1} - \tfrac{1}{2}(a_{k+1} + b_{k+1}) = \tfrac{1}{2}(a_{k+1} - b_{k+1}) > 0$$

and $b_{k+1} - b_{k+2} = b_{k+1} - \sqrt{a_{k+1}b_{k+1}} = \sqrt{b_{k+1}}\left(\sqrt{b_{k+1}} - \sqrt{a_{k+1}}\right) < 0 \Rightarrow$

$a_{k+1} > a_{k+2} > b_{k+2} > b_{k+1}$, so the assertion is true for $n = k + 1$. Thus, it is true for all n by mathematical induction.

(b) From part (a) we have $a > a_n > a_{n+1} > b_{n+1} > b_n > b$, which shows that both sequences, $\{a_n\}$ and $\{b_n\}$, are monotonic and bounded. So they are both convergent by the Monotonic Sequence Theorem.

(c) Let $\lim\limits_{n\to\infty} a_n = \alpha$ and $\lim\limits_{n\to\infty} b_n = \beta$. Then $\lim\limits_{n\to\infty} a_{n+1} = \lim\limits_{n\to\infty} \dfrac{a_n + b_n}{2} \Rightarrow \alpha = \dfrac{\alpha + \beta}{2} \Rightarrow$

$2\alpha = \alpha + \beta \Rightarrow \alpha = \beta$.

72. (a) Let $\varepsilon > 0$. Since $\lim\limits_{n\to\infty} a_{2n} = L$, there exists N_1 such that $|a_{2n} - L| < \varepsilon$ for $n > N_1$. Since $\lim\limits_{n\to\infty} a_{2n+1} = L$, there exists N_2 such that $|a_{2n+1} - L| < \varepsilon$ for $n > N_2$. Let $N = \max\{2N_1, 2N_2 + 1\}$ and let $n > N$. If n is even, then $n = 2m$ where $m > N_1$, so $|a_n - L| = |a_{2m} - L| < \varepsilon$. If n is odd, then $n = 2m + 1$, where $m > N_2$, so $|a_n - L| = |a_{2m+1} - L| < \varepsilon$. Therefore $\lim\limits_{n\to\infty} a_n = L$.

(b) $a_1 = 1$, $a_2 = 1 + \frac{1}{1+1} = \frac{3}{2} = 1.5$, $a_3 = 1 + \frac{1}{5/2} = \frac{7}{5} = 1.4$, $a_4 = 1 + \frac{1}{12/5} = \frac{17}{12} = 1.41\overline{6}$,

$a_5 = 1 + \frac{1}{29/12} = \frac{41}{29} \approx 1.413793$, $a_6 = 1 + \frac{1}{70/29} = \frac{99}{70} \approx 1.414286$, $a_7 = 1 + \frac{1}{169/70} = \frac{239}{169} \approx 1.414201$,

$a_8 = 1 + \frac{1}{408/169} = \frac{577}{408} \approx 1.414216$. Notice that $a_1 < a_3 < a_5 < a_7$ and $a_2 > a_4 > a_6 > a_8$. It appears that the odd terms are increasing and the even terms are decreasing. Let's prove that $a_{2n-2} > a_{2n}$ and $a_{2n-1} < a_{2n+1}$ by mathematical induction. Suppose that $a_{2k-2} > a_{2k}$. Then $1 + a_{2k-2} > 1 + a_{2k} \Rightarrow$

$$\frac{1}{1 + a_{2k-2}} < \frac{1}{1 + a_{2k}} \Rightarrow 1 + \frac{1}{1 + a_{2k-2}} < 1 + \frac{1}{1 + a_{2k}} \Rightarrow a_{2k-1} < a_{2k+1} \Rightarrow$$

$$1 + a_{2k-1} < 1 + a_{2k+1} \Rightarrow \frac{1}{1 + a_{2k-1}} > \frac{1}{1 + a_{2k+1}} \Rightarrow 1 + \frac{1}{1 + a_{2k-1}} > 1 + \frac{1}{1 + a_{2k+1}} \Rightarrow$$

$a_{2k} > a_{2k+2}$. We have thus shown, by induction, that the odd terms are increasing and the even terms are decreasing. Also all terms lie between 1 and 2, so both $\{a_n\}$ and $\{b_n\}$ are bounded monotonic sequences and are therefore convergent by Theorem 11. Let $\lim\limits_{n\to\infty} a_{2n} = L$. Then $\lim\limits_{n\to\infty} a_{2n+2} = L$ also. We have

$a_{n+2} = 1 + \dfrac{1}{1+1+1/(1+a_n)} = 1 + \dfrac{1}{(3+2a_n)/(1+a_n)} = \dfrac{4+3a_n}{3+2a_n}$, so $a_{2n+2} = \dfrac{4+3a_{2n}}{3+2a_{2n}}$. Taking limits of both sides, we get $L = \dfrac{4+3L}{3+2L} \;\Rightarrow\; 3L + 2L^2 = 4 + 3L \;\Rightarrow\; L^2 = 2 \;\Rightarrow\; L = \sqrt{2}$ (since $L > 0$). Thus, $\lim\limits_{n\to\infty} a_{2n} = \sqrt{2}$. Similarly we find that $\lim\limits_{n\to\infty} a_{2n+1} = \sqrt{2}$. So, by part (a), $\lim\limits_{n\to\infty} a_n = \sqrt{2}$.

73. (a) Suppose $\{p_n\}$ converges to p. Then $p_{n+1} = \dfrac{bp_n}{a+p_n} \;\Rightarrow\; \lim\limits_{n\to\infty} p_{n+1} = \dfrac{b \lim\limits_{n\to\infty} p_n}{a + \lim\limits_{n\to\infty} p_n} \;\Rightarrow$

$$p = \frac{bp}{a+p} \quad\Rightarrow\quad p^2 + ap = bp \quad\Rightarrow\quad p(p+a-b) = 0 \quad\Rightarrow\quad p = 0 \text{ or } p = b-a.$$

(b) $p_{n+1} = \dfrac{bp_n}{a+p_n} = \dfrac{\frac{b}{a}p_n}{1+\frac{p_n}{a}} < \dfrac{b}{a}p_n$ since $1 + \dfrac{p_n}{a} > 1$.

(c) By part (b), $p_1 < \left(\dfrac{b}{a}\right)p_0$, $p_2 < \left(\dfrac{b}{a}\right)p_1 < \left(\dfrac{b}{a}\right)^2 p_0$, $p_3 < \left(\dfrac{b}{a}\right)p_2 < \left(\dfrac{b}{a}\right)^3 p_0$, etc. In general, $p_n < \left(\dfrac{b}{a}\right)^n p_0$, so $\lim\limits_{n\to\infty} p_n \le \lim\limits_{n\to\infty} \left(\dfrac{b}{a}\right)^n \cdot p_0 = 0$ since $b < a$. [By result 8, $\lim\limits_{n\to\infty} r^n = 0$ if $-1 < r < 1$. Here $r = \dfrac{b}{a} \in (0,1)$.]

(d) Let $a < b$. We first show, by induction, that if $p_0 < b - a$, then $p_n < b - a$ and $p_{n+1} > p_n$.

For $n = 0$, we have $p_1 - p_0 = \dfrac{bp_0}{a+p_0} - p_0 = \dfrac{p_0(b-a-p_0)}{a+p_0} > 0$ since $p_0 < b - a$. So $p_1 > p_0$.

Now we suppose the assertion is true for $n = k$, that is, $p_k < b - a$ and $p_{k+1} > p_k$. Then

$$b - a - p_{k+1} = b - a - \frac{bp_k}{a+p_k} = \frac{a(b-a) + bp_k - ap_k - bp_k}{a+p_k} = \frac{a(b-a-p_k)}{a+p_k} > 0 \text{ because } p_k < b-a.$$

So $p_{k+1} < b - a$. And $p_{k+2} - p_{k+1} = \dfrac{bp_{k+1}}{a+p_{k+1}} - p_{k+1} = \dfrac{p_{k+1}(b-a-p_{k+1})}{a+p_{k+1}} > 0$ since $p_{k+1} < b - a$.

Therefore, $p_{k+2} > p_{k+1}$. Thus, the assertion is true for $n = k+1$. It is therefore true for all n by mathematical induction. A similar proof by induction shows that if $p_0 > b - a$, then $p_n > b - a$ and $\{p_n\}$ is decreasing.

In either case the sequence $\{p_n\}$ is bounded and monotonic, so it is convergent by the Monotonic Sequence Theorem. It then follows from part (a) that $\lim\limits_{n\to\infty} p_n = b - a$.

LABORATORY PROJECT Logistic Sequences

1. To write such a program in Maple it is best to calculate all the points first and then graph them. One possible sequence of commands [taking $p_0 = \frac{1}{2}$ and $k = 1.5$ for the difference equation] is

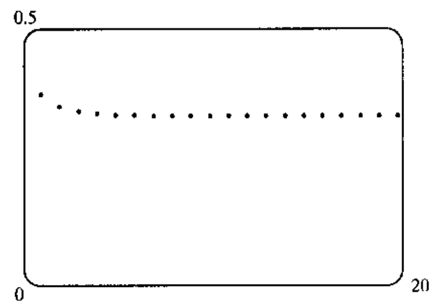
```
t:='t';p(0):=1/2;k:=1.5;
for j from 1 to 20  do p(j):=k*p(j-1)*(1-p(j-1)) od;
plot([seq([t,p(t)] t=0..20)],t=0..20,p=0..0.5,style=point);
```

In Mathematica, we can use the following program:

```
p[0]=1/2
k=1.5
p[j_]:=k*p[j-1]*(1-p[j-1])
P=Table[p[t],{t,20}]
ListPlot[P]
```

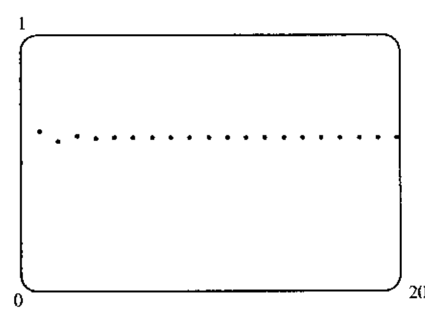
With $p_0 = \frac{1}{2}$ and $k = 1.5$:

n	p_n	n	p_n	n	p_n
0	0.5	7	0.3338465076	14	0.3333373303
1	0.375	8	0.3335895255	15	0.3333353318
2	0.3515625	9	0.3334613309	16	0.3333343326
3	0.3419494629	10	0.3333973076	17	0.3333338329
4	0.3375300416	11	0.3333653143	18	0.3333335831
5	0.3354052689	12	0.3333493223	19	0.3333334582
6	0.3343628617	13	0.3333413274	20	0.3333333958



With $p_0 = \frac{1}{2}$ and $k = 2.5$:

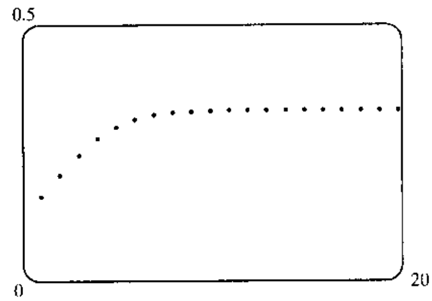
n	p_n	n	p_n	n	p_n
0	0.5	7	0.6004164790	14	0.5999967417
1	0.625	8	0.5997913269	15	0.6000016291
2	0.5859375	9	0.6001042277	16	0.5999991854
3	0.6065368651	10	0.5999478590	17	0.6000004073
4	0.5966247409	11	0.6000260637	18	0.5999997964
5	0.6016591486	12	0.5999869664	19	0.6000001018
6	0.5991635437	13	0.6000065164	20	0.5999999491



Both of these sequences seem to converge (the first to about $\frac{1}{3}$, the second to about 0.60).

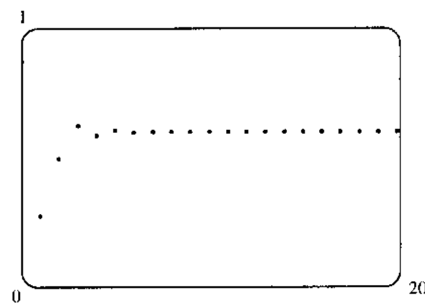
With $p_0 = \frac{7}{8}$ and $k = 1.5$:

n	p_n	n	p_n	n	p_n
0	0.875	7	0.3239166554	14	0.3332554829
1	0.1640625	8	0.3284919837	15	0.3332943990
2	0.2057189941	9	0.3308775005	16	0.3333138639
3	0.2450980344	10	0.3320963702	17	0.3333235980
4	0.2775374819	11	0.3327125567	18	0.3333284655
5	0.3007656421	12	0.3330223670	19	0.3333308994
6	0.3154585059	13	0.3331777051	20	0.3333321164



With $p_0 = \frac{7}{8}$ and $k = 2.5$:

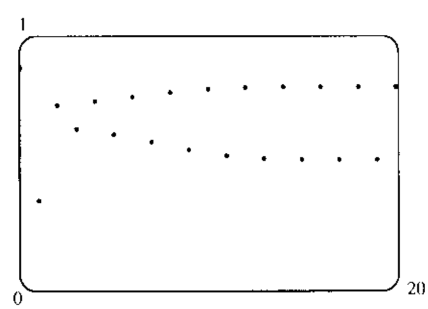
n	p_n	n	p_n	n	p_n
0	0.875	7	0.6016572368	14	0.5999869815
1	0.2734375	8	0.5991645155	15	0.6000065088
2	0.4966735840	9	0.6004159972	16	0.5999967455
3	0.6249723374	10	0.5997915688	17	0.6000016272
4	0.5859547872	11	0.6001041070	18	0.5999991864
5	0.6065294364	12	0.5999479194	19	0.6000004068
6	0.5966286980	13	0.6000260335	20	0.5999997966



The limit of the sequence seems to depend on k, but not on p_0.

2. With $p_0 = \frac{7}{8}$ and $k = 3.2$:

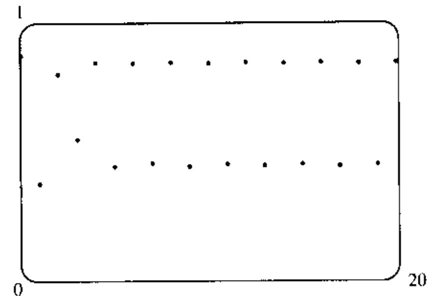
n	p_n	n	p_n	n	p_n
0	0.875	7	0.5830728495	14	0.7990633827
1	0.35	8	0.7779164854	15	0.5137954979
2	0.728	9	0.5528397669	16	0.7993909896
3	0.6336512	10	0.7910654689	17	0.5131681132
4	0.7428395416	11	0.5288988570	18	0.7994451225
5	0.6112926626	12	0.7973275394	19	0.5130643795
6	0.7603646184	13	0.5171082698	20	0.7994538304



It seems that eventually the terms fluctuate between two values (about 0.5 and 0.8 in this case).

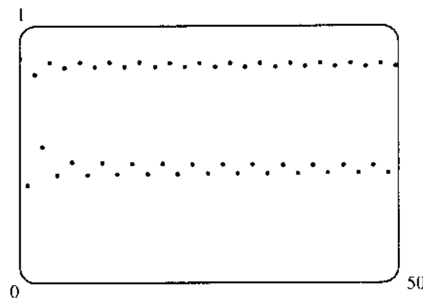
3. With $p_0 = \frac{7}{8}$ and $k = 3.42$:

n	p_n	n	p_n	n	p_n
0	0.875	7	0.4523028596	14	0.8442074951
1	0.3740625	8	0.8472194412	15	0.4498025048
2	0.8007579316	9	0.4426802161	16	0.8463823232
3	0.5456427596	10	0.8437633929	17	0.4446659586
4	0.8478752457	11	0.4508474156	18	0.8445284520
5	0.4411212220	12	0.8467373602	19	0.4490464985
6	0.8431438501	13	0.4438243545	20	0.8461207931



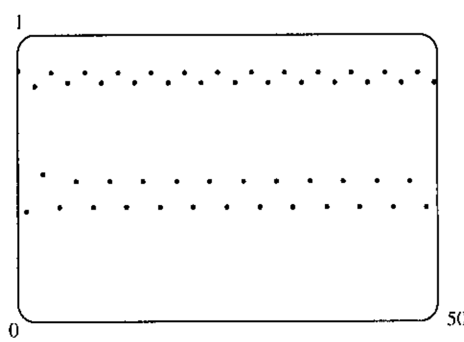
With $p_0 = \frac{7}{8}$ and $k = 3.45$:

n	p_n	n	p_n	n	p_n
0	0.875	7	0.4670259170	14	0.8403376122
1	0.37734375	8	0.8587488490	15	0.4628875685
2	0.8105962830	9	0.4184824586	16	0.8577482026
3	0.5296783241	10	0.8395743720	17	0.4209559716
4	0.8594612299	11	0.4646778983	18	0.8409445432
5	0.4167173034	12	0.8581956045	19	0.4614610237
6	0.8385707740	13	0.4198508858	20	0.8573758782

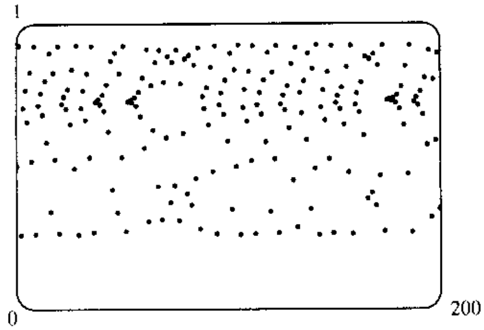


From the graphs above, it seems that for k between 3.4 and 3.5, the terms eventually fluctuate between four values. In the graph below, the pattern followed by the terms is $0.395, 0.832, 0.487, 0.869, 0.395, \ldots$. Note that even for $k = 3.42$ (as in the first graph), there are four distinct "branches; even after 1000 terms, the first and third terms in the pattern differ by about 2×10^{-9}, while the first and fifth terms differ by only 2×10^{-10}.

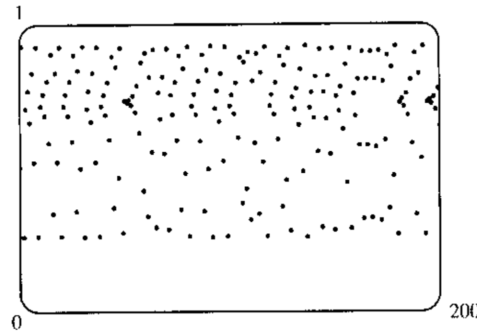
With $p_0 = \frac{7}{8}$ and $k = 3.48$:



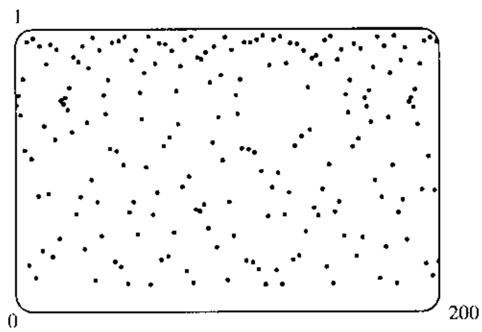
4.



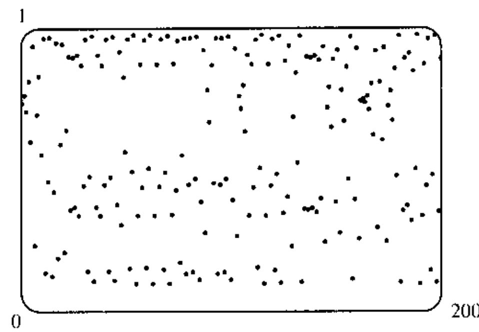
$p_0 = 0.5, k = 3.7$



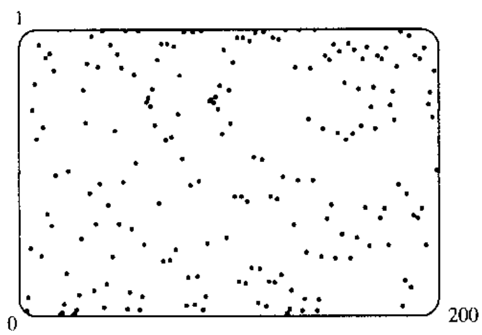
$p_0 = 0.501, k = 3.7$



$p_0 = 0.75, k = 3.9$



$p_0 = 0.749, k = 3.9$



$p_0 = 0.5, k = 3.999$

From the graphs, it seems that if p_0 is changed by 0.001, the whole graph changes completely. (Note, however, that this might be partially due to accumulated round-off error in the CAS. These graphs were generated by Maple with 100-digit accuracy, and different degrees of accuracy give different graphs.) There seem to be some some fleeting patterns in these graphs, but on the whole they are certainly very chaotic. As k increases, the graph spreads out vertically, with more extreme values close to 0 or 1.

12.2 Series

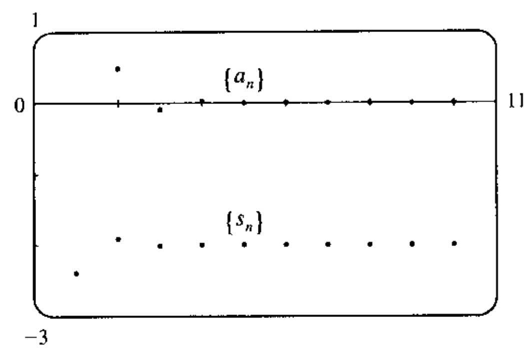
1. (a) A sequence is an ordered list of numbers whereas a series is the *sum* of a list of numbers.

(b) A series is convergent if the sequence of partial sums is a convergent sequence. A series is divergent if it is not convergent.

2. $\sum_{n=1}^{\infty} a_n = 5$ means that by adding sufficiently many terms of the series we can get as close as we like to the number 5. In other words, it means that $\lim_{n\to\infty} s_n = 5$, where s_n is the nth partial sum, that is, $\sum_{i=1}^{n} a_i$.

3.

n	s_n
1	−2.40000
2	−1.92000
3	−2.01600
4	−1.99680
5	−2.00064
6	−1.99987
7	−2.00003
8	−1.99999
9	−2.00000
10	−2.00000



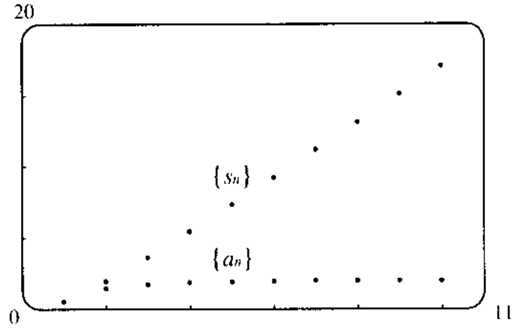
From the graph and the table, it seems that the series converges to −2. In fact, it is a geometric series with $a = -2.4$ and $r = -\frac{1}{5}$, so its sum is

$$\sum_{n=1}^{\infty} \frac{12}{(-5)^n} = \frac{-2.4}{1-\left(-\frac{1}{5}\right)} = \frac{-2.4}{1.2} = -2.$$ Note that the dot corresponding to $n = 1$ is part of both $\{a_n\}$ and $\{s_n\}$.

TI-86 Note: To graph $\{a_n\}$ and $\{s_n\}$, set your calculator to Param mode and DrawDot mode. (DrawDot is under GRAPH, MORE, FORMT (F3).) Now under `E(t)=` make the assignments: `xt1=t, yt1=12/(-5)^t, xt2=t, yt2=sum seq(yt1,t,1,t,1)`. (sum and seq are under LIST, OPS (F5), MORE.) Under WIND use `1,10,1,0,10,1,-3,1,1` to obtain a graph similar to the one above. Then use TRACE (F4) to see the values.

4.

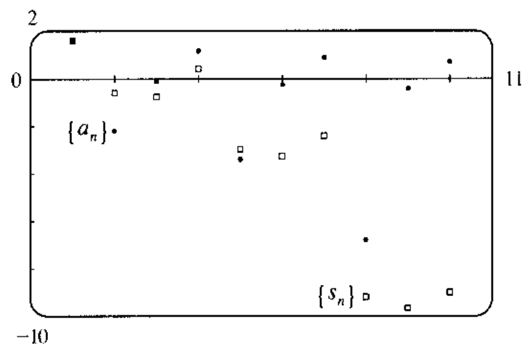
n	s_n
1	0.50000
2	1.90000
3	3.60000
4	5.42353
5	7.30814
6	9.22706
7	11.16706
8	13.12091
9	15.08432
10	17.05462



The series $\sum_{n=1}^{\infty} \frac{2n^2-1}{n^2+1}$ diverges, since its terms do not approach 0.

5.

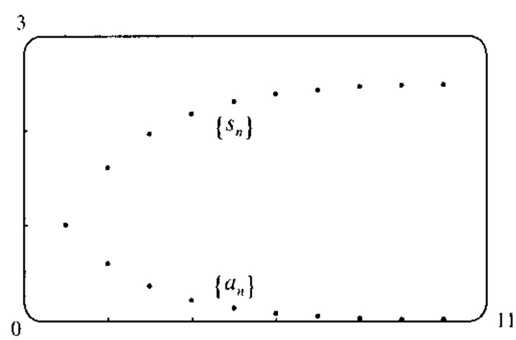
n	s_n
1	1.55741
2	−0.62763
3	−0.77018
4	0.38764
5	−2.99287
6	−3.28388
7	−2.41243
8	−9.21214
9	−9.66446
10	−9.01610



The series $\sum_{n=1}^{\infty} \tan n$ diverges, since its terms do not approach 0.

6.

n	s_n
1	1.00000
2	1.60000
3	1.96000
4	2.17600
5	2.30560
6	2.38336
7	2.43002
8	2.45801
9	2.47481
10	2.48488

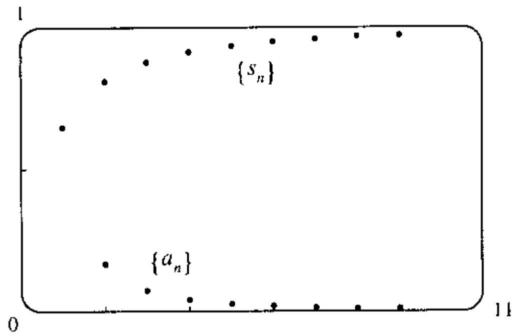


From the graph and the table, it seems that the series converges to 2.5. In fact, it is a geometric series with $a = 1$ and $r = 0.6$, so its sum is

$$\sum_{n=1}^{\infty} (0.6)^{n-1} = \frac{1}{1-0.6} = \frac{1}{2/5} = 2.5.$$

7.

n	s_n
1	0.64645
2	0.80755
3	0.87500
4	0.91056
5	0.93196
6	0.94601
7	0.95581
8	0.96296
9	0.96838
10	0.97259



From the graph, it seems that the series converges to 1. To find the sum, we write

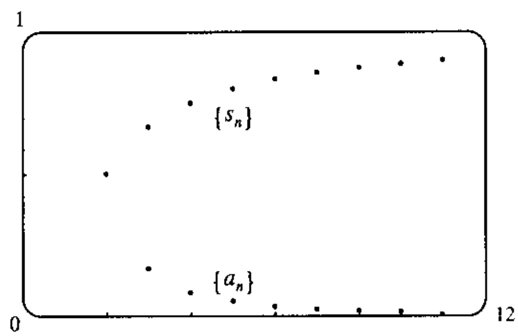
$$s_n = \sum_{i=1}^{n} \left(\frac{1}{i^{1.5}} - \frac{1}{(i+1)^{1.5}} \right)$$

$$= \left(1 - \frac{1}{2^{1.5}}\right) + \left(\frac{1}{2^{1.5}} - \frac{1}{3^{1.5}}\right) + \left(\frac{1}{3^{1.5}} - \frac{1}{4^{1.5}}\right) + \cdots + \left(\frac{1}{n^{1.5}} - \frac{1}{(n+1)^{1.5}}\right) = 1 - \frac{1}{(n+1)^{1.5}}$$

So the sum is $\lim_{n\to\infty} s_n = 1 - 0 = 1$.

8.

n	s_n
2	0.50000
3	0.66667
4	0.75000
5	0.80000
6	0.83333
7	0.85714
8	0.87500
9	0.88889
10	0.90000
11	0.90909
100	0.99000



From the graph and the table, it seems that the series converges to 1. To find the sum, we write

$$s_n = \sum_{i=2}^{n} \frac{1}{i(i-1)} = \sum_{i=2}^{n} \left(\frac{1}{i-1} - \frac{1}{i} \right) \qquad \text{[partial fractions]}$$

$$= \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{n-1} - \frac{1}{n}\right) = 1 - \frac{1}{n},$$

and so the sum is $\lim_{n\to\infty} s_n = 1 - 0 = 1$.

9. (a) $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \dfrac{2n}{3n+1} = \dfrac{2}{3}$, so the *sequence* $\{a_n\}$ is convergent by (12.1.1).

(b) Since $\lim_{n\to\infty} a_n = \frac{2}{3} \neq 0$, the *series* $\sum_{n=1}^{\infty} a_n$ is divergent by the Test for Divergence (7).

10. (a) Both $\sum_{i=1}^{n} a_i$ and $\sum_{j=1}^{n} a_j$ represent the sum of the first n terms of the sequence $\{a_n\}$, that is, the nth partial sum.

(b) $\sum_{i=1}^{n} a_j = \underbrace{a_j + a_j + \cdots + a_j}_{n \text{ terms}} = na_j$, which, in general, is not the same as $\sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n$.

11. $3 + 2 + \frac{4}{3} + \frac{8}{9} + \cdots$ is a geometric series with first term $a = 3$ and common ratio $r = \frac{2}{3}$. Since $|r| = \frac{2}{3} < 1$, the series converges to $\frac{a}{1-r} = \frac{3}{1-2/3} = \frac{3}{1/3} = 9$.

12. $\frac{1}{8} - \frac{1}{4} + \frac{1}{2} - 1 + \cdots$ is a geometric series with $r = -2$. Since $|r| = 2 > 1$, the series diverges.

13. $-2 + \frac{5}{2} - \frac{25}{8} + \frac{125}{32} - \cdots$ is a geometric series with $a = -2$ and $r = \frac{5/2}{-2} = -\frac{5}{4}$. Since $|r| = \frac{5}{4} > 1$, the series diverges by (4).

14. $1 + 0.4 + 0.16 + 0.064 + \cdots$ is a geometric series with ratio 0.4. The series converges to $\frac{a}{1-r} = \frac{1}{1-2/5} = \frac{5}{3}$ since $|r| = \frac{2}{5} < 1$.

15. $\sum_{n=1}^{\infty} 5\left(\frac{2}{3}\right)^{n-1}$ is a geometric series with $a = 5$ and $r = \frac{2}{3}$. Since $|r| = \frac{2}{3} < 1$, the series converges to $\frac{a}{1-r} = \frac{5}{1-2/3} = \frac{5}{1/3} = 15$.

16. $\sum_{n=1}^{\infty} \dfrac{(-6)^{n-1}}{5^{n-1}}$ is a geometric series with $a = 1$ and $r = -\frac{6}{5}$. The series diverges since $|r| = \frac{6}{5} > 1$.

17. $\sum_{n=1}^{\infty} \dfrac{(-3)^{n-1}}{4^n} = \dfrac{1}{4} \sum_{n=1}^{\infty} \left(-\dfrac{3}{4}\right)^{n-1}$. The latter series is geometric with $a = 1$ and $r = -\frac{3}{4}$. Since $|r| = \frac{3}{4} < 1$, it converges to $\frac{1}{1-(-3/4)} = \frac{4}{7}$. Thus, the given series converges to $\left(\frac{1}{4}\right)\left(\frac{4}{7}\right) = \frac{1}{7}$.

18. $\sum_{n=0}^{\infty} \dfrac{1}{(\sqrt{2})^n}$ is a geometric series with ratio $r = \dfrac{1}{\sqrt{2}}$. Since $|r| = \dfrac{1}{\sqrt{2}} < 1$, the series converges. Its sum is

$$\frac{1}{1 - 1/\sqrt{2}} = \frac{\sqrt{2}}{\sqrt{2}-1} = \frac{\sqrt{2}}{\sqrt{2}-1} \cdot \frac{\sqrt{2}+1}{\sqrt{2}+1} = \sqrt{2}(\sqrt{2}+1) = 2 + \sqrt{2}.$$

19. $\sum_{n=0}^{\infty} \dfrac{\pi^n}{3^{n+1}} = \dfrac{1}{3} \sum_{n=0}^{\infty} \left(\dfrac{\pi}{3}\right)^n$ is a geometric series with ratio $r = \dfrac{\pi}{3}$. Since $|r| > 1$, the series diverges.

20. $\sum_{n=1}^{\infty} \dfrac{e^n}{3^{n-1}} = 3 \sum_{n=1}^{\infty} \left(\dfrac{e}{3}\right)^n$ is a geometric series with first term $3(e/3) = e$ and ratio $r = \dfrac{e}{3}$. Since $|r| < 1$, the series converges. Its sum is $\dfrac{e}{1 - e/3} = \dfrac{3e}{3-e}$.

21. $\sum_{n=1}^{\infty} \dfrac{n}{n+5}$ diverges since $\lim_{n\to\infty} a_n = \lim_{n\to\infty} \dfrac{n}{n+5} = 1 \neq 0$. [Use (7), the Test for Divergence.]

22. $\sum_{n=1}^{\infty} \frac{3}{n} = 3 \sum_{n=1}^{\infty} \frac{1}{n}$ diverges since each of its partial sums is 3 times the corresponding partial sum of the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$, which diverges. [If $\sum_{n=1}^{\infty} \frac{3}{n}$ were to converge, then $\sum_{n=1}^{\infty} \frac{1}{n}$ would also have to converge by Theorem 8(i).] In general, constant multiples of divergent series are divergent.

23. Using partial fractions, the partial sums are

$$s_n = \sum_{i=2}^{n} \frac{2}{(i-1)(i+1)} = \sum_{i=2}^{n} \left(\frac{1}{i-1} - \frac{1}{i+1} \right)$$
$$= \left(1 - \frac{1}{3}\right) + \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \cdots + \left(\frac{1}{n-3} - \frac{1}{n-1}\right) + \left(\frac{1}{n-2} - \frac{1}{n}\right)$$

This sum is a telescoping series and $s_n = 1 + \frac{1}{2} - \frac{1}{n-1} - \frac{1}{n}$.

Thus, $\sum_{n=2}^{\infty} \frac{2}{n^2 - 1} = \lim_{n\to\infty} \left(1 + \frac{1}{2} - \frac{1}{n-1} - \frac{1}{n}\right) = \frac{3}{2}$.

24. $\sum_{n=1}^{\infty} \frac{(n+1)^2}{n(n+2)}$ diverges by (7), the Test for Divergence, since

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{n^2 + 2n + 1}{n^2 + 2n} = \lim_{n\to\infty} \left(1 + \frac{1}{n^2 + 2n}\right) = 1 \neq 0$.

25. $\sum_{k=2}^{\infty} \frac{k^2}{k^2 - 1}$ diverges by the Test for Divergence since $\lim_{k\to\infty} a_k = \lim_{k\to\infty} \frac{k^2}{k^2 - 1} = 1 \neq 0$.

26. Converges. $s_n = \sum_{i=1}^{n} \frac{2}{i^2 + 4i + 3} = \sum_{i=1}^{n} \left(\frac{1}{i+1} - \frac{1}{i+3}\right)$ (using partial fractions). The latter sum is

$\left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \left(\frac{1}{4} - \frac{1}{6}\right) + \left(\frac{1}{5} - \frac{1}{7}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+2}\right) + \left(\frac{1}{n+1} - \frac{1}{n+3}\right) = \frac{1}{2} + \frac{1}{3} - \frac{1}{n+2} - \frac{1}{n+3}$

(telescoping series). Thus, $\sum_{n=1}^{\infty} \frac{2}{n^2 + 4n + 3} = \lim_{n\to\infty} \left(\frac{1}{2} + \frac{1}{3} - \frac{1}{n+2} - \frac{1}{n+3}\right) = \frac{1}{2} + \frac{1}{3} = \frac{5}{6}$.

27. Converges. $\sum_{n=1}^{\infty} \frac{3^n + 2^n}{6^n} = \sum_{n=1}^{\infty} \left(\frac{3^n}{6^n} + \frac{2^n}{6^n}\right) = \sum_{n=1}^{\infty} \left[\left(\frac{1}{2}\right)^n + \left(\frac{1}{3}\right)^n\right] = \frac{1/2}{1 - 1/2} + \frac{1/3}{1 - 1/3} = 1 + \frac{1}{2} = \frac{3}{2}$

28. $\sum_{n=1}^{\infty} \left[(0.8)^{n-1} - (0.3)^n\right] = \sum_{n=1}^{\infty} (0.8)^{n-1} - \sum_{n=1}^{\infty} (0.3)^n$ [difference of two convergent geometric series]

$= \frac{1}{1 - 0.8} - \frac{0.3}{1 - 0.3} = 5 - \frac{3}{7} = \frac{32}{7}$.

29. $\sum_{n=1}^{\infty} \sqrt[n]{2} = 2 + \sqrt{2} + \sqrt[3]{2} + \sqrt[4]{2} + \cdots$ diverges by the Test for Divergence since

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \sqrt[n]{2} = \lim_{n\to\infty} 2^{1/n} = 2^0 = 1 \neq 0$.

30. $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \ln\left(\dfrac{n}{2n+5}\right) = \lim\limits_{n\to\infty} \ln\left(\dfrac{1}{2+5/n}\right) = \ln\frac{1}{2} \neq 0$, so the series diverges by the Test for Divergence.

31. $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \arctan n = \frac{\pi}{2} \neq 0$, so the series diverges by the Test for Divergence.

32. $\sum\limits_{k=1}^{\infty} (\cos 1)^k$ is a geometric series with ratio $r = \cos 1 \approx 0.540302$. It converges because $|r| < 1$. Its sum is $\dfrac{\cos 1}{1 - \cos 1} \approx 1.175343$.

33. The first series is a telescoping sum:

$$\begin{aligned}\sum_{n=1}^{\infty} \frac{3}{n(n+3)} &= \sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+3}\right) = \sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+1} + \frac{1}{n+1} - \frac{1}{n+2} + \frac{1}{n+2} - \frac{1}{n+3}\right)\\ &= \sum_{n=1}^{\infty}\left(\frac{1}{n} - \frac{1}{n+1}\right) + \sum_{n=1}^{\infty}\left(\frac{1}{n+1} - \frac{1}{n+2}\right) + \sum_{n=1}^{\infty}\left(\frac{1}{n+2} - \frac{1}{n+3}\right)\\ &= 1 + \frac{1}{2} + \frac{1}{3} = \frac{11}{6}\end{aligned}$$

The second series is geometric with first term $\frac{5}{4}$ and ratio $\frac{1}{4}$: $\sum\limits_{n=1}^{\infty} \dfrac{5}{4^n} = \dfrac{5/4}{1 - 1/4} = \dfrac{5}{3}$. Thus,

$$\sum_{n=1}^{\infty}\left(\frac{3}{n(n+3)} + \frac{5}{4^n}\right) = \sum_{n=1}^{\infty} \frac{3}{n(n+3)} + \sum_{n=1}^{\infty} \frac{5}{4^n} \text{ [sum of two convergent series]} = \frac{11}{6} + \frac{5}{3} = \frac{7}{2}.$$

34. $\sum\limits_{n=1}^{\infty}\left(\dfrac{3}{5^n} + \dfrac{2}{n}\right)$ diverges because $\sum\limits_{n=1}^{\infty} \dfrac{2}{n} = 2\sum\limits_{n=1}^{\infty} \dfrac{1}{n}$ diverges. (If it converged, then $\dfrac{1}{2} \cdot 2\sum\limits_{n=1}^{\infty} \dfrac{1}{n}$ would also converge by Theorem 8(i), but we know from Example 7 that the harmonic series $\sum\limits_{n=1}^{\infty} \dfrac{1}{n}$ diverges.) If the given series converges, then the difference $\sum\limits_{n=1}^{\infty}\left(\dfrac{3}{5^n} + \dfrac{2}{n}\right) - \sum\limits_{n=1}^{\infty} \dfrac{3}{5^n}$ must converge (since $\sum\limits_{n=1}^{\infty} \dfrac{3}{5^n}$ is a convergent geometric series) and equal $\sum\limits_{n=1}^{\infty} \dfrac{2}{n}$, but we have just seen that $\sum\limits_{n=1}^{\infty} \dfrac{2}{n}$ diverges, so the given series must also diverge.

35. $0.\overline{2} = \dfrac{2}{10} + \dfrac{2}{10^2} + \cdots$ is a geometric series with $a = \dfrac{2}{10}$ and $r = \dfrac{1}{10}$. It converges to $\dfrac{a}{1-r} = \dfrac{2/10}{1 - 1/10} = \dfrac{2}{9}$.

36. $0.\overline{73} = \dfrac{73}{10^2} + \dfrac{73}{10^4} + \cdots = \dfrac{73/10^2}{1 - 1/10^2} = \dfrac{73/100}{99/100} = \dfrac{73}{99}$

37. $3.\overline{417} = 3 + \dfrac{417}{10^3} + \dfrac{417}{10^6} + \cdots = 3 + \dfrac{417/10^3}{1 - 1/10^3} = 3 + \dfrac{417}{999} = \dfrac{3414}{999} = \dfrac{1138}{333}$

38. $6.2\overline{54} = 6.2 + \dfrac{54}{10^3} + \dfrac{54}{10^5} + \cdots = 6.2 + \dfrac{54/10^3}{1 - 1/10^2} = \dfrac{62}{10} + \dfrac{54}{990} = \dfrac{6192}{990} = \dfrac{344}{55}$

39. $0.123\overline{456} = \dfrac{123}{1000} + \dfrac{0.000456}{1-0.001} = \dfrac{123}{1000} + \dfrac{456}{999{,}000} = \dfrac{123{,}333}{999{,}000} = \dfrac{41{,}111}{333{,}000}$

40. $5.\overline{6021} = 5 + \dfrac{6021}{10^4} + \dfrac{6021}{10^8} + \cdots = 5 + \dfrac{6021/10^4}{1-1/10^4} = 5 + \dfrac{6021}{9999} = \dfrac{56{,}016}{9999} = \dfrac{6224}{1111}$

41. $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{3^n} = \sum_{n=1}^{\infty} \left(\frac{x}{3}\right)^n$ is a geometric series with $r = \dfrac{x}{3}$, so the series converges $\Leftrightarrow$ $|r| < 1$ $\Leftrightarrow$ $\dfrac{|x|}{3} < 1$ $\Leftrightarrow$ $|x| < 3$; that is, $-3 < x < 3$. In that case, the sum of the series is $\dfrac{a}{1-r} = \dfrac{x/3}{1-x/3} = \dfrac{x/3}{1-x/3} \cdot \dfrac{3}{3} = \dfrac{x}{3-x}$.

42. $\sum_{n=1}^{\infty} (x-4)^n$ is a geometric series with $r = x-4$, so the series converges $\Leftrightarrow$ $|r| < 1$ $\Leftrightarrow$ $|x-4| < 1$ $\Leftrightarrow$ $3 < x < 5$. In that case, the sum of the series is $\dfrac{x-4}{1-(x-4)} = \dfrac{x-4}{5-x}$.

43. $\sum_{n=0}^{\infty} 4^n x^n = \sum_{n=0}^{\infty} (4x)^n$ is a geometric series with $r = 4x$, so the series converges $\Leftrightarrow$ $|r| < 1$ $\Leftrightarrow$ $4\,|x| < 1$ $\Leftrightarrow$ $|x| < \frac{1}{4}$. In that case, the sum of the series is $\dfrac{1}{1-4x}$.

44. $\displaystyle\sum_{n=0}^{\infty} \frac{(x+3)^n}{2^n}$ is a geometric series with $r = \dfrac{x+3}{2}$, so the series converges $\Leftrightarrow$ $|r| < 1$ $\Leftrightarrow$ $\dfrac{|x+3|}{2} < 1$ $\Leftrightarrow$ $|x+3| < 2$ $\Leftrightarrow$ $-5 < x < -1$. For these values of x, the sum of the series is

$$\frac{1}{1-(x+3)/2} = \frac{2}{2-(x+3)} = -\frac{2}{x+1}.$$

45. $\displaystyle\sum_{n=0}^{\infty} \frac{\cos^n x}{2^n}$ is a geometric series with first term 1 and ratio $r = \dfrac{\cos x}{2}$, so it converges $\Leftrightarrow$ $|r| < 1$. But $|r| = \dfrac{|\cos x|}{2} \leq \dfrac{1}{2}$ for all x. Thus, the series converges for all real values of x and the sum of the series is

$$\frac{1}{1-(\cos x)/2} = \frac{2}{2-\cos x}.$$

46. Because $\dfrac{1}{n} \to 0$ and ln is continuous, we have $\displaystyle\lim_{n\to\infty} \ln\left(1+\frac{1}{n}\right) = \ln 1 = 0$. We now show that the series $\displaystyle\sum_{n=1}^{\infty} \ln\left(1+\frac{1}{n}\right) = \sum_{n=1}^{\infty} \ln\left(\frac{n+1}{n}\right) = \sum_{n=1}^{\infty} [\ln(n+1) - \ln n]$ diverges.

$s_n = (\ln 2 - \ln 1) + (\ln 3 - \ln 2) + \cdots + (\ln(n+1) - \ln n) = \ln(n+1) - \ln 1 = \ln(n+1)$. As $n \to \infty$, $s_n = \ln(n+1) \to \infty$, so the series diverges.

47. After defining f, We use `convert(f,parfrac);` in Maple, `Apart` in Mathematica, or `Expand Rational` and `Simplify` in Derive to find that the general term is $\dfrac{1}{(4n+1)(4n-3)} = -\dfrac{1/4}{4n+1} + \dfrac{1/4}{4n-3}$. So the

nth partial sum is

$$s_n = \sum_{k=1}^{n}\left(-\frac{1/4}{4k+1}+\frac{1/4}{4k-3}\right) = \frac{1}{4}\sum_{k=1}^{n}\left(\frac{1}{4k-3}-\frac{1}{4k+1}\right)$$

$$= \frac{1}{4}\left[\left(1-\frac{1}{5}\right)+\left(\frac{1}{5}-\frac{1}{9}\right)+\left(\frac{1}{9}-\frac{1}{13}\right)+\cdots+\left(\frac{1}{4n-3}-\frac{1}{4n+1}\right)\right] = \frac{1}{4}\left(1-\frac{1}{4n+1}\right)$$

The series converges to $\lim\limits_{n\to\infty} s_n = \frac{1}{4}$. This can be confirmed by directly computing the sum using `sum(f,1..infinity);` (in Maple), `Sum[f,{n,1,Infinity}]` (in Mathematica), or `Calculus Sum` (from 1 to ∞) and `Simplify` (in Derive).

48. See Exercise 47 for specific CAS commands. $\dfrac{n^2+3n+1}{(n^2+n)^2} = \dfrac{1}{n^2}+\dfrac{1}{n}-\dfrac{1}{(n+1)^2}-\dfrac{1}{n+1}$. So the nth partial sum is

$$s_n = \sum_{k=1}^{n}\left(\frac{1}{k^2}+\frac{1}{k}-\frac{1}{(k+1)^2}-\frac{1}{k+1}\right)$$

$$= \left(1+1-\frac{1}{2^2}-\frac{1}{2}\right)+\left(\frac{1}{2^2}+\frac{1}{2}-\frac{1}{3^2}-\frac{1}{3}\right)+\cdots+\left(\frac{1}{n^2}+\frac{1}{n}-\frac{1}{(n+1)^2}-\frac{1}{n+1}\right)$$

$$= 1+1-\frac{1}{(n+1)^2}-\frac{1}{n+1}$$

The series converges to $\lim\limits_{n\to\infty} s_n = 2$.

49. For $n=1$, $a_1 = 0$ since $s_1 = 0$. For $n>1$,

$$a_n = s_n - s_{n-1} = \frac{n-1}{n+1} - \frac{(n-1)-1}{(n-1)+1} = \frac{(n-1)n-(n+1)(n-2)}{(n+1)n} = \frac{2}{n(n+1)}$$

Also, $\sum\limits_{n=1}^{\infty} a_n = \lim\limits_{n\to\infty} s_n = \lim\limits_{n\to\infty}\dfrac{1-1/n}{1+1/n} = 1$.

50. $a_1 = s_1 = 3-\frac{1}{2} = \frac{5}{2}$. For $n \neq 1$,

$$a_n = s_n - s_{n-1} = \left(3-n2^{-n}\right)-\left[3-(n-1)2^{-(n-1)}\right] = -\frac{n}{2^n}+\frac{n-1}{2^{n-1}}\cdot\frac{2}{2} = \frac{2(n-1)}{2^n}-\frac{n}{2^n} = \frac{n-2}{2^n}$$

Also, $\sum\limits_{n=1}^{\infty} a_n = \lim\limits_{n\to\infty} s_n = \lim\limits_{n\to\infty}\left(3-\dfrac{n}{2^n}\right) = 3$ because $\lim\limits_{x\to\infty}\dfrac{x}{2^x} \overset{H}{=} \lim\limits_{x\to\infty}\dfrac{1}{2^x\ln 2} = 0$.

51. (a) The first step in the chain occurs when the local government spends D dollars. The people who receive it spend a fraction c of those D dollars, that is, Dc dollars. Those who receive the Dc dollars spend a fraction c of it, that is, Dc^2 dollars. Continuing in this way, we see that the total spending after n transactions is

$S_n = D + Dc + Dc^2 + \cdots + Dc^{n-1} = \dfrac{D(1-c^n)}{1-c}$ by (3).

(b) $\lim\limits_{n\to\infty} S_n = \lim\limits_{n\to\infty}\dfrac{D(1-c^n)}{1-c} = \dfrac{D}{1-c}\lim\limits_{n\to\infty}(1-c^n) = \dfrac{D}{1-c}$ (since $0<c<1 \Rightarrow \lim\limits_{n\to\infty} c^n = 0$)

$= \dfrac{D}{s}$ (since $c+s=1$) $= kD$ (since $k = 1/s$)

If $c = 0.8$, then $s = 1-c = 0.2$ and the multiplier is $k = 1/s = 5$.

52. (a) Initially, the ball falls a distance H, then rebounds a distance rH, falls rH, rebounds r^2H, falls r^2H, etc. The total distance it travels is

$$H + 2rH + 2r^2H + 2r^3H + \cdots = H\left(1 + 2r + 2r^2 + 2r^3 + \cdots\right)$$
$$= H\left[1 + 2r\left(1 + r + r^2 + \cdots\right)\right] = H\left[1 + 2r\left(\frac{1}{1-r}\right)\right] = H\left(\frac{1+r}{1-r}\right) \text{ meters}$$

(b) From Example 3 in Section 2.1, we know that a ball falls $\frac{1}{2}gt^2$ meters in t seconds, where g is the gravitational acceleration. Thus, a ball falls h meters in $t = \sqrt{2h/g}$ seconds. The total travel time in seconds is

$$\sqrt{\frac{2H}{g}} + 2\sqrt{\frac{2H}{g}r} + 2\sqrt{\frac{2H}{g}r^2} + 2\sqrt{\frac{2H}{g}r^3} + \cdots = \sqrt{\frac{2H}{g}}\left[1 + 2\sqrt{r} + 2\sqrt{r}^2 + 2\sqrt{r}^3 + \cdots\right]$$
$$= \sqrt{\frac{2H}{g}}\left(1 + 2\sqrt{r}\left[1 + \sqrt{r} + \sqrt{r}^2 + \cdots\right]\right) = \sqrt{\frac{2H}{g}}\left[1 + 2\sqrt{r}\left(\frac{1}{1-\sqrt{r}}\right)\right] = \sqrt{\frac{2H}{g}}\,\frac{1+\sqrt{r}}{1-\sqrt{r}}$$

(c) It will help to make a chart of the time for each descent and each rebound of the ball, together with the velocity just before and just after each bounce. Recall that the time in seconds needed to fall h meters is $\sqrt{2h/g}$. The ball hits the ground with velocity $-g\sqrt{2h/g} = -\sqrt{2hg}$ (taking the upward direction to be positive) and rebounds with velocity $kg\sqrt{2h/g} = k\sqrt{2hg}$, taking time $k\sqrt{2h/g}$ to reach the top of its bounce, where its velocity is 0. At that point, its height is k^2h. All these results follow from the formulas for vertical motion with gravitational acceleration $-g$: $\dfrac{d^2y}{dt^2} = -g \;\Rightarrow\; v = \dfrac{dy}{dt} = v_0 - gt \;\Rightarrow\; y = y_0 + v_0t - \frac{1}{2}gt^2$.

number of descent	time of descent	speed before bounce	speed after bounce	time of ascent	peak height
1	$\sqrt{2H/g}$	$\sqrt{2Hg}$	$k\sqrt{2Hg}$	$k\sqrt{2H/g}$	k^2H
2	$\sqrt{2k^2H/g}$	$\sqrt{2k^2Hg}$	$k\sqrt{2k^2Hg}$	$k\sqrt{2k^2H/g}$	k^4H
3	$\sqrt{2k^4H/g}$	$\sqrt{2k^4Hg}$	$k\sqrt{2k^4Hg}$	$k\sqrt{2k^4H/g}$	k^6H
...	...	...	...	...	...

The total travel time in seconds is

$$\sqrt{\frac{2H}{g}} + k\sqrt{\frac{2H}{g}} + k\sqrt{\frac{2H}{g}} + k^2\sqrt{\frac{2H}{g}} + k^2\sqrt{\frac{2H}{g}} + \cdots = \sqrt{\frac{2H}{g}}\left(1 + 2k + 2k^2 + 2k^3 + \cdots\right)$$
$$= \sqrt{\frac{2H}{g}}\left[1 + 2k\left(1 + k + k^2 + \cdots\right)\right] = \sqrt{\frac{2H}{g}}\left[1 + 2k\left(\frac{1}{1-k}\right)\right] = \sqrt{\frac{2H}{g}}\,\frac{1+k}{1-k}$$

Another method: We could use part (b). At the top of the bounce, the height is $k^2h = rh$, so $\sqrt{r} = k$ and the result follows from part (b).

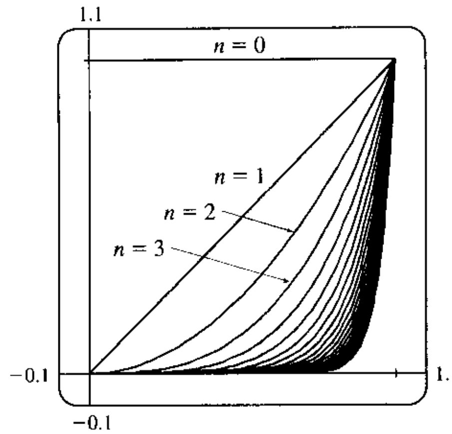
53. $\sum_{n=2}^{\infty}(1+c)^{-n}$ is a geometric series with $a = (1+c)^{-2}$ and $r = (1+c)^{-1}$, so the series converges when $\left|(1+c)^{-1}\right| < 1 \;\Leftrightarrow\; |1+c| > 1 \;\Leftrightarrow\; 1 + c > 1$ or $1 + c < -1 \;\Leftrightarrow\; c > 0$ or $c < -2$. We calculate the sum of the series and set it equal to 2: $\dfrac{(1+c)^{-2}}{1-(1+c)^{-1}} = 2 \;\Leftrightarrow\; \left(\dfrac{1}{1+c}\right)^2 = 2 - 2\left(\dfrac{1}{1+c}\right) \;\Leftrightarrow$

$1 = 2(1+c)^2 - 2(1+c) \Leftrightarrow 2c^2 + 2c - 1 = 0 \Leftrightarrow c = \frac{-2 \pm \sqrt{12}}{4} = \frac{\pm\sqrt{3}-1}{2}$. However, the negative root is inadmissible because $-2 < \frac{-\sqrt{3}-1}{2} < 0$. So $c = \frac{\sqrt{3}-1}{2}$.

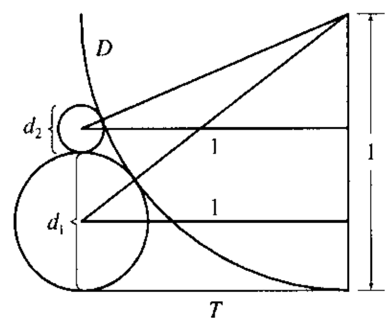
54. The area between $y = x^{n-1}$ and $y = x^n$ for $0 \le x \le 1$ is

$$\int_0^1 \left(x^{n-1} - x^n\right) dx = \left[\frac{x^n}{n} - \frac{x^{n+1}}{n+1}\right]_0^1 = \frac{1}{n} - \frac{1}{n+1}$$
$$= \frac{(n+1)-n}{n(n+1)} = \frac{1}{n(n+1)}$$

We can see from the diagram that as $n \to \infty$, the sum of the areas between the successive curves approaches the area of the unit square, that is, 1. So $\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = 1$.



55. Let d_n be the diameter of C_n. We draw lines from the centers of the C_i to the center of D (or C), and using the Pythagorean Theorem, we can write $1^2 + \left(1 - \frac{1}{2}d_1\right)^2 = \left(1 + \frac{1}{2}d_1\right)^2 \Leftrightarrow$
$1 = \left(1 + \frac{1}{2}d_1\right)^2 - \left(1 - \frac{1}{2}d_1\right)^2 = 2d_1$ (difference of squares)
$\Rightarrow d_1 = \frac{1}{2}$. Similarly,



$$1 = \left(1 + \tfrac{1}{2}d_2\right)^2 - \left(1 - d_1 - \tfrac{1}{2}d_2\right)^2 = 2d_2 + 2d_1 - d_1^2 - d_1 d_2$$
$$= (2 - d_1)(d_1 + d_2) \quad \Leftrightarrow$$

$d_2 = \dfrac{1}{2 - d_1} - d_1 = \dfrac{(1-d_1)^2}{2-d_1}$, $1 = \left(1 + \frac{1}{2}d_3\right)^2 - \left(1 - d_1 - d_2 - \frac{1}{2}d_3\right)^2 \Leftrightarrow d_3 = \dfrac{[1-(d_1+d_2)]^2}{2-(d_1+d_2)}$, and

in general, $d_{n+1} = \dfrac{\left(1 - \sum_{i=1}^{n} d_i\right)^2}{2 - \sum_{i=1}^{n} d_i}$. If we actually calculate d_2 and d_3 from the formulas above, we find that they are $\dfrac{1}{6} = \dfrac{1}{2\cdot 3}$ and $\dfrac{1}{12} = \dfrac{1}{3\cdot 4}$ respectively, so we suspect that in general, $d_n = \dfrac{1}{n(n+1)}$. To prove this, we use induction: Assume that for all $k \le n$, $d_k = \dfrac{1}{k(k+1)} = \dfrac{1}{k} - \dfrac{1}{k+1}$. Then

$\sum_{i=1}^{n} d_i = 1 - \dfrac{1}{n+1} = \dfrac{n}{n+1}$ (telescoping sum). Substituting this into our formula for d_{n+1}, we get

$$d_{n+1} = \frac{\left[1 - \frac{n}{n+1}\right]^2}{2 - \left(\frac{n}{n+1}\right)} = \frac{\frac{1}{(n+1)^2}}{\frac{n+2}{n+1}} = \frac{1}{(n+1)(n+2)},$$ and the induction is complete.

Now, we observe that the partial sums $\sum_{i=1}^{n} d_i$ of the diameters of the circles approach 1 as $n \to \infty$; that is,

$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \dfrac{1}{n(n+1)} = 1$, which is what we wanted to prove.

56. $|CD| = b\sin\theta$, $|DE| = |CD|\sin\theta = b\sin^2\theta$, $|EF| = |DE|\sin\theta = b\sin^3\theta, \ldots$. Therefore,

$|CD| + |DE| + |EF| + |FG| + \cdots = b\sum_{n=1}^{\infty} \sin^n\theta = b\left(\dfrac{\sin\theta}{1-\sin\theta}\right)$ since this is a geometric series with

$r = \sin\theta$ and $|\sin\theta| < 1$ (because $0 < \theta < \frac{\pi}{2}$).

57. The series $1 - 1 + 1 - 1 + 1 - 1 + \cdots$ diverges (geometric series with $r = -1$) so we cannot say that $0 = 1 - 1 + 1 - 1 + 1 - 1 + \cdots$.

58. If $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$ by Theorem 6, so $\lim_{n\to\infty} \dfrac{1}{a_n} \neq 0$, and so $\sum_{n=1}^{\infty} \dfrac{1}{a_n}$ is divergent by the Test for Divergence.

59. $\sum_{n=1}^{\infty} ca_n = \lim_{n\to\infty} \sum_{i=1}^{n} ca_i = \lim_{n\to\infty} c\sum_{i=1}^{n} a_i = c \lim_{n\to\infty} \sum_{i=1}^{n} a_i = c\sum_{n=1}^{\infty} a_n$, which exists by hypothesis.

60. If $\sum ca_n$ were convergent, then $\sum(1/c)(ca_n) = \sum a_n$ would be also, by Theorem 8. But this is not the case, so $\sum ca_n$ must diverge.

61. Suppose on the contrary that $\sum(a_n + b_n)$ converges. Then $\sum(a_n + b_n)$ and $\sum a_n$ are convergent series. So by Theorem 8, $\sum[(a_n + b_n) - a_n]$ would also be convergent. But $\sum[(a_n + b_n) - a_n] = \sum b_n$, a contradiction, since $\sum b_n$ is given to be divergent.

62. No. For example, take $\sum a_n = \sum n$ and $\sum b_n = \sum(-n)$, which both diverge, yet $\sum(a_n + b_n) = \sum 0$, which converges with sum 0.

63. The partial sums $\{s_n\}$ form an increasing sequence, since $s_n - s_{n-1} = a_n > 0$ for all n. Also, the sequence $\{s_n\}$ is bounded since $s_n \leq 1000$ for all n. So by Theorem 12.1.11, the sequence of partial sums converges, that is, the series $\sum a_n$ is convergent.

64. (a) $$\text{RHS} = \frac{1}{f_{n-1}f_n} - \frac{1}{f_n f_{n+1}} = \frac{f_n f_{n+1} - f_n f_{n-1}}{f_n^2 f_{n-1} f_{n+1}} = \frac{f_{n+1} - f_{n-1}}{f_n f_{n-1} f_{n+1}} = \frac{(f_{n-1} + f_n) - f_{n-1}}{f_n f_{n-1} f_{n+1}}$$

$$= \frac{1}{f_{n-1}f_{n+1}} = \text{LHS}$$

(b) $$\sum_{n=2}^{\infty} \frac{1}{f_{n-1}f_{n+1}} = \sum_{n=2}^{\infty}\left(\frac{1}{f_{n-1}f_n} - \frac{1}{f_n f_{n+1}}\right) \quad \text{[from part (a)]}$$

$$= \lim_{n\to\infty}\left[\left(\frac{1}{f_1 f_2} - \frac{1}{f_2 f_3}\right) + \left(\frac{1}{f_2 f_3} - \frac{1}{f_3 f_4}\right) + \left(\frac{1}{f_3 f_4} - \frac{1}{f_4 f_5}\right) + \cdots + \left(\frac{1}{f_{n-1}f_n} - \frac{1}{f_n f_{n+1}}\right)\right]$$

$$= \lim_{n\to\infty}\left(\frac{1}{f_1 f_2} - \frac{1}{f_n f_{n+1}}\right) = \frac{1}{f_1 f_2} - 0 = \frac{1}{1\cdot 1} = 1 \text{ because } f_n \to \infty \text{ as } n \to \infty.$$

(c) $$\sum_{n=2}^{\infty} \frac{f_n}{f_{n-1}f_{n+1}} = \sum_{n=2}^{\infty}\left(\frac{f_n}{f_{n-1}f_n} - \frac{f_n}{f_n f_{n+1}}\right) \quad \text{(as above)}$$
$$= \sum_{n=2}^{\infty}\left(\frac{1}{f_{n-1}} - \frac{1}{f_{n+1}}\right)$$
$$= \lim_{n\to\infty}\left[\left(\frac{1}{f_1} - \frac{1}{f_3}\right) + \left(\frac{1}{f_2} - \frac{1}{f_4}\right) + \left(\frac{1}{f_3} - \frac{1}{f_5}\right) + \left(\frac{1}{f_4} - \frac{1}{f_6}\right) + \cdots + \left(\frac{1}{f_{n-1}} - \frac{1}{f_{n+1}}\right)\right]$$
$$= \lim_{n\to\infty}\left(\frac{1}{f_1} + \frac{1}{f_2} - \frac{1}{f_n} - \frac{1}{f_{n+1}}\right) = 1 + 1 - 0 - 0 = 2 \text{ because } f_n \to \infty \text{ as } n \to \infty.$$

65. (a) At the first step, only the interval $\left(\frac{1}{3}, \frac{2}{3}\right)$ (length $\frac{1}{3}$) is removed. At the second step, we remove the intervals $\left(\frac{1}{9}, \frac{2}{9}\right)$ and $\left(\frac{7}{9}, \frac{8}{9}\right)$, which have a total length of $2 \cdot \left(\frac{1}{3}\right)^2$. At the third step, we remove 2^2 intervals, each of length $\left(\frac{1}{3}\right)^3$. In general, at the nth step we remove 2^{n-1} intervals, each of length $\left(\frac{1}{3}\right)^n$, for a length of $2^{n-1} \cdot \left(\frac{1}{3}\right)^n = \frac{1}{3}\left(\frac{2}{3}\right)^{n-1}$. Thus, the total length of all removed intervals is $\sum_{n=1}^{\infty} \frac{1}{3}\left(\frac{2}{3}\right)^{n-1} = \frac{1/3}{1-2/3} = 1$ (geometric series with $a = \frac{1}{3}$ and $r = \frac{2}{3}$). Notice that at the nth step, the leftmost interval that is removed is $\left(\left(\frac{1}{3}\right)^n, \left(\frac{2}{3}\right)^n\right)$, so we never remove 0, and 0 is in the Cantor set. Also, the rightmost interval removed is $\left(1 - \left(\frac{2}{3}\right)^n, 1 - \left(\frac{1}{3}\right)^n\right)$, so 1 is never removed. Some other numbers in the Cantor set are $\frac{1}{3}$, $\frac{2}{3}$, $\frac{1}{9}$, $\frac{2}{9}$, $\frac{7}{9}$, and $\frac{8}{9}$.

(b) The area removed at the first step is $\frac{1}{9}$; at the second step, $8 \cdot \left(\frac{1}{9}\right)^2$; at the third step, $(8)^2 \cdot \left(\frac{1}{9}\right)^3$. In general, the area removed at the nth step is $(8)^{n-1}\left(\frac{1}{9}\right)^n = \frac{1}{9}\left(\frac{8}{9}\right)^{n-1}$, so the total area of all removed squares is
$$\sum_{n=1}^{\infty} \frac{1}{9}\left(\frac{8}{9}\right)^{n-1} = \frac{1/9}{1 - 8/9} = 1.$$

66. (a)

a_1	1	2	4	1	1	1000
a_2	2	3	1	4	1000	1
a_3	1.5	2.5	2.5	2.5	500.5	500.5
a_4	1.75	2.75	1.75	3.25	750.25	250.75
a_5	1.625	2.625	2.125	2.875	625.375	375.625
a_6	1.6875	2.6875	1.9375	3.0625	687.813	313.188
a_7	1.65625	2.65625	2.03125	2.96875	656.594	344.406
a_8	1.67188	2.67188	1.98438	3.01563	672.203	328.797
a_9	1.66406	2.66406	2.00781	2.99219	664.398	336.602
a_{10}	1.66797	2.66797	1.99609	3.00391	668.301	332.699
a_{11}	1.66602	2.66602	2.00195	2.99805	666.350	334.650
a_{12}	1.66699	2.66699	1.99902	3.00098	667.325	333.675

The limits seem to be $\frac{5}{3}, \frac{8}{3}$, 2, 3, 667, and 334. Note that the limits appear to be "weighted" more toward a_2. In general, we guess that the limit is $\dfrac{a_1 + 2a_2}{3}$.

(b) $a_{n+1} - a_n = \frac{1}{2}(a_n + a_{n-1}) - a_n = -\frac{1}{2}(a_n - a_{n-1}) = -\frac{1}{2}\left[\frac{1}{2}(a_{n-1} + a_{n-2}) - a_{n-1}\right]$

$$= -\tfrac{1}{2}\left[-\tfrac{1}{2}(a_{n-1} - a_{n-2})\right] = \cdots = \left(-\tfrac{1}{2}\right)^{n-1}(a_2 - a_1)$$

Note that we have used the formula $a_k = \frac{1}{2}(a_{k-1} + a_{k-2})$ a total of $n-1$ times in this calculation, once for each k between 3 and $n+1$. Now we can write

$$\begin{aligned} a_n &= a_1 + (a_2 - a_1) + (a_3 - a_2) + \cdots + (a_{n-1} - a_{n-2}) + (a_n - a_{n-1}) \\ &= a_1 + \textstyle\sum_{k=1}^{n-1}(a_{k+1} - a_k) = a_1 + \sum_{k=1}^{n-1}\left(-\frac{1}{2}\right)^{k-1}(a_2 - a_1) \end{aligned}$$

and so

$$\begin{aligned} \lim_{n\to\infty} a_n &= a_1 + (a_2 - a_1)\textstyle\sum_{k=1}^{\infty}\left(-\frac{1}{2}\right)^{k-1} = a_1 + (a_2 - a_1)\left[\frac{1}{1-(-1/2)}\right] \\ &= a_1 + \tfrac{2}{3}(a_2 - a_1) = \frac{a_1 + 2a_2}{3} \end{aligned}$$

67. (a) For $\sum\limits_{n=1}^{\infty} \dfrac{n}{(n+1)!}$, $s_1 = \dfrac{1}{1\cdot 2} = \dfrac{1}{2}$, $s_2 = \dfrac{1}{2} + \dfrac{2}{1\cdot 2\cdot 3} = \dfrac{5}{6}$, $s_3 = \dfrac{5}{6} + \dfrac{3}{1\cdot 2\cdot 3\cdot 4} = \dfrac{23}{24}$, $s_4 = \dfrac{23}{24} + \dfrac{4}{1\cdot 2\cdot 3\cdot 4\cdot 5} = \dfrac{119}{120}$. The denominators are $(n+1)!$, so a guess would be $s_n = \dfrac{(n+1)! - 1}{(n+1)!}$.

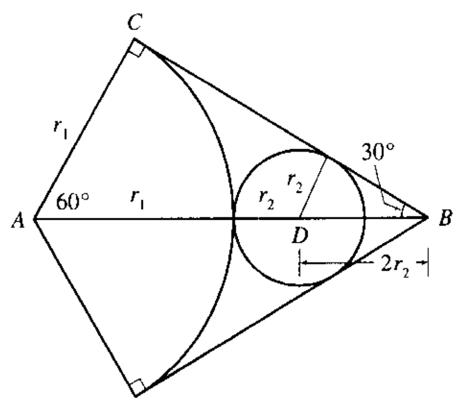
(b) For $n = 1$, $s_1 = \dfrac{1}{2} = \dfrac{2! - 1}{2!}$, so the formula holds for $n = 1$. Assume $s_k = \dfrac{(k+1)! - 1}{(k+1)!}$. Then

$$\begin{aligned} s_{k+1} &= \frac{(k+1)! - 1}{(k+1)!} + \frac{k+1}{(k+2)!} = \frac{(k+1)! - 1}{(k+1)!} + \frac{k+1}{(k+1)!(k+2)} \\ &= \frac{(k+2)! - (k+2) + k + 1}{(k+2)!} = \frac{(k+2)! - 1}{(k+2)!} \end{aligned}$$

Thus, the formula is true for $n = k+1$. So by induction, the guess is correct.

(c) $\lim\limits_{n\to\infty} s_n = \lim\limits_{n\to\infty} \dfrac{(n+1)! - 1}{(n+1)!} = \lim\limits_{n\to\infty}\left[1 - \dfrac{1}{(n+1)!}\right] = 1$ and so $\sum\limits_{n=1}^{\infty} \dfrac{n}{(n+1)!} = 1$.

68.



Let r_1 = radius of the large circle, r_2 = radius of next circle, and so on. From the figure we have $\angle BAC = 60°$ and $\cos 60° = r_1/|AB|$, so $|AB| = 2r_1$ and $|DB| = 2r_2$. Therefore, $2r_1 = r_1 + r_2 + 2r_2 = r_1 + 3r_2 \Rightarrow r_1 = 3r_2$. In general, we have $r_{n+1} = \frac{1}{3}r_n$, so the total area is

$$\begin{aligned} A &= \pi r_1^2 + 3\pi r_2^2 + 3\pi r_3^2 + \cdots \\ &= \pi r_1^2 + 3\pi r_2^2\left(1 + \frac{1}{3^2} + \frac{1}{3^4} + \frac{1}{3^6} + \cdots\right) \\ &= \pi r_1^2 + 3\pi r_2^2 \cdot \tfrac{1}{1 - 1/9} = \pi r_1^2 + \tfrac{27}{8}\pi r_2^2 \end{aligned}$$

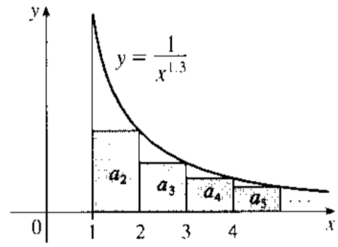
Since the sides of the triangle have length 1, $|BC| = \frac{1}{2}$ and $\tan 30° = \dfrac{r_1}{1/2}$. Thus, $r_1 = \dfrac{\tan 30°}{2} = \dfrac{1}{2\sqrt{3}} \Rightarrow$

$r_2 = \frac{1}{6\sqrt{3}}$, so $A = \pi\left(\frac{1}{2\sqrt{3}}\right)^2 + \frac{27\pi}{8}\left(\frac{1}{6\sqrt{3}}\right)^2 = \frac{\pi}{12} + \frac{\pi}{32} = \frac{11\pi}{96}$. The area of the triangle is $\frac{\sqrt{3}}{4}$, so the circles occupy about 83.1% of the area of the triangle.

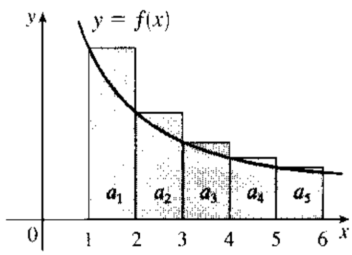
12.3 The Integral Test and Estimates of Sums

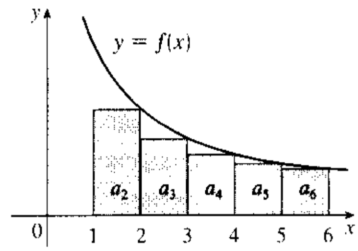
1. The picture shows that $a_2 = \dfrac{1}{2^{1.3}} < \displaystyle\int_1^2 \frac{1}{x^{1.3}}\,dx$,

$a_3 = \dfrac{1}{3^{1.3}} < \displaystyle\int_2^3 \frac{1}{x^{1.3}}\,dx$, and so on, so $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n^{1.3}} < \int_1^{\infty} \frac{1}{x^{1.3}}\,dx$. The integral converges by (8.8.2) with $p = 1.3 > 1$, so the series converges.



2. From the first figure, we see that $\int_1^6 f(x)dx < \sum_{i=1}^{5} a_i$. From the second figure, we see that $\sum_{i=2}^{6} a_i < \int_1^6 f(x)\,dx$. Thus, we have $\sum_{i=2}^{6} a_i < \int_1^6 f(x)\,dx < \sum_{i=1}^{5} a_i$.





3. The function $f(x) = 1/x^4$ is continuous, positive, and decreasing on $[1, \infty)$, so the Integral Test applies.
$\displaystyle\int_1^{\infty} \frac{1}{x^4}\,dx = \lim_{t\to\infty}\int_1^t x^{-4}\,dx = \lim_{t\to\infty}\left[\frac{x^{-3}}{-3}\right]_1^t = \lim_{t\to\infty}\left(-\frac{1}{3t^3} + \frac{1}{3}\right) = \frac{1}{3}$. Since this improper integral is convergent, the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^4}$ is also convergent by the Integral Test.

4. The function $f(x) = 1/\sqrt[4]{x} = x^{-1/4}$ is continuous, positive, and decreasing on $[1, \infty)$, so the Integral Test applies.
$\int_1^{\infty} x^{-1/4}\,dx = \lim\limits_{t\to\infty} \int_1^t x^{-1/4}\,dx = \lim\limits_{t\to\infty}\left[\frac{4}{3}x^{3/4}\right]_1^t = \lim\limits_{t\to\infty}\left(\frac{4}{3}t^{3/4} - \frac{4}{3}\right) = \infty$, so $\sum_{n=1}^{\infty} 1/\sqrt[4]{n}$ diverges.

5. The function $f(x) = 1/(3x+1)$ is continuous, positive, and decreasing on $[1, \infty)$, so the Integral Test applies.

$$\int_1^{\infty} \frac{dx}{3x+1} = \lim_{b\to\infty}\int_1^b \frac{dx}{3x+1} = \lim_{b\to\infty}\left[\tfrac{1}{3}\ln(3x+1)\right]_1^b = \lim_{b\to\infty}\left[\tfrac{1}{3}\ln(3b+1) - \tfrac{1}{3}\ln 4\right] = \infty$$

so the improper integral diverges, and so does the series $\sum_{n=1}^{\infty} 1/(3n+1)$.

6. The function $f(x) = e^{-x}$ is continuous, positive, and decreasing on $[1, \infty)$, so the Integral Test applies.
$\int_1^{\infty} e^{-x}\,dx = \lim\limits_{b\to\infty}\int_1^b e^{-x}\,dx = \lim\limits_{b\to\infty}\left[-e^{-x}\right]_1^b = \lim\limits_{b\to\infty}\left(-e^{-b} + e^{-1}\right) = e^{-1}$, so $\sum_{n=1}^{\infty} e^{-n}$ converges. *Note:* This is a geometric series, with first term $a = e^{-1}$ and ratio $r = e^{-1}$. Since $|r| < 1$, the series converges to $e^{-1}/(1 - e^{-1}) = 1/(e-1)$.

7. $f(x) = xe^{-x}$ is continuous and positive on $[1, \infty)$. $f'(x) = -xe^{-x} + e^{-x} = e^{-x}(1-x) < 0$ for $x > 1$, so f is decreasing on $[1, \infty)$. Thus, the Integral Test applies.

$$\int_1^{\infty} xe^{-x}\,dx = \lim_{b\to\infty}\int_1^b xe^{-x}\,dx = \lim_{b\to\infty}\left[-xe^{-x} - e^{-x}\right]_1^b \quad \text{(by parts)}$$
$$= \lim_{b\to\infty}\left[-be^{-b} - e^{-b} + e^{-1} + e^{-1}\right] = 2/e$$

since $\lim\limits_{b\to\infty} be^{-b} = \lim\limits_{b\to\infty}(b/e^b) \overset{H}{=} \lim\limits_{b\to\infty}(1/e^b) = 0$ and $\lim\limits_{b\to\infty} e^{-b} = 0$. Thus, $\sum_{n=1}^{\infty} ne^{-n}$ converges.

8. The function $f(x) = \dfrac{x+2}{x+1} = 1 + \dfrac{1}{x+1}$ is continuous, positive, and decreasing on $[1,\infty)$, so the Integral Test applies.

$\displaystyle\int_1^\infty f(x)\,dx = \lim_{t\to\infty}\int_1^t \left(1 + \frac{1}{x+1}\right)dx = \lim_{t\to\infty}[x + \ln(x+1)]_1^t = \lim_{t\to\infty}(t + \ln(t+1) - 1 - \ln 2) = \infty$, so $\displaystyle\int_1^\infty \frac{x+2}{x+1}\,dx$ is divergent and the series $\displaystyle\sum_{n=1}^\infty \frac{n+2}{n+1}$ is divergent. NOTE: $\displaystyle\lim_{n\to\infty}\frac{n+2}{n+1} = 1$, so the given series diverges by the Test for Divergence.

9. The series $\displaystyle\sum_{n=1}^\infty \frac{1}{n^{0.85}}$ is a p-series with $p = 0.85 \le 1$, so it diverges by (1). Therefore, the series $\displaystyle\sum_{n=1}^\infty \frac{2}{n^{0.85}}$ must also diverge, for if it converged, then $\displaystyle\sum_{n=1}^\infty \frac{1}{n^{0.85}}$ would have to converge (by Theorem 8(i) in Section 11.2).

10. $\displaystyle\sum_{n=1}^\infty n^{-1.4}$ and $\displaystyle\sum_{n=1}^\infty n^{-1.2}$ are p-series with $p > 1$, so they converge by (1). Thus, $\displaystyle\sum_{n=1}^\infty 3n^{-1.2}$ converges by Theorem 8(i) in Section 11.2. It follows from Theorem 8(ii) that the given series $\displaystyle\sum_{n=1}^\infty (n^{-1.4} + 3n^{-1.2})$ also converges.

11. $1 + \dfrac{1}{8} + \dfrac{1}{27} + \dfrac{1}{64} + \dfrac{1}{125} + \cdots = \displaystyle\sum_{n=1}^\infty \frac{1}{n^3}$. This is a p-series with $p = 3 > 1$, so it converges by (1).

12. $1 + \dfrac{1}{2\sqrt{2}} + \dfrac{1}{3\sqrt{3}} + \dfrac{1}{4\sqrt{4}} + \dfrac{1}{5\sqrt{5}} + \cdots = \displaystyle\sum_{n=1}^\infty \frac{1}{n\sqrt{n}} = \sum_{n=1}^\infty \frac{1}{n^{3/2}}$. This is a p-series with $p = \frac{3}{2} > 1$, so it converges by (1).

13. $\displaystyle\sum_{n=1}^\infty \frac{5 - 2\sqrt{n}}{n^3} = 5\sum_{n=1}^\infty \frac{1}{n^3} - 2\sum_{n=1}^\infty \frac{1}{n^{5/2}}$ by Theorem 12.2.8, since $\displaystyle\sum_{n=1}^\infty \frac{1}{n^3}$ and $\displaystyle\sum_{n=1}^\infty \frac{1}{n^{5/2}}$ both converge by (1) (with $p = 3 > 1$ and $p = \frac{5}{2} > 1$). Thus, $\displaystyle\sum_{n=1}^\infty \frac{5 - 2\sqrt{n}}{n^3}$ converges.

14. The function $f(x) = \dfrac{5}{x-2}$ is continuous, positive, and decreasing on $[3,\infty)$, so we can apply the Integral Test.

$\displaystyle\int_3^\infty \frac{5}{x-2}\,dx = \lim_{t\to\infty}\int_3^t \frac{5}{x-2}\,dx = \lim_{t\to\infty}[5\ln(x-2)]_3^t = \lim_{t\to\infty}[5\ln(t-2) - 0] = \infty$, so the series $\displaystyle\sum_{n=3}^\infty \frac{5}{n-2}$ diverges.

15. The function $f(x) = \dfrac{1}{x^2+4}$ is continuous, positive, and decreasing on $[1,\infty)$, so we can apply the Integral Test.

$$\begin{aligned}\int_1^\infty \frac{1}{x^2+4}\,dx &= \lim_{t\to\infty}\int_1^t \frac{1}{x^2+4}\,dx = \lim_{t\to\infty}\left[\frac{1}{2}\tan^{-1}\frac{x}{2}\right]_1^t = \frac{1}{2}\lim_{t\to\infty}\left[\tan^{-1}\left(\frac{t}{2}\right) - \tan^{-1}\left(\frac{1}{2}\right)\right] \\ &= \frac{1}{2}\left[\frac{\pi}{2} - \tan^{-1}\left(\frac{1}{2}\right)\right]\end{aligned}$$

Therefore, the series $\displaystyle\sum_{n=1}^\infty \frac{1}{n^2+4}$ converges.

16. The function $f(x)=\dfrac{3x+2}{x(x+1)}=\dfrac{2}{x}+\dfrac{1}{x+1}$ [by partial fractions] is continuous, positive, and decreasing on $[1,\infty)$ since it is the sum of two such functions. Thus, we can apply the Integral Test.

$$\int_1^\infty \frac{3x+2}{x(x+1)}\,dx=\lim_{t\to\infty}\int_1^t\left[\frac{2}{x}+\frac{1}{x+1}\right]dx=\lim_{t\to\infty}\left[2\ln x+\ln(x+1)\right]_1^t$$
$$=\lim_{t\to\infty}\left[2\ln t+\ln(t+1)-\ln 2\right]=\infty$$

Thus, the series $\displaystyle\sum_{n=1}^{\infty}\frac{3n+2}{n(n+1)}$ diverges.

17. $f(x)=\dfrac{x}{x^2+1}$ is continuous and positive on $[1,\infty)$, and since $f'(x)=\dfrac{1-x^2}{(x^2+1)^2}<0$ for $x>1$, f is also decreasing. Using the Integral Test, $\displaystyle\int_1^\infty\frac{x}{x^2+1}\,dx=\lim_{t\to\infty}\int_1^t\frac{x}{x^2+1}\,dx=\lim_{t\to\infty}\left[\frac{\ln(x^2+1)}{2}\right]_1^t=\frac{1}{2}\lim_{t\to\infty}\left[\ln(t^2+1)-\ln 2\right]=\infty$, so the series diverges.

18. The function $f(x)=\dfrac{1}{x^2-4x+5}=\dfrac{1}{(x-2)^2+1}$ is continuous, positive, and decreasing on $[2,\infty)$, so the Integral Test applies. $\displaystyle\int_2^\infty f(x)\,dx=\lim_{t\to\infty}\int_2^t f(x)\,dx=\lim_{t\to\infty}\int_2^t\frac{1}{(x-2)^2+1}\,dx=\lim_{t\to\infty}\left[\tan^{-1}(x-2)\right]_2^t=$ $\displaystyle\lim_{t\to\infty}\left[\tan^{-1}(t-2)-\tan^{-1}0\right]=\frac{\pi}{2}-0=\frac{\pi}{2}$, so the series $\displaystyle\sum_{n=2}^{\infty}\frac{1}{n^2-4n+5}$ converges. Of course this means that $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n^2-4n+5}$ converges too.

19. $f(x)=xe^{-x^2}$ is continuous and positive on $[1,\infty)$, and since $f'(x)=e^{-x^2}\left(1-2x^2\right)<0$ for $x>1$, f is decreasing as well. Thus, we can use the Integral Test. $\int_1^\infty xe^{-x^2}\,dx=\lim_{t\to\infty}\left[-\frac{1}{2}e^{-x^2}\right]_1^t=0-\left(-\frac{1}{2}e^{-1}\right)=1/(2e)$. Since the integral converges, the series converges.

20. $f(x)=\dfrac{\ln x}{x^2}$ is continuous and positive for $x\geq 2$, and $f'(x)=\dfrac{1-2\ln x}{x^3}<0$ for $x\geq 2$, so f is decreasing. $\displaystyle\int_2^\infty\frac{\ln x}{x^2}\,dx=\lim_{t\to\infty}\left[-\frac{\ln x}{x}-\frac{1}{x}\right]_2^t$ [by parts] $\overset{\text{H}}{=}1$. Thus, $\displaystyle\sum_{n=1}^{\infty}\frac{\ln n}{n^2}=\sum_{n=2}^{\infty}\frac{\ln n}{n^2}$ converges by the Integral Test.

21. $f(x)=\dfrac{1}{x\ln x}$ is continuous and positive on $[2,\infty)$, and also decreasing since $f'(x)=-\dfrac{1+\ln x}{x^2(\ln x)^2}<0$ for $x>2$, so we can use the Integral Test. $\displaystyle\int_2^\infty\frac{1}{x\ln x}\,dx=\lim_{t\to\infty}\left[\ln(\ln x)\right]_2^t=\lim_{t\to\infty}\left[\ln(\ln t)-\ln(\ln 2)\right]=\infty$, so the series diverges.

22. The function $f(x) = \dfrac{x}{x^4+1}$ is positive, continuous, and decreasing on $[1,\infty)$. [Note that $f'(x) = \dfrac{x^4+1-4x^4}{(x^4+1)^2} = \dfrac{1-3x^4}{(x^4+1)^2} < 0$ on $[1,\infty)$.] Thus, we can apply the Integral Test.

$$\int_1^\infty \frac{x}{x^4+1}\,dx = \lim_{t\to\infty}\int_1^t \frac{\frac{1}{2}(2x)}{1+(x^2)^2}\,dx = \lim_{t\to\infty}\left[\frac{1}{2}\tan^{-1}(x^2)\right]_1^t = \frac{1}{2}\lim_{t\to\infty}[\tan^{-1}(t^2) - \tan^{-1}1]$$
$$= \frac{1}{2}\left(\frac{\pi}{2} - \frac{\pi}{4}\right) = \frac{\pi}{8}$$

so the series $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^4+1}$ converges.

23. The function $f(x) = \dfrac{1}{x^3+x}$ is continuous, positive, and decreasing on $[1,\infty)$, so the Integral Test applies. We use partial fractions to evaluate the integral:

$$\int_1^\infty \frac{1}{x^3+x}\,dx = \lim_{t\to\infty}\int_1^t \left[\frac{1}{x} - \frac{x}{1+x^2}\right]dx = \lim_{t\to\infty}\left[\ln x - \frac{1}{2}\ln(1+x^2)\right]_1^t$$
$$= \lim_{t\to\infty}\left[\ln\frac{x}{\sqrt{1+x^2}}\right]_1^t = \lim_{t\to\infty}\left(\ln\frac{t}{\sqrt{1+t^2}} - \ln\frac{1}{\sqrt{2}}\right)$$
$$= \lim_{t\to\infty}\left(\ln\frac{1}{\sqrt{1+1/t^2}} + \frac{1}{2}\ln 2\right) = \frac{1}{2}\ln 2$$

so the series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^3+n}$ converges.

24. $f(x) = \dfrac{1}{x\ln x\ln(\ln x)}$ is positive and continuous on $[3,\infty)$, and is decreasing since x, $\ln x$, and $\ln(\ln x)$ are all increasing; so we can apply the Integral Test. $\displaystyle\int_3^\infty \frac{dx}{x\ln x\ln(\ln x)} = \lim_{t\to\infty}[\ln(\ln(\ln x))]_3^t = \infty$. The integral diverges, so $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n\ln n\ln(\ln n)}$ diverges.

25. We have already shown (in Exercise 21) that when $p = 1$ the series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^p}$ diverges, so assume that $p \neq 1$. $f(x) = \dfrac{1}{x(\ln x)^p}$ is continuous and positive on $[2,\infty)$, and $f'(x) = -\dfrac{p+\ln x}{x^2(\ln x)^{p+1}} < 0$ if $x > e^{-p}$, so that f is eventually decreasing and we can use the Integral Test.

$$\int_2^\infty \frac{1}{x(\ln x)^p}\,dx = \lim_{t\to\infty}\left[\frac{(\ln x)^{1-p}}{1-p}\right]_2^t \quad (\text{for } p \neq 1) \;= \lim_{t\to\infty}\left[\frac{(\ln t)^{1-p}}{1-p}\right] - \frac{(\ln 2)^{1-p}}{1-p}$$

This limit exists whenever $1 - p < 0 \;\Leftrightarrow\; p > 1$, so the series converges for $p > 1$.

26. As in Exercise 24, we can apply the Integral Test. $\displaystyle\int_3^\infty \frac{dx}{x\ln x\,(\ln\ln x)^p} = \lim_{t\to\infty}\left[\frac{(\ln\ln x)^{-p+1}}{-p+1}\right]_3^t$ (for $p \neq 1$; if $p = 1$ see Exercise 24) and $\displaystyle\lim_{t\to\infty}\frac{(\ln\ln t)^{-p+1}}{-p+1}$ exists whenever $-p+1 < 0 \;\Leftrightarrow\; p > 1$, so the series converges for $p > 1$.

27. Clearly the series cannot converge if $p \geq -\frac{1}{2}$, because then $\lim\limits_{n\to\infty} n(1+n^2)^p \neq 0$. Also, if $p = -1$ the series diverges (see Exercise 17). So assume $p < -\frac{1}{2}$, $p \neq -1$. Then $f(x) = x(1+x^2)^p$ is continuous, positive, and eventually decreasing on $[1, \infty)$, and we can use the Integral Test.

$\int_1^\infty x(1+x^2)^p dx = \lim\limits_{t\to\infty}\left[\frac{1}{2}\cdot\frac{(1+x^2)^{p+1}}{p+1}\right]_1^t = \lim\limits_{t\to\infty}\frac{1}{2}\cdot\frac{(1+t^2)^{p+1}}{p+1} - \frac{2^p}{p+1}$. This limit exists and is finite $\Leftrightarrow$ $p+1<0$ $\Leftrightarrow$ $p<-1$, so the series converges whenever $p<-1$.

28. If $p \leq 0$, $\lim\limits_{n\to\infty}\frac{\ln n}{n^p} = \infty$ and the series diverges, so assume $p > 0$. $f(x) = \frac{\ln x}{x^p}$ is positive and continuous and $f'(x) < 0$ for $x > e^{1/p}$, so f is eventually decreasing and we can use the Integral Test. Integration by parts gives

$\int_1^\infty \frac{\ln x}{x^p}dx = \lim\limits_{t\to\infty}\left[\frac{x^{1-p}[(1-p)\ln x - 1]}{(1-p)^2}\right]_1^t$ (for $p \neq 1$) $= \frac{1}{(1-p)^2}\left[\lim\limits_{t\to\infty} t^{1-p}[(1-p)\ln t - 1] + 1\right]$,

which exists whenever $1-p<0$ $\Leftrightarrow$ $p>1$. Since we have already done the case $p=1$ in Exercise 25 (set $p=-1$ in that exercise), $\sum\limits_{n=1}^{\infty}\frac{\ln n}{n^p}$ converges $\Leftrightarrow$ $p > 1$.

29. Since this is a p-series with $p = x$, $\zeta(x)$ is defined when $x > 1$. Unless specified otherwise, the domain of a function f is the set of numbers x such that the expression for $f(x)$ makes sense and defines a real number. So, in the case of a series, it's the set of numbers x such that the series is convergent.

30. (a) $f(x) = 1/x^4$ is positive and continuous and $f'(x) = -4/x^5$ is negative for $x > 0$, and so the Integral Test applies. $\sum\limits_{n=1}^{\infty}\frac{1}{n^4} \approx s_{10} = \frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\cdots+\frac{1}{10^4} \approx 1.082037$.

$R_{10} \leq \int_{10}^\infty \frac{1}{x^4}dx = \lim\limits_{t\to\infty}\left[\frac{1}{-3x^3}\right]_{10}^t = \lim\limits_{t\to\infty}\left(-\frac{1}{3t^3}+\frac{1}{3(10)^3}\right) = \frac{1}{3000}$, so the error is at most $0.000\overline{3}$.

(b) $s_{10} + \int_{11}^\infty \frac{1}{x^4}dx \leq s \leq s_{10} + \int_{10}^\infty \frac{1}{x^4}dx \Rightarrow s_{10} + \frac{1}{3(11)^3} \leq s \leq s_{10} + \frac{1}{3(10)^3} \Rightarrow$ $1.082037 + 0.000250 = 1.082287 \leq s \leq 1.082037 + 0.000333 = 1.082370$, so we get $s \approx 1.08233$ with error ≤ 0.00005.

(c) $R_n \leq \int_n^\infty \frac{1}{x^4}dx = \frac{1}{3n^3}$. So $R_n < 0.00001 \Rightarrow \frac{1}{3n^3} < \frac{1}{10^5} \Rightarrow 3n^3 > 10^5 \Rightarrow$ $n > \sqrt[3]{(10)^5/3} \approx 32.2$, that is, for $n > 32$.

31. (a) $f(x) = \frac{1}{x^2}$ is positive and continuous and $f'(x) = -\frac{2}{x^3}$ is negative for $x > 0$, and so the Integral Test applies. $\sum\limits_{n=1}^{\infty}\frac{1}{n^2} \approx s_{10} = \frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots+\frac{1}{10^2} \approx 1.549768$.

$R_{10} \leq \int_{10}^\infty \frac{1}{x^2}dx = \lim\limits_{t\to\infty}\left[\frac{-1}{x}\right]_{10}^t = \lim\limits_{t\to\infty}\left(-\frac{1}{t}+\frac{1}{10}\right) = \frac{1}{10}$, so the error is at most 0.1.

(b) $s_{10} + \int_{11}^\infty \frac{1}{x^2}dx \leq s \leq s_{10} + \int_{10}^\infty \frac{1}{x^2}dx \Rightarrow s_{10} + \frac{1}{11} \leq s \leq s_{10} + \frac{1}{10} \Rightarrow$ $1.549768 + 0.090909 = 1.640677 \leq s \leq 1.549768 + 0.1 = 1.649768$, so we get $s \approx 1.64522$ (the average of 1.640677 and 1.649768) with error ≤ 0.005 (the maximum of $1.649768 - 1.64522$ and $1.64522 - 1.640677$, rounded up).

(c) $R_n \leq \int_n^\infty \frac{1}{x^2}dx = \frac{1}{n}$. So $R_n < 0.001$ if $\frac{1}{n} < \frac{1}{1000} \Leftrightarrow n > 1000$.

32. $f(x) = 1/x^5$ is positive and continuous and $f'(x) = -5/x^6$ is negative for $x > 0$, and so the Integral Test applies. Using (3), $R_n \le \int_n^\infty x^{-5}\,dx = \lim_{t\to\infty}\left[\dfrac{-1}{4x^4}\right]_n^t = \dfrac{1}{4n^4}$. If we take $n = 5$, then $s_5 \approx 1.036662$ and $R_5 \le 0.0004$. So $s \approx s_5 \approx 1.037$.

33. $f(x) = x^{-3/2}$ is positive and continuous and $f'(x) = -\frac{3}{2}x^{-5/2}$ is negative for $x > 0$, so the Integral Test applies. From the end of Example 6, we see that the error is at most half the length of the interval. From (3), the interval is $\left(s_n + \int_{n+1}^\infty f(x)\,dx,\, s_n + \int_n^\infty f(x)\,dx\right)$, so its length is $\int_n^\infty f(x)\,dx - \int_{n+1}^\infty f(x)\,dx = \int_n^{n+1} f(x)\,dx$. Thus, we need n such that

$$0.01 > \frac{1}{2}\int_n^{n+1} x^{-3/2}\,dx = \frac{1}{2}\left[\frac{-2}{\sqrt{x}}\right]_n^{n+1} = \frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}}$$

$\Leftrightarrow$ $n > 13.08$ (use a graphing calculator to solve $1/\sqrt{x} - 1/\sqrt{x+1} < 0.01$). Again from the end of Example 6, we approximate s by the midpoint of this interval. In general, the midpoint is $\frac{1}{2}\left[\left(s_n + \int_{n+1}^\infty f(x)\,dx\right) + \left(s_n + \int_n^\infty f(x)\,dx\right)\right] = s_n + \frac{1}{2}\left(\int_{n+1}^\infty f(x)\,dx + \int_n^\infty f(x)\,dx\right)$. So using $n = 14$, we have $s \approx s_{14} + \frac{1}{2}\left(\int_{14}^\infty x^{-3/2}\,dx + \int_{15}^\infty x^{-3/2}\,dx\right) \approx 2.0872 + \frac{1}{\sqrt{14}} + \frac{1}{\sqrt{15}} \approx 2.6127 \approx 2.61$. Any larger value of n will also work. For instance, $s \approx s_{30} + \frac{1}{\sqrt{30}} + \frac{1}{\sqrt{31}} \approx 2.6124$.

34. $f(x) = \dfrac{1}{x(\ln x)^2}$ is positive and continuous and $f'(x) = -\dfrac{\ln x + 2}{x^2(\ln x)^3}$ is negative for $x > 1$, so the Integral Test applies. Using (2), we need $0.01 > \int_n^\infty \dfrac{dx}{x(\ln x)^2} = \lim_{t\to\infty}\left[\dfrac{-1}{\ln x}\right]_n^t = \dfrac{1}{\ln n}$. This is true for $n > e^{100}$, so we would have to take this many terms, which would be problematic because $e^{100} \approx 2.7 \times 10^{43}$.

35. $\sum_{n=1}^\infty n^{-1.001} = \sum_{n=1}^\infty \dfrac{1}{n^{1.001}}$ is a convergent p-series with $p = 1.001 > 1$. Using (2), we get

$$R_n \le \int_n^\infty x^{-1.001}\,dx = \lim_{t\to\infty}\left[\frac{x^{-0.001}}{-0.001}\right]_n^t = -1000\lim_{t\to\infty}\left[\frac{1}{x^{0.001}}\right]_n^t = -1000\left(-\frac{1}{n^{0.001}}\right) = \frac{1000}{n^{0.001}}.$$ We want

$$R_n < 0.000\,000\,005 \quad\Leftrightarrow\quad \frac{1000}{n^{0.001}} < 5\times10^{-9} \quad\Leftrightarrow\quad n^{0.001} > \frac{1000}{5\times10^{-9}} \quad\Leftrightarrow$$

$n > (2\times10^{11})^{1000} = 2^{1000}\times10^{11{,}000} \approx 1.07\times10^{301}\times10^{11{,}000} = 1.07\times10^{11{,}301}$.

36. (a) $f(x) = \left(\dfrac{\ln x}{x}\right)^2$ is continuous and positive for $x > 1$, and since $f'(x) = \dfrac{2\ln x\,(1-\ln x)}{x^3} < 0$ for $x > e$, we can apply the Integral Test. Using a CAS, we get $\int_1^\infty \left(\dfrac{\ln x}{x}\right)^2 dx = 2$, so the series also converges.

(b) Since the Integral Test applies, the error in $s \approx s_n$ is $R_n \le \int_n^\infty \left(\dfrac{\ln x}{x}\right)^2 dx = \dfrac{(\ln n)^2 + 2\ln n + 2}{n}$.

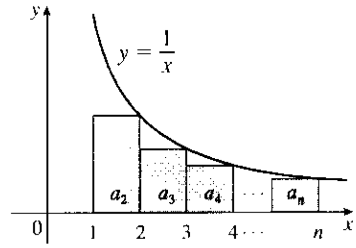
(c) By graphing the functions $y_1 = \dfrac{(\ln x)^2 + 2\ln x + 2}{x}$ and $y_2 = 0.05$, we see that $y_1 < y_2$ for $n \ge 1373$.

(d) Using the CAS to sum the first 1373 terms, we get $s_{1373} \approx 1.94$.

37. (a) From the figure, $a_2 + a_3 + \cdots + a_n \le \int_1^n f(x)\,dx$, so with

$f(x) = \dfrac{1}{x}$, $\dfrac{1}{2} + \dfrac{1}{3} + \dfrac{1}{4} + \cdots + \dfrac{1}{n} \le \displaystyle\int_1^n \frac{1}{x}\,dx = \ln n$. Thus,

$$s_n = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots + \frac{1}{n} \le 1 + \ln n.$$



(b) By part (a), $s_{10^6} \le 1 + \ln 10^6 \approx 14.82 < 15$ and $s_{10^9} \le 1 + \ln 10^9 \approx 21.72 < 22$.

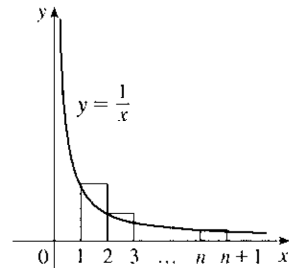
38. (a) The sum of the areas of the n rectangles in the graph to the right is

$1 + \dfrac{1}{2} + \dfrac{1}{3} + \cdots + \dfrac{1}{n}$. Now $\displaystyle\int_1^{n+1} \frac{dx}{x}$ is less than this sum because

the rectangles extend above the curve $y = 1/x$, so

$\displaystyle\int_1^{n+1} \frac{1}{x}\,dx = \ln(n+1) < 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$, and since

$\ln n < \ln(n+1)$, $0 < 1 + \dfrac{1}{2} + \dfrac{1}{3} + \cdots + \dfrac{1}{n} - \ln n = t_n$.



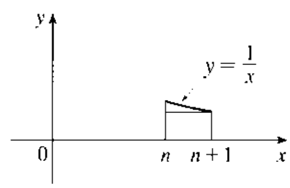
(b) The area under $f(x) = 1/x$ between $x = n$ and $x = n + 1$ is

$\displaystyle\int_n^{n+1} \frac{dx}{x} = \ln(n+1) - \ln n$, and this is clearly greater than the

area of the inscribed rectangle in the figure to the right

$\left[\text{which is } \dfrac{1}{n+1}\right]$, so $t_n - t_{n+1} = [\ln(n+1) - \ln n] - \dfrac{1}{n+1} > 0$,

and so $t_n > t_{n+1}$, so $\{t_n\}$ is a decreasing sequence.



(c) We have shown that $\{t_n\}$ is decreasing and that $t_n > 0$ for all n. Thus, $0 < t_n \le t_1 = 1$, so $\{t_n\}$ is a bounded monotonic sequence, and hence converges by Theorem 12.1.11.

39. $b^{\ln n} = \left(e^{\ln b}\right)^{\ln n} = \left(e^{\ln n}\right)^{\ln b} = n^{\ln b} = \dfrac{1}{n^{-\ln b}}$. This is a p-series, which converges for all b such that $-\ln b > 1$

$\Leftrightarrow \quad \ln b < -1 \quad \Leftrightarrow \quad b < e^{-1} \quad \Leftrightarrow \quad b < 1/e$ [with $b > 0$].

12.4 The Comparison Tests

1. (a) We cannot say anything about $\sum a_n$. If $a_n > b_n$ for all n and $\sum b_n$ is convergent, then $\sum a_n$ could be convergent or divergent. (See the note after Example 2.)

(b) If $a_n < b_n$ for all n, then $\sum a_n$ is convergent. [This is part (i) of the Comparison Test.]

2. (a) If $a_n > b_n$ for all n, then $\sum a_n$ is divergent. [This is part (ii) of the Comparison Test.]

(b) We cannot say anything about $\sum a_n$. If $a_n < b_n$ for all n and $\sum b_n$ is divergent, then $\sum a_n$ could be convergent or divergent.

3. $\dfrac{1}{n^2+n+1} < \dfrac{1}{n^2}$ for all $n \geq 1$, so $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^2+n+1}$ converges by comparison with $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^2}$, which converges because it is a p-series with $p = 2 > 1$.

4. $\dfrac{2}{n^3+4} < \dfrac{2}{n^3}$ for all $n \geq 1$, so $\sum\limits_{n=1}^{\infty} \dfrac{2}{n^3+4}$ converges by comparison with $\sum\limits_{n=1}^{\infty} \dfrac{2}{n^3} = 2\sum\limits_{n=1}^{\infty} \dfrac{1}{n^3}$, which converges because it is a constant multiple of a convergent p-series ($p = 3 > 1$).

5. $\dfrac{5}{2+3^n} < \dfrac{5}{3^n}$ for all $n \geq 1$, so $\sum\limits_{n=1}^{\infty} \dfrac{5}{2+3^n}$ converges by comparison with $\sum\limits_{n=1}^{\infty} \dfrac{5}{3^n} = 5\sum\limits_{n=1}^{\infty} \dfrac{1}{3^n}$, which converges because $\sum\limits_{n=1}^{\infty} \dfrac{1}{3^n}$ is a convergent geometric series with $r = \frac{1}{3}$ $(|r| < 1)$.

6. $\dfrac{1}{n-\sqrt{n}} > \dfrac{1}{n}$ for all $n \geq 2$, so $\sum\limits_{n=2}^{\infty} \dfrac{1}{n-\sqrt{n}}$ diverges by comparison with the divergent (partial) harmonic series $\sum\limits_{n=2}^{\infty} \dfrac{1}{n}$.

7. $\dfrac{n+1}{n^2} > \dfrac{n}{n^2} = \dfrac{1}{n}$ for all $n \geq 1$, so $\sum\limits_{n=1}^{\infty} \dfrac{n+1}{n^2}$ diverges by comparison with the harmonic series $\sum\limits_{n=1}^{\infty} \dfrac{1}{n}$.

8. $\dfrac{4+3^n}{2^n} > \dfrac{3^n}{2^n} = \left(\dfrac{3}{2}\right)^n$ for all $n \geq 1$, so $\sum\limits_{n=1}^{\infty} \dfrac{4+3^n}{2^n}$ diverges by comparison with the divergent geometric series $\sum_{n=1}^{\infty} \left(\frac{3}{2}\right)^n$.

9. $\dfrac{\cos^2 n}{n^2+1} \leq \dfrac{1}{n^2+1} < \dfrac{1}{n^2}$, so the series $\sum\limits_{n=1}^{\infty} \dfrac{\cos^2 n}{n^2+1}$ converges by comparison with the p-series $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^2}$ ($p = 2 > 1$).

10. $\dfrac{n^2-1}{3n^4+1} < \dfrac{n^2}{3n^4+1} < \dfrac{n^2}{3n^4} = \dfrac{1}{3}\dfrac{1}{n^2}$. $\sum\limits_{n=1}^{\infty} \dfrac{n^2-1}{3n^4+1}$ converges by comparison with $\sum\limits_{n=1}^{\infty} \dfrac{1}{3n^2}$, which converges because it is a constant multiple of a convergent p-series ($p = 2 > 1$). The terms of the given series are positive for $n > 1$, which is good enough.

11. If $a_n = \dfrac{n^2+1}{n^3-1}$ and $b_n = \dfrac{1}{n}$, then $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \lim\limits_{n\to\infty} \dfrac{n^3+n}{n^3-1} = \lim\limits_{n\to\infty} \dfrac{1+1/n^2}{1-1/n^3} = 1$, so $\sum\limits_{n=2}^{\infty} \dfrac{n^2+1}{n^3-1}$ diverges by the Limit Comparison Test with the divergent (partial) harmonic series $\sum\limits_{n=2}^{\infty} \dfrac{1}{n}$.

Or: Since $a_n = \dfrac{n^2+1}{n^3-1} > \dfrac{n^2+1}{n^3} > \dfrac{n^2}{n^3} = \dfrac{1}{n} = b_n$, we could use the Comparison Test.

12. $\dfrac{1+\sin n}{10^n} \leq \dfrac{2}{10^n}$ and $\sum\limits_{n=0}^{\infty} \dfrac{2}{10^n} = 2\sum\limits_{n=0}^{\infty} \left(\dfrac{1}{10}\right)^n$, so the given series converges by comparison with a constant multiple of a convergent geometric series.

13. $\dfrac{n-1}{n4^n}$ is positive for $n > 1$ and $\dfrac{n-1}{n4^n} < \dfrac{n}{n4^n} = \dfrac{1}{4^n}$, so $\sum\limits_{n=1}^{\infty} \dfrac{n-1}{n4^n}$ converges by comparison with the convergent geometric series $\sum\limits_{n=1}^{\infty} \left(\dfrac{1}{4}\right)^n$.

14. $\dfrac{\sqrt{n}}{n-1} > \dfrac{\sqrt{n}}{n} = \dfrac{1}{\sqrt{n}}$, so $\displaystyle\sum_{n=2}^{\infty} \frac{\sqrt{n}}{n-1}$ diverges by comparison with the divergent (partial) p-series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\sqrt{n}}$ $\left(p = \frac{1}{2} \leq 1\right)$.

15. $\dfrac{2+(-1)^n}{n\sqrt{n}} \leq \dfrac{3}{n\sqrt{n}}$, and $\displaystyle\sum_{n=1}^{\infty} \frac{3}{n\sqrt{n}}$ converges because it is a constant multiple of the convergent p-series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n\sqrt{n}}$ $\left(p = \frac{3}{2} > 1\right)$, so the given series converges by the Comparison Test.

16. $\dfrac{1}{\sqrt{n^3+1}} < \dfrac{1}{\sqrt{n^3}} = \dfrac{1}{n^{3/2}}$, so $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3+1}}$ converges by comparison with the convergent p-series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ $\left(p = \frac{3}{2} > 1\right)$.

17. Use the Limit Comparison Test with $a_n = \dfrac{1}{\sqrt{n^2+1}}$ and $b_n = \dfrac{1}{n}$:

$\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n}{\sqrt{n^2+1}} = \lim_{n\to\infty} \frac{1}{\sqrt{1+(1/n^2)}} = 1 > 0$. Since the harmonic series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, so does $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^2+1}}$.

18. Use the Limit Comparison Test with $a_n = \dfrac{1}{2n+3}$ and $b_n = \dfrac{1}{n}$:

$\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n}{2n+3} = \lim_{n\to\infty} \frac{1}{2+(3/n)} = \frac{1}{2} > 0$. Since the harmonic series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, so does $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n+3}$.

19. $\dfrac{2^n}{1+3^n} < \dfrac{2^n}{3^n} = \left(\dfrac{2}{3}\right)^n$. $\displaystyle\sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n$ is a convergent geometric series $\left(|r| = \frac{2}{3} < 1\right)$, so $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{1+3^n}$ converges by the Comparison Test.

20. Use the Limit Comparison Test with $a_n = \dfrac{1+2^n}{1+3^n}$ and $b_n = \dfrac{2^n}{3^n}$: $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{(1/2)^n+1}{(1/3)^n+1} = 1 > 0$. Since $\displaystyle\sum_{n=1}^{\infty} b_n$ converges (geometric series with $|r| = \frac{2}{3} < 1$), $\displaystyle\sum_{n=1}^{\infty} \frac{1+2^n}{1+3^n}$ also converges.

21. Use the Limit Comparison Test with $a_n = \dfrac{1}{1+\sqrt{n}}$ and $b_n = \dfrac{1}{\sqrt{n}}$: $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sqrt{n}}{1+\sqrt{n}} = 1 > 0$. Since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ is a divergent p-series $\left(p = \frac{1}{2} \leq 1\right)$, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{1+\sqrt{n}}$ also diverges.

22. Use the Limit Comparison Test with $a_n = \dfrac{n+2}{(n+1)^3}$ and $b_n = \dfrac{1}{n^2}$:

$\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^2(n+2)}{(n+1)^3} = \lim_{n\to\infty} \frac{1+\frac{2}{n}}{\left(1+\frac{1}{n}\right)^3} = 1 > 0$. Since $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n^2}$ is a convergent (partial) p-series $(p = 2 > 1)$, the series $\displaystyle\sum_{n=3}^{\infty} \frac{n+2}{(n+1)^3}$ also converges.

23. Use the Limit Comparison Test with $a_n = \dfrac{5+2n}{(1+n^2)^2}$ and $b_n = \dfrac{1}{n^3}$:

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{n^3(5+2n)}{(1+n^2)^2} = \lim_{n\to\infty}\frac{5n^3+2n^4}{(1+n^2)^2}\cdot\frac{1/n^4}{1/(n^2)^2} = \lim_{n\to\infty}\frac{\frac{5}{n}+2}{\left(\frac{1}{n^2}+1\right)^2} = 2 > 0.$$ Since $\sum_{n=1}^{\infty}\frac{1}{n^3}$ is a convergent p-series ($p = 3 > 1$), the series $\sum_{n=1}^{\infty}\frac{5+2n}{(1+n^2)^2}$ also converges.

24. If $a_n = \dfrac{n^2-5n}{n^3+n+1}$ and $b_n = \dfrac{1}{n}$, then $\lim\limits_{n\to\infty}\dfrac{a_n}{b_n} = \lim\limits_{n\to\infty}\dfrac{n^3-5n^2}{n^3+n+1} = \lim\limits_{n\to\infty}\dfrac{1-5/n}{1+1/n^2+1/n^3} = 1 > 0$, so $\sum_{n=1}^{\infty}\frac{n^2-5n}{n^3+n+1}$ diverges by the Limit Comparison Test with the divergent harmonic series $\sum_{n=1}^{\infty}\frac{1}{n}$. (Note that $a_n > 0$ for $n \geq 6$.)

25. If $a_n = \dfrac{1+n+n^2}{\sqrt{1+n^2+n^6}}$ and $b_n = \dfrac{1}{n}$, then

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{n+n^2+n^3}{\sqrt{1+n^2+n^6}} = \lim_{n\to\infty}\frac{1/n^2+1/n+1}{\sqrt{1/n^6+1/n^4+1}} = 1 > 0,$$ so $\sum_{n=1}^{\infty}\frac{1+n+n^2}{\sqrt{1+n^2+n^6}}$ diverges by the Limit Comparison Test with the divergent harmonic series $\sum_{n=1}^{\infty}\frac{1}{n}$.

26. If $a_n = \dfrac{n+5}{\sqrt[3]{n^7+n^2}}$ and $b_n = \dfrac{n}{\sqrt[3]{n^7}} = \dfrac{n}{n^{7/3}} = \dfrac{1}{n^{4/3}}$, then

$$\begin{aligned}\lim_{n\to\infty}\frac{a_n}{b_n} &= \lim_{n\to\infty}\frac{n^{7/3}+5n^{4/3}}{(n^7+n^2)^{1/3}}\cdot\frac{n^{-7/3}}{n^{-7/3}} = \lim_{n\to\infty}\frac{1+5/n}{\left[(n^7+n^2)/n^7\right]^{1/3}} \\ &= \lim_{n\to\infty}\frac{1+5/n}{(1+1/n^5)^{1/3}} = \frac{1+0}{(1+0)^{1/3}} = 1 > 0,\end{aligned}$$

so $\sum_{n=1}^{\infty}\frac{n+5}{\sqrt[3]{n^7+n^2}}$ converges by the Limit Comparison Test with the convergent p-series $\sum_{n=1}^{\infty}\frac{1}{n^{4/3}}$.

27. Use the Limit Comparison Test with $a_n = \left(1+\dfrac{1}{n}\right)^2 e^{-n}$ and $b_n = e^{-n}$: $\lim\limits_{n\to\infty}\dfrac{a_n}{b_n} = \lim\limits_{n\to\infty}\left(1+\dfrac{1}{n}\right)^2 = 1 > 0$. Since $\sum_{n=1}^{\infty}e^{-n} = \sum_{n=1}^{\infty}\frac{1}{e^n}$ is a convergent geometric series $\left(|r| = \frac{1}{e} < 1\right)$, the series $\sum_{n=1}^{\infty}\left(1+\frac{1}{n}\right)^2 e^{-n}$ also converges.

28. Use the Limit Comparison Test with $a_n = \dfrac{2n^2+7n}{3^n(n^2+5n-1)}$ and $b_n = \dfrac{1}{3^n}$.

$\lim\limits_{n\to\infty}\dfrac{a_n}{b_n} = \lim\limits_{n\to\infty}\dfrac{2n^2+7n}{n^2+5n-1} = 2 > 0$, and since $\sum_{n=1}^{\infty}b_n$ is a convergent geometric series ($|r| = \frac{1}{3} < 1$), $\sum_{n=1}^{\infty}\frac{2n^2+7n}{3^n(n^2+5n-1)}$ converges also.

29. Clearly $n! = n(n-1)(n-2)\cdots(3)(2) \geq 2\cdot 2\cdot 2\cdot\cdots\cdot 2\cdot 2 = 2^{n-1}$, so $\dfrac{1}{n!} \leq \dfrac{1}{2^{n-1}}$. $\sum_{n=1}^{\infty}\frac{1}{2^{n-1}}$ is a convergent geometric series ($|r| = \frac{1}{2} < 1$), so $\sum_{n=1}^{\infty}\frac{1}{n!}$ converges by the Comparison Test.

30. $\dfrac{n!}{n^n} = \dfrac{1\cdot 2\cdot 3\cdot\cdots\cdot(n-1)n}{n\cdot n\cdot n\cdot\cdots\cdot n\cdot n} \leq \dfrac{1}{n}\cdot\dfrac{2}{n}\cdot 1\cdot 1\cdot\cdots\cdot 1$ for $n \geq 2$, so since $\sum_{n=1}^{\infty}\frac{2}{n^2}$ converges ($p = 2 > 1$), $\sum_{n=1}^{\infty}\frac{n!}{n^n}$ converges also by the Comparison Test.

31. Use the Limit Comparison Test with $a_n = \sin\left(\frac{1}{n}\right)$ and $b_n = \frac{1}{n}$. Then $\sum a_n$ and $\sum b_n$ are series with positive terms and $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = \lim\limits_{n\to\infty} \frac{\sin(1/n)}{1/n} = \lim\limits_{\theta\to 0} \frac{\sin\theta}{\theta} = 1 > 0$. Since $\sum_{n=1}^{\infty} b_n$ is the divergent harmonic series, $\sum_{n=1}^{\infty} \sin(1/n)$ also diverges. (Note that we could also use l'Hospital's Rule to evaluate the limit: $\lim\limits_{x\to\infty} \frac{\sin(1/x)}{1/x} \overset{H}{=} \lim\limits_{x\to\infty} \frac{\cos(1/x)\cdot(-1/x^2)}{-1/x^2} = \lim\limits_{x\to\infty} \cos\frac{1}{x} = \cos 0 = 1$.)

32. Use the Limit Comparison Test with $a_n = \frac{1}{n^{1+1/n}}$ and $b_n = \frac{1}{n}$. $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = \lim\limits_{n\to\infty} \frac{n}{n^{1+1/n}} = \lim\limits_{n\to\infty} \frac{1}{n^{1/n}} = 1$ (since $\lim\limits_{x\to\infty} x^{1/x} = 1$ by l'Hospital's Rule), so $\sum\limits_{n=1}^{\infty} \frac{1}{n}$ diverges (harmonic series) $\Rightarrow$ $\sum\limits_{n=1}^{\infty} \frac{1}{n^{1+1/n}}$ diverges.

33. $\sum\limits_{n=1}^{10} \frac{1}{n^4+n^2} = \frac{1}{2} + \frac{1}{20} + \frac{1}{90} + \cdots + \frac{1}{10{,}100} \approx 0.567975$. Now $\frac{1}{n^4+n^2} < \frac{1}{n^4}$, so using the reasoning and notation of Example 5, the error is $R_{10} \le T_{10} = \sum\limits_{n=11}^{\infty} \frac{1}{n^4} \le \int_{10}^{\infty} \frac{dx}{x^4} = \lim\limits_{t\to\infty} \left[-\frac{x^{-3}}{3}\right]_{10}^{t} = \frac{1}{3000} = 0.000\overline{3}$.

34. $\sum\limits_{n=1}^{10} \frac{1+\cos n}{n^5} = 1 + \cos 1 + \frac{1+\cos 2}{32} + \frac{1+\cos 3}{243} + \cdots + \frac{1+\cos 10}{100{,}000} \approx 1.55972$. Now $\frac{1+\cos n}{n^5} \le \frac{2}{n^5}$, so as in Example 5, $R_{10} \le T_{10} \le \int_{10}^{\infty} \frac{2}{x^5}\,dx = 2\lim\limits_{t\to\infty} \left[-\frac{1}{4}x^{-4}\right]_{10}^{t} = 0.00005$.

35. $\sum\limits_{n=1}^{10} \frac{1}{1+2^n} = \frac{1}{3} + \frac{1}{5} + \frac{1}{9} + \cdots + \frac{1}{1025} \approx 0.76352$. Now $\frac{1}{1+2^n} < \frac{1}{2^n}$, so the error is $R_{10} \le T_{10} = \sum\limits_{n=11}^{\infty} \frac{1}{2^n} = \frac{1/2^{11}}{1-1/2}$ (geometric series) ≈ 0.00098.

36. $\sum\limits_{n=1}^{10} \frac{n}{(n+1)3^n} = \frac{1}{6} + \frac{2}{27} + \frac{3}{108} + \cdots + \frac{10}{649{,}539} \approx 0.283597$. Now $\frac{n}{(n+1)3^n} < \frac{n}{n\cdot 3^n} = \frac{1}{3^n}$, so the error is $R_{10} \le T_{10} = \sum\limits_{n=11}^{\infty} \frac{1}{3^n} = \frac{1/3^{11}}{1-1/3} \approx 0.0000085$.

37. Since $\frac{d_n}{10^n} \le \frac{9}{10^n}$ for each n, and since $\sum\limits_{n=1}^{\infty} \frac{9}{10^n}$ is a convergent geometric series ($|r| = \frac{1}{10} < 1$), $0.d_1d_2d_3\ldots = \sum\limits_{n=1}^{\infty} \frac{d_n}{10^n}$ will always converge by the Comparison Test.

38. Clearly, if $p < 0$ then the series diverges, since $\lim\limits_{n\to\infty} \frac{1}{n^p \ln n} = \infty$. If $0 \le p \le 1$, then $n^p \ln n \le n \ln n$ $\Rightarrow$ $\frac{1}{n^p \ln n} \ge \frac{1}{n \ln n}$ and $\sum\limits_{n=2}^{\infty} \frac{1}{n \ln n}$ diverges (Exercise 12.3.21), so $\sum\limits_{n=2}^{\infty} \frac{1}{n^p \ln n}$ diverges. If $p > 1$, use the Limit Comparison Test with $a_n = \frac{1}{n^p \ln n}$ and $b_n = \frac{1}{n^p}$. $\sum\limits_{n=2}^{\infty} b_n$ converges, and $\lim\limits_{n\to\infty} \frac{a_n}{b_n} = \lim\limits_{n\to\infty} \frac{1}{\ln n} = 0$, so $\sum\limits_{n=2}^{\infty} \frac{1}{n^p \ln n}$ also converges. (Or use the Comparison Test, since $n^p \ln n > n^p$ for $n > e$.) In summary, the series converges if and only if $p > 1$.

39. Since $\sum a_n$ converges, $\lim\limits_{n\to\infty} a_n = 0$, so there exists N such that $|a_n - 0| < 1$ for all $n > N$ $\Rightarrow$ $0 \le a_n < 1$ for all $n > N$ $\Rightarrow$ $0 \le a_n^2 \le a_n$. Since $\sum a_n$ converges, so does $\sum a_n^2$ by the Comparison Test.

40. (a) Since $\lim_{n\to\infty}(a_n/b_n)=0$, there is a number $N>0$ such that $|a_n/b_n-0|<1$ for all $n>N$, and so $a_n<b_n$ since a_n and b_n are positive. Thus, since $\sum b_n$ converges, so does $\sum a_n$ by the Comparison Test.

(b) (i) If $a_n=\dfrac{\ln n}{n^3}$ and $b_n=\dfrac{1}{n^2}$, then $\lim\limits_{n\to\infty}\dfrac{a_n}{b_n}=\lim\limits_{n\to\infty}\dfrac{\ln n}{n}=\lim\limits_{x\to\infty}\dfrac{\ln x}{x}\overset{H}{=}\lim\limits_{x\to\infty}\dfrac{1/x}{1}=0$, so $\sum\limits_{n=1}^{\infty}\dfrac{\ln n}{n^3}$ converges by part (a).

(ii) If $a_n=\dfrac{\ln n}{\sqrt{n}e^n}$ and $b_n=\dfrac{1}{e^n}$, then

$$\lim_{n\to\infty}\frac{a_n}{b_n}=\lim_{n\to\infty}\frac{\ln n}{\sqrt{n}}=\lim_{x\to\infty}\frac{\ln x}{\sqrt{x}}\overset{H}{=}\lim_{x\to\infty}\frac{1/x}{1/(2\sqrt{x})}=\lim_{x\to\infty}\frac{2}{\sqrt{x}}=0$$

Now $\sum b_n$ is a convergent geometric series with ratio $r=1/e$ ($|r|<1$), so $\sum a_n$ converges by part (a).

41. (a) Since $\lim\limits_{n\to\infty}\dfrac{a_n}{b_n}=\infty$, there is an integer N such that $\dfrac{a_n}{b_n}>1$ whenever $n>N$. (Take $M=1$ in Definition 12.1.5.) Then $a_n>b_n$ whenever $n>N$ and since $\sum b_n$ is divergent, $\sum a_n$ is also divergent by the Comparison Test.

(b) (i) If $a_n=\dfrac{1}{\ln n}$ and $b_n=\dfrac{1}{n}$ for $n\geq 2$, then

$\lim\limits_{n\to\infty}\dfrac{a_n}{b_n}=\lim\limits_{n\to\infty}\dfrac{n}{\ln n}=\lim\limits_{x\to\infty}\dfrac{x}{\ln x}\overset{H}{=}\lim\limits_{x\to\infty}\dfrac{1}{1/x}=\lim\limits_{x\to\infty}x=\infty$, so by part (a), $\sum\limits_{n=2}^{\infty}\dfrac{1}{\ln n}$ is divergent.

(ii) If $a_n=\dfrac{\ln n}{n}$ and $b_n=\dfrac{1}{n}$, then $\sum\limits_{n=1}^{\infty}b_n$ is the divergent harmonic series and

$\lim\limits_{n\to\infty}\dfrac{a_n}{b_n}=\lim\limits_{n\to\infty}\ln n=\lim\limits_{x\to\infty}\ln x=\infty$, so $\sum\limits_{n=1}^{\infty}a_n$ diverges by part (a).

42. Let $a_n=\dfrac{1}{n^2}$ and $b_n=\dfrac{1}{n}$. Then $\lim\limits_{n\to\infty}\dfrac{a_n}{b_n}=\lim\limits_{n\to\infty}\dfrac{1}{n}=0$, but $\sum b_n$ diverges while $\sum a_n$ converges.

43. $\lim\limits_{n\to\infty}na_n=\lim\limits_{n\to\infty}\dfrac{a_n}{1/n}$, so we apply the Limit Comparison Test with $b_n=\dfrac{1}{n}$. Since $\lim\limits_{n\to\infty}na_n>0$ we know that either both series converge or both series diverge, and we also know that $\sum\limits_{n=0}^{\infty}\dfrac{1}{n}$ diverges (p-series with $p=1$). Therefore, $\sum a_n$ must be divergent.

44. First we observe that, by l'Hospital's Rule, $\lim\limits_{x\to 0}\dfrac{\ln(1+x)}{x}=\lim\limits_{x\to 0}\dfrac{1}{1+x}=1$. Also, if $\sum a_n$ converges, then $\lim\limits_{n\to\infty}a_n=0$ by Theorem 12.2.6. Therefore, $\lim\limits_{n\to\infty}\dfrac{\ln(1+a_n)}{a_n}=\lim\limits_{x\to 0}\dfrac{\ln(1+x)}{x}=1>0$. We are given that $\sum a_n$ is convergent and $a_n>0$. Thus, $\sum\ln(1+a_n)$ is convergent by the Limit Comparison Test.

45. Yes. Since $\sum a_n$ is a convergent series with positive terms, $\lim\limits_{n\to\infty}a_n=0$ by Theorem 12.2.6, and $\sum b_n=\sum\sin(a_n)$ is a series with positive terms (for large enough n). We have $\lim\limits_{n\to\infty}\dfrac{b_n}{a_n}=\lim\limits_{n\to\infty}\dfrac{\sin(a_n)}{a_n}=1>0$ by Theorem 3.5.2. Thus, $\sum b_n$ is also convergent by the Limit Comparison Test.

46. Yes. Since $\sum a_n$ converges, its terms approach 0 as $n\to\infty$, so for some integer N, $a_n\leq 1$ for all $n\geq N$. But then $\sum_{n=1}^{\infty}a_nb_n=\sum_{n=1}^{N-1}a_nb_n+\sum_{n=N}^{\infty}a_nb_n\leq\sum_{n=1}^{N-1}a_nb_n+\sum_{n=N}^{\infty}b_n$. The first term is a finite sum, and the second term converges since $\sum_{n=1}^{\infty}b_n$ converges. So $\sum a_nb_n$ converges by the Comparison Test.

12.5 Alternating Series

1. (a) An alternating series is a series whose terms are alternately positive and negative.

(b) An alternating series $\sum_{n=1}^{\infty}(-1)^{n-1}b_n$ converges if $0 < b_{n+1} \leq b_n$ for all n and $\lim\limits_{n\to\infty} b_n = 0$. (This is the Alternating Series Test.)

(c) The error involved in using the partial sum s_n as an approximation to the total sum s is the remainder $R_n = s - s_n$ and the size of the error is smaller than b_{n+1}; that is, $|R_n| \leq b_{n+1}$. (This is the Alternating Series Estimation Theorem.)

2. $-\frac{1}{3}+\frac{2}{4}-\frac{3}{5}+\frac{4}{6}-\frac{5}{7}+\cdots=\sum\limits_{n=1}^{\infty}(-1)^n\frac{n}{n+2}$. Here $a_n=(-1)^n\frac{n}{n+2}$. Since $\lim\limits_{n\to\infty} a_n \neq 0$ (in fact the limit does not exist), the series diverges by the Test for Divergence.

3. $\frac{4}{7}-\frac{4}{8}+\frac{4}{9}-\frac{4}{10}+\frac{4}{11}-\cdots=\sum\limits_{n=1}^{\infty}(-1)^{n-1}\frac{4}{n+6}$. Now $b_n=\frac{4}{n+6}>0$, $\{b_n\}$ is decreasing, and $\lim\limits_{n\to\infty} b_n = 0$, so the series converges by the Alternating Series Test.

4. $\sum\limits_{n=2}^{\infty}(-1)^n\frac{1}{\ln n}$. $b_n=\frac{1}{\ln n}$ is positive and $\{b_n\}$ is decreasing; $\lim\limits_{n\to\infty}\frac{1}{\ln n}=0$, so the series converges by the Alternating Series Test.

5. $b_n=\frac{1}{\sqrt{n}}>0$, $\{b_n\}$ is decreasing, and $\lim\limits_{n\to\infty} b_n=0$, so the series $\sum\limits_{n=1}^{\infty}\frac{(-1)^{n-1}}{\sqrt{n}}$ converges by the Alternating Series Test.

6. $b_n=\frac{1}{3n-1}>0$, $\{b_n\}$ is decreasing, and $\lim\limits_{n\to\infty} b_n=0$, so the series $\sum\limits_{n=1}^{\infty}\frac{(-1)^{n-1}}{3n-1}$ converges by the Alternating Series Test.

7. $\sum\limits_{n=1}^{\infty}a_n=\sum\limits_{n=1}^{\infty}(-1)^n\frac{3n-1}{2n+1}=\sum\limits_{n=1}^{\infty}(-1)^n b_n$. Now $\lim\limits_{n\to\infty} b_n=\lim\limits_{n\to\infty}\frac{3-1/n}{2+1/n}=\frac{3}{2}\neq 0$. Since $\lim\limits_{n\to\infty} a_n\neq 0$ (in fact the limit does not exist), the series diverges by the Test for Divergence.

8. $b_n=\frac{2n}{4n^2+1}>0$, $\{b_n\}$ is decreasing [since

$$b_n-b_{n+1}=\frac{2n}{4n^2+1}-\frac{2n+2}{4n^2+8n+5}=\frac{8n^2+8n-2}{(4n^2+1)(4n^2+8n+5)}>0 \text{ for } n\geq 1], \text{ and}$$

$\lim\limits_{n\to\infty} b_n=\lim\limits_{n\to\infty}\frac{2/n}{4+1/n^2}=0$, so the series $\sum\limits_{n=1}^{\infty}(-1)^n\frac{2n}{4n^2+1}$ converges by the Alternating Series Test.

Alternatively, to show that $\{b_n\}$ is decreasing, we could verify that $\frac{d}{dx}\left(\frac{2x}{4x^2+1}\right)<0$ for $x\geq 1$.

9. $b_n=\frac{1}{4n^2+1}>0$, $\{b_n\}$ is decreasing, and $\lim\limits_{n\to\infty} b_n=0$, so the series $\sum\limits_{n=1}^{\infty}\frac{(-1)^{n+1}}{4n^2+1}$ converges by the Alternating Series Test.

10. $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} (-1)^n \frac{\sqrt{n}}{1+2\sqrt{n}} = \sum_{n=1}^{\infty} (-1)^n b_n$. Now $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{2+1/\sqrt{n}} = \frac{1}{2} \neq 0$. Since $\lim_{n\to\infty} a_n \neq 0$ (in fact the limit does not exist), the series diverges by the Test for Divergence.

11. $b_n = \frac{n^2}{n^3+4} > 0$ for $n \geq 1$. $\{b_n\}$ is decreasing for $n \geq 2$ since

$$\left(\frac{x^2}{x^3+4}\right)' = \frac{(x^3+4)(2x) - x^2(3x^2)}{(x^3+4)^2} = \frac{x(2x^3+8-3x^3)}{(x^3+4)^2} = \frac{x(8-x^3)}{(x^3+4)^2} < 0 \text{ for } x > 2. \text{ Also,}$$

$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1/n}{1+4/n^3} = 0$. Thus, the series $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{n^3+4}$ converges by the Alternating Series Test.

12. $b_n = \frac{e^{1/n}}{n} > 0$ for $n \geq 1$. $\{b_n\}$ is decreasing since

$$\left(\frac{e^{1/x}}{x}\right)' = \frac{x \cdot e^{1/x}(-1/x^2) - e^{1/x} \cdot 1}{x^2} = \frac{-e^{1/x}(1+x)}{x^3} < 0 \text{ for } x > 0. \text{ Also, } \lim_{n\to\infty} b_n = 0 \text{ since}$$

$\lim_{n\to\infty} e^{1/n} = 1$. Thus, the series $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{e^{1/n}}{n}$ converges by the Alternating Series Test.

13. $\sum_{n=2}^{\infty} (-1)^n \frac{n}{\ln n}$. $\lim_{n\to\infty} \frac{n}{\ln n} = \lim_{x\to\infty} \frac{x}{\ln x} \overset{H}{=} \lim_{x\to\infty} \frac{1}{1/x} = \infty$, so the series diverges by the Test for Divergence.

14. $\sum_{n=1}^{\infty} (-1)^{n-1} \left(\frac{\ln n}{n}\right) = 0 + \sum_{n=2}^{\infty} (-1)^{n-1} \left(\frac{\ln n}{n}\right)$. $b_n = \frac{\ln n}{n} > 0$ for $n \geq 2$, and if $f(x) = \frac{\ln x}{x}$,

then $f'(x) = \frac{1 - \ln x}{x^2} < 0$ for $x > e$, so $\{b_n\}$ is eventually decreasing. Also,

$\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{\ln n}{n} = \lim_{x\to\infty} \frac{\ln x}{x} \overset{H}{=} \lim_{x\to\infty} \frac{1/x}{1} = 0$, so the series converges by the Alternating Series Test.

15. $\sum_{n=1}^{\infty} \frac{\cos n\pi}{n^{3/4}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^{3/4}}$. $b_n = \frac{1}{n^{3/4}}$ is decreasing and positive and $\lim_{n\to\infty} \frac{1}{n^{3/4}} = 0$, so the series converges by the Alternating Series Test.

16. $\sin\left(\frac{n\pi}{2}\right) = 0$ if n is even and $(-1)^k$ if $n = 2k+1$, so the series is $\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}$. $b_n = \frac{1}{(2n+1)!} > 0$, $\{b_n\}$ is decreasing, and $\lim_{n\to\infty} \frac{1}{(2n+1)!} = 0$, so the series converges by the Alternating Series Test.

17. $\sum_{n=1}^{\infty} (-1)^n \sin\frac{\pi}{n}$. $b_n = \sin\frac{\pi}{n} > 0$ for $n \geq 2$ and $\sin\frac{\pi}{n} \geq \sin\frac{\pi}{n+1}$, and $\lim_{n\to\infty} \sin\frac{\pi}{n} = \sin 0 = 0$, so the series converges by the Alternating Series Test.

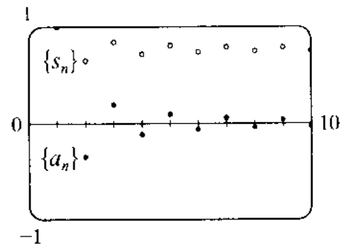
18. $\sum_{n=1}^{\infty} (-1)^n \cos\left(\frac{\pi}{n}\right)$. $\lim_{n\to\infty} \cos\left(\frac{\pi}{n}\right) = \cos(0) = 1$, so $\lim_{n\to\infty} (-1)^n \cos\left(\frac{\pi}{n}\right)$ does not exist and the series diverges by the Test for Divergence.

19. $\frac{n^n}{n!} = \frac{n \cdot n \cdot \cdots \cdot n}{1 \cdot 2 \cdot \cdots \cdot n} \geq n \quad \Rightarrow \quad \lim_{n\to\infty} \frac{n^n}{n!} = \infty \quad \Rightarrow \quad \lim_{n\to\infty} \frac{(-1)^n n^n}{n!}$ does not exist. So the series diverges by the Test for Divergence.

20. $\sum_{n=1}^{\infty}\left(-\frac{n}{5}\right)^n$ diverges by the Test for Divergence since $\lim_{n\to\infty}\left(\frac{n}{5}\right)^n=\infty \quad\Rightarrow\quad \lim_{n\to\infty}\left(-\frac{n}{5}\right)^n$ does not exist.

21.

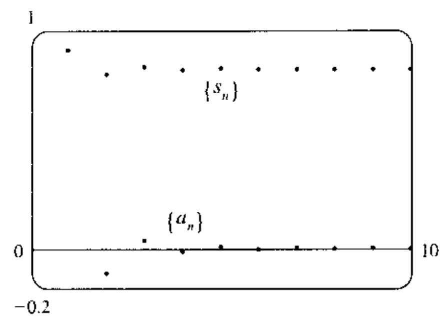
n	a_n	s_n
1	1	1
2	−0.35355	0.64645
3	0.19245	0.83890
4	−0.125	0.71390
5	0.08944	0.80334
6	−0.06804	0.73530
7	0.05399	0.78929
8	−0.04419	0.74510
9	0.03704	0.78214
10	−0.03162	0.75051



By the Alternating Series Estimation Theorem, the error in the approximation $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^{3/2}}\approx 0.75051$ is $|s-s_{10}|\le b_{11}=1/(11)^{3/2}\approx 0.0275$ (to four decimal places, rounded up).

22.

n	a_n	s_n
1	1	1
2	−0.125	0.875
3	0.03704	0.91204
4	−0.01563	0.89641
5	0.008	0.90441
6	−0.00463	0.89978
7	0.00292	0.90270
8	−0.00195	0.90074
9	0.00137	0.90212
10	−0.001	0.90112



By the Alternating Series Estimation Theorem, the error in the approximation $\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^3}\approx 0.90112$ is $|s-s_{10}|\le b_{11}=1/11^3\approx 0.0007513$.

23. The series $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{n^2}$ satisfies (i) of the Alternating Series Test because $\frac{1}{(n+1)^2}<\frac{1}{n^2}$ and (ii) $\lim_{n\to\infty}\frac{1}{n^2}=0$, so the series is convergent. Now $b_{10}=\frac{1}{10^2}=0.01$ and $b_{11}=\frac{1}{11^2}=\frac{1}{121}\approx 0.008<0.01$, so by the Alternating Series Estimation Theorem, $n=10$. (That is, since the 11th term is less than the desired error, we need to add the first 10 terms to get the sum to the desired accuracy.)

24. The series $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n^4}$ satisfies (i) of the Alternating Series Test because $\frac{1}{(n+1)^4}<\frac{1}{n^4}$ and (ii) $\lim_{n\to\infty}\frac{1}{n^4}=0$, so the series is convergent. Now $b_5=1/5^4=0.0016>0.001$ and $b_6=1/6^4\approx 0.00077<0.001$, so by the Alternating Series Estimation Theorem, $n=5$.

25. The series $\displaystyle\sum_{n=1}^{\infty}\frac{(-2)^n}{n!}=\sum_{n=1}^{\infty}(-1)^n\frac{2^n}{n!}$ satisfies (i) of the Alternating Series Test

because $b_{n+1}=\dfrac{2^{n+1}}{(n+1)!}=\dfrac{2\cdot 2^n}{(n+1)n!}=\dfrac{2}{n+1}\cdot\dfrac{2^n}{n!}=\dfrac{2}{n+1}\cdot b_n\leq b_n$ and (ii)

$\displaystyle\lim_{n\to\infty}\frac{2^n}{n!}=\frac{2}{n}\cdot\frac{2}{n-1}\cdot\cdots\cdot\frac{2}{2}\cdot\frac{2}{1}=0$, so the series is convergent. Now $b_7=2^7/7!\approx 0.025>0.01$ and $b_8=2^8/8!\approx 0.006<0.01$, so by the Alternating Series Estimation Theorem, $n=7$. (That is, since the 8th term is less than the desired error, we need to add the first 7 terms to get the sum to the desired accuracy.)

26. The series $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n n}{4^n}=\sum_{n=1}^{\infty}(-1)^n\frac{n}{4^n}$ satisfies (i) of the Alternating Series Test because

$b_{n+1}=\dfrac{n+1}{4^{n+1}}<\dfrac{n+3n}{4^n\cdot 4^1}=\dfrac{4n}{4\cdot 4^n}=\dfrac{n}{4^n}=b_n$ and (ii) $\displaystyle\lim_{n\to\infty}\frac{n}{4^n}=0$, so the series is convergent. Now $b_5=5/4^5\approx 0.0049>0.002$ and $b_6=6/4^6\approx 0.0015<0.002$, so by the Alternating Series Estimation Theorem, $n=5$.

27. $b_7=\dfrac{1}{7^5}=\dfrac{1}{16{,}807}\approx 0.000\,059\,5$, so

$\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^5}\approx s_6=\sum_{n=1}^{6}\frac{(-1)^{n+1}}{n^5}=1-\tfrac{1}{32}+\tfrac{1}{243}-\tfrac{1}{1024}+\tfrac{1}{3125}-\tfrac{1}{7776}\approx 0.972\,080$. Adding b_7 to s_6 does not change the fourth decimal place of s_6, so the sum of the series, correct to four decimal places, is 0.9721.

28. $b_6=\dfrac{6}{8^6}=\dfrac{6}{262{,}144}\approx 0.000\,023$, so

$\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n n}{8^n}\approx s_5=\sum_{n=1}^{5}\frac{(-1)^n n}{8^n}=-\tfrac{1}{8}+\tfrac{2}{64}-\tfrac{3}{512}+\tfrac{4}{4096}-\tfrac{5}{32{,}768}\approx -0.098\,785$. Adding b_6 to s_5 does not change the fourth decimal place of s_5, so the sum of the series, correct to four decimal places, is -0.0988.

29. $b_7=\dfrac{7^2}{10^7}=0.000\,004\,9$, so

$\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^{n-1}n^2}{10^n}\approx s_6=\sum_{n=1}^{6}\frac{(-1)^{n-1}n^2}{10^n}=\tfrac{1}{10}-\tfrac{4}{100}+\tfrac{9}{1000}-\tfrac{16}{10{,}000}+\tfrac{25}{100{,}000}-\tfrac{36}{1{,}000{,}000}=0.067\,614$.

Adding b_7 to s_6 does not change the fourth decimal place of s_6, so the sum of the series, correct to four decimal places, is 0.0676.

30. $b_6=\dfrac{1}{3^6\cdot 6!}=\dfrac{1}{524{,}880}\approx 0.000\,001\,9$, so

$\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{3^n n!}\approx s_5=\sum_{n=1}^{5}\frac{(-1)^n}{3^n n!}=-\tfrac{1}{3}+\tfrac{1}{18}-\tfrac{1}{162}+\tfrac{1}{1944}-\tfrac{1}{29{,}160}\approx -0.283\,471$. Adding b_6 to s_5 does not change the fourth decimal place of s_5, so the sum of the series, correct to four decimal places, is -0.2835.

31. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{49} - \frac{1}{50} + \frac{1}{51} - \frac{1}{52} + \cdots$. The 50th partial sum of this series is an underestimate, since $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = s_{50} + \left(\frac{1}{51} - \frac{1}{52}\right) + \left(\frac{1}{53} - \frac{1}{54}\right) + \cdots$, and the terms in parentheses are all positive. The result can be seen geometrically in Figure 1.

32. If $p > 0$, $\frac{1}{(n+1)^p} \leq \frac{1}{n^p}$ ($\{1/n^p\}$ is decreasing) and $\lim_{n\to\infty} \frac{1}{n^p} = 0$, so the series converges by the Alternating Series Test. If $p \leq 0$, $\lim_{n\to\infty} \frac{(-1)^{n-1}}{n^p}$ does not exist, so the series diverges by the Test for Divergence. Thus, $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^p}$ converges $\Leftrightarrow$ $p > 0$.

33. Clearly $b_n = \frac{1}{n+p}$ is decreasing and eventually positive and $\lim_{n\to\infty} b_n = 0$ for any p. So the series converges (by the Alternating Series Test) for any p for which every b_n is defined, that is, $n + p \neq 0$ for $n \geq 1$, or p is not a negative integer.

34. Let $f(x) = \frac{(\ln x)^p}{x}$. Then $f'(x) = \frac{(\ln x)^{p-1}(p - \ln x)}{x^2} < 0$ if $x > e^p$ so f is eventually decreasing for every p. Clearly $\lim_{n\to\infty} \frac{(\ln n)^p}{n} = 0$ if $p \leq 0$, and if $p > 0$ we can apply l'Hospital's Rule $[\![p+1]\!]$ times to get a limit of 0 as well. So the series converges for all p (by the Alternating Series Test).

35. $\sum b_{2n} = \sum 1/(2n)^2$ clearly converges (by comparison with the p-series for $p = 2$). So suppose that $\sum (-1)^{n-1} b_n$ converges. Then by Theorem 12.2.8(ii), so does $\sum \left[(-1)^{n-1} b_n + b_n\right] = 2\left(1 + \frac{1}{3} + \frac{1}{5} + \cdots\right) = 2\sum \frac{1}{2n-1}$. But this diverges by comparison with the harmonic series, a contradiction. Therefore, $\sum (-1)^{n-1} b_n$ must diverge. The Alternating Series Test does not apply since $\{b_n\}$ is not decreasing.

36. (a) We will prove this by induction. Let $P(n)$ be the proposition that $s_{2n} = h_{2n} - h_n$. $P(1)$ is the statement $s_2 = h_2 - h_1$, which is true since $1 - \frac{1}{2} = \left(1 + \frac{1}{2}\right) - 1$. So suppose that $P(n)$ is true. We will show that $P(n+1)$ must be true as a consequence.

$$h_{2n+2} - h_{n+1} = \left(h_{2n} + \frac{1}{2n+1} + \frac{1}{2n+2}\right) - \left(h_n + \frac{1}{n+1}\right) = (h_{2n} - h_n) + \frac{1}{2n+1} - \frac{1}{2n+2}$$

$$= s_{2n} + \frac{1}{2n+1} - \frac{1}{2n+2} = s_{2n+2}$$

which is $P(n+1)$, and proves that $s_{2n} = h_{2n} - h_n$ for all n.

(b) We know that $h_{2n} - \ln(2n) \to \gamma$ and $h_n - \ln n \to \gamma$ as $n \to \infty$. So $s_{2n} = h_{2n} - h_n = [h_{2n} - \ln(2n)] - (h_n - \ln n) + [\ln(2n) - \ln n]$, and $\lim_{n\to\infty} s_{2n} = \gamma - \gamma + \lim_{n\to\infty} [\ln(2n) - \ln n] = \lim_{n\to\infty} (\ln 2 + \ln n - \ln n) = \ln 2$.

12.6 Absolute Convergence and the Ratio and Root Tests

1. (a) Since $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = 8 > 1$, part (b) of the Ratio Test tells us that the series $\sum a_n$ is divergent.

(b) Since $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = 0.8 < 1$, part (a) of the Ratio Test tells us that the series $\sum a_n$ is absolutely convergent (and therefore convergent).

(c) Since $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = 1$, the Ratio Test fails and the series $\sum a_n$ might converge or it might diverge.

2. The series $\sum\limits_{n=1}^{\infty}\dfrac{n^2}{2^n}$ has positive terms and $\lim\limits_{n\to\infty}\dfrac{a_{n+1}}{a_n} = \lim\limits_{n\to\infty}\left[\dfrac{(n+1)^2}{2^{n+1}}\cdot\dfrac{2^n}{n^2}\right] = \lim\limits_{n\to\infty}\left(1+\dfrac{1}{n}\right)^2\cdot\dfrac{1}{2} = \dfrac{1}{2} < 1$, so the series is absolutely convergent by the Ratio Test.

3. $\sum\limits_{n=0}^{\infty}\dfrac{(-10)^n}{n!}$. Using the Ratio Test, $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\left|\dfrac{(-10)^{n+1}}{(n+1)!}\cdot\dfrac{n!}{(-10)^n}\right| = \lim\limits_{n\to\infty}\left|\dfrac{-10}{n+1}\right| = 0 < 1$, so the series is absolutely convergent.

4. $\sum\limits_{n=1}^{\infty}(-1)^{n-1}\dfrac{2^n}{n^4}$ diverges by the Test for Divergence. $\lim\limits_{n\to\infty}\dfrac{2^n}{n^4} = \infty$, so $\lim\limits_{n\to\infty}(-1)^{n-1}\dfrac{2^n}{n^4}$ does not exist.

5. $\sum\limits_{n=1}^{\infty}\dfrac{(-1)^{n+1}}{\sqrt[4]{n}}$ converges by the Alternating Series Test, but $\sum\limits_{n=1}^{\infty}\dfrac{1}{\sqrt[4]{n}}$ is a divergent p-series $\left(p = \frac{1}{4} \le 1\right)$, so the given series is conditionally convergent.

6. $\sum\limits_{n=1}^{\infty}\dfrac{1}{n^4}$ is a convergent p-series ($p = 4 > 1$), so $\sum\limits_{n=1}^{\infty}\dfrac{(-1)^n}{n^4}$ is absolutely convergent.

7. $\lim\limits_{n\to\infty}|a_n| = \lim\limits_{n\to\infty}\dfrac{n}{5+n} = \lim\limits_{n\to\infty}\dfrac{1}{5/n+1} = 1$, so $\lim\limits_{n\to\infty} a_n \ne 0$. Thus, the given series is divergent by the Test for Divergence.

8. $\sum\limits_{n=1}^{\infty}\dfrac{n}{n^2+1}$ diverges by the Limit Comparison Test with the harmonic series:

$\lim\limits_{n\to\infty}\dfrac{n/(n^2+1)}{1/n} = \lim\limits_{n\to\infty}\dfrac{n^2}{n^2+1} = 1$. But $\sum\limits_{n=1}^{\infty}(-1)^{n-1}\dfrac{n}{n^2+1}$ converges by the Alternating Series Test:

$\left\{\dfrac{n}{n^2+1}\right\}$ has positive terms, is decreasing since $\left(\dfrac{x}{x^2+1}\right)' = \dfrac{1-x^2}{(x^2+1)^2} \le 0$ for $x \ge 1$, and

$\lim\limits_{n\to\infty}\dfrac{n}{n^2+1} = 0$. Thus, $\sum\limits_{n=1}^{\infty}(-1)^{n-1}\dfrac{n}{n^2+1}$ is conditionally convergent.

9. $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\dfrac{1/(2n+2)!}{1/(2n)!} = \lim\limits_{n\to\infty}\dfrac{(2n)!}{(2n+2)!} = \lim\limits_{n\to\infty}\dfrac{(2n)!}{(2n+2)(2n+1)(2n)!}$

$= \lim\limits_{n\to\infty}\dfrac{1}{(2n+2)(2n+1)} = 0 < 1$, so the series $\sum\limits_{n=1}^{\infty}\dfrac{1}{(2n)!}$ is absolutely convergent by the Ratio Test. Of course, absolute convergence is the same as convergence for this series, since all of its terms are positive.

10. $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\left|\dfrac{(n+1)!/e^{n+1}}{n!/e^n}\right| = \dfrac{1}{e}\lim\limits_{n\to\infty}(n+1) = \infty$, so the series $\sum\limits_{n=1}^{\infty}e^{-n}n!$ diverges by the Ratio Test.

11. Since $0 \le \dfrac{e^{1/n}}{n^3} \le \dfrac{e}{n^3} = e\left(\dfrac{1}{n^3}\right)$ and $\sum_{n=1}^{\infty}\dfrac{1}{n^3}$ is a convergent p-series ($p = 3 > 1$), $\sum_{n=1}^{\infty}\dfrac{e^{1/n}}{n^3}$ converges, and so $\sum_{n=1}^{\infty}\dfrac{(-1)^n e^{1/n}}{n^3}$ is absolutely convergent.

12. $\left|\dfrac{\sin 4n}{4^n}\right| \le \dfrac{1}{4^n}$, so $\sum_{n=1}^{\infty}\left|\dfrac{\sin 4n}{4^n}\right|$ converges by comparison with the convergent geometric series $\sum_{n=1}^{\infty}\dfrac{1}{4^n}$ $\left(|r| = \frac{1}{4} < 1\right)$. Thus, $\sum_{n=1}^{\infty}\dfrac{\sin 4n}{4^n}$ is absolutely convergent.

13. $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\left[\dfrac{(n+1)\,3^{n+1}}{4^n}\cdot\dfrac{4^{n-1}}{n\cdot 3^n}\right] = \lim\limits_{n\to\infty}\left(\dfrac{3}{4}\cdot\dfrac{n+1}{n}\right) = \dfrac{3}{4} < 1$, so the series $\sum_{n=1}^{\infty}\dfrac{n(-3)^n}{4^{n-1}}$ is absolutely convergent by the Ratio Test.

14. $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\left[\dfrac{(n+1)^2\,2^{n+1}}{(n+1)!}\cdot\dfrac{n!}{n^2 2^n}\right] = \lim\limits_{n\to\infty}\left[\left(1+\dfrac{1}{n}\right)^2\cdot\dfrac{2}{n+1}\right] = 0$, so the series $\sum_{n=1}^{\infty}(-1)^{n+1}\dfrac{n^2 2^n}{n!}$ is absolutely convergent by the Ratio Test.

15. $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\left[\dfrac{10^{n+1}}{(n+2)\,4^{2n+3}}\cdot\dfrac{(n+1)\,4^{2n+1}}{10^n}\right] = \lim\limits_{n\to\infty}\left(\dfrac{10}{4^2}\cdot\dfrac{n+1}{n+2}\right) = \dfrac{5}{8} < 1$, so the series $\sum_{n=1}^{\infty}\dfrac{10^n}{(n+1)4^{2n+1}}$ is absolutely convergent by the Ratio Test. Since the terms of this series are positive, absolute convergence is the same as convergence.

16. $n^{2/3} - 2 > 0$ for $n \ge 3$, so $\dfrac{3-\cos n}{n^{2/3}-2} > \dfrac{1}{n^{2/3}-2} > \dfrac{1}{n^{2/3}}$ for $n \ge 3$. Since $\sum_{n=1}^{\infty}\dfrac{1}{n^{2/3}}$ diverges ($p = \frac{2}{3} \le 1$), so does $\sum_{n=1}^{\infty}\dfrac{3-\cos n}{n^{2/3}-2}$ by the Comparison Test.

17. $\sum_{n=2}^{\infty}\dfrac{(-1)^n}{\ln n}$ converges by the Alternating Series Test since $\lim\limits_{n\to\infty}\dfrac{1}{\ln n} = 0$ and $\left\{\dfrac{1}{\ln n}\right\}$ is decreasing. Now $\ln n < n$, so $\dfrac{1}{\ln n} > \dfrac{1}{n}$, and since $\sum_{n=2}^{\infty}\dfrac{1}{n}$ is the divergent (partial) harmonic series, $\sum_{n=2}^{\infty}\dfrac{1}{\ln n}$ diverges by the Comparison Test. Thus, $\sum_{n=2}^{\infty}\dfrac{(-1)^n}{\ln n}$ is conditionally convergent.

18. $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\dfrac{(n+1)!/(n+1)^{n+1}}{n!/n^n} = \lim\limits_{n\to\infty}\dfrac{n^n}{(n+1)^n} = \lim\limits_{n\to\infty}\dfrac{1}{(1+1/n)^n} = \dfrac{1}{e} < 1$, so the series $\sum_{n=1}^{\infty}\dfrac{n!}{n^n}$ converges absolutely by the Ratio Test.

19. $\dfrac{|\cos(n\pi/3)|}{n!} \le \dfrac{1}{n!}$ and $\sum_{n=1}^{\infty}\dfrac{1}{n!}$ converges (use the Ratio Test or the result of Exercise 12.4.29), so the series $\sum_{n=1}^{\infty}\dfrac{\cos(n\pi/3)}{n!}$ converges absolutely by the Comparison Test.

20. $\lim\limits_{n\to\infty}\sqrt[n]{|a_n|}=\lim\limits_{n\to\infty}\dfrac{1}{\ln n}=0<1$, so the series $\sum\limits_{n=2}^{\infty}\dfrac{(-1)^n}{(\ln n)^n}$ converges absolutely by the Root Test.

21. $\lim\limits_{n\to\infty}\sqrt[n]{|a_n|}=\lim\limits_{n\to\infty}\left(\dfrac{n^n}{3^{1+3n}}\right)^{1/n}=\lim\limits_{n\to\infty}\dfrac{n}{\sqrt[n]{3}\cdot 3^3}=\infty$, so the series $\sum\limits_{n=1}^{\infty}\dfrac{n^n}{3^{1+3n}}$ is divergent by the Root Test.

Or: $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right|=\lim\limits_{n\to\infty}\left[\dfrac{(n+1)^{n+1}}{3^{4+3n}}\cdot\dfrac{3^{1+3n}}{n^n}\right]=\lim\limits_{n\to\infty}\left[\dfrac{1}{3^3}\cdot\left(\dfrac{n+1}{n}\right)^n(n+1)\right]$

$$=\frac{1}{27}\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n\lim_{n\to\infty}(n+1)=\tfrac{1}{27}e\lim_{n\to\infty}(n+1)=\infty,$$

so the series is divergent by the Ratio Test.

22. Since $\left\{\dfrac{1}{n\ln n}\right\}$ is decreasing and $\lim\limits_{n\to\infty}\dfrac{1}{n\ln n}=0$, the series $\sum\limits_{n=2}^{\infty}\dfrac{(-1)^n}{n\ln n}$ converges by the Alternating Series Test. Since $\sum\limits_{n=2}^{\infty}\dfrac{1}{n\ln n}$ diverges by the Integral Test (Exercise 12.3.21), the series $\sum\limits_{n=2}^{\infty}\dfrac{(-1)^n}{n\ln n}$ is conditionally convergent.

23. $\lim\limits_{n\to\infty}\sqrt[n]{|a_n|}=\lim\limits_{n\to\infty}\dfrac{n^2+1}{2n^2+1}=\lim\limits_{n\to\infty}\dfrac{1+1/n^2}{2+1/n^2}=\dfrac{1}{2}<1$, so the series $\sum\limits_{n=1}^{\infty}\left(\dfrac{n^2+1}{2n^2+1}\right)^n$ is absolutely convergent by the Root Test.

24. $\lim\limits_{n\to\infty}\sqrt[n]{|a_n|}=\lim\limits_{n\to\infty}\dfrac{1}{\arctan n}=\dfrac{1}{\pi/2}=\dfrac{2}{\pi}<1$, so the series $\sum\limits_{n=1}^{\infty}\dfrac{(-1)^n}{(\arctan n)^n}$ is absolutely convergent by the Root Test.

25. Use the Ratio Test with the series $1-\dfrac{1\cdot 3}{3!}+\dfrac{1\cdot 3\cdot 5}{5!}-\dfrac{1\cdot 3\cdot 5\cdot 7}{7!}+\cdots+(-1)^{n-1}\dfrac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}{(2n-1)!}+\cdots$

$=\sum\limits_{n=1}^{\infty}(-1)^{n-1}\dfrac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}{(2n-1)!}$.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{(-1)^n\cdot 1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)[2(n+1)-1]}{[2(n+1)-1]!}\cdot\frac{(2n-1)!}{(-1)^{n-1}\cdot 1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}\right|$$

$$=\lim_{n\to\infty}\left|\frac{(-1)(2n+1)(2n-1)!}{(2n+1)(2n)(2n-1)!}\right|$$

$$=\lim_{n\to\infty}\frac{1}{2n}=0<1,$$

so the given series is absolutely convergent and therefore convergent.

26. Use the Ratio Test with the series $\dfrac{2}{5}+\dfrac{2\cdot 6}{5\cdot 8}+\dfrac{2\cdot 6\cdot 10}{5\cdot 8\cdot 11}+\dfrac{2\cdot 6\cdot 10\cdot 14}{5\cdot 8\cdot 11\cdot 14}+\cdots=\sum\limits_{n=1}^{\infty}\dfrac{2\cdot 6\cdot 10\cdot 14\cdot\cdots\cdot(4n-2)}{5\cdot 8\cdot 11\cdot 14\cdot\cdots\cdot(3n+2)}$.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{2\cdot 6\cdot 10\cdot\cdots\cdot(4n-2)[4(n+1)-2]}{5\cdot 8\cdot 11\cdot\cdots\cdot(3n+2)[3(n+1)+2]}\cdot\frac{5\cdot 8\cdot 11\cdot\cdots\cdot(3n+2)}{2\cdot 6\cdot 10\cdot\cdots\cdot(4n-2)}\right|$$

$$=\lim_{n\to\infty}\frac{4n+2}{3n+5}=\frac{4}{3}>1,$$

so the given series is divergent.

27. $\sum\limits_{n=1}^{\infty} \dfrac{2 \cdot 4 \cdot 6 \cdot \cdots \cdot (2n)}{n!} = \sum\limits_{n=1}^{\infty} \dfrac{(2 \cdot 1) \cdot (2 \cdot 2) \cdot (2 \cdot 3) \cdot \cdots \cdot (2 \cdot n)}{n!} = \sum\limits_{n=1}^{\infty} \dfrac{2^n n!}{n!} = \sum\limits_{n=1}^{\infty} 2^n$, which diverges by the Test for Divergence since $\lim\limits_{n\to\infty} 2^n = \infty$.

28. $\lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty} \left| \dfrac{\dfrac{2^{n+1}(n+1)!}{5 \cdot 8 \cdot 11 \cdot \cdots \cdot (3n+5)}}{\dfrac{2^n n!}{5 \cdot 8 \cdot 11 \cdot \cdots \cdot (3n+2)}} \right| = \lim\limits_{n\to\infty} \dfrac{2(n+1)}{3n+5} = \dfrac{2}{3} < 1$, so the series converges absolutely by the Ratio Test.

29. By the recursive definition, $\lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty} \left|\dfrac{5n+1}{4n+3}\right| = \dfrac{5}{4} > 1$, so the series diverges by the Ratio Test.

30. By the recursive definition, $\lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty} \left|\dfrac{2+\cos n}{\sqrt{n}}\right| = 0 < 1$, so the series converges absolutely by the Ratio Test.

31. (a) $\lim\limits_{n\to\infty} \left|\dfrac{1/(n+1)^3}{1/n^3}\right| = \lim\limits_{n\to\infty} \dfrac{n^3}{(n+1)^3} = \lim\limits_{n\to\infty} \dfrac{1}{(1+1/n)^3} = 1$. Inconclusive.

(b) $\lim\limits_{n\to\infty} \left|\dfrac{(n+1)}{2^{n+1}} \cdot \dfrac{2^n}{n}\right| = \lim\limits_{n\to\infty} \dfrac{n+1}{2n} = \lim\limits_{n\to\infty} \left(\dfrac{1}{2} + \dfrac{1}{2n}\right) = \dfrac{1}{2}$. Conclusive (convergent).

(c) $\lim\limits_{n\to\infty} \left|\dfrac{(-3)^n}{\sqrt{n+1}} \cdot \dfrac{\sqrt{n}}{(-3)^{n-1}}\right| = 3 \lim\limits_{n\to\infty} \sqrt{\dfrac{n}{n+1}} = 3 \lim\limits_{n\to\infty} \sqrt{\dfrac{1}{1+1/n}} = 3$. Conclusive (divergent).

(d) $\lim\limits_{n\to\infty} \left|\dfrac{\sqrt{n+1}}{1+(n+1)^2} \cdot \dfrac{1+n^2}{\sqrt{n}}\right| = \lim\limits_{n\to\infty} \left[\sqrt{1+\dfrac{1}{n}} \cdot \dfrac{1/n^2+1}{1/n^2+(1+1/n)^2}\right] = 1$. Inconclusive.

32. We use the Ratio Test:

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{[(n+1)!]^2}{[k(n+1)]!} \cdot \frac{(kn)!}{(n!)^2}\right| = \lim_{n\to\infty} \left|\frac{(n+1)^2}{[k(n+1)]\,[k(n+1)-1]\cdots[kn+1]}\right|$$

Now if $k = 1$, then this is equal to $\lim\limits_{n\to\infty} \left|\dfrac{(n+1)^2}{(n+1)}\right| = \infty$, so the series diverges; if $k = 2$, the limit is $\lim\limits_{n\to\infty} \left|\dfrac{(n+1)^2}{(2n+2)(2n+1)}\right| = \dfrac{1}{4} < 1$, so the series converges, and if $k > 2$, then the highest power of n in the denominator is larger than 2, and so the limit is 0, indicating convergence. So the series converges for $k \geq 2$.

33. (a) $\lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty} \left|\dfrac{x^{n+1}}{(n+1)!} \cdot \dfrac{n!}{x^n}\right| = \lim\limits_{n\to\infty} \left|\dfrac{x}{n+1}\right| = |x| \lim\limits_{n\to\infty} \dfrac{1}{n+1} = |x| \cdot 0 = 0 < 1$, so by the Ratio Test the series $\sum\limits_{n=0}^{\infty} \dfrac{x^n}{n!}$ converges for all x.

(b) Since the series of part (a) always converges, we must have $\lim\limits_{n\to\infty} \dfrac{x^n}{n!} = 0$ by Theorem 12.2.6.

34. (a) $R_n = a_{n+1} + a_{n+2} + a_{n+3} + a_{n+4} + \cdots = a_{n+1}\left(1 + \frac{a_{n+2}}{a_{n+1}} + \frac{a_{n+3}}{a_{n+1}} + \frac{a_{n+4}}{a_{n+1}} + \cdots\right)$

$$= a_{n+1}\left(1 + \frac{a_{n+2}}{a_{n+1}} + \frac{a_{n+3}}{a_{n+2}}\frac{a_{n+2}}{a_{n+1}} + \frac{a_{n+4}}{a_{n+3}}\frac{a_{n+3}}{a_{n+2}}\frac{a_{n+2}}{a_{n+1}} + \cdots\right)$$

$$= a_{n+1}(1 + r_{n+1} + r_{n+2}r_{n+1} + r_{n+3}r_{n+2}r_{n+1} + \cdots) \quad (\star)$$

$$\leq a_{n+1}(1 + r_{n+1} + r_{n+1}^2 + r_{n+1}^3 + \cdots) \quad \text{[since } \{r_n\} \text{ is decreasing]} \quad = \frac{a_{n+1}}{1 - r_{n+1}}$$

(b) Note that since $\{r_n\}$ is increasing and $r_n \to L$ as $n \to \infty$, we have $r_n < L$ for all n. So, starting with equation ($\star$),

$$R_n = a_{n+1}(1 + r_{n+1} + r_{n+2}r_{n+1} + r_{n+3}r_{n+2}r_{n+1} + \cdots) \leq a_{n+1}(1 + L + L^2 + L^3 + \cdots) = \frac{a_{n+1}}{1 - L}$$

35. (a) $s_5 = \sum_{n=1}^{5} \frac{1}{n2^n} = \frac{1}{2} + \frac{1}{8} + \frac{1}{24} + \frac{1}{64} + \frac{1}{160} = \frac{661}{960} \approx 0.68854$. Now the ratios

$r_n = \frac{a_{n+1}}{a_n} = \frac{n2^n}{(n+1)2^{n+1}} = \frac{n}{2(n+1)}$ form an increasing sequence, since

$r_{n+1} - r_n = \frac{n+1}{2(n+2)} - \frac{n}{2(n+1)} = \frac{(n+1)^2 - n(n+2)}{2(n+1)(n+2)} = \frac{1}{2(n+1)(n+2)} > 0$. So by Exercise 34(b),

the error in using s_5 is $R_5 \leq \frac{a_6}{1 - \lim_{n\to\infty} r_n} = \frac{1/(6 \cdot 2^6)}{1 - 1/2} = \frac{1}{192} \approx 0.00521$.

(b) The error in using s_n as an approximation to the sum is $R_n = \frac{a_{n+1}}{1 - \frac{1}{2}} = \frac{2}{(n+1)2^{n+1}}$. We want

$R_n < 0.00005 \quad \Leftrightarrow \quad \frac{1}{(n+1)2^n} < 0.00005 \quad \Leftrightarrow \quad (n+1)2^n > 20{,}000$. To find such an n we can use trial

and error or a graph. We calculate $(11+1)2^{11} = 24{,}576$, so $s_{11} = \sum_{n=1}^{11} \frac{1}{n2^n} \approx 0.693109$ is within 0.00005 of the actual sum.

36. $s_{10} = \sum_{n=1}^{10} \frac{n}{2^n} = \frac{1}{2} + \frac{2}{4} + \frac{3}{8} + \cdots + \frac{10}{1024} \approx 1.988$. The ratios

$r_n = \frac{a_{n+1}}{a_n} = \frac{n+1}{2^{n+1}} \cdot \frac{2^n}{n} = \frac{n+1}{2n} = \frac{1}{2}\left(1 + \frac{1}{n}\right)$ form a decreasing sequence, and

$r_{11} = \frac{11+1}{2(11)} = \frac{12}{22} = \frac{6}{11} < 1$, so by Exercise 34(a), the error in using s_{10} to approximate the sum of the series

$\sum_{n=1}^{\infty} \frac{n}{2^n}$ is $R_{10} \leq \frac{a_{11}}{1 - r_{11}} = \frac{\frac{11}{2048}}{1 - \frac{6}{11}} = \frac{121}{10{,}240} \approx 0.0118$.

37. Summing the inequalities $-|a_i| \leq a_i \leq |a_i|$ for $i = 1, 2, \ldots, n$, we get $-\sum_{i=1}^{n} |a_i| \leq \sum_{i=1}^{n} a_i \leq \sum_{i=1}^{n} |a_i|$

$\Rightarrow \quad -\lim_{n\to\infty} \sum_{i=1}^{n} |a_i| \leq \lim_{n\to\infty} \sum_{i=1}^{n} a_i \leq \lim_{n\to\infty} \sum_{i=1}^{n} |a_i| \quad \Rightarrow \quad -\sum_{n=1}^{\infty} |a_n| \leq \sum_{n=1}^{\infty} a_n \leq \sum_{n=1}^{\infty} |a_n| \quad \Rightarrow$

$\left|\sum_{n=1}^{\infty} a_n\right| \leq \sum_{n=1}^{\infty} |a_n|$.

38. (a) Following the hint, we get that $|a_n| < r^n$ for $n \ge N$, and so since the geometric series $\sum_{n=1}^{\infty} r^n$ converges $(0 < r < 1)$, the series $\sum_{n=N}^{\infty} |a_n|$ converges as well by the Comparison Test, and hence so does $\sum_{n=1}^{\infty} |a_n|$, so $\sum_{n=1}^{\infty} a_n$ is absolutely convergent.

(b) If $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} = L > 1$, then there is an integer N such that $\sqrt[n]{|a_n|} > 1$ for all $n \ge N$, so $|a_n| > 1$ for $n \ge N$. Thus, $\lim\limits_{n\to\infty} a_n \neq 0$, so $\sum_{n=1}^{\infty} a_n$ diverges by the Test for Divergence.

39. (a) Since $\sum a_n$ is absolutely convergent, and since $|a_n^+| \le |a_n|$ and $|a_n^-| \le |a_n|$ (because a_n^+ and a_n^- each equal either a_n or 0), we conclude by the Comparison Test that both $\sum a_n^+$ and $\sum a_n^-$ must be absolutely convergent. (Or use Theorem 12.2.8.)

(b) We will show by contradiction that both $\sum a_n^+$ and $\sum a_n^-$ must diverge. For suppose that $\sum a_n^+$ converged. Then so would $\sum\left(a_n^+ - \frac{1}{2}a_n\right)$ by Theorem 12.2.8. But $\sum\left(a_n^+ - \frac{1}{2}a_n\right) = \sum\left[\frac{1}{2}\left(a_n + |a_n|\right) - \frac{1}{2}a_n\right] = \frac{1}{2}\sum |a_n|$, which diverges because $\sum a_n$ is only conditionally convergent. Hence, $\sum a_n^+$ can't converge. Similarly, neither can $\sum a_n^-$.

40. Let $\sum b_n$ be the rearranged series constructed in the hint. [This series can be constructed by virtue of the result of Exercise 39(b).] This series will have partial sums s_n that oscillate in value back and forth across r. Since $\lim\limits_{n\to\infty} a_n = 0$ (by Theorem 12.2.6), and since the size of the oscillations $|s_n - r|$ is always less than $|a_n|$ because of the way $\sum b_n$ was constructed, we have that $\sum b_n = \lim\limits_{n\to\infty} s_n = r$.

12.7 Strategy for Testing Series

1. $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} \dfrac{n^2-1}{n^2+1} = \lim\limits_{n\to\infty} \dfrac{1-1/n^2}{1+1/n} = 1 \neq 0$, so the series $\sum\limits_{n=1}^{\infty} \dfrac{n^2-1}{n^2+1}$ diverges by the Test for Divergence.

2. If $a_n = \dfrac{n-1}{n^2+n}$ and $b_n = \dfrac{1}{n}$, then $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \lim\limits_{n\to\infty} \dfrac{n^2-n}{n^2+n} = \lim\limits_{n\to\infty} \dfrac{1-1/n}{1+1/n} = 1$, so the series $\sum\limits_{n=1}^{\infty} \dfrac{n-1}{n^2+n}$ diverges by the Limit Comparison Test with the harmonic series.

3. $\dfrac{1}{n^2+n} < \dfrac{1}{n^2}$ for all $n \ge 1$, so $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^2+n}$ converges by the Comparison Test with $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^2}$, a p-series that converges because $p = 2 > 1$.

4. Let $b_n = \dfrac{n-1}{n^2+n}$. Then $b_1 = 0$, and $b_2 = b_3 = \frac{1}{6}$, but $b_n > b_{n+1}$ for $n \ge 3$ since

$$\left(\frac{x-1}{x^2+x}\right)' = \frac{(x^2+x)-(x-1)(2x+1)}{(x^2+x)^2} = \frac{-x^2+2x+1}{(x^2+x)^2} = \frac{2-(x-1)^2}{(x^2+x)^2} < 0 \text{ for } x \ge 3. \text{ Thus,}$$

$\{b_n \mid n \geq 3\}$ is decreasing and $\lim_{n\to\infty} b_n = 0$, so $\sum_{n=3}^{\infty} (-1)^{n-1} \dfrac{n-1}{n^2+n}$ converges by the Alternating Series Test.

Hence, the full series $\sum_{n=1}^{\infty} (-1)^{n-1} \dfrac{n-1}{n^2+n}$ also converges.

5. $\lim_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\dfrac{(-3)^{n+2}}{2^{3(n+1)}} \cdot \dfrac{2^{3n}}{(-3)^{n+1}}\right| = \lim_{n\to\infty} \left|\dfrac{-3 \cdot 2^{3n}}{2^{3n} \cdot 2^3}\right| = \lim_{n\to\infty} \dfrac{3}{2^3} = \dfrac{3}{8} < 1$, so the series

$\sum_{n=1}^{\infty} \dfrac{(-3)^{n+1}}{2^{3n}}$ is absolutely convergent by the Ratio Test.

6. $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \dfrac{3n}{1+8n} = \lim_{n\to\infty} \dfrac{3}{1/n+8} = \dfrac{3}{8} < 1$, so $\sum_{n=1}^{\infty} \left(\dfrac{3n}{1+8n}\right)^n$ converges by the Root Test.

7. Let $f(x) = \dfrac{1}{x\sqrt{\ln x}}$. Then f is positive, continuous, and decreasing on $[2,\infty)$, so we can apply the Integral Test.

Since $\int \dfrac{1}{x\sqrt{\ln x}}\,dx \; \begin{bmatrix} u = \ln x, \\ du = dx/x \end{bmatrix} = \int u^{-1/2}\,du = 2u^{1/2} + C = 2\sqrt{\ln x} + C$, we find

$\int_2^\infty \dfrac{dx}{x\sqrt{\ln x}} = \lim_{t\to\infty} \int_2^t \dfrac{dx}{x\sqrt{\ln x}} = \lim_{t\to\infty} \left[2\sqrt{\ln x}\right]_2^t = \lim_{t\to\infty} \left(2\sqrt{\ln t} - 2\sqrt{\ln 2}\right) = \infty$. Since the integral

diverges, the given series $\sum_{n=2}^{\infty} \dfrac{1}{n\sqrt{\ln n}}$ diverges.

8. $\sum_{k=1}^{\infty} \dfrac{2^k k!}{(k+2)!} = \sum_{k=1}^{\infty} \dfrac{2^k}{(k+1)(k+2)}$. Using the Ratio Test, we get

$\lim_{k\to\infty} \left|\dfrac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty} \left|\dfrac{2^{k+1}}{(k+2)(k+3)} \cdot \dfrac{(k+1)(k+2)}{2^k}\right| = \lim_{k\to\infty} \left(2 \cdot \dfrac{k+1}{k+3}\right) = 2 > 1$, so the series diverges.

Or: Use the Test for Divergence.

9. $\sum_{k=1}^{\infty} k^2 e^{-k} = \sum_{k=1}^{\infty} \dfrac{k^2}{e^k}$. Using the Ratio Test, we get

$\lim_{k\to\infty} \left|\dfrac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty} \left|\dfrac{(k+1)^2}{e^{k+1}} \cdot \dfrac{e^k}{k^2}\right| = \lim_{k\to\infty} \left[\left(\dfrac{k+1}{k}\right)^2 \cdot \dfrac{1}{e}\right] = 1^2 \cdot \dfrac{1}{e} = \dfrac{1}{e} < 1$, so the series converges.

10. Let $f(x) = x^2 e^{-x^3}$. Then f is continuous and positive on $[1,\infty)$, and $f'(x) = \dfrac{x(2-3x^3)}{e^{x^3}} < 0$ for $x \geq 1$, so f is decreasing on $[1,\infty)$ as well, and we can apply the Integral Test. $\int_1^\infty x^2 e^{-x^3}\,dx = \lim_{t\to\infty} \left[-\frac{1}{3}e^{-x^3}\right]_1^t = \frac{1}{3e}$, so the integral converges, and hence, the series converges.

11. $b_n = \dfrac{1}{n \ln n} > 0$ for $n \geq 2$, $\{b_n\}$ is decreasing, and $\lim_{n\to\infty} b_n = 0$, so the given series $\sum_{n=2}^{\infty} \dfrac{(-1)^{n+1}}{n \ln n}$ converges by the Alternating Series Test.

12. Let $b_n = \dfrac{n}{n^2+25}$. Then $b_n > 0$, $\lim_{n\to\infty} b_n = 0$, and

$b_n - b_{n+1} = \dfrac{n}{n^2+25} - \dfrac{n+1}{n^2+2n+26} = \dfrac{n^2+n-25}{(n^2+25)(n^2+2n+26)}$, which is positive for $n \geq 5$, so the sequence $\{b_n\}$ decreases from $n=5$ on. Hence, the given series $\sum_{n=1}^{\infty}(-1)^n \dfrac{n}{n^2+25}$ converges by the Alternating Series Test.

13. $\lim_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\dfrac{3^{n+1}(n+1)^2}{(n+1)!}\cdot\dfrac{n!}{3^n n^2}\right| = \lim_{n\to\infty}\left[\dfrac{3(n+1)^2}{(n+1)n^2}\right] = 3\lim_{n\to\infty}\dfrac{n+1}{n^2} = 0 < 1$, so the series $\sum_{n=1}^{\infty}\dfrac{3^n n^2}{n!}$ converges by the Ratio Test.

14. The series $\sum_{n=1}^{\infty} \sin n$ diverges by the Test for Divergence since $\lim_{n\to\infty} \sin n$ does not exist.

15.
$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!}{2\cdot5\cdot8\cdot\cdots\cdot(3n+2)[3(n+1)+2]}\cdot\frac{2\cdot5\cdot8\cdot\cdots\cdot(3n+2)}{n!}\right|$$
$$= \lim_{n\to\infty}\frac{n+1}{3n+5} = \frac{1}{3} < 1$$
so the series $\sum_{n=0}^{\infty}\dfrac{n!}{2\cdot5\cdot8\cdot\cdots\cdot(3n+2)}$ converges by the Ratio Test.

16. Using the Limit Comparison Test with $a_n = \dfrac{n^2+1}{n^3+1}$ and $b_n = \dfrac{1}{n}$, we have

$\lim_{n\to\infty}\dfrac{a_n}{b_n} = \lim_{n\to\infty}\left(\dfrac{n^2+1}{n^3+1}\cdot\dfrac{n}{1}\right) = \lim_{n\to\infty}\dfrac{n^3+n}{n^3+1} = \lim_{n\to\infty}\dfrac{1+1/n^2}{1+1/n^3} = 1 > 0$. Since $\sum_{n=1}^{\infty} b_n$ is the divergent harmonic series, $\sum_{n=1}^{\infty} a_n$ is also divergent.

17. $\lim_{n\to\infty} 2^{1/n} = 2^0 = 1$, so $\lim_{n\to\infty}(-1)^n 2^{1/n}$ does not exist and the series $\sum_{n=1}^{\infty}(-1)^n 2^{1/n}$ diverges by the Test for Divergence.

18. $b_n = \dfrac{1}{\sqrt{n}-1}$ for $n \geq 2$. $\{b_n\}$ is a decreasing sequence of positive numbers and $\lim_{n\to\infty} b_n = 0$, so $\sum_{n=2}^{\infty}\dfrac{(-1)^{n-1}}{\sqrt{n}-1}$ converges by the Alternating Series Test.

19. Let $f(x) = \dfrac{\ln x}{\sqrt{x}}$. Then $f'(x) = \dfrac{2-\ln x}{2x^{3/2}} < 0$ when $\ln x > 2$ or $x > e^2$, so $\dfrac{\ln n}{\sqrt{n}}$ is decreasing for $n > e^2$.

By l'Hospital's Rule, $\lim_{n\to\infty}\dfrac{\ln n}{\sqrt{n}} = \lim_{n\to\infty}\dfrac{1/n}{1/(2\sqrt{n})} = \lim_{n\to\infty}\dfrac{2}{\sqrt{n}} = 0$, so the series $\sum_{n=1}^{\infty}(-1)^n\dfrac{\ln n}{\sqrt{n}}$ converges by the Alternating Series Test.

20. $\lim_{k\to\infty}\left|\frac{a_{k+1}}{a_k}\right| = \lim_{k\to\infty}\left|\frac{k+6}{5^{k+1}}\cdot\frac{5^k}{k+5}\right| = \frac{1}{5}\lim_{k\to\infty}\frac{k+6}{k+5} = \frac{1}{5} < 1$, so the series $\sum_{k=1}^{\infty}\frac{k+5}{5^k}$ converges by the Ratio Test.

21. $\sum_{n=1}^{\infty}\frac{(-2)^{2n}}{n^n} = \sum_{n=1}^{\infty}\left(\frac{4}{n}\right)^n$. $\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\frac{4}{n} = 0 < 1$, so the given series is absolutely convergent by the Root Test.

22. $\frac{\sqrt{n^2-1}}{n^3+2n^2+5} < \frac{n}{n^3+2n^2+5} < \frac{n}{n^3} = \frac{1}{n^2}$ for $n \ge 1$, so $\sum_{n=1}^{\infty}\frac{\sqrt{n^2-1}}{n^3+2n^2+5}$ converges by the Comparison Test with the convergent p-series $\sum_{n=1}^{\infty} 1/n^2$ $(p = 2 > 1)$.

23. Using the Limit Comparison Test with $a_n = \tan\left(\frac{1}{n}\right)$ and $b_n = \frac{1}{n}$, we have

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{\tan(1/n)}{1/n} = \lim_{x\to\infty}\frac{\tan(1/x)}{1/x} \overset{\text{H}}{=} \lim_{x\to\infty}\frac{\sec^2(1/x)\cdot(-1/x^2)}{-1/x^2} = \lim_{x\to\infty}\sec^2(1/x) = 1^2 = 1 > 0.$$

Since $\sum_{n=1}^{\infty} b_n$ is the divergent harmonic series, $\sum_{n=1}^{\infty} a_n$ is also divergent.

24. $\frac{|\cos(n/2)|}{n^2+4n} < \frac{1}{n^2+4n} < \frac{1}{n^2}$ and since $\sum_{n=1}^{\infty}\frac{1}{n^2}$ converges $(p = 2 > 1)$, $\sum_{n=1}^{\infty}\frac{\cos(n/2)}{n^2+4n}$ converges absolutely by the Comparison Test.

25. Use the Ratio Test. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)!}{e^{(n+1)^2}}\cdot\frac{e^{n^2}}{n!}\right| = \lim_{n\to\infty}\frac{(n+1)n!\cdot e^{n^2}}{e^{n^2+2n+1}n!} = \lim_{n\to\infty}\frac{n+1}{e^{2n+1}} = 0 < 1$, so $\sum_{n=1}^{\infty}\frac{n!}{e^{n^2}}$ converges.

26. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\left(\frac{n^2+2n+2}{5^{n+1}}\cdot\frac{5^n}{n^2+1}\right) = \lim_{n\to\infty}\left(\frac{1+2/n+2/n^2}{1+1/n^2}\cdot\frac{1}{5}\right) = \frac{1}{5} < 1$, so $\sum_{n=1}^{\infty}\frac{n^2+1}{5^n}$ converges by the Ratio Test.

27. $\int_2^{\infty}\frac{\ln x}{x^2}\,dx = \lim_{t\to\infty}\left[-\frac{\ln x}{x} - \frac{1}{x}\right]_1^t$ (using integration by parts) $\overset{\text{H}}{=} 1$. So $\sum_{n=1}^{\infty}\frac{\ln n}{n^2}$ converges by the Integral Test, and since $\frac{k\ln k}{(k+1)^3} < \frac{k\ln k}{k^3} = \frac{\ln k}{k^2}$, the given series $\sum_{k=1}^{\infty}\frac{k\ln k}{(k+1)^3}$ converges by the Comparison Test.

28. Since $\left\{\frac{1}{n}\right\}$ is a decreasing sequence, $e^{1/n} \le e^{1/1} = e$ for all $n \ge 1$, and $\sum_{n=1}^{\infty}\frac{e}{n^2}$ converges $(p = 2 > 1)$, so $\sum_{n=1}^{\infty}\frac{e^{1/n}}{n^2}$ converges by the Comparison Test. (Or use the Integral Test.)

29. $0 < \frac{\tan^{-1} n}{n^{3/2}} < \frac{\pi/2}{n^{3/2}}$. $\sum_{n=1}^{\infty}\frac{\pi/2}{n^{3/2}} = \frac{\pi}{2}\sum_{n=1}^{\infty}\frac{1}{n^{3/2}}$ which is a convergent p-series $(p = \frac{3}{2} > 1)$, so $\sum_{n=1}^{\infty}\frac{\tan^{-1} n}{n^{3/2}}$ converges by the Comparison Test.

30. Let $f(x) = \dfrac{\sqrt{x}}{x+5}$. Then $f(x)$ is continuous and positive on $[1, \infty)$, and since $f'(x) = \dfrac{5-x}{2\sqrt{x}\,(x+5)^2} < 0$ for $x > 5$, $f(x)$ is eventually decreasing, so we can use the Alternating Series Test.

$\displaystyle\lim_{n\to\infty} \frac{\sqrt{n}}{n+5} = \lim_{n\to\infty} \frac{1}{n^{1/2} + 5n^{-1/2}} = 0$, so the series $\displaystyle\sum_{j=1}^{\infty} (-1)^j \frac{\sqrt{j}}{j+5}$ converges.

31. $\displaystyle\lim_{k\to\infty} a_k = \lim_{k\to\infty} \frac{5^k}{3^k + 4^k} = [\text{divide by } 4^k]\ \lim_{k\to\infty} \frac{(5/4)^k}{(3/4)^k + 1} = \infty$ since $\displaystyle\lim_{k\to\infty}\left(\frac{3}{4}\right)^k = 0$ and $\displaystyle\lim_{k\to\infty}\left(\frac{5}{4}\right)^k = \infty$.

Thus, $\displaystyle\sum_{k=1}^{\infty} \frac{5^k}{3^k + 4^k}$ diverges by the Test for Divergence.

32. $\displaystyle\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \frac{2n}{n^2} = \lim_{n\to\infty} \frac{2}{n} = 0$, so the series $\displaystyle\sum_{n=1}^{\infty} \frac{(2n)^n}{n^{2n}}$ converges by the Root Test.

33. Let $a_n = \dfrac{\sin(1/n)}{\sqrt{n}}$ and $b_n = \dfrac{1}{n\sqrt{n}}$. Then $\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\sin(1/n)}{1/n} = 1 > 0$, so $\displaystyle\sum_{n=1}^{\infty} \frac{\sin(1/n)}{\sqrt{n}}$ converges by limit comparison with the convergent p-series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^{3/2}}$ $(p = 3/2 > 1)$.

34. $0 \le n\cos^2 n \le n$, so $\dfrac{1}{n + n\cos^2 n} \ge \dfrac{1}{n+n} = \dfrac{1}{2n}$. Thus, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n + n\cos^2 n}$ diverges by comparison with $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2n}$, which is a constant multiple of the (divergent) harmonic series.

35. $\displaystyle\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^{n^2/n} = \lim_{n\to\infty} \frac{1}{[(n+1)/n]^n} = \frac{1}{\lim\limits_{n\to\infty} (1 + 1/n)^n} = \frac{1}{e} < 1$, so the series $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n^2}$ converges by the Root Test.

36. Note that $(\ln n)^{\ln n} = \left(e^{\ln \ln n}\right)^{\ln n} = \left(e^{\ln n}\right)^{\ln \ln n} = n^{\ln \ln n}$ and $\ln \ln n \to \infty$ as $n \to \infty$, so $\ln \ln n > 2$ for sufficiently large n. For these n we have $(\ln n)^{\ln n} > n^2$, so $\dfrac{1}{(\ln n)^{\ln n}} < \dfrac{1}{n^2}$. Since $\sum\limits_{n=2}^{\infty} \dfrac{1}{n^2}$ converges $(p = 2 > 1)$, so does $\sum\limits_{n=2}^{\infty} \dfrac{1}{(\ln n)^{\ln n}}$ by the Comparison Test.

37. $\displaystyle\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \left(2^{1/n} - 1\right) = 1 - 1 = 0 < 1$, so the series $\displaystyle\sum_{n=1}^{\infty} \left(\sqrt[n]{2} - 1\right)^n$ converges by the Root Test.

38. Use the Limit Comparison Test with $a_n = \sqrt[n]{2} - 1$ and $b_n = 1/n$. Then $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \lim\limits_{n\to\infty} \dfrac{2^{1/n} - 1}{1/n}$

$= \lim\limits_{x\to\infty} \dfrac{2^{1/x} - 1}{1/x} \overset{\text{H}}{=} \lim\limits_{x\to\infty} \dfrac{2^{1/x} \cdot \ln 2 \cdot (-1/x^2)}{-1/x^2} = \lim\limits_{x\to\infty} (2^{1/x} \cdot \ln 2) = 1 \cdot \ln 2 = \ln 2 > 0$. So since $\sum_{n=1}^{\infty} b_n$

diverges (harmonic series), so does $\sum_{n=1}^{\infty} \left(\sqrt[n]{2} - 1\right)$.

Alternate Solution:

$\sqrt[n]{2} - 1 = \dfrac{1}{2^{(n-1)/n} + 2^{(n-2)/n} + 2^{(n-3)/n} + \cdots + 2^{1/n} + 1}$ [rationalize the numerator] $\geq \dfrac{1}{2n}$.

and since $\sum\limits_{n=1}^{\infty} \dfrac{1}{2n} = \dfrac{1}{2} \sum\limits_{n=1}^{\infty} \dfrac{1}{n}$ diverges (harmonic series), so does $\sum\limits_{n=1}^{\infty} \left(\sqrt[n]{2} - 1\right)$ by the Comparison Test.

12.8 Power Series

1. A power series is a series of the form $\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$, where x is a variable and the c_n's are constants called the coefficients of the series.

More generally, a series of the form $\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$ is called a power series in $(x-a)$ or a power series centered at a or a power series about a, where a is a constant.

2. (a) Given the power series $\sum_{n=0}^{\infty} c_n (x-a)^n$, the radius of convergence is:

(i) 0 if the series converges only when $x = a$

(ii) ∞ if the series converges for all x, or

(iii) a positive number R such that the series converges if $|x - a| < R$ and diverges if $|x - a| > R$.

In most cases, R can be found by using the Ratio Test.

(b) The interval of convergence of a power series is the interval that consists of all values of x for which the series converges. Corresponding to the cases in part (a), the interval of convergence is: (i) the single point $\{a\}$, (ii) all real numbers; that is, the real number line $(-\infty, \infty)$, or (iii) an interval with endpoints $a - R$ and $a + R$ which can contain neither, either, or both of the endpoints. In this case, we must test the series for convergence at each endpoint to determine the interval of convergence.

3. If $a_n = \dfrac{x^n}{\sqrt{n}}$, then $\lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty} \left|\dfrac{x^{n+1}}{\sqrt{n+1}} \cdot \dfrac{\sqrt{n}}{x}\right| = \lim\limits_{n\to\infty} \left|\dfrac{x}{\sqrt{n+1}/\sqrt{n}}\right| = \lim\limits_{n\to\infty} \dfrac{|x|}{\sqrt{1 + 1/n}} = |x|$.

By the Ratio Test, the series $\sum\limits_{n=1}^{\infty} \dfrac{x^n}{\sqrt{n}}$ converges when $|x| < 1$, so the radius of convergence $R = 1$. Now we'll check the endpoints, that is, $x = \pm 1$. When $x = 1$, the series $\sum\limits_{n=1}^{\infty} \dfrac{1}{\sqrt{n}}$ diverges because it is a p-series with $p = \frac{1}{2} \leq 1$. When $x = -1$, the series $\sum\limits_{n=1}^{\infty} \dfrac{(-1)^n}{\sqrt{n}}$ converges by the Alternating Series Test. Thus, the interval of convergence is $I = [-1, 1)$.

4. If $a_n = \dfrac{(-1)^n x^n}{n+1}$, then $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\left|\dfrac{x^{n+1}}{n+2}\cdot\dfrac{n+1}{x^n}\right| = \lim\limits_{n\to\infty}\dfrac{|x|}{1+1/(n+1)} = |x|$. By the Ratio Test, the series $\sum\limits_{n=0}^{\infty}\dfrac{(-1)^n x^n}{n+1}$ converges when $|x|<1$, so $R=1$. When $x=-1$, the series diverges because it is the harmonic series; when $x=1$, it is the alternating harmonic series, which converges by the Alternating Series Test. Thus, $I=(-1,1]$.

5. If $a_n = \dfrac{(-1)^{n-1}x^n}{n^3}$, then $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\left|\dfrac{(-1)^n x^{n+1}}{(n+1)^3}\cdot\dfrac{n^3}{(-1)^{n-1}x^n}\right| = \lim\limits_{n\to\infty}\left|\dfrac{(-1)xn^3}{(n+1)^3}\right|$

$= \lim\limits_{n\to\infty}\left[\left(\dfrac{n}{n+1}\right)^3 |x|\right] = 1^3\cdot|x| = |x|$. By the Ratio Test, the series $\sum\limits_{n=1}^{\infty}\dfrac{(-1)^{n-1}x^n}{n^3}$ converges when $|x|<1$, so the radius of convergence $R=1$. Now we'll check the endpoints, that is, $x=\pm1$. When $x=1$, the series $\sum\limits_{n=1}^{\infty}\dfrac{(-1)^{n-1}}{n^3}$ converges by the Alternating Series Test. When $x=-1$, the series

$\sum\limits_{n=1}^{\infty}\dfrac{(-1)^{n-1}(-1)^n}{n^3} = -\sum\limits_{n=1}^{\infty}\dfrac{1}{n^3}$ converges because it is a constant multiple of a convergent p-series ($p=3>1$).

Thus, the interval of convergence is $I=[-1,1]$.

6. $a_n = \sqrt{n}\,x^n$, so we need $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\dfrac{\sqrt{n+1}\,|x|^{n+1}}{\sqrt{n}\,|x|^n} = \lim\limits_{n\to\infty}\sqrt{1+\dfrac{1}{n}}\,|x| = |x|<1$ for convergence (by the Ratio Test), so $R=1$. When $x=\pm1$, $\lim\limits_{n\to\infty}|a_n| = \lim\limits_{n\to\infty}\sqrt{n}=\infty$, so the series diverges by the Test for Divergence. Thus, $I=(-1,1)$.

7. If $a_n=\dfrac{x^n}{n!}$, then $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\left|\dfrac{x^{n+1}}{(n+1)!}\cdot\dfrac{n!}{x^n}\right| = \lim\limits_{n\to\infty}\left|\dfrac{x}{n+1}\right| = |x|\lim\limits_{n\to\infty}\dfrac{1}{n+1} = |x|\cdot 0 = 0<1$ for *all* real x. So, by the Ratio Test, $R=\infty$, and $I=(-\infty,\infty)$.

8. Here the Root Test is easier. If $a_n=n^n x^n$ then $\lim\limits_{n\to\infty}\sqrt[n]{|a_n|} = \lim\limits_{n\to\infty} n\,|x| = \infty$ if $x\neq 0$, so $R=0$ and $I=\{0\}$.

9. $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\dfrac{(n+1)\,4^{n+1}\,|x|^{n+1}}{n4^n\,|x|^n} = \lim\limits_{n\to\infty}\left(1+\dfrac{1}{n}\right)4\,|x| = 4\,|x|$. Now $4\,|x|<1 \;\Leftrightarrow\; |x|<\frac14$, so by the Ratio Test, $R=\frac14$. When $x=\frac14$, we get the divergent series $\sum_{n=1}^{\infty}(-1)^n n$, and when $x=-\frac14$, we get the divergent series $\sum_{n=1}^{\infty} n$. Thus, $I=\left(-\frac14,\frac14\right)$.

10. If $a_n=\dfrac{x^n}{n3^n}$, then $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\left|\dfrac{x^{n-1}}{(n+1)3^{n+1}}\cdot\dfrac{n3^n}{x^n}\right| = \lim\limits_{n\to\infty}\left|\dfrac{xn}{(n+1)3}\right| = \dfrac{|x|}{3}\lim\limits_{n\to\infty}\dfrac{n}{n+1} = \dfrac{|x|}{3}$.

By the Ratio Test, the series converges when $\dfrac{|x|}{3}<1 \;\Leftrightarrow\; |x|<3$, so $R=3$. When $x=-3$, the series is the alternating harmonic series, which converges by the Alternating Series Test. When $x=3$, it is the harmonic series, which diverges. Thus, $I=[-3,3)$.

11. $a_n=\dfrac{(-2)^n x^n}{\sqrt[4]{n}}$, so $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\dfrac{2^{n+1}\,|x|^{n+1}}{\sqrt[4]{n+1}}\cdot\dfrac{\sqrt[4]{n}}{2^n\,|x|^n} = \lim\limits_{n\to\infty}2\,|x|\sqrt[4]{\dfrac{n}{n+1}} = 2\,|x|$, so by the Ratio Test, the series converges when $2\,|x|<1 \;\Leftrightarrow\; |x|<\frac12$, so $R=\frac12$. When $x=-\frac12$, we get the divergent

p-series $\sum_{n=1}^{\infty} \frac{1}{\sqrt[4]{n}}$ $\left(p = \frac{1}{4} \le 1\right)$. When $x = \frac{1}{2}$, we get the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt[4]{n}}$, which converges by the Alternating Series Test. Thus, $I = \left(-\frac{1}{2}, \frac{1}{2}\right]$.

12. $a_n = \frac{x^n}{5^n n^5}$, so $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{x^{n+1}}{5^{n+1}(n+1)^5} \cdot \frac{5^n n^5}{x^n}\right| = \lim_{n\to\infty} \frac{|x|}{5}\left(\frac{n}{n+1}\right)^5 = \frac{|x|}{5}$. By the Ratio Test, the series converges when $|x|/5 < 1 \quad\Leftrightarrow\quad |x| < 5$, so $R = 5$. When $x = -5$, we get the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^5}$, which converges by the Alternating Series Test. When $x = 5$, we get the convergent p-series $\sum_{n=1}^{\infty} \frac{1}{n^5}$ $(p = 5 > 1)$. Thus, $I = [-5, 5]$.

13. If $a_n = (-1)^n \frac{x^n}{4^n \ln n}$, then

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{x^{n+1}}{4^{n+1}\ln(n+1)} \cdot \frac{4^n \ln n}{x^n}\right| = \frac{|x|}{4} \lim_{n\to\infty} \frac{\ln n}{\ln(n+1)} = \frac{|x|}{4} \cdot 1 \text{ (by l'Hospital's Rule)} = \frac{|x|}{4}.$$

By the Ratio Test, the series converges when $\frac{|x|}{4} < 1 \quad\Leftrightarrow\quad |x| < 4$, so $R = 4$. When $x = -4$,

$\sum_{n=2}^{\infty}(-1)^n \frac{x^n}{4^n \ln n} = \sum_{n=2}^{\infty} \frac{[(-1)(-4)]^n}{4^n \ln n} = \sum_{n=2}^{\infty} \frac{1}{\ln n}$. Since $\ln n < n$ for $n \ge 2$, $\frac{1}{\ln n} > \frac{1}{n}$ and $\sum_{n=2}^{\infty} \frac{1}{n}$ is the divergent harmonic series (without the $n = 1$ term), $\sum_{n=2}^{\infty} \frac{1}{\ln n}$ is divergent by the Comparison Test. When $x = 4$,

$\sum_{n=2}^{\infty}(-1)^n \frac{x^n}{4^n \ln n} = \sum_{n=2}^{\infty}(-1)^n \frac{1}{\ln n}$, which converges by the Alternating Series Test. Thus, $I = (-4, 4]$.

14. $a_n = (-1)^n \frac{x^{2n}}{(2n)!}$, so $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{|x|^{2n+2}}{(2n+2)!} \cdot \frac{(2n)!}{|x|^{2n}} = \lim_{n\to\infty} \frac{|x|^2}{(2n+1)(2n+2)} = 0$. Thus, by the Ratio Test, the series converges for *all* real x and we have $R = \infty$ and $I = (-\infty, \infty)$.

15. If $a_n = \sqrt{n}\,(x-1)^n$, then $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{\sqrt{n+1}\,|x-1|^{n+1}}{\sqrt{n}\,|x-1|^n}\right| = \lim_{n\to\infty} \sqrt{1 + \frac{1}{n}}\,|x-1| = |x-1|$. By the Ratio Test, the series converges when $|x-1| < 1$ [so $R = 1$] $\quad\Leftrightarrow\quad -1 < x-1 < 1 \quad\Leftrightarrow\quad 0 < x < 2$. When $x = 0$, the series becomes $\sum_{n=0}^{\infty}(-1)^n\sqrt{n}$, which diverges by the Test for Divergence. When $x = 2$, the series becomes $\sum_{n=0}^{\infty}\sqrt{n}$, which also diverges by the Test for Divergence. Thus, $I = (0, 2)$.

16. If $a_n = n^3(x-5)^n$, $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{(n+1)^3(x-5)^{n+1}}{n^3(x-5)^n}\right| = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^3 |x-5| = |x-5|$. By the Ratio Test, the series converges when $|x-5| < 1 \quad\Leftrightarrow\quad -1 < x-5 < 1 \quad\Leftrightarrow\quad 4 < x < 6$. When $x = 4$, the series becomes $\sum_{n=0}^{\infty}(-1)^n n^3$, which diverges by the Test for Divergence. When $x = 6$, the series becomes $\sum_{n=0}^{\infty} n^3$, which also diverges by the Test for Divergence. Thus, $R = 1$ and $I = (4, 6)$.

17. If $a_n = (-1)^n \dfrac{(x+2)^n}{n2^n}$, then

$\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left[\frac{|x+2|^{n+1}}{(n+1)2^{n+1}}\cdot\frac{n2^n}{|x+2|^n}\right] = \lim_{n\to\infty}\frac{n}{n+1}\cdot\frac{|x+2|}{2} = \frac{|x+2|}{2}$. By the Ratio Test, the series converges when $\dfrac{|x+2|}{2} < 1 \;\Leftrightarrow\; |x+2| < 2$ [so $R = 2$] $\Leftrightarrow\; -2 < x+2 < 2 \;\Leftrightarrow\; -4 < x < 0$.

When $x = -4$, the series becomes $\displaystyle\sum_{n=1}^{\infty}(-1)^n\frac{(-2)^n}{n2^n} = \sum_{n=1}^{\infty}\frac{2^n}{n2^n} = \sum_{n=1}^{\infty}\frac{1}{n}$, which is the divergent harmonic series.

When $x = 0$, the series is $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$, the alternating harmonic series, which converges by the Alternating Series Test. Thus, $I = (-4, 0]$.

18. If $a_n = \dfrac{(-2)^n}{\sqrt{n}}(x+3)^n$, then

$\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-2)^{n+1}(x+3)^{n+1}}{\sqrt{n+1}}\cdot\frac{\sqrt{n}}{(-2)^n(x+3)^n}\right| = \lim_{n\to\infty}\frac{2\,|x+3|}{\sqrt{1+1/n}} = 2\,|x+3| < 1 \;\Leftrightarrow$

$|x+3| < \frac{1}{2}$ [so $R = \frac{1}{2}$] $\Leftrightarrow\; -\frac{7}{2} < x < -\frac{5}{2}$. When $x = -\frac{7}{2}$, the series becomes $\displaystyle\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$, which diverges because it is a p-series with $p = \frac{1}{2} \le 1$. When $x = -\frac{5}{2}$, the series becomes $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n}}$, which converges by the Alternating Series Test. Thus, $I = \left(-\frac{7}{2}, -\frac{5}{2}\right]$.

19. If $a_n = \dfrac{(x-2)^n}{n^n}$, then $\displaystyle\lim_{n\to\infty}\sqrt[n]{|a_n|} = \lim_{n\to\infty}\frac{|x-2|}{n} = 0$, so the series converges for all x (by the Root Test). $R = \infty$ and $I = (-\infty, \infty)$.

20. $\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(3x-2)^{n+1}}{(n+1)\,3^{n+1}}\cdot\frac{n3^n}{(3x-2)^n}\right| = \lim_{n\to\infty}\left(\frac{|3x-2|}{3}\cdot\frac{1}{1+1/n}\right) = \frac{|3x-2|}{3} = \left|x-\tfrac{2}{3}\right|$, so by the Ratio Test, the series converges when $\left|x-\frac{2}{3}\right| < 1 \;\Leftrightarrow\; -\frac{1}{3} < x < \frac{5}{3}$. $R = 1$. When $x = -\frac{1}{3}$, the series is $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n}{n}$, the convergent alternating harmonic series. When $x = \frac{5}{3}$, the series becomes the divergent harmonic series. Thus, $I = \left[-\frac{1}{3}, \frac{5}{3}\right)$.

21. $a_n = \dfrac{n}{b^n}(x-a)^n$, where $b > 0$.

$\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{(n+1)\,|x-a|^{n+1}}{b^{n+1}}\cdot\frac{b^n}{n\,|x-a|^n} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)\frac{|x-a|}{b} = \frac{|x-a|}{b}$.

By the Ratio Test, the series converges when $\dfrac{|x-a|}{b} < 1 \;\Leftrightarrow\; |x-a| < b$ [so $R = b$] $\Leftrightarrow$ $-b < x-a < b \;\Leftrightarrow\; a-b < x < a+b$. When $|x-a| = b$, $\displaystyle\lim_{n\to\infty}|a_n| = \lim_{n\to\infty} n = \infty$, so the series diverges. Thus, $I = (a-b, a+b)$.

22. $a_n = \dfrac{n(x-4)^n}{n^3+1}$, so

$\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{(n+1)\,|x-4|^{n+1}}{(n+1)^3+1}\cdot\frac{n^3+1}{n\,|x-4|^n} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)\frac{n^3+1}{n^3+3n^2+3n+2}\,|x-4| = |x-4|$.

By the Ratio Test, the series converges when $|x-4|<1$ [so $R=1$] $\Leftrightarrow$ $-1<x-4<1$ $\Leftrightarrow$ $3<x<5$. When $|x-4|=1$, $\sum\limits_{n=1}^{\infty}|a_n|=\sum\limits_{n=1}^{\infty}\frac{n}{n^3+1}$, which converges by comparison with the convergent p-series $\sum_{n=1}^{\infty}\frac{1}{n^2}$ $(p=2>1)$. Thus, $I=[3,5]$.

23. If $a_n=n!(2x-1)^n$, then $\lim\limits_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim\limits_{n\to\infty}\left|\frac{(n+1)!(2x-1)^{n+1}}{n!(2x-1)^n}\right|=\lim\limits_{n\to\infty}(n+1)\,|2x-1|\to\infty$ as $n\to\infty$ for all $x\neq\frac{1}{2}$. Since the series diverges for all $x\neq\frac{1}{2}$, $R=0$ and $I=\left\{\frac{1}{2}\right\}$.

24. $a_n=\frac{n^2x^n}{2\cdot4\cdot6\cdot\dots\cdot(2n)}=\frac{n^2x^n}{2^n n!}=\frac{nx^n}{2^n(n-1)!}$, so

$\lim\limits_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim\limits_{n\to\infty}\frac{(n+1)\,|x|^{n+1}}{2^{n+1}n!}\cdot\frac{2^n(n-1)!}{n\,|x|^n}=\lim\limits_{n\to\infty}\frac{n+1}{n^2}\frac{|x|}{2}=0$. Thus, by the Ratio Test, the series converges for *all* real x and we have $R=\infty$ and $I=(-\infty,\infty)$.

25. $\lim\limits_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim\limits_{n\to\infty}\left[\frac{|4x+1|^{n+1}}{(n+1)^2}\cdot\frac{n^2}{|4x+1|^n}\right]=\lim\limits_{n\to\infty}\frac{|4x+1|}{(1+1/n)^2}=|4x+1|$, so by the Ratio Test, the series converges when $|4x+1|<1$ $\Leftrightarrow$ $-1<4x+1<1$ $\Leftrightarrow$ $-2<4x<0$ $\Leftrightarrow$ $-\frac{1}{2}<x<0$, so $R=\frac{1}{4}$. When $x=-\frac{1}{2}$, the series becomes $\sum\limits_{n=1}^{\infty}\frac{(-1)^n}{n^2}$, which converges by the Alternating Series Test. When $x=0$, the series becomes $\sum\limits_{n=1}^{\infty}\frac{1}{n^2}$, a convergent p-series $(p=2>1)$. $I=\left[-\frac{1}{2},0\right]$.

26. If $a_n=\frac{(-1)^n(2x+3)^n}{n\ln n}$, then we need $\lim\limits_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=|2x+3|\lim\limits_{n\to\infty}\frac{n\ln n}{(n+1)\ln(n+1)}=|2x+3|<1$ for convergence, so $-2<x<-1$ and $R=\frac{1}{2}$. When $x=-2$, $\sum\limits_{n=2}^{\infty}a_n=\sum\limits_{n=2}^{\infty}\frac{1}{n\ln n}$, which diverges (Integral Test), and when $x=-1$, $\sum\limits_{n=2}^{\infty}a_n=\sum\limits_{n=2}^{\infty}\frac{(-1)^n}{n\ln n}$, which converges (Alternating Series Test), so $I=(-2,-1]$.

27. If $a_n=\frac{x^n}{(\ln n)^n}$, then $\lim\limits_{n\to\infty}\sqrt[n]{|a_n|}=\lim\limits_{n\to\infty}\frac{|x|}{\ln n}=0<1$ for all x, so $R=\infty$ and $I=(-\infty,\infty)$ by the Root Test.

28. If $a_n=\frac{2\cdot4\cdot6\cdot\dots\cdot(2n)\,x^n}{1\cdot3\cdot5\cdot\dots\cdot(2n-1)}$, then we need $\lim\limits_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim\limits_{n\to\infty}|x|\left(\frac{2n+2}{2n+1}\right)=|x|<1$ for convergence, so $R=1$. If $x=\pm1$, $|a_n|=\frac{2\cdot4\cdot6\cdot\dots\cdot(2n)}{1\cdot3\cdot5\cdot\dots\cdot(2n-1)}>1$ for all n since each integer in the numerator is larger than the corresponding one in the denominator, so $\sum a_n$ diverges in both cases by the Test for Divergence, and $I=(-1,1)$.

29. (a) We are given that the power series $\sum_{n=0}^{\infty}c_nx^n$ is convergent for $x=4$. So by Theorem 3, it must converge for at least $-4<x\le4$. In particular, it converges when $x=-2$; that is, $\sum_{n=0}^{\infty}c_n(-2)^n$ is convergent.

(b) It does not follow that $\sum_{n=0}^{\infty}c_n(-4)^n$ is necessarily convergent. [See the comments after Theorem 3 about convergence at the endpoint of an interval. An example is $c_n=(-1)^n/(n4^n)$.]

30. We are given that the power series $\sum_{n=0}^{\infty} c_n x^n$ is convergent for $x = -4$ and divergent when $x = 6$. So by Theorem 3 it converges for at least $-4 \le x < 4$ and diverges for at least $x \ge 6$ and $x < -6$. Therefore:

(a) It converges when $x = 1$; that is, $\sum c_n$ is convergent.

(b) It diverges when $x = 8$; that is, $\sum c_n 8^n$ is divergent.

(c) It converges when $x = -3$; that is, $\sum c_n (-3)^n$ is convergent.

(d) It diverges when $x = -9$; that is, $\sum c_n(-9)^n = \sum(-1)^n c_n 9^n$ is divergent.

31. If $a_n = \dfrac{(n!)^k}{(kn)!} x^n$, then

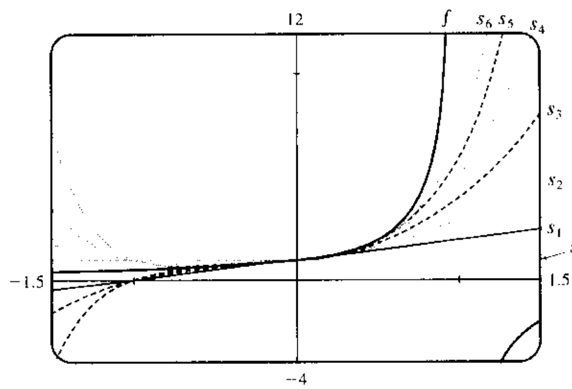
$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{[(n+1)!]^k\,(kn)!}{(n!)^k\,[k(n+1)]!}|x| = \lim_{n\to\infty}\frac{(n+1)^k}{(kn+k)(kn+k-1)\cdots(kn+2)(kn+1)}|x|$$

$$= \lim_{n\to\infty}\left[\frac{(n+1)}{(kn+1)}\frac{(n+1)}{(kn+2)}\cdots\frac{(n+1)}{(kn+k)}\right]|x|$$

$$= \lim_{n\to\infty}\left[\frac{n+1}{kn+1}\right]\lim_{n\to\infty}\left[\frac{n+1}{kn+2}\right]\cdots\lim_{n\to\infty}\left[\frac{n+1}{kn+k}\right]|x| = \left(\frac{1}{k}\right)^k|x| < 1 \quad\Leftrightarrow$$

$|x| < k^k$ for convergence, and the radius of convergence is $R = k^k$.

32. The partial sums of the series $\sum_{n=0}^{\infty} x^n$ definitely do not converge to $f(x) = 1/(1-x)$ for $x \ge 1$, since f is undefined at $x = 1$ and negative on $(1, \infty)$, while all the partial sums are positive on this interval. The partial sums also fail to converge to f for $x \le -1$, since $0 < f(x) < 1$ on this interval, while the partial sums are either larger than 1 or less than 0. The partial sums seem to converge to f on $(-1, 1)$. This graphical evidence is consistent with what we know about geometric series: convergence for $|x| < 1$, divergence for $|x| \ge 1$ (see Example 12.2.5).



33. (a) If $a_n = \dfrac{(-1)^n x^{2n+1}}{n!(n+1)!\,2^{2n+1}}$, then

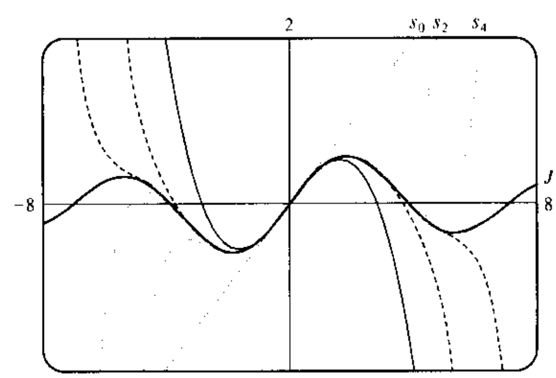
$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{2n+3}}{(n+1)!(n+2)!\,2^{2n+3}}\cdot\frac{n!(n+1)!\,2^{2n+1}}{x^{2n+1}}\right| = \left(\frac{x}{2}\right)^2\lim_{n\to\infty}\frac{1}{(n+1)(n+2)} = 0 \text{ for}$$

all x. So $J_1(x)$ converges for all x and its domain is $(-\infty, \infty)$.

(b), (c) The initial terms of $J_1(x)$ up to $n = 5$ are

$a_0 = \dfrac{x}{2}$, $a_1 = -\dfrac{x^3}{16}$, $a_2 = \dfrac{x^5}{384}$, $a_3 = -\dfrac{x^7}{18{,}432}$,

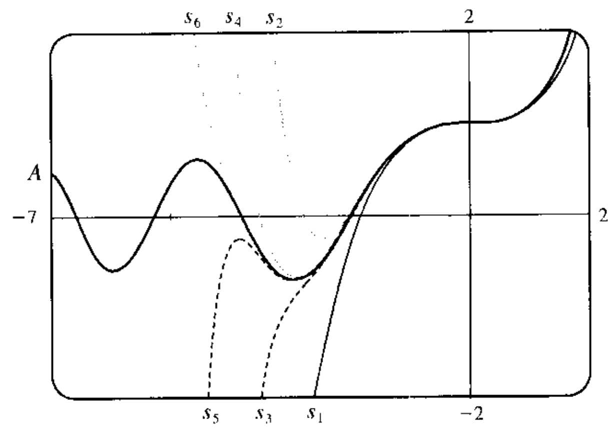
$a_4 = \dfrac{x^9}{1{,}474{,}560}$, and $a_5 = -\dfrac{x^{11}}{176{,}947{,}200}$. The partial sums seem to approximate $J_1(x)$ well near the origin, but as $|x|$ increases, we need to take a large number of terms to get a good approximation.



34. (a) $A(x) = 1 + \sum_{n=1}^{\infty} a_n$, where $a_n = \dfrac{x^{3n}}{2 \cdot 3 \cdot 5 \cdot 6 \cdot \cdots \cdot (3n-1)(3n)}$, so

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = |x|^3 \lim_{n\to\infty} \frac{1}{(3n+2)(3n+3)} = 0 \text{ for all } x \text{, so the domain is } \mathbb{R}.$$

(b), (c)



$s_0 = 1$ has been omitted from the graph. The partial sums seem to approximate $A(x)$ well near the origin, but as $|x|$ increases, we need to take a large number of terms to get a good approximation.

To plot A, we must first define $A(x)$ for the CAS. Note that for $n \geq 1$, the denominator of a_n is

$$2 \cdot 3 \cdot 5 \cdot 6 \cdot \cdots \cdot (3n-1) \cdot 3n = \frac{(3n)!}{1 \cdot 4 \cdot 7 \cdot \cdots \cdot (3n-2)} = \frac{(3n)!}{\prod_{k=1}^{n}(3k-2)}, \text{ so } a_n = \frac{\prod_{k=1}^{n}(3k-2)}{(3n)!} x^{3n} \text{ and}$$

thus $A(x) = 1 + \sum_{n=1}^{\infty} \dfrac{\prod_{k=1}^{n}(3k-2)}{(3n)!} x^{3n}$. Both Maple and Mathematica are able to plot A if we define it this way, and Derive is able to produce a similar graph using a suitable partial sum of $A(x)$.

Derive, Maple and Mathematica all have two initially known Airy functions, called `AI_SERIES(z,m)` and `BI_SERIES(z,m)` from `BESSEL.MTH` in Derive and `AiryAi` and `AiryBi` in Maple and Mathematica (just `Ai` and `Bi` in older versions of Maple). However, it is very difficult to solve for A in terms of the CAS's Airy functions, although in fact $A(x) = \dfrac{\sqrt{3}\,\texttt{AiryAi}(x) + \texttt{AiryBi}(x)}{\sqrt{3}\,\texttt{AiryAi}(0) + \texttt{AiryBi}(0)}$.

35.
$$\begin{aligned} s_{2n-1} &= 1 + 2x + x^2 + 2x^3 + x^4 + 2x^5 + \cdots + x^{2n-2} + 2x^{2n-1} \\ &= 1(1+2x) + x^2(1+2x) + x^4(1+2x) + \cdots + x^{2n-2}(1+2x) \\ &= (1+2x)\left(1 + x^2 + x^4 + \cdots + x^{2n-2}\right) \\ &= (1+2x)\frac{1-x^{2n}}{1-x^2} \ [\text{by (12.2.3)] with } r = x^2] \to \frac{1+2x}{1-x^2} \text{ as } n \to \infty \ [\text{by (12.2.4)}], \end{aligned}$$

when $|x| < 1$. Also $s_{2n} = s_{2n-1} + x^{2n} \to \dfrac{1+2x}{1-x^2}$ since $x^{2n} \to 0$ for $|x| < 1$. Therefore,

$s_n \to \dfrac{1+2x}{1-x^2}$ since s_{2n} and s_{2n-1} both approach $\dfrac{1+2x}{1-x^2}$ as $n \to \infty$. Thus, the interval of convergence is $(-1, 1)$ and $f(x) = \dfrac{1+2x}{1-x^2}$.

36. $s_{4n-1} = c_0 + c_1x + c_2x^2 + c_3x^3 + c_0x^4 + c_1x^5 + c_2x^6 + c_3x^7 + \cdots + c_3x^{4n-1}$

$$= (c_0 + c_1x + c_2x^2 + c_3x^3)(1 + x^4 + x^8 + \cdots + x^{4n-4}) \to \frac{c_0 + c_1x + c_2x^2 + c_3x^3}{1 - x^4} \text{ as } n \to \infty$$

[by (12.2.4) with $r = x^4$] for $|x^4| < 1 \Leftrightarrow |x| < 1$. Also $s_{4n}, s_{4n+1}, s_{4n+2}$ have the same limits (for example, $s_{4n} = s_{4n-1} + c_0x^{4n}$ and $x^{4n} \to 0$ for $|x| < 1$). So if at least one of c_0, c_1, c_2, and c_3 is nonzero, then the interval of convergence is $(-1, 1)$ and $f(x) = \dfrac{c_0 + c_1x + c_2x^2 + c_3x^3}{1 - x^4}$.

37. We use the Root Test on the series $\sum c_nx^n$. We need $\lim\limits_{n\to\infty} \sqrt[n]{|c_nx^n|} = |x| \lim\limits_{n\to\infty} \sqrt[n]{|c_n|} = c|x| < 1$ for convergence, or $|x| < 1/c$, so $R = 1/c$.

38. Suppose $c_n \neq 0$. Applying the Ratio Test to the series $\sum c_n(x-a)^n$, we find that

$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{c_{n+1}(x-a)^{n+1}}{c_n(x-a)^n}\right| = \lim_{n\to\infty} \frac{|x-a|}{|c_n/c_{n+1}|} \, (*) = \frac{|x-a|}{\lim\limits_{n\to\infty} |c_n/c_{n+1}|}$$

(if $\lim\limits_{n\to\infty} |c_n/c_{n+1}| \neq 0$), so the series converges when $\dfrac{|x-a|}{\lim\limits_{n\to\infty} |c_n/c_{n-1}|} < 1 \Leftrightarrow |x-a| < \lim\limits_{n\to\infty} \left|\dfrac{c_n}{c_{n+1}}\right|$. Thus, $R = \lim\limits_{n\to\infty} \left|\dfrac{c_n}{c_{n-1}}\right|$. If $\lim\limits_{n\to\infty} \left|\dfrac{c_n}{c_{n+1}}\right| = 0$ and $|x-a| \neq 0$, then $(*)$ shows that $L = \infty$ and so the series diverges, and hence, $R = 0$. Thus, in all cases, $R = \lim\limits_{n\to\infty} \left|\dfrac{c_n}{c_{n+1}}\right|$.

39. For $2 < x < 3$, $\sum c_nx^n$ diverges and $\sum d_nx^n$ converges. By Exercise 12.2.61, $\sum (c_n + d_n)x^n$ diverges. Since both series converge for $|x| < 2$, the radius of convergence of $\sum (c_n + d_n)x^n$ is 2.

40. Since $\sum c_nx^n$ converges whenever $|x| < R$, $\sum c_nx^{2n} = \sum c_n(x^2)^n$ converges whenever $|x^2| < R \Leftrightarrow |x| < \sqrt{R}$, so the second series has radius of convergence $\sqrt{R}$.

12.9 Representations of Functions as Power Series

1. If $f(x) = \sum\limits_{n=0}^{\infty} c_nx^n$ has radius of convergence 10, then $f'(x) = \sum\limits_{n=1}^{\infty} nc_nx^{n-1}$ also has radius of convergence 10 by Theorem 2.

2. If $f(x) = \sum\limits_{n=0}^{\infty} b_nx^n$ converges on $(-2, 2)$, then $\int f(x)dx = C + \sum\limits_{n=0}^{\infty} \dfrac{b_n}{n+1}x^{n+1}$ has the same radius of convergence (by Theorem 2), but may not have the same interval of convergence—it may happen that the integrated series converges at an endpoint (or both endpoints).

3. Our goal is to write the function in the form $\dfrac{1}{1-r}$, and then use Equation (1) to represent the function as a sum of a power series. $f(x) = \dfrac{1}{1+x} = \dfrac{1}{1-(-x)} = \sum\limits_{n=0}^{\infty} (-x)^n = \sum\limits_{n=0}^{\infty} (-1)^nx^n$ with $|-x| < 1 \Leftrightarrow |x| < 1$, so $R = 1$ and $I = (-1, 1)$.

4. $f(x) = \dfrac{3}{1-x^4} = 3\left(\dfrac{1}{1-x^4}\right) = 3(1 + x^4 + x^8 + x^{12} + \cdots) = 3\sum_{n=0}^{\infty}(x^4)^n = \sum_{n=0}^{\infty} 3x^{4n}$ with

$|x^4| < 1 \quad\Leftrightarrow\quad |x| < 1$, so $R = 1$ and $I = (-1, 1)$. [Note that $3\sum_{n=0}^{\infty}(x^4)^n$ converges $\Leftrightarrow$ $\sum_{n=0}^{\infty}(x^4)^n$ converges, so the appropriate condition (from equation (1)) is $|x^4| < 1$.]

5. Replacing x with x^3 in (1) gives $f(x) = \dfrac{1}{1-x^3} = \sum_{n=0}^{\infty}(x^3)^n = \sum_{n=0}^{\infty} x^{3n}$. The series converges when $|x^3| < 1$ $\Leftrightarrow\quad |x|^3 < 1 \quad\Leftrightarrow\quad |x| < \sqrt[3]{1} \quad\Leftrightarrow\quad |x| < 1$. Thus, $R = 1$ and $I = (-1, 1)$.

6. $f(x) = \dfrac{1}{1+9x^2} = \dfrac{1}{1-(-9x^2)} = \sum_{n=0}^{\infty}(-9x^2)^n = \sum_{n=0}^{\infty}(-1)^n 3^{2n} x^{2n}$. The series converges when $|-9x^2| < 1$; that is, when $|x| < \frac{1}{3}$, so $I = \left(-\frac{1}{3}, \frac{1}{3}\right)$.

7. $f(x) = \dfrac{1}{x-5} = -\dfrac{1}{5}\left(\dfrac{1}{1-x/5}\right) = -\dfrac{1}{5}\sum_{n=0}^{\infty}\left(\dfrac{x}{5}\right)^n$ or equivalently, $-\sum_{n=0}^{\infty}\dfrac{1}{5^{n+1}}x^n$. The series converges when $\left|\dfrac{x}{5}\right| < 1$; that is, when $|x| < 5$, so $I = (-5, 5)$.

8. $f(x) = \dfrac{x}{4x+1} = x \cdot \dfrac{1}{1-(-4x)} = x\sum_{n=0}^{\infty}(-4x)^n = \sum_{n=0}^{\infty}(-1)^n 2^{2n} x^{n+1}$. The series converges when $|-4x| < 1$; that is, when $|x| < \frac{1}{4}$, so $I = \left(-\frac{1}{4}, \frac{1}{4}\right)$.

9. $f(x) = \dfrac{x}{9+x^2} = \dfrac{x}{9}\left[\dfrac{1}{1+(x/3)^2}\right] = \dfrac{x}{9}\left[\dfrac{1}{1-\{-(x/3)^2\}}\right] = \dfrac{x}{9}\sum_{n=0}^{\infty}\left[-\left(\dfrac{x}{3}\right)^2\right]^n$

$= \dfrac{x}{9}\sum_{n=0}^{\infty}(-1)^n\dfrac{x^{2n}}{9^n} = \sum_{n=0}^{\infty}(-1)^n\dfrac{x^{2n+1}}{9^{n+1}}$. The geometric series $\sum_{n=0}^{\infty}\left[-\left(\dfrac{x}{3}\right)^2\right]^n$ converges when

$\left|-\left(\dfrac{x}{3}\right)^2\right| < 1 \quad\Leftrightarrow\quad \dfrac{|x^2|}{9} < 1 \quad\Leftrightarrow\quad |x|^2 < 9 \quad\Leftrightarrow\quad |x| < 3$, so $R = 3$ and $I = (-3, 3)$.

10. $f(x) = \dfrac{x^2}{a^3 - x^3} = \dfrac{x^2}{a^3}\cdot\dfrac{1}{1-x^3/a^3} = \dfrac{x^2}{a^3}\sum_{n=0}^{\infty}\left(\dfrac{x^3}{a^3}\right)^n = \sum_{n=0}^{\infty}\dfrac{x^{3n+2}}{a^{3n+3}}$. The series converges when $|x^3/a^3| < 1 \quad\Leftrightarrow\quad |x^3| < |a^3| \quad\Leftrightarrow\quad |x| < |a|$, so $R = |a|$ and $I = (-|a|, |a|)$.

11. $f(x) = \dfrac{3}{x^2+x-2} = \dfrac{3}{(x+2)(x-1)} = \dfrac{A}{x+2} + \dfrac{B}{x-1} \quad\Rightarrow\quad 3 = A(x-1) + B(x+2)$. Taking $x = -2$, we get $A = -1$. Taking $x = 1$, we get $B = 1$. Thus,

$$\frac{3}{x^2+x-2} = \frac{1}{x-1} - \frac{1}{x+2} = -\frac{1}{1-x} - \frac{1}{2}\frac{1}{1+x/2} = -\sum_{n=0}^{\infty}x^n - \frac{1}{2}\sum_{n=0}^{\infty}\left(-\frac{x}{2}\right)^n$$

$$= \sum_{n=0}^{\infty}\left[-1 - \tfrac{1}{2}\left(-\tfrac{1}{2}\right)^n\right]x^n = \sum_{n=0}^{\infty}\left[-1 + \left(-\tfrac{1}{2}\right)^{n+1}\right]x^n = \sum_{n=0}^{\infty}\left[\frac{(-1)^{n+1}}{2^{n+1}} - 1\right]x^n$$

We represented the given function as the sum of two geometric series; the first converges for $x \in (-1, 1)$ and the second converges for $x \in (-2, 2)$. Thus, the sum converges for $x \in (-1, 1) = I$.

12. $f(x) = \dfrac{7x-1}{3x^2+2x-1} = \dfrac{7x-1}{(3x-1)(x+1)} = \dfrac{A}{3x-1} + \dfrac{B}{x+1} = \dfrac{1}{3x-1} + \dfrac{2}{x+1} = 2 \cdot \dfrac{1}{1-(-x)} - \dfrac{1}{1-3x}$

$= 2\sum_{n=0}^{\infty}(-x)^n - \sum_{n=0}^{\infty}(3x)^n = \sum_{n=0}^{\infty}\left[2(-1)^n - 3^n\right]x^n$

The series $\sum(-x)^n$ converges for $x \in (-1, 1)$ and the series $\sum(3x)^n$ converges for $x \in \left(-\frac{1}{3}, \frac{1}{3}\right)$, so their sum converges for $x \in \left(-\frac{1}{3}, \frac{1}{3}\right) = I$.

13. (a) $f(x) = \dfrac{1}{(1+x)^2} = \dfrac{d}{dx}\left(\dfrac{-1}{1+x}\right) = -\dfrac{d}{dx}\left[\sum_{n=0}^{\infty}(-1)^n x^n\right]$ [from Exercise 3]

$= \sum_{n=1}^{\infty}(-1)^{n+1}nx^{n-1}$ [from Theorem 2(i)] $= \sum_{n=0}^{\infty}(-1)^n(n+1)x^n$ with $R = 1$.

In the last step, note that we *decreased* the initial value of the summation variable n by 1, and then *increased* each occurrence of n in the term by 1 [also note that $(-1)^{n+2} = (-1)^n$].

(b) $f(x) = \dfrac{1}{(1+x)^3} = -\dfrac{1}{2}\dfrac{d}{dx}\left[\dfrac{1}{(1+x)^2}\right] = -\dfrac{1}{2}\dfrac{d}{dx}\left[\sum_{n=0}^{\infty}(-1)^n(n+1)x^n\right]$ [from part (a)]

$= -\frac{1}{2}\sum_{n=1}^{\infty}(-1)^n(n+1)nx^{n-1} = \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n(n+2)(n+1)x^n$ with $R = 1$.

(c) $f(x) = \dfrac{x^2}{(1+x)^3} = x^2 \cdot \dfrac{1}{(1+x)^3} = x^2 \cdot \dfrac{1}{2}\sum_{n=0}^{\infty}(-1)^n(n+2)(n+1)x^n$ [from part (b)]

$= \dfrac{1}{2}\sum_{n=0}^{\infty}(-1)^n(n+2)(n+1)x^{n+2}$. To write the power series with x^n rather than x^{n+2}, we will *decrease* each occurrence of n in the term by 2 and *increase* the initial value of the summation variable by 2. This gives us $\dfrac{1}{2}\sum_{n=2}^{\infty}(-1)^n(n)(n-1)x^n$.

14. (a) $\dfrac{1}{1+x} = \dfrac{1}{1-(-x)} = \sum_{n=0}^{\infty}(-1)^n x^n$ [geometric series with $R = 1$], so

$$f(x) = \ln(1+x) = \int \frac{dx}{1+x} = \int\left[\sum_{n=0}^{\infty}(-1)^n x^n\right]dx = C + \sum_{n=0}^{\infty}(-1)^n\frac{x^{n+1}}{n+1}$$

$$= \sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^n}{n} \quad [C = 0 \text{ since } f(0) = \ln 1 = 0], \text{ with } R = 1$$

(b) $f(x) = x\ln(1+x) = x\left[\sum_{n=1}^{\infty}\dfrac{(-1)^{n-1}x^n}{n}\right]$ [by part (a)] $= \sum_{n=1}^{\infty}\dfrac{(-1)^{n-1}x^{n+1}}{n} = \sum_{n=2}^{\infty}\dfrac{(-1)^n x^n}{n-1}$

with $R = 1$.

(c) $f(x) = \ln(x^2+1) = \sum_{n=1}^{\infty}\dfrac{(-1)^{n-1}(x^2)^n}{n}$ [by part (a)] $= \sum_{n=1}^{\infty}\dfrac{(-1)^{n-1}x^{2n}}{n}$ with $R = 1$.

15. $f(x) = \ln(5-x) = -\displaystyle\int\frac{dx}{5-x} = -\frac{1}{5}\int\frac{dx}{1-x/5}$

$= -\dfrac{1}{5}\displaystyle\int\left[\sum_{n=0}^{\infty}\left(\frac{x}{5}\right)^n\right]dx = C - \frac{1}{5}\sum_{n=0}^{\infty}\frac{x^{n+1}}{5^n(n+1)} = C - \sum_{n=1}^{\infty}\frac{x^n}{n5^n}$

Putting $x = 0$, we get $C = \ln 5$. The series converges for $|x/5| < 1 \Leftrightarrow |x| < 5$, so $R = 5$.

16. We know that $\dfrac{1}{1-2x} = \sum\limits_{n=0}^{\infty} (2x)^n$. Differentiating, we get $\dfrac{2}{(1-2x)^2} = \sum\limits_{n=1}^{\infty} 2^n n x^{n-1} = \sum\limits_{n=0}^{\infty} 2^{n+1}(n+1)x^n$, so

$$f(x) = \frac{x^2}{(1-2x)^2} = \frac{x^2}{2} \cdot \frac{2}{(1-2x)^2} = \frac{x^2}{2} \sum_{n=0}^{\infty} 2^{n+1}(n+1)x^n = \sum_{n=0}^{\infty} 2^n(n+1)x^{n+2} \text{ or } \sum_{n=2}^{\infty} 2^{n-2}(n-1)x^n,$$

with $R = \frac{1}{2}$.

17. $\dfrac{1}{2-x} = \dfrac{1}{2(1-x/2)} = \dfrac{1}{2}\sum\limits_{n=0}^{\infty}\left(\dfrac{x}{2}\right)^n = \sum\limits_{n=0}^{\infty} \dfrac{1}{2^{n+1}} x^n$ for $\left|\dfrac{x}{2}\right| < 1 \;\Leftrightarrow\; |x| < 2$. Now

$$\frac{1}{(x-2)^2} = \frac{d}{dx}\left(\frac{1}{2-x}\right) = \frac{d}{dx}\left(\sum_{n=0}^{\infty} \frac{1}{2^{n+1}} x^n\right) = \sum_{n=1}^{\infty} \frac{n}{2^{n+1}} x^{n-1} = \sum_{n=0}^{\infty} \frac{n+1}{2^{n+2}} x^n. \text{ So}$$

$$f(x) = \frac{x^3}{(x-2)^2} = x^3 \sum_{n=0}^{\infty} \frac{n+1}{2^{n+2}} x^n = \sum_{n=0}^{\infty} \frac{n+1}{2^{n+2}} x^{n+3} \text{ or } \sum_{n=3}^{\infty} \frac{n-2}{2^{n-1}} x^n \text{ for } |x| < 2. \text{ Thus, } R = 2 \text{ and}$$

$I = (-2, 2)$.

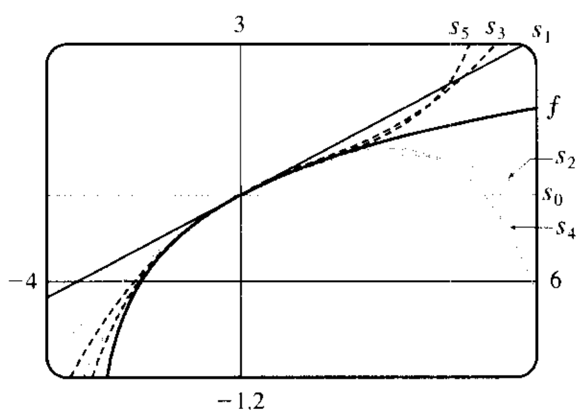
18. From Example 7, $g(x) = \arctan x = \sum\limits_{n=0}^{\infty} (-1)^n \dfrac{x^{2n+1}}{2n+1}$. Thus,

$$f(x) = \arctan(x/3) = \sum_{n=0}^{\infty} (-1)^n \frac{(x/3)^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} (-1)^n \frac{1}{3^{2n+1}(2n+1)} x^{2n+1} \text{ for } \left|\frac{x}{3}\right| < 1 \;\Leftrightarrow\; |x| < 3,$$

so $R = 3$.

19.
$$f(x) = \ln(3+x) = \int \frac{dx}{3+x} = \frac{1}{3}\int \frac{dx}{1+x/3} = \frac{1}{3}\int \frac{dx}{1-(-x/3)} = \frac{1}{3}\int \sum_{n=0}^{\infty}\left(-\frac{x}{3}\right)^n dx$$
$$= C + \frac{1}{3}\sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)3^n} x^{n+1} = \ln 3 + \frac{1}{3}\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n 3^{n-1}} x^n \quad [C = f(0) = \ln 3]$$
$$= \ln 3 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n 3^n} x^n. \text{ The series converges when } |-x/3| < 1 \;\Leftrightarrow\; |x| < 3, \text{ so } R = 3.$$

The terms of the series are $a_0 = \ln 3$, $a_1 = \dfrac{x}{3}$, $a_2 = -\dfrac{x^2}{18}$, $a_3 = \dfrac{x^3}{81}$, $a_4 = -\dfrac{x^4}{324}$, $a_5 = \dfrac{x^5}{1215}$, ...

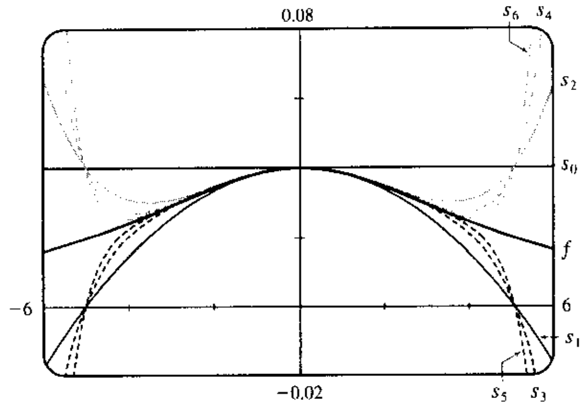


As n increases, $s_n(x)$ approximates f better on the interval of convergence, which is $(-3, 3)$.

20. $f(x) = \dfrac{1}{x^2+25} = \dfrac{1}{25}\left(\dfrac{1}{1+x^2/25}\right) = \dfrac{1}{25}\left(\dfrac{1}{1-(-x^2/25)}\right) = \dfrac{1}{25}\sum_{n=0}^{\infty}\left(-\dfrac{x^2}{25}\right)^n = \dfrac{1}{25}\sum_{n=0}^{\infty}(-1)^n\left(\dfrac{x}{5}\right)^{2n}$.

The series converges when $\left|-x^2/25\right| < 1 \quad \Leftrightarrow \quad x^2 < 25 \quad \Leftrightarrow \quad |x| < 5$, so $R = 5$. The terms of the series are

$a_0 = \dfrac{1}{25}, a_1 = -\dfrac{x^2}{625}, a_2 = \dfrac{x^4}{15{,}625}, \ldots$.



As n increases, $s_n(x)$ approximates f better on the interval of convergence, which is $(-5, 5)$.

21. $f(x) = \ln\left(\dfrac{1+x}{1-x}\right) = \ln(1+x) - \ln(1-x) = \displaystyle\int \frac{dx}{1+x} + \int \frac{dx}{1-x}$

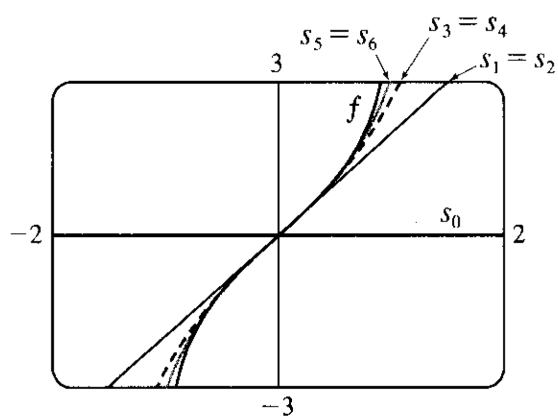
$$= \int \frac{dx}{1-(-x)} + \int \frac{dx}{1-x} = \int \left[\sum_{n=0}^{\infty}(-1)^n x^n + \sum_{n=0}^{\infty} x^n\right] dx$$

$$= \int \left[\left(1 - x + x^2 - x^3 + x^4 - \cdots\right) + \left(1 + x + x^2 + x^3 + x^4 + \cdots\right)\right] dx$$

$$= \int \left(2 + 2x^2 + 2x^4 + \cdots\right) dx = \int \sum_{n=0}^{\infty} 2x^{2n}\, dx = C + \sum_{n=0}^{\infty} \frac{2x^{2n+1}}{2n+1}$$

But $f(0) = \ln\frac{1}{1} = 0$, so $C = 0$ and we have $f(x) = \displaystyle\sum_{n=0}^{\infty}\frac{2x^{2n+1}}{2n+1}$ with $R = 1$. If $x = \pm 1$, then

$f(x) = \pm 2\displaystyle\sum_{n=0}^{\infty}\frac{1}{2n+1}$, which both diverge by the Limit Comparison Test with $b_n = \dfrac{1}{n}$.

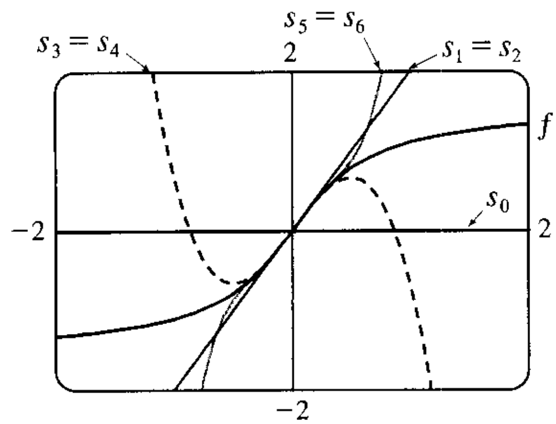


As n increases, $s_n(x)$ approximates f better on the interval of convergence, which is $(-1, 1)$.

22. $f(x) = \tan^{-1}(2x) = 2\int \frac{dx}{1+4x^2} = 2\int \sum_{n=0}^{\infty}(-1)^n (4x^2)^n \, dx = 2\int \sum_{n=0}^{\infty}(-1)^n 4^n x^{2n}\, dx$

$= C + 2\sum_{n=0}^{\infty} \frac{(-1)^n 4^n x^{2n+1}}{2n+1} = \sum_{n=0}^{\infty}\frac{(-1)^n 2^{2n+1} x^{2n+1}}{2n+1}$ $[f(0) = \tan^{-1} 0 = 0$, so $C = 0]$.

The series converges when $|4x^2| < 1 \Leftrightarrow |x| < \frac{1}{2}$, so $R = \frac{1}{2}$. If $x = \pm\frac{1}{2}$, then $f(x) = \sum_{n=0}^{\infty}(-1)^n \frac{1}{2n+1}$ and $f(x) = \sum_{n=0}^{\infty}(-1)^{n+1}\frac{1}{2n+1}$, respectively. Both series converge by the Alternating Series Test.



As n increases, $s_n(x)$ approximates f better on the interval of convergence, which is $\left[-\frac{1}{2}, \frac{1}{2}\right]$.

23. $\frac{t}{1-t^8} = t\cdot\frac{1}{1-t^8} = t\sum_{n=0}^{\infty}(t^8)^n = \sum_{n=0}^{\infty} t^{8n+1} \Rightarrow \int \frac{t}{1-t^8}\,dt = C + \sum_{n=0}^{\infty}\frac{t^{8n+2}}{8n+2}$. The series for $\frac{1}{1-t^8}$ converges when $|t^8| < 1 \Leftrightarrow |t| < 1$, so $R = 1$ for that series and also the series for $t/(1-t^8)$. By Theorem 2, the series for $\int \frac{t}{1-t^8}\,dt$ also has $R = 1$.

24. By Example 6, $\ln(1-t) = -\sum_{n=1}^{\infty}\frac{t^n}{n}$ for $|t| < 1$, so $\frac{\ln(1-t)}{t} = -\sum_{n=1}^{\infty}\frac{t^{n-1}}{n}$ and

$\int \frac{\ln(1-t)}{t}\,dt = C - \sum_{n=1}^{\infty}\frac{t^n}{n^2}$. By Theorem 2, $R = 1$.

25. By Example 7, $\tan^{-1} x = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n+1}}{2n+1}$ with $R = 1$, so

$x - \tan^{-1} x = x - \left(x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots\right) = \frac{x^3}{3} - \frac{x^5}{5} + \frac{x^7}{7} - \cdots = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^{2n+1}}{2n+1}$

and $\frac{x - \tan^{-1} x}{x^3} = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^{2n-2}}{2n+1}$, so

$\int \frac{x - \tan^{-1} x}{x^3}\,dx = C + \sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^{2n-1}}{(2n+1)(2n-1)} = C + \sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^{2n-1}}{4n^2-1}$. By Theorem 2, $R = 1$.

26. By Example 7, $\int \tan^{-1}(x^2)\,dx = \int \sum_{n=0}^{\infty}(-1)^n \frac{(x^2)^{2n+1}}{2n+1}\,dx = C + \sum_{n=0}^{\infty}(-1)^n \frac{x^{4n+3}}{(2n+1)(4n+3)}$ with $R = 1$.

27. $\frac{1}{1+x^5} = \frac{1}{1-(-x^5)} = \sum_{n=0}^{\infty}(-x^5)^n = \sum_{n=0}^{\infty}(-1)^n x^{5n} \Rightarrow$

$\int \frac{1}{1+x^5}\,dx = \int \sum_{n=0}^{\infty}(-1)^n x^{5n}\,dx = C + \sum_{n=0}^{\infty}(-1)^n \frac{x^{5n+1}}{5n+1}$. Thus,

$I = \int_0^{0.2} \frac{1}{1+x^5}\,dx = \left[x - \frac{x^6}{6} + \frac{x^{11}}{11} - \cdots\right]_0^{0.2} = 0.2 - \frac{(0.2)^6}{6} + \frac{(0.2)^{11}}{11} - \cdots$. The series is alternating, so if we use the first two terms, the error is at most $(0.2)^{11}/11 \approx 1.9 \times 10^{-9}$. So $I \approx 0.2 - (0.2)^6/6 \approx 0.199989$ to six decimal places.

28. From Example 6 we know $\ln(1-x) = -\sum_{n=1}^{\infty}\frac{x^n}{n}$, so

$\ln(1+x^4) = \ln[1-(-x^4)] = -\sum_{n=1}^{\infty}\frac{(-x^4)^n}{n} = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^{4n}}{n} \Rightarrow$

$\int \ln(1+x^4)\,dx = \int \sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^{4n}}{n}\,dx = C + \sum_{n=1}^{\infty}(-1)^{n+1}\frac{x^{4n+1}}{n(4n+1)}$. Thus,

$I = \int_0^{0.4} \ln(1+x^4)\,dx = \left[\frac{x^5}{5} - \frac{x^9}{18} + \frac{x^{13}}{39} - \frac{x^{17}}{68} + \cdots\right]_0^{0.4} = \frac{(0.4)^5}{5} - \frac{(0.4)^9}{18} + \frac{(0.4)^{13}}{39} - \frac{(0.4)^{17}}{68} + \cdots$.

The series is alternating, so if we use the first three terms, the error is at most $(0.4)^{17}/68 \approx 2.5 \times 10^{-9}$. So $I \approx (0.4)^5/5 - (0.4)^9/18 + (0.9)^{13}/39 \approx 0.002034$ to six decimal places.

29. We substitute x^4 for x in Example 7, and find that

$$\int x^2 \tan^{-1}(x^4)\,dx = \int x^2 \sum_{n=0}^{\infty}(-1)^n \frac{(x^4)^{2n+1}}{2n+1}\,dx$$

$$= \int \sum_{n=0}^{\infty}(-1)^n \frac{x^{8n+6}}{2n+1}\,dx = C + \sum_{n=0}^{\infty}(-1)^n \frac{x^{8n+7}}{(2n+1)(8n+7)}$$

So $\int_0^{1/3} x^2 \tan^{-1}(x^4)\,dx = \left[\frac{x^7}{7} - \frac{x^{15}}{45} + \cdots\right]_0^{1/3} = \frac{1}{7\cdot 3^7} - \frac{1}{45\cdot 3^{15}} + \cdots$. The series is alternating, so if we use only one term, the error is at most $1/(45\cdot 3^{15}) \approx 1.5 \times 10^{-9}$. So $\int_0^{1/3} x^2 \tan^{-1}(x^4)\,dx \approx 1/(7\cdot 3^7) \approx 0.000065$ to six decimal places.

30. $\int_0^{0.5} \frac{dx}{1+x^6} = \int_0^{0.5} \sum_{n=0}^{\infty}(-1)^n x^{6n}\,dx = \sum_{n=0}^{\infty}\left[\frac{(-1)^n x^{6n+1}}{6n+1}\right]_0^{1/2} = \sum_{n=0}^{\infty}\frac{(-1)^n}{(6n+1)2^{6n+1}}$

$= \frac{1}{2} - \frac{1}{7\cdot 2^7} + \frac{1}{13\cdot 2^{13}} - \frac{1}{19\cdot 2^{19}} + \cdots$

The series is alternating, so if we use only three terms, the error is at most $\frac{1}{19\cdot 2^{19}} \approx 1.0 \times 10^{-7}$. So, to six decimal places, $\int_0^{0.5}\frac{dx}{1+x^6} \approx \frac{1}{2} - \frac{1}{7\cdot 2^7} + \frac{1}{13\cdot 2^{13}} \approx 0.498893$.

31. Using the result of Example 6, $\ln(1-x) = -\sum_{n=1}^{\infty} \frac{x^n}{n}$, with $x = -0.1$, we have

$\ln 1.1 = \ln[1-(-0.1)] = 0.1 - \frac{0.01}{2} + \frac{0.001}{3} - \frac{0.0001}{4} + \frac{0.00001}{5} - \cdots$. The series is alternating, so if we use only the first four terms, the error is at most $\frac{0.00001}{5} = 0.000002$. So

$\ln 1.1 \approx 0.1 - \frac{0.01}{2} + \frac{0.001}{3} - \frac{0.0001}{4} \approx 0.09531$.

32. $f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \quad\Rightarrow\quad f'(x) = \sum_{n=1}^{\infty} \frac{(-1)^n 2n x^{2n-1}}{(2n)!}$ [the first term disappears], so

$$f''(x) = \sum_{n=1}^{\infty} \frac{(-1)^n (2n)(2n-1)x^{2n-2}}{(2n)!} = \sum_{n=1}^{\infty} \frac{(-1)^n x^{2(n-1)}}{[2(n-1)]!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^{n+1} x^{2n}}{(2n)!} \quad [\text{substituting } n+1 \text{ for } n]$$

$$= -\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = -f(x) \quad\Rightarrow\quad f''(x) + f(x) = 0.$$

33. (a) $J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}$, $J_0'(x) = \sum_{n=1}^{\infty} \frac{(-1)^n 2n x^{2n-1}}{2^{2n}(n!)^2}$, and $J_0''(x) = \sum_{n=1}^{\infty} \frac{(-1)^n 2n(2n-1)x^{2n-2}}{2^{2n}(n!)^2}$, so

$$x^2 J_0''(x) + xJ_0'(x) + x^2 J_0(x) = \sum_{n=1}^{\infty} \frac{(-1)^n 2n(2n-1)x^{2n}}{2^{2n}(n!)^2} + \sum_{n=1}^{\infty} \frac{(-1)^n 2n x^{2n}}{2^{2n}(n!)^2} + \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{2^{2n}(n!)^2}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n 2n(2n-1)x^{2n}}{2^{2n}(n!)^2} + \sum_{n=1}^{\infty} \frac{(-1)^n 2n x^{2n}}{2^{2n}(n!)^2} + \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^{2n}}{2^{2n-2}[(n-1)!]^2}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n 2n(2n-1)x^{2n}}{2^{2n}(n!)^2} + \sum_{n=1}^{\infty} \frac{(-1)^n 2n x^{2n}}{2^{2n}(n!)^2} + \sum_{n=1}^{\infty} \frac{(-1)^n(-1)^{-1}2^2 n^2 x^{2n}}{2^{2n}(n!)^2}$$

$$= \sum_{n=1}^{\infty} (-1)^n \left[\frac{2n(2n-1) + 2n - 2^2 n^2}{2^{2n}(n!)^2}\right] x^{2n} = \sum_{n=1}^{\infty} (-1)^n \left[\frac{4n^2 - 2n + 2n - 4n^2}{2^{2n}(n!)^2}\right] x^{2n} = 0$$

(b) $$\int_0^1 J_0(x)\,dx = \int_0^1 \left[\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n!)^2}\right] dx = \int_0^1 \left(1 - \frac{x^2}{4} + \frac{x^4}{64} - \frac{x^6}{2304} + \cdots\right) dx$$

$$= \left[x - \frac{x^3}{3\cdot 4} + \frac{x^5}{5\cdot 64} - \frac{x^7}{7\cdot 2304} + \cdots\right]_0^1 = 1 - \frac{1}{12} + \frac{1}{320} - \frac{1}{16{,}128} + \cdots$$

Since $\frac{1}{16{,}128} \approx 0.000062$, it follows from The Alternating Series Estimation Theorem that, correct to three decimal places, $\int_0^1 J_0(x)\,dx \approx 1 - \frac{1}{12} + \frac{1}{320} \approx 0.920$.

34. (a) $J_1(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{n!\,(n+1)!\,2^{2n+1}}$, $J_1'(x) = \sum_{n=0}^{\infty} \frac{(-1)^n\,(2n+1)\,x^{2n}}{n!\,(n+1)!\,2^{2n+1}}$, and

$$J_1''(x) = \sum_{n=1}^{\infty} \frac{(-1)^n\,(2n+1)\,(2n)\,x^{2n-1}}{n!\,(n+1)!\,2^{2n+1}}.$$

$$x^2 J_1''(x) + xJ_1'(x) + (x^2-1)\,J_1(x)$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n\,(2n+1)(2n)x^{2n+1}}{n!\,(n+1)!\,2^{2n+1}} + \sum_{n=0}^{\infty} \frac{(-1)^n\,(2n+1)x^{2n+1}}{n!\,(n+1)!\,2^{2n+1}} + \sum_{n=0}^{\infty} \frac{(-1)^n\,x^{2n+3}}{n!\,(n+1)!\,2^{2n+1}} - \sum_{n=0}^{\infty} \frac{(-1)^n\,x^{2n+1}}{n!\,(n+1)!\,2^{2n+1}}$$

$$= \sum_{n=1}^{\infty} \frac{(-1)^n\,(2n+1)(2n)x^{2n+1}}{n!\,(n+1)!\,2^{2n+1}} + \sum_{n=0}^{\infty} \frac{(-1)^n\,(2n+1)x^{2n+1}}{n!\,(n+1)!\,2^{2n+1}} - \sum_{n=1}^{\infty} \frac{(-1)^n\,x^{2n+1}}{(n-1)!\,n!\,2^{2n-1}} - \sum_{n=0}^{\infty} \frac{(-1)^n\,x^{2n+1}}{n!\,(n+1)!\,2^{2n+1}}$$

[Replace n with $n-1$ in the third term]

$$= \frac{x}{2} - \frac{x}{2} + \sum_{n=1}^{\infty} (-1)^n \left[\frac{(2n+1)(2n) + (2n+1) - (n)(n+1)2^2 - 1}{n!\,(n+1)!\,2^{2n+1}}\right] x^{2n+1} = 0$$

(b) $J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n\,x^{2n}}{2^{2n}\,(n!)^2} \quad\Rightarrow$

$$J_0'(x) = \sum_{n=1}^{\infty} \frac{(-1)^n\,(2n)x^{2n-1}}{2^{2n}\,(n!)^2} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1}\,2(n+1)x^{2n+1}}{2^{2n+2}\,[(n+1)!]^2} \quad \text{[Replace } n \text{ with } n+1\text{]}$$

$$= -\sum_{n=0}^{\infty} \frac{(-1)^n\,x^{2n+1}}{2^{2n+1}(n+1)!\,n!} \quad \text{[cancel 2 and } n+1\text{; take } -1 \text{ outside sum]} \quad = -J_1(x)$$

35. (a) $f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad\Rightarrow\quad f'(x) = \sum_{n=1}^{\infty} \frac{nx^{n-1}}{n!} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{(n-1)!} = \sum_{n=0}^{\infty} \frac{x^n}{n!} = f(x)$

(b) By Theorem 10.4.2, the only solution to the differential equation $df(x)/dx = f(x)$ is $f(x) = Ke^x$, but $f(0) = 1$, so $K = 1$ and $f(x) = e^x$.
Or: We could solve the equation $df(x)/dx = f(x)$ as a separable differential equation.

36. $\frac{|\sin nx|}{n^2} \le \frac{1}{n^2}$, so $\sum_{n=1}^{\infty} \frac{\sin nx}{n^2}$ converges by the Comparison Test. $\frac{d}{dx}\left(\frac{\sin nx}{n^2}\right) = \frac{\cos nx}{n}$, so when $x = 2k\pi$ (k an integer), $\sum_{n=1}^{\infty} f_n'(x) = \sum_{n=1}^{\infty} \frac{\cos(2kn\pi)}{n} = \sum_{n=1}^{\infty} \frac{1}{n}$, which diverges (harmonic series). $f_n''(x) = -\sin nx$, so $\sum_{n=1}^{\infty} f_n''(x) = -\sum_{n=1}^{\infty} \sin nx$, which converges only if $\sin nx = 0$, or $x = k\pi$ (k an integer).

37. If $a_n = \frac{x^n}{n^2}$, then by the Ratio Test, $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{x^{n+1}}{(n+1)^2} \cdot \frac{n^2}{x^n}\right| = |x| \lim_{n\to\infty} \left(\frac{n}{n+1}\right)^2 = |x| < 1$ for convergence, so $R = 1$. When $x = \pm 1$, $\sum_{n=1}^{\infty} \left|\frac{x^n}{n^2}\right| = \sum_{n=1}^{\infty} \frac{1}{n^2}$ which is a convergent p-series ($p = 2 > 1$), so the interval of convergence for f is $[-1, 1]$. By Theorem 2, the radii of convergence of f' and f'' are both 1, so we need only check the endpoints. $f(x) = \sum\limits_{n=1}^{\infty} \dfrac{x^n}{n^2} \Rightarrow f'(x) = \sum\limits_{n=1}^{\infty} \dfrac{nx^{n-1}}{n^2} = \sum\limits_{n=0}^{\infty} \dfrac{x^n}{n+1}$, and this series diverges for $x = 1$ (harmonic series) and converges for $x = -1$ (Alternating Series Test), so the interval of convergence is $[-1, 1)$. $f''(x) = \sum\limits_{n=1}^{\infty} \dfrac{nx^{n-1}}{n+1}$ diverges at both 1 and -1 (Test for Divergence) since $\lim\limits_{n\to\infty} \dfrac{n}{n+1} = 1 \neq 0$, so its interval of convergence is $(-1, 1)$.

38. (a) $\sum\limits_{n=1}^{\infty} nx^{n-1} = \sum\limits_{n=0}^{\infty} \dfrac{d}{dx} x^n = \dfrac{d}{dx}\left[\sum\limits_{n=0}^{\infty} x_n\right] = \dfrac{d}{dx}\left[\dfrac{1}{1-x}\right] = -\dfrac{1}{(1-x)^2}(-1) = \dfrac{1}{(1-x)^2}$, $|x| < 1$.

(b) (i) $\sum\limits_{n=1}^{\infty} nx^n = x\sum\limits_{n=1}^{\infty} nx^{n-1} = x\left[\dfrac{1}{(1-x)^2}\right]$ [from part (a)] $= \dfrac{x}{(1-x)^2}$ for $|x| < 1$.

(ii) Put $x = \frac{1}{2}$ in (i): $\sum\limits_{n=1}^{\infty} \dfrac{n}{2^n} = \sum\limits_{n=1}^{\infty} n\left(\frac{1}{2}\right)^n = \dfrac{1/2}{(1-1/2)^2} = 2.$

(c) (i) $\sum\limits_{n=2}^{\infty} n(n-1)x^n = x^2\sum\limits_{n=2}^{\infty} n(n-1)x^{n-2} = x^2 \dfrac{d}{dx}\left[\sum\limits_{n=1}^{\infty} nx^{n-1}\right] = x^2 \dfrac{d}{dx}\dfrac{1}{(1-x)^2}$

$$= x^2\frac{2}{(1-x)^3} = \frac{2x^2}{(1-x)^3} \text{ for } |x| < 1.$$

(ii) Put $x = \frac{1}{2}$ in (i): $\sum\limits_{n=2}^{\infty} \dfrac{n^2-n}{2^n} = \sum\limits_{n=2}^{\infty} n(n-1)\left(\frac{1}{2}\right)^n = \dfrac{2(1/2)^2}{(1-1/2)^3} = 4.$

(iii) From (b)(ii) and (c)(ii), we have $\sum\limits_{n=1}^{\infty} \dfrac{n^2}{2^n} = \sum\limits_{n=1}^{\infty} \dfrac{n^2-n}{2^n} + \sum\limits_{n=1}^{\infty} \dfrac{n}{2^n} = 4 + 2 = 6.$

39. By Example 7, $\tan^{-1} x = \sum\limits_{n=0}^{\infty} (-1)^n \dfrac{x^{2n+1}}{2n+1}$ for $|x| < 1$. In particular, for $x = \dfrac{1}{\sqrt{3}}$, we have

$$\frac{\pi}{6} = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \sum_{n=0}^{\infty} (-1)^n \frac{(1/\sqrt{3})^{2n+1}}{2n+1} = \sum_{n=0}^{\infty} (-1)^n \left(\frac{1}{3}\right)^n \frac{1}{\sqrt{3}}\frac{1}{2n+1}, \text{ so}$$

$$\pi = \frac{6}{\sqrt{3}}\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)3^n} = 2\sqrt{3}\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)3^n}.$$

40. (a) $\displaystyle\int_0^{1/2} \frac{dx}{x^2-x+1} = \int_0^{1/2} \frac{dx}{(x-1/2)^2+3/4}$ $\left[\begin{array}{l} x - 1/2 = (\sqrt{3}/2)u,\ u = (2/\sqrt{3})(x-1/2) \\ dx = (\sqrt{3}/2)\,du \end{array}\right]$

$$= \int_{-1/\sqrt{3}}^{0} \frac{(\sqrt{3}/2)\,du}{(3/4)(u^2+1)} = \frac{2\sqrt{3}}{3}\left[\tan^{-1} u\right]_{-1/\sqrt{3}}^{0} = \frac{2}{\sqrt{3}}\left[0 - \left(-\frac{\pi}{6}\right)\right] = \frac{\pi}{3\sqrt{3}}$$

(b) $\dfrac{1}{x^3+1} = \dfrac{1}{(x+1)(x^2-x+1)} \Rightarrow \dfrac{1}{x^2-x+1} = (x+1)\left(\dfrac{1}{1+x^3}\right) = (x+1)\dfrac{1}{1-(-x^3)}$

$$= (x+1)\sum_{n=0}^{\infty} (-1)^n x^{3n} = \sum_{n=0}^{\infty} (-1)^n x^{3n+1} + \sum_{n=0}^{\infty} (-1)^n x^{3n} \text{ for } |x| < 1 \Rightarrow \int \frac{dx}{x^2-x+1}$$

$$= C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{3n+2}}{3n+2} + \sum_{n=0}^{\infty} (-1)^n \frac{x^{3n+1}}{3n+1} \text{ for } |x| < 1 \Rightarrow \int_0^{1/2} \frac{dx}{x^2-x+1}$$

$$= \sum_{n=0}^{\infty} (-1)^n \left[\frac{1}{4\cdot 8^n(3n+2)} + \frac{1}{2\cdot 8^n(3n+1)}\right] = \frac{1}{4}\sum_{n=0}^{\infty} \frac{(-1)^n}{8^n}\left(\frac{2}{3n+1} + \frac{1}{3n+2}\right)$$

By part (a), this equals $\dfrac{\pi}{3\sqrt{3}}$, so $\pi = \dfrac{3\sqrt{3}}{4}\sum\limits_{n=0}^{\infty} \dfrac{(-1)^n}{8^n}\left(\dfrac{2}{3n+1} + \dfrac{1}{3n+2}\right).$

12.10 Taylor and Maclaurin Series

1. Using Theorem 5 with $\sum\limits_{n=0}^{\infty} b_n(x-5)^n$, $b_n = \dfrac{f^{(n)}(a)}{n!}$, so $b_8 = \dfrac{f^{(8)}(5)}{8!}$.

2. (a) Using Formula 6, a power series expansion of f at 1 must have the form $f(1) + f'(1)(x-1) + \cdots$. Comparing to the given series, $1.6 - 0.8(x-1) + \cdots$, we must have $f'(1) = -0.8$. But from the graph, $f'(1)$ is positive. Hence, the given series is *not* the Taylor series of f centered at 1.

(b) A power series expansion of f at 2 must have the form $f(2) + f'(2)(x-2) + \frac{1}{2}f''(2)(x-2)^2 + \cdots$. Comparing to the given series, $2.8 + 0.5(x-2) + 1.5(x-2)^2 - 0.1(x-2)^3 + \cdots$, we must have $\frac{1}{2}f''(2) = 1.5$; that is, $f''(2)$ is positive. But from the graph, f is concave downward near $x = 2$, so $f''(2)$ must be negative. Hence, the given series is *not* the Taylor series of f centered at 2.

3.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\cos x$	1
1	$-\sin x$	0
2	$-\cos x$	-1
3	$\sin x$	0
4	$\cos x$	1
$\vdots$	$\vdots$	$\vdots$

We use Equation 7 with $f(x) = \cos x$.

$$\cos x = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f^{(3)}(0)}{3!}x^3 + \frac{f^{(4)}(0)}{4!}x^4 + \cdots$$

$$= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

If $a_n = \dfrac{(-1)^n x^{2n}}{(2n)!}$, then

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{2n+2}}{(2n+2)!}\cdot\frac{(2n)!}{x^{2n}}\right| = x^2 \lim_{n\to\infty}\frac{1}{(2n+2)(2n+1)} = 0 < 1 \text{ for all } x.$$

So $R = \infty$ (Ratio Test).

4.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\sin 2x$	0
1	$2\cos 2x$	2
2	$-2^2 \sin 2x$	0
3	$-2^3 \cos 2x$	-2^3
4	$2^4 \sin 2x$	0
$\vdots$	$\vdots$	$\vdots$

$f^{(n)}(0) = 0$ if n is even and $f^{(2n+1)}(0) = (-1)^n 2^{2n+1}$, so

$$\sin 2x = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = \sum_{n=0}^{\infty} \frac{f^{(2n+1)}(0)}{(2n+1)!}x^{2n+1}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1} x^{2n+1}}{(2n+1)!}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{2^2|x|^2}{(2n+3)(2n+2)} = 0 < 1 \text{ for all } x,$$

so $R = \infty$ (Ratio Test).

5.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$(1+x)^{-3}$	1
1	$-3(1+x)^{-4}$	-3
2	$12(1+x)^{-5}$	12
3	$-60(1+x)^{-6}$	-60
4	$360(1+x)^{-7}$	360
$\vdots$	$\vdots$	$\vdots$

$$\begin{aligned}(1+x)^{-3} &= f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\frac{f'''(0)}{3!}x^3+\frac{f^{(4)}(0)}{4!}x^4+\cdots\\ &= 1-3x+\frac{4\cdot 3}{2!}x^2-\frac{5\cdot 4\cdot 3}{3!}x^3+\frac{6\cdot 5\cdot 4\cdot 3}{4!}x^4-\cdots\\ &= 1-3x+\frac{4\cdot 3\cdot 2}{2\cdot 2!}x^2-\frac{5\cdot 4\cdot 3\cdot 2}{2\cdot 3!}x^3+\frac{6\cdot 5\cdot 4\cdot 3\cdot 2}{2\cdot 4!}x^4-\cdots\\ &= \sum_{n=0}^{\infty}\frac{(-1)^n(n+2)!\,x^n}{2(n!)}=\sum_{n=0}^{\infty}\frac{(-1)^n(n+2)(n+1)x^n}{2}\end{aligned}$$

$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{(n+3)(n+2)x^{n+1}}{2}\cdot\frac{2}{(n+2)(n+1)x^n}\right|=|x|\lim_{n\to\infty}\frac{n+3}{n+1}=|x|<1$ for convergence, so $R=1$ (Ratio Test).

6.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\ln(1+x)$	0
1	$(1+x)^{-1}$	1
2	$-(1+x)^{-2}$	-1
3	$2(1+x)^{-3}$	2
4	$-6(1+x)^{-4}$	-6
5	$24(1+x)^{-5}$	24
$\vdots$	$\vdots$	$\vdots$

$$\begin{aligned}\ln(1+x) &= f(0)+f'(0)x+\frac{f''(0)}{2!}x^2+\frac{f'''(0)}{3!}x^3\\ &\quad+\frac{f^{(4)}(0)}{4!}x^4+\frac{f^{(5)}(0)}{5!}x^5+\cdots\\ &= x-\tfrac{1}{2}x^2+\tfrac{2}{6}x^3-\tfrac{6}{24}x^4+\tfrac{24}{120}x^5-\cdots\\ &= x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\frac{x^5}{5}-\cdots=\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n}x^n\end{aligned}$$

$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim_{n\to\infty}\left|\frac{x^{n-1}}{n+1}\cdot\frac{n}{x^n}\right|=\lim_{n\to\infty}\frac{|x|}{1+1/n}=|x|<1$ for convergence, so $R=1$.

7.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	e^{5x}	1
1	$5e^{5x}$	5
2	5^2e^{5x}	25
3	5^3e^{5x}	125
4	5^4e^{5x}	625
$\vdots$	$\vdots$	$\vdots$

$$e^{5x}=\sum_{n=0}^{\infty}\frac{f^{(n)}(0)}{n!}x^n=\sum_{n=0}^{\infty}\frac{5^n}{n!}x^n.$$

$$\begin{aligned}\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\left[\frac{5^{n+1}|x|^{n+1}}{(n+1)!}\cdot\frac{n!}{5^n|x|^n}\right]\\ &= \lim_{n\to\infty}\frac{5|x|}{n+1}=0<1 \text{ for all } x, \text{ so } R=\infty.\end{aligned}$$

8.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	xe^x	0
1	$(x+1)e^x$	1
2	$(x+2)e^x$	2
3	$(x+3)e^x$	3
$\vdots$	$\vdots$	$\vdots$

$$xe^x = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{n}{n!} x^n = \sum_{n=1}^{\infty} \frac{n}{n!} x^n = \sum_{n=1}^{\infty} \frac{x^n}{(n-1)!}.$$

$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left[\frac{|x|^{n+1}}{n!} \cdot \frac{(n-1)!}{|x|^n}\right] = \lim_{n\to\infty} \frac{|x|}{n} = 0 < 1$ for all x, so $R = \infty$.

9.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\sinh x$	0
1	$\cosh x$	1
2	$\sinh x$	0
3	$\cosh x$	1
4	$\sinh x$	0
$\vdots$	$\vdots$	$\vdots$

$f^{(n)}(0) = \begin{cases} 0 & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd} \end{cases}$ so $\sinh x = \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$.

Use the Ratio Test to find R. If $a_n = \frac{x^{2n+1}}{(2n+1)!}$, then

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{x^{2n+3}}{(2n+3)!} \cdot \frac{(2n+1)!}{x^{2n+1}}\right| = x^2 \cdot \lim_{n\to\infty} \frac{1}{(2n+3)(2n+2)} = 0 < 1$$

for all x, so $R = \infty$.

10.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\cosh x$	1
1	$\sinh x$	0
2	$\cosh x$	1
3	$\sinh x$	0
$\vdots$	$\vdots$	$\vdots$

$f^{(n)}(0) = \begin{cases} 1 & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd} \end{cases}$ so $\cosh x = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}$.

Use the Ratio Test to find R. If $a_n = \frac{x^{2n}}{(2n)!}$, then

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{x^{2n+2}}{(2n+2)!} \cdot \frac{(2n)!}{x^{2n}}\right| = x^2 \cdot \lim_{n\to\infty} \frac{1}{(2n+2)(2n+1)} = 0 < 1$$

for all x, so $R = \infty$.

11.

n	$f^{(n)}(x)$	$f^{(n)}(2)$
0	$1+x+x^2$	7
1	$1+2x$	5
2	2	2
3	0	0
4	0	0
$\vdots$	$\vdots$	$\vdots$

$$f(x) = 7 + 5(x-2) + \frac{2}{2!}(x-2)^2 + \sum_{n=3}^{\infty} \frac{0}{n!}(x-2)^n = 7 + 5(x-2) + (x-2)^2$$

Since $a_n = 0$ for large n, $R = \infty$.

12.

n	$f^{(n)}(x)$	$f^{(n)}(-1)$
0	x^3	-1
1	$3x^2$	3
2	$6x$	-6
3	6	6
4	0	0
5	0	0
$\vdots$	$\vdots$	$\vdots$

$$f(x) = -1 + 3(x+1) - \tfrac{6}{2!}(x+1)^2 + \tfrac{6}{3!}(x+1)^3$$
$$= -1 + 3(x+1) - 3(x+1)^2 + (x+1)^3$$

Since $a_n = 0$ for large n, $R = \infty$.

13. Clearly, $f^{(n)}(x) = e^x$, so $f^{(n)}(3) = e^3$ and $e^x = \sum\limits_{n=0}^{\infty} \dfrac{e^3}{n!}(x-3)^n$. If $a_n = \dfrac{e^3}{n!}(x-3)^n$, then

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{e^3(x-3)^{n+1}}{(n+1)!}\cdot\frac{n!}{e^3(x-3)^n}\right| = \lim_{n\to\infty}\frac{|x-3|}{n+1} = 0 < 1 \text{ for all } x\text{, so } R = \infty.$$

14.

n	$f^{(n)}(x)$	$f^{(n)}(2)$
0	$\ln x$	$\ln 2$
1	x^{-1}	$\frac{1}{2}$
2	$-x^{-2}$	$-\frac{1}{4}$
3	$2x^{-3}$	$\frac{2}{8}$
4	$-3\cdot 2x^{-4}$	$-\frac{3\cdot 2}{16}$
$\vdots$	$\vdots$	$\vdots$

$f^{(n)}(2) = \dfrac{(-1)^{n-1}(n-1)!}{2^n}$ for $n \geq 1$, so $\ln x = \ln 2 + \sum\limits_{n=1}^{\infty}\dfrac{(-1)^{n-1}(x-2)^n}{n\cdot 2^n}$.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{|x-2|}{2}\lim_{n\to\infty}\frac{n}{n+1} = \frac{|x-2|}{2} < 1 \text{ for convergence, so } |x-2| < 2 \;\Rightarrow\; R = 2.$$

15.

n	$f^{(n)}(x)$	$f^{(n)}(\pi)$
0	$\cos x$	-1
1	$-\sin x$	0
2	$-\cos x$	1
3	$\sin x$	0
4	$\cos x$	-1
$\vdots$	$\vdots$	$\vdots$

$$\cos x = \sum_{k=0}^{\infty}\frac{f^{(k)}(\pi)}{k!}(x-\pi)^k = -1 + \frac{(x-\pi)^2}{2!} - \frac{(x-\pi)^4}{4!} + \frac{(x-\pi)^6}{6!} - \cdots = \sum_{n=0}^{\infty}(-1)^{n+1}\frac{(x-\pi)^{2n}}{(2n)!}.$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left[\frac{|x-\pi|^{2n+2}}{(2n+2)!}\cdot\frac{(2n)!}{|x-\pi|^{2n}}\right] = \lim_{n\to\infty}\frac{|x-\pi|^2}{(2n+2)(2n+1)} = 0 < 1 \text{ for all } x\text{, so } R = \infty.$$

16.

n	$f^{(n)}(x)$	$f^{(n)}(\pi/2)$
0	$\sin x$	1
1	$\cos x$	0
2	$-\sin x$	-1
3	$-\cos x$	0
4	$\sin x$	1
$\vdots$	$\vdots$	$\vdots$

$$\sin x = \sum_{k=0}^{\infty} \frac{f^{(k)}(\pi/2)}{k!}\left(x - \frac{\pi}{2}\right)^k$$
$$= 1 - \frac{(x-\pi/2)^2}{2!} + \frac{(x-\pi/2)^4}{4!} - \frac{(x-\pi/2)^6}{6!} + \cdots$$
$$= \sum_{n=0}^{\infty} (-1)^n \frac{(x-\pi/2)^{2n}}{(2n)!}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left[\frac{|x-\pi/2|^{2n+2}}{(2n+2)!}\cdot\frac{(2n)!}{|x-\pi/2|^{2n}}\right] = \lim_{n\to\infty}\frac{|x-\pi/2|^2}{(2n+2)(2n+1)} = 0 < 1 \text{ for all } x,$$

so $R = \infty$.

17.

n	$f^{(n)}(x)$	$f^{(n)}(9)$
0	$x^{-1/2}$	$\frac{1}{3}$
1	$-\frac{1}{2}x^{-3/2}$	$-\frac{1}{2}\cdot\frac{1}{3^3}$
2	$\frac{3}{4}x^{-5/2}$	$-\frac{1}{2}\cdot\left(-\frac{3}{2}\right)\cdot\frac{1}{3^5}$
3	$-\frac{15}{8}x^{-7/2}$	$-\frac{1}{2}\cdot\left(-\frac{3}{2}\right)\cdot\left(-\frac{5}{2}\right)\cdot\frac{1}{3^7}$
$\vdots$	$\vdots$	$\vdots$

$$\frac{1}{\sqrt{x}} = \frac{1}{3} - \frac{1}{2\cdot 3^3}(x-9) + \frac{3}{2^2\cdot 3^5}\frac{(x-9)^2}{2!} - \frac{3\cdot 5}{2^3\cdot 3^7}\frac{(x-9)^3}{3!} + \cdots$$
$$= \sum_{n=0}^{\infty} (-1)^n \frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}{2^n\cdot 3^{2n+1}\cdot n!}(x-9)^n.$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left[\frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)[2(n+1)-1]\,|x-9|^{n+1}}{2^{n+1}\cdot 3^{[2(n+1)+1]}\cdot(n+1)!}\cdot\frac{2^n\cdot 3^{2n+1}\cdot n!}{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)\,|x-9|^n}\right]$$
$$= \lim_{n\to\infty}\left[\frac{(2n+1)\,|x-9|}{2\cdot 3^2(n+1)}\right] = \frac{1}{9}\,|x-9| < 1$$

for convergence, so $|x-9| < 9$ and $R = 9$.

18.

n	$f^{(n)}(x)$	$f^{(n)}(1)$
0	x^{-2}	1
1	$-2x^{-3}$	-2
2	$6x^{-4}$	6
3	$-24x^{-5}$	-24
4	$120x^{-6}$	120
$\vdots$	$\vdots$	$\vdots$

$$x^{-2} = 1 - 2(x-1) + 6\cdot\frac{(x-1)^2}{2!} - 24\cdot\frac{(x-1)^3}{3!} + 120\cdot\frac{(x-1)^4}{4!} - \cdots$$
$$= 1 - 2(x-1) + 3(x-1)^2 - 4(x-1)^3 + 5(x-1)^4 - \cdots$$
$$= \sum_{n=0}^{\infty} (-1)^n (n+1)(x-1)^n.$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\frac{(n+2)\,|x-1|^{n+1}}{(n+1)\,|x-1|^n} = \lim_{n\to\infty}\left[\frac{n+2}{n+1}\cdot|x-1|\right] = |x-1| < 1 \text{ for convergence, so } R = 1.$$

19. If $f(x) = \cos x$, then $f^{(n+1)}(x) = \pm\sin x$ or $\pm\cos x$. In each case, $\left|f^{(n+1)}(x)\right| \le 1$, so by Formula 9 with $a = 0$ and $M = 1$, $|R_n(x)| \le \dfrac{1}{(n+1)!}|x|^{n+1}$. Thus, $|R_n(x)| \to 0$ as $n \to \infty$ by Equation 10. So $\lim\limits_{n\to\infty} R_n(x) = 0$ and, by Theorem 8, the series in Exercise 3 represents $\cos x$ for all x.

20. If $f(x) = \sin x$, then $f^{(n+1)}(x) = \pm\sin x$ or $\pm\cos x$. In each case, $\left|f^{(n+1)}(x)\right| \le 1$, so by Formula 9 with $a = 0$ and $M = 1$, $|R_n(x)| \le \dfrac{1}{(n+1)!}\left|x - \dfrac{\pi}{2}\right|^{n+1}$. Thus, $|R_n(x)| \to 0$ as $n \to \infty$ by Equation 10. So $\lim\limits_{n\to\infty} R_n(x) = 0$ and, by Theorem 8, the series in Exercise 16 represents $\sin x$ for all x.

21. If $f(x) = \sinh x$, then for all n, $f^{(n+1)}(x) = \cosh x$ or $\sinh x$. Since $|\sinh x| < |\cosh x| = \cosh x$ for all x, we have $\left|f^{(n+1)}(x)\right| \le \cosh x$ for all n. If d is any positive number and $|x| \le d$, then $\left|f^{(n+1)}(x)\right| \le \cosh x \le \cosh d$, so by Formula 9 with $a = 0$ and $M = \cosh d$, we have $|R_n(x)| \le \dfrac{\cosh d}{(n+1)!}|x|^{n+1}$. It follows that $|R_n(x)| \to 0$ as $n \to \infty$ for $|x| \le d$ (by Equation 10). But d was an arbitrary positive number. So by Theorem 8, the series represents $\sinh x$ for all x.

22. If $f(x) = \cosh x$, then for all n, $f^{(n+1)}(x) = \cosh x$ or $\sinh x$. Since $|\sinh x| < |\cosh x| = \cosh x$ for all x, we have $\left|f^{(n+1)}(x)\right| \le \cosh x$ for all n. If d is any positive number and $|x| \le d$, then $\left|f^{(n+1)}(x)\right| \le \cosh x \le \cosh d$, so by Formula 9 with $a = 0$ and $M = \cosh d$, we have $|R_n(x)| \le \dfrac{\cosh d}{(n+1)!}|x|^{n+1}$. It follows that $|R_n(x)| \to 0$ as $n \to \infty$ for $|x| \le d$ (by Equation 10). But d was an arbitrary positive number. So by Theorem 8, the series represents $\cosh x$ for all x.

23. $\cos x = \sum\limits_{n=0}^{\infty}(-1)^n\dfrac{x^{2n}}{(2n)!} \;\Rightarrow\; f(x) = \cos(\pi x) = \sum\limits_{n=0}^{\infty}\dfrac{(-1)^n(\pi x)^{2n}}{(2n)!} = \sum\limits_{n=0}^{\infty}\dfrac{(-1)^n\pi^{2n}x^{2n}}{(2n)!}$, $R = \infty$

24. $e^x = \sum\limits_{n=0}^{\infty}\dfrac{x^n}{n!} \;\Rightarrow\; f(x) = e^{-x/2} = \sum\limits_{n=0}^{\infty}\dfrac{(-x/2)^n}{n!} = \sum\limits_{n=0}^{\infty}\dfrac{(-1)^n}{2^n\,n!}x^n$, $R = \infty$

25. $\tan^{-1}x = \sum\limits_{n=0}^{\infty}(-1)^n\dfrac{x^{2n+1}}{2n+1} \;\Rightarrow\; f(x) = x\tan^{-1}x = x\sum\limits_{n=0}^{\infty}(-1)^n\dfrac{x^{2n+1}}{2n+1} = \sum\limits_{n=0}^{\infty}(-1)^n\dfrac{x^{2n+2}}{2n+1}$, $R = 1$

26. $\sin x = \sum\limits_{n=0}^{\infty}(-1)^n\dfrac{x^{2n+1}}{(2n+1)!} \;\Rightarrow\; f(x) = \sin(x^4) = \sum\limits_{n=0}^{\infty}(-1)^n\dfrac{(x^4)^{2n+1}}{(2n+1)!} = \sum\limits_{n=0}^{\infty}\dfrac{(-1)^n}{(2n+1)!}x^{8n+4}$, $R = \infty$

27. $e^x = \sum\limits_{n=0}^{\infty}\dfrac{x^n}{n!} \;\Rightarrow\; f(x) = x^2e^{-x} = x^2\sum\limits_{n=0}^{\infty}\dfrac{(-x)^n}{n!} = \sum\limits_{n=0}^{\infty}\dfrac{(-1)^n\,x^{n+2}}{n!}$, $R = \infty$

28. $\cos x = \sum\limits_{n=0}^{\infty}(-1)^n\dfrac{x^{2n}}{(2n)!} \;\Rightarrow\; \cos 2x = \sum\limits_{n=0}^{\infty}(-1)^n\dfrac{(2x)^{2n}}{(2n)!} = \sum\limits_{n=0}^{\infty}\dfrac{(-1)^n\,2^{2n}}{(2n)!}x^{2n} \;\Rightarrow$

$f(x) = x\cos 2x = \sum\limits_{n=0}^{\infty}\dfrac{(-1)^n\,2^{2n}}{(2n)!}x^{2n+1}$, $R = \infty$

29. $\sin^2 x = \dfrac{1}{2}(1 - \cos 2x) = \dfrac{1}{2}\left[1 - \sum\limits_{n=0}^{\infty}\dfrac{(-1)^n(2x)^{2n}}{(2n)!}\right] = \dfrac{1}{2}\left[1 - 1 - \sum\limits_{n=1}^{\infty}\dfrac{(-1)^n(2x)^{2n}}{(2n)!}\right]$

$= \sum\limits_{n=1}^{\infty}\dfrac{(-1)^{n+1}2^{2n-1}x^{2n}}{(2n)!}$, $R = \infty$

30. $\cos^2 x = \frac{1}{2}(1+\cos 2x) = \frac{1}{2}\left[1+\sum_{n=0}^{\infty}\frac{(-1)^n(2x)^{2n}}{(2n)!}\right] = \frac{1}{2}\left[1+1+\sum_{n=1}^{\infty}\frac{(-1)^n 2^{2n}x^{2n}}{(2n)!}\right]$

$= 1+\sum_{n=1}^{\infty}\frac{(-1)^n 2^{2n-1}x^{2n}}{(2n)!}$, $R=\infty$

Another method: Use $\cos^2 x = 1-\sin^2 x$ and Exercise 29.

31. $\frac{\sin x}{x} = \frac{1}{x}\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n+1)!}$ and this series also gives the required value at $x=0$ (namely 1); $R=\infty$.

32. $\frac{x-\sin x}{x^3} = \frac{1}{x^3}\left[x-\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!}\right] = \frac{1}{x^3}\left[x-x-\sum_{n=1}^{\infty}\frac{(-1)^n x^{2n+1}}{(2n+1)!}\right] = \frac{1}{x^3}\left[-\sum_{n=0}^{\infty}\frac{(-1)^{n+1}x^{2n+3}}{(2n+3)!}\right]$

$= \frac{1}{x^3}\sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+3}}{(2n+3)!} = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n}}{(2n+3)!}$ and this series also gives the required value at $x=0$ (namely $1/6$); $R=\infty$.

33.

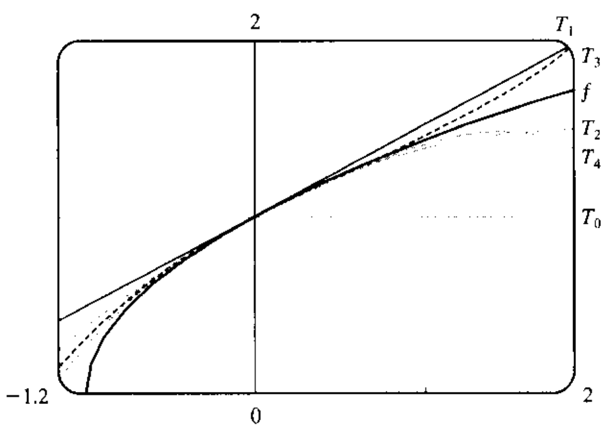
n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$(1+x)^{1/2}$	1
1	$\frac{1}{2}(1+x)^{-1/2}$	$\frac{1}{2}$
2	$-\frac{1}{4}(1+x)^{-3/2}$	$-\frac{1}{4}$
3	$\frac{3}{8}(1+x)^{-5/2}$	$\frac{3}{8}$
4	$-\frac{15}{16}(1+x)^{-7/2}$	$-\frac{15}{16}$
⋮	⋮	⋮

So $f^{(n)}(0) = \frac{(-1)^{n-1}\,1\cdot 3\cdot 5\cdot\cdots\cdot(2n-3)}{2^n}$ for $n\geq 2$, and

$\sqrt{1+x} = 1+\frac{x}{2}+\sum_{n=2}^{\infty}\frac{(-1)^{n-1}1\cdot 3\cdot 5\cdot\cdots\cdot(2n-3)}{2^n n!}x^n$. If $a_n = \frac{(-1)^{n-1}1\cdot 3\cdot 5\cdot\cdots\cdot(2n-3)}{2^n n!}x^n$,

then $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-3)(2n-1)x^{n+1}}{2^{n+1}(n+1)!}\cdot\frac{2^n n!}{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-3)x^n}\right|$

$= \frac{|x|}{2}\lim_{n\to\infty}\frac{2n-1}{n+1} = \frac{|x|}{2}\cdot 2 = |x| < 1$ for convergence, so $R=1$.

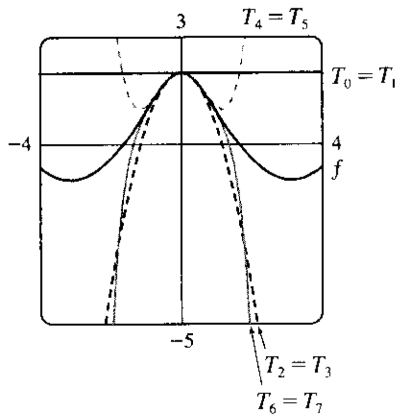


Notice that, as n increases, $T_n(x)$ becomes a better approximation to $f(x)$ for $-1<x<1$.

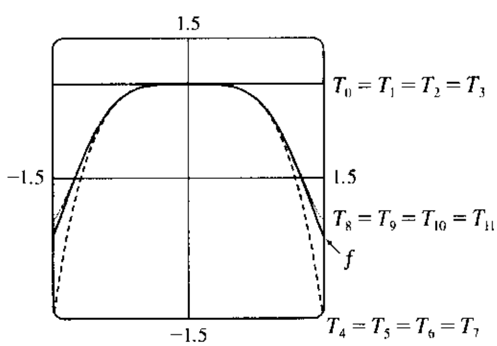
34. $e^x \overset{(11)}{=} \sum_{n=0}^{\infty} \frac{x^n}{n!}$, so $e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{n!}$. Also, $\cos x \overset{(16)}{=} \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}$, so

$$f(x) = e^{-x^2} + \cos x = \sum_{n=0}^{\infty} (-1)^n \left(\frac{1}{n!} + \frac{1}{(2n)!} \right) x^{2n} = 2 - \frac{3}{2}x^2 + \frac{13}{24}x^4 - \frac{121}{720}x^6 + \cdots.$$

The series for e^x and $\cos x$ converge for all x, so the same is true of the series for $f(x)$; that is, $R = \infty$. From the graphs of f and the first few Taylor polynomials, we see that $T_n(x)$ provides a closer fit to $f(x)$ near 0 as n increases.



35. $\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \Rightarrow f(x) = \cos(x^2) = \sum_{n=0}^{\infty} \frac{(-1)^n (x^2)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{4n}}{(2n)!}$, $R = \infty$



Notice that, as n increases, $T_n(x)$ becomes a better approximation to $f(x)$.

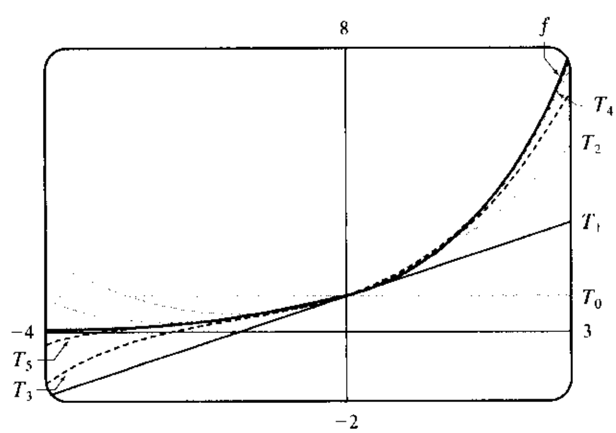
36. $2^x = (e^{\ln 2})^x$

$= e^{x \ln 2}$

$= \sum_{n=0}^{\infty} \frac{(x \ln 2)^n}{n!}$

$= \sum_{n=0}^{\infty} \frac{(\ln 2)^n x^n}{n!}$, $\quad R = \infty$.

Notice that, as n increases, $T_n(x)$ becomes a better approximation to $f(x)$.



37. $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, so

$$e^{-0.2} = \sum_{n=0}^{\infty} \frac{(-0.2)^n}{n!} = 1 - 0.2 + \frac{1}{2!}(0.2)^2 - \frac{1}{3!}(0.2)^3 + \frac{1}{4!}(0.2)^4 - \frac{1}{5!}(0.2)^5 + \frac{1}{6!}(0.2)^6 - \cdots.$$ But

$\frac{1}{6!}(0.2)^6 = 8.\overline{8} \times 10^{-8}$, so by the Alternating Series Estimation Theorem, $e^{-0.2} \approx \sum_{n=0}^{5} \frac{(-0.2)^n}{n!} \approx 0.81873$,

correct to five decimal places.

38. $3^\circ = \frac{\pi}{60}$ radians and $\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!}$, so

$$\sin\frac{\pi}{60} = \frac{\pi}{60} - \frac{\left(\frac{\pi}{60}\right)^3}{3!} + \frac{\left(\frac{\pi}{60}\right)^5}{5!} - \cdots = \frac{\pi}{60} - \frac{\pi^3}{1{,}296{,}000} + \frac{\pi^5}{93{,}312{,}000{,}000} - \cdots.$$

But $\frac{\pi^5}{93{,}312{,}000{,}000} < 10^{-8}$, so by the Alternating Series Estimation Theorem,

$\sin\frac{\pi}{60} \approx \frac{\pi}{60} - \frac{\pi^3}{1{,}296{,}000} \approx 0.05234$.

39. $\cos x \overset{(16)}{=} \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \Rightarrow \cos(x^3) = \sum_{n=0}^{\infty} (-1)^n \frac{(x^3)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{6n}}{(2n)!} \Rightarrow$

$x\cos(x^3) = \sum_{n=0}^{\infty} (-1)^n \frac{x^{6n+1}}{(2n)!} \Rightarrow \int x\cos(x^3)\,dx = C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{6n+2}}{(6n+2)(2n)!}$, with $R = \infty$.

40. $\frac{\sin x}{x} = \frac{1}{x}\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!}$, so

$$\int \frac{\sin x}{x}\,dx = \int \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!}\,dx = C + \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)(2n+1)!}$$

41. Using the series from Exercise 33 and substituting x^3 for x, we get

$$\int \sqrt{x^3+1}\,dx = \int \left[1 + \frac{x^3}{2} + \sum_{n=2}^{\infty} \frac{(-1)^{n-1}\,1\cdot 3\cdot 5\cdot \cdots \cdot (2n-3)}{2^n n!} x^{3n}\right] dx$$

$$= C + x + \frac{x^4}{8} + \sum_{n=2}^{\infty} \frac{(-1)^{n-1}\,1\cdot 3\cdot 5\cdot \cdots \cdot (2n-3)}{2^n n!(3n+1)} x^{3n+1}$$

42. $e^x \overset{(11)}{=} \sum_{n=0}^{\infty} \frac{x^n}{n!} \Rightarrow e^x - 1 = \sum_{n=1}^{\infty} \frac{x^n}{n!} \Rightarrow \frac{e^x - 1}{x} = \sum_{n=1}^{\infty} \frac{x^{n-1}}{n!} \Rightarrow$

$\int \frac{e^x - 1}{x}\,dx = C + \sum_{n=1}^{\infty} \frac{x^n}{n \cdot n!}$, with $R = \infty$.

43. By Exercise 39, $\int x\cos(x^3)\,dx = C + \sum_{n=0}^{\infty}(-1)^n \frac{x^{6n+2}}{(6n+2)(2n)!}$, so $\int_0^1 x\cos(x^3)\,dx$

$$= \left[\sum_{n=0}^{\infty}(-1)^n \frac{x^{6n+2}}{(6n+2)(2n)!}\right]_0^1 = \sum_{n=0}^{\infty}\frac{(-1)^n}{(6n+2)(2n)!} = \frac{1}{2} - \frac{1}{8\cdot 2!} + \frac{1}{14\cdot 4!} - \frac{1}{20\cdot 6!} + \cdots, \text{ but}$$

$\frac{1}{20\cdot 6!} = \frac{1}{14{,}400} \approx 0.000\,069$, so $\int_0^1 x\cos(x^3)\,dx \approx \frac{1}{2} - \frac{1}{16} + \frac{1}{336} \approx 0.440$ (correct to three decimal places) by the Alternating Series Estimation Theorem.

44. From the table of Maclaurin series in Section 12.10, we see that

$\tan^{-1}x = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{2n+1}$ for x in $[-1,1]$ and $\sin x = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n+1}}{(2n+1)!}$ for all real numbers x, so

$\tan^{-1}(x^3) + \sin(x^3) = \sum_{n=0}^{\infty}(-1)^n \frac{x^{6n+3}}{2n+1} + \sum_{n=0}^{\infty}(-1)^n \frac{x^{6n+3}}{(2n+1)!}$ for x^3 in $[-1,1]$ $\Leftrightarrow$ x in $[-1,1]$. Thus,

$$I = \int_0^{0.2}\left[\tan^{-1}(x^3) + \sin(x^3)\right]dx = \int_0^{0.2}\sum_{n=0}^{\infty}(-1)^n x^{6n+3}\left(\frac{1}{2n+1} + \frac{1}{(2n+1)!}\right)dx$$

$$= \left[\sum_{n=0}^{\infty}(-1)^n \frac{x^{6n+4}}{6n+4}\left(\frac{1}{2n+1} + \frac{1}{(2n+1)!}\right)\right]_0^{0.2}$$

$$= \sum_{n=0}^{\infty}(-1)^n \frac{(0.2)^{6n+4}}{6n+4}\left(\frac{1}{2n+1} + \frac{1}{(2n+1)!}\right) = \frac{(0.2)^4}{4}(1+1) - \frac{(0.2)^{10}}{10}\left(\frac{1}{3} + \frac{1}{3!}\right) + \cdots.$$

But $\frac{(0.2)^{10}}{10}\left(\frac{1}{3} + \frac{1}{3!}\right) = \frac{(0.2)^{10}}{20} = 5.12\times 10^{-9}$, so by the Alternating Series Estimation Theorem,

$I \approx \frac{(0.2)^4}{2} = 0.000\,80$ (correct to five decimal places). [Actually, the value is $0.000\,800\,0$, correct to seven decimal places.]

45. We first find a series representation for $f(x) = (1+x)^{-1/2}$, and then substitute.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$(1+x)^{-1/2}$	1
1	$-\frac{1}{2}(1+x)^{-3/2}$	$-\frac{1}{2}$
2	$\frac{3}{4}(1+x)^{-5/2}$	$\frac{3}{4}$
3	$-\frac{15}{8}(1+x)^{-7/2}$	$-\frac{15}{8}$
⋮	⋮	⋮

$$\frac{1}{\sqrt{1+x}} = 1 - \frac{x}{2} + \frac{3}{4}\left(\frac{x^2}{2!}\right) - \frac{15}{8}\left(\frac{x^3}{3!}\right) + \cdots \quad\Rightarrow\quad \frac{1}{\sqrt{1+x^3}} = 1 - \frac{1}{2}x^3 + \frac{3}{8}x^6 - \frac{5}{16}x^9 + \cdots \quad\Rightarrow$$

$\int_0^{0.1} \frac{dx}{\sqrt{1+x^3}} = \left[x - \frac{1}{8}x^4 + \frac{3}{56}x^7 - \frac{1}{32}x^{10} + \cdots\right]_0^{0.1} \approx (0.1) - \frac{1}{8}(0.1)^4$, by the Alternating Series Estimation Theorem, since $\frac{3}{56}(0.1)^7 \approx 0.000\,000\,005\,4 < 10^{-8}$, which is the maximum desired error. Therefore, $\int_0^{0.1} \frac{dx}{\sqrt{1+x^3}} \approx 0.099\,987\,50$.

46. $\int_0^{0.5} x^2 e^{-x^2}\,dx = \int_0^{0.5} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{n!}\,dx = \sum_{n=0}^{\infty}\left[\frac{(-1)^n x^{2n+3}}{n!(2n+3)}\right]_0^{1/2} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(2n+3)2^{2n+3}}$ and since the term with $n = 2$ is $\frac{1}{1792} < 0.001$, we use $\sum_{n=0}^{1} \frac{(-1)^n}{n!(2n+3)2^{2n+3}} = \frac{1}{24} - \frac{1}{160} \approx 0.0354$.

47.
$$\lim_{x\to 0} \frac{x - \tan^{-1}x}{x^3} = \lim_{x\to 0} \frac{x - \left(x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \cdots\right)}{x^3} = \lim_{x\to 0} \frac{\frac{1}{3}x^3 - \frac{1}{5}x^5 + \frac{1}{7}x^7 - \cdots}{x^3}$$
$$= \lim_{x\to 0}\left(\tfrac{1}{3} - \tfrac{1}{5}x^2 + \tfrac{1}{7}x^4 - \cdots\right) = \tfrac{1}{3}$$

since power series are continuous functions.

48.
$$\lim_{x\to 0} \frac{1 - \cos x}{1 + x - e^x} = \lim_{x\to 0} \frac{1 - \left(1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots\right)}{1 + x - \left(1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \frac{1}{5!}x^5 + \frac{1}{6!}x^6 + \cdots\right)}$$
$$= \lim_{x\to 0} \frac{\frac{1}{2!}x^2 - \frac{1}{4!}x^4 + \frac{1}{6!}x^6 - \cdots}{-\frac{1}{2!}x^2 - \frac{1}{3!}x^3 - \frac{1}{4!}x^4 - \frac{1}{5!}x^5 - \frac{1}{6!}x^6 - \cdots}$$
$$= \lim_{x\to 0} \frac{\frac{1}{2!} - \frac{1}{4!}x^2 + \frac{1}{6!}x^4 - \cdots}{-\frac{1}{2!} - \frac{1}{3!}x - \frac{1}{4!}x^2 - \frac{1}{5!}x^3 - \frac{1}{6!}x^4 - \cdots} = \frac{\frac{1}{2} - 0}{-\frac{1}{2} - 0} = -1$$

since power series are continuous functions.

49.
$$\lim_{x\to 0} \frac{\sin x - x + \frac{1}{6}x^3}{x^5} = \lim_{x\to 0} \frac{\left(x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots\right) - x + \frac{1}{6}x^3}{x^5}$$
$$= \lim_{x\to 0} \frac{\frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots}{x^5} = \lim_{x\to 0}\left(\frac{1}{5!} - \frac{x^2}{7!} + \frac{x^4}{9!} - \cdots\right) = \frac{1}{5!} = \frac{1}{120}$$

since power series are continuous functions.

50. $\lim_{x\to 0} \frac{\tan x - x}{x^3} = \lim_{x\to 0} \frac{\left(x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \cdots\right) - x}{x^3} = \lim_{x\to 0} \frac{\frac{1}{3}x^3 + \frac{2}{15}x^5 + \cdots}{x^3} = \lim_{x\to 0}\left(\frac{1}{3} + \frac{2}{15}x^2 + \cdots\right) = \frac{1}{3}$

since power series are continuous functions.

51. As in Example 8(a), we have $e^{-x^2} = 1 - \frac{x^2}{1!} + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots$ and we know that $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots$ from Equation 16. Therefore, $e^{-x^2}\cos x = \left(1 - x^2 + \frac{1}{2}x^4 - \cdots\right)\left(1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \cdots\right)$. Writing only the terms with degree ≤ 4, we get $e^{-x^2}\cos x = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - x^2 + \frac{1}{2}x^4 + \frac{1}{2}x^4 + \cdots = 1 - \frac{3}{2}x^2 + \frac{25}{24}x^4 + \cdots$.

52.

$$
\begin{array}{r|l}
 & 1 + \tfrac{1}{2}x^2 + \tfrac{5}{24}x^4 + \cdots \\ \hline
1 - \tfrac{1}{2}x^2 + \tfrac{1}{24}x^4 - \cdots & 1 \\
 & 1 - \tfrac{1}{2}x^2 + \tfrac{1}{24}x^4 - \cdots \\ \cline{2-2}
 & \tfrac{1}{2}x^2 - \tfrac{1}{24}x^4 + \cdots \\
 & \tfrac{1}{2}x^2 - \tfrac{1}{4}x^4 + \cdots \\ \cline{2-2}
 & \tfrac{5}{24}x^4 + \cdots \\
 & \tfrac{5}{24}x^4 + \cdots \\ \cline{2-2}
 & \cdots
\end{array}
$$

$\sec x = \dfrac{1}{\cos x} \overset{(16)}{=} \dfrac{1}{1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 - \cdots}$. From the long division above, $\sec x = 1 + \frac{1}{2}x^2 + \frac{5}{24}x^4 + \cdots$.

53.

$$
\begin{array}{r|l}
 & 1 + \tfrac{1}{6}x^2 + \tfrac{7}{360}x^4 + \cdots \\ \hline
x - \tfrac{1}{6}x^3 + \tfrac{1}{120}x^5 - \cdots & x \\
 & x - \tfrac{1}{6}x^3 + \tfrac{1}{120}x^5 - \cdots \\ \cline{2-2}
 & \tfrac{1}{6}x^3 - \tfrac{1}{120}x^5 + \cdots \\
 & \tfrac{1}{6}x^3 - \tfrac{1}{36}x^5 + \cdots \\ \cline{2-2}
 & \tfrac{7}{360}x^5 + \cdots \\
 & \tfrac{7}{360}x^5 + \cdots \\ \cline{2-2}
 & \cdots
\end{array}
$$

$\dfrac{x}{\sin x} \overset{(15)}{=} \dfrac{x}{x - \frac{1}{6}x^3 + \frac{1}{120}x^5 - \cdots}$. From the long division above, $\dfrac{x}{\sin x} = 1 + \frac{1}{6}x^2 + \frac{7}{360}x^4 + \cdots$.

54. From Example 6 in Section 12.9, we have $\ln(1-x) = -x - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \cdots$, $|x| < 1$. Therefore,

$$
\begin{aligned}
e^x \ln(1-x) &= \left(1 + x + \tfrac{1}{2}x^2 + \cdots\right)\left(-x - \tfrac{1}{2}x^2 - \tfrac{1}{3}x^3 - \cdots\right) \\
&= -x - \tfrac{1}{2}x^2 - \tfrac{1}{3}x^3 - x^2 - \tfrac{1}{2}x^3 - \tfrac{1}{2}x^3 - \cdots \\
&= -x - \tfrac{3}{2}x^2 - \tfrac{4}{3}x^3 - \cdots, \ |x| < 1
\end{aligned}
$$

55. $\displaystyle\sum_{n=0}^{\infty} (-1)^n \frac{x^{4n}}{n!} = \sum_{n=0}^{\infty} \frac{\left(-x^4\right)^n}{n!} = e^{-x^4}$, by (11).

56. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{6^{2n}(2n)!} = \sum_{n=0}^{\infty} (-1)^n \frac{\left(\frac{\pi}{6}\right)^{2n}}{(2n)!} = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$, by (16).

57. $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n+1}}{4^{2n+1}(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n \left(\frac{\pi}{4}\right)^{2n+1}}{(2n+1)!} = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}}$, by (15).

58. $\sum_{n=0}^{\infty} \frac{3^n}{5^n\, n!} = \sum_{n=0}^{\infty} \frac{(3/5)^n}{n!} = e^{3/5}$, by (11).

59. $3 + \frac{9}{2!} + \frac{27}{3!} + \frac{81}{4!} + \cdots = \frac{3^1}{1!} + \frac{3^2}{2!} + \frac{3^3}{3!} + \frac{3^4}{4!} + \cdots = \sum_{n=1}^{\infty} \frac{3^n}{n!} = \sum_{n=0}^{\infty} \frac{3^n}{n!} - 1 = e^3 - 1$, by (11).

60. $1 - \ln 2 + \frac{(\ln 2)^2}{2!} - \frac{(\ln 2)^3}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{(-\ln 2)^n}{n!} = e^{-\ln 2} = \left(e^{\ln 2}\right)^{-1} = 2^{-1} = \frac{1}{2}$, by (11).

61. Assume that $|f'''(x)| \le M$, so $f'''(x) \le M$ for $a \le x \le a + d$. Now $\int_a^x f'''(t)\,dt \le \int_a^x M\,dt \;\Rightarrow$

$f''(x) - f''(a) \le M(x-a) \;\Rightarrow\; f''(x) \le f''(a) + M(x-a)$. Thus, $\int_a^x f''(t)\,dt \le \int_a^x [f''(a) + M(t-a)]\,dt$

$\Rightarrow\; f'(x) - f'(a) \le f''(a)(x-a) + \frac{1}{2}M(x-a)^2 \;\Rightarrow\; f'(x) \le f'(a) + f''(a)(x-a) + \frac{1}{2}M(x-a)^2 \;\Rightarrow$

$\int_a^x f'(t)\,dt \le \int_a^x \left[f'(a) + f''(a)(t-a) + \frac{1}{2}M(t-a)^2\right] dt \;\Rightarrow$

$f(x) - f(a) \le f'(a)(x-a) + \frac{1}{2}f''(a)(x-a)^2 + \frac{1}{6}M(x-a)^3$. So

$f(x) - f(a) - f'(a)(x-a) - \frac{1}{2}f''(a)(x-a)^2 \le \frac{1}{6}M(x-a)^3$. But

$R_2(x) = f(x) - T_2(x) = f(x) - f(a) - f'(a)(x-a) - \frac{1}{2}f''(a)(x-a)^2$, so $R_2(x) \le \frac{1}{6}M(x-a)^3$.

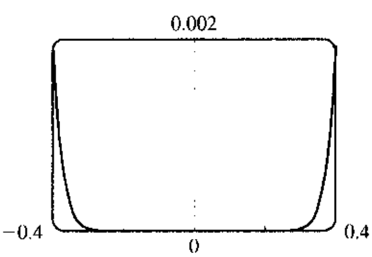
A similar argument using $f'''(x) \ge -M$ shows that $R_2(x) \ge -\frac{1}{6}M(x-a)^3$. So $|R_2(x_2)| \le \frac{1}{6}M\,|x-a|^3$.

Although we have assumed that $x > a$, a similar calculation shows that this inequality is also true if $x < a$.

62. (a) $f(x) = \begin{cases} e^{-1/x^2} & \text{if } x \ne 0 \\ 0 & \text{if } x = 0 \end{cases}$ so

$f'(0) = \lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \frac{e^{-1/x^2}}{x} = \lim_{x\to 0} \frac{1/x}{e^{1/x^2}} = \lim_{x\to 0} \frac{x}{2e^{1/x^2}} = 0$ (using l'Hospital's Rule and simplifying in the penultimate step). Similarly, we can use the definition of the derivative and l'Hospital's Rule to show that $f''(0) = 0$, $f^{(3)}(0) = 0, \ldots, f^{(n)}(0) = 0$, so that the Maclaurin series for f consists entirely of zero terms. But since $f(x) \ne 0$ except for $x = 0$, we see that f cannot equal its Maclaurin series except at $x = 0$.

(b)



From the graph, it seems that the function is extremely flat at the origin. In fact, it could be said to be "infinitely flat" at $x = 0$, since all of its derivatives are 0 there.

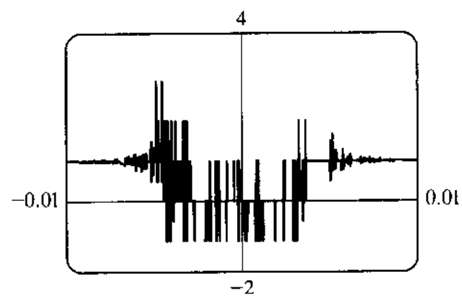
LABORATORY PROJECT An Elusive Limit

1. $f(x) = \dfrac{n(x)}{d(x)} = \dfrac{\sin(\tan x) - \tan(\sin x)}{\arcsin(\arctan x) - \arctan(\arcsin x)}$

x	$f(x)$
1	1.1838
0.1	0.9821
0.01	2.0000
0.001	3.3333
0.0001	3.3333

The table of function values were obtained using Maple with 10 digits of precision. The results of this project will vary depending on the CAS and precision level. It appears that as $x \to 0^+$, $f(x) \to \frac{10}{3}$. Since f is an even function, we have $f(x) \to \frac{10}{3}$ as $x \to 0$.

2. The graph is inconclusive about the limit of f as $x \to 0$.



3. The limit has the indeterminate form $\frac{0}{0}$. Applying l'Hospital's Rule, we obtain the form $\frac{0}{0}$ six times. Finally, on the seventh application we obtain $\lim\limits_{x \to 0} \dfrac{n^{(7)}(x)}{d^{(7)}(x)} = \dfrac{-168}{-168} = 1$.

4.
$$\lim_{x \to 0} f(x) = \lim_{x \to 0} \frac{n(x)}{d(x)} \overset{\text{CAS}}{=} \lim_{x \to 0} \frac{-\frac{1}{30}x^7 - \frac{29}{756}x^9 + \cdots}{-\frac{1}{30}x^7 + \frac{13}{756}x^9 + \cdots}$$
$$= \lim_{x \to 0} \frac{\left(-\frac{1}{30}x^7 - \frac{29}{756}x^9 + \cdots\right)/x^7}{\left(-\frac{1}{30}x^7 + \frac{13}{756}x^9 + \cdots\right)/x^7}$$
$$= \lim_{x \to 0} \frac{-\frac{1}{30} - \frac{29}{756}x^2 + \cdots}{-\frac{1}{30} + \frac{13}{756}x^2 + \cdots} = \frac{-\frac{1}{30}}{-\frac{1}{30}} = 1$$

Note that $n^{(7)}(x) = d^{(7)}(x) = -\frac{7!}{30} = -\frac{5040}{30} = -168$, which agrees with the result in Problem 3.

5. The limit command gives the result that $\lim\limits_{x \to 0} f(x) = 1$.

6. The strange results (with only 10 digits of precision) must be due to the fact that the terms being subtracted in the numerator and denominator are very close in value when $|x|$ is small. Thus, the differences are imprecise (have few correct digits).

12.11 The Binomial Series

1. The general binomial series in (2) is

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots.$$

$$\begin{aligned}(1+x)^{1/2} &= \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} x^n = 1 + \left(\tfrac{1}{2}\right)x + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!}x^2 + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}x^3 + \cdots \\ &= 1 + \frac{x}{2} - \frac{x^2}{2^2 \cdot 2!} + \frac{1 \cdot 3 \cdot x^3}{2^3 \cdot 3!} - \frac{1 \cdot 3 \cdot 5 \cdot x^4}{2^4 \cdot 4!} + \cdots \\ &= 1 + \frac{x}{2} + \sum_{n=2}^{\infty} \frac{(-1)^{n-1}\, 1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-3)x^n}{2^n \cdot n!} \quad \text{for } |x| < 1, \text{ so } R = 1\end{aligned}$$

2. $\dfrac{1}{(1+x)^4} = (1+x)^{-4} = \sum_{n=0}^{\infty} \binom{-4}{n} x^n$. The binomial coefficient is

$$\begin{aligned}\binom{-4}{n} &= \frac{(-4)(-5)(-6)\cdot \cdots \cdot(-4-n+1)}{n!} = \frac{(-4)(-5)(-6)\cdot \cdots \cdot[-(n+3)]}{n!} \\ &= \frac{(-1)^n \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot \cdots \cdot (n+1)(n+2)(n+3)}{2 \cdot 3 \cdot n!} = \frac{(-1)^n (n+1)(n+2)(n+3)}{6}\end{aligned}$$

Thus, $\dfrac{1}{(1+x)^4} = \sum_{n=0}^{\infty} \dfrac{(-1)^n (n+1)(n+2)(n+3)}{6} x^n$ for $|x| < 1$, so $R = 1$.

3. $\dfrac{1}{(2+x)^3} = \dfrac{1}{[2(1+x/2)]^3} = \dfrac{1}{8}\left(1+\dfrac{x}{2}\right)^{-3} = \dfrac{1}{8}\sum_{n=0}^{\infty} \binom{-3}{n}\left(\dfrac{x}{2}\right)^n$. The binomial coefficient is

$$\begin{aligned}\binom{-3}{n} &= \frac{(-3)(-4)(-5)\cdot \cdots \cdot(-3-n+1)}{n!} = \frac{(-3)(-4)(-5)\cdot \cdots \cdot[-(n+2)]}{n!} \\ &= \frac{(-1)^n \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot \cdots \cdot (n+1)(n+2)}{2 \cdot n!} = \frac{(-1)^n (n+1)(n+2)}{2}\end{aligned}$$

Thus, $\dfrac{1}{(2+x)^3} = \dfrac{1}{8}\sum_{n=0}^{\infty} \dfrac{(-1)^n (n+1)(n+2)}{2} \dfrac{x^n}{2^n} = \sum_{n=0}^{\infty} \dfrac{(-1)^n (n+1)(n+2)x^n}{2^{n+4}}$ for $\left|\dfrac{x}{2}\right| < 1 \;\Leftrightarrow$ $|x| < 2$, so $R = 2$.

4.
$$\begin{aligned}(1-x)^{2/3} &= \sum_{n=0}^{\infty} \binom{\frac{2}{3}}{n} (-x)^n \\ &= 1 + \tfrac{2}{3}(-x) + \frac{\frac{2}{3}\left(-\frac{1}{3}\right)}{2!}(-x)^2 + \frac{\frac{2}{3}\left(-\frac{1}{3}\right)\left(-\frac{4}{3}\right)}{3!}(-x)^3 + \cdots \\ &= 1 - \tfrac{2}{3}x + \sum_{n=2}^{\infty} \frac{(-1)^{n-1}(-1)^n \cdot 2 \cdot [1 \cdot 4 \cdot 7 \cdot \cdots \cdot (3n-5)]}{3^n \cdot n!}x^n \\ &= 1 - \tfrac{2}{3}x - 2\sum_{n=2}^{\infty} \frac{1 \cdot 4 \cdot 7 \cdot \cdots \cdot (3n-5)}{3^n \cdot n!}x^n\end{aligned}$$

and $|-x| < 1 \;\Leftrightarrow\; |x| < 1$, so $R = 1$.

5. $$\sqrt[4]{1-8x} = (1-8x)^{1/4} = \sum_{n=0}^{\infty} \binom{\frac{1}{4}}{n} (-8x)^n$$

$$= 1 + \tfrac{1}{4}(-8x) + \frac{\frac{1}{4}\left(-\frac{3}{4}\right)}{2!}(-8x)^2 + \frac{\left(\frac{1}{4}\right)\left(-\frac{3}{4}\right)\left(-\frac{7}{4}\right)}{3!}(-8x)^3 + \cdots$$

$$= 1 - 2x + \sum_{n=2}^{\infty} \frac{(-1)^n (-1)^{n-1} \cdot 3 \cdot 7 \cdot \cdots \cdot (4n-5)\, 8^n}{4^n \cdot n!} x^n$$

$$= 1 - 2x - \sum_{n=2}^{\infty} \frac{3 \cdot 7 \cdot \cdots \cdot (4n-5)\, 2^n}{n!} x^n$$

and $|-8x| < 1 \quad \Leftrightarrow \quad |x| < \frac{1}{8}$, so $R = \frac{1}{8}$.

6. $$\frac{1}{\sqrt[5]{32-x}} = \frac{1}{2\sqrt[5]{1-x/32}} = \frac{1}{2}\left(1 - \frac{x}{32}\right)^{-1/5} = \frac{1}{2}\sum_{n=0}^{\infty}\binom{-\frac{1}{5}}{n}\left(-\frac{x}{32}\right)^n = \frac{1}{2}\sum_{n=0}^{\infty}\binom{-\frac{1}{5}}{n}\frac{(-1)^n x^n}{2^{5n}}$$

$$= \frac{1}{2}\left[1 + \left(-\frac{1}{5}\right)\left(-\frac{x}{2^5}\right) + \frac{\left(-\frac{1}{5}\right)\left(-\frac{6}{5}\right)}{2!}\frac{x^2}{2^{10}} + \frac{\left(-\frac{1}{5}\right)\left(-\frac{6}{5}\right)\left(-\frac{11}{5}\right)}{3!}\left(-\frac{x^3}{2^{15}}\right) + \cdots\right]$$

$$= \frac{1}{2} + \frac{1}{5 \cdot 2^6}x + \frac{1 \cdot 6}{5^2 \cdot 2! \cdot 2^{11}}x^2 + \frac{1 \cdot 6 \cdot 11}{5^3 \cdot 3! \cdot 2^{16}}x^3 + \cdots = \frac{1}{2} + \sum_{n=1}^{\infty}\frac{1 \cdot 6 \cdot \cdots \cdot (5n-4)}{5^n 2^{5n+1}\, n!}x^n$$

The radius of convergence is 32.

7. We must write the binomial in the form (1+ expression), so we'll factor out a 4.

$$\frac{x}{\sqrt{4+x^2}} = \frac{x}{\sqrt{4(1+x^2/4)}} = \frac{x}{2\sqrt{1+x^2/4}} = \frac{x}{2}\left(1 + \frac{x^2}{4}\right)^{-1/2} = \frac{x}{2}\sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n}\left(\frac{x^2}{4}\right)^n$$

$$= \frac{x}{2}\left[1 + \left(-\tfrac{1}{2}\right)\frac{x^2}{4} + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}\left(\frac{x^2}{4}\right)^2 + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3!}\left(\frac{x^2}{4}\right)^3 + \cdots\right]$$

$$= \frac{x}{2} + \frac{x}{2}\sum_{n=1}^{\infty}(-1)^n\frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{2^n \cdot 4^n \cdot n!}x^{2n}$$

$$= \frac{x}{2} + \sum_{n=1}^{\infty}(-1)^n\frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{n!\, 2^{3n+1}}x^{2n+1} \text{ and } \frac{x^2}{4} < 1 \quad \Leftrightarrow \quad \frac{|x|}{2} < 1 \quad \Leftrightarrow$$

$|x| < 2$, so $R = 2$.

8. $$\frac{x^2}{\sqrt{2+x}} = \frac{x^2}{\sqrt{2(1+x/2)}} = \frac{x^2}{\sqrt{2}}\left(1 + \frac{x}{2}\right)^{-1/2} = \frac{x^2}{\sqrt{2}}\sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n}\left(\frac{x}{2}\right)^n$$

$$= \frac{x^2}{\sqrt{2}}\left[1 + \left(-\tfrac{1}{2}\right)\left(\frac{x}{2}\right) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}\left(\frac{x}{2}\right)^2 + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3!}\left(\frac{x}{2}\right)^3 + \cdots\right]$$

$$= \frac{x^2}{\sqrt{2}} + \frac{x^2}{\sqrt{2}}\sum_{n=1}^{\infty}(-1)^n\frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{n!\, 2^{2n}}x^n$$

$$= \frac{x^2}{\sqrt{2}} + \sum_{n=1}^{\infty}(-1)^n\frac{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)}{n!\, 2^{2n+1/2}}x^{n+2} \text{ and } \left|\frac{x}{2}\right| < 1 \quad \Leftrightarrow \quad |x| < 2, \text{ so } R = 2.$$

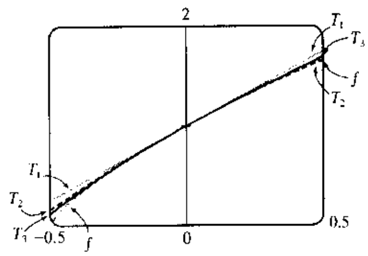
9. $(1+2x)^{3/4} = 1 + \frac{3}{4}(2x) + \frac{\left(\frac{3}{4}\right)\left(-\frac{1}{4}\right)}{2!}(2x)^2 + \frac{\left(\frac{3}{4}\right)\left(-\frac{1}{4}\right)\left(-\frac{5}{4}\right)}{3!}(2x)^3 + \cdots$

$$= 1 + \frac{3}{2}x + 3\sum_{n=2}^{\infty}(-1)^{n+1}\frac{1\cdot 5\cdot 9\cdot\cdots\cdot(4n-7)}{4^n\cdot n!}\cdot 2^n x^n$$

$$= 1 + \frac{3}{2}x + 3\sum_{n=2}^{\infty}(-1)^{n+1}\frac{1\cdot 5\cdot 9\cdot\cdots\cdot(4n-7)}{2^n\cdot n!}x^n \text{ and } |2x|<1 \quad\Leftrightarrow\quad |x|<\frac{1}{2}, \text{ so } R=\frac{1}{2}.$$

The three Taylor polynomials are $T_1(x) = 1 + \frac{3}{2}x$, $T_2(x) = 1 + \frac{3}{2}x - \frac{3}{8}x^2$, and

$T_3(x) = 1 + \frac{3}{2}x - \frac{3}{8}x^2 + \frac{5}{16}x^3$.



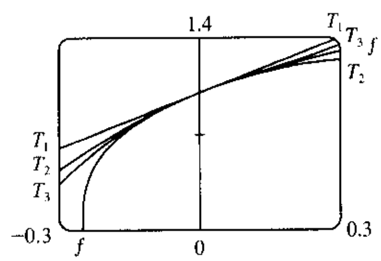
10. $\sqrt[3]{1+4x} = (1+4x)^{1/3}$

$$= 1 + \frac{1}{3}(4x) + \frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)}{2!}(4x)^2 + \frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)}{3!}(4x)^3 + \cdots$$

$$= 1 + \frac{4}{3}x + \sum_{n=2}^{\infty}(-1)^{n+1}\frac{2\cdot 5\cdot 8\cdot\cdots\cdot(3n-4)}{3^n\cdot n!}(4x)^n \text{ and } |4x|<1 \quad\Leftrightarrow\quad |x|<\frac{1}{4}, \text{ so } R=\frac{1}{4}.$$

The three Taylor polynomials are $T_1(x) = 1 + \frac{4}{3}x$, $T_2(x) = 1 + \frac{4}{3}x - \frac{16}{9}x^2$, and

$T_3(x) = 1 + \frac{4}{3}x - \frac{16}{9}x^2 + \frac{320}{81}x^3$.



11. (a) $1/\sqrt{1-x^2} = \left[1+(-x^2)\right]^{-1/2}$

$$= 1 + \left(-\tfrac{1}{2}\right)(-x^2) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}(-x^2)^2 + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3!}(-x^2)^3 + \cdots$$

$$= 1 + \sum_{n=1}^{\infty}\frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}{2^n\cdot n!}x^{2n}$$

(b) $\sin^{-1}x = \int\frac{1}{\sqrt{1-x^2}}\,dx = C + x + \sum_{n=1}^{\infty}\frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}{(2n+1)2^n\cdot n!}x^{2n+1}$

$$= x + \sum_{n=1}^{\infty}\frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}{(2n+1)2^n\cdot n!}x^{2n+1} \quad \text{since } 0 = \sin^{-1}0 = C.$$

12. (a) $\left(1+x^2\right)^{-1/2}=\sum\limits_{n=0}^{\infty}\binom{-1/2}{n}x^{2n}=1+\sum\limits_{n=1}^{\infty}\dfrac{(-1)^n\cdot1\cdot3\cdot5\cdot\cdots\cdot(2n-1)\,x^{2n}}{2^n\cdot n!}$

(b) $\sinh^{-1}x=\displaystyle\int\frac{dx}{\sqrt{1+x^2}}=C+x+\sum_{n=1}^{\infty}\frac{(-1)^n\cdot1\cdot3\cdot5\cdot\cdots\cdot(2n-1)\,x^{2n+1}}{2^n\cdot n!\,(2n+1)}$, but $C=0$ since $\sinh^{-1}0=0$, so $\sinh^{-1}x=x+\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n\cdot1\cdot3\cdot5\cdot\cdots\cdot(2n-1)\,x^{2n+1}}{2^n\cdot n!\,(2n+1)}$, $R=1$.

13. (a) $\sqrt[3]{1+x}=(1+x)^{1/3}=\displaystyle\sum_{n=0}^{\infty}\binom{\frac{1}{3}}{n}x^n$

$$=1+\frac{1}{3}x+\frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)}{2!}x^2+\frac{\left(\frac{1}{3}\right)\left(-\frac{2}{3}\right)\left(-\frac{5}{3}\right)}{3!}x^3+\cdots$$

$$=1+\frac{x}{3}+\sum_{n=2}^{\infty}(-1)^{n+1}\frac{2\cdot5\cdot8\cdot\cdots\cdot(3n-4)}{3^n\cdot n!}x^n$$

(b) $\sqrt[3]{1+x}=1+\frac{1}{3}x-\frac{1}{9}x^2+\frac{5}{81}x^3-\cdots$. $\sqrt[3]{1.01}=\sqrt[3]{1+0.01}$, so let $x=0.01$. The sum of the first two terms is then $1+\frac{1}{3}(0.01)\approx1.0033$. The third term is $\frac{1}{9}(0.01)^2\approx0.000\,01$, which does not affect the fourth decimal place of the sum, so we have $\sqrt[3]{1.01}\approx1.0033$.

14. (a) $1/\sqrt[4]{1+x}=(1+x)^{-1/4}=\displaystyle\sum_{n=0}^{\infty}\binom{-\frac{1}{4}}{n}x^n$

$$=1-\frac{1}{4}x+\frac{\left(-\frac{1}{4}\right)\left(-\frac{5}{4}\right)}{2!}x^2+\frac{\left(-\frac{1}{4}\right)\left(-\frac{5}{4}\right)\left(-\frac{9}{4}\right)}{3!}x^3+\cdots$$

$$=1-\frac{1}{4}x+\sum_{n=2}^{\infty}(-1)^n\frac{1\cdot5\cdot9\cdot\cdots\cdot(4n-3)}{4^n\cdot n!}x^n$$

(b) $1/\sqrt[4]{1+x}=1-\frac{1}{4}x+\frac{5}{32}x^2-\frac{15}{128}x^3+\frac{195}{2048}x^4-\cdots$. $1/\sqrt[4]{1.1}=1/\sqrt[4]{1+0.1}$, so let $x=0.1$. The sum of the first four terms is then $1-\frac{1}{4}(0.1)+\frac{5}{32}(0.1)^2-\frac{15}{128}(0.1)^3\approx0.976$. The fifth term is $\frac{195}{2048}(0.1)^4\approx0.000\,009\,5$, which does not affect the third decimal place of the sum, so we have $1/\sqrt[4]{1.1}\approx0.976$. (Note that the third decimal place of the sum of the first three terms is affected by the fourth term, so we need to use more than three terms for the sum.)

15. (a) $[1+(-x)]^{-2}=1+(-2)(-x)+\dfrac{(-2)(-3)}{2!}(-x)^2+\dfrac{(-2)(-3)(-4)}{3!}(-x)^3+\cdots$

$$=1+2x+3x^2+4x^3+\cdots=\sum_{n=0}^{\infty}(n+1)x^n,$$

so $\dfrac{x}{(1-x)^2}=x\displaystyle\sum_{n=0}^{\infty}(n+1)x^n=\sum_{n=0}^{\infty}(n+1)x^{n+1}=\sum_{n=1}^{\infty}nx^n$.

(b) With $x=\frac{1}{2}$ in part (a), we have $\displaystyle\sum_{n=1}^{\infty}n\left(\tfrac{1}{2}\right)^n=\sum_{n=1}^{\infty}\frac{n}{2^n}=\frac{\frac{1}{2}}{\left(1-\frac{1}{2}\right)^2}=\frac{\frac{1}{2}}{\frac{1}{4}}=2$.

16. (a) $[1+(-x)]^{-3} = \sum_{n=0}^{\infty} \binom{-3}{n} (-x)^n$

$$= 1 + (-3)(-x) + \frac{(-3)(-4)}{2!}(-x)^2 + \frac{(-3)(-4)(-5)}{3!}(-x)^3 + \cdots$$

$$= 1 + \sum_{n=1}^{\infty} \frac{3 \cdot 4 \cdot 5 \cdot \cdots \cdot (n+2)}{n!} x^n = \sum_{n=0}^{\infty} \frac{2 \cdot 3 \cdot 4 \cdot 5 \cdot \cdots \cdot (n+2)}{2 \cdot n!} x^n$$

$$= \sum_{n=0}^{\infty} \frac{(n+1)(n+2)}{2} x^n \quad \Rightarrow$$

$$(x+x^2)[1+(-x)]^{-3} = x\,[1+(-x)]^{-3} + x^2\,[1+(-x)]^{-3}$$

$$= \sum_{n=0}^{\infty} \frac{(n+1)(n+2)}{2} x^{n+1} + \sum_{n=0}^{\infty} \frac{(n+1)(n+2)}{2} x^{n+2}$$

$$= \sum_{n=1}^{\infty} \frac{n(n+1)}{2} x^n + \sum_{n=1}^{\infty} \frac{n(n+1)}{2} x^{n+1}$$

$$= x + \sum_{n=2}^{\infty} \frac{n(n+1)}{2} x^n + \sum_{n=2}^{\infty} \frac{(n-1)n}{2} x^n = x + \sum_{n=2}^{\infty} \left[\frac{n(n+1)}{2} + \frac{(n-1)n}{2}\right] x^n$$

$$= x + \sum_{n=2}^{\infty} n^2 x^n = \sum_{n=1}^{\infty} n^2 x^n, \; -1 < x < 1$$

(b) Setting $x = \frac{1}{2}$ in the last series above gives the required series, so $\sum_{n=1}^{\infty} \frac{n^2}{2^n} = \frac{\frac{1}{2} + \left(\frac{1}{2}\right)^2}{\left(1 - \frac{1}{2}\right)^3} = \frac{\frac{3}{4}}{\frac{1}{8}} = 6.$

17. (a) $\left(1+x^2\right)^{1/2} = 1 + \left(\frac{1}{2}\right)x^2 + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!}\left(x^2\right)^2 + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}\left(x^2\right)^3 + \cdots$

$$= 1 + \frac{x^2}{2} + \sum_{n=2}^{\infty} \frac{(-1)^{n-1}\, 1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-3)}{2^n \cdot n!} x^{2n}$$

(b) The coefficient of x^{10} (corresponding to $n = 5$) in the above Maclaurin series is $\frac{f^{(10)}(0)}{10!}$, so

$$\frac{f^{(10)}(0)}{10!} = \frac{(-1)^4 \cdot 1 \cdot 3 \cdot 5 \cdot 7}{2^5 \cdot 5!} \quad \Rightarrow \quad f^{(10)}(0) = 10!\left(\frac{1 \cdot 3 \cdot 5 \cdot 7}{2^5 \cdot 5!}\right) = 99{,}225.$$

18. (a) $\left(1+x^3\right)^{-1/2} = \sum_{n=0}^{\infty} \binom{-\frac{1}{2}}{n} \left(x^3\right)^n$

$$= 1 + \left(-\tfrac{1}{2}\right)\left(x^3\right) + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}\left(x^3\right)^2 + \frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3!}\left(x^3\right)^3 + \cdots$$

$$= 1 + \sum_{n=1}^{\infty} \frac{(-1)^n\, 1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)\, x^{3n}}{2^n \cdot n!}$$

(b) The coefficient of x^9 (corresponding to $n = 3$) in the preceding series is

$$\frac{f^{(9)}(0)}{9!}, \text{ so } \frac{f^{(9)}(0)}{9!} = \frac{(-1)^3\, 1 \cdot 3 \cdot 5}{2^3 \cdot 3!} \quad \Rightarrow \quad f^{(9)}(0) = -\frac{9! \cdot 5}{8 \cdot 2} = -113{,}400.$$

19. (a) $g(x)=\sum_{n=0}^{\infty}\binom{k}{n}x^n \quad\Rightarrow\quad g'(x)=\sum_{n=1}^{\infty}\binom{k}{n}nx^{n-1}$, so

$$(1+x)g'(x)=(1+x)\sum_{n=1}^{\infty}\binom{k}{n}nx^{n-1}=\sum_{n=1}^{\infty}\binom{k}{n}nx^{n-1}+\sum_{n=1}^{\infty}\binom{k}{n}nx^{n}$$

$$=\sum_{n=0}^{\infty}\binom{k}{n+1}(n+1)x^n+\sum_{n=0}^{\infty}\binom{k}{n}nx^n \qquad \begin{bmatrix}\text{Replace } n \text{ with } n+1\\ \text{in the first series}\end{bmatrix}$$

$$=\sum_{n=0}^{\infty}(n+1)\frac{k(k-1)(k-2)\cdots(k-n+1)(k-n)}{(n+1)!}x^n$$

$$+\sum_{n=0}^{\infty}\left[(n)\frac{k(k-1)(k-2)\cdots(k-n+1)}{n!}\right]x^n$$

$$=\sum_{n=0}^{\infty}\frac{(n+1)k(k-1)(k-2)\cdots(k-n+1)}{(n+1)!}\left[(k-n)+n\right]x^n$$

$$=k\sum_{n=0}^{\infty}\frac{k(k-1)(k-2)\cdots(k-n+1)}{n!}x^n=k\sum_{n=0}^{\infty}\binom{k}{n}x^n=kg(x)$$

Thus, $g'(x)=\dfrac{kg(x)}{1+x}$.

(b) $h(x)=(1+x)^{-k}g(x) \quad\Rightarrow$

$h'(x)=-k(1+x)^{-k-1}g(x)+(1+x)^{-k}g'(x)$ [Product Rule]

$=-k(1+x)^{-k-1}g(x)+(1+x)^{-k}\dfrac{kg(x)}{1+x}$ [from part (a)]

$=-k(1+x)^{-k-1}g(x)+k(1+x)^{-k-1}g(x)=0$

(c) From part (b) we see that $h(x)$ must be constant for $x\in(-1,1)$, so $h(x)=h(0)=1$ for $x\in(-1,1)$.

Thus, $h(x)=1=(1+x)^{-k}g(x) \quad\Leftrightarrow\quad g(x)=(1+x)^k$ for $x\in(-1,1)$.

20. By Exercise 12.11.1, $\sqrt{1+x}=1+\dfrac{x}{2}+\sum_{n=2}^{\infty}\dfrac{(-1)^{n-1}1\cdot3\cdot5\cdot\cdots\cdot(2n-3)x^n}{2^n\cdot n!}$, so

$(1-x^2)^{1/2}=1-\dfrac{1}{2}x^2-\sum_{n=2}^{\infty}\dfrac{1\cdot3\cdot5\cdot\cdots\cdot(2n-3)}{2^n\cdot n!}x^{2n}$ and

$\sqrt{1-e^2\sin^2\theta}=1-\dfrac{1}{2}e^2\sin^2\theta-\sum_{n=2}^{\infty}\dfrac{1\cdot3\cdot5\cdot\cdots\cdot(2n-3)}{2^n\cdot n!}e^{2n}\sin^{2n}\theta$. Thus,

$$L = 4a\int_0^{\pi/2}\sqrt{1-e^2\sin^2\theta}\,d\theta = 4a\int_0^{\pi/2}\left(1-\frac{1}{2}e^2\sin^2\theta-\sum_{n=2}^{\infty}\frac{1\cdot3\cdot5\cdot\cdots\cdot(2n-3)}{2^n\cdot n!}e^{2n}\sin^{2n}\theta\right)d\theta$$

$$= 4a\left[\frac{\pi}{2}-\frac{e^2}{2}S_1-\sum_{n=2}^{\infty}\frac{1\cdot3\cdot5\cdot\cdots\cdot(2n-3)}{n!}\left(\frac{e^2}{2}\right)^n S_n\right]$$

where $S_n = \int_0^{\pi/2}\sin^{2n}\theta\,d\theta = \frac{1\cdot3\cdot5\cdot\cdots\cdot(2n-1)}{2\cdot4\cdot6\cdot\cdots\cdot2n}\frac{\pi}{2}$ by Exercise 44 of 8.1.

$$L = 4a\left(\frac{\pi}{2}\right)\left[1-\frac{e^2}{2}\cdot\frac{1}{2}-\sum_{n=2}^{\infty}\frac{1\cdot3\cdot5\cdot\cdots\cdot(2n-3)}{n!}\left(\frac{e^2}{2}\right)^n\frac{1\cdot3\cdot5\cdot\cdots\cdot(2n-1)}{2\cdot4\cdot6\cdot\cdots\cdot2n}\right]$$

$$= 2\pi a\left[1-\frac{e^2}{4}-\sum_{n=2}^{\infty}\frac{e^{2n}}{2^n}\cdot\frac{1^2\cdot3^2\cdot5^2\cdot\cdots\cdot(2n-3)^2(2n-1)}{n!\cdot2^n\cdot n!}\right]$$

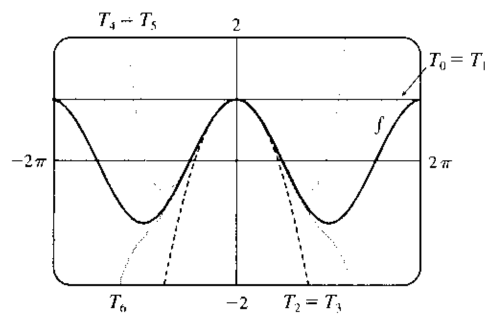
$$= 2\pi a\left[1-\frac{e^2}{4}-\sum_{n=2}^{\infty}\frac{e^{2n}}{4^n}\left(\frac{1\cdot3\cdot\cdots\cdot(2n-3)}{n!}\right)^2(2n-1)\right]$$

$$= 2\pi a\left[1-\frac{e^2}{4}-\frac{3e^4}{64}-\frac{5e^6}{256}-\cdots\right] = \frac{\pi a}{128}(256-64e^2-12e^4-5e^6-\cdots)$$

12.12 Applications of Taylor Polynomials

1. (a)

n	$f^{(n)}(x)$	$f^{(n)}(0)$	$T_n(x)$
0	$\cos x$	1	1
1	$-\sin x$	0	1
2	$-\cos x$	-1	$1-\frac{1}{2}x^2$
3	$\sin x$	0	$1-\frac{1}{2}x^2$
4	$\cos x$	1	$1-\frac{1}{2}x^2+\frac{1}{24}x^4$
5	$-\sin x$	0	$1-\frac{1}{2}x^2+\frac{1}{24}x^4$
6	$-\cos x$	-1	$1-\frac{1}{2}x^2+\frac{1}{24}x^4-\frac{1}{720}x^6$



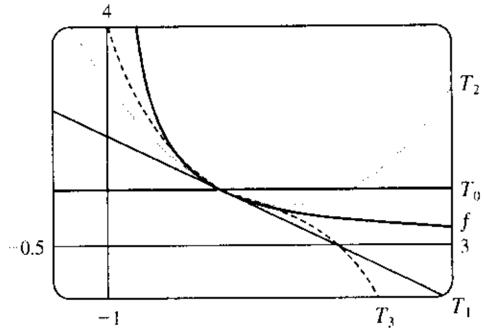
(b)

x	f	$T_0=T_1$	$T_2=T_3$	$T_4=T_5$	T_6
$\frac{\pi}{4}$	0.7071	1	0.6916	0.7074	0.7071
$\frac{\pi}{2}$	0	1	-0.2337	0.0200	-0.0009
π	-1	1	-3.9348	0.1239	-1.2114

(c) As n increases, $T_n(x)$ is a good approximation to $f(x)$ on a larger and larger interval.

2. (a)

n	$f^{(n)}(x)$	$f^{(n)}(1)$	$T_n(x)$
0	x^{-1}	1	1
1	$-x^{-2}$	-1	$1-(x-1)=2-x$
2	$2x^{-3}$	2	$1-(x-1)+(x-1)^2=x^2-3x+3$
3	$-6x^{-4}$	-6	$1-(x-1)+(x-1)^2-(x-1)^3=-x^3+4x^2-6x+4$



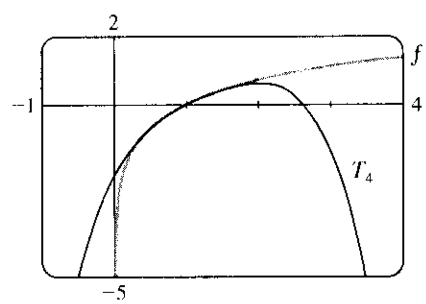
(b)

x	f	T_0	T_1	T_2	T_3
0.9	$1.\overline{1}$	1	1.1	1.11	1.111
1.3	0.7692	1	0.7	0.79	0.763

(c) As n increases, $T_n(x)$ is a good approximation to $f(x)$ on a larger and larger interval.

3.

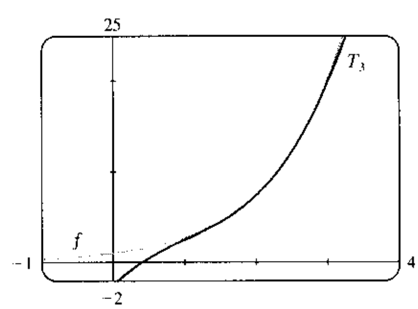
n	$f^{(n)}(x)$	$f^{(n)}(1)$
0	$\ln x$	0
1	$1/x$	1
2	$-1/x^2$	-1
3	$2/x^3$	2
4	$-6/x^4$	-6



$$T_4(x)=\sum_{n=0}^{4}\frac{f^{(n)}(1)}{n!}(x-1)^n=0+(x-1)-\tfrac{1}{2}(x-1)^2+\tfrac{1}{3}(x-1)^3-\tfrac{1}{4}(x-1)^4$$

4.

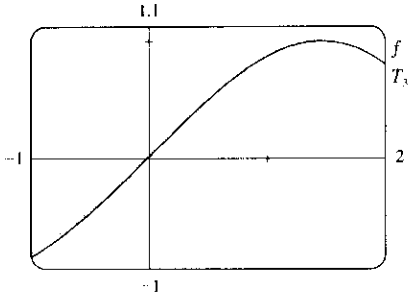
n	$f^{(n)}(x)$	$f^{(n)}(2)$
0	e^x	e^2
1	e^x	e^2
2	e^x	e^2
3	e^x	e^2



$$T_3(x)=\sum_{n=0}^{3}\frac{f^{(n)}(2)}{n!}(x-2)^n=e^2+e^2(x-2)+\frac{e^2}{2}(x-2)^2+\frac{e^2}{6}(x-2)^3$$

5.

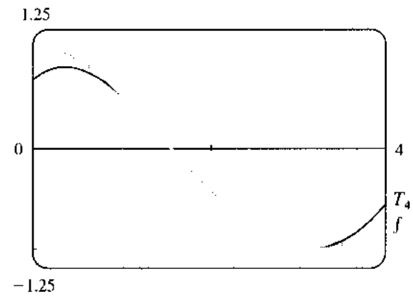
n	$f^{(n)}(x)$	$f^{(n)}\left(\frac{\pi}{6}\right)$
0	$\sin x$	$\frac{1}{2}$
1	$\cos x$	$\frac{\sqrt{3}}{2}$
2	$-\sin x$	$-\frac{1}{2}$
3	$-\cos x$	$-\frac{\sqrt{3}}{2}$



$$T_3(x) = \sum_{n=0}^{3} \frac{f^{(n)}\left(\frac{\pi}{6}\right)}{n!}\left(x - \frac{\pi}{6}\right)^n = \frac{1}{2} + \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{6}\right) - \frac{1}{4}\left(x - \frac{\pi}{6}\right)^2 - \frac{\sqrt{3}}{12}\left(x - \frac{\pi}{6}\right)^3$$

6.

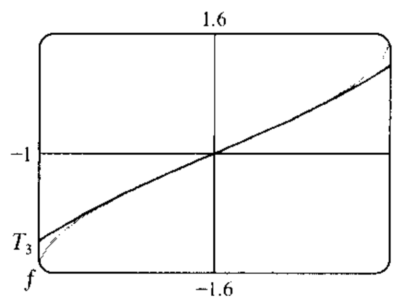
n	$f^{(n)}(x)$	$f^{(n)}\left(\frac{2\pi}{3}\right)$
0	$\cos x$	$-\frac{1}{2}$
1	$-\sin x$	$-\frac{\sqrt{3}}{2}$
2	$-\cos x$	$\frac{1}{2}$
3	$\sin x$	$\frac{\sqrt{3}}{2}$
4	$\cos x$	$-\frac{1}{2}$



$$T_4(x) = \sum_{n=0}^{4} \frac{f^{(n)}\left(\frac{2\pi}{3}\right)}{n!}\left(x - \frac{2\pi}{3}\right)^n = -\frac{1}{2} - \frac{\sqrt{3}}{2}\left(x - \frac{2\pi}{3}\right) + \frac{1}{4}\left(x - \frac{2\pi}{3}\right)^2 + \frac{\sqrt{3}}{12}\left(x - \frac{2\pi}{3}\right)^3 - \frac{1}{48}\left(x - \frac{2\pi}{3}\right)^4$$

7.

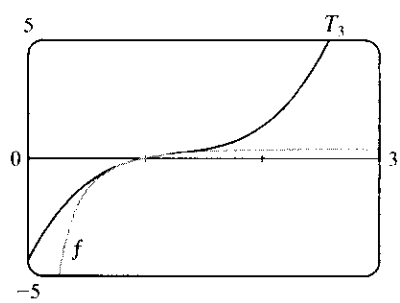
n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\arcsin x$	0
1	$1/\sqrt{1-x^2}$	1
2	$x/(1-x^2)^{3/2}$	0
3	$(2x^2+1)/(1-x^2)^{5/2}$	1



$$T_3(x) = \sum_{n=0}^{3} \frac{f^{(n)}(0)}{n!}x^n = x + \frac{x^3}{6}$$

8.

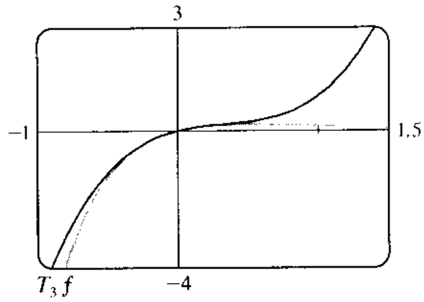
n	$f^{(n)}(x)$	$f^{(n)}(1)$
0	$(\ln x)/x$	0
1	$(1-\ln x)/x^2$	1
2	$(-3+2\ln x)/x^3$	-3
3	$(11-6\ln x)/x^4$	11



$$T_3(x) = \sum_{n=0}^{3} \frac{f^{(n)}(1)}{n!}(x-1)^n = (x-1) - \frac{3}{2}(x-1)^2 + \frac{11}{6}(x-1)^3$$

9.

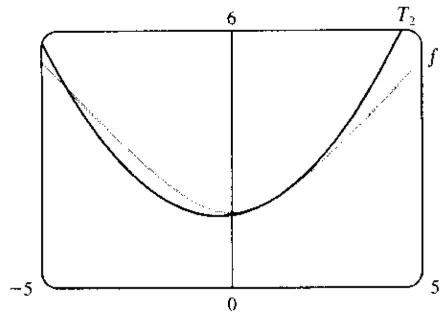
n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	xe^{-2x}	0
1	$(1-2x)e^{-2x}$	1
2	$4(x-1)e^{-2x}$	-4
3	$4(3-2x)e^{-2x}$	12



$$T_3(x)=\sum_{n=0}^{3}\frac{f^{(n)}(0)}{n!}x^n=\tfrac{0}{1}\cdot 1+\tfrac{1}{1}x^1+\tfrac{-4}{2}x^2+\tfrac{12}{6}x^3=x-2x^2+2x^3$$

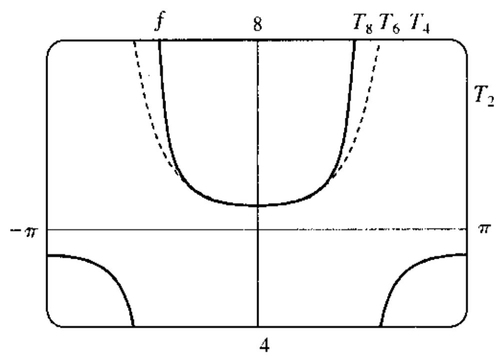
10.

n	$f^{(n)}(x)$	$f^{(n)}(1)$
0	$(3+x^2)^{1/2}$	2
1	$x(3+x^2)^{-1/2}$	$\frac{1}{2}$
2	$3(3+x^2)^{-3/2}$	$\frac{3}{8}$

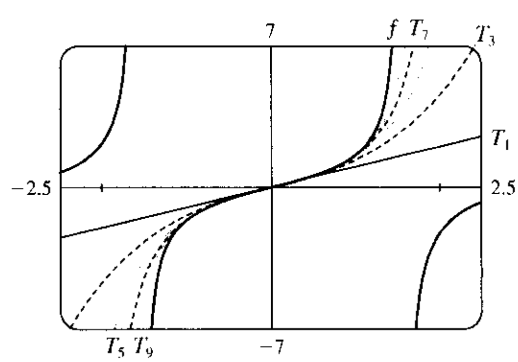


$$T_2(x)=\sum_{n=0}^{2}\frac{f^{(n)}(1)}{n!}(x-1)^n=2+\tfrac{1}{2}(x-1)+\tfrac{3/8}{2}(x-1)^2=2+\tfrac{1}{2}(x-1)+\tfrac{3}{16}(x-1)^2$$

11. In Maple, we can find the Taylor polynomials by the following method: first define `f:=sec(x);` and then set `T2:=convert(taylor(f,x=0,3),polynom);`, `T4:=convert(taylor(f,x=0,5),polynom);`, etc. (The third argument in the `taylor` function is one more than the degree of the desired polynomial). We must `convert` to the type `polynom` because the output of the `taylor` function contains an error term which we do not want. In Mathematica, we use `Tn:=Normal[Series[f,{x,0,n}]]`, with `n=2, 4`, etc. Note that in Mathematica, the "degree" argument is the same as the degree of the desired polynomial. In Derive, author $\sec x$, then enter `Calculus, Taylor, 8, 0`; and then simplify the expression. The eighth Taylor polynomial is $T_8(x)=1+\frac{1}{2}x^2+\frac{5}{24}x^4+\frac{61}{720}x^6+\frac{277}{8064}x^8$.



12. See Exercise 11 for the CAS commands used to generate the Taylor polynomials. The ninth Taylor polynomial for $\tan x$ is $T_9(x)=x+\frac{1}{3}x^3+\frac{2}{15}x^5+\frac{17}{315}x^7+\frac{62}{2835}x^9$.



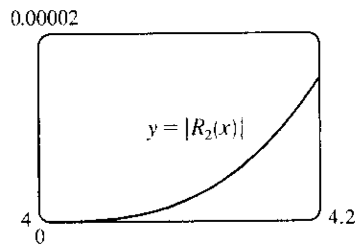
13.

n	$f^{(n)}(x)$	$f^{(n)}(4)$
0	$\sqrt{x}$	2
1	$\frac{1}{2}x^{-1/2}$	$\frac{1}{4}$
2	$-\frac{1}{4}x^{-3/2}$	$-\frac{1}{32}$
3	$\frac{3}{8}x^{-5/2}$	

(a) $f(x) = \sqrt{x} \approx T_2(x) = 2 + \frac{1}{4}(x-4) - \frac{1/32}{2!}(x-4)^2 = 2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2$

(b) $|R_2(x)| \le \dfrac{M}{3!}|x-4|^3$, where $|f'''(x)| \le M$. Now $4 \le x \le 4.2 \Rightarrow |x-4| \le 0.2 \Rightarrow$ $|x-4|^3 \le 0.008$. Since $f'''(x)$ is decreasing on $[4, 4.2]$, we can take $M = |f'''(4)| = \frac{3}{8}4^{-5/2} = \frac{3}{256}$, so $|R_2(x)| \le \frac{3/256}{6}(0.008) = \frac{0.008}{512} = 0.000015625$.

(c) From the graph of $|R_2(x)| = |\sqrt{x} - T_2(x)|$, it seems that the error is less than 1.52×10^{-5} on $[4, 4.2]$.



14.

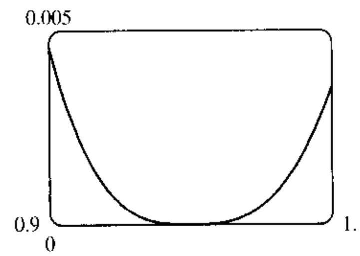
n	$f^{(n)}(x)$	$f^{(n)}(1)$
0	x^{-2}	1
1	$-2x^{-3}$	-2
2	$6x^{-4}$	6
3	$-24x^{-5}$	

(a) $f(x) = x^{-2} \approx T_2(x)$
$= 1 - 2(x-1) + \frac{6}{2!}(x-1)^2$
$= 1 - 2(x-1) + 3(x-1)^2$

(b) $|R_2(x)| \le \dfrac{M}{3!}|x-1|^3$, where $|f'''(x)| \le M$. Now $0.9 \le x \le 1.1 \Rightarrow |x-1| \le 0.1 \Rightarrow$ $|x-1|^3 \le 0.001$. Since $f'''(x)$ is decreasing on $[0.9, 1.1]$, we can take $M = |f'''(0.9)| = \frac{24}{(0.9)^5}$, so

$|R_2(x)| \le \frac{24/(0.9)^5}{6}(0.001) = \frac{0.004}{0.59049}$
≈ 0.00677404

(c)



From the graph of $|R_2(x)| = |x^{-2} - T_2(x)|$, it seems that the error is less than 0.0046 on $[0.9, 1.1]$.

15.

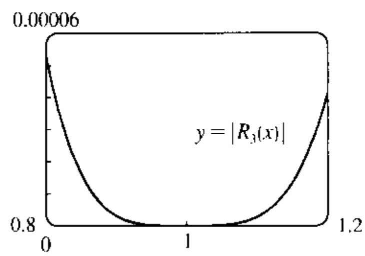
n	$f^{(n)}(x)$	$f^{(n)}(1)$
0	$x^{2/3}$	1
1	$\frac{2}{3}x^{-1/3}$	$\frac{2}{3}$
2	$-\frac{2}{9}x^{-4/3}$	$-\frac{2}{9}$
3	$\frac{8}{27}x^{-7/3}$	$\frac{8}{27}$
4	$-\frac{56}{81}x^{-10/3}$	

(a) $f(x) = x^{2/3} \approx T_3(x) = 1 + \frac{2}{3}(x-1) - \frac{2/9}{2!}(x-1)^2 + \frac{8/27}{3!}(x-1)^3$

$$= 1 + \tfrac{2}{3}(x-1) - \tfrac{1}{9}(x-1)^2 + \tfrac{4}{81}(x-1)^3$$

(b) $|R_3(x)| \le \frac{M}{4!}|x-1|^4$, where $\left|f^{(4)}(x)\right| \le M$. Now $0.8 \le x \le 1.2 \;\Rightarrow\; |x-1| \le 0.2 \;\Rightarrow$ $|x-1|^4 \le 0.0016$. Since $\left|f^{(4)}(x)\right|$ is decreasing on $[0.8, 1.2]$, we can take $M = \left|f^{(4)}(0.8)\right| = \frac{56}{81}(0.8)^{-10/3}$, so $|R_3(x)| \le \dfrac{\frac{56}{81}(0.8)^{-10/3}}{24}(0.0016) \approx 0.000\,096\,97$.

(c) From the graph of $|R_3(x)| = \left|x^{2/3} - T_3(x)\right|$, it seems that the error is less than $0.000\,053\,3$ on $[0.8, 1.2]$.



16.

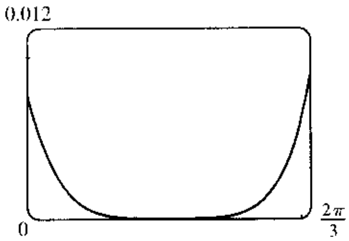
n	$f^{(n)}(x)$	$f^{(n)}\left(\frac{\pi}{3}\right)$
0	$\cos x$	$\frac{1}{2}$
1	$-\sin x$	$-\frac{\sqrt{3}}{2}$
2	$-\cos x$	$-\frac{1}{2}$
3	$\sin x$	$\frac{\sqrt{3}}{2}$
4	$\cos x$	$\frac{1}{2}$
5	$-\sin x$	

(a) $f(x) = \cos x \approx T_4(x)$

$$= \tfrac{1}{2} - \tfrac{\sqrt{3}}{2}\left(x - \tfrac{\pi}{3}\right) - \tfrac{1}{4}\left(x - \tfrac{\pi}{3}\right)^2 + \tfrac{\sqrt{3}}{12}\left(x - \tfrac{\pi}{3}\right)^3 + \tfrac{1}{48}\left(x - \tfrac{\pi}{3}\right)^4$$

(b) $|R_4(x)| \le \dfrac{M}{5!}\left|x - \frac{\pi}{3}\right|^5$, where $\left|f^{(5)}(x)\right| \le M$. Now $0 \le x \le \frac{2\pi}{3} \;\Rightarrow\; \left(x - \frac{\pi}{3}\right)^5 \le \left(\frac{\pi}{3}\right)^5$, and letting $x = \frac{\pi}{2}$ gives $M = 1$, so $|R_4(x)| \le \frac{1}{5!}\left(\frac{\pi}{3}\right)^5 \approx 0.0105$.

(c)



From the graph of $|R_4(x)| = |\cos x - T_4(x)|$, it seems that the error is less than 0.01 on $\left[0, \frac{2\pi}{3}\right]$.

17.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\tan x$	0
1	$\sec^2 x$	1
2	$2\sec^2 x \tan x$	0
3	$4\sec^2 x \tan^2 x + 2\sec^4 x$	2
4	$8\sec^2 x \tan^3 x + 16\sec^4 x \tan x$	

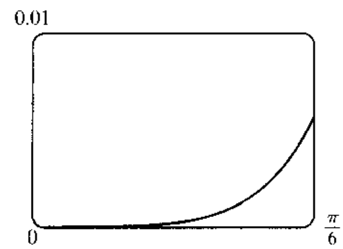
(a) $f(x) = \tan x \approx T_3(x) = x + \frac{1}{3}x^3$

(b) $|R_3(x)| \le \dfrac{M}{4!}|x|^4$, where $\left|f^{(4)}(x)\right| \le M$. Now $0 \le x \le \frac{\pi}{6} \;\Rightarrow\; x^4 \le \left(\frac{\pi}{6}\right)^4$, and letting $x = \frac{\pi}{6}$ $\left[\text{since } f^{(4)} \text{ is increasing on } \left(0, \frac{\pi}{6}\right)\right]$ gives

$$|R_3(x)| \le \frac{8\left(\frac{2}{\sqrt{3}}\right)^2\left(\frac{1}{\sqrt{3}}\right)^3 + 16\left(\frac{2}{\sqrt{3}}\right)^4\left(\frac{1}{\sqrt{3}}\right)}{4!}\left(\frac{\pi}{6}\right)^4$$

$$= \tfrac{4\sqrt{3}}{9}\left(\tfrac{\pi}{6}\right)^4 \approx 0.057859$$

(c)



From the graph of $|R_3(x)| = |\tan x - T_3(3)|$, it seems that the error is less than 0.006 on $[0, \pi/6]$.

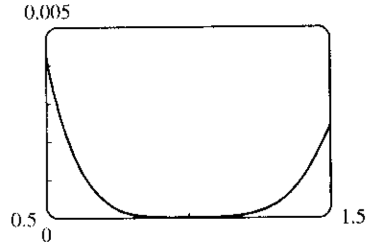
18.

n	$f^{(n)}(x)$	$f^{(n)}(1)$
0	$\ln(1+2x)$	$\ln 3$
1	$2/(1+2x)$	$\frac{2}{3}$
2	$-4/(1+2x)^2$	$-\frac{4}{9}$
3	$16/(1+2x)^3$	$\frac{16}{27}$
4	$-96/(1+2x)^4$	

(a) $f(x) = \ln(1+2x) \approx T_3(x) = \ln 3 + \frac{2}{3}(x-1) - \frac{4/9}{2!}(x-1)^2 + \frac{16/27}{3!}(x-1)^3$

(b) $|R_3(x)| \le \dfrac{M}{4!}|x-1|^4$, where $\left|f^{(4)}(x)\right| \le M$. Now $0.5 \le x \le 1.5 \;\Rightarrow\; -0.5 \le x - 1 \le 0.5 \;\Rightarrow\; |x-1| \le 0.5 \;\Rightarrow\; |x-1|^4 \le \frac{1}{16}$, and letting $x = 0.5$ gives $M = 6$, so $|R_3(x)| \le \dfrac{6}{4!}\cdot\dfrac{1}{16} = \dfrac{1}{64} = 0.015625$.

(c)



From the graph of $|R_3(x)| = |\ln(1+2x) - T_3(x)|$, it seems that the error is less than 0.005 on $[0.5, 1.5]$.

19.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	e^{x^2}	1
1	$e^{x^2}(2x)$	0
2	$e^{x^2}(2+4x^2)$	2
3	$e^{x^2}(12x+8x^3)$	0
4	$e^{x^2}(12+48x^2+16x^4)$	

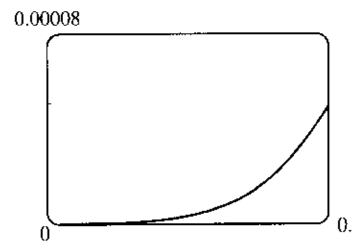
(a) $f(x) = e^{x^2} \approx T_3(x) = 1 + \frac{2}{2!}x^2 = 1 + x^2$

(b) $|R_3(x)| \le \dfrac{M}{4!}|x|^4$, where $\left|f^{(4)}(x)\right| \le M$.

Now $0 \le x \le 0.1 \;\Rightarrow\; x^4 \le (0.1)^4$, and letting $x = 0.1$ gives

$|R_3(x)| \le \dfrac{e^{0.01}(12+0.48+0.0016)}{24}(0.1)^4 \approx$ 0.00006.

(c)



From the graph of $|R_3(x)| = \left|e^{x^2} - (1+x^2)\right|$, it appears that the error is less than 0.000051 on $[0, 0.1]$.

20.

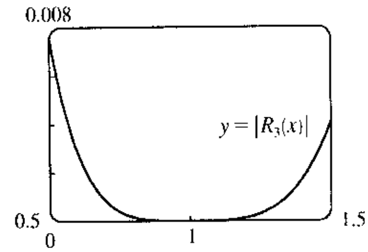
n	$f^{(n)}(x)$	$f^{(n)}(1)$
0	$x\ln x$	0
1	$\ln x + 1$	1
2	$1/x$	1
3	$-1/x^2$	-1
4	$2/x^3$	

(a) $f(x) = x\ln x \approx T_3(x) = (x-1) + \frac{1}{2}(x-1)^2 - \frac{1}{6}(x-1)^3$

(b) $|R_3(x)| \le \frac{M}{4!}|x-1|^4$, where $\left|f^{(4)}(x)\right| \le M$. Now

$0.5 \le x \le 1.5 \;\Rightarrow\; |x-1| \le \frac{1}{2} \;\Rightarrow\; |x-1|^4 \le \frac{1}{16}$. Since $\left|f^{(4)}(x)\right|$ is decreasing on $[0.5, 1.5]$, we can take $M = \left|f^{(4)}(0.5)\right| = 2/(0.5)^3 = 16$, so $|R_3(x)| \le \frac{16}{24}(1/16) = \frac{1}{24} = 0.041\overline{6}$.

(c) From the graph of $|R_3(x)| = |x \ln x - T_3(x)|$, it seems that the error is less than 0.0076 on $[0.5, 1.5]$.



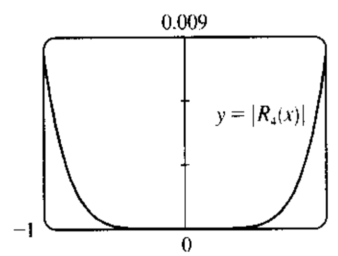
21.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$x \sin x$	0
1	$\sin x + x \cos x$	0
2	$2 \cos x - x \sin x$	2
3	$-3 \sin x - x \cos x$	0
4	$-4 \cos x + x \sin x$	-4
5	$5 \sin x + x \cos x$	

(a) $f(x) = x \sin x \approx T_4(x) = \frac{2}{2!}(x-0)^2 + \frac{-4}{4!}(x-0)^4 = x^2 - \frac{1}{6}x^4$

(b) $|R_4(x)| \le \frac{M}{5!}|x|^5$, where $\left|f^{(5)}(x)\right| \le M$. Now $-1 \le x \le 1 \;\Rightarrow\; |x| \le 1$, and a graph of $f^{(5)}(x)$ shows that $\left|f^{(5)}(x)\right| \le 5$ for $-1 \le x \le 1$. Thus, we can take $M = 5$ and get $|R_4(x)| \le \frac{5}{5!} \cdot 1^5 = \dfrac{1}{24} = 0.041\overline{6}$.

(c) From the graph of $|R_4(x)| = |x \sin x - T_4(x)|$, it seems that the error is less than 0.0082 on $[-1, 1]$.



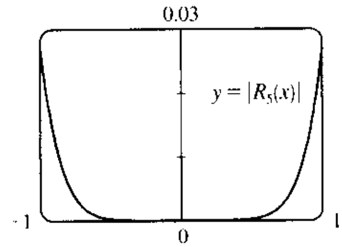
22.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\sinh 2x$	0
1	$2 \cosh 2x$	2
2	$4 \sinh 2x$	0
3	$8 \cosh 2x$	8
4	$16 \sinh 2x$	0
5	$32 \cosh 2x$	32
6	$64 \sinh 2x$	

(a) $f(x) = \sinh 2x \approx T_5(x) = 2x + \frac{8}{3!}x^3 + \frac{32}{5!}x^5 = 2x + \frac{4}{3}x^3 + \frac{4}{15}x^5$

(b) $|R_5(x)| \le \frac{M}{6!}|x|^6$, where $\left|f^{(6)}(x)\right| \le M$. For x in $[-1,1]$, we have $|x| \le 1$. Since $f^{(6)}(x)$ is an increasing odd function on $[-1,1]$, we see that $\left|f^{(6)}(x)\right| \le f^{(6)}(1) = 64\sinh 2 = 32(e^2 - e^{-2}) \approx 232.119$, so we can take $M = 232.12$ and get $|R_5(x)| \le \frac{232.12}{720} \cdot 1^6 \approx 0.3224$.

(c) From the graph of $|R_5(x)| = |\sinh 2x - T_5(x)|$, it seems that the error is less than 0.027 on $[-1,1]$.



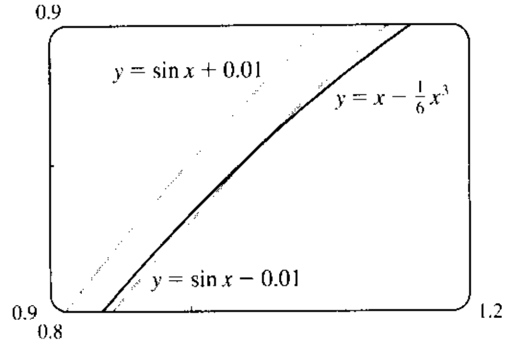
23. From Exercise 5, $\sin x = \frac{1}{2} + \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{6}\right) - \frac{1}{4}\left(x - \frac{\pi}{6}\right)^2 - \frac{\sqrt{3}}{12}\left(x - \frac{\pi}{6}\right)^3 + R_3(x)$, where $|R_3(x)| \le \frac{M}{4!}\left|x - \frac{\pi}{6}\right|^4$ with $\left|f^{(4)}(x)\right| = |\sin x| \le M = 1$. Now $x = 35^\circ = (30^\circ + 5^\circ) = \left(\frac{\pi}{6} + \frac{\pi}{36}\right)$ radians, so the error is $\left|R_3\left(\frac{\pi}{36}\right)\right| \le \dfrac{\left(\frac{\pi}{36}\right)^4}{4!} < 0.000003$. Therefore, to five decimal places, $\sin 35^\circ \approx \frac{1}{2} + \frac{\sqrt{3}}{2}\left(\frac{\pi}{36}\right) - \frac{1}{4}\left(\frac{\pi}{36}\right)^2 - \frac{\sqrt{3}}{12}\left(\frac{\pi}{36}\right)^3 \approx 0.57358$.

24. From Exercise 16, $\cos x = \frac{1}{2} - \frac{\sqrt{3}}{2}\left(x - \frac{\pi}{3}\right) - \frac{1}{4}\left(x - \frac{\pi}{3}\right)^2 + \frac{\sqrt{3}}{12}\left(x - \frac{\pi}{3}\right)^3 + \frac{1}{48}\left(x - \frac{\pi}{3}\right)^4 + R_4(x)$. Now since $x = 69^\circ = (60^\circ + 9^\circ) = \left(\frac{\pi}{3} + \frac{\pi}{20}\right)$ radians, the error is $|R_4(x)| \le \dfrac{\left(\frac{\pi}{20}\right)^5}{5!} < 8 \times 10^{-7}$. Therefore, to five decimal places, $\cos 69^\circ \approx \frac{1}{2} - \frac{\sqrt{3}}{2}\left(\frac{\pi}{20}\right) - \frac{1}{4}\left(\frac{\pi}{20}\right)^2 + \frac{\sqrt{3}}{12}\left(\frac{\pi}{20}\right)^3 + \frac{1}{48}\left(\frac{\pi}{20}\right)^4 \approx 0.35837$.

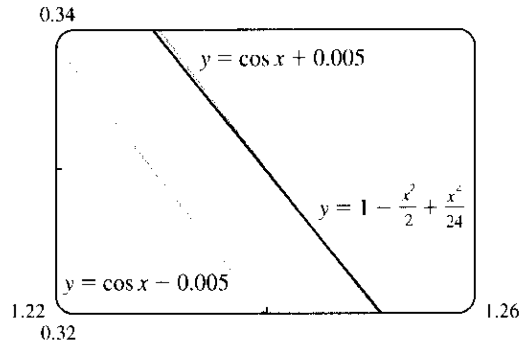
25. All derivatives of e^x are e^x, so $|R_n(x)| \le \dfrac{e^x}{(n+1)!}|x|^{n+1}$, where $0 < x < 0.1$. Letting $x = 0.1$, $R_n(0.1) \le \dfrac{e^{0.1}}{(n+1)!}(0.1)^{n+1} < 0.00001$, and by trial and error we find that $n = 3$ satisfies this inequality since $R_3(0.1) < 0.0000046$. Thus, by adding the four terms of the Maclaurin series for e^x corresponding to $n = 0, 1, 2$, and 3, we can estimate $e^{0.1}$ to within 0.00001. (In fact, this sum is $1.1051\overline{6}$ and $e^{0.1} \approx 1.10517$.)

26. Example 6 in Section 12.9 gives the Maclaurin series for $\ln(1-x)$ as $-\sum\limits_{n=1}^{\infty}\dfrac{x^n}{n}$ for $|x| < 1$. Thus, $\ln 1.4 = \ln[1 - (-0.4)] = -\sum\limits_{n=1}^{\infty}\dfrac{(-0.4)^n}{n} = \sum\limits_{n=1}^{\infty}(-1)^{n+1}\dfrac{(0.4)^n}{n}$. Since this is an alternating series, the error is less than the first neglected term by the Alternating Series Estimation Theorem, and we find that $|a_6| = (0.4)^6/6 \approx 0.0007 < 0.001$. So we need the first five (non-zero) terms of the Maclaurin series for the desired accuracy. (In fact, this sum is approximately 0.33698 and $\ln 1.4 \approx 0.33647$.)

27. $\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \cdots$. By the Alternating Series Estimation Theorem, the error in the approximation $\sin x = x - \frac{1}{3!}x^3$ is less than $\left|\frac{1}{5!}x^5\right| < 0.01 \Leftrightarrow \left|x^5\right| < 120(0.01) \Leftrightarrow |x| < (1.2)^{1/5} \approx 1.037$. The curves $y = x - \frac{1}{6}x^3$ and $y = \sin x - 0.01$ intersect at $x \approx 1.043$, so the graph confirms our estimate. Since both the sine function and the given approximation are odd functions, we need to check the estimate only for $x > 0$. Thus, the desired range of values for x is $-1.037 < x < 1.037$.



28. $\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots$. By the Alternating Series Estimation Theorem, the error is less than $\left|-\frac{1}{6!}x^6\right| < 0.005 \Leftrightarrow x^6 < 720(0.005) \Leftrightarrow |x| < (3.6)^{1/6} \approx 1.238$. The curves $y = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4$ and $y = \cos x + 0.005$ intersect at $x \approx 1.244$, so the graph confirms our estimate. Since both the cosine function and the given approximation are even functions, we need to check the estimate only for $x > 0$. Thus, the desired range of values for x is $-1.238 < x < 1.238$.



29. Let $s(t)$ be the position function of the car, and for convenience set $s(0) = 0$. The velocity of the car is $v(t) = s'(t)$ and the acceleration is $a(t) = s''(t)$, so the second degree Taylor polynomial is $T_2(t) = s(0) + v(0)t + \dfrac{a(0)}{2}t^2 = 20t + t^2$. We estimate the distance travelled during the next second to be $s(1) \approx T_2(1) = 20 + 1 = 21$ m. The function $T_2(t)$ would not be accurate over a full minute, since the car could not possibly maintain an acceleration of 2 m/s^2 for that long (if it did, its final speed would be 140 m/s ≈ 313 mi/h!)

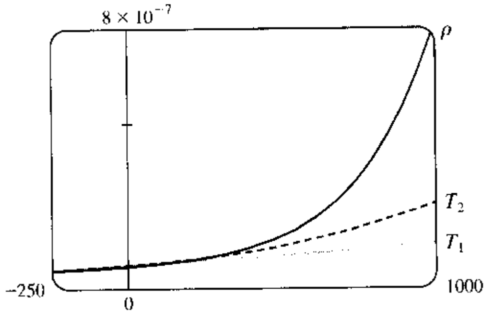
30. (a)

n	$\rho^{(n)}(t)$	$\rho^{(n)}(20)$
0	$\rho_{20}e^{\alpha(t-20)}$	ρ_{20}
1	$\alpha\rho_{20}e^{\alpha(t-20)}$	$\alpha\rho_{20}$
2	$\alpha^2\rho_{20}e^{\alpha(t-20)}$	$\alpha^2\rho_{20}$

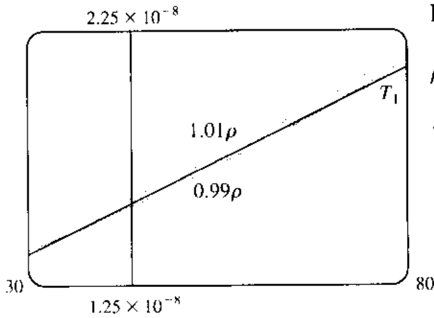
The linear approximation is $T_1(t) = \rho(20) + \rho'(20)(t-20) = \rho_{20}\left[1 + \alpha(t-20)\right]$. The quadratic approximation is

$$T_2(t) = \rho(20) + \rho'(20)(t-20) + \frac{\rho''(20)}{2}(t-20)^2 = \rho_{20}\left[1 + \alpha(t-20) + \tfrac{1}{2}\alpha^2(t-20)^2\right]$$

(b)



(c)



From the graph, it seems that $T_1(t)$ is within 1% of $\rho(t)$, that is, $0.99\rho(t) \leq T_1(t) \leq 1.01\rho(t)$, for $-14\ ^\circ\text{C} \leq t \leq 58\ ^\circ\text{C}$.

31. $E = \dfrac{q}{D^2} - \dfrac{q}{(D+d)^2} = \dfrac{q}{D^2} - \dfrac{q}{D^2(1+d/D)^2} = \dfrac{q}{D^2}\left[1 - \left(1 + \dfrac{d}{D}\right)^{-2}\right].$

We use the Binomial Series to expand $(1 + d/D)^{-2}$:

$$E = \frac{q}{D^2}\left[1 - \left(1 - 2\left(\frac{d}{D}\right) + \frac{2\cdot 3}{2!}\left(\frac{d}{D}\right)^2 - \frac{2\cdot 3\cdot 4}{3!}\left(\frac{d}{D}\right)^3 + \cdots\right)\right]$$

$$= \frac{q}{D^2}\left[2\left(\frac{d}{D}\right) - 3\left(\frac{d}{D}\right)^2 + 4\left(\frac{d}{D}\right)^3 - \cdots\right] \approx \frac{q}{D^2}\cdot 2\left(\frac{d}{D}\right) = 2qd\cdot\frac{1}{D^3}$$

when D is much larger than d; that is, when P is far away from the dipole.

32. (a) $\dfrac{n_1}{\ell_o} + \dfrac{n_2}{\ell_i} = \dfrac{1}{R}\left(\dfrac{n_2 s_i}{\ell_i} - \dfrac{n_1 s_o}{\ell_o}\right)$ (Equation 1) where

$$\ell_o = \sqrt{R^2 + (s_o+R)^2 - 2R(s_o+R)\cos\phi} \quad \text{and} \quad \ell_i = \sqrt{R^2 + (s_i-R)^2 + 2R(s_i-R)\cos\phi} \quad (2)$$

Using $\cos\phi \approx 1$ gives

$$\ell_o = \sqrt{R^2 + (s_o+R)^2 - 2R(s_o+R)} = \sqrt{R^2 + s_o^2 + 2Rs_o + R^2 - 2Rs_o - 2R^2} = \sqrt{s_o^2} = s_o$$

and similarly, $\ell_i = s_i$. Thus, Equation 1 becomes

$$\frac{n_1}{s_o} + \frac{n_2}{s_i} = \frac{1}{R}\left(\frac{n_2 s_i}{s_i} - \frac{n_1 s_o}{s_o}\right) \quad \Rightarrow \quad \frac{n_1}{s_o} + \frac{n_2}{s_i} = \frac{n_2 - n_1}{R}$$

(b) Using $\cos\phi \approx 1 - \frac{1}{2}\phi^2$ in (2) gives us

$$\ell_o = \sqrt{R^2 + (s_o + R)^2 - 2R(s_o + R)\left(1 - \tfrac{1}{2}\phi^2\right)}$$
$$= \sqrt{R^2 + s_o^2 + 2Rs_o + R^2 - 2Rs_o + Rs_o\phi^2 - 2R^2 + R^2\phi^2} = \sqrt{s_o^2 + Rs_o\phi^2 + R^2\phi^2}$$

Anticipating that we will use the binomial series expansion $(1+x)^k \approx 1 + kx$, we can write the last expression for ℓ_o as $s_o\sqrt{1 + \phi^2\left(\frac{R}{s_o} + \frac{R^2}{s_o^2}\right)}$ and similarly, $\ell_i = s_i\sqrt{1 - \phi^2\left(\frac{R}{s_i} - \frac{R^2}{s_i^2}\right)}$. Thus, from Equation 1,

$$\frac{n_1}{\ell_o} + \frac{n_2}{\ell_i} = \frac{1}{R}\left(\frac{n_2 s_i}{\ell_i} - \frac{n_1 s_o}{\ell_o}\right) \quad\Leftrightarrow\quad n_1\ell_o^{-1} + n_2\ell_i^{-1} = \frac{n_2}{R}\cdot\frac{s_i}{\ell_i} - \frac{n_1}{R}\cdot\frac{s_o}{\ell_o} \quad\Leftrightarrow$$

$$\frac{n_1}{s_o}\left[1 + \phi^2\left(\frac{R}{s_o} + \frac{R^2}{s_o^2}\right)\right]^{-1/2} + \frac{n_2}{s_i}\left[1 - \phi^2\left(\frac{R}{s_i} - \frac{R^2}{s_i^2}\right)\right]^{-1/2}$$
$$= \frac{n_2}{R}\left[1 - \phi^2\left(\frac{R}{s_i} - \frac{R^2}{s_i^2}\right)\right]^{-1/2} - \frac{n_1}{R}\left[1 + \phi^2\left(\frac{R}{s_o} + \frac{R^2}{s_o^2}\right)\right]^{-1/2}$$

Approximating the expressions for ℓ_o^{-1} and ℓ_i^{-1} by the first two terms in their binomial series, we get

$$\frac{n_1}{s_o}\left[1 - \tfrac{1}{2}\phi^2\left(\frac{R}{s_o} + \frac{R^2}{s_o^2}\right)\right] + \frac{n_2}{s_i}\left[1 + \tfrac{1}{2}\phi^2\left(\frac{R}{s_i} - \frac{R^2}{s_i^2}\right)\right]$$
$$= \frac{n_2}{R}\left[1 + \tfrac{1}{2}\phi^2\left(\frac{R}{s_i} - \frac{R^2}{s_i^2}\right)\right] - \frac{n_1}{R}\left[1 - \tfrac{1}{2}\phi^2\left(\frac{R}{s_o} + \frac{R^2}{s_o^2}\right)\right] \quad\Leftrightarrow$$

$$\frac{n_1}{s_o} - \frac{n_1\phi^2}{2s_o}\left(\frac{R}{s_o} + \frac{R^2}{s_o^2}\right) + \frac{n_2}{s_i} + \frac{n_2\phi^2}{2s_i}\left(\frac{R}{s_i} - \frac{R^2}{s_i^2}\right)$$
$$= \frac{n_2}{R} + \frac{n_2\phi^2}{2R}\left(\frac{R}{s_i} - \frac{R^2}{s_i^2}\right) - \frac{n_1}{R} + \frac{n_1\phi^2}{2R}\left(\frac{R}{s_o} + \frac{R^2}{s_o^2}\right) \quad\Leftrightarrow$$

$$\frac{n_1}{s_o} + \frac{n_2}{s_i} = \frac{n_2}{R} - \frac{n_1}{R} + \frac{n_1\phi^2}{2s_o}\left(\frac{R}{s_o} + \frac{R^2}{s_o^2}\right) + \frac{n_1\phi^2}{2R}\left(\frac{R}{s_o} + \frac{R^2}{s_o^2}\right) + \frac{n_2\phi^2}{2R}\left(\frac{R}{s_i} - \frac{R^2}{s_i^2}\right) - \frac{n_2\phi^2}{2s_i}\left(\frac{R}{s_i} - \frac{R^2}{s_i^2}\right)$$
$$= \frac{n_2 - n_1}{R} + \frac{n_1\phi^2}{2}\left(\frac{R}{s_o} + \frac{R^2}{s_o^2}\right)\left(\frac{1}{s_o} + \frac{1}{R}\right) + \frac{n_2\phi^2}{2}\left(\frac{R}{s_i} - \frac{R^2}{s_i^2}\right)\left(\frac{1}{R} - \frac{1}{s_i}\right)$$
$$= \frac{n_2 - n_1}{R} + \frac{n_1\phi^2 R^2}{2s_o}\left(\frac{1}{R} + \frac{1}{s_o}\right)\left(\frac{1}{R} + \frac{1}{s_o}\right) + \frac{n_2\phi^2 R^2}{2s_i}\left(\frac{1}{R} - \frac{1}{s_i}\right)\left(\frac{1}{R} - \frac{1}{s_i}\right)$$
$$= \frac{n_2 - n_1}{R} + \phi^2 R^2\left[\frac{n_1}{2s_o}\left(\frac{1}{R} + \frac{1}{s_o}\right)^2 + \frac{n_2}{2s_i}\left(\frac{1}{R} - \frac{1}{s_i}\right)^2\right]$$

From Figure 8, we see that $\sin\phi = h/R$. So if we approximate $\sin\phi$ with ϕ, we get $h = R\phi$ and $h^2 = \phi^2 R^2$ and hence, Equation 4, as desired.

33. (a) If the water is deep, then $2\pi d/L$ is large, and we know that $\tanh x \to 1$ as $x \to \infty$. So we can approximate $\tanh(2\pi d/L) \approx 1$, and so $v^2 \approx gL/(2\pi) \quad \Leftrightarrow \quad v \approx \sqrt{gL/(2\pi)}$.

(b) From the table, the first term in the Maclaurin series of $\tanh x$ is x, so if the water is shallow, we can approximate $\tanh \dfrac{2\pi d}{L} \approx \dfrac{2\pi d}{L}$, and so $v^2 \approx \dfrac{gL}{2\pi} \cdot \dfrac{2\pi d}{L} \quad \Leftrightarrow \quad v \approx \sqrt{gd}$.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\tanh x$	0
1	$\operatorname{sech}^2 x$	1
2	$-2\operatorname{sech}^2 x \tanh x$	0
3	$2\operatorname{sech}^2 x\,(3\tanh^2 x - 1)$	-2

(c) Since $\tanh x$ is an odd function, its Maclaurin series is alternating, so the error in the approximation

$\tanh \dfrac{2\pi d}{L} \approx \dfrac{2\pi d}{L}$ is less than the first neglected term, which is $\dfrac{|f'''(0)|}{3!}\left(\dfrac{2\pi d}{L}\right)^3 = \dfrac{1}{3}\left(\dfrac{2\pi d}{L}\right)^3$.

If $L > 10d$, then $\dfrac{1}{3}\left(\dfrac{2\pi d}{L}\right)^3 < \dfrac{1}{3}\left(2\pi \cdot \dfrac{1}{10}\right)^3 = \dfrac{\pi^3}{375}$, so the error in the approximation $v^2 = gd$ is less

than $\dfrac{gL}{2\pi} \cdot \dfrac{\pi^3}{375} \approx 0.0132gL$.

34. (a)

$$4\sqrt{\frac{L}{g}}\int_0^{\pi/2}\frac{dx}{\sqrt{1-k^2\sin^2 x}} = 4\sqrt{\frac{L}{g}}\int_0^{\pi/2}\left[1+(-k^2\sin^2 x)\right]^{-1/2}dx$$

$$= 4\sqrt{\frac{L}{g}}\int_0^{\pi/2}\left[1-\frac{1}{2}(-k^2\sin^2 x)+\frac{\frac{1}{2}\cdot\frac{3}{2}}{2!}(-k^2\sin^2 x)^2-\frac{\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}}{3!}(-k^2\sin^2 x)^3+\cdots\right]dx$$

$$= 4\sqrt{\frac{L}{g}}\int_0^{\pi/2}\left[1+\left(\frac{1}{2}\right)k^2\sin^2 x+\left(\frac{1\cdot 3}{2\cdot 4}\right)k^4\sin^4 x+\left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)k^6\sin^6 x+\cdots\right]dx$$

$$= 4\sqrt{\frac{L}{g}}\left[\frac{\pi}{2}+\left(\frac{1}{2}\right)\left(\frac{1}{2}\cdot\frac{\pi}{2}\right)k^2+\left(\frac{1\cdot 3}{2\cdot 4}\right)\left(\frac{1\cdot 3}{2\cdot 4}\cdot\frac{\pi}{2}\right)k^4\right.$$
$$\left.+\left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\right)\left(\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot\frac{\pi}{2}\right)k^6+\cdots\right]$$

[split up the integral and use the result from Exercise 8.1.44]

$$= 2\pi\sqrt{\frac{L}{g}}\left[1+\frac{1^2}{2^2}k^2+\frac{1^2\cdot 3^2}{2^2\cdot 4^2}k^4+\frac{1^2\cdot 3^2\cdot 5^2}{2^2\cdot 4^2\cdot 6^2}k^6+\cdots\right]$$

(b) The first of the two inequalities is true because all of the terms in the series are positive. For the second,

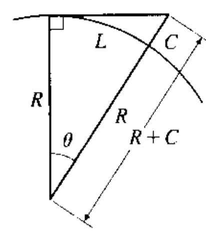
$$T = 2\pi\sqrt{\frac{L}{g}}\left[1+\frac{1^2}{2^2}k^2+\frac{1^2\cdot 3^2}{2^2\cdot 4^2}k^4+\frac{1^2\cdot 3^2\cdot 5^2}{2^2\cdot 4^2\cdot 6^2}k^6+\frac{1^2\cdot 3^2\cdot 5^2\cdot 7^2}{2^2\cdot 4^2\cdot 6^2\cdot 8^2}k^8+\cdots\right]$$
$$\le 2\pi\sqrt{\frac{L}{g}}\left[1+\frac{1}{4}k^2+\frac{1}{4}k^4+\frac{1}{4}k^6+\frac{1}{4}k^8+\cdots\right]$$

The terms in brackets (after the first) form a geometric series with $a = \frac{1}{4}k^2$ and $r = k^2 = \sin^2\left(\frac{1}{2}\theta_0\right) < 1$.

So $T \le 2\pi\sqrt{\dfrac{L}{g}}\left[1+\dfrac{k^2/4}{1-k^2}\right] = 2\pi\sqrt{\dfrac{L}{g}}\,\dfrac{4-3k^2}{4-4k^2}$.

(c) We substitute $L = 1$, $g = 9.8$, and $k = \sin(10°/2) \approx 0.08716$, and the inequality from part (b) becomes $2.01090 \leq T \leq 2.01093$, so $T \approx 2.0109$. The estimate $T \approx 2\pi\sqrt{L/g} \approx 2.0071$ differs by about 0.2%. If $\theta_0 = 42°$, then $k \approx 0.35837$ and the inequality becomes $2.07153 \leq T \leq 2.08103$, so $T \approx 2.0763$. The one-term estimate is the same, and the discrepancy between the two estimates increases to about 3.4%.

35. (a) L is the length of the arc subtended by the angle θ, so $L = R\theta \;\Rightarrow$ $\theta = L/R$. Now $\sec\theta = (R+C)/R \;\Rightarrow\; R\sec\theta = R + C \;\Rightarrow$ $C = R\sec\theta - R = R\sec(L/R) - R$.



(b) From Exercise 11, $\sec x \approx T_4(x) = 1 + \frac{1}{2}x^2 + \frac{5}{24}x^4$. By part (a),

$$C \approx R\left[1 + \frac{1}{2}\left(\frac{L}{R}\right)^2 + \frac{5}{24}\left(\frac{L}{R}\right)^4\right] - R = R + \frac{1}{2}R\cdot\frac{L^2}{R^2} + \frac{5}{24}R\cdot\frac{L^4}{R^4} - R = \frac{L^2}{2R} + \frac{5L^4}{24R^3}.$$

(c) Taking $L = 100$ km and $R = 6370$ km, the formula in part (a) says that

$$C = R\sec(L/R) - R = 6370\sec(100/6370) - 6370 \approx 0.785\,009\,965\,44 \text{ km}$$

The formula in part (b) says that

$$C \approx \frac{L^2}{2R} + \frac{5L^4}{24R^3} = \frac{100^2}{2\cdot 6370} + \frac{5\cdot 100^4}{24\cdot 6370^3} \approx 0.785\,009\,957\,36 \text{ km}.$$

The difference between these two results is only 0.000 000 008 08 km, or 0.000 008 08 m!

36. $T_n(x) = f(a) + \dfrac{f'(a)}{1!}(x-a) + \dfrac{f''(a)}{2!}(x-a)^2 + \cdots + \dfrac{f^{(n)}(a)}{n!}(x-a)^n$. Let $0 \leq m \leq n$. Then

$$T_n^{(m)}(x) = m!\frac{f^{(m)}(a)}{m!}(x-a)^0 + (m+1)(m)\cdots(2)\frac{f^{(m+1)}(a)}{(m+1)!}(x-a)^1 + \cdots$$
$$+ n(n-1)\cdots(n-m+1)\frac{f^{(n)}(a)}{n!}(x-a)^{n-m}$$

For $x = a$, all terms in this sum except the first one are 0, so $T_n^{(m)}(a) = \dfrac{m!\,f^{(m)}(a)}{m!} = f^{(m)}(a)$.

37. Using $f(x) = T_n(x) + R_n(x)$ with $n = 1$ and $x = r$, we have $f(r) = T_1(r) + R_1(r)$, where T_1 is the first-degree Taylor polynomial of f at a. Because $a = x_n$, $f(r) = f(x_n) + f'(x_n)(r - x_n) + R_1(r)$. But r is a root of f, so $f(r) = 0$ and we have $0 = f(x_n) + f'(x_n)(r - x_n) + R_1(r)$. Taking the first two terms to the left side gives us $f'(x_n)(x_n - r) - f(x_n) = R_1(r)$. Dividing by $f'(x_n)$, we get $x_n - r - \dfrac{f(x_n)}{f'(x_n)} = \dfrac{R_1(r)}{f'(x_n)}$. By the formula for Newton's method, the left side of the preceding equation is $x_{n+1} - r$, so $|x_{n+1} - r| = \left|\dfrac{R_1(r)}{f'(x_n)}\right|$. Taylor's Inequality gives us $|R_1(r)| \leq \dfrac{|f''(r)|}{2!}|r - x_n|^2$. Combining this inequality with the facts $|f''(x)| \leq M$ and $|f'(x)| \geq K$ gives us $|x_{n+1} - r| \leq \dfrac{M}{2K}|x_n - r|^2$.

APPLIED PROJECT Radiation from the Stars

1. If we write $f(\lambda) = \dfrac{8\pi hc\lambda^{-5}}{e^{hc/(\lambda kT)} - 1} = \dfrac{a\lambda^{-5}}{e^{b/(\lambda T)} - 1}$, then as $\lambda \to 0^+$, it is of the form ∞/∞, and as $\lambda \to \infty$ it is of the form $0/0$, so in either case we can use l'Hospital's Rule. First of all,

$$\lim_{\lambda\to\infty} f(\lambda) \overset{\text{H}}{=} \lim_{\lambda\to\infty} \frac{a\left(-5\lambda^{-6}\right)}{-\dfrac{bT}{(\lambda T)^2} e^{b/(\lambda T)}} = 5\frac{aT}{b} \lim_{\lambda\to\infty} \frac{\lambda^2\lambda^{-6}}{e^{b/(\lambda T)}} = 5\frac{aT}{b} \lim_{\lambda\to\infty} \frac{\lambda^{-4}}{e^{b/(\lambda T)}} = 0$$

Also,

$$\lim_{\lambda\to0^+} f(\lambda) \overset{\text{H}}{=} 5\frac{aT}{b} \lim_{\lambda\to0^+} \frac{\lambda^{-4}}{e^{b/(\lambda T)}} \overset{\text{H}}{=} 5\frac{aT}{b} \lim_{\lambda\to0^+} \frac{-4\lambda^{-5}}{-\dfrac{bT}{(\lambda T)^2} e^{b/(\lambda T)}} = 20\frac{aT^2}{b^2} \lim_{\lambda\to0^+} \frac{\lambda^{-3}}{e^{b/(\lambda T)}}$$

This is still indeterminate, but note that each time we use l'Hospital's Rule, we gain a factor of λ in the numerator, as well as a constant factor, and the denominator is unchanged. So if we use l'Hospital's Rule three more times, the exponent of λ in the numerator will become 0. That is, for some $\{k_i\}$, all constant,

$$\lim_{\lambda\to0^+} f(\lambda) \overset{\text{H}}{=} k_1 \lim_{\lambda\to0^+} \frac{\lambda^{-3}}{e^{b/(\lambda T)}} \overset{\text{H}}{=} k_2 \lim_{\lambda\to0^+} \frac{\lambda^{-2}}{e^{b/(\lambda T)}} \overset{\text{H}}{=} k_3 \lim_{\lambda\to0^+} \frac{\lambda^{-1}}{e^{b/(\lambda T)}} \overset{\text{H}}{=} k_4 \lim_{\lambda\to0^+} \frac{1}{e^{b/(\lambda T)}} = 0$$

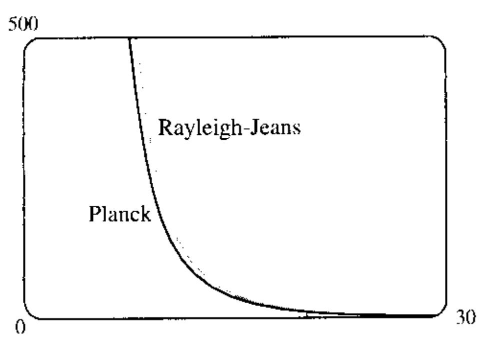
2. We expand the denominator of Planck's Law using the Taylor series $e^x = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \cdots$ with $x = \dfrac{hc}{\lambda kT}$, and use the fact that if λ is large, then all subsequent terms in the Taylor expansion are very small compared to the first one, so we can approximate using the Taylor polynomial T_1:

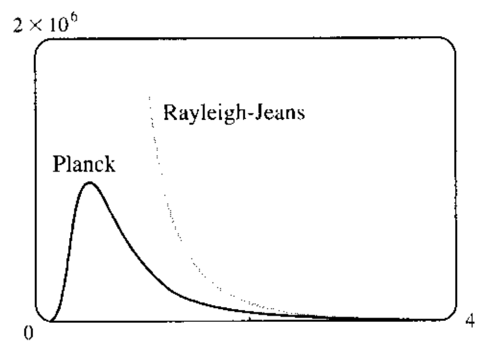
$$f(\lambda) = \frac{8\pi hc\lambda^{-5}}{e^{hc/(\lambda kT)} - 1} = \frac{8\pi hc\lambda^{-5}}{\left[1 + \dfrac{hc}{\lambda kT} + \dfrac{1}{2!}\left(\dfrac{hc}{\lambda kT}\right)^2 + \dfrac{1}{3!}\left(\dfrac{hc}{\lambda kT}\right)^3 + \cdots\right] - 1}$$

$$\approx \frac{8\pi hc\lambda^{-5}}{\left(1 + \dfrac{hc}{\lambda kT}\right) - 1} = \frac{8\pi kT}{\lambda^4}$$

which is the Rayleigh-Jeans Law.

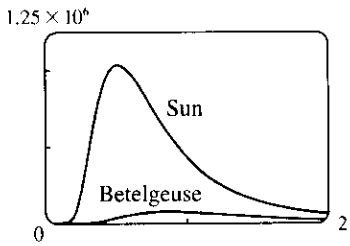
3. To convert to μm, we substitute $\lambda/10^6$ for λ in both laws. The first figure shows that the two laws are similar for large λ. The second figure shows that the two laws are very different for short wavelengths (Planck's Law gives a maximum at $\lambda \approx 0.51\ \mu$m; the Rayleigh-Jeans Law gives no minimum or maximum.).

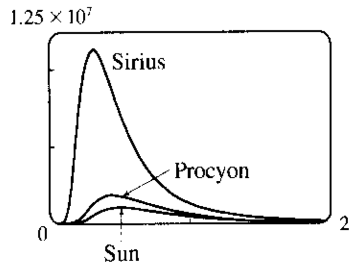




4. From the graph in Problem 3, $f(\lambda)$ has a maximum under Planck's Law at $\lambda \approx 0.51\ \mu$m.

5.





As T gets larger, the total area under the curve increases, as we would expect: the hotter the star, the more energy it emits. Also, as T increases, the λ-value of the maximum decreases, so the higher the temperature, the shorter the peak wavelength (and consequently the average wavelength) of light emitted. This is why Sirius is a blue star and Betelgeuse is a red star: most of Sirius's light is of a fairly short wavelength; that is, a higher frequency, toward the blue end of the spectrum, whereas most of Betelgeuse's light is of a lower frequency, toward the red end of the spectrum.

12 Review

CONCEPT CHECK

1. (a) See Definition 12.1.1.

(b) See Definition 12.2.2.

(c) The terms of the sequence $\{a_n\}$ approach 3 as n becomes large.

(d) By adding sufficiently many terms of the series, we can make the partial sums as close to 3 as we like.

2. (a) See Definition 12.1.10.

(b) A sequence is monotonic if it is either increasing or decreasing.

(c) By Theorem 12.1.11, every bounded, monotonic sequence is convergent.

3. (a) See (4) in Section 12.2.

(b) The p-series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$.

4. If $\sum a_n = 3$, then $\lim\limits_{n\to\infty} a_n = 0$ and $\lim\limits_{n\to\infty} s_n = 3$.

5. (a) See the Test for Divergence on page 754.

(b) See the Integral Test on page 760.

(c) See the Comparison Test on page 767.

(d) See the Limit Comparison Test on page 768.

(e) See the Alternating Series Test on page 772.

(f) See the Ratio Test on page 778.

(g) See the Root Test on page 780.

6. (a) A series $\sum a_n$ is called *absolutely convergent* if the series of absolute values $\sum |a_n|$ is convergent.

(b) If a series $\sum a_n$ is absolutely convergent, then it is convergent.

(c) A series $\sum a_n$ is called *conditionally convergent* if it is convergent but not absolutely convergent.

7. (a) Use (3) in Section 12.3.

(b) See Example 5 in Section 12.4.

(c) By adding terms until you reach the desired accuracy given by the Alternating Series Estimation Theorem on page 774.

8. (a) $\sum_{n=0}^{\infty} c_n(x-a)^n$

(b) Given the power series $\sum_{n=0}^{\infty} c_n(x-a)^n$, the radius of convergence is:

(i) 0 if the series converges only when $x = a$

(ii) ∞ if the series converges for all x, or

(iii) a positive number R such that the series converges if $|x-a| < R$ and diverges if $|x-a| > R$.

(c) The interval of convergence of a power series is the interval that consists of all values of x for which the series converges. Corresponding to the cases in part (b), the interval of convergence is: (i) the single point $\{a\}$, (ii) all real numbers, that is, the real number line $(-\infty, \infty)$, or (iii) an interval with endpoints $a - R$ and $a + R$ which can contain neither, either, or both of the endpoints. In this case, we must test the series for convergence at each endpoint to determine the interval of convergence.

9. (a), (b) See Theorem 12.9.2.

10. (a) $T_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i$

(b) $\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$

(c) $\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n$ [$a = 0$ in part (b)]

(d) See Theorem 12.10.8.

(e) See Taylor's Inequality (12.10.9).

11. (a) – (e) See the table on page 803.

12. See the Binomial Series (12.11.2) for the expansion. The radius of convergence for the binomial series is 1.

TRUE-FALSE QUIZ

1. False. See Note 2 after Theorem 12.2.6.

2. False. The series $\sum_{n=1}^{\infty} n^{-\sin 1} = \sum_{n=1}^{\infty} \frac{1}{n^{\sin 1}}$ is a p-series with $p = \sin 1 \approx 0.84 \leq 1$, so the series diverges.

3. True. If $\lim_{n\to\infty} a_n = L$, then given any $\varepsilon > 0$, we can find a positive integer N such that $|a_n - L| < \varepsilon$ whenever $n > N$. If $n > N$, then $2n+1 > N$ and $|a_{2n+1} - L| < \varepsilon$. Thus, $\lim_{n\to\infty} a_{2n+1} = L$.

4. True by Theorem 12.8.3.
Or: Use the Comparison Test to show that $\sum c_n(-2)^n$ converges absolutely.

5. False. For example, take $c_n = (-1)^n / (n6^n)$.

6. True by Theorem 12.8.3.

7. False, since $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\left|\dfrac{1}{(n+1)^3}\cdot\dfrac{n^3}{1}\right| = \lim\limits_{n\to\infty}\left|\dfrac{n^3}{(n+1)^3}\cdot\dfrac{1/n^3}{1/n^3}\right| = \lim\limits_{n\to\infty}\dfrac{1}{(1+1/n)^3} = 1.$

8. True, since $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\left|\dfrac{1}{(n+1)!}\cdot\dfrac{n!}{1}\right| = \lim\limits_{n\to\infty}\dfrac{1}{n+1} = 0 < 1.$

9. False. See the note after Example 2 in Section 12.4.

10. True, since $\dfrac{1}{e} = e^{-1}$ and $e^x = \sum\limits_{n=0}^{\infty}\dfrac{x^n}{n!}$, so $e^{-1} = \sum\limits_{n=0}^{\infty}\dfrac{(-1)^n}{n!}$.

11. True. See (8) in Section 12.1.

12. True, because if $\sum|a_n|$ is convergent, then so is $\sum a_n$ by Theorem 12.6.3.

13. True. By Theorem 12.10.5 the coefficient of x^3 is $\dfrac{f'''(0)}{3!} = \dfrac{1}{3} \;\Rightarrow\; f'''(0) = 2.$

Or: Use Theorem 12.9.2 to differentiate f three times.

14. False. Let $a_n = n$ and $b_n = -n$. Then $\{a_n\}$ and $\{b_n\}$ are divergent, but $a_n + b_n = 0$, so $\{a_n + b_n\}$ is convergent.

15. False. For example, let $a_n = b_n = (-1)^n$. Then $\{a_n\}$ and $\{b_n\}$ are divergent, but $a_n b_n = 1$, so $\{a_n b_n\}$ is convergent.

16. True by the Monotonic Sequence Theorem, since $\{a_n\}$ is decreasing and $0 < a_n \le a_1$ for all $n \;\Rightarrow\; \{a_n\}$ is bounded.

17. True by Theorem 12.6.3. [$\sum(-1)^n a_n$ is absolutely convergent and hence convergent.]

18. True. $\lim\limits_{n\to\infty}\dfrac{a_{n+1}}{a_n} < 1 \;\Rightarrow\; \sum a_n$ converges (Ratio Test) $\;\Rightarrow\; \lim\limits_{n\to\infty} a_n = 0$ [Theorem 12.2.6].

EXERCISES

1. $\left\{\dfrac{2+n^3}{1+2n^3}\right\}$ converges since $\lim\limits_{n\to\infty}\dfrac{2+n^3}{1+2n^3} = \lim\limits_{n\to\infty}\dfrac{2/n^3+1}{1/n^3+2} = \dfrac{1}{2}.$

2. $a_n = \dfrac{9^{n+1}}{10^n} = 9\cdot\left(\frac{9}{10}\right)^n$, so $\lim\limits_{n\to\infty} a_n = 9\lim\limits_{n\to\infty}\left(\frac{9}{10}\right)^n = 9\cdot 0 = 0$ by (12.1.8).

3. $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty}\dfrac{n^3}{1+n^2} = \lim\limits_{n\to\infty}\dfrac{n}{1/n^2+1} = \infty$, so the sequence diverges.

4. $a_n = \cos(n\pi/2)$, so $a_n = 0$ if n is odd and $a_n = \pm 1$ if n is even. As n increases, a_n keeps cycling through the values $0, 1, 0, -1$, so the sequence $\{a_n\}$ is divergent.

5. $|a_n| = \left|\dfrac{n\sin n}{n^2+1}\right| \le \dfrac{n}{n^2+1} < \dfrac{1}{n}$, so $|a_n| \to 0$ as $n \to \infty$. Thus, $\lim\limits_{n\to\infty} a_n = 0$. The sequence $\{a_n\}$ is convergent.

6. $a_n = \dfrac{\ln n}{\sqrt{n}}$. Let $f(x) = \dfrac{\ln x}{\sqrt{x}}$ for $x > 0$. Then $\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to\infty}\dfrac{\ln x}{\sqrt{x}} \overset{\text{H}}{=} \lim\limits_{x\to\infty}\dfrac{1/x}{1/(2\sqrt{x})} = \lim\limits_{x\to\infty}\dfrac{2}{\sqrt{x}} = 0.$

Thus, by Theorem 3 in Section 12.1, $\{a_n\}$ converges and $\lim\limits_{n\to\infty} a_n = 0$.

7. $\left\{\left(1+\frac{3}{n}\right)^{4n}\right\}$ is convergent. Let $y=\left(1+\frac{3}{x}\right)^{4x}$. Then

$$\lim_{x\to\infty}\ln y=\lim_{x\to\infty}4x\ln(1+3/x)=\lim_{x\to\infty}\frac{\ln(1+3/x)}{1/(4x)}\overset{\text{H}}{=}\lim_{x\to\infty}\frac{\frac{1}{1+3/x}\left(-\frac{3}{x^2}\right)}{-1/(4x^2)}=\lim_{x\to\infty}\frac{12}{1+3/x}=12$$

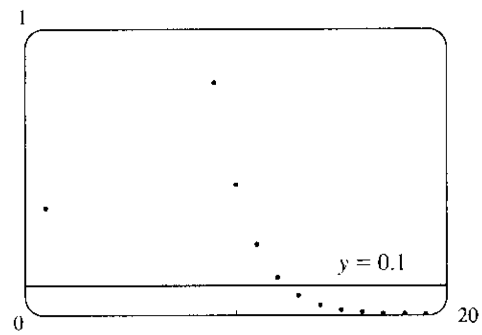
so $\lim\limits_{x\to\infty}y=\lim\limits_{n\to\infty}\left(1+\frac{3}{n}\right)^{4n}=e^{12}$.

Or: Use Exercise 7.7.54.

8. $\left\{\frac{(-10)^n}{n!}\right\}$ converges, since $\frac{10^n}{n!}=\frac{10\cdot10\cdot10\cdot\dots\cdot10}{1\cdot2\cdot3\cdot\dots\cdot10}\cdot\frac{10\cdot10\cdot\dots\cdot10}{11\cdot12\cdot\dots\cdot n}\leq10^{10}\left(\frac{10}{11}\right)^{n-10}\to0$ as $n\to\infty$, so $\lim\limits_{n\to\infty}\frac{(-10)^n}{n!}=0$ (Squeeze Theorem). *Or:* Use (12.10.10).

9. We use induction, hypothesizing that $a_{n-1}<a_n<2$. Note first that $1<a_2=\frac{1}{3}(1+4)=\frac{5}{3}<2$, so the hypothesis holds for $n=2$. Now assume that $a_{k-1}<a_k<2$. Then $a_k=\frac{1}{3}(a_{k-1}+4)<\frac{1}{3}(a_k+4)<\frac{1}{3}(2+4)=2$. So $a_k<a_{k+1}<2$, and the induction is complete. To find the limit of the sequence, we note that $L=\lim\limits_{n\to\infty}a_n=\lim\limits_{n\to\infty}a_{n+1}\Rightarrow L=\frac{1}{3}(L+4)\Rightarrow L=2$.

10. $\lim\limits_{x\to\infty}\frac{x^4}{e^x}\overset{\text{H}}{=}\lim\limits_{x\to\infty}\frac{4x^3}{e^x}\overset{\text{H}}{=}\lim\limits_{x\to\infty}\frac{12x^2}{e^x}\overset{\text{H}}{=}\lim\limits_{x\to\infty}\frac{24x}{e^x}\overset{\text{H}}{=}\lim\limits_{x\to\infty}\frac{24}{e^x}=0$



Then we conclude from Theorem 12.1.3 that $\lim\limits_{n\to\infty}n^4e^{-n}=0$.

From the graph, it seems that $12^4e^{-12}>0.1$, but $n^4e^{-n}<0.1$ whenever $n>12$. So the smallest value of N corresponding to $\varepsilon=0.1$ in the definition of the limit is $N=12$.

11. $\frac{n}{n^3+1}<\frac{n}{n^3}=\frac{1}{n^2}$, so $\sum\limits_{n=1}^{\infty}\frac{n}{n^3+1}$ converges by the Comparison Test with the convergent p-series $\sum\limits_{n=1}^{\infty}\frac{1}{n^2}$ $(p=2>1)$.

12. Let $a_n=\frac{n^2+1}{n^3+1}$ and $b_n=\frac{1}{n}$, so $\lim\limits_{n\to\infty}\frac{a_n}{b_n}=\lim\limits_{n\to\infty}\frac{n^3+n}{n^3+1}=\lim\limits_{n\to\infty}\frac{1+1/n^2}{1+1/n^3}=1>0$. Since $\sum_{n=1}^{\infty}b_n$ is the divergent harmonic series, $\sum_{n=1}^{\infty}a_n$ also diverges by the Limit Comparison Test.

13. $\lim\limits_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|=\lim\limits_{n\to\infty}\left[\frac{(n+1)^3}{5^{n+1}}\cdot\frac{5^n}{n^3}\right]=\lim\limits_{n\to\infty}\left(1+\frac{1}{n}\right)^3\cdot\frac{1}{5}=\frac{1}{5}<1$, so $\sum\limits_{n=1}^{\infty}\frac{n^3}{5^n}$ converges by the Ratio Test.

14. Let $b_n=\frac{1}{\sqrt{n+1}}$. Then b_n is positive for $n\geq1$, the sequence $\{b_n\}$ is decreasing, and $\lim\limits_{n\to\infty}b_n=0$, so the series $\sum\limits_{n=1}^{\infty}\frac{(-1)^n}{\sqrt{n+1}}$ converges by the Alternating Series Test.

15. Let $f(x) = \dfrac{1}{x\sqrt{\ln x}}$. Then f is continuous, positive, and decreasing on $[2, \infty)$, so the Integral Test applies.

$$\int_2^\infty f(x)\,dx = \lim_{t\to\infty} \int_2^t \frac{1}{x\sqrt{\ln x}}\,dx \quad \begin{bmatrix} u = \ln x, \\ du = \frac{1}{x}\,dx \end{bmatrix} \quad = \lim_{t\to\infty} \int_{\ln 2}^{\ln t} u^{-1/2}du = \lim_{t\to\infty} \left[2\sqrt{u}\right]_{\ln 2}^{\ln t}$$

$= \lim\limits_{t\to\infty} (2\sqrt{\ln t} - 2\sqrt{\ln 2}) = \infty$, so the series $\displaystyle\sum_{n=2}^\infty \frac{1}{n\sqrt{\ln n}}$ diverges.

16. $\lim\limits_{n\to\infty} \dfrac{n}{3n+1} = \dfrac{1}{3}$, so $\lim\limits_{n\to\infty} \ln\left(\dfrac{n}{3n+1}\right) = \ln \frac{1}{3} \neq 0$. Thus, the series $\sum\limits_{n=1}^\infty \ln\left(\dfrac{n}{3n+1}\right)$ diverges by the Test for Divergence.

17. $|a_n| = \left|\dfrac{\cos 3n}{1+(1.2)^n}\right| \le \dfrac{1}{1+(1.2)^n} < \dfrac{1}{(1.2)^n} = \left(\dfrac{5}{6}\right)^n$, so $\displaystyle\sum_{n=1}^\infty |a_n|$ converges by comparison with the convergent geometric series $\displaystyle\sum_{n=1}^\infty \left(\frac{5}{6}\right)^n$ $\left(r = \frac{5}{6} < 1\right)$. It follows that $\displaystyle\sum_{n=1}^\infty a_n$ converges (by Theorem 3 in Section 12.6).

18. $\lim\limits_{n\to\infty} \sqrt[n]{|a_n|} = \lim\limits_{n\to\infty} \sqrt[n]{\left|\dfrac{n^{2n}}{(1+2n^2)^n}\right|} = \lim\limits_{n\to\infty} \dfrac{n^2}{1+2n^2} = \lim\limits_{n\to\infty} \dfrac{1}{1/n^2+2} = \dfrac{1}{2} < 1$, so $\displaystyle\sum_{n=1}^\infty \frac{n^{2n}}{(1+2n^2)^n}$ converges by the Root Test.

19. $\lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty} \dfrac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)(2n+1)}{5^{n+1}\,(n+1)!} \cdot \dfrac{5^n n!}{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)} = \lim\limits_{n\to\infty} \dfrac{2n+1}{5(n+1)} = \dfrac{2}{5} < 1$, so the series converges by the Ratio Test.

20. $\displaystyle\sum_{n=1}^\infty \frac{(-5)^{2n}}{n^2 9^n} = \sum_{n=1}^\infty \frac{1}{n^2}\left(\frac{25}{9}\right)^n$. Now

$\lim\limits_{n\to\infty} \left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty} \dfrac{25^{n+1}}{(n+1)^2\cdot 9^{n+1}} \cdot \dfrac{n^2\cdot 9^n}{25^n} = \lim\limits_{n\to\infty} \dfrac{25n^2}{9(n+1)^2} = \dfrac{25}{9} > 1$, so the series diverges by the Ratio Test.

21. $b_n = \dfrac{\sqrt{n}}{n+1} > 0$, $\{b_n\}$ is decreasing, and $\lim\limits_{n\to\infty} b_n = 0$, so the series $\displaystyle\sum_{n=1}^\infty (-1)^{n-1}\frac{\sqrt{n}}{n+1}$ converges by the Alternating Series Test.

22. Use the Limit Comparison Test with $a_n = \dfrac{\sqrt{n+1}-\sqrt{n-1}}{n} = \dfrac{2}{n\left(\sqrt{n+1}+\sqrt{n-1}\right)}$ (rationalizing the numerator) and $b_n = \dfrac{1}{n^{3/2}}$. $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \lim\limits_{n\to\infty} \dfrac{2\sqrt{n}}{\sqrt{n+1}+\sqrt{n-1}} = 1$, so since $\sum_{n=1}^\infty b_n$ converges $\left(p = \frac{3}{2} > 1\right)$, $\sum_{n=1}^\infty a_n$ converges also.

23. Consider the series of absolute values: $\sum_{n=1}^\infty n^{-1/3}$ is a p-series with $p = \frac{1}{3} \le 1$ and is therefore divergent. But if we apply the Alternating Series Test, we see that $b_n = \dfrac{1}{\sqrt[3]{n}} > 0$, $\{b_n\}$ is decreasing, and $\lim\limits_{n\to\infty} b_n = 0$, so the series $\displaystyle\sum_{n=1}^\infty (-1)^{n-1}n^{-1/3}$ converges. Thus, $\displaystyle\sum_{n=1}^\infty (-1)^{n-1}n^{-1/3}$ is conditionally convergent.

24. $\sum_{n=1}^\infty \left|(-1)^{n-1}n^{-3}\right| = \sum_{n=1}^\infty n^{-3}$ is a convergent p-series ($p = 3 > 1$.) Therefore, $\sum_{n=1}^\infty (-1)^{n-1}n^{-3}$ is absolutely convergent.

25. $\left|\dfrac{a_{n+1}}{a_n}\right| = \left|\dfrac{(-1)^{n+1}(n+2)\,3^{n+1}}{2^{2n+3}} \cdot \dfrac{2^{2n+1}}{(-1)^n (n+1)\,3^n}\right| = \dfrac{n+2}{n+1}\cdot\dfrac{3}{4} = \dfrac{1+(2/n)}{1+(1/n)}\cdot\dfrac{3}{4} \to \dfrac{3}{4} < 1$ as $n\to\infty$, so by the Ratio Test, $\displaystyle\sum_{n=1}^{\infty}\frac{(-1)^n(n+1)\,3^n}{2^{2n+1}}$ is absolutely convergent.

26. $\displaystyle\lim_{x\to\infty}\frac{\sqrt{x}}{\ln x} \overset{H}{=} \lim_{x\to\infty}\frac{1/(2\sqrt{x})}{1/x} = \lim_{x\to\infty}\frac{\sqrt{x}}{2} = \infty$. Therefore, $\displaystyle\lim_{n\to\infty}\frac{(-1)^n\sqrt{n}}{\ln n} \neq 0$, so the given series is divergent by the Test for Divergence.

27. $\dfrac{2^{2n+1}}{5^n} = \dfrac{2^{2n}\cdot 2^1}{5^n} = \dfrac{(2^2)^n\cdot 2}{5^n} = 2\left(\dfrac{4}{5}\right)^n$, so $\displaystyle\sum_{n=1}^{\infty}\frac{2^{2n+1}}{5^n} = 2\sum_{n=1}^{\infty}\left(\frac{4}{5}\right)^n$ is a geometric series with $a=\dfrac{8}{5}$ and $r=\dfrac{4}{5}$. Since $|r| = \dfrac{4}{5} < 1$, the series converges to $\dfrac{a}{1-r} = \dfrac{8/5}{1-4/5} = \dfrac{8/5}{1/5} = 8$.

28. $\displaystyle\sum_{n=1}^{\infty}\frac{1}{n(n+3)} = \sum_{n=1}^{\infty}\left[\frac{1}{3n} - \frac{1}{3(n+3)}\right]$ (partial fractions).

$\displaystyle s_n = \sum_{i=1}^{n}\left[\frac{1}{3i} - \frac{1}{3(i+3)}\right] = \frac{1}{3} + \frac{1}{6} + \frac{1}{9} - \frac{1}{3(n+1)} - \frac{1}{3(n+2)} - \frac{1}{3(n+3)}$ (telescoping sum), so

$\displaystyle\sum_{n=1}^{\infty}\frac{1}{n(n+3)} = \lim_{n\to\infty}s_n = \frac{1}{3} + \frac{1}{6} + \frac{1}{9} = \frac{11}{18}.$

29.
$$\begin{aligned}\textstyle\sum_{n=1}^{\infty}\left[\tan^{-1}(n+1) - \tan^{-1}n\right] &= \lim_{n\to\infty}s_n = \lim_{n\to\infty}\left[(\tan^{-1}2 - \tan^{-1}1) + (\tan^{-1}3 - \tan^{-1}2) + \cdots \right.\\ &\qquad\left. + (\tan^{-1}(n+1) - \tan^{-1}n)\right]\\ &= \lim_{n\to\infty}\left[\tan^{-1}(n+1) - \tan^{-1}1\right] = \tfrac{\pi}{2} - \tfrac{\pi}{4} = \tfrac{\pi}{4}\end{aligned}$$

30. $\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n\pi^n}{3^{2n}(2n)!} = \sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n)!}\cdot\frac{\pi^n}{3^{2n}} = \sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n)!}\cdot\left(\frac{\sqrt{\pi}}{3}\right)^{2n} = \cos\left(\frac{\sqrt{\pi}}{3}\right)$ since

$\displaystyle\cos x = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{(2n)!}$ for all x.

31. $\displaystyle 1 - e + \frac{e^2}{2!} - \frac{e^3}{3!} + \frac{e^4}{4!} - \cdots = \sum_{n=0}^{\infty}(-1)^n\frac{e^n}{n!} = \sum_{n=0}^{\infty}\frac{(-e)^n}{n!} = e^{-e}$ since $\displaystyle e^x = \sum_{n=0}^{\infty}\frac{x^n}{n!}$ for all x.

32. $\displaystyle 4.17\overline{326} = 4.17 + \frac{326}{10^5} + \frac{326}{10^8} + \cdots = 4.17 + \frac{326/10^5}{1-1/10^3} = \frac{417}{100} + \frac{326}{99{,}900} = \frac{416{,}909}{99{,}900}$

33.
$$\begin{aligned}\cosh x &= \frac{1}{2}(e^x + e^{-x}) = \frac{1}{2}\left(\sum_{n=0}^{\infty}\frac{x^n}{n!} + \sum_{n=0}^{\infty}\frac{(-x)^n}{n!}\right)\\ &= \frac{1}{2}\left[\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots\right) + \left(1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} - \cdots\right)\right]\\ &= \frac{1}{2}\left(2 + 2\cdot\frac{x^2}{2!} + 2\cdot\frac{x^4}{4!} + \cdots\right)\\ &= 1 + \frac{1}{2}x^2 + \sum_{n=2}^{\infty}\frac{x^{2n}}{(2n)!}\\ &\geq 1 + \frac{1}{2}x^2 \quad \text{for all } x\end{aligned}$$

34. $\sum_{n=1}^{\infty}(\ln x)^n$ is a geometric series which converges whenever $|\ln x| < 1 \;\Rightarrow\; -1 < \ln x < 1 \;\Rightarrow\; e^{-1} < x < e$.

35. $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^5} = 1 - \frac{1}{32} + \frac{1}{243} - \frac{1}{1024} + \frac{1}{3125} - \frac{1}{7776} + \frac{1}{16{,}807} - \frac{1}{32{,}768} + \cdots$.

Since $b_8 = \frac{1}{8^5} = \frac{1}{32{,}768} < 0.000031$, $\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^5} \approx \sum_{n=1}^{7}\frac{(-1)^{n+1}}{n^5} \approx 0.9721$.

36. (a) $s_5 = \sum_{n=1}^{5}\frac{1}{n^6} = 1 + \frac{1}{2^6} + \cdots + \frac{1}{5^6} \approx 1.017305$. The series $\sum_{n=1}^{\infty}\frac{1}{n^6}$ converges by the Integral Test, so we estimate the remainder R_5 with (12.3.2): $R_5 \le \int_5^{\infty}\frac{dx}{x^6} = \left[-\frac{x^{-5}}{5}\right]_5^{\infty} = \frac{5^{-5}}{5} = 0.000064$. So the error is at most 0.000064.

(b) In general, $R_n \le \int_n^{\infty}\frac{dx}{x^6} = \frac{1}{5n^5}$. If we take $n = 9$, then $s_9 \approx 1.01734$ and $R_9 \le \frac{1}{5\cdot 9^5} \approx 3.4 \times 10^{-6}$. So to five decimal places, $\sum_{n=1}^{\infty}\frac{1}{n^5} \approx \sum_{n=1}^{9}\frac{1}{n^5} \approx 1.01734$.

Another Method: Use (12.3.3) instead of (12.3.2).

37. $\sum_{n=1}^{\infty}\frac{1}{2+5^n} \approx \sum_{n=1}^{8}\frac{1}{2+5^n} \approx 0.18976224$. To estimate the error, note that $\frac{1}{2+5^n} < \frac{1}{5^n}$, so the remainder term is $R_8 = \sum_{n=9}^{\infty}\frac{1}{2+5^n} < \sum_{n=9}^{\infty}\frac{1}{5^n} = \frac{1/5^9}{1-1/5} = 6.4 \times 10^{-7}$ (geometric series with $a = \frac{1}{5^9}$ and $r = \frac{1}{5}$).

38. (a) $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(n+1)^{n+1}}{[2(n+1)]!}\cdot\frac{(2n)!}{n^n}\right| = \lim_{n\to\infty}\frac{(n+1)^n(n+1)^1}{(2n+2)(2n+1)n^n} = \lim_{n\to\infty}\left(\frac{n+1}{n}\right)^n\frac{1}{2(2n+1)}$

$$= \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n\frac{1}{2(2n+1)} = e\cdot 0 = 0 < 1$$

so the series converges by the Ratio Test.

(b) The series in part (a) is convergent, so $\lim_{n\to\infty} a_n = 0$ by Theorem 12.2.6.

39. Use the Limit Comparison Test. $\lim_{n\to\infty}\left|\frac{\left(\frac{n+1}{n}\right)a_n}{a_n}\right| = \lim_{n\to\infty}\frac{n+1}{n} = \lim_{n\to\infty}\left(1+\frac{1}{n}\right) = 1 > 0$.

Since $\sum|a_n|$ is convergent, so is $\sum\left|\left(\frac{n+1}{n}\right)a_n\right|$, by the Limit Comparison Test.

40. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{x^{n+1}}{(n+1)^2 5^{n+1}}\cdot\frac{n^2 5^n}{x^n}\right| = \lim_{n\to\infty}\frac{1}{(1+1/n)^2}\frac{|x|}{5} = \frac{|x|}{5}$, so by the Ratio Test, $\sum_{n=1}^{\infty}(-1)^n\frac{x^n}{n^2 5^n}$ converges when $\frac{|x|}{5} < 1 \;\Leftrightarrow\; |x| < 5$. $R = 5$. When $x = -5$, the series becomes the convergent p-series $\sum_{n=1}^{\infty}\frac{1}{n^2}$ with $p = 2 > 1$. When $x = 5$, the series becomes $\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$, which converges by the Alternating Series Test. Thus, $I = [-5, 5]$.

41. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left[\frac{|x+2|^{n+1}}{(n+1)\,4^{n+1}}\cdot\frac{n4^n}{|x+2|^n}\right] = \lim_{n\to\infty}\left[\frac{n}{n+1}\,\frac{|x+2|}{4}\right] = \frac{|x+2|}{4} < 1 \quad\Leftrightarrow\quad |x+2| < 4,$

so $R = 4$. $|x+2| < 4 \quad\Leftrightarrow\quad -4 < x+2 < 4 \quad\Leftrightarrow\quad -6 < x < 2$. If $x = -6$, then the series $\sum\limits_{n=1}^{\infty}\frac{(x+2)^n}{n4^n}$ becomes $\sum\limits_{n=1}^{\infty}\frac{(-4)^n}{n4^n} = \sum\limits_{n=1}^{\infty}\frac{(-1)^n}{n}$, the alternating harmonic series, which converges by the Alternating Series Test. When $x = 2$, the series becomes the harmonic series $\sum\limits_{n=1}^{\infty}\frac{1}{n}$, which diverges. Thus, $I = [-6, 2)$.

42. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{2^{n+1}(x-2)^{n+1}}{(n+3)!}\cdot\frac{(n+2)!}{2^n(x-2)^n}\right| = \lim_{n\to\infty}\frac{2}{n+3}\,|x-2| = 0 < 1$, so the series $\sum\limits_{n=1}^{\infty}\frac{2^n(x-2)^n}{(n+2)!}$ converges for all x. $R = \infty$ and $I = (-\infty, \infty)$.

43. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{2^{n+1}(x-3)^{n+1}}{\sqrt{n+4}}\cdot\frac{\sqrt{n+3}}{2^n(x-3)^n}\right| = 2\,|x-3|\lim_{n\to\infty}\sqrt{\frac{n+3}{n+4}} = 2\,|x-3| < 1 \quad\Leftrightarrow$

$|x-3| < \frac{1}{2}$, so $R = \frac{1}{2}$. $|x-3| < \frac{1}{2} \quad\Leftrightarrow\quad -\frac{1}{2} < x-3 < \frac{1}{2} \quad\Leftrightarrow\quad \frac{5}{2} < x < \frac{7}{2}$. For $x = \frac{7}{2}$, the series $\sum\limits_{n=1}^{\infty}\frac{2^n(x-3)^n}{\sqrt{n+3}}$ becomes $\sum\limits_{n=0}^{\infty}\frac{1}{\sqrt{n+3}} = \sum\limits_{n=3}^{\infty}\frac{1}{n^{1/2}}$, which diverges ($p = \frac{1}{2} \le 1$), but for $x = \frac{5}{2}$, we get $\sum\limits_{n=0}^{\infty}\frac{(-1)^n}{\sqrt{n+3}}$, which is a convergent alternating series, so $I = \left[\frac{5}{2}, \frac{7}{2}\right)$.

44. $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(2n+2)!x^{n+1}}{[(n+1)!]^2}\cdot\frac{(n!)^2}{(2n)!x^n}\right| = \lim_{n\to\infty}\frac{(2n+2)(2n+1)}{(n+1)(n+1)}\,|x| = 4\,|x|.$

To converge, we must have $4\,|x| < 1 \quad\Leftrightarrow\quad |x| < \frac{1}{4}$, so $R = \frac{1}{4}$.

45.

n	$f^{(n)}(x)$	$f^{(n)}\left(\frac{\pi}{6}\right)$
0	$\sin x$	$\frac{1}{2}$
1	$\cos x$	$\frac{\sqrt{3}}{2}$
2	$-\sin x$	$-\frac{1}{2}$
3	$-\cos x$	$-\frac{\sqrt{3}}{2}$
4	$\sin x$	$\frac{1}{2}$
$\vdots$	$\vdots$	$\vdots$

$$\sin x = f\left(\frac{\pi}{6}\right) + f'\left(\frac{\pi}{6}\right)\left(x-\frac{\pi}{6}\right) + \frac{f''\left(\frac{\pi}{6}\right)}{2!}\left(x-\frac{\pi}{6}\right)^2 + \frac{f^{(3)}\left(\frac{\pi}{6}\right)}{3!}\left(x-\frac{\pi}{6}\right)^3 + \frac{f^{(4)}\left(\frac{\pi}{6}\right)}{4!}\left(x-\frac{\pi}{6}\right)^4 + \cdots$$

$$= \frac{1}{2}\left[1 - \frac{1}{2!}\left(x-\frac{\pi}{6}\right)^2 + \frac{1}{4!}\left(x-\frac{\pi}{6}\right)^4 - \cdots\right] + \frac{\sqrt{3}}{2}\left[\left(x-\frac{\pi}{6}\right) - \frac{1}{3!}\left(x-\frac{\pi}{6}\right)^3 + \cdots\right]$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n)!}\left(x-\frac{\pi}{6}\right)^{2n} + \frac{\sqrt{3}}{2}\sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n+1)!}\left(x-\frac{\pi}{6}\right)^{2n+1}$$

46.

n	$f^{(n)}(x)$	$f^{(n)}\left(\frac{\pi}{3}\right)$
0	$\cos x$	$\frac{1}{2}$
1	$-\sin x$	$-\frac{\sqrt{3}}{2}$
2	$-\cos x$	$-\frac{1}{2}$
3	$\sin x$	$\frac{\sqrt{3}}{2}$
4	$\cos x$	$\frac{1}{2}$
$\vdots$	$\vdots$	$\vdots$

$$\cos x = f\left(\frac{\pi}{3}\right) + f'\left(\frac{\pi}{3}\right)\left(x-\frac{\pi}{3}\right) + \frac{f''\left(\frac{\pi}{3}\right)}{2!}\left(x-\frac{\pi}{3}\right)^2 + \frac{f^{(3)}\left(\frac{\pi}{3}\right)}{3!}\left(x-\frac{\pi}{3}\right)^3 + \frac{f^{(4)}\left(\frac{\pi}{3}\right)}{4!}\left(x-\frac{\pi}{3}\right)^4 + \cdots$$

$$= \frac{1}{2}\left[1 - \frac{1}{2!}\left(x-\frac{\pi}{3}\right)^2 + \frac{1}{4!}\left(x-\frac{\pi}{3}\right)^4 - \cdots\right] + \frac{\sqrt{3}}{2}\left[-\left(x-\frac{\pi}{3}\right) + \frac{1}{3!}\left(x-\frac{\pi}{3}\right)^3 - \cdots\right]$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}(-1)^n\frac{1}{(2n)!}\left(x-\frac{\pi}{3}\right)^{2n} + \frac{\sqrt{3}}{2}\sum_{n=0}^{\infty}(-1)^{n+1}\frac{1}{(2n+1)!}\left(x-\frac{\pi}{3}\right)^{2n+1}$$

47. $\dfrac{1}{1+x} = \dfrac{1}{1-(-x)} = \sum\limits_{n=0}^{\infty}(-x)^n = \sum\limits_{n=0}^{\infty}(-1)^n x^n$ for $|x|<1$ $\Rightarrow$ $\dfrac{x^2}{1+x} = \sum\limits_{n=0}^{\infty}(-1)^n x^{n+2}$ with $R=1$.

48. $\tan^{-1}x = \sum\limits_{n=0}^{\infty}(-1)^n\dfrac{x^{2n+1}}{2n+1}$ with interval of convergence $[-1,1]$, so

$\tan^{-1}(x^2) = \sum\limits_{n=0}^{\infty}(-1)^n\dfrac{(x^2)^{2n+1}}{2n+1} = \sum\limits_{n=0}^{\infty}(-1)^n\dfrac{x^{4n+2}}{2n+1}$, which converges when $x^2\in[-1,1]$ $\Leftrightarrow$

$x\in[-1,1]$. Therefore, $R=1$.

49. $\dfrac{1}{1-x} = \sum\limits_{n=0}^{\infty}x^n$ for $|x|<1$ $\Rightarrow$ $\ln(1-x) = -\displaystyle\int\frac{dx}{1-x} = -\int\sum_{n=0}^{\infty}x^n\,dx = C - \sum_{n=0}^{\infty}\frac{x^{n+1}}{n+1}$.

$\ln(1-0) = C-0$ $\Rightarrow$ $C=0$ $\Rightarrow$ $\ln(1-x) = -\sum\limits_{n=0}^{\infty}\dfrac{x^{n+1}}{n+1} = \sum\limits_{n=1}^{\infty}\dfrac{-x^n}{n}$ with $R=1$.

50. $e^x = \sum\limits_{n=0}^{\infty}\dfrac{x^n}{n!}$ $\Rightarrow$ $e^{2x} = \sum\limits_{n=0}^{\infty}\dfrac{(2x)^n}{n!}$ $\Rightarrow$ $xe^{2x} = x\sum\limits_{n=0}^{\infty}\dfrac{2^n x^n}{n!} = \sum\limits_{n=0}^{\infty}\dfrac{2^n x^{n+1}}{n!}$, $R=\infty$

51. $\sin x = \sum\limits_{n=0}^{\infty}\dfrac{(-1)^n x^{2n+1}}{(2n+1)!}$ $\Rightarrow$ $\sin(x^4) = \sum\limits_{n=0}^{\infty}\dfrac{(-1)^n (x^4)^{2n+1}}{(2n+1)!} = \sum\limits_{n=0}^{\infty}\dfrac{(-1)^n x^{8n+4}}{(2n+1)!}$ for all x, so the radius of convergence is ∞.

52. $e^x = \sum\limits_{n=0}^{\infty}\dfrac{x^n}{n!}$ $\Rightarrow$ $10^x = e^{(\ln 10)x} = \sum\limits_{n=0}^{\infty}\dfrac{[(\ln 10)\,x]^n}{n!} = \sum\limits_{n=0}^{\infty}\dfrac{(\ln 10)^n x^n}{n!}$, $R=\infty$

53. $f(x) = 1 / \sqrt[4]{16 - x} = 1 / \left(\sqrt[4]{16} \sqrt[4]{1 - \frac{1}{16}x} \right) = \frac{1}{2}\left(1 - \frac{1}{16}x\right)^{-1/4}$

$$= \frac{1}{2}\left[1 + \left(-\frac{1}{4}\right)\left(-\frac{x}{16}\right) + \frac{\left(-\frac{1}{4}\right)\left(-\frac{5}{4}\right)}{2!}\left(-\frac{x}{16}\right)^2 + \frac{\left(-\frac{1}{4}\right)\left(-\frac{5}{4}\right)\left(-\frac{9}{4}\right)}{3!}\left(-\frac{x}{16}\right)^3 + \cdots\right]$$

$$= \frac{1}{2} + \sum_{n=1}^{\infty} \frac{1 \cdot 5 \cdot 9 \cdot \cdots \cdot (4n-3)}{2 \cdot 4^n \cdot n! \cdot 16^n} x^n = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{1 \cdot 5 \cdot 9 \cdot \cdots \cdot (4n-3)}{2^{6n+1}\, n!} x^n$$

for $\left|-\frac{x}{16}\right| < 1 \quad \Leftrightarrow \quad |x| < 16$, so $R = 16$.

54. $(1 - 3x)^{-5} = \sum_{n=0}^{\infty} \binom{-5}{n} (-3x)^n = 1 + (-5)(-3x) + \frac{(-5)(-6)}{2!}(-3x)^2 + \frac{(-5)(-6)(-7)}{3!}(-3x)^3 + \cdots$

$$= 1 + \sum_{n=1}^{\infty} \frac{5 \cdot 6 \cdot 7 \cdot \cdots \cdot (n+4) \cdot 3^n x^n}{n!} \quad \text{for } |-3x| < 1 \quad \Leftrightarrow \quad |x| < \tfrac{1}{3}, \text{ so } R = \tfrac{1}{3}.$$

55. $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$, so $\frac{e^x}{x} = \frac{1}{x} + \sum_{n=1}^{\infty} \frac{x^{n-1}}{n!}$ and $\int \frac{e^x}{x}\,dx = C + \ln|x| + \sum_{n=1}^{\infty} \frac{x^n}{n \cdot n!}$.

56. $(1 + x^4)^{1/2} = \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} (x^4)^n = 1 + \left(\frac{1}{2}\right)x^4 + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)}{2!}(x^4)^2 + \frac{\left(\frac{1}{2}\right)\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{3!}(x^4)^3 + \cdots$

$$= 1 + \tfrac{1}{2}x^4 - \tfrac{1}{8}x^8 + \tfrac{1}{16}x^{12} - \cdots$$

so $\int_0^1 (1 + x^4)^{1/2}\,dx = \left[x + \frac{1}{10}x^5 - \frac{1}{72}x^9 + \frac{1}{208}x^{13} - \cdots\right]_0^1 = 1 + \frac{1}{10} - \frac{1}{72} + \frac{1}{208} - \cdots$.

This is an alternating series, so by the Alternating Series Test, the error in the approximation

$\int_0^1 (1 + x^4)^{1/2}\,dx \approx 1 + \frac{1}{10} - \frac{1}{72} \approx 1.086$ is less than $\frac{1}{208}$, sufficient for the desired accuracy.

Thus, correct to two decimal places, $\int_0^1 (1 + x^4)^{1/2}\,dx \approx 1.09$.

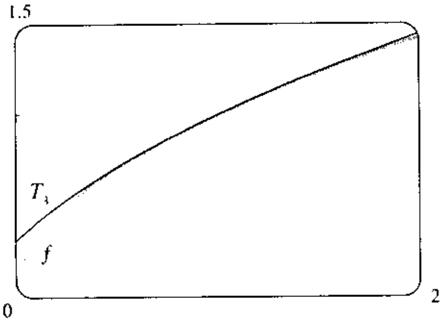
57. (a)

n	$f^{(n)}(x)$	$f^{(n)}(1)$
0	$x^{1/2}$	1
1	$\frac{1}{2}x^{-1/2}$	$\frac{1}{2}$
2	$-\frac{1}{4}x^{-3/2}$	$-\frac{1}{4}$
3	$\frac{3}{8}x^{-5/2}$	$\frac{3}{8}$
4	$-\frac{15}{16}x^{-7/2}$	$-\frac{15}{16}$
$\vdots$	$\vdots$	$\vdots$

$$\sqrt{x} \approx T_3(x) = 1 + \frac{1/2}{1!}(x-1) - \frac{1/4}{2!}(x-1)^2 + \frac{3/8}{3!}(x-1)^3$$

$$= 1 + \tfrac{1}{2}(x-1) - \tfrac{1}{8}(x-1)^2 + \tfrac{1}{16}(x-1)^3$$

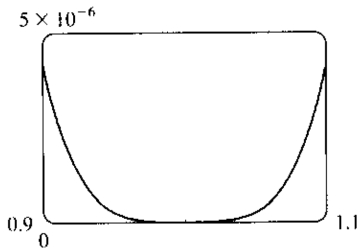
(b)



(c) $|R_3(x)| \le \dfrac{M}{4!}|x-1|^4$, where $\left|f^{(4)}(x)\right| \le M$ with $f^{(4)}(x) = -\frac{15}{16}x^{-7/2}$. Now $0.9 \le x \le 1.1 \Rightarrow -0.1 \le x-1 \le 0.1 \Rightarrow (x-1)^4 \le (0.1)^4$, and letting $x = 0.9$ gives $M = \dfrac{15}{16(0.9)^{7/2}}$, so

$$|R_3(x)| \le \frac{15}{16(0.9)^{7/2}4!}(0.1)^4 \approx 0.000005648.$$

(d)



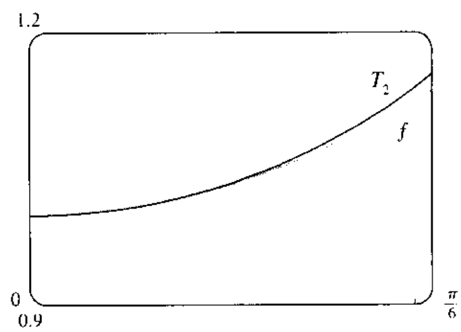
From the graph of $|R_3(x)| = |\sqrt{x} - T_3(x)|$, it appears that the error is less than 5×10^{-6} on $[0.9, 1.1]$.

58. (a)

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	$\sec x$	1
1	$\sec x \tan x$	0
2	$\sec x \tan^2 x + \sec^3 x$	1
3	$\sec x \tan^3 x + 5\sec^3 x \tan x$	0
$\vdots$	$\vdots$	$\vdots$

$\sec x \approx T_2(x) = 1 + \frac{1}{2}x^2$

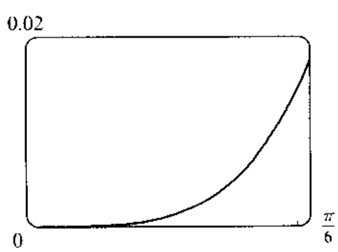
(b)



(c) $|R_2(x)| \le \dfrac{M}{3!}|x|^3$, where $\left|f^{(3)}(x)\right| \le M$ with $f^{(3)}(x) = \sec x \tan^3 x + 5\sec^3 x \tan x$. Now $0 \le x \le \frac{\pi}{6} \Rightarrow x^3 \le \left(\frac{\pi}{6}\right)^3$, and letting $x = \frac{\pi}{6}$ gives $M = \frac{14}{3}$, so

$|R_2(x)| \le \frac{14}{3 \cdot 6}\left(\frac{\pi}{6}\right)^3 \approx 0.111648.$

(d)



From the graph of $|R_2(x)| = |\sec x - T_2(x)|$, it appears that the error is less than 0.02 on $\left[0, \frac{\pi}{6}\right]$.

59. $\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$, so $\sin x - x = -\frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$ and

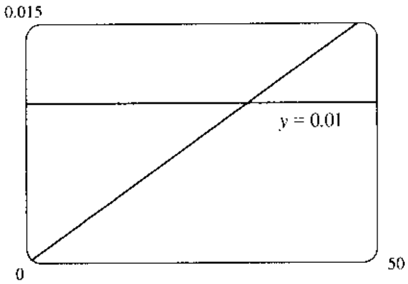
$\frac{\sin x - x}{x^3} = -\frac{1}{3!} + \frac{x^2}{5!} - \frac{x^4}{7!} + \cdots$. Thus, $\lim_{x \to 0} \frac{\sin x - x}{x^3} = \lim_{x \to 0}\left(-\frac{1}{6} + \frac{x^2}{120} - \frac{x^4}{5040} + \cdots\right) = -\frac{1}{6}$.

60. (a) $F = \frac{mgR^2}{(R+h)^2} = \frac{mg}{(1+h/R)^2} = mg \sum_{n=0}^{\infty} \binom{-2}{n} \left(\frac{h}{R}\right)^n$ (Binomial Series)

(b) We expand $F = mg\left[1 - 2(h/R) + 3(h/R)^2 - \cdots\right]$.

This is an alternating series, so by the Alternating Series Estimation Theorem, the error in the approximation $F = mg$ is less than $2mgh/R$, so for accuracy within 1% we want

$\left|\frac{2mgh/R}{mgR^2/(R+h)^2}\right| < 0.01 \quad \Leftrightarrow \quad \frac{2h(R+h)^2}{R^3} < 0.01$. This inequality would be difficult to solve for h, so we substitute $R = 6{,}400$ km and plot both sides of the inequality. It appears that the approximation is accurate to within 1% for $h < 31$ km.



61. $f(x) = \sum_{n=0}^{\infty} c_n x^n \quad \Rightarrow \quad f(-x) = \sum_{n=0}^{\infty} c_n(-x)^n = \sum_{n=0}^{\infty}(-1)^n c_n x^n$

(a) If f is an odd function, then $f(-x) = -f(x) \quad \Rightarrow \quad \sum_{n=0}^{\infty}(-1)^n c_n x^n = \sum_{n=0}^{\infty} -c_n x^n$. The coefficients of any power series are uniquely determined (by Theorem 12.10.5), so $(-1)^n c_n = -c_n$. If n is even, then $(-1)^n = 1$, so $c_n = -c_n \quad \Rightarrow \quad 2c_n = 0 \quad \Rightarrow \quad c_n = 0$. Thus, all even coefficients are 0, that is, $c_0 = c_2 = c_4 = \cdots = 0$.

(b) If f is even, then $f(-x) = f(x) \quad \Rightarrow \quad \sum_{n=0}^{\infty}(-1)^n c_n x^n = \sum_{n=0}^{\infty} c_n x^n \quad \Rightarrow \quad (-1)^n c_n = c_n$.

If n is odd, then $(-1)^n = -1$, so $-c_n = c_n \quad \Rightarrow \quad 2c_n = 0 \quad \Rightarrow \quad c_n = 0$. Thus, all odd coefficients are 0, that is, $c_1 = c_3 = c_5 = \cdots = 0$.

62. $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \quad \Rightarrow \quad f(x) = e^{x^2} = \sum_{n=0}^{\infty} \frac{(x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{x^{2n}}{n!} = \sum_{n=0}^{\infty} \frac{1}{n!} x^{2n}$. By Theorem 12.10.6 with $a = 0$, we also have $f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k$. Comparing coefficients for $k = 2n$, we have $\frac{f^{(2n)}(0)}{(2n)!} = \frac{1}{n!} \quad \Rightarrow$

$f^{(2n)}(0) = \frac{(2n)!}{n!}$.

PROBLEMS PLUS

1. It would be far too much work to compute 15 derivatives of f. The key idea is to remember that $f^{(n)}(0)$ occurs in the coefficient of x^n in the Maclaurin series of f. We start with the Maclaurin series for sin: $\sin x = x - \dfrac{x^3}{3!} + \dfrac{x^5}{5!} - \cdots$. Then $\sin(x^3) = x^3 - \dfrac{x^9}{3!} + \dfrac{x^{15}}{5!} - \cdots$, and so the coefficient of x^{15} is $\dfrac{f^{(15)}(0)}{15!} = \dfrac{1}{5!}$. Therefore, $f^{(15)}(0) = \dfrac{15!}{5!} = 6 \cdot 7 \cdot 8 \cdot 9 \cdot 10 \cdot 11 \cdot 12 \cdot 13 \cdot 14 \cdot 15 = 10{,}897{,}286{,}400$.

2. We use the problem-solving strategy of taking cases:

Case (i): If $|x| < 1$, then $0 \le x^2 < 1$, so $\lim\limits_{n\to\infty} x^{2n} = 0$ (see Example 9 in Section 12.1)

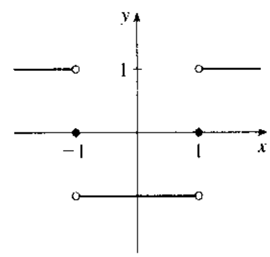
$$\text{and } f(x) = \lim_{n\to\infty} \frac{x^{2n} - 1}{x^{2n} + 1} = \frac{0 - 1}{0 + 1} = -1.$$

Case (ii): If $|x| = 1$, that is, $x = \pm 1$, then $x^2 = 1$, so $f(x) = \lim\limits_{n\to\infty} \dfrac{x^{2n} - 1}{x^{2n} + 1} = \lim\limits_{n\to\infty} \dfrac{1 - 1}{1 + 1} = 0$.

Case (iii): If $|x| > 1$, then $x^2 > 1$, so $\lim\limits_{n\to\infty} x^{2n} = \infty$ and

$$f(x) = \lim_{n\to\infty} \frac{x^{2n} - 1}{x^{2n} + 1} = \lim_{n\to\infty} \frac{1 - (1/x^{2n})}{1 + (1/x^{2n})} = \frac{1 - 0}{1 + 0} = 1.$$

$$\text{Thus, } f(x) = \begin{cases} 1 & \text{if } x < -1 \\ 0 & \text{if } x = -1 \\ -1 & \text{if } -1 < x < 1 \\ 0 & \text{if } x = 1 \\ 1 & \text{if } x > 1 \end{cases}$$



The graph shows that f is continuous everywhere except at $x = \pm 1$.

3. (a) From Formula 14a in Appendix D, with $x = y = \theta$, we get $\tan 2\theta = \dfrac{2\tan\theta}{1 - \tan^2\theta}$, so $\cot 2\theta = \dfrac{1 - \tan^2\theta}{2\tan\theta}$ $\Rightarrow$ $2\cot 2\theta = \dfrac{1 - \tan^2\theta}{\tan\theta} = \cot\theta - \tan\theta$. Replacing θ by $\frac{1}{2}x$, we get $2\cot x = \cot\frac{1}{2}x - \tan\frac{1}{2}x$, or

$$\tan\tfrac{1}{2}x = \cot\tfrac{1}{2}x - 2\cot x$$

(b) From part (a) with $\dfrac{x}{2^{n-1}}$ in place of x, $\tan\dfrac{x}{2^n} = \cot\dfrac{x}{2^n} - 2\cot\dfrac{x}{2^{n-1}}$, so the nth partial sum of $\displaystyle\sum_{n=1}^{\infty} \frac{1}{2^n}\tan\frac{x}{2^n}$ is

$$\begin{aligned} s_n &= \frac{\tan(x/2)}{2} + \frac{\tan(x/4)}{4} + \frac{\tan(x/8)}{8} + \cdots + \frac{\tan(x/2^n)}{2^n} \\ &= \left[\frac{\cot(x/2)}{2} - \cot x\right] + \left[\frac{\cot(x/4)}{4} - \frac{\cot(x/2)}{2}\right] + \left[\frac{\cot(x/8)}{8} - \frac{\cot(x/4)}{4}\right] + \cdots \\ &\quad + \left[\frac{\cot(x/2^n)}{2^n} - \frac{\cot(x/2^{n-1})}{2^{n-1}}\right] = -\cot x + \frac{\cot(x/2^n)}{2^n} \quad \text{(telescoping sum)} \end{aligned}$$

Now $\dfrac{\cot(x/2^n)}{2^n} = \dfrac{\cos(x/2^n)}{2^n \sin(x/2^n)} = \dfrac{\cos(x/2^n)}{x} \cdot \dfrac{x/2^n}{\sin(x/2^n)} \to \dfrac{1}{x} \cdot 1 = \dfrac{1}{x}$ as $n \to \infty$ since $x/2^n \to 0$

for $x \neq 0$. Therefore, if $x \neq 0$ and $x \neq k\pi$ where k is any integer, then

$$\sum_{n=1}^{\infty} \frac{1}{2^n} \tan \frac{x}{2^n} = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \left(-\cot x + \frac{1}{2^n} \cot \frac{x}{2^n} \right) = -\cot x + \frac{1}{x}$$

If $x = 0$, then all terms in the series are 0, so the sum is 0.

4. $|AP_2|^2 = 2$, $|AP_3|^2 = 2 + 2^2$, $|AP_4|^2 = 2 + 2^2 + (2^2)^2$, $|AP_5|^2 = 2 + 2^2 + (2^2)^2 + (2^3)^2, \ldots,$

$$|AP_n|^2 = 2 + 2^2 + (2^2)^2 + \cdots + (2^{n-2})^2 \quad [\text{for } n \geq 3] \quad = 2 + (4 + 4^2 + 4^3 + \cdots + 4^{n-2})$$

$$= 2 + \frac{4(4^{n-2} - 1)}{4 - 1} \quad [\text{finite geometric sum with } a = 4, r = 4] \quad = \frac{6}{3} + \frac{4^{n-1} - 4}{3} = \frac{2}{3} + \frac{4^{n-1}}{3}$$

So $\tan \angle P_n A P_{n+1} = \dfrac{|P_n P_{n+1}|}{|AP_n|} = \dfrac{2^{n-1}}{\sqrt{\frac{2}{3} + \frac{4^{n-1}}{3}}} = \dfrac{\sqrt{4^{n-1}}}{\sqrt{\frac{2}{3} + \frac{4^{n-1}}{3}}} = \dfrac{1}{\sqrt{\frac{2}{3 \cdot 4^{n-1}} + \frac{1}{3}}} \to \sqrt{3}$ as $n \to \infty$.

Thus, $\angle P_n A P_{n+1} \to \frac{\pi}{3}$ as $n \to \infty$.

5. (a) At each stage, each side is replaced by four shorter sides, each of length $\frac{1}{3}$ of the side length at the preceding stage. Writing s_0 and ℓ_0 for the number of sides and the length of the side of the initial triangle, we generate the table at right. In general, we have $s_n = 3 \cdot 4^n$ and $\ell_n = \left(\frac{1}{3}\right)^n$, so the length of the perimeter at the nth stage of construction is $p_n = s_n \ell_n = 3 \cdot 4^n \cdot \left(\frac{1}{3}\right)^n = 3 \cdot \left(\frac{4}{3}\right)^n$.

$s_0 = 3$	$\ell_0 = 1$
$s_1 = 3 \cdot 4$	$\ell_1 = 1/3$
$s_2 = 3 \cdot 4^2$	$\ell_2 = 1/3^2$
$s_3 = 3 \cdot 4^3$	$\ell_3 = 1/3^3$
$\cdots$	$\cdots$

(b) $p_n = \dfrac{4^n}{3^{n-1}} = 4\left(\dfrac{4}{3}\right)^{n-1}$. Since $\frac{4}{3} > 1$, $p_n \to \infty$ as $n \to \infty$.

(c) The area of each of the small triangles added at a given stage is one-ninth of the area of the triangle added at the preceding stage. Let a be the area of the original triangle. Then the area a_n of each of the small triangles added at stage n is $a_n = a \cdot \dfrac{1}{9^n} = \dfrac{a}{9^n}$. Since a small triangle is added to each side at every stage, it follows that the total area A_n added to the figure at the nth stage is $A_n = s_{n-1} \cdot a_n = 3 \cdot 4^{n-1} \cdot \dfrac{a}{9^n} = a \cdot \dfrac{4^{n-1}}{3^{2n-1}}$. Then the total area enclosed by the snowflake curve is $A = a + A_1 + A_2 + A_3 + \cdots = a + a \cdot \dfrac{1}{3} + a \cdot \dfrac{4}{3^3} + a \cdot \dfrac{4^2}{3^5} + a \cdot \dfrac{4^3}{3^7} + \cdots$. After the first term, this is a geometric series with common ratio $\frac{4}{9}$, so $A = a + \dfrac{a/3}{1 - \frac{4}{9}} = a + \dfrac{a}{3} \cdot \dfrac{9}{5} = \dfrac{8a}{5}$. But the area of the original equilateral triangle with side 1 is $a = \frac{1}{2} \cdot 1 \cdot \sin \frac{\pi}{3} = \frac{\sqrt{3}}{4}$. So the area enclosed by the snowflake curve is $\frac{8}{5} \cdot \frac{\sqrt{3}}{4} = \frac{2\sqrt{3}}{5}$.

6. Let the series $S = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \frac{1}{9} + \frac{1}{12} + \cdots$. Then every term in S is of the form $\dfrac{1}{2^m 3^n}$, $m, n \geq 0$, and furthermore each term occurs only once. So we can write

$$S = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{1}{2^m 3^n} = \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} \frac{1}{2^m} \frac{1}{3^n} = \sum_{m=0}^{\infty} \frac{1}{2^m} \sum_{n=0}^{\infty} \frac{1}{3^n} = \frac{1}{1 - \frac{1}{2}} \cdot \frac{1}{1 - \frac{1}{3}} = 2 \cdot \tfrac{3}{2} = 3$$

7. (a) Let $a = \arctan x$ and $b = \arctan y$. Then, from Formula 14b in Appendix D,

$$\tan(a-b) = \frac{\tan a - \tan b}{1 + \tan a \tan b} = \frac{\tan(\arctan x) - \tan(\arctan y)}{1 + \tan(\arctan x)\tan(\arctan y)} = \frac{x-y}{1+xy}.$$

Now $\arctan x - \arctan y = a - b = \arctan(\tan(a-b)) = \arctan \dfrac{x-y}{1+xy}$ since $-\frac{\pi}{2} < a - b < \frac{\pi}{2}$.

(b) From part (a) we have

$$\arctan \tfrac{120}{119} - \arctan \tfrac{1}{239} = \arctan \frac{\frac{120}{119} - \frac{1}{239}}{1 + \frac{120}{119}\cdot\frac{1}{239}} = \arctan \frac{\frac{28,561}{28,441}}{\frac{28,561}{28,441}} = \arctan 1 = \tfrac{\pi}{4}$$

(c) Replacing y by $-y$ in the formula of part (a), we get $\arctan x + \arctan y = \arctan \dfrac{x+y}{1-xy}$. So

$$4\arctan \tfrac{1}{5} = 2\left(\arctan \tfrac{1}{5} + \arctan \tfrac{1}{5}\right) = 2\arctan \frac{\frac{1}{5}+\frac{1}{5}}{1 - \frac{1}{5}\cdot\frac{1}{5}} = 2\arctan \tfrac{5}{12} = \arctan \tfrac{5}{12} + \arctan \tfrac{5}{12}$$

$$= \arctan \frac{\frac{5}{12}+\frac{5}{12}}{1 - \frac{5}{12}\cdot\frac{5}{12}} = \arctan \tfrac{120}{119}$$

Thus, from part (b), we have $4\arctan \frac{1}{5} - \arctan \frac{1}{239} = \arctan \frac{120}{119} - \arctan \frac{1}{239} = \frac{\pi}{4}$.

(d) From Example 7 in Section 12.9 we have $\arctan x = x - \dfrac{x^3}{3} + \dfrac{x^5}{5} - \dfrac{x^7}{7} + \dfrac{x^9}{9} - \dfrac{x^{11}}{11} + \cdots$, so

$$\arctan \frac{1}{5} = \frac{1}{5} - \frac{1}{3\cdot 5^3} + \frac{1}{5\cdot 5^5} - \frac{1}{7\cdot 5^7} + \frac{1}{9\cdot 5^9} - \frac{1}{11\cdot 5^{11}} + \cdots$$

This is an alternating series and the size of the terms decreases to 0, so by the Alternating Series Estimation Theorem, the sum lies between s_5 and s_6, that is, $0.197395560 < \arctan \frac{1}{5} < 0.197395562$.

(e) From the series in part (d) we get $\arctan \dfrac{1}{239} = \dfrac{1}{239} - \dfrac{1}{3\cdot 239^3} + \dfrac{1}{5\cdot 239^5} - \cdots$. The third term is less than 2.6×10^{-13}, so by the Alternating Series Estimation Theorem, we have, to nine decimal places, $\arctan \frac{1}{239} \approx s_2 \approx 0.004184076$. Thus, $0.004184075 < \arctan \frac{1}{239} < 0.004184077$.

(f) From part (c) we have $\pi = 16\arctan \frac{1}{5} - 4\arctan \frac{1}{239}$, so from parts (d) and (e) we have

$16(0.197395560) - 4(0.004184077) < \pi < 16(0.197395562) - 4(0.004184075) \Rightarrow$

$3.141592652 < \pi < 3.141592692$. So, to 7 decimal places, $\pi \approx 3.1415927$.

8. (a) Let $a = \operatorname{arccot} x$ and $b = \operatorname{arccot} y$ where $0 < a - b < \pi$. Then

$$\cot(a-b) = \frac{1}{\tan(a-b)} = \frac{1 + \tan a \tan b}{\tan a - \tan b} = \frac{1 + \frac{1}{\cot a}\cdot\frac{1}{\cot b}}{\frac{1}{\cot a} - \frac{1}{\cot b}} \cdot \frac{\cot a \cot b}{\cot a \cot b}$$

$$= \frac{\cot a \cot b + 1}{\cot b - \cot a}, \text{ so}$$

$$\cot(a-b) = \frac{1 + \cot a \cot b}{\cot b - \cot a} = \frac{1 + \cot(\operatorname{arccot} x)\cot(\operatorname{arccot} y)}{\cot(\operatorname{arccot} y) - \cot(\operatorname{arccot} x)} = \frac{1+xy}{y-x}.$$

Now $\operatorname{arccot} x - \operatorname{arccot} y = a - b = \operatorname{arccot}(\cot(a-b))$

$$= \operatorname{arccot} \frac{1+xy}{y-x} \text{ since } 0 < a - b < \pi.$$

(b) Applying the identity in part (a) with $x = n$ and $y = n + 1$, we have

$$\operatorname{arccot}(n^2 + n + 1) = \operatorname{arccot}(1 + n(n+1)) = \operatorname{arccot} \frac{1 + n(n+1)}{(n+1) - n} = \operatorname{arccot} n - \operatorname{arccot}(n+1)$$

Thus, we have a telescoping series with nth partial sum

$$\begin{aligned} s_n &= [\operatorname{arccot} 0 - \operatorname{arccot} 1] + [\operatorname{arccot} 1 - \operatorname{arccot} 2] + \cdots + [\operatorname{arccot} n - \operatorname{arccot}(n+1)] \\ &= \operatorname{arccot} 0 - \operatorname{arccot}(n+1) \end{aligned}$$

Thus, $\sum_{n=0}^{\infty} \operatorname{arccot}(n^2 + n + 1) = \lim\limits_{n\to\infty} s_n = \lim\limits_{n\to\infty} [\operatorname{arccot} 0 - \operatorname{arccot}(n+1)] = \frac{\pi}{2} - 0 = \frac{\pi}{2}$.

9. We start with the geometric series $\sum\limits_{n=0}^{\infty} x^n = \dfrac{1}{1-x}$, $|x| < 1$, and differentiate:

$$\sum_{n=1}^{\infty} nx^{n-1} = \frac{d}{dx}\left(\sum_{n=0}^{\infty} x^n\right) = \frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{1}{(1-x)^2} \text{ for } |x| < 1 \quad \Rightarrow$$

$\sum\limits_{n=1}^{\infty} nx^n = x \sum\limits_{n=1}^{\infty} nx^{n-1} = \dfrac{x}{(1-x)^2}$ for $|x| < 1$. Differentiate again:

$$\sum_{n=1}^{\infty} n^2 x^{n-1} = \frac{d}{dx} \frac{x}{(1-x)^2} = \frac{(1-x)^2 - x \cdot 2(1-x)(-1)}{(1-x)^4} = \frac{x+1}{(1-x)^3} \quad \Rightarrow \quad \sum_{n=1}^{\infty} n^2 x^n = \frac{x^2 + x}{(1-x)^3} \quad \Rightarrow$$

$$\sum_{n=1}^{\infty} n^3 x^{n-1} = \frac{d}{dx} \frac{x^2 + x}{(1-x)^3} = \frac{(1-x)^3 (2x+1) - (x^2 + x)\, 3(1-x)^2 (-1)}{(1-x)^6} = \frac{x^2 + 4x + 1}{(1-x)^4} \quad \Rightarrow$$

$\sum\limits_{n=1}^{\infty} n^3 x^n = \dfrac{x^3 + 4x^2 + x}{(1-x)^4}$, $|x| < 1$. The radius of convergence is 1 because that is the radius of convergence for the geometric series we started with. If $x = \pm 1$, the series is $\sum n^3 (\pm 1)^n$, which diverges by the Test For Divergence, so the interval of convergence is $(-1, 1)$.

10. Let's first try the case $k = 1$: $a_0 + a_1 = 0 \quad \Rightarrow \quad a_1 = -a_0 \quad \Rightarrow$

$$\lim_{n\to\infty} \left(a_0\sqrt{n} + a_1\sqrt{n+1}\right) = \lim_{n\to\infty} \left(a_0\sqrt{n} - a_0\sqrt{n+1}\right) = a_0 \lim_{n\to\infty} \left(\sqrt{n} - \sqrt{n+1}\right) \frac{\sqrt{n} + \sqrt{n+1}}{\sqrt{n} + \sqrt{n+1}}$$

$$= a_0 \lim_{n\to\infty} \frac{-1}{\sqrt{n} + \sqrt{n+1}} = 0$$

In general we have $a_0 + a_1 + \cdots + a_k = 0 \quad \Rightarrow \quad a_k = -a_0 - a_1 - \cdots - a_{k-1} \quad \Rightarrow$

$$\lim_{n\to\infty} \left(a_0\sqrt{n} + a_1\sqrt{n+1} + a_2\sqrt{n+2} + \cdots + a_k\sqrt{n+k}\right)$$

$$= \lim_{n\to\infty} \left(a_0\sqrt{n} + a_1\sqrt{n+1} + \cdots + a_{k-1}\sqrt{n+k-1} - a_0\sqrt{n+k} - a_1\sqrt{n+k} - \cdots - a_{k-1}\sqrt{n+k}\right)$$

$$= a_0 \lim_{n\to\infty} \left(\sqrt{n} - \sqrt{n+k}\right) + a_1 \lim_{n\to\infty} \left(\sqrt{n+1} - \sqrt{n+k}\right) + \cdots + a_{k-1} \lim_{n\to\infty} \left(\sqrt{n+k-1} - \sqrt{n+k}\right)$$

Each of these limits is 0 by the same type of simplification as in the case $k = 1$. So we have

$$\lim_{n\to\infty} \left(a_0\sqrt{n} + a_1\sqrt{n+1} + a_2\sqrt{n+2} + \cdots + a_k\sqrt{n+k}\right) = a_0(0) + a_1(0) + \cdots + a_{k-1}(0) = 0$$

11. $\ln\left(1-\frac{1}{n^2}\right)=\ln\left(\frac{n^2-1}{n^2}\right)=\ln\frac{(n+1)(n-1)}{n^2}=\ln[(n+1)(n-1)]-\ln n^2$

$$=\ln(n+1)+\ln(n-1)-2\ln n$$

$$=\ln(n-1)-\ln n-\ln n+\ln(n+1)$$

$$=\ln\frac{n-1}{n}-[\ln n-\ln(n+1)]=\ln\frac{n-1}{n}-\ln\frac{n}{n+1}.$$

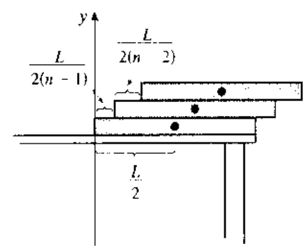
Let $s_k=\sum_{n=2}^{k}\ln\left(1-\frac{1}{n^2}\right)=\sum_{n=2}^{k}\left(\ln\frac{n-1}{n}-\ln\frac{n}{n+1}\right)$ for $k\geq 2$. Then

$$s_k=\left(\ln\frac{1}{2}-\ln\frac{2}{3}\right)+\left(\ln\frac{2}{3}-\ln\frac{3}{4}\right)+\cdots+\left(\ln\frac{k-1}{k}-\ln\frac{k}{k+1}\right)=\ln\frac{1}{2}-\ln\frac{k}{k+1},\text{ so}$$

$$\sum_{n=2}^{\infty}\ln\left(1-\frac{1}{n^2}\right)=\lim_{k\to\infty}s_k=\lim_{k\to\infty}\left(\ln\frac{1}{2}-\ln\frac{k}{k+1}\right)=\ln\frac{1}{2}-\ln 1=\ln 1-\ln 2-\ln 1=-\ln 2.$$

12. Place the y-axis as shown and let the length of each book be L. We want to show that the center of mass of the system of n books lies above the table, that is, $\overline{x}<L$. The x-coordinates of the centers of mass of the books are

$x_1=\frac{L}{2}$, $x_2=\frac{L}{2(n-1)}+\frac{L}{2}$, $x_3=\frac{L}{2(n-1)}+\frac{L}{2(n-2)}+\frac{L}{2}$, and so on.



Each book has the same mass m, so if there are n books, then

$$\overline{x}=\frac{mx_1+mx_2+\cdots+mx_n}{mn}=\frac{x_1+x_2+\cdots+x_n}{n}$$

$$=\frac{1}{n}\left[\frac{L}{2}+\left(\frac{L}{2(n-1)}+\frac{L}{2}\right)+\left(\frac{L}{2(n-1)}+\frac{L}{2(n-2)}+\frac{L}{2}\right)+\cdots\right.$$

$$\left.+\left(\frac{L}{2(n-1)}+\frac{L}{2(n-2)}+\cdots+\frac{L}{4}+\frac{L}{2}+\frac{L}{2}\right)\right]$$

$$=\frac{L}{n}\left[\frac{n-1}{2(n-1)}+\frac{n-2}{2(n-2)}+\cdots+\frac{2}{4}+\frac{1}{2}+\frac{n}{2}\right]=\frac{L}{n}\left[(n-1)\frac{1}{2}+\frac{n}{2}\right]=\frac{2n-1}{2n}L<L$$

This shows that, no matter how many books are added according to the given scheme, the center of mass lies above the table. It remains to observe that the series $\frac{1}{2}+\frac{1}{4}+\frac{1}{6}+\frac{1}{8}+\cdots=\frac{1}{2}\sum(1/n)$ is divergent (harmonic series), so we can make the top book extend as far as we like beyond the edge of the table if we add enough books.

13. $u=1+\frac{x^3}{3!}+\frac{x^6}{6!}+\frac{x^9}{9!}+\cdots$, $v=x+\frac{x^4}{4!}+\frac{x^7}{7!}+\frac{x^{10}}{10!}+\cdots$, $w=\frac{x^2}{2!}+\frac{x^5}{5!}+\frac{x^8}{8!}+\cdots$.

Use the Ratio Test to show that the series for u, v, and w have positive radii of convergence (∞ in each case), so Theorem 12.9.2 applies, and hence, we may differentiate each of these series:

$$\frac{du}{dx}=\frac{3x^2}{3!}+\frac{6x^5}{6!}+\frac{9x^8}{9!}+\cdots=\frac{x^2}{2!}+\frac{x^5}{5!}+\frac{x^8}{8!}+\cdots=w$$

Similarly, $\dfrac{dv}{dx} = 1 + \dfrac{x^3}{3!} + \dfrac{x^6}{6!} + \dfrac{x^9}{9!} + \cdots = u$, and $\dfrac{dw}{dx} = x + \dfrac{x^4}{4!} + \dfrac{x^7}{7!} + \dfrac{x^{10}}{10!} + \cdots = v$.

So $u' = w$, $v' = u$, and $w' = v$. Now differentiate the left hand side of the desired equation:

$$\frac{d}{dx}\left(u^3 + v^3 + w^3 - 3uvw\right) = 3u^2u' + 3v^2v' + 3w^2w' - 3(u'vw + uv'w + uvw')$$

$$= 3u^2w + 3v^2u + 3w^2v - 3(vw^2 + u^2w + uv^2) = 0 \quad \Rightarrow$$

$u^3 + v^3 + w^3 - 3uvw = C$. To find the value of the constant C, we put $x = 0$ in the last equation and get $1^3 + 0^3 + 0^3 - 3(1 \cdot 0 \cdot 0) = C \quad \Rightarrow \quad C = 1$, so $u^3 + v^3 + w^3 - 3uvw = 1$.

14. First notice that both series are absolutely convergent (p-series with $p > 1$.) Let the given expression be called x. Then

$$x = \frac{1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots}{1 - \frac{1}{2^p} + \frac{1}{3^p} - \frac{1}{4^p} + \cdots} = \frac{1 + \left(2 \cdot \frac{1}{2^p} - \frac{1}{2^p}\right) + \frac{1}{3^p} + \left(2 \cdot \frac{1}{4^p} - \frac{1}{4^p}\right) + \cdots}{1 - \frac{1}{2^p} + \frac{1}{3^p} - \frac{1}{4^p} + \cdots}$$

$$= \frac{\left(1 - \frac{1}{2^p} + \frac{1}{3^p} - \frac{1}{4^p} + \cdots\right) + \left(2 \cdot \frac{1}{2^p} + 2 \cdot \frac{1}{4^p} + 2 \cdot \frac{1}{6^p} + \cdots\right)}{1 - \frac{1}{2^p} + \frac{1}{3^p} - \frac{1}{4^p} + \cdots}$$

$$= 1 + \frac{2\left(\frac{1}{2^p} + \frac{1}{4^p} + \frac{1}{6^p} + \frac{1}{8^p} + \cdots\right)}{1 - \frac{1}{2^p} + \frac{1}{3^p} - \frac{1}{4^p} + \cdots} = 1 + \frac{\frac{1}{2^{p-1}}\left(1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots\right)}{1 - \frac{1}{2^p} + \frac{1}{3^p} - \frac{1}{4^p} + \cdots} = 1 + 2^{1-p}x$$

Therefore, $x = 1 + 2^{1-p}x \quad \Leftrightarrow \quad x - 2^{1-p}x = 1 \quad \Leftrightarrow \quad x(1 - 2^{1-p}) = 1 \quad \Leftrightarrow \quad x = \dfrac{1}{1 - 2^{1-p}}$.

15. If L is the length of a side of the equilateral triangle, then the area is $A = \frac{1}{2}L \cdot \frac{\sqrt{3}}{2}L = \frac{\sqrt{3}}{4}L^2$ and so $L^2 = \frac{4}{\sqrt{3}}A$.

Let r be the radius of one of the circles. When there are n rows of circles, the figure shows that

$L = \sqrt{3}r + r + (n-2)(2r) + r + \sqrt{3}r = r(2n - 2 + 2\sqrt{3})$, so $r = \dfrac{L}{2(n + \sqrt{3} - 1)}$.

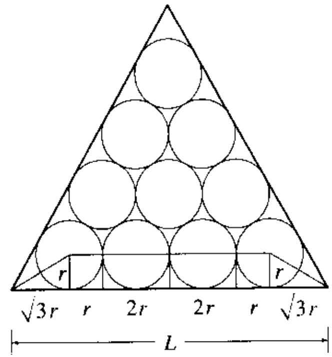
The number of circles is $1 + 2 + \cdots + n = \dfrac{n(n+1)}{2}$, and so the total area of the circles is

$$A_n = \frac{n(n+1)}{2}\pi r^2 = \frac{n(n+1)}{2}\pi\frac{L^2}{4(n + \sqrt{3} - 1)^2} = \frac{n(n+1)}{2}\pi\frac{4A/\sqrt{3}}{4(n + \sqrt{3} - 1)^2}$$

$$= \frac{n(n+1)}{(n + \sqrt{3} - 1)^2}\frac{\pi A}{2\sqrt{3}} \quad \Rightarrow$$

$$\frac{A_n}{A} = \frac{n(n+1)}{\left(n+\sqrt{3}-1\right)^2} \frac{\pi}{2\sqrt{3}}$$

$$= \frac{1+1/n}{\left[1+\left(\sqrt{3}-1\right)/n\right]^2} \frac{\pi}{2\sqrt{3}} \to \frac{\pi}{2\sqrt{3}} \text{ as } n \to \infty$$



16. Given $a_0 = a_1 = 1$ and $a_n = \dfrac{(n-1)(n-2)a_{n-1} - (n-3)a_{n-2}}{n(n-1)}$, we calculate the next few terms of the sequence: $a_2 = \dfrac{1 \cdot 0 \cdot a_1 - (-1)a_0}{2 \cdot 1} = \dfrac{1}{2}$, $a_3 = \dfrac{2 \cdot 1 \cdot a_2 - 0 \cdot a_1}{3 \cdot 2} = \dfrac{1}{6}$, $a_4 = \dfrac{3 \cdot 2 \cdot a_3 - 1 \cdot a_2}{4 \cdot 3} = \dfrac{1}{24}$.

It seems that $a_n = \dfrac{1}{n!}$, so we try to prove this by induction. The first step is done, so assume $a_k = \dfrac{1}{k!}$ and $a_{k-1} = \dfrac{1}{(k-1)!}$. Then

$$a_{k+1} = \frac{k(k-1)a_k - (k-2)a_{k-1}}{(k+1)k} = \frac{\dfrac{k(k-1)}{k!} - \dfrac{k-2}{(k-1)!}}{(k+1)k} = \frac{(k-1)-(k-2)}{[(k+1)(k)]\,(k-1)!} = \frac{1}{(k+1)!}$$

and the induction is complete. Therefore, $\sum_{n=0}^{\infty} a_n = \sum_{n=0}^{\infty} 1/n! = e$.

17. As in Section 12.9 we have to integrate the function x^x by integrating series. Writing $x^x = (e^{\ln x})^x = e^{x \ln x}$ and using the Maclaurin series for e^x, we have $x^x = (e^{\ln x})^x = e^{x \ln x} = \sum_{n=0}^{\infty} \dfrac{(x \ln x)^n}{n!} = \sum_{n=0}^{\infty} \dfrac{x^n (\ln x)^n}{n!}$.

As with power series, we can integrate this series term-by-term:

$$\int_0^1 x^x\,dx = \sum_{n=0}^{\infty} \int_0^1 \frac{x^n (\ln x)^n}{n!}\,dx = \sum_{n=0}^{\infty} \frac{1}{n!} \int_0^1 x^n (\ln x)^n\,dx$$

We integrate by parts with $u = (\ln x)^n$, $dv = x^n\,dx$, so $du = \dfrac{n(\ln x)^{n-1}}{x}\,dx$ and $v = \dfrac{x^{n+1}}{n+1}$:

$$\int_0^1 x^n (\ln x)^n\,dx = \lim_{t \to 0^+} \int_t^1 x^n (\ln x)^n\,dx = \lim_{t \to 0^+} \left[\frac{x^{n+1}}{n+1} (\ln x)^n\right]_t^1 - \lim_{t \to 0^+} \int_t^1 \frac{n}{n+1} x^n (\ln x)^{n-1}\,dx$$

$$= 0 - \frac{n}{n+1} \int_0^1 x^n (\ln x)^{n-1}\,dx$$

(where l'Hospital's Rule was used to help evaluate the first limit).

Further integration by parts gives $\int_0^1 x^n (\ln x)^k\,dx = -\dfrac{k}{n+1} \int_0^1 x^n (\ln x)^{k-1}\,dx$ and, combining these steps, we get $\int_0^1 x^n (\ln x)^n\,dx = \dfrac{(-1)^n\, n!}{(n+1)^n} \int_0^1 x^n\,dx = \dfrac{(-1)^n\, n!}{(n+1)^{n+1}} \quad \Rightarrow$

$$\int_0^1 x^x\,dx = \sum_{n=0}^{\infty} \frac{1}{n!} \int_0^1 x^n (\ln x)^n\,dx = \sum_{n=0}^{\infty} \frac{1}{n!} \frac{(-1)^n\, n!}{(n+1)^{n+1}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)^{n+1}} = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^n}.$$

18. (a) Since P_n is defined as the midpoint of $P_{n-4}P_{n-3}$, $x_n = \frac{1}{2}(x_{n-4} + x_{n-3})$ for $n \geq 5$. So we prove by induction that $\frac{1}{2}x_n + x_{n+1} + x_{n+2} + x_{n+3} = 2$. The case $n = 1$ is immediate, since $\frac{1}{2} \cdot 0 + 1 + 1 + 0 = 2$. Assume that the result holds for $n = k - 1$, that is, $\frac{1}{2}x_{k-1} + x_k + x_{k+1} + x_{k+2} = 2$. Then for $n = k$,

$$\begin{aligned}\tfrac{1}{2}x_k + x_{k+1} + x_{k+2} + x_{k+3} &= \tfrac{1}{2}x_k + x_{k+1} + x_{k+2} + \tfrac{1}{2}(x_{k+3-4} + x_{k+3-3}) \quad \text{(by above)}\\ &= \tfrac{1}{2}x_{k-1} + x_k + x_{k+1} + x_{k+2} = 2 \quad \text{(by the induction hypothesis)}\end{aligned}$$

Similarly, for $n \geq 5$, $y_n = \frac{1}{2}(y_{n-4} + y_{n-3})$, so the same argument as above holds for y, with 2 replaced by $\frac{1}{2}y_1 + y_2 + y_3 + y_4 = \frac{1}{2} \cdot 1 + 1 + 0 + 0 = \frac{3}{2}$. So $\frac{1}{2}y_n + y_{n+1} + y_{n+2} + y_{n+3} = \frac{3}{2}$ for all n.

(b) $\lim\limits_{n\to\infty}\left(\frac{1}{2}x_n + x_{n+1} + x_{n+2} + x_{n+3}\right) = \frac{1}{2}\lim\limits_{n\to\infty} x_n + \lim\limits_{n\to\infty} x_{n+1} + \lim\limits_{n\to\infty} x_{n+2} + \lim\limits_{n\to\infty} x_{n+3} = 2$. Since all the limits on the left hand side are the same, we get $\frac{7}{2}\lim\limits_{n\to\infty} x_n = 2 \Rightarrow \lim\limits_{n\to\infty} x_n = \frac{4}{7}$. In the same way, $\frac{7}{2}\lim\limits_{n\to\infty} y_n = \frac{3}{2} \Rightarrow \lim\limits_{n\to\infty} y_n = \frac{3}{7}$, so $P = \left(\frac{4}{7}, \frac{3}{7}\right)$.

19. Let $f(x) = \sum_{m=0}^{\infty} c_m x^m$ and $g(x) = e^{f(x)} = \sum_{n=0}^{\infty} d_n x^n$. Then $g'(x) = \sum_{n=0}^{\infty} n d_n x^{n-1}$, so $n d_n$ occurs as the coefficient of x^{n-1}. But also

$$\begin{aligned}g'(x) &= e^{f(x)} f'(x) = \left(\textstyle\sum_{n=0}^{\infty} d_n x^n\right)\left(\textstyle\sum_{m=1}^{\infty} m c_m x^{m-1}\right)\\ &= \left(d_0 + d_1 x + d_2 x^2 + \cdots + d_{n-1}x^{n-1} + \cdots\right)\left(c_1 + 2c_2 x + 3c_3 x^2 + \cdots + nc_n x^{n-1} + \cdots\right)\end{aligned}$$

so the coefficient of x^{n-1} is $c_1 d_{n-1} + 2c_2 d_{n-2} + 3c_3 d_{n-3} + \cdots + nc_n d_0 = \sum_{i=1}^{n} i c_i d_{n-i}$. Therefore, $n d_n = \sum_{i=1}^{n} i c_i d_{n-i}$.

20. Suppose the base of the first right triangle has length a. Then by repeated use of the Pythagorean theorem, we find that the base of the second right triangle has length $\sqrt{1 + a^2}$, the base of the third right triangle has length $\sqrt{2 + a^2}$, and in general, the nth right triangle has base of length $\sqrt{n - 1 + a^2}$ and hypotenuse of length $\sqrt{n + a^2}$. Thus, $\theta_n = \tan^{-1}\left(1/\sqrt{n - 1 + a^2}\right)$ and $\sum\limits_{n=1}^{\infty} \theta_n = \sum\limits_{n=1}^{\infty} \tan^{-1}\left(\dfrac{1}{\sqrt{n - 1 + a^2}}\right) = \sum\limits_{n=0}^{\infty} \tan^{-1}\left(\dfrac{1}{\sqrt{n + a^2}}\right)$. We wish to show that this series diverges.

First notice that the series $\sum\limits_{n=1}^{\infty} \dfrac{1}{\sqrt{n + a^2}}$ diverges by the Limit Comparison Test with the divergent p-series $\sum\limits_{n=1}^{\infty} \dfrac{1}{\sqrt{n}}$ $\left(p = \frac{1}{2} \leq 1\right)$ since

$$\lim_{n\to\infty} \frac{1/\sqrt{n + a^2}}{1/\sqrt{n}} = \lim_{n\to\infty} \frac{\sqrt{n}}{\sqrt{n + a^2}} = \lim_{n\to\infty} \sqrt{\frac{n}{n + a^2}} = \lim_{n\to\infty} \sqrt{\frac{1}{1 + a^2/n}} = 1 > 0.$$ Thus, $\sum\limits_{n=0}^{\infty} \dfrac{1}{\sqrt{n + a^2}}$ also diverges. Now $\sum\limits_{n=0}^{\infty} \tan^{-1}\left(\dfrac{1}{\sqrt{n + a^2}}\right)$ diverges by the Limit Comparison Test with $\sum\limits_{n=0}^{\infty} \dfrac{1}{\sqrt{n + a^2}}$ since

$$\lim_{n\to\infty}\frac{\tan^{-1}(1/\sqrt{n+a^2})}{1/\sqrt{n+a^2}}=\lim_{x\to\infty}\frac{\tan^{-1}(1/\sqrt{x+a^2})}{1/\sqrt{x+a^2}}=\lim_{y\to\infty}\frac{\tan^{-1}(1/y)}{1/y}\quad\left[y=\sqrt{x+a^2}\right]$$
$$=\lim_{z\to 0^+}\frac{\tan^{-1}z}{z}\quad[z=1/y]\overset{\text{H}}{=}\lim_{z\to 0^+}\frac{1/(1+z^2)}{1}=1>0$$

Thus, $\sum\limits_{n=1}^{\infty}\theta_n$ is a divergent series.

21. Call the series S. We group the terms according to the number of digits in their denominators:

$$S=\underbrace{\left(\tfrac{1}{1}+\tfrac{1}{2}+\cdots+\tfrac{1}{8}+\tfrac{1}{9}\right)}_{g_1}+\underbrace{\left(\tfrac{1}{11}+\cdots+\tfrac{1}{99}\right)}_{g_2}+\underbrace{\left(\tfrac{1}{111}+\cdots+\tfrac{1}{999}\right)}_{g_3}+\cdots$$

Now in the group g_n, since we have 9 choices for each of the n digits in the denominator, there are 9^n terms. Furthermore, each term in g_n is less than $\frac{1}{10^{n-1}}$ [except for the first term in g_1]. So $g_n<9^n\cdot\frac{1}{10^{n-1}}=9\left(\frac{9}{10}\right)^{n-1}$. Now $\sum_{n=1}^{\infty}9\left(\frac{9}{10}\right)^{n-1}$ is a geometric series with $a=9$ and $r=\frac{9}{10}<1$. Therefore, by the Comparison Test, $S=\sum_{n=1}^{\infty}g_n<\sum_{n=1}^{\infty}9\left(\frac{9}{10}\right)^{n-1}=\frac{9}{1-9/10}=90$.

22. (a) Let $f(x)=\dfrac{x}{1-x-x^2}=\sum\limits_{n=0}^{\infty}c_nx^n=c_0+c_1x+c_2x^2+c_3x^3+\cdots$. Then

$$x=(1-x-x^2)(c_0+c_1x+c_2x^2+c_3x^3+\cdots)$$

$$\begin{aligned}x=c_0+c_1x+c_2x^2+c_3x^3+c_4x^4+c_5x^5+\cdots\\ -c_0x-c_1x^2-c_2x^3-c_3x^4-c_4x^5-\cdots\\ -c_0x^2-c_1x^3-c_2x^4-c_3x^5-\cdots\end{aligned}$$

$$x=c_0+(c_1-c_0)x+(c_2-c_1-c_0)x^2+(c_3-c_2-c_1)x^3+\cdots$$

Comparing coefficients of powers of x gives us $c_0=0$ and

$$c_1-c_0=1\quad\Rightarrow\quad c_1=c_0+1=1$$

$$c_2-c_1-c_0=0\quad\Rightarrow\quad c_2=c_1+c_0=1+0=1$$

$$c_3-c_2-c_1=0\quad\Rightarrow\quad c_3=c_2+c_1=1+1=2$$

In general, we have $c_n=c_{n-1}+c_{n-2}$ for $n\geq 3$. Each c_n is equal to the nth Fibonacci number; that is,

$$\sum_{n=0}^{\infty}c_nx^n=\sum_{n=1}^{\infty}c_nx^n=\sum_{n=1}^{\infty}f_nx^n$$

(b) Completing the square on x^2+x-1 gives us

$$\left(x^2+x+\frac{1}{4}\right)-1-\frac{1}{4}=\left(x+\frac{1}{2}\right)^2-\frac{5}{4}=\left(x+\frac{1}{2}\right)^2-\left(\frac{\sqrt{5}}{2}\right)^2$$
$$=\left(x+\frac{1}{2}+\frac{\sqrt{5}}{2}\right)\left(x+\frac{1}{2}-\frac{\sqrt{5}}{2}\right)=\left(x+\frac{1+\sqrt{5}}{2}\right)\left(x+\frac{1-\sqrt{5}}{2}\right)$$

So $\dfrac{x}{1-x-x^2} = \dfrac{-x}{x^2+x-1} = \dfrac{-x}{\left(x+\frac{1+\sqrt{5}}{2}\right)\left(x+\frac{1-\sqrt{5}}{2}\right)}$. The factors in the denominator are linear, so the partial fraction decomposition is

$$\frac{-x}{\left(x+\frac{1+\sqrt{5}}{2}\right)\left(x+\frac{1-\sqrt{5}}{2}\right)} = \frac{A}{x+\frac{1+\sqrt{5}}{2}} + \frac{B}{x+\frac{1-\sqrt{5}}{2}}$$

$$-x = A\left(x+\tfrac{1-\sqrt{5}}{2}\right) + B\left(x+\tfrac{1+\sqrt{5}}{2}\right)$$

If $x = \frac{-1+\sqrt{5}}{2}$, then $-\frac{-1+\sqrt{5}}{2} = B\sqrt{5} \;\Rightarrow\; B = \frac{1-\sqrt{5}}{2\sqrt{5}}$.

If $x = \frac{-1-\sqrt{5}}{2}$, then $-\frac{-1-\sqrt{5}}{2} = A(-\sqrt{5}) \;\Rightarrow\; A = \frac{1+\sqrt{5}}{-2\sqrt{5}}$. Thus,

$$\begin{aligned}
\frac{x}{1-x-x^2} &= \frac{\frac{1+\sqrt{5}}{-2\sqrt{5}}}{x+\frac{1+\sqrt{5}}{2}} + \frac{\frac{1-\sqrt{5}}{2\sqrt{5}}}{x+\frac{1-\sqrt{5}}{2}} \\
&= \frac{\frac{1+\sqrt{5}}{-2\sqrt{5}}}{x+\frac{1+\sqrt{5}}{2}} \cdot \frac{\frac{2}{1+\sqrt{5}}}{\frac{2}{1+\sqrt{5}}} + \frac{\frac{1-\sqrt{5}}{2\sqrt{5}}}{x+\frac{1-\sqrt{5}}{2}} \cdot \frac{\frac{2}{1-\sqrt{5}}}{\frac{2}{1-\sqrt{5}}} \\
&= \frac{-1/\sqrt{5}}{1+\frac{2}{1+\sqrt{5}}x} + \frac{1/\sqrt{5}}{1+\frac{2}{1-\sqrt{5}}x} \\
&= -\frac{1}{\sqrt{5}}\sum_{n=0}^{\infty}\left(-\frac{2}{1+\sqrt{5}}x\right)^n + \frac{1}{\sqrt{5}}\sum_{n=0}^{\infty}\left(-\frac{2}{1-\sqrt{5}}x\right)^n \\
&= \frac{1}{\sqrt{5}}\sum_{n=0}^{\infty}\left[\left(\frac{-2}{1-\sqrt{5}}\right)^n - \left(\frac{-2}{1+\sqrt{5}}\right)^n\right]x^n \\
&= \frac{1}{\sqrt{5}}\sum_{n=1}^{\infty}\left[\frac{(-2)^n(1+\sqrt{5})^n - (-2)^n(1-\sqrt{5})^n}{(1-\sqrt{5})^n(1+\sqrt{5})^n}\right]x^n \qquad [\text{the } n=0 \text{ term is } 0] \\
&= \frac{1}{\sqrt{5}}\sum_{n=1}^{\infty}\left[\frac{(-2)^n\left((1+\sqrt{5})^n - (1-\sqrt{5})^n\right)}{(1-5)^n}\right]x^n \\
&= \frac{1}{\sqrt{5}}\sum_{n=1}^{\infty}\left[\frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2^n}\right]x^n \qquad [(-4)^n = (-2)^n \cdot 2^n]
\end{aligned}$$

From part (a), this series must equal $\sum_{n=1}^{\infty} f_n x^n$, so $f_n = \dfrac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{2^n\sqrt{5}}$, which is an explicit formula for the nth Fibonacci number.