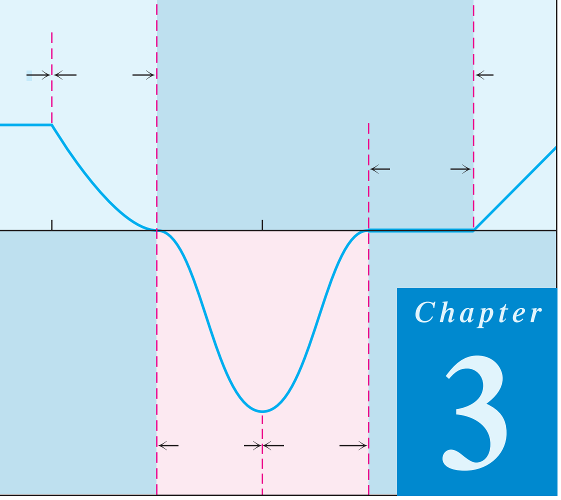


Chapter 3

DIFFERENTIATION

OVERVIEW In Chapter 2, we defined the slope of a curve at a point as the limit of secant slopes. This limit, called a derivative, measures the rate at which a function changes, and it is one of the most important ideas in calculus. Derivatives are used to calculate velocity and acceleration, to estimate the rate of spread of a disease, to set levels of production so as to maximize efficiency, to find the best dimensions of a cylindrical can, to find the age of a prehistoric artifact, and for many other applications. In this chapter, we develop techniques to calculate derivatives easily and learn how to use derivatives to approximate complicated functions.

3.1 The Derivative as a Function

HISTORICAL ESSAY

The Derivative

History

At the end of Chapter 2, we defined the slope of a curve $y = f(x)$ at the point where $x = x_0$ to be

$$\lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h}.$$

We called this limit, when it existed, the derivative of f at x_0. We now investigate the derivative as a *function* derived from f by considering the limit at each point of the domain of f.

DEFINITION Derivative Function

The **derivative** of the function $f(x)$ with respect to the variable x is the function f' whose value at x is

$$f'(x) = \lim_{h \to 0} \frac{f(x + h) - f(x)}{h},$$

provided the limit exists.

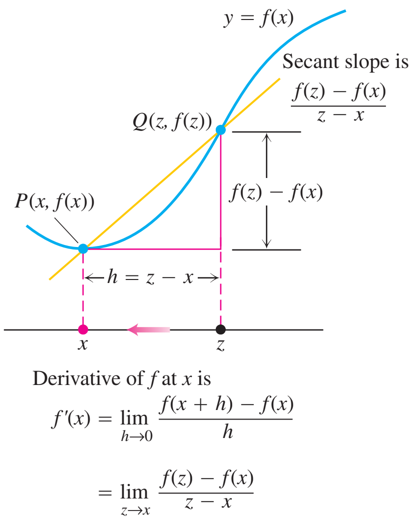


FIGURE 3.1 The way we write the difference quotient for the derivative of a function f depends on how we label the points involved.

We use the notation $f(x)$ rather than simply f in the definition to emphasize the independent variable x, which we are differentiating with respect to. The domain of f' is the set of points in the domain of f for which the limit exists, and the domain may be the same or smaller than the domain of f. If f' exists at a particular x, we say that f is **differentiable (has a derivative)** at x. If f' exists at every point in the domain of f, we call f **differentiable**.

If we write $z = x + h$, then $h = z - x$ and h approaches 0 if and only if z approaches x. Therefore, an equivalent definition of the derivative is as follows (see Figure 3.1).

Alternative Formula for the Derivative

$$f'(x) = \lim_{z \to x} \frac{f(z) - f(x)}{z - x}.$$

Calculating Derivatives from the Definition

The process of calculating a derivative is called **differentiation**. To emphasize the idea that differentiation is an operation performed on a function $y = f(x)$, we use the notation

$$\frac{d}{dx} f(x)$$

as another way to denote the derivative $f'(x)$. Examples 2 and 3 of Section 2.7 illustrate the differentiation process for the functions $y = mx + b$ and $y = 1/x$. Example 2 shows that

$$\frac{d}{dx}(mx + b) = m.$$

For instance,

$$\frac{d}{dx}\left(\frac{3}{2}x - 4\right) = \frac{3}{2}.$$

In Example 3, we see that

$$\frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}.$$

Here are two more examples.

EXAMPLE 1 Applying the Definition

You Try It

Differentiate $f(x) = \dfrac{x}{x - 1}$.

Solution Here we have $f(x) = \dfrac{x}{x - 1}$

and

$$f(x+h) = \frac{(x+h)}{(x+h)-1}, \text{ so}$$

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} \\ &= \frac{\dfrac{x+h}{x+h-1} - \dfrac{x}{x-1}}{h} \\ &= \lim_{h\to 0} \frac{1}{h} \cdot \frac{(x+h)(x-1) - x(x+h-1)}{(x+h-1)(x-1)} \\ &= \lim_{h\to 0} \frac{1}{h} \cdot \frac{-h}{(x+h-1)(x-1)} \\ &= \lim_{h\to 0} \frac{-1}{(x+h-1)(x-1)} = \frac{-1}{(x-1)^2}. \end{aligned}$$

$\frac{a}{b} - \frac{c}{d} = \frac{ad - cb}{bd}$

■

EXAMPLE 2 Derivative of the Square Root Function

(a) Find the derivative of $y = \sqrt{x}$ for $x > 0$.

(b) Find the tangent line to the curve $y = \sqrt{x}$ at $x = 4$.

You will often need to know the derivative of $\sqrt{x}$ for $x > 0$:

$$\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}.$$

Solution

(a) We use the equivalent form to calculate f':

$$\begin{aligned} f'(x) &= \lim_{z\to x} \frac{f(z)-f(x)}{z-x} \\ &= \lim_{z\to x} \frac{\sqrt{z}-\sqrt{x}}{z-x} \\ &= \lim_{z\to x} \frac{\sqrt{z}-\sqrt{x}}{\left(\sqrt{z}-\sqrt{x}\right)\left(\sqrt{z}+\sqrt{x}\right)} \\ &= \lim_{z\to x} \frac{1}{\sqrt{z}+\sqrt{x}} = \frac{1}{2\sqrt{x}}. \end{aligned}$$

(b) The slope of the curve at $x = 4$ is

$$f'(4) = \frac{1}{2\sqrt{4}} = \frac{1}{4}.$$

The tangent is the line through the point (4, 2) with slope 1/4 (Figure 3.2):

$$y = 2 + \frac{1}{4}(x - 4)$$

$$y = \frac{1}{4}x + 1.$$

■

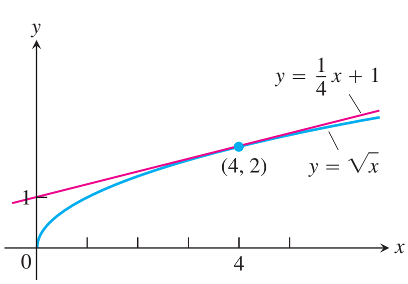


FIGURE 3.2 The curve $y = \sqrt{x}$ and its tangent at (4, 2). The tangent's slope is found by evaluating the derivative at $x = 4$ (Example 2).

We consider the derivative of $y = \sqrt{x}$ when $x = 0$ in Example 6.

Notations

There are many ways to denote the derivative of a function $y = f(x)$, where the independent variable is x and the dependent variable is y. Some common alternative notations for the derivative are

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx}f(x) = D(f)(x) = D_x f(x).$$

The symbols d/dx and D indicate the operation of differentiation and are called **differentiation operators**. We read dy/dx as "the derivative of y with respect to x," and df/dx and $(d/dx)f(x)$ as "the derivative of f with respect to x." The "prime" notations y' and f' come from notations that Newton used for derivatives. The d/dx notations are similar to those used by Leibniz. The symbol dy/dx should not be regarded as a ratio (until we introduce the idea of "differentials" in Section 3.8).

Be careful not to confuse the notation $D(f)$ as meaning the domain of the function f instead of the derivative function f'. The distinction should be clear from the context.

To indicate the value of a derivative at a specified number $x = a$, we use the notation

$$f'(a) = \left.\frac{dy}{dx}\right|_{x=a} = \left.\frac{df}{dx}\right|_{x=a} = \left.\frac{d}{dx}f(x)\right|_{x=a}.$$

For instance, in Example 2b we could write

$$f'(4) = \left.\frac{d}{dx}\sqrt{x}\right|_{x=4} = \left.\frac{1}{2\sqrt{x}}\right|_{x=4} = \frac{1}{2\sqrt{4}} = \frac{1}{4}.$$

To evaluate an expression, we sometimes use the right bracket] in place of the vertical bar $|$.

Graphing the Derivative

We can often make a reasonable plot of the derivative of $y = f(x)$ by estimating the slopes on the graph of f. That is, we plot the points $(x, f'(x))$ in the xy-plane and connect them with a smooth curve, which represents $y = f'(x)$.

EXAMPLE 3 Graphing a Derivative

Graph the derivative of the function $y = f(x)$ in Figure 3.3a.

Solution We sketch the tangents to the graph of f at frequent intervals and use their slopes to estimate the values of $f'(x)$ at these points. We plot the corresponding $(x, f'(x))$ pairs and connect them with a smooth curve as sketched in Figure 3.3b. ■

What can we learn from the graph of $y = f'(x)$? At a glance we can see

1. where the rate of change of f is positive, negative, or zero;
2. the rough size of the growth rate at any x and its size in relation to the size of $f(x)$;
3. where the rate of change itself is increasing or decreasing.

Here's another example.

EXAMPLE 4 Concentration of Blood Sugar

On April 23, 1988, the human-powered airplane *Daedalus* flew a record-breaking 119 km from Crete to the island of Santorini in the Aegean Sea, southeast of mainland Greece. Dur-

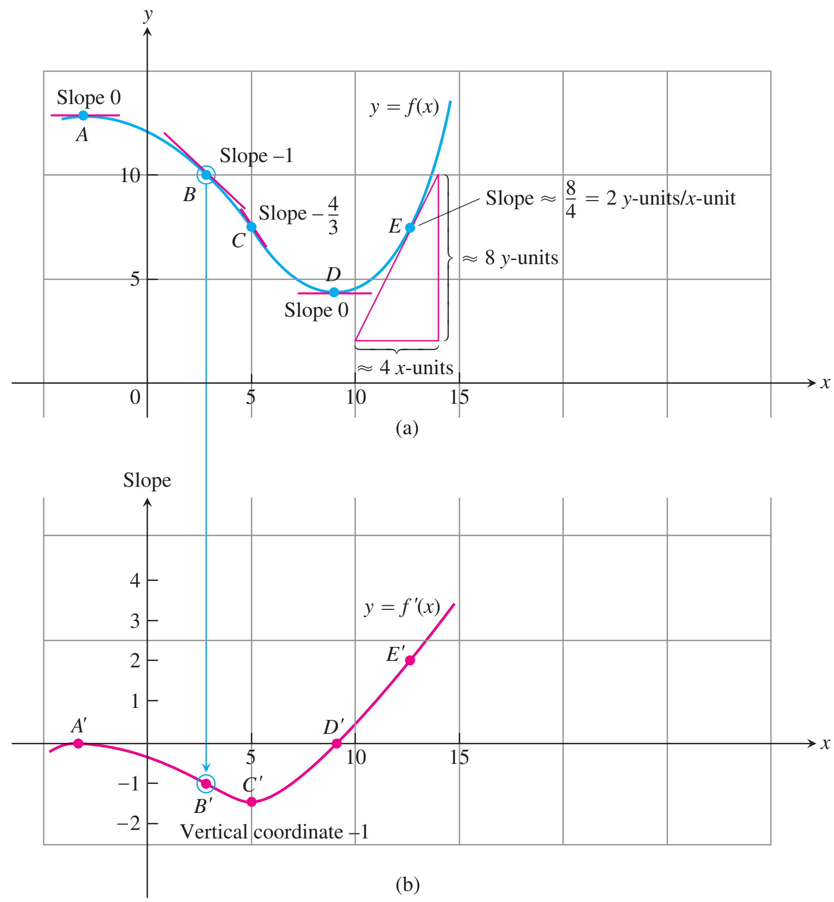


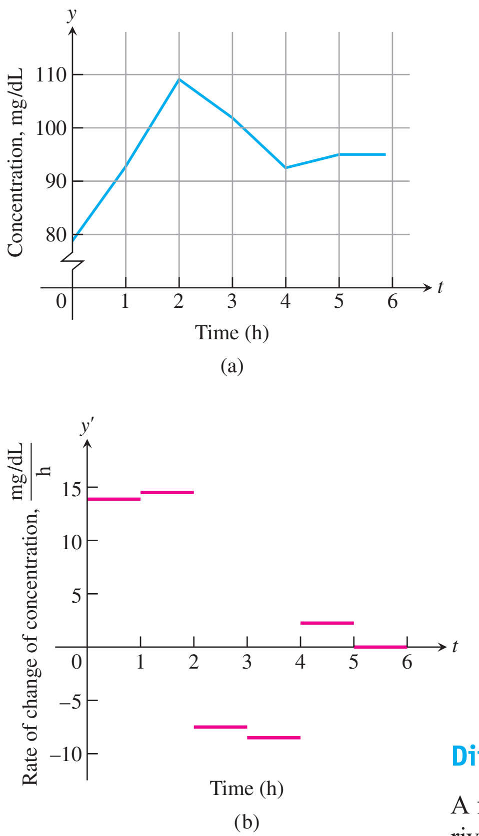
FIGURE 3.3 We made the graph of $y = f'(x)$ in (b) by plotting slopes from the graph of $y = f(x)$ in (a). The vertical coordinate of B' is the slope at B and so on. The graph of f' is a visual record of how the slope of f changes with x.

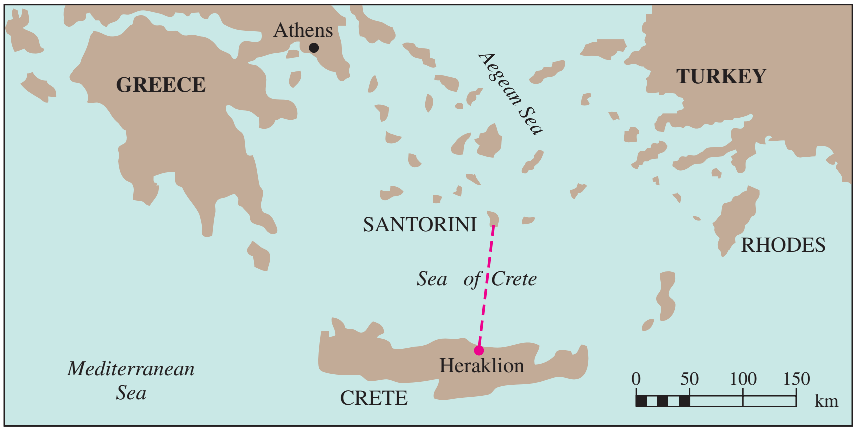
ing the 6-hour endurance tests before the flight, researchers monitored the prospective pilots' blood-sugar concentrations. The concentration graph for one of the athlete-pilots is shown in Figure 3.4a, where the concentration in milligrams/deciliter is plotted against time in hours.

The graph consists of line segments connecting data points. The constant slope of each segment gives an estimate of the derivative of the concentration between measurements. We calculated the slope of each segment from the coordinate grid and plotted the derivative as a step function in Figure 3.4b. To make the plot for the first hour, for instance, we observed that the concentration increased from about 79 mg/dL to 93 mg/dL. The net increase was $\Delta y = 93 - 79 = 14$ mg/dL. Dividing this by $\Delta t = 1$ hour gave the rate of change as

$$\frac{\Delta y}{\Delta t} = \frac{14}{1} = 14 \text{ mg/dL per hour}.$$

Notice that we can make no estimate of the concentration's rate of change at times $t = 1, 2, \ldots, 5$, where the graph we have drawn for the concentration has a corner and no slope. The derivative step function is not defined at these times. ■





Daedalus's flight path on April 23, 1988

FIGURE 3.4 (a) Graph of the sugar concentration in the blood of a *Daedalus* pilot during a 6-hour preflight endurance test. (b) The derivative of the pilot's blood-sugar concentration shows how rapidly the concentration rose and fell during various portions of the test.

Differentiable on an Interval; One-Sided Derivatives

A function $y = f(x)$ is **differentiable** on an open interval (finite or infinite) if it has a derivative at each point of the interval. It is differentiable on a closed interval $[a, b]$ if it is differentiable on the interior (a, b) and if the limits

$$\lim_{h\to 0^+} \frac{f(a+h) - f(a)}{h} \qquad \textbf{Right-hand derivative at } \boldsymbol{a}$$

$$\lim_{h\to 0^-} \frac{f(b+h) - f(b)}{h} \qquad \textbf{Left-hand derivative at } \boldsymbol{b}$$

exist at the endpoints (Figure 3.5).

Right-hand and left-hand derivatives may be defined at any point of a function's domain. The usual relation between one-sided and two-sided limits holds for these derivatives. Because of Theorem 6, Section 2.4, a function has a derivative at a point if and only if it has left-hand and right-hand derivatives there, and these one-sided derivatives are equal.

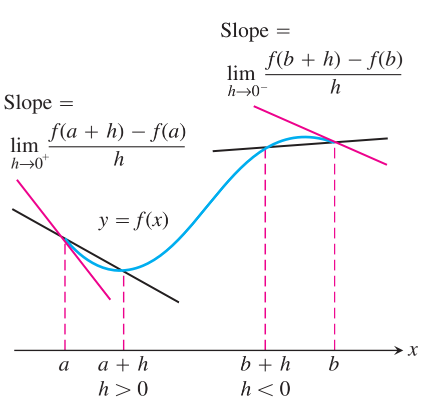


FIGURE 3.5 Derivatives at endpoints are one-sided limits.

EXAMPLE 5 $y = |x|$ Is Not Differentiable at the Origin

Show that the function $y = |x|$ is differentiable on $(-\infty, 0)$ and $(0, \infty)$ but has no derivative at $x = 0$.

Solution To the right of the origin,

$$\frac{d}{dx}(|x|) = \frac{d}{dx}(x) = \frac{d}{dx}(1 \cdot x) = 1. \qquad \frac{d}{dx}(mx + b) = m, |x| = x$$

To the left,

$$\frac{d}{dx}(|x|) = \frac{d}{dx}(-x) = \frac{d}{dx}(-1 \cdot x) = -1 \qquad |x| = -x$$

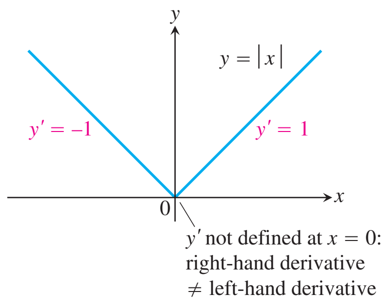


FIGURE 3.6 The function $y = |x|$ is not differentiable at the origin where the graph has a "corner."

(Figure 3.6). There can be no derivative at the origin because the one-sided derivatives differ there:

$$\begin{aligned}
\text{Right-hand derivative of } |x| \text{ at zero} &= \lim_{h\to 0^+} \frac{|0+h| - |0|}{h} = \lim_{h\to 0^+} \frac{|h|}{h} \\
&= \lim_{h\to 0^+} \frac{h}{h} \qquad |h| = h \text{ when } h > 0. \\
&= \lim_{h\to 0^+} 1 = 1 \\
\text{Left-hand derivative of } |x| \text{ at zero} &= \lim_{h\to 0^-} \frac{|0+h| - |0|}{h} = \lim_{h\to 0^-} \frac{|h|}{h} \\
&= \lim_{h\to 0^-} \frac{-h}{h} \qquad |h| = -h \text{ when } h < 0. \\
&= \lim_{h\to 0^-} -1 = -1.
\end{aligned}$$

■

EXAMPLE 6 $y = \sqrt{x}$ Is Not Differentiable at $x = 0$

In Example 2 we found that for $x > 0$,

$$\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}.$$

We apply the definition to examine if the derivative exists at $x = 0$:

$$\lim_{h\to 0^+} \frac{\sqrt{0+h} - \sqrt{0}}{h} = \lim_{h\to 0^+} \frac{1}{\sqrt{h}} = \infty.$$

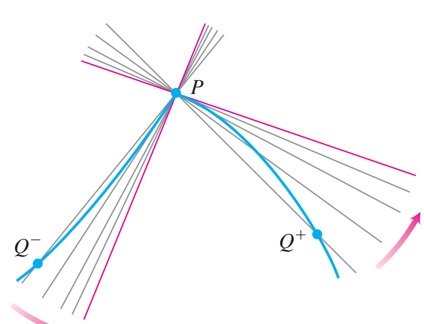
Since the (right-hand) limit is not finite, there is no derivative at $x = 0$. Since the slopes of the secant lines joining the origin to the points $(h, \sqrt{h})$ on a graph of $y = \sqrt{x}$ approach ∞, the graph has a *vertical tangent* at the origin. ■

When Does a Function *Not* Have a Derivative at a Point?

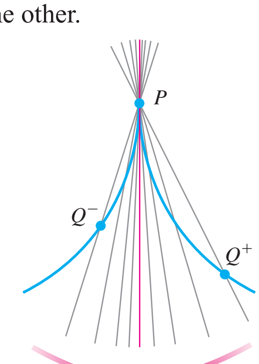
A function has a derivative at a point x_0 if the slopes of the secant lines through $P(x_0, f(x_0))$ and a nearby point Q on the graph approach a limit as Q approaches P. Whenever the secants fail to take up a limiting position or become vertical as Q approaches P, the derivative does not exist. Thus differentiability is a "smoothness" condition on the graph of f. A function whose graph is otherwise smooth will fail to have a derivative at a point for several reasons, such as at points where the graph has

You Try It

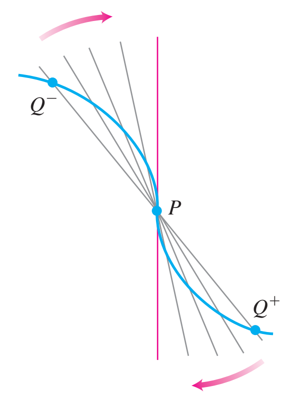
1. a *corner*, where the one-sided derivatives differ.



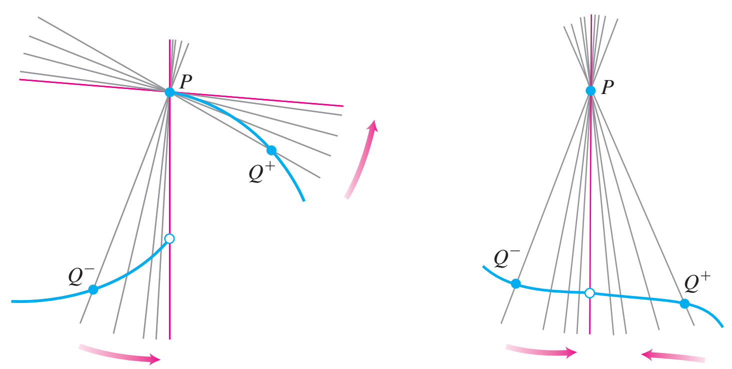
2. a *cusp*, where the slope of PQ approaches ∞ from one side and $-\infty$ from the other.



3. a *vertical tangent*, where the slope of PQ approaches ∞ from both sides or approaches $-\infty$ from both sides (here, $-\infty$).



4. a *discontinuity*.



Differentiable Functions Are Continuous

A function is continuous at every point where it has a derivative.

> **THEOREM 1 Differentiability Implies Continuity**
> If f has a derivative at $x = c$, then f is continuous at $x = c$.

Proof Given that $f'(c)$ exists, we must show that $\lim_{x\to c} f(x) = f(c)$, or equivalently, that $\lim_{h\to 0} f(c+h) = f(c)$. If $h \neq 0$, then

$$\begin{aligned} f(c+h) &= f(c) + (f(c+h) - f(c)) \\ &= f(c) + \frac{f(c+h) - f(c)}{h} \cdot h. \end{aligned}$$

Now take limits as $h \to 0$. By Theorem 1 of Section 2.2,

$$\begin{aligned}\lim_{h\to 0} f(c+h) &= \lim_{h\to 0} f(c) + \lim_{h\to 0} \frac{f(c+h)-f(c)}{h} \cdot \lim_{h\to 0} h \\ &= f(c) + f'(c)\cdot 0 \\ &= f(c) + 0 \\ &= f(c).\end{aligned}$$

■

Similar arguments with one-sided limits show that if f has a derivative from one side (right or left) at $x = c$ then f is continuous from that side at $x = c$.

Theorem 1 on page 154 says that if a function has a discontinuity at a point (for instance, a jump discontinuity), then it cannot be differentiable there. The greatest integer function $y = \lfloor x \rfloor = \operatorname{int} x$ fails to be differentiable at every integer $x = n$ (Example 4, Section 2.6).

CAUTION The converse of Theorem 1 is false. A function need not have a derivative at a point where it is continuous, as we saw in Example 5.

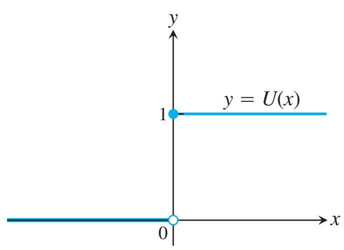


FIGURE 3.7 The unit step function does not have the Intermediate Value Property and cannot be the derivative of a function on the real line.

The Intermediate Value Property of Derivatives

Not every function can be some function's derivative, as we see from the following theorem.

> **THEOREM 2**
> If a and b are any two points in an interval on which f is differentiable, then f' takes on every value between $f'(a)$ and $f'(b)$.

Theorem 2 (which we will not prove) says that a function cannot *be* a derivative on an interval unless it has the Intermediate Value Property there. For example, the unit step function in Figure 3.7 cannot be the derivative of any real-valued function on the real line. In Chapter 5 we will see that every continuous function is a derivative of some function.

In Section 4.4, we invoke Theorem 2 to analyze what happens at a point on the graph of a twice-differentiable function where it changes its "bending" behavior.

EXERCISES 3.1

Finding Derivative Functions and Values

Using the definition, calculate the derivatives of the functions in Exercises 1–6. Then find the values of the derivatives as specified.

1. $f(x) = 4 - x^2; \quad f'(-3), f'(0), f'(1)$
2. $F(x) = (x - 1)^2 + 1; \quad F'(-1), F'(0), F'(2)$
3. $g(t) = \dfrac{1}{t^2}; \quad g'(-1), g'(2), g'\left(\sqrt{3}\right)$
4. $k(z) = \dfrac{1 - z}{2z}; \quad k'(-1), k'(1), k'\left(\sqrt{2}\right)$
5. $p(\theta) = \sqrt{3\theta}; \quad p'(1), p'(3), p'(2/3)$
6. $r(s) = \sqrt{2s + 1}; \quad r'(0), r'(1), r'(1/2)$

In Exercises 7–12, find the indicated derivatives.

7. $\dfrac{dy}{dx}$ if $y = 2x^3$
8. $\dfrac{dr}{ds}$ if $r = \dfrac{s^3}{2} + 1$

9. $\dfrac{ds}{dt}$ if $s = \dfrac{t}{2t + 1}$

10. $\dfrac{dv}{dt}$ if $v = t - \dfrac{1}{t}$

11. $\dfrac{dp}{dq}$ if $p = \dfrac{1}{\sqrt{q + 1}}$

12. $\dfrac{dz}{dw}$ if $z = \dfrac{1}{\sqrt{3w - 2}}$

Slopes and Tangent Lines

In Exercises 13–16, differentiate the functions and find the slope of the tangent line at the given value of the independent variable.

13. $f(x) = x + \dfrac{9}{x}, \quad x = -3$

14. $k(x) = \dfrac{1}{2 + x}, \quad x = 2$

15. $s = t^3 - t^2, \quad t = -1$

16. $y = (x + 1)^3, \quad x = -2$

In Exercises 17–18, differentiate the functions. Then find an equation of the tangent line at the indicated point on the graph of the function.

17. $y = f(x) = \dfrac{8}{\sqrt{x - 2}}, \quad (x, y) = (6, 4)$

18. $w = g(z) = 1 + \sqrt{4 - z}, \quad (z, w) = (3, 2)$

In Exercises 19–22, find the values of the derivatives.

19. $\left.\dfrac{ds}{dt}\right|_{t=-1}$ if $s = 1 - 3t^2$

20. $\left.\dfrac{dy}{dx}\right|_{x=\sqrt{3}}$ if $y = 1 - \dfrac{1}{x}$

21. $\left.\dfrac{dr}{d\theta}\right|_{\theta=0}$ if $r = \dfrac{2}{\sqrt{4 - \theta}}$

22. $\left.\dfrac{dw}{dz}\right|_{z=4}$ if $w = z + \sqrt{z}$

Using the Alternative Formula for Derivatives

Use the formula

$$f'(x) = \lim_{z \to x} \frac{f(z) - f(x)}{z - x}$$

to find the derivative of the functions in Exercises 23–26.

23. $f(x) = \dfrac{1}{x + 2}$

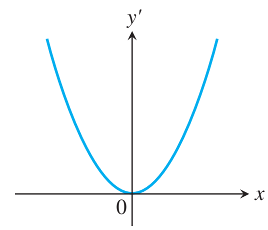
24. $f(x) = \dfrac{1}{(x - 1)^2}$

25. $g(x) = \dfrac{x}{x - 1}$

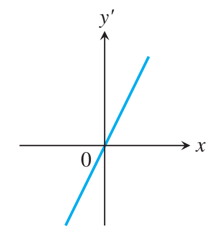
26. $g(x) = 1 + \sqrt{x}$

Graphs

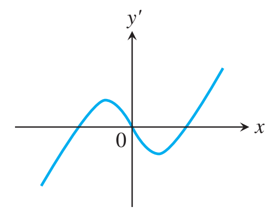
Match the functions graphed in Exercises 27–30 with the derivatives graphed in the accompanying figures (a)–(d).



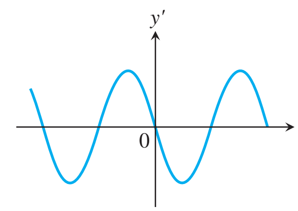
(a)



(b)

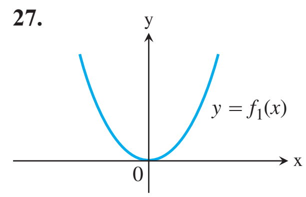


(c)



(d)

27.



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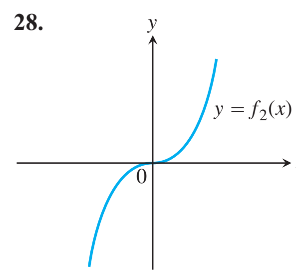

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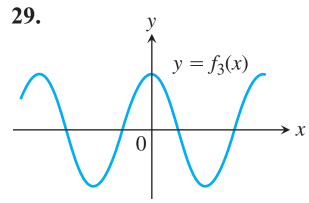
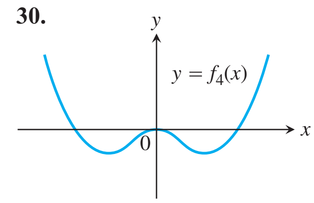
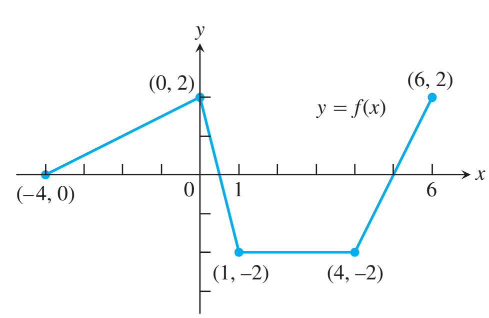


30.



31. a. The graph in the accompanying figure is made of line segments joined end to end. At which points of the interval $[-4, 6]$ is f' not defined? Give reasons for your answer.



Exercises

b. Graph the derivative of f.
The graph should show a step function.

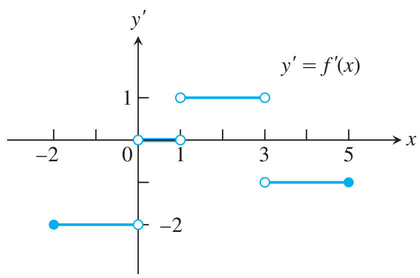
32. Recovering a function from its derivative

a. Use the following information to graph the function f over the closed interval $[-2, 5]$.

i) The graph of f is made of closed line segments joined end to end.

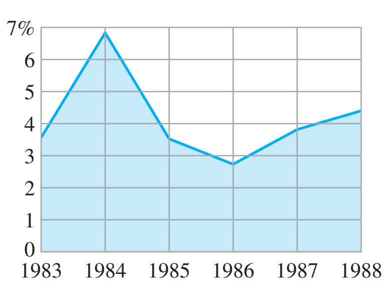
ii) The graph starts at the point $(-2, 3)$.

iii) The derivative of f is the step function in the figure shown here.



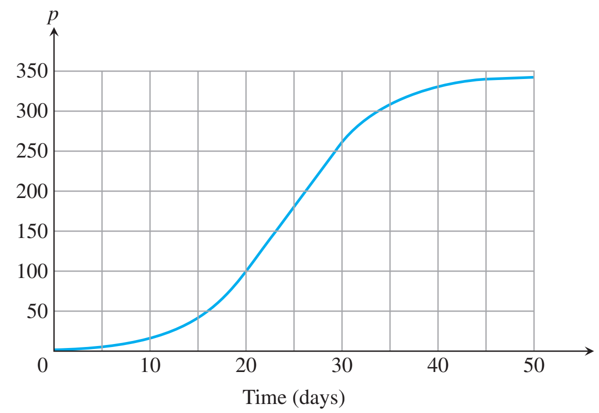
b. Repeat part (a) assuming that the graph starts at $(-2, 0)$ instead of $(-2, 3)$.

33. Growth in the economy The graph in the accompanying figure shows the average annual percentage change $y = f(t)$ in the U.S. gross national product (GNP) for the years 1983–1988. Graph dy/dt (where defined). (*Source*: *Statistical Abstracts of the United States*, 110th Edition, U.S. Department of Commerce, p. 427.)



34. Fruit flies (*Continuation of Example 3, Section 2.1.*) Populations starting out in closed environments grow slowly at first, when there are relatively few members, then more rapidly as the number of reproducing individuals increases and resources are still abundant, then slowly again as the population reaches the carrying capacity of the environment.

a. Use the graphical technique of Example 3 to graph the derivative of the fruit fly population introduced in Section 2.1. The graph of the population is reproduced here.

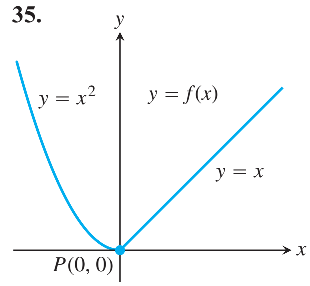


b. During what days does the population seem to be increasing fastest? Slowest?

One-Sided Derivatives

Compare the right-hand and left-hand derivatives to show that the functions in Exercises 35–38 are not differentiable at the point P.

35.



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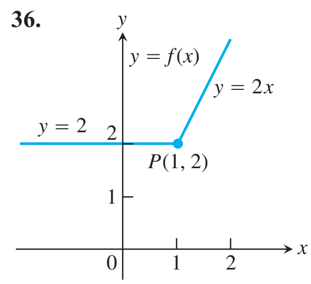

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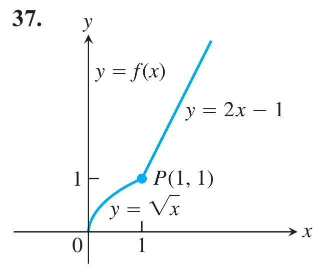

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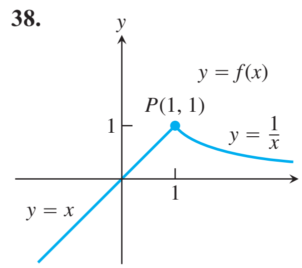


Differentiability and Continuity on an Interval

Each figure in Exercises 39–44 shows the graph of a function over a closed interval D. At what domain points does the function appear to be

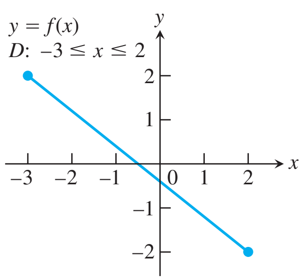
a. differentiable?

b. continuous but not differentiable?

c. neither continuous nor differentiable?

Give reasons for your answers.

39.



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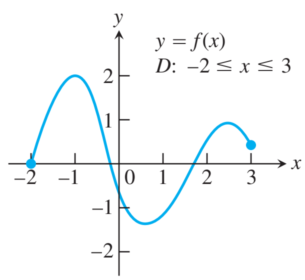

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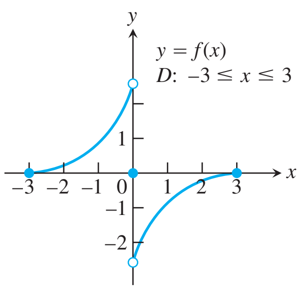

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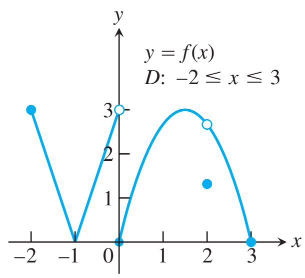
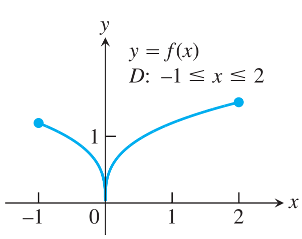
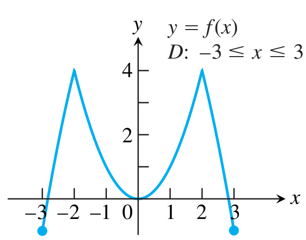


43.



44.



Theory and Examples

In Exercises 45–48,

a. Find the derivative $f'(x)$ of the given function $y = f(x)$.

b. Graph $y = f(x)$ and $y = f'(x)$ side by side using separate sets of coordinate axes, and answer the following questions.

c. For what values of x, if any, is f' positive? Zero? Negative?

d. Over what intervals of x-values, if any, does the function $y = f(x)$ increase as x increases? Decrease as x increases? How is this related to what you found in part (c)? (We will say more about this relationship in Chapter 4.)

45. $y = -x^2$ **46.** $y = -1/x$

47. $y = x^3/3$ **48.** $y = x^4/4$

49. Does the curve $y = x^3$ ever have a negative slope? If so, where? Give reasons for your answer.

50. Does the curve $y = 2\sqrt{x}$ have any horizontal tangents? If so, where? Give reasons for your answer.

51. Tangent to a parabola Does the parabola $y = 2x^2 - 13x + 5$ have a tangent whose slope is -1? If so, find an equation for the line and the point of tangency. If not, why not?

52. Tangent to $y = \sqrt{x}$ Does any tangent to the curve $y = \sqrt{x}$ cross the x-axis at $x = -1$? If so, find an equation for the line and the point of tangency. If not, why not?

53. Greatest integer in x Does any function differentiable on $(-\infty, \infty)$ have $y = \operatorname{int} x$, the greatest integer in x (see Figure 2.55), as its derivative? Give reasons for your answer.

54. Derivative of $y = |x|$ Graph the derivative of $f(x) = |x|$. Then graph $y = (|x| - 0)/(x - 0) = |x|/x$. What can you conclude?

55. Derivative of $-f$ Does knowing that a function $f(x)$ is differentiable at $x = x_0$ tell you anything about the differentiability of the function $-f$ at $x = x_0$? Give reasons for your answer.

56. Derivative of multiples Does knowing that a function $g(t)$ is differentiable at $t = 7$ tell you anything about the differentiability of the function $3g$ at $t = 7$? Give reasons for your answer.

57. Limit of a quotient Suppose that functions $g(t)$ and $h(t)$ are defined for all values of t and $g(0) = h(0) = 0$. Can $\lim_{t\to 0} (g(t))/(h(t))$ exist? If it does exist, must it equal zero? Give reasons for your answers.

58. **a.** Let $f(x)$ be a function satisfying $|f(x)| \le x^2$ for $-1 \le x \le 1$. Show that f is differentiable at $x = 0$ and find $f'(0)$.

b. Show that

$$f(x) = \begin{cases} x^2 \sin \dfrac{1}{x}, & x \ne 0 \\ 0, & x = 0 \end{cases}$$

is differentiable at $x = 0$ and find $f'(0)$.

T **59.** Graph $y = 1/(2\sqrt{x})$ in a window that has $0 \le x \le 2$. Then, on the same screen, graph

$$y = \frac{\sqrt{x+h} - \sqrt{x}}{h}$$

for $h = 1, 0.5, 0.1$. Then try $h = -1, -0.5, -0.1$. Explain what is going on.

T **60.** Graph $y = 3x^2$ in a window that has $-2 \le x \le 2, 0 \le y \le 3$. Then, on the same screen, graph

$$y = \frac{(x+h)^3 - x^3}{h}$$

for $h = 2, 1, 0.2$. Then try $h = -2, -1, -0.2$. Explain what is going on.

T **61. Weierstrass's nowhere differentiable continuous function** The sum of the first eight terms of the Weierstrass function $f(x) = \sum_{n=0}^{\infty} (2/3)^n \cos(9^n \pi x)$ is

$$g(x) = \cos(\pi x) + (2/3)^1 \cos(9\pi x) + (2/3)^2 \cos(9^2 \pi x) + (2/3)^3 \cos(9^3 \pi x) + \cdots + (2/3)^7 \cos(9^7 \pi x).$$

Graph this sum. Zoom in several times. How wiggly and bumpy is this graph? Specify a viewing window in which the displayed portion of the graph is smooth.

COMPUTER EXPLORATIONS

Use a CAS to perform the following steps for the functions in Exercises 62–67.

a. Plot $y = f(x)$ to see that function's global behavior.

b. Define the difference quotient q at a general point x, with general step size h.

c. Take the limit as $h \to 0$. What formula does this give?

d. Substitute the value $x = x_0$ and plot the function $y = f(x)$ together with its tangent line at that point.

e. Substitute various values for x larger and smaller than x_0 into the formula obtained in part (c). Do the numbers make sense with your picture?

f. Graph the formula obtained in part (c). What does it mean when its values are negative? Zero? Positive? Does this make sense with your plot from part (a)? Give reasons for your answer.

62. $f(x) = x^3 + x^2 - x, \quad x_0 = 1$

63. $f(x) = x^{1/3} + x^{2/3}, \quad x_0 = 1$

64. $f(x) = \dfrac{4x}{x^2 + 1}, \quad x_0 = 2$

65. $f(x) = \dfrac{x - 1}{3x^2 + 1}, \quad x_0 = -1$

66. $f(x) = \sin 2x, \quad x_0 = \pi/2$

67. $f(x) = x^2 \cos x, \quad x_0 = \pi/4$

3.2 Differentiation Rules

This section introduces a few rules that allow us to differentiate a great variety of functions. By proving these rules here, we can differentiate functions without having to apply the definition of the derivative each time.

Powers, Multiples, Sums, and Differences

The first rule of differentiation is that the derivative of every constant function is zero.

RULE 1 Derivative of a Constant Function

If f has the constant value $f(x) = c$, then

$$\frac{df}{dx} = \frac{d}{dx}(c) = 0.$$

EXAMPLE 1

If f has the constant value $f(x) = 8$, then

$$\frac{df}{dx} = \frac{d}{dx}(8) = 0.$$

Similarly,

$$\frac{d}{dx}\left(-\frac{\pi}{2}\right) = 0 \quad \text{and} \quad \frac{d}{dx}\left(\sqrt{3}\right) = 0.$$

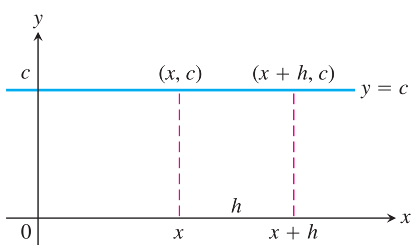


FIGURE 3.8 The rule $(d/dx)(c) = 0$ is another way to say that the values of constant functions never change and that the slope of a horizontal line is zero at every point.

Proof of Rule 1 We apply the definition of derivative to $f(x) = c$, the function whose outputs have the constant value c (Figure 3.8). At every value of x, we find that

$$f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h} = \lim_{h\to 0}\frac{c - c}{h} = \lim_{h\to 0} 0 = 0.$$

The second rule tells how to differentiate x^n if n is a positive integer.

RULE 2 Power Rule for Positive Integers

If n is a positive integer, then

$$\frac{d}{dx}x^n = nx^{n-1}.$$

To apply the Power Rule, we subtract 1 from the original exponent (n) and multiply the result by n.

EXAMPLE 2 Interpreting Rule 2

f	x	x^2	x^3	x^4	$\cdots$
f'	1	$2x$	$3x^2$	$4x^3$	$\cdots$

HISTORICAL BIOGRAPHY

Richard Courant
(1888–1972)

Biography

First Proof of Rule 2 The formula

$$z^n - x^n = (z - x)(z^{n-1} + z^{n-2}x + \cdots + zx^{n-2} + x^{n-1})$$

can be verified by multiplying out the right-hand side. Then from the alternative form for the definition of the derivative,

$$\begin{aligned} f'(x) &= \lim_{z \to x} \frac{f(z) - f(x)}{z - x} = \lim_{z \to x} \frac{z^n - x^n}{z - x} \\ &= \lim_{z \to x}(z^{n-1} + z^{n-2}x + \cdots + zx^{n-2} + x^{n-1}) \\ &= nx^{n-1} \end{aligned}$$

Second Proof of Rule 2 If $f(x) = x^n$, then $f(x + h) = (x + h)^n$. Since n is a positive integer, we can expand $(x + h)^n$ by the Binomial Theorem to get

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^n - x^n}{h} \\ &= \lim_{h \to 0} \frac{\left[x^n + nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h^2 + \cdots + nxh^{n-1} + h^n\right] - x^n}{h} \\ &= \lim_{h \to 0} \frac{nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h^2 + \cdots + nxh^{n-1} + h^n}{h} \\ &= \lim_{h \to 0} \left[nx^{n-1} + \frac{n(n-1)}{2}x^{n-2}h + \cdots + nxh^{n-2} + h^{n-1}\right] \\ &= nx^{n-1} \end{aligned}$$

The third rule says that when a differentiable function is multiplied by a constant, its derivative is multiplied by the same constant.

RULE 3 Constant Multiple Rule
If u is a differentiable function of x, and c is a constant, then

$$\frac{d}{dx}(cu) = c\frac{du}{dx}.$$

In particular, if n is a positive integer, then

$$\frac{d}{dx}(cx^n) = cnx^{n-1}.$$

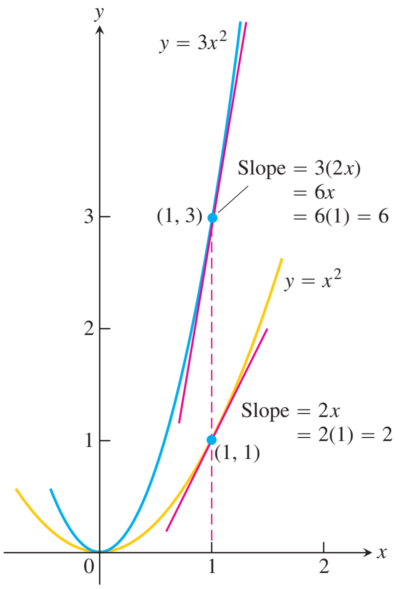


FIGURE 3.9 The graphs of $y = x^2$ and $y = 3x^2$. Tripling the y-coordinates triples the slope (Example 3).

EXAMPLE 3

(a) The derivative formula

$$\frac{d}{dx}(3x^2) = 3 \cdot 2x = 6x$$

says that if we rescale the graph of $y = x^2$ by multiplying each y-coordinate by 3, then we multiply the slope at each point by 3 (Figure 3.9).

(b) A useful special case

The derivative of the negative of a differentiable function u is the negative of the function's derivative. Rule 3 with $c = -1$ gives

$$\frac{d}{dx}(-u) = \frac{d}{dx}(-1 \cdot u) = -1 \cdot \frac{d}{dx}(u) = -\frac{du}{dx}.$$ ■

Proof of Rule 3

$$\begin{aligned}
\frac{d}{dx}cu &= \lim_{h\to 0}\frac{cu(x+h) - cu(x)}{h} && \text{Derivative definition with } f(x) = cu(x)\\
&= c\lim_{h\to 0}\frac{u(x+h) - u(x)}{h} && \text{Limit property}\\
&= c\frac{du}{dx} && u \text{ is differentiable.}
\end{aligned}$$ ■

The next rule says that the derivative of the sum of two differentiable functions is the sum of their derivatives.

Denoting Functions by u and v
The functions we are working with when we need a differentiation formula are likely to be denoted by letters like f and g. When we apply the formula, we do not want to find it using these same letters in some other way. To guard against this problem, we denote the functions in differentiation rules by letters like u and v that are not likely to be already in use.

RULE 4 Derivative Sum Rule
If u and v are differentiable functions of x, then their sum $u + v$ is differentiable at every point where u and v are both differentiable. At such points,

$$\frac{d}{dx}(u + v) = \frac{du}{dx} + \frac{dv}{dx}.$$

EXAMPLE 4 Derivative of a Sum

$$
\begin{aligned}
y &= x^4 + 12x \\
\frac{dy}{dx} &= \frac{d}{dx}(x^4) + \frac{d}{dx}(12x) \\
&= 4x^3 + 12
\end{aligned}
$$

Proof of Rule 4 We apply the definition of derivative to $f(x) = u(x) + v(x)$:

$$
\begin{aligned}
\frac{d}{dx}[u(x) + v(x)] &= \lim_{h\to 0} \frac{[u(x+h) + v(x+h)] - [u(x) + v(x)]}{h} \\
&= \lim_{h\to 0} \left[\frac{u(x+h) - u(x)}{h} + \frac{v(x+h) - v(x)}{h}\right] \\
&= \lim_{h\to 0} \frac{u(x+h) - u(x)}{h} + \lim_{h\to 0} \frac{v(x+h) - v(x)}{h} = \frac{du}{dx} + \frac{dv}{dx}.
\end{aligned}
$$

Combining the Sum Rule with the Constant Multiple Rule gives the **Difference Rule,** which says that the derivative of a *difference* of differentiable functions is the difference of their derivatives.

$$
\frac{d}{dx}(u - v) = \frac{d}{dx}[u + (-1)v] = \frac{du}{dx} + (-1)\frac{dv}{dx} = \frac{du}{dx} - \frac{dv}{dx}
$$

The Sum Rule also extends to sums of more than two functions, as long as there are only finitely many functions in the sum. If $u_1, u_2, \ldots, u_n$ are differentiable at x, then so is $u_1 + u_2 + \cdots + u_n$, and

$$
\frac{d}{dx}(u_1 + u_2 + \cdots + u_n) = \frac{du_1}{dx} + \frac{du_2}{dx} + \cdots + \frac{du_n}{dx}.
$$

EXAMPLE 5 Derivative of a Polynomial

$$
\begin{aligned}
y &= x^3 + \frac{4}{3}x^2 - 5x + 1 \\
\frac{dy}{dx} &= \frac{d}{dx}x^3 + \frac{d}{dx}\left(\frac{4}{3}x^2\right) - \frac{d}{dx}(5x) + \frac{d}{dx}(1) \\
&= 3x^2 + \frac{4}{3} \cdot 2x - 5 + 0 \\
&= 3x^2 + \frac{8}{3}x - 5
\end{aligned}
$$

Notice that we can differentiate any polynomial term by term, the way we differentiated the polynomial in Example 5. All polynomials are differentiable everywhere.

Proof of the Sum Rule for Sums of More Than Two Functions We prove the statement

$$
\frac{d}{dx}(u_1 + u_2 + \cdots + u_n) = \frac{du_1}{dx} + \frac{du_2}{dx} + \cdots + \frac{du_n}{dx}
$$

by mathematical induction (see Appendix 1). The statement is true for $n = 2$, as was just proved. This is Step 1 of the induction proof.

Step 2 is to show that if the statement is true for any positive integer $n = k$, where $k \geq n_0 = 2$, then it is also true for $n = k + 1$. So suppose that

$$\frac{d}{dx}(u_1 + u_2 + \cdots + u_k) = \frac{du_1}{dx} + \frac{du_2}{dx} + \cdots + \frac{du_k}{dx}. \tag{1}$$

Then

$$\frac{d}{dx}(\underbrace{u_1 + u_2 + \cdots + u_k}_{\text{Call the function defined by this sum } u.} + \underbrace{u_{k+1}}_{\text{Call this function } v.})$$

$$= \frac{d}{dx}(u_1 + u_2 + \cdots + u_k) + \frac{du_{k+1}}{dx} \qquad \text{Rule 4 for } \frac{d}{dx}(u + v)$$

$$= \frac{du_1}{dx} + \frac{du_2}{dx} + \cdots + \frac{du_k}{dx} + \frac{du_{k+1}}{dx}. \qquad \text{Eq. (1)}$$

With these steps verified, the mathematical induction principle now guarantees the Sum Rule for every integer $n \geq 2$. ■

EXAMPLE 6 Finding Horizontal Tangents

Does the curve $y = x^4 - 2x^2 + 2$ have any horizontal tangents? If so, where?

Solution The horizontal tangents, if any, occur where the slope dy/dx is zero. We have,

$$\frac{dy}{dx} = \frac{d}{dx}(x^4 - 2x^2 + 2) = 4x^3 - 4x.$$

Now solve the equation $\frac{dy}{dx} = 0$ for x:

$$\begin{aligned} 4x^3 - 4x &= 0 \\ 4x(x^2 - 1) &= 0 \\ x &= 0, 1, -1. \end{aligned}$$

The curve $y = x^4 - 2x^2 + 2$ has horizontal tangents at $x = 0, 1,$ and -1. The corresponding points on the curve are $(0, 2)$, $(1, 1)$ and $(-1, 1)$. See Figure 3.10. ■

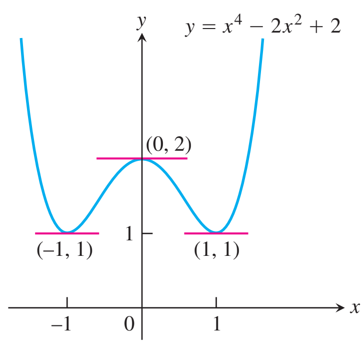


FIGURE 3.10 The curve $y = x^4 - 2x^2 + 2$ and its horizontal tangents (Example 6).

Products and Quotients

While the derivative of the sum of two functions is the sum of their derivatives, the derivative of the product of two functions is *not* the product of their derivatives. For instance,

$$\frac{d}{dx}(x \cdot x) = \frac{d}{dx}(x^2) = 2x, \qquad \text{while} \qquad \frac{d}{dx}(x) \cdot \frac{d}{dx}(x) = 1 \cdot 1 = 1.$$

The derivative of a product of two functions is the sum of *two* products, as we now explain.

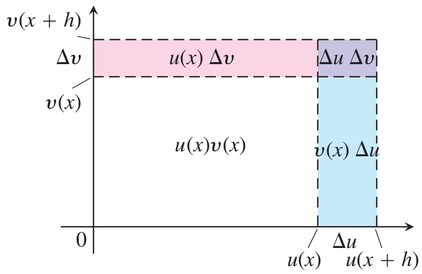
RULE 5 Derivative Product Rule

If u and v are differentiable at x, then so is their product uv, and

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}.$$

Picturing the Product Rule

If $u(x)$ and $v(x)$ are positive and increase when x increases, and if $h > 0$,



then the total shaded area in the picture is

$$u(x+h)v(x+h) - u(x)v(x)$$
$$= u(x+h)\,\Delta v + v(x+h)\,\Delta u - \Delta u \Delta v.$$

Dividing both sides of this equation by h gives

$$\frac{u(x+h)v(x+h) - u(x)v(x)}{h} = u(x+h)\frac{\Delta v}{h} + v(x+h)\frac{\Delta u}{h} - \Delta u \frac{\Delta v}{h}.$$

As $h \to 0^+$,

$$\Delta u \cdot \frac{\Delta v}{h} \to 0 \cdot \frac{dv}{dx} = 0,$$

leaving

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}.$$

The derivative of the product uv is u times the derivative of v plus v times the derivative of u. In *prime notation*, $(uv)' = uv' + vu'$. In function notation,

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x).$$

EXAMPLE 7 Using the Product Rule

Find the derivative of

$$y = \frac{1}{x}\left(x^2 + \frac{1}{x}\right).$$

Solution We apply the Product Rule with $u = 1/x$ and $v = x^2 + (1/x)$:

$$\frac{d}{dx}\left[\frac{1}{x}\left(x^2 + \frac{1}{x}\right)\right] = \frac{1}{x}\left(2x - \frac{1}{x^2}\right) + \left(x^2 + \frac{1}{x}\right)\left(-\frac{1}{x^2}\right)$$
$$= 2 - \frac{1}{x^3} - 1 - \frac{1}{x^3}$$
$$= 1 - \frac{2}{x^3}.$$

$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$, and $\frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}$ by Example 3, Section 2.7.

■

Proof of Rule 5

$$\frac{d}{dx}(uv) = \lim_{h\to 0}\frac{u(x+h)v(x+h) - u(x)v(x)}{h}$$

To change this fraction into an equivalent one that contains difference quotients for the derivatives of u and v, we subtract and add $u(x+h)v(x)$ in the numerator:

$$\frac{d}{dx}(uv) = \lim_{h\to 0}\frac{u(x+h)v(x+h) - u(x+h)v(x) + u(x+h)v(x) - u(x)v(x)}{h}$$
$$= \lim_{h\to 0}\left[u(x+h)\frac{v(x+h) - v(x)}{h} + v(x)\frac{u(x+h) - u(x)}{h}\right]$$
$$= \lim_{h\to 0} u(x+h) \cdot \lim_{h\to 0}\frac{v(x+h) - v(x)}{h} + v(x) \cdot \lim_{h\to 0}\frac{u(x+h) - u(x)}{h}.$$

As h approaches zero, $u(x+h)$ approaches $u(x)$ because u, being differentiable at x, is continuous at x. The two fractions approach the values of dv/dx at x and du/dx at x. In short,

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}.$$

■

In the following example, we have only numerical values with which to work.

EXAMPLE 8 Derivative from Numerical Values

Let $y = uv$ be the product of the functions u and v. Find $y'(2)$ if

$$u(2) = 3, \quad u'(2) = -4, \quad v(2) = 1, \quad \text{and} \quad v'(2) = 2.$$

Solution From the Product Rule, in the form

$$y' = (uv)' = uv' + vu',$$

we have

$$\begin{aligned} y'(2) &= u(2)v'(2) + v(2)u'(2) \\ &= (3)(2) + (1)(-4) = 6 - 4 = 2. \end{aligned}$$

■

EXAMPLE 9 Differentiating a Product in Two Ways

Find the derivative of $y = (x^2 + 1)(x^3 + 3)$.

Solution

(a) From the Product Rule with $u = x^2 + 1$ and $v = x^3 + 3$, we find

$$\begin{aligned} \frac{d}{dx}\left[\left(x^2 + 1\right)\left(x^3 + 3\right)\right] &= (x^2 + 1)(3x^2) + (x^3 + 3)(2x) \\ &= 3x^4 + 3x^2 + 2x^4 + 6x \\ &= 5x^4 + 3x^2 + 6x. \end{aligned}$$

(b) This particular product can be differentiated as well (perhaps better) by multiplying out the original expression for y and differentiating the resulting polynomial:

$$\begin{aligned} y &= (x^2 + 1)(x^3 + 3) = x^5 + x^3 + 3x^2 + 3 \\ \frac{dy}{dx} &= 5x^4 + 3x^2 + 6x. \end{aligned}$$

This is in agreement with our first calculation. ■

Just as the derivative of the product of two differentiable functions is not the product of their derivatives, the derivative of the quotient of two functions is not the quotient of their derivatives. What happens instead is the Quotient Rule.

RULE 6 **Derivative Quotient Rule**

If u and v are differentiable at x and if $v(x) \neq 0$, then the quotient u/v is differentiable at x, and

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}.$$

In function notation,

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)f'(x) - f(x)g'(x)}{g^2(x)}.$$

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EXAMPLE 10 Using the Quotient Rule

Find the derivative of

$$y = \frac{t^2 - 1}{t^2 + 1}.$$

Solution

We apply the Quotient Rule with $u = t^2 - 1$ and $v = t^2 + 1$:

$$\begin{aligned}\frac{dy}{dt} &= \frac{(t^2+1)\cdot 2t - (t^2-1)\cdot 2t}{(t^2+1)^2} \qquad \frac{d}{dt}\left(\frac{u}{v}\right) = \frac{v(du/dt) - u(dv/dt)}{v^2}\\ &= \frac{2t^3 + 2t - 2t^3 + 2t}{(t^2+1)^2}\\ &= \frac{4t}{(t^2+1)^2}.\end{aligned}$$

Proof of Rule 6

$$\begin{aligned}\frac{d}{dx}\left(\frac{u}{v}\right) &= \lim_{h\to 0}\frac{\dfrac{u(x+h)}{v(x+h)} - \dfrac{u(x)}{v(x)}}{h}\\ &= \lim_{h\to 0}\frac{v(x)u(x+h) - u(x)v(x+h)}{hv(x+h)v(x)}\end{aligned}$$

To change the last fraction into an equivalent one that contains the difference quotients for the derivatives of u and v, we subtract and add $v(x)u(x)$ in the numerator. We then get

$$\begin{aligned}\frac{d}{dx}\left(\frac{u}{v}\right) &= \lim_{h\to 0}\frac{v(x)u(x+h) - v(x)u(x) + v(x)u(x) - u(x)v(x+h)}{hv(x+h)v(x)}\\ &= \lim_{h\to 0}\frac{v(x)\dfrac{u(x+h)-u(x)}{h} - u(x)\dfrac{v(x+h)-v(x)}{h}}{v(x+h)v(x)}.\end{aligned}$$

Taking the limit in the numerator and denominator now gives the Quotient Rule. ■

Negative Integer Powers of *x*

The Power Rule for negative integers is the same as the rule for positive integers.

RULE 7 Power Rule for Negative Integers

If n is a negative integer and $x \neq 0$, then

$$\frac{d}{dx}(x^n) = nx^{n-1}.$$

EXAMPLE 11

(a) $\dfrac{d}{dx}\left(\dfrac{1}{x}\right) = \dfrac{d}{dx}(x^{-1}) = (-1)x^{-2} = -\dfrac{1}{x^2}$ Agrees with Example 3, Section 2.7

(b) $\dfrac{d}{dx}\left(\dfrac{4}{x^3}\right) = 4\dfrac{d}{dx}(x^{-3}) = 4(-3)x^{-4} = -\dfrac{12}{x^4}$

Proof of Rule 7 The proof uses the Quotient Rule. If n is a negative integer, then $n = -m$, where m is a positive integer. Hence, $x^n = x^{-m} = 1/x^m$, and

$$\begin{aligned}
\frac{d}{dx}(x^n) &= \frac{d}{dx}\left(\frac{1}{x^m}\right) \\
&= \frac{x^m \cdot \frac{d}{dx}(1) - 1 \cdot \frac{d}{dx}(x^m)}{(x^m)^2} && \text{Quotient Rule with } u = 1 \text{ and } v = x^m \\
&= \frac{0 - mx^{m-1}}{x^{2m}} && \text{Since } m > 0, \ \frac{d}{dx}(x^m) = mx^{m-1} \\
&= -mx^{-m-1} \\
&= nx^{n-1}. && \text{Since } -m = n
\end{aligned}$$

■

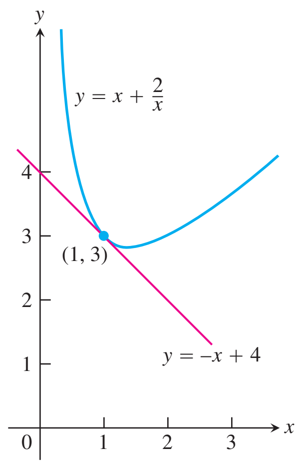

FIGURE 3.11 The tangent to the curve $y = x + (2/x)$ at $(1, 3)$ in Example 12. The curve has a third-quadrant portion not shown here. We see how to graph functions like this one in Chapter 4.

EXAMPLE 12 Tangent to a Curve

Find an equation for the tangent to the curve

$$y = x + \frac{2}{x}$$

at the point (1, 3) (Figure 3.11).

Solution The slope of the curve is

$$\frac{dy}{dx} = \frac{d}{dx}(x) + 2\frac{d}{dx}\left(\frac{1}{x}\right) = 1 + 2\left(-\frac{1}{x^2}\right) = 1 - \frac{2}{x^2}.$$

The slope at $x = 1$ is

$$\left.\frac{dy}{dx}\right|_{x=1} = \left[1 - \frac{2}{x^2}\right]_{x=1} = 1 - 2 = -1.$$

The line through (1, 3) with slope $m = -1$ is

$$\begin{aligned}
y - 3 &= (-1)(x - 1) && \text{Point-slope equation} \\
y &= -x + 1 + 3 \\
y &= -x + 4.
\end{aligned}$$

■

The choice of which rules to use in solving a differentiation problem can make a difference in how much work you have to do. Here is an example.

EXAMPLE 13 Choosing Which Rule to Use

Rather than using the Quotient Rule to find the derivative of

$$y = \frac{(x-1)(x^2 - 2x)}{x^4},$$

expand the numerator and divide by x^4:

$$y = \frac{(x-1)(x^2 - 2x)}{x^4} = \frac{x^3 - 3x^2 + 2x}{x^4} = x^{-1} - 3x^{-2} + 2x^{-3}.$$

Then use the Sum and Power Rules:

$$\frac{dy}{dx} = -x^{-2} - 3(-2)x^{-3} + 2(-3)x^{-4}$$
$$= -\frac{1}{x^2} + \frac{6}{x^3} - \frac{6}{x^4}.$$

■

Second- and Higher-Order Derivatives

If $y = f(x)$ is a differentiable function, then its derivative $f'(x)$ is also a function. If f' is also differentiable, then we can differentiate f' to get a new function of x denoted by f''. So $f'' = (f')'$. The function f'' is called the **second derivative** of f because it is the derivative of the first derivative. Notationally,

$$f''(x) = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{dy'}{dx} = y'' = D^2(f)(x) = D_x^{\,2} f(x).$$

The symbol D^2 means the operation of differentiation is performed twice.

If $y = x^6$, then $y' = 6x^5$ and we have

$$y'' = \frac{dy'}{dx} = \frac{d}{dx}\left(6x^5\right) = 30x^4.$$

Thus $D^2\left(x^6\right) = 30x^4$.

If y'' is differentiable, its derivative, $y''' = dy''/dx = d^3y/dx^3$ is the **third derivative** of y with respect to x. The names continue as you imagine, with

$$y^{(n)} = \frac{d}{dx}y^{(n-1)} = \frac{d^ny}{dx^n} = D^ny$$

denoting the ***n*th derivative** of y with respect to x for any positive integer n.

We can interpret the second derivative as the rate of change of the slope of the tangent to the graph of $y = f(x)$ at each point. You will see in the next chapter that the second derivative reveals whether the graph bends upward or downward from the tangent line as we move off the point of tangency. In the next section, we interpret both the second and third derivatives in terms of motion along a straight line.

How to Read the Symbols for Derivatives

y'	"y prime"
y''	"y double prime"
$\frac{d^2y}{dx^2}$	"d squared y dx squared"
y'''	"y triple prime"
$y^{(n)}$	"y super n"
$\frac{d^ny}{dx^n}$	"d to the n of y by dx to the n"
D^n	"D to the n"

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EXAMPLE 14 Finding Higher Derivatives

The first four derivatives of $y = x^3 - 3x^2 + 2$ are

First derivative:	$y' = 3x^2 - 6x$
Second derivative:	$y'' = 6x - 6$
Third derivative:	$y''' = 6$
Fourth derivative:	$y^{(4)} = 0.$

The function has derivatives of all orders, the fifth and later derivatives all being zero.

■

EXERCISES 3.2

Derivative Calculations

In Exercises 1–12, find the first and second derivatives.

1. $y = -x^2 + 3$

2. $y = x^2 + x + 8$

3. $s = 5t^3 - 3t^5$

4. $w = 3z^7 - 7z^3 + 21z^2$

5. $y = \frac{4x^3}{3} - x$

6. $y = \frac{x^3}{3} + \frac{x^2}{2} + \frac{x}{4}$

7. $w = 3z^{-2} - \frac{1}{z}$

8. $s = -2t^{-1} + \frac{4}{t^2}$

9. $y = 6x^2 - 10x - 5x^{-2}$

10. $y = 4 - 2x - x^{-3}$

11. $r = \frac{1}{3s^2} - \frac{5}{2s}$

12. $r = \frac{12}{\theta} - \frac{4}{\theta^3} + \frac{1}{\theta^4}$

In Exercises 13–16, find y' **(a)** by applying the Product Rule and **(b)** by multiplying the factors to produce a sum of simpler terms to differentiate.

13. $y = (3 - x^2)(x^3 - x + 1)$

14. $y = (x - 1)(x^2 + x + 1)$

15. $y = (x^2 + 1)\left(x + 5 + \frac{1}{x}\right)$

16. $y = \left(x + \frac{1}{x}\right)\left(x - \frac{1}{x} + 1\right)$

Find the derivatives of the functions in Exercises 17–28.

17. $y = \frac{2x + 5}{3x - 2}$

18. $z = \frac{2x + 1}{x^2 - 1}$

19. $g(x) = \frac{x^2 - 4}{x + 0.5}$

20. $f(t) = \frac{t^2 - 1}{t^2 + t - 2}$

21. $v = (1 - t)(1 + t^2)^{-1}$

22. $w = (2x - 7)^{-1}(x + 5)$

23. $f(s) = \frac{\sqrt{s} - 1}{\sqrt{s} + 1}$

24. $u = \frac{5x + 1}{2\sqrt{x}}$

25. $v = \frac{1 + x - 4\sqrt{x}}{x}$

26. $r = 2\left(\frac{1}{\sqrt{\theta}} + \sqrt{\theta}\right)$

27. $y = \frac{1}{(x^2 - 1)(x^2 + x + 1)}$

28. $y = \frac{(x + 1)(x + 2)}{(x - 1)(x - 2)}$

Find the derivatives of all orders of the functions in Exercises 29 and 30.

29. $y = \frac{x^4}{2} - \frac{3}{2}x^2 - x$

30. $y = \frac{x^5}{120}$

Find the first and second derivatives of the functions in Exercises 31–38.

31. $y = \frac{x^3 + 7}{x}$

32. $s = \frac{t^2 + 5t - 1}{t^2}$

33. $r = \frac{(\theta - 1)(\theta^2 + \theta + 1)}{\theta^3}$

34. $u = \frac{(x^2 + x)(x^2 - x + 1)}{x^4}$

35. $w = \left(\frac{1 + 3z}{3z}\right)(3 - z)$

36. $w = (z + 1)(z - 1)(z^2 + 1)$

37. $p = \left(\frac{q^2 + 3}{12q}\right)\left(\frac{q^4 - 1}{q^3}\right)$

38. $p = \frac{q^2 + 3}{(q - 1)^3 + (q + 1)^3}$

Using Numerical Values

39. Suppose u and v are functions of x that are differentiable at $x = 0$ and that

$$u(0) = 5, \quad u'(0) = -3, \quad v(0) = -1, \quad v'(0) = 2.$$

Find the values of the following derivatives at $x = 0$.

a. $\frac{d}{dx}(uv)$ **b.** $\frac{d}{dx}\left(\frac{u}{v}\right)$ **c.** $\frac{d}{dx}\left(\frac{v}{u}\right)$ **d.** $\frac{d}{dx}(7v - 2u)$

40. Suppose u and v are differentiable functions of x and that

$$u(1) = 2, \quad u'(1) = 0, \quad v(1) = 5, \quad v'(1) = -1.$$

Find the values of the following derivatives at $x = 1$.

a. $\frac{d}{dx}(uv)$ **b.** $\frac{d}{dx}\left(\frac{u}{v}\right)$ **c.** $\frac{d}{dx}\left(\frac{v}{u}\right)$ **d.** $\frac{d}{dx}(7v - 2u)$

Slopes and Tangents

41. a. Normal to a curve Find an equation for the line perpendicular to the tangent to the curve $y = x^3 - 4x + 1$ at the point (2, 1).

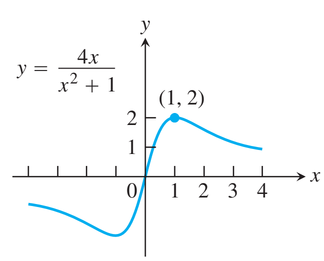
b. Smallest slope What is the smallest slope on the curve? At what point on the curve does the curve have this slope?

c. Tangents having specified slope Find equations for the tangents to the curve at the points where the slope of the curve is 8.

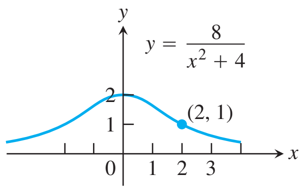
42. a. Horizontal tangents Find equations for the horizontal tangents to the curve $y = x^3 - 3x - 2$. Also find equations for the lines that are perpendicular to these tangents at the points of tangency.

b. Smallest slope What is the smallest slope on the curve? At what point on the curve does the curve have this slope? Find an equation for the line that is perpendicular to the curve's tangent at this point.

43. Find the tangents to *Newton's serpentine* (graphed here) at the origin and the point (1, 2).



44. Find the tangent to the *Witch of Agnesi* (graphed here) at the point (2, 1).



45. Quadratic tangent to identity function The curve $y = ax^2 + bx + c$ passes through the point (1, 2) and is tangent to the line $y = x$ at the origin. Find a, b, and c.

46. Quadratics having a common tangent The curves $y = x^2 + ax + b$ and $y = cx - x^2$ have a common tangent line at the point (1, 0). Find a, b, and c.

47. a. Find an equation for the line that is tangent to the curve $y = x^3 - x$ at the point $(-1, 0)$.

T **b.** Graph the curve and tangent line together. The tangent intersects the curve at another point. Use Zoom and Trace to estimate the point's coordinates.

T **c.** Confirm your estimates of the coordinates of the second intersection point by solving the equations for the curve and tangent simultaneously (Solver key).

48. a. Find an equation for the line that is tangent to the curve $y = x^3 - 6x^2 + 5x$ at the origin.

T **b.** Graph the curve and tangent together. The tangent intersects the curve at another point. Use Zoom and Trace to estimate the point's coordinates.

T **c.** Confirm your estimates of the coordinates of the second intersection point by solving the equations for the curve and tangent simultaneously (Solver key).

Theory and Examples

49. The general polynomial of degree n has the form

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0$$

where $a_n \neq 0$. Find $P'(x)$.

50. The body's reaction to medicine The reaction of the body to a dose of medicine can sometimes be represented by an equation of the form

$$R = M^2\left(\frac{C}{2} - \frac{M}{3}\right),$$

where C is a positive constant and M is the amount of medicine absorbed in the blood. If the reaction is a change in blood pressure, R is measured in millimeters of mercury. If the reaction is a change in temperature, R is measured in degrees, and so on.

Find dR/dM. This derivative, as a function of M, is called the sensitivity of the body to the medicine. In Section 4.5, we will see how to find the amount of medicine to which the body is most sensitive.

51. Suppose that the function v in the Product Rule has a constant value c. What does the Product Rule then say? What does this say about the Constant Multiple Rule?

52. The Reciprocal Rule

a. The *Reciprocal Rule* says that at any point where the function $v(x)$ is differentiable and different from zero,

$$\frac{d}{dx}\left(\frac{1}{v}\right) = -\frac{1}{v^2}\frac{dv}{dx}.$$

Show that the Reciprocal Rule is a special case of the Quotient Rule.

b. Show that the Reciprocal Rule and the Product Rule together imply the Quotient Rule.

53. Generalizing the Product Rule The Product Rule gives the formula

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$$

for the derivative of the product uv of two differentiable functions of x.

a. What is the analogous formula for the derivative of the product uvw of *three* differentiable functions of x?

b. What is the formula for the derivative of the product $u_1u_2u_3u_4$ of *four* differentiable functions of x?

c. What is the formula for the derivative of a product $u_1u_2u_3 \ldots u_n$ of a finite number n of differentiable functions of x?

54. Rational Powers

a. Find $\frac{d}{dx}\left(x^{3/2}\right)$ by writing $x^{3/2}$ as $x \cdot x^{1/2}$ and using the Product Rule. Express your answer as a rational number times a rational power of x. Work parts (b) and (c) by a similar method.

b. Find $\frac{d}{dx}(x^{5/2})$.

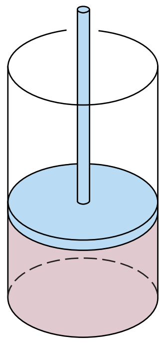
c. Find $\frac{d}{dx}(x^{7/2})$.

d. What patterns do you see in your answers to parts (a), (b), and (c)? Rational powers are one of the topics in Section 3.6.

55. Cylinder pressure If gas in a cylinder is maintained at a constant temperature T, the pressure P is related to the volume V by a formula of the form

$$P = \frac{nRT}{V - nb} - \frac{an^2}{V^2},$$

in which a, b, n, and R are constants. Find dP/dV. (See accompanying figure.)



56. The best quantity to order One of the formulas for inventory management says that the average weekly cost of ordering, paying for, and holding merchandise is

$$A(q) = \frac{km}{q} + cm + \frac{hq}{2},$$

where q is the quantity you order when things run low (shoes, radios, brooms, or whatever the item might be); k is the cost of placing an order (the same, no matter how often you order); c is the cost of one item (a constant); m is the number of items sold each week (a constant); and h is the weekly holding cost per item (a constant that takes into account things such as space, utilities, insurance, and security). Find dA/dq and d^2A/dq^2.

3.3 The Derivative as a Rate of Change

In Section 2.1, we initiated the study of average and instantaneous rates of change. In this section, we continue our investigations of applications in which derivatives are used to model the rates at which things change in the world around us. We revisit the study of motion along a line and examine other applications.

It is natural to think of change as change with respect to time, but other variables can be treated in the same way. For example, a physician may want to know how change in dosage affects the body's response to a drug. An economist may want to study how the cost of producing steel varies with the number of tons produced.

Instantaneous Rates of Change

If we interpret the difference quotient $(f(x+h)-f(x))/h$ as the average rate of change in f over the interval from x to $x+h$, we can interpret its limit as $h \to 0$ as the rate at which f is changing at the point x.

DEFINITION Instantaneous Rate of Change

The **instantaneous rate of change** of f with respect to x at x_0 is the derivative

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0+h)-f(x_0)}{h},$$

provided the limit exists.

Thus, instantaneous rates are limits of average rates.

It is conventional to use the word *instantaneous* even when x does not represent time. The word is, however, frequently omitted. When we say *rate of change*, we mean *instantaneous rate of change*.

EXAMPLE 1 How a Circle's Area Changes with Its Diameter

The area A of a circle is related to its diameter by the equation

$$A = \frac{\pi}{4} D^2.$$

How fast does the area change with respect to the diameter when the diameter is 10 m?

Solution The rate of change of the area with respect to the diameter is

$$\frac{dA}{dD} = \frac{\pi}{4} \cdot 2D = \frac{\pi D}{2}.$$

When $D = 10$ m, the area is changing at rate $(\pi/2)10 = 5\pi \text{ m}^2/\text{m}$. ■

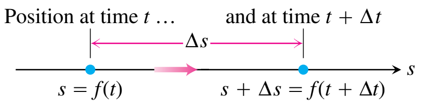


FIGURE 3.12 The positions of a body moving along a coordinate line at time t and shortly later at time $t + \Delta t$.

Motion Along a Line: Displacement, Velocity, Speed, Acceleration, and Jerk

Suppose that an object is moving along a coordinate line (say an s-axis) so that we know its position s on that line as a function of time t:

$$s = f(t).$$

The **displacement** of the object over the time interval from t to $t + \Delta t$ (Figure 3.12) is

$$\Delta s = f(t + \Delta t) - f(t),$$

and the **average velocity** of the object over that time interval is

$$v_{av} = \frac{\text{displacement}}{\text{travel time}} = \frac{\Delta s}{\Delta t} = \frac{f(t + \Delta t) - f(t)}{\Delta t}.$$

To find the body's velocity at the exact instant t, we take the limit of the average velocity over the interval from t to $t + \Delta t$ as Δt shrinks to zero. This limit is the derivative of f with respect to t.

DEFINITION **Velocity**

Velocity (instantaneous velocity) is the derivative of position with respect to time. If a body's position at time t is $s = f(t)$, then the body's velocity at time t is

$$v(t) = \frac{ds}{dt} = \lim_{\Delta t \to 0} \frac{f(t + \Delta t) - f(t)}{\Delta t}.$$

EXAMPLE 2 Finding the Velocity of a Race Car

Figure 3.13 shows the time-to-distance graph of a 1996 Riley & Scott Mk III-Olds WSC race car. The slope of the secant PQ is the average velocity for the 3-sec interval from $t = 2$ to $t = 5$ sec; in this case, it is about 100 ft/sec or 68 mph.

The slope of the tangent at P is the speedometer reading at $t = 2$ sec, about 57 ft/sec or 39 mph. The acceleration for the period shown is a nearly constant 28.5 ft/sec^2 during

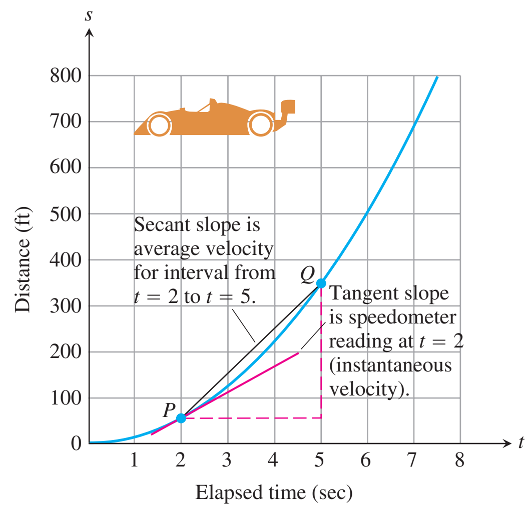


FIGURE 3.13 The time-to-distance graph for Example 2. The slope of the tangent line at P is the instantaneous velocity at $t = 2$ sec.

each second, which is about $0.89g$, where g is the acceleration due to gravity. The race car's top speed is an estimated 190 mph. (*Source: Road and Track*, March 1997.) ■

Besides telling how fast an object is moving, its velocity tells the direction of motion. When the object is moving forward (s increasing), the velocity is positive; when the body is moving backward (s decreasing), the velocity is negative (Figure 3.14).

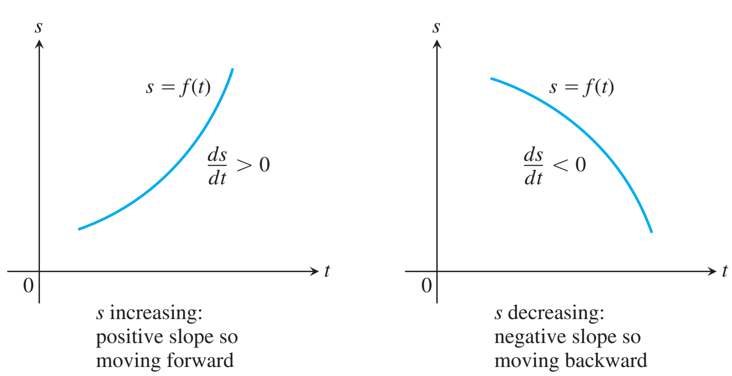


FIGURE 3.14 For motion $s = f(t)$ along a straight line, $v = ds/dt$ is positive when s increases and negative when s decreases.

If we drive to a friend's house and back at 30 mph, say, the speedometer will show 30 on the way over but it will not show -30 on the way back, even though our distance from home is decreasing. The speedometer always shows *speed*, which is the absolute value of velocity. Speed measures the rate of progress regardless of direction.

DEFINITION Speed

Speed is the absolute value of velocity.

$$\text{Speed} = |v(t)| = \left|\frac{ds}{dt}\right|$$

EXAMPLE 3 Horizontal Motion

Figure 3.15 shows the velocity $v = f'(t)$ of a particle moving on a coordinate line. The particle moves forward for the first 3 sec, moves backward for the next 2 sec, stands still for a second, and moves forward again. The particle achieves its greatest speed at time $t = 4$, while moving backward. ■

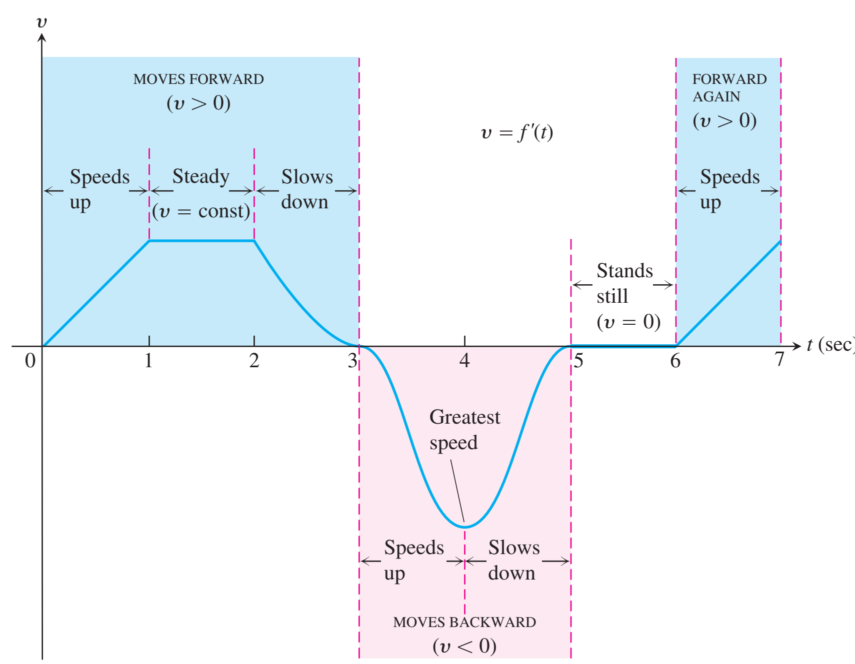


FIGURE 3.15 The velocity graph for Example 3.

HISTORICAL BIOGRAPHY

Bernard Bolzano
(1781–1848)

The rate at which a body's velocity changes is the body's *acceleration*. The acceleration measures how quickly the body picks up or loses speed.

A sudden change in acceleration is called a *jerk*. When a ride in a car or a bus is jerky, it is not that the accelerations involved are necessarily large but that the changes in acceleration are abrupt.

DEFINITIONS Acceleration, Jerk

Acceleration is the derivative of velocity with respect to time. If a body's position at time t is $s = f(t)$, then the body's acceleration at time t is

$$a(t) = \frac{dv}{dt} = \frac{d^2s}{dt^2}.$$

Jerk is the derivative of acceleration with respect to time:

$$j(t) = \frac{da}{dt} = \frac{d^3s}{dt^3}.$$

Near the surface of the Earth all bodies fall with the same constant acceleration. Galileo's experiments with free fall (Example 1, Section 2.1) lead to the equation

$$s = \frac{1}{2}gt^2,$$

where s is distance and g is the acceleration due to Earth's gravity. This equation holds in a vacuum, where there is no air resistance, and closely models the fall of dense, heavy objects, such as rocks or steel tools, for the first few seconds of their fall, before air resistance starts to slow them down.

The value of g in the equation $s = (1/2)gt^2$ depends on the units used to measure t and s. With t in seconds (the usual unit), the value of g determined by measurement at sea level is approximately 32 ft/sec^2 (feet per second squared) in English units, and $g = 9.8$ m/sec^2 (meters per second squared) in metric units. (These gravitational constants depend on the distance from Earth's center of mass, and are slightly lower on top of Mt. Everest, for example.)

The jerk of the constant acceleration of gravity ($g = 32$ ft/sec^2) is zero:

$$j = \frac{d}{dt}(g) = 0.$$

An object does not exhibit jerkiness during free fall.

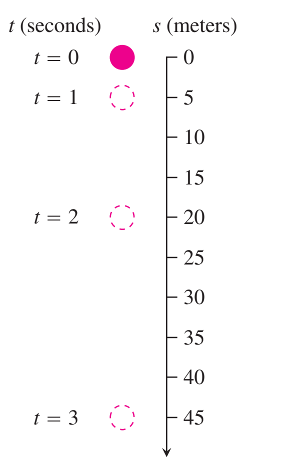


FIGURE 3.16 A ball bearing falling from rest (Example 4).

EXAMPLE 4 Modeling Free Fall

Figure 3.16 shows the free fall of a heavy ball bearing released from rest at time $t = 0$ sec.

(a) How many meters does the ball fall in the first 2 sec?

(b) What is its velocity, speed, and acceleration then?

Solution

(a) The metric free-fall equation is $s = 4.9t^2$. During the first 2 sec, the ball falls

$$s(2) = 4.9(2)^2 = 19.6 \text{ m}.$$

(b) At any time t, *velocity* is the derivative of position:

$$v(t) = s'(t) = \frac{d}{dt}(4.9t^2) = 9.8t.$$

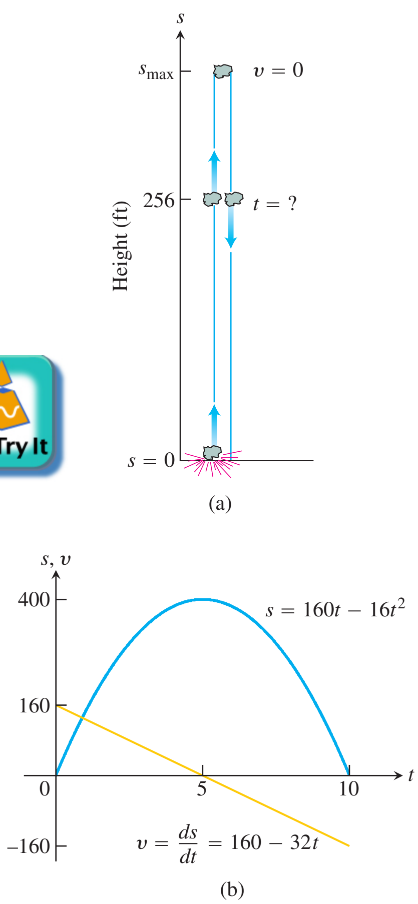


FIGURE 3.17 (a) The rock in Example 5. (b) The graphs of s and v as functions of time; s is largest when $v = ds/dt = 0$. The graph of s is *not* the path of the rock: It is a plot of height versus time. The slope of the plot is the rock's velocity, graphed here as a straight line.

At $t = 2$, the velocity is

$$v(2) = 19.6 \text{ m/sec}$$

in the downward (increasing s) direction. The *speed* at $t = 2$ is

$$\text{Speed} = |v(2)| = 19.6 \text{ m/sec}.$$

The *acceleration* at any time t is

$$a(t) = v'(t) = s''(t) = 9.8 \text{ m/sec}^2.$$

At $t = 2$, the acceleration is 9.8 m/sec^2. ■

EXAMPLE 5 Modeling Vertical Motion

A dynamite blast blows a heavy rock straight up with a launch velocity of 160 ft/sec (about 109 mph) (Figure 3.17a). It reaches a height of $s = 160t - 16t^2$ ft after t sec.

(a) How high does the rock go?

(b) What are the velocity and speed of the rock when it is 256 ft above the ground on the way up? On the way down?

(c) What is the acceleration of the rock at any time t during its flight (after the blast)?

(d) When does the rock hit the ground again?

Solution

(a) In the coordinate system we have chosen, s measures height from the ground up, so the velocity is positive on the way up and negative on the way down. The instant the rock is at its highest point is the one instant during the flight when the velocity is 0. To find the maximum height, all we need to do is to find when $v = 0$ and evaluate s at this time.

At any time t, the velocity is

$$v = \frac{ds}{dt} = \frac{d}{dt}(160t - 16t^2) = 160 - 32t \text{ ft/sec}.$$

The velocity is zero when

$$160 - 32t = 0 \qquad \text{or} \qquad t = 5 \text{ sec}.$$

The rock's height at $t = 5$ sec is

$$s_{max} = s(5) = 160(5) - 16(5)^2 = 800 - 400 = 400 \text{ ft}.$$

See Figure 3.17b.

(b) To find the rock's velocity at 256 ft on the way up and again on the way down, we first find the two values of t for which

$$s(t) = 160t - 16t^2 = 256.$$

To solve this equation, we write

$$\begin{aligned} 16t^2 - 160t + 256 &= 0 \\ 16(t^2 - 10t + 16) &= 0 \\ (t - 2)(t - 8) &= 0 \\ t = 2 \text{ sec}, t &= 8 \text{ sec}. \end{aligned}$$

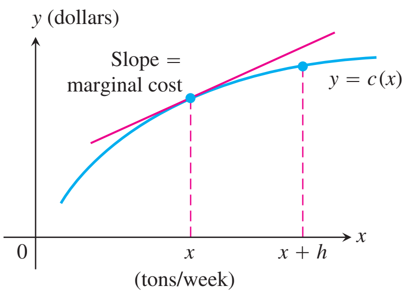


FIGURE 3.18 Weekly steel production: $c(x)$ is the cost of producing x tons per week. The cost of producing an additional h tons is $c(x + h) - c(x)$.

The rock is 256 ft above the ground 2 sec after the explosion and again 8 sec after the explosion. The rock's velocities at these times are

$$v(2) = 160 - 32(2) = 160 - 64 = 96 \text{ ft/sec.}$$
$$v(8) = 160 - 32(8) = 160 - 256 = -96 \text{ ft/sec.}$$

At both instants, the rock's speed is 96 ft/sec. Since $v(2) > 0$, the rock is moving upward (s is increasing) at $t = 2$ sec; it is moving downward (s is decreasing) at $t = 8$ because $v(8) < 0$.

(c) At any time during its flight following the explosion, the rock's acceleration is a constant

$$a = \frac{dv}{dt} = \frac{d}{dt}(160 - 32t) = -32 \text{ ft/sec}^2.$$

The acceleration is always downward. As the rock rises, it slows down; as it falls, it speeds up.

(d) The rock hits the ground at the positive time t for which $s = 0$. The equation $160t - 16t^2 = 0$ factors to give $16t(10 - t) = 0$, so it has solutions $t = 0$ and $t = 10$. At $t = 0$, the blast occurred and the rock was thrown upward. It returned to the ground 10 sec later. ■

Derivatives in Economics

Engineers use the terms *velocity* and *acceleration* to refer to the derivatives of functions describing motion. Economists, too, have a specialized vocabulary for rates of change and derivatives. They call them *marginals*.

In a manufacturing operation, the *cost of production* $c(x)$ is a function of x, the number of units produced. The **marginal cost of production** is the rate of change of cost with respect to level of production, so it is dc/dx.

Suppose that $c(x)$ represents the dollars needed to produce x tons of steel in one week. It costs more to produce $x + h$ units per week, and the cost difference, divided by h, is the average cost of producing each additional ton:

$$\frac{c(x + h) - c(x)}{h} = \begin{array}{l}\text{average cost of each of the additional}\\ h \text{ tons of steel produced.}\end{array}$$

The limit of this ratio as $h \to 0$ is the *marginal cost* of producing more steel per week when the current weekly production is x tons (Figure 3.18).

$$\frac{dc}{dx} = \lim_{h \to 0} \frac{c(x + h) - c(x)}{h} = \text{marginal cost of production.}$$

Sometimes the marginal cost of production is loosely defined to be the extra cost of producing one unit:

$$\frac{\Delta c}{\Delta x} = \frac{c(x + 1) - c(x)}{1},$$

which is approximated by the value of dc/dx at x. This approximation is acceptable if the slope of the graph of c does not change quickly near x. Then the difference quotient will be close to its limit dc/dx, which is the rise in the tangent line if $\Delta x = 1$ (Figure 3.19). The approximation works best for large values of x.

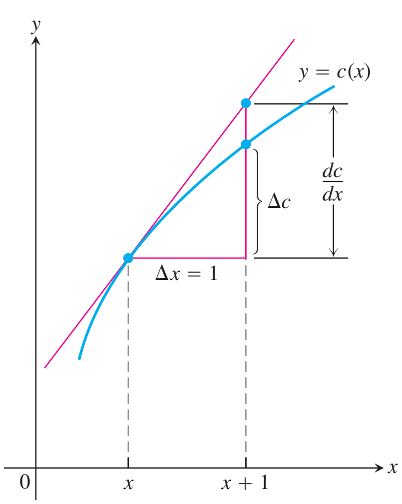


FIGURE 3.19 The marginal cost dc/dx is approximately the extra cost Δc of producing $\Delta x = 1$ more unit.

Economists often represent a total cost function by a cubic polynomial

$$c(x) = \alpha x^3 + \beta x^2 + \gamma x + \delta$$

where δ represents *fixed costs* such as rent, heat, equipment capitalization, and management costs. The other terms represent *variable costs* such as the costs of raw materials, taxes, and labor. Fixed costs are independent of the number of units produced, whereas variable costs depend on the quantity produced. A cubic polynomial is usually complicated enough to capture the cost behavior on a relevant quantity interval.

EXAMPLE 6 Marginal Cost and Marginal Revenue

Suppose that it costs

$$c(x) = x^3 - 6x^2 + 15x$$

dollars to produce x radiators when 8 to 30 radiators are produced and that

$$r(x) = x^3 - 3x^2 + 12x$$

gives the dollar revenue from selling x radiators. Your shop currently produces 10 radiators a day. About how much extra will it cost to produce one more radiator a day, and what is your estimated increase in revenue for selling 11 radiators a day?

Solution The cost of producing one more radiator a day when 10 are produced is about $c'(10)$:

$$c'(x) = \frac{d}{dx}\left(x^3 - 6x^2 + 15x\right) = 3x^2 - 12x + 15$$

$$c'(10) = 3(100) - 12(10) + 15 = 195.$$

The additional cost will be about \$195. The marginal revenue is

$$r'(x) = \frac{d}{dx}\left(x^3 - 3x^2 + 12x\right) = 3x^2 - 6x + 12.$$

The marginal revenue function estimates the increase in revenue that will result from selling one additional unit. If you currently sell 10 radiators a day, you can expect your revenue to increase by about

$$r'(10) = 3(100) - 6(10) + 12 = \$252$$

if you increase sales to 11 radiators a day. ■

EXAMPLE 7 Marginal Tax Rate

To get some feel for the language of marginal rates, consider marginal tax rates. If your marginal income tax rate is 28% and your income increases by \$1000, you can expect to pay an extra \$280 in taxes. This does not mean that you pay 28% of your entire income in taxes. It just means that at your current income level I, the rate of increase of taxes T with respect to income is $dT/dI = 0.28$. You will pay \$0.28 out of every extra dollar you earn in taxes. Of course, if you earn a lot more, you may land in a higher tax bracket and your marginal rate will increase. ■

Sensitivity to Change

When a small change in x produces a large change in the value of a function $f(x)$, we say that the function is relatively **sensitive** to changes in x. The derivative $f'(x)$ is a measure of this sensitivity.

EXAMPLE 8 Genetic Data and Sensitivity to Change

The Austrian monk Gregor Johann Mendel (1822–1884), working with garden peas and other plants, provided the first scientific explanation of hybridization.

His careful records showed that if p (a number between 0 and 1) is the frequency of the gene for smooth skin in peas (dominant) and $(1 - p)$ is the frequency of the gene for wrinkled skin in peas, then the proportion of smooth-skinned peas in the next generation will be

$$y = 2p(1 - p) + p^2 = 2p - p^2.$$

The graph of y versus p in Figure 3.20a suggests that the value of y is more sensitive to a change in p when p is small than when p is large. Indeed, this fact is borne out by the derivative graph in Figure 3.20b, which shows that dy/dp is close to 2 when p is near 0 and close to 0 when p is near 1.

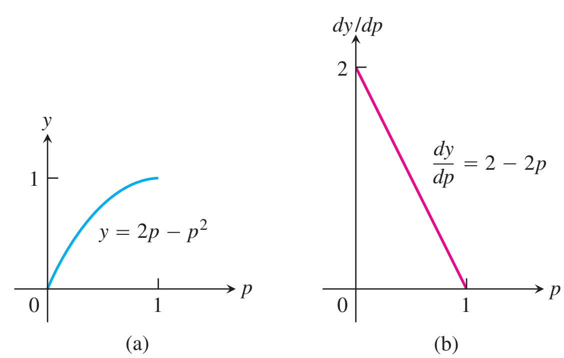


FIGURE 3.20 (a) The graph of $y = 2p - p^2$, describing the proportion of smooth-skinned peas. (b) The graph of dy/dp (Example 8).

The implication for genetics is that introducing a few more dominant genes into a highly recessive population (where the frequency of wrinkled skin peas is small) will have a more dramatic effect on later generations than will a similar increase in a highly dominant population. ■

EXERCISES 3.3

Motion Along a Coordinate Line

Exercises 1–6 give the positions $s = f(t)$ of a body moving on a coordinate line, with s in meters and t in seconds.

a. Find the body's displacement and average velocity for the given time interval.

b. Find the body's speed and acceleration at the endpoints of the interval.

c. When, if ever, during the interval does the body change direction?

1. $s = t^2 - 3t + 2, \quad 0 \le t \le 2$

2. $s = 6t - t^2, \quad 0 \le t \le 6$

3. $s = -t^3 + 3t^2 - 3t, \quad 0 \le t \le 3$

4. $s = (t^4/4) - t^3 + t^2, \quad 0 \le t \le 3$

5. $s = \dfrac{25}{t^2} - \dfrac{5}{t}, \quad 1 \le t \le 5$

6. $s = \dfrac{25}{t+5}, \quad -4 \le t \le 0$

7. **Particle motion** At time t, the position of a body moving along the s-axis is $s = t^3 - 6t^2 + 9t$ m.
   - **a.** Find the body's acceleration each time the velocity is zero.
   - **b.** Find the body's speed each time the acceleration is zero.
   - **c.** Find the total distance traveled by the body from $t = 0$ to $t = 2$.

8. **Particle motion** At time $t \ge 0$, the velocity of a body moving along the s-axis is $v = t^2 - 4t + 3$.
   - **a.** Find the body's acceleration each time the velocity is zero.
   - **b.** When is the body moving forward? Backward?
   - **c.** When is the body's velocity increasing? Decreasing?

Free-Fall Applications

9. **Free fall on Mars and Jupiter** The equations for free fall at the surfaces of Mars and Jupiter (s in meters, t in seconds) are $s = 1.86t^2$ on Mars and $s = 11.44t^2$ on Jupiter. How long does it take a rock falling from rest to reach a velocity of 27.8 m/sec (about 100 km/h) on each planet?

10. **Lunar projectile motion** A rock thrown vertically upward from the surface of the moon at a velocity of 24 m/sec (about 86 km/h) reaches a height of $s = 24t - 0.8t^2$ meters in t sec.
    - **a.** Find the rock's velocity and acceleration at time t. (The acceleration in this case is the acceleration of gravity on the moon.)
    - **b.** How long does it take the rock to reach its highest point?
    - **c.** How high does the rock go?
    - **d.** How long does it take the rock to reach half its maximum height?
    - **e.** How long is the rock aloft?

11. **Finding g on a small airless planet** Explorers on a small airless planet used a spring gun to launch a ball bearing vertically upward from the surface at a launch velocity of 15 m/sec. Because the acceleration of gravity at the planet's surface was g_s m/sec^2, the explorers expected the ball bearing to reach a height of $s = 15t - (1/2)g_s t^2$ meters t sec later. The ball bearing reached its maximum height 20 sec after being launched. What was the value of g_s?

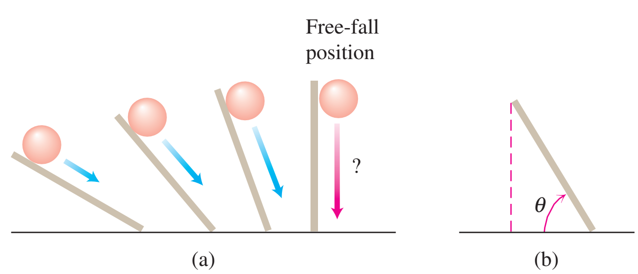
12. **Speeding bullet** A 45-caliber bullet fired straight up from the surface of the moon would reach a height of $s = 832t - 2.6t^2$ feet after t sec. On Earth, in the absence of air, its height would be $s = 832t - 16t^2$ ft after t sec. How long will the bullet be aloft in each case? How high will the bullet go?

13. **Free fall from the Tower of Pisa** Had Galileo dropped a cannonball from the Tower of Pisa, 179 ft above the ground, the ball's height above ground t sec into the fall would have been $s = 179 - 16t^2$.
    - **a.** What would have been the ball's velocity, speed, and acceleration at time t?
    - **b.** About how long would it have taken the ball to hit the ground?
    - **c.** What would have been the ball's velocity at the moment of impact?

14. **Galileo's free-fall formula** Galileo developed a formula for a body's velocity during free fall by rolling balls from rest down increasingly steep inclined planks and looking for a limiting formula that would predict a ball's behavior when the plank was vertical and the ball fell freely; see part (a) of the accompanying figure. He found that, for any given angle of the plank, the ball's velocity t sec into motion was a constant multiple of t. That is, the velocity was given by a formula of the form $v = kt$. The value of the constant k depended on the inclination of the plank.

    In modern notation—part (b) of the figure—with distance in meters and time in seconds, what Galileo determined by experiment was that, for any given angle θ, the ball's velocity t sec into the roll was

$$v = 9.8(\sin\theta)t \text{ m/sec}.$$

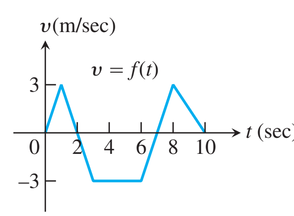


(a) (b)

   - **a.** What is the equation for the ball's velocity during free fall?
   - **b.** Building on your work in part (a), what constant acceleration does a freely falling body experience near the surface of Earth?

Conclusions About Motion from Graphs

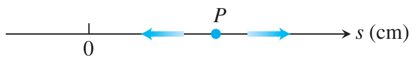
15. The accompanying figure shows the velocity $v = ds/dt = f(t)$ (m/sec) of a body moving along a coordinate line.



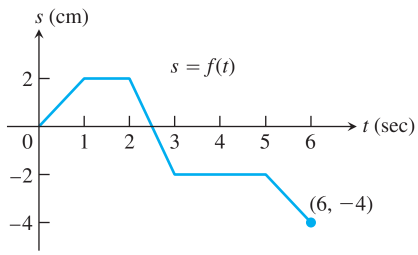
   - **a.** When does the body reverse direction?
   - **b.** When (approximately) is the body moving at a constant speed?
   - **c.** Graph the body's speed for $0 \le t \le 10$.
   - **d.** Graph the acceleration, where defined.

Exercises

16. A particle P moves on the number line shown in part (a) of the accompanying figure. Part (b) shows the position of P as a function of time t.



(a)



(b)

a. When is P moving to the left? Moving to the right? Standing still?

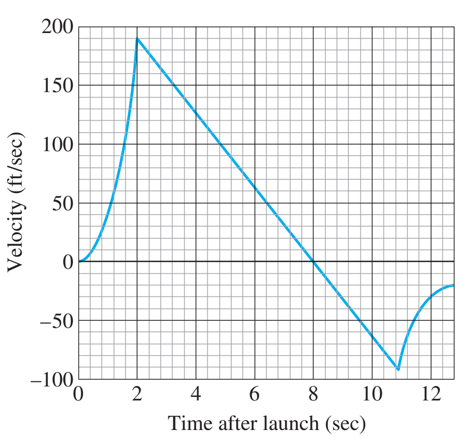
b. Graph the particle's velocity and speed (where defined).

17. Launching a rocket When a model rocket is launched, the propellant burns for a few seconds, accelerating the rocket upward. After burnout, the rocket coasts upward for a while and then begins to fall. A small explosive charge pops out a parachute shortly after the rocket starts down. The parachute slows the rocket to keep it from breaking when it lands.

The figure here shows velocity data from the flight of the model rocket. Use the data to answer the following.

a. How fast was the rocket climbing when the engine stopped?

b. For how many seconds did the engine burn?



c. When did the rocket reach its highest point? What was its velocity then?

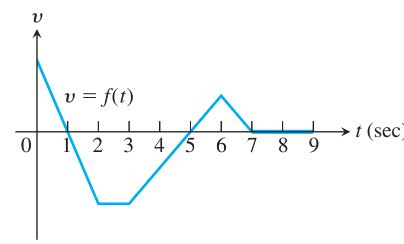
d. When did the parachute pop out? How fast was the rocket falling then?

e. How long did the rocket fall before the parachute opened?

f. When was the rocket's acceleration greatest?

g. When was the acceleration constant? What was its value then (to the nearest integer)?

18. The accompanying figure shows the velocity $v = f(t)$ of a particle moving on a coordinate line.



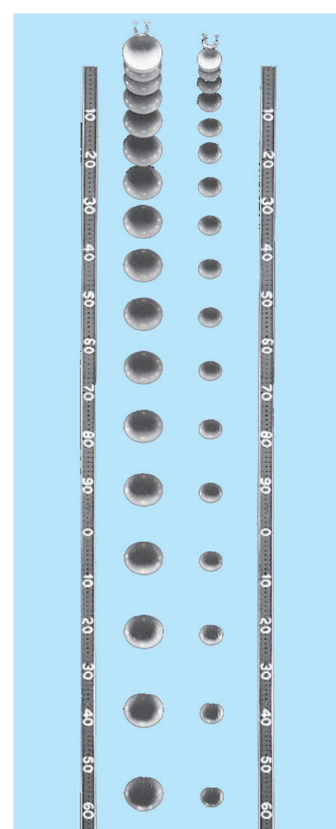
a. When does the particle move forward? Move backward? Speed up? Slow down?

b. When is the particle's acceleration positive? Negative? Zero?

c. When does the particle move at its greatest speed?

d. When does the particle stand still for more than an instant?

19. Two falling balls The multiflash photograph in the accompanying figure shows two balls falling from rest. The vertical rulers are marked in centimeters. Use the equation $s = 490t^2$ (the free-fall equation for s in centimeters and t in seconds) to answer the following questions.



a. How long did it take the balls to fall the first 160 cm? What was their average velocity for the period?

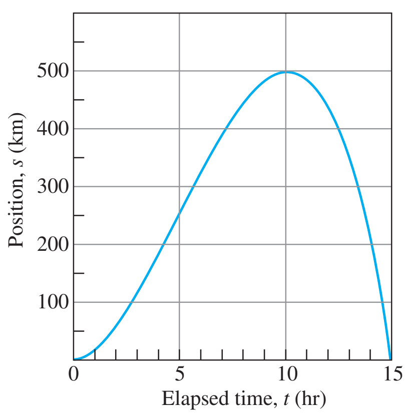
b. How fast were the balls falling when they reached the 160-cm mark? What was their acceleration then?

c. About how fast was the light flashing (flashes per second)?

20. A traveling truck The accompanying graph shows the position s of a truck traveling on a highway. The truck starts at $t = 0$ and returns 15 h later at $t = 15$.

a. Use the technique described in Section 3.1, Example 3, to graph the truck's velocity $v = ds/dt$ for $0 \le t \le 15$. Then repeat the process, with the velocity curve, to graph the truck's acceleration dv/dt.

b. Suppose that $s = 15t^2 - t^3$. Graph ds/dt and d^2s/dt^2 and compare your graphs with those in part (a).



21. The graphs in Figure 3.21 show the position s, velocity $v = ds/dt$, and acceleration $a = d^2s/dt^2$ of a body moving along a coordinate line as functions of time t. Which graph is which? Give reasons for your answers.

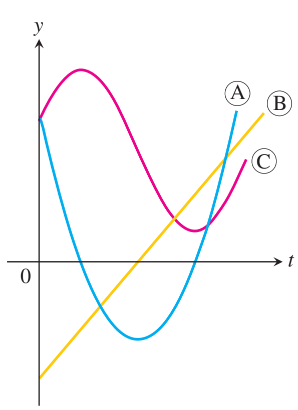


FIGURE 3.21 The graphs for Exercise 21.

22. The graphs in Figure 3.22 show the position s, the velocity $v = ds/dt$, and the acceleration $a = d^2s/dt^2$ of a body moving along the coordinate line as functions of time t. Which graph is which? Give reasons for your answers.

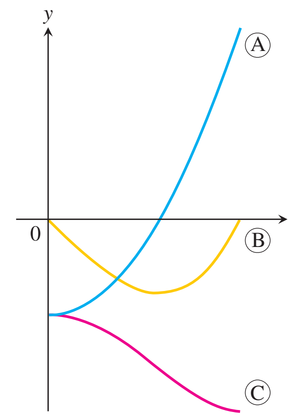


FIGURE 3.22 The graphs for Exercise 22.

Economics

23. Marginal cost Suppose that the dollar cost of producing x washing machines is $c(x) = 2000 + 100x - 0.1x^2$.

a. Find the average cost per machine of producing the first 100 washing machines.

b. Find the marginal cost when 100 washing machines are produced.

c. Show that the marginal cost when 100 washing machines are produced is approximately the cost of producing one more washing machine after the first 100 have been made, by calculating the latter cost directly.

24. Marginal revenue Suppose that the revenue from selling x washing machines is

$$r(x) = 20{,}000\left(1 - \frac{1}{x}\right)$$

dollars.

a. Find the marginal revenue when 100 machines are produced.

b. Use the function $r'(x)$ to estimate the increase in revenue that will result from increasing production from 100 machines a week to 101 machines a week.

c. Find the limit of $r'(x)$ as $x \to \infty$. How would you interpret this number?

Additional Applications

25. Bacterium population When a bactericide was added to a nutrient broth in which bacteria were growing, the bacterium population continued to grow for a while, but then stopped growing and began to decline. The size of the population at time t (hours) was $b = 10^6 + 10^4t - 10^3t^2$. Find the growth rates at

a. $t = 0$ hours.

b. $t = 5$ hours.

c. $t = 10$ hours.

26. Draining a tank The number of gallons of water in a tank t minutes after the tank has started to drain is $Q(t) = 200(30 - t)^2$. How fast is the water running out at the end of 10 min? What is the average rate at which the water flows out during the first 10 min?

27. Draining a tank It takes 12 hours to drain a storage tank by opening the valve at the bottom. The depth y of fluid in the tank t hours after the valve is opened is given by the formula

$$y = 6\left(1 - \frac{t}{12}\right)^2 \text{ m}.$$

a. Find the rate dy/dt (m/h) at which the tank is draining at time t.

b. When is the fluid level in the tank falling fastest? Slowest? What are the values of dy/dt at these times?

c. Graph y and dy/dt together and discuss the behavior of y in relation to the signs and values of dy/dt.

28. Inflating a balloon The volume $V = (4/3)\pi r^3$ of a spherical balloon changes with the radius.

a. At what rate (ft^3/ft) does the volume change with respect to the radius when $r = 2$ ft?

b. By approximately how much does the volume increase when the radius changes from 2 to 2.2 ft?

29. Airplane takeoff Suppose that the distance an aircraft travels along a runway before takeoff is given by $D = (10/9)t^2$, where D is measured in meters from the starting point and t is measured in seconds from the time the brakes are released. The aircraft will become airborne when its speed reaches 200 km/h. How long will it take to become airborne, and what distance will it travel in that time?

30. Volcanic lava fountains Although the November 1959 Kilauea Iki eruption on the island of Hawaii began with a line of fountains along the wall of the crater, activity was later confined to a single vent in the crater's floor, which at one point shot lava 1900 ft straight into the air (a world record). What was the lava's exit velocity in feet per second? In miles per hour? (*Hint*: If v_0 is the exit velocity of a particle of lava, its height t sec later will be $s = v_0 t - 16t^2$ ft. Begin by finding the time at which $ds/dt = 0$. Neglect air resistance.)

Exercises 31–34 give the position function $s = f(t)$ of a body moving along the s-axis as a function of time t. Graph f together with the velocity function $v(t) = ds/dt = f'(t)$ and the acceleration function $a(t) = d^2s/dt^2 = f''(t)$. Comment on the body's behavior in relation to the signs and values of v and a. Include in your commentary such topics as the following:

a. When is the body momentarily at rest?

b. When does it move to the left (down) or to the right (up)?

c. When does it change direction?

d. When does it speed up and slow down?

e. When is it moving fastest (highest speed)? Slowest?

f. When is it farthest from the axis origin?

31. $s = 200t - 16t^2, \quad 0 \le t \le 12.5$ (a heavy object fired straight up from Earth's surface at 200 ft/sec)

32. $s = t^2 - 3t + 2, \quad 0 \le t \le 5$

33. $s = t^3 - 6t^2 + 7t, \quad 0 \le t \le 4$

34. $s = 4 - 7t + 6t^2 - t^3, \quad 0 \le t \le 4$

35. Thoroughbred racing A racehorse is running a 10-furlong race. (A furlong is 220 yards, although we will use furlongs and seconds as our units in this exercise.) As the horse passes each furlong marker (F), a steward records the time elapsed (t) since the beginning of the race, as shown in the table:

F	0	1	2	3	4	5	6	7	8	9	10
t	0	20	33	46	59	73	86	100	112	124	135

a. How long does it take the horse to finish the race?

b. What is the average speed of the horse over the first 5 furlongs?

c. What is the approximate speed of the horse as it passes the 3-furlong marker?

d. During which portion of the race is the horse running the fastest?

e. During which portion of the race is the horse accelerating the fastest?

3.4 Derivatives of Trigonometric Functions

Many of the phenomena we want information about are approximately periodic (electromagnetic fields, heart rhythms, tides, weather). The derivatives of sines and cosines play a key role in describing periodic changes. This section shows how to differentiate the six basic trigonometric functions.

Derivative of the Sine Function

To calculate the derivative of $f(x) = \sin x$, for x measured in radians, we combine the limits in Example 5a and Theorem 7 in Section 2.4 with the angle sum identity for the sine:

$$\sin(x + h) = \sin x \cos h + \cos x \sin h.$$

If $f(x) = \sin x$, then

$$\begin{aligned}
f'(x) &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h} && \text{Derivative definition} \\
&= \lim_{h \to 0} \frac{(\sin x \cos h + \cos x \sin h) - \sin x}{h} && \text{Sine angle sum identity} \\
&= \lim_{h \to 0} \frac{\sin x (\cos h - 1) + \cos x \sin h}{h} \\
&= \lim_{h \to 0} \left(\sin x \cdot \frac{\cos h - 1}{h} \right) + \lim_{h \to 0} \left(\cos x \cdot \frac{\sin h}{h} \right) \\
&= \sin x \cdot \lim_{h \to 0} \frac{\cos h - 1}{h} + \cos x \cdot \lim_{h \to 0} \frac{\sin h}{h} \\
&= \sin x \cdot 0 + \cos x \cdot 1 && \text{Example 5(a) and Theorem 7, Section 2.4} \\
&= \cos x.
\end{aligned}$$

The derivative of the sine function is the cosine function:

$$\frac{d}{dx}(\sin x) = \cos x.$$

EXAMPLE 1 Derivatives Involving the Sine

You Try It

Video

(a) $y = x^2 - \sin x$:

$$\begin{aligned}
\frac{dy}{dx} &= 2x - \frac{d}{dx}(\sin x) && \text{Difference Rule} \\
&= 2x - \cos x.
\end{aligned}$$

(b) $y = x^2 \sin x$:

$$\begin{aligned}
\frac{dy}{dx} &= x^2 \frac{d}{dx}(\sin x) + 2x \sin x && \text{Product Rule} \\
&= x^2 \cos x + 2x \sin x.
\end{aligned}$$

(c) $y = \dfrac{\sin x}{x}$:

$$\begin{aligned}
\frac{dy}{dx} &= \frac{x \cdot \frac{d}{dx}(\sin x) - \sin x \cdot 1}{x^2} && \text{Quotient Rule} \\
&= \frac{x \cos x - \sin x}{x^2}.
\end{aligned}$$

■

Derivative of the Cosine Function

With the help of the angle sum formula for the cosine,

$$\cos(x + h) = \cos x \cos h - \sin x \sin h,$$

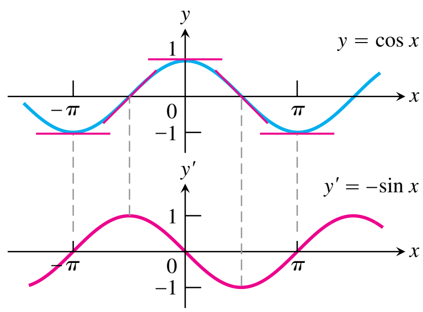


FIGURE 3.23 The curve $y' = -\sin x$ as the graph of the slopes of the tangents to the curve $y = \cos x$.

we have

$$\frac{d}{dx}(\cos x) = \lim_{h\to 0}\frac{\cos(x+h)-\cos x}{h}$$ Derivative definition

$$= \lim_{h\to 0}\frac{(\cos x\cos h - \sin x\sin h)-\cos x}{h}$$ Cosine angle sum identity

$$= \lim_{h\to 0}\frac{\cos x(\cos h - 1)-\sin x\sin h}{h}$$

$$= \lim_{h\to 0}\cos x\cdot\frac{\cos h-1}{h} - \lim_{h\to 0}\sin x\cdot\frac{\sin h}{h}$$

$$= \cos x\cdot\lim_{h\to 0}\frac{\cos h-1}{h} - \sin x\cdot\lim_{h\to 0}\frac{\sin h}{h}$$

$$= \cos x\cdot 0 - \sin x\cdot 1$$ Example 5(a) and Theorem 7, Section 2.4

$$= -\sin x.$$

The derivative of the cosine function is the negative of the sine function:

$$\frac{d}{dx}(\cos x) = -\sin x$$

Figure 3.23 shows a way to visualize this result.

EXAMPLE 2 Derivatives Involving the Cosine

You Try It

(a) $y = 5x + \cos x$:

$$\frac{dy}{dx} = \frac{d}{dx}(5x) + \frac{d}{dx}(\cos x)$$ Sum Rule

$$= 5 - \sin x.$$

Video

(b) $y = \sin x\cos x$:

$$\frac{dy}{dx} = \sin x\frac{d}{dx}(\cos x) + \cos x\frac{d}{dx}(\sin x)$$ Product Rule

$$= \sin x(-\sin x) + \cos x(\cos x)$$

$$= \cos^2 x - \sin^2 x.$$

Video

(c) $y = \dfrac{\cos x}{1-\sin x}$:

$$\frac{dy}{dx} = \frac{(1-\sin x)\frac{d}{dx}(\cos x) - \cos x\frac{d}{dx}(1-\sin x)}{(1-\sin x)^2}$$ Quotient Rule

$$= \frac{(1-\sin x)(-\sin x) - \cos x(0-\cos x)}{(1-\sin x)^2}$$

$$= \frac{1-\sin x}{(1-\sin x)^2}$$ $\sin^2 x + \cos^2 x = 1$

$$= \frac{1}{1-\sin x}.$$ ■

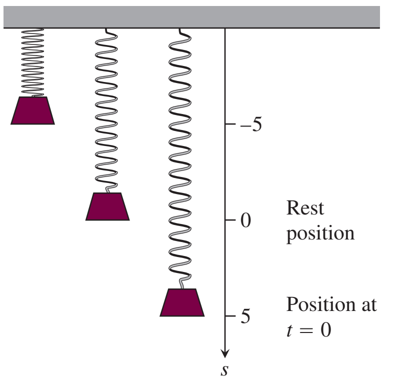


FIGURE 3.24 A body hanging from a vertical spring and then displaced oscillates above and below its rest position. Its motion is described by trigonometric functions (Example 3).

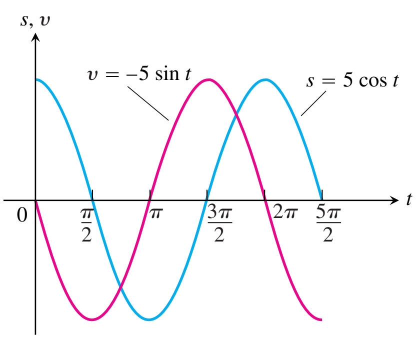


FIGURE 3.25 The graphs of the position and velocity of the body in Example 3.

Simple Harmonic Motion

The motion of a body bobbing freely up and down on the end of a spring or bungee cord is an example of *simple harmonic motion*. The next example describes a case in which there are no opposing forces such as friction or buoyancy to slow the motion down.

EXAMPLE 3 Motion on a Spring

A body hanging from a spring (Figure 3.24) is stretched 5 units beyond its rest position and released at time $t = 0$ to bob up and down. Its position at any later time t is

$$s = 5\cos t.$$

What are its velocity and acceleration at time t?

Solution We have

Position: $s = 5\cos t$

Velocity: $v = \dfrac{ds}{dt} = \dfrac{d}{dt}(5\cos t) = -5\sin t$

Acceleration: $a = \dfrac{dv}{dt} = \dfrac{d}{dt}(-5\sin t) = -5\cos t.$

Notice how much we can learn from these equations:

1. As time passes, the weight moves down and up between $s = -5$ and $s = 5$ on the s-axis. The amplitude of the motion is 5. The period of the motion is 2π.
2. The velocity $v = -5\sin t$ attains its greatest magnitude, 5, when $\cos t = 0$, as the graphs show in Figure 3.25. Hence, the speed of the weight, $|v| = 5|\sin t|$, is greatest when $\cos t = 0$, that is, when $s = 0$ (the rest position). The speed of the weight is zero when $\sin t = 0$. This occurs when $s = 5\cos t = \pm 5$, at the endpoints of the interval of motion.
3. The acceleration value is always the exact opposite of the position value. When the weight is above the rest position, gravity is pulling it back down; when the weight is below the rest position, the spring is pulling it back up.
4. The acceleration, $a = -5\cos t$, is zero only at the rest position, where $\cos t = 0$ and the force of gravity and the force from the spring offset each other. When the weight is anywhere else, the two forces are unequal and acceleration is nonzero. The acceleration is greatest in magnitude at the points farthest from the rest position, where $\cos t = \pm 1$. ■

EXAMPLE 4 Jerk

The jerk of the simple harmonic motion in Example 3 is

$$j = \frac{da}{dt} = \frac{d}{dt}(-5\cos t) = 5\sin t.$$

It has its greatest magnitude when $\sin t = \pm 1$, not at the extremes of the displacement but at the rest position, where the acceleration changes direction and sign. ■

Derivatives of the Other Basic Trigonometric Functions

Because $\sin x$ and $\cos x$ are differentiable functions of x, the related functions

$$\tan x = \frac{\sin x}{\cos x}, \quad \cot x = \frac{\cos x}{\sin x}, \quad \sec x = \frac{1}{\cos x}, \quad \text{and} \quad \csc x = \frac{1}{\sin x}$$

are differentiable at every value of x at which they are defined. Their derivatives, calculated from the Quotient Rule, are given by the following formulas. Notice the negative signs in the derivative formulas for the cofunctions.

Derivatives of the Other Trigonometric Functions

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\cot x) = -\csc^2 x$$

$$\frac{d}{dx}(\csc x) = -\csc x \cot x$$

To show a typical calculation, we derive the derivative of the tangent function. The other derivations are left to Exercise 50.

EXAMPLE 5

Find $d(\tan x)/dx$.

Solution

$$\frac{d}{dx}(\tan x) = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) = \frac{\cos x \frac{d}{dx}(\sin x) - \sin x \frac{d}{dx}(\cos x)}{\cos^2 x} \qquad \text{Quotient Rule}$$

$$= \frac{\cos x \cos x - \sin x(-\sin x)}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x} = \sec^2 x$$

■

EXAMPLE 6

Find y'' if $y = \sec x$.

Solution

$$y = \sec x$$

$$y' = \sec x \tan x$$

$$y'' = \frac{d}{dx}(\sec x \tan x)$$

$$= \sec x \frac{d}{dx}(\tan x) + \tan x \frac{d}{dx}(\sec x) \qquad \text{Product Rule}$$

$$= \sec x(\sec^2 x) + \tan x(\sec x \tan x)$$

$$= \sec^3 x + \sec x \tan^2 x$$

■

The differentiability of the trigonometric functions throughout their domains gives another proof of their continuity at every point in their domains (Theorem 1, Section 3.1). So we can calculate limits of algebraic combinations and composites of trigonometric functions by direct substitution.

EXAMPLE 7 Finding a Trigonometric Limit

$$\lim_{x \to 0} \frac{\sqrt{2 + \sec x}}{\cos(\pi - \tan x)} = \frac{\sqrt{2 + \sec 0}}{\cos(\pi - \tan 0)} = \frac{\sqrt{2 + 1}}{\cos(\pi - 0)} = \frac{\sqrt{3}}{-1} = -\sqrt{3}$$ ■

EXERCISES 3.4

Derivatives

In Exercises 1–12, find dy/dx.

1. $y = -10x + 3\cos x$
2. $y = \dfrac{3}{x} + 5\sin x$
3. $y = \csc x - 4\sqrt{x} + 7$
4. $y = x^2 \cot x - \dfrac{1}{x^2}$
5. $y = (\sec x + \tan x)(\sec x - \tan x)$
6. $y = (\sin x + \cos x)\sec x$
7. $y = \dfrac{\cot x}{1 + \cot x}$
8. $y = \dfrac{\cos x}{1 + \sin x}$
9. $y = \dfrac{4}{\cos x} + \dfrac{1}{\tan x}$
10. $y = \dfrac{\cos x}{x} + \dfrac{x}{\cos x}$
11. $y = x^2 \sin x + 2x\cos x - 2\sin x$
12. $y = x^2 \cos x - 2x\sin x - 2\cos x$

In Exercises 13–16, find ds/dt.

13. $s = \tan t - t$
14. $s = t^2 - \sec t + 1$
15. $s = \dfrac{1 + \csc t}{1 - \csc t}$
16. $s = \dfrac{\sin t}{1 - \cos t}$

In Exercises 17–20, find $dr/d\theta$.

17. $r = 4 - \theta^2 \sin\theta$
18. $r = \theta\sin\theta + \cos\theta$
19. $r = \sec\theta\csc\theta$
20. $r = (1 + \sec\theta)\sin\theta$

In Exercises 21–24, find dp/dq.

21. $p = 5 + \dfrac{1}{\cot q}$
22. $p = (1 + \csc q)\cos q$
23. $p = \dfrac{\sin q + \cos q}{\cos q}$
24. $p = \dfrac{\tan q}{1 + \tan q}$

25. Find y'' if
    a. $y = \csc x$. **b.** $y = \sec x$.

26. Find $y^{(4)} = d^4y/dx^4$ if
    a. $y = -2\sin x$. **b.** $y = 9\cos x$.

Tangent Lines

In Exercises 27–30, graph the curves over the given intervals, together with their tangents at the given values of x. Label each curve and tangent with its equation.

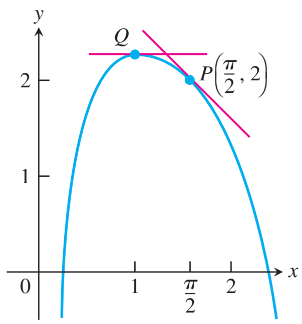
27. $y = \sin x, \quad -3\pi/2 \le x \le 2\pi$
    $x = -\pi, 0, 3\pi/2$
28. $y = \tan x, \quad -\pi/2 < x < \pi/2$
    $x = -\pi/3, 0, \pi/3$
29. $y = \sec x, \quad -\pi/2 < x < \pi/2$
    $x = -\pi/3, \pi/4$
30. $y = 1 + \cos x, \quad -3\pi/2 \le x \le 2\pi$
    $x = -\pi/3, 3\pi/2$

T Do the graphs of the functions in Exercises 31–34 have any horizontal tangents in the interval $0 \le x \le 2\pi$? If so, where? If not, why not? Visualize your findings by graphing the functions with a grapher.

31. $y = x + \sin x$
32. $y = 2x + \sin x$
33. $y = x - \cot x$
34. $y = x + 2\cos x$
35. Find all points on the curve $y = \tan x$, $-\pi/2 < x < \pi/2$, where the tangent line is parallel to the line $y = 2x$. Sketch the curve and tangent(s) together, labeling each with its equation.
36. Find all points on the curve $y = \cot x$, $0 < x < \pi$, where the tangent line is parallel to the line $y = -x$. Sketch the curve and tangent(s) together, labeling each with its equation.

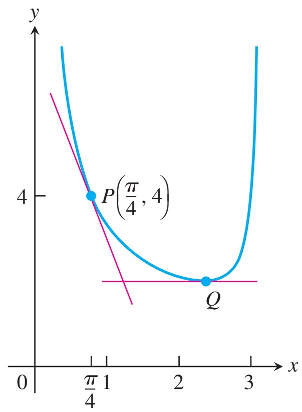
In Exercises 37 and 38, find an equation for **(a)** the tangent to the curve at P and **(b)** the horizontal tangent to the curve at Q.

37.



$y = 4 + \cot x - 2\csc x$

38.



$y = 1 + \sqrt{2}\csc x + \cot x$

Trigonometric Limits

Find the limits in Exercises 39–44.

39. $\lim_{x \to 2} \sin\left(\frac{1}{x} - \frac{1}{2}\right)$

40. $\lim_{x \to -\pi/6} \sqrt{1 + \cos(\pi \csc x)}$

41. $\lim_{x \to 0} \sec\left[\cos x + \pi \tan\left(\frac{\pi}{4 \sec x}\right) - 1\right]$

42. $\lim_{x \to 0} \sin\left(\frac{\pi + \tan x}{\tan x - 2 \sec x}\right)$

43. $\lim_{t \to 0} \tan\left(1 - \frac{\sin t}{t}\right)$

44. $\lim_{\theta \to 0} \cos\left(\frac{\pi\theta}{\sin\theta}\right)$

Simple Harmonic Motion

The equations in Exercises 45 and 46 give the position $s = f(t)$ of a body moving on a coordinate line (s in meters, t in seconds). Find the body's velocity, speed, acceleration, and jerk at time $t = \pi/4$ sec.

45. $s = 2 - 2\sin t$

46. $s = \sin t + \cos t$

Theory and Examples

47. Is there a value of c that will make

$$f(x) = \begin{cases} \frac{\sin^2 3x}{x^2}, & x \neq 0 \\ c, & x = 0 \end{cases}$$

continuous at $x = 0$? Give reasons for your answer.

48. Is there a value of b that will make

$$g(x) = \begin{cases} x + b, & x < 0 \\ \cos x, & x \geq 0 \end{cases}$$

continuous at $x = 0$? Differentiable at $x = 0$? Give reasons for your answers.

49. Find $d^{999}/dx^{999}(\cos x)$.

50. Derive the formula for the derivative with respect to x of

a. $\sec x$. **b.** $\csc x$. **c.** $\cot x$.

T **51.** Graph $y = \cos x$ for $-\pi \leq x \leq 2\pi$. On the same screen, graph

$$y = \frac{\sin(x + h) - \sin x}{h}$$

for $h = 1, 0.5, 0.3$, and 0.1. Then, in a new window, try $h = -1, -0.5$, and -0.3. What happens as $h \to 0^+$? As $h \to 0^-$? What phenomenon is being illustrated here?

T **52.** Graph $y = -\sin x$ for $-\pi \leq x \leq 2\pi$. On the same screen, graph

$$y = \frac{\cos(x + h) - \cos x}{h}$$

for $h = 1, 0.5, 0.3$, and 0.1. Then, in a new window, try $h = -1, -0.5$, and -0.3. What happens as $h \to 0^+$? As $h \to 0^-$? What phenomenon is being illustrated here?

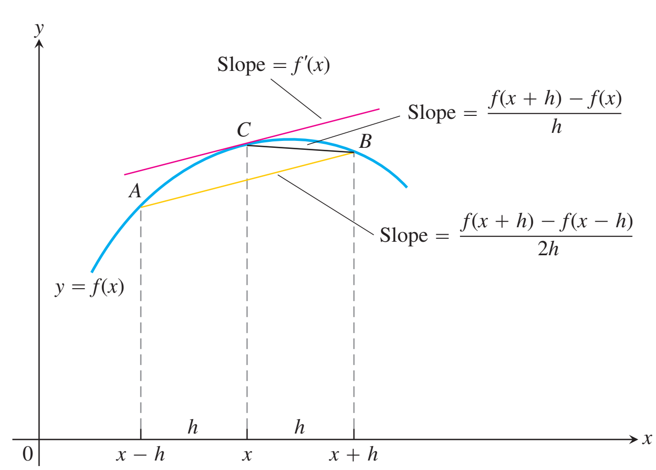
T **53. Centered difference quotients** The *centered difference quotient*

$$\frac{f(x + h) - f(x - h)}{2h}$$

is used to approximate $f'(x)$ in numerical work because (1) its limit as $h \to 0$ equals $f'(x)$ when $f'(x)$ exists, and (2) it usually gives a better approximation of $f'(x)$ for a given value of h than Fermat's difference quotient

$$\frac{f(x + h) - f(x)}{h}.$$

See the accompanying figure.



a. To see how rapidly the centered difference quotient for $f(x) = \sin x$ converges to $f'(x) = \cos x$, graph $y = \cos x$ together with

$$y = \frac{\sin(x+h) - \sin(x-h)}{2h}$$

over the interval $[-\pi, 2\pi]$ for $h = 1, 0.5$, and 0.3. Compare the results with those obtained in Exercise 51 for the same values of h.

b. To see how rapidly the centered difference quotient for $f(x) = \cos x$ converges to $f'(x) = -\sin x$, graph $y = -\sin x$ together with

$$y = \frac{\cos(x+h) - \cos(x-h)}{2h}$$

over the interval $[-\pi, 2\pi]$ for $h = 1, 0.5$, and 0.3. Compare the results with those obtained in Exercise 52 for the same values of h.

54. A caution about centered difference quotients (*Continuation of Exercise 53.*) The quotient

$$\frac{f(x+h) - f(x-h)}{2h}$$

may have a limit as $h \to 0$ when f has no derivative at x. As a case in point, take $f(x) = |x|$ and calculate

$$\lim_{h \to 0} \frac{|0+h| - |0-h|}{2h}.$$

As you will see, the limit exists even though $f(x) = |x|$ has no derivative at $x = 0$. *Moral:* Before using a centered difference quotient, be sure the derivative exists.

T **55. Slopes on the graph of the tangent function** Graph $y = \tan x$ and its derivative together on $(-\pi/2, \pi/2)$. Does the graph of the tangent function appear to have a smallest slope? a largest slope? Is the slope ever negative? Give reasons for your answers.

T **56. Slopes on the graph of the cotangent function** Graph $y = \cot x$ and its derivative together for $0 < x < \pi$. Does the graph of the cotangent function appear to have a smallest slope? A largest slope? Is the slope ever positive? Give reasons for your answers.

T **57. Exploring $(\sin kx)/x$** Graph $y = (\sin x)/x$, $y = (\sin 2x)/x$, and $y = (\sin 4x)/x$ together over the interval $-2 \le x \le 2$. Where does each graph appear to cross the y-axis? Do the graphs really intersect the axis? What would you expect the graphs of $y = (\sin 5x)/x$ and $y = (\sin(-3x))/x$ to do as $x \to 0$? Why? What about the graph of $y = (\sin kx)/x$ for other values of k? Give reasons for your answers.

T **58. Radians versus degrees: degree mode derivatives** What happens to the derivatives of $\sin x$ and $\cos x$ if x is measured in degrees instead of radians? To find out, take the following steps.

a. With your graphing calculator or computer grapher in *degree mode*, graph

$$f(h) = \frac{\sin h}{h}$$

and estimate $\lim_{h \to 0} f(h)$. Compare your estimate with $\pi/180$. Is there any reason to believe the limit *should* be $\pi/180$?

b. With your grapher still in degree mode, estimate

$$\lim_{h \to 0} \frac{\cos h - 1}{h}.$$

c. Now go back to the derivation of the formula for the derivative of $\sin x$ in the text and carry out the steps of the derivation using degree-mode limits. What formula do you obtain for the derivative?

d. Work through the derivation of the formula for the derivative of $\cos x$ using degree-mode limits. What formula do you obtain for the derivative?

e. The disadvantages of the degree-mode formulas become apparent as you start taking derivatives of higher order. Try it. What are the second and third degree-mode derivatives of $\sin x$ and $\cos x$?

3.5 The Chain Rule and Parametric Equations

We know how to differentiate $y = f(u) = \sin u$ and $u = g(x) = x^2 - 4$, but how do we differentiate a composite like $F(x) = f(g(x)) = \sin(x^2 - 4)$? The differentiation formulas we have studied so far do not tell us how to calculate $F'(x)$. So how do we find the derivative of $F = f \circ g$? The answer is, with the Chain Rule, which says that the derivative of the composite of two differentiable functions is the product of their derivatives evaluated at appropriate points. The Chain Rule is one of the most important and widely used rules of differentiation. This section describes the rule and how to use it. We then apply the rule to describe curves in the plane and their tangent lines in another way.

Derivative of a Composite Function

We begin with examples.

EXAMPLE 1 Relating Derivatives

The function $y = \frac{3}{2}x = \frac{1}{2}(3x)$ is the composite of the functions $y = \frac{1}{2}u$ and $u = 3x$. How are the derivatives of these functions related?

Solution We have

$$\frac{dy}{dx} = \frac{3}{2}, \qquad \frac{dy}{du} = \frac{1}{2}, \qquad \text{and} \qquad \frac{du}{dx} = 3.$$

Since $\frac{3}{2} = \frac{1}{2} \cdot 3$, we see that

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}.$$

Is it an accident that

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}?$$

If we think of the derivative as a rate of change, our intuition allows us to see that this relationship is reasonable. If $y = f(u)$ changes half as fast as u and $u = g(x)$ changes three times as fast as x, then we expect y to change $3/2$ times as fast as x. This effect is much like that of a multiple gear train (Figure 3.26). ■

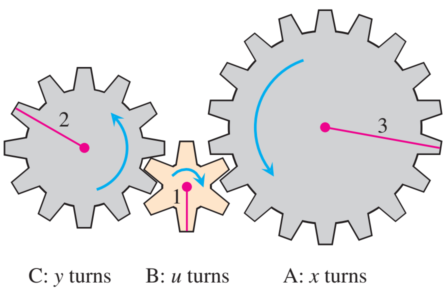


FIGURE 3.26 When gear A makes x turns, gear B makes u turns and gear C makes y turns. By comparing circumferences or counting teeth, we see that $y = u/2$ (C turns one-half turn for each B turn) and $u = 3x$ (B turns three times for A's one), so $y = 3x/2$. Thus, $dy/dx = 3/2 = (1/2)(3) = (dy/du)(du/dx)$.

EXAMPLE 2

The function

$$y = 9x^4 + 6x^2 + 1 = (3x^2 + 1)^2$$

is the composite of $y = u^2$ and $u = 3x^2 + 1$. Calculating derivatives, we see that

$$\begin{aligned}\frac{dy}{du} \cdot \frac{du}{dx} &= 2u \cdot 6x \\ &= 2(3x^2 + 1) \cdot 6x \\ &= 36x^3 + 12x.\end{aligned}$$

Calculating the derivative from the expanded formula, we get

$$\begin{aligned}\frac{dy}{dx} &= \frac{d}{dx}\left(9x^4 + 6x^2 + 1\right) \\ &= 36x^3 + 12x.\end{aligned}$$

Once again,

$$\frac{dy}{du} \cdot \frac{du}{dx} = \frac{dy}{dx}.$$ ■

The derivative of the composite function $f(g(x))$ at x is the derivative of f at $g(x)$ times the derivative of g at x. This is known as the Chain Rule (Figure 3.27).

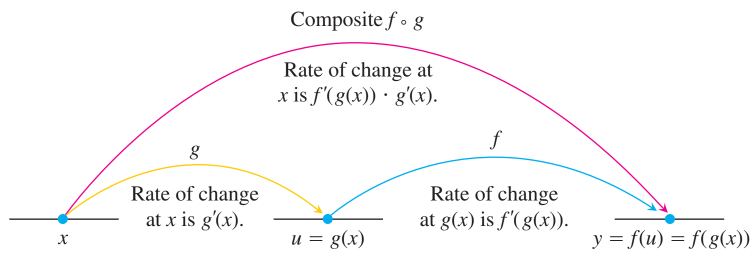


FIGURE 3.27 Rates of change multiply: The derivative of $f \circ g$ at x is the derivative of f at $g(x)$ times the derivative of g at x.

THEOREM 3 The Chain Rule

If $f(u)$ is differentiable at the point $u = g(x)$ and $g(x)$ is differentiable at x, then the composite function $(f \circ g)(x) = f(g(x))$ is differentiable at x, and

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x).$$

In Leibniz's notation, if $y = f(u)$ and $u = g(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx},$$

where dy/du is evaluated at $u = g(x)$.

Intuitive "Proof" of the Chain Rule:

Let Δu be the change in u corresponding to a change of Δx in x, that is

$$\Delta u = g(x + \Delta x) - g(x)$$

Then the corresponding change in y is

$$\Delta y = f(u + \Delta u) - f(u).$$

It would be tempting to write

$$\frac{\Delta y}{\Delta x} = \frac{\Delta y}{\Delta u} \cdot \frac{\Delta u}{\Delta x} \tag{1}$$

and take the limit as $\Delta x \to 0$:

$$\begin{aligned}
\frac{dy}{dx} &= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta u} \cdot \frac{\Delta u}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta u} \cdot \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} \\
&= \lim_{\Delta u \to 0} \frac{\Delta y}{\Delta u} \cdot \lim_{\Delta x \to 0} \frac{\Delta u}{\Delta x} \\
&= \frac{dy}{du} \frac{du}{dx}.
\end{aligned}$$

(Note that $\Delta u \to 0$ as $\Delta x \to 0$ since g is continuous.)

The only flaw in this reasoning is that in Equation (1) it might happen that $\Delta u = 0$ (even when $\Delta x \neq 0$) and, of course, we can't divide by 0. The proof requires a different approach to overcome this flaw, and we give a precise proof in Section 3.8. ■

EXAMPLE 3 Applying the Chain Rule

An object moves along the x-axis so that its position at any time $t \geq 0$ is given by $x(t) = \cos(t^2 + 1)$. Find the velocity of the object as a function of t.

Solution We know that the velocity is dx/dt. In this instance, x is a composite function: $x = \cos(u)$ and $u = t^2 + 1$. We have

$$\frac{dx}{du} = -\sin(u) \qquad x = \cos(u)$$

$$\frac{du}{dt} = 2t. \qquad u = t^2 + 1$$

By the Chain Rule,

$$\begin{aligned} \frac{dx}{dt} &= \frac{dx}{du} \cdot \frac{du}{dt} \\ &= -\sin(u) \cdot 2t \qquad \frac{dx}{du} \text{ evaluated at } u \\ &= -\sin(t^2 + 1) \cdot 2t \\ &= -2t\sin(t^2 + 1). \end{aligned}$$

■

As we see from Example 3, a difficulty with the Leibniz notation is that it doesn't state specifically where the derivatives are supposed to be evaluated.

"Outside-Inside" Rule

It sometimes helps to think about the Chain Rule this way: If $y = f(g(x))$, then

$$\frac{dy}{dx} = f'(g(x)) \cdot g'(x).$$

In words, differentiate the "outside" function f and evaluate it at the "inside" function $g(x)$ left alone; then multiply by the derivative of the "inside function."

You Try It

EXAMPLE 4 Differentiating from the Outside In

Differentiate $\sin(x^2 + x)$ with respect to x.

Solution

$$\frac{d}{dx}\sin\underbrace{(x^2 + x)}_{\text{inside}} = \cos\underbrace{(x^2 + x)}_{\text{inside left alone}} \cdot \underbrace{(2x + 1)}_{\text{derivative of the inside}}$$

■

Repeated Use of the Chain Rule

We sometimes have to use the Chain Rule two or more times to find a derivative. Here is an example.

HISTORICAL BIOGRAPHY

Johann Bernoulli
(1667–1748)

EXAMPLE 5 A Three-Link "Chain"

Find the derivative of $g(t) = \tan(5 - \sin 2t)$.

Solution Notice here that the tangent is a function of $5 - \sin 2t$, whereas the sine is a function of $2t$, which is itself a function of t. Therefore, by the Chain Rule,

$$g'(t) = \frac{d}{dt}\left(\tan\left(5 - \sin 2t\right)\right)$$

$$= \sec^2(5 - \sin 2t) \cdot \frac{d}{dt}\left(5 - \sin 2t\right) \qquad \text{Derivative of } \tan u \text{ with } u = 5 - \sin 2t$$

$$= \sec^2(5 - \sin 2t) \cdot \left(0 - \cos 2t \cdot \frac{d}{dt}(2t)\right) \qquad \text{Derivative of } 5 - \sin u \text{ with } u = 2t$$

$$= \sec^2(5 - \sin 2t) \cdot (-\cos 2t) \cdot 2$$

$$= -2(\cos 2t)\sec^2(5 - \sin 2t).$$

■

The Chain Rule with Powers of a Function

If f is a differentiable function of u and if u is a differentiable function of x, then substituting $y = f(u)$ into the Chain Rule formula

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

leads to the formula

$$\frac{d}{dx} f(u) = f'(u)\frac{du}{dx}.$$

Here's an example of how it works: If n is a positive or negative integer and $f(u) = u^n$, the Power Rules (Rules 2 and 7) tell us that $f'(u) = nu^{n-1}$. If u is a differentiable function of x, then we can use the Chain Rule to extend this to the **Power Chain Rule**:

$$\frac{d}{dx} u^n = nu^{n-1}\frac{du}{dx}. \qquad \frac{d}{du}\left(u^n\right) = nu^{n-1}$$

EXAMPLE 6 Applying the Power Chain Rule

(a)
$$\frac{d}{dx}(5x^3 - x^4)^7 = 7(5x^3 - x^4)^6 \frac{d}{dx}\left(5x^3 - x^4\right) \qquad \text{Power Chain Rule with } u = 5x^3 - x^4, n = 7$$

$$= 7(5x^3 - x^4)^6(5 \cdot 3x^2 - 4x^3)$$

$$= 7(5x^3 - x^4)^6(15x^2 - 4x^3)$$

(b)
$$\frac{d}{dx}\left(\frac{1}{3x - 2}\right) = \frac{d}{dx}(3x - 2)^{-1}$$

$$= -1(3x - 2)^{-2}\frac{d}{dx}(3x - 2) \qquad \text{Power Chain Rule with } u = 3x - 2, n = -1$$

$$= -1(3x - 2)^{-2}(3)$$

$$= -\frac{3}{(3x - 2)^2}$$

In part (b) we could also have found the derivative with the Quotient Rule. ■

$\sin^n x$ means $(\sin x)^n$, $n \neq -1$.

EXAMPLE 7 Finding Tangent Slopes

(a) Find the slope of the line tangent to the curve $y = \sin^5 x$ at the point where $x = \pi/3$.

(b) Show that the slope of every line tangent to the curve $y = 1/(1 - 2x)^3$ is positive.

Solution

(a)
$$\frac{dy}{dx} = 5\sin^4 x \cdot \frac{d}{dx}\sin x \qquad \text{Power Chain Rule with } u = \sin x,\ n = 5$$
$$= 5\sin^4 x \cos x$$

The tangent line has slope

$$\left.\frac{dy}{dx}\right|_{x=\pi/3} = 5\left(\frac{\sqrt{3}}{2}\right)^4\left(\frac{1}{2}\right) = \frac{45}{32}.$$

(b)
$$\frac{dy}{dx} = \frac{d}{dx}(1 - 2x)^{-3}$$
$$= -3(1 - 2x)^{-4} \cdot \frac{d}{dx}(1 - 2x) \qquad \text{Power Chain Rule with } u = (1 - 2x),\ n = -3$$
$$= -3(1 - 2x)^{-4} \cdot (-2)$$
$$= \frac{6}{(1 - 2x)^4}$$

At any point (x, y) on the curve, $x \neq 1/2$ and the slope of the tangent line is

$$\frac{dy}{dx} = \frac{6}{(1 - 2x)^4},$$

the quotient of two positive numbers. ■

EXAMPLE 8 Radians Versus Degrees

It is important to remember that the formulas for the derivatives of both $\sin x$ and $\cos x$ were obtained under the assumption that x is measured in radians, *not* degrees. The Chain Rule gives us new insight into the difference between the two. Since $180° = \pi$ radians, $x° = \pi x/180$ radians where $x°$ means the angle x measured in degrees.

By the Chain Rule,

$$\frac{d}{dx}\sin(x°) = \frac{d}{dx}\sin\left(\frac{\pi x}{180}\right) = \frac{\pi}{180}\cos\left(\frac{\pi x}{180}\right) = \frac{\pi}{180}\cos(x°).$$

See Figure 3.28. Similarly, the derivative of $\cos(x°)$ is $-(\pi/180)\sin(x°)$.

The factor $\pi/180$, annoying in the first derivative, would compound with repeated differentiation. We see at a glance the compelling reason for the use of radian measure. ■

Parametric Equations

Instead of describing a curve by expressing the y-coordinate of a point $P(x, y)$ on the curve as a function of x, it is sometimes more convenient to describe the curve by expressing *both* coordinates as functions of a third variable t. Figure 3.29 shows the path of a moving particle described by a pair of equations, $x = f(t)$ and $y = g(t)$. For studying motion,

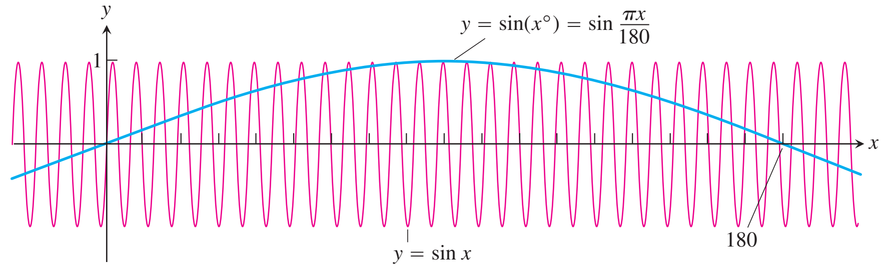


FIGURE 3.28 Sin $(x°)$ oscillates only $\pi/180$ times as often as $\sin x$ oscillates. Its maximum slope is $\pi/180$ at $x = 0$ (Example 8).

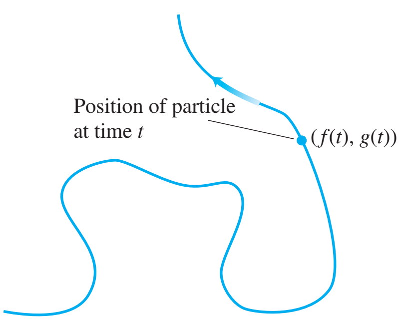


FIGURE 3.29 The path traced by a particle moving in the xy-plane is not always the graph of a function of x or a function of y.

t usually denotes time. Equations like these are better than a Cartesian formula because they tell us the particle's position $(x, y) = (f(t), g(t))$ at any time t.

DEFINITION Parametric Curve

If x and y are given as functions

$$x = f(t), \qquad y = g(t)$$

over an interval of t-values, then the set of points $(x, y) = (f(t), g(t))$ defined by these equations is a **parametric curve**. The equations are **parametric equations** for the curve.

The variable t is a **parameter** for the curve, and its domain I is the **parameter interval**. If I is a closed interval, $a \le t \le b$, the point $(f(a), g(a))$ is the **initial point** of the curve. The point $(f(b), g(b))$ is the **terminal point**. When we give parametric equations and a parameter interval for a curve, we say that we have **parametrized** the curve. The equations and interval together constitute a **parametrization** of the curve.

EXAMPLE 9 Moving Counterclockwise on a Circle

Graph the parametric curves

(a) $x = \cos t, \qquad y = \sin t, \qquad 0 \le t \le 2\pi.$

(b) $x = a\cos t, \qquad y = a\sin t, \qquad 0 \le t \le 2\pi.$

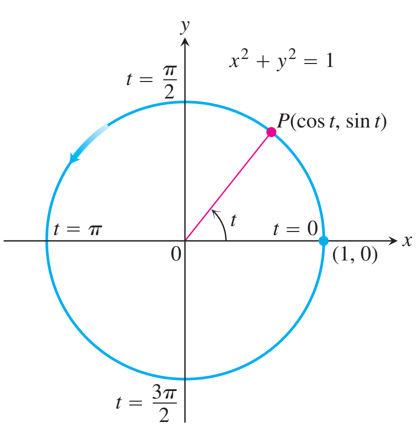


FIGURE 3.30 The equations $x = \cos t$ and $y = \sin t$ describe motion on the circle $x^2 + y^2 = 1$. The arrow shows the direction of increasing t (Example 9).

Solution

(a) Since $x^2 + y^2 = \cos^2 t + \sin^2 t = 1$, the parametric curve lies along the unit circle $x^2 + y^2 = 1$. As t increases from 0 to 2π, the point $(x, y) = (\cos t, \sin t)$ starts at $(1, 0)$ and traces the entire circle once counterclockwise (Figure 3.30).

(b) For $x = a\cos t$, $y = a\sin t$, $0 \le t \le 2\pi$, we have $x^2 + y^2 = a^2\cos^2 t + a^2\sin^2 t = a^2$. The parametrization describes a motion that begins at the point $(a, 0)$ and traverses the circle $x^2 + y^2 = a^2$ once counterclockwise, returning to $(a, 0)$ at $t = 2\pi$. ■

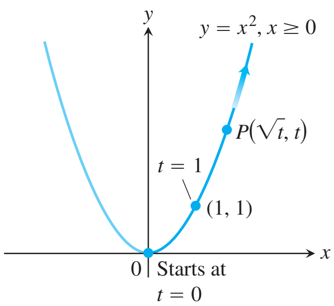


FIGURE 3.31 The equations $x = \sqrt{t}$ and $y = t$ and the interval $t \geq 0$ describe the motion of a particle that traces the right-hand half of the parabola $y = x^2$ (Example 10).

EXAMPLE 10 Moving Along a Parabola

The position $P(x, y)$ of a particle moving in the xy-plane is given by the equations and parameter interval

$$x = \sqrt{t}, \qquad y = t, \qquad t \geq 0.$$

Identify the path traced by the particle and describe the motion.

Solution We try to identify the path by eliminating t between the equations $x = \sqrt{t}$ and $y = t$. With any luck, this will produce a recognizable algebraic relation between x and y. We find that

$$y = t = \left(\sqrt{t}\right)^2 = x^2.$$

Thus, the particle's position coordinates satisfy the equation $y = x^2$, so the particle moves along the parabola $y = x^2$.

It would be a mistake, however, to conclude that the particle's path is the entire parabola $y = x^2$; it is only half the parabola. The particle's x-coordinate is never negative. The particle starts at (0, 0) when $t = 0$ and rises into the first quadrant as t increases (Figure 3.31). The parameter interval is $[0, \infty)$ and there is no terminal point. ■

EXAMPLE 11 Parametrizing a Line Segment

Find a parametrization for the line segment with endpoints $(-2, 1)$ and (3, 5).

Solution Using $(-2, 1)$ we create the parametric equations

$$x = -2 + at, \qquad y = 1 + bt.$$

These represent a line, as we can see by solving each equation for t and equating to obtain

$$\frac{x + 2}{a} = \frac{y - 1}{b}.$$

This line goes through the point $(-2, 1)$ when $t = 0$. We determine a and b so that the line goes through (3, 5) when $t = 1$.

$$3 = -2 + a \quad \Rightarrow \quad a = 5 \qquad x = 3 \text{ when } t = 1.$$

$$5 = 1 + b \quad \Rightarrow \quad b = 4 \qquad y = 5 \text{ when } t = 1.$$

Therefore,

$$x = -2 + 5t, \qquad y = 1 + 4t, \qquad 0 \leq t \leq 1$$

is a parametrization of the line segment with initial point $(-2, 1)$ and terminal point (3, 5). ■

Slopes of Parametrized Curves

A parametrized curve $x = f(t)$ and $y = g(t)$ is **differentiable** at t if f and g are differentiable at t. At a point on a differentiable parametrized curve where y is also a differentiable function of x, the derivatives dy/dt, dx/dt, and dy/dx are related by the Chain Rule:

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}.$$

If $dx/dt \neq 0$, we may divide both sides of this equation by dx/dt to solve for dy/dx.

Parametric Formula for dy/dx

If all three derivatives exist and $dx/dt \neq 0$,

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}. \tag{2}$$

EXAMPLE 12 Differentiating with a Parameter

If $x = 2t + 3$ and $y = t^2 - 1$, find the value of dy/dx at $t = 6$.

Solution Equation (2) gives dy/dx as a function of t:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2t}{2} = t = \frac{x - 3}{2}.$$

When $t = 6$, $dy/dx = 6$. Notice that we are also able to find the derivative dy/dx as a function of x. ■

EXAMPLE 13 Moving Along the Ellipse $x^2/a^2 + y^2/b^2 = 1$

Describe the motion of a particle whose position $P(x, y)$ at time t is given by

$$x = a\cos t, \qquad y = b\sin t, \qquad 0 \leq t \leq 2\pi.$$

Find the line tangent to the curve at the point $\left(a/\sqrt{2}, b/\sqrt{2}\right)$, where $t = \pi/4$. (The constants a and b are both positive.)

Solution We find a Cartesian equation for the particle's coordinates by eliminating t between the equations

$$\cos t = \frac{x}{a}, \qquad \sin t = \frac{y}{b}.$$

The identity $\cos^2 t + \sin^2 t = 1$, yields

$$\left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1, \qquad \text{or} \qquad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$

The particle's coordinates (x, y) satisfy the equation $(x^2/a^2) + (y^2/b^2) = 1$, so the particle moves along this ellipse. When $t = 0$, the particle's coordinates are

$$x = a\cos(0) = a, \qquad y = b\sin(0) = 0,$$

so the motion starts at $(a, 0)$. As t increases, the particle rises and moves toward the left, moving counterclockwise. It traverses the ellipse once, returning to its starting position $(a, 0)$ at $t = 2\pi$.

The slope of the tangent line to the ellipse when $t = \pi/4$ is

$$\begin{aligned} \left.\frac{dy}{dx}\right|_{t=\pi/4} &= \left.\frac{dy/dt}{dx/dt}\right|_{t=\pi/4} \\ &= \left.\frac{b\cos t}{-a\sin t}\right|_{t=\pi/4} \\ &= \frac{b/\sqrt{2}}{-a/\sqrt{2}} = -\frac{b}{a}. \end{aligned}$$

The tangent line is

$$y - \frac{b}{\sqrt{2}} = -\frac{b}{a}\left(x - \frac{a}{\sqrt{2}}\right)$$
$$y = \frac{b}{\sqrt{2}} - \frac{b}{a}\left(x - \frac{a}{\sqrt{2}}\right)$$

or

$$y = -\frac{b}{a}x + \sqrt{2}b.$$

■

If parametric equations define y as a twice-differentiable function of x, we can apply Equation (2) to the function $dy/dx = y'$ to calculate d^2y/dx^2 as a function of t:

$$\frac{d^2y}{dx^2} = \frac{d}{dx}(y') = \frac{dy'/dt}{dx/dt}. \qquad \text{Eq. (2) with } y' \text{ in place of } y$$

Parametric Formula for d^2y/dx^2

If the equations $x = f(t)$, $y = g(t)$ define y as a twice-differentiable function of x, then at any point where $dx/dt \neq 0$,

$$\frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt}. \tag{3}$$

EXAMPLE 14 Finding d^2y/dx^2 for a Parametrized Curve

You Try It

Find d^2y/dx^2 as a function of t if $x = t - t^2$, $y = t - t^3$.

Finding d^2y/dx^2 in Terms of t

1. Express $y' = dy/dx$ in terms of t.
2. Find dy'/dt.
3. Divide dy'/dt by dx/dt.

Solution

1. Express $y' = dy/dx$ in terms of t.

$$y' = \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{1 - 3t^2}{1 - 2t}$$

2. Differentiate y' with respect to t.

$$\frac{dy'}{dt} = \frac{d}{dt}\left(\frac{1 - 3t^2}{1 - 2t}\right) = \frac{2 - 6t + 6t^2}{(1 - 2t)^2} \qquad \text{Quotient Rule}$$

3. Divide dy'/dt by dx/dt.

$$\frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt} = \frac{(2 - 6t + 6t^2)/(1 - 2t)^2}{1 - 2t} = \frac{2 - 6t + 6t^2}{(1 - 2t)^3} \qquad \text{Eq. (3)}$$

■

EXAMPLE 15 Dropping Emergency Supplies

A Red Cross aircraft is dropping emergency food and medical supplies into a disaster area. If the aircraft releases the supplies immediately above the edge of an open field 700 ft long and if the cargo moves along the path

$$x = 120t \quad \text{and} \quad y = -16t^2 + 500, \quad t \geq 0$$

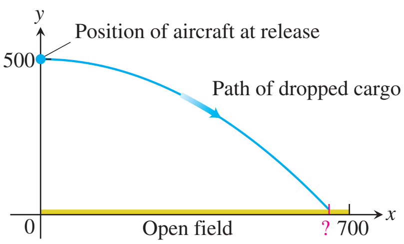


FIGURE 3.32 The path of the dropped cargo of supplies in Example 15.

does the cargo land in the field? The coordinates x and y are measured in feet, and the parameter t (time since release) in seconds. Find a Cartesian equation for the path of the falling cargo (Figure 3.32) and the cargo's rate of descent relative to its forward motion when it hits the ground.

Solution The cargo hits the ground when $y = 0$, which occurs at time t when

$$-16t^2 + 500 = 0 \qquad \text{Set } y = 0.$$

$$t = \sqrt{\frac{500}{16}} = \frac{5\sqrt{5}}{2} \text{ sec.} \qquad t \geq 0$$

The x-coordinate at the time of the release is $x = 0$. At the time the cargo hits the ground, the x-coordinate is

$$x = 120t = 120\left(\frac{5\sqrt{5}}{2}\right) = 300\sqrt{5} \text{ ft.}$$

Since $300\sqrt{5} \approx 670.8 < 700$, the cargo does land in the field.

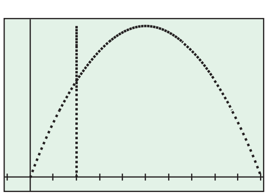
We find a Cartesian equation for the cargo's coordinates by eliminating t between the parametric equations:

$$\begin{aligned} y &= -16t^2 + 500 && \text{Parametric equation for } y \\ &= -16\left(\frac{x}{120}\right)^2 + 500 && \text{Substitute for } t \text{ from the equation } x = 120t. \\ &= -\frac{1}{900}x^2 + 500. && \text{A parabola} \end{aligned}$$

The rate of descent relative to its forward motion when the cargo hits the ground is

$$\begin{aligned} \left.\frac{dy}{dx}\right|_{t=5\sqrt{5}/2} &= \left.\frac{dy/dt}{dx/dt}\right|_{t=5\sqrt{5}/2} \\ &= \left.\frac{-32t}{120}\right|_{t=5\sqrt{5}/2} \\ &= -\frac{2\sqrt{5}}{3} \approx -1.49. \end{aligned}$$

Thus, it is falling about 1.5 feet for every foot of forward motion when it hits the ground.



$\begin{cases} x(t) = 2 \\ y(t) = 160t - 16t^2 \end{cases}$

and

$\begin{cases} x(t) = t \\ y(t) = 160t - 16t^2 \end{cases}$

in dot mode

USING TECHNOLOGY **Simulation of Motion on a Vertical Line**

The parametric equations

$$x(t) = c, \qquad y(t) = f(t)$$

will illuminate pixels along the vertical line $x = c$. If $f(t)$ denotes the height of a moving body at time t, graphing $(x(t), y(t)) = (c, f(t))$ will simulate the actual motion. Try it for the rock in Example 5, Section 3.3 with $x(t) = 2$, say, and $y(t) = 160t - 16t^2$, in dot mode with t Step $= 0.1$. Why does the spacing of the dots vary? Why does the grapher seem to stop after it reaches the top? (Try the plots for $0 \leq t \leq 5$ and $5 \leq t \leq 10$ separately.)

For a second experiment, plot the parametric equations

$$x(t) = t, \qquad y(t) = 160t - 16t^2$$

together with the vertical line simulation of the motion, again in dot mode. Use what you know about the behavior of the rock from the calculations of Example 5 to select a window size that will display all the interesting behavior.

Standard Parametrizations and Derivative Rules

CIRCLE $x^2 + y^2 = a^2$:

$$x = a\cos t$$
$$y = a\sin t$$
$$0 \le t \le 2\pi$$

ELLIPSE $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$:

$$x = a\cos t$$
$$y = b\sin t$$
$$0 \le t \le 2\pi$$

FUNCTION $y = f(x)$:

$$x = t$$
$$y = f(t)$$

DERIVATIVES

$$y' = \frac{dy}{dx} = \frac{dy/dt}{dx/dt}, \qquad \frac{d^2y}{dx^2} = \frac{dy'/dt}{dx/dt}$$

EXERCISES 3.5

Derivative Calculations

In Exercises 1–8, given $y = f(u)$ and $u = g(x)$, find $dy/dx = f'(g(x))g'(x)$.

1. $y = 6u - 9, \quad u = (1/2)x^4$
2. $y = 2u^3, \quad u = 8x - 1$
3. $y = \sin u, \quad u = 3x + 1$
4. $y = \cos u, \quad u = -x/3$
5. $y = \cos u, \quad u = \sin x$
6. $y = \sin u, \quad u = x - \cos x$
7. $y = \tan u, \quad u = 10x - 5$
8. $y = -\sec u, \quad u = x^2 + 7x$

In Exercises 9–18, write the function in the form $y = f(u)$ and $u = g(x)$. Then find dy/dx as a function of x.

9. $y = (2x + 1)^5$
10. $y = (4 - 3x)^9$
11. $y = \left(1 - \dfrac{x}{7}\right)^{-7}$
12. $y = \left(\dfrac{x}{2} - 1\right)^{-10}$
13. $y = \left(\dfrac{x^2}{8} + x - \dfrac{1}{x}\right)^4$
14. $y = \left(\dfrac{x}{5} + \dfrac{1}{5x}\right)^5$
15. $y = \sec(\tan x)$
16. $y = \cot\left(\pi - \dfrac{1}{x}\right)$
17. $y = \sin^3 x$
18. $y = 5\cos^{-4} x$

Find the derivatives of the functions in Exercises 19–38.

19. $p = \sqrt{3 - t}$
20. $q = \sqrt{2r - r^2}$
21. $s = \dfrac{4}{3\pi}\sin 3t + \dfrac{4}{5\pi}\cos 5t$
22. $s = \sin\left(\dfrac{3\pi t}{2}\right) + \cos\left(\dfrac{3\pi t}{2}\right)$
23. $r = (\csc\theta + \cot\theta)^{-1}$
24. $r = -(\sec\theta + \tan\theta)^{-1}$
25. $y = x^2\sin^4 x + x\cos^{-2} x$
26. $y = \dfrac{1}{x}\sin^{-5} x - \dfrac{x}{3}\cos^3 x$
27. $y = \dfrac{1}{21}(3x - 2)^7 + \left(4 - \dfrac{1}{2x^2}\right)^{-1}$
28. $y = (5 - 2x)^{-3} + \dfrac{1}{8}\left(\dfrac{2}{x} + 1\right)^4$
29. $y = (4x + 3)^4(x + 1)^{-3}$
30. $y = (2x - 5)^{-1}(x^2 - 5x)^6$
31. $h(x) = x\tan\left(2\sqrt{x}\right) + 7$
32. $k(x) = x^2\sec\left(\dfrac{1}{x}\right)$
33. $f(\theta) = \left(\dfrac{\sin\theta}{1 + \cos\theta}\right)^2$
34. $g(t) = \left(\dfrac{1 + \cos t}{\sin t}\right)^{-1}$
35. $r = \sin(\theta^2)\cos(2\theta)$
36. $r = \sec\sqrt{\theta}\tan\left(\dfrac{1}{\theta}\right)$
37. $q = \sin\left(\dfrac{t}{\sqrt{t + 1}}\right)$
38. $q = \cot\left(\dfrac{\sin t}{t}\right)$

In Exercises 39–48, find dy/dt.

39. $y = \sin^2(\pi t - 2)$
40. $y = \sec^2 \pi t$
41. $y = (1 + \cos 2t)^{-4}$
42. $y = (1 + \cot(t/2))^{-2}$

43. $y = \sin(\cos(2t - 5))$

44. $y = \cos\left(5 \sin\left(\frac{t}{3}\right)\right)$

45. $y = \left(1 + \tan^4\left(\frac{t}{12}\right)\right)^3$

46. $y = \frac{1}{6}\left(1 + \cos^2(7t)\right)^3$

47. $y = \sqrt{1 + \cos(t^2)}$

48. $y = 4 \sin\left(\sqrt{1 + \sqrt{t}}\right)$

Second Derivatives

Find y'' in Exercises 49–52.

49. $y = \left(1 + \frac{1}{x}\right)^3$

50. $y = \left(1 - \sqrt{x}\right)^{-1}$

51. $y = \frac{1}{9}\cot(3x - 1)$

52. $y = 9 \tan\left(\frac{x}{3}\right)$

Finding Numerical Values of Derivatives

In Exercises 53–58, find the value of $(f \circ g)'$ at the given value of x.

53. $f(u) = u^5 + 1, \quad u = g(x) = \sqrt{x}, \quad x = 1$

54. $f(u) = 1 - \frac{1}{u}, \quad u = g(x) = \frac{1}{1 - x}, \quad x = -1$

55. $f(u) = \cot\frac{\pi u}{10}, \quad u = g(x) = 5\sqrt{x}, \quad x = 1$

56. $f(u) = u + \frac{1}{\cos^2 u}, \quad u = g(x) = \pi x, \quad x = 1/4$

57. $f(u) = \frac{2u}{u^2 + 1}, \quad u = g(x) = 10x^2 + x + 1, \quad x = 0$

58. $f(u) = \left(\frac{u - 1}{u + 1}\right)^2, \quad u = g(x) = \frac{1}{x^2} - 1, \quad x = -1$

59. Suppose that functions f and g and their derivatives with respect to x have the following values at $x = 2$ and $x = 3$.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
2	8	2	1/3	−3
3	3	−4	2π	5

Find the derivatives with respect to x of the following combinations at the given value of x.

a. $2f(x), \quad x = 2$

b. $f(x) + g(x), \quad x = 3$

c. $f(x) \cdot g(x), \quad x = 3$

d. $f(x)/g(x), \quad x = 2$

e. $f(g(x)), \quad x = 2$

f. $\sqrt{f(x)}, \quad x = 2$

g. $1/g^2(x), \quad x = 3$

h. $\sqrt{f^2(x) + g^2(x)}, \quad x = 2$

60. Suppose that the functions f and g and their derivatives with respect to x have the following values at $x = 0$ and $x = 1$.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
0	1	1	5	1/3
1	3	−4	−1/3	−8/3

Find the derivatives with respect to x of the following combinations at the given value of x,

a. $5f(x) - g(x), \quad x = 1$

b. $f(x)g^3(x), \quad x = 0$

c. $\frac{f(x)}{g(x) + 1}, \quad x = 1$

d. $f(g(x)), \quad x = 0$

e. $g(f(x)), \quad x = 0$

f. $(x^{11} + f(x))^{-2}, \quad x = 1$

g. $f(x + g(x)), \quad x = 0$

61. Find ds/dt when $\theta = 3\pi/2$ if $s = \cos\theta$ and $d\theta/dt = 5$.

62. Find dy/dt when $x = 1$ if $y = x^2 + 7x - 5$ and $dx/dt = 1/3$.

Choices in Composition

What happens if you can write a function as a composite in different ways? Do you get the same derivative each time? The Chain Rule says you should. Try it with the functions in Exercises 63 and 64.

63. Find dy/dx if $y = x$ by using the Chain Rule with y as a composite of

a. $y = (u/5) + 7 \quad$ and $\quad u = 5x - 35$

b. $y = 1 + (1/u) \quad$ and $\quad u = 1/(x - 1)$.

64. Find dy/dx if $y = x^{3/2}$ by using the Chain Rule with y as a composite of

a. $y = u^3 \quad$ and $\quad u = \sqrt{x}$

b. $y = \sqrt{u} \quad$ and $\quad u = x^3$.

Tangents and Slopes

65. a. Find the tangent to the curve $y = 2\tan(\pi x/4)$ at $x = 1$.

b. Slopes on a tangent curve What is the smallest value the slope of the curve can ever have on the interval $-2 < x < 2$? Give reasons for your answer.

66. Slopes on sine curves

a. Find equations for the tangents to the curves $y = \sin 2x$ and $y = -\sin(x/2)$ at the origin. Is there anything special about how the tangents are related? Give reasons for your answer.

b. Can anything be said about the tangents to the curves $y = \sin mx$ and $y = -\sin(x/m)$ at the origin (m a constant $\neq 0$)? Give reasons for your answer.

c. For a given m, what are the largest values the slopes of the curves $y = \sin mx$ and $y = -\sin(x/m)$ can ever have? Give reasons for your answer.

d. The function $y = \sin x$ completes one period on the interval $[0, 2\pi]$, the function $y = \sin 2x$ completes two periods, the function $y = \sin(x/2)$ completes half a period, and so on. Is there any relation between the number of periods $y = \sin mx$ completes on $[0, 2\pi]$ and the slope of the curve $y = \sin mx$ at the origin? Give reasons for your answer.

Finding Cartesian Equations from Parametric Equations

Exercises 67–78 give parametric equations and parameter intervals for the motion of a particle in the xy-plane. Identify the particle's path by finding a Cartesian equation for it. Graph the Cartesian equation. (The graphs will vary with the equation used.) Indicate the portion of the graph traced by the particle and the direction of motion.

67. $x = \cos 2t, \quad y = \sin 2t, \quad 0 \le t \le \pi$

68. $x = \cos(\pi - t), \quad y = \sin(\pi - t), \quad 0 \le t \le \pi$

69. $x = 4\cos t, \quad y = 2\sin t, \quad 0 \le t \le 2\pi$

70. $x = 4\sin t, \quad y = 5\cos t, \quad 0 \le t \le 2\pi$

71. $x = 3t, \quad y = 9t^2, \quad -\infty < t < \infty$

72. $x = -\sqrt{t}, \quad y = t, \quad t \ge 0$

73. $x = 2t - 5, \quad y = 4t - 7, \quad -\infty < t < \infty$

74. $x = 3 - 3t, \quad y = 2t, \quad 0 \le t \le 1$

75. $x = t, \quad y = \sqrt{1 - t^2}, \quad -1 \le t \le 0$

76. $x = \sqrt{t + 1}, \quad y = \sqrt{t}, \quad t \ge 0$

77. $x = \sec^2 t - 1, \quad y = \tan t, \quad -\pi/2 < t < \pi/2$

78. $x = -\sec t, \quad y = \tan t, \quad -\pi/2 < t < \pi/2$

Determining Parametric Equations

79. Find parametric equations and a parameter interval for the motion of a particle that starts at $(a, 0)$ and traces the circle $x^2 + y^2 = a^2$

a. once clockwise. **b.** once counterclockwise.

c. twice clockwise. **d.** twice counterclockwise.

(There are many ways to do these, so your answers may not be the same as the ones in the back of the book.)

80. Find parametric equations and a parameter interval for the motion of a particle that starts at $(a, 0)$ and traces the ellipse $(x^2/a^2) + (y^2/b^2) = 1$

a. once clockwise. **b.** once counterclockwise.

c. twice clockwise. **d.** twice counterclockwise.

(As in Exercise 79, there are many correct answers.)

In Exercises 81–86, find a parametrization for the curve.

81. the line segment with endpoints $(-1, -3)$ and $(4, 1)$

82. the line segment with endpoints $(-1, 3)$ and $(3, -2)$

83. the lower half of the parabola $x - 1 = y^2$

84. the left half of the parabola $y = x^2 + 2x$

85. the ray (half line) with initial point $(2, 3)$ that passes through the point $(-1, -1)$

86. the ray (half line) with initial point $(-1, 2)$ that passes through the point $(0, 0)$

Tangents to Parametrized Curves

In Exercises 87–94, find an equation for the line tangent to the curve at the point defined by the given value of t. Also, find the value of d^2y/dx^2 at this point.

87. $x = 2\cos t, \quad y = 2\sin t, \quad t = \pi/4$

88. $x = \cos t, \quad y = \sqrt{3}\cos t, \quad t = 2\pi/3$

89. $x = t, \quad y = \sqrt{t}, \quad t = 1/4$

90. $x = -\sqrt{t + 1}, \quad y = \sqrt{3t}, \quad t = 3$

91. $x = 2t^2 + 3, \quad y = t^4, \quad t = -1$

92. $x = t - \sin t, \quad y = 1 - \cos t, \quad t = \pi/3$

93. $x = \cos t, \quad y = 1 + \sin t, \quad t = \pi/2$

94. $x = \sec^2 t - 1, \quad y = \tan t, \quad t = -\pi/4$

Theory, Examples, and Applications

95. Running machinery too fast Suppose that a piston is moving straight up and down and that its position at time t sec is

$$s = A\cos(2\pi bt),$$

with A and b positive. The value of A is the amplitude of the motion, and b is the frequency (number of times the piston moves up and down each second). What effect does doubling the frequency have on the piston's velocity, acceleration, and jerk? (Once you find out, you will know why machinery breaks when you run it too fast.)

96. Temperatures in Fairbanks, Alaska The graph in Figure 3.33 shows the average Fahrenheit temperature in Fairbanks, Alaska, during a typical 365-day year. The equation that approximates the temperature on day x is

$$y = 37\sin\left[\frac{2\pi}{365}(x - 101)\right] + 25.$$

a. On what day is the temperature increasing the fastest?

b. About how many degrees per day is the temperature increasing when it is increasing at its fastest?

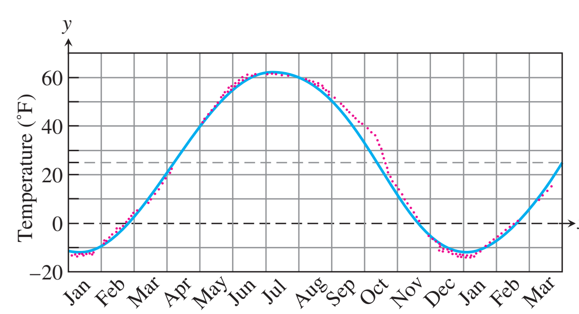


FIGURE 3.33 Normal mean air temperatures at Fairbanks, Alaska, plotted as data points, and the approximating sine function (Exercise 96).

97. Particle motion The position of a particle moving along a coordinate line is $s = \sqrt{1 + 4t}$, with s in meters and t in seconds. Find the particle's velocity and acceleration at $t = 6$ sec.

98. Constant acceleration Suppose that the velocity of a falling body is $v = k\sqrt{s}$ m/sec (k a constant) at the instant the body has fallen s m from its starting point. Show that the body's acceleration is constant.

99. **Falling meteorite** The velocity of a heavy meteorite entering Earth's atmosphere is inversely proportional to $\sqrt{s}$ when it is s km from Earth's center. Show that the meteorite's acceleration is inversely proportional to s^2.

100. **Particle acceleration** A particle moves along the x-axis with velocity $dx/dt = f(x)$. Show that the particle's acceleration is $f(x)f'(x)$.

101. **Temperature and the period of a pendulum** For oscillations of small amplitude (short swings), we may safely model the relationship between the period T and the length L of a simple pendulum with the equation

$$T = 2\pi\sqrt{\frac{L}{g}},$$

where g is the constant acceleration of gravity at the pendulum's location. If we measure g in centimeters per second squared, we measure L in centimeters and T in seconds. If the pendulum is made of metal, its length will vary with temperature, either increasing or decreasing at a rate that is roughly proportional to L. In symbols, with u being temperature and k the proportionality constant,

$$\frac{dL}{du} = kL.$$

Assuming this to be the case, show that the rate at which the period changes with respect to temperature is $kT/2$.

102. **Chain Rule** Suppose that $f(x) = x^2$ and $g(x) = |x|$. Then the composites

$$(f \circ g)(x) = |x|^2 = x^2 \quad \text{and} \quad (g \circ f)(x) = |x^2| = x^2$$

are both differentiable at $x = 0$ even though g itself is not differentiable at $x = 0$. Does this contradict the Chain Rule? Explain.

103. **Tangents** Suppose that $u = g(x)$ is differentiable at $x = 1$ and that $y = f(u)$ is differentiable at $u = g(1)$. If the graph of $y = f(g(x))$ has a horizontal tangent at $x = 1$, can we conclude anything about the tangent to the graph of g at $x = 1$ or the tangent to the graph of f at $u = g(1)$? Give reasons for your answer.

104. Suppose that $u = g(x)$ is differentiable at $x = -5$, $y = f(u)$ is differentiable at $u = g(-5)$, and $(f \circ g)'(-5)$ is negative. What, if anything, can be said about the values of $g'(-5)$ and $f'(g(-5))$?

T 105. **The derivative of sin 2x** Graph the function $y = 2\cos 2x$ for $-2 \le x \le 3.5$. Then, on the same screen, graph

$$y = \frac{\sin 2(x + h) - \sin 2x}{h}$$

for $h = 1.0, 0.5$, and 0.2. Experiment with other values of h, including negative values. What do you see happening as $h \to 0$? Explain this behavior.

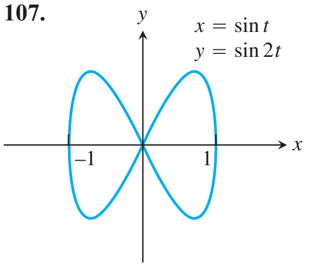
T 106. **The derivative of cos (x^2)** Graph $y = -2x\sin(x^2)$ for $-2 \le x \le 3$. Then, on the same screen, graph

$$y = \frac{\cos((x + h)^2) - \cos(x^2)}{h}$$

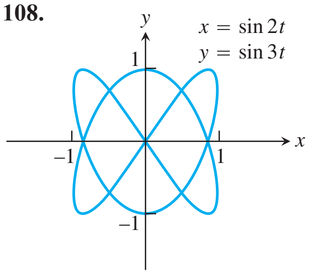
for $h = 1.0, 0.7$, and 0.3. Experiment with other values of h. What do you see happening as $h \to 0$? Explain this behavior.

T The curves in Exercises 107 and 108 are called *Bowditch curves* or *Lissajous figures*. In each case, find the point in the interior of the first quadrant where the tangent to the curve is horizontal, and find the equations of the two tangents at the origin.

107.



108.



Using the Chain Rule, show that the power rule $(d/dx)x^n = nx^{n-1}$ holds for the functions x^n in Exercises 109 and 110.

109. $x^{1/4} = \sqrt{\sqrt{x}}$

110. $x^{3/4} = \sqrt{x\sqrt{x}}$

COMPUTER EXPLORATIONS

Trigonometric Polynomials

111. As Figure 3.34 shows, the trigonometric "polynomial"

$$s = f(t) = 0.78540 - 0.63662\cos 2t - 0.07074\cos 6t$$
$$-0.02546\cos 10t - 0.01299\cos 14t$$

gives a good approximation of the sawtooth function $s = g(t)$ on the interval $[-\pi, \pi]$. How well does the derivative of f approximate the derivative of g at the points where dg/dt is defined? To find out, carry out the following steps.

a. Graph dg/dt (where defined) over $[-\pi, \pi]$.

b. Find df/dt.

c. Graph df/dt. Where does the approximation of dg/dt by df/dt seem to be best? Least good? Approximations by trigonometric polynomials are important in the theories of heat and oscillation, but we must not expect too much of them, as we see in the next exercise.

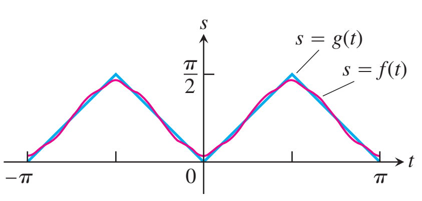


FIGURE 3.34 The approximation of a sawtooth function by a trigonometric "polynomial" (Exercise 111).

112. (*Continuation of Exercise 111.*) In Exercise 111, the trigonometric polynomial $f(t)$ that approximated the sawtooth function $g(t)$ on $[-\pi, \pi]$ had a derivative that approximated the derivative of the sawtooth function. It is possible, however, for a trigonometric polynomial to approximate a function in a reasonable way without its derivative approximating the function's derivative at all well. As a case in point, the "polynomial"

$$s = h(t) = 1.2732 \sin 2t + 0.4244 \sin 6t + 0.25465 \sin 10t + 0.18189 \sin 14t + 0.14147 \sin 18t$$

graphed in Figure 3.35 approximates the step function $s = k(t)$ shown there. Yet the derivative of h is nothing like the derivative of k.

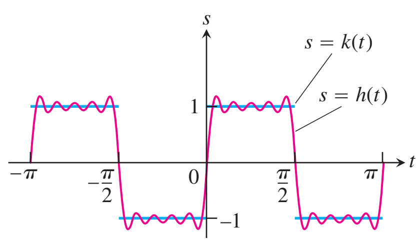


FIGURE 3.35 The approximation of a step function by a trigonometric "polynomial" (Exercise 112).

a. Graph dk/dt (where defined) over $[-\pi, \pi]$.

b. Find dh/dt.

c. Graph dh/dt to see how badly the graph fits the graph of dk/dt. Comment on what you see.

Parametrized Curves

Use a CAS to perform the following steps on the parametrized curves in Exercises 113–116.

a. Plot the curve for the given interval of t values.

b. Find dy/dx and d^2y/dx^2 at the point t_0.

c. Find an equation for the tangent line to the curve at the point defined by the given value t_0. Plot the curve together with the tangent line on a single graph.

113. $x = \frac{1}{3}t^3, \quad y = \frac{1}{2}t^2, \quad 0 \le t \le 1, \quad t_0 = 1/2$

114. $x = 2t^3 - 16t^2 + 25t + 5, \quad y = t^2 + t - 3, \quad 0 \le t \le 6, \quad t_0 = 3/2$

115. $x = t - \cos t, \quad y = 1 + \sin t, \quad -\pi \le t \le \pi, \quad t_0 = \pi/4$

116. $x = e^t \cos t, \quad y = e^t \sin t, \quad 0 \le t \le \pi, \quad t_0 = \pi/2$

3.6 Implicit Differentiation

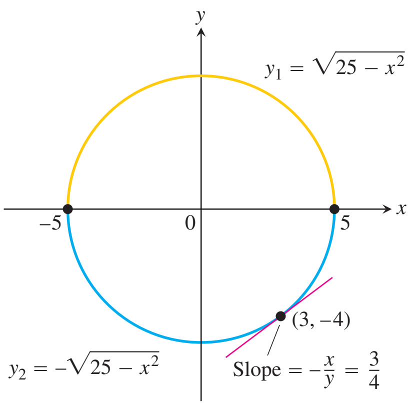


FIGURE 3.36 The circle combines the graphs of two functions. The graph of y_2 is the lower semicircle and passes through $(3, -4)$.

Most of the functions we have dealt with so far have been described by an equation of the form $y = f(x)$ that expresses y explicitly in terms of the variable x. We have learned rules for differentiating functions defined in this way. In Section 3.5 we also learned how to find the derivative dy/dx when a curve is defined parametrically by equations $x = x(t)$ and $y = y(t)$. A third situation occurs when we encounter equations like

$$x^2 + y^2 - 25 = 0, \qquad y^2 - x = 0, \qquad \text{or} \quad x^3 + y^3 - 9xy = 0.$$

(See Figures 3.36, 3.37, and 3.38.) These equations define an *implicit* relation between the variables x and y. In some cases we may be able to solve such an equation for y as an explicit function (or even several functions) of x. When we cannot put an equation $F(x, y) = 0$ in the form $y = f(x)$ to differentiate it in the usual way, we may still be able to find dy/dx by *implicit differentiation*. This consists of differentiating both sides of the equation with respect to x and then solving the resulting equation for y'. This section describes the technique and uses it to extend the Power Rule for differentiation to include rational exponents. In the examples and exercises of this section it is always assumed that the given equation determines y implicitly as a differentiable function of x.

Implicitly Defined Functions

We begin with an example.

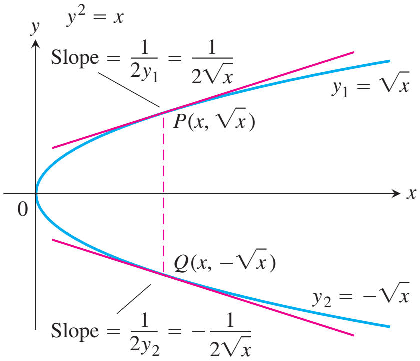


FIGURE 3.37 The equation $y^2 - x = 0$, or $y^2 = x$ as it is usually written, defines two differentiable functions of x on the interval $x \geq 0$. Example 1 shows how to find the derivatives of these functions without solving the equation $y^2 = x$ for y.

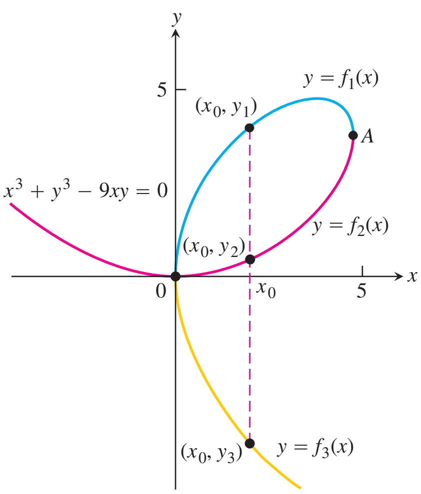


FIGURE 3.38 The curve $x^3 + y^3 - 9xy = 0$ is not the graph of any one function of x. The curve can, however, be divided into separate arcs that *are* the graphs of functions of x. This particular curve, called a *folium*, dates to Descartes in 1638.

EXAMPLE 1 Differentiating Implicitly

Find dy/dx if $y^2 = x$.

Solution The equation $y^2 = x$ defines two differentiable functions of x that we can actually find, namely $y_1 = \sqrt{x}$ and $y_2 = -\sqrt{x}$ (Figure 3.37). We know how to calculate the derivative of each of these for $x > 0$:

$$\frac{dy_1}{dx} = \frac{1}{2\sqrt{x}} \quad \text{and} \quad \frac{dy_2}{dx} = -\frac{1}{2\sqrt{x}}.$$

But suppose that we knew only that the equation $y^2 = x$ defined y as one or more differentiable functions of x for $x > 0$ without knowing exactly what these functions were. Could we still find dy/dx?

The answer is yes. To find dy/dx, we simply differentiate both sides of the equation $y^2 = x$ with respect to x, treating $y = f(x)$ as a differentiable function of x:

$$y^2 = x$$

$$2y\frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{2y}.$$

The Chain Rule gives $\frac{d}{dx}(y^2) = \frac{d}{dx}[f(x)]^2 = 2f(x)f'(x) = 2y\frac{dy}{dx}$.

This one formula gives the derivatives we calculated for *both* explicit solutions $y_1 = \sqrt{x}$ and $y_2 = -\sqrt{x}$:

$$\frac{dy_1}{dx} = \frac{1}{2y_1} = \frac{1}{2\sqrt{x}} \quad \text{and} \quad \frac{dy_2}{dx} = \frac{1}{2y_2} = \frac{1}{2(-\sqrt{x})} = -\frac{1}{2\sqrt{x}}.$$

■

EXAMPLE 2 Slope of a Circle at a Point

Find the slope of circle $x^2 + y^2 = 25$ at the point $(3, -4)$.

Solution The circle is not the graph of a single function of x. Rather it is the combined graphs of two differentiable functions, $y_1 = \sqrt{25 - x^2}$ and $y_2 = -\sqrt{25 - x^2}$ (Figure 3.36). The point $(3, -4)$ lies on the graph of y_2, so we can find the slope by calculating explicitly:

$$\left.\frac{dy_2}{dx}\right|_{x=3} = -\left.\frac{-2x}{2\sqrt{25 - x^2}}\right|_{x=3} = -\frac{-6}{2\sqrt{25 - 9}} = \frac{3}{4}.$$

But we can also solve the problem more easily by differentiating the given equation of the circle implicitly with respect to x:

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = \frac{d}{dx}(25)$$

$$2x + 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{x}{y}.$$

The slope at $(3, -4)$ is $-\left.\frac{x}{y}\right|_{(3,-4)} = -\frac{3}{-4} = \frac{3}{4}$.

Notice that unlike the slope formula for dy_2/dx, which applies only to points below the x-axis, the formula $dy/dx = -x/y$ applies everywhere the circle has a slope. Notice also that the derivative involves *both* variables x and y, not just the independent variable x. ■

To calculate the derivatives of other implicitly defined functions, we proceed as in Examples 1 and 2: We treat y as a differentiable implicit function of x and apply the usual rules to differentiate both sides of the defining equation.

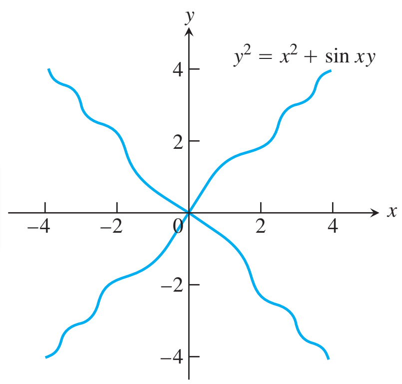


FIGURE 3.39 The graph of $y^2 = x^2 + \sin xy$ in Example 3. The example shows how to find slopes on this implicitly defined curve.

EXAMPLE 3 Differentiating Implicitly

Find dy/dx if $y^2 = x^2 + \sin xy$ (Figure 3.39).

Solution

$$y^2 = x^2 + \sin xy$$

$$\frac{d}{dx}\left(y^2\right) = \frac{d}{dx}\left(x^2\right) + \frac{d}{dx}\left(\sin xy\right)$$ Differentiate both sides with respect to x . . .

$$2y\frac{dy}{dx} = 2x + (\cos xy)\frac{d}{dx}(xy)$$ . . . treating y as a function of x and using the Chain Rule.

$$2y\frac{dy}{dx} = 2x + (\cos xy)\left(y + x\frac{dy}{dx}\right)$$ Treat xy as a product.

$$2y\frac{dy}{dx} - (\cos xy)\left(x\frac{dy}{dx}\right) = 2x + (\cos xy)y$$ Collect terms with dy/dx . . .

$$(2y - x\cos xy)\frac{dy}{dx} = 2x + y\cos xy$$ . . . and factor out dy/dx.

$$\frac{dy}{dx} = \frac{2x + y\cos xy}{2y - x\cos xy}$$ Solve for dy/dx by dividing.

Notice that the formula for dy/dx applies everywhere that the implicitly defined curve has a slope. Notice again that the derivative involves *both* variables x and y, not just the independent variable x. ■

Implicit Differentiation

1. Differentiate both sides of the equation with respect to x, treating y as a differentiable function of x.
2. Collect the terms with dy/dx on one side of the equation.
3. Solve for dy/dx.

Lenses, Tangents, and Normal Lines

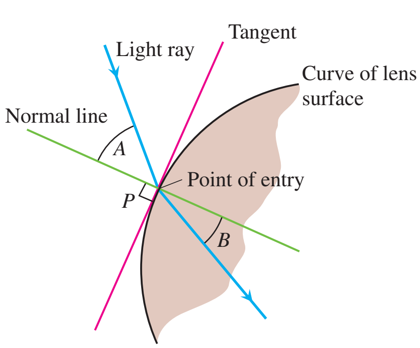


FIGURE 3.40 The profile of a lens, showing the bending (refraction) of a ray of light as it passes through the lens surface.

In the law that describes how light changes direction as it enters a lens, the important angles are the angles the light makes with the line perpendicular to the surface of the lens at the point of entry (angles A and B in Figure 3.40). This line is called the *normal* to the surface at the point of entry. In a profile view of a lens like the one in Figure 3.40, the **normal** is the line perpendicular to the tangent to the profile curve at the point of entry.

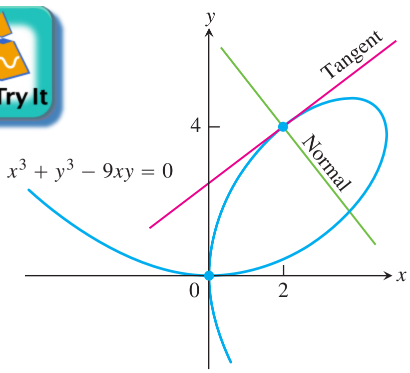


FIGURE 3.41 Example 4 shows how to find equations for the tangent and normal to the folium of Descartes at (2, 4).

EXAMPLE 4 Tangent and Normal to the Folium of Descartes

Show that the point (2, 4) lies on the curve $x^3 + y^3 - 9xy = 0$. Then find the tangent and normal to the curve there (Figure 3.41).

Solution The point (2, 4) lies on the curve because its coordinates satisfy the equation given for the curve: $2^3 + 4^3 - 9(2)(4) = 8 + 64 - 72 = 0$.

To find the slope of the curve at (2, 4), we first use implicit differentiation to find a formula for dy/dx:

$$x^3 + y^3 - 9xy = 0$$

$$\frac{d}{dx}\left(x^3\right) + \frac{d}{dx}\left(y^3\right) - \frac{d}{dx}\left(9xy\right) = \frac{d}{dx}\left(0\right) \qquad \text{Differentiate both sides with respect to } x.$$

$$3x^2 + 3y^2\frac{dy}{dx} - 9\left(x\frac{dy}{dx} + y\frac{dx}{dx}\right) = 0 \qquad \text{Treat } xy \text{ as a product and } y \text{ as a function of } x.$$

$$(3y^2 - 9x)\frac{dy}{dx} + 3x^2 - 9y = 0$$

$$3(y^2 - 3x)\frac{dy}{dx} = 9y - 3x^2$$

$$\frac{dy}{dx} = \frac{3y - x^2}{y^2 - 3x}. \qquad \text{Solve for } dy/dx.$$

We then evaluate the derivative at $(x, y) = (2, 4)$:

$$\left.\frac{dy}{dx}\right|_{(2,4)} = \left.\frac{3y - x^2}{y^2 - 3x}\right|_{(2,4)} = \frac{3(4) - 2^2}{4^2 - 3(2)} = \frac{8}{10} = \frac{4}{5}.$$

The tangent at (2, 4) is the line through (2, 4) with slope $4/5$:

$$y = 4 + \frac{4}{5}\left(x - 2\right)$$

$$y = \frac{4}{5}x + \frac{12}{5}.$$

The normal to the curve at (2, 4) is the line perpendicular to the tangent there, the line through (2, 4) with slope $-5/4$:

$$y = 4 - \frac{5}{4}(x - 2)$$

$$y = -\frac{5}{4}x + \frac{13}{2}.$$ ■

The quadratic formula enables us to solve a second-degree equation like $y^2 - 2xy + 3x^2 = 0$ for y in terms of x. There is a formula for the three roots of a cubic equation that is like the quadratic formula but much more complicated. If this formula is used to solve the equation $x^3 + y^3 = 9xy$ for y in terms of x, then three functions determined by the equation are

$$y = f(x) = \sqrt[3]{-\frac{x^3}{2} + \sqrt{\frac{x^6}{4} - 27x^3}} + \sqrt[3]{-\frac{x^3}{2} - \sqrt{\frac{x^6}{4} - 27x^3}}$$

and

$$y = \frac{1}{2}\left[-f(x) \pm \sqrt{-3}\left(\sqrt[3]{-\frac{x^3}{2} + \sqrt{\frac{x^6}{4} - 27x^3}} - \sqrt[3]{-\frac{x^3}{2} - \sqrt{\frac{x^6}{4} - 27x^3}}\right)\right].$$

Using implicit differentiation in Example 4 was much simpler than calculating dy/dx directly from any of the above formulas. Finding slopes on curves defined by higher-degree equations usually requires implicit differentiation.

Derivatives of Higher Order

Implicit differentiation can also be used to find higher derivatives. Here is an example.

You Try It

EXAMPLE 5 Finding a Second Derivative Implicitly

Find d^2y/dx^2 if $2x^3 - 3y^2 = 8$.

Solution To start, we differentiate both sides of the equation with respect to x in order to find $y' = dy/dx$.

$$\frac{d}{dx}\left(2x^3 - 3y^2\right) = \frac{d}{dx}(8)$$

$$6x^2 - 6yy' = 0 \qquad \text{Treat } y \text{ as a function of } x.$$

$$x^2 - yy' = 0$$

$$y' = \frac{x^2}{y}, \quad \text{when } y \neq 0 \qquad \text{Solve for } y'.$$

We now apply the Quotient Rule to find y''.

$$y'' = \frac{d}{dx}\left(\frac{x^2}{y}\right) = \frac{2xy - x^2y'}{y^2} = \frac{2x}{y} - \frac{x^2}{y^2}\cdot y'$$

Finally, we substitute $y' = x^2/y$ to express y'' in terms of x and y.

$$y'' = \frac{2x}{y} - \frac{x^2}{y^2}\left(\frac{x^2}{y}\right) = \frac{2x}{y} - \frac{x^4}{y^3}, \quad \text{when } y \neq 0$$

■

Rational Powers of Differentiable Functions

We know that the rule

$$\frac{d}{dx}x^n = nx^{n-1}$$

holds when n is an integer. Using implicit differentiation we can show that it holds when n is any rational number.

THEOREM 4 **Power Rule for Rational Powers**

If p/q is a rational number, then $x^{p/q}$ is differentiable at every interior point of the domain of $x^{(p/q)-1}$, and

$$\frac{d}{dx}x^{p/q} = \frac{p}{q}x^{(p/q)-1}.$$

EXAMPLE 6 Using the Rational Power Rule

(a) $\frac{d}{dx}\left(x^{1/2}\right) = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$ for $x > 0$

(b) $\frac{d}{dx}\left(x^{2/3}\right) = \frac{2}{3}x^{-1/3}$ for $x \neq 0$

(c) $\frac{d}{dx}\left(x^{-4/3}\right) = -\frac{4}{3}x^{-7/3}$ for $x \neq 0$ ■

Proof of Theorem 4 Let p and q be integers with $q > 0$ and suppose that $y = \sqrt[q]{x^p} = x^{p/q}$. Then

$$y^q = x^p.$$

Since p and q are integers (for which we already have the Power Rule), and assuming that y is a differentiable function of x, we can differentiate both sides of the equation with respect to x and get

$$qy^{q-1}\frac{dy}{dx} = px^{p-1}.$$

If $y \neq 0$, we can divide both sides of the equation by qy^{q-1} to solve for dy/dx, obtaining

$$\begin{aligned}
\frac{dy}{dx} &= \frac{px^{p-1}}{qy^{q-1}} \\
&= \frac{p}{q}\cdot\frac{x^{p-1}}{(x^{p/q})^{q-1}} && y = x^{p/q} \\
&= \frac{p}{q}\cdot\frac{x^{p-1}}{x^{p-p/q}} && \frac{p}{q}(q-1) = p - \frac{p}{q} \\
&= \frac{p}{q}\cdot x^{(p-1)-(p-p/q)} && \text{A law of exponents} \\
&= \frac{p}{q}\cdot x^{(p/q)-1},
\end{aligned}$$

which proves the rule. ■

We will drop the assumption of differentiability used in the proof of Theorem 4 in Chapter 7, where we prove the Power Rule for any nonzero real exponent. (See Section 7.3.)

By combining the result of Theorem 4 with the Chain Rule, we get an extension of the Power Chain Rule to rational powers of u: If p/q is a rational number and u is a differentiable function of x, then $u^{p/q}$ is a differentiable function of x and

$$\frac{d}{dx}u^{p/q} = \frac{p}{q}u^{(p/q)-1}\frac{du}{dx},$$

provided that $u \neq 0$ if $(p/q) < 1$. This restriction is necessary because 0 might be in the domain of $u^{p/q}$ but not in the domain of $u^{(p/q)-1}$, as we see in the next example.

EXAMPLE 7 Using the Rational Power and Chain Rules

(a) $$\frac{d}{dx}\underbrace{\left(1-x^2\right)^{1/4}}_{\text{function defined on } [-1,\,1]} = \frac{1}{4}\left(1-x^2\right)^{-3/4}(-2x)$$ Power Chain Rule with $u = 1 - x^2$

$$= \frac{-x}{\underbrace{2\left(1-x^2\right)^{3/4}}_{\text{derivative defined only on } (-1,\,1)}}$$

(b) $$\frac{d}{dx}(\cos x)^{-1/5} = -\frac{1}{5}(\cos x)^{-6/5}\frac{d}{dx}(\cos x)$$

$$= -\frac{1}{5}(\cos x)^{-6/5}(-\sin x)$$

$$= \frac{1}{5}(\sin x)(\cos x)^{-6/5}$$ ■

EXERCISES 3.6

Derivatives of Rational Powers

Find dy/dx in Exercises 1-10.

1. $y = x^{9/4}$

2. $y = x^{-3/5}$

3. $y = \sqrt[3]{2x}$

4. $y = \sqrt[4]{5x}$

5. $y = 7\sqrt{x+6}$

6. $y = -2\sqrt{x-1}$

7. $y = (2x+5)^{-1/2}$

8. $y = (1-6x)^{2/3}$

9. $y = x(x^2+1)^{1/2}$

10. $y = x(x^2+1)^{-1/2}$

Find the first derivatives of the functions in Exercises 11-18.

11. $s = \sqrt[7]{t^2}$

12. $r = \sqrt[4]{\theta^{-3}}$

13. $y = \sin\left[(2t+5)^{-2/3}\right]$

14. $z = \cos\left[(1-6t)^{2/3}\right]$

15. $f(x) = \sqrt{1-\sqrt{x}}$

16. $g(x) = 2(2x^{-1/2}+1)^{-1/3}$

17. $h(\theta) = \sqrt[3]{1+\cos(2\theta)}$

18. $k(\theta) = (\sin(\theta+5))^{5/4}$

Differentiating Implicitly

Use implicit differentiation to find dy/dx in Exercises 19-32.

19. $x^2y + xy^2 = 6$

20. $x^3 + y^3 = 18xy$

21. $2xy + y^2 = x + y$

22. $x^3 - xy + y^3 = 1$

23. $x^2(x-y)^2 = x^2 - y^2$

24. $(3xy+7)^2 = 6y$

25. $y^2 = \dfrac{x-1}{x+1}$

26. $x^2 = \dfrac{x-y}{x+y}$

27. $x = \tan y$

28. $xy = \cot(xy)$

29. $x + \tan(xy) = 0$

30. $x + \sin y = xy$

31. $y\sin\left(\dfrac{1}{y}\right) = 1 - xy$

32. $y^2\cos\left(\dfrac{1}{y}\right) = 2x + 2y$

Find $dr/d\theta$ in Exercises 33-36.

33. $\theta^{1/2} + r^{1/2} = 1$

34. $r - 2\sqrt{\theta} = \dfrac{3}{2}\theta^{2/3} + \dfrac{4}{3}\theta^{3/4}$

35. $\sin(r\theta) = \dfrac{1}{2}$

36. $\cos r + \cot\theta = r\theta$

Second Derivatives

In Exercises 37-42, use implicit differentiation to find dy/dx and then d^2y/dx^2.

37. $x^2 + y^2 = 1$

38. $x^{2/3} + y^{2/3} = 1$

39. $y^2 = x^2 + 2x$

40. $y^2 - 2x = 1 - 2y$

41. $2\sqrt{y} = x - y$

42. $xy + y^2 = 1$

43. If $x^3 + y^3 = 16$, find the value of d^2y/dx^2 at the point $(2, 2)$.

44. If $xy + y^2 = 1$, find the value of d^2y/dx^2 at the point $(0, -1)$.

Slopes, Tangents, and Normals

In Exercises 45 and 46, find the slope of the curve at the given points.

45. $y^2 + x^2 = y^4 - 2x$ at $(-2, 1)$ and $(-2, -1)$

46. $(x^2+y^2)^2 = (x-y)^2$ at $(1, 0)$ and $(1, -1)$

In Exercises 47-56, verify that the given point is on the curve and find the lines that are **(a)** tangent and **(b)** normal to the curve at the given point.

47. $x^2 + xy - y^2 = 1, \quad (2, 3)$

48. $x^2 + y^2 = 25, \quad (3, -4)$

49. $x^2y^2 = 9, \quad (-1, 3)$

50. $y^2 - 2x - 4y - 1 = 0, \quad (-2, 1)$

51. $6x^2 + 3xy + 2y^2 + 17y - 6 = 0, \quad (-1, 0)$

52. $x^2 - \sqrt{3}xy + 2y^2 = 5, \quad (\sqrt{3}, 2)$

53. $2xy + \pi \sin y = 2\pi, \quad (1, \pi/2)$

54. $x \sin 2y = y \cos 2x, \quad (\pi/4, \pi/2)$

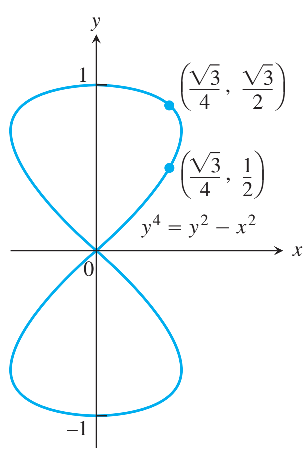
55. $y = 2 \sin (\pi x - y), \quad (1, 0)$

56. $x^2 \cos^2 y - \sin y = 0, \quad (0, \pi)$

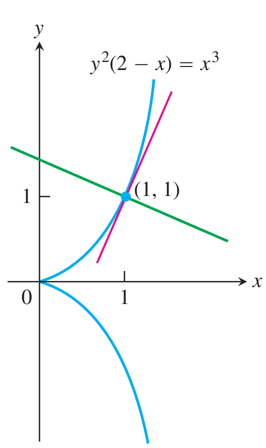
57. Parallel tangents Find the two points where the curve $x^2 + xy + y^2 = 7$ crosses the x-axis, and show that the tangents to the curve at these points are parallel. What is the common slope of these tangents?

58. Tangents parallel to the coordinate axes Find points on the curve $x^2 + xy + y^2 = 7$ **(a)** where the tangent is parallel to the x-axis and **(b)** where the tangent is parallel to the y-axis. In the latter case, dy/dx is not defined, but dx/dy is. What value does dx/dy have at these points?

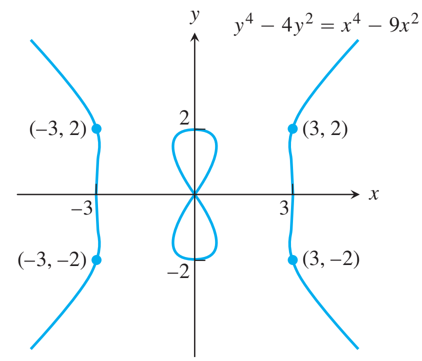
59. The eight curve Find the slopes of the curve $y^4 = y^2 - x^2$ at the two points shown here.



60. The cissoid of Diocles (from about 200 B.C.) Find equations for the tangent and normal to the cissoid of Diocles $y^2(2 - x) = x^3$ at (1, 1).



61. The devil's curve (Gabriel Cramer [the Cramer of Cramer's rule], 1750) Find the slopes of the devil's curve $y^4 - 4y^2 = x^4 - 9x^2$ at the four indicated points.



62. The folium of Descartes (See Figure 3.38)

a. Find the slope of the folium of Descartes, $x^3 + y^3 - 9xy = 0$ at the points (4, 2) and (2, 4).

b. At what point other than the origin does the folium have a horizontal tangent?

c. Find the coordinates of the point A in Figure 3.38, where the folium has a vertical tangent.

Implicitly Defined Parametrizations

Assuming that the equations in Exercises 63–66 define x and y implicitly as differentiable functions $x = f(t), y = g(t)$, find the slope of the curve $x = f(t), y = g(t)$ at the given value of t.

63. $x^2 - 2tx + 2t^2 = 4, \quad 2y^3 - 3t^2 = 4, \quad t = 2$

64. $x = \sqrt{5 - \sqrt{t}}, \quad y(t - 1) = \sqrt{t}, \quad t = 4$

65. $x + 2x^{3/2} = t^2 + t, \quad y\sqrt{t + 1} + 2t\sqrt{y} = 4, \quad t = 0$

66. $x \sin t + 2x = t, \quad t \sin t - 2t = y, \quad t = \pi$

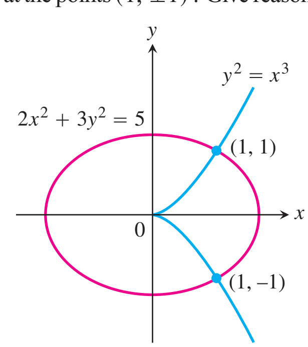
Theory and Examples

67. Which of the following could be true if $f''(x) = x^{-1/3}$?

a. $f(x) = \frac{3}{2}x^{2/3} - 3$ **b.** $f(x) = \frac{9}{10}x^{5/3} - 7$

c. $f'''(x) = -\frac{1}{3}x^{-4/3}$ **d.** $f'(x) = \frac{3}{2}x^{2/3} + 6$

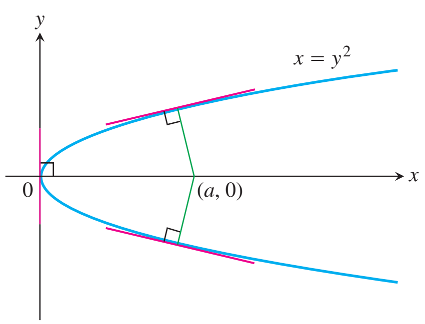
68. Is there anything special about the tangents to the curves $y^2 = x^3$ and $2x^2 + 3y^2 = 5$ at the points $(1, \pm 1)$? Give reasons for your answer.



69. Intersecting normal The line that is normal to the curve $x^2 + 2xy - 3y^2 = 0$ at (1, 1) intersects the curve at what other point?

70. Normals parallel to a line Find the normals to the curve $xy + 2x - y = 0$ that are parallel to the line $2x + y = 0$.

71. Normals to a parabola Show that if it is possible to draw three normals from the point $(a, 0)$ to the parabola $x = y^2$ shown here, then a must be greater than 1/2. One of the normals is the x-axis. For what value of a are the other two normals perpendicular?



72. What is the geometry behind the restrictions on the domains of the derivatives in Example 6(b) and Example 7(a)?

T In Exercises 73 and 74, find both dy/dx (treating y as a differentiable function of x) and dx/dy (treating x as a differentiable function of y). How do dy/dx and dx/dy seem to be related? Explain the relationship geometrically in terms of the graphs.

73. $xy^3 + x^2y = 6$

74. $x^3 + y^2 = \sin^2 y$

COMPUTER EXPLORATIONS

75. a. Given that $x^4 + 4y^2 = 1$, find dy/dx two ways: (1) by solving for y and differentiating the resulting functions in the usual way and (2) by implicit differentiation. Do you get the same result each way?

b. Solve the equation $x^4 + 4y^2 = 1$ for y and graph the resulting functions together to produce a complete graph of the equation $x^4 + 4y^2 = 1$. Then add the graphs of the first derivatives of these functions to your display. Could you have predicted the general behavior of the derivative graphs from looking at the graph of $x^4 + 4y^2 = 1$? Could you have predicted the general behavior of the graph of $x^4 + 4y^2 = 1$ by looking at the derivative graphs? Give reasons for your answers.

76. a. Given that $(x - 2)^2 + y^2 = 4$ find dy/dx two ways: (1) by solving for y and differentiating the resulting functions with respect to x and (2) by implicit differentiation. Do you get the same result each way?

b. Solve the equation $(x - 2)^2 + y^2 = 4$ for y and graph the resulting functions together to produce a complete graph of the equation $(x - 2)^2 + y^2 = 4$. Then add the graphs of the functions' first derivatives to your picture. Could you have predicted the general behavior of the derivative graphs from looking at the graph of $(x - 2)^2 + y^2 = 4$? Could you have predicted the general behavior of the graph of $(x - 2)^2 + y^2 = 4$ by looking at the derivative graphs? Give reasons for your answers.

Use a CAS to perform the following steps in Exercises 77–84.

a. Plot the equation with the implicit plotter of a CAS. Check to see that the given point P satisfies the equation.

b. Using implicit differentiation, find a formula for the derivative dy/dx and evaluate it at the given point P.

c. Use the slope found in part (b) to find an equation for the tangent line to the curve at P. Then plot the implicit curve and tangent line together on a single graph.

77. $x^3 - xy + y^3 = 7, \quad P(2, 1)$

78. $x^5 + y^3x + yx^2 + y^4 = 4, \quad P(1, 1)$

79. $y^2 + y = \dfrac{2 + x}{1 - x}, \quad P(0, 1)$

80. $y^3 + \cos xy = x^2, \quad P(1, 0)$

81. $x + \tan\left(\dfrac{y}{x}\right) = 2, \quad P\left(1, \dfrac{\pi}{4}\right)$

82. $xy^3 + \tan(x + y) = 1, \quad P\left(\dfrac{\pi}{4}, 0\right)$

83. $2y^2 + (xy)^{1/3} = x^2 + 2, \quad P(1, 1)$

84. $x\sqrt{1 + 2y} + y = x^2, \quad P(1, 0)$

3.7 Related Rates

In this section we look at problems that ask for the rate at which some variable changes. In each case the rate is a derivative that has to be computed from the rate at which some other variable (or perhaps several variables) is known to change. To find it, we write an equation that relates the variables involved and differentiate it to get an equation that relates the rate we seek to the rates we know. The problem of finding a rate you cannot measure easily from some other rates that you can is called a *related rates problem*.

Related Rates Equations

Suppose we are pumping air into a spherical balloon. Both the volume and radius of the balloon are increasing over time. If V is the volume and r is the radius of the balloon at an instant of time, then

$$V = \frac{4}{3}\pi r^3.$$

Using the Chain Rule, we differentiate to find the related rates equation

$$\frac{dV}{dt} = \frac{dV}{dr}\frac{dr}{dt} = 4\pi r^2 \frac{dr}{dt}.$$

So if we know the radius r of the balloon and the rate dV/dt at which the volume is increasing at a given instant of time, then we can solve this last equation for dr/dt to find how fast the radius is increasing at that instant. Note that it is easier to measure directly the rate of increase of the volume than it is to measure the increase in the radius. The related rates equation allows us to calculate dr/dt from dV/dt.

Very often the key to relating the variables in a related rates problem is drawing a picture that shows the geometric relations between them, as illustrated in the following example.

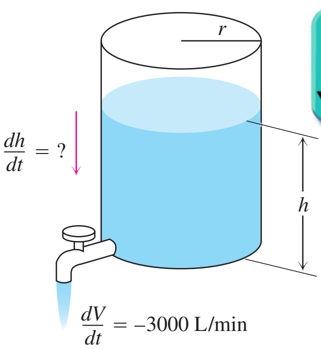


FIGURE 3.42 The rate of change of fluid volume in a cylindrical tank is related to the rate of change of fluid level in the tank (Example 1).

EXAMPLE 1 Pumping Out a Tank

How rapidly will the fluid level inside a vertical cylindrical tank drop if we pump the fluid out at the rate of 3000 L/min?

Solution We draw a picture of a partially filled vertical cylindrical tank, calling its radius r and the height of the fluid h (Figure 3.42). Call the volume of the fluid V.

As time passes, the radius remains constant, but V and h change. We think of V and h as differentiable functions of time and use t to represent time. We are told that

$$\frac{dV}{dt} = -3000. \qquad \text{We pump out at the rate of 3000 L/min. The rate is negative because the volume is decreasing.}$$

We are asked to find

$$\frac{dh}{dt}. \qquad \text{How fast will the fluid level drop?}$$

To find dh/dt, we first write an equation that relates h to V. The equation depends on the units chosen for V, r, and h. With V in liters and r and h in meters, the appropriate equation for the cylinder's volume is

$$V = 1000\pi r^2 h$$

because a cubic meter contains 1000 L.

Since V and h are differentiable functions of t, we can differentiate both sides of the equation $V = 1000\pi r^2 h$ with respect to t to get an equation that relates dh/dt to dV/dt:

$$\frac{dV}{dt} = 1000\pi r^2 \frac{dh}{dt}. \qquad r \text{ is a constant.}$$

We substitute the known value $dV/dt = -3000$ and solve for dh/dt:

$$\frac{dh}{dt} = \frac{-3000}{1000\pi r^2} = -\frac{3}{\pi r^2}.$$

The fluid level will drop at the rate of $3/(\pi r^2)$ m/min.

The equation $dh/dt = -3/\pi r^2$ shows how the rate at which the fluid level drops depends on the tank's radius. If r is small, dh/dt will be large; if r is large, dh/dt will be small.

$$\text{If } r = 1 \text{ m:} \quad \frac{dh}{dt} = -\frac{3}{\pi} \approx -0.95 \text{ m/min} = -95 \text{ cm/min}.$$

$$\text{If } r = 10 \text{ m:} \quad \frac{dh}{dt} = -\frac{3}{100\pi} \approx -0.0095 \text{ m/min} = -0.95 \text{ cm/min}. \quad ■$$

Related Rates Problem Strategy

1. *Draw a picture and name the variables and constants*. Use t for time. Assume that all variables are differentiable functions of t.
2. *Write down the numerical information* (in terms of the symbols you have chosen).
3. *Write down what you are asked to find* (usually a rate, expressed as a derivative).
4. *Write an equation that relates the variables*. You may have to combine two or more equations to get a single equation that relates the variable whose rate you want to the variables whose rates you know.
5. *Differentiate with respect to t*. Then express the rate you want in terms of the rate and variables whose values you know.
6. *Evaluate*. Use known values to find the unknown rate.

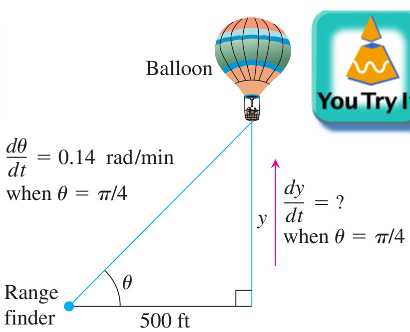


FIGURE 3.43 The rate of change of the balloon's height is related to the rate of change of the angle the range finder makes with the ground (Example 2).

EXAMPLE 2 A Rising Balloon

A hot air balloon rising straight up from a level field is tracked by a range finder 500 ft from the liftoff point. At the moment the range finder's elevation angle is $\pi/4$, the angle is increasing at the rate of 0.14 rad/min. How fast is the balloon rising at that moment?

Solution We answer the question in six steps.

1. *Draw a picture and name the variables and constants* (Figure 3.43). The variables in the picture are

   $\theta =$ the angle in radians the range finder makes with the ground.

   $y =$ the height in feet of the balloon.

We let t represent time in minutes and assume that θ and y are differentiable functions of t.

The one constant in the picture is the distance from the range finder to the liftoff point (500 ft). There is no need to give it a special symbol.

2. *Write down the additional numerical information.*

$$\frac{d\theta}{dt} = 0.14 \text{ rad/min} \qquad \text{when} \qquad \theta = \frac{\pi}{4}$$

3. *Write down what we are to find*. We want dy/dt when $\theta = \pi/4$.

4. *Write an equation that relates the variables y and θ.*

$$\frac{y}{500} = \tan\theta \qquad \text{or} \qquad y = 500\tan\theta$$

5. *Differentiate with respect to t using the Chain Rule.* The result tells how dy/dt (which we want) is related to $d\theta/dt$ (which we know).

$$\frac{dy}{dt} = 500\,(\sec^2\theta)\,\frac{d\theta}{dt}$$

6. *Evaluate with $\theta = \pi/4$ and $d\theta/dt = 0.14$ to find dy/dt.*

$$\frac{dy}{dt} = 500\left(\sqrt{2}\right)^2(0.14) = 140 \qquad \sec\frac{\pi}{4} = \sqrt{2}$$

At the moment in question, the balloon is rising at the rate of 140 ft/min. ■

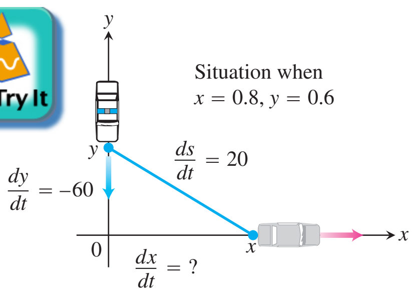


FIGURE 3.44 The speed of the car is related to the speed of the police cruiser and the rate of change of the distance between them (Example 3).

EXAMPLE 3 A Highway Chase

A police cruiser, approaching a right-angled intersection from the north, is chasing a speeding car that has turned the corner and is now moving straight east. When the cruiser is 0.6 mi north of the intersection and the car is 0.8 mi to the east, the police determine with radar that the distance between them and the car is increasing at 20 mph. If the cruiser is moving at 60 mph at the instant of measurement, what is the speed of the car?

Solution We picture the car and cruiser in the coordinate plane, using the positive x-axis as the eastbound highway and the positive y-axis as the southbound highway (Figure 3.44). We let t represent time and set

$$x = \text{position of car at time } t$$
$$y = \text{position of cruiser at time } t$$
$$s = \text{distance between car and cruiser at time } t.$$

We assume that x, y, and s are differentiable functions of t.

We want to find dx/dt when

$$x = 0.8 \text{ mi}, \qquad y = 0.6 \text{ mi}, \qquad \frac{dy}{dt} = -60 \text{ mph}, \qquad \frac{ds}{dt} = 20 \text{ mph}.$$

Note that dy/dt is negative because y is decreasing.

We differentiate the distance equation

$$s^2 = x^2 + y^2$$

(we could also use $s = \sqrt{x^2 + y^2}$), and obtain

$$2s\frac{ds}{dt} = 2x\frac{dx}{dt} + 2y\frac{dy}{dt}$$

$$\frac{ds}{dt} = \frac{1}{s}\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right)$$

$$= \frac{1}{\sqrt{x^2 + y^2}}\left(x\frac{dx}{dt} + y\frac{dy}{dt}\right).$$

Finally, use $x = 0.8$, $y = 0.6$, $dy/dt = -60$, $ds/dt = 20$, and solve for dx/dt.

$$20 = \frac{1}{\sqrt{(0.8)^2 + (0.6)^2}}\left(0.8\frac{dx}{dt} + (0.6)(-60)\right)$$

$$\frac{dx}{dt} = \frac{20\sqrt{(0.8)^2 + (0.6)^2} + (0.6)(60)}{0.8} = 70$$

At the moment in question, the car's speed is 70 mph. ■

EXAMPLE 4 Filling a Conical Tank

Water runs into a conical tank at the rate of $9 \text{ ft}^3/\text{min}$. The tank stands point down and has a height of 10 ft and a base radius of 5 ft. How fast is the water level rising when the water is 6 ft deep?

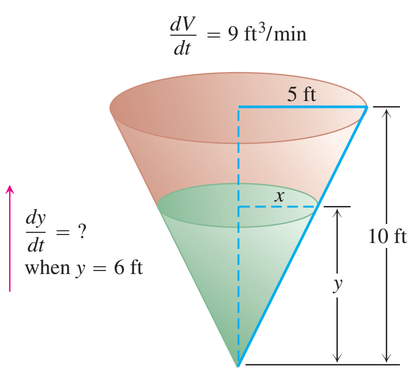


FIGURE 3.45 The geometry of the conical tank and the rate at which water fills the tank determine how fast the water level rises (Example 4).

Solution Figure 3.45 shows a partially filled conical tank. The variables in the problem are

V = volume (ft^3) of the water in the tank at time t (min)

x = radius (ft) of the surface of the water at time t

y = depth (ft) of water in tank at time t.

We assume that V, x, and y are differentiable functions of t. The constants are the dimensions of the tank. We are asked for dy/dt when

$$y = 6 \text{ ft} \quad \text{and} \quad \frac{dV}{dt} = 9 \text{ ft}^3/\text{min}.$$

The water forms a cone with volume

$$V = \frac{1}{3}\pi x^2 y.$$

This equation involves x as well as V and y. Because no information is given about x and dx/dt at the time in question, we need to eliminate x. The similar triangles in Figure 3.45 give us a way to express x in terms of y:

$$\frac{x}{y} = \frac{5}{10} \quad \text{or} \quad x = \frac{y}{2}.$$

Therefore,

$$V = \frac{1}{3}\pi\left(\frac{y}{2}\right)^2 y = \frac{\pi}{12}y^3$$

to give the derivative

$$\frac{dV}{dt} = \frac{\pi}{12} \cdot 3y^2\frac{dy}{dt} = \frac{\pi}{4}y^2\frac{dy}{dt}.$$

Finally, use $y = 6$ and $dV/dt = 9$ to solve for dy/dt.

$$9 = \frac{\pi}{4}(6)^2\frac{dy}{dt}$$

$$\frac{dy}{dt} = \frac{1}{\pi} \approx 0.32$$

At the moment in question, the water level is rising at about 0.32 ft/min. ■

EXERCISES 3.7

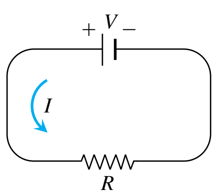
1. **Area** Suppose that the radius r and area $A = \pi r^2$ of a circle are differentiable functions of t. Write an equation that relates dA/dt to dr/dt.

2. **Surface area** Suppose that the radius r and surface area $S = 4\pi r^2$ of a sphere are differentiable functions of t. Write an equation that relates dS/dt to dr/dt.

3. **Volume** The radius r and height h of a right circular cylinder are related to the cylinder's volume V by the formula $V = \pi r^2 h$.
   a. How is dV/dt related to dh/dt if r is constant?
   b. How is dV/dt related to dr/dt if h is constant?
   c. How is dV/dt related to dr/dt and dh/dt if neither r nor h is constant?

4. **Volume** The radius r and height h of a right circular cone are related to the cone's volume V by the equation $V = (1/3)\pi r^2 h$.
   a. How is dV/dt related to dh/dt if r is constant?
   b. How is dV/dt related to dr/dt if h is constant?
   c. How is dV/dt related to dr/dt and dh/dt if neither r nor h is constant?

5. **Changing voltage** The voltage V (volts), current I (amperes), and resistance R (ohms) of an electric circuit like the one shown here are related by the equation $V = IR$. Suppose that V is increasing at the rate of 1 volt/sec while I is decreasing at the rate of 1/3 amp/sec. Let t denote time in seconds.



   a. What is the value of dV/dt?
   b. What is the value of dI/dt?
   c. What equation relates dR/dt to dV/dt and dI/dt?
   d. Find the rate at which R is changing when $V = 12$ volts and $I = 2$ amp. Is R increasing, or decreasing?

6. **Electrical power** The power P (watts) of an electric circuit is related to the circuit's resistance R (ohms) and current I (amperes) by the equation $P = RI^2$.
   a. How are dP/dt, dR/dt, and dI/dt related if none of P, R, and I are constant?
   b. How is dR/dt related to dI/dt if P is constant?

7. **Distance** Let x and y be differentiable functions of t and let $s = \sqrt{x^2 + y^2}$ be the distance between the points $(x, 0)$ and $(0, y)$ in the xy-plane.
   a. How is ds/dt related to dx/dt if y is constant?
   b. How is ds/dt related to dx/dt and dy/dt if neither x nor y is constant?
   c. How is dx/dt related to dy/dt if s is constant?

8. **Diagonals** If x, y, and z are lengths of the edges of a rectangular box, the common length of the box's diagonals is $s = \sqrt{x^2 + y^2 + z^2}$.
   a. Assuming that x, y, and z are differentiable functions of t, how is ds/dt related to dx/dt, dy/dt, and dz/dt?
   b. How is ds/dt related to dy/dt and dz/dt if x is constant?
   c. How are dx/dt, dy/dt, and dz/dt related if s is constant?

9. **Area** The area A of a triangle with sides of lengths a and b enclosing an angle of measure θ is
$$A = \frac{1}{2}ab\sin\theta.$$
   a. How is dA/dt related to $d\theta/dt$ if a and b are constant?
   b. How is dA/dt related to $d\theta/dt$ and da/dt if only b is constant?
   c. How is dA/dt related to $d\theta/dt$, da/dt, and db/dt if none of a, b, and θ are constant?

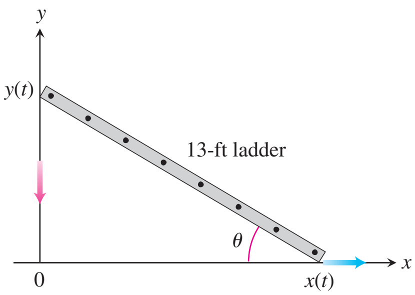
10. **Heating a plate** When a circular plate of metal is heated in an oven, its radius increases at the rate of 0.01 cm/min. At what rate is the plate's area increasing when the radius is 50 cm?

11. **Changing dimensions in a rectangle** The length l of a rectangle is decreasing at the rate of 2 cm/sec while the width w is increasing at the rate of 2 cm/sec. When $l = 12$ cm and $w = 5$ cm, find the rates of change of **(a)** the area, **(b)** the perimeter, and **(c)** the lengths of the diagonals of the rectangle. Which of these quantities are decreasing, and which are increasing?

Exercises

12. **Changing dimensions in a rectangular box** Suppose that the edge lengths x, y, and z of a closed rectangular box are changing at the following rates:
$$\frac{dx}{dt} = 1\text{ m/sec}, \quad \frac{dy}{dt} = -2\text{ m/sec}, \quad \frac{dz}{dt} = 1\text{ m/sec}.$$
Find the rates at which the box's **(a)** volume, **(b)** surface area, and **(c)** diagonal length $s = \sqrt{x^2 + y^2 + z^2}$ are changing at the instant when $x = 4$, $y = 3$, and $z = 2$.

13. **A sliding ladder** A 13-ft ladder is leaning against a house when its base starts to slide away. By the time the base is 12 ft from the house, the base is moving at the rate of 5 ft/sec.
   a. How fast is the top of the ladder sliding down the wall then?
   b. At what rate is the area of the triangle formed by the ladder, wall, and ground changing then?
   c. At what rate is the angle θ between the ladder and the ground changing then?



14. **Commercial air traffic** Two commercial airplanes are flying at 40,000 ft along straight-line courses that intersect at right angles. Plane A is approaching the intersection point at a speed of 442 knots (nautical miles per hour; a nautical mile is 2000 yd). Plane B is approaching the intersection at 481 knots. At what rate is the distance between the planes changing when A is 5 nautical miles from the intersection point and B is 12 nautical miles from the intersection point?

15. **Flying a kite** A girl flies a kite at a height of 300 ft, the wind carrying the kite horizontally away from her at a rate of 25 ft/sec. How fast must she let out the string when the kite is 500 ft away from her?

16. **Boring a cylinder** The mechanics at Lincoln Automotive are reboring a 6-in.-deep cylinder to fit a new piston. The machine they are using increases the cylinder's radius one-thousandth of an inch every 3 min. How rapidly is the cylinder volume increasing when the bore (diameter) is 3.800 in.?

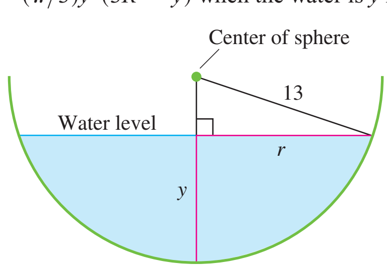
17. **A growing sand pile** Sand falls from a conveyor belt at the rate of 10 m^3/min onto the top of a conical pile. The height of the pile is always three-eighths of the base diameter. How fast are the **(a)** height and **(b)** radius changing when the pile is 4 m high? Answer in centimeters per minute.

18. **A draining conical reservoir** Water is flowing at the rate of 50 m^3/min from a shallow concrete conical reservoir (vertex down) of base radius 45 m and height 6 m.

   a. How fast (centimeters per minute) is the water level falling when the water is 5 m deep?

   b. How fast is the radius of the water's surface changing then? Answer in centimeters per minute.

19. **A draining hemispherical reservoir** Water is flowing at the rate of 6 m^3/min from a reservoir shaped like a hemispherical bowl of radius 13 m, shown here in profile. Answer the following questions, given that the volume of water in a hemispherical bowl of radius R is $V = (\pi/3)y^2(3R - y)$ when the water is y meters deep.



   a. At what rate is the water level changing when the water is 8 m deep?

   b. What is the radius r of the water's surface when the water is y m deep?

   c. At what rate is the radius r changing when the water is 8 m deep?

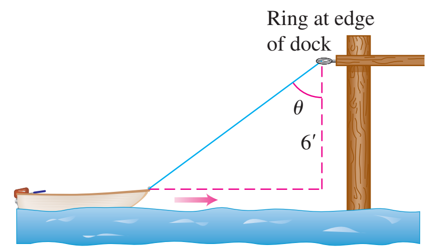
20. **A growing raindrop** Suppose that a drop of mist is a perfect sphere and that, through condensation, the drop picks up moisture at a rate proportional to its surface area. Show that under these circumstances the drop's radius increases at a constant rate.

21. **The radius of an inflating balloon** A spherical balloon is inflated with helium at the rate of 100π ft^3/min. How fast is the balloon's radius increasing at the instant the radius is 5 ft? How fast is the surface area increasing?

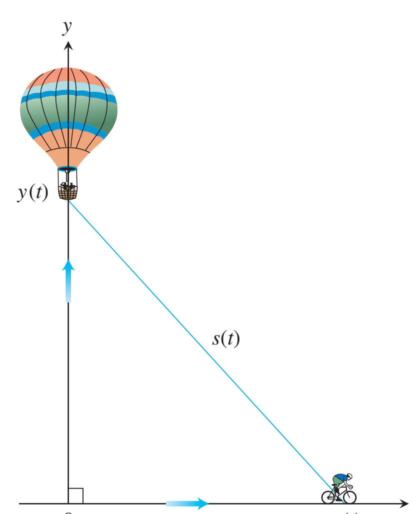
22. **Hauling in a dinghy** A dinghy is pulled toward a dock by a rope from the bow through a ring on the dock 6 ft above the bow. The rope is hauled in at the rate of 2 ft/sec.

   a. How fast is the boat approaching the dock when 10 ft of rope are out?

   b. At what rate is the angle θ changing then (see the figure)?



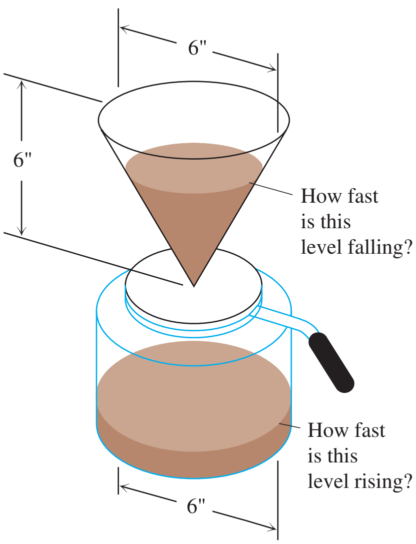
23. **A balloon and a bicycle** A balloon is rising vertically above a level, straight road at a constant rate of 1 ft/sec. Just when the balloon is 65 ft above the ground, a bicycle moving at a constant rate of 17 ft/sec passes under it. How fast is the distance $s(t)$ between the bicycle and balloon increasing 3 sec later?



24. Making coffee Coffee is draining from a conical filter into a cylindrical coffeepot at the rate of 10 in^3/min.

a. How fast is the level in the pot rising when the coffee in the cone is 5 in. deep?

b. How fast is the level in the cone falling then?



25. Cardiac output In the late 1860s, Adolf Fick, a professor of physiology in the Faculty of Medicine in Würzberg, Germany, developed one of the methods we use today for measuring how much blood your heart pumps in a minute. Your cardiac output as you read this sentence is probably about 7 L/min. At rest it is likely to be a bit under 6 L/min. If you are a trained marathon runner running a marathon, your cardiac output can be as high as 30 L/min.

Your cardiac output can be calculated with the formula

$$y = \frac{Q}{D},$$

where Q is the number of milliliters of CO_2 you exhale in a minute and D is the difference between the CO_2 concentration (ml/L) in the blood pumped to the lungs and the CO_2 concentration in the blood returning from the lungs. With $Q = 233$ ml/min and $D = 97 - 56 = 41$ ml/L,

$$y = \frac{233 \text{ ml/min}}{41 \text{ ml/L}} \approx 5.68 \text{ L/min},$$

fairly close to the 6 L/min that most people have at basal (resting) conditions. (Data courtesy of J. Kenneth Herd, M.D., Quillan College of Medicine, East Tennessee State University.)

Suppose that when $Q = 233$ and $D = 41$, we also know that D is decreasing at the rate of 2 units a minute but that Q remains unchanged. What is happening to the cardiac output?

26. Cost, revenue, and profit A company can manufacture x items at a cost of $c(x)$ thousand dollars, a sales revenue of $r(x)$ thousand dollars, and a profit of $p(x) = r(x) - c(x)$ thousand dollars. Find dc/dt, dr/dt, and dp/dt for the following values of x and dx/dt.

a. $r(x) = 9x$, $c(x) = x^3 - 6x^2 + 15x$, and $dx/dt = 0.1$ when $x = 2$

b. $r(x) = 70x$, $c(x) = x^3 - 6x^2 + 45/x$, and $dx/dt = 0.05$ when $x = 1.5$

27. Moving along a parabola A particle moves along the parabola $y = x^2$ in the first quadrant in such a way that its x-coordinate (measured in meters) increases at a steady 10 m/sec. How fast is the angle of inclination θ of the line joining the particle to the origin changing when $x = 3$ m?

Exercises

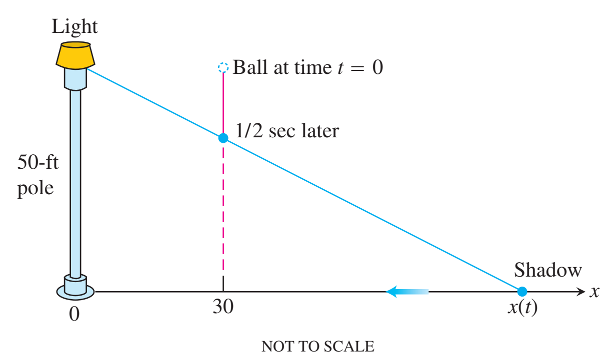
28. Moving along another parabola A particle moves from right to left along the parabolic curve $y = \sqrt{-x}$ in such a way that its x-coordinate (measured in meters) decreases at the rate of 8 m/sec. How fast is the angle of inclination θ of the line joining the particle to the origin changing when $x = -4$?

29. Motion in the plane The coordinates of a particle in the metric xy-plane are differentiable functions of time t with $dx/dt = -1$ m/sec and $dy/dt = -5$ m/sec. How fast is the particle's distance from the origin changing as it passes through the point (5, 12)?

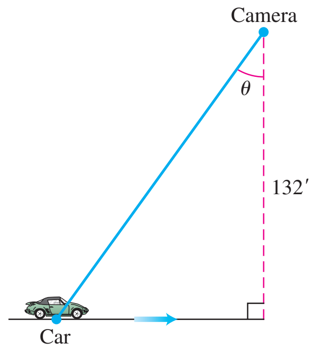
Exercises

30. A moving shadow A man 6 ft tall walks at the rate of 5 ft/sec toward a streetlight that is 16 ft above the ground. At what rate is the tip of his shadow moving? At what rate is the length of his shadow changing when he is 10 ft from the base of the light?

31. Another moving shadow A light shines from the top of a pole 50 ft high. A ball is dropped from the same height from a point 30 ft away from the light. (See accompanying figure.) How fast is the shadow of the ball moving along the ground 1/2 sec later? (Assume the ball falls a distance $s = 16t^2$ ft in t sec.)



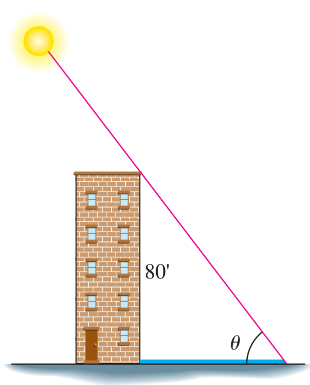
32. Videotaping a moving car You are videotaping a race from a stand 132 ft from the track, following a car that is moving at 180 mi/h (264 ft/sec). How fast will your camera angle θ be changing when the car is right in front of you? A half second later?



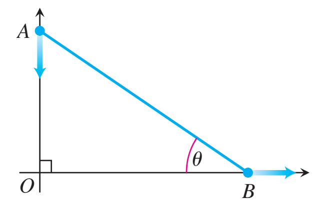
33. A melting ice layer A spherical iron ball 8 in. in diameter is coated with a layer of ice of uniform thickness. If the ice melts at the rate of $10\ \text{in}^3/\text{min}$, how fast is the thickness of the ice decreasing when it is 2 in. thick? How fast is the outer surface area of ice decreasing?

34. Highway patrol A highway patrol plane flies 3 mi above a level, straight road at a steady 120 mi/h. The pilot sees an oncoming car and with radar determines that at the instant the line-of-sight distance from plane to car is 5 mi, the line-of-sight distance is decreasing at the rate of 160 mi/h. Find the car's speed along the highway.

35. A building's shadow On a morning of a day when the sun will pass directly overhead, the shadow of an 80-ft building on level ground is 60 ft long. At the moment in question, the angle θ the sun makes with the ground is increasing at the rate of $0.27°/\text{min}$. At what rate is the shadow decreasing? (Remember to use radians. Express your answer in inches per minute, to the nearest tenth.)



36. Walkers A and B are walking on straight streets that meet at right angles. A approaches the intersection at 2 m/sec; B moves away from the intersection 1 m/sec. At what rate is the angle θ changing when A is 10 m from the intersection and B is 20 m from the intersection? Express your answer in degrees per second to the nearest degree.

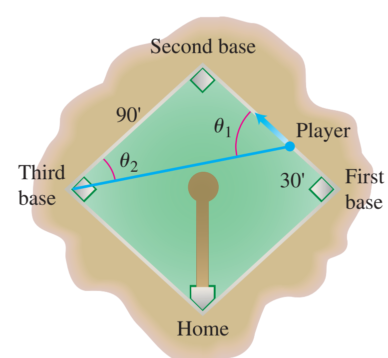


37. Baseball players A baseball diamond is a square 90 ft on a side. A player runs from first base to second at a rate of 16 ft/sec.

a. At what rate is the player's distance from third base changing when the player is 30 ft from first base?

b. At what rates are angles θ_1 and θ_2 (see the figure) changing at that time?

c. The player slides into second base at the rate of 15 ft/sec. At what rates are angles θ_1 and θ_2 changing as the player touches base?



38. Ships Two ships are steaming straight away from a point O along routes that make a 120° angle. Ship A moves at 14 knots (nautical miles per hour; a nautical mile is 2000 yd). Ship B moves at 21 knots. How fast are the ships moving apart when $OA = 5$ and $OB = 3$ nautical miles?

3.8 Linearization and Differentials

Sometimes we can approximate complicated functions with simpler ones that give the accuracy we want for specific applications and are easier to work with. The approximating functions discussed in this section are called *linearizations*, and they are based on tangent lines. Other approximating functions, such as polynomials, are discussed in Chapter 11.

We introduce new variables dx and dy, called *differentials*, and define them in a way that makes Leibniz's notation for the derivative dy/dx a true ratio. We use dy to estimate error in measurement and sensitivity of a function to change. Application of these ideas then provides for a precise proof of the Chain Rule (Section 3.5).

Linearization

As you can see in Figure 3.46, the tangent to the curve $y = x^2$ lies close to the curve near the point of tangency. For a brief interval to either side, the y-values along the tangent line give good approximations to the y-values on the curve. We observe this phenomenon by zooming in on the two graphs at the point of tangency or by looking at tables of values for the difference between $f(x)$ and its tangent line near the x-coordinate of the point of tangency. Locally, every differentiable curve behaves like a straight line.

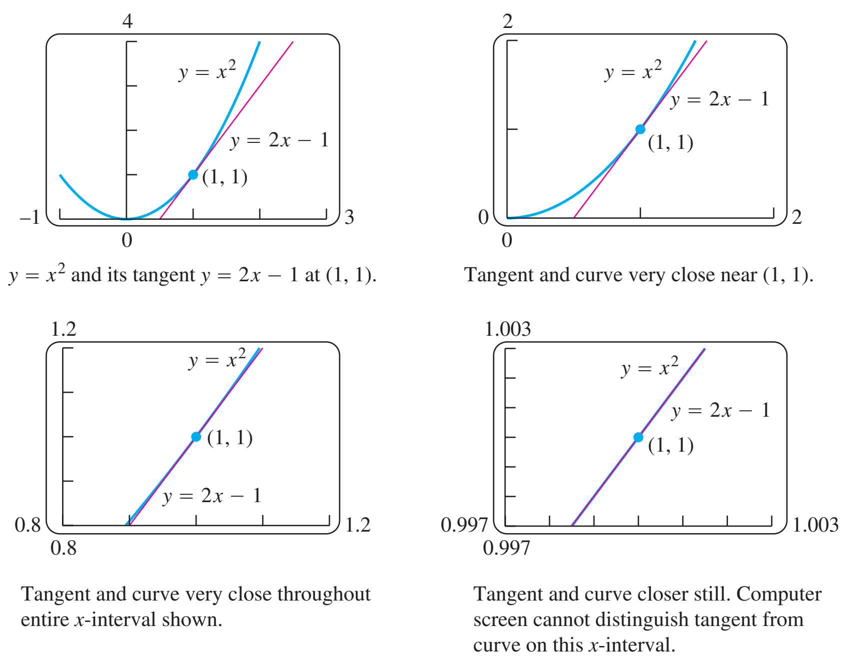


FIGURE 3.46 The more we magnify the graph of a function near a point where the function is differentiable, the flatter the graph becomes and the more it resembles its tangent.

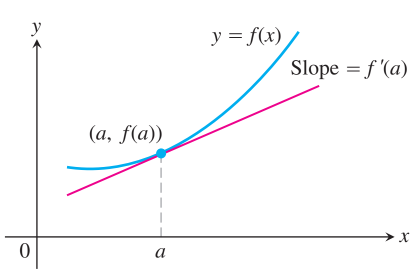


FIGURE 3.47 The tangent to the curve $y = f(x)$ at $x = a$ is the line $L(x) = f(a) + f'(a)(x - a)$.

In general, the tangent to $y = f(x)$ at a point $x = a$, where f is differentiable (Figure 3.47), passes through the point $(a, f(a))$, so its point-slope equation is

$$y = f(a) + f'(a)(x - a).$$

Thus, this tangent line is the graph of the linear function

$$L(x) = f(a) + f'(a)(x - a).$$

For as long as this line remains close to the graph of f, $L(x)$ gives a good approximation to $f(x)$.

DEFINITIONS **Linearization, Standard Linear Approximation**

If f is differentiable at $x = a$, then the approximating function

$$L(x) = f(a) + f'(a)(x - a)$$

is the **linearization** of f at a. The approximation

$$f(x) \approx L(x)$$

of f by L is the **standard linear approximation** of f at a. The point $x = a$ is the **center** of the approximation.

You Try It

EXAMPLE 1 Finding a Linearization

Find the linearization of $f(x) = \sqrt{1 + x}$ at $x = 0$ (Figure 3.48).

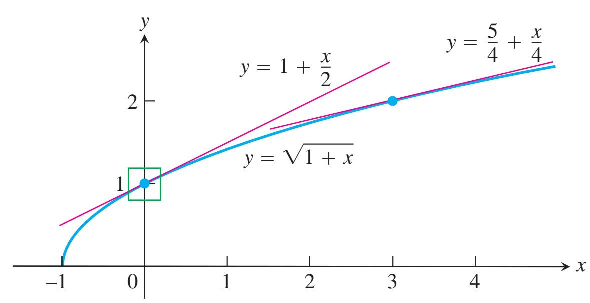


FIGURE 3.48 The graph of $y = \sqrt{1 + x}$ and its linearizations at $x = 0$ and $x = 3$. Figure 3.49 shows a magnified view of the small window about 1 on the y-axis.

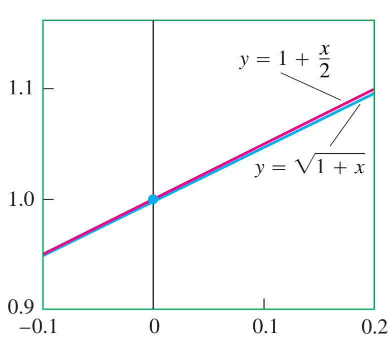


FIGURE 3.49 Magnified view of the window in Figure 3.48.

Solution Since

$$f'(x) = \frac{1}{2}\left(1 + x\right)^{-1/2},$$

we have $f(0) = 1$ and $f'(0) = 1/2$, giving the linearization

$$L(x) = f(a) + f'(a)(x - a) = 1 + \frac{1}{2}\left(x - 0\right) = 1 + \frac{x}{2}.$$

See Figure 3.49. ■

Look at how accurate the approximation $\sqrt{1 + x} \approx 1 + (x/2)$ from Example 1 is for values of x near 0.

As we move away from zero, we lose accuracy. For example, for $x = 2$, the linearization gives 2 as the approximation for $\sqrt{3}$, which is not even accurate to one decimal place.

Do not be misled by the preceding calculations into thinking that whatever we do with a linearization is better done with a calculator. In practice, we would never use a linearization to find a particular square root. The utility of a linearization is its ability to replace a complicated formula by a simpler one over an entire interval of values. If we have to work with $\sqrt{1 + x}$ for x close to 0 and can tolerate the small amount of error involved, we can

Approximation	True value	$\lvert$True value $-$ approximation$\rvert$
$\sqrt{1.2} \approx 1 + \frac{0.2}{2} = 1.10$	1.095445	$< 10^{-2}$
$\sqrt{1.05} \approx 1 + \frac{0.05}{2} = 1.025$	1.024695	$< 10^{-3}$
$\sqrt{1.005} \approx 1 + \frac{0.005}{2} = 1.00250$	1.002497	$< 10^{-5}$

work with $1 + (x/2)$ instead. Of course, we then need to know how much error there is. We have more to say on the estimation of error in Chapter 11.

A linear approximation normally loses accuracy away from its center. As Figure 3.48 suggests, the approximation $\sqrt{1+x} \approx 1 + (x/2)$ will probably be too crude to be useful near $x = 3$. There, we need the linearization at $x = 3$.

Video

EXAMPLE 2 Finding a Linearization at Another Point

Find the linearization of $f(x) = \sqrt{1+x}$ at $x = 3$.

Solution We evaluate the equation defining $L(x)$ at $a = 3$. With

$$f(3) = 2, \qquad f'(3) = \frac{1}{2}(1+x)^{-1/2}\bigg|_{x=3} = \frac{1}{4},$$

we have

$$L(x) = 2 + \frac{1}{4}(x - 3) = \frac{5}{4} + \frac{x}{4}.$$

At $x = 3.2$, the linearization in Example 2 gives

$$\sqrt{1+x} = \sqrt{1+3.2} \approx \frac{5}{4} + \frac{3.2}{4} = 1.250 + 0.800 = 2.050,$$

which differs from the true value $\sqrt{4.2} \approx 2.04939$ by less than one one-thousandth. The linearization in Example 1 gives

$$\sqrt{1+x} = \sqrt{1+3.2} \approx 1 + \frac{3.2}{2} = 1 + 1.6 = 2.6,$$

a result that is off by more than 25%.

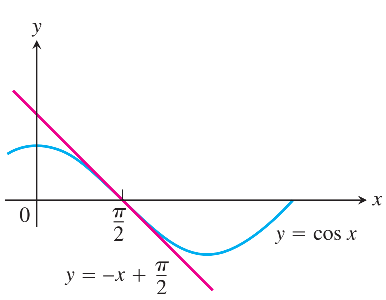


FIGURE 3.50 The graph of $f(x) = \cos x$ and its linearization at $x = \pi/2$. Near $x = \pi/2$, $\cos x \approx -x + (\pi/2)$ (Example 3).

EXAMPLE 3 Finding a Linearization for the Cosine Function

You Try It

Find the linearization of $f(x) = \cos x$ at $x = \pi/2$ (Figure 3.50).

Solution Since $f(\pi/2) = \cos(\pi/2) = 0$, $f'(x) = -\sin x$, and $f'(\pi/2) = -\sin(\pi/2) = -1$, we have

$$\begin{aligned} L(x) &= f(a) + f'(a)(x - a) \\ &= 0 + (-1)\left(x - \frac{\pi}{2}\right) \\ &= -x + \frac{\pi}{2}. \end{aligned}$$

An important linear approximation for roots and powers is

$$(1 + x)^k \approx 1 + kx \qquad (x \text{ near } 0; \text{ any number } k)$$

(Exercise 15). This approximation, good for values of x sufficiently close to zero, has broad application. For example, when x is small,

$$\sqrt{1 + x} \approx 1 + \frac{1}{2}x \qquad k = 1/2$$

$$\frac{1}{1 - x} = (1 - x)^{-1} \approx 1 + (-1)(-x) = 1 + x \qquad k = -1; \text{ replace } x \text{ by } -x.$$

$$\sqrt[3]{1 + 5x^4} = (1 + 5x^4)^{1/3} \approx 1 + \frac{1}{3}\left(5x^4\right) = 1 + \frac{5}{3}x^4 \qquad k = 1/3; \text{ replace } x \text{ by } 5x^4.$$

$$\frac{1}{\sqrt{1 - x^2}} = (1 - x^2)^{-1/2} \approx 1 + \left(-\frac{1}{2}\right)(-x^2) = 1 + \frac{1}{2}x^2 \qquad k = -1/2; \text{ replace } x \text{ by } -x^2.$$

Differentials

We sometimes use the Leibniz notation dy/dx to represent the derivative of y with respect to x. Contrary to its appearance, it is not a ratio. We now introduce two new variables dx and dy with the property that if their ratio exists, it will be equal to the derivative.

DEFINITION Differential

Let $y = f(x)$ be a differentiable function. The **differential dx** is an independent variable. The **differential dy** is

$$dy = f'(x)\,dx.$$

Unlike the independent variable dx, the variable dy is always a dependent variable. It depends on both x and dx. If dx is given a specific value and x is a particular number in the domain of the function f, then the numerical value of dy is determined.

EXAMPLE 4 Finding the Differential dy

(a) Find dy if $y = x^5 + 37x$.

(b) Find the value of dy when $x = 1$ and $dx = 0.2$.

Solution

(a) $dy = (5x^4 + 37)\,dx$

(b) Substituting $x = 1$ and $dx = 0.2$ in the expression for dy, we have

$$dy = (5 \cdot 1^4 + 37)0.2 = 8.4.$$

■

The geometric meaning of differentials is shown in Figure 3.51. Let $x = a$ and set $dx = \Delta x$. The corresponding change in $y = f(x)$ is

$$\Delta y = f(a + dx) - f(a).$$

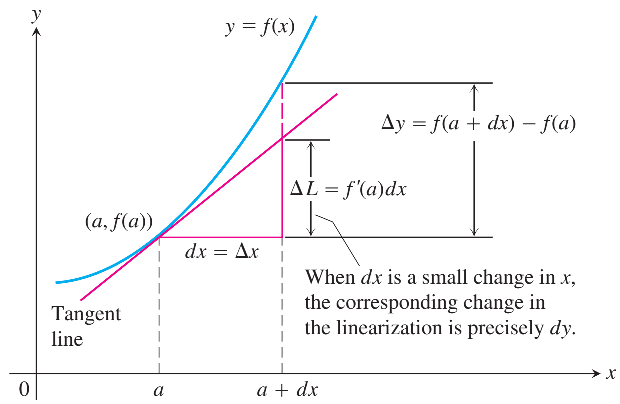


FIGURE 3.51 Geometrically, the differential dy is the change ΔL in the linearization of f when $x = a$ changes by an amount $dx = \Delta x$.

The corresponding change in the tangent line L is

$$\begin{aligned}\Delta L &= L(a + dx) - L(a)\\ &= \underbrace{f(a) + f'(a)[(a + dx) - a]}_{L(a+dx)} - \underbrace{f(a)}_{L(a)}\\ &= f'(a)\,dx.\end{aligned}$$

That is, the change in the linearization of f is precisely the value of the differential dy when $x = a$ and $dx = \Delta x$. Therefore, dy represents the amount the tangent line rises or falls when x changes by an amount $dx = \Delta x$.

If $dx \neq 0$, then the quotient of the differential dy by the differential dx is equal to the derivative $f'(x)$ because

$$dy \div dx = \frac{f'(x)\,dx}{dx} = f'(x) = \frac{dy}{dx}.$$

We sometimes write

$$df = f'(x)\,dx$$

in place of $dy = f'(x)\,dx$, calling df the **differential of f**. For instance, if $f(x) = 3x^2 - 6$, then

$$df = d(3x^2 - 6) = 6x\,dx.$$

Every differentiation formula like

$$\frac{d(u + v)}{dx} = \frac{du}{dx} + \frac{dv}{dx} \qquad \text{or} \qquad \frac{d(\sin u)}{dx} = \cos u\,\frac{du}{dx}$$

has a corresponding differential form like

$$d(u + v) = du + dv \qquad \text{or} \qquad d(\sin u) = \cos u\,du.$$

EXAMPLE 5 Finding Differentials of Functions

(a) $d(\tan 2x) = \sec^2(2x)\, d(2x) = 2\sec^2 2x\, dx$

(b) $d\left(\dfrac{x}{x+1}\right) = \dfrac{(x+1)\,dx - x\,d(x+1)}{(x+1)^2} = \dfrac{x\,dx + dx - x\,dx}{(x+1)^2} = \dfrac{dx}{(x+1)^2}$ ■

Estimating with Differentials

Suppose we know the value of a differentiable function $f(x)$ at a point a and want to predict how much this value will change if we move to a nearby point $a + dx$. If dx is small, then we can see from Figure 3.51 that Δy is approximately equal to the differential dy. Since

$$f(a + dx) = f(a) + \Delta y,$$

the differential approximation gives

$$f(a + dx) \approx f(a) + dy$$

where $dx = \Delta x$. Thus the approximation $\Delta y \approx dy$ can be used to calculate $f(a + dx)$ when $f(a)$ is known and dx is small.

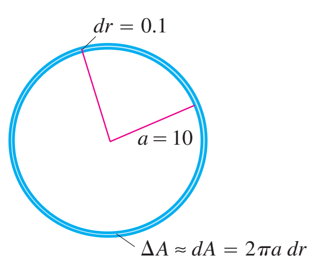


FIGURE 3.52 When dr is small compared with a, as it is when $dr = 0.1$ and $a = 10$, the differential $dA = 2\pi a\, dr$ gives a way to estimate the area of the circle with radius $r = a + dr$ (Example 6).

EXAMPLE 6 Estimating with Differentials

The radius r of a circle increases from $a = 10$ m to 10.1 m (Figure 3.52). Use dA to estimate the increase in the circle's area A. Estimate the area of the enlarged circle and compare your estimate to the true area.

Solution Since $A = \pi r^2$, the estimated increase is

$$dA = A'(a)\, dr = 2\pi a\, dr = 2\pi(10)(0.1) = 2\pi \text{ m}^2.$$

Thus,

$$\begin{aligned} A(10 + 0.1) &\approx A(10) + 2\pi \\ &= \pi(10)^2 + 2\pi = 102\pi. \end{aligned}$$

The area of a circle of radius 10.1 m is approximately 102π m^2.

The true area is

$$\begin{aligned} A(10.1) &= \pi(10.1)^2 \\ &= 102.01\pi \text{ m}^2. \end{aligned}$$

The error in our estimate is 0.01π m^2, which is the difference $\Delta A - dA$. ■

Error in Differential Approximation

Let $f(x)$ be differentiable at $x = a$ and suppose that $dx = \Delta x$ is an increment of x. We have two ways to describe the change in f as x changes from a to $a + \Delta x$:

The true change:	$\Delta f = f(a + \Delta x) - f(a)$
The differential estimate:	$df = f'(a)\, \Delta x.$

How well does df approximate Δf?

We measure the approximation error by subtracting df from Δf:

$$\begin{aligned}
\text{Approximation error} &= \Delta f - df \\
&= \Delta f - f'(a)\Delta x \\
&= \underbrace{f(a+\Delta x) - f(a)}_{\Delta f} - f'(a)\Delta x \\
&= \underbrace{\left(\frac{f(a+\Delta x) - f(a)}{\Delta x} - f'(a)\right)}_{\text{Call this part } \epsilon} \cdot \Delta x \\
&= \epsilon \cdot \Delta x.
\end{aligned}$$

As $\Delta x \to 0$, the difference quotient

$$\frac{f(a+\Delta x) - f(a)}{\Delta x}$$

approaches $f'(a)$ (remember the definition of $f'(a)$), so the quantity in parentheses becomes a very small number (which is why we called it ϵ). In fact, $\epsilon \to 0$ as $\Delta x \to 0$. When Δx is small, the approximation error $\epsilon\, \Delta x$ is smaller still.

$$\underbrace{\Delta f}_{\substack{\text{true}\\ \text{change}}} = \underbrace{f'(a)\Delta x}_{\substack{\text{estimated}\\ \text{change}}} + \underbrace{\epsilon\, \Delta x}_{\text{error}}$$

Although we do not know exactly how small the error is and will not be able to make much progress on this front until Chapter 11, there is something worth noting here, namely the *form* taken by the equation.

Change in $y = f(x)$ near $x = a$

If $y = f(x)$ is differentiable at $x = a$ and x changes from a to $a + \Delta x$, the change Δy in f is given by an equation of the form

$$\Delta y = f'(a)\, \Delta x + \epsilon\, \Delta x \tag{1}$$

in which $\epsilon \to 0$ as $\Delta x \to 0$.

In Example 6 we found that

$$\Delta A = \pi(10.1)^2 - \pi(10)^2 = (102.01 - 100)\pi = (\underbrace{2\pi}_{dA} + \underbrace{0.01\pi}_{\text{error}})\ \mathrm{m}^2$$

so the approximation error is $\Delta A - dA = \epsilon\, \Delta r = 0.01\pi$ and $\epsilon = 0.01\pi/\Delta r = 0.01\pi/0.1 = 0.1\pi$ m.

Equation (1) enables us to bring the proof of the Chain Rule to a successful conclusion.

Proof of the Chain Rule

Our goal is to show that if $f(u)$ is a differentiable function of u and $u = g(x)$ is a differentiable function of x, then the composite $y = f(g(x))$ is a differentiable function of x.

More precisely, if g is differentiable at x_0 and f is differentiable at $g(x_0)$, then the composite is differentiable at x_0 and

$$\left.\frac{dy}{dx}\right|_{x=x_0} = f'(g(x_0)) \cdot g'(x_0).$$

Let Δx be an increment in x and let Δu and Δy be the corresponding increments in u and y. Applying Equation (1) we have,

$$\Delta u = g'(x_0)\Delta x + \epsilon_1 \Delta x = (g'(x_0) + \epsilon_1)\Delta x,$$

where $\epsilon_1 \to 0$ as $\Delta x \to 0$. Similarly,

$$\Delta y = f'(u_0)\Delta u + \epsilon_2 \Delta u = (f'(u_0) + \epsilon_2)\Delta u,$$

where $\epsilon_2 \to 0$ as $\Delta u \to 0$. Notice also that $\Delta u \to 0$ as $\Delta x \to 0$. Combining the equations for Δu and Δy gives

$$\Delta y = (f'(u_0) + \epsilon_2)(g'(x_0) + \epsilon_1)\Delta x,$$

so

$$\frac{\Delta y}{\Delta x} = f'(u_0)g'(x_0) + \epsilon_2 g'(x_0) + f'(u_0)\epsilon_1 + \epsilon_2\epsilon_1.$$

Since ϵ_1 and ϵ_2 go to zero as Δx goes to zero, three of the four terms on the right vanish in the limit, leaving

$$\left.\frac{dy}{dx}\right|_{x=x_0} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = f'(u_0)g'(x_0) = f'(g(x_0)) \cdot g'(x_0).$$

This concludes the proof. ■

Sensitivity to Change

The equation $df = f'(x)\,dx$ tells how *sensitive* the output of f is to a change in input at different values of x. The larger the value of f' at x, the greater the effect of a given change dx. As we move from a to a nearby point $a + dx$, we can describe the change in f in three ways:

	True	**Estimated**
Absolute change	$\Delta f = f(a + dx) - f(a)$	$df = f'(a)\,dx$
Relative change	$\frac{\Delta f}{f(a)}$	$\frac{df}{f(a)}$
Percentage change	$\frac{\Delta f}{f(a)} \times 100$	$\frac{df}{f(a)} \times 100$

EXAMPLE 7 Finding the Depth of a Well

You want to calculate the depth of a well from the equation $s = 16t^2$ by timing how long it takes a heavy stone you drop to splash into the water below. How sensitive will your calculations be to a 0.1-sec error in measuring the time?

Solution The size of ds in the equation

$$ds = 32t\,dt$$

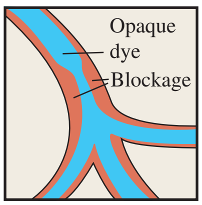
depends on how big t is. If $t = 2$ sec, the change caused by $dt = 0.1$ is about

$$ds = 32(2)(0.1) = 6.4 \text{ ft}.$$

Three seconds later at $t = 5$ sec, the change caused by the same dt is

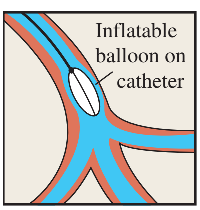
$$ds = 32(5)(0.1) = 16 \text{ ft}.$$

The estimated depth of the well differs from its true depth by a greater distance the longer the time it takes the stone to splash into the water below, for a given error in measuring the time. ■



Angiography

An opaque dye is injected into a partially blocked artery to make the inside visible under X-rays. This reveals the location and severity of the blockage.



Angioplasty

A balloon-tipped catheter is inflated inside the artery to widen it at the blockage site.

You Try It

EXAMPLE 8 Unclogging Arteries

In the late 1830s, French physiologist Jean Poiseuille ("pwa-ZOY") discovered the formula we use today to predict how much the radius of a partially clogged artery has to be expanded to restore normal flow. His formula,

$$V = kr^4,$$

says that the volume V of fluid flowing through a small pipe or tube in a unit of time at a fixed pressure is a constant times the fourth power of the tube's radius r. How will a 10% increase in r affect V?

Solution The differentials of r and V are related by the equation

$$dV = \frac{dV}{dr}\,dr = 4kr^3\,dr.$$

The relative change in V is

$$\frac{dV}{V} = \frac{4kr^3\,dr}{kr^4} = 4\frac{dr}{r}.$$

The relative change in V is 4 times the relative change in r, so a 10% increase in r will produce a 40% increase in the flow. ■

EXAMPLE 9 Converting Mass to Energy

Newton's second law,

$$F = \frac{d}{dt}(mv) = m\frac{dv}{dt} = ma,$$

is stated with the assumption that mass is constant, but we know this is not strictly true because the mass of a body increases with velocity. In Einstein's corrected formula, mass has the value

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}},$$

where the "rest mass" m_0 represents the mass of a body that is not moving and c is the speed of light, which is about 300,000 km/sec. Use the approximation

$$\frac{1}{\sqrt{1 - x^2}} \approx 1 + \frac{1}{2}x^2 \qquad (2)$$

to estimate the increase Δm in mass resulting from the added velocity v.

Solution When v is very small compared with c, v^2/c^2 is close to zero and it is safe to use the approximation

$$\frac{1}{\sqrt{1 - v^2/c^2}} \approx 1 + \frac{1}{2}\left(\frac{v^2}{c^2}\right) \qquad \text{Eq. (2) with } x = \frac{v}{c}$$

to obtain

$$m = \frac{m_0}{\sqrt{1 - v^2/c^2}} \approx m_0\left[1 + \frac{1}{2}\left(\frac{v^2}{c^2}\right)\right] = m_0 + \frac{1}{2}m_0 v^2\left(\frac{1}{c^2}\right),$$

or

$$m \approx m_0 + \frac{1}{2}m_0 v^2\left(\frac{1}{c^2}\right). \tag{3}$$

Equation (3) expresses the increase in mass that results from the added velocity v.

Energy Interpretation

In Newtonian physics, $(1/2)m_0 v^2$ is the kinetic energy (KE) of the body, and if we rewrite Equation (3) in the form

$$(m - m_0)c^2 \approx \frac{1}{2}m_0 v^2,$$

we see that

$$(m - m_0)c^2 \approx \frac{1}{2}m_0 v^2 = \frac{1}{2}m_0 v^2 - \frac{1}{2}m_0(0)^2 = \Delta(\text{KE}),$$

or

$$(\Delta m)c^2 \approx \Delta(\text{KE}).$$

So the change in kinetic energy $\Delta(\text{KE})$ in going from velocity 0 to velocity v is approximately equal to $(\Delta m)c^2$, the change in mass times the square of the speed of light. Using $c \approx 3 \times 10^8$ m/sec, we see that a small change in mass can create a large change in energy. ■

EXERCISES 3.8

Finding Linearizations

In Exercises 1–4, find the linearization $L(x)$ of $f(x)$ at $x = a$.

1. $f(x) = x^3 - 2x + 3, \quad a = 2$
2. $f(x) = \sqrt{x^2 + 9}, \quad a = -4$
3. $f(x) = x + \dfrac{1}{x}, \quad a = 1$
4. $f(x) = \sqrt[3]{x}, \quad a = -8$

Linearization for Approximation

You want linearizations that will replace the functions in Exercises 5–10 over intervals that include the given points x_0. To make your subsequent work as simple as possible, you want to center each linearization not at x_0 but at a nearby integer $x = a$ at which the given function and its derivative are easy to evaluate. What linearization do you use in each case?

5. $f(x) = x^2 + 2x, \quad x_0 = 0.1$
6. $f(x) = x^{-1}, \quad x_0 = 0.9$
7. $f(x) = 2x^2 + 4x - 3, \quad x_0 = -0.9$
8. $f(x) = 1 + x, \quad x_0 = 8.1$
9. $f(x) = \sqrt[3]{x}, \quad x_0 = 8.5$
10. $f(x) = \dfrac{x}{x + 1}, \quad x_0 = 1.3$

Linearizing Trigonometric Functions

In Exercises 11–14, find the linearization of f at $x = a$. Then graph the linearization and f together.

11. $f(x) = \sin x$ at (a) $x = 0$, (b) $x = \pi$

12. $f(x) = \cos x$ at (a) $x = 0$, (b) $x = -\pi/2$

13. $f(x) = \sec x$ at (a) $x = 0$, (b) $x = -\pi/3$

14. $f(x) = \tan x$ at (a) $x = 0$, (b) $x = \pi/4$

The Approximation $(1 + x)^k \approx 1 + kx$

15. Show that the linearization of $f(x) = (1 + x)^k$ at $x = 0$ is $L(x) = 1 + kx$.

16. Use the linear approximation $(1 + x)^k \approx 1 + kx$ to find an approximation for the function $f(x)$ for values of x near zero.

a. $f(x) = (1 - x)^6$ **b.** $f(x) = \dfrac{2}{1 - x}$

c. $f(x) = \dfrac{1}{\sqrt{1 + x}}$ **d.** $f(x) = \sqrt{2 + x^2}$

e. $f(x) = (4 + 3x)^{1/3}$ **f.** $f(x) = \sqrt[3]{\left(1 - \dfrac{1}{2 + x}\right)^2}$

17. Faster than a calculator Use the approximation $(1 + x)^k \approx 1 + kx$ to estimate the following.

a. $(1.0002)^{50}$ **b.** $\sqrt[3]{1.009}$

18. Find the linearization of $f(x) = \sqrt{x + 1} + \sin x$ at $x = 0$. How is it related to the individual linearizations of $\sqrt{x + 1}$ and $\sin x$ at $x = 0$?

Derivatives in Differential Form

In Exercises 19–30, find dy.

19. $y = x^3 - 3\sqrt{x}$ **20.** $y = x\sqrt{1 - x^2}$

21. $y = \dfrac{2x}{1 + x^2}$ **22.** $y = \dfrac{2\sqrt{x}}{3(1 + \sqrt{x})}$

23. $2y^{3/2} + xy - x = 0$ **24.** $xy^2 - 4x^{3/2} - y = 0$

25. $y = \sin(5\sqrt{x})$ **26.** $y = \cos(x^2)$

27. $y = 4\tan(x^3/3)$ **28.** $y = \sec(x^2 - 1)$

29. $y = 3\csc(1 - 2\sqrt{x})$ **30.** $y = 2\cot\left(\dfrac{1}{\sqrt{x}}\right)$

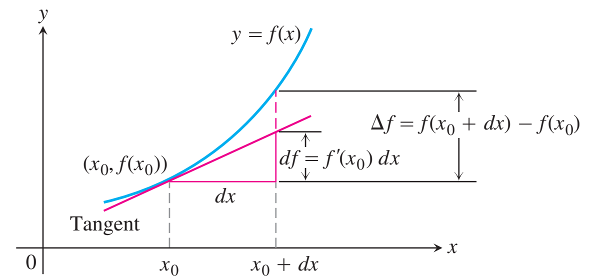
Approximation Error

In Exercises 31–36, each function $f(x)$ changes value when x changes from x_0 to $x_0 + dx$. Find

a. the change $\Delta f = f(x_0 + dx) - f(x_0)$;

b. the value of the estimate $df = f'(x_0)\,dx$; and

c. the approximation error $|\Delta f - df|$.



31. $f(x) = x^2 + 2x$, $x_0 = 1$, $dx = 0.1$

32. $f(x) = 2x^2 + 4x - 3$, $x_0 = -1$, $dx = 0.1$

33. $f(x) = x^3 - x$, $x_0 = 1$, $dx = 0.1$

34. $f(x) = x^4$, $x_0 = 1$, $dx = 0.1$

35. $f(x) = x^{-1}$, $x_0 = 0.5$, $dx = 0.1$

36. $f(x) = x^3 - 2x + 3$, $x_0 = 2$, $dx = 0.1$

Differential Estimates of Change

In Exercises 37–42, write a differential formula that estimates the given change in volume or surface area.

37. The change in the volume $V = (4/3)\pi r^3$ of a sphere when the radius changes from r_0 to $r_0 + dr$

38. The change in the volume $V = x^3$ of a cube when the edge lengths change from x_0 to $x_0 + dx$

39. The change in the surface area $S = 6x^2$ of a cube when the edge lengths change from x_0 to $x_0 + dx$

40. The change in the lateral surface area $S = \pi r\sqrt{r^2 + h^2}$ of a right circular cone when the radius changes from r_0 to $r_0 + dr$ and the height does not change

41. The change in the volume $V = \pi r^2 h$ of a right circular cylinder when the radius changes from r_0 to $r_0 + dr$ and the height does not change

42. The change in the lateral surface area $S = 2\pi rh$ of a right circular cylinder when the height changes from h_0 to $h_0 + dh$ and the radius does not change

Applications

43. The radius of a circle is increased from 2.00 to 2.02 m.

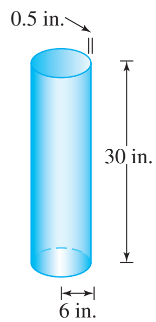
a. Estimate the resulting change in area.

b. Express the estimate as a percentage of the circle's original area.

44. The diameter of a tree was 10 in. During the following year, the circumference increased 2 in. About how much did the tree's diameter increase? The tree's cross-section area?

45. Estimating volume Estimate the volume of material in a cylindrical shell with height 30 in., radius 6 in., and shell thickness 0.5 in.

Exercises



46. Estimating height of a building A surveyor, standing 30 ft from the base of a building, measures the angle of elevation to the top of the building to be 75°. How accurately must the angle be measured for the percentage error in estimating the height of the building to be less than 4%?

47. Tolerance The height and radius of a right circular cylinder are equal, so the cylinder's volume is $V = \pi h^3$. The volume is to be calculated with an error of no more than 1% of the true value. Find approximately the greatest error that can be tolerated in the measurement of *h*, expressed as a percentage of *h*.

48. Tolerance

a. About how accurately must the interior diameter of a 10-m-high cylindrical storage tank be measured to calculate the tank's volume to within 1% of its true value?

b. About how accurately must the tank's exterior diameter be measured to calculate the amount of paint it will take to paint the side of the tank to within 5% of the true amount?

49. Minting coins A manufacturer contracts to mint coins for the federal government. How much variation dr in the radius of the coins can be tolerated if the coins are to weigh within 1/1000 of their ideal weight? Assume that the thickness does not vary.

50. Sketching the change in a cube's volume The volume $V = x^3$ of a cube with edges of length x increases by an amount ΔV when x increases by an amount Δx. Show with a sketch how to represent ΔV geometrically as the sum of the volumes of

a. three slabs of dimensions x by x by Δx

b. three bars of dimensions x by Δx by Δx

c. one cube of dimensions Δx by Δx by Δx.

The differential formula $dV = 3x^2\, dx$ estimates the change in V with the three slabs.

51. The effect of flight maneuvers on the heart The amount of work done by the heart's main pumping chamber, the left ventricle, is given by the equation

$$W = PV + \frac{V\delta v^2}{2g},$$

where W is the work per unit time, P is the average blood pressure, V is the volume of blood pumped out during the unit of time, δ ("delta") is the weight density of the blood, v is the average velocity of the exiting blood, and g is the acceleration of gravity.

When P, V, δ, and v remain constant, W becomes a function of g, and the equation takes the simplified form

$$W = a + \frac{b}{g} \quad (a, b \text{ constant}).$$

As a member of NASA's medical team, you want to know how sensitive W is to apparent changes in g caused by flight maneuvers, and this depends on the initial value of g. As part of your investigation, you decide to compare the effect on W of a given change dg on the moon, where $g = 5.2$ ft/sec^2, with the effect the same change dg would have on Earth, where $g = 32$ ft/sec^2. Use the simplified equation above to find the ratio of dW_{moon} to dW_{Earth}.

52. Measuring acceleration of gravity When the length L of a clock pendulum is held constant by controlling its temperature, the pendulum's period T depends on the acceleration of gravity g. The period will therefore vary slightly as the clock is moved from place to place on the earth's surface, depending on the change in g. By keeping track of ΔT, we can estimate the variation in g from the equation $T = 2\pi(L/g)^{1/2}$ that relates T, g, and L.

a. With L held constant and g as the independent variable, calculate dT and use it to answer parts (b) and (c).

b. If g increases, will T increase or decrease? Will a pendulum clock speed up or slow down? Explain.

c. A clock with a 100-cm pendulum is moved from a location where $g = 980$ cm/sec^2 to a new location. This increases the period by $dT = 0.001$ sec. Find dg and estimate the value of g at the new location.

53. The edge of a cube is measured as 10 cm with an error of 1%. The cube's volume is to be calculated from this measurement. Estimate the percentage error in the volume calculation.

54. About how accurately should you measure the side of a square to be sure of calculating the area within 2% of its true value?

55. The diameter of a sphere is measured as 100 ± 1 cm and the volume is calculated from this measurement. Estimate the percentage error in the volume calculation.

56. Estimate the allowable percentage error in measuring the diameter D of a sphere if the volume is to be calculated correctly to within 3%.

57. (*Continuation of Example 7.*) Show that a 5% error in measuring t will cause about a 10% error in calculating s from the equation $s = 16t^2$.

58. (*Continuation of Example 8.*) By what percentage should r be increased to increase V by 50%?

Theory and Examples

59. Show that the approximation of $\sqrt{1 + x}$ by its linearization at the origin must improve as $x \to 0$ by showing that

$$\lim_{x \to 0} \frac{\sqrt{1 + x}}{1 + (x/2)} = 1.$$

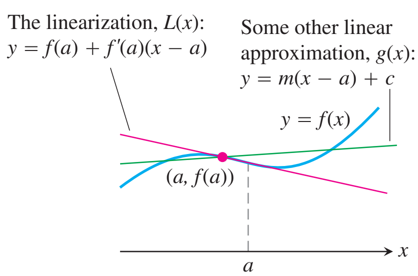
60. Show that the approximation of tan x by its linearization at the origin must improve as $x \to 0$ by showing that

$$\lim_{x\to 0} \frac{\tan x}{x} = 1.$$

61. The linearization is the best linear approximation (This is why we use the linearization.) Suppose that $y = f(x)$ is differentiable at $x = a$ and that $g(x) = m(x - a) + c$ is a linear function in which m and c are constants. If the error $E(x) = f(x) - g(x)$ were small enough near $x = a$, we might think of using g as a linear approximation of f instead of the linearization $L(x) = f(a) + f'(a)(x - a)$. Show that if we impose on g the conditions

1. $E(a) = 0$ — The approximation error is zero at $x = a$.
2. $\lim_{x\to a} \frac{E(x)}{x - a} = 0$ — The error is negligible when compared with $x - a$.

then $g(x) = f(a) + f'(a)(x - a)$. Thus, the linearization $L(x)$ gives the only linear approximation whose error is both zero at $x = a$ and negligible in comparison with $x - a$.



62. Quadratic approximations

a. Let $Q(x) = b_0 + b_1(x - a) + b_2(x - a)^2$ be a quadratic approximation to $f(x)$ at $x = a$ with the properties:

i. $Q(a) = f(a)$

ii. $Q'(a) = f'(a)$

iii. $Q''(a) = f''(a)$

Determine the coefficients b_0, b_1, and b_2.

b. Find the quadratic approximation to $f(x) = 1/(1 - x)$ at $x = 0$.

T **c.** Graph $f(x) = 1/(1 - x)$ and its quadratic approximation at $x = 0$. Then zoom in on the two graphs at the point (0, 1). Comment on what you see.

T **d.** Find the quadratic approximation to $g(x) = 1/x$ at $x = 1$. Graph g and its quadratic approximation together. Comment on what you see.

T **e.** Find the quadratic approximation to $h(x) = \sqrt{1 + x}$ at $x = 0$. Graph h and its quadratic approximation together. Comment on what you see.

f. What are the linearizations of f, g, and h at the respective points in parts (b), (d), and (e)?

T **63. Reading derivatives from graphs** The idea that differentiable curves flatten out when magnified can be used to estimate the values of the derivatives of functions at particular points. We magnify the curve until the portion we see looks like a straight line through the point in question, and then we use the screen's coordinate grid to read the slope of the curve as the slope of the line it resembles.

a. To see how the process works, try it first with the function $y = x^2$ at $x = 1$. The slope you read should be 2.

b. Then try it with the curve $y = e^x$ at $x = 1$, $x = 0$, and $x = -1$. In each case, compare your estimate of the derivative with the value of e^x at the point. What pattern do you see? Test it with other values of x. Chapter 7 will explain what is going on.

64. Suppose that the graph of a differentiable function $f(x)$ has a horizontal tangent at $x = a$. Can anything be said about the linearization of f at $x = a$? Give reasons for your answer.

65. To what relative speed should a body at rest be accelerated to increase its mass by 1%?

T **66. Repeated root-taking**

a. Enter 2 in your calculator and take successive square roots by pressing the square root key repeatedly (or raising the displayed number repeatedly to the 0.5 power). What pattern do you see emerging? Explain what is going on. What happens if you take successive tenth roots instead?

b. Repeat the procedure with 0.5 in place of 2 as the original entry. What happens now? Can you use any positive number x in place of 2? Explain what is going on.

COMPUTER EXPLORATIONS

Comparing Functions with Their Linearizations

In Exercises 67–70, use a CAS to estimate the magnitude of the error in using the linearization in place of the function over a specified interval I. Perform the following steps:

a. Plot the function f over I.

b. Find the linearization L of the function at the point a.

c. Plot f and L together on a single graph.

d. Plot the absolute error $|f(x) - L(x)|$ over I and find its maximum value.

e. From your graph in part (d), estimate as large a $\delta > 0$ as you can, satisfying

$$|x - a| < \delta \quad \Rightarrow \quad |f(x) - L(x)| < \epsilon$$

for $\epsilon = 0.5, 0.1,$ and 0.01. Then check graphically to see if your δ-estimate holds true.

67. $f(x) = x^3 + x^2 - 2x, \quad [-1, 2], \quad a = 1$

68. $f(x) = \frac{x - 1}{4x^2 + 1}, \quad \left[-\frac{3}{4}, 1\right], \quad a = \frac{1}{2}$

69. $f(x) = x^{2/3}(x - 2), \quad [-2, 3], \quad a = 2$

70. $f(x) = \sqrt{x} - \sin x, \quad [0, 2\pi], \quad a = 2$

Chapter 3 Additional and Advanced Exercises

1. An equation like $\sin^2\theta + \cos^2\theta = 1$ is called an **identity** because it holds for all values of θ. An equation like $\sin\theta = 0.5$ is not an identity because it holds only for selected values of θ, not all. If you differentiate both sides of a trigonometric identity in θ with respect to θ, the resulting new equation will also be an identity.

   Differentiate the following to show that the resulting equations hold for all θ.

   a. $\sin 2\theta = 2\sin\theta\cos\theta$

   b. $\cos 2\theta = \cos^2\theta - \sin^2\theta$

2. If the identity $\sin(x + a) = \sin x\cos a + \cos x\sin a$ is differentiated with respect to x, is the resulting equation also an identity? Does this principle apply to the equation $x^2 - 2x - 8 = 0$? Explain.

3. **a.** Find values for the constants a, b, and c that will make

$$f(x) = \cos x \quad \text{and} \quad g(x) = a + bx + cx^2$$

   satisfy the conditions

$$f(0) = g(0), \quad f'(0) = g'(0), \quad \text{and} \quad f''(0) = g''(0).$$

b. Find values for b and c that will make

$$f(x) = \sin(x + a) \quad \text{and} \quad g(x) = b \sin x + c \cos x$$

satisfy the conditions

$$f(0) = g(0) \quad \text{and} \quad f'(0) = g'(0).$$

c. For the determined values of a, b, and c, what happens for the third and fourth derivatives of f and g in each of parts (a) and (b)?

4. Solutions to differential equations

a. Show that $y = \sin x$, $y = \cos x$, and $y = a \cos x + b \sin x$ (a and b constants) all satisfy the equation

$$y'' + y = 0.$$

b. How would you modify the functions in part (a) to satisfy the equation

$$y'' + 4y = 0?$$

Generalize this result.

5. An osculating circle Find the values of h, k, and a that make the circle $(x - h)^2 + (y - k)^2 = a^2$ tangent to the parabola $y = x^2 + 1$ at the point (1, 2) and that also make the second derivatives d^2y/dx^2 have the same value on both curves there. Circles like this one that are tangent to a curve and have the same second derivative as the curve at the point of tangency are called *osculating circles* (from the Latin *osculari*, meaning "to kiss"). We encounter them again in Chapter 13.

6. Marginal revenue A bus will hold 60 people. The number x of people per trip who use the bus is related to the fare charged (p dollars) by the law $p = [3 - (x/40)]^2$. Write an expression for the total revenue $r(x)$ per trip received by the bus company. What number of people per trip will make the marginal revenue dr/dx equal to zero? What is the corresponding fare? (This fare is the one that maximizes the revenue, so the bus company should probably rethink its fare policy.)

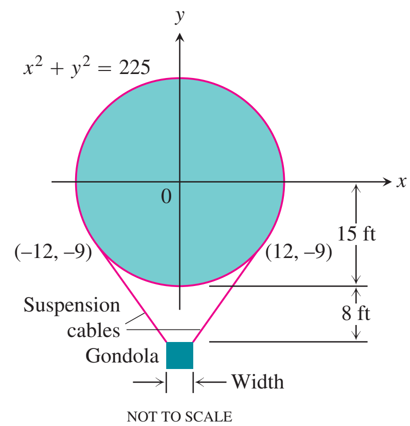
7. Industrial production

a. Economists often use the expression "rate of growth" in relative rather than absolute terms. For example, let $u = f(t)$ be the number of people in the labor force at time t in a given industry. (We treat this function as though it were differentiable even though it is an integer-valued step function.)

Let $v = g(t)$ be the average production per person in the labor force at time t. The total production is then $y = uv$. If the labor force is growing at the rate of 4% per year ($du/dt = 0.04u$) and the production per worker is growing at the rate of 5% per year ($dv/dt = 0.05v$), find the rate of growth of the total production, y.

b. Suppose that the labor force in part (a) is decreasing at the rate of 2% per year while the production per person is increasing at the rate of 3% per year. Is the total production increasing, or is it decreasing, and at what rate?

8. Designing a gondola The designer of a 30-ft-diameter spherical hot air balloon wants to suspend the gondola 8 ft below the bottom of the balloon with cables tangent to the surface of the balloon, as shown. Two of the cables are shown running from the top edges of the gondola to their points of tangency, $(-12, -9)$ and $(12, -9)$. How wide should the gondola be?



9. Pisa by parachute The photograph shows Mike McCarthy parachuting from the top of the Tower of Pisa on August 5, 1988. Make a rough sketch to show the shape of the graph of his speed during the jump.

Photograph is not available.

Mike McCarthy of London jumped from the Tower of Pisa and then opened his parachute in what he said was a world record low-level parachute jump of 179 ft. (*Source: Boston Globe*, Aug. 6, 1988.)

10. Motion of a particle The position at time $t \geq 0$ of a particle moving along a coordinate line is

$$s = 10\cos(t + \pi/4).$$

a. What is the particle's starting position $(t = 0)$?

b. What are the points farthest to the left and right of the origin reached by the particle?

c. Find the particle's velocity and acceleration at the points in part (b).

d. When does the particle first reach the origin? What are its velocity, speed, and acceleration then?

11. Shooting a paper clip On Earth, you can easily shoot a paper clip 64 ft straight up into the air with a rubber band. In t sec after firing, the paper clip is $s = 64t - 16t^2$ ft above your hand.

a. How long does it take the paper clip to reach its maximum height? With what velocity does it leave your hand?

b. On the moon, the same acceleration will send the paper clip to a height of $s = 64t - 2.6t^2$ ft in t sec. About how long will it take the paper clip to reach its maximum height, and how high will it go?

12. Velocities of two particles At time t sec, the positions of two particles on a coordinate line are $s_1 = 3t^3 - 12t^2 + 18t + 5$ m and $s_2 = -t^3 + 9t^2 - 12t$ m. When do the particles have the same velocities?

13. Velocity of a particle A particle of constant mass m moves along the x-axis. Its velocity v and position x satisfy the equation

$$\frac{1}{2}m(v^2 - {v_0}^2) = \frac{1}{2}k({x_0}^2 - x^2),$$

where k, v_0, and x_0 are constants. Show that whenever $v \neq 0$,

$$m\frac{dv}{dt} = -kx.$$

14. Average and instantaneous velocity

a. Show that if the position x of a moving point is given by a quadratic function of t, $x = At^2 + Bt + C$, then the average velocity over any time interval $[t_1, t_2]$ is equal to the instantaneous velocity at the midpoint of the time interval.

b. What is the geometric significance of the result in part (a)?

15. Find all values of the constants m and b for which the function

$$y = \begin{cases} \sin x, & x < \pi \\ mx + b, & x \geq \pi \end{cases}$$

is

a. continuous at $x = \pi$.

b. differentiable at $x = \pi$.

16. Does the function

$$f(x) = \begin{cases} \dfrac{1 - \cos x}{x}, & x \neq 0 \\ 0, & x = 0 \end{cases}$$

have a derivative at $x = 0$? Explain.

17. a. For what values of a and b will

$$f(x) = \begin{cases} ax, & x < 2 \\ ax^2 - bx + 3, & x \geq 2 \end{cases}$$

be differentiable for all values of x?

b. Discuss the geometry of the resulting graph of f.

18. a. For what values of a and b will

$$g(x) = \begin{cases} ax + b, & x \leq -1 \\ ax^3 + x + 2b, & x > -1 \end{cases}$$

be differentiable for all values of x?

b. Discuss the geometry of the resulting graph of g.

19. Odd differentiable functions Is there anything special about the derivative of an odd differentiable function of x? Give reasons for your answer.

20. Even differentiable functions Is there anything special about the derivative of an even differentiable function of x? Give reasons for your answer.

21. Suppose that the functions f and g are defined throughout an open interval containing the point x_0, that f is differentiable at x_0, that $f(x_0) = 0$, and that g is continuous at x_0. Show that the product fg is differentiable at x_0. This process shows, for example, that although $|x|$ is not differentiable at $x = 0$, the product $x|x|$ *is* differentiable at $x = 0$.

22. *(Continuation of Exercise 21.)* Use the result of Exercise 21 to show that the following functions are differentiable at $x = 0$.

a. $|x|\sin x$ **b.** $x^{2/3}\sin x$ **c.** $\sqrt[3]{x}(1 - \cos x)$

d. $h(x) = \begin{cases} x^2\sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$

23. Is the derivative of

$$h(x) = \begin{cases} x^2\sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

continuous at $x = 0$? How about the derivative of $k(x) = xh(x)$? Give reasons for your answers.

24. Suppose that a function f satisfies the following conditions for all real values of x and y:

i. $f(x + y) = f(x) \cdot f(y)$.

ii. $f(x) = 1 + xg(x)$, where $\lim_{x\to 0} g(x) = 1$.

Show that the derivative $f'(x)$ exists at every value of x and that $f'(x) = f(x)$.

25. The generalized product rule Use mathematical induction to prove that if $y = u_1u_2\cdots u_n$ is a finite product of differentiable functions, then y is differentiable on their common domain and

$$\frac{dy}{dx} = \frac{du_1}{dx}u_2\cdots u_n + u_1\frac{du_2}{dx}\cdots u_n + \cdots + u_1u_2\cdots u_{n-1}\frac{du_n}{dx}.$$

26. Leibniz's rule for higher-order derivatives of products Leibniz's rule for higher-order derivatives of products of differentiable functions says that

a. $\dfrac{d^2(uv)}{dx^2} = \dfrac{d^2u}{dx^2}v + 2\dfrac{du}{dx}\dfrac{dv}{dx} + u\dfrac{d^2v}{dx^2}$

b. $\dfrac{d^3(uv)}{dx^3} = \dfrac{d^3u}{dx^3}v + 3\dfrac{d^2u}{dx^2}\dfrac{dv}{dx} + 3\dfrac{du}{dx}\dfrac{d^2v}{dx^2} + u\dfrac{d^3v}{dx^3}$

c. $$\frac{d^n(uv)}{dx^n} = \frac{d^nu}{dx^n}v + n\frac{d^{n-1}u}{dx^{n-1}}\frac{dv}{dx} + \cdots + \frac{n(n-1)\cdots(n-k+1)}{k!}\frac{d^{n-k}u}{dx^{n-k}}\frac{d^kv}{dx^k} + \cdots + u\frac{d^nv}{dx^n}.$$

The equations in parts (a) and (b) are special cases of the equation in part (c). Derive the equation in part (c) by mathematical induction, using

$$\binom{m}{k} + \binom{m}{k+1} = \frac{m!}{k!(m-k)!} + \frac{m!}{(k+1)!(m-k-1)!}.$$

27. The period of a clock pendulum The period T of a clock pendulum (time for one full swing and back) is given by the formula $T^2 = 4\pi^2 L/g$, where T is measured in seconds, $g = 32.2$ ft/sec^2, and L, the length of the pendulum, is measured in feet. Find approximately

a. the length of a clock pendulum whose period is $T = 1$ sec.

b. the change dT in T if the pendulum in part (a) is lengthened 0.01 ft.

c. the amount the clock gains or loses in a day as a result of the period's changing by the amount dT found in part (b).

28. The melting ice cube Assume an ice cube retains its cubical shape as it melts. If we call its edge length s, its volume is $V = s^3$ and its surface area is $6s^2$. We assume that V and s are differentiable functions of time t. We assume also that the cube's volume decreases at a rate that is proportional to its surface area. (This latter assumption seems reasonable enough when we think that the melting takes place at the surface: Changing the amount of surface changes the amount of ice exposed to melt.) In mathematical terms,

$$\frac{dV}{dt} = -k(6s^2), \qquad k > 0.$$

The minus sign indicates that the volume is decreasing. We assume that the proportionality factor k is constant. (It probably depends on many things, such as the relative humidity of the surrounding air, the air temperature, and the incidence or absence of sunlight, to name only a few.) Assume a particular set of conditions in which the cube lost $1/4$ of its volume during the first hour, and that the volume is V_0 when $t = 0$. How long will it take the ice cube to melt?

Chapter 3 Practice Exercises

Derivatives of Functions

Find the derivatives of the functions in Exercises 1-40.

1. $y = x^5 - 0.125x^2 + 0.25x$

2. $y = 3 - 0.7x^3 + 0.3x^7$

3. $y = x^3 - 3(x^2 + \pi^2)$

4. $y = x^7 + \sqrt{7}x - \dfrac{1}{\pi + 1}$

5. $y = (x + 1)^2(x^2 + 2x)$

6. $y = (2x - 5)(4 - x)^{-1}$

7. $y = (\theta^2 + \sec\theta + 1)^3$

8. $y = \left(-1 - \dfrac{\csc\theta}{2} - \dfrac{\theta^2}{4}\right)^2$

9. $s = \dfrac{\sqrt{t}}{1 + \sqrt{t}}$

10. $s = \dfrac{1}{\sqrt{t} - 1}$

11. $y = 2\tan^2 x - \sec^2 x$

12. $y = \frac{1}{\sin^2 x} - \frac{2}{\sin x}$

13. $s = \cos^4(1 - 2t)$

14. $s = \cot^3\left(\frac{2}{t}\right)$

15. $s = (\sec t + \tan t)^5$

16. $s = \csc^5(1 - t + 3t^2)$

17. $r = \sqrt{2\theta \sin\theta}$

18. $r = 2\theta\sqrt{\cos\theta}$

19. $r = \sin\sqrt{2\theta}$

20. $r = \sin\left(\theta + \sqrt{\theta + 1}\right)$

21. $y = \frac{1}{2}x^2 \csc\frac{2}{x}$

22. $y = 2\sqrt{x}\sin\sqrt{x}$

23. $y = x^{-1/2}\sec(2x)^2$

24. $y = \sqrt{x}\csc(x + 1)^3$

25. $y = 5\cot x^2$

26. $y = x^2\cot 5x$

27. $y = x^2\sin^2(2x^2)$

28. $y = x^{-2}\sin^2(x^3)$

29. $s = \left(\frac{4t}{t + 1}\right)^{-2}$

30. $s = \frac{-1}{15(15t - 1)^3}$

31. $y = \left(\frac{\sqrt{x}}{1 + x}\right)^2$

32. $y = \left(\frac{2\sqrt{x}}{2\sqrt{x} + 1}\right)^2$

33. $y = \sqrt{\frac{x^2 + x}{x^2}}$

34. $y = 4x\sqrt{x + \sqrt{x}}$

35. $r = \left(\frac{\sin\theta}{\cos\theta - 1}\right)^2$

36. $r = \left(\frac{1 + \sin\theta}{1 - \cos\theta}\right)^2$

37. $y = (2x + 1)\sqrt{2x + 1}$

38. $y = 20(3x - 4)^{1/4}(3x - 4)^{-1/5}$

39. $y = \frac{3}{(5x^2 + \sin 2x)^{3/2}}$

40. $y = (3 + \cos^3 3x)^{-1/3}$

Implicit Differentiation

In Exercises 41–48, find dy/dx.

41. $xy + 2x + 3y = 1$

42. $x^2 + xy + y^2 - 5x = 2$

43. $x^3 + 4xy - 3y^{4/3} = 2x$

44. $5x^{4/5} + 10y^{6/5} = 15$

45. $\sqrt{xy} = 1$

46. $x^2y^2 = 1$

47. $y^2 = \frac{x}{x + 1}$

48. $y^2 = \sqrt{\frac{1 + x}{1 - x}}$

In Exercises 49 and 50, find dp/dq.

49. $p^3 + 4pq - 3q^2 = 2$

50. $q = (5p^2 + 2p)^{-3/2}$

In Exercises 51 and 52, find dr/ds.

51. $r\cos 2s + \sin^2 s = \pi$

52. $2rs - r - s + s^2 = -3$

53. Find d^2y/dx^2 by implicit differentiation:

a. $x^3 + y^3 = 1$

b. $y^2 = 1 - \frac{2}{x}$

54. **a.** By differentiating $x^2 - y^2 = 1$ implicitly, show that $dy/dx = x/y$.

b. Then show that $d^2y/dx^2 = -1/y^3$.

Numerical Values of Derivatives

55. Suppose that functions $f(x)$ and $g(x)$ and their first derivatives have the following values at $x = 0$ and $x = 1$.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
0	1	1	−3	1/2
1	3	5	1/2	−4

Find the first derivatives of the following combinations at the given value of x.

a. $6f(x) - g(x), \quad x = 1$

b. $f(x)g^2(x), \quad x = 0$

c. $\frac{f(x)}{g(x) + 1}, \quad x = 1$

d. $f(g(x)), \quad x = 0$

e. $g(f(x)), \quad x = 0$

f. $(x + f(x))^{3/2}, \quad x = 1$

g. $f(x + g(x)), \quad x = 0$

56. Suppose that the function $f(x)$ and its first derivative have the following values at $x = 0$ and $x = 1$.

x	$f(x)$	$f'(x)$
0	9	−2
1	−3	1/5

Find the first derivatives of the following combinations at the given value of x.

a. $\sqrt{x}\,f(x), \quad x = 1$

b. $\sqrt{f(x)}, \quad x = 0$

c. $f(\sqrt{x}), \quad x = 1$

d. $f(1 - 5\tan x), \quad x = 0$

e. $\frac{f(x)}{2 + \cos x}, \quad x = 0$

f. $10\sin\left(\frac{\pi x}{2}\right)f^2(x), \quad x = 1$

57. Find the value of dy/dt at $t = 0$ if $y = 3\sin 2x$ and $x = t^2 + \pi$.

58. Find the value of ds/du at $u = 2$ if $s = t^2 + 5t$ and $t = (u^2 + 2u)^{1/3}$.

59. Find the value of dw/ds at $s = 0$ if $w = \sin\left(\sqrt{r} - 2\right)$ and $r = 8\sin(s + \pi/6)$.

60. Find the value of dr/dt at $t = 0$ if $r = (\theta^2 + 7)^{1/3}$ and $\theta^2 t + \theta = 1$.

61. If $y^3 + y = 2\cos x$, find the value of d^2y/dx^2 at the point (0, 1).

62. If $x^{1/3} + y^{1/3} = 4$, find d^2y/dx^2 at the point (8, 8).

Derivative Definition

In Exercises 63 and 64, find the derivative using the definition.

63. $f(t) = \frac{1}{2t + 1}$

64. $g(x) = 2x^2 + 1$

65. **a.** Graph the function

$$f(x) = \begin{cases} x^2, & -1 \le x < 0 \\ -x^2, & 0 \le x \le 1. \end{cases}$$

b. Is f continuous at $x = 0$?

c. Is f differentiable at $x = 0$?

Give reasons for your answers.

66. a. Graph the function

$$f(x) = \begin{cases} x, & -1 \le x < 0 \\ \tan x, & 0 \le x \le \pi/4. \end{cases}$$

b. Is f continuous at $x = 0$?

c. Is f differentiable at $x = 0$?

Give reasons for your answers.

67. a. Graph the function

$$f(x) = \begin{cases} x, & 0 \le x \le 1 \\ 2 - x, & 1 < x \le 2. \end{cases}$$

b. Is f continuous at $x = 1$?

c. Is f differentiable at $x = 1$?

Give reasons for your answers.

68. For what value or values of the constant m, if any, is

$$f(x) = \begin{cases} \sin 2x, & x \le 0 \\ mx, & x > 0 \end{cases}$$

a. continuous at $x = 0$?

b. differentiable at $x = 0$?

Give reasons for your answers.

Slopes, Tangents, and Normals

69. Tangents with specified slope Are there any points on the curve $y = (x/2) + 1/(2x - 4)$ where the slope is $-3/2$? If so, find them.

70. Tangents with specified slope Are there any points on the curve $y = x - 1/(2x)$ where the slope is 3? If so, find them.

71. Horizontal tangents Find the points on the curve $y = 2x^3 - 3x^2 - 12x + 20$ where the tangent is parallel to the x-axis.

72. Tangent intercepts Find the x- and y-intercepts of the line that is tangent to the curve $y = x^3$ at the point $(-2, -8)$.

73. Tangents perpendicular or parallel to lines Find the points on the curve $y = 2x^3 - 3x^2 - 12x + 20$ where the tangent is

a. perpendicular to the line $y = 1 - (x/24)$.

b. parallel to the line $y = \sqrt{2} - 12x$.

74. Intersecting tangents Show that the tangents to the curve $y = (\pi \sin x)/x$ at $x = \pi$ and $x = -\pi$ intersect at right angles.

75. Normals parallel to a line Find the points on the curve $y = \tan x$, $-\pi/2 < x < \pi/2$, where the normal is parallel to the line $y = -x/2$. Sketch the curve and normals together, labeling each with its equation.

76. Tangent and normal lines Find equations for the tangent and normal to the curve $y = 1 + \cos x$ at the point $(\pi/2, 1)$. Sketch the curve, tangent, and normal together, labeling each with its equation.

77. Tangent parabola The parabola $y = x^2 + C$ is to be tangent to the line $y = x$. Find C.

78. Slope of tangent Show that the tangent to the curve $y = x^3$ at any point (a, a^3) meets the curve again at a point where the slope is four times the slope at (a, a^3).

79. Tangent curve For what value of c is the curve $y = c/(x + 1)$ tangent to the line through the points $(0, 3)$ and $(5, -2)$?

80. Normal to a circle Show that the normal line at any point of the circle $x^2 + y^2 = a^2$ passes through the origin.

Tangents and Normals to Implicitly Defined Curves

In Exercises 81–86, find equations for the lines that are tangent and normal to the curve at the given point.

81. $x^2 + 2y^2 = 9, \quad (1, 2)$

82. $x^3 + y^2 = 2, \quad (1, 1)$

83. $xy + 2x - 5y = 2, \quad (3, 2)$

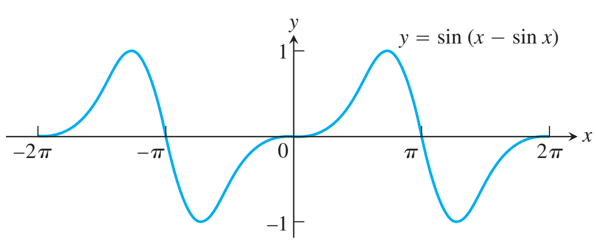
84. $(y - x)^2 = 2x + 4, \quad (6, 2)$

85. $x + \sqrt{xy} = 6, \quad (4, 1)$

86. $x^{3/2} + 2y^{3/2} = 17, \quad (1, 4)$

87. Find the slope of the curve $x^3y^3 + y^2 = x + y$ at the points $(1, 1)$ and $(1, -1)$.

88. The graph shown suggests that the curve $y = \sin(x - \sin x)$ might have horizontal tangents at the x-axis. Does it? Give reasons for your answer.



Tangents to Parametrized Curves

In Exercises 89 and 90, find an equation for the line in the xy-plane that is tangent to the curve at the point corresponding to the given value of t. Also, find the value of d^2y/dx^2 at this point.

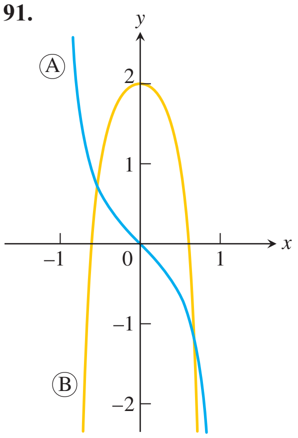
89. $x = (1/2)\tan t, \quad y = (1/2)\sec t, \quad t = \pi/3$

90. $x = 1 + 1/t^2, \quad y = 1 - 3/t, \quad t = 2$

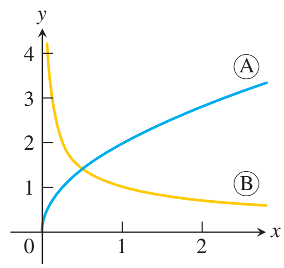
Analyzing Graphs

Each of the figures in Exercises 91 and 92 shows two graphs, the graph of a function $y = f(x)$ together with the graph of its derivative $f'(x)$. Which graph is which? How do you know?

91.



92.

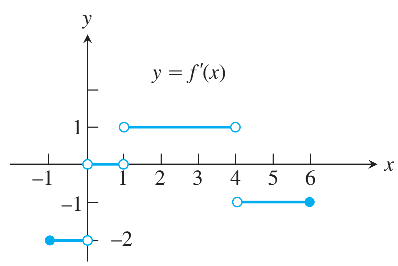


93. Use the following information to graph the function $y = f(x)$ for $-1 \le x \le 6$.

i. The graph of f is made of line segments joined end to end.

ii. The graph starts at the point $(-1, 2)$.

iii. The derivative of f, where defined, agrees with the step function shown here.



94. Repeat Exercise 93, supposing that the graph starts at $(-1, 0)$ instead of $(-1, 2)$.

Exercises 95 and 96 are about the graphs in Figure 3.53 (right-hand column). The graphs in part (a) show the numbers of rabbits and foxes in a small arctic population. They are plotted as functions of time for 200 days. The number of rabbits increases at first, as the rabbits reproduce. But the foxes prey on rabbits and, as the number of foxes increases, the rabbit population levels off and then drops. Figure 3.53b shows the graph of the derivative of the rabbit population. We made it by plotting slopes.

95. a. What is the value of the derivative of the rabbit population in Figure 3.53 when the number of rabbits is largest? Smallest?

b. What is the size of the rabbit population in Figure 3.53 when its derivative is largest? Smallest (negative value)?

96. In what units should the slopes of the rabbit and fox population curves be measured?

Trigonometric Limits

97. $\lim_{x \to 0} \dfrac{\sin x}{2x^2 - x}$

98. $\lim_{x \to 0} \dfrac{3x - \tan 7x}{2x}$

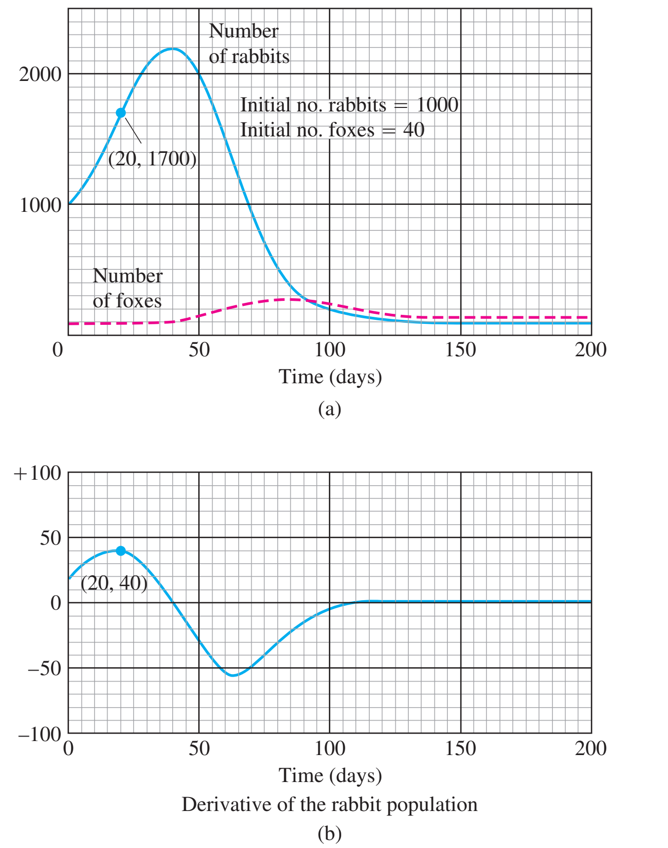


FIGURE 3.53 Rabbits and foxes in an arctic predator-prey food chain.

99. $\lim_{r \to 0} \dfrac{\sin r}{\tan 2r}$

100. $\lim_{\theta \to 0} \dfrac{\sin(\sin\theta)}{\theta}$

101. $\lim_{\theta \to (\pi/2)^-} \dfrac{4\tan^2\theta + \tan\theta + 1}{\tan^2\theta + 5}$

102. $\lim_{\theta \to 0^+} \dfrac{1 - 2\cot^2\theta}{5\cot^2\theta - 7\cot\theta - 8}$

103. $\lim_{x \to 0} \dfrac{x \sin x}{2 - 2\cos x}$

104. $\lim_{\theta \to 0} \dfrac{1 - \cos\theta}{\theta^2}$

Show how to extend the functions in Exercises 105 and 106 to be continuous at the origin.

105. $g(x) = \dfrac{\tan(\tan x)}{\tan x}$

106. $f(x) = \dfrac{\tan(\tan x)}{\sin(\sin x)}$

Related Rates

107. Right circular cylinder The total surface area S of a right circular cylinder is related to the base radius r and height h by the equation $S = 2\pi r^2 + 2\pi r h$.

a. How is dS/dt related to dr/dt if h is constant?

b. How is dS/dt related to dh/dt if r is constant?

c. How is dS/dt related to dr/dt and dh/dt if neither r nor h is constant?

d. How is dr/dt related to dh/dt if S is constant?

108. Right circular cone The lateral surface area S of a right circular cone is related to the base radius r and height h by the equation $S = \pi r\sqrt{r^2 + h^2}$.

a. How is dS/dt related to dr/dt if h is constant?

b. How is dS/dt related to dh/dt if r is constant?

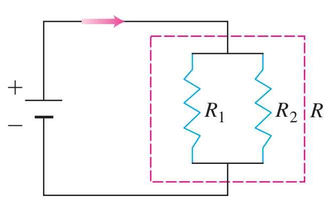
c. How is dS/dt related to dr/dt and dh/dt if neither r nor h is constant?

109. Circle's changing area The radius of a circle is changing at the rate of $-2/\pi$ m/sec. At what rate is the circle's area changing when $r = 10$ m?

110. Cube's changing edges The volume of a cube is increasing at the rate of 1200 cm^3/min at the instant its edges are 20 cm long. At what rate are the lengths of the edges changing at that instant?

111. Resistors connected in parallel If two resistors of R_1 and R_2 ohms are connected in parallel in an electric circuit to make an R-ohm resistor, the value of R can be found from the equation

$$\frac{1}{R} = \frac{1}{R_1} + \frac{1}{R_2}.$$



If R_1 is decreasing at the rate of 1 ohm/sec and R_2 is increasing at the rate of 0.5 ohm/sec, at what rate is R changing when $R_1 = 75$ ohms and $R_2 = 50$ ohms?

112. Impedance in a series circuit The impedance Z (ohms) in a series circuit is related to the resistance R (ohms) and reactance X (ohms) by the equation $Z = \sqrt{R^2 + X^2}$. If R is increasing at 3 ohms/sec and X is decreasing at 2 ohms/sec, at what rate is Z changing when $R = 10$ ohms and $X = 20$ ohms?

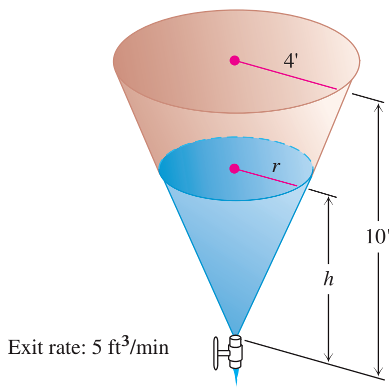
113. Speed of moving particle The coordinates of a particle moving in the metric xy-plane are differentiable functions of time t with $dx/dt = 10$ m/sec and $dy/dt = 5$ m/sec. How fast is the particle moving away from the origin as it passes through the point $(3, -4)$?

114. Motion of a particle A particle moves along the curve $y = x^{3/2}$ in the first quadrant in such a way that its distance from the origin increases at the rate of 11 units per second. Find dx/dt when $x = 3$.

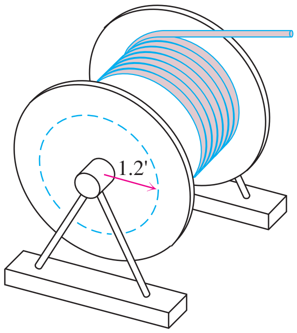
115. Draining a tank Water drains from the conical tank shown in the accompanying figure at the rate of 5 ft^3/min.

a. What is the relation between the variables h and r in the figure?

b. How fast is the water level dropping when $h = 6$ ft?



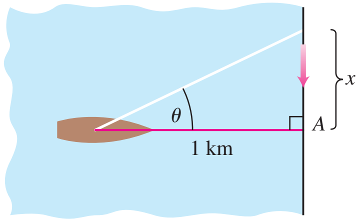
116. Rotating spool As television cable is pulled from a large spool to be strung from the telephone poles along a street, it unwinds from the spool in layers of constant radius (see accompanying figure). If the truck pulling the cable moves at a steady 6 ft/sec (a touch over 4 mph), use the equation $s = r\theta$ to find how fast (radians per second) the spool is turning when the layer of radius 1.2 ft is being unwound.



117. Moving searchlight beam The figure shows a boat 1 km offshore, sweeping the shore with a searchlight. The light turns at a constant rate, $d\theta/dt = -0.6$ rad/sec.

a. How fast is the light moving along the shore when it reaches point A?

b. How many revolutions per minute is 0.6 rad/sec?



118. Points moving on coordinate axes Points A and B move along the x- and y-axes, respectively, in such a way that the distance r (meters) along the perpendicular from the origin to the line AB remains constant. How fast is OA changing, and is it increasing, or decreasing, when $OB = 2r$ and B is moving toward O at the rate of $0.3r$ m/sec?

Linearization

119. Find the linearizations of

a. $\tan x$ at $x = -\pi/4$ **b.** $\sec x$ at $x = -\pi/4$.

Graph the curves and linearizations together.

120. We can obtain a useful linear approximation of the function $f(x) = 1/(1 + \tan x)$ at $x = 0$ by combining the approximations

$$\frac{1}{1+x} \approx 1 - x \quad \text{and} \quad \tan x \approx x$$

to get

$$\frac{1}{1+\tan x} \approx 1 - x.$$

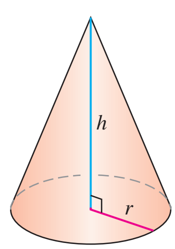
Show that this result is the standard linear approximation of $1/(1 + \tan x)$ at $x = 0$.

121. Find the linearization of $f(x) = \sqrt{1+x} + \sin x - 0.5$ at $x = 0$.

122. Find the linearization of $f(x) = 2/(1 - x) + \sqrt{1+x} - 3.1$ at $x = 0$.

Differential Estimates of Change

123. Surface area of a cone Write a formula that estimates the change that occurs in the lateral surface area of a right circular cone when the height changes from h_0 to $h_0 + dh$ and the radius does not change.



$V = \frac{1}{3}\pi r^2 h$

$S = \pi r\sqrt{r^2 + h^2}$

(Lateral surface area)

124. Controlling error

a. How accurately should you measure the edge of a cube to be reasonably sure of calculating the cube's surface area with an error of no more than 2%?

b. Suppose that the edge is measured with the accuracy required in part (a). About how accurately can the cube's volume be calculated from the edge measurement? To find out, estimate the percentage error in the volume calculation that might result from using the edge measurement.

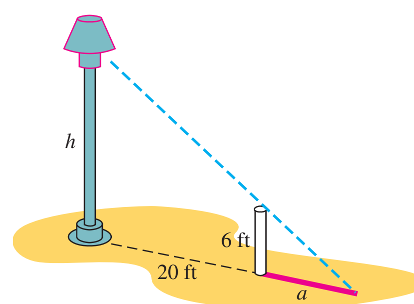
125. Compounding error The circumference of the equator of a sphere is measured as 10 cm with a possible error of 0.4 cm. This measurement is then used to calculate the radius. The radius is then used to calculate the surface area and volume of the sphere. Estimate the percentage errors in the calculated values of

a. the radius.

b. the surface area.

c. the volume.

126. Finding height To find the height of a lamppost (see accompanying figure), you stand a 6 ft pole 20 ft from the lamp and measure the length a of its shadow, finding it to be 15 ft, give or take an inch. Calculate the height of the lamppost using the value $a = 15$ and estimate the possible error in the result.



Chapter 3 Questions to Guide Your Review

1. What is the derivative of a function f? How is its domain related to the domain of f? Give examples.
2. What role does the derivative play in defining slopes, tangents, and rates of change?
3. How can you sometimes graph the derivative of a function when all you have is a table of the function's values?
4. What does it mean for a function to be differentiable on an open interval? On a closed interval?
5. How are derivatives and one-sided derivatives related?
6. Describe geometrically when a function typically does *not* have a derivative at a point.
7. How is a function's differentiability at a point related to its continuity there, if at all?
8. Could the unit step function

$$U(x) = \begin{cases} 0, & x < 0 \\ 1, & x \geq 0 \end{cases}$$

possibly be the derivative of some other function on $[-1, 1]$? Explain.
9. What rules do you know for calculating derivatives? Give some examples.
10. Explain how the three formulas

a. $\frac{d}{dx}(x^n) = nx^{n-1}$

b. $\frac{d}{dx}(cu) = c\frac{du}{dx}$

c. $\frac{d}{dx}(u_1 + u_2 + \cdots + u_n) = \frac{du_1}{dx} + \frac{du_2}{dx} + \cdots + \frac{du_n}{dx}$

enable us to differentiate any polynomial.
11. What formula do we need, in addition to the three listed in Question 10, to differentiate rational functions?
12. What is a second derivative? A third derivative? How many derivatives do the functions you know have? Give examples.
13. What is the relationship between a function's average and instantaneous rates of change? Give an example.
14. How do derivatives arise in the study of motion? What can you learn about a body's motion along a line by examining the derivatives of the body's position function? Give examples.
15. How can derivatives arise in economics?
16. Give examples of still other applications of derivatives.
17. What do the limits $\lim_{h\to 0}((\sin h)/h)$ and $\lim_{h\to 0}((\cos h - 1)/h)$ have to do with the derivatives of the sine and cosine functions? What *are* the derivatives of these functions?
18. Once you know the derivatives of $\sin x$ and $\cos x$, how can you find the derivatives of $\tan x$, $\cot x$, $\sec x$, and $\csc x$? What *are* the derivatives of these functions?
19. At what points are the six basic trigonometric functions continuous? How do you know?
20. What is the rule for calculating the derivative of a composite of two differentiable functions? How is such a derivative evaluated? Give examples.
21. What is the formula for the slope dy/dx of a parametrized curve $x = f(t), y = g(t)$? When does the formula apply? When can you expect to be able to find d^2y/dx^2 as well? Give examples.
22. If u is a differentiable function of x, how do you find $(d/dx)(u^n)$ if n is an integer? If n is a rational number? Give examples.
23. What is implicit differentiation? When do you need it? Give examples.
24. How do related rates problems arise? Give examples.
25. Outline a strategy for solving related rates problems. Illustrate with an example.
26. What is the linearization $L(x)$ of a function $f(x)$ at a point $x = a$? What is required of f at a for the linearization to exist? How are linearizations used? Give examples.
27. If x moves from a to a nearby value $a + dx$, how do you estimate the corresponding change in the value of a differentiable function $f(x)$? How do you estimate the relative change? The percentage change? Give an example.

Chapter 3 Technology Application Projects

Mathematica/Maple Module

Convergence of Secant Slopes to the Derivative Function

You will visualize the secant line between successive points on a curve and observe what happens as the distance between them becomes small. The function, sample points, and secant lines are plotted on a single graph, while a second graph compares the slopes of the secant lines with the derivative function.

Mathematica/Maple Module

Derivatives, Slopes, Tangent Lines, and Making Movies

Parts I–III. You will visualize the derivative at a point, the linearization of a function, and the derivative of a function. You learn how to plot the function and selected tangents on the same graph.

Part IV (Plotting Many Tangents)

Part V (Making Movies). Parts IV and V of the module can be used to animate tangent lines as one moves along the graph of a function.

Mathematica/Maple Module

Convergence of Secant Slopes to the Derivative Function

You will visualize right-hand and left-hand derivatives.

Mathematica/Maple Module

Motion Along a Straight Line: Position → Velocity → Acceleration

Observe dramatic animated visualizations of the derivative relations among the position, velocity, and acceleration functions. Figures in the text can be animated.