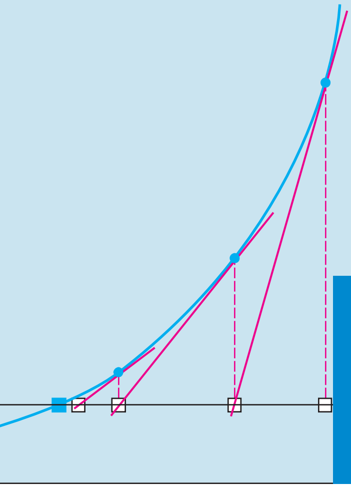


Chapter 4

APPLICATIONS OF DERIVATIVES

OVERVIEW This chapter studies some of the important applications of derivatives. We learn how derivatives are used to find extreme values of functions, to determine and analyze the shapes of graphs, to calculate limits of fractions whose numerators and denominators both approach zero or infinity, and to find numerically where a function equals zero. We also consider the process of recovering a function from its derivative. The key to many of these accomplishments is the Mean Value Theorem, a theorem whose corollaries provide the gateway to integral calculus in Chapter 5.

4.1 Extreme Values of Functions

This section shows how to locate and identify extreme values of a continuous function from its derivative. Once we can do this, we can solve a variety of *optimization problems* in which we find the optimal (best) way to do something in a given situation.

> **DEFINITIONS Absolute Maximum, Absolute Minimum**
>
> Let f be a function with domain D. Then f has an **absolute maximum** value on D at a point c if
>
> $$f(x) \leq f(c) \qquad \text{for all } x \text{ in } D$$
>
> and an **absolute minimum** value on D at c if
>
> $$f(x) \geq f(c) \qquad \text{for all } x \text{ in } D.$$

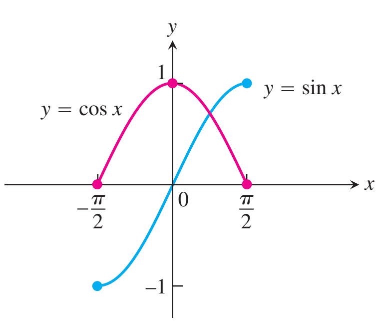


FIGURE 4.1 Absolute extrema for the sine and cosine functions on $[-\pi/2, \pi/2]$. These values can depend on the domain of a function.

Absolute maximum and minimum values are called absolute **extrema** (plural of the Latin *extremum*). Absolute extrema are also called **global** extrema, to distinguish them from *local extrema* defined below.

For example, on the closed interval $[-\pi/2, \pi/2]$ the function $f(x) = \cos x$ takes on an absolute maximum value of 1 (once) and an absolute minimum value of 0 (twice). On the same interval, the function $g(x) = \sin x$ takes on a maximum value of 1 and a minimum value of -1 (Figure 4.1).

Functions with the same defining rule can have different extrema, depending on the domain.

EXAMPLE 1 Exploring Absolute Extrema

The absolute extrema of the following functions on their domains can be seen in Figure 4.2.

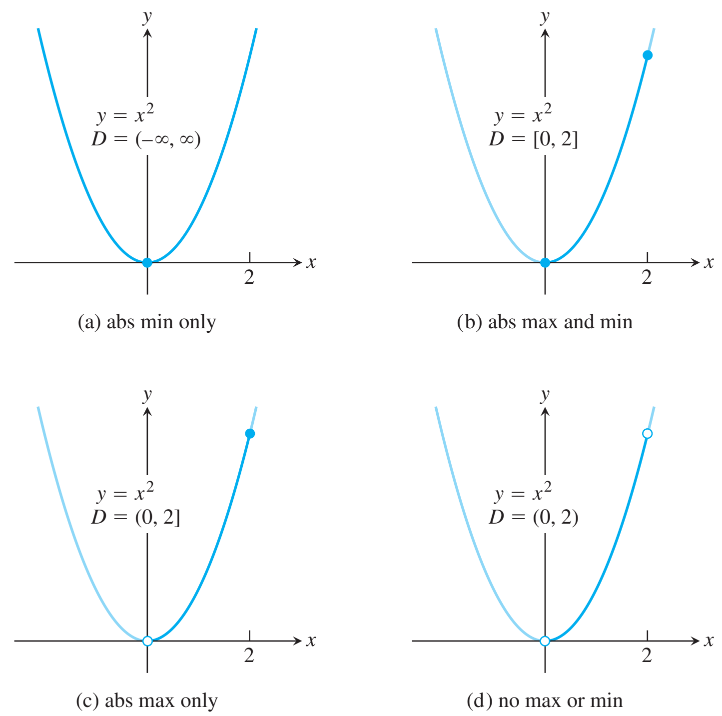


FIGURE 4.2 Graphs for Example 1.

Function rule	**Domain *D***	**Absolute extrema on *D***
(a) $y = x^2$	$(-\infty, \infty)$	No absolute maximum. Absolute minimum of 0 at $x = 0$.
(b) $y = x^2$	$[0, 2]$	Absolute maximum of 4 at $x = 2$. Absolute minimum of 0 at $x = 0$.
(c) $y = x^2$	$(0, 2]$	Absolute maximum of 4 at $x = 2$. No absolute minimum.
(d) $y = x^2$	$(0, 2)$	No absolute extrema.

■

HISTORICAL BIOGRAPHY

Daniel Bernoulli
(1700–1789)

The following theorem asserts that a function which is continuous at every point of a closed interval $[a, b]$ has an absolute maximum and an absolute minimum value on the interval. We always look for these values when we graph a function.

THEOREM 1 The Extreme Value Theorem

If f is continuous on a closed interval $[a, b]$, then f attains both an absolute maximum value M and an absolute minimum value m in $[a, b]$. That is, there are numbers x_1 and x_2 in $[a, b]$ with $f(x_1) = m$, $f(x_2) = M$, and $m \leq f(x) \leq M$ for every other x in $[a, b]$ (Figure 4.3).

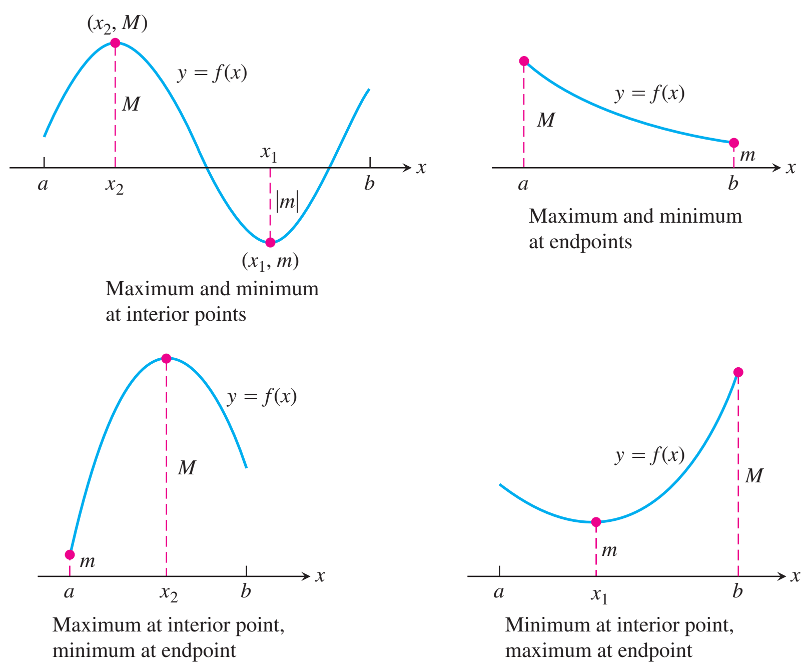


FIGURE 4.3 Some possibilities for a continuous function's maximum and minimum on a closed interval $[a, b]$.

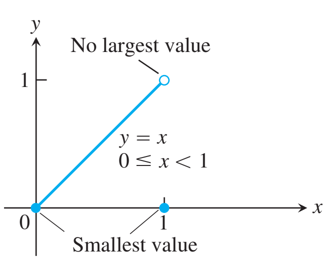


FIGURE 4.4 Even a single point of discontinuity can keep a function from having either a maximum or minimum value on a closed interval. The function

$$y = \begin{cases} x, & 0 \leq x < 1 \\ 0, & x = 1 \end{cases}$$

is continuous at every point of $[0, 1]$ except $x = 1$, yet its graph over $[0, 1]$ does not have a highest point.

The proof of The Extreme Value Theorem requires a detailed knowledge of the real number system (see Appendix 4) and we will not give it here. Figure 4.3 illustrates possible locations for the absolute extrema of a continuous function on a closed interval $[a, b]$. As we observed for the function $y = \cos x$, it is possible that an absolute minimum (or absolute maximum) may occur at two or more different points of the interval.

The requirements in Theorem 1 that the interval be closed and finite, and that the function be continuous, are key ingredients. Without them, the conclusion of the theorem need not hold. Example 1 shows that an absolute extreme value may not exist if the interval fails to be both closed and finite. Figure 4.4 shows that the continuity requirement cannot be omitted.

Local (Relative) Extreme Values

Figure 4.5 shows a graph with five points where a function has extreme values on its domain $[a, b]$. The function's absolute minimum occurs at a even though at e the function's value is

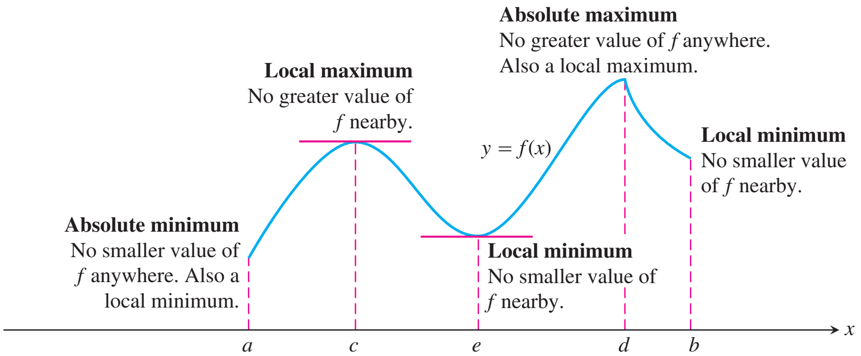


FIGURE 4.5 How to classify maxima and minima.

smaller than at any other point *nearby*. The curve rises to the left and falls to the right around c, making $f(c)$ a maximum locally. The function attains its absolute maximum at d.

DEFINITIONS Local Maximum, Local Minimum

A function f has a **local maximum** value at an interior point c of its domain if

$$f(x) \leq f(c) \qquad \text{for all } x \text{ in some open interval containing } c.$$

A function f has a **local minimum** value at an interior point c of its domain if

$$f(x) \geq f(c) \qquad \text{for all } x \text{ in some open interval containing } c.$$

We can extend the definitions of local extrema to the endpoints of intervals by defining f to have a **local maximum** or **local minimum** value *at an endpoint* c if the appropriate inequality holds for all x in some half-open interval in its domain containing c. In Figure 4.5, the function f has local maxima at c and d and local minima at a, e, and b. Local extrema are also called **relative extrema**.

An absolute maximum is also a local maximum. Being the largest value overall, it is also the largest value in its immediate neighborhood. Hence, *a list of all local maxima will automatically include the absolute maximum if there is one*. Similarly, *a list of all local minima will include the absolute minimum if there is one*.

Finding Extrema

The next theorem explains why we usually need to investigate only a few values to find a function's extrema.

THEOREM 2 The First Derivative Theorem for Local Extreme Values

If f has a local maximum or minimum value at an interior point c of its domain, and if f' is defined at c, then

$$f'(c) = 0.$$

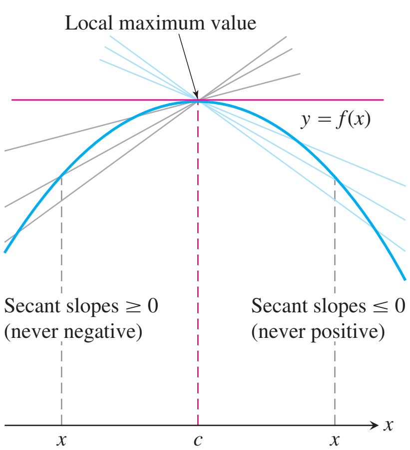


FIGURE 4.6 A curve with a local maximum value. The slope at c, simultaneously the limit of nonpositive numbers and nonnegative numbers, is zero.

Proof To prove that $f'(c)$ is zero at a local extremum, we show first that $f'(c)$ cannot be positive and second that $f'(c)$ cannot be negative. The only number that is neither positive nor negative is zero, so that is what $f'(c)$ must be.

To begin, suppose that f has a local maximum value at $x = c$ (Figure 4.6) so that $f(x) - f(c) \leq 0$ for all values of x near enough to c. Since c is an interior point of f's domain, $f'(c)$ is defined by the two-sided limit

$$\lim_{x \to c} \frac{f(x) - f(c)}{x - c}.$$

This means that the right-hand and left-hand limits both exist at $x = c$ and equal $f'(c)$. When we examine these limits separately, we find that

$$f'(c) = \lim_{x \to c^+} \frac{f(x) - f(c)}{x - c} \leq 0. \qquad \text{Because } (x - c) > 0 \text{ and } f(x) \leq f(c) \qquad (1)$$

Similarly,

$$f'(c) = \lim_{x \to c^-} \frac{f(x) - f(c)}{x - c} \geq 0. \qquad \text{Because } (x - c) < 0 \text{ and } f(x) \leq f(c) \qquad (2)$$

Together, Equations (1) and (2) imply $f'(c) = 0$.

This proves the theorem for local maximum values. To prove it for local minimum values, we simply use $f(x) \geq f(c)$, which reverses the inequalities in Equations (1) and (2). ■

Theorem 2 says that a function's first derivative is always zero at an interior point where the function has a local extreme value and the derivative is defined. Hence the only places where a function f can possibly have an extreme value (local or global) are

1. interior points where $f' = 0$,
2. interior points where f' is undefined,
3. endpoints of the domain of f.

The following definition helps us to summarize.

DEFINITION Critical Point

An interior point of the domain of a function f where f' is zero or undefined is a **critical point** of f.

Thus the only domain points where a function can assume extreme values are critical points and endpoints.

Be careful not to misinterpret Theorem 2 because its converse is false. A differentiable function may have a critical point at $x = c$ without having a local extreme value there. For instance, the function $f(x) = x^3$ has a critical point at the origin and zero value there, but is positive to the right of the origin and negative to the left. So it cannot have a local extreme value at the origin. Instead, it has a *point of inflection* there. This idea is defined and discussed further in Section 4.4.

Most quests for extreme values call for finding the absolute extrema of a continuous function on a closed and finite interval. Theorem 1 assures us that such values exist; Theorem 2 tells us that they are taken on only at critical points and endpoints. Often we can simply list these points and calculate the corresponding function values to find what the largest and smallest values are, and where they are located.

How to Find the Absolute Extrema of a Continuous Function f on a Finite Closed Interval

1. Evaluate f at all critical points and endpoints.
2. Take the largest and smallest of these values.

EXAMPLE 2 Finding Absolute Extrema

Find the absolute maximum and minimum values of $f(x) = x^2$ on $[-2, 1]$.

Solution The function is differentiable over its entire domain, so the only critical point is where $f'(x) = 2x = 0$, namely $x = 0$. We need to check the function's values at $x = 0$ and at the endpoints $x = -2$ and $x = 1$:

Critical point value: $f(0) = 0$

Endpoint values: $f(-2) = 4$

$f(1) = 1$

The function has an absolute maximum value of 4 at $x = -2$ and an absolute minimum value of 0 at $x = 0$. ■

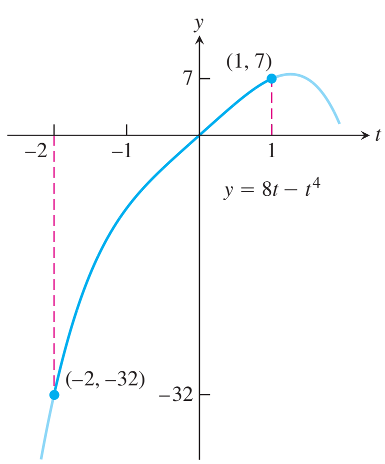


FIGURE 4.7 The extreme values of $g(t) = 8t - t^4$ on $[-2, 1]$ (Example 3).

EXAMPLE 3 Absolute Extrema at Endpoints

Find the absolute extrema values of $g(t) = 8t - t^4$ on $[-2, 1]$.

Solution The function is differentiable on its entire domain, so the only critical points occur where $g'(t) = 0$. Solving this equation gives

$$8 - 4t^3 = 0 \quad \text{or} \quad t = \sqrt[3]{2} > 1,$$

a point not in the given domain. The function's absolute extrema therefore occur at the endpoints, $g(-2) = -32$ (absolute minimum), and $g(1) = 7$ (absolute maximum). See Figure 4.7. ■

EXAMPLE 4 Finding Absolute Extrema on a Closed Interval

Find the absolute maximum and minimum values of $f(x) = x^{2/3}$ on the interval $[-2, 3]$.

Solution We evaluate the function at the critical points and endpoints and take the largest and smallest of the resulting values.

The first derivative

$$f'(x) = \frac{2}{3}x^{-1/3} = \frac{2}{3\sqrt[3]{x}}$$

has no zeros but is undefined at the interior point $x = 0$. The values of f at this one critical point and at the endpoints are

Critical point value: $f(0) = 0$

Endpoint values: $f(-2) = (-2)^{2/3} = \sqrt[3]{4}$

$f(3) = (3)^{2/3} = \sqrt[3]{9}$.

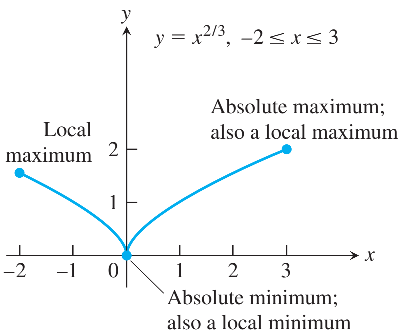


FIGURE 4.8 The extreme values of $f(x) = x^{2/3}$ on $[-2, 3]$ occur at $x = 0$ and $x = 3$ (Example 4).

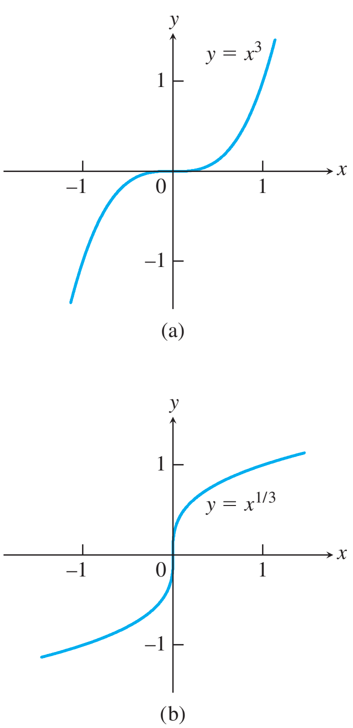


FIGURE 4.9 Critical points without extreme values. (a) $y' = 3x^2$ is 0 at $x = 0$, but $y = x^3$ has no extremum there. (b) $y' = (1/3)x^{-2/3}$ is undefined at $x = 0$, but $y = x^{1/3}$ has no extremum there.

We can see from this list that the function's absolute maximum value is $\sqrt[3]{9} \approx 2.08$, and it occurs at the right endpoint $x = 3$. The absolute minimum value is 0, and it occurs at the interior point $x = 0$. (Figure 4.8). ■

While a function's extrema can occur only at critical points and endpoints, not every critical point or endpoint signals the presence of an extreme value. Figure 4.9 illustrates this for interior points.

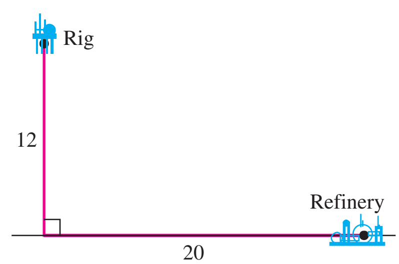
We complete this section with an example illustrating how the concepts we studied are used to solve a real-world optimization problem.

EXAMPLE 5 Piping Oil from a Drilling Rig to a Refinery

A drilling rig 12 mi offshore is to be connected by pipe to a refinery onshore, 20 mi straight down the coast from the rig. If underwater pipe costs \$500,000 per mile and land-based pipe costs \$300,000 per mile, what combination of the two will give the least expensive connection?

Solution We try a few possibilities to get a feel for the problem:

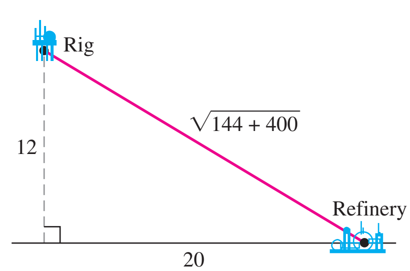
(a) *Smallest amount of underwater pipe*



Underwater pipe is more expensive, so we use as little as we can. We run straight to shore (12 mi) and use land pipe for 20 mi to the refinery.

$$\begin{aligned}\text{Dollar cost} &= 12(500{,}000) + 20(300{,}000)\\ &= 12{,}000{,}000\end{aligned}$$

(b) *All pipe underwater (most direct route)*

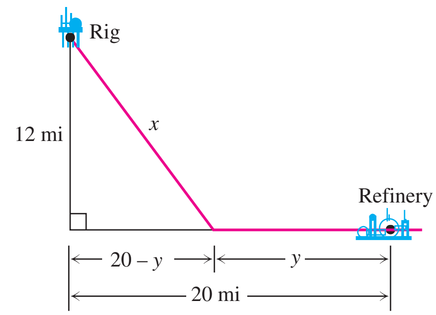


We go straight to the refinery underwater.

$$\begin{aligned}\text{Dollar cost} &= \sqrt{544}\,(500{,}000)\\ &\approx 11{,}661{,}900\end{aligned}$$

This is less expensive than plan (a).

(c) *Something in between*



Now we introduce the length x of underwater pipe and the length y of land-based pipe as variables. The right angle opposite the rig is the key to expressing the relationship between x and y, for the Pythagorean theorem gives

$$x^2 = 12^2 + (20 - y)^2$$
$$x = \sqrt{144 + (20 - y)^2}. \quad (3)$$

Only the positive root has meaning in this model.

The dollar cost of the pipeline is

$$c = 500{,}000x + 300{,}000y.$$

To express c as a function of a single variable, we can substitute for x, using Equation (3):

$$c(y) = 500{,}000\sqrt{144 + (20 - y)^2} + 300{,}000y.$$

Our goal now is to find the minimum value of $c(y)$ on the interval $0 \le y \le 20$. The first derivative of $c(y)$ with respect to y according to the Chain Rule is

$$c'(y) = 500{,}000 \cdot \frac{1}{2} \cdot \frac{2(20 - y)(-1)}{\sqrt{144 + (20 - y)^2}} + 300{,}000$$
$$= -500{,}000 \frac{20 - y}{\sqrt{144 + (20 - y)^2}} + 300{,}000.$$

Setting c' equal to zero gives

$$500{,}000\,(20 - y) = 300{,}000\sqrt{144 + (20 - y)^2}$$
$$\frac{5}{3}\left(20 - y\right) = \sqrt{144 + (20 - y)^2}$$
$$\frac{25}{9}\left(20 - y\right)^2 = 144 + (20 - y)^2$$
$$\frac{16}{9}\left(20 - y\right)^2 = 144$$
$$(20 - y) = \pm\frac{3}{4} \cdot 12 = \pm 9$$
$$y = 20 \pm 9$$
$$y = 11 \quad \text{or} \quad y = 29.$$

Only $y = 11$ lies in the interval of interest. The values of c at this one critical point and at the endpoints are

$$\begin{aligned} c(11) &= 10{,}800{,}000 \\ c(0) &= 11{,}661{,}900 \\ c(20) &= 12{,}000{,}000 \end{aligned}$$

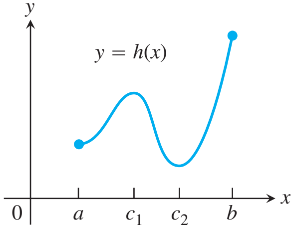
The least expensive connection costs $10,800,000, and we achieve it by running the line underwater to the point on shore 11 mi from the refinery. ■

EXERCISES 4.1

Finding Extrema from Graphs

In Exercises 1–6, determine from the graph whether the function has any absolute extreme values on $[a, b]$. Then explain how your answer is consistent with Theorem 1.

1.



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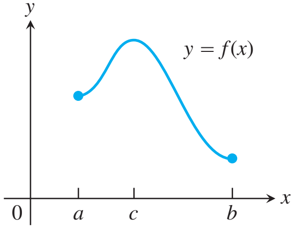

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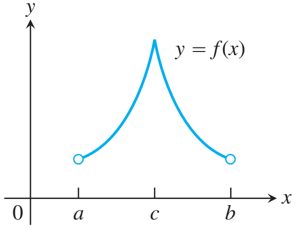

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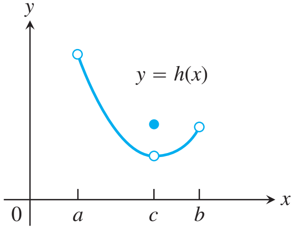

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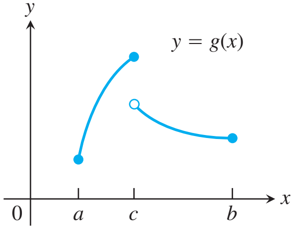
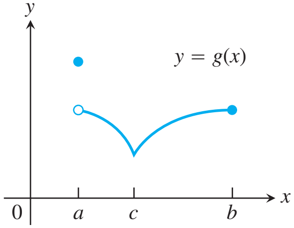
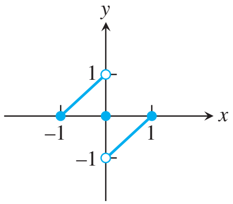


6.



In Exercises 7–10, find the extreme values and where they occur.

7.



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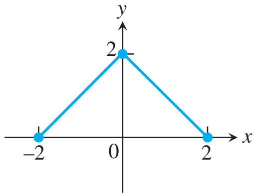

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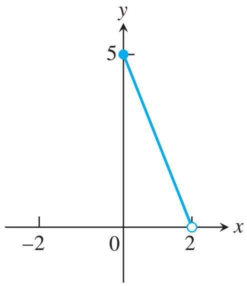
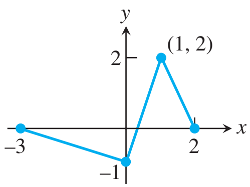


10.



In Exercises 11-14, match the table with a graph.

11.

x	$f'(x)$
a	0
b	0
c	5

12.

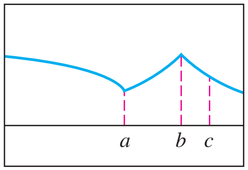
x	$f'(x)$
a	0
b	0
c	-5

13.

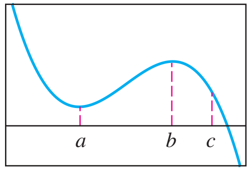
x	$f'(x)$
a	does not exist
b	0
c	-2

14.

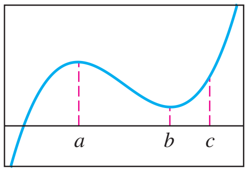
x	$f'(x)$
a	does not exist
b	does not exist
c	-1.7



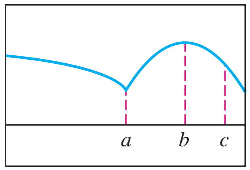
(a)



(b)



(c)



(d)

Absolute Extrema on Finite Closed Intervals

In Exercises 15–30, find the absolute maximum and minimum values of each function on the given interval. Then graph the function. Identify the points on the graph where the absolute extrema occur, and include their coordinates.

15. $f(x) = \frac{2}{3}x - 5, \quad -2 \le x \le 3$

16. $f(x) = -x - 4, \quad -4 \le x \le 1$

17. $f(x) = x^2 - 1, \quad -1 \le x \le 2$

18. $f(x) = 4 - x^2, \quad -3 \le x \le 1$

19. $F(x) = -\frac{1}{x^2}, \quad 0.5 \le x \le 2$

20. $F(x) = -\frac{1}{x}, \quad -2 \le x \le -1$

21. $h(x) = \sqrt[3]{x}, \quad -1 \le x \le 8$

22. $h(x) = -3x^{2/3}, \quad -1 \le x \le 1$

23. $g(x) = \sqrt{4 - x^2}, \quad -2 \le x \le 1$

24. $g(x) = -\sqrt{5 - x^2}, \quad -\sqrt{5} \le x \le 0$

25. $f(\theta) = \sin\theta, \quad -\frac{\pi}{2} \le \theta \le \frac{5\pi}{6}$

26. $f(\theta) = \tan\theta, \quad -\frac{\pi}{3} \le \theta \le \frac{\pi}{4}$

27. $g(x) = \csc x, \quad \frac{\pi}{3} \le x \le \frac{2\pi}{3}$

28. $g(x) = \sec x, \quad -\frac{\pi}{3} \le x \le \frac{\pi}{6}$

29. $f(t) = 2 - |t|, \quad -1 \le t \le 3$

30. $f(t) = |t - 5|, \quad 4 \le t \le 7$

In Exercises 31–34, find the function's absolute maximum and minimum values and say where they are assumed.

31. $f(x) = x^{4/3}, \quad -1 \le x \le 8$

32. $f(x) = x^{5/3}, \quad -1 \le x \le 8$

33. $g(\theta) = \theta^{3/5}, \quad -32 \le \theta \le 1$

34. $h(\theta) = 3\theta^{2/3}, \quad -27 \le \theta \le 8$

Finding Extreme Values

In Exercises 35–44, find the extreme values of the function and where they occur.

35. $y = 2x^2 - 8x + 9$

36. $y = x^3 - 2x + 4$

37. $y = x^3 + x^2 - 8x + 5$

38. $y = x^3 - 3x^2 + 3x - 2$

39. $y = \sqrt{x^2 - 1}$

40. $y = \frac{1}{\sqrt{1 - x^2}}$

41. $y = \frac{1}{\sqrt[3]{1 - x^2}}$

42. $y = \sqrt{3 + 2x - x^2}$

43. $y = \frac{x}{x^2 + 1}$

44. $y = \frac{x + 1}{x^2 + 2x + 2}$

Local Extrema and Critical Points

In Exercises 45–52, find the derivative at each critical point and determine the local extreme values.

45. $y = x^{2/3}(x + 2)$

46. $y = x^{2/3}(x^2 - 4)$

47. $y = x\sqrt{4 - x^2}$

48. $y = x^2\sqrt{3 - x}$

49. $y = \begin{cases} 4 - 2x, & x \le 1 \\ x + 1, & x > 1 \end{cases}$

50. $y = \begin{cases} 3 - x, & x < 0 \\ 3 + 2x - x^2, & x \ge 0 \end{cases}$

51. $y = \begin{cases} -x^2 - 2x + 4, & x \le 1 \\ -x^2 + 6x - 4, & x > 1 \end{cases}$

52. $y = \begin{cases} -\frac{1}{4}x^2 - \frac{1}{2}x + \frac{15}{4}, & x \le 1 \\ x^3 - 6x^2 + 8x, & x > 1 \end{cases}$

In Exercises 53 and 54, give reasons for your answers.

53. Let $f(x) = (x - 2)^{2/3}$.

a. Does $f'(2)$ exist?

b. Show that the only local extreme value of f occurs at $x = 2$.

c. Does the result in part (b) contradict the Extreme Value Theorem?

d. Repeat parts (a) and (b) for $f(x) = (x - a)^{2/3}$, replacing 2 by a.

54. Let $f(x) = |x^3 - 9x|$.

a. Does $f'(0)$ exist?

b. Does $f'(3)$ exist?

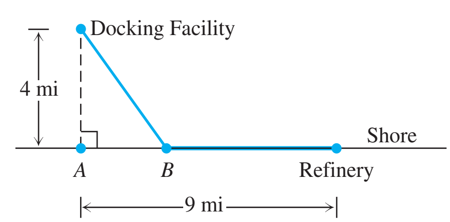
c. Does $f'(-3)$ exist?

d. Determine all extrema of f.

Optimization Applications

Whenever you are maximizing or minimizing a function of a single variable, we urge you to graph the function over the domain that is appropriate to the problem you are solving. The graph will provide insight before you begin to calculate and will furnish a visual context for understanding your answer.

55. Constructing a pipeline Supertankers off-load oil at a docking facility 4 mi offshore. The nearest refinery is 9 mi east of the shore point nearest the docking facility. A pipeline must be constructed connecting the docking facility with the refinery. The pipeline costs \$300,000 per mile if constructed underwater and \$200,000 per mile if overland.

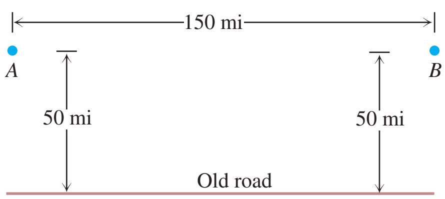


a. Locate Point B to minimize the cost of the construction.

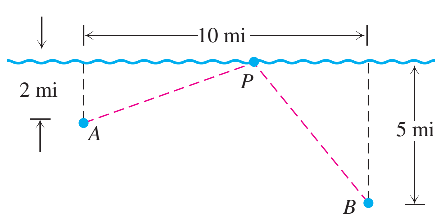
Exercises

b. The cost of underwater construction is expected to increase, whereas the cost of overland construction is expected to stay constant. At what cost does it become optimal to construct the pipeline directly to Point A?

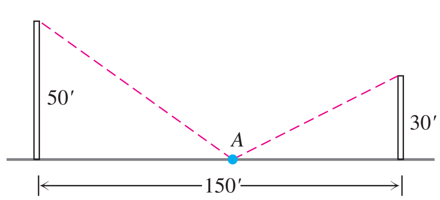
56. Upgrading a highway A highway must be constructed to connect Village A with Village B. There is a rudimentary roadway that can be upgraded 50 mi south of the line connecting the two villages. The cost of upgrading the existing roadway is \$300,000 per mile, whereas the cost of constructing a new highway is \$500,000 per mile. Find the combination of upgrading and new construction that minimizes the cost of connecting the two villages. Clearly define the location of the proposed highway.



57. Locating a pumping station Two towns lie on the south side of a river. A pumping station is to be located to serve the two towns. A pipeline will be constructed from the pumping station to each of the towns along the line connecting the town and the pumping station. Locate the pumping station to minimize the amount of pipeline that must be constructed.



58. Length of a guy wire One tower is 50 ft high and another tower is 30 ft high. The towers are 150 ft apart. A guy wire is to run from Point A to the top of each tower.



a. Locate Point A so that the total length of guy wire is minimal.

b. Show in general that regardless of the height of the towers, the length of guy wire is minimized if the angles at A are equal.

59. The function

$$V(x) = x(10 - 2x)(16 - 2x), \qquad 0 < x < 5,$$

models the volume of a box.

a. Find the extreme values of V.

b. Interpret any values found in part (a) in terms of volume of the box.

60. The function

$$P(x) = 2x + \frac{200}{x}, \qquad 0 < x < \infty,$$

models the perimeter of a rectangle of dimensions x by $100/x$.

a. Find any extreme values of P.

b. Give an interpretation in terms of perimeter of the rectangle for any values found in part (a).

61. Area of a right triangle What is the largest possible area for a right triangle whose hypotenuse is 5 cm long?

62. Area of an athletic field An athletic field is to be built in the shape of a rectangle x units long capped by semicircular regions of radius r at the two ends. The field is to be bounded by a 400-m racetrack.

a. Express the area of the rectangular portion of the field as a function of x alone or r alone (your choice).

b. What values of x and r give the rectangular portion the largest possible area?

63. Maximum height of a vertically moving body The height of a body moving vertically is given by

$$s = -\frac{1}{2}gt^2 + v_0 t + s_0, \qquad g > 0,$$

with s in meters and t in seconds. Find the body's maximum height.

64. Peak alternating current Suppose that at any given time t (in seconds) the current i (in amperes) in an alternating current circuit is $i = 2\cos t + 2\sin t$. What is the peak current for this circuit (largest magnitude)?

Theory and Examples

65. A minimum with no derivative The function $f(x) = |x|$ has an absolute minimum value at $x = 0$ even though f is not differentiable at $x = 0$. Is this consistent with Theorem 2? Give reasons for your answer.

66. Even functions If an even function $f(x)$ has a local maximum value at $x = c$, can anything be said about the value of f at $x = -c$? Give reasons for your answer.

67. Odd functions If an odd function $g(x)$ has a local minimum value at $x = c$, can anything be said about the value of g at $x = -c$? Give reasons for your answer.

68. We know how to find the extreme values of a continuous function $f(x)$ by investigating its values at critical points and endpoints. But what if there *are* no critical points or endpoints? What happens then? Do such functions really exist? Give reasons for your answers.

69. Cubic functions Consider the cubic function

$$f(x) = ax^3 + bx^2 + cx + d.$$

a. Show that f can have 0, 1, or 2 critical points. Give examples and graphs to support your argument.

b. How many local extreme values can f have?

T **70. Functions with no extreme values at endpoints**

a. Graph the function

$$f(x) = \begin{cases} \sin \frac{1}{x}, & x > 0 \\ 0, & x = 0. \end{cases}$$

Explain why $f(0) = 0$ is not a local extreme value of f.

b. Construct a function of your own that fails to have an extreme value at a domain endpoint.

T Graph the functions in Exercises 71–74. Then find the extreme values of the function on the interval and say where they occur.

71. $f(x) = |x - 2| + |x + 3|, \quad -5 \le x \le 5$

72. $g(x) = |x - 1| - |x - 5|, \quad -2 \le x \le 7$

73. $h(x) = |x + 2| - |x - 3|, \quad -\infty < x < \infty$

74. $k(x) = |x + 1| + |x - 3|, \quad -\infty < x < \infty$

COMPUTER EXPLORATIONS

In Exercises 75–80, you will use a CAS to help find the absolute extrema of the given function over the specified closed interval. Perform the following steps.

a. Plot the function over the interval to see its general behavior there.

b. Find the interior points where $f' = 0$. (In some exercises, you may have to use the numerical equation solver to approximate a solution.) You may want to plot f' as well.

c. Find the interior points where f' does not exist.

d. Evaluate the function at all points found in parts (b) and (c) and at the endpoints of the interval.

e. Find the function's absolute extreme values on the interval and identify where they occur.

75. $f(x) = x^4 - 8x^2 + 4x + 2, \quad [-20/25, 64/25]$

76. $f(x) = -x^4 + 4x^3 - 4x + 1, \quad [-3/4, 3]$

77. $f(x) = x^{2/3}(3 - x), \quad [-2, 2]$

78. $f(x) = 2 + 2x - 3x^{2/3}, \quad [-1, 10/3]$

79. $f(x) = \sqrt{x} + \cos x, \quad [0, 2\pi]$

80. $f(x) = x^{3/4} - \sin x + \frac{1}{2}, \quad [0, 2\pi]$

4.2 The Mean Value Theorem

We know that constant functions have zero derivatives, but could there be a complicated function, with many terms, the derivatives of which all cancel to give zero? What is the relationship between two functions that have identical derivatives over an interval? What we are really asking here is what functions can have a particular *kind* of derivative. These and many other questions we study in this chapter are answered by applying the Mean Value Theorem. To arrive at this theorem we first need Rolle's Theorem.

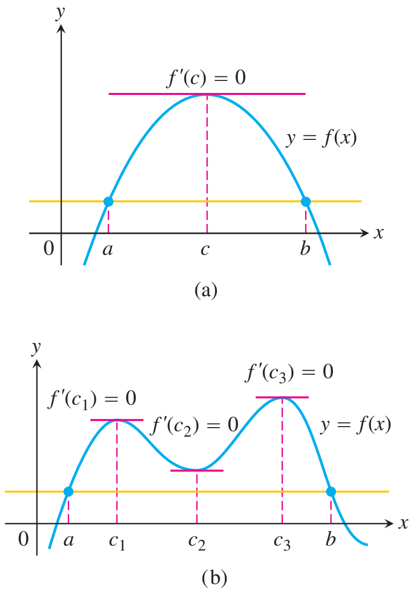


FIGURE 4.10 Rolle's Theorem says that a differentiable curve has at least one horizontal tangent between any two points where it crosses a horizontal line. It may have just one (a), or it may have more (b).

Rolle's Theorem

Drawing the graph of a function gives strong geometric evidence that between any two points where a differentiable function crosses a horizontal line there is at least one point on the curve where the tangent is horizontal (Figure 4.10). More precisely, we have the following theorem.

THEOREM 3 Rolle's Theorem
Suppose that $y = f(x)$ is continuous at every point of the closed interval $[a, b]$ and differentiable at every point of its interior (a, b). If

$$f(a) = f(b),$$

then there is at least one number c in (a, b) at which

$$f'(c) = 0.$$

Proof Being continuous, f assumes absolute maximum and minimum values on $[a, b]$. These can occur only

HISTORICAL BIOGRAPHY

Michel Rolle
(1652–1719)

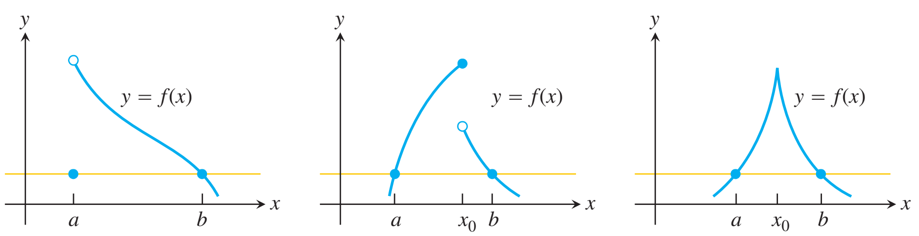
1. at interior points where f' is zero,
2. at interior points where f' does not exist,
3. at the endpoints of the function's domain, in this case a and b.

By hypothesis, f has a derivative at every interior point. That rules out possibility (2), leaving us with interior points where $f' = 0$ and with the two endpoints a and b.

If either the maximum or the minimum occurs at a point c between a and b, then $f'(c) = 0$ by Theorem 2 in Section 4.1, and we have found a point for Rolle's theorem.

If both the absolute maximum and the absolute minimum occur at the endpoints, then because $f(a) = f(b)$ it must be the case that f is a constant function with $f(x) = f(a) = f(b)$ for every $x \in [a, b]$. Therefore $f'(x) = 0$ and the point c can be taken anywhere in the interior (a, b). ∎

The hypotheses of Theorem 3 are essential. If they fail at even one point, the graph may not have a horizontal tangent (Figure 4.11).



(a) Discontinuous at an endpoint of $[a, b]$

(b) Discontinuous at an interior point of $[a, b]$

(c) Continuous on $[a, b]$ but not differentiable at an interior point

FIGURE 4.11 There may be no horizontal tangent if the hypotheses of Rolle's Theorem do not hold.

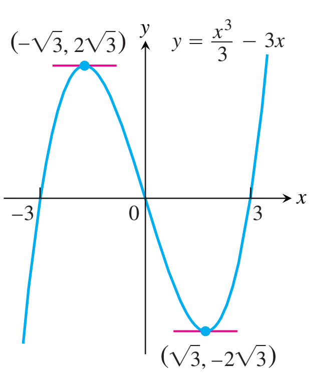


FIGURE 4.12 As predicted by Rolle's Theorem, this curve has horizontal tangents between the points where it crosses the x-axis (Example 1).

EXAMPLE 1 Horizontal Tangents of a Cubic Polynomial

The polynomial function

$$f(x) = \frac{x^3}{3} - 3x$$

graphed in Figure 4.12 is continuous at every point of $[-3, 3]$ and is differentiable at every point of $(-3, 3)$. Since $f(-3) = f(3) = 0$, Rolle's Theorem says that f' must be zero at least once in the open interval between $a = -3$ and $b = 3$. In fact, $f'(x) = x^2 - 3$ is zero twice in this interval, once at $x = -\sqrt{3}$ and again at $x = \sqrt{3}$. ∎

EXAMPLE 2 Solution of an Equation $f(x) = 0$

Show that the equation

$$x^3 + 3x + 1 = 0$$

has exactly one real solution.

Solution Let

$$y = f(x) = x^3 + 3x + 1.$$

Then the derivative

$$f'(x) = 3x^2 + 3$$

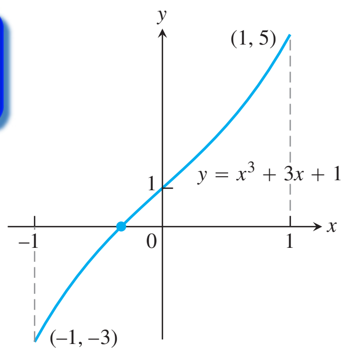


FIGURE 4.13 The only real zero of the polynomial $y = x^3 + 3x + 1$ is the one shown here where the curve crosses the x-axis between -1 and 0 (Example 2).

is never zero (because it is always positive). Now, if there were even two points $x = a$ and $x = b$ where $f(x)$ was zero, Rolle's Theorem would guarantee the existence of a point $x = c$ in between them where f' was zero. Therefore, f has no more than one zero. It does in fact have one zero, because the Intermediate Value Theorem tells us that the graph of $y = f(x)$ crosses the x-axis somewhere between $x = -1$ (where $y = -3$) and $x = 0$ (where $y = 1$). (See Figure 4.13.) ■

Our main use of Rolle's Theorem is in proving the Mean Value Theorem.

The Mean Value Theorem

The Mean Value Theorem, which was first stated by Joseph-Louis Lagrange, is a slanted version of Rolle's Theorem (Figure 4.14). There is a point where the tangent is parallel to chord AB.

THEOREM 4 The Mean Value Theorem

Suppose $y = f(x)$ is continuous on a closed interval $[a, b]$ and differentiable on the interval's interior (a, b). Then there is at least one point c in (a, b) at which

$$\frac{f(b) - f(a)}{b - a} = f'(c). \tag{1}$$

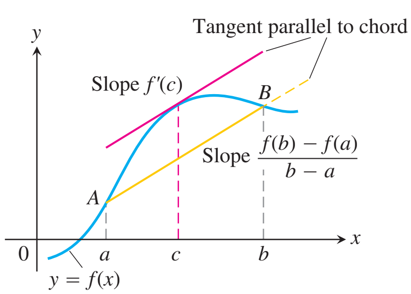


FIGURE 4.14 Geometrically, the Mean Value Theorem says that somewhere between A and B the curve has at least one tangent parallel to chord AB.

Proof We picture the graph of f as a curve in the plane and draw a line through the points $A(a, f(a))$ and $B(b, f(b))$ (see Figure 4.15). The line is the graph of the function

$$g(x) = f(a) + \frac{f(b) - f(a)}{b - a}(x - a) \tag{2}$$

(point-slope equation). The vertical difference between the graphs of f and g at x is

$$\begin{aligned} h(x) &= f(x) - g(x) \\ &= f(x) - f(a) - \frac{f(b) - f(a)}{b - a}(x - a). \end{aligned} \tag{3}$$

Figure 4.16 shows the graphs of f, g, and h together.

The function h satisfies the hypotheses of Rolle's Theorem on $[a, b]$. It is continuous on $[a, b]$ and differentiable on (a, b) because both f and g are. Also, $h(a) = h(b) = 0$ because the graphs of f and g both pass through A and B. Therefore $h'(c) = 0$ at some point $c \in (a, b)$. This is the point we want for Equation (1).

To verify Equation (1), we differentiate both sides of Equation (3) with respect to x and then set $x = c$:

$$h'(x) = f'(x) - \frac{f(b) - f(a)}{b - a} \qquad \text{Derivative of Eq. (3)} \ldots$$

$$h'(c) = f'(c) - \frac{f(b) - f(a)}{b - a} \qquad \ldots \text{with } x = c$$

$$0 = f'(c) - \frac{f(b) - f(a)}{b - a} \qquad h'(c) = 0$$

$$f'(c) = \frac{f(b) - f(a)}{b - a}, \qquad \text{Rearranged}$$

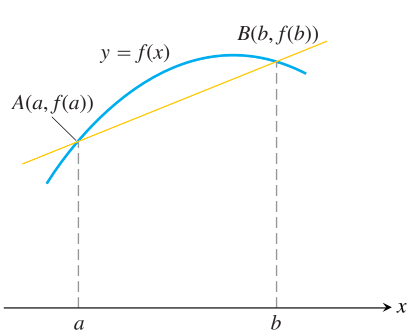


FIGURE 4.15 The graph of f and the chord AB over the interval $[a, b]$.

which is what we set out to prove. ■

Historical Biography

Joseph-Louis Lagrange
(1736–1813)

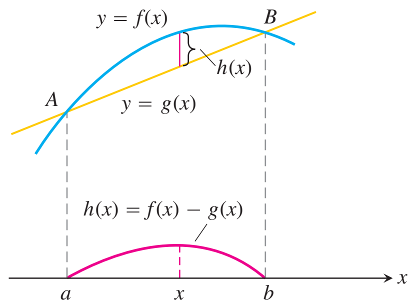


FIGURE 4.16 The chord AB is the graph of the function $g(x)$. The function $h(x) = f(x) - g(x)$ gives the vertical distance between the graphs of f and g at x.

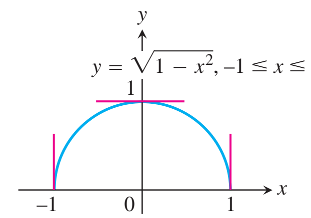


FIGURE 4.17 The function $f(x) = \sqrt{1 - x^2}$ satisfies the hypotheses (and conclusion) of the Mean Value Theorem on $[-1, 1]$ even though f is not differentiable at -1 and 1.

The hypotheses of the Mean Value Theorem do not require f to be differentiable at either a or b. Continuity at a and b is enough (Figure 4.17).

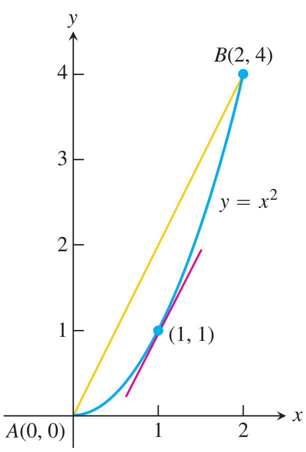


FIGURE 4.18 As we find in Example 3, $c = 1$ is where the tangent is parallel to the chord.

You Try It

EXAMPLE 3 The function $f(x) = x^2$ (Figure 4.18) is continuous for $0 \le x \le 2$ and differentiable for $0 < x < 2$. Since $f(0) = 0$ and $f(2) = 4$, the Mean Value Theorem says that at some point c in the interval, the derivative $f'(x) = 2x$ must have the value $(4 - 0)/(2 - 0) = 2$. In this (exceptional) case we can identify c by solving the equation $2c = 2$ to get $c = 1$. ■

A Physical Interpretation

If we think of the number $(f(b) - f(a))/(b - a)$ as the average change in f over $[a, b]$ and $f'(c)$ as an instantaneous change, then the Mean Value Theorem says that at some interior point the instantaneous change must equal the average change over the entire interval.

EXAMPLE 4 If a car accelerating from zero takes 8 sec to go 352 ft, its average velocity for the 8-sec interval is $352/8 = 44$ ft/sec. At some point during the acceleration, the Mean Value Theorem says, the speedometer must read exactly 30 mph (44 ft/sec) (Figure 4.19). ■

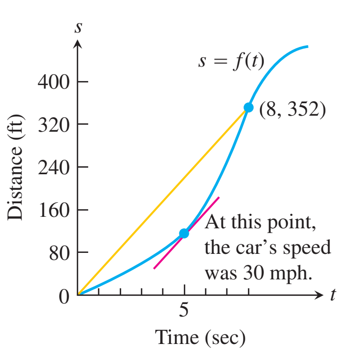


FIGURE 4.19 Distance versus elapsed time for the car in Example 4.

Mathematical Consequences

At the beginning of the section, we asked what kind of function has a zero derivative over an interval. The first corollary of the Mean Value Theorem provides the answer.

COROLLARY 1 Functions with Zero Derivatives Are Constant
If $f'(x) = 0$ at each point x of an open interval (a, b), then $f(x) = C$ for all $x \in (a, b)$, where C is a constant.

Proof We want to show that f has a constant value on the interval (a, b). We do so by showing that if x_1 and x_2 are any two points in (a, b), then $f(x_1) = f(x_2)$. Numbering x_1 and x_2 from left to right, we have $x_1 < x_2$. Then f satisfies the hypotheses of the Mean Value Theorem on $[x_1, x_2]$: It is differentiable at every point of $[x_1, x_2]$ and hence continuous at every point as well. Therefore,

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = f'(c)$$

at some point c between x_1 and x_2. Since $f' = 0$ throughout (a, b), this equation translates successively into

$$\frac{f(x_2) - f(x_1)}{x_2 - x_1} = 0, \qquad f(x_2) - f(x_1) = 0, \qquad \text{and} \qquad f(x_1) = f(x_2). \qquad \blacksquare$$

At the beginning of this section, we also asked about the relationship between two functions that have identical derivatives over an interval. The next corollary tells us that their values on the interval have a constant difference.

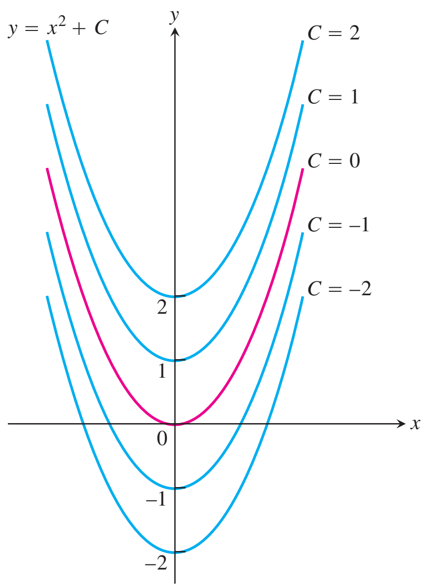


FIGURE 4.20 From a geometric point of view, Corollary 2 of the Mean Value Theorem says that the graphs of functions with identical derivatives on an interval can differ only by a vertical shift there. The graphs of the functions with derivative $2x$ are the parabolas $y = x^2 + C$, shown here for selected values of C.

COROLLARY 2 Functions with the Same Derivative Differ by a Constant
If $f'(x) = g'(x)$ at each point x in an open interval (a, b), then there exists a constant C such that $f(x) = g(x) + C$ for all $x \in (a, b)$. That is, $f - g$ is a constant on (a, b).

Proof At each point $x \in (a, b)$ the derivative of the difference function $h = f - g$ is

$$h'(x) = f'(x) - g'(x) = 0.$$

Thus, $h(x) = C$ on (a, b) by Corollary 1. That is, $f(x) - g(x) = C$ on (a, b), so $f(x) = g(x) + C$. $\blacksquare$

Corollaries 1 and 2 are also true if the open interval (a, b) fails to be finite. That is, they remain true if the interval is (a, ∞), $(-\infty, b)$, or $(-\infty, \infty)$.

Corollary 2 plays an important role when we discuss antiderivatives in Section 4.8. It tells us, for instance, that since the derivative of $f(x) = x^2$ on $(-\infty, \infty)$ is $2x$, any other function with derivative $2x$ on $(-\infty, \infty)$ must have the formula $x^2 + C$ for some value of C (Figure 4.20).

EXAMPLE 5 Find the function $f(x)$ whose derivative is $\sin x$ and whose graph passes through the point $(0, 2)$.

Solution Since $f(x)$ has the same derivative as $g(x) = -\cos x$, we know that $f(x) = -\cos x + C$ for some constant C. The value of C can be determined from the condition that $f(0) = 2$ (the graph of f passes through $(0, 2)$):

$$f(0) = -\cos(0) + C = 2, \qquad \text{so} \qquad C = 3.$$

The function is $f(x) = -\cos x + 3$. $\blacksquare$

Finding Velocity and Position from Acceleration

Here is how to find the velocity and displacement functions of a body falling freely from rest with acceleration 9.8 m/sec^2.

We know that $v(t)$ is some function whose derivative is 9.8. We also know that the derivative of $g(t) = 9.8t$ is 9.8. By Corollary 2,

$$v(t) = 9.8t + C$$

for some constant C. Since the body falls from rest, $v(0) = 0$. Thus

$$9.8(0) + C = 0, \quad \text{and} \quad C = 0.$$

The velocity function must be $v(t) = 9.8t$. How about the position function $s(t)$?

We know that $s(t)$ is some function whose derivative is $9.8t$. We also know that the derivative of $f(t) = 4.9t^2$ is $9.8t$. By Corollary 2,

$$s(t) = 4.9t^2 + C$$

for some constant C. If the initial height is $s(0) = h$, measured positive downward from the rest position, then

$$4.9(0)^2 + C = h, \quad \text{and} \quad C = h.$$

The position function must be $s(t) = 4.9t^2 + h$.

The ability to find functions from their rates of change is one of the very powerful tools of calculus. As we will see, it lies at the heart of the mathematical developments in Chapter 5.

EXERCISES 4.2

Finding *c* in the Mean Value Theorem

Find the value or values of c that satisfy the equation

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

in the conclusion of the Mean Value Theorem for the functions and intervals in Exercises 1–4.

1. $f(x) = x^2 + 2x - 1, \quad [0, 1]$

2. $f(x) = x^{2/3}, \quad [0, 1]$

3. $f(x) = x + \frac{1}{x}, \quad \left[\frac{1}{2}, 2\right]$

4. $f(x) = \sqrt{x - 1}, \quad [1, 3]$

Checking and Using Hypotheses

Which of the functions in Exercises 5–8 satisfy the hypotheses of the Mean Value Theorem on the given interval, and which do not? Give reasons for your answers.

5. $f(x) = x^{2/3}, \quad [-1, 8]$

6. $f(x) = x^{4/5}, \quad [0, 1]$

7. $f(x) = \sqrt{x(1 - x)}, \quad [0, 1]$

8. $f(x) = \begin{cases} \dfrac{\sin x}{x}, & -\pi \le x < 0 \\ 0, & x = 0 \end{cases}$

9. The function

$$f(x) = \begin{cases} x, & 0 \le x < 1 \\ 0, & x = 1 \end{cases}$$

is zero at $x = 0$ and $x = 1$ and differentiable on $(0, 1)$, but its derivative on $(0, 1)$ is never zero. How can this be? Doesn't Rolle's Theorem say the derivative has to be zero somewhere in $(0, 1)$? Give reasons for your answer.

10. For what values of a, m and b does the function

$$f(x) = \begin{cases} 3, & x = 0 \\ -x^2 + 3x + a, & 0 < x < 1 \\ mx + b, & 1 \le x \le 2 \end{cases}$$

satisfy the hypotheses of the Mean Value Theorem on the interval $[0, 2]$?

Roots (Zeros)

11. a. Plot the zeros of each polynomial on a line together with the zeros of its first derivative.

i) $y = x^2 - 4$

ii) $y = x^2 + 8x + 15$

iii) $y = x^3 - 3x^2 + 4 = (x + 1)(x - 2)^2$

iv) $y = x^3 - 33x^2 + 216x = x(x - 9)(x - 24)$

b. Use Rolle's Theorem to prove that between every two zeros of $x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ there lies a zero of
$$nx^{n-1} + (n-1)a_{n-1}x^{n-2} + \cdots + a_1.$$

12. Suppose that f'' is continuous on $[a, b]$ and that f has three zeros in the interval. Show that f'' has at least one zero in (a, b). Generalize this result.

13. Show that if $f'' > 0$ throughout an interval $[a, b]$, then f' has at most one zero in $[a, b]$. What if $f'' < 0$ throughout $[a, b]$ instead?

14. Show that a cubic polynomial can have at most three real zeros.

Show that the functions in Exercises 15–22 have exactly one zero in the given interval.

15. $f(x) = x^4 + 3x + 1, \quad [-2, -1]$

16. $f(x) = x^3 + \dfrac{4}{x^2} + 7, \quad (-\infty, 0)$

17. $g(t) = \sqrt{t} + \sqrt{1+t} - 4, \quad (0, \infty)$

18. $g(t) = \dfrac{1}{1-t} + \sqrt{1+t} - 3.1, \quad (-1, 1)$

19. $r(\theta) = \theta + \sin^2\left(\dfrac{\theta}{3}\right) - 8, \quad (-\infty, \infty)$

20. $r(\theta) = 2\theta - \cos^2\theta + \sqrt{2}, \quad (-\infty, \infty)$

21. $r(\theta) = \sec\theta - \dfrac{1}{\theta^3} + 5, \quad (0, \pi/2)$

22. $r(\theta) = \tan\theta - \cot\theta - \theta, \quad (0, \pi/2)$

Finding Functions from Derivatives

23. Suppose that $f(-1) = 3$ and that $f'(x) = 0$ for all x. Must $f(x) = 3$ for all x? Give reasons for your answer.

24. Suppose that $f(0) = 5$ and that $f'(x) = 2$ for all x. Must $f(x) = 2x + 5$ for all x. Give reasons for your answer.

25. Suppose that $f'(x) = 2x$ for all x. Find $f(2)$ if

a. $f(0) = 0$ **b.** $f(1) = 0$ **c.** $f(-2) = 3$.

26. What can be said about functions whose derivatives are constant? Give reasons for your answer.

In Exercises 27–32, find all possible functions with the given derivative.

27. **a.** $y' = x$ **b.** $y' = x^2$ **c.** $y' = x^3$

28. **a.** $y' = 2x$ **b.** $y' = 2x - 1$ **c.** $y' = 3x^2 + 2x - 1$

29. **a.** $y' = -\dfrac{1}{x^2}$ **b.** $y' = 1 - \dfrac{1}{x^2}$ **c.** $y' = 5 + \dfrac{1}{x^2}$

30. **a.** $y' = \dfrac{1}{2\sqrt{x}}$ **b.** $y' = \dfrac{1}{\sqrt{x}}$ **c.** $y' = 4x - \dfrac{1}{\sqrt{x}}$

31. **a.** $y' = \sin 2t$ **b.** $y' = \cos\dfrac{t}{2}$ **c.** $y' = \sin 2t + \cos\dfrac{t}{2}$

32. **a.** $y' = \sec^2\theta$ **b.** $y' = \sqrt{\theta}$ **c.** $y' = \sqrt{\theta} - \sec^2\theta$

In Exercises 33–36, find the function with the given derivative whose graph passes through the point P.

33. $f'(x) = 2x - 1, \quad P(0, 0)$

34. $g'(x) = \dfrac{1}{x^2} + 2x, \quad P(-1, 1)$

35. $r'(\theta) = 8 - \csc^2\theta, \quad P\left(\dfrac{\pi}{4}, 0\right)$

36. $r'(t) = \sec t \tan t - 1, \quad P(0, 0)$

Finding Position from Velocity

Exercises 37–40 give the velocity $v = ds/dt$ and initial position of a body moving along a coordinate line. Find the body's position at time t.

37. $v = 9.8t + 5, \quad s(0) = 10$

38. $v = 32t - 2, \quad s(0.5) = 4$

39. $v = \sin\pi t, \quad s(0) = 0$

40. $v = \dfrac{2}{\pi}\cos\dfrac{2t}{\pi}, \quad s(\pi^2) = 1$

Finding Position from Acceleration

Exercises 41–44 give the acceleration $a = d^2s/dt^2$, initial velocity, and initial position of a body moving on a coordinate line. Find the body's position at time t.

41. $a = 32, \quad v(0) = 20, \quad s(0) = 5$

42. $a = 9.8, \quad v(0) = -3, \quad s(0) = 0$

43. $a = -4\sin 2t, \quad v(0) = 2, \quad s(0) = -3$

44. $a = \dfrac{9}{\pi^2}\cos\dfrac{3t}{\pi}, \quad v(0) = 0, \quad s(0) = -1$

Applications

45. Temperature change It took 14 sec for a mercury thermometer to rise from −19°C to 100°C when it was taken from a freezer and placed in boiling water. Show that somewhere along the way the mercury was rising at the rate of 8.5°C/sec.

46. A trucker handed in a ticket at a toll booth showing that in 2 hours she had covered 159 mi on a toll road with speed limit 65 mph. The trucker was cited for speeding. Why?

47. Classical accounts tell us that a 170-oar trireme (ancient Greek or Roman warship) once covered 184 sea miles in 24 hours. Explain why at some point during this feat the trireme's speed exceeded 7.5 knots (sea miles per hour).

48. A marathoner ran the 26.2-mi New York City Marathon in 2.2 hours. Show that at least twice the marathoner was running at exactly 11 mph.

49. Show that at some instant during a 2-hour automobile trip the car's speedometer reading will equal the average speed for the trip.

50. Free fall on the moon On our moon, the acceleration of gravity is 1.6 m/sec^2. If a rock is dropped into a crevasse, how fast will it be going just before it hits bottom 30 sec later?

Theory and Examples

51. The geometric mean of *a* and *b* The *geometric mean* of two positive numbers a and b is the number $\sqrt{ab}$. Show that the value of c in the conclusion of the Mean Value Theorem for $f(x) = 1/x$ on an interval of positive numbers $[a, b]$ is $c = \sqrt{ab}$.

52. The arithmetic mean of *a* and *b* The *arithmetic mean* of two numbers a and b is the number $(a + b)/2$. Show that the value of c in the conclusion of the Mean Value Theorem for $f(x) = x^2$ on any interval $[a, b]$ is $c = (a + b)/2$.

T **53.** Graph the function

$$f(x) = \sin x \sin(x + 2) - \sin^2(x + 1).$$

What does the graph do? Why does the function behave this way? Give reasons for your answers.

54. Rolle's Theorem

a. Construct a polynomial $f(x)$ that has zeros at $x = -2, -1, 0, 1$, and 2.

b. Graph f and its derivative f' together. How is what you see related to Rolle's Theorem?

c. Do $g(x) = \sin x$ and its derivative g' illustrate the same phenomenon?

55. Unique solution Assume that f is continuous on $[a, b]$ and differentiable on (a, b). Also assume that $f(a)$ and $f(b)$ have opposite signs and that $f' \neq 0$ between a and b. Show that $f(x) = 0$ exactly once between a and b.

56. Parallel tangents Assume that f and g are differentiable on $[a, b]$ and that $f(a) = g(a)$ and $f(b) = g(b)$. Show that there is at least one point between a and b where the tangents to the graphs of f and g are parallel or the same line. Illustrate with a sketch.

57. If the graphs of two differentiable functions $f(x)$ and $g(x)$ start at the same point in the plane and the functions have the same rate of change at every point, do the graphs have to be identical? Give reasons for your answer.

58. Show that for any numbers a and b, the inequality $|\sin b - \sin a| \leq |b - a|$ is true.

59. Assume that f is differentiable on $a \leq x \leq b$ and that $f(b) < f(a)$. Show that f' is negative at some point between a and b.

60. Let f be a function defined on an interval $[a, b]$. What conditions could you place on f to guarantee that

$$\min f' \leq \frac{f(b) - f(a)}{b - a} \leq \max f',$$

where $\min f'$ and $\max f'$ refer to the minimum and maximum values of f' on $[a, b]$? Give reasons for your answers.

T **61.** Use the inequalities in Exercise 60 to estimate $f(0.1)$ if $f'(x) = 1/(1 + x^4 \cos x)$ for $0 \leq x \leq 0.1$ and $f(0) = 1$.

T **62.** Use the inequalities in Exercise 60 to estimate $f(0.1)$ if $f'(x) = 1/(1 - x^4)$ for $0 \leq x \leq 0.1$ and $f(0) = 2$.

63. Let f be differentiable at every value of x and suppose that $f(1) = 1$, that $f' < 0$ on $(-\infty, 1)$, and that $f' > 0$ on $(1, \infty)$.

a. Show that $f(x) \geq 1$ for all x.

b. Must $f'(1) = 0$? Explain.

64. Let $f(x) = px^2 + qx + r$ be a quadratic function defined on a closed interval $[a, b]$. Show that there is exactly one point c in (a, b) at which f satisfies the conclusion of the Mean Value Theorem.

4.3 Monotonic Functions and The First Derivative Test

In sketching the graph of a differentiable function it is useful to know where it increases (rises from left to right) and where it decreases (falls from left to right) over an interval. This section defines precisely what it means for a function to be increasing or decreasing over an interval, and gives a test to determine where it increases and where it decreases. We also show how to test the critical points of a function for the presence of local extreme values.

Increasing Functions and Decreasing Functions

What kinds of functions have positive derivatives or negative derivatives? The answer, provided by the Mean Value Theorem's third corollary, is this: The only functions with positive derivatives are increasing functions; the only functions with negative derivatives are decreasing functions.

DEFINITIONS Increasing, Decreasing Function

Let f be a function defined on an interval I and let x_1 and x_2 be any two points in I.

1. If $f(x_1) < f(x_2)$ whenever $x_1 < x_2$, then f is said to be **increasing** on I.
2. If $f(x_2) < f(x_1)$ whenever $x_1 < x_2$, then f is said to be **decreasing** on I.

A function that is increasing or decreasing on I is called **monotonic** on I.

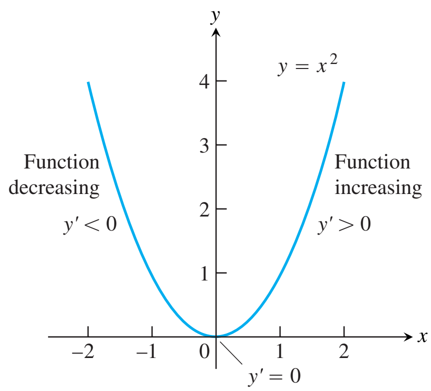


FIGURE 4.21 The function $f(x) = x^2$ is monotonic on the intervals $(-\infty, 0]$ and $[0, \infty)$, but it is not monotonic on $(-\infty, \infty)$.

It is important to realize that the definitions of increasing and decreasing functions must be satisfied for *every* pair of points x_1 and x_2 in I with $x_1 < x_2$. Because of the inequality $<$ comparing the function values, and not $\leq$, some books say that f is *strictly* increasing or decreasing on I. The interval I may be finite or infinite.

The function $f(x) = x^2$ decreases on $(-\infty, 0]$ and increases on $[0, \infty)$ as can be seen from its graph (Figure 4.21). The function f is monotonic on $(-\infty, 0]$ and $[0, \infty)$, but it is not monotonic on $(-\infty, \infty)$. Notice that on the interval $(-\infty, 0)$ the tangents have negative slopes, so the first derivative is always negative there; for $(0, \infty)$ the tangents have positive slopes and the first derivative is positive. The following result confirms these observations.

COROLLARY 3 First Derivative Test for Monotonic Functions

Suppose that f is continuous on $[a, b]$ and differentiable on (a, b).

If $f'(x) > 0$ at each point $x \in (a, b)$, then f is increasing on $[a, b]$.
If $f'(x) < 0$ at each point $x \in (a, b)$, then f is decreasing on $[a, b]$.

Proof Let x_1 and x_2 be any two points in $[a, b]$ with $x_1 < x_2$. The Mean Value Theorem applied to f on $[x_1, x_2]$ says that

$$f(x_2) - f(x_1) = f'(c)(x_2 - x_1)$$

for some c between x_1 and x_2. The sign of the right-hand side of this equation is the same as the sign of $f'(c)$ because $x_2 - x_1$ is positive. Therefore, $f(x_2) > f(x_1)$ if f' is positive on (a, b) and $f(x_2) < f(x_1)$ if f' is negative on (a, b). ■

Here is how to apply the First Derivative Test to find where a function is increasing and decreasing. If $a < b$ are two critical points for a function f, and if f' exists but is not zero on the interval (a, b), then f' must be positive on (a, b) or negative there (Theorem 2, Section 3.1). One way we can determine the sign of f' on the interval is simply by evaluating f' for some point x in (a, b). Then we apply Corollary 3.

EXAMPLE 1 Using the First Derivative Test for Monotonic Functions

Find the critical points of $f(x) = x^3 - 12x - 5$ and identify the intervals on which f is increasing and decreasing.

Solution The function f is everywhere continuous and differentiable. The first derivative

$$\begin{aligned} f'(x) &= 3x^2 - 12 = 3(x^2 - 4) \\ &= 3(x + 2)(x - 2) \end{aligned}$$

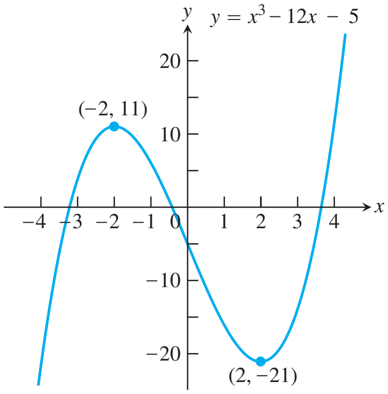


FIGURE 4.22 The function $f(x) = x^3 - 12x - 5$ is monotonic on three separate intervals (Example 1).

is zero at $x = -2$ and $x = 2$. These critical points subdivide the domain of f into intervals $(-\infty, -2)$, $(-2, 2)$, and $(2, \infty)$ on which f' is either positive or negative. We determine the sign of f' by evaluating f at a convenient point in each subinterval. The behavior of f is determined by then applying Corollary 3 to each subinterval. The results are summarized in the following table, and the graph of f is given in Figure 4.22.

Intervals	$-\infty < x < -2$	$-2 < x < 2$	$2 < x < \infty$
f' Evaluated	$f'(-3) = 15$	$f'(0) = -12$	$f'(3) = 15$
Sign of f'	$+$	$-$	$+$
Behavior of f	increasing	decreasing	increasing

Corollary 3 is valid for infinite as well as finite intervals, and we used that fact in our analysis in Example 1.

Knowing where a function increases and decreases also tells us how to test for the nature of local extreme values.

HISTORICAL BIOGRAPHY

Edmund Halley
(1656–1742)

First Derivative Test for Local Extrema

In Figure 4.23, at the points where f has a minimum value, $f' < 0$ immediately to the left and $f' > 0$ immediately to the right. (If the point is an endpoint, there is only one side to consider.) Thus, the function is decreasing on the left of the minimum value and it is increasing on its right. Similarly, at the points where f has a maximum value, $f' > 0$ immediately to the left and $f' < 0$ immediately to the right. Thus, the function is increasing on the left of the maximum value and decreasing on its right. In summary, at a local extreme point, the sign of $f'(x)$ changes.

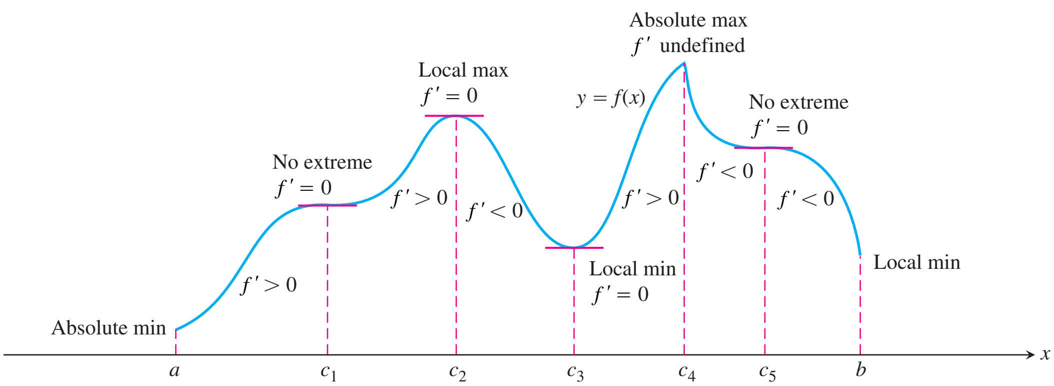


FIGURE 4.23 A function's first derivative tells how the graph rises and falls.

These observations lead to a test for the presence and nature of local extreme values of differentiable functions.

First Derivative Test for Local Extrema

Suppose that c is a critical point of a continuous function f, and that f is differentiable at every point in some interval containing c except possibly at c itself. Moving across c from left to right,

1. if f' changes from negative to positive at c, then f has a local minimum at c;
2. if f' changes from positive to negative at c, then f has a local maximum at c;
3. if f' does not change sign at c (that is, f' is positive on both sides of c or negative on both sides), then f has no local extremum at c.

The test for local extrema at endpoints is similar, but there is only one side to consider.

Proof Part (1). Since the sign of f' changes from negative to positive at c, these are numbers a and b such that $f' < 0$ on (a, c) and $f' > 0$ on (c, b). If $x \in (a, c)$, then $f(c) < f(x)$ because $f' < 0$ implies that f is decreasing on $[a, c]$. If $x \in (c, b)$, then $f(c) < f(x)$ because $f' > 0$ implies that f is increasing on $[c, b]$. Therefore, $f(x) \geq f(c)$ for every $x \in (a, b)$. By definition, f has a local minimum at c.

Parts (2) and (3) are proved similarly. ■

EXAMPLE 2 Using the First Derivative Test for Local Extrema

Find the critical points of

$$f(x) = x^{1/3}(x - 4) = x^{4/3} - 4x^{1/3}.$$

Identify the intervals on which f is increasing and decreasing. Find the function's local and absolute extreme values.

Solution The function f is continuous at all x since it is the product of two continuous functions, $x^{1/3}$ and $(x - 4)$. The first derivative

$$f'(x) = \frac{d}{dx}\left(x^{4/3} - 4x^{1/3}\right) = \frac{4}{3}x^{1/3} - \frac{4}{3}x^{-2/3}$$

$$= \frac{4}{3}x^{-2/3}\left(x - 1\right) = \frac{4(x - 1)}{3x^{2/3}}$$

is zero at $x = 1$ and undefined at $x = 0$. There are no endpoints in the domain, so the critical points $x = 0$ and $x = 1$ are the only places where f might have an extreme value.

The critical points partition the x-axis into intervals on which f' is either positive or negative. The sign pattern of f' reveals the behavior of f between and at the critical points. We can display the information in a table like the following:

Intervals	$x < 0$	$0 < x < 1$	$x > 1$
Sign of f'	$-$	$-$	$+$
Behavior of f	decreasing	decreasing	increasing

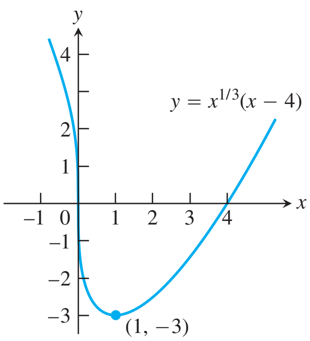


FIGURE 4.24 The function $f(x) = x^{1/3}(x - 4)$ decreases when $x < 1$ and increases when $x > 1$ (Example 2).

Corollary 3 to the Mean Value Theorem tells us that f decreases on $(-\infty, 0)$, decreases on (0, 1), and increases on $(1, \infty)$. The First Derivative Test for Local Extrema tells us that f does not have an extreme value at $x = 0$ (f' does not change sign) and that f has a local minimum at $x = 1$ (f' changes from negative to positive).

The value of the local minimum is $f(1) = 1^{1/3}(1 - 4) = -3$. This is also an absolute minimum because the function's values fall toward it from the left and rise away from it on the right. Figure 4.24 shows this value in relation to the function's graph.

Note that $\lim_{x \to 0} f'(x) = -\infty$, so the graph of f has a vertical tangent at the origin. ■

EXERCISES 4.3

Analyzing f Given f'

Answer the following questions about the functions whose derivatives are given in Exercises 1–8:

a. What are the critical points of f?

b. On what intervals is f increasing or decreasing?

c. At what points, if any, does f assume local maximum and minimum values?

1. $f'(x) = x(x - 1)$

2. $f'(x) = (x - 1)(x + 2)$

3. $f'(x) = (x - 1)^2(x + 2)$

4. $f'(x) = (x - 1)^2(x + 2)^2$

5. $f'(x) = (x - 1)(x + 2)(x - 3)$

6. $f'(x) = (x - 7)(x + 1)(x + 5)$

7. $f'(x) = x^{-1/3}(x + 2)$

8. $f'(x) = x^{-1/2}(x - 3)$

Extremes of Given Functions

In Exercises 9–28:

a. Find the intervals on which the function is increasing and decreasing.

b. Then identify the function's local extreme values, if any, saying where they are taken on.

c. Which, if any, of the extreme values are absolute?

T **d.** Support your findings with a graphing calculator or computer grapher.

9. $g(t) = -t^2 - 3t + 3$

10. $g(t) = -3t^2 + 9t + 5$

11. $h(x) = -x^3 + 2x^2$

12. $h(x) = 2x^3 - 18x$

13. $f(\theta) = 3\theta^2 - 4\theta^3$

14. $f(\theta) = 6\theta - \theta^3$

15. $f(r) = 3r^3 + 16r$

16. $h(r) = (r + 7)^3$

17. $f(x) = x^4 - 8x^2 + 16$

18. $g(x) = x^4 - 4x^3 + 4x^2$

19. $H(t) = \frac{3}{2}t^4 - t^6$

20. $K(t) = 15t^3 - t^5$

21. $g(x) = x\sqrt{8 - x^2}$

22. $g(x) = x^2\sqrt{5 - x}$

23. $f(x) = \dfrac{x^2 - 3}{x - 2}, \quad x \neq 2$

24. $f(x) = \dfrac{x^3}{3x^2 + 1}$

25. $f(x) = x^{1/3}(x + 8)$

26. $g(x) = x^{2/3}(x + 5)$

27. $h(x) = x^{1/3}(x^2 - 4)$

28. $k(x) = x^{2/3}(x^2 - 4)$

Extreme Values on Half-Open Intervals

In Exercises 29–36:

a. Identify the function's local extreme values in the given domain, and say where they are assumed.

b. Which of the extreme values, if any, are absolute?

T **c.** Support your findings with a graphing calculator or computer grapher.

29. $f(x) = 2x - x^2, \quad -\infty < x \leq 2$

30. $f(x) = (x + 1)^2, \quad -\infty < x \leq 0$

31. $g(x) = x^2 - 4x + 4, \quad 1 \leq x < \infty$

32. $g(x) = -x^2 - 6x - 9, \quad -4 \leq x < \infty$

33. $f(t) = 12t - t^3, \quad -3 \leq t < \infty$

34. $f(t) = t^3 - 3t^2, \quad -\infty < t \leq 3$

35. $h(x) = \dfrac{x^3}{3} - 2x^2 + 4x, \quad 0 \leq x < \infty$

36. $k(x) = x^3 + 3x^2 + 3x + 1, \quad -\infty < x \leq 0$

Graphing Calculator or Computer Grapher

In Exercises 37–40:

a. Find the local extrema of each function on the given interval, and say where they are assumed.

T **b.** Graph the function and its derivative together. Comment on the behavior of f in relation to the signs and values of f'.

37. $f(x) = \frac{x}{2} - 2\sin\frac{x}{2}, \quad 0 \le x \le 2\pi$

38. $f(x) = -2\cos x - \cos^2 x, \quad -\pi \le x \le \pi$

39. $f(x) = \csc^2 x - 2\cot x, \quad 0 < x < \pi$

40. $f(x) = \sec^2 x - 2\tan x, \quad \frac{-\pi}{2} < x < \frac{\pi}{2}$

Theory and Examples

Show that the functions in Exercises 41 and 42 have local extreme values at the given values of θ, and say which kind of local extreme the function has.

41. $h(\theta) = 3\cos\frac{\theta}{2}, \quad 0 \le \theta \le 2\pi, \quad$ at $\theta = 0$ and $\theta = 2\pi$

42. $h(\theta) = 5\sin\frac{\theta}{2}, \quad 0 \le \theta \le \pi, \quad$ at $\theta = 0$ and $\theta = \pi$

43. Sketch the graph of a differentiable function $y = f(x)$ through the point $(1, 1)$ if $f'(1) = 0$ and

a. $f'(x) > 0$ for $x < 1$ and $f'(x) < 0$ for $x > 1$;

b. $f'(x) < 0$ for $x < 1$ and $f'(x) > 0$ for $x > 1$;

c. $f'(x) > 0$ for $x \ne 1$;

d. $f'(x) < 0$ for $x \ne 1$.

44. Sketch the graph of a differentiable function $y = f(x)$ that has

a. a local minimum at (1, 1) and a local maximum at (3, 3);

b. a local maximum at (1, 1) and a local minimum at (3, 3);

c. local maxima at (1, 1) and (3, 3);

d. local minima at (1, 1) and (3, 3).

45. Sketch the graph of a continuous function $y = g(x)$ such that

a. $g(2) = 2, \quad 0 < g' < 1$ for $x < 2, \quad g'(x) \to 1^-$ as $x \to 2^-, \quad -1 < g' < 0$ for $x > 2$, and $g'(x) \to -1^+$ as $x \to 2^+$;

b. $g(2) = 2, \quad g' < 0$ for $x < 2, \quad g'(x) \to -\infty$ as $x \to 2^-$, $g' > 0$ for $x > 2$, and $g'(x) \to \infty$ as $x \to 2^+$.

46. Sketch the graph of a continuous function $y = h(x)$ such that

a. $h(0) = 0$, $-2 \le h(x) \le 2$ for all x, $h'(x) \to \infty$ as $x \to 0^-$, and $h'(x) \to \infty$ as $x \to 0^+$;

b. $h(0) = 0$, $-2 \le h(x) \le 0$ for all x, $h'(x) \to \infty$ as $x \to 0^-$, and $h'(x) \to -\infty$ as $x \to 0^+$.

47. As x moves from left to right through the point $c = 2$, is the graph of $f(x) = x^3 - 3x + 2$ rising, or is it falling? Give reasons for your answer.

48. Find the intervals on which the function $f(x) = ax^2 + bx + c$, $a \ne 0$, is increasing and decreasing. Describe the reasoning behind your answer.

4.4 Concavity and Curve Sketching

In Section 4.3 we saw how the first derivative tells us where a function is increasing and where it is decreasing. At a critical point of a differentiable function, the First Derivative Test tells us whether there is a local maximum or a local minimum, or whether the graph just continues to rise or fall there.

In this section we see how the second derivative gives information about the way the graph of a differentiable function bends or turns. This additional information enables us to capture key aspects of the behavior of a function and its graph, and then present these features in a sketch of the graph.

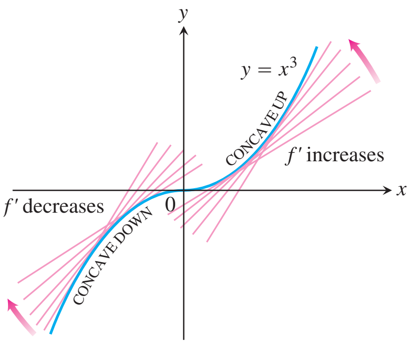


FIGURE 4.25 The graph of $f(x) = x^3$ is concave down on $(-\infty, 0)$ and concave up on $(0, \infty)$ (Example 1a).

Concavity

As you can see in Figure 4.25, the curve $y = x^3$ rises as x increases, but the portions defined on the intervals $(-\infty, 0)$ and $(0, \infty)$ turn in different ways. As we approach the origin from the left along the curve, the curve turns to our right and falls below its tangents. The slopes of the tangents are decreasing on the interval $(-\infty, 0)$. As we move away from the origin along the curve to the right, the curve turns to our left and rises above its tangents. The slopes of the tangents are increasing on the interval $(0, \infty)$. This turning or bending behavior defines the *concavity* of the curve.

DEFINITION Concave Up, Concave Down

The graph of a differentiable function $y = f(x)$ is

(a) concave up on an open interval I if f' is increasing on I

(b) concave down on an open interval I if f' is decreasing on I.

If $y = f(x)$ has a second derivative, we can apply Corollary 3 of the Mean Value Theorem to conclude that f' increases if $f'' > 0$ on I, and decreases if $f'' < 0$.

The Second Derivative Test for Concavity

Let $y = f(x)$ be twice-differentiable on an interval I.

1. If $f'' > 0$ on I, the graph of f over I is concave up.
2. If $f'' < 0$ on I, the graph of f over I is concave down.

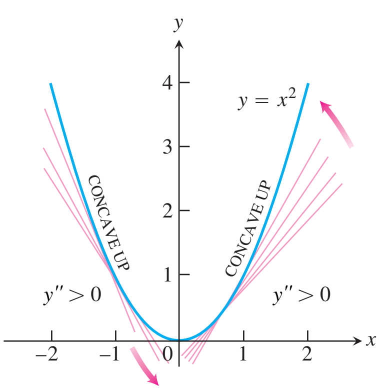


FIGURE 4.26 The graph of $f(x) = x^2$ is concave up on every interval (Example 1b).

If $y = f(x)$ is twice-differentiable, we will use the notations f'' and y'' interchangeably when denoting the second derivative.

EXAMPLE 1 Applying the Concavity Test

(a) The curve $y = x^3$ (Figure 4.25) is concave down on $(-\infty, 0)$ where $y'' = 6x < 0$ and concave up on $(0, \infty)$ where $y'' = 6x > 0$.

(b) The curve $y = x^2$ (Figure 4.26) is concave up on $(-\infty, \infty)$ because its second derivative $y'' = 2$ is always positive. ∎

EXAMPLE 2 Determining Concavity

Determine the concavity of $y = 3 + \sin x$ on $[0, 2\pi]$.

Solution The graph of $y = 3 + \sin x$ is concave down on $(0, \pi)$, where $y'' = -\sin x$ is negative. It is concave up on $(\pi, 2\pi)$, where $y'' = -\sin x$ is positive (Figure 4.27). ∎

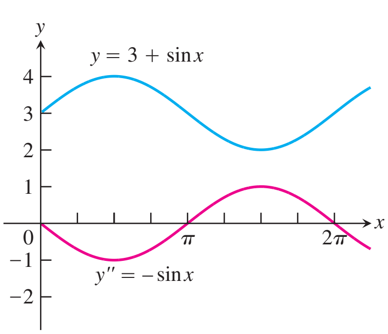


FIGURE 4.27 Using the graph of y'' to determine the concavity of y (Example 2).

Points of Inflection

The curve $y = 3 + \sin x$ in Example 2 changes concavity at the point $(\pi, 3)$. We call $(\pi, 3)$ a *point of inflection* of the curve.

DEFINITION Point of Inflection

A point where the graph of a function has a tangent line and where the concavity changes is a **point of inflection**.

A point on a curve where y'' is positive on one side and negative on the other is a point of inflection. At such a point, y'' is either zero (because derivatives have the Intermediate Value Property) or undefined. If y is a twice-differentiable function, $y'' = 0$ at a point of inflection and y' has a local maximum or minimum.

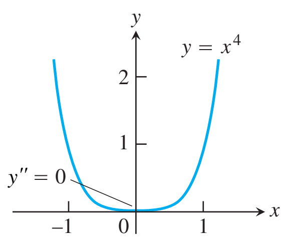


FIGURE 4.28 The graph of $y = x^4$ has no inflection point at the origin, even though $y'' = 0$ there (Example 3).

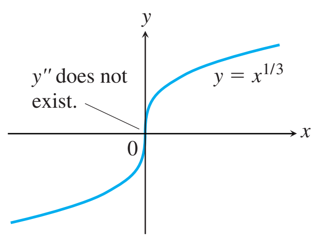


FIGURE 4.29 A point where y'' fails to exist can be a point of inflection (Example 4).

EXAMPLE 3 An Inflection Point May Not Exist Where $y'' = 0$

The curve $y = x^4$ has no inflection point at $x = 0$ (Figure 4.28). Even though $y'' = 12x^2$ is zero there, it does not change sign. ■

EXAMPLE 4 An Inflection Point May Occur Where y'' Does Not Exist

The curve $y = x^{1/3}$ has a point of inflection at $x = 0$ (Figure 4.29), but y'' does not exist there.

$$y'' = \frac{d^2}{dx^2}\left(x^{1/3}\right) = \frac{d}{dx}\left(\frac{1}{3}x^{-2/3}\right) = -\frac{2}{9}x^{-5/3}.$$ ■

We see from Example 3 that a zero second derivative does not always produce a point of inflection. From Example 4, we see that inflection points can also occur where there *is* no second derivative.

To study the motion of a body moving along a line as a function of time, we often are interested in knowing when the body's acceleration, given by the second derivative, is positive or negative. The points of inflection on the graph of the body's position function reveal where the acceleration changes sign.

EXAMPLE 5 Studying Motion Along a Line

A particle is moving along a horizontal line with position function

$$s(t) = 2t^3 - 14t^2 + 22t - 5, \qquad t \geq 0.$$

Find the velocity and acceleration, and describe the motion of the particle.

Solution The velocity is

$$v(t) = s'(t) = 6t^2 - 28t + 22 = 2(t-1)(3t-11),$$

and the acceleration is

$$a(t) = v'(t) = s''(t) = 12t - 28 = 4(3t-7).$$

When the function $s(t)$ is increasing, the particle is moving to the right; when $s(t)$ is decreasing, the particle is moving to the left.

Notice that the first derivative ($v = s'$) is zero when $t = 1$ and $t = 11/3$.

Intervals	$0 < t < 1$	$1 < t < 11/3$	$11/3 < t$
Sign of $v = s'$	$+$	$-$	$+$
Behavior of s	increasing	decreasing	increasing
Particle motion	right	left	right

The particle is moving to the right in the time intervals $[0, 1)$ and $(11/3, \infty)$, and moving to the left in $(1, 11/3)$. It is momentarily stationary (at rest), at $t = 1$ and $t = 11/3$.

The acceleration $a(t) = s''(t) = 4(3t - 7)$ is zero when $t = 7/3$.

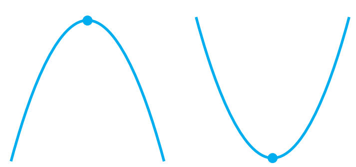
Intervals	$0 < t < 7/3$	$7/3 < t$
Sign of $a = s''$	$-$	$+$
Graph of s	concave down	concave up

The accelerating force is directed toward the left during the time interval [0, 7/3], is momentarily zero at $t = 7/3$, and is directed toward the right thereafter. ■

Second Derivative Test for Local Extrema

Instead of looking for sign changes in f' at critical points, we can sometimes use the following test to determine the presence and character of local extrema.

> **THEOREM 5 Second Derivative Test for Local Extrema**
>
> Suppose f'' is continuous on an open interval that contains $x = c$.
>
> **1.** If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at $x = c$.
>
> **2.** If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at $x = c$.
>
> **3.** If $f'(c) = 0$ and $f''(c) = 0$, then the test fails. The function f may have a local maximum, a local minimum, or neither.



$f' = 0, f'' < 0 \Rightarrow$ local max $\quad$ $f' = 0, f'' > 0 \Rightarrow$ local min

Proof Part (1). If $f''(c) < 0$, then $f''(x) < 0$ on some open interval I containing the point c, since f'' is continuous. Therefore, f' is decreasing on I. Since $f'(c) = 0$, the sign of f' changes from positive to negative at c so f has a local maximum at c by the First Derivative Test.

The proof of Part (2) is similar.

For Part (3), consider the three functions $y = x^4$, $y = -x^4$, and $y = x^3$. For each function, the first and second derivatives are zero at $x = 0$. Yet the function $y = x^4$ has a local minimum there, $y = -x^4$ has a local maximum, and $y = x^3$ is increasing in any open interval containing $x = 0$ (having neither a maximum nor a minimum there). Thus the test fails. ■

This test requires us to know f'' *only at c itself* and not in an interval about c. This makes the test easy to apply. That's the good news. The bad news is that the test is inconclusive if $f'' = 0$ or if f'' does not exist at $x = c$. When this happens, use the First Derivative Test for local extreme values.

Together f' and f'' tell us the shape of the function's graph, that is, where the critical points are located and what happens at a critical point, where the function is increasing and where it is decreasing, and how the curve is turning or bending as defined by its concavity. We use this information to sketch a graph of the function that captures its key features.

You Try It You Try It

EXAMPLE 6 Using f' and f'' to Graph f

Sketch a graph of the function

$$f(x) = x^4 - 4x^3 + 10$$

using the following steps.

(a) Identify where the extrema of f occur.

(b) Find the intervals on which f is increasing and the intervals on which f is decreasing.

(c) Find where the graph of f is concave up and where it is concave down.

(d) Sketch the general shape of the graph for f.

(e) Plot some specific points, such as local maximum and minimum points, points of inflection, and intercepts. Then sketch the curve.

Solution f is continuous since $f'(x) = 4x^3 - 12x^2$ exists. The domain of f is $(-\infty, \infty)$, and the domain of f' is also $(-\infty, \infty)$. Thus, the critical points of f occur only at the zeros of f'. Since

$$f'(x) = 4x^3 - 12x^2 = 4x^2(x - 3)$$

the first derivative is zero at $x = 0$ and $x = 3$.

Intervals	$x < 0$	$0 < x < 3$	$3 < x$
Sign of f'	$-$	$-$	$+$
Behavior of f	decreasing	decreasing	increasing

(a) Using the First Derivative Test for local extrema and the table above, we see that there is no extremum at $x = 0$ and a local minimum at $x = 3$.

(b) Using the table above, we see that f is decreasing on $(-\infty, 0]$ and $[0, 3]$, and increasing on $[3, \infty)$.

(c) $f''(x) = 12x^2 - 24x = 12x(x - 2)$ is zero at $x = 0$ and $x = 2$.

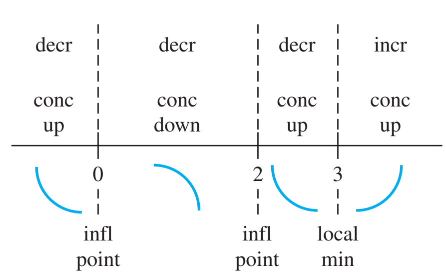
Intervals	$x < 0$	$0 < x < 2$	$2 < x$
Sign of f'	$+$	$-$	$+$
Behavior of f	concave up	concave down	concave up

We see that f is concave up on the intervals $(-\infty, 0)$ and $(2, \infty)$, and concave down on $(0, 2)$.

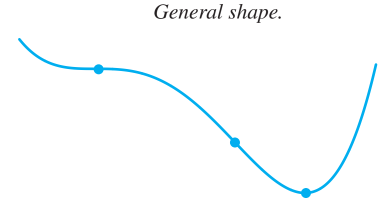
(d) Summarizing the information in the two tables above, we obtain

$x < 0$	$0 < x < 2$	$2 < x < 3$	$3 < x$
decreasing concave up	decreasing concave down	decreasing concave up	increasing concave up

The general shape of the curve is



General shape.



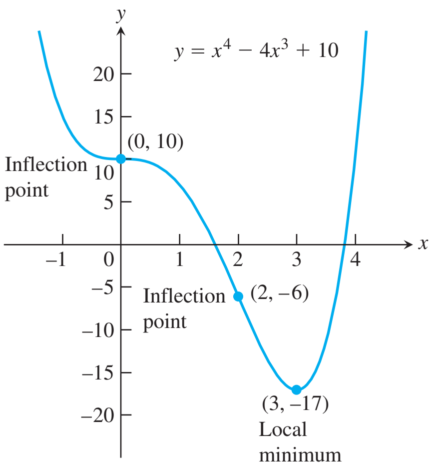


FIGURE 4.30 The graph of $f(x) = x^4 - 4x^3 + 10$ (Example 6).

(e) Plot the curve's intercepts (if possible) and the points where y' and y'' are zero. Indicate any local extreme values and inflection points. Use the general shape as a guide to sketch the curve. (Plot additional points as needed.) Figure 4.30 shows the graph of f. ■

The steps in Example 6 help in giving a procedure for graphing to capture the key features of a function and its graph.

Strategy for Graphing $y = f(x)$

1. Identify the domain of f and any symmetries the curve may have.
2. Find y' and y''.
3. Find the critical points of f, and identify the function's behavior at each one.
4. Find where the curve is increasing and where it is decreasing.
5. Find the points of inflection, if any occur, and determine the concavity of the curve.
6. Identify any asymptotes.
7. Plot key points, such as the intercepts and the points found in Steps 3–5, and sketch the curve.

EXAMPLE 7 Using the Graphing Strategy

Sketch the graph of $f(x) = \dfrac{(x+1)^2}{1+x^2}$.

Solution

1. The domain of f is $(-\infty, \infty)$ and there are no symmetries about either axis or the origin (Section 1.4).
2. *Find f' and f''.*

$$f(x) = \frac{(x+1)^2}{1+x^2}$$

x-intercept at $x = -1$, y-intercept ($y = 1$) at $x = 0$

$$f'(x) = \frac{(1+x^2)\cdot 2(x+1) - (x+1)^2 \cdot 2x}{(1+x^2)^2}$$

$$= \frac{2(1-x^2)}{(1+x^2)^2}$$

Critical points: $x = -1, x = 1$

$$f''(x) = \frac{(1+x^2)^2 \cdot 2(-2x) - 2(1-x^2)[2(1+x^2)\cdot 2x]}{(1+x^2)^4}$$

$$= \frac{4x(x^2-3)}{(1+x^2)^3}$$

After some algebra

3. *Behavior at critical points.* The critical points occur only at $x = \pm 1$ where $f'(x) = 0$ (Step 2) since f' exists everywhere over the domain of f. At $x = -1$, $f''(-1) = 1 > 0$ yielding a relative minimum by the Second Derivative Test. At $x = 1$, $f''(1) = -1 < 0$ yielding a relative maximum by the Second Derivative Test. We will see in Step 6 that both are absolute extrema as well.

4. *Increasing and decreasing.* We see that on the interval $(-\infty, -1)$ the derivative $f'(x) < 0$, and the curve is decreasing. On the interval $(-1, 1)$, $f'(x) > 0$ and the curve is increasing; it is decreasing on $(1, \infty)$ where $f'(x) < 0$ again.
5. *Inflection points.* Notice that the denominator of the second derivative (Step 2) is always positive. The second derivative f'' is zero when $x = -\sqrt{3}, 0$, and $\sqrt{3}$. The second derivative changes sign at each of these points: negative on $\left(-\infty, -\sqrt{3}\right)$, positive on $\left(-\sqrt{3}, 0\right)$, negative on $\left(0, \sqrt{3}\right)$, and positive again on $\left(\sqrt{3}, \infty\right)$. Thus each point is a point of inflection. The curve is concave down on the interval $\left(-\infty, -\sqrt{3}\right)$, concave up on $\left(-\sqrt{3}, 0\right)$, concave down on $\left(0, \sqrt{3}\right)$, and concave up again on $\left(\sqrt{3}, \infty\right)$.
6. *Asymptotes.* Expanding the numerator of $f(x)$ and then dividing both numerator and denominator by x^2 gives

$$f(x) = \frac{(x+1)^2}{1+x^2} = \frac{x^2+2x+1}{1+x^2} \qquad \text{Expanding numerator}$$

$$= \frac{1 + (2/x) + (1/x^2)}{(1/x^2) + 1}. \qquad \text{Dividing by } x^2$$

We see that $f(x) \rightarrow 1^+$ as $x \rightarrow \infty$ and that $f(x) \rightarrow 1^-$ as $x \rightarrow -\infty$. Thus, the line $y = 1$ is a horizontal asymptote.

Since f decreases on $(-\infty, -1)$ and then increases on $(-1, 1)$, we know that $f(-1) = 0$ is a local minimum. Although f decreases on $(1, \infty)$, it never crosses the horizontal asymptote $y = 1$ on that interval (it approaches the asymptote from above). So the graph never becomes negative, and $f(-1) = 0$ is an absolute minimum as well. Likewise, $f(1) = 2$ is an absolute maximum because the graph never crosses the asymptote $y = 1$ on the interval $(-\infty, -1)$, approaching it from below. Therefore, there are no vertical asymptotes (the range of f is $0 \leq y \leq 2$).
7. The graph of f is sketched in Figure 4.31. Notice how the graph is concave down as it approaches the horizontal asymptote $y = 1$ as $x \rightarrow -\infty$, and concave up in its approach to $y = 1$ as $x \rightarrow \infty$. ■

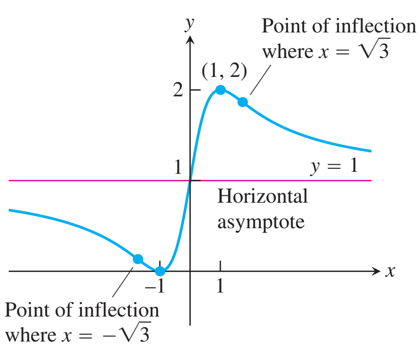
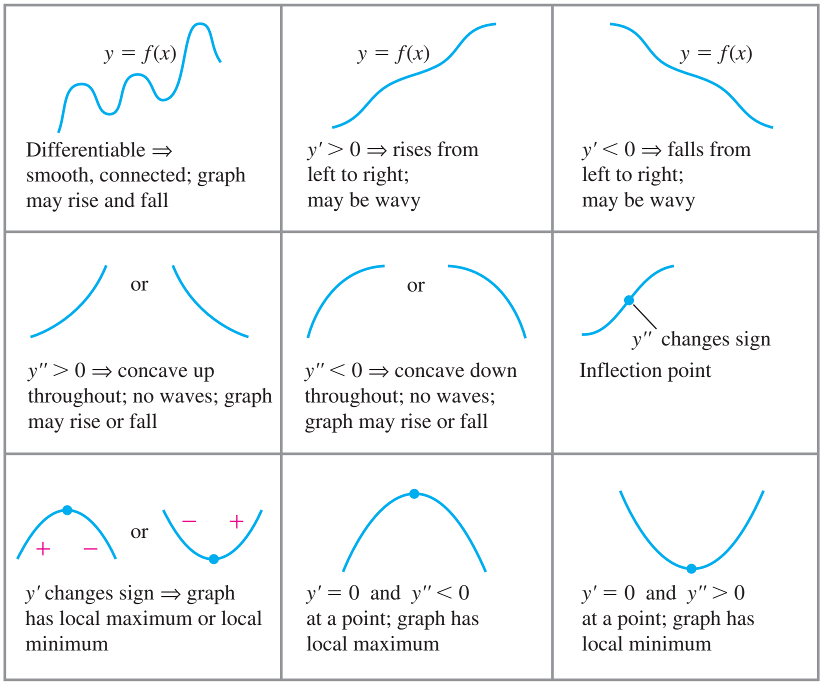


FIGURE 4.31 The graph of $y = \dfrac{(x+1)^2}{1+x^2}$ (Example 7).

Learning About Functions from Derivatives

As we saw in Examples 6 and 7, we can learn almost everything we need to know about a twice-differentiable function $y = f(x)$ by examining its first derivative. We can find where the function's graph rises and falls and where any local extrema are assumed. We can differentiate y' to learn how the graph bends as it passes over the intervals of rise and fall. We can determine the shape of the function's graph. Information we cannot get from the derivative is how to place the graph in the xy-plane. But, as we discovered in Section 4.2, the only additional information we need to position the graph is the value of f at one point. The derivative does not give us information about the asymptotes, which are found using limits (Sections 2.4 and 2.5).



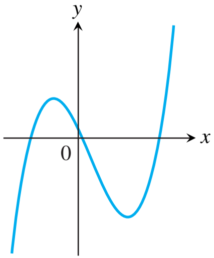
$y = f(x)$
Differentiable ⇒ smooth, connected; graph may rise and fall
$y = f(x)$
$y' > 0$ ⇒ rises from left to right; may be wavy
$y = f(x)$
$y' < 0$ ⇒ falls from left to right; may be wavy
or
$y'' > 0$ ⇒ concave up throughout; no waves; graph may rise or fall
or
$y'' < 0$ ⇒ concave down throughout; no waves; graph may rise or fall
y'' changes sign
Inflection point
or
+
−
−
+
y' changes sign ⇒ graph has local maximum or local minimum
$y' = 0$ and $y'' < 0$ at a point; graph has local maximum
$y' = 0$ and $y'' > 0$ at a point; graph has local minimum

EXERCISES 4.4

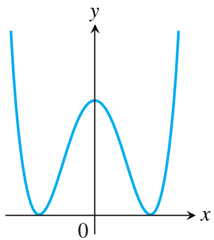
Analyzing Graphed Functions

Identify the inflection points and local maxima and minima of the functions graphed in Exercises 1–8. Identify the intervals on which the functions are concave up and concave down.

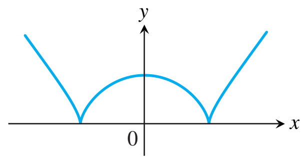
1. $y = \frac{x^3}{3} - \frac{x^2}{2} - 2x + \frac{1}{3}$



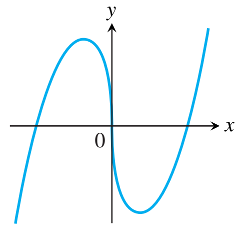
2. $y = \frac{x^4}{4} - 2x^2 + 4$



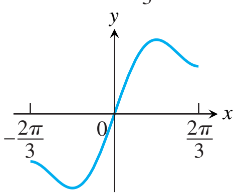
3. $y = \frac{3}{4}(x^2 - 1)^{2/3}$



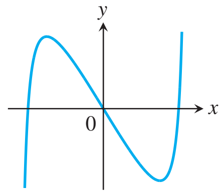
4. $y = \frac{9}{14}x^{1/3}(x^2 - 7)$



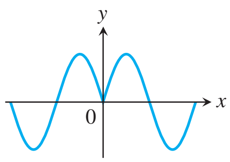
5. $y = x + \sin 2x, -\frac{2\pi}{3} \le x \le \frac{2\pi}{3}$



6. $y = \tan x - 4x, -\frac{\pi}{2} < x < \frac{\pi}{2}$

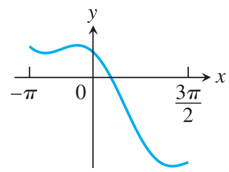


7. $y = \sin|x|, -2\pi \le x \le 2\pi$



NOT TO SCALE

8. $y = 2\cos x - \sqrt{2}x, -\pi \le x \le \frac{3\pi}{2}$



Graphing Equations

Use the steps of the graphing procedure on page 272 to graph the equations in Exercises 9–40. Include the coordinates of any local extreme points and inflection points.

9. $y = x^2 - 4x + 3$

10. $y = 6 - 2x - x^2$

11. $y = x^3 - 3x + 3$

12. $y = x(6 - 2x)^2$

13. $y = -2x^3 + 6x^2 - 3$

14. $y = 1 - 9x - 6x^2 - x^3$

15. $y = (x - 2)^3 + 1$

16. $y = 1 - (x + 1)^3$

17. $y = x^4 - 2x^2 = x^2(x^2 - 2)$

18. $y = -x^4 + 6x^2 - 4 = x^2(6 - x^2) - 4$

19. $y = 4x^3 - x^4 = x^3(4 - x)$

20. $y = x^4 + 2x^3 = x^3(x + 2)$

21. $y = x^5 - 5x^4 = x^4(x - 5)$

22. $y = x\left(\frac{x}{2} - 5\right)^4$

23. $y = x + \sin x, \quad 0 \le x \le 2\pi$

24. $y = x - \sin x, \quad 0 \le x \le 2\pi$

25. $y = x^{1/5}$

26. $y = x^{3/5}$

27. $y = x^{2/5}$

28. $y = x^{4/5}$

29. $y = 2x - 3x^{2/3}$

30. $y = 5x^{2/5} - 2x$

31. $y = x^{2/3}\left(\frac{5}{2} - x\right)$

32. $y = x^{2/3}(x - 5)$

33. $y = x\sqrt{8 - x^2}$

34. $y = (2 - x^2)^{3/2}$

35. $y = \frac{x^2 - 3}{x - 2}, \quad x \ne 2$

36. $y = \frac{x^3}{3x^2 + 1}$

37. $y = |x^2 - 1|$

38. $y = |x^2 - 2x|$

39. $y = \sqrt{|x|} = \begin{cases} \sqrt{-x}, & x \le 0 \\ \sqrt{x}, & x > 0 \end{cases}$

40. $y = \sqrt{|x - 4|}$

Sketching the General Shape Knowing y'

Each of Exercises 41–62 gives the first derivative of a continuous function $y = f(x)$. Find y'' and then use steps 2–4 of the graphing procedure on page 272 to sketch the general shape of the graph of f.

41. $y' = 2 + x - x^2$

42. $y' = x^2 - x - 6$

43. $y' = x(x - 3)^2$

44. $y' = x^2(2 - x)$

45. $y' = x(x^2 - 12)$

46. $y' = (x - 1)^2(2x + 3)$

47. $y' = (8x - 5x^2)(4 - x)^2$

48. $y' = (x^2 - 2x)(x - 5)^2$

49. $y' = \sec^2 x, \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$

50. $y' = \tan x, \quad -\frac{\pi}{2} < x < \frac{\pi}{2}$

51. $y' = \cot\frac{\theta}{2}, \quad 0 < \theta < 2\pi$

52. $y' = \csc^2\frac{\theta}{2}, \quad 0 < \theta < 2\pi$

53. $y' = \tan^2\theta - 1, \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$

54. $y' = 1 - \cot^2\theta, \quad 0 < \theta < \pi$

55. $y' = \cos t, \quad 0 \le t \le 2\pi$

56. $y' = \sin t, \quad 0 \le t \le 2\pi$

57. $y' = (x + 1)^{-2/3}$

58. $y' = (x - 2)^{-1/3}$

59. $y' = x^{-2/3}(x - 1)$

60. $y' = x^{-4/5}(x + 1)$

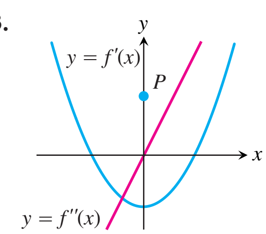
61. $y' = 2|x| = \begin{cases} -2x, & x \le 0 \\ 2x, & x > 0 \end{cases}$

62. $y' = \begin{cases} -x^2, & x \le 0 \\ x^2, & x > 0 \end{cases}$

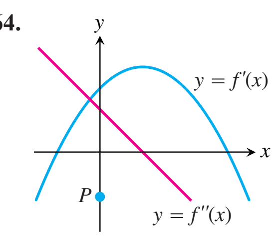
Sketching y from Graphs of y' and y''

Each of Exercises 63–66 shows the graphs of the first and second derivatives of a function $y = f(x)$. Copy the picture and add to it a sketch of the approximate graph of f, given that the graph passes through the point P.

63.



64.



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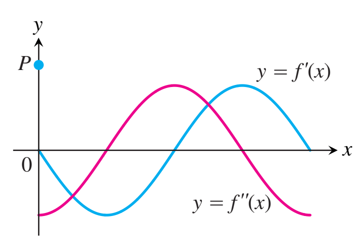
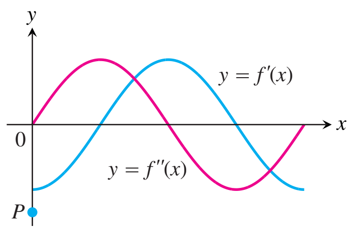
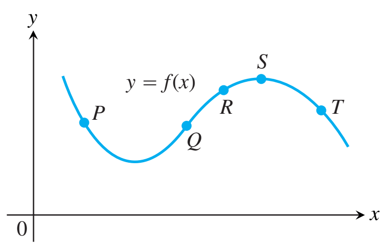


66.



Theory and Examples

67. The accompanying figure shows a portion of the graph of a twice-differentiable function $y = f(x)$. At each of the five labeled points, classify y' and y'' as positive, negative, or zero.



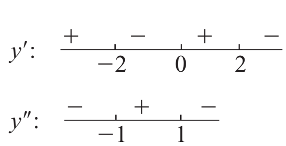
68. Sketch a smooth connected curve $y = f(x)$ with

$$\begin{array}{ll} f(-2) = 8, & f'(2) = f'(-2) = 0, \\ f(0) = 4, & f'(x) < 0 \quad \text{for} \quad |x| < 2, \\ f(2) = 0, & f''(x) < 0 \quad \text{for} \quad x < 0, \\ f'(x) > 0 \quad \text{for} \quad |x| > 2, & f''(x) > 0 \quad \text{for} \quad x > 0. \end{array}$$

69. Sketch the graph of a twice-differentiable function $y = f(x)$ with the following properties. Label coordinates where possible.

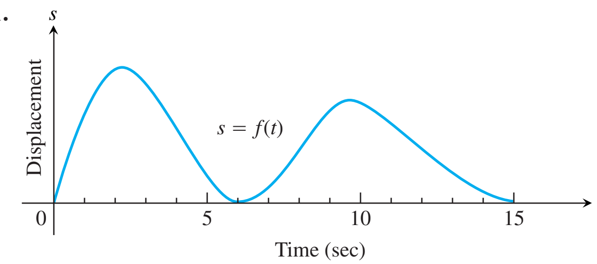
x	y	Derivatives
$x < 2$		$y' < 0,\quad y'' > 0$
2	1	$y' = 0,\quad y'' > 0$
$2 < x < 4$		$y' > 0,\quad y'' > 0$
4	4	$y' > 0,\quad y'' = 0$
$4 < x < 6$		$y' > 0,\quad y'' < 0$
6	7	$y' = 0,\quad y'' < 0$
$x > 6$		$y' < 0,\quad y'' < 0$

70. Sketch the graph of a twice-differentiable function $y = f(x)$ that passes through the points $(-2, 2), (-1, 1), (0, 0), (1, 1)$ and $(2, 2)$ and whose first two derivatives have the following sign patterns:

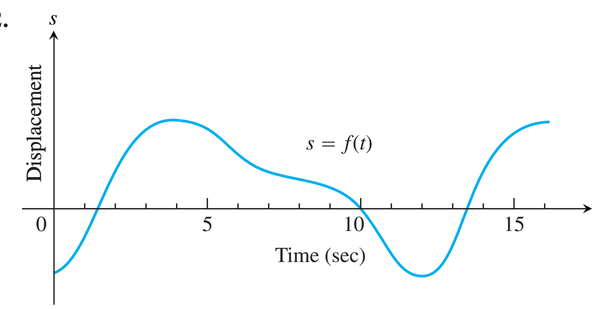


Motion Along a Line The graphs in Exercises 71 and 72 show the position $s = f(t)$ of a body moving back and forth on a coordinate line. (**a**) When is the body moving away from the origin? Toward the origin? At approximately what times is the (**b**) velocity equal to zero? (**c**) Acceleration equal to zero? (**d**) When is the acceleration positive? Negative?

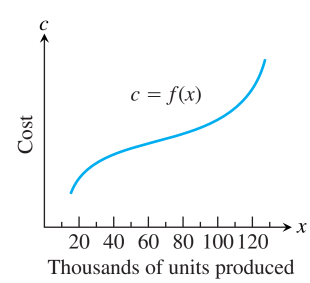
71.



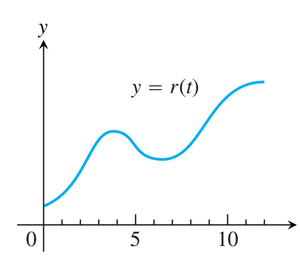
72.



73. Marginal cost The accompanying graph shows the hypothetical cost $c = f(x)$ of manufacturing x items. At approximately what production level does the marginal cost change from decreasing to increasing?



74. The accompanying graph shows the monthly revenue of the Widget Corporation for the last 12 years. During approximately what time intervals was the marginal revenue increasing? decreasing?



75. Suppose the derivative of the function $y = f(x)$ is

$$y' = (x - 1)^2(x - 2).$$

At what points, if any, does the graph of f have a local minimum, local maximum, or point of inflection? (*Hint:* Draw the sign pattern for y'.)

76. Suppose the derivative of the function $y = f(x)$ is

$$y' = (x - 1)^2(x - 2)(x - 4).$$

At what points, if any, does the graph of f have a local minimum, local maximum, or point of inflection?

77. For $x > 0$, sketch a curve $y = f(x)$ that has $f(1) = 0$ and $f'(x) = 1/x$. Can anything be said about the concavity of such a curve? Give reasons for your answer.

78. Can anything be said about the graph of a function $y = f(x)$ that has a continuous second derivative that is never zero? Give reasons for your answer.

79. If b, c, and d are constants, for what value of b will the curve $y = x^3 + bx^2 + cx + d$ have a point of inflection at $x = 1$? Give reasons for your answer.

80. Horizontal tangents True, or false? Explain.

a. The graph of every polynomial of even degree (largest exponent even) has at least one horizontal tangent.

b. The graph of every polynomial of odd degree (largest exponent odd) has at least one horizontal tangent.

81. Parabolas

a. Find the coordinates of the vertex of the parabola $y = ax^2 + bx + c, a \neq 0$.

b. When is the parabola concave up? Concave down? Give reasons for your answers.

82. Is it true that the concavity of the graph of a twice-differentiable function $y = f(x)$ changes every time $f''(x) = 0$? Give reasons for your answer.

83. Quadratic curves What can you say about the inflection points of a quadratic curve $y = ax^2 + bx + c, a \neq 0$? Give reasons for your answer.

84. Cubic curves What can you say about the inflection points of a cubic curve $y = ax^3 + bx^2 + cx + d, a \neq 0$? Give reasons for your answer.

COMPUTER EXPLORATIONS

In Exercises 85–88, find the inflection points (if any) on the graph of the function and the coordinates of the points on the graph where the function has a local maximum or local minimum value. Then graph the function in a region large enough to show all these points simultaneously. Add to your picture the graphs of the function's first and second derivatives. How are the values at which these graphs intersect the x-axis related to the graph of the function? In what other ways are the graphs of the derivatives related to the graph of the function?

85. $y = x^5 - 5x^4 - 240$

86. $y = x^3 - 12x^2$

87. $y = \frac{4}{5}x^5 + 16x^2 - 25$

88. $y = \frac{x^4}{4} - \frac{x^3}{3} - 4x^2 + 12x + 20$

89. Graph $f(x) = 2x^4 - 4x^2 + 1$ and its first two derivatives together. Comment on the behavior of f in relation to the signs and values of f' and f''.

90. Graph $f(x) = x \cos x$ and its second derivative together for $0 \leq x \leq 2\pi$. Comment on the behavior of the graph of f in relation to the signs and values of f''.

91. a. On a common screen, graph $f(x) = x^3 + kx$ for $k = 0$ and nearby positive and negative values of k. How does the value of k seem to affect the shape of the graph?

b. Find $f'(x)$. As you will see, $f'(x)$ is a quadratic function of x. Find the discriminant of the quadratic (the discriminant of $ax^2 + bx + c$ is $b^2 - 4ac$). For what values of k is the discriminant positive? Zero? Negative? For what values of k does f' have two zeros? One or no zeros? Now explain what the value of k has to do with the shape of the graph of f.

c. Experiment with other values of k. What appears to happen as $k \to -\infty$? as $k \to \infty$?

92. a. On a common screen, graph $f(x) = x^4 + kx^3 + 6x^2$, $-2 \leq x \leq 2$ for $k = -4$, and some nearby integer values of k. How does the value of k seem to affect the shape of the graph?

b. Find $f''(x)$. As you will see, $f''(x)$ is a quadratic function of x. What is the discriminant of this quadratic (see Exercise 91(b))? For what values of k is the discriminant positive? Zero? Negative? For what values of k does $f''(x)$ have two zeros? One or no zeros? Now explain what the value of k has to do with the shape of the graph of f.

93. a. Graph $y = x^{2/3}(x^2 - 2)$ for $-3 \leq x \leq 3$. Then use calculus to confirm what the screen shows about concavity, rise, and fall. (Depending on your grapher, you may have to enter $x^{2/3}$ as $(x^2)^{1/3}$ to obtain a plot for negative values of x.)

b. Does the curve have a cusp at $x = 0$, or does it just have a corner with different right-hand and left-hand derivatives?

94. a. Graph $y = 9x^{2/3}(x - 1)$ for $-0.5 \leq x \leq 1.5$. Then use calculus to confirm what the screen shows about concavity, rise, and fall. What concavity does the curve have to the left of the origin? (Depending on your grapher, you may have to enter $x^{2/3}$ as $(x^2)^{1/3}$ to obtain a plot for negative values of x.)

b. Does the curve have a cusp at $x = 0$, or does it just have a corner with different right-hand and left-hand derivatives?

95. Does the curve $y = x^2 + 3 \sin 2x$ have a horizontal tangent near $x = -3$? Give reasons for your answer.

4.5 Applied Optimization Problems

To optimize something means to maximize or minimize some aspect of it. What are the dimensions of a rectangle with fixed perimeter having maximum area? What is the least expensive shape for a cylindrical can? What is the size of the most profitable production run? The differential calculus is a powerful tool for solving problems that call for maximizing or minimizing a function. In this section we solve a variety of optimization problems from business, mathematics, physics, and economics.

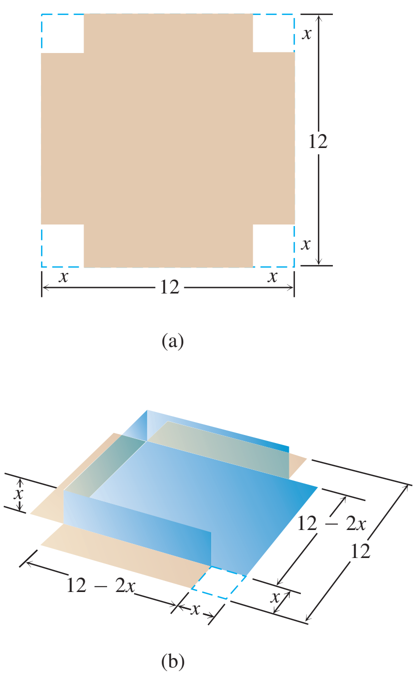


FIGURE 4.32 An open box made by cutting the corners from a square sheet of tin. What size corners maximize the box's volume (Example 1)?

Examples from Business and Industry

EXAMPLE 1 Fabricating a Box

An open-top box is to be made by cutting small congruent squares from the corners of a 12-in.-by-12-in. sheet of tin and bending up the sides. How large should the squares cut from the corners be to make the box hold as much as possible?

Solution We start with a picture (Figure 4.32). In the figure, the corner squares are x in. on a side. The volume of the box is a function of this variable:

$$V(x) = x(12 - 2x)^2 = 144x - 48x^2 + 4x^3. \qquad V = hlw$$

Since the sides of the sheet of tin are only 12 in. long, $x \le 6$ and the domain of V is the interval $0 \le x \le 6$.

A graph of V (Figure 4.33) suggests a minimum value of 0 at $x = 0$ and $x = 6$ and a maximum near $x = 2$. To learn more, we examine the first derivative of V with respect to x:

$$\frac{dV}{dx} = 144 - 96x + 12x^2 = 12(12 - 8x + x^2) = 12(2 - x)(6 - x).$$

Of the two zeros, $x = 2$ and $x = 6$, only $x = 2$ lies in the interior of the function's domain and makes the critical-point list. The values of V at this one critical point and two endpoints are

Critical-point value: $V(2) = 128$

Endpoint values: $V(0) = 0, \qquad V(6) = 0.$

The maximum volume is 128 in.^{3}. The cutout squares should be 2 in. on a side. ■

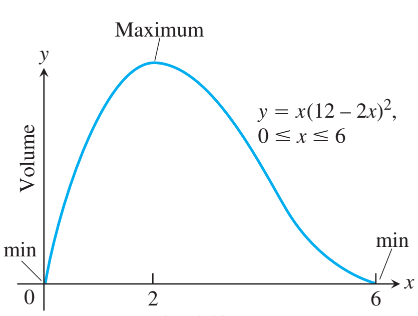


FIGURE 4.33 The volume of the box in Figure 4.32 graphed as a function of x.

EXAMPLE 2 Designing an Efficient Cylindrical Can

You have been asked to design a 1-liter can shaped like a right circular cylinder (Figure 4.34). What dimensions will use the least material?

Solution *Volume of can:* If r and h are measured in centimeters, then the volume of the can in cubic centimeters is

$$\pi r^2 h = 1000. \qquad 1 \text{ liter} = 1000 \text{ cm}^3$$

Surface area of can: $$A = \underbrace{2\pi r^2}_{\text{circular ends}} + \underbrace{2\pi r h}_{\text{circular wall}}$$

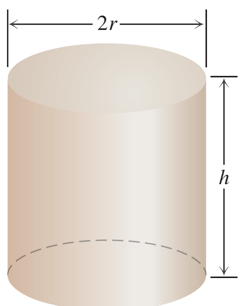


FIGURE 4.34 This 1-L can uses the least material when $h = 2r$ (Example 2).

How can we interpret the phrase "least material"? First, it is customary to ignore the thickness of the material and the waste in manufacturing. Then we ask for dimensions r and h that make the total surface area as small as possible while satisfying the constraint $\pi r^2 h = 1000$.

To express the surface area as a function of one variable, we solve for one of the variables in $\pi r^2 h = 1000$ and substitute that expression into the surface area formula. Solving for h is easier:

$$h = \frac{1000}{\pi r^2}.$$

Thus,

$$\begin{aligned} A &= 2\pi r^2 + 2\pi r h \\ &= 2\pi r^2 + 2\pi r\left(\frac{1000}{\pi r^2}\right) \\ &= 2\pi r^2 + \frac{2000}{r}. \end{aligned}$$

Our goal is to find a value of $r > 0$ that minimizes the value of A. Figure 4.35 suggests that such a value exists.

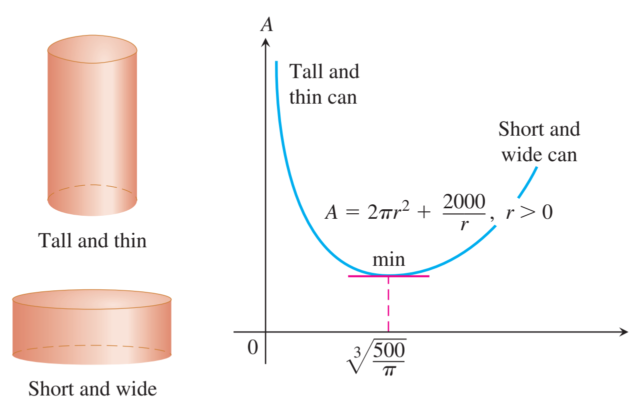


FIGURE 4.35 The graph of $A = 2\pi r^2 + 2000/r$ is concave up.

Notice from the graph that for small r (a tall thin container, like a piece of pipe), the term $2000/r$ dominates and A is large. For large r (a short wide container, like a pizza pan), the term $2\pi r^2$ dominates and A again is large.

Since A is differentiable on $r > 0$, an interval with no endpoints, it can have a minimum value only where its first derivative is zero.

$$\begin{aligned} \frac{dA}{dr} &= 4\pi r - \frac{2000}{r^2} \\ 0 &= 4\pi r - \frac{2000}{r^2} && \text{Set } dA/dr = 0. \\ 4\pi r^3 &= 2000 && \text{Multiply by } r^2. \\ r &= \sqrt[3]{\frac{500}{\pi}} \approx 5.42 && \text{Solve for } r. \end{aligned}$$

What happens at $r = \sqrt[3]{500/\pi}$?

The second derivative

$$\frac{d^2A}{dr^2} = 4\pi + \frac{4000}{r^3}$$

is positive throughout the domain of A. The graph is therefore everywhere concave up and the value of A at $r = \sqrt[3]{500/\pi}$ an absolute minimum.

The corresponding value of h (after a little algebra) is

$$h = \frac{1000}{\pi r^2} = 2\sqrt[3]{\frac{500}{\pi}} = 2r.$$

The 1-L can that uses the least material has height equal to the diameter, here with $r \approx 5.42$ cm and $h \approx 10.84$ cm. ■

Solving Applied Optimization Problems

1. *Read the problem.* Read the problem until you understand it. What is given? What is the unknown quantity to be optimized?
2. *Draw a picture.* Label any part that may be important to the problem.
3. *Introduce variables.* List every relation in the picture and in the problem as an equation or algebraic expression, and identify the unknown variable.
4. *Write an equation for the unknown quantity.* If you can, express the unknown as a function of a single variable or in two equations in two unknowns. This may require considerable manipulation.
5. *Test the critical points and endpoints in the domain of the unknown.* Use what you know about the shape of the function's graph. Use the first and second derivatives to identify and classify the function's critical points.

Examples from Mathematics and Physics

EXAMPLE 3 Inscribing Rectangles

A rectangle is to be inscribed in a semicircle of radius 2. What is the largest area the rectangle can have, and what are its dimensions?

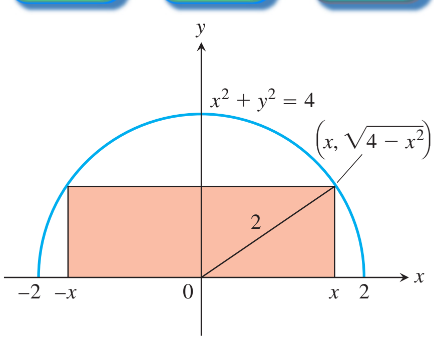


FIGURE 4.36 The rectangle inscribed in the semicircle in Example 3.

Solution Let $(x, \sqrt{4 - x^2})$ be the coordinates of the corner of the rectangle obtained by placing the circle and rectangle in the coordinate plane (Figure 4.36). The length, height, and area of the rectangle can then be expressed in terms of the position x of the lower right-hand corner:

$$\text{Length: } 2x, \qquad \text{Height: } \sqrt{4 - x^2}, \qquad \text{Area: } 2x \cdot \sqrt{4 - x^2}.$$

Notice that the values of x are to be found in the interval $0 \le x \le 2$, where the selected corner of the rectangle lies.

Our goal is to find the absolute maximum value of the function

$$A(x) = 2x\sqrt{4 - x^2}$$

on the domain $[0, 2]$.

The derivative

$$\frac{dA}{dx} = \frac{-2x^2}{\sqrt{4 - x^2}} + 2\sqrt{4 - x^2}$$

is not defined when $x = 2$ and is equal to zero when

$$\begin{aligned} \frac{-2x^2}{\sqrt{4 - x^2}} + 2\sqrt{4 - x^2} &= 0 \\ -2x^2 + 2(4 - x^2) &= 0 \\ 8 - 4x^2 &= 0 \\ x^2 = 2 \text{ or } x &= \pm\sqrt{2}. \end{aligned}$$

Of the two zeros, $x = \sqrt{2}$ and $x = -\sqrt{2}$, only $x = \sqrt{2}$ lies in the interior of A's domain and makes the critical-point list. The values of A at the endpoints and at this one critical point are

$$\begin{aligned} &\text{Critical-point value:} \quad A\left(\sqrt{2}\right) = 2\sqrt{2}\sqrt{4 - 2} = 4 \\ &\text{Endpoint values:} \qquad A(0) = 0, \quad A(2) = 0. \end{aligned}$$

The area has a maximum value of 4 when the rectangle is $\sqrt{4 - x^2} = \sqrt{2}$ units high and $2x = 2\sqrt{2}$ unit long. ■

HISTORICAL BIOGRAPHY

Willebrord Snell van Royen (1580–1626)

EXAMPLE 4 Fermat's Principle and Snell's Law

The speed of light depends on the medium through which it travels, and is generally slower in denser media.

Fermat's principle in optics states that light travels from one point to another along a path for which the time of travel is a minimum. Find the path that a ray of light will follow in going from a point A in a medium where the speed of light is c_1 to a point B in a second medium where its speed is c_2.

Solution Since light traveling from A to B follows the quickest route, we look for a path that will minimize the travel time. We assume that A and B lie in the xy-plane and that the line separating the two media is the x-axis (Figure 4.37).

In a uniform medium, where the speed of light remains constant, "shortest time" means "shortest path," and the ray of light will follow a straight line. Thus the path from A to B will consist of a line segment from A to a boundary point P, followed by another line segment from P to B. Distance equals rate times time, so

$$\text{Time} = \frac{\text{distance}}{\text{rate}}.$$

The time required for light to travel from A to P is

$$t_1 = \frac{AP}{c_1} = \frac{\sqrt{a^2 + x^2}}{c_1}.$$

From P to B, the time is

$$t_2 = \frac{PB}{c_2} = \frac{\sqrt{b^2 + (d - x)^2}}{c_2}.$$

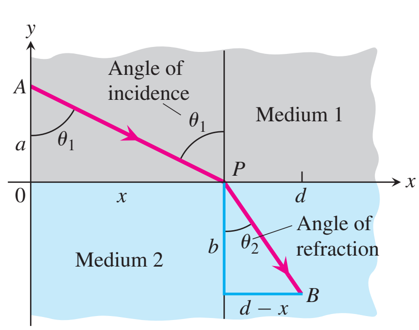


FIGURE 4.37 A light ray refracted (deflected from its path) as it passes from one medium to a denser medium (Example 4).

The time from A to B is the sum of these:

$$t = t_1 + t_2 = \frac{\sqrt{a^2 + x^2}}{c_1} + \frac{\sqrt{b^2 + (d - x)^2}}{c_2}.$$

This equation expresses t as a differentiable function of x whose domain is $[0, d]$. We want to find the absolute minimum value of t on this closed interval. We find the derivative

$$\frac{dt}{dx} = \frac{x}{c_1\sqrt{a^2 + x^2}} - \frac{d - x}{c_2\sqrt{b^2 + (d - x)^2}}.$$

In terms of the angles θ_1 and θ_2 in Figure 4.37,

$$\frac{dt}{dx} = \frac{\sin\theta_1}{c_1} - \frac{\sin\theta_2}{c_2}.$$

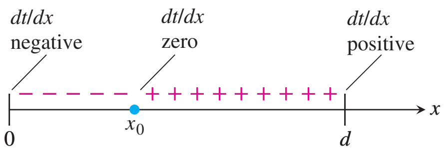


FIGURE 4.38 The sign pattern of dt/dx in Example 4.

If we restrict x to the interval $0 \le x \le d$, then t has a negative derivative at $x = 0$ and a positive derivative at $x = d$. By the Intermediate Value Theorem for Derivatives (Section 3.1), there is a point $x_0 \in [0, d]$ where $dt/dx = 0$ (Figure 4.38). There is only one such point because dt/dx is an increasing function of x (Exercise 54). At this point

$$\frac{\sin\theta_1}{c_1} = \frac{\sin\theta_2}{c_2}.$$

This equation is **Snell's Law** or the **Law of Refraction**, and is an important principle in the theory of optics. It describes the path the ray of light follows. ■

Examples from Economics

In these examples we point out two ways that calculus makes a contribution to economics. The first has to do with maximizing profit. The second has to do with minimizing average cost.

Suppose that

$$\begin{aligned} r(x) &= \text{the revenue from selling } x \text{ items} \\ c(x) &= \text{the cost of producing the } x \text{ items} \\ p(x) = r(x) - c(x) &= \text{the profit from producing and selling } x \text{ items}. \end{aligned}$$

The **marginal revenue, marginal cost,** and **marginal profit** when producing and selling x items are

$$\frac{dr}{dx} = \text{marginal revenue},$$

$$\frac{dc}{dx} = \text{marginal cost},$$

$$\frac{dp}{dx} = \text{marginal profit}.$$

The first observation is about the relationship of p to these derivatives.

If $r(x)$ and $c(x)$ are differentiable for all $x > 0$, and if $p(x) = r(x) - c(x)$ has a maximum value, it occurs at a production level at which $p'(x) = 0$. Since $p'(x) = r'(x) - c'(x)$, $p'(x) = 0$ implies that

$$r'(x) - c'(x) = 0 \qquad \text{or} \qquad r'(x) = c'(x).$$

Therefore

> At a production level yielding maximum profit, marginal revenue equals marginal cost (Figure 4.39).

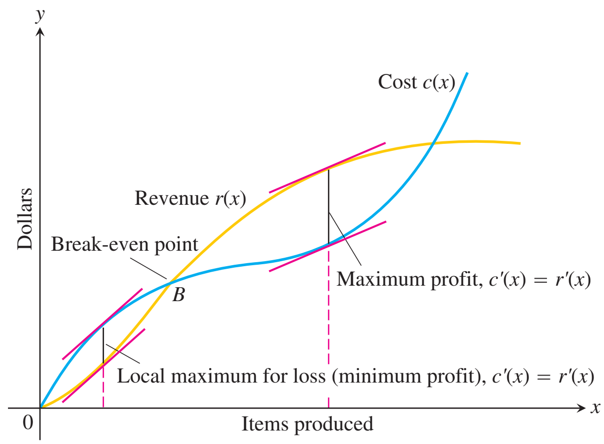


FIGURE 4.39 The graph of a typical cost function starts concave down and later turns concave up. It crosses the revenue curve at the break-even point B. To the left of B, the company operates at a loss. To the right, the company operates at a profit, with the maximum profit occurring where $c'(x) = r'(x)$. Farther to the right, cost exceeds revenue (perhaps because of a combination of rising labor and material costs and market saturation) and production levels become unprofitable again.

EXAMPLE 5 Maximizing Profit

Suppose that $r(x) = 9x$ and $c(x) = x^3 - 6x^2 + 15x$, where x represents thousands of units. Is there a production level that maximizes profit? If so, what is it?

Solution Notice that $r'(x) = 9$ and $c'(x) = 3x^2 - 12x + 15$.

$$3x^2 - 12x + 15 = 9 \qquad \text{Set } c'(x) = r'(x).$$
$$3x^2 - 12x + 6 = 0$$

The two solutions of the quadratic equation are

$$x_1 = \frac{12 - \sqrt{72}}{6} = 2 - \sqrt{2} \approx 0.586 \quad \text{and}$$
$$x_2 = \frac{12 + \sqrt{72}}{6} = 2 + \sqrt{2} \approx 3.414.$$

The possible production levels for maximum profit are $x \approx 0.586$ thousand units or $x \approx 3.414$ thousand units. The second derivative of $p(x) = r(x) - c(x)$ is $p''(x) = -c''(x)$ since $r''(x)$ is everywhere zero. Thus, $p''(x) = 6(2 - x)$ which is negative at $x = 2 + \sqrt{2}$ and positive at $x = 2 - \sqrt{2}$. By the Second Derivative Test, a maximum profit occurs at about $x = 3.414$ (where revenue exceeds costs) and maximum loss occurs at about $x = 0.586$. The graph of $r(x)$ is shown in Figure 4.40. ■

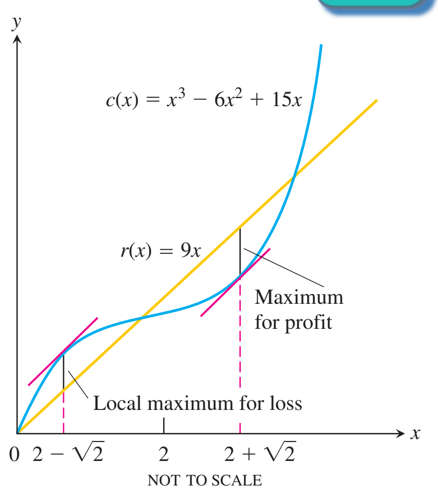


FIGURE 4.40 The cost and revenue curves for Example 5.

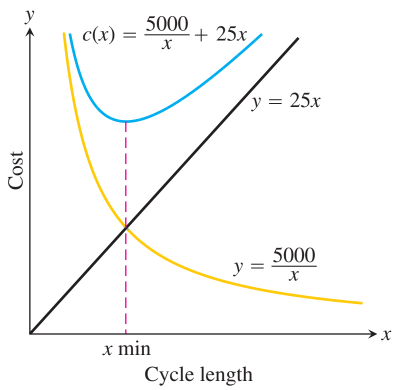


FIGURE 4.41 The average daily cost $c(x)$ is the sum of a hyperbola and a linear function (Example 6).

EXAMPLE 6 Minimizing Costs

A cabinetmaker uses plantation-farmed mahogany to produce 5 furnishings each day. Each delivery of one container of wood is \$5000, whereas the storage of that material is \$10 per day per unit stored, where a unit is the amount of material needed by her to produce 1 furnishing. How much material should be ordered each time and how often should the material be delivered to minimize her average daily cost in the production cycle between deliveries?

Solution If she asks for a delivery every x days, then she must order $5x$ units to have enough material for that delivery cycle. The *average* amount in storage is approximately one-half of the delivery amount, or $5x/2$. Thus, the cost of delivery and storage for each cycle is approximately

$$\text{Cost per cycle} = \text{delivery costs} + \text{storage costs}$$

$$\text{Cost per cycle} = \underbrace{5000}_{\substack{\text{delivery}\\ \text{cost}}} + \underbrace{\left(\frac{5x}{2}\right)}_{\substack{\text{average}\\ \text{amount stored}}} \cdot \underbrace{x}_{\substack{\text{number of}\\ \text{days stored}}} \cdot \underbrace{10}_{\substack{\text{storage cost}\\ \text{per day}}}$$

We compute the *average daily cost* $c(x)$ by dividing the cost per cycle by the number of days x in the cycle (see Figure 4.41).

$$c(x) = \frac{5000}{x} + 25x, \qquad x > 0.$$

As $x \to 0$ and as $x \to \infty$, the average daily cost becomes large. So we expect a minimum to exist, but where? Our goal is to determine the number of days x between deliveries that provides the absolute minimum cost.

We find the critical points by determining where the derivative is equal to zero:

$$c'(x) = -\frac{5000}{x^2} + 25 = 0$$
$$x = \pm\sqrt{200} \approx \pm 14.14.$$

Of the two critical points, only $\sqrt{200}$ lies in the domain of $c(x)$. The critical-point value of the average daily cost is

$$c\left(\sqrt{200}\right) = \frac{5000}{\sqrt{200}} + 25\sqrt{200} = 500\sqrt{2} \approx \$707.11.$$

We note that $c(x)$ is defined over the open interval $(0, \infty)$ with $c''(x) = 10000/x^3 > 0$. Thus, an absolute minimum exists at $x = \sqrt{200} \approx 14.14$ days.

The cabinetmaker should schedule a delivery of $5(14) = 70$ units of the exotic wood every 14 days. ■

In Examples 5 and 6 we allowed the number of items x to be any positive real number. In reality it usually only makes sense for x to be a positive integer (or zero). If we must round our answers, should we round up or down?

EXAMPLE 7 Sensitivity of the Minimum Cost

Should we round the number of days between deliveries up or down for the best solution in Example 6?

Solution The average daily cost will increase by about \$0.03 if we round down from 14.14 to 14 days:

$$c(14) = \frac{5000}{14} + 25(14) = \$707.14$$

and

$$c(14) - c(14.14) = \$707.14 - \$707.11 = \$0.03.$$

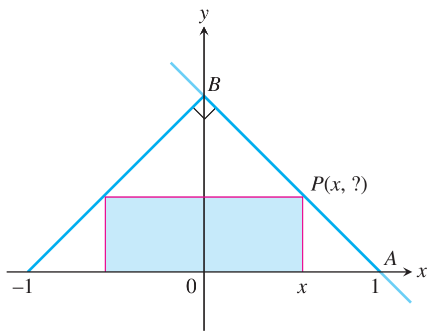
On the other hand, $c(15) = \$708.33$, and our cost would increase by $\$708.33 - \$707.11 = \$1.22$ if we round up. Thus, it is better that we round x down to 14 days. ■

EXERCISES 4.5

Whenever you are maximizing or minimizing a function of a single variable, we urge you to graph it over the domain that is appropriate to the problem you are solving. The graph will provide insight before you calculate and will furnish a visual context for understanding your answer.

Applications in Geometry

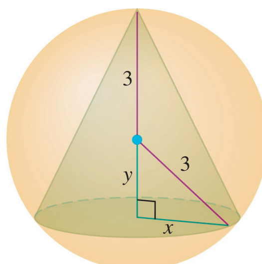
1. **Minimizing perimeter** What is the smallest perimeter possible for a rectangle whose area is 16 in.^{2}, and what are its dimensions?
2. Show that among all rectangles with an 8-m perimeter, the one with largest area is a square.
3. The figure shows a rectangle inscribed in an isosceles right triangle whose hypotenuse is 2 units long.
   a. Express the y-coordinate of P in terms of x. (*Hint*: Write an equation for the line AB.)
   b. Express the area of the rectangle in terms of x.
   c. What is the largest area the rectangle can have, and what are its dimensions?



4. A rectangle has its base on the x-axis and its upper two vertices on the parabola $y = 12 - x^2$. What is the largest area the rectangle can have, and what are its dimensions?
5. You are planning to make an open rectangular box from an 8-in.-by-15-in. piece of cardboard by cutting congruent squares from the corners and folding up the sides. What are the dimensions of the box of largest volume you can make this way, and what is its volume?
6. You are planning to close off a corner of the first quadrant with a line segment 20 units long running from $(a, 0)$ to $(0, b)$. Show that the area of the triangle enclosed by the segment is largest when $a = b$.
7. **The best fencing plan** A rectangular plot of farmland will be bounded on one side by a river and on the other three sides by a single-strand electric fence. With 800m of wire at your disposal, what is the largest area you can enclose, and what are its dimensions?
8. **The shortest fence** A 216 m^2 rectangular pea patch is to be enclosed by a fence and divided into two equal parts by another fence parallel to one of the sides. What dimensions for the outer rectangle will require the smallest total length of fence? How much fence will be needed?
9. **Designing a tank** Your iron works has contracted to design and build a 500 ft^3, square-based, open-top, rectangular steel holding tank for a paper company. The tank is to be made by welding thin stainless steel plates together along their edges. As the production engineer, your job is to find dimensions for the base and height that will make the tank weigh as little as possible.
   a. What dimensions do you tell the shop to use?
   b. Briefly describe how you took weight into account.
10. **Catching rainwater** A 1125 ft^3 open-top rectangular tank with a square base x ft on a side and y ft deep is to be built with its top flush with the ground to catch runoff water. The costs associated with the tank involve not only the material from which the tank is made but also an excavation charge proportional to the product xy.
    a. If the total cost is
    $$c = 5(x^2 + 4xy) + 10xy,$$
    what values of x and y will minimize it?
    b. Give a possible scenario for the cost function in part (a).

11. Designing a poster You are designing a rectangular poster to contain 50 in.^{2} of printing with a 4-in. margin at the top and bottom and a 2-in. margin at each side. What overall dimensions will minimize the amount of paper used?

12. Find the volume of the largest right circular cone that can be inscribed in a sphere of radius 3.



13. Two sides of a triangle have lengths a and b, and the angle between them is θ. What value of θ will maximize the triangle's area? (*Hint:* $A = (1/2)ab \sin \theta$.)

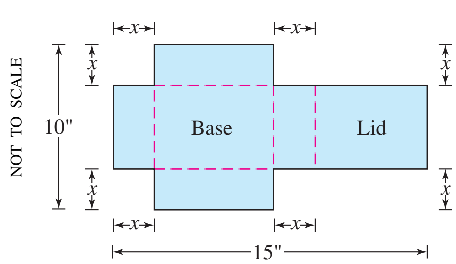
14. Designing a can What are the dimensions of the lightest open-top right circular cylindrical can that will hold a volume of 1000 cm^3? Compare the result here with the result in Example 2.

15. Designing a can You are designing a 1000 cm^3 right circular cylindrical can whose manufacture will take waste into account. There is no waste in cutting the aluminum for the side, but the top and bottom of radius r will be cut from squares that measure $2r$ units on a side. The total amount of aluminum used up by the can will therefore be

$$A = 8r^2 + 2\pi rh$$

rather than the $A = 2\pi r^2 + 2\pi rh$ in Example 2. In Example 2, the ratio of h to r for the most economical can was 2 to 1. What is the ratio now?

16. Designing a box with a lid A piece of cardboard measures 10 in. by 15 in. Two equal squares are removed from the corners of a 10-in. side as shown in the figure. Two equal rectangles are removed from the other corners so that the tabs can be folded to form a rectangular box with lid.



a. Write a formula $V(x)$ for the volume of the box.

b. Find the domain of V for the problem situation and graph V over this domain.

c. Use a graphical method to find the maximum volume and the value of x that gives it.

d. Confirm your result in part (c) analytically.

17. Designing a suitcase A 24-in.-by-36-in. sheet of cardboard is folded in half to form a 24-in.-by-18-in. rectangle as shown in the accompanying figure. Then four congruent squares of side length x are cut from the corners of the folded rectangle. The sheet is unfolded, and the six tabs are folded up to form a box with sides and a lid.

a. Write a formula $V(x)$ for the volume of the box.

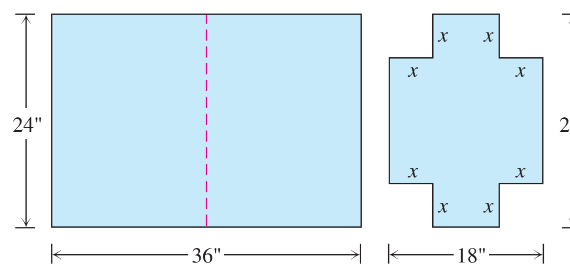
b. Find the domain of V for the problem situation and graph V over this domain.

c. Use a graphical method to find the maximum volume and the value of x that gives it.

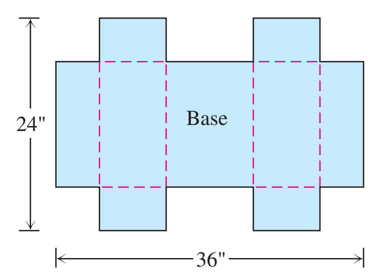
d. Confirm your result in part (c) analytically.

e. Find a value of x that yields a volume of 1120 in.^{3}.

f. Write a paragraph describing the issues that arise in part (b).



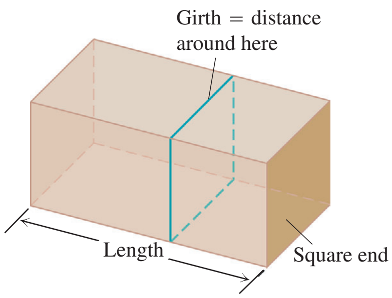
The sheet is then unfolded.



18. A rectangle is to be inscribed under the arch of the curve $y = 4 \cos (0.5x)$ from $x = -\pi$ to $x = \pi$. What are the dimensions of the rectangle with largest area, and what is the largest area?

19. Find the dimensions of a right circular cylinder of maximum volume that can be inscribed in a sphere of radius 10 cm. What is the maximum volume?

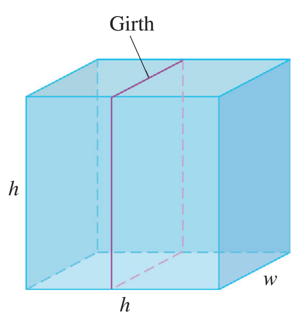
20. a. The U.S. Postal Service will accept a box for domestic shipment only if the sum of its length and girth (distance around) does not exceed 108 in. What dimensions will give a box with a square end the largest possible volume?



b. Graph the volume of a 108-in. box (length plus girth equals 108 in.) as a function of its length and compare what you see with your answer in part (a).

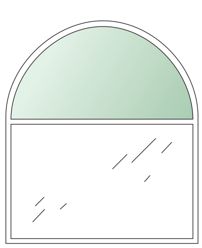
21. (*Continuation of Exercise 20.*)

a. Suppose that instead of having a box with square ends you have a box with square sides so that its dimensions are h by h by w and the girth is $2h + 2w$. What dimensions will give the box its largest volume now?



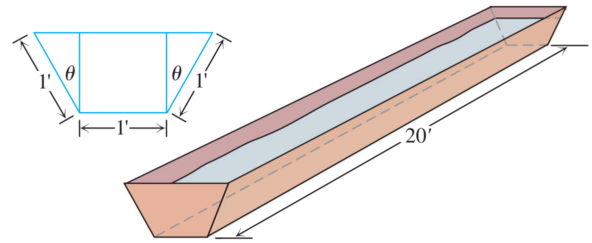
b. Graph the volume as a function of h and compare what you see with your answer in part (a).

22. A window is in the form of a rectangle surmounted by a semicircle. The rectangle is of clear glass, whereas the semicircle is of tinted glass that transmits only half as much light per unit area as clear glass does. The total perimeter is fixed. Find the proportions of the window that will admit the most light. Neglect the thickness of the frame.



23. A silo (base not included) is to be constructed in the form of a cylinder surmounted by a hemisphere. The cost of construction per square unit of surface area is twice as great for the hemisphere as it is for the cylindrical sidewall. Determine the dimensions to be used if the volume is fixed and the cost of construction is to be kept to a minimum. Neglect the thickness of the silo and waste in construction.

24. The trough in the figure is to be made to the dimensions shown. Only the angle θ can be varied. What value of θ will maximize the trough's volume?

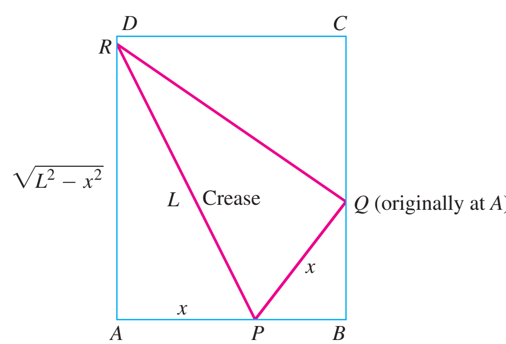


25. Paper folding A rectangular sheet of 8.5-in.-by-11-in. paper is placed on a flat surface. One of the corners is placed on the opposite longer edge, as shown in the figure, and held there as the paper is smoothed flat. The problem is to make the length of the crease as small as possible. Call the length L. Try it with paper.

a. Show that $L^2 = 2x^3/(2x - 8.5)$.

b. What value of x minimizes L^2?

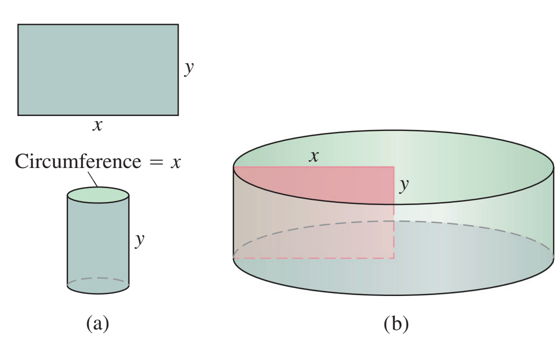
c. What is the minimum value of L?



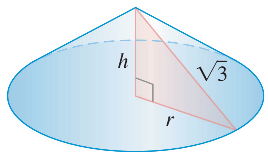
26. Constructing cylinders Compare the answers to the following two construction problems.

a. A rectangular sheet of perimeter 36 cm and dimensions x cm by y cm to be rolled into a cylinder as shown in part (a) of the figure. What values of x and y give the largest volume?

b. The same sheet is to be revolved about one of the sides of length y to sweep out the cylinder as shown in part (b) of the figure. What values of x and y give the largest volume?



27. Constructing cones A right triangle whose hypotenuse is $\sqrt{3}$ m long is revolved about one of its legs to generate a right circular cone. Find the radius, height, and volume of the cone of greatest volume that can be made this way.



28. What value of a makes $f(x) = x^2 + (a/x)$ have

a. a local minimum at $x = 2$?

b. a point of inflection at $x = 1$?

29. Show that $f(x) = x^2 + (a/x)$ cannot have a local maximum for any value of a.

30. What values of a and b make $f(x) = x^3 + ax^2 + bx$ have

a. a local maximum at $x = -1$ and a local minimum at $x = 3$?

b. a local minimum at $x = 4$ and a point of inflection at $x = 1$?

Physical Applications

31. Vertical motion The height of an object moving vertically is given by

$$s = -16t^2 + 96t + 112,$$

with s in feet and t in seconds. Find

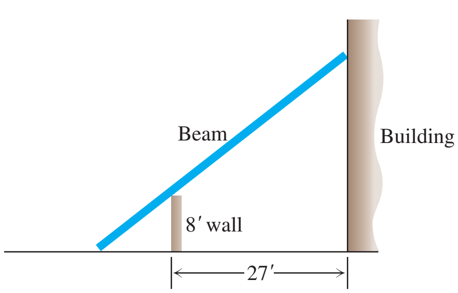
a. the object's velocity when $t = 0$

b. its maximum height and when it occurs

c. its velocity when $s = 0$.

32. Quickest route Jane is 2 mi offshore in a boat and wishes to reach a coastal village 6 mi down a straight shoreline from the point nearest the boat. She can row 2 mph and can walk 5 mph. Where should she land her boat to reach the village in the least amount of time?

33. Shortest beam The 8-ft wall shown here stands 27 ft from the building. Find the length of the shortest straight beam that will reach to the side of the building from the ground outside the wall.

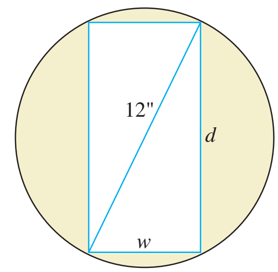


T 34. Strength of a beam The strength S of a rectangular wooden beam is proportional to its width times the square of its depth. (See accompanying figure.)

a. Find the dimensions of the strongest beam that can be cut from a 12-in.-diameter cylindrical log.

b. Graph S as a function of the beam's width w, assuming the proportionality constant to be $k = 1$. Reconcile what you see with your answer in part (a).

c. On the same screen, graph S as a function of the beam's depth d, again taking $k = 1$. Compare the graphs with one another and with your answer in part (a). What would be the effect of changing to some other value of k? Try it.



T 35. Stiffness of a beam The stiffness S of a rectangular beam is proportional to its width times the cube of its depth.

a. Find the dimensions of the stiffest beam that can be cut from a 12-in.-diameter cylindrical log.

b. Graph S as a function of the beam's width w, assuming the proportionality constant to be $k = 1$. Reconcile what you see with your answer in part (a).

c. On the same screen, graph S as a function of the beam's depth d, again taking $k = 1$. Compare the graphs with one another and with your answer in part (a). What would be the effect of changing to some other value of k? Try it.

36. Motion on a line The positions of two particles on the s-axis are $s_1 = \sin t$ and $s_2 = \sin(t + \pi/3)$, with s_1 and s_2 in meters and t in seconds.

a. At what time(s) in the interval $0 \le t \le 2\pi$ do the particles meet?

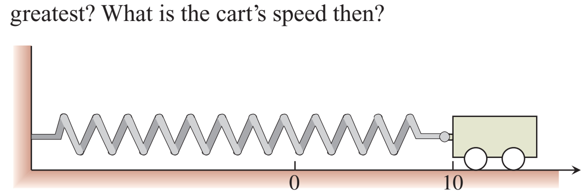
b. What is the farthest apart that the particles ever get?

c. When in the interval $0 \le t \le 2\pi$ is the distance between the particles changing the fastest?

37. Frictionless cart A small frictionless cart, attached to the wall by a spring, is pulled 10 cm from its rest position and released at time $t = 0$ to roll back and forth for 4 sec. Its position at time t is $s = 10 \cos \pi t$.

a. What is the cart's maximum speed? When is the cart moving that fast? Where is it then? What is the magnitude of the acceleration then?

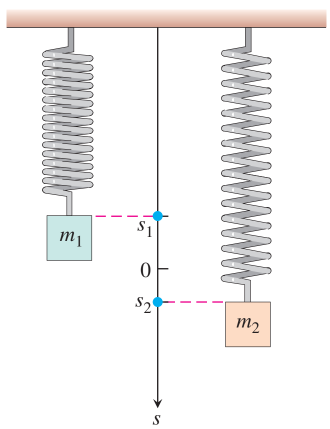
b. Where is the cart when the magnitude of the acceleration is greatest? What is the cart's speed then?



38. Two masses hanging side by side from springs have positions $s_1 = 2 \sin t$ and $s_2 = \sin 2t$, respectively.

a. At what times in the interval $0 < t$ do the masses pass each other? (*Hint:* $\sin 2t = 2 \sin t \cos t$.)

b. When in the interval $0 \le t \le 2\pi$ is the vertical distance between the masses the greatest? What is this distance? (*Hint:* $\cos 2t = 2 \cos^2 t - 1$.)



39. Distance between two ships At noon, ship A was 12 nautical miles due north of ship B. Ship A was sailing south at 12 knots (nautical miles per hour; a nautical mile is 2000 yd) and continued to do so all day. Ship B was sailing east at 8 knots and continued to do so all day.

a. Start counting time with $t = 0$ at noon and express the distance s between the ships as a function of t.

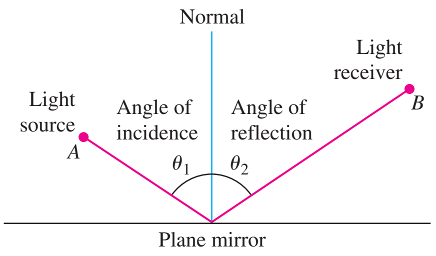
b. How rapidly was the distance between the ships changing at noon? One hour later?

c. The visibility that day was 5 nautical miles. Did the ships ever sight each other?

T **d.** Graph s and ds/dt together as functions of t for $-1 \le t \le 3$, using different colors if possible. Compare the graphs and reconcile what you see with your answers in parts (b) and (c).

e. The graph of ds/dt looks as if it might have a horizontal asymptote in the first quadrant. This in turn suggests that ds/dt approaches a limiting value as $t \to \infty$. What is this value? What is its relation to the ships' individual speeds?

40. Fermat's principle in optics Fermat's principle in optics states that light always travels from one point to another along a path that minimizes the travel time. Light from a source A is reflected by a plane mirror to a receiver at point B, as shown in the figure. Show that for the light to obey Fermat's principle, the angle of incidence must equal the angle of reflection, both measured from the line normal to the reflecting surface. (This result can also be derived without calculus. There is a purely geometric argument, which you may prefer.)



41. Tin pest When metallic tin is kept below 13.2°C, it slowly becomes brittle and crumbles to a gray powder. Tin objects eventually crumble to this gray powder spontaneously if kept in a cold climate for years. The Europeans who saw tin organ pipes in their churches crumble away years ago called the change *tin pest* because it seemed to be contagious, and indeed it was, for the gray powder is a catalyst for its own formation.

A *catalyst* for a chemical reaction is a substance that controls the rate of reaction without undergoing any permanent change in itself. An *autocatalytic reaction* is one whose product is a catalyst for its own formation. Such a reaction may proceed slowly at first if the amount of catalyst present is small and slowly again at the end, when most of the original substance is used up. But in between, when both the substance and its catalyst product are abundant, the reaction proceeds at a faster pace.

In some cases, it is reasonable to assume that the rate $v = dx/dt$ of the reaction is proportional both to the amount of the original substance present and to the amount of product. That is, v may be considered to be a function of x alone, and

$$v = kx(a - x) = kax - kx^2,$$

where

$x =$ the amount of product

$a =$ the amount of substance at the beginning

$k =$ a positive constant.

At what value of x does the rate v have a maximum? What is the maximum value of v?

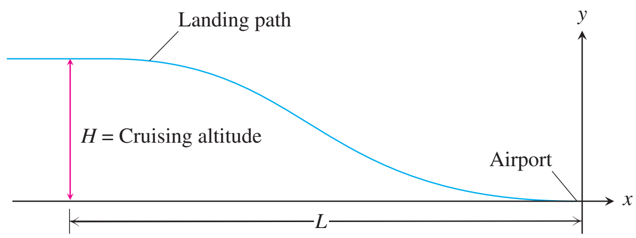
42. Airplane landing path An airplane is flying at altitude H when it begins its descent to an airport runway that is at horizontal ground distance L from the airplane, as shown in the figure. Assume that the landing path of the airplane is the graph of a cubic polynomial function $y = ax^3 + bx^2 + cx + d$, where $y(-L) = H$ and $y(0) = 0$.

a. What is dy/dx at $x = 0$?

b. What is dy/dx at $x = -L$?

c. Use the values for dy/dx at $x = 0$ and $x = -L$ together with $y(0) = 0$ and $y(-L) = H$ to show that

$$y(x) = H\left[2\left(\frac{x}{L}\right)^3 + 3\left(\frac{x}{L}\right)^2\right].$$



Business and Economics

43. It costs you c dollars each to manufacture and distribute backpacks. If the backpacks sell at x dollars each, the number sold is given by

$$n = \frac{a}{x - c} + b(100 - x),$$

where a and b are positive constants. What selling price will bring a maximum profit?

44. You operate a tour service that offers the following rates:

$200 per person if 50 people (the minimum number to book the tour) go on the tour.

For each additional person, up to a maximum of 80 people total, the rate per person is reduced by $2.

It costs $6000 (a fixed cost) plus $32 per person to conduct the tour. How many people does it take to maximize your profit?

45. Wilson lot size formula One of the formulas for inventory management says that the average weekly cost of ordering, paying for, and holding merchandise is

$$A(q) = \frac{km}{q} + cm + \frac{hq}{2},$$

where q is the quantity you order when things run low (shoes, radios, brooms, or whatever the item might be), k is the cost of placing an order (the same, no matter how often you order), c is the cost of one item (a constant), m is the number of items sold each week (a constant), and h is the weekly holding cost per item (a constant that takes into account things such as space, utilities, insurance, and security).

a. Your job, as the inventory manager for your store, is to find the quantity that will minimize $A(q)$. What is it? (The formula you get for the answer is called the *Wilson lot size formula*.)

b. Shipping costs sometimes depend on order size. When they do, it is more realistic to replace k by $k + bq$, the sum of k and a constant multiple of q. What is the most economical quantity to order now?

46. Production level Prove that the production level (if any) at which average cost is smallest is a level at which the average cost equals marginal cost.

47. Show that if $r(x) = 6x$ and $c(x) = x^3 - 6x^2 + 15x$ are your revenue and cost functions, then the best you can do is break even (have revenue equal cost).

48. Production level Suppose that $c(x) = x^3 - 20x^2 + 20{,}000x$ is the cost of manufacturing x items. Find a production level that will minimize the average cost of making x items.

49. Average daily cost In Example 6, assume for any material that a cost of d is incurred per delivery, the storage cost is s dollars per unit stored per day, and the production rate is p units per day.

a. How much should be delivered every x days?

b. Show that

$$\text{cost per cycle} = d + \frac{px}{2}sx.$$

c. Find the time between deliveries x^* and the amount to deliver that minimizes the *average daily cost* of delivery and storage.

d. Show that x^* occurs at the intersection of the hyperbola $y = d/x$ and the line $y = psx/2$.

50. Minimizing average cost Suppose that $c(x) = 2000 + 96x + 4x^{3/2}$, where x represents thousands of units. Is there a production level that minimizes average cost? If so, what is it?

Medicine

51. Sensitivity to medicine (*Continuation of Exercise 50, Section 3.2.*) Find the amount of medicine to which the body is most sensitive by finding the value of M that maximizes the derivative dR/dM, where

$$R = M^2\left(\frac{C}{2} - \frac{M}{3}\right)$$

and C is a constant.

52. How we cough

a. When we cough, the trachea (windpipe) contracts to increase the velocity of the air going out. This raises the questions of how much it should contract to maximize the velocity and whether it really contracts that much when we cough.

Under reasonable assumptions about the elasticity of the tracheal wall and about how the air near the wall is slowed by friction, the average flow velocity v can be modeled by the equation

$$v = c(r_0 - r)r^2 \text{ cm/sec}, \qquad \frac{r_0}{2} \le r \le r_0,$$

where r_0 is the rest radius of the trachea in centimeters and c is a positive constant whose value depends in part on the length of the trachea.

Show that v is greatest when $r = (2/3)r_0$, that is, when the trachea is about 33% contracted. The remarkable fact is that X-ray photographs confirm that the trachea contracts about this much during a cough.

T **b.** Take r_0 to be 0.5 and c to be 1 and graph v over the interval $0 \le r \le 0.5$. Compare what you see with the claim that v is at a maximum when $r = (2/3)r_0$.

Theory and Examples

53. An inequality for positive integers Show that if a, b, c, and d are positive integers, then

$$\frac{(a^2+1)(b^2+1)(c^2+1)(d^2+1)}{abcd} \geq 16.$$

54. The derivative dt/dx in Example 4

a. Show that

$$f(x) = \frac{x}{\sqrt{a^2+x^2}}$$

is an increasing function of x.

b. Show that

$$g(x) = \frac{d-x}{\sqrt{b^2+(d-x)^2}}$$

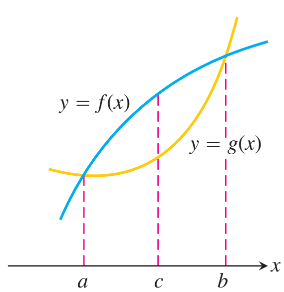
is a decreasing function of x.

c. Show that

$$\frac{dt}{dx} = \frac{x}{c_1\sqrt{a^2+x^2}} - \frac{d-x}{c_2\sqrt{b^2+(d-x)^2}}$$

is an increasing function of x.

55. Let $f(x)$ and $g(x)$ be the differentiable functions graphed here. Point c is the point where the vertical distance between the curves is the greatest. Is there anything special about the tangents to the two curves at c? Give reasons for your answer.



56. You have been asked to determine whether the function $f(x) = 3 + 4\cos x + \cos 2x$ is ever negative.

a. Explain why you need to consider values of x only in the interval $[0, 2\pi]$.

b. Is f ever negative? Explain.

57. a. The function $y = \cot x - \sqrt{2}\csc x$ has an absolute maximum value on the interval $0 < x < \pi$. Find it.

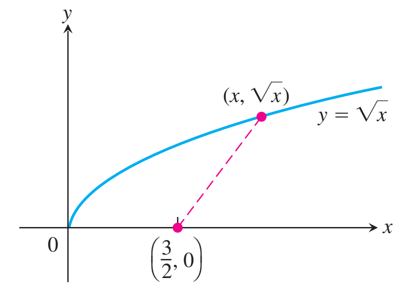
T **b.** Graph the function and compare what you see with your answer in part (a).

58. a. The function $y = \tan x + 3\cot x$ has an absolute minimum value on the interval $0 < x < \pi/2$. Find it.

T **b.** Graph the function and compare what you see with your answer in part (a).

59. a. How close does the curve $y = \sqrt{x}$ come to the point $(3/2, 0)$? (*Hint*: If you minimize the *square* of the distance, you can avoid square roots.)

T **b.** Graph the distance function and $y = \sqrt{x}$ together and reconcile what you see with your answer in part (a).



60. a. How close does the semicircle $y = \sqrt{16 - x^2}$ come to the point $\left(1, \sqrt{3}\right)$?

T **b.** Graph the distance function and $y = \sqrt{16 - x^2}$ together and reconcile what you see with your answer in part (a).

COMPUTER EXPLORATIONS

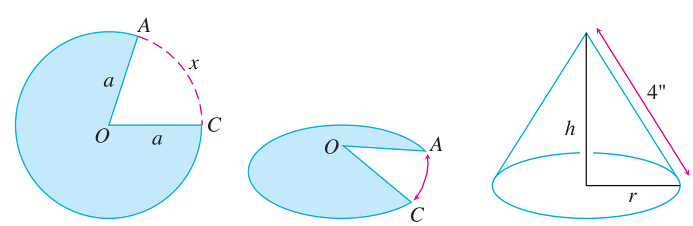
In Exercises 61 and 62, you may find it helpful to use a CAS.

61. Generalized cone problem A cone of height h and radius r is constructed from a flat, circular disk of radius a in. by removing a sector AOC of arc length x in. and then connecting the edges OA and OC.

a. Find a formula for the volume V of the cone in terms of x and a.

b. Find r and h in the cone of maximum volume for $a = 4, 5, 6, 8$.

c. Find a simple relationship between r and h that is independent of a for the cone of maximum volume. Explain how you arrived at your relationship.

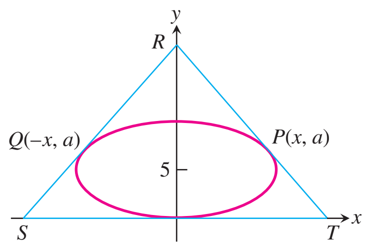


NOT TO SCALE

62. Circumscribing an ellipse Let $P(x, a)$ and $Q(-x, a)$ be two points on the upper half of the ellipse

$$\frac{x^2}{100} + \frac{(y-5)^2}{25} = 1$$

centered at $(0, 5)$. A triangle RST is formed by using the tangent lines to the ellipse at Q and P as shown in the figure.



a. Show that the area of the triangle is

$$A(x) = -f'(x)\left[x - \frac{f(x)}{f'(x)}\right]^2,$$

where $y = f(x)$ is the function representing the upper half of the ellipse.

b. What is the domain of A? Draw the graph of A. How are the asymptotes of the graph related to the problem situation?

c. Determine the height of the triangle with minimum area. How is it related to the y coordinate of the center of the ellipse?

d. Repeat parts (a) through (c) for the ellipse

$$\frac{x^2}{C^2} + \frac{(y - B)^2}{B^2} = 1$$

centered at $(0, B)$. Show that the triangle has minimum area when its height is $3B$.

4.6 Indeterminate Forms and L'Hôpital's Rule

HISTORICAL BIOGRAPHY

Guillaume François Antoine de l'Hôpital (1661–1704)

Biography

John Bernoulli discovered a rule for calculating limits of fractions whose numerators and denominators both approach zero or $+\infty$. The rule is known today as **l'Hôpital's Rule**, after Guillaume de l'Hôpital. He was a French nobleman who wrote the first introductory differential calculus text, where the rule first appeared in print.

Indeterminate Form 0/0

If the continuous functions $f(x)$ and $g(x)$ are both zero at $x = a$, then

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

cannot be found by substituting $x = a$. The substitution produces $0/0$, a meaningless expression, which we cannot evaluate. We use $0/0$ as a notation for an expression known as an **indeterminate form**. Sometimes, but not always, limits that lead to indeterminate forms may be found by cancellation, rearrangement of terms, or other algebraic manipulations. This was our experience in Chapter 2. It took considerable analysis in Section 2.4 to find $\lim_{x \to 0} (\sin x)/x$. But we have had success with the limit

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a},$$

from which we calculate derivatives and which always produces the equivalent of $0/0$ when we substitute $x = a$. L'Hôpital's Rule enables us to draw on our success with derivatives to evaluate limits that otherwise lead to indeterminate forms.

THEOREM 6 L'Hôpital's Rule (First Form)

Suppose that $f(a) = g(a) = 0$, that $f'(a)$ and $g'(a)$ exist, and that $g'(a) \neq 0$. Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}.$$

Caution
To apply l'Hôpital's Rule to f/g, divide the derivative of f by the derivative of g. Do not fall into the trap of taking the derivative of f/g. The quotient to use is f'/g', not $(f/g)'$.

Proof Working backward from $f'(a)$ and $g'(a)$, which are themselves limits, we have

$$\frac{f'(a)}{g'(a)} = \frac{\lim_{x\to a} \dfrac{f(x)-f(a)}{x-a}}{\lim_{x\to a} \dfrac{g(x)-g(a)}{x-a}} = \lim_{x\to a} \frac{\dfrac{f(x)-f(a)}{x-a}}{\dfrac{g(x)-g(a)}{x-a}}$$

$$= \lim_{x\to a} \frac{f(x)-f(a)}{g(x)-g(a)} = \lim_{x\to a} \frac{f(x)-0}{g(x)-0} = \lim_{x\to a} \frac{f(x)}{g(x)}.$$ ■

EXAMPLE 1 Using L'Hôpital's Rule

(a) $\lim_{x\to 0} \dfrac{3x - \sin x}{x} = \dfrac{3 - \cos x}{1}\Bigg|_{x=0} = 2$

(b) $\lim_{x\to 0} \dfrac{\sqrt{1+x} - 1}{x} = \dfrac{\dfrac{1}{2\sqrt{1+x}}}{1}\Bigg|_{x=0} = \dfrac{1}{2}$ ■

Sometimes after differentiation, the new numerator and denominator both equal zero at $x = a$, as we see in Example 2. In these cases, we apply a stronger form of l'Hôpital's Rule.

> **THEOREM 7 L'Hôpital's Rule (Stronger Form)**
> Suppose that $f(a) = g(a) = 0$, that f and g are differentiable on an open interval I containing a, and that $g'(x) \neq 0$ on I if $x \neq a$. Then
> $$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)},$$
> assuming that the limit on the right side exists.

Before we give a proof of Theorem 7, let's consider an example.

EXAMPLE 2 Applying the Stronger Form of L'Hôpital's Rule

(a) $\lim_{x\to 0} \dfrac{\sqrt{1+x} - 1 - x/2}{x^2}$ $\quad \frac{0}{0}$

$$= \lim_{x\to 0} \frac{(1/2)(1+x)^{-1/2} - 1/2}{2x} \qquad \text{Still } \tfrac{0}{0}\text{; differentiate again.}$$

$$= \lim_{x\to 0} \frac{-(1/4)(1+x)^{-3/2}}{2} = -\frac{1}{8} \qquad \text{Not } \tfrac{0}{0}\text{; limit is found.}$$

(b) $\lim_{x\to 0} \dfrac{x - \sin x}{x^3}$ $\quad \frac{0}{0}$

$$= \lim_{x\to 0} \frac{1 - \cos x}{3x^2} \qquad \text{Still } \tfrac{0}{0}$$

$$= \lim_{x\to 0} \frac{\sin x}{6x} \qquad \text{Still } \tfrac{0}{0}$$

$$= \lim_{x\to 0} \frac{\cos x}{6} = \frac{1}{6} \qquad \text{Not } \tfrac{0}{0}\text{; limit is found.}$$ ■

The proof of the stronger form of l'Hôpital's Rule is based on Cauchy's Mean Value Theorem, a Mean Value Theorem that involves two functions instead of one. We prove Cauchy's Theorem first and then show how it leads to l'Hôpital's Rule.

HISTORICAL BIOGRAPHY

Augustin-Louis Cauchy (1789–1857)

THEOREM 8 Cauchy's Mean Value Theorem

Suppose functions f and g are continuous on $[a, b]$ and differentiable throughout (a, b) and also suppose $g'(x) \neq 0$ throughout (a, b). Then there exists a number c in (a, b) at which

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

Proof We apply the Mean Value Theorem of Section 4.2 twice. First we use it to show that $g(a) \neq g(b)$. For if $g(b)$ did equal $g(a)$, then the Mean Value Theorem would give

$$g'(c) = \frac{g(b) - g(a)}{b - a} = 0$$

for some c between a and b, which cannot happen because $g'(x) \neq 0$ in (a, b).

We next apply the Mean Value Theorem to the function

$$F(x) = f(x) - f(a) - \frac{f(b) - f(a)}{g(b) - g(a)}[g(x) - g(a)].$$

This function is continuous and differentiable where f and g are, and $F(b) = F(a) = 0$. Therefore, there is a number c between a and b for which $F'(c) = 0$. When expressed in terms of f and g, this equation becomes

$$F'(c) = f'(c) - \frac{f(b) - f(a)}{g(b) - g(a)}[g'(c)] = 0$$

or

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

■

Notice that the Mean Value Theorem in Section 4.2 is Theorem 8 with $g(x) = x$.

Cauchy's Mean Value Theorem has a geometric interpretation for a curve C defined by the parametric equations $x = g(t)$ and $y = f(t)$. From Equation (2) in Section 3.5, the slope of the parametric curve at t is given by

$$\frac{dy/dt}{dx/dt} = \frac{f'(t)}{g'(t)},$$

so $f'(c)/g'(c)$ is the slope of the tangent to the curve when $t = c$. The secant line joining the two points $(g(a), f(a))$ and $(g(b), f(b))$ on C has slope

$$\frac{f(b) - f(a)}{g(b) - g(a)}.$$

Theorem 8 says that there is a parameter value c in the interval (a, b) for which the slope of the tangent to the curve at the point $(g(c), f(c))$ is the same as the slope of the secant line joining the points $(g(a), f(a))$ and $(g(b), f(b))$. This geometric result is shown in Figure 4.42. Note that more than one such value c of the parameter may exist.

We now prove Theorem 7.

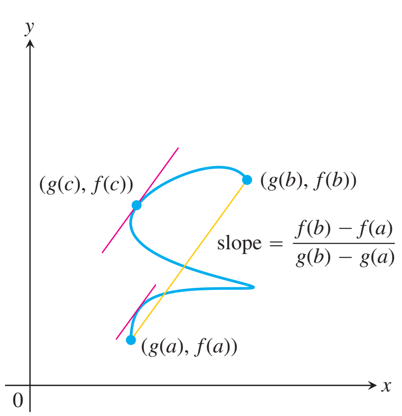


FIGURE 4.42 There is at least one value of the parameter $t = c$, $a < c < b$, for which the slope of the tangent to the curve at $(g(c), f(c))$ is the same as the slope of the secant line joining the points $(g(a), f(a))$ and $(g(b), f(b))$.

Proof of the Stronger Form of l'Hôpital's Rule We first establish the limit equation for the case $x \to a^+$. The method needs almost no change to apply to $x \to a^-$, and the combination of these two cases establishes the result.

Suppose that x lies to the right of a. Then $g'(x) \neq 0$, and we can apply Cauchy's Mean Value Theorem to the closed interval from a to x. This step produces a number c between a and x such that

$$\frac{f'(c)}{g'(c)} = \frac{f(x) - f(a)}{g(x) - g(a)}.$$

But $f(a) = g(a) = 0$, so

$$\frac{f'(c)}{g'(c)} = \frac{f(x)}{g(x)}.$$

As x approaches a, c approaches a because it always lies between a and x. Therefore,

$$\lim_{x \to a^+} \frac{f(x)}{g(x)} = \lim_{c \to a^+} \frac{f'(c)}{g'(c)} = \lim_{x \to a^+} \frac{f'(x)}{g'(x)},$$

which establishes l'Hôpital's Rule for the case where x approaches a from above. The case where x approaches a from below is proved by applying Cauchy's Mean Value Theorem to the closed interval $[x, a]$, $x < a$. ■

Most functions encountered in the real world and most functions in this book satisfy the conditions of l'Hôpital's Rule.

Using L'Hôpital's Rule

To find

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

by l'Hôpital's Rule, continue to differentiate f and g, so long as we still get the form $0/0$ at $x = a$. But as soon as one or the other of these derivatives is different from zero at $x = a$ we stop differentiating. L'Hôpital's Rule does not apply when either the numerator or denominator has a finite nonzero limit.

EXAMPLE 3 Incorrectly Applying the Stronger Form of L'Hôpital's Rule

$$\lim_{x \to 0} \frac{1 - \cos x}{x + x^2} \qquad \frac{0}{0}$$

$$= \lim_{x \to 0} \frac{\sin x}{1 + 2x} = \frac{0}{1} = 0 \qquad \text{Not } \frac{0}{0}\text{; limit is found.}$$

Up to now the calculation is correct, but if we continue to differentiate in an attempt to apply l'Hôpital's Rule once more, we get

$$\lim_{x \to 0} \frac{1 - \cos x}{x + x^2} = \lim_{x \to 0} \frac{\sin x}{1 + 2x} = \lim_{x \to 0} \frac{\cos x}{2} = \frac{1}{2},$$

which is wrong. L'Hôpital's Rule can only be applied to limits which give indeterminate forms, and $0/1$ is not an indeterminate form. ■

L'Hôpital's Rule applies to one-sided limits as well, which is apparent from the proof of Theorem 7.

EXAMPLE 4 Using L'Hôpital's Rule with One-Sided Limits

Recall that ∞ and $+\infty$ mean the same thing.

(a) $\displaystyle \lim_{x \to 0^+} \frac{\sin x}{x^2}$ $\quad \frac{0}{0}$

$\displaystyle = \lim_{x \to 0^+} \frac{\cos x}{2x} = \infty$ $\quad$ Positive for $x > 0$.

(b) $\displaystyle \lim_{x \to 0^-} \frac{\sin x}{x^2}$ $\quad \frac{0}{0}$

$\displaystyle = \lim_{x \to 0^-} \frac{\cos x}{2x} = -\infty$ $\quad$ Negative for $x < 0$. ■

Indeterminate Forms ∞/∞, $\infty \cdot 0$, $\infty - \infty$

Sometimes when we try to evaluate a limit as $x \to a$ by substituting $x = a$ we get an ambiguous expression like ∞/∞, $\infty \cdot 0$, or $\infty - \infty$, instead of $0/0$. We first consider the form ∞/∞.

In more advanced books it is proved that l'Hôpital's Rule applies to the indeterminate form ∞/∞ as well as to $0/0$. If $f(x) \to \pm\infty$ and $g(x) \to \pm\infty$ as $x \to a$, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

provided the limit on the right exists. In the notation $x \to a$, a may be either finite or infinite. Moreover $x \to a$ may be replaced by the one-sided limits $x \to a^+$ or $x \to a^-$.

EXAMPLE 5 Working with the Indeterminate Form ∞/∞

Find

(a) $\displaystyle \lim_{x \to \pi/2} \frac{\sec x}{1 + \tan x}$

(b) $\displaystyle \lim_{x \to \infty} \frac{x - 2x^2}{3x^2 + 5x}$

Solution

(a) The numerator and denominator are discontinuous at $x = \pi/2$, so we investigate the one-sided limits there. To apply l'Hôpital's Rule, we can choose I to be any open interval with $x = \pi/2$ as an endpoint.

$\displaystyle \lim_{x \to (\pi/2)^-} \frac{\sec x}{1 + \tan x}$ $\quad \frac{\infty}{\infty}$ from the left

$$= \lim_{x \to (\pi/2)^-} \frac{\sec x \tan x}{\sec^2 x} = \lim_{x \to (\pi/2)^-} \sin x = 1$$

The right-hand limit is 1 also, with $(-\infty)/(-\infty)$ as the indeterminate form. Therefore, the two-sided limit is equal to 1.

(b) $\displaystyle \lim_{x \to \infty} \frac{x - 2x^2}{3x^2 + 5x} = \lim_{x \to \infty} \frac{1 - 4x}{6x + 5} = \lim_{x \to \infty} \frac{-4}{6} = -\frac{2}{3}.$ ■

Next we turn our attention to the indeterminate forms $\infty \cdot 0$ and $\infty - \infty$. Sometimes these forms can be handled by using algebra to convert them to a $0/0$ or ∞/∞ form. Here again we do not mean to suggest that $\infty \cdot 0$ or $\infty - \infty$ is a number. They are only notations for functional behaviors when considering limits. Here are examples of how we might work with these indeterminate forms.

EXAMPLE 6 Working with the Indeterminate Form $\infty \cdot 0$

Find

$$\lim_{x \to \infty} \left(x \sin \frac{1}{x} \right)$$

Solution

$$\begin{aligned} &\lim_{x \to \infty} \left(x \sin \frac{1}{x} \right) && \infty \cdot 0 \\ &\quad = \lim_{h \to 0^+} \left(\frac{1}{h} \sin h \right) && \text{Let } h = 1/x. \\ &\quad = 1 \end{aligned}$$

■

EXAMPLE 7 Working with the Indeterminate Form $\infty - \infty$

Find

$$\lim_{x \to 0} \left(\frac{1}{\sin x} - \frac{1}{x} \right).$$

Solution If $x \to 0^+$, then $\sin x \to 0^+$ and

$$\frac{1}{\sin x} - \frac{1}{x} \to \infty - \infty.$$

Similarly, if $x \to 0^-$, then $\sin x \to 0^-$ and

$$\frac{1}{\sin x} - \frac{1}{x} \to -\infty - (-\infty) = -\infty + \infty.$$

Neither form reveals what happens in the limit. To find out, we first combine the fractions:

$$\frac{1}{\sin x} - \frac{1}{x} = \frac{x - \sin x}{x \sin x} \qquad \text{Common denominator is } x \sin x$$

Then apply l'Hôpital's Rule to the result:

$$\begin{aligned} \lim_{x \to 0} \left(\frac{1}{\sin x} - \frac{1}{x} \right) &= \lim_{x \to 0} \frac{x - \sin x}{x \sin x} && \frac{0}{0} \\ &= \lim_{x \to 0} \frac{1 - \cos x}{\sin x + x \cos x} && \text{Still } \frac{0}{0} \\ &= \lim_{x \to 0} \frac{\sin x}{2 \cos x - x \sin x} = \frac{0}{2} = 0. \end{aligned}$$

■

EXERCISES 4.6

Finding Limits

In Exercises 1–6, use l'Hôpital's Rule to evaluate the limit. Then evaluate the limit using a method studied in Chapter 2.

1. $\lim_{x\to 2} \dfrac{x-2}{x^2-4}$

2. $\lim_{x\to 0} \dfrac{\sin 5x}{x}$

3. $\lim_{x\to\infty} \dfrac{5x^2-3x}{7x^2+1}$

4. $\lim_{x\to 1} \dfrac{x^3-1}{4x^3-x-3}$

5. $\lim_{x\to 0} \dfrac{1-\cos x}{x^2}$

6. $\lim_{x\to\infty} \dfrac{2x^2+3x}{x^3+x+1}$

Applying l'Hôpital's Rule

Use l'Hôpital's Rule to find the limits in Exercises 7–26.

7. $\lim_{t\to 0} \dfrac{\sin t^2}{t}$

8. $\lim_{x\to\pi/2} \dfrac{2x-\pi}{\cos x}$

9. $\lim_{\theta\to\pi} \dfrac{\sin\theta}{\pi-\theta}$

10. $\lim_{x\to\pi/2} \dfrac{1-\sin x}{1+\cos 2x}$

11. $\lim_{x\to\pi/4} \dfrac{\sin x-\cos x}{x-\pi/4}$

12. $\lim_{x\to\pi/3} \dfrac{\cos x-0.5}{x-\pi/3}$

13. $\lim_{x\to(\pi/2)^-} \left(x-\dfrac{\pi}{2}\right)\tan x$

14. $\lim_{x\to 0} \dfrac{2x}{x+7\sqrt{x}}$

15. $\lim_{x\to 1} \dfrac{2x^2-(3x+1)\sqrt{x}+2}{x-1}$

16. $\lim_{x\to 2} \dfrac{\sqrt{x^2+5}-3}{x^2-4}$

17. $\lim_{x\to 0} \dfrac{\sqrt{a(a+x)}-a}{x}, \quad a>0$

18. $\lim_{t\to 0} \dfrac{10(\sin t-t)}{t^3}$

19. $\lim_{x\to 0} \dfrac{x(\cos x-1)}{\sin x-x}$

20. $\lim_{h\to 0} \dfrac{\sin(a+h)-\sin a}{h}$

21. $\lim_{r\to 1} \dfrac{a(r^n-1)}{r-1}, \quad n$ a positive integer

22. $\lim_{x\to 0^+} \left(\dfrac{1}{x}-\dfrac{1}{\sqrt{x}}\right)$

23. $\lim_{x\to\infty} \left(x-\sqrt{x^2+x}\right)$

24. $\lim_{x\to\infty} x\tan\dfrac{1}{x}$

25. $\lim_{x\to\pm\infty} \dfrac{3x-5}{2x^2-x+2}$

26. $\lim_{x\to 0} \dfrac{\sin 7x}{\tan 11x}$

Theory and Applications

L'Hôpital's Rule does not help with the limits in Exercises 27–30. Try it; you just keep on cycling. Find the limits some other way.

27. $\lim_{x\to\infty} \dfrac{\sqrt{9x+1}}{\sqrt{x+1}}$

28. $\lim_{x\to 0^+} \dfrac{\sqrt{x}}{\sqrt{\sin x}}$

29. $\lim_{x\to(\pi/2)^-} \dfrac{\sec x}{\tan x}$

30. $\lim_{x\to 0^+} \dfrac{\cot x}{\csc x}$

31. Which one is correct, and which one is wrong? Give reasons for your answers.

a. $\lim_{x\to 3} \dfrac{x-3}{x^2-3} = \lim_{x\to 3} \dfrac{1}{2x} = \dfrac{1}{6}$

b. $\lim_{x\to 3} \dfrac{x-3}{x^2-3} = \dfrac{0}{6} = 0$

32. **∞/∞ Form** Give an example of two differentiable functions f and g with $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} g(x) = \infty$ that satisfy the following.

a. $\lim_{x\to\infty} \dfrac{f(x)}{g(x)} = 3$

b. $\lim_{x\to\infty} \dfrac{f(x)}{g(x)} = 0$

c. $\lim_{x\to\infty} \dfrac{f(x)}{g(x)} = \infty$

33. **Continuous extension** Find a value of c that makes the function

$$f(x) = \begin{cases} \dfrac{9x-3\sin 3x}{5x^3}, & x\neq 0 \\ c, & x=0 \end{cases}$$

continuous at $x=0$. Explain why your value of c works.

34. Let

$$f(x) = \begin{cases} x+2, & x\neq 0 \\ 0, & x=0 \end{cases} \quad\text{and}\quad g(x) = \begin{cases} x+1, & x\neq 0 \\ 0, & x=0. \end{cases}$$

a. Show that

$$\lim_{x\to 0} \frac{f'(x)}{g'(x)} = 1 \quad\text{but}\quad \lim_{x\to 0} \frac{f(x)}{g(x)} = 2.$$

b. Explain why this does not contradict l'Hôpital's Rule.

T **35.** **0/0 Form** Estimate the value of

$$\lim_{x\to 1} \frac{2x^2-(3x+1)\sqrt{x}+2}{x-1}$$

by graphing. Then confirm your estimate with l'Hôpital's Rule.

T **36.** **$\infty-\infty$ Form**

a. Estimate the value of

$$\lim_{x\to\infty} \left(x-\sqrt{x^2+x}\right)$$

by graphing $f(x) = x-\sqrt{x^2+x}$ over a suitably large interval of x-values.

b. Now confirm your estimate by finding the limit with l'Hôpital's Rule. As the first step, multiply $f(x)$ by the fraction $\left(x+\sqrt{x^2+x}\right)/\left(x+\sqrt{x^2+x}\right)$ and simplify the new numerator.

T **37.** Let

$$f(x) = \frac{1 - \cos x^6}{x^{12}}.$$

Explain why some graphs of f may give false information about $\lim_{x \to 0} f(x)$. (*Hint:* Try the window $[-1, 1]$ by $[-0.5, 1]$.)

38. Find all values of c, that satisfy the conclusion of Cauchy's Mean Value Theorem for the given functions and interval.

a. $f(x) = x, \quad g(x) = x^2, \quad (a, b) = (-2, 0)$

b. $f(x) = x, \quad g(x) = x^2, \quad (a, b)$ arbitrary

c. $f(x) = x^3/3 - 4x, \quad g(x) = x^2, \quad (a, b) = (0, 3)$

39. In the accompanying figure, the circle has radius OA equal to 1, and AB is tangent to the circle at A. The arc AC has radian measure θ and the segment AB also has length θ. The line through B and C crosses the x-axis at $P(x, 0)$.

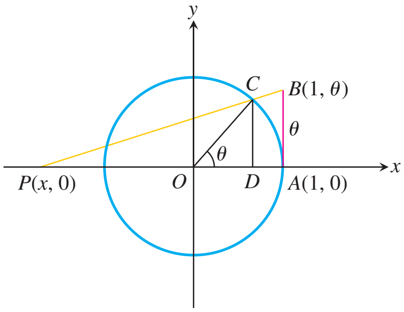
a. Show that the length of PA is

$$1 - x = \frac{\theta(1 - \cos\theta)}{\theta - \sin\theta}.$$

b. Find $\lim_{\theta \to 0} (1 - x)$.

c. Show that $\lim_{\theta \to \infty} [(1 - x) - (1 - \cos\theta)] = 0$.

Interpret this geometrically.

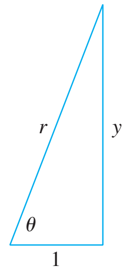


40. A right triangle has one leg of length 1, another of length y, and a hypotenuse of length r. The angle opposite y has radian measure θ. Find the limits as $\theta \to \pi/2$ of

a. $r - y$.

b. $r^2 - y^2$.

c. $r^3 - y^3$.



4.7 Newton's Method

HISTORICAL BIOGRAPHY

Niels Henrik Abel
(1802–1829)

One of the basic problems of mathematics is solving equations. Using the quadratic root formula, we know how to find a point (solution) where $x^2 - 3x + 2 = 0$. There are more complicated formulas to solve cubic or quartic equations (polynomials of degree 3 or 4), but the Norwegian mathematician Niels Abel showed that no simple formulas exist to solve polynomials of degree equal to five. There is also no simple formula for solving equations like $\sin x = x^2$, which involve transcendental functions as well as polynomials or other algebraic functions.

In this section we study a numerical method, called *Newton's method* or the *Newton–Raphson method*, which is a technique to approximate the solution to an equation $f(x) = 0$. Essentially it uses tangent lines in place of the graph of $y = f(x)$ near the points where f is zero. (A value of x where f is zero is a *root* of the function f and a *solution* of the equation $f(x) = 0$.)

Procedure for Newton's Method

The goal of Newton's method for estimating a solution of an equation $f(x) = 0$ is to produce a sequence of approximations that approach the solution. We pick the first number x_0 of the sequence. Then, under favorable circumstances, the method does the rest by moving step by step toward a point where the graph of f crosses the x-axis (Figure 4.43). At each

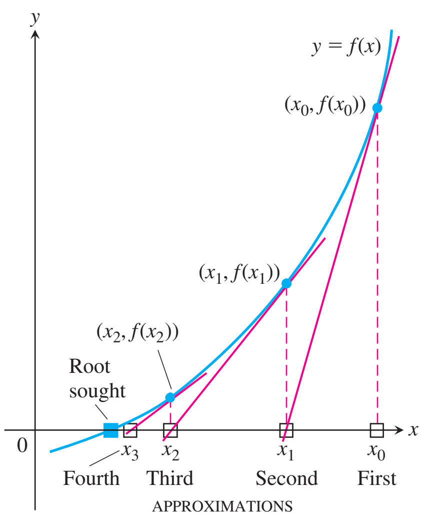


FIGURE 4.43 Newton's method starts with an initial guess x_0 and (under favorable circumstances) improves the guess one step at a time.

step the method approximates a zero of f with a zero of one of its linearizations. Here is how it works.

The initial estimate, x_0, may be found by graphing or just plain guessing. The method then uses the tangent to the curve $y = f(x)$ at $(x_0, f(x_0))$ to approximate the curve, calling the point x_1 where the tangent meets the x-axis (Figure 4.43). The number x_1 is usually a better approximation to the solution than is x_0. The point x_2 where the tangent to the curve at $(x_1, f(x_1))$ crosses the x-axis is the next approximation in the sequence. We continue on, using each approximation to generate the next, until we are close enough to the root to stop.

We can derive a formula for generating the successive approximations in the following way. Given the approximation x_n, the point-slope equation for the tangent to the curve at $(x_n, f(x_n))$ is

$$y = f(x_n) + f'(x_n)(x - x_n).$$

We can find where it crosses the x-axis by setting $y = 0$ (Figure 4.44).

$$0 = f(x_n) + f'(x_n)(x - x_n)$$

$$-\frac{f(x_n)}{f'(x_n)} = x - x_n$$

$$x = x_n - \frac{f(x_n)}{f'(x_n)} \qquad \text{If } f'(x_n) \neq 0$$

This value of x is the next approximation x_{n+1}. Here is a summary of Newton's method.

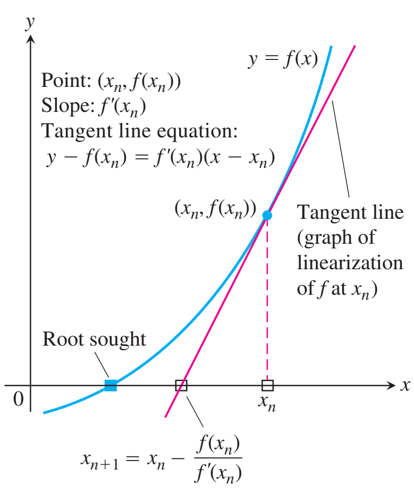


FIGURE 4.44 The geometry of the successive steps of Newton's method. From x_n we go up to the curve and follow the tangent line down to find x_{n+1}.

Procedure for Newton's Method

1. Guess a first approximation to a solution of the equation $f(x) = 0$. A graph of $y = f(x)$ may help.
2. Use the first approximation to get a second, the second to get a third, and so on, using the formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \qquad \text{if } f'(x_n) \neq 0 \tag{1}$$

Applying Newton's Method

Applications of Newton's method generally involve many numerical computations, making them well suited for computers or calculators. Nevertheless, even when the calculations are done by hand (which may be very tedious), they give a powerful way to find solutions of equations.

In our first example, we find decimal approximations to $\sqrt{2}$ by estimating the positive root of the equation $f(x) = x^2 - 2 = 0$.

Video You Try It

EXAMPLE 1 Finding the Square Root of 2

Find the positive root of the equation

$$f(x) = x^2 - 2 = 0.$$

Solution With $f(x) = x^2 - 2$ and $f'(x) = 2x$, Equation (1) becomes

$$\begin{aligned} x_{n+1} &= x_n - \frac{x_n{}^2 - 2}{2x_n} \\ &= x_n - \frac{x_n}{2} + \frac{1}{x_n} \\ &= \frac{x_n}{2} + \frac{1}{x_n}. \end{aligned}$$

The equation

$$x_{n+1} = \frac{x_n}{2} + \frac{1}{x_n}$$

enables us to go from each approximation to the next with just a few keystrokes. With the starting value $x_0 = 1$, we get the results in the first column of the following table. (To five decimal places, $\sqrt{2} = 1.41421$.)

	Error	Number of correct digits
$x_0 = 1$	-0.41421	1
$x_1 = 1.5$	0.08579	1
$x_2 = 1.41667$	0.00246	3
$x_3 = 1.41422$	0.00001	5

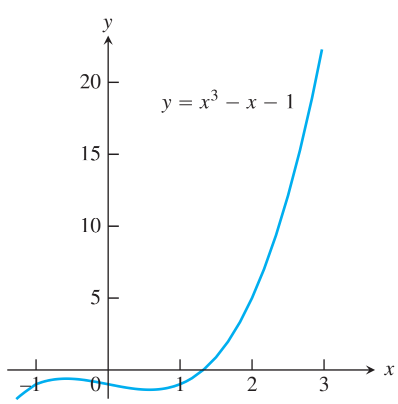


FIGURE 4.45 The graph of $f(x) = x^3 - x - 1$ crosses the x-axis once; this is the root we want to find (Example 2).

Newton's method is the method used by most calculators to calculate roots because it converges so fast (more about this later). If the arithmetic in the table in Example 1 had been carried to 13 decimal places instead of 5, then going one step further would have given $\sqrt{2}$ correctly to more than 10 decimal places.

EXAMPLE 2 Using Newton's Method

Find the x-coordinate of the point where the curve $y = x^3 - x$ crosses the horizontal line $y = 1$.

Solution The curve crosses the line when $x^3 - x = 1$ or $x^3 - x - 1 = 0$. When does $f(x) = x^3 - x - 1$ equal zero? Since $f(1) = -1$ and $f(2) = 5$, we know by the Intermediate Value Theorem there is a root in the interval (1, 2) (Figure 4.45).

We apply Newton's method to f with the starting value $x_0 = 1$. The results are displayed in Table 4.1 and Figure 4.46.

At $n = 5$, we come to the result $x_6 = x_5 = 1.3247\ 17957$. When $x_{n+1} = x_n$, Equation (1) shows that $f(x_n) = 0$. We have found a solution of $f(x) = 0$ to nine decimals.

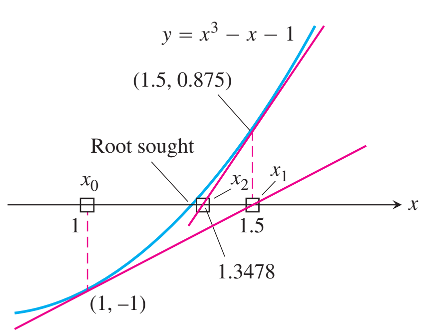


FIGURE 4.46 The first three x-values in Table 4.1 (four decimal places).

In Figure 4.47 we have indicated that the process in Example 2 might have started at the point $B_0(3, 23)$ on the curve, with $x_0 = 3$. Point B_0 is quite far from the x-axis, but the tangent at B_0 crosses the x-axis at about (2.12, 0), so x_1 is still an improvement over x_0. If we use Equation (1) repeatedly as before, with $f(x) = x^3 - x - 1$ and $f'(x) = 3x^2 - 1$, we confirm the nine-place solution $x_7 = x_6 = 1.3247\ 17957$ in seven steps.

TABLE 4.1 The result of applying Newton's method to $f(x) = x^3 - x - 1$ with $x_0 = 1$

n	x_n	$f(x_n)$	$f'(x_n)$	$x_{n+1} = x_n - \dfrac{f(x_n)}{f'(x_n)}$
0	1	−1	2	1.5
1	1.5	0.875	5.75	1.3478 26087
2	1.3478 26087	0.1006 82173	4.4499 05482	1.3252 00399
3	1.3252 00399	0.0020 58362	4.2684 68292	1.3247 18174
4	1.3247 18174	0.0000 00924	4.2646 34722	1.3247 17957
5	1.3247 17957	−1.8672E-13	4.2646 32999	1.3247 17957

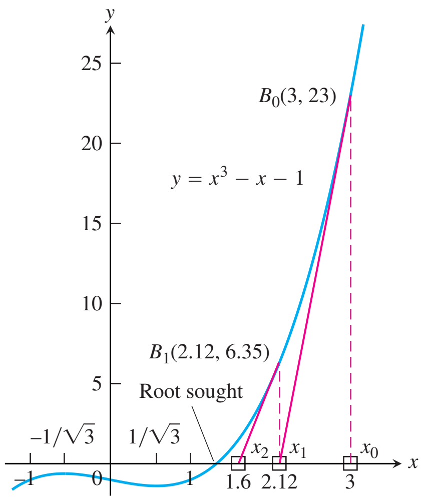


FIGURE 4.47 Any starting value x_0 to the right of $x = 1/\sqrt{3}$ will lead to the root.

The curve in Figure 4.47 has a local maximum at $x = -1/\sqrt{3}$ and a local minimum at $x = 1/\sqrt{3}$. We would not expect good results from Newton's method if we were to start with x_0 between these points, but we can start any place to the right of $x = 1/\sqrt{3}$ and get the answer. It would not be very clever to do so, but we could even begin far to the right of B_0, for example with $x_0 = 10$. It takes a bit longer, but the process still converges to the same answer as before.

Convergence of Newton's Method

In practice, Newton's method usually converges with impressive speed, but this is not guaranteed. One way to test convergence is to begin by graphing the function to estimate a good starting value for x_0. You can test that you are getting closer to a zero of the function by evaluating $|f(x_n)|$ and check that the method is converging by evaluating $|x_n - x_{n+1}|$.

Theory does provide some help. A theorem from advanced calculus says that if

$$\left|\frac{f(x)f''(x)}{(f'(x))^2}\right| < 1 \tag{2}$$

for all x in an interval about a root r, then the method will converge to r for any starting value x_0 in that interval. Note that this condition is satisfied if the graph of f is not too horizontal near where it crosses the x-axis.

Newton's method always converges if, between r and x_0, the graph of f is concave up when $f(x_0) > 0$ and concave down when $f(x_0) < 0$. (See Figure 4.48.) In most cases, the speed of the convergence to the root r is expressed by the advanced calculus formula

$$\underbrace{|x_{n+1} - r|}_{\text{error } e_{n+1}} \le \frac{\max|f''|}{2\min|f'|}|x_n - r|^2 = \text{constant} \cdot \underbrace{|x_n - r|^2}_{\text{error } e_n}, \tag{3}$$

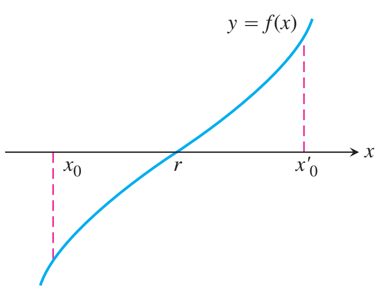


FIGURE 4.48 Newton's method will converge to r from either starting point.

where max and min refer to the maximum and minimum values in an interval surrounding r. The formula says that the error in step $n + 1$ is no greater than a constant times the square of the error in step n. This may not seem like much, but think of what it says. If the constant is less than or equal to 1 and $|x_n - r| < 10^{-3}$, then $|x_{n+1} - r| < 10^{-6}$. *In a single step*, the method moves from three decimal places of accuracy to six, and the number of decimals of accuracy continues to double with each successive step.

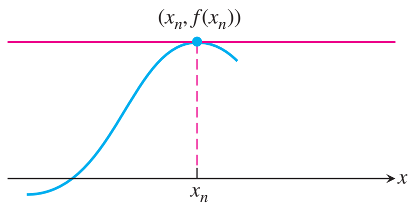


FIGURE 4.49 If $f'(x_n) = 0$, there is no intersection point to define x_{n+1}.

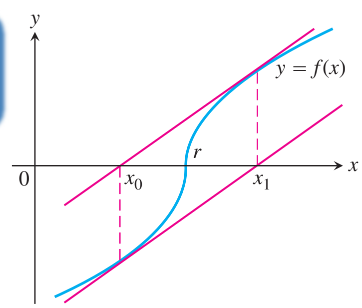


FIGURE 4.50 Newton's method fails to converge. You go from x_0 to x_1 and back to x_0, never getting any closer to r.

But Things Can Go Wrong

Newton's method stops if $f'(x_n) = 0$ (Figure 4.49). In that case, try a new starting point. Of course, f and f' may have the same root. To detect whether this is so, you could first find the solutions of $f'(x) = 0$ and check f at those values, or you could graph f and f' together.

Newton's method does not always converge. For instance, if

$$f(x) = \begin{cases} -\sqrt{r - x}, & x < r \\ \sqrt{x - r}, & x \geq r, \end{cases}$$

the graph will be like the one in Figure 4.50. If we begin with $x_0 = r - h$, we get $x_1 = r + h$, and successive approximations go back and forth between these two values. No amount of iteration brings us closer to the root than our first guess.

If Newton's method does converge, it converges to a root. Be careful, however. There are situations in which the method appears to converge but there is no root there. Fortunately, such situations are rare.

When Newton's method converges to a root, it may not be the root you have in mind. Figure 4.51 shows two ways this can happen.

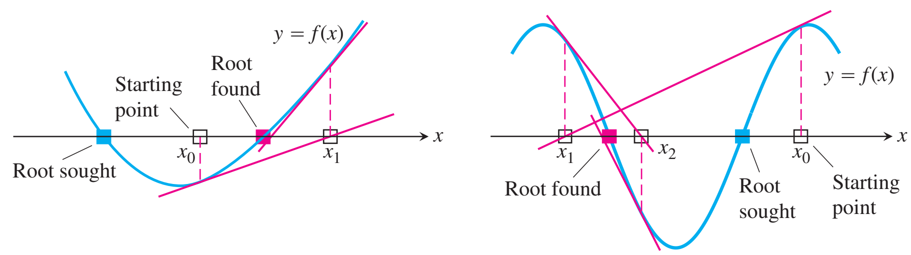


FIGURE 4.51 If you start too far away, Newton's method may miss the root you want.

Fractal Basins and Newton's Method

The process of finding roots by Newton's method can be uncertain in the sense that for some equations, the final outcome can be extremely sensitive to the starting value's location.

The equation $4x^4 - 4x^2 = 0$ is a case in point (Figure 4.52a). Starting values in the blue zone on the x-axis lead to root A. Starting values in the black lead to root B, and starting values in the red zone lead to root C. The points $\pm\sqrt{2}/2$ give horizontal tangents. The points $\pm\sqrt{21}/7$ "cycle," each leading to the other, and back (Figure 4.52b).

The interval between $\sqrt{21}/7$ and $\sqrt{2}/2$ contains infinitely many open intervals of points leading to root A, alternating with intervals of points leading to root C (Figure 4.52c). The boundary points separating consecutive intervals (there are infinitely many) do not lead to roots, but cycle back and forth from one to another. Moreover, as we select points that approach $\sqrt{21}/7$ from the right, it becomes increasingly difficult to distinguish which lead to root A and which to root C. On the same side of $\sqrt{21}/7$, we find arbitrarily close together points whose ultimate destinations are far apart.

If we think of the roots as "attractors" of other points, the coloring in Figure 4.52 shows the intervals of the points they attract (the "intervals of attraction"). You might think that points between roots A and B would be attracted to either A or B, but, as we see, that is not the case. Between A and B there are infinitely many intervals of points attracted to C. Similarly between B and C lie infinitely many intervals of points attracted to A.

We encounter an even more dramatic example of such behavior when we apply Newton's method to solve the complex-number equation $z^6 - 1 = 0$. It has six solutions: $1, -1$, and the four numbers $\pm(1/2) \pm \left(\sqrt{3}/2\right)i$. As Figure 4.53 suggests, each of the

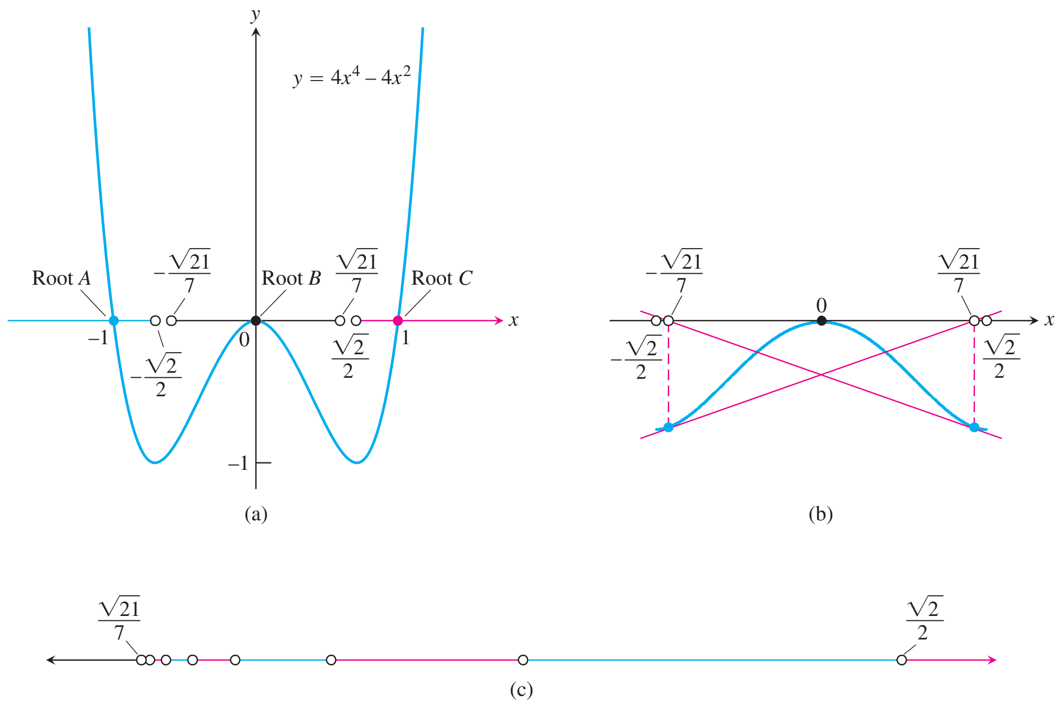


FIGURE 4.52 (a) Starting values in $(-\infty, -\sqrt{2}/2)$, $(-\sqrt{21}/7, \sqrt{21}/7)$, and $(\sqrt{2}/2, \infty)$ lead respectively to roots A, B, and C. (b) The values $x = \pm\sqrt{21}/7$ lead only to each other. (c) Between $\sqrt{21}/7$ and $\sqrt{2}/2$, there are infinitely many open intervals of points attracted to A alternating with open intervals of points attracted to C. This behavior is mirrored in the interval $(-\sqrt{2}/2, -\sqrt{21}/7)$.

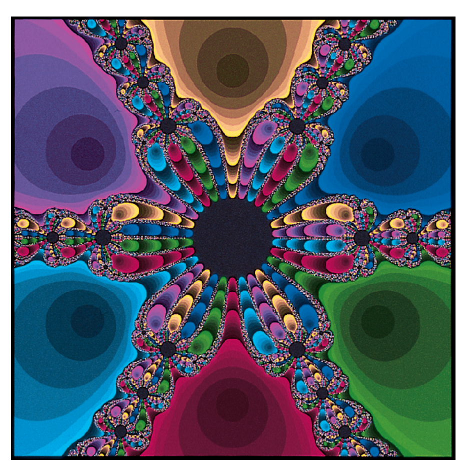


FIGURE 4.53 This computer-generated initial value portrait uses color to show where different points in the complex plane end up when they are used as starting values in applying Newton's method to solve the equation $z^6 - 1 = 0$. Red points go to 1, green points to $(1/2) + (\sqrt{3}/2)i$, dark blue points to $(-1/2) + (\sqrt{3}/2)i$, and so on. Starting values that generate sequences that do not arrive within 0.1 unit of a root after 32 steps are colored black.

six roots has infinitely many "basins" of attraction in the complex plane (Appendix 5). Starting points in red basins are attracted to the root 1, those in the green basin to the root $(1/2) + \left(\sqrt{3}/2\right)i$, and so on. Each basin has a boundary whose complicated pattern repeats without end under successive magnifications. These basins are called **fractal basins**.

EXERCISES 4.7

Root-Finding

Exercises

1. Use Newton's method to estimate the solutions of the equation $x^2 + x - 1 = 0$. Start with $x_0 = -1$ for the left-hand solution and with $x_0 = 1$ for the solution on the right. Then, in each case, find x_2.
2. Use Newton's method to estimate the one real solution of $x^3 + 3x + 1 = 0$. Start with $x_0 = 0$ and then find x_2.
3. Use Newton's method to estimate the two zeros of the function $f(x) = x^4 + x - 3$. Start with $x_0 = -1$ for the left-hand zero and with $x_0 = 1$ for the zero on the right. Then, in each case, find x_2.
4. Use Newton's method to estimate the two zeros of the function $f(x) = 2x - x^2 + 1$. Start with $x_0 = 0$ for the left-hand zero and with $x_0 = 2$ for the zero on the right. Then, in each case, find x_2.
5. Use Newton's method to find the positive fourth root of 2 by solving the equation $x^4 - 2 = 0$. Start with $x_0 = 1$ and find x_2.
6. Use Newton's method to find the negative fourth root of 2 by solving the equation $x^4 - 2 = 0$. Start with $x_0 = -1$ and find x_2.

Theory, Examples, and Applications

7. **Guessing a root** Suppose that your first guess is lucky, in the sense that x_0 is a root of $f(x) = 0$. Assuming that $f'(x_0)$ is defined and not 0, what happens to x_1 and later approximations?
8. **Estimating pi** You plan to estimate $\pi/2$ to five decimal places by using Newton's method to solve the equation $\cos x = 0$. Does it matter what your starting value is? Give reasons for your answer.
9. **Oscillation** Show that if $h > 0$, applying Newton's method to

$$f(x) = \begin{cases} \sqrt{x}, & x \geq 0 \\ \sqrt{-x}, & x < 0 \end{cases}$$

leads to $x_1 = -h$ if $x_0 = h$ and to $x_1 = h$ if $x_0 = -h$. Draw a picture that shows what is going on.

10. **Approximations that get worse and worse** Apply Newton's method to $f(x) = x^{1/3}$ with $x_0 = 1$ and calculate x_1, x_2, x_3, and x_4. Find a formula for $|x_n|$. What happens to $|x_n|$ as $n \to \infty$? Draw a picture that shows what is going on.
11. Explain why the following four statements ask for the same information:
    i) Find the roots of $f(x) = x^3 - 3x - 1$.
    ii) Find the x-coordinates of the intersections of the curve $y = x^3$ with the line $y = 3x + 1$.
    iii) Find the x-coordinates of the points where the curve $y = x^3 - 3x$ crosses the horizontal line $y = 1$.
    iv) Find the values of x where the derivative of $g(x) = (1/4)x^4 - (3/2)x^2 - x + 5$ equals zero.
12. **Locating a planet** To calculate a planet's space coordinates, we have to solve equations like $x = 1 + 0.5 \sin x$. Graphing the function $f(x) = x - 1 - 0.5 \sin x$ suggests that the function has a root near $x = 1.5$. Use one application of Newton's method to improve this estimate. That is, start with $x_0 = 1.5$ and find x_1. (The value of the root is 1.49870 to five decimal places.) Remember to use radians.

T 13. **A program for using Newton's method on a grapher** Let $f(x) = x^3 + 3x + 1$. Here is a home screen program to perform the computations in Newton's method.
   a. Let $y_0 = f(x)$ and $y_1 = \text{NDER } f(x)$.
   b. Store $x_0 = -0.3$ into x.
   c. Then store $x - (y_0/y_1)$ into x and press the Enter key over and over. Watch as the numbers converge to the zero of f.
   d. Use different values for x_0 and repeat steps (b) and (c).
   e. Write your own equation and use this approach to solve it using Newton's method. Compare your answer with the answer given by the built-in feature of your calculator that gives zeros of functions.

T 14. (*Continuation of Exercise 11.*)
   a. Use Newton's method to find the two negative zeros of $f(x) = x^3 - 3x - 1$ to five decimal places.
   b. Graph $f(x) = x^3 - 3x - 1$ for $-2 \leq x \leq 2.5$. Use the Zoom and Trace features to estimate the zeros of f to five decimal places.
   c. Graph $g(x) = 0.25x^4 - 1.5x^2 - x + 5$. Use the Zoom and Trace features with appropriate rescaling to find, to five decimal places, the values of x where the graph has horizontal tangents.

T 15. **Intersecting curves** The curve $y = \tan x$ crosses the line $y = 2x$ between $x = 0$ and $x = \pi/2$. Use Newton's method to find where.

T 16. **Real solutions of a quartic** Use Newton's method to find the two real solutions of the equation $x^4 - 2x^3 - x^2 - 2x + 2 = 0$.

T 17. a. How many solutions does the equation $\sin 3x = 0.99 - x^2$ have?
   b. Use Newton's method to find them.

T **18. Intersection of curves**

a. Does $\cos 3x$ ever equal x? Give reasons for your answer.

b. Use Newton's method to find where.

T **19.** Find the four real zeros of the function $f(x) = 2x^4 - 4x^2 + 1$.

T **20. Estimating pi** Estimate π to as many decimal places as your calculator will display by using Newton's method to solve the equation $\tan x = 0$ with $x_0 = 3$.

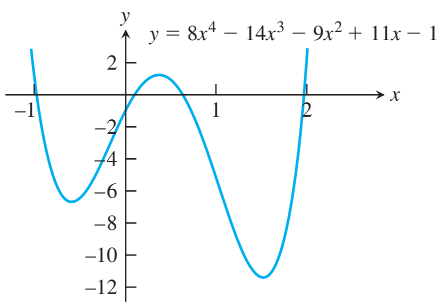
21. At what values(s) of x does $\cos x = 2x$?

22. At what value(s) of x does $\cos x = -x$?

23. Use the Intermediate Value Theorem from Section 2.6 to show that $f(x) = x^3 + 2x - 4$ has a root between $x = 1$ and $x = 2$. Then find the root to five decimal places.

24. Factoring a quartic Find the approximate values of r_1 through r_4 in the factorization

$$8x^4 - 14x^3 - 9x^2 + 11x - 1 = 8(x - r_1)(x - r_2)(x - r_3)(x - r_4).$$



T **25. Converging to different zeros** Use Newton's method to find the zeros of $f(x) = 4x^4 - 4x^2$ using the given starting values (Figure 4.52).

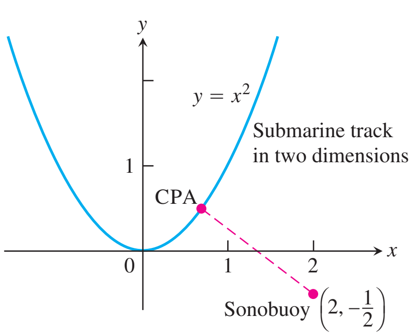
a. $x_0 = -2$ and $x_0 = -0.8$, lying in $\left(-\infty, -\sqrt{2}/2\right)$

b. $x_0 = -0.5$ and $x_0 = 0.25$, lying in $\left(-\sqrt{21}/7, \sqrt{21}/7\right)$

c. $x_0 = 0.8$ and $x_0 = 2$, lying in $\left(\sqrt{2}/2, \infty\right)$

d. $x_0 = -\sqrt{21}/7$ and $x_0 = \sqrt{21}/7$

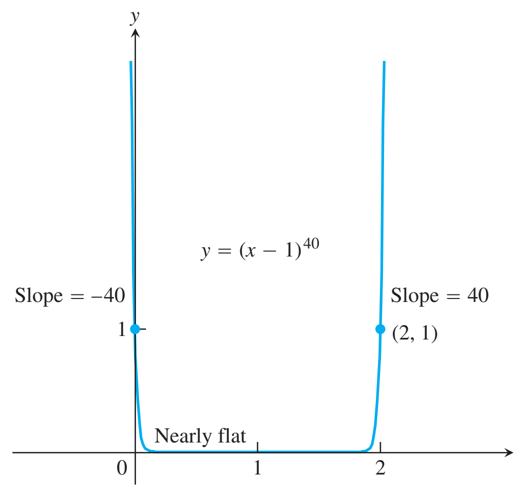
26. The sonobuoy problem In submarine location problems, it is often necessary to find a submarine's closest point of approach (CPA) to a sonobuoy (sound detector) in the water. Suppose that the submarine travels on the parabolic path $y = x^2$ and that the buoy is located at the point $(2, -1/2)$.



a. Show that the value of x that minimizes the distance between the submarine and the buoy is a solution of the equation $x = 1/(x^2 + 1)$.

b. Solve the equation $x = 1/(x^2 + 1)$ with Newton's method.

27. Curves that are nearly flat at the root Some curves are so flat that, in practice, Newton's method stops too far from the root to give a useful estimate. Try Newton's method on $f(x) = (x - 1)^{40}$ with a starting value of $x_0 = 2$ to see how close your machine comes to the root $x = 1$.



28. Finding a root different from the one sought All three roots of $f(x) = 4x^4 - 4x^2$ can be found by starting Newton's method near $x = \sqrt{21}/7$. Try it. (See Figure 4.52.)

29. Finding an ion concentration While trying to find the acidity of a saturated solution of magnesium hydroxide in hydrochloric acid, you derive the equation

$$\frac{3.64 \times 10^{-11}}{[H_3O^+]^2} = [H_3O^+] + 3.6 \times 10^{-4}$$

for the hydronium ion concentration $[H_3O^+]$. To find the value of $[H_3O^+]$, you set $x = 10^4[H_3O^+]$ and convert the equation to

$$x^3 + 3.6x^2 - 36.4 = 0.$$

You then solve this by Newton's method. What do you get for x? (Make it good to two decimal places.) For $[H_3O^+]$?

T **30. Complex roots** If you have a computer or a calculator that can be programmed to do complex-number arithmetic, experiment with Newton's method to solve the equation $z^6 - 1 = 0$. The recursion relation to use is

$$z_{n+1} = z_n - \frac{z_n^{\,6} - 1}{6z_n^{\,5}} \quad \text{or} \quad z_{n+1} = \frac{5}{6}z_n + \frac{1}{6z_n^{\,5}}.$$

Try these starting values (among others): 2, i, $\sqrt{3} + i$.

4.8 Antiderivatives

We have studied how to find the derivative of a function. However, many problems require that we recover a function from its known derivative (from its known rate of change). For instance, we may know the velocity function of an object falling from an initial height and need to know its height at any time over some period. More generally, we want to find a function F from its derivative f. If such a function F exists, it is called an *antiderivative* of f.

Finding Antiderivatives

DEFINITION Antiderivative

A function F is an **antiderivative** of f on an interval I if $F'(x) = f(x)$ for all x in I.

The process of recovering a function $F(x)$ from its derivative $f(x)$ is called *antidifferentiation*. We use capital letters such as F to represent an antiderivative of a function f, G to represent an antiderivative of g, and so forth.

EXAMPLE 1 Finding Antiderivatives

Find an antiderivative for each of the following functions.

(a) $f(x) = 2x$

(b) $g(x) = \cos x$

(c) $h(x) = 2x + \cos x$

Solution

(a) $F(x) = x^2$

(b) $G(x) = \sin x$

(c) $H(x) = x^2 + \sin x$

Each answer can be checked by differentiating. The derivative of $F(x) = x^2$ is $2x$. The derivative of $G(x) = \sin x$ is $\cos x$ and the derivative of $H(x) = x^2 + \sin x$ is $2x + \cos x$. ■

The function $F(x) = x^2$ is not the only function whose derivative is $2x$. The function $x^2 + 1$ has the same derivative. So does $x^2 + C$ for any constant C. Are there others?

Corollary 2 of the Mean Value Theorem in Section 4.2 gives the answer: Any two antiderivatives of a function differ by a constant. So the functions $x^2 + C$, where C is an **arbitrary constant**, form *all* the antiderivatives of $f(x) = 2x$. More generally, we have the following result.

> If F is an antiderivative of f on an interval I, then the most general antiderivative of f on I is
> $$F(x) + C$$
> where C is an arbitrary constant.

Thus the most general antiderivative of f on I is a *family* of functions $F(x) + C$ whose graphs are vertical translates of one another. We can select a particular antiderivative from this family by assigning a specific value to C. Here is an example showing how such an assignment might be made.

EXAMPLE 2 Finding a Particular Antiderivative

Find an antiderivative of $f(x) = \sin x$ that satisfies $F(0) = 3$.

Solution Since the derivative of $-\cos x$ is $\sin x$, the general antiderivative

$$F(x) = -\cos x + C$$

gives all the antiderivatives of $f(x)$. The condition $F(0) = 3$ determines a specific value for C. Substituting $x = 0$ into $F(x) = -\cos x + C$ gives

$$F(0) = -\cos 0 + C = -1 + C.$$

Since $F(0) = 3$, solving for C gives $C = 4$. So

$$F(x) = -\cos x + 4$$

is the antiderivative satisfying $F(0) = 3$. ■

By working backward from assorted differentiation rules, we can derive formulas and rules for antiderivatives. In each case there is an arbitrary constant C in the general expression representing all antiderivatives of a given function. Table 4.2 gives antiderivative formulas for a number of important functions.

TABLE 4.2 Antiderivative formulas

	Function	General antiderivative
1.	x^n	$\dfrac{x^{n+1}}{n+1} + C, \quad n \neq -1$, n rational
2.	$\sin kx$	$-\dfrac{\cos kx}{k} + C, \quad k$ a constant, $k \neq 0$
3.	$\cos kx$	$\dfrac{\sin kx}{k} + C, \quad k$ a constant, $k \neq 0$
4.	$\sec^2 x$	$\tan x + C$
5.	$\csc^2 x$	$-\cot x + C$
6.	$\sec x \tan x$	$\sec x + C$
7.	$\csc x \cot x$	$-\csc x + C$

The rules in Table 4.2 are easily verified by differentiating the general antiderivative formula to obtain the function to its left. For example, the derivative of $\tan x + C$ is $\sec^2 x$, whatever the value of the constant C, and this establishes the formula for the most general antiderivative of $\sec^2 x$.

EXAMPLE 3 Finding Antiderivatives Using Table 4.2

Find the general antiderivative of each of the following functions.

(a) $f(x) = x^5$

(b) $g(x) = \dfrac{1}{\sqrt{x}}$

(c) $h(x) = \sin 2x$

(d) $i(x) = \cos \dfrac{x}{2}$

Solution

(a) $F(x) = \dfrac{x^6}{6} + C$ — Formula 1 with $n = 5$

(b) $g(x) = x^{-1/2}$, so

$$G(x) = \frac{x^{1/2}}{1/2} + C = 2\sqrt{x} + C$$ — Formula 1 with $n = -1/2$

(c) $H(x) = \dfrac{-\cos 2x}{2} + C$ — Formula 2 with $k = 2$

(d) $I(x) = \dfrac{\sin (x/2)}{1/2} + C = 2 \sin \dfrac{x}{2} + C$ — Formula 3 with $k = 1/2$ ■

Other derivative rules also lead to corresponding antiderivative rules. We can add and subtract antiderivatives, and multiply them by constants.

The formulas in Table 4.3 are easily proved by differentiating the antiderivatives and verifying that the result agrees with the original function. Formula 2 is the special case $k = -1$ in Formula 1.

TABLE 4.3 Antiderivative linearity rules

		Function	General antiderivative
1.	*Constant Multiple Rule:*	$kf(x)$	$kF(x) + C$, k a constant
2.	*Negative Rule:*	$-f(x)$	$-F(x) + C$,
3.	*Sum or Difference Rule:*	$f(x) \pm g(x)$	$F(x) \pm G(x) + C$

EXAMPLE 4 Using the Linearity Rules for Antiderivatives

Find the general antiderivative of

$$f(x) = \frac{3}{\sqrt{x}} + \sin 2x.$$

Solution We have that $f(x) = 3g(x) + h(x)$ for the functions g and h in Example 3. Since $G(x) = 2\sqrt{x}$ is an antiderivative of $g(x)$ from Example 3b, it follows from the Constant Multiple Rule for antiderivatives that $3G(x) = 3 \cdot 2\sqrt{x} = 6\sqrt{x}$ is an antiderivative of $3g(x) = 3/\sqrt{x}$. Likewise, from Example 3c we know that $H(x) = (-1/2)\cos 2x$ is an antiderivative of $h(x) = \sin 2x$. From the Sum Rule for antiderivatives, we then get that

$$\begin{aligned} F(x) &= 3G(x) + H(x) + C \\ &= 6\sqrt{x} - \frac{1}{2}\cos 2x + C \end{aligned}$$

is the general antiderivative formula for $f(x)$, where C is an arbitrary constant. ■

Antiderivatives play several important roles, and methods and techniques for finding them are a major part of calculus. (This is the subject of Chapter 8.)

Initial Value Problems and Differential Equations

Finding an antiderivative for a function $f(x)$ is the same problem as finding a function $y(x)$ that satisfies the equation

$$\frac{dy}{dx} = f(x).$$

This is called a **differential equation**, since it is an equation involving an unknown function y that is being differentiated. To solve it, we need a function $y(x)$ that satisfies the equation. This function is found by taking the antiderivative of $f(x)$. We fix the arbitrary constant arising in the antidifferentiation process by specifying an initial condition

$$y(x_0) = y_0.$$

This condition means the function $y(x)$ has the value y_0 when $x = x_0$. The combination of a differential equation and an initial condition is called an **initial value problem**. Such problems play important roles in all branches of science. Here's an example of solving an initial value problem.

EXAMPLE 5 Finding a Curve from Its Slope Function and a Point

Find the curve whose slope at the point (x, y) is $3x^2$ if the curve is required to pass through the point $(1, -1)$.

Solution In mathematical language, we are asked to solve the initial value problem that consists of the following.

The differential equation: $\dfrac{dy}{dx} = 3x^2$ The curve's slope is $3x^2$.

The initial condition: $y(1) = -1$

1. *Solve the differential equation*: The function y is an antiderivative of $f(x) = 3x^2$, so

$$y = x^3 + C.$$

This result tells us that y equals $x^3 + C$ for some value of C. We find that value from the initial condition $y(1) = -1$.

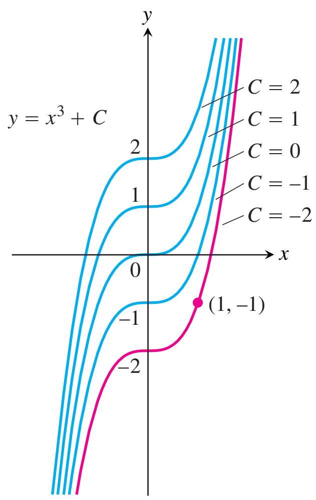


FIGURE 4.54 The curves $y = x^3 + C$ fill the coordinate plane without overlapping. In Example 5, we identify the curve $y = x^3 - 2$ as the one that passes through the given point $(1, -1)$.

2. *Evaluate C:*

$$\begin{aligned} y &= x^3 + C \\ -1 &= (1)^3 + C \qquad \text{Initial condition } y(1) = -1 \\ C &= -2. \end{aligned}$$

The curve we want is $y = x^3 - 2$ (Figure 4.54). ■

The most general antiderivative $F(x) + C$ (which is $x^3 + C$ in Example 5) of the function $f(x)$ gives the **general solution** $y = F(x) + C$ of the differential equation $dy/dx = f(x)$. The general solution gives *all* the solutions of the equation (there are infinitely many, one for each value of C). We **solve** the differential equation by finding its general solution. We then solve the initial value problem by finding the **particular solution** that satisfies the initial condition $y(x_0) = y_0$.

Antiderivatives and Motion

We have seen that the derivative of the position of an object gives its velocity, and the derivative of its velocity gives its acceleration. If we know an object's acceleration, then by finding an antiderivative we can recover the velocity, and from an antiderivative of the velocity we can recover its position function. This procedure was used as an application of Corollary 2 in Section 4.2. Now that we have a terminology and conceptual framework in terms of antiderivatives, we revisit the problem from the point of view of differential equations.

EXAMPLE 6 Dropping a Package from an Ascending Balloon

A balloon ascending at the rate of 12 ft/sec is at a height 80 ft above the ground when a package is dropped. How long does it take the package to reach the ground?

Solution Let $v(t)$ denote the velocity of the package at time t, and let $s(t)$ denote its height above the ground. The acceleration of gravity near the surface of the earth is 32 ft/sec^2. Assuming no other forces act on the dropped package, we have

$$\frac{dv}{dt} = -32. \qquad \text{Negative because gravity acts in the direction of decreasing } s.$$

This leads to the initial value problem.

$$\textit{Differential equation:} \quad \frac{dv}{dt} = -32$$

$$\textit{Initial condition:} \quad v(0) = 12,$$

which is our mathematical model for the package's motion. We solve the initial value problem to obtain the velocity of the package.

1. *Solve the differential equation*: The general formula for an antiderivative of -32 is

$$v = -32t + C.$$

Having found the general solution of the differential equation, we use the initial condition to find the particular solution that solves our problem.

2. *Evaluate C:*

$$\begin{aligned} 12 &= -32(0) + C \qquad \text{Initial condition } v(0) = 12 \\ C &= 12. \end{aligned}$$

The solution of the initial value problem is

$$v = -32t + 12.$$

Since velocity is the derivative of height and the height of the package is 80 ft at the time $t = 0$ when it is dropped, we now have a second initial value problem.

$$\textit{Differential equation:} \quad \frac{ds}{dt} = -32t + 12 \qquad \text{Set } v = ds/dt \text{ in the last equation.}$$

$$\textit{Initial condition:} \quad s(0) = 80$$

We solve this initial value problem to find the height as a function of t.

1. *Solve the differential equation*: Finding the general antiderivative of $-32t + 12$ gives

$$s = -16t^2 + 12t + C.$$

2. *Evaluate C*:

$$80 = -16(0)^2 + 12(0) + C \qquad \text{Initial condition } s(0) = 80$$

$$C = 80.$$

The package's height above ground at time t is

$$s = -16t^2 + 12t + 80.$$

Use the solution: To find how long it takes the package to reach the ground, we set s equal to 0 and solve for t:

$$-16t^2 + 12t + 80 = 0$$

$$-4t^2 + 3t + 20 = 0$$

$$t = \frac{-3 \pm \sqrt{329}}{-8} \qquad \text{Quadratic formula}$$

$$t \approx -1.89, \qquad t \approx 2.64.$$

The package hits the ground about 2.64 sec after it is dropped from the balloon. (The negative root has no physical meaning.) ■

Indefinite Integrals

A special symbol is used to denote the collection of all antiderivatives of a function f.

DEFINITION Indefinite Integral, Integrand

The set of all antiderivatives of f is the **indefinite integral** of f with respect to x, denoted by

$$\int f(x)\, dx.$$

The symbol $\int$ is an **integral sign**. The function f is the **integrand** of the integral, and x is the **variable of integration**.

Using this notation, we restate the solutions of Example 1, as follows:

$$\int 2x\,dx = x^2 + C,$$

$$\int \cos x\,dx = \sin x + C,$$

$$\int (2x + \cos x)\,dx = x^2 + \sin x + C.$$

This notation is related to the main application of antiderivatives, which will be explored in Chapter 5. Antiderivatives play a key role in computing limits of infinite sums, an unexpected and wonderfully useful role that is described in a central result of Chapter 5, called the Fundamental Theorem of Calculus.

You Try It

EXAMPLE 7 Indefinite Integration Done Term-by-Term and Rewriting the Constant of Integration

Evaluate

$$\int (x^2 - 2x + 5)\,dx.$$

Solution If we recognize that $(x^3/3) - x^2 + 5x$ is an antiderivative of $x^2 - 2x + 5$, we can evaluate the integral as

$$\int (x^2 - 2x + 5)\,dx = \overbrace{\frac{x^3}{3} - x^2 + 5x}^{\text{antiderivative}} + \underbrace{C}_{\text{arbitrary constant}}.$$

If we do not recognize the antiderivative right away, we can generate it term-by-term with the Sum, Difference, and Constant Multiple Rules:

$$\begin{aligned}
\int (x^2 - 2x + 5)\,dx &= \int x^2\,dx - \int 2x\,dx + \int 5\,dx \\
&= \int x^2\,dx - 2\int x\,dx + 5\int 1\,dx \\
&= \left(\frac{x^3}{3} + C_1\right) - 2\left(\frac{x^2}{2} + C_2\right) + 5(x + C_3) \\
&= \frac{x^3}{3} + C_1 - x^2 - 2C_2 + 5x + 5C_3.
\end{aligned}$$

This formula is more complicated than it needs to be. If we combine C_1, $-2C_2$, and $5C_3$ into a single arbitrary constant $C = C_1 - 2C_2 + 5C_3$, the formula simplifies to

$$\frac{x^3}{3} - x^2 + 5x + C$$

and *still* gives all the antiderivatives there are. For this reason, we recommend that you go right to the final form even if you elect to integrate term-by-term. Write

$$\int (x^2 - 2x + 5)\,dx = \int x^2\,dx - \int 2x\,dx + \int 5\,dx$$

$$= \frac{x^3}{3} - x^2 + 5x + C.$$

Find the simplest antiderivative you can for each part and add the arbitrary constant of integration at the end. ■

EXERCISES 4.8

Finding Antiderivatives

In Exercises 1–16, find an antiderivative for each function. Do as many as you can mentally. Check your answers by differentiation.

1. **a.** $2x$ **b.** x^2 **c.** $x^2 - 2x + 1$

2. **a.** $6x$ **b.** x^7 **c.** $x^7 - 6x + 8$

3. **a.** $-3x^{-4}$ **b.** x^{-4} **c.** $x^{-4} + 2x + 3$

4. **a.** $2x^{-3}$ **b.** $\dfrac{x^{-3}}{2} + x^2$ **c.** $-x^{-3} + x - 1$

5. **a.** $\dfrac{1}{x^2}$ **b.** $\dfrac{5}{x^2}$ **c.** $2 - \dfrac{5}{x^2}$

6. **a.** $-\dfrac{2}{x^3}$ **b.** $\dfrac{1}{2x^3}$ **c.** $x^3 - \dfrac{1}{x^3}$

7. **a.** $\dfrac{3}{2}\sqrt{x}$ **b.** $\dfrac{1}{2\sqrt{x}}$ **c.** $\sqrt{x} + \dfrac{1}{\sqrt{x}}$

8. **a.** $\dfrac{4}{3}\sqrt[3]{x}$ **b.** $\dfrac{1}{3\sqrt[3]{x}}$ **c.** $\sqrt[3]{x} + \dfrac{1}{\sqrt[3]{x}}$

9. **a.** $\dfrac{2}{3}x^{-1/3}$ **b.** $\dfrac{1}{3}x^{-2/3}$ **c.** $-\dfrac{1}{3}x^{-4/3}$

10. **a.** $\dfrac{1}{2}x^{-1/2}$ **b.** $-\dfrac{1}{2}x^{-3/2}$ **c.** $-\dfrac{3}{2}x^{-5/2}$

11. **a.** $-\pi \sin \pi x$ **b.** $3 \sin x$ **c.** $\sin \pi x - 3 \sin 3x$

12. **a.** $\pi \cos \pi x$ **b.** $\dfrac{\pi}{2} \cos \dfrac{\pi x}{2}$ **c.** $\cos \dfrac{\pi x}{2} + \pi \cos x$

13. **a.** $\sec^2 x$ **b.** $\dfrac{2}{3} \sec^2 \dfrac{x}{3}$ **c.** $-\sec^2 \dfrac{3x}{2}$

14. **a.** $\csc^2 x$ **b.** $-\dfrac{3}{2} \csc^2 \dfrac{3x}{2}$ **c.** $1 - 8 \csc^2 2x$

15. **a.** $\csc x \cot x$ **b.** $-\csc 5x \cot 5x$ **c.** $-\pi \csc \dfrac{\pi x}{2} \cot \dfrac{\pi x}{2}$

16. **a.** $\sec x \tan x$ **b.** $4 \sec 3x \tan 3x$ **c.** $\sec \dfrac{\pi x}{2} \tan \dfrac{\pi x}{2}$

Finding Indefinite Integrals

In Exercises 17–54, find the most general antiderivative or indefinite integral. Check your answers by differentiation.

17. $\displaystyle\int (x + 1)\, dx$

18. $\displaystyle\int (5 - 6x)\, dx$

19. $\displaystyle\int \left(3t^2 + \frac{t}{2}\right) dt$

20. $\displaystyle\int \left(\frac{t^2}{2} + 4t^3\right) dt$

21. $\displaystyle\int (2x^3 - 5x + 7)\, dx$

22. $\displaystyle\int (1 - x^2 - 3x^5)\, dx$

23. $\displaystyle\int \left(\frac{1}{x^2} - x^2 - \frac{1}{3}\right) dx$

24. $\displaystyle\int \left(\frac{1}{5} - \frac{2}{x^3} + 2x\right) dx$

25. $\displaystyle\int x^{-1/3}\, dx$

26. $\displaystyle\int x^{-5/4}\, dx$

27. $\displaystyle\int \left(\sqrt{x} + \sqrt[3]{x}\right) dx$

28. $\displaystyle\int \left(\frac{\sqrt{x}}{2} + \frac{2}{\sqrt{x}}\right) dx$

29. $\displaystyle\int \left(8y - \frac{2}{y^{1/4}}\right) dy$

30. $\displaystyle\int \left(\frac{1}{7} - \frac{1}{y^{5/4}}\right) dy$

31. $\displaystyle\int 2x(1 - x^{-3})\, dx$

32. $\displaystyle\int x^{-3}(x + 1)\, dx$

33. $\displaystyle\int \frac{t\sqrt{t} + \sqrt{t}}{t^2}\, dt$

34. $\displaystyle\int \frac{4 + \sqrt{t}}{t^3}\, dt$

35. $\displaystyle\int (-2 \cos t)\, dt$

36. $\displaystyle\int (-5 \sin t)\, dt$

37. $\displaystyle\int 7 \sin \frac{\theta}{3}\, d\theta$

38. $\displaystyle\int 3 \cos 5\theta\, d\theta$

39. $\displaystyle\int (-3 \csc^2 x)\, dx$

40. $\displaystyle\int \left(-\frac{\sec^2 x}{3}\right) dx$

41. $\displaystyle\int \frac{\csc \theta \cot \theta}{2}\, d\theta$

42. $\displaystyle\int \frac{2}{5} \sec \theta \tan \theta\, d\theta$

Exercises

43. $\int (4 \sec x \tan x - 2 \sec^2 x)\, dx$

44. $\int \frac{1}{2}(\csc^2 x - \csc x \cot x)\, dx$

45. $\int (\sin 2x - \csc^2 x)\, dx$

46. $\int (2 \cos 2x - 3 \sin 3x)\, dx$

47. $\int \frac{1 + \cos 4t}{2}\, dt$

48. $\int \frac{1 - \cos 6t}{2}\, dt$

49. $\int (1 + \tan^2 \theta)\, d\theta$

(*Hint:* $1 + \tan^2 \theta = \sec^2 \theta$)

50. $\int (2 + \tan^2 \theta)\, d\theta$

51. $\int \cot^2 x\, dx$

(*Hint:* $1 + \cot^2 x = \csc^2 x$)

52. $\int (1 - \cot^2 x)\, dx$

53. $\int \cos \theta (\tan \theta + \sec \theta)\, d\theta$

54. $\int \frac{\csc \theta}{\csc \theta - \sin \theta}\, d\theta$

Checking Antiderivative Formulas

Verify the formulas in Exercises 55–60 by differentiation.

55. $\int (7x - 2)^3\, dx = \frac{(7x - 2)^4}{28} + C$

56. $\int (3x + 5)^{-2}\, dx = -\frac{(3x + 5)^{-1}}{3} + C$

57. $\int \sec^2 (5x - 1)\, dx = \frac{1}{5} \tan (5x - 1) + C$

58. $\int \csc^2 \left(\frac{x - 1}{3}\right) dx = -3 \cot \left(\frac{x - 1}{3}\right) + C$

59. $\int \frac{1}{(x + 1)^2}\, dx = -\frac{1}{x + 1} + C$

60. $\int \frac{1}{(x + 1)^2}\, dx = \frac{x}{x + 1} + C$

61. Right, or wrong? Say which for each formula and give a brief reason for each answer.

a. $\int x \sin x\, dx = \frac{x^2}{2} \sin x + C$

b. $\int x \sin x\, dx = -x \cos x + C$

c. $\int x \sin x\, dx = -x \cos x + \sin x + C$

62. Right, or wrong? Say which for each formula and give a brief reason for each answer.

a. $\int \tan \theta \sec^2 \theta\, d\theta = \frac{\sec^3 \theta}{3} + C$

b. $\int \tan \theta \sec^2 \theta\, d\theta = \frac{1}{2} \tan^2 \theta + C$

c. $\int \tan \theta \sec^2 \theta\, d\theta = \frac{1}{2} \sec^2 \theta + C$

63. Right, or wrong? Say which for each formula and give a brief reason for each answer.

a. $\int (2x + 1)^2\, dx = \frac{(2x + 1)^3}{3} + C$

b. $\int 3(2x + 1)^2\, dx = (2x + 1)^3 + C$

c. $\int 6(2x + 1)^2\, dx = (2x + 1)^3 + C$

64. Right, or wrong? Say which for each formula and give a brief reason for each answer.

a. $\int \sqrt{2x + 1}\, dx = \sqrt{x^2 + x + C}$

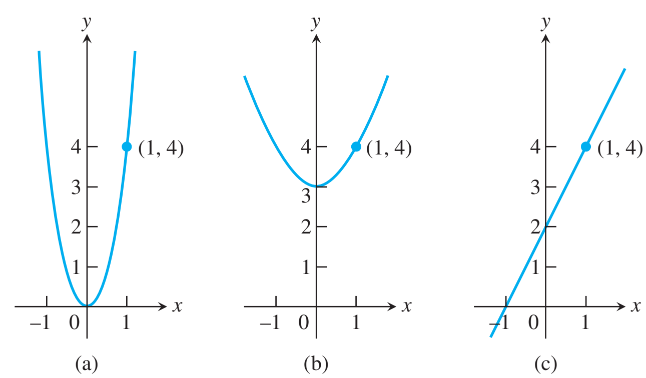
b. $\int \sqrt{2x + 1}\, dx = \sqrt{x^2 + x} + C$

c. $\int \sqrt{2x + 1}\, dx = \frac{1}{3}\left(\sqrt{2x + 1}\right)^3 + C$

Initial Value Problems

65. Which of the following graphs shows the solution of the initial value problem

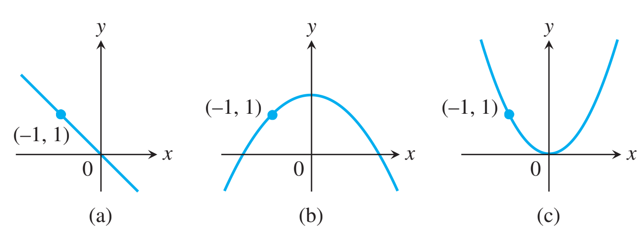
$$\frac{dy}{dx} = 2x, \quad y = 4 \text{ when } x = 1?$$



Give reasons for your answer.

66. Which of the following graphs shows the solution of the initial value problem

$$\frac{dy}{dx} = -x, \quad y = 1 \text{ when } x = -1?$$



Give reasons for your answer.

Solve the initial value problems in Exercises 67–86.

67. $\dfrac{dy}{dx} = 2x - 7, \quad y(2) = 0$

68. $\dfrac{dy}{dx} = 10 - x, \quad y(0) = -1$

69. $\dfrac{dy}{dx} = \dfrac{1}{x^2} + x, \quad x > 0; \quad y(2) = 1$

70. $\dfrac{dy}{dx} = 9x^2 - 4x + 5, \quad y(-1) = 0$

71. $\dfrac{dy}{dx} = 3x^{-2/3}, \quad y(-1) = -5$

72. $\dfrac{dy}{dx} = \dfrac{1}{2\sqrt{x}}, \quad y(4) = 0$

73. $\dfrac{ds}{dt} = 1 + \cos t, \quad s(0) = 4$

74. $\dfrac{ds}{dt} = \cos t + \sin t, \quad s(\pi) = 1$

75. $\dfrac{dr}{d\theta} = -\pi \sin \pi\theta, \quad r(0) = 0$

76. $\dfrac{dr}{d\theta} = \cos \pi\theta, \quad r(0) = 1$

77. $\dfrac{dv}{dt} = \dfrac{1}{2}\sec t \tan t, \quad v(0) = 1$

78. $\dfrac{dv}{dt} = 8t + \csc^2 t, \quad v\left(\dfrac{\pi}{2}\right) = -7$

79. $\dfrac{d^2y}{dx^2} = 2 - 6x; \quad y'(0) = 4, \quad y(0) = 1$

80. $\dfrac{d^2y}{dx^2} = 0; \quad y'(0) = 2, \quad y(0) = 0$

81. $\dfrac{d^2r}{dt^2} = \dfrac{2}{t^3}; \quad \left.\dfrac{dr}{dt}\right|_{t=1} = 1, \quad r(1) = 1$

82. $\dfrac{d^2s}{dt^2} = \dfrac{3t}{8}; \quad \left.\dfrac{ds}{dt}\right|_{t=4} = 3, \quad s(4) = 4$

83. $\dfrac{d^3y}{dx^3} = 6; \quad y''(0) = -8, \quad y'(0) = 0, \quad y(0) = 5$

84. $\dfrac{d^3\theta}{dt^3} = 0; \quad \theta''(0) = -2, \quad \theta'(0) = -\dfrac{1}{2}, \quad \theta(0) = \sqrt{2}$

85. $y^{(4)} = -\sin t + \cos t;$
$y'''(0) = 7, \quad y''(0) = y'(0) = -1, \quad y(0) = 0$

86. $y^{(4)} = -\cos x + 8 \sin 2x;$
$y'''(0) = 0, \quad y''(0) = y'(0) = 1, \quad y(0) = 3$

Finding Curves

87. Find the curve $y = f(x)$ in the xy-plane that passes through the point (9, 4) and whose slope at each point is $3\sqrt{x}$.

88. a. Find a curve $y = f(x)$ with the following properties:

i) $\dfrac{d^2y}{dx^2} = 6x$

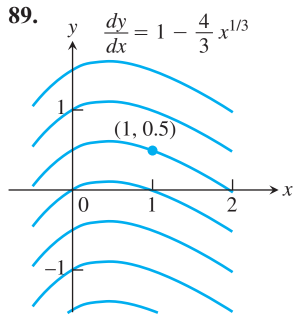
ii) Its graph passes through the point (0, 1), and has a horizontal tangent there.

b. How many curves like this are there? How do you know?

Solution (Integral) Curves

Exercises 89–92 show solution curves of differential equations. In each exercise, find an equation for the curve through the labeled point.

89.



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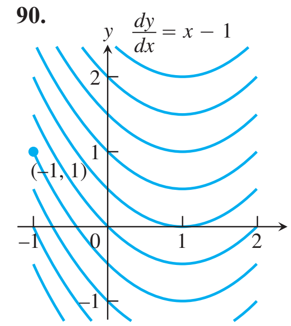

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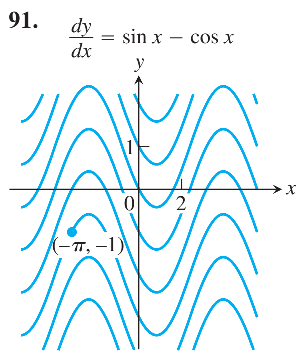
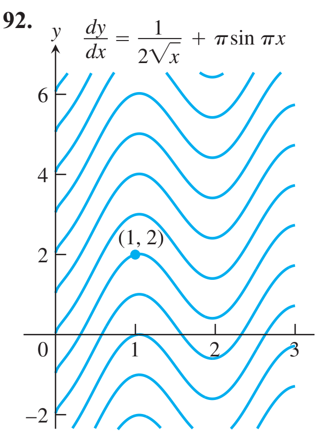


92.



Applications

93. Finding displacement from an antiderivative of velocity

a. Suppose that the velocity of a body moving along the s-axis is

$$\frac{ds}{dt} = v = 9.8t - 3.$$

i) Find the body's displacement over the time interval from $t = 1$ to $t = 3$ given that $s = 5$ when $t = 0$.

ii) Find the body's displacement from $t = 1$ to $t = 3$ given that $s = -2$ when $t = 0$.

iii) Now find the body's displacement from $t = 1$ to $t = 3$ given that $s = s_0$ when $t = 0$.

Exercises

b. Suppose that the position s of a body moving along a coordinate line is a differentiable function of time t. Is it true that once you know an antiderivative of the velocity function ds/dt you can find the body's displacement from $t = a$ to $t = b$ even if you do not know the body's exact position at either of those times? Give reasons for your answer.

94. Liftoff from Earth A rocket lifts off the surface of Earth with a constant acceleration of 20 m/sec^2. How fast will the rocket be going 1 min later?

95. Stopping a car in time You are driving along a highway at a steady 60 mph (88 ft/sec) when you see an accident ahead and slam on the brakes. What constant deceleration is required to stop your car in 242 ft? To find out, carry out the following steps.

1. Solve the initial value problem

$$\text{Differential equation:}\quad \frac{d^2s}{dt^2} = -k \qquad (k \text{ constant})$$

$$\text{Initial conditions:}\quad \frac{ds}{dt} = 88 \text{ and } s = 0 \text{ when } t = 0.$$

Measuring time and distance from when the brakes are applied.

2. Find the value of t that makes $ds/dt = 0$. (The answer will involve k.)

3. Find the value of k that makes $s = 242$ for the value of t you found in Step 2.

96. Stopping a motorcycle The State of Illinois Cycle Rider Safety Program requires riders to be able to brake from 30 mph (44 ft/sec) to 0 in 45 ft. What constant deceleration does it take to do that?

97. Motion along a coordinate line A particle moves on a coordinate line with acceleration $a = d^2s/dt^2 = 15\sqrt{t} - \left(3/\sqrt{t}\right)$, subject to the conditions that $ds/dt = 4$ and $s = 0$ when $t = 1$. Find

a. the velocity $v = ds/dt$ in terms of t

b. the position s in terms of t.

T **98. The hammer and the feather** When *Apollo 15* astronaut David Scott dropped a hammer and a feather on the moon to demonstrate that in a vacuum all bodies fall with the same (constant) acceleration, he dropped them from about 4 ft above the ground. The television footage of the event shows the hammer and the feather falling more slowly than on Earth, where, in a vacuum, they would have taken only half a second to fall the 4 ft. How long did it take the hammer and feather to fall 4 ft on the moon? To find out, solve the following initial value problem for s as a function of t. Then find the value of t that makes s equal to 0.

$$\text{Differential equation:}\quad \frac{d^2s}{dt^2} = -5.2 \text{ ft/sec}^2$$

$$\text{Initial conditions:}\quad \frac{ds}{dt} = 0 \text{ and } s = 4 \text{ when } t = 0$$

99. Motion with constant acceleration The standard equation for the position s of a body moving with a constant acceleration a along a coordinate line is

$$s = \frac{a}{2}t^2 + v_0 t + s_0, \tag{1}$$

where v_0 and s_0 are the body's velocity and position at time $t = 0$. Derive this equation by solving the initial value problem

$$\text{Differential equation:}\quad \frac{d^2s}{dt^2} = a$$

$$\text{Initial conditions:}\quad \frac{ds}{dt} = v_0 \text{ and } s = s_0 \text{ when } t = 0.$$

100. Free fall near the surface of a planet For free fall near the surface of a planet where the acceleration due to gravity has a constant magnitude of g length-units/sec^2, Equation (1) in Exercise 99 takes the form

$$s = -\frac{1}{2}gt^2 + v_0 t + s_0, \tag{2}$$

where s is the body's height above the surface. The equation has a minus sign because the acceleration acts downward, in the direction of decreasing s. The velocity v_0 is positive if the object is rising at time $t = 0$ and negative if the object is falling.

Instead of using the result of Exercise 99, you can derive Equation (2) directly by solving an appropriate initial value problem. What initial value problem? Solve it to be sure you have the right one, explaining the solution steps as you go along.

Theory and Examples

101. Suppose that

$$f(x) = \frac{d}{dx}\left(1 - \sqrt{x}\right) \quad \text{and} \quad g(x) = \frac{d}{dx}(x + 2).$$

Find:

a. $\int f(x)\,dx$ **b.** $\int g(x)\,dx$

c. $\int [-f(x)]\,dx$ **d.** $\int [-g(x)]\,dx$

e. $\int [f(x) + g(x)]\,dx$ **f.** $\int [f(x) - g(x)]\,dx$

102. Uniqueness of solutions If differentiable functions $y = F(x)$ and $y = G(x)$ both solve the initial value problem

$$\frac{dy}{dx} = f(x), \qquad y(x_0) = y_0,$$

on an interval I, must $F(x) = G(x)$ for every x in I? Give reasons for your answer.

COMPUTER EXPLORATIONS

Use a CAS to solve the initial problems in Exercises 103–106. Plot the solution curves.

103. $y' = \cos^2 x + \sin x, \quad y(\pi) = 1$

104. $y' = \frac{1}{x} + x, \quad y(1) = -1$

105. $y' = \frac{1}{\sqrt{4 - x^2}}, \quad y(0) = 2$

106. $y'' = \frac{2}{x} + \sqrt{x}, \quad y(1) = 0, \quad y'(1) = 0$

Chapter 4 Additional and Advanced Exercises

1. What can you say about a function whose maximum and minimum values on an interval are equal? Give reasons for your answer.

2. Is it true that a discontinuous function cannot have both an absolute maximum and an absolute minimum value on a closed interval? Give reasons for your answer.

3. Can you conclude anything about the extreme values of a continuous function on an open interval? On a half-open interval? Give reasons for your answer.

4. **Local extrema** Use the sign pattern for the derivative

$$\frac{df}{dx} = 6(x-1)(x-2)^2(x-3)^3(x-4)^4$$

to identify the points where f has local maximum and minimum values.

5. **Local extrema**

   a. Suppose that the first derivative of $y = f(x)$ is

   $$y' = 6(x+1)(x-2)^2.$$

   At what points, if any, does the graph of f have a local maximum, local minimum, or point of inflection?

   b. Suppose that the first derivative of $y = f(x)$ is

   $$y' = 6x(x+1)(x-2).$$

   At what points, if any, does the graph of f have a local maximum, local minimum, or point of inflection?

6. If $f'(x) \le 2$ for all x, what is the most the values of f can increase on $[0, 6]$? Give reasons for your answer.

7. **Bounding a function** Suppose that f is continuous on $[a, b]$ and that c is an interior point of the interval. Show that if $f'(x) \le 0$ on $[a, c)$ and $f'(x) \ge 0$ on $(c, b]$, then $f(x)$ is never less than $f(c)$ on $[a, b]$.

8. **An inequality**

   a. Show that $-1/2 \le x/(1+x^2) \le 1/2$ for every value of x.

   b. Suppose that f is a function whose derivative is $f'(x) = x/(1+x^2)$. Use the result in part (a) to show that

   $$|f(b) - f(a)| \le \frac{1}{2}|b - a|$$

   for any a and b.

9. The derivative of $f(x) = x^2$ is zero at $x = 0$, but f is not a constant function. Doesn't this contradict the corollary of the Mean Value Theorem that says that functions with zero derivatives are constant? Give reasons for your answer.

10. **Extrema and inflection points** Let $h = fg$ be the product of two differentiable functions of x.

    a. If f and g are positive, with local maxima at $x = a$, and if f' and g' change sign at a, does h have a local maximum at a?

    b. If the graphs of f and g have inflection points at $x = a$, does the graph of h have an inflection point at a?

    In either case, if the answer is yes, give a proof. If the answer is no, give a counterexample.

11. **Finding a function** Use the following information to find the values of a, b, and c in the formula $f(x) = (x+a)/(bx^2 + cx + 2)$.

    i) The values of a, b, and c are either 0 or 1.

    ii) The graph of f passes through the point $(-1, 0)$.

    iii) The line $y = 1$ is an asymptote of the graph of f.

12. **Horizontal tangent** For what value or values of the constant k will the curve $y = x^3 + kx^2 + 3x - 4$ have exactly one horizontal tangent?

13. **Largest inscribed triangle** Points A and B lie at the ends of a diameter of a unit circle and point C lies on the circumference. Is it true that the area of triangle ABC is largest when the triangle is isosceles? How do you know?

14. **Proving the second derivative test** The Second Derivative Test for Local Maxima and Minima (Section 4.4) says:

    a. f has a local maximum value at $x = c$ if $f'(c) = 0$ and $f''(c) < 0$

    b. f has a local minimum value at $x = c$ if $f'(c) = 0$ and $f''(c) > 0$.

    To prove statement (a), let $\epsilon = (1/2)|f''(c)|$. Then use the fact that

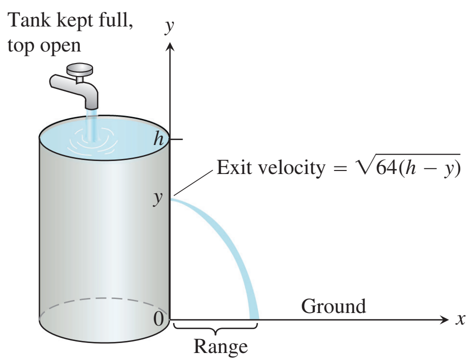
    $$f''(c) = \lim_{h\to 0} \frac{f'(c+h) - f'(c)}{h} = \lim_{h\to 0} \frac{f'(c+h)}{h}$$

    to conclude that for some $\delta > 0$,

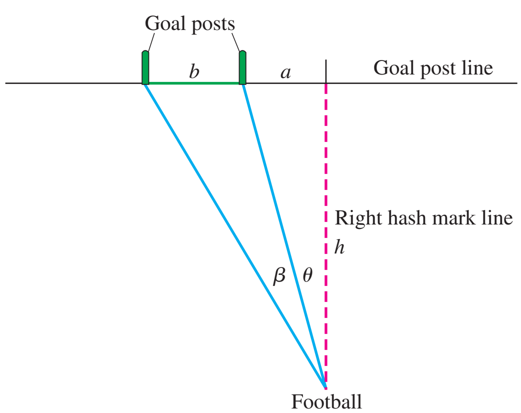
    $$0 < |h| < \delta \quad\Rightarrow\quad \frac{f'(c+h)}{h} < f''(c) + \epsilon < 0.$$

    Thus, $f'(c+h)$ is positive for $-\delta < h < 0$ and negative for $0 < h < \delta$. Prove statement (b) in a similar way.

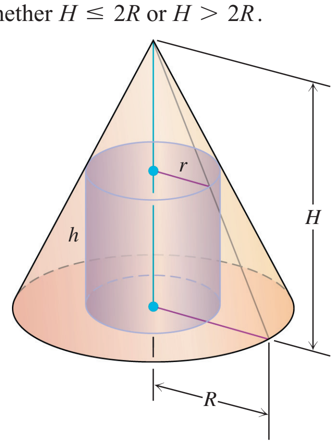
15. **Hole in a water tank** You want to bore a hole in the side of the tank shown here at a height that will make the stream of water coming out hit the ground as far from the tank as possible. If you drill the hole near the top, where the pressure is low, the water will exit slowly but spend a relatively long time in the air. If you drill the hole near the bottom, the water will exit at a higher velocity but have only a short time to fall. Where is the best place, if any, for the hole? (*Hint:* How long will it take an exiting particle of water to fall from height y to the ground?)



16. Kicking a field goal An American football player wants to kick a field goal with the ball being on a right hash mark. Assume that the goal posts are b feet apart and that the hash mark line is a distance $a > 0$ feet from the right goal post. (See the accompanying figure.) Find the distance h from the goal post line that gives the kicker his largest angle β. Assume that the football field is flat.



17. A max-min problem with a variable answer Sometimes the solution of a max-min problem depends on the proportions of the shapes involved. As a case in point, suppose that a right circular cylinder of radius r and height h is inscribed in a right circular cone of radius R and height H, as shown here. Find the value of r (in terms of R and H) that maximizes the total surface area of the cylinder (including top and bottom). As you will see, the solution depends on whether $H \le 2R$ or $H > 2R$.



18. Minimizing a parameter Find the smallest value of the positive constant m that will make $mx - 1 + (1/x)$ greater than or equal to zero for all positive values of x.

19. Evaluate the following limits.

a. $\displaystyle\lim_{x\to 0} \frac{2\sin 5x}{3x}$ **b.** $\displaystyle\lim_{x\to 0} \sin 5x \cot 3x$

c. $\displaystyle\lim_{x\to 0} x \csc^2 \sqrt{2x}$ **d.** $\displaystyle\lim_{x\to \pi/2} (\sec x - \tan x)$

e. $\displaystyle\lim_{x\to 0} \frac{x - \sin x}{x - \tan x}$ **f.** $\displaystyle\lim_{x\to 0} \frac{\sin x^2}{x \sin x}$

g. $\displaystyle\lim_{x\to 0} \frac{\sec x - 1}{x^2}$ **h.** $\displaystyle\lim_{x\to 2} \frac{x^3 - 8}{x^2 - 4}$

20. L'Hôpital's Rule does not help with the following limits. Find them some other way.

a. $\displaystyle\lim_{x\to\infty} \frac{\sqrt{x+5}}{\sqrt{x}+5}$ **b.** $\displaystyle\lim_{x\to\infty} \frac{2x}{x + 7\sqrt{x}}$

21. Suppose that it costs a company $y = a + bx$ dollars to produce x units per week. It can sell x units per week at a price of $P = c - ex$ dollars per unit. Each of a, b, c, and e represents a positive constant. **(a)** What production level maximizes the profit? **(b)** What is the corresponding price? **(c)** What is the weekly profit at this level of production? **(d)** At what price should each item be sold to maximize profits if the government imposes a tax of t dollars per item sold? Comment on the difference between this price and the price before the tax.

22. Estimating reciprocals without division You can estimate the value of the reciprocal of a number a without ever dividing by a if you apply Newton's method to the function $f(x) = (1/x) - a$. For example, if $a = 3$, the function involved is $f(x) = (1/x) - 3$.

a. Graph $y = (1/x) - 3$. Where does the graph cross the x-axis?

b. Show that the recursion formula in this case is

$$x_{n+1} = x_n(2 - 3x_n),$$

so there is no need for division.

23. To find $x = \sqrt[q]{a}$, we apply Newton's method to $f(x) = x^q - a$. Here we assume that a is a positive real number and q is a positive integer. Show that x_1 is a "weighted average" of x_0 and a/x_0^{q-1}, and find the coefficients m_0, m_1 such that

$$x_1 = m_0 x_0 + m_1\left(\frac{a}{x_0^{q-1}}\right), \qquad \begin{aligned} m_0 > 0,\ m_1 > 0,\\ m_0 + m_1 = 1. \end{aligned}$$

What conclusion would you reach if x_0 and a/x_0^{q-1} were equal? What would be the value of x_1 in that case?

24. The family of straight lines $y = ax + b$ (a, b arbitrary constants) can be characterized by the relation $y'' = 0$. Find a similar relation satisfied by the family of all circles

$$(x - h)^2 + (y - h)^2 = r^2,$$

where h and r are arbitrary constants. (*Hint:* Eliminate h and r from the set of three equations including the given one and two obtained by successive differentiation.)

25. Assume that the brakes of an automobile produce a constant deceleration of k ft/sec^2. **(a)** Determine what k must be to bring an automobile traveling 60 mi/hr (88 ft/sec) to rest in a distance of 100 ft from the point where the brakes are applied. **(b)** With the same k, how far would a car traveling 30 mi/hr travel before being brought to a stop?

26. Let $f(x)$, $g(x)$ be two continuously differentiable functions satisfying the relationships $f'(x) = g(x)$ and $f''(x) = -f(x)$. Let $h(x) = f^2(x) + g^2(x)$. If $h(0) = 5$, find $h(10)$.

27. Can there be a curve satisfying the following conditions? d^2y/dx^2 is everywhere equal to zero and, when $x = 0$, $y = 0$ and $dy/dx = 1$. Give a reason for your answer.

28. Find the equation for the curve in the xy-plane that passes through the point $(1, -1)$ if its slope at x is always $3x^2 + 2$.

29. A particle moves along the x-axis. Its acceleration is $a = -t^2$. At $t = 0$, the particle is at the origin. In the course of its motion, it reaches the point $x = b$, where $b > 0$, but no point beyond b. Determine its velocity at $t = 0$.

30. A particle moves with acceleration $a = \sqrt{t} - \left(1/\sqrt{t}\right)$. Assuming that the velocity $v = 4/3$ and the position $s = -4/15$ when $t = 0$, find

a. the velocity v in terms of t.

b. the position s in terms of t.

31. Given $f(x) = ax^2 + 2bx + c$ with $a > 0$. By considering the minimum, prove that $f(x) \geq 0$ for all real x if, and only if, $b^2 - ac \leq 0$.

32. Schwarz's inequality

a. In Exercise 31, let

$$f(x) = (a_1x + b_1)^2 + (a_2x + b_2)^2 + \cdots + (a_nx + b_n)^2,$$

and deduce Schwarz's inequality:

$$(a_1b_1 + a_2b_2 + \cdots + a_nb_n)^2 \leq (a_1^2 + a_2^2 + \cdots + a_n^2)(b_1^2 + b_2^2 + \cdots + b_n^2).$$

b. Show that equality holds in Schwarz's inequality only if there exists a real number x that makes a_ix equal $-b_i$ for every value of i from 1 to n.

Chapter 4 Practice Exercises

Existence of Extreme Values

1. Does $f(x) = x^3 + 2x + \tan x$ have any local maximum or minimum values? Give reasons for your answer.

2. Does $g(x) = \csc x + 2 \cot x$ have any local maximum values? Give reasons for your answer.

3. Does $f(x) = (7 + x)(11 - 3x)^{1/3}$ have an absolute minimum value? An absolute maximum? If so, find them or give reasons why they fail to exist. List all critical points of f.

4. Find values of a and b such that the function

$$f(x) = \frac{ax + b}{x^2 - 1}$$

has a local extreme value of 1 at $x = 3$. Is this extreme value a local maximum, or a local minimum? Give reasons for your answer.

5. The greatest integer function $f(x) = \lfloor x \rfloor$, defined for all values of x, assumes a local maximum value of 0 at each point of $[0, 1)$. Could any of these local maximum values also be local minimum values of f? Give reasons for your answer.

6. a. Give an example of a differentiable function f whose first derivative is zero at some point c even though f has neither a local maximum nor a local minimum at c.

b. How is this consistent with Theorem 2 in Section 4.1? Give reasons for your answer.

7. The function $y = 1/x$ does not take on either a maximum or a minimum on the interval $0 < x < 1$ even though the function is continuous on this interval. Does this contradict the Extreme Value Theorem for continuous functions? Why?

8. What are the maximum and minimum values of the function $y = |x|$ on the interval $-1 \leq x < 1$? Notice that the interval is not closed. Is this consistent with the Extreme Value Theorem for continuous functions? Why?

T **9.** A graph that is large enough to show a function's global behavior may fail to reveal important local features. The graph of $f(x) = (x^8/8) - (x^6/2) - x^5 + 5x^3$ is a case in point.

a. Graph f over the interval $-2.5 \leq x \leq 2.5$. Where does the graph appear to have local extreme values or points of inflection?

b. Now factor $f'(x)$ and show that f has a local maximum at $x = \sqrt[3]{5} \approx 1.70998$ and local minima at $x = \pm\sqrt{3} \approx \pm 1.73205$.

c. Zoom in on the graph to find a viewing window that shows the presence of the extreme values at $x = \sqrt[3]{5}$ and $x = \sqrt{3}$.

The moral here is that without calculus the existence of two of the three extreme values would probably have gone unnoticed. On any normal graph of the function, the values would lie close enough together to fall within the dimensions of a single pixel on the screen.

(*Source*: *Uses of Technology in the Mathematics Curriculum*, by Benny Evans and Jerry Johnson, Oklahoma State University, published in 1990 under National Science Foundation Grant USE-8950044.)

T **10.** (*Continuation of Exercise 9.*)

a. Graph $f(x) = (x^8/8) - (2/5)x^5 - 5x - (5/x^2) + 11$ over the interval $-2 \leq x \leq 2$. Where does the graph appear to have local extreme values or points of inflection?

b. Show that f has a local maximum value at $x = \sqrt[7]{5} \approx 1.2585$ and a local minimum value at $x = \sqrt[3]{2} \approx 1.2599$.

c. Zoom in to find a viewing window that shows the presence of the extreme values at $x = \sqrt[7]{5}$ and $x = \sqrt[3]{2}$.

The Mean Value Theorem

11. a. Show that $g(t) = \sin^2 t - 3t$ decreases on every interval in its domain.

b. How many solutions does the equation $\sin^2 t - 3t = 5$ have? Give reasons for your answer.

12. a. Show that $y = \tan\theta$ increases on every interval in its domain.

b. If the conclusion in part (a) is really correct, how do you explain the fact that $\tan\pi = 0$ is less than $\tan(\pi/4) = 1$?

13. a. Show that the equation $x^4 + 2x^2 - 2 = 0$ has exactly one solution on $[0, 1]$.

T **b.** Find the solution to as many decimal places as you can.

14. a. Show that $f(x) = x/(x + 1)$ increases on every interval in its domain.

b. Show that $f(x) = x^3 + 2x$ has no local maximum or minimum values.

15. Water in a reservoir As a result of a heavy rain, the volume of water in a reservoir increased by 1400 acre-ft in 24 hours. Show that at some instant during that period the reservoir's volume was increasing at a rate in excess of 225,000 gal/min. (An acre-foot is 43,560 ft^3, the volume that would cover 1 acre to the depth of 1 ft. A cubic foot holds 7.48 gal.)

16. The formula $F(x) = 3x + C$ gives a different function for each value of C. All of these functions, however, have the same derivative with respect to x, namely $F'(x) = 3$. Are these the only differentiable functions whose derivative is 3? Could there be any others? Give reasons for your answers.

17. Show that

$$\frac{d}{dx}\left(\frac{x}{x + 1}\right) = \frac{d}{dx}\left(-\frac{1}{x + 1}\right)$$

even though

$$\frac{x}{x + 1} \neq -\frac{1}{x + 1}.$$

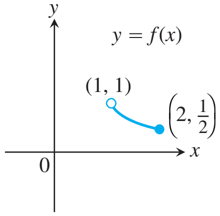
Doesn't this contradict Corollary 2 of the Mean Value Theorem? Give reasons for your answer.

18. Calculate the first derivatives of $f(x) = x^2/(x^2 + 1)$ and $g(x) = -1/(x^2 + 1)$. What can you conclude about the graphs of these functions?

Conclusions from Graphs

In Exercises 19 and 20, use the graph to answer the questions.

19. Identify any global extreme values of f and the values of x at which they occur.

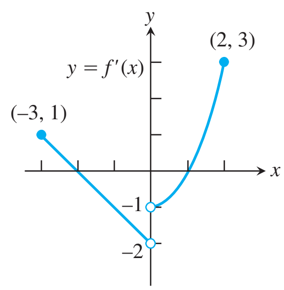


20. Estimate the intervals on which the function $y = f(x)$ is

a. increasing.

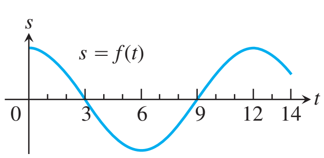
b. decreasing.

c. Use the given graph of f' to indicate where any local extreme values of the function occur, and whether each extreme is a relative maximum or minimum.

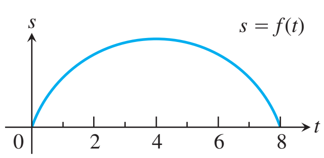


Each of the graphs in Exercises 21 and 22 is the graph of the position function $s = f(t)$ of a body moving on a coordinate line (t represents time). At approximately what times (if any) is each body's **(a)** velocity equal to zero? **(b)** Acceleration equal to zero? During approximately what time intervals does the body move **(c)** forward? **(d)** Backward?

21.



22.



Graphs and Graphing

Graph the curves in Exercises 23–32.

23. $y = x^2 - (x^3/6)$

24. $y = x^3 - 3x^2 + 3$

25. $y = -x^3 + 6x^2 - 9x + 3$

26. $y = (1/8)(x^3 + 3x^2 - 9x - 27)$

27. $y = x^3(8 - x)$

28. $y = x^2(2x^2 - 9)$

29. $y = x - 3x^{2/3}$

30. $y = x^{1/3}(x - 4)$

31. $y = x\sqrt{3 - x}$

32. $y = x\sqrt{4 - x^2}$

Each of Exercises 33–38 gives the first derivative of a function $y = f(x)$. **(a)** At what points, if any, does the graph of f have a local maximum, local minimum, or inflection point? **(b)** Sketch the general shape of the graph.

33. $y' = 16 - x^2$

34. $y' = x^2 - x - 6$

35. $y' = 6x(x + 1)(x - 2)$

36. $y' = x^2(6 - 4x)$

37. $y' = x^4 - 2x^2$

38. $y' = 4x^2 - x^4$

In Exercises 39–42, graph each function. Then use the function's first derivative to explain what you see.

39. $y = x^{2/3} + (x - 1)^{1/3}$

40. $y = x^{2/3} + (x - 1)^{2/3}$

41. $y = x^{1/3} + (x - 1)^{1/3}$

42. $y = x^{2/3} - (x - 1)^{1/3}$

Sketch the graphs of the functions in Exercises 43–50.

43. $y = \dfrac{x + 1}{x - 3}$

44. $y = \dfrac{2x}{x + 5}$

45. $y = \dfrac{x^2 + 1}{x}$

46. $y = \dfrac{x^2 - x + 1}{x}$

47. $y = \dfrac{x^3 + 2}{2x}$

48. $y = \dfrac{x^4 - 1}{x^2}$

49. $y = \dfrac{x^2 - 4}{x^2 - 3}$

50. $y = \dfrac{x^2}{x^2 - 4}$

Applying l'Hôpital's Rule

Use l'Hôpital's Rule to find the limits in Exercises 51–62.

51. $\lim\limits_{x \to 1} \dfrac{x^2 + 3x - 4}{x - 1}$

52. $\lim\limits_{x \to 1} \dfrac{x^a - 1}{x^b - 1}$

53. $\lim\limits_{x \to \pi} \dfrac{\tan x}{x}$

54. $\lim\limits_{x \to 0} \dfrac{\tan x}{x + \sin x}$

55. $\lim\limits_{x \to 0} \dfrac{\sin^2 x}{\tan(x^2)}$

56. $\lim\limits_{x \to 0} \dfrac{\sin mx}{\sin nx}$

57. $\lim\limits_{x \to \pi/2^-} \sec 7x \cos 3x$

58. $\lim\limits_{x \to 0^+} \sqrt{x} \sec x$

59. $\lim\limits_{x \to 0} (\csc x - \cot x)$

60. $\lim\limits_{x \to 0} \left(\dfrac{1}{x^4} - \dfrac{1}{x^2}\right)$

61. $\lim\limits_{x \to \infty} \left(\sqrt{x^2 + x + 1} - \sqrt{x^2 - x}\right)$

62. $\lim\limits_{x \to \infty} \left(\dfrac{x^3}{x^2 - 1} - \dfrac{x^3}{x^2 + 1}\right)$

Optimization

63. The sum of two nonnegative numbers is 36. Find the numbers if

a. the difference of their square roots is to be as large as possible.

b. the sum of their square roots is to be as large as possible.

64. The sum of two nonnegative numbers is 20. Find the numbers

a. if the product of one number and the square root of the other is to be as large as possible.

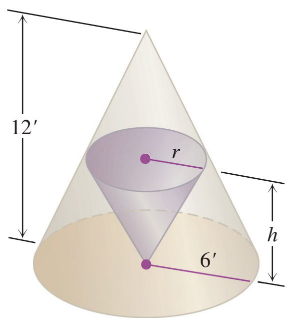
b. if one number plus the square root of the other is to be as large as possible.

65. An isosceles triangle has its vertex at the origin and its base parallel to the x-axis with the vertices above the axis on the curve $y = 27 - x^2$. Find the largest area the triangle can have.

66. A customer has asked you to design an open-top rectangular stainless steel vat. It is to have a square base and a volume of 32 ft^3, to be welded from quarter-inch plate, and to weigh no more than necessary. What dimensions do you recommend?

67. Find the height and radius of the largest right circular cylinder that can be put in a sphere of radius $\sqrt{3}$.

68. The figure here shows two right circular cones, one upside down inside the other. The two bases are parallel, and the vertex of the smaller cone lies at the center of the larger cone's base. What values of r and h will give the smaller cone the largest possible volume?



69. Manufacturing tires Your company can manufacture x hundred grade A tires and y hundred grade B tires a day, where $0 \le x \le 4$ and

$$y = \frac{40 - 10x}{5 - x}.$$

Your profit on a grade A tire is twice your profit on a grade B tire. What is the most profitable number of each kind to make?

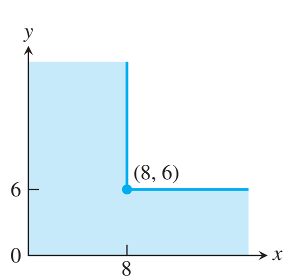
70. Particle motion The positions of two particles on the s-axis are $s_1 = \cos t$ and $s_2 = \cos(t + \pi/4)$.

a. What is the farthest apart the particles ever get?

b. When do the particles collide?

T **71. Open-top box** An open-top rectangular box is constructed from a 10-in.-by-16-in. piece of cardboard by cutting squares of equal side length from the corners and folding up the sides. Find analytically the dimensions of the box of largest volume and the maximum volume. Support your answers graphically.

72. The ladder problem What is the approximate length (in feet) of the longest ladder you can carry horizontally around the corner of the corridor shown here? Round your answer down to the nearest foot.



Newton's Method

73. Let $f(x) = 3x - x^3$. Show that the equation $f(x) = -4$ has a solution in the interval [2, 3] and use Newton's method to find it.

74. Let $f(x) = x^4 - x^3$. Show that the equation $f(x) = 75$ has a solution in the interval [3, 4] and use Newton's method to find it.

Finding Indefinite Integrals

Find the indefinite integrals (most general antiderivatives) in Exercises 75–90. Check your answers by differentiation.

75. $\int (x^3 + 5x - 7)\, dx$

76. $\int \left(8t^3 - \frac{t^2}{2} + t\right) dt$

77. $\int \left(3\sqrt{t} + \frac{4}{t^2}\right) dt$

78. $\int \left(\frac{1}{2\sqrt{t}} - \frac{3}{t^4}\right) dt$

79. $\int \frac{dr}{(r + 5)^2}$

80. $\int \frac{6\, dr}{\left(r - \sqrt{2}\right)^3}$

81. $\int 3\theta\sqrt{\theta^2 + 1}\, d\theta$

82. $\int \frac{\theta}{\sqrt{7 + \theta^2}}\, d\theta$

83. $\int x^3(1 + x^4)^{-1/4}\, dx$

84. $\int (2 - x)^{3/5}\, dx$

85. $\int \sec^2 \frac{s}{10}\, ds$

86. $\int \csc^2 \pi s\, ds$

87. $\int \csc \sqrt{2}\theta \cot \sqrt{2}\theta\, d\theta$

88. $\int \sec \frac{\theta}{3} \tan \frac{\theta}{3}\, d\theta$

89. $\int \sin^2 \frac{x}{4}\, dx$

90. $\int \cos^2 \frac{x}{2}\, dx \quad \left(\textit{Hint}: \cos^2\theta = \frac{1 + \cos 2\theta}{2}\right)$

Initial Value Problems

Solve the initial value problems in Exercises 91–94.

91. $\frac{dy}{dx} = \frac{x^2 + 1}{x^2}, \quad y(1) = -1$

92. $\frac{dy}{dx} = \left(x + \frac{1}{x}\right)^2, \quad y(1) = 1$

93. $\frac{d^2r}{dt^2} = 15\sqrt{t} + \frac{3}{\sqrt{t}}; \quad r'(1) = 8, \quad r(1) = 0$

94. $\frac{d^3r}{dt^3} = -\cos t; \quad r''(0) = r'(0) = 0, \quad r(0) = -1$

Chapter 4 Questions to Guide Your Review

1. What can be said about the extreme values of a function that is continuous on a closed interval?
2. What does it mean for a function to have a local extreme value on its domain? An absolute extreme value? How are local and absolute extreme values related, if at all? Give examples.
3. How do you find the absolute extrema of a continuous function on a closed interval? Give examples.
4. What are the hypotheses and conclusion of Rolle's Theorem? Are the hypotheses really necessary? Explain.
5. What are the hypotheses and conclusion of the Mean Value Theorem? What physical interpretations might the theorem have?
6. State the Mean Value Theorem's three corollaries.
7. How can you sometimes identify a function $f(x)$ by knowing f' and knowing the value of f at a point $x = x_0$? Give an example.
8. What is the First Derivative Test for Local Extreme Values? Give examples of how it is applied.
9. How do you test a twice-differentiable function to determine where its graph is concave up or concave down? Give examples.
10. What is an inflection point? Give an example. What physical significance do inflection points sometimes have?
11. What is the Second Derivative Test for Local Extreme Values? Give examples of how it is applied.
12. What do the derivatives of a function tell you about the shape of its graph?
13. List the steps you would take to graph a polynomial function. Illustrate with an example.
14. What is a cusp? Give examples.
15. List the steps you would take to graph a rational function. Illustrate with an example.
16. Outline a general strategy for solving max-min problems. Give examples.
17. Describe l'Hôpital's Rule. How do you know when to use the rule and when to stop? Give an example.
18. How can you sometimes handle limits that lead to indeterminate forms ∞/∞, $\infty \cdot 0$, and $\infty - \infty$. Give examples.
19. Describe Newton's method for solving equations. Give an example. What is the theory behind the method? What are some of the things to watch out for when you use the method?
20. Can a function have more than one antiderivative? If so, how are the antiderivatives related? Explain.
21. What is an indefinite integral? How do you evaluate one? What general formulas do you know for finding indefinite integrals?
22. How can you sometimes solve a differential equation of the form $dy/dx = f(x)$?
23. What is an initial value problem? How do you solve one? Give an example.
24. If you know the acceleration of a body moving along a coordinate line as a function of time, what more do you need to know to find the body's position function? Give an example.

Chapter 4 Technology Application Projects

Mathematica/Maple Module

Motion Along a Straight Line: Position → Velocity → Acceleration

You will observe the shape of a graph through dramatic animated visualizations of the derivative relations among the position, velocity, and acceleration. Figures in the text can be animated.

Mathematica/Maple Module

Newton's Method: Estimate π to How Many Places?

Plot a function, observe a root, pick a starting point near the root, and use Newton's Iteration Procedure to approximate the root to a desired accuracy. The numbers π, e, and $\sqrt{2}$ are approximated.