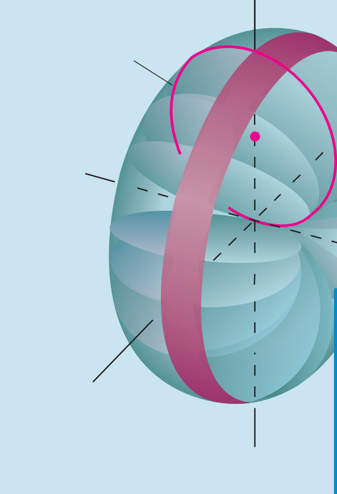


Chapter

6

APPLICATIONS OF DEFINITE INTEGRALS

OVERVIEW In Chapter 5 we discovered the connection between Riemann sums

$$S_P = \sum_{k=1}^{n} f(c_k)\,\Delta x_k$$

associated with a partition P of the finite closed interval $[a, b]$ and the process of integration. We found that for a continuous function f on $[a, b]$, the limit of S_P as the norm of the partition $\|P\|$ approaches zero is the number

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where F is any antiderivative of f. We applied this to the problems of computing the area between the x-axis and the graph of $y = f(x)$ for $a \le x \le b$, and to finding the area between two curves.

In this chapter we extend the applications to finding volumes, lengths of plane curves, centers of mass, areas of surfaces of revolution, work, and fluid forces against planar walls. We define all these as limits of Riemann sums of continuous functions on closed intervals—that is, as definite integrals which can be evaluated using the Fundamental Theorem of Calculus.

6.1 Volumes by Slicing and Rotation About an Axis

In this section we define volumes of solids whose cross-sections are plane regions. A **cross-section** of a solid S is the plane region formed by intersecting S with a plane (Figure 6.1).

Suppose we want to find the volume of a solid S like the one in Figure 6.1. We begin by extending the definition of a cylinder from classical geometry to cylindrical solids with arbitrary bases (Figure 6.2). If the cylindrical solid has a known base area A and height h, then the volume of the cylindrical solid is

$$\text{Volume} = \text{area} \times \text{height} = A \cdot h.$$

This equation forms the basis for defining the volumes of many solids that are not cylindrical by the *method of slicing*.

If the cross-section of the solid S at each point x in the interval $[a, b]$ is a region $R(x)$ of area $A(x)$, and A is a continuous function of x, we can define and calculate the volume of the solid S as a definite integral in the following way.

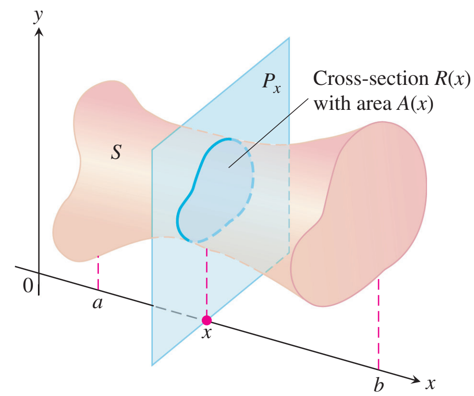


FIGURE 6.1 A cross-section of the solid S formed by intersecting S with a plane P_x perpendicular to the x-axis through the point x in the interval $[a, b]$.

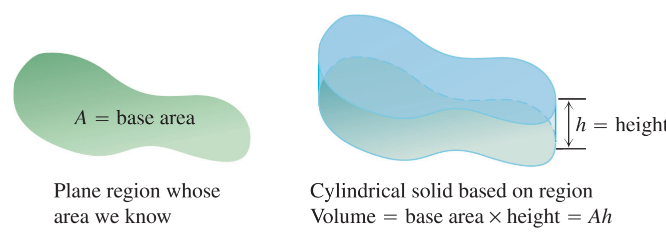


FIGURE 6.2 The volume of a cylindrical solid is always defined to be its base area times its height.

We partition $[a, b]$ into subintervals of width (length) Δx_k and slice the solid, as we would a loaf of bread, by planes perpendicular to the x-axis at the partition points $a = x_0 < x_1 < \cdots < x_n = b$. The planes P_{x_k}, perpendicular to the x-axis at the partition points, slice S into thin "slabs" (like thin slices of a loaf of bread). A typical slab is shown in Figure 6.3. We approximate the slab between the plane at x_{k-1} and the plane at x_k by a cylindrical solid with base area $A(x_k)$ and height $\Delta x_k = x_k - x_{k-1}$ (Figure 6.4). The volume V_k of this cylindrical solid is $A(x_k) \cdot \Delta x_k$, which is approximately the same volume as that of the slab:

$$\text{Volume of the } k\text{th slab} \approx V_k = A(x_k)\, \Delta x_k.$$

The volume V of the entire solid S is therefore approximated by the sum of these cylindrical volumes,

$$V \approx \sum_{k=1}^{n} V_k = \sum_{k=1}^{n} A(x_k)\, \Delta x_k.$$

This is a Riemann sum for the function $A(x)$ on $[a, b]$. We expect the approximations from these sums to improve as the norm of the partition of $[a, b]$ goes to zero, so we define their limiting definite integral to be the volume of the solid S.

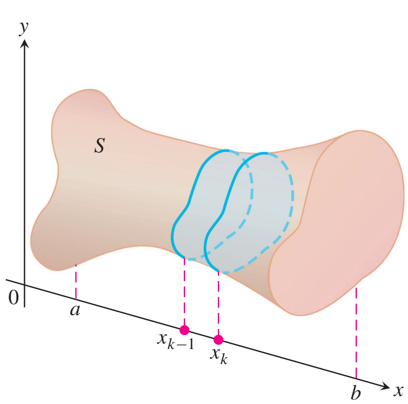


FIGURE 6.3 A typical thin slab in the solid S.

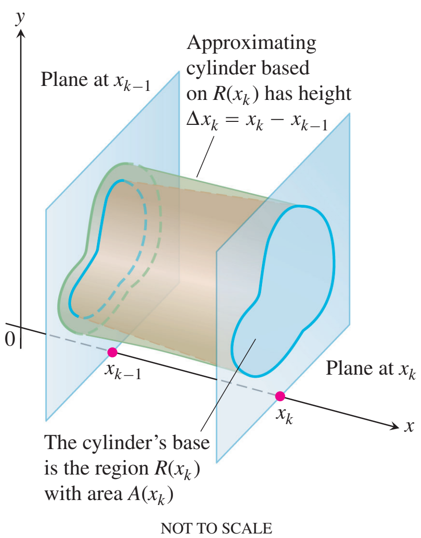


FIGURE 6.4 The solid thin slab in Figure 6.3 is approximated by the cylindrical solid with base $R(x_k)$ having area $A(x_k)$ and height $\Delta x_k = x_k - x_{k-1}$.

DEFINITION Volume

The **volume** of a solid of known integrable cross-sectional area $A(x)$ from $x = a$ to $x = b$ is the integral of A from a to b,

$$V = \int_a^b A(x)\, dx.$$

This definition applies whenever $A(x)$ is continuous, or more generally, when it is integrable. To apply the formula in the definition to calculate the volume of a solid, take the following steps:

Calculating the Volume of a Solid

1. *Sketch the solid and a typical cross-section.*
2. *Find a formula for* $A(x)$, the area of a typical cross-section.
3. *Find the limits of integration.*
4. *Integrate* $A(x)$ using the Fundamental Theorem.

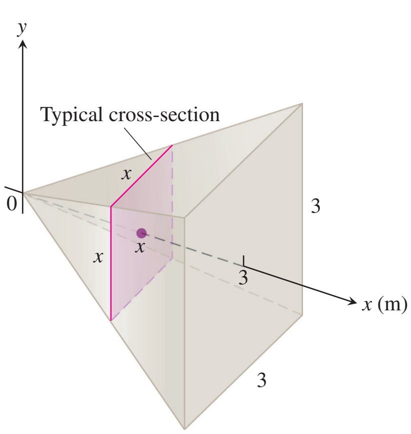


FIGURE 6.5 The cross-sections of the pyramid in Example 1 are squares.

EXAMPLE 1 Volume of a Pyramid

A pyramid 3 m high has a square base that is 3 m on a side. The cross-section of the pyramid perpendicular to the altitude x m down from the vertex is a square x m on a side. Find the volume of the pyramid.

Solution

1. *A sketch*. We draw the pyramid with its altitude along the x-axis and its vertex at the origin and include a typical cross-section (Figure 6.5).
2. *A formula for* $A(x)$. The cross-section at x is a square x meters on a side, so its area is

$$A(x) = x^2.$$

3. *The limits of integration*. The squares lie on the planes from $x = 0$ to $x = 3$.
4. *Integrate to find the volume.*

$$V = \int_0^3 A(x)\, dx = \int_0^3 x^2\, dx = \frac{x^3}{3}\bigg]_0^3 = 9 \text{ m}^3$$ ■

HISTORICAL BIOGRAPHY

Bonaventura Cavalieri
(1598–1647)

EXAMPLE 2 Cavalieri's Principle

Cavalieri's principle says that solids with equal altitudes and identical cross-sectional areas at each height have the same volume (Figure 6.6). This follows immediately from the definition of volume, because the cross-sectional area function $A(x)$ and the interval $[a, b]$ are the same for both solids. ■

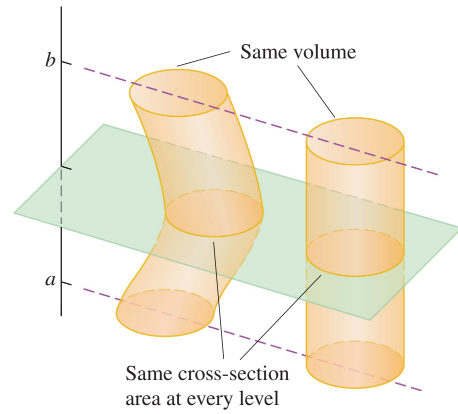


FIGURE 6.6 *Cavalieri's Principle:* These solids have the same volume, which can be illustrated with stacks of coins (Example 2).

EXAMPLE 3 Volume of a Wedge

A curved wedge is cut from a cylinder of radius 3 by two planes. One plane is perpendicular to the axis of the cylinder. The second plane crosses the first plane at a 45° angle at the center of the cylinder. Find the volume of the wedge.

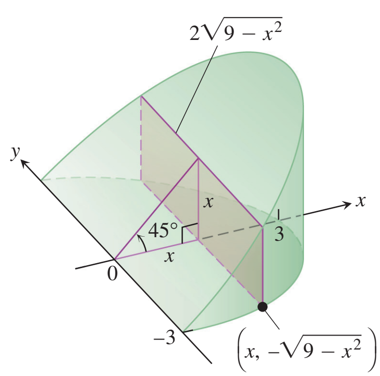


FIGURE 6.7 The wedge of Example 3, sliced perpendicular to the x-axis. The cross-sections are rectangles.

Solution We draw the wedge and sketch a typical cross-section perpendicular to the x-axis (Figure 6.7). The cross-section at x is a rectangle of area

$$\begin{aligned} A(x) &= (\text{height})(\text{width}) = (x)\left(2\sqrt{9-x^2}\right) \\ &= 2x\sqrt{9-x^2}. \end{aligned}$$

The rectangles run from $x = 0$ to $x = 3$, so we have

$$\begin{aligned} V &= \int_a^b A(x)\,dx = \int_0^3 2x\sqrt{9-x^2}\,dx \\ &= -\frac{2}{3}(9-x^2)^{3/2}\Big]_0^3 \\ &= 0 + \frac{2}{3}(9)^{3/2} \\ &= 18. \end{aligned}$$

Let $u = 9 - x^2$, $du = -2x\,dx$, integrate, and substitute back.

■

Solids of Revolution: The Disk Method

The solid generated by rotating a plane region about an axis in its plane is called a **solid of revolution**. To find the volume of a solid like the one shown in Figure 6.8, we need only observe that the cross-sectional area $A(x)$ is the area of a disk of radius $R(x)$, the distance of the planar region's boundary from the axis of revolution. The area is then

$$A(x) = \pi(\text{radius})^2 = \pi[R(x)]^2.$$

So the definition of volume gives

$$\boxed{V = \int_a^b A(x)\,dx = \int_a^b \pi[R(x)]^2\,dx.}$$

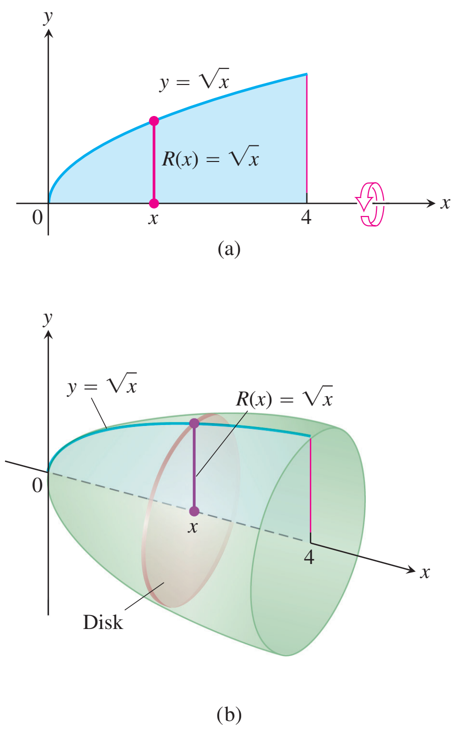


FIGURE 6.8 The region (a) and solid of revolution (b) in Example 4.

This method for calculating the volume of a solid of revolution is often called the **disk method** because a cross-section is a circular disk of radius $R(x)$.

EXAMPLE 4 A Solid of Revolution (Rotation About the *x*-Axis)

The region between the curve $y = \sqrt{x}$, $0 \le x \le 4$, and the x-axis is revolved about the x-axis to generate a solid. Find its volume.

Solution We draw figures showing the region, a typical radius, and the generated solid (Figure 6.8). The volume is

$$\begin{aligned} V &= \int_a^b \pi[R(x)]^2\,dx \\ &= \int_0^4 \pi\left[\sqrt{x}\right]^2 dx \qquad R(x) = \sqrt{x} \\ &= \pi \int_0^4 x\,dx = \pi \frac{x^2}{2}\bigg]_0^4 = \pi \frac{(4)^2}{2} = 8\pi. \end{aligned}$$

■

EXAMPLE 5 Volume of a Sphere

The circle

$$x^2 + y^2 = a^2$$

is rotated about the x-axis to generate a sphere. Find its volume.

Solution We imagine the sphere cut into thin slices by planes perpendicular to the x-axis (Figure 6.9). The cross-sectional area at a typical point x between $-a$ and a is

$$A(x) = \pi y^2 = \pi(a^2 - x^2).$$

Therefore, the volume is

$$V = \int_{-a}^{a} A(x)\,dx = \int_{-a}^{a} \pi(a^2 - x^2)\,dx = \pi\left[a^2 x - \frac{x^3}{3}\right]_{-a}^{a} = \frac{4}{3}\pi a^3.$$

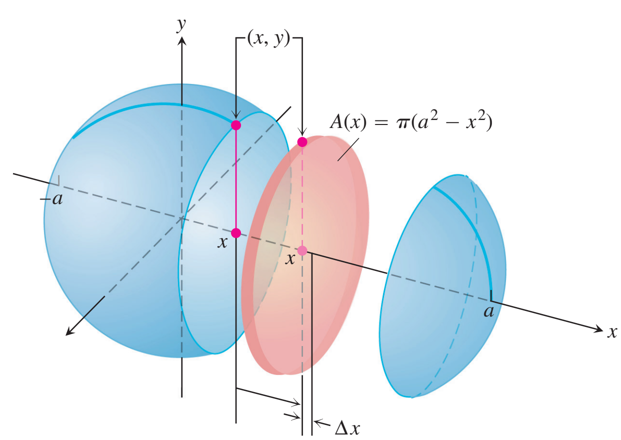


FIGURE 6.9 The sphere generated by rotating the circle $x^2 + y^2 = a^2$ about the x-axis. The radius is $R(x) = y = \sqrt{a^2 - x^2}$ (Example 5).

■

The axis of revolution in the next example is not the x-axis, but the rule for calculating the volume is the same: Integrate $\pi(\text{radius})^2$ between appropriate limits.

EXAMPLE 6 A Solid of Revolution (Rotation About the Line $y = 1$)

Find the volume of the solid generated by revolving the region bounded by $y = \sqrt{x}$ and the lines $y = 1$, $x = 4$ about the line $y = 1$.

Solution We draw figures showing the region, a typical radius, and the generated solid (Figure 6.10). The volume is

$$\begin{aligned} V &= \int_1^4 \pi[R(x)]^2\,dx \\ &= \int_1^4 \pi\left[\sqrt{x} - 1\right]^2 dx \\ &= \pi\int_1^4 \left[x - 2\sqrt{x} + 1\right] dx \\ &= \pi\left[\frac{x^2}{2} - 2\cdot\frac{2}{3}x^{3/2} + x\right]_1^4 = \frac{7\pi}{6}. \end{aligned}$$

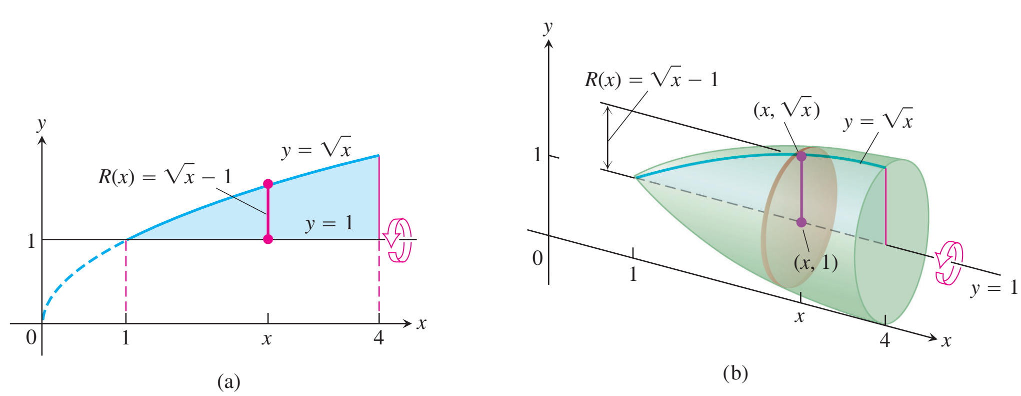


FIGURE 6.10 The region (a) and solid of revolution (b) in Example 6. ■

To find the volume of a solid generated by revolving a region between the y-axis and a curve $x = R(y)$, $c \le y \le d$, about the y-axis, we use the same method with x replaced by y. In this case, the circular cross-section is

$$A(y) = \pi[\text{radius}]^2 = \pi[R(y)]^2.$$

EXAMPLE 7 Rotation About the y-Axis

Find the volume of the solid generated by revolving the region between the y-axis and the curve $x = 2/y$, $1 \le y \le 4$, about the y-axis.

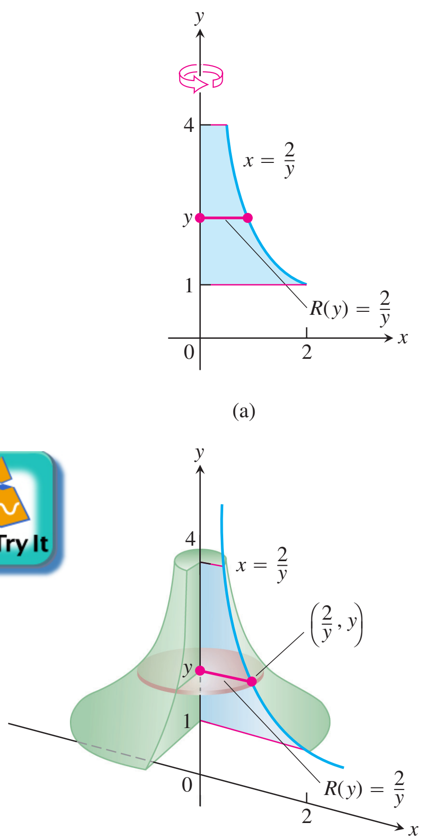


FIGURE 6.11 The region (a) and part of the solid of revolution (b) in Example 7.

Solution We draw figures showing the region, a typical radius, and the generated solid (Figure 6.11). The volume is

$$
\begin{aligned}
V &= \int_1^4 \pi[R(y)]^2\,dy \\
&= \int_1^4 \pi\left(\frac{2}{y}\right)^2 dy \\
&= \pi\int_1^4 \frac{4}{y^2}\,dy = 4\pi\left[-\frac{1}{y}\right]_1^4 = 4\pi\left[\frac{3}{4}\right] \\
&= 3\pi.
\end{aligned}
$$

EXAMPLE 8 Rotation About a Vertical Axis

Find the volume of the solid generated by revolving the region between the parabola $x = y^2 + 1$ and the line $x = 3$ about the line $x = 3$.

Solution We draw figures showing the region, a typical radius, and the generated solid (Figure 6.12). Note that the cross-sections are perpendicular to the line $x = 3$. The volume is

$$
\begin{aligned}
V &= \int_{-\sqrt{2}}^{\sqrt{2}} \pi[R(y)]^2\,dy \\
&= \int_{-\sqrt{2}}^{\sqrt{2}} \pi[2 - y^2]^2\,dy \qquad R(y) = 3 - (y^2 + 1) = 2 - y^2 \\
&= \pi\int_{-\sqrt{2}}^{\sqrt{2}} [4 - 4y^2 + y^4]\,dy \\
&= \pi\left[4y - \frac{4}{3}y^3 + \frac{y^5}{5}\right]_{-\sqrt{2}}^{\sqrt{2}} \\
&= \frac{64\pi\sqrt{2}}{15}.
\end{aligned}
$$

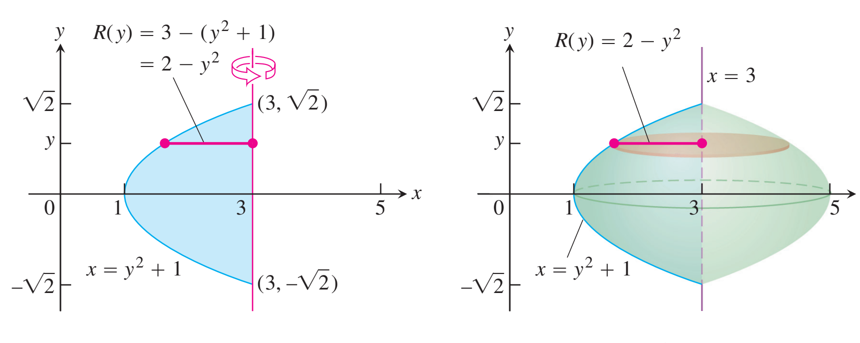


FIGURE 6.12 The region (a) and solid of revolution (b) in Example 8.

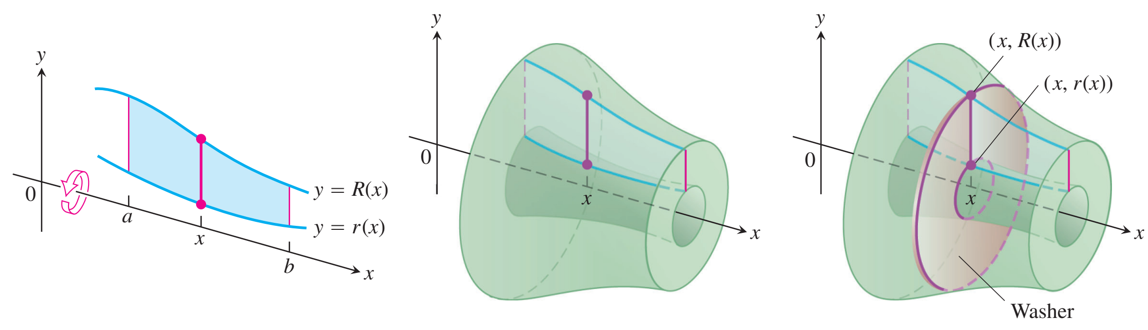


FIGURE 6.13 The cross-sections of the solid of revolution generated here are washers, not disks, so the integral $\int_a^b A(x)\,dx$ leads to a slightly different formula.

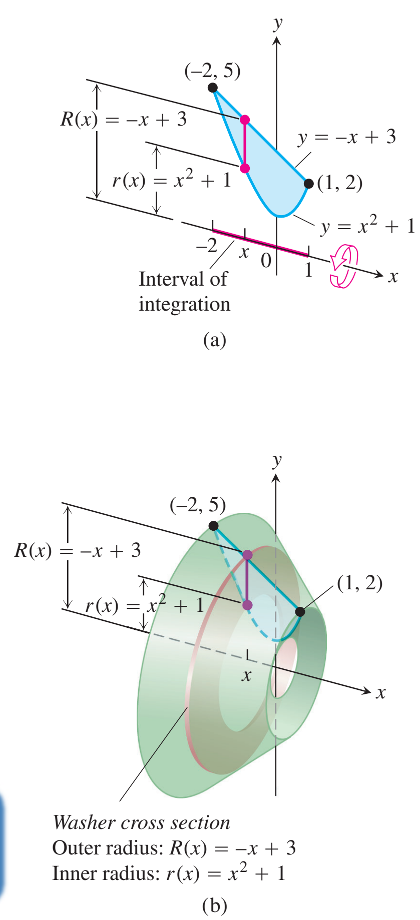


FIGURE 6.14 (a) The region in Example 9 spanned by a line segment perpendicular to the axis of revolution. (b) When the region is revolved about the x-axis, the line segment generates a washer.

Solids of Revolution: The Washer Method

If the region we revolve to generate a solid does not border on or cross the axis of revolution, the solid has a hole in it (Figure 6.13). The cross-sections perpendicular to the axis of revolution are washers (the purplish circular surface in Figure 6.13) instead of disks. The dimensions of a typical washer are

Outer radius:	$R(x)$
Inner radius:	$r(x)$

The washer's area is

$$A(x) = \pi[R(x)]^2 - \pi[r(x)]^2 = \pi([R(x)]^2 - [r(x)]^2).$$

Consequently, the definition of volume gives

$$\boxed{V = \int_a^b A(x)\,dx = \int_a^b \pi([R(x)]^2 - [r(x)]^2)\,dx.}$$

This method for calculating the volume of a solid of revolution is called the **washer method** because a slab is a circular washer of outer radius $R(x)$ and inner radius $r(x)$.

EXAMPLE 9 A Washer Cross-Section (Rotation About the x-Axis)

The region bounded by the curve $y = x^2 + 1$ and the line $y = -x + 3$ is revolved about the x-axis to generate a solid. Find the volume of the solid.

Solution

1. Draw the region and sketch a line segment across it perpendicular to the axis of revolution (the red segment in Figure 6.14).
2. Find the outer and inner radii of the washer that would be swept out by the line segment if it were revolved about the x-axis along with the region.

Video

These radii are the distances of the ends of the line segment from the axis of revolution (Figure 6.14).

$$\text{Outer radius:} \quad R(x) = -x + 3$$
$$\text{Inner radius:} \quad r(x) = x^2 + 1$$

3. Find the limits of integration by finding the x-coordinates of the intersection points of the curve and line in Figure 6.14a.

$$\begin{aligned} x^2 + 1 &= -x + 3 \\ x^2 + x - 2 &= 0 \\ (x + 2)(x - 1) &= 0 \\ x = -2, \quad x &= 1 \end{aligned}$$

4. Evaluate the volume integral.

$$\begin{aligned} V &= \int_a^b \pi([R(x)]^2 - [r(x)]^2)\,dx \\ &= \int_{-2}^{1} \pi((-x + 3)^2 - (x^2 + 1)^2)\,dx \qquad \text{Values from Steps 2 and 3} \\ &= \int_{-2}^{1} \pi(8 - 6x - x^2 - x^4)\,dx \\ &= \pi\left[8x - 3x^2 - \frac{x^3}{3} - \frac{x^5}{5}\right]_{-2}^{1} = \frac{117\pi}{5} \end{aligned}$$ ■

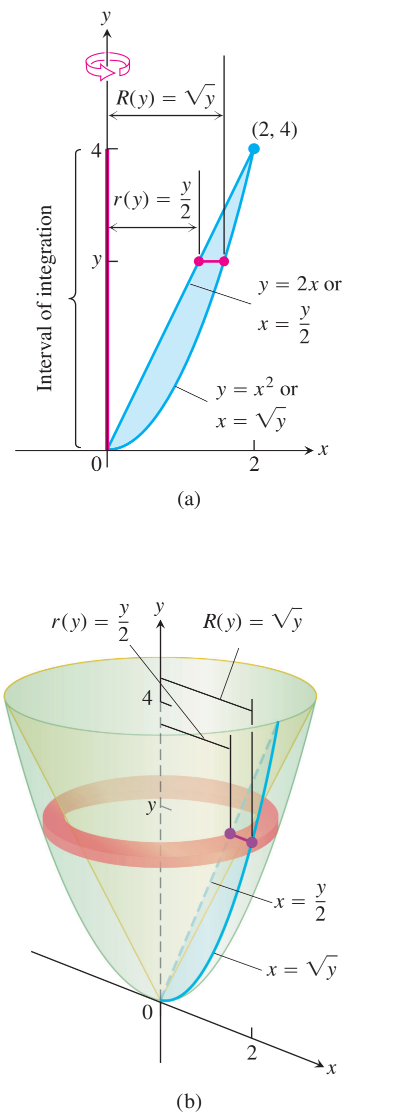


FIGURE 6.15 (a) The region being rotated about the y-axis, the washer radii, and limits of integration in Example 10. (b) The washer swept out by the line segment in part (a).

To find the volume of a solid formed by revolving a region about the y-axis, we use the same procedure as in Example 9, but integrate with respect to y instead of x. In this situation the line segment sweeping out a typical washer is perpendicular to the y-axis (the axis of revolution), and the outer and inner radii of the washer are functions of y.

EXAMPLE 10 A Washer Cross-Section (Rotation About the y-Axis)

The region bounded by the parabola $y = x^2$ and the line $y = 2x$ in the first quadrant is revolved about the y-axis to generate a solid. Find the volume of the solid.

Solution First we sketch the region and draw a line segment across it perpendicular to the axis of revolution (the y-axis). See Figure 6.15a.

The radii of the washer swept out by the line segment are $R(y) = \sqrt{y}$, $r(y) = y/2$ (Figure 6.15).

The line and parabola intersect at $y = 0$ and $y = 4$, so the limits of integration are $c = 0$ and $d = 4$. We integrate to find the volume:

$$\begin{aligned} V &= \int_c^d \pi([R(y)]^2 - [r(y)]^2)\,dy \\ &= \int_0^4 \pi\left(\left[\sqrt{y}\right]^2 - \left[\frac{y}{2}\right]^2\right) dy \\ &= \pi\int_0^4 \left(y - \frac{y^2}{4}\right) dy = \pi\left[\frac{y^2}{2} - \frac{y^3}{12}\right]_0^4 = \frac{8}{3}\pi. \end{aligned}$$ ■

Summary

In all of our volume examples, no matter how the cross-sectional area $A(x)$ of a typical slab is determined, the definition of volume as the definite integral $V = \int_a^b A(x)\,dx$ is the heart of the calculations we made.

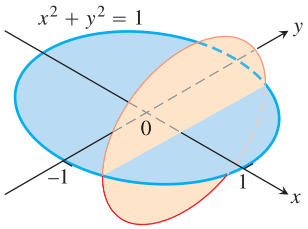
EXERCISES 6.1

Cross-Sectional Areas

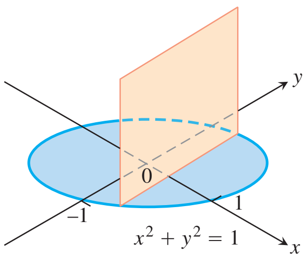
In Exercises 1 and 2, find a formula for the area $A(x)$ of the cross-sections of the solid perpendicular to the x-axis.

1. The solid lies between planes perpendicular to the x-axis at $x = -1$ and $x = 1$. In each case, the cross-sections perpendicular to the x-axis between these planes run from the semicircle $y = -\sqrt{1 - x^2}$ to the semicircle $y = \sqrt{1 - x^2}$.

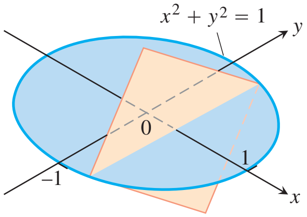
a. The cross-sections are circular disks with diameters in the xy-plane.



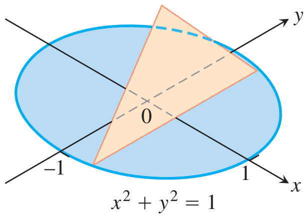
b. The cross-sections are squares with bases in the xy-plane.



c. The cross-sections are squares with diagonals in the xy-plane. (The length of a square's diagonal is $\sqrt{2}$ times the length of its sides.)

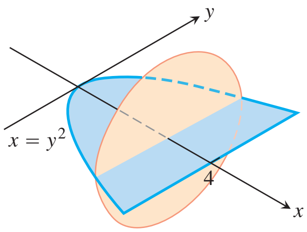


d. The cross-sections are equilateral triangles with bases in the xy-plane.

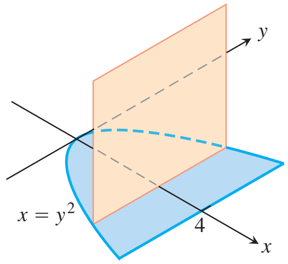


2. The solid lies between planes perpendicular to the x-axis at $x = 0$ and $x = 4$. The cross-sections perpendicular to the x-axis between these planes run from the parabola $y = -\sqrt{x}$ to the parabola $y = \sqrt{x}$.

a. The cross-sections are circular disks with diameters in the xy-plane.



b. The cross-sections are squares with bases in the xy-plane.



c. The cross-sections are squares with diagonals in the xy-plane.

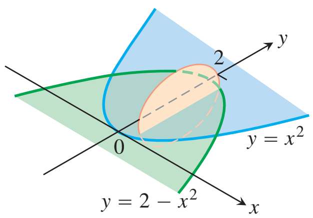
d. The cross-sections are equilateral triangles with bases in the xy-plane.

Volumes by Slicing

Find the volumes of the solids in Exercises 3–10.

3. The solid lies between planes perpendicular to the x-axis at $x = 0$ and $x = 4$. The cross-sections perpendicular to the axis on the interval $0 \le x \le 4$ are squares whose diagonals run from the parabola $y = -\sqrt{x}$ to the parabola $y = \sqrt{x}$.

4. The solid lies between planes perpendicular to the x-axis at $x = -1$ and $x = 1$. The cross-sections perpendicular to the x-axis are circular disks whose diameters run from the parabola $y = x^2$ to the parabola $y = 2 - x^2$.

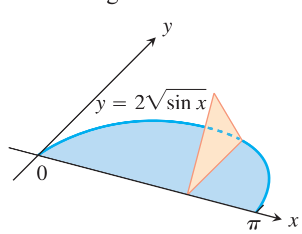


5. The solid lies between planes perpendicular to the x-axis at $x = -1$ and $x = 1$. The cross-sections perpendicular to the x-axis between these planes are squares whose bases run from the semicircle $y = -\sqrt{1 - x^2}$ to the semicircle $y = \sqrt{1 - x^2}$.

6. The solid lies between planes perpendicular to the x-axis at $x = -1$ and $x = 1$. The cross-sections perpendicular to the x-axis between these planes are squares whose diagonals run from the semicircle $y = -\sqrt{1 - x^2}$ to the semicircle $y = \sqrt{1 - x^2}$.

7. The base of a solid is the region between the curve $y = 2\sqrt{\sin x}$ and the interval $[0, \pi]$ on the x-axis. The cross-sections perpendicular to the x-axis are

a. equilateral triangles with bases running from the x-axis to the curve as shown in the figure.



b. squares with bases running from the x-axis to the curve.

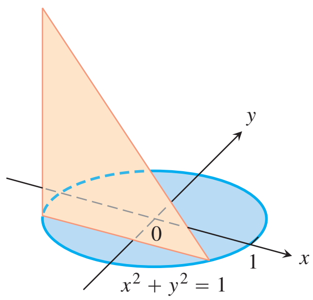
8. The solid lies between planes perpendicular to the x-axis at $x = -\pi/3$ and $x = \pi/3$. The cross-sections perpendicular to the x-axis are

a. circular disks with diameters running from the curve $y = \tan x$ to the curve $y = \sec x$.

b. squares whose bases run from the curve $y = \tan x$ to the curve $y = \sec x$.

9. The solid lies between planes perpendicular to the y-axis at $y = 0$ and $y = 2$. The cross-sections perpendicular to the y-axis are circular disks with diameters running from the y-axis to the parabola $x = \sqrt{5}y^2$.

10. The base of the solid is the disk $x^2 + y^2 \le 1$. The cross-sections by planes perpendicular to the y-axis between $y = -1$ and $y = 1$ are isosceles right triangles with one leg in the disk.

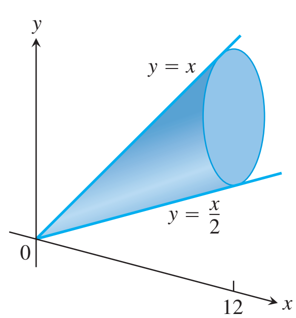


11. A twisted solid A square of side length s lies in a plane perpendicular to a line L. One vertex of the square lies on L. As this square moves a distance h along L, the square turns one revolution about L to generate a corkscrew-like column with square cross-sections.

a. Find the volume of the column.

b. What will the volume be if the square turns twice instead of once? Give reasons for your answer.

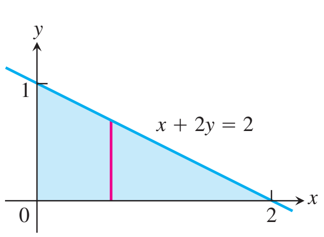
12. Cavalieri's Principle A solid lies between planes perpendicular to the x-axis at $x = 0$ and $x = 12$. The cross-sections by planes perpendicular to the x-axis are circular disks whose diameters run from the line $y = x/2$ to the line $y = x$ as shown in the accompanying figure. Explain why the solid has the same volume as a right circular cone with base radius 3 and height 12.



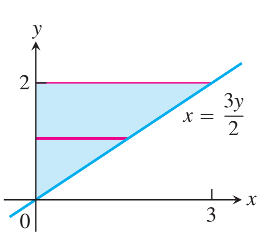
Volumes by the Disk Method

In Exercises 13–16, find the volume of the solid generated by revolving the shaded region about the given axis.

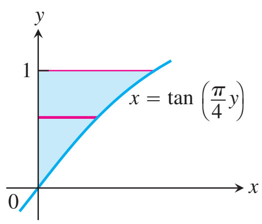
13. About the x-axis



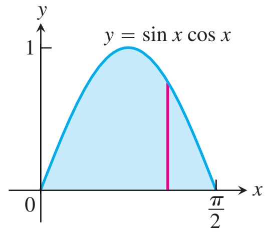
14. About the y-axis



15. About the y-axis



16. About the x-axis



Find the volumes of the solids generated by revolving the regions bounded by the lines and curves in Exercises 17–22 about the x-axis.

17. $y = x^2, \quad y = 0, \quad x = 2$

18. $y = x^3, \quad y = 0, \quad x = 2$

19. $y = \sqrt{9 - x^2}, \quad y = 0$

20. $y = x - x^2, \quad y = 0$

21. $y = \sqrt{\cos x}, \quad 0 \le x \le \pi/2, \quad y = 0, \quad x = 0$

22. $y = \sec x, \quad y = 0, \quad x = -\pi/4, \quad x = \pi/4$

In Exercises 23 and 24, find the volume of the solid generated by revolving the region about the given line.

23. The region in the first quadrant bounded above by the line $y = \sqrt{2}$, below by the curve $y = \sec x \tan x$, and on the left by the y-axis, about the line $y = \sqrt{2}$

24. The region in the first quadrant bounded above by the line $y = 2$, below by the curve $y = 2\sin x$, $0 \le x \le \pi/2$, and on the left by the y-axis, about the line $y = 2$

Find the volumes of the solids generated by revolving the regions bounded by the lines and curves in Exercises 25–30 about the y-axis.

25. The region enclosed by $x = \sqrt{5}y^2, \quad x = 0, \quad y = -1, \quad y = 1$

26. The region enclosed by $x = y^{3/2}, \quad x = 0, \quad y = 2$

27. The region enclosed by $x = \sqrt{2\sin 2y}, \quad 0 \le y \le \pi/2, \quad x = 0$

28. The region enclosed by $x = \sqrt{\cos(\pi y/4)}, \quad -2 \le y \le 0, \quad x = 0$

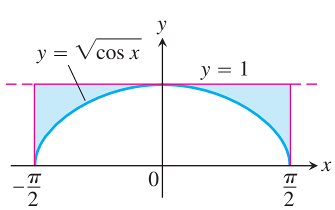
29. $x = 2/(y + 1), \quad x = 0, \quad y = 0, \quad y = 3$

30. $x = \sqrt{2y}/(y^2 + 1), \quad x = 0, \quad y = 1$

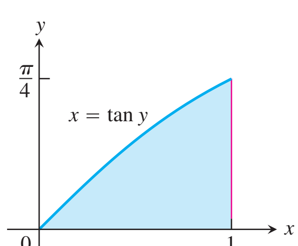
Volumes by the Washer Method

Find the volumes of the solids generated by revolving the shaded regions in Exercises 31 and 32 about the indicated axes.

31. The x-axis



32. The y-axis



Find the volumes of the solids generated by revolving the regions bounded by the lines and curves in Exercises 33–38 about the x-axis.

33. $y = x, \quad y = 1, \quad x = 0$

34. $y = 2\sqrt{x}, \quad y = 2, \quad x = 0$

35. $y = x^2 + 1, \quad y = x + 3$

36. $y = 4 - x^2, \quad y = 2 - x$

37. $y = \sec x, \quad y = \sqrt{2}, \quad -\pi/4 \le x \le \pi/4$

38. $y = \sec x, \quad y = \tan x, \quad x = 0, \quad x = 1$

In Exercises 39–42, find the volume of the solid generated by revolving each region about the y-axis.

39. The region enclosed by the triangle with vertices (1, 0), (2, 1), and (1, 1)

40. The region enclosed by the triangle with vertices (0, 1), (1, 0), and (1, 1)

41. The region in the first quadrant bounded above by the parabola $y = x^2$, below by the x-axis, and on the right by the line $x = 2$

42. The region in the first quadrant bounded on the left by the circle $x^2 + y^2 = 3$, on the right by the line $x = \sqrt{3}$, and above by the line $y = \sqrt{3}$

In Exercises 43 and 44, find the volume of the solid generated by revolving each region about the given axis.

43. The region in the first quadrant bounded above by the curve $y = x^2$, below by the x-axis, and on the right by the line $x = 1$, about the line $x = -1$

44. The region in the second quadrant bounded above by the curve $y = -x^3$, below by the x-axis, and on the left by the line $x = -1$, about the line $x = -2$

Volumes of Solids of Revolution

45. Find the volume of the solid generated by revolving the region bounded by $y = \sqrt{x}$ and the lines $y = 2$ and $x = 0$ about

a. the x-axis. **b.** the y-axis.

c. the line $y = 2$. **d.** the line $x = 4$.

46. Find the volume of the solid generated by revolving the triangular region bounded by the lines $y = 2x$, $y = 0$, and $x = 1$ about

a. the line $x = 1$. **b.** the line $x = 2$.

47. Find the volume of the solid generated by revolving the region bounded by the parabola $y = x^2$ and the line $y = 1$ about

a. the line $y = 1$. **b.** the line $y = 2$.

c. the line $y = -1$.

48. By integration, find the volume of the solid generated by revolving the triangular region with vertices $(0, 0)$, $(b, 0)$, $(0, h)$ about

a. the x-axis. **b.** the y-axis.

Theory and Applications

49. The volume of a torus The disk $x^2 + y^2 \le a^2$ is revolved about the line $x = b$ $(b > a)$ to generate a solid shaped like a doughnut and called a *torus*. Find its volume. (*Hint:* $\int_{-a}^{a}\sqrt{a^2-y^2}\,dy = \pi a^2/2$, since it is the area of a semicircle of radius a.)

50. Volume of a bowl A bowl has a shape that can be generated by revolving the graph of $y = x^2/2$ between $y = 0$ and $y = 5$ about the y-axis.

a. Find the volume of the bowl.

b. Related rates If we fill the bowl with water at a constant rate of 3 cubic units per second, how fast will the water level in the bowl be rising when the water is 4 units deep?

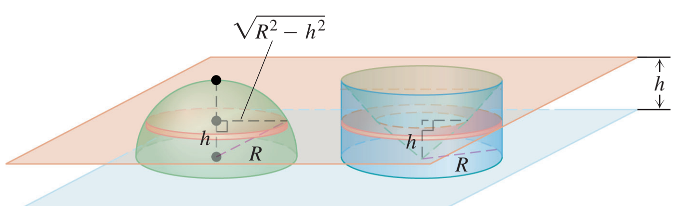
51. Volume of a bowl

a. A hemispherical bowl of radius a contains water to a depth h. Find the volume of water in the bowl.

b. Related rates Water runs into a sunken concrete hemispherical bowl of radius 5 m at the rate of $0.2\ \text{m}^3/\text{sec}$. How fast is the water level in the bowl rising when the water is 4 m deep?

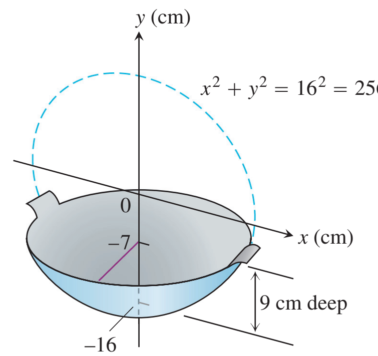
52. Explain how you could estimate the volume of a solid of revolution by measuring the shadow cast on a table parallel to its axis of revolution by a light shining directly above it.

53. Volume of a hemisphere Derive the formula $V = (2/3)\pi R^3$ for the volume of a hemisphere of radius R by comparing its cross-sections with the cross-sections of a solid right circular cylinder of radius R and height R from which a solid right circular cone of base radius R and height R has been removed as suggested by the accompanying figure.

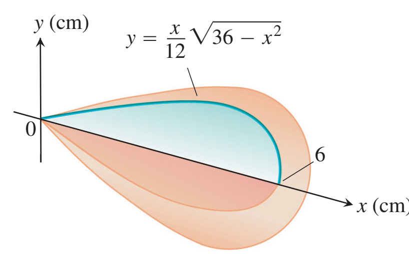


54. Volume of a cone Use calculus to find the volume of a right circular cone of height h and base radius r.

55. Designing a wok You are designing a wok frying pan that will be shaped like a spherical bowl with handles. A bit of experimentation at home persuades you that you can get one that holds about 3 L if you make it 9 cm deep and give the sphere a radius of 16 cm. To be sure, you picture the wok as a solid of revolution, as shown here, and calculate its volume with an integral. To the nearest cubic centimeter, what volume do you really get? $(1\ \text{L} = 1000\ \text{cm}^3.)$



56. Designing a plumb bob Having been asked to design a brass plumb bob that will weigh in the neighborhood of 190 g, you decide to shape it like the solid of revolution shown here. Find the plumb bob's volume. If you specify a brass that weighs $8.5\ \text{g/cm}^3$, how much will the plumb bob weigh (to the nearest gram)?



57. Max-min The arch $y = \sin x$, $0 \le x \le \pi$, is revolved about the line $y = c$, $0 \le c \le 1$, to generate the solid in Figure 6.16.

a. Find the value of c that minimizes the volume of the solid. What is the minimum volume?

b. What value of c in $[0, 1]$ maximizes the volume of the solid?

T **c.** Graph the solid's volume as a function of c, first for $0 \le c \le 1$ and then on a larger domain. What happens to the volume of the solid as c moves away from $[0, 1]$? Does this make sense physically? Give reasons for your answers.

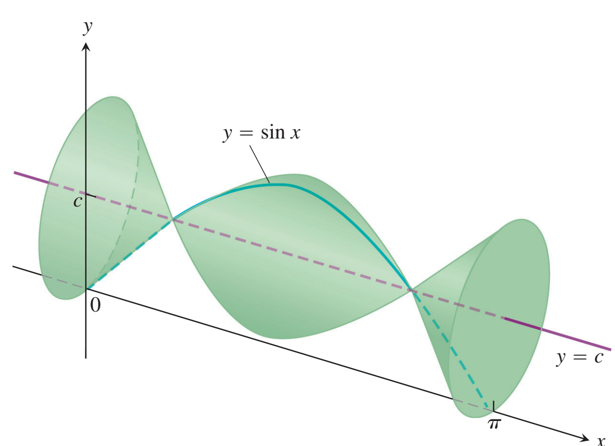


FIGURE 6.16

58. An auxiliary fuel tank You are designing an auxiliary fuel tank that will fit under a helicopter's fuselage to extend its range. After some experimentation at your drawing board, you decide to shape the tank like the surface generated by revolving the curve $y = 1 - (x^2/16)$, $-4 \le x \le 4$, about the x-axis (dimensions in feet).

a. How many cubic feet of fuel will the tank hold (to the nearest cubic foot)?

b. A cubic foot holds 7.481 gal. If the helicopter gets 2 mi to the gallon, how many additional miles will the helicopter be able to fly once the tank is installed (to the nearest mile)?

6.2 Volumes by Cylindrical Shells

In Section 6.1 we defined the volume of a solid S as the definite integral

$$V = \int_a^b A(x)\,dx,$$

where $A(x)$ is an integrable cross-sectional area of S from $x = a$ to $x = b$. The area $A(x)$ was obtained by slicing through the solid with a plane perpendicular to the x-axis. In this section we use the same integral definition for volume, but obtain the area by slicing through the solid in a different way. Now we slice through the solid using circular cylinders of increasing radii, like cookie cutters. We slice straight down through the solid perpendicular to the x-axis, with the axis of the cylinder parallel to the y-axis. The vertical axis of each cylinder is the same line, but the radii of the cylinders increase with each slice. In this way the solid S is sliced up into thin cylindrical shells of constant thickness that grow outward from their common axis, like circular tree rings. Unrolling a cylindrical shell shows that its volume is approximately that of a rectangular slab with area $A(x)$ and thickness Δx. This allows us to apply the same integral definition for volume as before. Before describing the method in general, let's look at an example to gain some insight.

Video

You Try It

EXAMPLE 1 Finding a Volume Using Shells

The region enclosed by the x-axis and the parabola $y = f(x) = 3x - x^2$ is revolved about the vertical line $x = -1$ to generate the shape of a solid (Figure 6.17). Find the volume of the solid.

Solution Using the washer method from Section 6.1 would be awkward here because we would need to express the x-values of the left and right branches of the parabola in terms

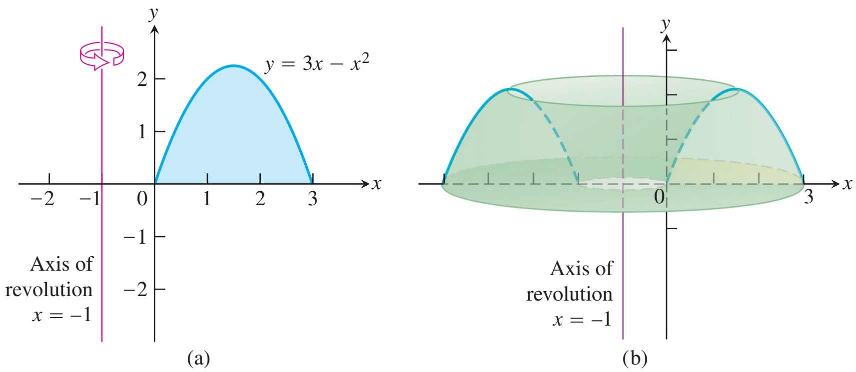


FIGURE 6.17 (a) The graph of the region in Example 1, before revolution. (b) The solid formed when the region in part (a) is revolved about the axis of revolution $x = -1$.

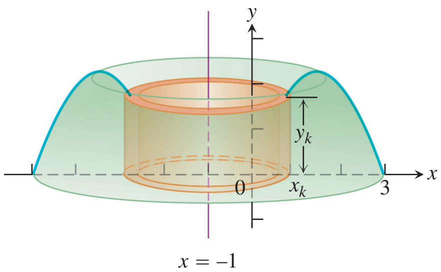


FIGURE 6.18 A cylindrical shell of height y_k obtained by rotating a vertical strip of thickness Δx about the line $x = -1$. The outer radius of the cylinder occurs at x_k, where the height of the parabola is $y_k = 3x_k - x_k^2$ (Example 1).

of y. (These x-values are the inner and outer radii for a typical washer, leading to complicated formulas.) Instead of rotating a horizontal strip of thickness Δy, we rotate a *vertical strip* of thickness Δx. This rotation produces a *cylindrical shell* of height y_k above a point x_k within the base of the vertical strip, and of thickness Δx. An example of a cylindrical shell is shown as the orange-shaded region in Figure 6.18. We can think of the cylindrical shell shown in the figure as approximating a slice of the solid obtained by cutting straight down through it, parallel to the axis of revolution, all the way around close to the inside hole. We then cut another cylindrical slice around the enlarged hole, then another, and so on, obtaining n cylinders. The radii of the cylinders gradually increase, and the heights of the cylinders follow the contour of the parabola: shorter to taller, then back to shorter (Figure 6.17a).

Each slice is sitting over a subinterval of the x-axis of length (width) Δx. Its radius is approximately $(1 + x_k)$, and its height is approximately $3x_k - x_k^2$. If we unroll the cylinder at x_k and flatten it out, it becomes (approximately) a rectangular slab with thickness Δx (Figure 6.19). The outer circumference of the kth cylinder is $2\pi \cdot \text{radius} = 2\pi(1 + x_k)$, and this is the length of the rolled-out rectangular slab. Its volume is approximated by that of a rectangular solid,

$$\begin{aligned}\Delta V_k &= \text{circumference} \times \text{height} \times \text{thickness} \\ &= 2\pi(1 + x_k) \cdot \left(3x_k - x_k^2\right) \cdot \Delta x.\end{aligned}$$

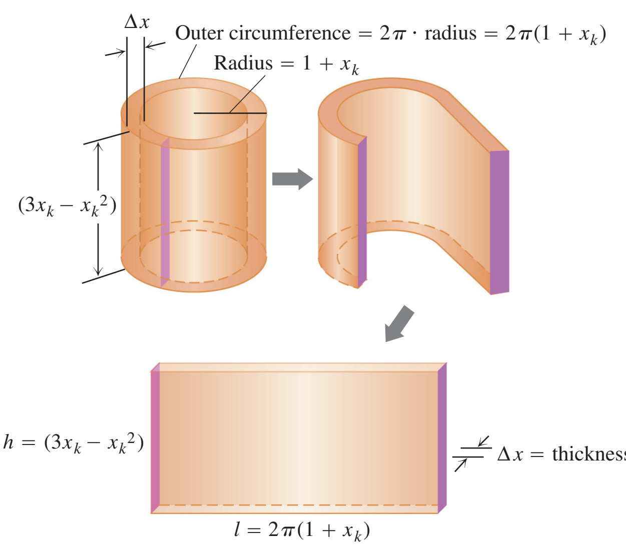


FIGURE 6.19 Imagine cutting and unrolling a cylindrical shell to get a flat (nearly) rectangular solid (Example 1).

Summing together the volumes ΔV_k of the individual cylindrical shells over the interval $[0, 3]$ gives the Riemann sum

$$\sum_{k=1}^{n} \Delta V_k = \sum_{k=1}^{n} 2\pi(x_k + 1)\left(3x_k - x_k^2\right) \Delta x.$$

Taking the limit as the thickness $\Delta x \to 0$ gives the volume integral

$$\begin{aligned} V &= \int_0^3 2\pi(x+1)(3x-x^2)\,dx \\ &= \int_0^3 2\pi(3x^2+3x-x^3-x^2)\,dx \\ &= 2\pi\int_0^3 (2x^2+3x-x^3)\,dx \\ &= 2\pi\left[\frac{2}{3}x^3+\frac{3}{2}x^2-\frac{1}{4}x^4\right]_0^3 \\ &= \frac{45\pi}{2}. \end{aligned}$$

We now generalize the procedure used in Example 1.

The Shell Method

Suppose the region bounded by the graph of a nonnegative continuous function $y = f(x)$ and the x-axis over the finite closed interval $[a, b]$ lies to the right of the vertical line $x = L$ (Figure 6.20a). We assume $a \geq L$, so the vertical line may touch the region, but not pass through it. We generate a solid S by rotating this region about the vertical line L.

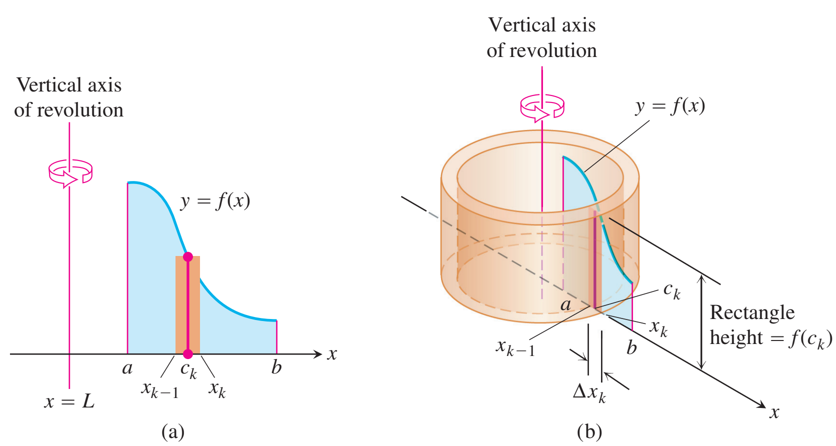


FIGURE 6.20 When the region shown in (a) is revolved about the vertical line $x = L$, a solid is produced which can be sliced into cylindrical shells. A typical shell is shown in (b).

Let P be a partition of the interval $[a, b]$ by the points $a = x_0 < x_1 < \cdots < x_n = b$, and let c_k be the midpoint of the kth subinterval $[x_{k-1}, x_k]$. We approximate the region in Figure 6.20a with rectangles based on this partition of $[a, b]$. A typical approximating rectangle has height $f(c_k)$ and width $\Delta x_k = x_k - x_{k-1}$. If this rectangle is rotated about the vertical line $x = L$, then a shell is swept out, as in Figure 6.20b. A formula from geometry tells us that the volume of the shell swept out by the rectangle is

$$\begin{aligned} \Delta V_k &= 2\pi \times \text{average shell radius} \times \text{shell height} \times \text{thickness} \\ &= 2\pi \cdot (c_k - L) \cdot f(c_k) \cdot \Delta x_k. \end{aligned}$$

We approximate the volume of the solid S by summing the volumes of the shells swept out by the n rectangles based on P:

$$V \approx \sum_{k=1}^{n} \Delta V_k .$$

The limit of this Riemann sum as $\|P\| \to 0$ gives the volume of the solid as a definite integral:

$$\begin{aligned} V &= \int_a^b 2\pi(\text{shell radius})(\text{shell height})\, dx . \\ &= \int_a^b 2\pi(x - L) f(x)\, dx . \end{aligned}$$

We refer to the variable of integration, here x, as the **thickness variable**. We use the first integral, rather than the second containing a formula for the integrand, to emphasize the *process* of the shell method. This will allow for rotations about a horizontal line L as well.

Shell Formula for Revolution About a Vertical Line

The volume of the solid generated by revolving the region between the x-axis and the graph of a continuous function $y = f(x) \geq 0$, $L \leq a \leq x \leq b$, about a vertical line $x = L$ is

$$V = \int_a^b 2\pi \begin{pmatrix} \text{shell} \\ \text{radius} \end{pmatrix} \begin{pmatrix} \text{shell} \\ \text{height} \end{pmatrix} dx .$$

Video Video You Try It

EXAMPLE 2 Cylindrical Shells Revolving About the y-Axis

The region bounded by the curve $y = \sqrt{x}$, the x-axis, and the line $x = 4$ is revolved about the y-axis to generate a solid. Find the volume of the solid.

Solution Sketch the region and draw a line segment across it *parallel* to the axis of revolution (Figure 6.21a). Label the segment's height (shell height) and distance from the axis of revolution (shell radius). (We drew the shell in Figure 6.21b, but you need not do that.)

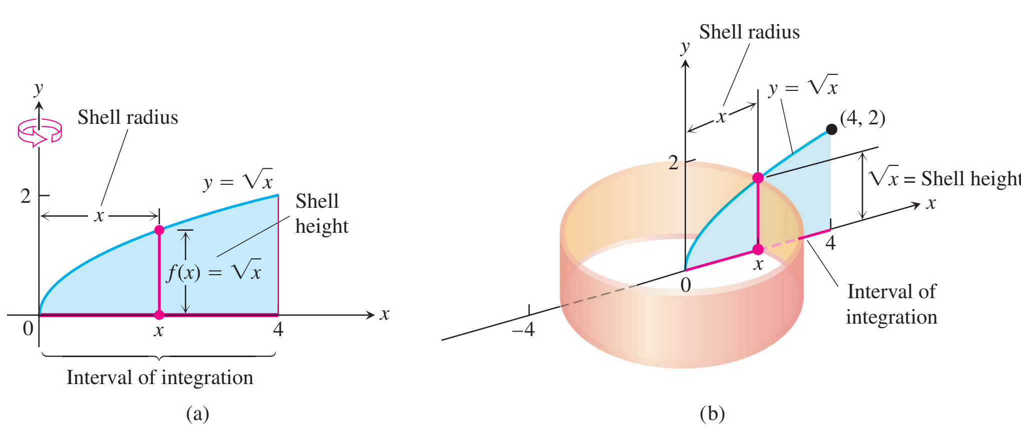


FIGURE 6.21 (a) The region, shell dimensions, and interval of integration in Example 2. (b) The shell swept out by the vertical segment in part (a) with a width Δx.

The shell thickness variable is x, so the limits of integration for the shell formula are $a = 0$ and $b = 4$ (Figure 6.20). The volume is then

$$\begin{aligned} V &= \int_a^b 2\pi \begin{pmatrix}\text{shell}\\\text{radius}\end{pmatrix}\begin{pmatrix}\text{shell}\\\text{height}\end{pmatrix} dx \\ &= \int_0^4 2\pi(x)\left(\sqrt{x}\right) dx \\ &= 2\pi \int_0^4 x^{3/2}\, dx = 2\pi\left[\frac{2}{5}x^{5/2}\right]_0^4 = \frac{128\pi}{5}. \end{aligned}$$

So far, we have used vertical axes of revolution. For horizontal axes, we replace the x's with y's.

Video

You Try It

EXAMPLE 3 Cylindrical Shells Revolving About the x-Axis

The region bounded by the curve $y = \sqrt{x}$, the x-axis, and the line $x = 4$ is revolved about the x-axis to generate a solid. Find the volume of the solid.

Solution Sketch the region and draw a line segment across it *parallel* to the axis of revolution (Figure 6.22a). Label the segment's length (shell height) and distance from the axis of revolution (shell radius). (We drew the shell in Figure 6.22b, but you need not do that.)

In this case, the shell thickness variable is y, so the limits of integration for the shell formula method are $a = 0$ and $b = 2$ (along the y-axis in Figure 6.22). The volume of the solid is

$$\begin{aligned} V &= \int_a^b 2\pi \begin{pmatrix}\text{shell}\\\text{radius}\end{pmatrix}\begin{pmatrix}\text{shell}\\\text{height}\end{pmatrix} dy \\ &= \int_0^2 2\pi(y)(4 - y^2)\, dy \\ &= \int_0^2 2\pi(4y - y^3)\, dy \\ &= 2\pi\left[2y^2 - \frac{y^4}{4}\right]_0^2 = 8\pi. \end{aligned}$$

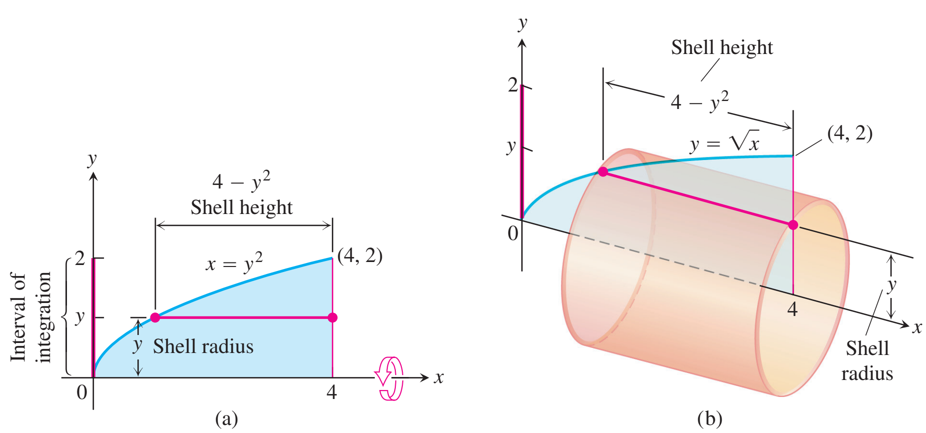


FIGURE 6.22 (a) The region, shell dimensions, and interval of integration in Example 3. (b) The shell swept out by the horizontal segment in part (a) with a width Δy.

Summary of the Shell Method

Regardless of the position of the axis of revolution (horizontal or vertical), the steps for implementing the shell method are these.

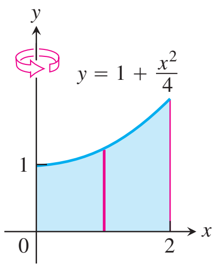
1. *Draw the region and sketch a line segment* across it *parallel* to the axis of revolution. *Label* the segment's height or length (shell height) and distance from the axis of revolution (shell radius).
2. *Find* the limits of integration for the thickness variable.
3. *Integrate* the product 2π (shell radius) (shell height) with respect to the thickness variable (x or y) to find the volume.

The shell method gives the same answer as the washer method when both are used to calculate the volume of a region. We do not prove that result here, but it is illustrated in Exercises 33 and 34. Both volume formulas are actually special cases of a general volume formula we look at in studying double and triple integrals in Chapter 15. That general formula also allows for computing volumes of solids other than those swept out by regions of revolution.

EXERCISES 6.2

In Exercises 1–6, use the shell method to find the volumes of the solids generated by revolving the shaded region about the indicated axis.

1.



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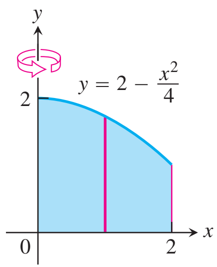
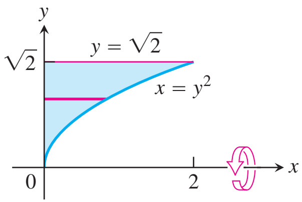


3.



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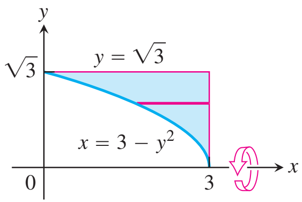
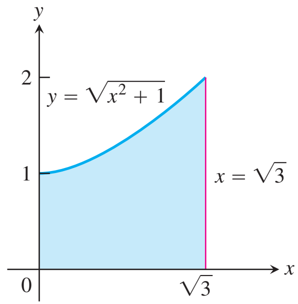
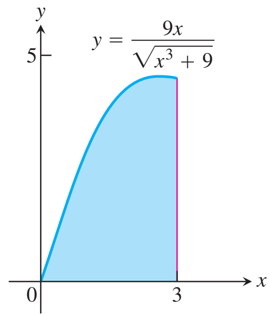


5. The y-axis



6. The y-axis



Revolution About the y-Axis

Use the shell method to find the volumes of the solids generated by revolving the regions bounded by the curves and lines in Exercises 7–14 about the y-axis.

7. $y = x, \quad y = -x/2, \quad x = 2$

8. $y = 2x, \quad y = x/2, \quad x = 1$

9. $y = x^2, \quad y = 2 - x, \quad x = 0$, for $x \geq 0$

10. $y = 2 - x^2, \quad y = x^2, \quad x = 0$

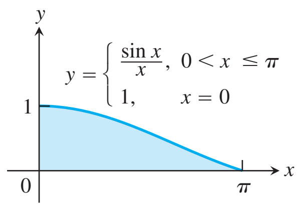
11. $y = 2x - 1, \quad y = \sqrt{x}, \quad x = 0$

12. $y = 3/(2\sqrt{x}), \quad y = 0, \quad x = 1, \quad x = 4$

13. Let $f(x) = \begin{cases} (\sin x)/x, & 0 < x \le \pi \\ 1, & x = 0 \end{cases}$

a. Show that $x f(x) = \sin x,\ 0 \le x \le \pi$.

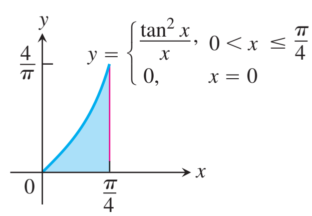
b. Find the volume of the solid generated by revolving the shaded region about the y-axis.



14. Let $g(x) = \begin{cases} (\tan x)^2/x, & 0 < x \le \pi/4 \\ 0, & x = 0 \end{cases}$

a. Show that $x g(x) = (\tan x)^2,\ 0 \le x \le \pi/4$.

b. Find the volume of the solid generated by revolving the shaded region about the y-axis.



Revolution About the x-Axis

Use the shell method to find the volumes of the solids generated by revolving the regions bounded by the curves and lines in Exercises 15–22 about the x-axis.

15. $x = \sqrt{y}, \quad x = -y, \quad y = 2$

16. $x = y^2, \quad x = -y, \quad y = 2, \quad y \ge 0$

17. $x = 2y - y^2, \quad x = 0$

18. $x = 2y - y^2, \quad x = y$

19. $y = |x|, \quad y = 1$

20. $y = x, \quad y = 2x, \quad y = 2$

21. $y = \sqrt{x}, \quad y = 0, \quad y = x - 2$

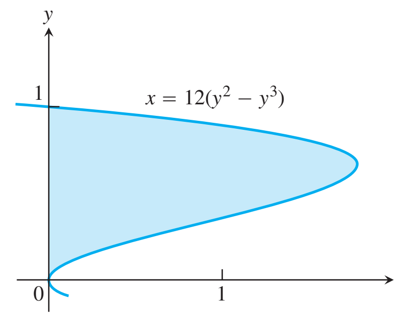
22. $y = \sqrt{x}, \quad y = 0, \quad y = 2 - x$

Revolution About Horizontal Lines

In Exercises 23 and 24, use the shell method to find the volumes of the solids generated by revolving the shaded regions about the indicated axes.

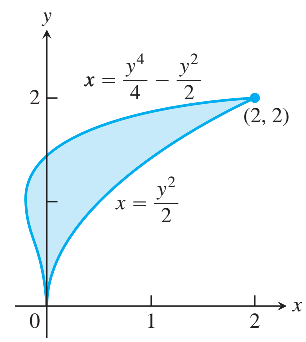
23. **a.** The x-axis **b.** The line $y = 1$

c. The line $y = 8/5$ **d.** The line $y = -2/5$



24. **a.** The x-axis **b.** The line $y = 2$

c. The line $y = 5$ **d.** The line $y = -5/8$



Comparing the Washer and Shell Models

For some regions, both the washer and shell methods work well for the solid generated by revolving the region about the coordinate axes, but this is not always the case. When a region is revolved about the y-axis, for example, and washers are used, we must integrate with respect to y. It may not be possible, however, to express the integrand in terms of y. In such a case, the shell method allows us to integrate with respect to x instead. Exercises 25 and 26 provide some insight.

25. Compute the volume of the solid generated by revolving the region bounded by $y = x$ and $y = x^2$ about each coordinate axis using

a. the shell method. **b.** the washer method.

26. Compute the volume of the solid generated by revolving the triangular region bounded by the lines $2y = x + 4$, $y = x$, and $x = 0$ about

a. the x-axis using the washer method.

b. the y-axis using the shell method.

c. the line $x = 4$ using the shell method.

d. the line $y = 8$ using the washer method.

Choosing Shells or Washers

In Exercises 27–32, find the volumes of the solids generated by revolving the regions about the given axes. If you think it would be better to use washers in any given instance, feel free to do so.

27. The triangle with vertices (1, 1), (1, 2), and (2, 2) about

a. the x-axis **b.** the y-axis

c. the line $x = 10/3$ **d.** the line $y = 1$

28. The region bounded by $y = \sqrt{x}$, $y = 2$, $x = 0$ about

a. the x-axis **b.** the y-axis

c. the line $x = 4$ **d.** the line $y = 2$

29. The region in the first quadrant bounded by the curve $x = y - y^3$ and the y-axis about

a. the x-axis **b.** the line $y = 1$

30. The region in the first quadrant bounded by $x = y - y^3$, $x = 1$, and $y = 1$ about

a. the x-axis **b.** the y-axis

c. the line $x = 1$ **d.** the line $y = 1$

31. The region bounded by $y = \sqrt{x}$ and $y = x^2/8$ about

a. the x-axis **b.** the y-axis

32. The region bounded by $y = 2x - x^2$ and $y = x$ about

a. the y-axis **b.** the line $x = 1$

33. The region in the first quadrant that is bounded above by the curve $y = 1/x^{1/4}$, on the left by the line $x = 1/16$, and below by the line $y = 1$ is revolved about the x-axis to generate a solid. Find the volume of the solid by

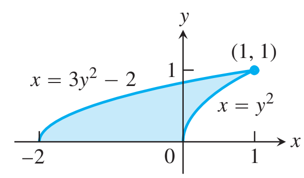
a. the washer method. **b.** the shell method.

34. The region in the first quadrant that is bounded above by the curve $y = 1/\sqrt{x}$, on the left by the line $x = 1/4$, and below by the line $y = 1$ is revolved about the y-axis to generate a solid. Find the volume of the solid by

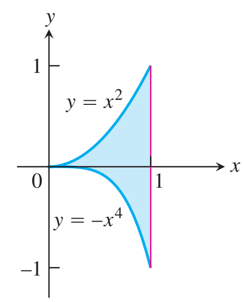
a. the washer method. **b.** the shell method.

Choosing Disks, Washers, or Shells

35. The region shown here is to be revolved about the x-axis to generate a solid. Which of the methods (disk, washer, shell) could you use to find the volume of the solid? How many integrals would be required in each case? Explain.



36. The region shown here is to be revolved about the y-axis to generate a solid. Which of the methods (disk, washer, shell) could you use to find the volume of the solid? How many integrals would be required in each case? Give reasons for your answers.



6.3 Lengths of Plane Curves

HISTORICAL BIOGRAPHY

Archimedes
(287–212 B.C.)

We know what is meant by the length of a straight line segment, but without calculus, we have no precise notion of the length of a general winding curve. The idea of approximating the length of a curve running from point A to point B by subdividing the curve into many pieces and joining successive points of division by straight line segments dates back to the ancient Greeks. Archimedes used this method to approximate the circumference of a circle by inscribing a polygon of n sides and then using geometry to compute its perimeter

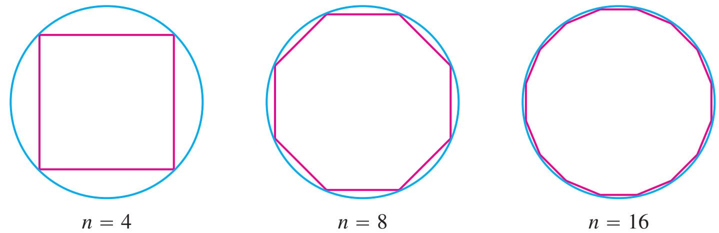


FIGURE 6.23 Archimedes used the perimeters of inscribed polygons to approximate the circumference of a circle. For $n = 96$ the approximation method gives $\pi \approx 3.14103$ as the circumference of the unit circle.

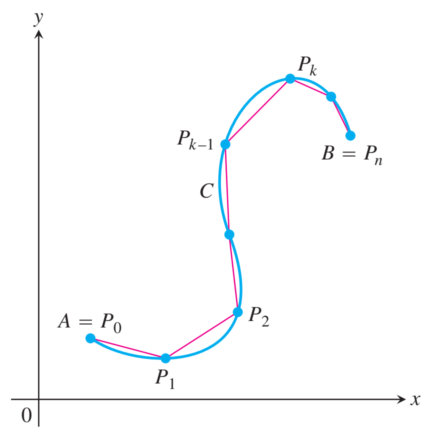


FIGURE 6.24 The curve C defined parametrically by the equations $x = f(t)$ and $y = g(t)$, $a \le t \le b$. The length of the curve from A to B is approximated by the sum of the lengths of the polygonal path (straight line segments) starting at $A = P_0$, then to P_1, and so on, ending at $B = P_n$.

(Figure 6.23). The extension of this idea to a more general curve is displayed in Figure 6.24, and we now describe how that method works.

Length of a Parametrically Defined Curve

Let C be a curve given parametrically by the equations

$$x = f(t) \quad \text{and} \quad y = g(t), \quad a \le t \le b.$$

We assume the functions f and g have continuous derivatives on the interval $[a, b]$ that are not simultaneously zero. Such functions are said to be **continuously differentiable**, and the curve C defined by them is called a **smooth curve**. It may be helpful to imagine the curve as the path of a particle moving from point $A = (f(a), g(a))$ at time $t = a$ to point $B = (f(b), g(b))$ in Figure 6.24. We subdivide the path (or arc) AB into n pieces at points $A = P_0, P_1, P_2, \ldots, P_n = B$. These points correspond to a partition of the interval $[a, b]$ by $a = t_0 < t_1 < t_2 < \cdots < t_n = b$, where $P_k = (f(t_k), g(t_k))$. Join successive points of this subdivision by straight line segments (Figure 6.24). A representative line segment has length

$$\begin{aligned} L_k &= \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2} \\ &= \sqrt{[f(t_k) - f(t_{k-1})]^2 + [g(t_k) - g(t_{k-1})]^2} \end{aligned}$$

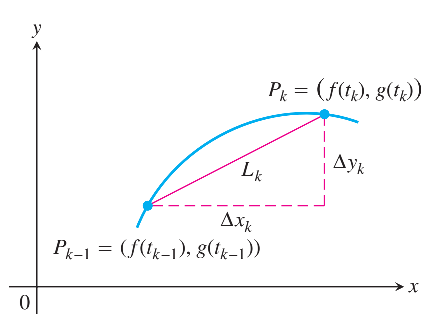


FIGURE 6.25 The arc $P_{k-1}P_k$ is approximated by the straight line segment shown here, which has length $L_k = \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}$.

(see Figure 6.25). If Δt_k is small, the length L_k is approximately the length of arc $P_{k-1}P_k$. By the Mean Value Theorem there are numbers t_k^* and t_k^{**} in $[t_{k-1}, t_k]$ such that

$$\begin{aligned} \Delta x_k &= f(t_k) - f(t_{k-1}) = f'(t_k^*)\,\Delta t_k, \\ \Delta y_k &= g(t_k) - g(t_{k-1}) = g'(t_k^{**})\,\Delta t_k. \end{aligned}$$

Assuming the path from A to B is traversed exactly once as t increases from $t = a$ to $t = b$, with no doubling back or retracing, an intuitive approximation to the "length" of the curve AB is the sum of all the lengths L_k:

$$\begin{aligned} \sum_{k=1}^{n} L_k &= \sum_{k=1}^{n} \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2} \\ &= \sum_{k=1}^{n} \sqrt{[f'(t_k^*)]^2 + [g'(t_k^{**})]^2}\,\Delta t_k. \end{aligned}$$

Although this last sum on the right is not exactly a Riemann sum (because f' and g' are evaluated at different points), a theorem in advanced calculus guarantees its limit, as the norm of the partition tends to zero, to be the definite integral

$$\int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\, dt.$$

Therefore, it is reasonable to define the length of the curve from A to B as this integral.

DEFINITION Length of a Parametric Curve

If a curve C is defined parametrically by $x = f(t)$ and $y = g(t)$, $a \leq t \leq b$, where f' and g' are continuous and not simultaneously zero on $[a, b]$, and C is traversed exactly once as t increases from $t = a$ to $t = b$, then **the length of C** is the definite integral

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}\, dt.$$

A smooth curve C does not double back or reverse the direction of motion over the time interval $[a, b]$ since $(f')^2 + (g')^2 > 0$ throughout the interval.

If $x = f(t)$ and $y = g(t)$, then using the Leibniz notation we have the following result for arc length:

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt. \tag{1}$$

What if there are two different parametrizations for a curve C whose length we want to find; does it matter which one we use? The answer, from advanced calculus, is no, as long as the parametrization we choose meets the conditions stated in the definition of the length of C (see Exercise 29).

You Try It

EXAMPLE 1 The Circumference of a Circle

Find the length of the circle of radius r defined parametrically by

$$x = r\cos t \quad \text{and} \quad y = r\sin t, \qquad 0 \leq t \leq 2\pi.$$

Solution As t varies from 0 to 2π, the circle is traversed exactly once, so the circumference is

$$L = \int_0^{2\pi} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

We find

$$\frac{dx}{dt} = -r\sin t, \quad \frac{dy}{dt} = r\cos t$$

and

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = r^2(\sin^2 t + \cos^2 t) = r^2.$$

So

$$L = \int_0^{2\pi} \sqrt{r^2}\, dt = r\Big[t\Big]_0^{2\pi} = 2\pi r.$$

EXAMPLE 2 Applying the Parametric Formula for Length of a Curve

Find the length of the astroid (Figure 6.26)

$$x = \cos^3 t, \qquad y = \sin^3 t, \qquad 0 \le t \le 2\pi.$$

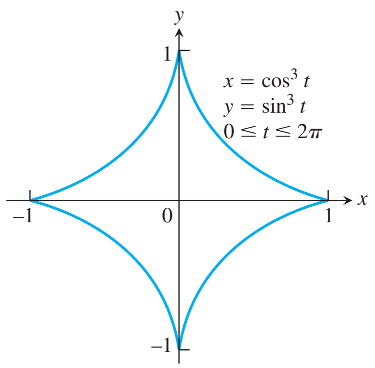


FIGURE 6.26 The astroid in Example 2.

Solution Because of the curve's symmetry with respect to the coordinate axes, its length is four times the length of the first-quadrant portion. We have

$$x = \cos^3 t, \qquad y = \sin^3 t$$

$$\left(\frac{dx}{dt}\right)^2 = [3\cos^2 t(-\sin t)]^2 = 9\cos^4 t \sin^2 t$$

$$\left(\frac{dy}{dt}\right)^2 = [3\sin^2 t(\cos t)]^2 = 9\sin^4 t \cos^2 t$$

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{9\cos^2 t \sin^2 t\underbrace{(\cos^2 t + \sin^2 t)}_{1}}$$

$$= \sqrt{9\cos^2 t \sin^2 t}$$

$$= 3\,|\cos t \sin t|$$

$$= 3\cos t \sin t. \qquad \text{$\cos t \sin t \ge 0$ for $0 \le t \le \pi/2$}$$

Therefore,

$$\text{Length of first-quadrant portion} = \int_0^{\pi/2} 3\cos t \sin t\, dt$$

$$= \frac{3}{2}\int_0^{\pi/2} \sin 2t\, dt \qquad \text{$\cos t \sin t = (1/2)\sin 2t$}$$

$$= -\frac{3}{4}\cos 2t\Big]_0^{\pi/2} = \frac{3}{2}.$$

The length of the astroid is four times this: $4(3/2) = 6$.

HISTORICAL BIOGRAPHY

Gregory St. Vincent
(1584–1667)

Length of a Curve $y = f(x)$

Given a continuously differentiable function $y = f(x)$, $a \le x \le b$, we can assign $x = t$ as a parameter. The graph of the function f is then the curve C defined parametrically by

$$x = t \quad \text{and} \quad y = f(t), \qquad a \le t \le b,$$

a special case of what we considered before. Then,

$$\frac{dx}{dt} = 1 \quad \text{and} \quad \frac{dy}{dt} = f'(t).$$

From our calculations in Section 3.5, we have

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = f'(t)$$

giving

$$\begin{aligned}\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 &= 1 + [f'(t)]^2\\ &= 1 + \left(\frac{dy}{dx}\right)^2\\ &= 1 + [f'(x)]^2.\end{aligned}$$

Substitution into Equation (1) gives the arc length formula for the graph of $y = f(x)$.

Formula for the Length of $y = f(x)$, $a \le x \le b$

If f is continuously differentiable on the closed interval $[a, b]$, the length of the curve (graph) $y = f(x)$ from $x = a$ to $x = b$ is

$$L = \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx = \int_a^b \sqrt{1 + [f'(x)]^2}\,dx. \tag{2}$$

EXAMPLE 3 Applying the Arc Length Formula for a Graph

Find the length of the curve

$$y = \frac{4\sqrt{2}}{3}x^{3/2} - 1, \qquad 0 \le x \le 1.$$

Solution We use Equation (2) with $a = 0$, $b = 1$, and

$$\begin{aligned}y &= \frac{4\sqrt{2}}{3}x^{3/2} - 1\\ \frac{dy}{dx} &= \frac{4\sqrt{2}}{3}\cdot\frac{3}{2}x^{1/2} = 2\sqrt{2}x^{1/2}\\ \left(\frac{dy}{dx}\right)^2 &= \left(2\sqrt{2}x^{1/2}\right)^2 = 8x.\end{aligned}$$

The length of the curve from $x = 0$ to $x = 1$ is

$$\begin{aligned}L &= \int_0^1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx = \int_0^1 \sqrt{1 + 8x}\,dx && \text{Eq. (2) with } a = 0,\ b = 1\\ &= \frac{2}{3}\cdot\frac{1}{8}(1 + 8x)^{3/2}\Big]_0^1 = \frac{13}{6}. && \text{Let } u = 1 + 8x \text{, integrate, and replace } u \text{ by } 1 + 8x.\end{aligned}$$

■

Dealing with Discontinuities in dy/dx

At a point on a curve where dy/dx fails to exist, dx/dy may exist and we may be able to find the curve's length by expressing x as a function of y and applying the following analogue of Equation (2):

Formula for the Length of $x = g(y)$, $c \le y \le d$

If g is continuously differentiable on $[c, d]$, the length of the curve $x = g(y)$ from $y = c$ to $y = d$ is

$$L = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \int_c^d \sqrt{1 + [g'(y)]^2}\, dy. \tag{3}$$

EXAMPLE 4 Length of a Graph Which Has a Discontinuity in dy/dx

Find the length of the curve $y = (x/2)^{2/3}$ from $x = 0$ to $x = 2$.

Solution The derivative

$$\frac{dy}{dx} = \frac{2}{3}\left(\frac{x}{2}\right)^{-1/3}\left(\frac{1}{2}\right) = \frac{1}{3}\left(\frac{2}{x}\right)^{1/3}$$

is not defined at $x = 0$, so we cannot find the curve's length with Equation (2).

We therefore rewrite the equation to express x in terms of y:

$$y = \left(\frac{x}{2}\right)^{2/3}$$

$$y^{3/2} = \frac{x}{2} \qquad \text{Raise both sides to the power 3/2.}$$

$$x = 2y^{3/2}. \qquad \text{Solve for } x.$$

From this we see that the curve whose length we want is also the graph of $x = 2y^{3/2}$ from $y = 0$ to $y = 1$ (Figure 6.27).

The derivative

$$\frac{dx}{dy} = 2\left(\frac{3}{2}\right)y^{1/2} = 3y^{1/2}$$

is continuous on $[0, 1]$. We may therefore use Equation (3) to find the curve's length:

$$L = \int_c^d \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy = \int_0^1 \sqrt{1 + 9y}\, dy \qquad \text{Eq. (3) with } c = 0, d = 1.$$

$$= \frac{1}{9} \cdot \frac{2}{3}(1 + 9y)^{3/2}\Big]_0^1 \qquad \text{Let } u = 1 + 9y,\ du/9 = dy, \text{ integrate, and substitute back.}$$

$$= \frac{2}{27}\left(10\sqrt{10} - 1\right) \approx 2.27.$$

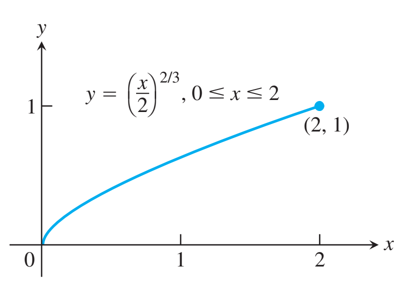


FIGURE 6.27 The graph of $y = (x/2)^{2/3}$ from $x = 0$ to $x = 2$ is also the graph of $x = 2y^{3/2}$ from $y = 0$ to $y = 1$ (Example 4).

HISTORICAL BIOGRAPHY

James Gregory
(1638–1675)

The Short Differential Formula

Equation (1) is frequently written in terms of differentials in place of derivatives. This is done formally by writing $(dt)^2$ under the radical in place of the dt outside the radical, and then writing

$$\left(\frac{dx}{dt}\right)^2 (dt)^2 = \left(\frac{dx}{dt}\,dt\right)^2 = (dx)^2$$

and

$$\left(\frac{dy}{dt}\right)^2 (dt)^2 = \left(\frac{dy}{dt}\,dt\right)^2 = (dy)^2.$$

It is also customary to eliminate the parentheses in $(dx)^2$ and write dx^2 instead, so that Equation (1) is written

$$L = \int \sqrt{dx^2 + dy^2}. \tag{4}$$

We can think of these differentials as a way to summarize and simplify the properties of integrals. Differentials are given a precise mathematical definition in a more advanced text.

To do an integral computation, dx and dy must both be expressed in terms of one and the same variable, and appropriate limits must be supplied in Equation (4).

A useful way to remember Equation (4) is to write

$$ds = \sqrt{dx^2 + dy^2} \tag{5}$$

and treat ds as the differential of arc length, which can be integrated between appropriate limits to give the total length of a curve. Figure 6.28a gives the exact interpretation of ds corresponding to Equation (5). Figure 6.28b is not strictly accurate but is to be thought of as a simplified approximation of Figure 6.28a.

With Equation (5) in mind, the quickest way to recall the formulas for arc length is to remember the equation

$$\text{Arc length} = \int ds.$$

If we write $L = \int ds$ and have the graph of $y = f(x)$, we can rewrite Equation (5) to get

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{dx^2 + \frac{dy^2}{dx^2}\,dx^2} = \sqrt{1 + \frac{dy^2}{dx^2}}\,dx = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx,$$

resulting in Equation (2). If we have instead $x = g(y)$, we rewrite Equation (5)

$$ds = \sqrt{dx^2 + dy^2} = \sqrt{dy^2 + \frac{dx^2}{dy^2}\,dy^2} = \sqrt{1 + \frac{dx^2}{dy^2}}\,dy = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\,dy,$$

and obtain Equation (3).

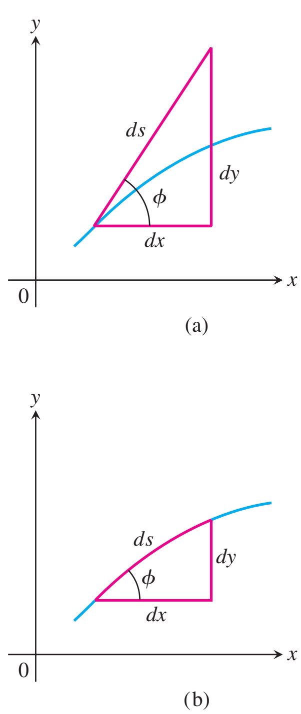


FIGURE 6.28 Diagrams for remembering the equation $ds = \sqrt{dx^2 + dy^2}$.

EXERCISES 6.3

Lengths of Parametrized Curves

Find the lengths of the curves in Exercises 1–6.

1. $x = 1 - t, \quad y = 2 + 3t, \quad -2/3 \le t \le 1$
2. $x = \cos t, \quad y = t + \sin t, \quad 0 \le t \le \pi$
3. $x = t^3, \quad y = 3t^2/2, \quad 0 \le t \le \sqrt{3}$
4. $x = t^2/2, \quad y = (2t + 1)^{3/2}/3, \quad 0 \le t \le 4$
5. $x = (2t + 3)^{3/2}/3, \quad y = t + t^2/2, \quad 0 \le t \le 3$
6. $x = 8\cos t + 8t \sin t, \quad y = 8 \sin t - 8t \cos t, \quad 0 \le t \le \pi/2$

Finding Lengths of Curves

Find the lengths of the curves in Exercises 7–16. If you have a grapher, you may want to graph these curves to see what they look like.

7. $y = (1/3)(x^2 + 2)^{3/2}$ from $x = 0$ to $x = 3$
8. $y = x^{3/2}$ from $x = 0$ to $x = 4$
9. $x = (y^3/3) + 1/(4y)$ from $y = 1$ to $y = 3$
   (*Hint:* $1 + (dx/dy)^2$ is a perfect square.)
10. $x = (y^{3/2}/3) - y^{1/2}$ from $y = 1$ to $y = 9$
   (*Hint:* $1 + (dx/dy)^2$ is a perfect square.)
11. $x = (y^4/4) + 1/(8y^2)$ from $y = 1$ to $y = 2$
   (*Hint:* $1 + (dx/dy)^2$ is a perfect square.)
12. $x = (y^3/6) + 1/(2y)$ from $y = 2$ to $y = 3$
   (*Hint:* $1 + (dx/dy)^2$ is a perfect square.)
13. $y = (3/4)x^{4/3} - (3/8)x^{2/3} + 5, \quad 1 \le x \le 8$
14. $y = (x^3/3) + x^2 + x + 1/(4x + 4), \quad 0 \le x \le 2$
15. $x = \int_0^y \sqrt{\sec^4 t - 1}\, dt, \quad -\pi/4 \le y \le \pi/4$
16. $y = \int_{-2}^x \sqrt{3t^4 - 1}\, dt, \quad -2 \le x \le -1$

T Finding Integrals for Lengths of Curves

In Exercises 17–24, do the following.

a. Set up an integral for the length of the curve.

b. Graph the curve to see what it looks like.

c. Use your grapher's or computer's integral evaluator to find the curve's length numerically.

17. $y = x^2, \quad -1 \le x \le 2$
18. $y = \tan x, \quad -\pi/3 \le x \le 0$
19. $x = \sin y, \quad 0 \le y \le \pi$
20. $x = \sqrt{1 - y^2}, \quad -1/2 \le y \le 1/2$
21. $y^2 + 2y = 2x + 1$ from $(-1, -1)$ to $(7, 3)$
22. $y = \sin x - x \cos x, \quad 0 \le x \le \pi$
23. $y = \int_0^x \tan t\, dt, \quad 0 \le x \le \pi/6$
24. $x = \int_0^y \sqrt{\sec^2 t - 1}\, dt, \quad -\pi/3 \le y \le \pi/4$

Theory and Applications

25. Is there a smooth (continuously differentiable) curve $y = f(x)$ whose length over the interval $0 \le x \le a$ is always $\sqrt{2}a$? Give reasons for your answer.

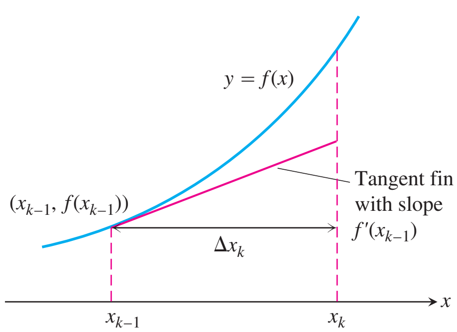
26. **Using tangent fins to derive the length formula for curves** Assume that f is smooth on $[a, b]$ and partition the interval $[a, b]$ in the usual way. In each subinterval $[x_{k-1}, x_k]$, construct the *tangent fin* at the point $(x_{k-1}, f(x_{k-1}))$, as shown in the accompanying figure.

   a. Show that the length of the kth tangent fin over the interval $[x_{k-1}, x_k]$ equals $\sqrt{(\Delta x_k)^2 + (f'(x_{k-1})\,\Delta x_k)^2}$.

   b. Show that

$$\lim_{n\to\infty} \sum_{k=1}^{n} (\text{length of } k\text{th tangent fin}) = \int_a^b \sqrt{1 + (f'(x))^2}\, dx,$$

   which is the length L of the curve $y = f(x)$ from a to b.



27. **a.** Find a curve through the point $(1, 1)$ whose length integral is

$$L = \int_1^4 \sqrt{1 + \frac{1}{4x}}\, dx.$$

   b. How many such curves are there? Give reasons for your answer.

28. **a.** Find a curve through the point $(0, 1)$ whose length integral is

$$L = \int_1^2 \sqrt{1 + \frac{1}{y^4}}\, dy.$$

   b. How many such curves are there? Give reasons for your answer.

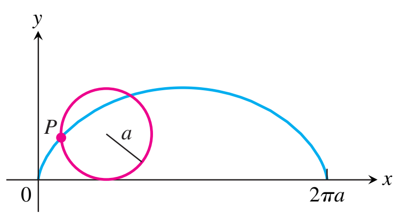
29. **Length is independent of parametrization** To illustrate the fact that the numbers we get for length do not depend on the way

we parametrize our curves (except for the mild restrictions preventing doubling back mentioned earlier), calculate the length of the semicircle $y = \sqrt{1 - x^2}$ with these two different parametrizations:

a. $x = \cos 2t, \quad y = \sin 2t, \quad 0 \le t \le \pi/2$

b. $x = \sin \pi t, \quad y = \cos \pi t, \quad -1/2 \le t \le 1/2$

30. Find the length of one arch of the cycloid $x = a(\theta - \sin\theta)$, $y = a(1 - \cos\theta)$, $0 \le \theta \le 2\pi$, shown in the accompanying figure. A **cycloid** is the curve traced out by a point P on the circumference of a circle rolling along a straight line, such as the x-axis.



COMPUTER EXPLORATIONS

In Exercises 31–36, use a CAS to perform the following steps for the given curve over the closed interval.

a. Plot the curve together with the polygonal path approximations for $n = 2, 4, 8$ partition points over the interval. (See Figure 6.24.)

b. Find the corresponding approximation to the length of the curve by summing the lengths of the line segments.

c. Evaluate the length of the curve using an integral. Compare your approximations for $n = 2, 4, 8$ with the actual length given by the integral. How does the actual length compare with the approximations as n increases? Explain your answer.

31. $f(x) = \sqrt{1 - x^2}, \quad -1 \le x \le 1$

32. $f(x) = x^{1/3} + x^{2/3}, \quad 0 \le x \le 2$

33. $f(x) = \sin(\pi x^2), \quad 0 \le x \le \sqrt{2}$

34. $f(x) = x^2 \cos x, \quad 0 \le x \le \pi$

35. $f(x) = \dfrac{x - 1}{4x^2 + 1}, \quad -\dfrac{1}{2} \le x \le 1$

36. $f(x) = x^3 - x^2, \quad -1 \le x \le 1$

37. $x = \dfrac{1}{3}t^3, \quad y = \dfrac{1}{2}t^2, \quad 0 \le t \le 1$

38. $x = 2t^3 - 16t^2 + 25t + 5, \quad y = t^2 + t - 3,$
$0 \le t \le 6$

39. $x = t - \cos t, \quad y = 1 + \sin t, \quad -\pi \le t \le \pi$

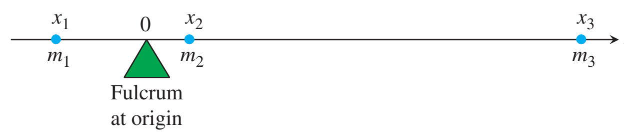
40. $x = e^t \cos t, \quad y = e^t \sin t, \quad 0 \le t \le \pi$

6.4 Moments and Centers of Mass

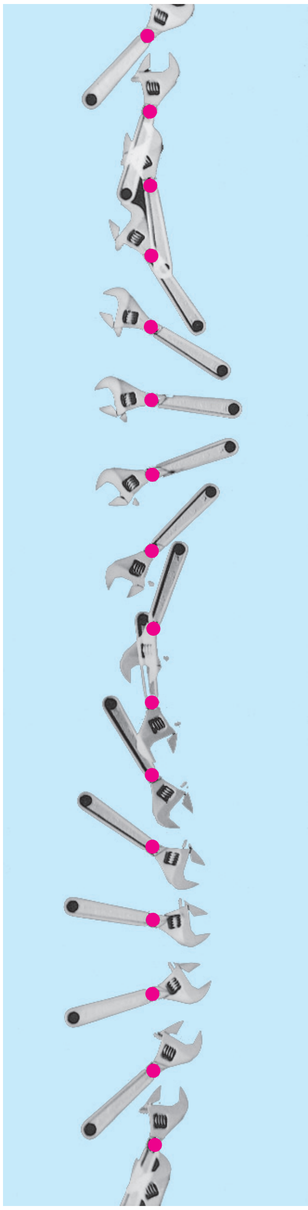
Many structures and mechanical systems behave as if their masses were concentrated at a single point, called the *center of mass* (Figure 6.29). It is important to know how to locate this point, and doing so is basically a mathematical enterprise. For the moment, we deal with one- and two-dimensional objects. Three-dimensional objects are best done with the multiple integrals of Chapter 15.

Masses Along a Line

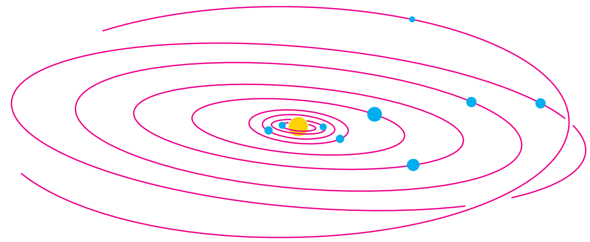
We develop our mathematical model in stages. The first stage is to imagine masses m_1, m_2, and m_3 on a rigid x-axis supported by a fulcrum at the origin.



The resulting system might balance, or it might not. It depends on how large the masses are and how they are arranged.



(a)



(b)

FIGURE 6.29 (a) The motion of this wrench gliding on ice seems haphazard until we notice that the wrench is simply turning about its center of mass as the center glides in a straight line. (b) The planets, asteroids, and comets of our solar system revolve about their collective center of mass. (It lies inside the sun.)

Each mass m_k exerts a downward force $m_k g$ (the weight of m_k) equal to the magnitude of the mass times the acceleration of gravity. Each of these forces has a tendency to turn the axis about the origin, the way you turn a seesaw. This turning effect, called a **torque**, is measured by multiplying the force $m_k g$ by the signed distance x_k from the point of application to the origin. Masses to the left of the origin exert negative (counterclockwise) torque. Masses to the right of the origin exert positive (clockwise) torque.

The sum of the torques measures the tendency of a system to rotate about the origin. This sum is called the **system torque**.

$$\text{System torque} = m_1 g x_1 + m_2 g x_2 + m_3 g x_3 \tag{1}$$

The system will balance if and only if its torque is zero.

If we factor out the g in Equation (1), we see that the system torque is

$$\underbrace{g}_{\substack{\text{a feature of the}\\ \text{environment}}} \cdot \underbrace{(m_1 x_1 + m_2 x_2 + m_3 x_3)}_{\substack{\text{a feature of}\\ \text{the system}}}$$

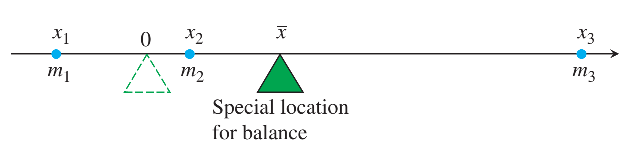
Thus, the torque is the product of the gravitational acceleration g, which is a feature of the environment in which the system happens to reside, and the number $(m_1 x_1 + m_2 x_2 + m_3 x_3)$, which is a feature of the system itself, a constant that stays the same no matter where the system is placed.

The number $(m_1 x_1 + m_2 x_2 + m_3 x_3)$ is called the **moment of the system about the origin**. It is the sum of the **moments** $m_1 x_1, m_2 x_2, m_3 x_3$ of the individual masses.

$$M_0 = \text{Moment of system about origin} = \sum m_k x_k.$$

(We shift to sigma notation here to allow for sums with more terms.)

We usually want to know where to place the fulcrum to make the system balance, that is, at what point $\bar{x}$ to place it to make the torques add to zero.



The torque of each mass about the fulcrum in this special location is

$$\begin{aligned}\text{Torque of } m_k \text{ about } \bar{x} &= \begin{pmatrix}\text{signed distance}\\ \text{of } m_k \text{ from } \bar{x}\end{pmatrix}\begin{pmatrix}\text{downward}\\ \text{force}\end{pmatrix}\\ &= (x_k - \bar{x})m_k g.\end{aligned}$$

When we write the equation that says that the sum of these torques is zero, we get an equation we can solve for $\bar{x}$:

$$\begin{aligned}\sum (x_k - \bar{x})m_k g &= 0 && \text{Sum of the torques equals zero}\\ g\sum (x_k - \bar{x})m_k &= 0 && \text{Constant Multiple Rule for Sums}\\ \sum (m_k x_k - \bar{x}m_k) &= 0 && g \text{ divided out, } m_k \text{ distributed}\\ \sum m_k x_k - \sum \bar{x}m_k &= 0 && \text{Difference Rule for Sums}\\ \sum m_k x_k &= \bar{x}\sum m_k && \text{Rearranged, Constant Multiple Rule again}\\ \bar{x} &= \frac{\sum m_k x_k}{\sum m_k}. && \text{Solved for } \bar{x}\end{aligned}$$

This last equation tells us to find $\bar{x}$ by dividing the system's moment about the origin by the system's total mass:

You Try It

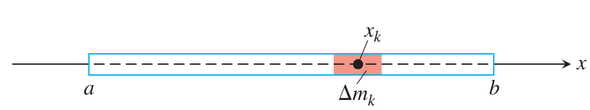
$$\bar{x} = \frac{\sum m_k x_k}{\sum m_k} = \frac{\text{system moment about origin}}{\text{system mass}}.$$

The point $\bar{x}$ is called the system's **center of mass**.

Wires and Thin Rods

In many applications, we want to know the center of mass of a rod or a thin strip of metal. In cases like these where we can model the distribution of mass with a continuous function, the summation signs in our formulas become integrals in a manner we now describe.

Imagine a long, thin strip lying along the x-axis from $x = a$ to $x = b$ and cut into small pieces of mass Δm_k by a partition of the interval $[a, b]$. Choose x_k to be any point in the kth subinterval of the partition.



The kth piece is Δx_k units long and lies approximately x_k units from the origin. Now observe three things.

First, the strip's center of mass $\bar{x}$ is nearly the same as that of the system of point masses we would get by attaching each mass Δm_k to the point x_k:

$$\bar{x} \approx \frac{\text{system moment}}{\text{system mass}}.$$

Density
A material's density is its mass per unit volume. In practice, however, we tend to use units we can conveniently measure. For wires, rods, and narrow strips, we use mass per unit length. For flat sheets and plates, we use mass per unit area.

Second, the moment of each piece of the strip about the origin is approximately $x_k \,\Delta m_k$, so the system moment is approximately the sum of the $x_k \,\Delta m_k$:

$$\text{System moment} \approx \sum x_k \,\Delta m_k .$$

Third, if the density of the strip at x_k is $\delta(x_k)$, expressed in terms of mass per unit length and if δ is continuous, then Δm_k is approximately equal to $\delta(x_k)\,\Delta x_k$ (mass per unit length times length):

$$\Delta m_k \approx \delta(x_k)\,\Delta x_k .$$

Combining these three observations gives

$$\bar{x} \approx \frac{\text{system moment}}{\text{system mass}} \approx \frac{\sum x_k \,\Delta m_k}{\sum \Delta m_k} \approx \frac{\sum x_k \,\delta(x_k)\,\Delta x_k}{\sum \delta(x_k)\,\Delta x_k}. \tag{2}$$

The sum in the last numerator in Equation (2) is a Riemann sum for the continuous function $x\delta(x)$ over the closed interval $[a, b]$. The sum in the denominator is a Riemann sum for the function $\delta(x)$ over this interval. We expect the approximations in Equation (2) to improve as the strip is partitioned more finely, and we are led to the equation

$$\bar{x} = \frac{\displaystyle\int_a^b x\delta(x)\,dx}{\displaystyle\int_a^b \delta(x)\,dx}.$$

This is the formula we use to find $\bar{x}$.

Moment, Mass, and Center of Mass of a Thin Rod or Strip Along the x-Axis with Density Function $\delta(x)$

Moment about the origin:	$M_0 = \displaystyle\int_a^b x\delta(x)\,dx$	(3a)
Mass:	$M = \displaystyle\int_a^b \delta(x)\,dx$	(3b)
Center of mass:	$\bar{x} = \dfrac{M_0}{M}$	(3c)

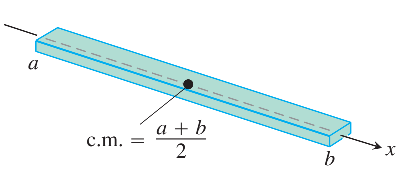


FIGURE 6.30 The center of mass of a straight, thin rod or strip of constant density lies halfway between its ends (Example 1).

EXAMPLE 1 Strips and Rods of Constant Density

Show that the center of mass of a straight, thin strip or rod of constant density lies halfway between its two ends.

Solution We model the strip as a portion of the x-axis from $x = a$ to $x = b$ (Figure 6.30). Our goal is to show that $\bar{x} = (a + b)/2$, the point halfway between a and b.

The key is the density's having a constant value. This enables us to regard the function $\delta(x)$ in the integrals in Equation (3) as a constant (call it δ), with the result that

$$M_0 = \int_a^b \delta x\, dx = \delta \int_a^b x\, dx = \delta\left[\frac{1}{2}x^2\right]_a^b = \frac{\delta}{2}(b^2 - a^2)$$

$$M = \int_a^b \delta\, dx = \delta \int_a^b dx = \delta\Big[x\Big]_a^b = \delta(b - a)$$

$$\bar{x} = \frac{M_0}{M} = \frac{\frac{\delta}{2}(b^2 - a^2)}{\delta(b - a)}$$

$$= \frac{a + b}{2}.$$

The δ's cancel in the formula for $\bar{x}$.

EXAMPLE 2 Variable-Density Rod

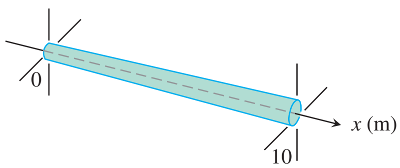


FIGURE 6.31 We can treat a rod of variable thickness as a rod of variable density (Example 2).

The 10-m-long rod in Figure 6.31 thickens from left to right so that its density, instead of being constant, is $\delta(x) = 1 + (x/10)$ kg/m. Find the rod's center of mass.

Solution The rod's moment about the origin (Equation 3a) is

$$M_0 = \int_0^{10} x\delta(x)\, dx = \int_0^{10} x\left(1 + \frac{x}{10}\right) dx = \int_0^{10} \left(x + \frac{x^2}{10}\right) dx$$

$$= \left[\frac{x^2}{2} + \frac{x^3}{30}\right]_0^{10} = 50 + \frac{100}{3} = \frac{250}{3}\ \text{kg}\cdot\text{m}.$$

The units of a moment are mass $\times$ length.

The rod's mass (Equation 3b) is

$$M = \int_0^{10} \delta(x)\, dx = \int_0^{10} \left(1 + \frac{x}{10}\right) dx = \left[x + \frac{x^2}{20}\right]_0^{10} = 10 + 5 = 15\ \text{kg}.$$

The center of mass (Equation 3c) is located at the point

$$\bar{x} = \frac{M_0}{M} = \frac{250}{3}\cdot\frac{1}{15} = \frac{50}{9} \approx 5.56\ \text{m}.$$

Masses Distributed over a Plane Region

Suppose that we have a finite collection of masses located in the plane, with mass m_k at the point (x_k, y_k) (see Figure 6.32). The mass of the system is

$$\text{System mass: } M = \sum m_k.$$

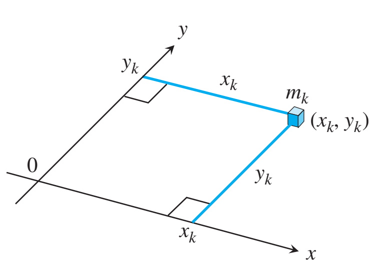


FIGURE 6.32 Each mass m_k has a moment about each axis.

Each mass m_k has a moment about each axis. Its moment about the x-axis is $m_k y_k$, and its moment about the y-axis is $m_k x_k$. The moments of the entire system about the two axes are

$$\text{Moment about } x\text{-axis:} \qquad M_x = \sum m_k y_k,$$

$$\text{Moment about } y\text{-axis:} \qquad M_y = \sum m_k x_k.$$

The x-coordinate of the system's center of mass is defined to be

$$\bar{x} = \frac{M_y}{M} = \frac{\sum m_k x_k}{\sum m_k}. \qquad (4)$$

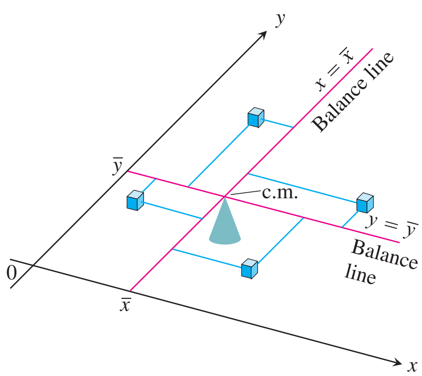


FIGURE 6.33 A two-dimensional array of masses balances on its center of mass.

With this choice of $\bar{x}$, as in the one-dimensional case, the system balances about the line $x = \bar{x}$ (Figure 6.33).

The y-coordinate of the system's center of mass is defined to be

$$\bar{y} = \frac{M_x}{M} = \frac{\sum m_k y_k}{\sum m_k}. \tag{5}$$

With this choice of $\bar{y}$, the system balances about the line $y = \bar{y}$ as well. The torques exerted by the masses about the line $y = \bar{y}$ cancel out. Thus, as far as balance is concerned, the system behaves as if all its mass were at the single point $(\bar{x}, \bar{y})$. We call this point the system's **center of mass**.

Thin, Flat Plates

In many applications, we need to find the center of mass of a thin, flat plate: a disk of aluminum, say, or a triangular sheet of steel. In such cases, we assume the distribution of mass to be continuous, and the formulas we use to calculate $\bar{x}$ and $\bar{y}$ contain integrals instead of finite sums. The integrals arise in the following way.

Imagine the plate occupying a region in the xy-plane, cut into thin strips parallel to one of the axes (in Figure 6.34, the y-axis). The center of mass of a typical strip is $(\tilde{x}, \tilde{y})$. We treat the strip's mass Δm as if it were concentrated at $(\tilde{x}, \tilde{y})$. The moment of the strip about the y-axis is then $\tilde{x}\,\Delta m$. The moment of the strip about the x-axis is $\tilde{y}\,\Delta m$. Equations (4) and (5) then become

$$\bar{x} = \frac{M_y}{M} = \frac{\sum \tilde{x}\,\Delta m}{\sum \Delta m}, \qquad \bar{y} = \frac{M_x}{M} = \frac{\sum \tilde{y}\,\Delta m}{\sum \Delta m}.$$

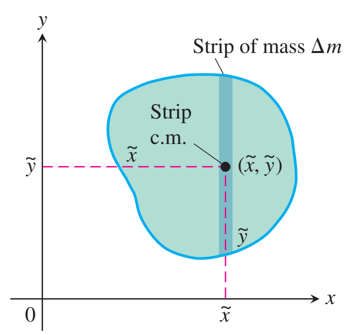


FIGURE 6.34 A plate cut into thin strips parallel to the y-axis. The moment exerted by a typical strip about each axis is the moment its mass Δm would exert if concentrated at the strip's center of mass $(\tilde{x}, \tilde{y})$.

As in the one-dimensional case, the sums are Riemann sums for integrals and approach these integrals as limiting values as the strips into which the plate is cut become narrower and narrower. We write these integrals symbolically as

$$\bar{x} = \frac{\int \tilde{x}\,dm}{\int dm} \quad \text{and} \quad \bar{y} = \frac{\int \tilde{y}\,dm}{\int dm}.$$

Moments, Mass, and Center of Mass of a Thin Plate Covering a Region in the xy-Plane

Moment about the x-axis: $\quad M_x = \int \tilde{y}\,dm$

Moment about the y-axis: $\quad M_y = \int \tilde{x}\,dm \qquad (6)$

Mass: $\quad M = \int dm$

Center of mass: $\quad \bar{x} = \frac{M_y}{M}, \quad \bar{y} = \frac{M_x}{M}$

To evaluate these integrals, we picture the plate in the coordinate plane and sketch a strip of mass parallel to one of the coordinates axes. We then express the strip's mass dm and the coordinates $(\tilde{x}, \tilde{y})$ of the strip's center of mass in terms of x or y. Finally, we integrate $\tilde{y}\,dm$, $\tilde{x}\,dm$, and dm between limits of integration determined by the plate's location in the plane.

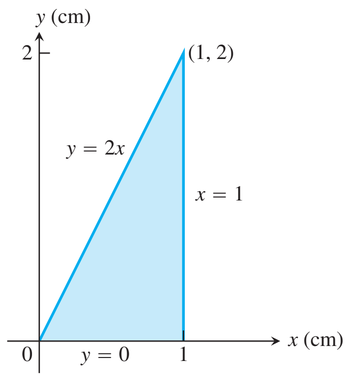


FIGURE 6.35 The plate in Example 3.

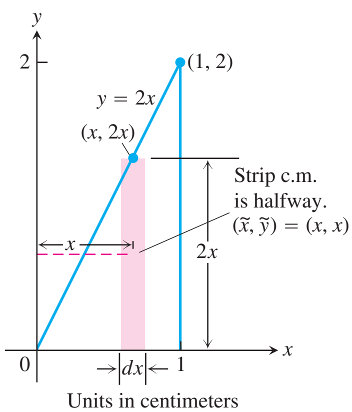


FIGURE 6.36 Modeling the plate in Example 3 with vertical strips.

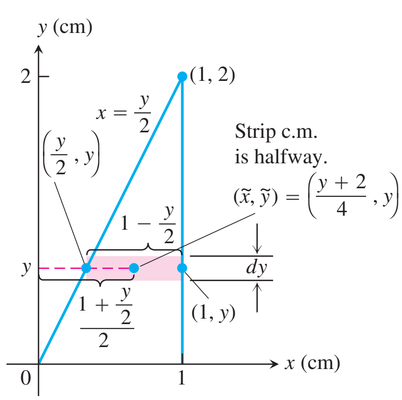


FIGURE 6.37 Modeling the plate in Example 3 with horizontal strips.

EXAMPLE 3 Constant-Density Plate

The triangular plate shown in Figure 6.35 has a constant density of $\delta = 3 \text{ g/cm}^2$. Find

(a) the plate's moment M_y about the y-axis.

(b) the plate's mass M.

(c) the x-coordinate of the plate's center of mass (c.m.).

Solution

Method 1: Vertical Strips (Figure 6.36)

(a) The moment M_y: The typical vertical strip has

center of mass (c.m.): $(\tilde{x}, \tilde{y}) = (x, x)$
length: $2x$
width: dx
area: $dA = 2x\,dx$
mass: $dm = \delta\,dA = 3 \cdot 2x\,dx = 6x\,dx$
distance of c.m. from y-axis: $\tilde{x} = x$.

The moment of the strip about the y-axis is

$$\tilde{x}\,dm = x \cdot 6x\,dx = 6x^2\,dx.$$

The moment of the plate about the y-axis is therefore

$$M_y = \int \tilde{x}\,dm = \int_0^1 6x^2\,dx = 2x^3\Big]_0^1 = 2 \text{ g} \cdot \text{cm}.$$

(b) The plate's mass:

$$M = \int dm = \int_0^1 6x\,dx = 3x^2\Big]_0^1 = 3 \text{ g}.$$

(c) The x-coordinate of the plate's center of mass:

$$\bar{x} = \frac{M_y}{M} = \frac{2 \text{ g} \cdot \text{cm}}{3 \text{ g}} = \frac{2}{3} \text{ cm}.$$

By a similar computation, we could find M_x and $\bar{y} = M_x/M$.

Method 2: Horizontal Strips (Figure 6.37)

(a) The moment M_y: The y-coordinate of the center of mass of a typical horizontal strip is y (see the figure), so

$$\tilde{y} = y.$$

The x-coordinate is the x-coordinate of the point halfway across the triangle. This makes it the average of $y/2$ (the strip's left-hand x-value) and 1 (the strip's right-hand x-value):

$$\tilde{x} = \frac{(y/2) + 1}{2} = \frac{y}{4} + \frac{1}{2} = \frac{y + 2}{4}.$$

We also have

$$\text{length:} \quad 1 - \frac{y}{2} = \frac{2-y}{2}$$

$$\text{width:} \quad dy$$

$$\text{area:} \quad dA = \frac{2-y}{2}\,dy$$

$$\text{mass:} \quad dm = \delta\, dA = 3 \cdot \frac{2-y}{2}\,dy$$

$$\text{distance of c.m. to } y\text{-axis:} \quad \tilde{x} = \frac{y+2}{4}.$$

The moment of the strip about the y-axis is

$$\tilde{x}\,dm = \frac{y+2}{4} \cdot 3 \cdot \frac{2-y}{2}\,dy = \frac{3}{8}(4-y^2)\,dy.$$

The moment of the plate about the y-axis is

$$M_y = \int \tilde{x}\,dm = \int_0^2 \frac{3}{8}(4-y^2)\,dy = \frac{3}{8}\left[4y - \frac{y^3}{3}\right]_0^2 = \frac{3}{8}\left(\frac{16}{3}\right) = 2\text{ g}\cdot\text{cm}.$$

(b) The plate's mass:

$$M = \int dm = \int_0^2 \frac{3}{2}(2-y)\,dy = \frac{3}{2}\left[2y - \frac{y^2}{2}\right]_0^2 = \frac{3}{2}(4-2) = 3\text{ g}.$$

(c) The x-coordinate of the plate's center of mass:

$$\bar{x} = \frac{M_y}{M} = \frac{2\text{ g}\cdot\text{cm}}{3\text{ g}} = \frac{2}{3}\text{ cm}.$$

By a similar computation, we could find M_x and $\bar{y}$. ■

How to Find a Plate's Center of Mass

1. Picture the plate in the xy-plane.
2. Sketch a strip of mass parallel to one of the coordinate axes and find its dimensions.
3. Find the strip's mass dm and center of mass $(\tilde{x}, \tilde{y})$.
4. Integrate $\tilde{y}\,dm$, $\tilde{x}\,dm$, and dm to find M_x, M_y, and M.
5. Divide the moments by the mass to calculate $\bar{x}$ and $\bar{y}$.

If the distribution of mass in a thin, flat plate has an axis of symmetry, the center of mass will lie on this axis. If there are two axes of symmetry, the center of mass will lie at their intersection. These facts often help to simplify our work.

You Try It

EXAMPLE 4 Constant-Density Plate

Find the center of mass of a thin plate of constant density δ covering the region bounded above by the parabola $y = 4 - x^2$ and below by the x-axis (Figure 6.38).

Solution Since the plate is symmetric about the y-axis and its density is constant, the distribution of mass is symmetric about the y-axis and the center of mass lies on the y-axis. Thus, $\bar{x} = 0$. It remains to find $\bar{y} = M_x/M$.

A trial calculation with horizontal strips (Figure 6.38a) leads to an inconvenient integration

$$M_x = \int_0^4 2\delta y\sqrt{4-y}\,dy.$$

We therefore model the distribution of mass with vertical strips instead (Figure 6.38b).

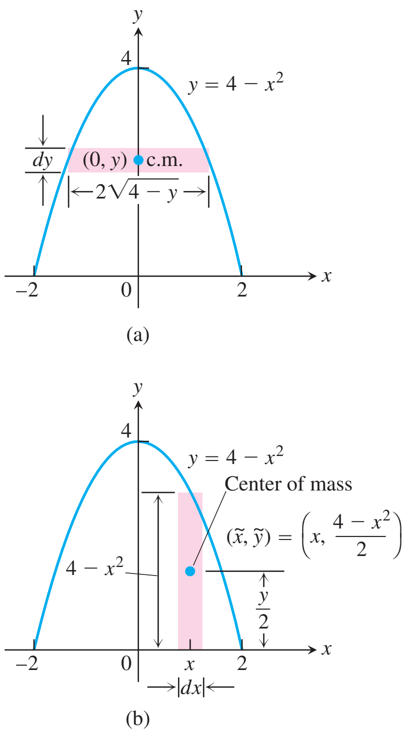


FIGURE 6.38 Modeling the plate in Example 4 with (a) horizontal strips leads to an inconvenient integration, so we model with (b) vertical strips instead.

The typical vertical strip has

center of mass (c.m.): $(\tilde{x}, \tilde{y}) = \left(x, \dfrac{4 - x^2}{2}\right)$

length: $4 - x^2$

width: dx

area: $dA = (4 - x^2)\,dx$

mass: $dm = \delta\,dA = \delta(4 - x^2)\,dx$

distance from c.m. to x-axis: $\tilde{y} = \dfrac{4 - x^2}{2}.$

The moment of the strip about the x-axis is

$$\tilde{y}\,dm = \frac{4 - x^2}{2} \cdot \delta(4 - x^2)\,dx = \frac{\delta}{2}(4 - x^2)^2\,dx.$$

The moment of the plate about the x-axis is

$$\begin{aligned} M_x &= \int \tilde{y}\,dm = \int_{-2}^{2} \frac{\delta}{2}(4 - x^2)^2\,dx \\ &= \frac{\delta}{2}\int_{-2}^{2}(16 - 8x^2 + x^4)\,dx = \frac{256}{15}\delta. \end{aligned} \tag{7}$$

The mass of the plate is

$$M = \int dm = \int_{-2}^{2} \delta(4 - x^2)\,dx = \frac{32}{3}\delta. \tag{8}$$

Therefore,

$$\bar{y} = \frac{M_x}{M} = \frac{(256/15)\,\delta}{(32/3)\,\delta} = \frac{8}{5}.$$

The plate's center of mass is the point

$$(\bar{x}, \bar{y}) = \left(0, \frac{8}{5}\right).$$

■

You Try It

EXAMPLE 5 Variable-Density Plate

Find the center of mass of the plate in Example 4 if the density at the point (x, y) is $\delta = 2x^2$, twice the square of the distance from the point to the y-axis.

Solution The mass distribution is still symmetric about the y-axis, so $\bar{x} = 0$. With $\delta = 2x^2$, Equations (7) and (8) become

$$M_x = \int \tilde{y}\, dm = \int_{-2}^{2} \frac{\delta}{2}(4 - x^2)^2\, dx = \int_{-2}^{2} x^2(4 - x^2)^2\, dx$$

$$= \int_{-2}^{2} (16x^2 - 8x^4 + x^6)\, dx = \frac{2048}{105} \tag{7'}$$

$$M = \int dm = \int_{-2}^{2} \delta(4 - x^2)\, dx = \int_{-2}^{2} 2x^2(4 - x^2)\, dx$$

$$= \int_{-2}^{2} (8x^2 - 2x^4)\, dx = \frac{256}{15}. \tag{8'}$$

Therefore,

$$\bar{y} = \frac{M_x}{M} = \frac{2048}{105} \cdot \frac{15}{256} = \frac{8}{7}.$$

The plate's new center of mass is

$$(\bar{x}, \bar{y}) = \left(0, \frac{8}{7}\right).$$

Video

EXAMPLE 6 Constant-Density Wire

Find the center of mass of a wire of constant density δ shaped like a semicircle of radius a.

Solution We model the wire with the semicircle $y = \sqrt{a^2 - x^2}$ (Figure 6.39). The distribution of mass is symmetric about the y-axis, so $\bar{x} = 0$. To find $\bar{y}$, we imagine the wire divided into short segments. The typical segment (Figure 6.39a) has

length: $ds = a\, d\theta$

mass: $dm = \delta\, ds = \delta a\, d\theta$ Mass per unit length times length

distance of c.m. to x-axis: $\tilde{y} = a \sin\theta$.

Hence,

$$\bar{y} = \frac{\int \tilde{y}\, dm}{\int dm} = \frac{\int_0^{\pi} a \sin\theta \cdot \delta a\, d\theta}{\int_0^{\pi} \delta a\, d\theta} = \frac{\delta a^2 \big[-\cos\theta\big]_0^{\pi}}{\delta a \pi} = \frac{2}{\pi} a.$$

The center of mass lies on the axis of symmetry at the point $(0, 2a/\pi)$, about two-thirds of the way up from the origin (Figure 6.39b).

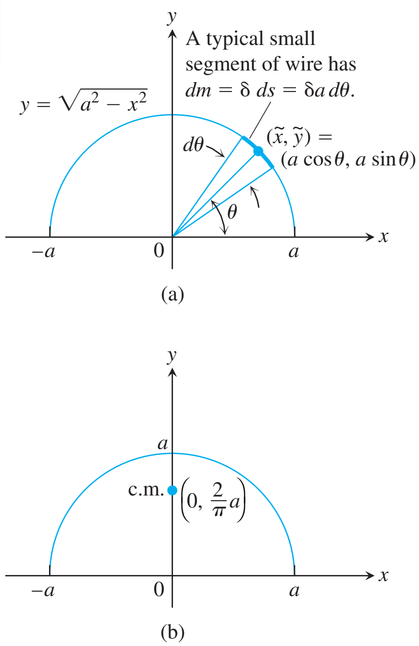


FIGURE 6.39 The semicircular wire in Example 6. (a) The dimensions and variables used in finding the center of mass. (b) The center of mass does not lie on the wire.

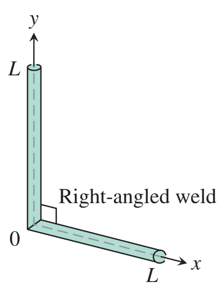
Centroids

When the density function is constant, it cancels out of the numerator and denominator of the formulas for $\bar{x}$ and $\bar{y}$. This happened in nearly every example in this section. As far as $\bar{x}$ and $\bar{y}$ were concerned, δ might as well have been 1. Thus, when the density is constant, the location of the center of mass is a feature of the geometry of the object and not of the material from which it is made. In such cases, engineers may call the center of mass the **centroid** of the shape, as in "Find the centroid of a triangle or a solid cone." To do so, just set δ equal to 1 and proceed to find $\bar{x}$ and $\bar{y}$ as before, by dividing moments by masses.

EXERCISES 6.4

Thin Rods

1. An 80-lb child and a 100-lb child are balancing on a seesaw. The 80-lb child is 5 ft from the fulcrum. How far from the fulcrum is the 100-lb child?
2. The ends of a log are placed on two scales. One scale reads 100 kg and the other 200 kg. Where is the log's center of mass?
3. The ends of two thin steel rods of equal length are welded together to make a right-angled frame. Locate the frame's center of mass. (*Hint:* Where is the center of mass of each rod?)



4. You weld the ends of two steel rods into a right-angled frame. One rod is twice the length of the other. Where is the frame's center of mass? (*Hint:* Where is the center of mass of each rod?)

Exercises 5–12 give density functions of thin rods lying along various intervals of the x-axis. Use Equations (3a) through (3c) to find each rod's moment about the origin, mass, and center of mass.

5. $\delta(x) = 4, \quad 0 \le x \le 2$
6. $\delta(x) = 4, \quad 1 \le x \le 3$
7. $\delta(x) = 1 + (x/3), \quad 0 \le x \le 3$
8. $\delta(x) = 2 - (x/4), \quad 0 \le x \le 4$
9. $\delta(x) = 1 + \left(1/\sqrt{x}\right), \quad 1 \le x \le 4$
10. $\delta(x) = 3(x^{-3/2} + x^{-5/2}), \quad 0.25 \le x \le 1$
11. $\delta(x) = \begin{cases} 2 - x, & 0 \le x < 1 \\ x, & 1 \le x \le 2 \end{cases}$
12. $\delta(x) = \begin{cases} x + 1, & 0 \le x < 1 \\ 2, & 1 \le x \le 2 \end{cases}$

Thin Plates with Constant Density

In Exercises 13–24, find the center of mass of a thin plate of constant density δ covering the given region.

13. The region bounded by the parabola $y = x^2$ and the line $y = 4$
14. The region bounded by the parabola $y = 25 - x^2$ and the x-axis
15. The region bounded by the parabola $y = x - x^2$ and the line $y = -x$
16. The region enclosed by the parabolas $y = x^2 - 3$ and $y = -2x^2$
17. The region bounded by the y-axis and the curve $x = y - y^3$, $0 \le y \le 1$
18. The region bounded by the parabola $x = y^2 - y$ and the line $y = x$
19. The region bounded by the x-axis and the curve $y = \cos x$, $-\pi/2 \le x \le \pi/2$
20. The region between the x-axis and the curve $y = \sec^2 x$, $-\pi/4 \le x \le \pi/4$
21. The region bounded by the parabolas $y = 2x^2 - 4x$ and $y = 2x - x^2$
22. **a.** The region cut from the first quadrant by the circle $x^2 + y^2 = 9$

    b. The region bounded by the x-axis and the semicircle $y = \sqrt{9 - x^2}$

    Compare your answer in part (b) with the answer in part (a).
23. The "triangular" region in the first quadrant between the circle $x^2 + y^2 = 9$ and the lines $x = 3$ and $y = 3$. (*Hint:* Use geometry to find the area.)
24. The region bounded above by the curve $y = 1/x^3$, below by the curve $y = -1/x^3$, and on the left and right by the lines $x = 1$ and $x = a > 1$. Also, find $\lim_{a\to\infty} \bar{x}$.

Thin Plates with Varying Density

25. Find the center of mass of a thin plate covering the region between the x-axis and the curve $y = 2/x^2$, $1 \le x \le 2$, if the plate's density at the point (x, y) is $\delta(x) = x^2$.
26. Find the center of mass of a thin plate covering the region bounded below by the parabola $y = x^2$ and above by the line $y = x$ if the plate's density at the point (x, y) is $\delta(x) = 12x$.
27. The region bounded by the curves $y = \pm 4/\sqrt{x}$ and the lines $x = 1$ and $x = 4$ is revolved about the y-axis to generate a solid.

    a. Find the volume of the solid.

    b. Find the center of mass of a thin plate covering the region if the plate's density at the point (x, y) is $\delta(x) = 1/x$.

    c. Sketch the plate and show the center of mass in your sketch.
28. The region between the curve $y = 2/x$ and the x-axis from $x = 1$ to $x = 4$ is revolved about the x-axis to generate a solid.

    a. Find the volume of the solid.

    b. Find the center of mass of a thin plate covering the region if the plate's density at the point (x, y) is $\delta(x) = \sqrt{x}$.

    c. Sketch the plate and show the center of mass in your sketch.

Centroids of Triangles

29. **The centroid of a triangle lies at the intersection of the triangle's medians** (*Figure 6.40a*) You may recall that the point

inside a triangle that lies one-third of the way from each side toward the opposite vertex is the point where the triangle's three medians intersect. Show that the centroid lies at the intersection of the medians by showing that it too lies one-third of the way from each side toward the opposite vertex. To do so, take the following steps.

i. Stand one side of the triangle on the x-axis as in Figure 6.40b. Express dm in terms of L and dy.

ii. Use similar triangles to show that $L = (b/h)(h - y)$. Substitute this expression for L in your formula for dm.

iii. Show that $\bar{y} = h/3$.

iv. Extend the argument to the other sides.

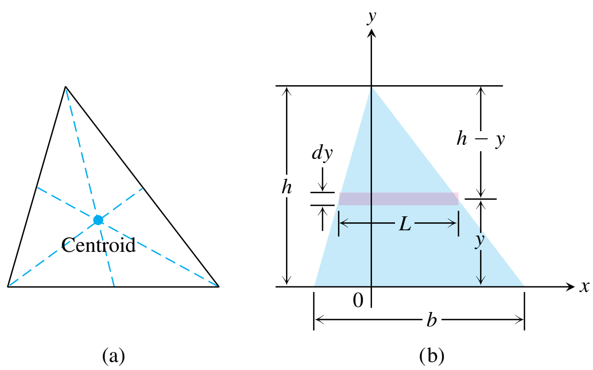


FIGURE 6.40 The triangle in Exercise 29. (a) The centroid. (b) The dimensions and variables to use in locating the center of mass.

Use the result in Exercise 29 to find the centroids of the triangles whose vertices appear in Exercises 30–34. Assume $a, b > 0$.

30. $(-1, 0), (1, 0), (0, 3)$

31. $(0, 0), (1, 0), (0, 1)$

32. $(0, 0), (a, 0), (0, a)$

33. $(0, 0), (a, 0), (0, b)$

34. $(0, 0), (a, 0), (a/2, b)$

Thin Wires

Exercises

35. Constant density Find the moment about the x-axis of a wire of constant density that lies along the curve $y = \sqrt{x}$ from $x = 0$ to $x = 2$.

36. Constant density Find the moment about the x-axis of a wire of constant density that lies along the curve $y = x^3$ from $x = 0$ to $x = 1$.

37. Variable density Suppose that the density of the wire in Example 6 is $\delta = k \sin \theta$ (k constant). Find the center of mass.

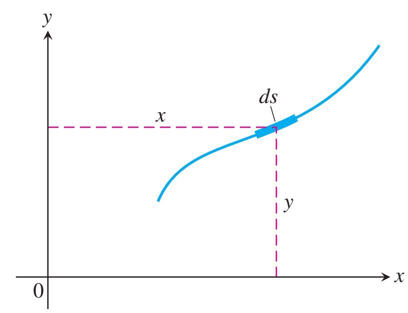
38. Variable density Suppose that the density of the wire in Example 6 is $\delta = 1 + k|\cos \theta|$ (k constant). Find the center of mass.

Engineering Formulas

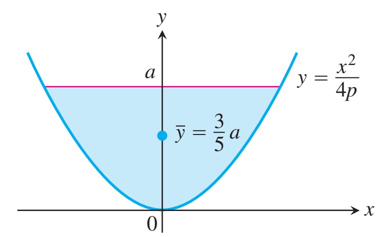
Verify the statements and formulas in Exercises 39–42.

39. The coordinates of the centroid of a differentiable plane curve are

$$\bar{x} = \frac{\int x\, ds}{\text{length}}, \qquad \bar{y} = \frac{\int y\, ds}{\text{length}}.$$

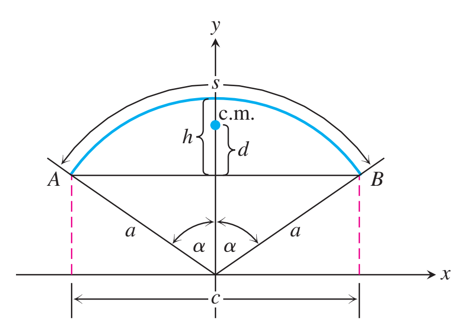


40. Whatever the value of $p > 0$ in the equation $y = x^2/(4p)$, the y-coordinate of the centroid of the parabolic segment shown here is $\bar{y} = (3/5)a$.



41. For wires and thin rods of constant density shaped like circular arcs centered at the origin and symmetric about the y-axis, the y-coordinate of the center of mass is

$$\bar{y} = \frac{a \sin \alpha}{\alpha} = \frac{ac}{s}.$$



42. (*Continuation of Exercise 41.*)

a. Show that when α is small, the distance d from the centroid to chord AB is about $2h/3$ (in the notation of the figure here) by taking the following steps.

i. Show that

$$\frac{d}{h} = \frac{\sin \alpha - \alpha \cos \alpha}{\alpha - \alpha \cos \alpha}. \qquad (9)$$

T **ii.** Graph

$$f(\alpha) = \frac{\sin \alpha - \alpha \cos \alpha}{\alpha - \alpha \cos \alpha}$$

and use the trace feature to show that $\lim_{\alpha \to 0^+} f(\alpha) \approx 2/3$.

b. The error (difference between d and $2h/3$) is small even for angles greater that 45°. See for yourself by evaluating the right-hand side of Equation (9) for $\alpha = 0.2, 0.4, 0.6, 0.8$, and 1.0 rad.

6.5 Areas of Surfaces of Revolution and the Theorems of Pappus

When you jump rope, the rope sweeps out a surface in the space around you called a *surface of revolution*. The "area" of this surface depends on the length of the rope and the distance of each of its segments from the axis of revolution. In this section we define areas of surfaces of revolution. More complicated surfaces are treated in Chapter 16.

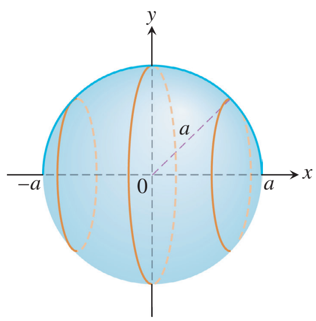


FIGURE 6.41 Rotating the semicircle $y = \sqrt{a^2 - x^2}$ of radius a with center at the origin generates a spherical surface with area $4\pi a^2$.

Defining Surface Area

We want our definition of the area of a surface of revolution to be consistent with known results from classical geometry for the surface areas of spheres, circular cylinders, and cones. So if the jump rope discussed in the introduction takes the shape of a semicircle with radius a rotated about the x-axis (Figure 6.41), it generates a sphere with surface area $4\pi a^2$.

Before considering general curves, we begin by rotating horizontal and slanted line segments about the x-axis. If we rotate the horizontal line segment AB having length Δx about the x-axis (Figure 6.42a), we generate a cylinder with surface area $2\pi y \Delta x$. This area is the same as that of a rectangle with side lengths Δx and $2\pi y$ (Figure 6.42b). The length $2\pi y$ is the circumference of the circle of radius y generated by rotating the point (x, y) on the line AB about the x-axis.

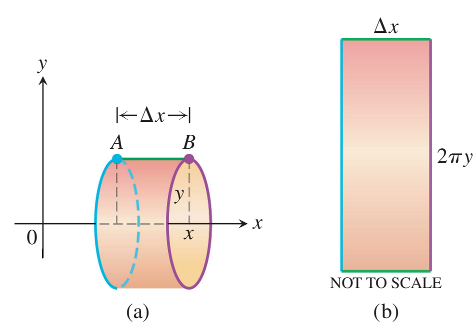


FIGURE 6.42 (a) A cylindrical surface generated by rotating the horizontal line segment AB of length Δx about the x-axis has area $2\pi y \Delta x$. (b) The cut and rolled out cylindrical surface as a rectangle.

Suppose the line segment AB has length Δs and is slanted rather than horizontal. Now when AB is rotated about the x-axis, it generates a frustum of a cone (Figure 6.43a). From classical geometry, the surface area of this frustum is $2\pi y^* \Delta s$, where $y^* = (y_1 + y_2)/2$ is the average height of the slanted segment AB above the x-axis. This surface area is the same as that of a rectangle with side lengths Δs and $2\pi y^*$ (Figure 6.43b).

Let's build on these geometric principles to define the area of a surface swept out by revolving more general curves about the x-axis. Suppose we want to find the area of the surface swept out by revolving the graph of a nonnegative continuous function $y = f(x)$, $a \le x \le b$, about the x-axis. We partition the closed interval $[a, b]$ in the usual way and use the points in the partition to subdivide the graph into short arcs. Figure 6.44 shows a typical arc PQ and the band it sweeps out as part of the graph of f.

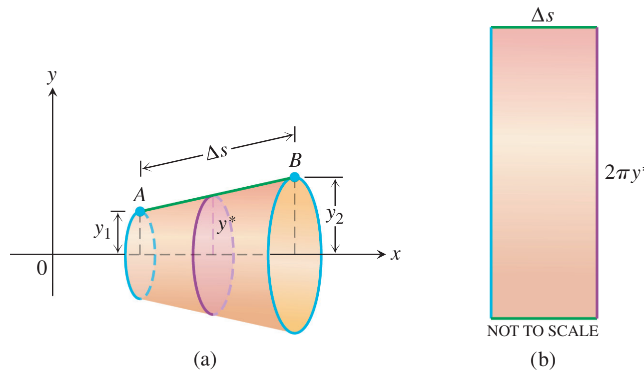


FIGURE 6.43 (a) The frustum of a cone generated by rotating the slanted line segment AB of length Δs about the x-axis has area $2\pi y^* \Delta s$. (b) The area of the rectangle for $y^* = \dfrac{y_1 + y_2}{2}$, the average height of AB above the x-axis.

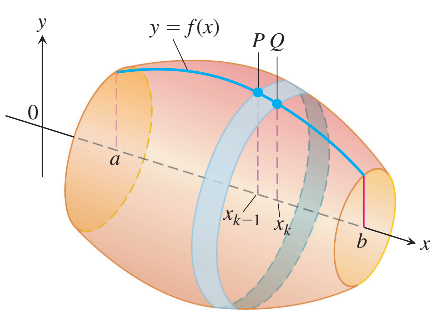


FIGURE 6.44 The surface generated by revolving the graph of a nonnegative function $y = f(x)$, $a \le x \le b$, about the x-axis. The surface is a union of bands like the one swept out by the arc PQ.

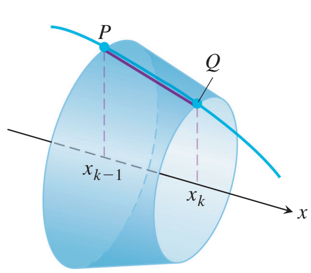


FIGURE 6.45 The line segment joining P and Q sweeps out a frustum of a cone.

As the arc PQ revolves about the x-axis, the line segment joining P and Q sweeps out a frustum of a cone whose axis lies along the x-axis (Figure 6.45). The surface area of this frustum approximates the surface area of the band swept out by the arc PQ. The surface area of the frustum of the cone shown in Figure 6.45 is $2\pi y^* L$, where y^* is the average height of the line segment joining P and Q, and L is its length (just as before). Since $f \ge 0$, from Figure 6.46 we see that the average height of the line segment is $y^* = (f(x_{k-1}) + f(x_k))/2$, and the slant length is $L = \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}$. Therefore,

$$\begin{aligned}\text{Frustum surface area} &= 2\pi \cdot \frac{f(x_{k-1}) + f(x_k)}{2} \cdot \sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}\\ &= \pi(f(x_{k-1}) + f(x_k))\sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}.\end{aligned}$$

The area of the original surface, being the sum of the areas of the bands swept out by arcs like arc PQ, is approximated by the frustum area sum

$$\sum_{k=1}^{n} \pi(f(x_{k-1}) + f(x_k))\sqrt{(\Delta x_k)^2 + (\Delta y_k)^2}. \tag{1}$$

We expect the approximation to improve as the partition of $[a, b]$ becomes finer. Moreover, if the function f is differentiable, then by the Mean Value Theorem, there is a point $(c_k, f(c_k))$ on the curve between P and Q where the tangent is parallel to the segment PQ (Figure 6.47). At this point,

$$f'(c_k) = \frac{\Delta y_k}{\Delta x_k},$$

$$\Delta y_k = f'(c_k)\,\Delta x_k.$$

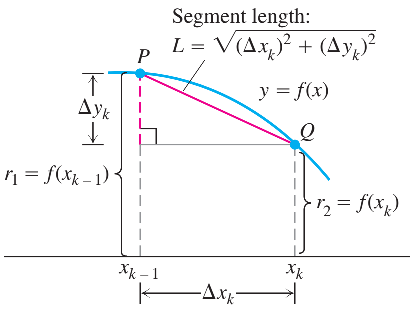


FIGURE 6.46 Dimensions associated with the arc and line segment PQ.

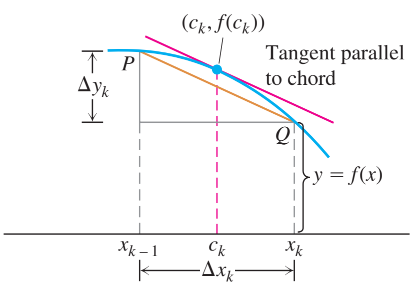


FIGURE 6.47 If f is smooth, the Mean Value Theorem guarantees the existence of a point c_k where the tangent is parallel to segment PQ.

With this substitution for Δy_k, the sums in Equation (1) take the form

$$\begin{aligned}\sum_{k=1}^{n} \pi(f(x_{k-1}) + f(x_k))\sqrt{(\Delta x_k)^2 + (f'(c_k)\,\Delta x_k)^2} \\ = \sum_{k=1}^{n} \pi(f(x_{k-1}) + f(x_k))\sqrt{1 + (f'(c_k))^2}\,\Delta x_k. \qquad (2)\end{aligned}$$

These sums are not the Riemann sums of any function because the points x_{k-1}, x_k, and c_k are not the same. However, a theorem from advanced calculus assures us that as the norm of the partition of $[a, b]$ goes to zero, the sums in Equation (2) converge to the integral

$$\int_a^b 2\pi f(x)\sqrt{1 + (f'(x))^2}\,dx.$$

We therefore define this integral to be the area of the surface swept out by the graph of f from a to b.

DEFINITION Surface Area for Revolution About the x-Axis

If the function $f(x) \geq 0$ is continuously differentiable on $[a, b]$, the **area** of the surface generated by revolving the curve $y = f(x)$ about the x-axis is

$$S = \int_a^b 2\pi y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx = \int_a^b 2\pi f(x)\sqrt{1 + (f'(x))^2}\,dx. \qquad (3)$$

The square root in Equation (3) is the same one that appears in the formula for the length of the generating curve in Equation (2) of Section 6.3.

Video You Try It

EXAMPLE 1 Applying the Surface Area Formula

Find the area of the surface generated by revolving the curve $y = 2\sqrt{x}$, $1 \leq x \leq 2$, about the x-axis (Figure 6.48).

Solution We evaluate the formula

$$S = \int_a^b 2\pi y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\,dx \qquad \text{Eq. (3)}$$

with

$$a = 1, \qquad b = 2, \qquad y = 2\sqrt{x}, \qquad \frac{dy}{dx} = \frac{1}{\sqrt{x}},$$

$$\begin{aligned}\sqrt{1 + \left(\frac{dy}{dx}\right)^2} &= \sqrt{1 + \left(\frac{1}{\sqrt{x}}\right)^2} \\ &= \sqrt{1 + \frac{1}{x}} = \sqrt{\frac{x+1}{x}} = \frac{\sqrt{x+1}}{\sqrt{x}}.\end{aligned}$$

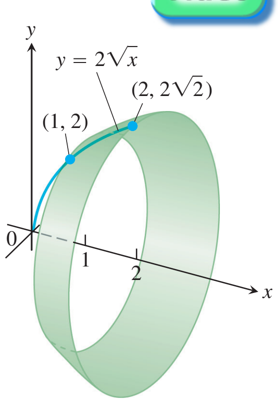


FIGURE 6.48 In Example 1 we calculate the area of this surface.

With these substitutions,

$$S = \int_1^2 2\pi \cdot 2\sqrt{x}\frac{\sqrt{x+1}}{\sqrt{x}}\,dx = 4\pi\int_1^2 \sqrt{x+1}\,dx$$
$$= 4\pi \cdot \frac{2}{3}(x+1)^{3/2}\Big]_1^2 = \frac{8\pi}{3}\left(3\sqrt{3} - 2\sqrt{2}\right).$$ ■

Revolution About the *y*-Axis

For revolution about the y-axis, we interchange x and y in Equation (3).

> **Surface Area for Revolution About the *y*-Axis**
> If $x = g(y) \geq 0$ is continuously differentiable on $[c, d]$, the area of the surface generated by revolving the curve $x = g(y)$ about the y-axis is
> $$S = \int_c^d 2\pi x\sqrt{1 + \left(\frac{dx}{dy}\right)^2}\,dy = \int_c^d 2\pi g(y)\sqrt{1 + (g'(y))^2}\,dy. \qquad (4)$$

You Try It

Video

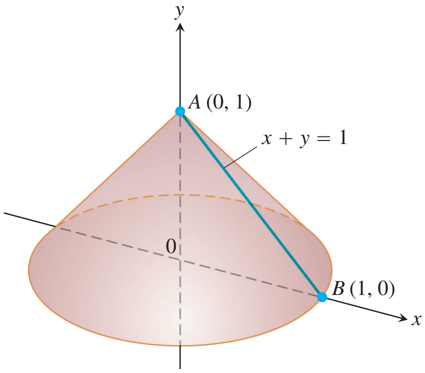


FIGURE 6.49 Revolving line segment AB about the y-axis generates a cone whose lateral surface area we can now calculate in two different ways (Example 2).

EXAMPLE 2 Finding Area for Revolution about the *y*-Axis

The line segment $x = 1 - y, 0 \leq y \leq 1$, is revolved about the y-axis to generate the cone in Figure 6.49. Find its lateral surface area (which excludes the base area).

Solution Here we have a calculation we can check with a formula from geometry:

$$\text{Lateral surface area} = \frac{\text{base circumference}}{2} \times \text{slant height} = \pi\sqrt{2}.$$

To see how Equation (4) gives the same result, we take

$$c = 0, \quad d = 1, \quad x = 1 - y, \quad \frac{dx}{dy} = -1,$$
$$\sqrt{1 + \left(\frac{dx}{dy}\right)^2} = \sqrt{1 + (-1)^2} = \sqrt{2}$$

and calculate

$$S = \int_c^d 2\pi x\sqrt{1 + \left(\frac{dx}{dy}\right)^2}\,dy = \int_0^1 2\pi(1 - y)\sqrt{2}\,dy$$
$$= 2\pi\sqrt{2}\left[y - \frac{y^2}{2}\right]_0^1 = 2\pi\sqrt{2}\left(1 - \frac{1}{2}\right)$$
$$= \pi\sqrt{2}.$$

The results agree, as they should. ■

Parametrized Curves

Regardless of the coordinate axis of revolution, the square roots appearing in Equations (3) and (4) are the same ones that appear in the formulas for arc length in Section 6.3. If the curve is parametrized by the equations $x = f(t)$ and $y = g(t)$, $a \le t \le b$, where f and g are continuously differentiable on $[a, b]$, then the corresponding square root appearing in the arc length formula is

$$\sqrt{[f'(t)]^2 + [g'(t)]^2} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}.$$

This observation leads to the following formulas for area of surfaces of revolution for smooth parametrized curves.

Surface Area of Revolution for Parametrized Curves

If a smooth curve $x = f(t)$, $y = g(t)$, $a \le t \le b$, is traversed exactly once as t increases from a to b, then the areas of the surfaces generated by revolving the curve about the coordinate axes are as follows.

1. Revolution about the x-axis $(y \ge 0)$:

$$S = \int_a^b 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \qquad (5)$$

2. Revolution about the y-axis $(x \ge 0)$:

$$S = \int_a^b 2\pi x \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \qquad (6)$$

As with length, we can calculate surface area from any convenient parametrization that meets the stated criteria.

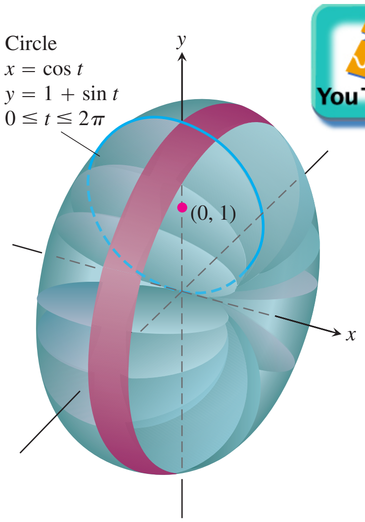


FIGURE 6.50 In Example 3 we calculate the area of the surface of revolution swept out by this parametrized curve.

EXAMPLE 3 Applying Surface Area Formula

The standard parametrization of the circle of radius 1 centered at the point (0, 1) in the xy-plane is

$$x = \cos t, \qquad y = 1 + \sin t, \qquad 0 \le t \le 2\pi.$$

Use this parametrization to find the area of the surface swept out by revolving the circle about the x-axis (Figure 6.50).

Solution We evaluate the formula

$$S = \int_a^b 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Eq. (5) for revolution about the x-axis; $y = 1 + \sin t > 0$

$$= \int_0^{2\pi} 2\pi(1 + \sin t)\sqrt{\underbrace{(-\sin t)^2 + (\cos t)^2}_{1}}\, dt$$

$$= 2\pi \int_0^{2\pi} (1 + \sin t)\, dt$$

$$= 2\pi \Big[t - \cos t\Big]_0^{2\pi} = 4\pi^2.$$

The Differential Form

The equations

$$S = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \quad \text{and} \quad S = \int_c^d 2\pi x \sqrt{\left(\frac{dx}{dy}\right)^2}\, dy$$

are often written in terms of the arc length differential $ds = \sqrt{dx^2 + dy^2}$ as

$$S = \int_a^b 2\pi y\, ds \quad \text{and} \quad S = \int_c^d 2\pi x\, ds.$$

In the first of these, y is the distance from the x-axis to an element of arc length ds. In the second, x is the distance from the y-axis to an element of arc length ds. Both integrals have the form

$$\boxed{S = \int 2\pi(\text{radius})(\text{band width}) = \int 2\pi\rho\, ds} \tag{7}$$

where ρ is the radius from the axis of revolution to an element of arc length ds (Figure 6.51).

In any particular problem, you would then express the radius function ρ and the arc length differential ds in terms of a common variable and supply limits of integration for that variable.

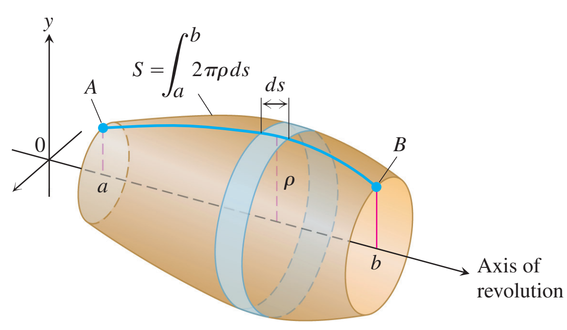


FIGURE 6.51 The area of the surface swept out by revolving arc AB about the axis shown here is $\int_a^b 2\pi\rho\, ds$. The exact expression depends on the formulas for ρ and ds.

EXAMPLE 4 Using the Differential Form for Surface Areas

Find the area of the surface generated by revolving the curve $y = x^3$, $0 \le x \le 1/2$, about the x-axis (Figure 6.52).

Solution We start with the short differential form:

$$\begin{aligned} S &= \int 2\pi\rho\, ds \\ &= \int 2\pi y\, ds && \text{For revolution about the } x\text{-axis, the radius function is } \rho = y > 0 \text{ on } 0 \le x \le 1/2. \\ &= \int 2\pi y\sqrt{dx^2 + dy^2}. && ds = \sqrt{dx^2 + dy^2} \end{aligned}$$

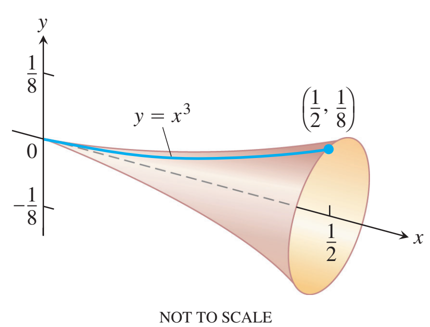


FIGURE 6.52 The surface generated by revolving the curve $y = x^3$, $0 \le x \le 1/2$, about the x-axis could be the design for a champagne glass (Example 4).

We then decide whether to express dy in terms of dx or dx in terms of dy. The original form of the equation, $y = x^3$, makes it easier to express dy in terms of dx, so we continue the calculation with

$$y = x^3, \qquad dy = 3x^2\,dx, \qquad \text{and} \qquad \sqrt{dx^2 + dy^2} = \sqrt{dx^2 + (3x^2\,dx)^2}$$
$$= \sqrt{1 + 9x^4}\,dx.$$

With these substitutions, x becomes the variable of integration and

$$\begin{aligned}
S &= \int_{x=0}^{x=1/2} 2\pi y\sqrt{dx^2 + dy^2} \\
&= \int_0^{1/2} 2\pi x^3\sqrt{1 + 9x^4}\,dx \\
&= 2\pi\left(\frac{1}{36}\right)\left(\frac{2}{3}\right)(1 + 9x^4)^{3/2}\Big]_0^{1/2} \\
&= \frac{\pi}{27}\left[\left(1 + \frac{9}{16}\right)^{3/2} - 1\right] \\
&= \frac{\pi}{27}\left[\left(\frac{25}{16}\right)^{3/2} - 1\right] = \frac{\pi}{27}\left(\frac{125}{64} - 1\right) \\
&= \frac{61\pi}{1728}.
\end{aligned}$$

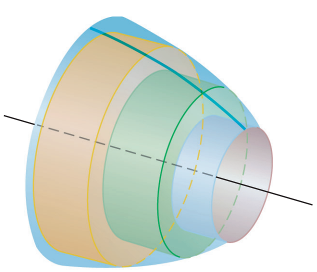
Substitute $u = 1 + 9x^4$, $du/36 = x^3\,dx$; integrate, and substitute back.

Cylindrical Versus Conical Bands

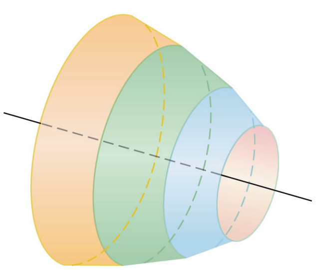
Why not find the surface area by approximating with cylindrical bands instead of conical bands, as suggested in Figure 6.53? The Riemann sums we get this way converge just as nicely as the ones based on conical bands, and the resulting integral is simpler. For revolution about the x-axis in this case, the radius in Equation (7) is $\rho = y$ and the band width is $ds = dx$. This leads to the integral formula

$$S = \int_a^b 2\pi f(x)\,dx \tag{8}$$

rather than the defining Equation (3). The problem with this new formula is that it fails to give results consistent with the surface area formulas from classical geometry, and that was one of our stated goals at the outset. Just because we end up with a nice-looking integral from a Riemann sum derivation does not mean it will calculate what we intend. (See Exercise 40.)



(a)



(b)

FIGURE 6.53 Why not use (a) cylindrical bands instead of (b) conical bands to approximate surface area?

CAUTION Do not use Equation (8) to calculate surface area. It does *not* give the correct result.

The Theorems of Pappus

In the third century, an Alexandrian Greek named Pappus discovered two formulas that relate centroids to surfaces and solids of revolution. The formulas provide shortcuts to a number of otherwise lengthy calculations.

THEOREM 1 Pappus's Theorem for Volumes

If a plane region is revolved once about a line in the plane that does not cut through the region's interior, then the volume of the solid it generates is equal to the region's area times the distance traveled by the region's centroid during the revolution. If ρ is the distance from the axis of revolution to the centroid, then

$$V = 2\pi\rho A. \tag{9}$$

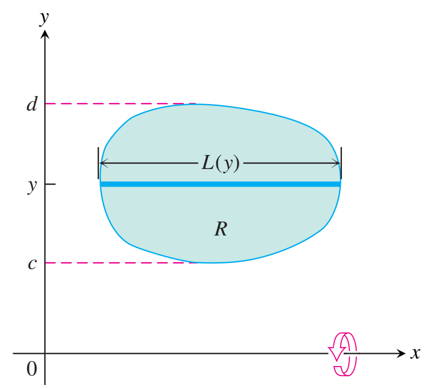


FIGURE 6.54 The region R is to be revolved (once) about the x-axis to generate a solid. A 1700-year-old theorem says that the solid's volume can be calculated by multiplying the region's area by the distance traveled by its centroid during the revolution.

Proof We draw the axis of revolution as the x-axis with the region R in the first quadrant (Figure 6.54). We let $L(y)$ denote the length of the cross-section of R perpendicular to the y-axis at y. We assume $L(y)$ to be continuous.

By the method of cylindrical shells, the volume of the solid generated by revolving the region about the x-axis is

$$V = \int_c^d 2\pi(\text{shell radius})(\text{shell height})\,dy = 2\pi\int_c^d y\,L(y)\,dy. \tag{10}$$

The y-coordinate of R's centroid is

$$\bar{y} = \frac{\int_c^d \tilde{y}\,dA}{A} = \frac{\int_c^d y\,L(y)\,dy}{A}, \qquad \tilde{y} = y,\ dA = L(y)dy$$

so that

$$\int_c^d y\,L(y)\,dy = A\bar{y}.$$

Substituting $A\bar{y}$ for the last integral in Equation (10) gives $V = 2\pi\bar{y}A$. With ρ equal to $\bar{y}$, we have $V = 2\pi\rho A$. ■

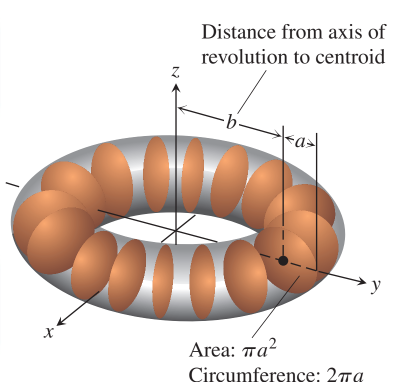


FIGURE 6.55 With Pappus's first theorem, we can find the volume of a torus without having to integrate (Example 5).

EXAMPLE 5 Volume of a Torus

The volume of the torus (doughnut) generated by revolving a circular disk of radius a about an axis in its plane at a distance $b \geq a$ from its center (Figure 6.55) is

$$V = 2\pi(b)(\pi a^2) = 2\pi^2 ba^2.$$ ■

EXAMPLE 6 Locate the Centroid of a Semicircular Region

Solution We model the region as the region between the semicircle $y = \sqrt{a^2 - x^2}$ (Figure 6.56) and the x-axis and imagine revolving the region about the x-axis to generate a solid sphere. By symmetry, the x-coordinate of the centroid is $\bar{x} = 0$. With $\bar{y} = \rho$ in Equation (9), we have

$$\bar{y} = \frac{V}{2\pi A} = \frac{(4/3)\pi a^3}{2\pi(1/2)\pi a^2} = \frac{4}{3\pi}a.$$ ■

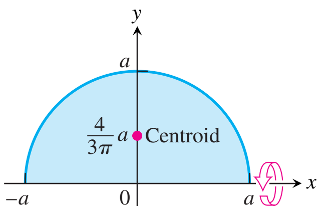


FIGURE 6.56 With Pappus's first theorem, we can locate the centroid of a semicircular region without having to integrate (Example 6).

THEOREM 2 Pappus's Theorem for Surface Areas

If an arc of a smooth plane curve is revolved once about a line in the plane that does not cut through the arc's interior, then the area of the surface generated by the arc equals the length of the arc times the distance traveled by the arc's centroid during the revolution. If ρ is the distance from the axis of revolution to the centroid, then

$$S = 2\pi\rho L. \tag{11}$$

The proof we give assumes that we can model the axis of revolution as the x-axis and the arc as the graph of a continuously differentiable function of x.

Proof We draw the axis of revolution as the x-axis with the arc extending from $x = a$ to $x = b$ in the first quadrant (Figure 6.57). The area of the surface generated by the arc is

$$S = \int_{x=a}^{x=b} 2\pi y\, ds = 2\pi \int_{x=a}^{x=b} y\, ds. \tag{12}$$

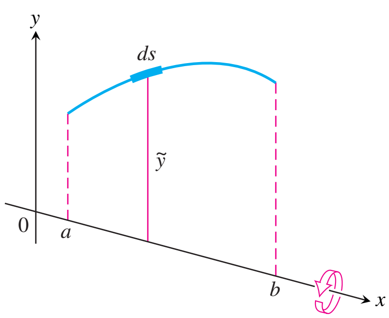


FIGURE 6.57 Figure for proving Pappus's area theorem.

The y-coordinate of the arc's centroid is

$$\bar{y} = \frac{\int_{x=a}^{x=b} \tilde{y}\, ds}{\int_{x=a}^{x=b} ds} = \frac{\int_{x=a}^{x=b} y\, ds}{L}.$$

$L = \int ds$ is the arc's length and $\tilde{y} = y$.

Hence

$$\int_{x=a}^{x=b} y\, ds = \bar{y}L.$$

Substituting $\bar{y}L$ for the last integral in Equation (12) gives $S = 2\pi\bar{y}L$. With ρ equal to $\bar{y}$, we have $S = 2\pi\rho L$. ■

EXAMPLE 7 Surface Area of a Torus

You Try It

The surface area of the torus in Example 5 is

$$S = 2\pi(b)(2\pi a) = 4\pi^2 ba.$$ ■

EXERCISES 6.5

Finding Integrals for Surface Area

In Exercises 1–8:

a. Set up an integral for the area of the surface generated by revolving the given curve about the indicated axis.

T **b.** Graph the curve to see what it looks like. If you can, graph the surface, too.

T **c.** Use your grapher's or computer's integral evaluator to find the surface's area numerically.

Exercises

1. $y = \tan x, \quad 0 \le x \le \pi/4; \quad x\text{-axis}$

2. $y = x^2, \quad 0 \le x \le 2; \quad x\text{-axis}$

3. $xy = 1, \quad 1 \le y \le 2; \quad y\text{-axis}$

4. $x = \sin y, \quad 0 \le y \le \pi; \quad y\text{-axis}$

5. $x^{1/2} + y^{1/2} = 3$ from $(4, 1)$ to $(1, 4)$; x-axis

6. $y + 2\sqrt{y} = x, \quad 1 \le y \le 2; \quad y\text{-axis}$

7. $x = \displaystyle\int_0^y \tan t\, dt, \quad 0 \le y \le \pi/3; \quad y\text{-axis}$

8. $y = \displaystyle\int_1^x \sqrt{t^2 - 1}\, dt, \quad 1 \le x \le \sqrt{5}; \quad x\text{-axis}$

Finding Surface Areas

Exercises

9. Find the lateral (side) surface area of the cone generated by revolving the line segment $y = x/2, 0 \le x \le 4$, about the x-axis. Check your answer with the geometry formula

$$\text{Lateral surface area} = \frac{1}{2} \times \text{base circumference} \times \text{slant height}.$$

10. Find the lateral surface area of the cone generated by revolving the line segment $y = x/2, 0 \le x \le 4$ about the y-axis. Check your answer with the geometry formula

$$\text{Lateral surface area} = \frac{1}{2} \times \text{base circumference} \times \text{slant height}.$$

11. Find the surface area of the cone frustum generated by revolving the line segment $y = (x/2) + (1/2), 1 \le x \le 3$, about the x-axis. Check your result with the geometry formula

$$\text{Frustum surface area} = \pi(r_1 + r_2) \times \text{slant height}.$$

12. Find the surface area of the cone frustum generated by revolving the line segment $y = (x/2) + (1/2), 1 \le x \le 3$, about the y-axis. Check your result with the geometry formula

$$\text{Frustum surface area} = \pi(r_1 + r_2) \times \text{slant height}.$$

Find the areas of the surfaces generated by revolving the curves in Exercises 13–22 about the indicated axes. If you have a grapher, you may want to graph these curves to see what they look like.

Exercises

13. $y = x^3/9, \quad 0 \le x \le 2; \quad x$-axis

14. $y = \sqrt{x}, \quad 3/4 \le x \le 15/4; \quad x$-axis

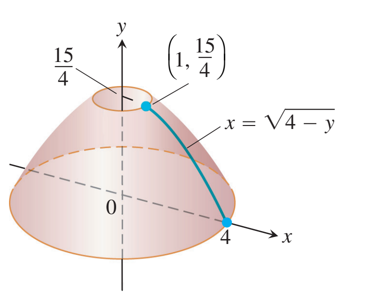
15. $y = \sqrt{2x - x^2}, \quad 0.5 \le x \le 1.5; \quad x$-axis

16. $y = \sqrt{x + 1}, \quad 1 \le x \le 5; \quad x$-axis

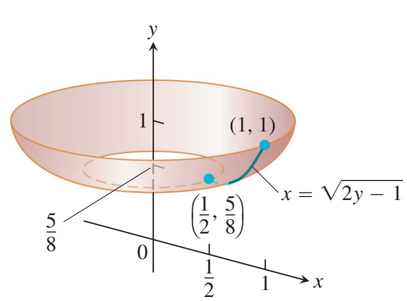
17. $x = y^3/3, \quad 0 \le y \le 1; \quad y$-axis

18. $x = (1/3)y^{3/2} - y^{1/2}, \quad 1 \le y \le 3; \quad y$-axis

19. $x = 2\sqrt{4 - y}, \quad 0 \le y \le 15/4; \quad y$-axis



20. $x = \sqrt{2y - 1}, \quad 5/8 \le y \le 1; \quad y$-axis



21. $x = (y^4/4) + 1/(8y^2), \quad 1 \le y \le 2; \quad x$-axis (*Hint:* Express $ds = \sqrt{dx^2 + dy^2}$ in terms of dy, and evaluate the integral $S = \int 2\pi y \, ds$ with appropriate limits.)

Exercises

22. $y = (1/3)(x^2 + 2)^{3/2}, \quad 0 \le x \le \sqrt{2}; \quad y$-axis (*Hint:* Express $ds = \sqrt{dx^2 + dy^2}$ in terms of dx, and evaluate the integral $S = \int 2\pi x \, ds$ with appropriate limits.)

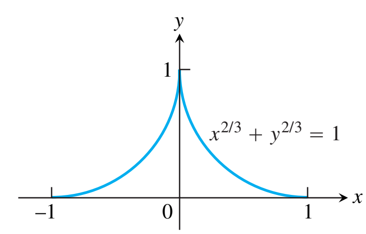
23. Testing the new definition Show that the surface area of a sphere of radius a is still $4\pi a^2$ by using Equation (3) to find the area of the surface generated by revolving the curve $y = \sqrt{a^2 - x^2}, -a \le x \le a$, about the x-axis.

24. Testing the new definition The lateral (side) surface area of a cone of height h and base radius r should be $\pi r\sqrt{r^2 + h^2}$, the semiperimeter of the base times the slant height. Show that this is still the case by finding the area of the surface generated by revolving the line segment $y = (r/h)x, 0 \le x \le h$, about the x-axis.

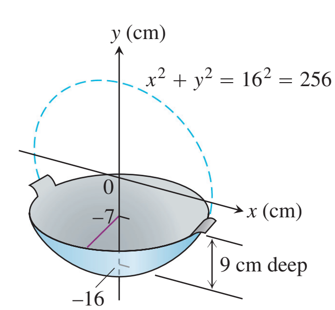
Exercises

25. Write an integral for the area of the surface generated by revolving the curve $y = \cos x, -\pi/2 \le x \le \pi/2$, about the x-axis. In Section 8.5 we will see how to evaluate such integrals.

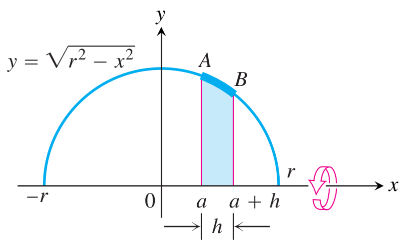
26. The surface of an astroid Find the area of the surface generated by revolving about the x-axis the portion of the astroid $x^{2/3} + y^{2/3} = 1$ shown here. (*Hint:* Revolve the first-quadrant portion $y = (1 - x^{2/3})^{3/2}, 0 \le x \le 1$, about the x-axis and double your result.)



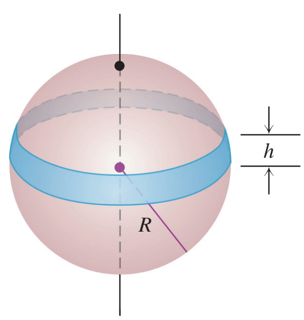
T **27. Enameling woks** Your company decided to put out a deluxe version of the successful wok you designed in Section 6.1, Exercise 55. The plan is to coat it inside with white enamel and outside with blue enamel. Each enamel will be sprayed on 0.5 mm thick before baking. (See diagram here.) Your manufacturing department wants to know how much enamel to have on hand for a production run of 5000 woks. What do you tell them? (Neglect waste and unused material and give your answer in liters. Remember that $1 \text{ cm}^3 = 1 \text{ mL}$, so $1 \text{ L} = 1000 \text{ cm}^3$.)



28. Slicing bread Did you know that if you cut a spherical loaf of bread into slices of equal width, each slice will have the same amount of crust? To see why, suppose the semicircle $y = \sqrt{r^2 - x^2}$ shown here is revolved about the x-axis to generate a sphere. Let AB be an arc of the semicircle that lies above an interval of length h on the x-axis. Show that the area swept out by AB does not depend on the location of the interval. (It does depend on the length of the interval.)



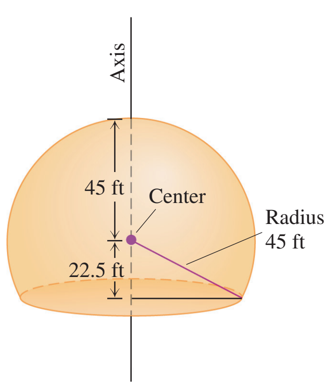
29. The shaded band shown here is cut from a sphere of radius R by parallel planes h units apart. Show that the surface area of the band is $2\pi Rh$.



30. Here is a schematic drawing of the 90-ft dome used by the U.S. National Weather Service to house radar in Bozeman, Montana.

a. How much outside surface is there to paint (not counting the bottom)?

T **b.** Express the answer to the nearest square foot.



31. Surfaces generated by curves that cross the axis of revolution The surface area formula in Equation (3) was developed under the assumption that the function f whose graph generated the surface was nonnegative over the interval $[a, b]$. For curves that cross the axis of revolution, we replace Equation (3) with the absolute value formula

$$S = \int 2\pi\rho\, ds = \int 2\pi\,|f(x)|\, ds. \tag{13}$$

Use Equation (13) to find the surface area of the double cone generated by revolving the line segment $y = x$, $-1 \le x \le 2$, about the x-axis.

32. (*Continuation of Exercise 31.*) Find the area of the surface generated by revolving the curve $y = x^3/9$, $-\sqrt{3} \le x \le \sqrt{3}$, about the x-axis. What do you think will happen if you drop the absolute value bars from Equation (13) and attempt to find the surface area with the formula $S = \int 2\pi f(x)\, ds$ instead? Try it.

Parametrizations

Find the areas of the surfaces generated by revolving the curves in Exercises 33–35 about the indicated axes.

33. $x = \cos t$, $y = 2 + \sin t$, $0 \le t \le 2\pi$; x-axis

34. $x = (2/3)t^{3/2}$, $y = 2\sqrt{t}$, $0 \le t \le \sqrt{3}$; y-axis

35. $x = t + \sqrt{2}$, $y = (t^2/2) + \sqrt{2}t$, $-\sqrt{2} \le t \le \sqrt{2}$; y-axis

36. Set up, but do not evaluate, an integral that represents the area of the surface obtained by rotating the curve $x = a(t - \sin t)$, $y = a(1 - \cos t)$, $0 \le t \le 2\pi$, about the x-axis.

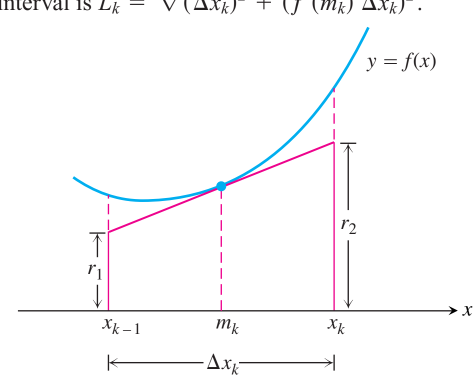
37. A cone frustum The line segment joining the points (0, 1) and (2, 2) is revolved about the x-axis to generate a frustum of a cone. Find the surface area of the frustum using the parametrization $x = 2t$, $y = t + 1$, $0 \le t \le 1$. Check your result with the geometry formula: Area $= \pi(r_1 + r_2)$(slant height).

38. A cone The line segment joining the origin to the point (h, r) is revolved about the x-axis to generate a cone of height h and base radius r. Find the cone's surface area with the parametric equations $x = ht$, $y = rt$, $0 \le t \le 1$. Check your result with the geometry formula: Area $= \pi r$(slant height).

39. An alternative derivation of the surface area formula Assume f is smooth on $[a, b]$ and partition $[a, b]$ in the usual way. In the kth subinterval $[x_{k-1}, x_k]$ construct the tangent line to the curve at the midpoint $m_k = (x_{k-1} + x_k)/2$, as in the figure here.

a. Show that $r_1 = f(m_k) - f'(m_k)\dfrac{\Delta x_k}{2}$ and $r_2 = f(m_k) + f'(m_k)\dfrac{\Delta x_k}{2}$.

b. Show that the length L_k of the tangent line segment in the kth subinterval is $L_k = \sqrt{(\Delta x_k)^2 + (f'(m_k)\,\Delta x_k)^2}$.

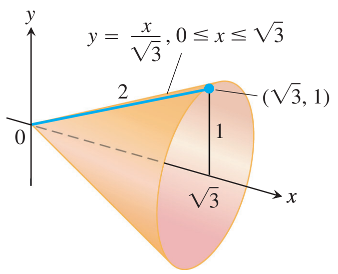


c. Show that the lateral surface area of the frustum of the cone swept out by the tangent line segment as it revolves about the x-axis is $2\pi f(m_k)\sqrt{1 + (f'(m_k))^2}\ \Delta x_k$.

d. Show that the area of the surface generated by revolving $y = f(x)$ about the x-axis over $[a, b]$ is

$$\lim_{n\to\infty} \sum_{k=1}^{n} \begin{pmatrix}\text{lateral surface area}\\ \text{of } k\text{th frustum}\end{pmatrix} = \int_a^b 2\pi f(x)\sqrt{1 + (f'(x))^2}\,dx.$$

40. Modeling surface area The lateral surface area of the cone swept out by revolving the line segment $y = x/\sqrt{3},\ 0 \le x \le \sqrt{3}$, about the x-axis should be $(1/2)(\text{base circumference})(\text{slant height}) = (1/2)(2\pi)(2) = 2\pi$. What do you get if you use Equation (8) with $f(x) = x/\sqrt{3}$?



The Theorems of Pappus

41. The square region with vertices (0, 2), (2, 0), (4, 2), and (2, 4) is revolved about the x-axis to generate a solid. Find the volume and surface area of the solid.

42. Use a theorem of Pappus to find the volume generated by revolving about the line $x = 5$ the triangular region bounded by the coordinate axes and the line $2x + y = 6$. (As you saw in Exercise 29 of Section 6.4, the centroid of a triangle lies at the intersection of the medians, one-third of the way from the midpoint of each side toward the opposite vertex.)

43. Find the volume of the torus generated by revolving the circle $(x - 2)^2 + y^2 = 1$ about the y-axis.

44. Use the theorems of Pappus to find the lateral surface area and the volume of a right circular cone.

45. Use the Second Theorem of Pappus and the fact that the surface area of a sphere of radius a is $4\pi a^2$ to find the centroid of the semicircle $y = \sqrt{a^2 - x^2}$.

46. As found in Exercise 45, the centroid of the semicircle $y = \sqrt{a^2 - x^2}$ lies at the point $(0, 2a/\pi)$. Find the area of the surface swept out by revolving the semicircle about the line $y = a$.

47. The area of the region R enclosed by the semiellipse $y = (b/a)\sqrt{a^2 - x^2}$ and the x-axis is $(1/2)\pi ab$ and the volume of the ellipsoid generated by revolving R about the x-axis is $(4/3)\pi ab^2$. Find the centroid of R. Notice that the location is independent of a.

48. As found in Example 6, the centroid of the region enclosed by the x-axis and the semicircle $y = \sqrt{a^2 - x^2}$ lies at the point $(0, 4a/3\pi)$. Find the volume of the solid generated by revolving this region about the line $y = -a$.

49. The region of Exercise 48 is revolved about the line $y = x - a$ to generate a solid. Find the volume of the solid.

50. As found in Exercise 45, the centroid of the semicircle $y = \sqrt{a^2 - x^2}$ lies at the point $(0, 2a/\pi)$. Find the area of the surface generated by revolving the semicircle about the line $y = x - a$.

51. Find the moment about the x-axis of the semicircular region in Example 6. If you use results already known, you will not need to integrate.

6.6 Work

In everyday life, *work* means an activity that requires muscular or mental effort. In science, the term refers specifically to a force acting on a body and the body's subsequent displacement. This section shows how to calculate work. The applications run from compressing railroad car springs and emptying subterranean tanks to forcing electrons together and lifting satellites into orbit.

Work Done by a Constant Force

When a body moves a distance d along a straight line as a result of being acted on by a force of constant magnitude F in the direction of motion, we define the **work** W done by the force on the body with the formula

$$W = Fd \qquad \text{(Constant-force formula for work).} \tag{1}$$

Joules
The joule, abbreviated J and pronounced "jewel," is named after the English physicist James Prescott Joule (1818–1889). The defining equation is

$$1 \text{ joule} = (1 \text{ newton})(1 \text{ meter}).$$

In symbols, $1 \text{ J} = 1 \text{ N} \cdot \text{m}$.

From Equation (1) we see that the unit of work in any system is the unit of force multiplied by the unit of distance. In SI units (SI stands for *Système International*, or International System), the unit of force is a newton, the unit of distance is a meter, and the unit of work is a newton-meter $(\text{N} \cdot \text{m})$. This combination appears so often it has a special name, the **joule**. In the British system, the unit of work is the foot-pound, a unit frequently used by engineers.

EXAMPLE 1 Jacking Up a Car

If you jack up the side of a 2000-lb car 1.25 ft to change a tire (you have to apply a constant vertical force of about 1000 lb) you will perform $1000 \times 1.25 = 1250$ ft-lb of work on the car. In SI units, you have applied a force of 4448 N through a distance of 0.381 m to do $4448 \times 0.381 \approx 1695$ J of work. ■

Work Done by a Variable Force Along a Line

If the force you apply varies along the way, as it will if you are compressing a spring, the formula $W = Fd$ has to be replaced by an integral formula that takes the variation in F into account.

Suppose that the force performing the work acts along a line that we take to be the x-axis and that its magnitude F is a continuous function of the position. We want to find the work done over the interval from $x = a$ to $x = b$. We partition $[a, b]$ in the usual way and choose an arbitrary point c_k in each subinterval $[x_{k-1}, x_k]$. If the subinterval is short enough, F, being continuous, will not vary much from x_{k-1} to x_k. The amount of work done across the interval will be about $F(c_k)$ times the distance Δx_k, the same as it would be if F were constant and we could apply Equation (1). The total work done from a to b is therefore approximated by the Riemann sum

$$\text{Work} \approx \sum_{k=1}^{n} F(c_k)\, \Delta x_k .$$

We expect the approximation to improve as the norm of the partition goes to zero, so we define the work done by the force from a to b to be the integral of F from a to b.

DEFINITION Work
The **work** done by a variable force $F(x)$ directed along the x-axis from $x = a$ to $x = b$ is

$$W = \int_a^b F(x)\, dx . \qquad (2)$$

The units of the integral are joules if F is in newtons and x is in meters, and foot-pounds if F is in pounds and x in feet. So, the work done by a force of $F(x) = 1/x^2$ newtons along the x-axis from $x = 1$ m to $x = 10$ m is

$$W = \int_1^{10} \frac{1}{x^2}\, dx = -\frac{1}{x}\bigg]_1^{10} = -\frac{1}{10} + 1 = 0.9 \text{ J}.$$

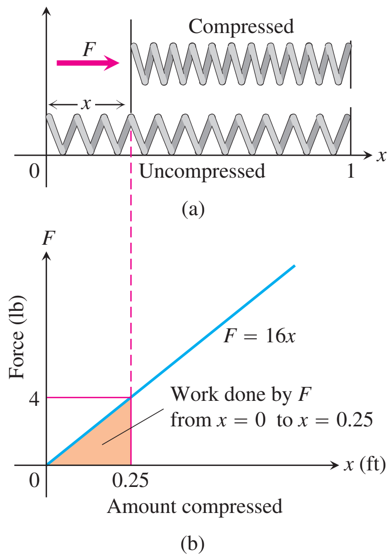


FIGURE 6.58 The force F needed to hold a spring under compression increases linearly as the spring is compressed (Example 2).

Hooke's Law for Springs: $F = kx$

Hooke's Law says that the force it takes to stretch or compress a spring x length units from its natural (unstressed) length is proportional to x. In symbols,

$$F = kx. \tag{3}$$

The constant k, measured in force units per unit length, is a characteristic of the spring, called the **force constant** (or **spring constant**) of the spring. Hooke's Law, Equation (3), gives good results as long as the force doesn't distort the metal in the spring. We assume that the forces in this section are too small to do that.

EXAMPLE 2 Compressing a Spring

Find the work required to compress a spring from its natural length of 1 ft to a length of 0.75 ft if the force constant is $k = 16$ lb/ft.

Solution We picture the uncompressed spring laid out along the x-axis with its movable end at the origin and its fixed end at $x = 1$ ft (Figure 6.58). This enables us to describe the force required to compress the spring from 0 to x with the formula $F = 16x$. To compress the spring from 0 to 0.25 ft, the force must increase from

$$F(0) = 16 \cdot 0 = 0 \text{ lb} \qquad \text{to} \qquad F(0.25) = 16 \cdot 0.25 = 4 \text{ lb}.$$

The work done by F over this interval is

$$W = \int_0^{0.25} 16x\,dx = 8x^2\Big]_0^{0.25} = 0.5 \text{ ft-lb}. \qquad \begin{array}{l}\text{Eq. (2) with}\\ a = 0,\ b = 0.25,\\ F(x) = 16x\end{array}$$

EXAMPLE 3 Stretching a Spring

A spring has a natural length of 1 m. A force of 24 N stretches the spring to a length of 1.8 m.

(a) Find the force constant k.

(b) How much work will it take to stretch the spring 2 m beyond its natural length?

(c) How far will a 45-N force stretch the spring?

Solution

(a) *The force constant.* We find the force constant from Equation (3). A force of 24 N stretches the spring 0.8 m, so

$$\begin{aligned} 24 &= k(0.8) \\ k &= 24/0.8 = 30 \text{ N/m}. \end{aligned} \qquad \begin{array}{l}\text{Eq. (3) with}\\ F = 24,\ x = 0.8\end{array}$$

(b) *The work to stretch the spring* 2 m. We imagine the unstressed spring hanging along the x-axis with its free end at $x = 0$ (Figure 6.59). The force required to stretch the spring x m beyond its natural length is the force required to pull the free end of the spring x units from the origin. Hooke's Law with $k = 30$ says that this force is

$$F(x) = 30x.$$

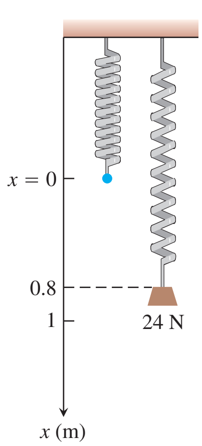


FIGURE 6.59 A 24-N weight stretches this spring 0.8 m beyond its unstressed length (Example 3).

The work done by F on the spring from $x = 0$ m to $x = 2$ m is

$$W = \int_0^2 30x\,dx = 15x^2\Big]_0^2 = 60\text{ J}.$$

(c) *How far will a* 45-N *force stretch the spring?* We substitute $F = 45$ in the equation $F = 30x$ to find

$$45 = 30x, \qquad \text{or} \qquad x = 1.5\text{ m}.$$

A 45-N force will stretch the spring 1.5 m. No calculus is required to find this. ■

The work integral is useful to calculate the work done in lifting objects whose weights vary with their elevation.

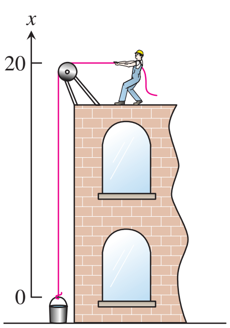


FIGURE 6.60 Lifting the bucket in Example 4.

EXAMPLE 4 Lifting a Rope and Bucket

A 5-lb bucket is lifted from the ground into the air by pulling in 20 ft of rope at a constant speed (Figure 6.60). The rope weighs 0.08 lb/ft. How much work was spent lifting the bucket and rope?

Solution The bucket has constant weight so the work done lifting it alone is weight $\times$ distance $= 5 \cdot 20 = 100$ ft-lb.

The weight of the rope varies with the bucket's elevation, because less of it is freely hanging. When the bucket is x ft off the ground, the remaining proportion of the rope still being lifted weighs $(0.08) \cdot (20 - x)$ lb. So the work in lifting the rope is

$$\begin{aligned}\text{Work on rope} &= \int_0^{20} (0.08)(20 - x)\,dx = \int_0^{20} (1.6 - 0.08x)\,dx \\ &= \left[1.6x - 0.04x^2\right]_0^{20} = 32 - 16 = 16\text{ ft-lb}.\end{aligned}$$

The total work for the bucket and rope combined is

$$100 + 16 = 116\text{ ft-lb}.$$

■

Pumping Liquids from Containers

How much work does it take to pump all or part of the liquid from a container? To find out, we imagine lifting the liquid out one thin horizontal slab at a time and applying the equation $W = Fd$ to each slab. We then evaluate the integral this leads to as the slabs become thinner and more numerous. The integral we get each time depends on the weight of the liquid and the dimensions of the container, but the way we find the integral is always the same. The next examples show what to do.

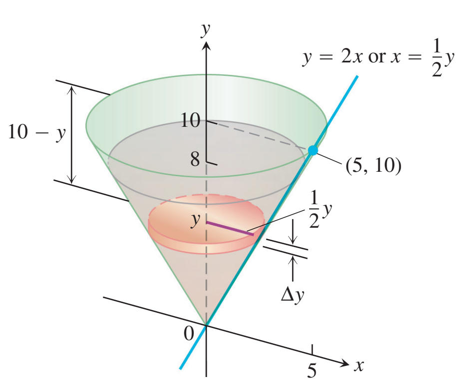


FIGURE 6.61 The olive oil and tank in Example 5.

EXAMPLE 5 Pumping Oil from a Conical Tank

The conical tank in Figure 6.61 is filled to within 2 ft of the top with olive oil weighing 57 lb/ft^3. How much work does it take to pump the oil to the rim of the tank?

Solution We imagine the oil divided into thin slabs by planes perpendicular to the y-axis at the points of a partition of the interval [0, 8].

The typical slab between the planes at y and $y + \Delta y$ has a volume of about

$$\Delta V = \pi(\text{radius})^2(\text{thickness}) = \pi\left(\frac{1}{2}y\right)^2 \Delta y = \frac{\pi}{4}y^2\,\Delta y\text{ ft}^3.$$

The force $F(y)$ required to lift this slab is equal to its weight,

$$F(y) = 57\,\Delta V = \frac{57\pi}{4} y^2 \Delta y \text{ lb}.$$

Weight = weight per unit volume × volume

The distance through which $F(y)$ must act to lift this slab to the level of the rim of the cone is about $(10 - y)$ ft, so the work done lifting the slab is about

$$\Delta W = \frac{57\pi}{4}(10 - y)y^2 \Delta y \text{ ft-lb}.$$

Assuming there are n slabs associated with the partition of $[0, 8]$, and that $y = y_k$ denotes the plane associated with the kth slab of thickness Δy_k, we can approximate the work done lifting all of the slabs with the Riemann sum

$$W \approx \sum_{k=1}^{n} \frac{57\pi}{4}(10 - y_k){y_k}^2 \Delta y_k \text{ ft-lb}.$$

The work of pumping the oil to the rim is the limit of these sums as the norm of the partition goes to zero.

$$\begin{aligned} W &= \int_0^8 \frac{57\pi}{4}(10 - y)y^2\, dy \\ &= \frac{57\pi}{4}\int_0^8 (10y^2 - y^3)\, dy \\ &= \frac{57\pi}{4}\left[\frac{10y^3}{3} - \frac{y^4}{4}\right]_0^8 \approx 30{,}561 \text{ ft-lb}. \end{aligned}$$

■

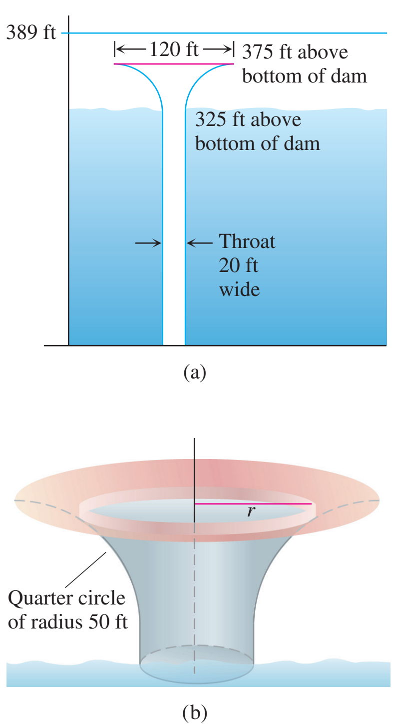


FIGURE 6.62 (a) Cross-section of the glory hole for a dam and (b) the top of the glory hole (Example 6).

EXAMPLE 6 Pumping Water from a Glory Hole

A glory hole is a vertical drain pipe that keeps the water behind a dam from getting too high. The top of the glory hole for a dam is 14 ft below the top of the dam and 375 ft above the bottom (Figure 6.62). The hole needs to be pumped out from time to time to permit the removal of seasonal debris.

From the cross-section in Figure 6.62a, we see that the glory hole is a funnel-shaped drain. The throat of the funnel is 20 ft wide and the head is 120 ft across. The outside boundary of the head cross-section are quarter circles formed with 50-ft radii, shown in Figure 6.62b. The glory hole is formed by rotating a cross-section around its center. Consequently, all horizontal cross-sections are circular disks throughout the entire glory hole. We calculate the work required to pump water from

(a) the throat of the hole.

(b) the funnel portion.

Solution

(a) *Pumping from the throat.* A typical slab in the throat between the planes at y and $y + \Delta y$ has a volume of about

$$\Delta V = \pi(\text{radius})^2(\text{thickness}) = \pi(10)^2 \Delta y \text{ ft}^3.$$

The force $F(y)$ required to lift this slab is equal to its weight (about 62.4 lb/ft^3 for water),

$$F(y) = 62.4\,\Delta V = 6240\pi\,\Delta y \text{ lb}.$$

The distance through which $F(y)$ must act to lift this slab to the top of the hole is $(375 - y)$ ft, so the work done lifting the slab is

$$\Delta W = 6240\pi(375 - y)\,\Delta y \text{ ft-lb}.$$

We can approximate the work done in pumping the water from the throat by summing the work done lifting all the slabs individually, and then taking the limit of this Riemann sum as the norm of the partition goes to zero. This gives the integral

$$\begin{aligned} W &= \int_0^{325} 6240\pi(375 - y)\, dy \\ &= 6240\pi\left[375y - \frac{y^2}{2}\right]_0^{325} \\ &\approx 1{,}353{,}869{,}354 \text{ ft-lb}. \end{aligned}$$

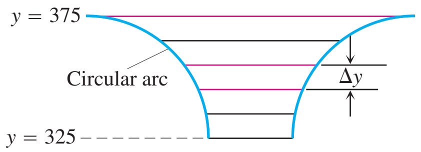


FIGURE 6.63 The glory hole funnel portion.

(b) *Pumping from the funnel*. To compute the work necessary to pump water from the funnel portion of the glory hole, from $y = 325$ to $y = 375$, we need to compute ΔV for approximating elements in the funnel as shown in Figure 6.63. As can be seen from the figure, the radii of the slabs vary with height y.

In Exercises 33 and 34, you are asked to complete the analysis to determine the total work required to pump the water and to find the power of the pumps necessary to pump out the glory hole. ■

EXERCISES 6.6

Springs

1. **Spring constant** It took 1800 J of work to stretch a spring from its natural length of 2 m to a length of 5 m. Find the spring's force constant.

2. **Stretching a spring** A spring has a natural length of 10 in. An 800-lb force stretches the spring to 14 in.
   - **a.** Find the force constant.
   - **b.** How much work is done in stretching the spring from 10 in. to 12 in.?
   - **c.** How far beyond its natural length will a 1600-lb force stretch the spring?

3. **Stretching a rubber band** A force of 2 N will stretch a rubber band 2 cm (0.02 m). Assuming that Hooke's Law applies, how far will a 4-N force stretch the rubber band? How much work does it take to stretch the rubber band this far?

4. **Stretching a spring** If a force of 90 N stretches a spring 1 m beyond its natural length, how much work does it take to stretch the spring 5 m beyond its natural length?

5. **Subway car springs** It takes a force of 21,714 lb to compress a coil spring assembly on a New York City Transit Authority subway car from its free height of 8 in. to its fully compressed height of 5 in.
   - **a.** What is the assembly's force constant?
   - **b.** How much work does it take to compress the assembly the first half inch? the second half inch? Answer to the nearest in.-lb.

   (Data courtesy of Bombardier, Inc., Mass Transit Division, for spring assemblies in subway cars delivered to the New York City Transit Authority from 1985 to 1987.)

6. **Bathroom scale** A bathroom scale is compressed $1/16$ in. when a 150-lb person stands on it. Assuming that the scale behaves like a spring that obeys Hooke's Law, how much does someone who compresses the scale $1/8$ in. weigh? How much work is done compressing the scale $1/8$ in.?

Work Done By a Variable Force

7. **Lifting a rope** A mountain climber is about to haul up a 50 m length of hanging rope. How much work will it take if the rope weighs 0.624 N/m?

8. **Leaky sandbag** A bag of sand originally weighing 144 lb was lifted at a constant rate. As it rose, sand also leaked out at a constant rate. The sand was half gone by the time the bag had been

lifted to 18 ft. How much work was done lifting the sand this far? (Neglect the weight of the bag and lifting equipment.)

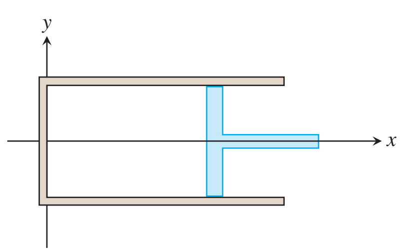
9. **Lifting an elevator cable** An electric elevator with a motor at the top has a multistrand cable weighing 4.5 lb/ft. When the car is at the first floor, 180 ft of cable are paid out, and effectively 0 ft are out when the car is at the top floor. How much work does the motor do just lifting the cable when it takes the car from the first floor to the top?

10. **Force of attraction** When a particle of mass m is at $(x, 0)$, it is attracted toward the origin with a force whose magnitude is k/x^2. If the particle starts from rest at $x = b$ and is acted on by no other forces, find the work done on it by the time it reaches $x = a$, $0 < a < b$.

11. **Compressing gas** Suppose that the gas in a circular cylinder of cross-sectional area A is being compressed by a piston. If p is the pressure of the gas in pounds per square inch and V is the volume in cubic inches, show that the work done in compressing the gas from state (p_1, V_1) to state (p_2, V_2) is given by the equation

$$\text{Work} = \int_{(p_1, V_1)}^{(p_2, V_2)} p\, dV.$$

(*Hint:* In the coordinates suggested in the figure here, $dV = A\, dx$. The force against the piston is pA.)



12. (*Continuation of Exercise 11.*) Use the integral in Exercise 11 to find the work done in compressing the gas from $V_1 = 243$ in.^{3} to $V_2 = 32$ in.^{3} if $p_1 = 50$ lb/in.^{3} and p and V obey the gas law $pV^{1.4} =$ constant (for adiabatic processes).

13. **Leaky bucket** Assume the bucket in Example 4 is leaking. It starts with 2 gal of water (16 lb) and leaks at a constant rate. It finishes draining just as it reaches the top. How much work was spent lifting the water alone? (*Hint:* Do not include the rope and bucket, and find the proportion of water left at elevation x ft.)

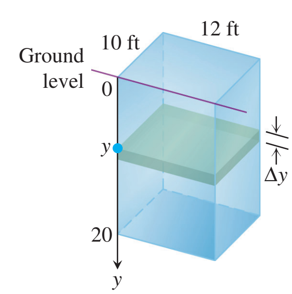
14. (*Continuation of Exercise 13.*) The workers in Example 4 and Exercise 13 changed to a larger bucket that held 5 gal (40 lb) of water, but the new bucket had an even larger leak so that it, too, was empty by the time it reached the top. Assuming that the water leaked out at a steady rate, how much work was done lifting the water alone? (Do not include the rope and bucket.)

Pumping Liquids from Containers

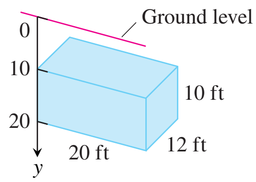
The Weight of Water

Because of Earth's rotation and variations in its gravitational field, the weight of a cubic foot of water at sea level can vary from about 62.26 lb at the equator to as much as 62.59 lb near the poles, a variation of about 0.5%. A cubic foot that weighs about 62.4 lb in Melbourne and New York City will weigh 62.5 lb in Juneau and Stockholm. Although 62.4 is a typical figure and common textbook value, there is considerable variation.

15. **Pumping water** The rectangular tank shown here, with its top at ground level, is used to catch runoff water. Assume that the water weighs 62.4 lb/ft^3.
    a. How much work does it take to empty the tank by pumping the water back to ground level once the tank is full?
    b. If the water is pumped to ground level with a (5/11)-horsepower (hp) motor (work output 250 ft-lb/sec), how long will it take to empty the full tank (to the nearest minute)?
    c. Show that the pump in part (b) will lower the water level 10 ft (halfway) during the first 25 min of pumping.
    d. **The weight of water** What are the answers to parts (a) and (b) in a location where water weighs 62.26 lb/ft^3? 62.59 lb/ft^3?



16. **Emptying a cistern** The rectangular cistern (storage tank for rainwater) shown below has its top 10 ft below ground level. The cistern, currently full, is to be emptied for inspection by pumping its contents to ground level.
    a. How much work will it take to empty the cistern?
    b. How long will it take a 1/2 hp pump, rated at 275 ft-lb/sec, to pump the tank dry?
    c. How long will it take the pump in part (b) to empty the tank halfway? (It will be less than half the time required to empty the tank completely.)
    d. **The weight of water** What are the answers to parts (a) through (c) in a location where water weighs 62.26 lb/ft^3? 62.59 lb/ft^3?

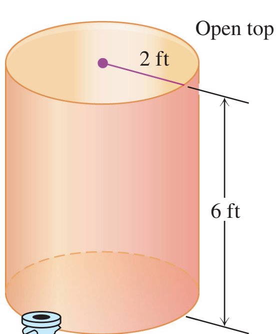


17. Pumping oil How much work would it take to pump oil from the tank in Example 5 to the level of the top of the tank if the tank were completely full?

18. Pumping a half-full tank Suppose that, instead of being full, the tank in Example 5 is only half full. How much work does it take to pump the remaining oil to a level 4 ft above the top of the tank?

19. Emptying a tank A vertical right circular cylindrical tank measures 30 ft high and 20 ft in diameter. It is full of kerosene weighing 51.2 lb/ft^3. How much work does it take to pump the kerosene to the level of the top of the tank?

20. The cylindrical tank shown here can be filled by pumping water from a lake 15 ft below the bottom of the tank. There are two ways to go about it. One is to pump the water through a hose attached to a valve in the bottom of the tank. The other is to attach the hose to the rim of the tank and let the water pour in. Which way will be faster? Give reasons for your answer.

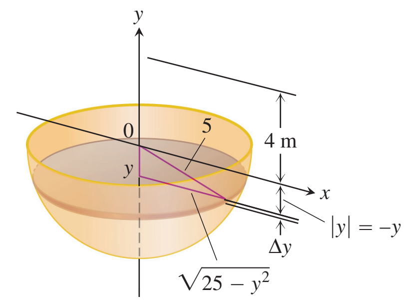


21. a. Pumping milk Suppose that the conical container in Example 5 contains milk (weighing 64.5 lb/ft^3) instead of olive oil. How much work will it take to pump the contents to the rim?

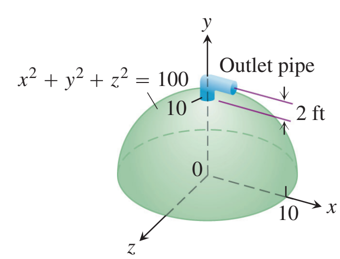
b. Pumping oil How much work will it take to pump the oil in Example 5 to a level 3 ft above the cone's rim?

22. Pumping seawater To design the interior surface of a huge stainless-steel tank, you revolve the curve $y = x^2, 0 \le x \le 4$, about the y-axis. The container, with dimensions in meters, is to be filled with seawater, which weighs 10,000 N/m^3. How much work will it take to empty the tank by pumping the water to the tank's top?

23. Emptying a water reservoir We model pumping from spherical containers the way we do from other containers, with the axis of integration along the vertical axis of the sphere. Use the figure here to find how much work it takes to empty a full hemispherical water reservoir of radius 5 m by pumping the water to a height of 4 m above the top of the reservoir. Water weighs 9800 N/m^3.



24. You are in charge of the evacuation and repair of the storage tank shown here. The tank is a hemisphere of radius 10 ft and is full of benzene weighing 56 lb/ft^3. A firm you contacted says it can empty the tank for $1/2$¢ per foot-pound of work. Find the work required to empty the tank by pumping the benzene to an outlet 2 ft above the top of the tank. If you have \$5000 budgeted for the job, can you afford to hire the firm?



Work and Kinetic Energy

25. Kinetic energy If a variable force of magnitude $F(x)$ moves a body of mass m along the x-axis from x_1 to x_2, the body's velocity v can be written as dx/dt (where t represents time). Use Newton's second law of motion $F = m(dv/dt)$ and the Chain Rule

$$\frac{dv}{dt} = \frac{dv}{dx}\frac{dx}{dt} = v\frac{dv}{dx}$$

to show that the net work done by the force in moving the body from x_1 to x_2 is

$$W = \int_{x_1}^{x_2} F(x)\,dx = \frac{1}{2}mv_2{}^2 - \frac{1}{2}mv_1{}^2,$$

where v_1 and v_2 are the body's velocities at x_1 and x_2. In physics, the expression $(1/2)mv^2$ is called the *kinetic energy* of a body of mass m moving with velocity v. Therefore, *the work done by the force equals the change in the body's kinetic energy*, and we can find the work by calculating this change.

In Exercises 26–32, use the result of Exercise 25.

26. Tennis A 2-oz tennis ball was served at 160 ft/sec (about 109 mph). How much work was done on the ball to make it go this fast? (To find the ball's mass from its weight, express the weight in pounds and divide by 32 ft/sec^2, the acceleration of gravity.)

27. Baseball How many foot-pounds of work does it take to throw a baseball 90 mph? A baseball weighs 5 oz, or 0.3125 lb.

28. Golf A 1.6-oz golf ball is driven off the tee at a speed of 280 ft/sec (about 191 mph). How many foot-pounds of work are done on the ball getting it into the air?

29. Tennis During the match in which Pete Sampras won the 1990 U.S. Open men's tennis championship, Sampras hit a serve that was clocked at a phenomenal 124 mph. How much work did Sampras have to do on the 2-oz ball to get it to that speed?

30. Football A quarterback threw a 14.5-oz football 88 ft/sec (60 mph). How many foot-pounds of work were done on the ball to get it to this speed?

31. Softball How much work has to be performed on a 6.5-oz softball to pitch it 132 ft/sec (90 mph)?

32. A ball bearing A 2-oz steel ball bearing is placed on a vertical spring whose force constant is $k = 18$ lb/ft. The spring is compressed 2 in. and released. About how high does the ball bearing go?

33. Pumping the funnel of the glory hole (*Continuation of Example 6.*)

a. Find the radius of the cross-section (funnel portion) of the glory hole in Example 6 as a function of the height y above the floor of the dam (from $y = 325$ to $y = 375$).

b. Find ΔV for the funnel section of the glory hole (from $y = 325$ to $y = 375$).

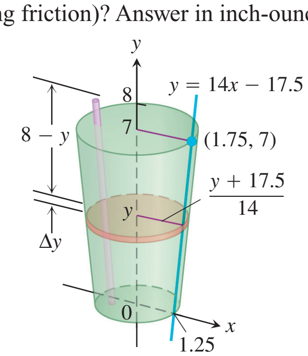
c. Find the work necessary to pump out the funnel section by formulating and evaluating the appropriate definite integral.

34. Pumping water from a glory hole (*Continuation of Exercise 33.*)

a. Find the total work necessary to pump out the glory hole, by adding the work necessary to pump both the throat and funnel sections.

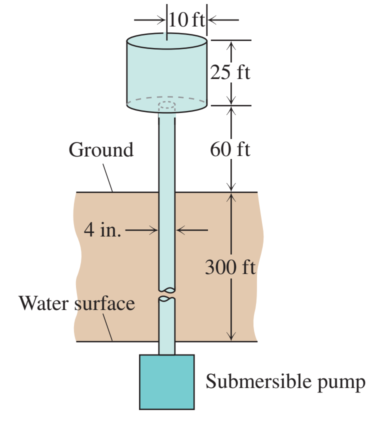
b. Your answer to part (a) is in foot-pounds. A more useful form is horsepower-hours, since motors are rated in horsepower. To convert from foot-pounds to horsepower-hours, divide by 1.98×10^6. How many hours would it take a 1000-horsepower motor to pump out the glory hole, assuming that the motor was fully efficient?

35. Drinking a milkshake The truncated conical container shown here is full of strawberry milkshake that weighs 4/9 oz/in.^{3} As you can see, the container is 7 in. deep, 2.5 in. across at the base, and 3.5 in. across at the top (a standard size at Brigham's in Boston). The straw sticks up an inch above the top. About how much work does it take to suck up the milkshake through the straw (neglecting friction)? Answer in inch-ounces.



Dimensions in inches

36. Water tower Your town has decided to drill a well to increase its water supply. As the town engineer, you have determined that a water tower will be necessary to provide the pressure needed for distribution, and you have designed the system shown here. The water is to be pumped from a 300 ft well through a vertical 4 in. pipe into the base of a cylindrical tank 20 ft in diameter and 25 ft high. The base of the tank will be 60 ft aboveground. The pump is a 3 hp pump, rated at 1650 ft · lb/sec. To the nearest hour, how long will it take to fill the tank the first time? (Include the time it takes to fill the pipe.) Assume that water weighs 62.4 lb/ft^3.



NOT TO SCALE

37. Putting a satellite in orbit The strength of Earth's gravitational field varies with the distance r from Earth's center, and the magnitude of the gravitational force experienced by a satellite of mass m during and after launch is

$$F(r) = \frac{mMG}{r^2}.$$

Here, $M = 5.975 \times 10^{24}$ kg is Earth's mass, $G = 6.6720 \times 10^{-11}\ \mathrm{N \cdot m^2\, kg^{-2}}$ is the universal gravitational constant, and r is measured in meters. The work it takes to lift a 1000-kg satellite from Earth's surface to a circular orbit 35,780 km above Earth's center is therefore given by the integral

$$\text{Work} = \int_{6{,}370{,}000}^{35{,}780{,}000} \frac{1000MG}{r^2}\,dr \text{ joules}.$$

Evaluate the integral. The lower limit of integration is Earth's radius in meters at the launch site. (This calculation does not take into account energy spent lifting the launch vehicle or energy spent bringing the satellite to orbit velocity.)

38. Forcing electrons together Two electrons r meters apart repel each other with a force of

$$F = \frac{23 \times 10^{-29}}{r^2} \text{ newtons}.$$

a. Suppose one electron is held fixed at the point (1, 0) on the x-axis (units in meters). How much work does it take to move a second electron along the x-axis from the point $(-1, 0)$ to the origin?

b. Suppose an electron is held fixed at each of the points $(-1, 0)$ and (1, 0). How much work does it take to move a third electron along the x-axis from (5, 0) to (3, 0)?

6.7 Fluid Pressures and Forces

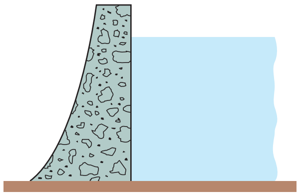


FIGURE 6.64 To withstand the increasing pressure, dams are built thicker as they go down.

We make dams thicker at the bottom than at the top (Figure 6.64) because the pressure against them increases with depth. The pressure at any point on a dam depends only on how far below the surface the point is and not on how much the surface of the dam happens to be tilted at that point. The pressure, in pounds per square foot at a point h feet below the surface, is always $62.4h$. The number 62.4 is the weight-density of water in pounds per cubic foot. The pressure h feet below the surface of any fluid is the fluid's *weight-density* times h.

The Pressure-Depth Equation

In a fluid that is standing still, the pressure p at depth h is the fluid's weight-density w times h:

$$p = wh. \tag{1}$$

In this section we use the equation $p = wh$ to derive a formula for the total force exerted by a fluid against all or part of a vertical or horizontal containing wall.

Weight-density

A fluid's weight-density is its weight per unit volume. Typical values (lb/ft^3) are

Gasoline	42
Mercury	849
Milk	64.5
Molasses	100
Olive oil	57
Seawater	64
Water	62.4

The Constant-Depth Formula for Fluid Force

In a container of fluid with a flat horizontal base, the total force exerted by the fluid against the base can be calculated by multiplying the area of the base by the pressure at the base. We can do this because total force equals force per unit area (pressure) times area. (See Figure 6.65.) If F, p, and A are the total force, pressure, and area, then

$$\begin{aligned} F &= \text{total force} = \text{force per unit area} \times \text{area} \\ &= \text{pressure} \times \text{area} = pA \\ &= whA. \qquad && p = wh \text{ from Eq. (1)} \end{aligned}$$

Fluid Force on a Constant-Depth Surface

$$F = pA = whA \tag{2}$$

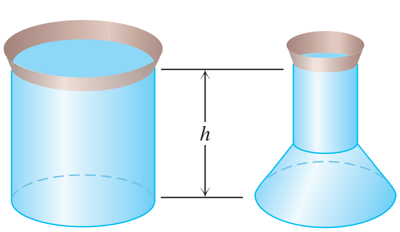


FIGURE 6.65 These containers are filled with water to the same depth and have the same base area. The total force is therefore the same on the bottom of each container. The containers' shapes do not matter here.

For example, the weight-density of water is 62.4 lb/ft^3, so the fluid force at the bottom of a $10 \text{ ft} \times 20 \text{ ft}$ rectangular swimming pool 3 ft deep is

$$\begin{aligned} F &= whA = (62.4 \text{ lb/ft}^3)(3 \text{ ft})(10 \cdot 20 \text{ ft}^2) \\ &= 37{,}440 \text{ lb}. \end{aligned}$$

For a flat plate submerged *horizontally*, like the bottom of the swimming pool just discussed, the downward force acting on its upper face due to liquid pressure is given by Equation (2). If the plate is submerged *vertically*, however, then the pressure against it will be different at different depths and Equation (2) no longer is usable in that form (because h varies). By dividing the plate into many narrow horizontal bands or strips, we can create a Riemann sum whose limit is the fluid force against the side of the submerged vertical plate. Here is the procedure.

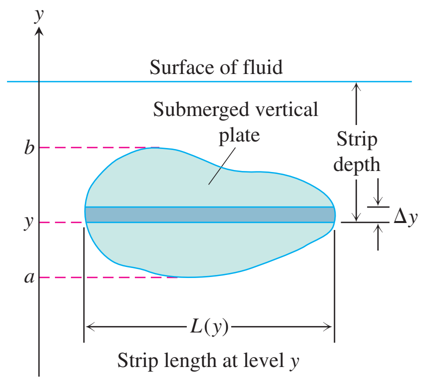


FIGURE 6.66 The force exerted by a fluid against one side of a thin, flat horizontal strip is about $\Delta F =$ pressure $\times$ area $= w \times$ (strip depth) $\times L(y)\,\Delta y$.

The Variable-Depth Formula

Suppose we want to know the force exerted by a fluid against one side of a vertical plate submerged in a fluid of weight-density w. To find it, we model the plate as a region extending from $y = a$ to $y = b$ in the xy-plane (Figure 6.66). We partition $[a, b]$ in the usual way and imagine the region to be cut into thin horizontal strips by planes perpendicular to the y-axis at the partition points. The typical strip from y to $y + \Delta y$ is Δy units wide by $L(y)$ units long. We assume $L(y)$ to be a continuous function of y.

The pressure varies across the strip from top to bottom. If the strip is narrow enough, however, the pressure will remain close to its bottom-edge value of $w \times$ (strip depth). The force exerted by the fluid against one side of the strip will be about

$$\begin{aligned}\Delta F &= (\text{pressure along bottom edge}) \times (\text{area}) \\ &= w \cdot (\text{strip depth}) \cdot L(y)\,\Delta y.\end{aligned}$$

Assume there are n strips associated with the partition of $a \leq y \leq b$ and that y_k is the bottom edge of the kth strip having length $L(y_k)$ and width Δy_k. The force against the entire plate is approximated by summing the forces against each strip, giving the Riemann sum

$$F \approx \sum_{k=1}^{n} (w \cdot (\text{strip depth})_k \cdot L(y_k))\,\Delta y_k. \tag{3}$$

The sum in Equation (3) is a Riemann sum for a continuous function on $[a, b]$, and we expect the approximations to improve as the norm of the partition goes to zero. The force against the plate is the limit of these sums.

The Integral for Fluid Force Against a Vertical Flat Plate

Suppose that a plate submerged vertically in fluid of weight-density w runs from $y = a$ to $y = b$ on the y-axis. Let $L(y)$ be the length of the horizontal strip measured from left to right along the surface of the plate at level y. Then the force exerted by the fluid against one side of the plate is

$$F = \int_a^b w \cdot (\text{strip depth}) \cdot L(y)\,dy. \tag{4}$$

EXAMPLE 1 Applying the Integral for Fluid Force

A flat isosceles right triangular plate with base 6 ft and height 3 ft is submerged vertically, base up, 2 ft below the surface of a swimming pool. Find the force exerted by the water against one side of the plate.

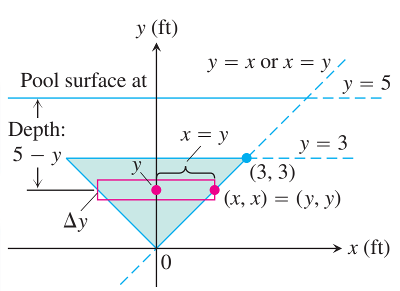


FIGURE 6.67 To find the force on one side of the submerged plate in Example 1, we can use a coordinate system like the one here.

Solution We establish a coordinate system to work in by placing the origin at the plate's bottom vertex and running the y-axis upward along the plate's axis of symmetry (Figure 6.67). The surface of the pool lies along the line $y = 5$ and the plate's top edge along the line $y = 3$. The plate's right-hand edge lies along the line $y = x$, with the upper right vertex at (3, 3). The length of a thin strip at level y is

$$L(y) = 2x = 2y.$$

The depth of the strip beneath the surface is $(5 - y)$. The force exerted by the water against one side of the plate is therefore

$$\begin{aligned} F &= \int_a^b w \cdot \begin{pmatrix}\text{strip}\\ \text{depth}\end{pmatrix} \cdot L(y)\, dy \qquad \text{Eq. (4)} \\ &= \int_0^3 62.4(5 - y)2y\, dy \\ &= 124.8 \int_0^3 (5y - y^2)\, dy \\ &= 124.8\left[\frac{5}{2}y^2 - \frac{y^3}{3}\right]_0^3 = 1684.8 \text{ lb}. \end{aligned}$$

Fluid Forces and Centroids

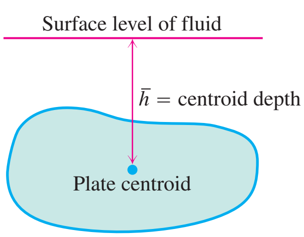


FIGURE 6.68 The force against one side of the plate is $w \cdot \bar{h} \cdot$ plate area.

If we know the location of the centroid of a submerged flat vertical plate (Figure 6.68), we can take a shortcut to find the force against one side of the plate. From Equation (4),

$$\begin{aligned} F &= \int_a^b w \times (\text{strip depth}) \times L(y)\, dy \\ &= w \int_a^b (\text{strip depth}) \times L(y)\, dy \\ &= w \times (\text{moment about surface level line of region occupied by plate}) \\ &= w \times (\text{depth of plate's centroid}) \times (\text{area of plate}). \end{aligned}$$

> **Fluid Forces and Centroids**
> The force of a fluid of weight-density w against one side of a submerged flat vertical plate is the product of w, the distance $\bar{h}$ from the plate's centroid to the fluid surface, and the plate's area:
> $$F = w\bar{h}A. \tag{5}$$

EXAMPLE 2 Finding Fluid Force Using Equation (5)

Use Equation (5) to find the force in Example 1.

Solution The centroid of the triangle (Figure 6.67) lies on the y-axis, one-third of the way from the base to the vertex, so $\bar{h} = 3$. The triangle's area is

$$\begin{aligned} A &= \frac{1}{2}(\text{base})(\text{height}) \\ &= \frac{1}{2}(6)(3) = 9. \end{aligned}$$

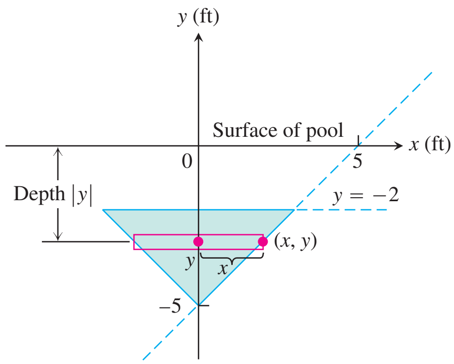
Hence,

$$\begin{aligned} F &= w\bar{h}A = (62.4)(3)(9) \\ &= 1684.8 \text{ lb}. \end{aligned}$$

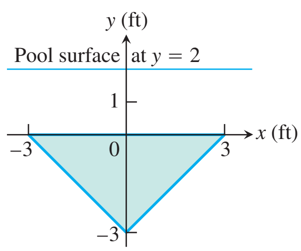
EXERCISES 6.7

The weight-densities of the fluids in the following exercises can be found in the table on page 456.

1. Triangular plate Calculate the fluid force on one side of the plate in Example 1 using the coordinate system shown here.



2. Triangular plate Calculate the fluid force on one side of the plate in Example 1 using the coordinate system shown here.



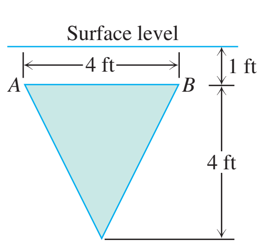
3. Lowered triangular plate The plate in Example 1 is lowered another 2 ft into the water. What is the fluid force on one side of the plate now?

4. Raised triangular plate The plate in Example 1 is raised to put its top edge at the surface of the pool. What is the fluid force on one side of the plate now?

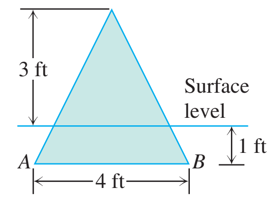
5. Triangular plate The isosceles triangular plate shown here is submerged vertically 1 ft below the surface of a freshwater lake.

a. Find the fluid force against one face of the plate.

b. What would be the fluid force on one side of the plate if the water were seawater instead of freshwater?



6. Rotated triangular plate The plate in Exercise 5 is revolved 180° about line AB so that part of the plate sticks out of the lake, as shown here. What force does the water exert on one face of the plate now?



7. New England Aquarium The viewing portion of the rectangular glass window in a typical fish tank at the New England Aquarium in Boston is 63 in. wide and runs from 0.5 in. below the water's surface to 33.5 in. below the surface. Find the fluid force against this portion of the window. The weight-density of seawater is 64 lb/ft^3. (In case you were wondering, the glass is 3/4 in. thick and the tank walls extend 4 in. above the water to keep the fish from jumping out.)

8. Fish tank A horizontal rectangular freshwater fish tank with base 2 ft × 4 ft and height 2 ft (interior dimensions) is filled to within 2 in. of the top.

a. Find the fluid force against each side and end of the tank.

b. If the tank is sealed and stood on end (without spilling), so that one of the square ends is the base, what does that do to the fluid forces on the rectangular sides?

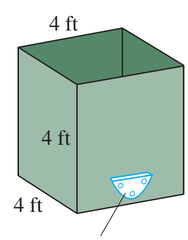
9. Semicircular plate A semicircular plate 2 ft in diameter sticks straight down into freshwater with the diameter along the surface. Find the force exerted by the water on one side of the plate.

10. Milk truck A tank truck hauls milk in a 6-ft-diameter horizontal right circular cylindrical tank. How much force does the milk exert on each end of the tank when the tank is half full?

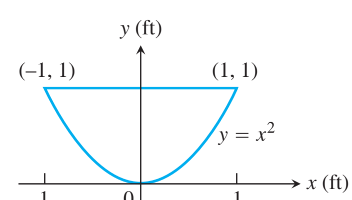
11. The cubical metal tank shown here has a parabolic gate, held in place by bolts and designed to withstand a fluid force of 160 lb without rupturing. The liquid you plan to store has a weight-density of 50 lb/ft^3.

a. What is the fluid force on the gate when the liquid is 2 ft deep?

b. What is the maximum height to which the container can be filled without exceeding its design limitation?



Parabolic gate

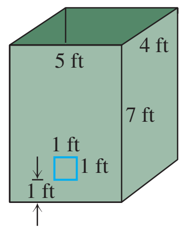


Enlarged view of parabolic gate

12. The rectangular tank shown here has a 1 ft × 1 ft square window 1 ft above the base. The window is designed to withstand a fluid force of 312 lb without cracking.

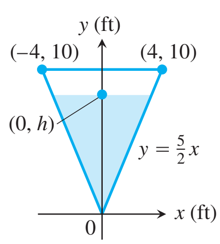
a. What fluid force will the window have to withstand if the tank is filled with water to a depth of 3 ft?

b. To what level can the tank be filled with water without exceeding the window's design limitation?

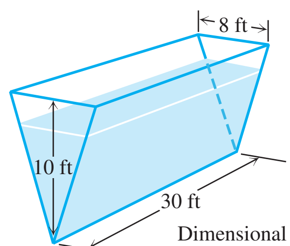


Exercises

13. The end plates of the trough shown here were designed to withstand a fluid force of 6667 lb. How many cubic feet of water can the tank hold without exceeding this limitation? Round down to the nearest cubic foot.



End view of trough

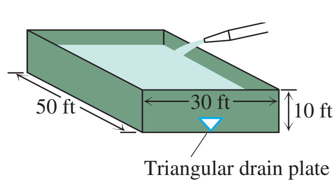


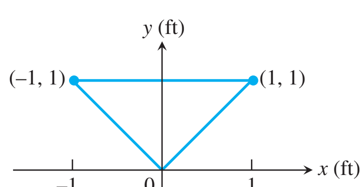
Dimensional view of trough

14. Water is running into the rectangular swimming pool shown here at the rate of $1000 \text{ ft}^3/\text{h}$.

a. Find the fluid force against the triangular drain plate after 9 h of filling.

b. The drain plate is designed to withstand a fluid force of 520 lb. How high can you fill the pool without exceeding this limitation?



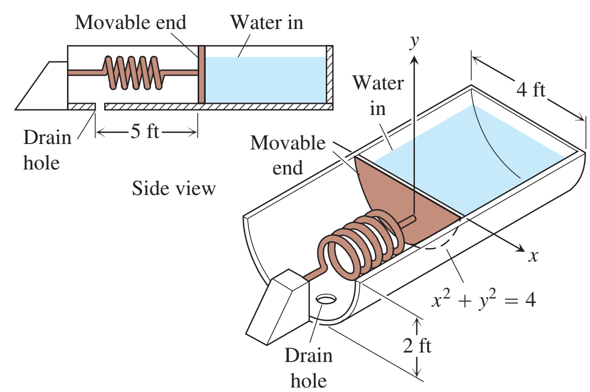


Enlarged view of drain plate

15. A vertical rectangular plate a units long by b units wide is submerged in a fluid of weight-density w with its long edges parallel to the fluid's surface. Find the average value of the pressure along the vertical dimension of the plate. Explain your answer.

16. (*Continuation of Exercise 15.*) Show that the force exerted by the fluid on one side of the plate is the average value of the pressure (found in Exercise 15) times the area of the plate.

17. Water pours into the tank here at the rate of $4 \text{ ft}^3/\text{min}$. The tank's cross-sections are 4-ft-diameter semicircles. One end of the tank is movable, but moving it to increase the volume compresses a spring. The spring constant is $k = 100 \text{ lb/ft}$. If the end of the tank moves 5 ft against the spring, the water will drain out of a safety hole in the bottom at the rate of $5 \text{ ft}^3/\text{min}$. Will the movable end reach the hole before the tank overflows?



18. Watering trough The vertical ends of a watering trough are squares 3 ft on a side.

a. Find the fluid force against the ends when the trough is full.

b. How many inches do you have to lower the water level in the trough to reduce the fluid force by 25%?

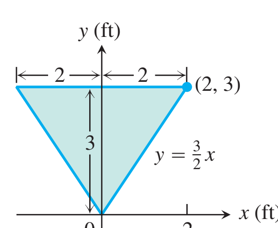
19. Milk carton A rectangular milk carton measures 3.75 in. × 3.75 in. at the base and is 7.75 in. tall. Find the force of the milk on one side when the carton is full.

Exercises

20. Olive oil can A standard olive oil can measures 5.75 in. × 3.5 in. at the base and is 10 in. tall. Find the fluid force against the base and each side when the can is full.

21. Watering trough The vertical ends of a watering trough are isosceles triangles like the one shown here (dimensions in feet).

Exercises



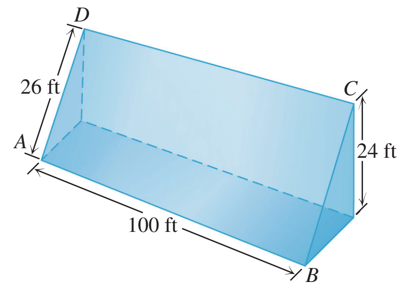
a. Find the fluid force against the ends when the trough is full.

Exercises

b. How many inches do you have to lower the water level in the trough to cut the fluid force on the ends in half? (Answer to the nearest half-inch.)

c. Does it matter how long the trough is? Give reasons for your answer.

22. The face of a dam is a rectangle, *ABCD*, of dimensions $AB = CD = 100$ ft, $AD = BC = 26$ ft. Instead of being vertical, the plane *ABCD* is inclined as indicated in the accompanying figure, so that the top of the dam is 24 ft higher than the bottom. Find the force due to water pressure on the dam when the surface of the water is level with the top of the dam.



Chapter 6 Additional and Advanced Exercises

Volume and Length

1. A solid is generated by revolving about the x-axis the region bounded by the graph of the positive continuous function $y = f(x)$, the x-axis, and the fixed line $x = a$ and the variable line $x = b$, $b > a$. Its volume, for all b, is $b^2 - ab$. Find $f(x)$.

2. A solid is generated by revolving about the x-axis the region bounded by the graph of the positive continuous function $y = f(x)$, the x-axis, and the lines $x = 0$ and $x = a$. Its volume, for all $a > 0$, is $a^2 + a$. Find $f(x)$.

3. Suppose that the increasing function $f(x)$ is smooth for $x \geq 0$ and that $f(0) = a$. Let $s(x)$ denote the length of the graph of f from $(0, a)$ to $(x, f(x))$, $x > 0$. Find $f(x)$ if $s(x) = Cx$ for some constant C. What are the allowable values for C?

4. **a.** Show that for $0 < \alpha \leq \pi/2$,

$$\int_0^{\alpha} \sqrt{1 + \cos^2 \theta}\, d\theta > \sqrt{\alpha^2 + \sin^2 \alpha}.$$

   b. Generalize the result in part (a).

Moments and Centers of Mass

5. Find the centroid of the region bounded below by the x-axis and above by the curve $y = 1 - x^n$, n an even positive integer. What is the limiting position of the centroid as $n \to \infty$?

6. If you haul a telephone pole on a two-wheeled carriage behind a truck, you want the wheels to be 3 ft or so behind the pole's center of mass to provide an adequate "tongue" weight. NYNEX's class 1.40-ft wooden poles have a 27-in. circumference at the top and a 43.5-in. circumference at the base. About how far from the top is the center of mass?

7. Suppose that a thin metal plate of area A and constant density δ occupies a region R in the xy-plane, and let M_y be the plate's moment about the y-axis. Show that the plate's moment about the line $x = b$ is

   a. $M_y - b\delta A$ if the plate lies to the right of the line, and

   b. $b\delta A - M_y$ if the plate lies to the left of the line.

8. Find the center of mass of a thin plate covering the region bounded by the curve $y^2 = 4ax$ and the line $x = a$, $a =$ positive constant, if the density at (x, y) is directly proportional to **(a)** x, **(b)** $|y|$.

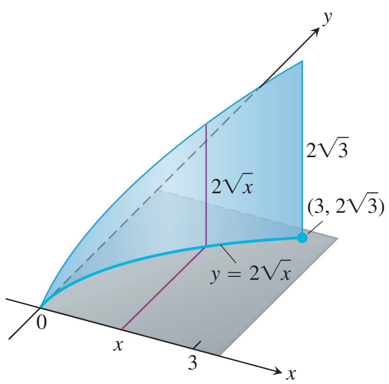
9. **a.** Find the centroid of the region in the first quadrant bounded by two concentric circles and the coordinate axes, if the circles have radii a and b, $0 < a < b$, and their centers are at the origin.

   b. Find the limits of the coordinates of the centroid as a approaches b and discuss the meaning of the result.

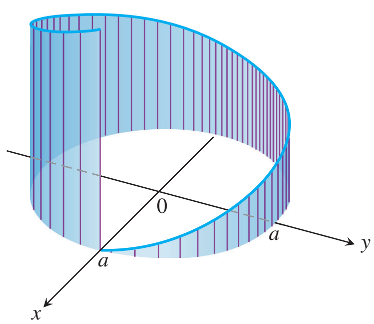
10. A triangular corner is cut from a square 1 ft on a side. The area of the triangle removed is 36 in.^{2}. If the centroid of the remaining region is 7 in. from one side of the original square, how far is it from the remaining sides?

Surface Area

11. At points on the curve $y = 2\sqrt{x}$, line segments of length $h = y$ are drawn perpendicular to the xy-plane. (See accompanying figure.) Find the area of the surface formed by these perpendiculars from $(0, 0)$ to $(3, 2\sqrt{3})$.



12. At points on a circle of radius a, line segments are drawn perpendicular to the plane of the circle, the perpendicular at each point P being of length ks, where s is the length of the arc of the circle measured counterclockwise from $(a, 0)$ to P and k is a positive constant, as shown here. Find the area of the surface formed by the perpendiculars along the arc beginning at $(a, 0)$ and extending once around the circle.



Work

13. A particle of mass m starts from rest at time $t = 0$ and is moved along the x-axis with constant acceleration a from $x = 0$ to $x = h$ against a variable force of magnitude $F(t) = t^2$. Find the work done.

14. **Work and kinetic energy** Suppose a 1.6-oz golf ball is placed on a vertical spring with force constant $k = 2$ lb/in. The spring is compressed 6 in. and released. About how high does the ball go (measured from the spring's rest position)?

Fluid Force

15. A triangular plate ABC is submerged in water with its plane vertical. The side AB, 4 ft long, is 6 ft below the surface of the water, while the vertex C is 2 ft below the surface. Find the force exerted by the water on one side of the plate.

16. A vertical rectangular plate is submerged in a fluid with its top edge parallel to the fluid's surface. Show that the force exerted by the fluid on one side of the plate equals the average value of the pressure up and down the plate times the area of the plate.

17. The *center of pressure* on one side of a plane region submerged in a fluid is defined to be the point at which the total force exerted by the fluid can be applied without changing its total moment about any axis in the plane. Find the depth to the center of pressure **(a)** on a vertical rectangle of height h and width b if its upper edge is in the surface of the fluid; **(b)** on a vertical triangle of height h and base b if the vertex opposite b is a ft and the base b is $(a + h)$ ft below the surface of the fluid.

Chapter 6 Practice Exercises

Volumes

Find the volumes of the solids in Exercises 1–16.

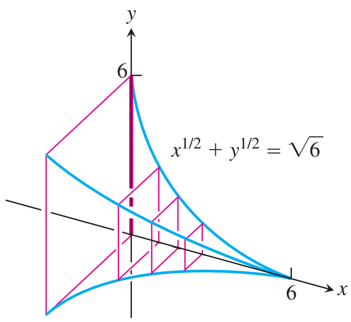
1. The solid lies between planes perpendicular to the x-axis at $x = 0$ and $x = 1$. The cross-sections perpendicular to the x-axis between these planes are circular disks whose diameters run from the parabola $y = x^2$ to the parabola $y = \sqrt{x}$.

2. The base of the solid is the region in the first quadrant between the line $y = x$ and the parabola $y = 2\sqrt{x}$. The cross-sections of

the solid perpendicular to the x-axis are equilateral triangles whose bases stretch from the line to the curve.

3. The solid lies between planes perpendicular to the x-axis at $x = \pi/4$ and $x = 5\pi/4$. The cross-sections between these planes are circular disks whose diameters run from the curve $y = 2\cos x$ to the curve $y = 2\sin x$.

4. The solid lies between planes perpendicular to the x-axis at $x = 0$ and $x = 6$. The cross-sections between these planes are squares whose bases run from the x-axis up to the curve $x^{1/2} + y^{1/2} = \sqrt{6}$.



5. The solid lies between planes perpendicular to the x-axis at $x = 0$ and $x = 4$. The cross-sections of the solid perpendicular to the x-axis between these planes are circular disks whose diameters run from the curve $x^2 = 4y$ to the curve $y^2 = 4x$.

6. The base of the solid is the region bounded by the parabola $y^2 = 4x$ and the line $x = 1$ in the xy-plane. Each cross-section perpendicular to the x-axis is an equilateral triangle with one edge in the plane. (The triangles all lie on the same side of the plane.)

7. Find the volume of the solid generated by revolving the region bounded by the x-axis, the curve $y = 3x^4$, and the lines $x = 1$ and $x = -1$ about **(a)** the x-axis; **(b)** the y-axis; **(c)** the line $x = 1$; **(d)** the line $y = 3$.

8. Find the volume of the solid generated by revolving the "triangular" region bounded by the curve $y = 4/x^3$ and the lines $x = 1$ and $y = 1/2$ about **(a)** the x-axis; **(b)** the y-axis; **(c)** the line $x = 2$; **(d)** the line $y = 4$.

9. Find the volume of the solid generated by revolving the region bounded on the left by the parabola $x = y^2 + 1$ and on the right by the line $x = 5$ about **(a)** the x-axis; **(b)** the y-axis; **(c)** the line $x = 5$.

10. Find the volume of the solid generated by revolving the region bounded by the parabola $y^2 = 4x$ and the line $y = x$ about **(a)** the x-axis; **(b)** the y-axis; **(c)** the line $x = 4$; **(d)** the line $y = 4$.

11. Find the volume of the solid generated by revolving the "triangular" region bounded by the x-axis, the line $x = \pi/3$, and the curve $y = \tan x$ in the first quadrant about the x-axis.

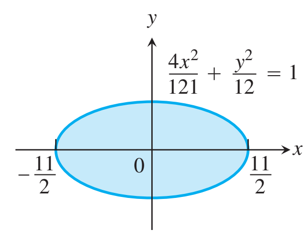
12. Find the volume of the solid generated by revolving the region bounded by the curve $y = \sin x$ and the lines $x = 0$, $x = \pi$, and $y = 2$ about the line $y = 2$.

13. Find the volume of the solid generated by revolving the region between the x-axis and the curve $y = x^2 - 2x$ about **(a)** the x-axis; **(b)** the line $y = -1$; **(c)** the line $x = 2$; **(d)** the line $y = 2$.

14. Find the volume of the solid generated by revolving about the x-axis the region bounded by $y = 2\tan x$, $y = 0$, $x = -\pi/4$, and $x = \pi/4$. (The region lies in the first and third quadrants and resembles a skewed bowtie.)

15. **Volume of a solid sphere hole** A round hole of radius $\sqrt{3}$ ft is bored through the center of a solid sphere of a radius 2 ft. Find the volume of material removed from the sphere.

16. **Volume of a football** The profile of a football resembles the ellipse shown here. Find the football's volume to the nearest cubic inch.



Lengths of Curves

Find the lengths of the curves in Exercises 17–23.

17. $y = x^{1/2} - (1/3)x^{3/2}, \quad 1 \le x \le 4$

18. $x = y^{2/3}, \quad 1 \le y \le 8$

19. $y = (5/12)x^{6/5} - (5/8)x^{4/5}, \quad 1 \le x \le 32$

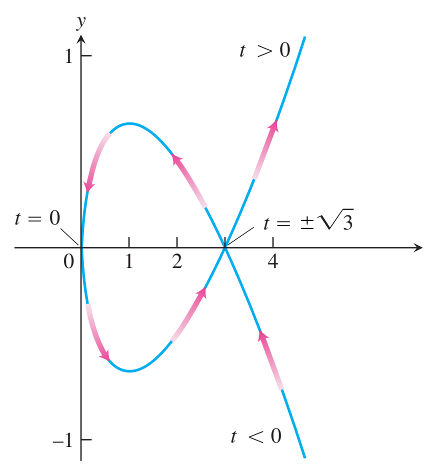
20. $x = (y^3/12) + (1/y), \quad 1 \le y \le 2$

21. $x = 5\cos t - \cos 5t, \quad y = 5\sin t - \sin 5t, \quad 0 \le t \le \pi/2$

22. $x = t^3 - 6t^2, \quad y = t^3 + 6t^2, \quad 0 \le t \le 1$

23. $x = 3\cos\theta, \quad y = 3\sin\theta, \quad 0 \le \theta \le \dfrac{3\pi}{2}$

24. Find the length of the enclosed loop $x = t^2$, $y = (t^3/3) - t$ shown here. The loop starts at $t = -\sqrt{3}$ and ends at $t = \sqrt{3}$.



Centroids and Centers of Mass

25. Find the centroid of a thin, flat plate covering the region enclosed by the parabolas $y = 2x^2$ and $y = 3 - x^2$.

26. Find the centroid of a thin, flat plate covering the region enclosed by the x-axis, the lines $x = 2$ and $x = -2$, and the parabola $y = x^2$.

27. Find the centroid of a thin, flat plate covering the "triangular" region in the first quadrant bounded by the y-axis, the parabola $y = x^2/4$, and the line $y = 4$.

28. Find the centroid of a thin, flat plate covering the region enclosed by the parabola $y^2 = x$ and the line $x = 2y$.

29. Find the center of mass of a thin, flat plate covering the region enclosed by the parabola $y^2 = x$ and the line $x = 2y$ if the density function is $\delta(y) = 1 + y$. (Use horizontal strips.)

30. **a.** Find the center of mass of a thin plate of constant density covering the region between the curve $y = 3/x^{3/2}$ and the x-axis from $x = 1$ to $x = 9$.

b. Find the plate's center of mass if, instead of being constant, the density is $\delta(x) = x$. (Use vertical strips.)

Areas of Surfaces of Revolution

In Exercises 31–36, find the areas of the surfaces generated by revolving the curves about the given axes.

31. $y = \sqrt{2x + 1}, \quad 0 \le x \le 3; \quad x\text{-axis}$

32. $y = x^3/3, \quad 0 \le x \le 1; \quad x\text{-axis}$

33. $x = \sqrt{4y - y^2}, \quad 1 \le y \le 2; \quad y\text{-axis}$

34. $x = \sqrt{y}, \quad 2 \le y \le 6; \quad y\text{-axis}$

35. $x = t^2/2, \quad y = 2t, \quad 0 \le t \le \sqrt{5}; \quad x\text{-axis}$

36. $x = t^2 + 1/(2t), \quad y = 4\sqrt{t}, \quad 1/\sqrt{2} \le t \le 1; \quad y\text{-axis}$

Work

37. Lifting equipment A rock climber is about to haul up 100 N (about 22.5 lb) of equipment that has been hanging beneath her on 40 m of rope that weighs 0.8 newton per meter. How much work will it take? (*Hint:* Solve for the rope and equipment separately, then add.)

38. Leaky tank truck You drove an 800-gal tank truck of water from the base of Mt. Washington to the summit and discovered on arrival that the tank was only half full. You started with a full tank, climbed at a steady rate, and accomplished the 4750-ft elevation change in 50 min. Assuming that the water leaked out at a steady rate, how much work was spent in carrying water to the top? Do not count the work done in getting yourself and the truck there. Water weights 8 lb/U.S. gal.

39. Stretching a spring If a force of 20 lb is required to hold a spring 1 ft beyond its unstressed length, how much work does it take to stretch the spring this far? An additional foot?

40. Garage door spring A force of 200 N will stretch a garage door spring 0.8 m beyond its unstressed length. How far will a 300-N force stretch the spring? How much work does it take to stretch the spring this far from its unstressed length?

41. Pumping a reservoir A reservoir shaped like a right circular cone, point down, 20 ft across the top and 8 ft deep, is full of water. How much work does it take to pump the water to a level 6 ft above the top?

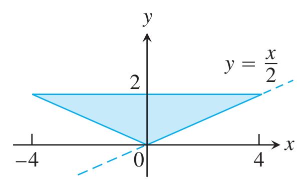
42. Pumping a reservoir (*Continuation of Exercise 41.*) The reservoir is filled to a depth of 5 ft, and the water is to be pumped to the same level as the top. How much work does it take?

43. Pumping a conical tank A right circular conical tank, point down, with top radius 5 ft and height 10 ft is filled with a liquid whose weight-density is 60 lb/ft^3. How much work does it take to pump the liquid to a point 2 ft above the tank? If the pump is driven by a motor rated at 275 ft-lb/sec (1/2 hp), how long will it take to empty the tank?

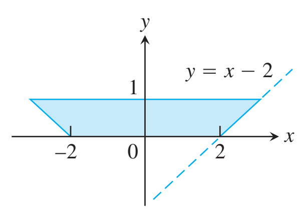
44. Pumping a cylindrical tank A storage tank is a right circular cylinder 20 ft long and 8 ft in diameter with its axis horizontal. If the tank is half full of olive oil weighing 57 lb/ft^3, find the work done in emptying it through a pipe that runs from the bottom of the tank to an outlet that is 6 ft above the top of the tank.

Fluid Force

45. Trough of water The vertical triangular plate shown here is the end plate of a trough full of water ($w = 62.4$). What is the fluid force against the plate?



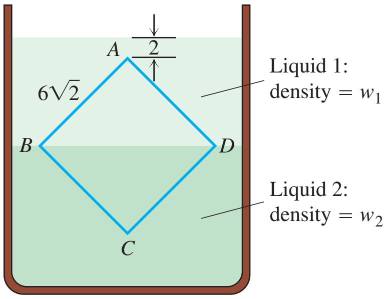
46. Trough of maple syrup The vertical trapezoid plate shown here is the end plate of a trough full of maple syrup weighing 75 lb/ft^3. What is the force exerted by the syrup against the end plate of the trough when the syrup is 10 in. deep?



47. Force on a parabolic gate A flat vertical gate in the face of a dam is shaped like the parabolic region between the curve $y = 4x^2$ and the line $y = 4$, with measurements in feet. The top of the gate lies 5 ft below the surface of the water. Find the force exerted by the water against the gate ($w = 62.4$).

T **48.** You plan to store mercury ($w = 849$ lb/ft^3) in a vertical rectangular tank with a 1 ft square base side whose interior side wall can withstand a total fluid force of 40,000 lb. About how many cubic feet of mercury can you store in the tank at any one time?

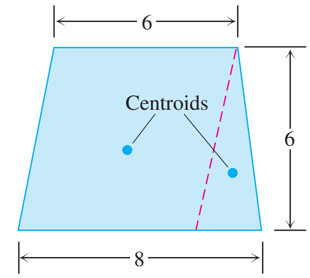
49. The container profiled in the accompanying figure is filled with two nonmixing liquids of weight-density w_1 and w_2. Find the fluid force on one side of the vertical square plate $ABCD$. The points B and D lie in the boundary layer and the square is $6\sqrt{2}$ ft on a side.



50. The isosceles trapezoidal plate shown here is submerged vertically in water ($w = 62.4$) with its upper edge 4 ft below the surface. Find the fluid force on one side of the plate in two different ways:

a. By evaluating an integral.

b. By dividing the plate into a parallelogram and an isosceles triangle, locating their centroids, and using the equation $F = w\bar{h}A$ from Section 6.7.



Chapter 6 Questions to Guide Your Review

1. How do you define and calculate the volumes of solids by the method of slicing? Give an example.
2. How are the disk and washer methods for calculating volumes derived from the method of slicing? Give examples of volume calculations by these methods.
3. Describe the method of cylindrical shells. Give an example.
4. How do you define the length of a smooth parametrized curve $x = f(t), y = g(t), a \leq t \leq b$? What does smoothness have to do with length? What else do you need to know about the parametrization to find the curve's length? Give examples.
5. How do you find the length of the graph of a smooth function over a closed interval? Give an example. What about functions that do not have continuous first derivatives?
6. What is a center of mass?
7. How do you locate the center of mass of a straight, narrow rod or strip of material? Give an example. If the density of the material is constant, you can tell right away where the center of mass is. Where is it?
8. How do you locate the center of mass of a thin flat plate of material? Give an example.
9. How do you define and calculate the area of the surface swept out by revolving the graph of a smooth function $y = f(x)$, $a \leq x \leq b$, about the x-axis? Give an example.
10. Under what conditions can you find the area of the surface generated by revolving a curve $x = f(t), y = g(t), a \leq t \leq b$, about the x-axis? The y-axis? Give examples.
11. What do Pappus's two theorems say? Give examples of how they are used to calculate surface areas and volumes and to locate centroids.
12. How do you define and calculate the work done by a variable force directed along a portion of the x-axis? How do you calculate the work it takes to pump a liquid from a tank? Give examples.
13. How do you calculate the force exerted by a liquid against a portion of a vertical wall? Give an example.

Chapter 6 Technology Application Projects

Mathematica/Maple Module

Using Riemann Sums to Estimate Areas, Volumes, and Lengths of Curves

Visualize and approximate areas and volumes in **Part I** and **Part II**: Volumes of Revolution; and **Part III**: Lengths of Curves.

Project Project

Mathematica/Maple Module

Modeling a Bungee Cord Jump

Collect data (or use data previously collected) to build and refine a model for the force exerted by a jumper's bungee cord. Use the work-energy theorem to compute the distance fallen for a given jumper and a given length of bungee cord.

Project Project Project Project