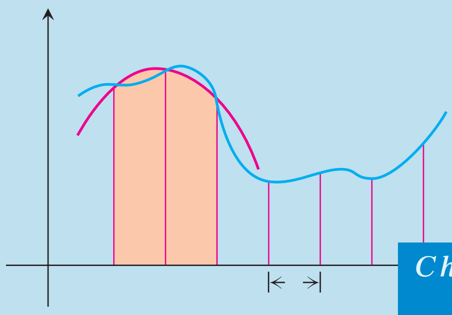


Chapter 8

TECHNIQUES OF INTEGRATION

OVERVIEW The Fundamental Theorem connects antiderivatives and the definite integral. Evaluating the indefinite integral

$$\int f(x)\,dx$$

is equivalent to finding a function F such that $F'(x) = f(x)$, and then adding an arbitrary constant C:

$$\int f(x)\,dx = F(x) + C.$$

In this chapter we study a number of important techniques for finding indefinite integrals of more complicated functions than those seen before. The goal of this chapter is to show how to change unfamiliar integrals into integrals we can recognize, find in a table, or evaluate with a computer. We also extend the idea of the definite integral to *improper integrals* for which the integrand may be unbounded over the interval of integration, or the interval itself may no longer be finite.

8.1 Basic Integration Formulas

To help us in the search for finding indefinite integrals, it is useful to build up a table of integral formulas by inverting formulas for derivatives, as we have done in previous chapters. Then we try to match any integral that confronts us against one of the standard types. This usually involves a certain amount of algebraic manipulation as well as use of the Substitution Rule.

Recall the Substitution Rule from Section 5.5:

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du$$

where $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I. Success in integration often hinges on the ability to spot what part of the integrand should be called u in order that one will also have du, so that a known formula can be applied. This means that the first requirement for skill in integration is a thorough mastery of the formulas for differentiation.

Table 8.1 shows the basic forms of integrals we have evaluated so far. In this section we present several algebraic or substitution methods to help us use this table. There is a more extensive table at the back of the book; we discuss its use in Section 8.6.

TABLE 8.1 Basic integration formulas

1. $\int du = u + C$
2. $\int k\,du = ku + C \qquad$ (any number k)
3. $\int (du + dv) = \int du + \int dv$
4. $\int u^n\,du = \dfrac{u^{n+1}}{n+1} + C \qquad (n \neq -1)$
5. $\int \dfrac{du}{u} = \ln |u| + C$
6. $\int \sin u\,du = -\cos u + C$
7. $\int \cos u\,du = \sin u + C$
8. $\int \sec^2 u\,du = \tan u + C$
9. $\int \csc^2 u\,du = -\cot u + C$
10. $\int \sec u \tan u\,du = \sec u + C$
11. $\int \csc u \cot u\,du = -\csc u + C$
12. $\int \tan u\,du = -\ln |\cos u| + C = \ln |\sec u| + C$
13. $\int \cot u\,du = \ln |\sin u| + C = -\ln |\csc u| + C$
14. $\int e^u\,du = e^u + C$
15. $\int a^u\,du = \dfrac{a^u}{\ln a} + C \qquad (a > 0, a \neq 1)$
16. $\int \sinh u\,du = \cosh u + C$
17. $\int \cosh u\,du = \sinh u + C$
18. $\int \dfrac{du}{\sqrt{a^2 - u^2}} = \sin^{-1}\left(\dfrac{u}{a}\right) + C$
19. $\int \dfrac{du}{a^2 + u^2} = \dfrac{1}{a}\tan^{-1}\left(\dfrac{u}{a}\right) + C$
20. $\int \dfrac{du}{u\sqrt{u^2 - a^2}} = \dfrac{1}{a}\sec^{-1}\left|\dfrac{u}{a}\right| + C$
21. $\int \dfrac{du}{\sqrt{a^2 + u^2}} = \sinh^{-1}\left(\dfrac{u}{a}\right) + C \qquad (a > 0)$
22. $\int \dfrac{du}{\sqrt{u^2 - a^2}} = \cosh^{-1}\left(\dfrac{u}{a}\right) + C \qquad (u > a > 0)$

We often have to rewrite an integral to match it to a standard formula.

Video

You Try It

EXAMPLE 1 Making a Simplifying Substitution

Evaluate

$$\int \frac{2x - 9}{\sqrt{x^2 - 9x + 1}}\,dx.$$

Solution

$$\int \frac{2x - 9}{\sqrt{x^2 - 9x + 1}}\,dx = \int \frac{du}{\sqrt{u}}$$

$u = x^2 - 9x + 1$,
$du = (2x - 9)\,dx$.

$$= \int u^{-1/2}\,du$$

$$= \frac{u^{(-1/2)+1}}{(-1/2) + 1} + C$$

Table 8.1 Formula 4,
with $n = -1/2$

$$= 2u^{1/2} + C$$

$$= 2\sqrt{x^2 - 9x + 1} + C$$

EXAMPLE 2 Completing the Square

You Try It

Evaluate

$$\int \frac{dx}{\sqrt{8x - x^2}}.$$

Solution We complete the square to simplify the denominator:

$$\begin{aligned} 8x - x^2 &= -(x^2 - 8x) = -(x^2 - 8x + 16 - 16) \\ &= -(x^2 - 8x + 16) + 16 = 16 - (x - 4)^2. \end{aligned}$$

Then

$$\int \frac{dx}{\sqrt{8x - x^2}} = \int \frac{dx}{\sqrt{16 - (x - 4)^2}}$$

$$= \int \frac{du}{\sqrt{a^2 - u^2}}$$

$a = 4$, $u = (x - 4)$,
$du = dx$

$$= \sin^{-1}\left(\frac{u}{a}\right) + C$$

Table 8.1, Formula 18

$$= \sin^{-1}\left(\frac{x - 4}{4}\right) + C.$$

EXAMPLE 3 Expanding a Power and Using a Trigonometric Identity

Video

Evaluate

$$\int (\sec x + \tan x)^2\,dx.$$

Solution We expand the integrand and get

$$(\sec x + \tan x)^2 = \sec^2 x + 2\sec x \tan x + \tan^2 x.$$

The first two terms on the right-hand side of this equation are familiar; we can integrate them at once. How about $\tan^2 x$? There is an identity that connects it with $\sec^2 x$:

$$\tan^2 x + 1 = \sec^2 x, \qquad \tan^2 x = \sec^2 x - 1.$$

We replace $\tan^2 x$ by $\sec^2 x - 1$ and get

$$\begin{aligned}
\int (\sec x + \tan x)^2\, dx &= \int (\sec^2 x + 2 \sec x \tan x + \sec^2 x - 1)\, dx \\
&= 2\int \sec^2 x\, dx + 2\int \sec x \tan x\, dx - \int 1\, dx \\
&= 2 \tan x + 2 \sec x - x + C.
\end{aligned}$$

EXAMPLE 4 Eliminating a Square Root

Evaluate

$$\int_0^{\pi/4} \sqrt{1 + \cos 4x}\, dx.$$

Solution We use the identity

$$\cos^2 \theta = \frac{1 + \cos 2\theta}{2}, \qquad \text{or} \qquad 1 + \cos 2\theta = 2 \cos^2 \theta.$$

With $\theta = 2x$, this identity becomes

$$1 + \cos 4x = 2 \cos^2 2x.$$

Hence,

$$\begin{aligned}
\int_0^{\pi/4} \sqrt{1 + \cos 4x}\, dx &= \int_0^{\pi/4} \sqrt{2}\, \sqrt{\cos^2 2x}\, dx \\
&= \sqrt{2} \int_0^{\pi/4} |\cos 2x|\, dx && \sqrt{u^2} = |u| \\
&= \sqrt{2} \int_0^{\pi/4} \cos 2x\, dx && \text{On } [0, \pi/4], \cos 2x \ge 0, \text{ so } |\cos 2x| = \cos 2x. \\
&= \sqrt{2} \left[\frac{\sin 2x}{2} \right]_0^{\pi/4} && \text{Table 8.1, Formula 7, with } u = 2x \text{ and } du = 2\, dx \\
&= \sqrt{2} \left[\frac{1}{2} - 0 \right] = \frac{\sqrt{2}}{2}.
\end{aligned}$$

EXAMPLE 5 Reducing an Improper Fraction

Evaluate

$$\int \frac{3x^2 - 7x}{3x + 2}\, dx.$$

$$\begin{array}{r}
x - 3 \\
3x + 2 \,\big)\, \overline{3x^2 - 7x} \\
\underline{3x^2 + 2x} \\
-9x \\
\underline{-9x - 6} \\
+ 6
\end{array}$$

Solution The integrand is an improper fraction (degree of numerator greater than or equal to degree of denominator). To integrate it, we divide first, getting a quotient plus a remainder that is a proper fraction:

$$\frac{3x^2 - 7x}{3x + 2} = x - 3 + \frac{6}{3x + 2}.$$

Therefore,

$$\int \frac{3x^2 - 7x}{3x + 2}\, dx = \int \left(x - 3 + \frac{6}{3x + 2} \right) dx = \frac{x^2}{2} - 3x + 2 \ln |3x + 2| + C.$$

Reducing an improper fraction by long division (Example 5) does not always lead to an expression we can integrate directly. We see what to do about that in Section 8.5.

EXAMPLE 6 Separating a Fraction

Evaluate

$$\int \frac{3x+2}{\sqrt{1-x^2}}\, dx.$$

Solution We first separate the integrand to get

$$\int \frac{3x+2}{\sqrt{1-x^2}}\, dx = 3\int \frac{x\, dx}{\sqrt{1-x^2}} + 2\int \frac{dx}{\sqrt{1-x^2}}.$$

In the first of these new integrals, we substitute

$$u = 1 - x^2, \qquad du = -2x\, dx, \qquad \text{and} \qquad x\, dx = -\frac{1}{2}\, du.$$

$$\begin{aligned} 3\int \frac{x\, dx}{\sqrt{1-x^2}} &= 3\int \frac{(-1/2)\, du}{\sqrt{u}} = -\frac{3}{2}\int u^{-1/2}\, du \\ &= -\frac{3}{2}\cdot\frac{u^{1/2}}{1/2} + C_1 = -3\sqrt{1-x^2} + C_1 \end{aligned}$$

The second of the new integrals is a standard form,

$$2\int \frac{dx}{\sqrt{1-x^2}} = 2\sin^{-1} x + C_2.$$

Combining these results and renaming $C_1 + C_2$ as C gives

$$\int \frac{3x+2}{\sqrt{1-x^2}}\, dx = -3\sqrt{1-x^2} + 2\sin^{-1} x + C.$$ ■

The final example of this section calculates an important integral by the algebraic technique of multiplying the integrand by a form of 1 to change the integrand into one we can integrate.

HISTORICAL BIOGRAPHY

George David Birkhoff (1884–1944)

EXAMPLE 7 Integral of $y = \sec x$—Multiplying by a Form of 1

Evaluate

$$\int \sec x\, dx.$$

Solution

$$\begin{aligned} \int \sec x\, dx &= \int (\sec x)(1)\, dx = \int \sec x \cdot \frac{\sec x + \tan x}{\sec x + \tan x}\, dx \\ &= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}\, dx \\ &= \int \frac{du}{u} \qquad \begin{aligned} &u = \tan x + \sec x, \\ &du = (\sec^2 x + \sec x \tan x)\, dx \end{aligned} \\ &= \ln|u| + C = \ln|\sec x + \tan x| + C. \end{aligned}$$ ■

With cosecants and cotangents in place of secants and tangents, the method of Example 7 leads to a companion formula for the integral of the cosecant (see Exercise 95).

TABLE 8.2 The secant and cosecant integrals

1. $\displaystyle\int \sec u\,du = \ln|\sec u + \tan u| + C$
2. $\displaystyle\int \csc u\,du = -\ln|\csc u + \cot u| + C$

Procedures for Matching Integrals to Basic Formulas

PROCEDURE	EXAMPLE
Making a simplifying substitution	$\dfrac{2x-9}{\sqrt{x^2-9x+1}}\,dx = \dfrac{du}{\sqrt{u}}$
Completing the square	$\sqrt{8x-x^2} = \sqrt{16-(x-4)^2}$
Using a trigonometric identity	$\begin{aligned}(\sec x + \tan x)^2 &= \sec^2 x + 2\sec x\tan x + \tan^2 x\\ &= \sec^2 x + 2\sec x\tan x + (\sec^2 x - 1)\\ &= 2\sec^2 x + 2\sec x\tan x - 1\end{aligned}$
Eliminating a square root	$\sqrt{1+\cos 4x} = \sqrt{2\cos^2 2x} = \sqrt{2}\,|\cos 2x|$
Reducing an improper fraction	$\dfrac{3x^2-7x}{3x+2} = x - 3 + \dfrac{6}{3x+2}$
Separating a fraction	$\dfrac{3x+2}{\sqrt{1-x^2}} = \dfrac{3x}{\sqrt{1-x^2}} + \dfrac{2}{\sqrt{1-x^2}}$
Multiplying by a form of 1	$\begin{aligned}\sec x &= \sec x \cdot \dfrac{\sec x + \tan x}{\sec x + \tan x}\\ &= \dfrac{\sec^2 x + \sec x\tan x}{\sec x + \tan x}\end{aligned}$

EXERCISES 8.1

Basic Substitutions

Evaluate each integral in Exercises 1–36 by using a substitution to reduce it to standard form.

1. $\displaystyle\int \frac{16x\,dx}{\sqrt{8x^2+1}}$

2. $\displaystyle\int \frac{3\cos x\,dx}{\sqrt{1+3\sin x}}$

3. $\displaystyle\int 3\sqrt{\sin v}\cos v\,dv$

4. $\displaystyle\int \cot^3 y\csc^2 y\,dy$

5. $\displaystyle\int_0^1 \frac{16x\,dx}{8x^2+2}$

6. $\displaystyle\int_{\pi/4}^{\pi/3} \frac{\sec^2 z}{\tan z}\,dz$

7. $\int \frac{dx}{\sqrt{x}(\sqrt{x}+1)}$

8. $\int \frac{dx}{x-\sqrt{x}}$

9. $\int \cot(3-7x)\,dx$

10. $\int \csc(\pi x - 1)\,dx$

11. $\int e^{\theta}\csc(e^{\theta}+1)\,d\theta$

12. $\int \frac{\cot(3+\ln x)}{x}\,dx$

13. $\int \sec\frac{t}{3}\,dt$

14. $\int x\sec(x^2-5)\,dx$

15. $\int \csc(s-\pi)\,ds$

16. $\int \frac{1}{\theta^2}\csc\frac{1}{\theta}\,d\theta$

17. $\int_0^{\sqrt{\ln 2}} 2x\,e^{x^2}\,dx$

18. $\int_{\pi/2}^{\pi}(\sin y)e^{\cos y}\,dy$

19. $\int e^{\tan v}\sec^2 v\,dv$

20. $\int \frac{e^{\sqrt{t}}\,dt}{\sqrt{t}}$

21. $\int 3^{x+1}\,dx$

22. $\int \frac{2^{\ln x}}{x}\,dx$

23. $\int \frac{2^{\sqrt{w}}\,dw}{2\sqrt{w}}$

24. $\int 10^{2\theta}\,d\theta$

25. $\int \frac{9\,du}{1+9u^2}$

26. $\int \frac{4\,dx}{1+(2x+1)^2}$

27. $\int_0^{1/6}\frac{dx}{\sqrt{1-9x^2}}$

28. $\int_0^1 \frac{dt}{\sqrt{4-t^2}}$

29. $\int \frac{2s\,ds}{\sqrt{1-s^4}}$

30. $\int \frac{2\,dx}{x\sqrt{1-4\ln^2 x}}$

31. $\int \frac{6\,dx}{x\sqrt{25x^2-1}}$

32. $\int \frac{dr}{r\sqrt{r^2-9}}$

33. $\int \frac{dx}{e^x+e^{-x}}$

34. $\int \frac{dy}{\sqrt{e^{2y}-1}}$

35. $\int_1^{e^{\pi/3}}\frac{dx}{x\cos(\ln x)}$

36. $\int \frac{\ln x\,dx}{x+4x\ln^2 x}$

Completing the Square

Evaluate each integral in Exercises 37–42 by completing the square and using a substitution to reduce it to standard form.

37. $\int_1^2 \frac{8\,dx}{x^2-2x+2}$

38. $\int_2^4 \frac{2\,dx}{x^2-6x+10}$

39. $\int \frac{dt}{\sqrt{-t^2+4t-3}}$

40. $\int \frac{d\theta}{\sqrt{2\theta-\theta^2}}$

41. $\int \frac{dx}{(x+1)\sqrt{x^2+2x}}$

42. $\int \frac{dx}{(x-2)\sqrt{x^2-4x+3}}$

Trigonometric Identities

Evaluate each integral in Exercises 43–46 by using trigonometric identities and substitutions to reduce it to standard form.

43. $\int (\sec x + \cot x)^2\,dx$

44. $\int (\csc x - \tan x)^2\,dx$

45. $\int \csc x \sin 3x\,dx$

46. $\int (\sin 3x\cos 2x - \cos 3x\sin 2x)\,dx$

Improper Fractions

Evaluate each integral in Exercises 47–52 by reducing the improper fraction and using a substitution (if necessary) to reduce it to standard form.

47. $\int \frac{x}{x+1}\,dx$

48. $\int \frac{x^2}{x^2+1}\,dx$

49. $\int_{\sqrt{2}}^{3}\frac{2x^3}{x^2-1}\,dx$

50. $\int_{-1}^{3}\frac{4x^2-7}{2x+3}\,dx$

51. $\int \frac{4t^3-t^2+16t}{t^2+4}\,dt$

52. $\int \frac{2\theta^3-7\theta^2+7\theta}{2\theta-5}\,d\theta$

Separating Fractions

Evaluate each integral in Exercises 53–56 by separating the fraction and using a substitution (if necessary) to reduce it to standard form.

53. $\int \frac{1-x}{\sqrt{1-x^2}}\,dx$

54. $\int \frac{x+2\sqrt{x-1}}{2x\sqrt{x-1}}\,dx$

55. $\int_0^{\pi/4}\frac{1+\sin x}{\cos^2 x}\,dx$

56. $\int_0^{1/2}\frac{2-8x}{1+4x^2}\,dx$

Multiplying by a Form of 1

Evaluate each integral in Exercises 57–62 by multiplying by a form of 1 and using a substitution (if necessary) to reduce it to standard form.

57. $\int \frac{1}{1+\sin x}\,dx$

58. $\int \frac{1}{1+\cos x}\,dx$

59. $\int \frac{1}{\sec\theta+\tan\theta}\,d\theta$

60. $\int \frac{1}{\csc\theta+\cot\theta}\,d\theta$

61. $\int \frac{1}{1-\sec x}\,dx$

62. $\int \frac{1}{1-\csc x}\,dx$

Eliminating Square Roots

Evaluate each integral in Exercises 63–70 by eliminating the square root.

63. $\int_0^{2\pi}\sqrt{\frac{1-\cos x}{2}}\,dx$

64. $\int_0^{\pi}\sqrt{1-\cos 2x}\,dx$

65. $\displaystyle\int_{\pi/2}^{\pi} \sqrt{1 + \cos 2t}\, dt$

66. $\displaystyle\int_{-\pi}^{0} \sqrt{1 + \cos t}\, dt$

67. $\displaystyle\int_{-\pi}^{0} \sqrt{1 - \cos^2 \theta}\, d\theta$

68. $\displaystyle\int_{\pi/2}^{\pi} \sqrt{1 - \sin^2 \theta}\, d\theta$

69. $\displaystyle\int_{-\pi/4}^{\pi/4} \sqrt{1 + \tan^2 y}\, dy$

70. $\displaystyle\int_{-\pi/4}^{0} \sqrt{\sec^2 y - 1}\, dy$

Assorted Integrations

Evaluate each integral in Exercises 71–82 by using any technique you think is appropriate.

71. $\displaystyle\int_{\pi/4}^{3\pi/4} (\csc x - \cot x)^2\, dx$

72. $\displaystyle\int_{0}^{\pi/4} (\sec x + 4\cos x)^2\, dx$

73. $\displaystyle\int \cos\theta \csc(\sin\theta)\, d\theta$

74. $\displaystyle\int \left(1 + \frac{1}{x}\right)\cot(x + \ln x)\, dx$

75. $\displaystyle\int (\csc x - \sec x)(\sin x + \cos x)\, dx$

76. $\displaystyle\int 3\sinh\left(\frac{x}{2} + \ln 5\right) dx$

77. $\displaystyle\int \frac{6\, dy}{\sqrt{y}(1 + y)}$

78. $\displaystyle\int \frac{dx}{x\sqrt{4x^2 - 1}}$

79. $\displaystyle\int \frac{7\, dx}{(x - 1)\sqrt{x^2 - 2x - 48}}$

80. $\displaystyle\int \frac{dx}{(2x + 1)\sqrt{4x^2 + 4x}}$

81. $\displaystyle\int \sec^2 t \tan(\tan t)\, dt$

82. $\displaystyle\int \frac{dx}{x\sqrt{3 + x^2}}$

Trigonometric Powers

83. **a.** Evaluate $\int \cos^3\theta\, d\theta$. (*Hint:* $\cos^2\theta = 1 - \sin^2\theta$.)

b. Evaluate $\int \cos^5\theta\, d\theta$.

c. Without actually evaluating the integral, explain how you would evaluate $\int \cos^9\theta\, d\theta$.

84. **a.** Evaluate $\int \sin^3\theta\, d\theta$. (*Hint:* $\sin^2\theta = 1 - \cos^2\theta$.)

b. Evaluate $\int \sin^5\theta\, d\theta$.

c. Evaluate $\int \sin^7\theta\, d\theta$.

d. Without actually evaluating the integral, explain how you would evaluate $\int \sin^{13}\theta\, d\theta$.

85. **a.** Express $\int \tan^3\theta\, d\theta$ in terms of $\int \tan\theta\, d\theta$. Then evaluate $\int \tan^3\theta\, d\theta$. (*Hint:* $\tan^2\theta = \sec^2\theta - 1$.)

b. Express $\int \tan^5\theta\, d\theta$ in terms of $\int \tan^3\theta\, d\theta$.

c. Express $\int \tan^7\theta\, d\theta$ in terms of $\int \tan^5\theta\, d\theta$.

d. Express $\int \tan^{2k+1}\theta\, d\theta$, where k is a positive integer, in terms of $\int \tan^{2k-1}\theta\, d\theta$.

86. **a.** Express $\int \cot^3\theta\, d\theta$ in terms of $\int \cot\theta\, d\theta$. Then evaluate $\int \cot^3\theta\, d\theta$. (*Hint:* $\cot^2\theta = \csc^2\theta - 1$.)

b. Express $\int \cot^5\theta\, d\theta$ in terms of $\int \cot^3\theta\, d\theta$.

c. Express $\int \cot^7\theta\, d\theta$ in terms of $\int \cot^5\theta\, d\theta$.

d. Express $\int \cot^{2k+1}\theta\, d\theta$, where k is a positive integer, in terms of $\int \cot^{2k-1}\theta\, d\theta$.

Theory and Examples

87. Area Find the area of the region bounded above by $y = 2\cos x$ and below by $y = \sec x$, $-\pi/4 \le x \le \pi/4$.

88. Area Find the area of the "triangular" region that is bounded from above and below by the curves $y = \csc x$ and $y = \sin x$, $\pi/6 \le x \le \pi/2$, and on the left by the line $x = \pi/6$.

89. Volume Find the volume of the solid generated by revolving the region in Exercise 87 about the x-axis.

90. Volume Find the volume of the solid generated by revolving the region in Exercise 88 about the x-axis.

91. Arc length Find the length of the curve $y = \ln(\cos x)$, $0 \le x \le \pi/3$.

92. Arc length Find the length of the curve $y = \ln(\sec x)$, $0 \le x \le \pi/4$.

93. Centroid Find the centroid of the region bounded by the x-axis, the curve $y = \sec x$, and the lines $x = -\pi/4$, $x = \pi/4$.

94. Centroid Find the centroid of the region that is bounded by the x-axis, the curve $y = \csc x$, and the lines $x = \pi/6$, $x = 5\pi/6$.

95. The integral of csc *x* Repeat the derivation in Example 7, using cofunctions, to show that

$$\int \csc x\, dx = -\ln|\csc x + \cot x| + C.$$

96. Using different substitutions Show that the integral

$$\int ((x^2 - 1)(x + 1))^{-2/3}\, dx$$

can be evaluated with any of the following substitutions.

a. $u = 1/(x + 1)$

b. $u = ((x - 1)/(x + 1))^k$ for $k = 1, 1/2, 1/3, -1/3, -2/3$, and -1

c. $u = \tan^{-1} x$

d. $u = \tan^{-1}\sqrt{x}$

e. $u = \tan^{-1}((x - 1)/2)$

f. $u = \cos^{-1} x$

g. $u = \cosh^{-1} x$

What is the value of the integral? (*Source:* "Problems and Solutions," *College Mathematics Journal*, Vol. 21, No. 5 (Nov. 1990), pp. 425–426.)

8.2 Integration by Parts

Since

$$\int x\,dx = \frac{1}{2}x^2 + C$$

and

$$\int x^2\,dx = \frac{1}{3}x^3 + C,$$

it is apparent that

$$\int x \cdot x\,dx \neq \int x\,dx \cdot \int x\,dx.$$

In other words, the integral of a product is generally *not* the product of the individual-integrals:

$$\int f(x)g(x)\,dx \text{ is not equal to } \int f(x)\,dx \cdot \int g(x)\,dx.$$

Integration by parts is a technique for simplifying integrals of the form

$$\int f(x)g(x)\,dx.$$

It is useful when f can be differentiated repeatedly and g can be integrated repeatedly without difficulty. The integral

$$\int xe^x\,dx$$

is such an integral because $f(x) = x$ can be differentiated twice to become zero and $g(x) = e^x$ can be integrated repeatedly without difficulty. Integration by parts also applies to integrals like

$$\int e^x \sin x\,dx$$

in which each part of the integrand appears again after repeated differentiation or integration.

In this section, we describe integration by parts and show how to apply it.

Product Rule in Integral Form

If f and g are differentiable functions of x, the Product Rule says

$$\frac{d}{dx}[f(x)g(x)] = f'(x)g(x) + f(x)g'(x).$$

In terms of indefinite integrals, this equation becomes

$$\int \frac{d}{dx}[f(x)g(x)]\,dx = \int [f'(x)g(x) + f(x)g'(x)]\,dx$$

or

$$\int \frac{d}{dx}[f(x)g(x)]\,dx = \int f'(x)g(x)\,dx + \int f(x)g'(x)\,dx.$$

Rearranging the terms of this last equation, we get

$$\int f(x)g'(x)\,dx = \int \frac{d}{dx}[f(x)g(x)]\,dx - \int f'(x)g(x)\,dx$$

leading to the **integration by parts** formula

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx \tag{1}$$

Sometimes it is easier to remember the formula if we write it in differential form. Let $u = f(x)$ and $v = g(x)$. Then $du = f'(x)\,dx$ and $dv = g'(x)\,dx$. Using the Substitution Rule, the integration by parts formula becomes

Integration by Parts Formula

$$\int u\,dv = uv - \int v\,du \tag{2}$$

This formula expresses one integral, $\int u\,dv$, in terms of a second integral, $\int v\,du$. With a proper choice of u and v, the second integral may be easier to evaluate than the first. In using the formula, various choices may be available for u and dv. The next examples illustrate the technique.

Video

EXAMPLE 1 Using Integration by Parts

Find

$$\int x\cos x\,dx.$$

Solution We use the formula $\displaystyle\int u\,dv = uv - \int v\,du$ with

$$\begin{aligned} u &= x, & dv &= \cos x\,dx, \\ du &= dx, & v &= \sin x. \end{aligned}$$ Simplest antiderivative of $\cos x$

Then

$$\int x\cos x\,dx = x\sin x - \int \sin x\,dx = x\sin x + \cos x + C.$$ ■

Let us examine the choices available for u and dv in Example 1.

EXAMPLE 2 Example 1 Revisited

To apply integration by parts to

$$\int x\cos x\,dx = \int u\,dv$$

we have four possible choices:

1. Let $u = 1$ and $dv = x \cos x \, dx$.
2. Let $u = x$ and $dv = \cos x \, dx$.
3. Let $u = x \cos x$ and $dv = dx$.
4. Let $u = \cos x$ and $dv = x \, dx$.

Let's examine these one at a time.

Choice 1 won't do because we don't know how to integrate $dv = x \cos x \, dx$ to get v.

Choice 2 works well, as we saw in Example 1.

Choice 3 leads to

$$\begin{aligned} u &= x \cos x, & dv &= dx, \\ du &= (\cos x - x \sin x) \, dx, & v &= x, \end{aligned}$$

and the new integral

$$\int v \, du = \int (x \cos x - x^2 \sin x) \, dx.$$

This is worse than the integral we started with.

Choice 4 leads to

$$\begin{aligned} u &= \cos x, & dv &= x \, dx, \\ du &= -\sin x \, dx, & v &= x^2/2, \end{aligned}$$

so the new integral is

$$\int v \, du = -\int \frac{x^2}{2} \sin x \, dx.$$

This, too, is worse. ■

The goal of integration by parts is to go from an integral $\int u \, dv$ that we don't see how to evaluate to an integral $\int v \, du$ that we can evaluate. Generally, you choose dv first to be as much of the integrand, including dx, as you can readily integrate; u is the leftover part. Keep in mind that integration by parts does not always work.

EXAMPLE 3 Integral of the Natural Logarithm

Find

$$\int \ln x \, dx.$$

Solution Since $\int \ln x \, dx$ can be written as $\int \ln x \cdot 1 \, dx$, we use the formula $\int u \, dv = uv - \int v \, du$ with

$u = \ln x$ Simplifies when differentiated $\qquad dv = dx$ Easy to integrate

$du = \frac{1}{x} dx,$ $\qquad v = x.$ Simplest antiderivative

Then

$$\int \ln x \, dx = x \ln x - \int x \cdot \frac{1}{x} dx = x \ln x - \int dx = x \ln x - x + C.$$ ■

Sometimes we have to use integration by parts more than once.

EXAMPLE 4 Repeated Use of Integration by Parts

Evaluate

$$\int x^2 e^x \, dx.$$

Solution With $u = x^2$, $dv = e^x \, dx$, $du = 2x \, dx$, and $v = e^x$, we have

$$\int x^2 e^x \, dx = x^2 e^x - 2 \int x e^x \, dx.$$

The new integral is less complicated than the original because the exponent on x is reduced by one. To evaluate the integral on the right, we integrate by parts again with $u = x$, $dv = e^x \, dx$. Then $du = dx$, $v = e^x$, and

$$\int x e^x \, dx = x e^x - \int e^x \, dx = x e^x - e^x + C.$$

Hence,

$$\begin{aligned} \int x^2 e^x \, dx &= x^2 e^x - 2 \int x e^x \, dx \\ &= x^2 e^x - 2x e^x + 2e^x + C. \end{aligned}$$

The technique of Example 4 works for any integral $\int x^n e^x \, dx$ in which n is a positive integer, because differentiating x^n will eventually lead to zero and integrating e^x is easy. We say more about this later in this section when we discuss *tabular integration.*

Integrals like the one in the next example occur in electrical engineering. Their evaluation requires two integrations by parts, followed by solving for the unknown integral.

EXAMPLE 5 Solving for the Unknown Integral

Evaluate

$$\int e^x \cos x \, dx.$$

Solution Let $u = e^x$ and $dv = \cos x \, dx$. Then $du = e^x \, dx$, $v = \sin x$, and

$$\int e^x \cos x \, dx = e^x \sin x - \int e^x \sin x \, dx.$$

The second integral is like the first except that it has $\sin x$ in place of $\cos x$. To evaluate it, we use integration by parts with

$$u = e^x, \qquad dv = \sin x \, dx, \qquad v = -\cos x, \qquad du = e^x \, dx.$$

Then

$$\begin{aligned} \int e^x \cos x \, dx &= e^x \sin x - \left(-e^x \cos x - \int (-\cos x)(e^x \, dx) \right) \\ &= e^x \sin x + e^x \cos x - \int e^x \cos x \, dx. \end{aligned}$$

The unknown integral now appears on both sides of the equation. Adding the integral to both sides and adding the constant of integration gives

$$2\int e^x \cos x\, dx = e^x \sin x + e^x \cos x + C_1.$$

Dividing by 2 and renaming the constant of integration gives

$$\int e^x \cos x\, dx = \frac{e^x \sin x + e^x \cos x}{2} + C.$$ ■

Evaluating Definite Integrals by Parts

The integration by parts formula in Equation (1) can be combined with Part 2 of the Fundamental Theorem in order to evaluate definite integrals by parts. Assuming that both f' and g' are continuous over the interval $[a, b]$, Part 2 of the Fundamental Theorem gives

Integration by Parts Formula for Definite Integrals

$$\int_a^b f(x)g'(x)\, dx = f(x)g(x)\Big]_a^b - \int_a^b f'(x)g(x)\, dx \qquad (3)$$

In applying Equation (3), we normally use the u and v notation from Equation (2) because it is easier to remember. Here is an example.

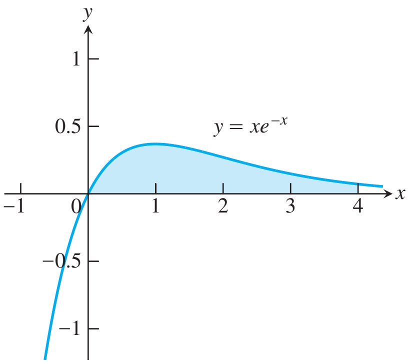


FIGURE 8.1 The region in Example 6.

EXAMPLE 6 Finding Area

Find the area of the region bounded by the curve $y = xe^{-x}$ and the x-axis from $x = 0$ to $x = 4$.

Solution The region is shaded in Figure 8.1. Its area is

$$\int_0^4 xe^{-x}\, dx.$$

Let $u = x$, $dv = e^{-x}\, dx$, $v = -e^{-x}$, and $du = dx$. Then,

$$\begin{aligned}\int_0^4 xe^{-x}\, dx &= -xe^{-x}\Big]_0^4 - \int_0^4 (-e^{-x})\, dx \\ &= [-4e^{-4} - (0)] + \int_0^4 e^{-x}\, dx \\ &= -4e^{-4} - e^{-x}\Big]_0^4 \\ &= -4e^{-4} - e^{-4} - (-e^0) = 1 - 5e^{-4} \approx 0.91.\end{aligned}$$ ■

Tabular Integration

We have seen that integrals of the form $\int f(x)g(x)\, dx$, in which f can be differentiated repeatedly to become zero and g can be integrated repeatedly without difficulty, are natural candidates for integration by parts. However, if many repetitions are required, the calculations can be cumbersome. In situations like this, there is a way to organize the calculations that saves a great deal of work. It is called **tabular integration** and is illustrated in the following examples.

EXAMPLE 7 Using Tabular Integration

Evaluate

$$\int x^2 e^x \, dx.$$

Solution With $f(x) = x^2$ and $g(x) = e^x$, we list:

$f(x)$ and its derivatives		**$g(x)$ and its integrals**
x^2	$(+)$	e^x
$2x$	$(-)$	e^x
2	$(+)$	e^x
0		e^x

We combine the products of the functions connected by the arrows according to the operation signs above the arrows to obtain

$$\int x^2 e^x \, dx = x^2 e^x - 2xe^x + 2e^x + C.$$

Compare this with the result in Example 4. ■

EXAMPLE 8 Using Tabular Integration

Evaluate

$$\int x^3 \sin x \, dx.$$

Solution With $f(x) = x^3$ and $g(x) = \sin x$, we list:

$f(x)$ and its derivatives		**$g(x)$ and its integrals**
x^3	$(+)$	$\sin x$
$3x^2$	$(-)$	$-\cos x$
$6x$	$(+)$	$-\sin x$
6	$(-)$	$\cos x$
0		$\sin x$

Again we combine the products of the functions connected by the arrows according to the operation signs above the arrows to obtain

$$\int x^3 \sin x \, dx = -x^3 \cos x + 3x^2 \sin x + 6x \cos x - 6 \sin x + C.$$ ■

The Additional Exercises at the end of this chapter show how tabular integration can be used when neither function f nor g can be differentiated repeatedly to become zero.

Summary

When substitution doesn't work, try integration by parts. Start with an integral in which the integrand is the product of two functions,

$$\int f(x)g(x)\,dx.$$

(Remember that g may be the constant function 1, as in Example 3.) Match the integral with the form

$$\int u\,dv$$

by choosing dv to be part of the integrand including dx and either $f(x)$ or $g(x)$. Remember that we must be able to readily integrate dv to get v in order to obtain the right side of the formula

$$\int u\,dv = uv - \int v\,du.$$

If the new integral on the right side is more complex than the original one, try a different choice for u and dv.

EXAMPLE 9 A Reduction Formula

Obtain a "reduction" formula that expresses the integral

$$\int \cos^n x\,dx$$

in terms of an integral of a lower power of $\cos x$.

Solution We may think of $\cos^n x$ as $\cos^{n-1} x \cdot \cos x$. Then we let

$$u = \cos^{n-1} x \qquad \text{and} \qquad dv = \cos x\,dx,$$

so that

$$du = (n-1)\cos^{n-2} x\,(-\sin x\,dx) \qquad \text{and} \qquad v = \sin x.$$

Hence

$$\begin{aligned}\int \cos^n x\,dx &= \cos^{n-1} x \sin x + (n-1)\int \sin^2 x \cos^{n-2} x\,dx \\ &= \cos^{n-1} x \sin x + (n-1)\int (1-\cos^2 x)\cos^{n-2} x\,dx, \\ &= \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x\,dx - (n-1)\int \cos^n x\,dx.\end{aligned}$$

If we add

$$(n-1)\int \cos^n x\,dx$$

to both sides of this equation, we obtain

$$n\int \cos^n x\,dx = \cos^{n-1} x \sin x + (n-1)\int \cos^{n-2} x\,dx.$$

We then divide through by n, and the final result is

$$\int \cos^n x\,dx = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n}\int \cos^{n-2} x\,dx.$$

This allows us to reduce the exponent on $\cos x$ by 2 and is a very useful formula. When n is a positive integer, we may apply the formula repeatedly until the remaining integral is either

$$\int \cos x\,dx = \sin x + C \qquad \text{or} \qquad \int \cos^0 x\,dx = \int dx = x + C.$$

■

EXAMPLE 10 Using a Reduction Formula

Evaluate

$$\int \cos^3 x\,dx.$$

Solution From the result in Example 9,

$$\begin{aligned}\int \cos^3 x\,dx &= \frac{\cos^2 x \sin x}{3} + \frac{2}{3}\int \cos x\,dx \\ &= \frac{1}{3}\cos^2 x \sin x + \frac{2}{3}\sin x + C.\end{aligned}$$

■

EXERCISES 8.2

Integration by Parts

Evaluate the integrals in Exercises 1–24.

1. $\int x \sin \frac{x}{2}\, dx$
2. $\int \theta \cos \pi\theta\, d\theta$
3. $\int t^2 \cos t\, dt$
4. $\int x^2 \sin x\, dx$
5. $\int_1^2 x \ln x\, dx$
6. $\int_1^e x^3 \ln x\, dx$
7. $\int \tan^{-1} y\, dy$
8. $\int \sin^{-1} y\, dy$
9. $\int x \sec^2 x\, dx$
10. $\int 4x \sec^2 2x\, dx$
11. $\int x^3 e^x\, dx$
12. $\int p^4 e^{-p}\, dp$
13. $\int (x^2 - 5x)e^x\, dx$
14. $\int (r^2 + r + 1)e^r\, dr$
15. $\int x^5 e^x\, dx$
16. $\int t^2 e^{4t}\, dt$
17. $\int_0^{\pi/2} \theta^2 \sin 2\theta\, d\theta$
18. $\int_0^{\pi/2} x^3 \cos 2x\, dx$
19. $\int_{2/\sqrt{3}}^{2} t \sec^{-1} t\, dt$
20. $\int_0^{1/\sqrt{2}} 2x \sin^{-1}(x^2)\, dx$
21. $\int e^\theta \sin \theta\, d\theta$
22. $\int e^{-y} \cos y\, dy$
23. $\int e^{2x} \cos 3x\, dx$
24. $\int e^{-2x} \sin 2x\, dx$

Substitution and Integration by Parts

Evaluate the integrals in Exercises 25–30 by using a substitution prior to integration by parts.

25. $\int e^{\sqrt{3s+9}}\,ds$

26. $\int_0^1 x\sqrt{1-x}\,dx$

27. $\int_0^{\pi/3} x\tan^2 x\,dx$

28. $\int \ln(x+x^2)\,dx$

29. $\int \sin(\ln x)\,dx$

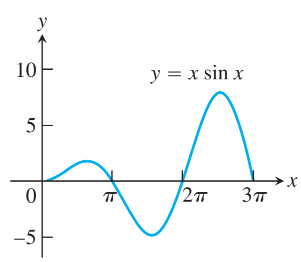
30. $\int z(\ln z)^2\,dz$

Theory and Examples

31. Finding area Find the area of the region enclosed by the curve $y = x\sin x$ and the x-axis (see the accompanying figure) for

a. $0 \le x \le \pi$ **b.** $\pi \le x \le 2\pi$ **c.** $2\pi \le x \le 3\pi$.

d. What pattern do you see here? What is the area between the curve and the x-axis for $n\pi \le x \le (n+1)\pi$, n an arbitrary nonnegative integer? Give reasons for your answer.



32. Finding area Find the area of the region enclosed by the curve $y = x\cos x$ and the x-axis (see the accompanying figure) for

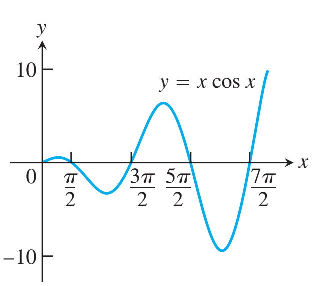
a. $\pi/2 \le x \le 3\pi/2$ **b.** $3\pi/2 \le x \le 5\pi/2$

c. $5\pi/2 \le x \le 7\pi/2$.

d. What pattern do you see? What is the area between the curve and the x-axis for

$$\left(\frac{2n-1}{2}\right)\pi \le x \le \left(\frac{2n+1}{2}\right)\pi,$$

n an arbitrary positive integer? Give reasons for your answer.



33. Finding volume Find the volume of the solid generated by revolving the region in the first quadrant bounded by the coordinate axes, the curve $y = e^x$, and the line $x = \ln 2$ about the line $x = \ln 2$.

34. Finding volume Find the volume of the solid generated by revolving the region in the first quadrant bounded by the coordinate axes, the curve $y = e^{-x}$, and the line $x = 1$

a. about the y-axis. **b.** about the line $x = 1$.

35. Finding volume Find the volume of the solid generated by revolving the region in the first quadrant bounded by the coordinate axes and the curve $y = \cos x$, $0 \le x \le \pi/2$, about

a. the y-axis. **b.** the line $x = \pi/2$.

36. Finding volume Find the volume of the solid generated by revolving the region bounded by the x-axis and the curve $y = x\sin x$, $0 \le x \le \pi$, about

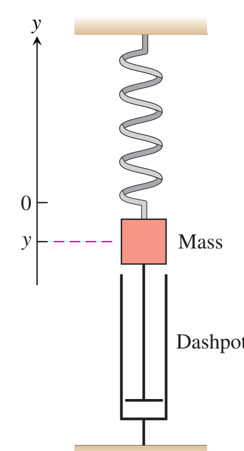
a. the y-axis. **b.** the line $x = \pi$.

(See Exercise 31 for a graph.)

37. Average value A retarding force, symbolized by the dashpot in the figure, slows the motion of the weighted spring so that the mass's position at time t is

$$y = 2e^{-t}\cos t, \qquad t \ge 0.$$

Find the average value of y over the interval $0 \le t \le 2\pi$.



38. Average value In a mass-spring-dashpot system like the one in Exercise 37, the mass's position at time t is

$$y = 4e^{-t}(\sin t - \cos t), \qquad t \ge 0.$$

Find the average value of y over the interval $0 \le t \le 2\pi$.

Reduction Formulas

In Exercises 39–42, use integration by parts to establish the *reduction formula*.

39. $\int x^n \cos x\,dx = x^n \sin x - n\int x^{n-1}\sin x\,dx$

40. $\int x^n \sin x\,dx = -x^n \cos x + n\int x^{n-1}\cos x\,dx$

41. $\int x^n e^{ax}\,dx = \dfrac{x^n e^{ax}}{a} - \dfrac{n}{a}\int x^{n-1}e^{ax}\,dx, \quad a \neq 0$

42. $\int (\ln x)^n\,dx = x(\ln x)^n - n\int (\ln x)^{n-1}\,dx$

Integrating Inverses of Functions

Integration by parts leads to a rule for integrating inverses that usually gives good results:

$$\begin{aligned}\int f^{-1}(x)\,dx &= \int y f'(y)\,dy && y = f^{-1}(x),\quad x = f(y),\ dx = f'(y)\,dy\\ &= yf(y) - \int f(y)\,dy && \text{Integration by parts with } u = y,\ dv = f'(y)\,dy\\ &= xf^{-1}(x) - \int f(y)\,dy\end{aligned}$$

The idea is to take the most complicated part of the integral, in this case $f^{-1}(x)$, and simplify it first. For the integral of $\ln x$, we get

$$\begin{aligned}\int \ln x\,dx &= \int ye^y\,dy && y = \ln x,\quad x = e^y,\ dx = e^y\,dy\\ &= ye^y - e^y + C\\ &= x\ln x - x + C.\end{aligned}$$

For the integral of $\cos^{-1} x$ we get

$$\begin{aligned}\int \cos^{-1} x\,dx &= x\cos^{-1} x - \int \cos y\,dy && y = \cos^{-1} x\\ &= x\cos^{-1} x - \sin y + C\\ &= x\cos^{-1} x - \sin(\cos^{-1} x) + C.\end{aligned}$$

Use the formula

$$\int f^{-1}(x)\,dx = xf^{-1}(x) - \int f(y)\,dy \qquad y = f^{-1}(x) \tag{4}$$

to evaluate the integrals in Exercises 43–46. Express your answers in terms of x.

43. $\int \sin^{-1} x\,dx$

44. $\int \tan^{-1} x\,dx$

45. $\int \sec^{-1} x\,dx$

46. $\int \log_2 x\,dx$

Another way to integrate $f^{-1}(x)$ (when f^{-1} is integrable, of course) is to use integration by parts with $u = f^{-1}(x)$ and $dv = dx$ to rewrite the integral of f^{-1} as

$$\int f^{-1}(x)\,dx = xf^{-1}(x) - \int x\left(\frac{d}{dx}f^{-1}(x)\right)dx. \tag{5}$$

Exercises 47 and 48 compare the results of using Equations (4) and (5).

47. Equations (4) and (5) give different formulas for the integral of $\cos^{-1} x$:

a. $\int \cos^{-1} x\,dx = x\cos^{-1} x - \sin(\cos^{-1} x) + C$ Eq. (4)

b. $\int \cos^{-1} x\,dx = x\cos^{-1} x - \sqrt{1 - x^2} + C$ Eq. (5)

Can both integrations be correct? Explain.

48. Equations (4) and (5) lead to different formulas for the integral of $\tan^{-1} x$:

a. $\int \tan^{-1} x\,dx = x\tan^{-1} x - \ln\sec(\tan^{-1} x) + C$ Eq. (4)

b. $\int \tan^{-1} x\,dx = x\tan^{-1} x - \ln\sqrt{1 + x^2} + C$ Eq. (5)

Can both integrations be correct? Explain.

Evaluate the integrals in Exercises 49 and 50 with **(a)** Eq. (4) and **(b)** Eq. (5). In each case, check your work by differentiating your answer with respect to x.

49. $\int \sinh^{-1} x\,dx$

50. $\int \tanh^{-1} x\,dx$

8.3 Integration of Rational Functions by Partial Fractions

This section shows how to express a rational function (a quotient of polynomials) as a sum of simpler fractions, called *partial fractions*, which are easily integrated. For instance, the rational function $(5x - 3)/(x^2 - 2x - 3)$ can be rewritten as

$$\frac{5x - 3}{x^2 - 2x - 3} = \frac{2}{x + 1} + \frac{3}{x - 3},$$

which can be verified algebraically by placing the fractions on the right side over a common denominator $(x+1)(x-3)$. The skill acquired in writing rational functions as such a sum is useful in other settings as well (for instance, when using certain transform methods to solve differential equations). To integrate the rational function $(5x-3)/(x+1)(x-3)$ on the left side of our previous expression, we simply sum the integrals of the fractions on the right side:

$$\int \frac{5x-3}{(x+1)(x-3)}\,dx = \int \frac{2}{x+1}\,dx + \int \frac{3}{x-3}\,dx$$
$$= 2\ln|x+1| + 3\ln|x-3| + C.$$

The method for rewriting rational functions as a sum of simpler fractions is called **the method of partial fractions**. In the case of the above example, it consists of finding constants A and B such that

$$\frac{5x-3}{x^2-2x-3} = \frac{A}{x+1} + \frac{B}{x-3}. \tag{1}$$

(Pretend for a moment that we do not know that $A = 2$ and $B = 3$ will work.) We call the fractions $A/(x+1)$ and $B/(x-3)$ **partial fractions** because their denominators are only part of the original denominator $x^2 - 2x - 3$. We call A and B **undetermined coefficients** until proper values for them have been found.

To find A and B, we first clear Equation (1) of fractions, obtaining

$$5x - 3 = A(x-3) + B(x+1) = (A+B)x - 3A + B.$$

This will be an identity in x if and only if the coefficients of like powers of x on the two sides are equal:

You Try It

$$A + B = 5, \qquad -3A + B = -3.$$

Solving these equations simultaneously gives $A = 2$ and $B = 3$.

General Description of the Method

Success in writing a rational function $f(x)/g(x)$ as a sum of partial fractions depends on two things:

- *The degree of $f(x)$ must be less than the degree of $g(x)$.* That is, the fraction must be proper. If it isn't, divide $f(x)$ by $g(x)$ and work with the remainder term. See Example 3 of this section.
- *We must know the factors of $g(x)$.* In theory, any polynomial with real coefficients can be written as a product of real linear factors and real quadratic factors. In practice, the factors may be hard to find.

Here is how we find the partial fractions of a proper fraction $f(x)/g(x)$ when the factors of g are known.

Method of Partial Fractions ($f(x)/g(x)$ Proper)

1. Let $x - r$ be a linear factor of $g(x)$. Suppose that $(x - r)^m$ is the highest power of $x - r$ that divides $g(x)$. Then, to this factor, assign the sum of the m partial fractions:
$$\frac{A_1}{x - r} + \frac{A_2}{(x - r)^2} + \cdots + \frac{A_m}{(x - r)^m}.$$
Do this for each distinct linear factor of $g(x)$.
2. Let $x^2 + px + q$ be a quadratic factor of $g(x)$. Suppose that $(x^2 + px + q)^n$ is the highest power of this factor that divides $g(x)$. Then, to this factor, assign the sum of the n partial fractions:
$$\frac{B_1x + C_1}{x^2 + px + q} + \frac{B_2x + C_2}{(x^2 + px + q)^2} + \cdots + \frac{B_nx + C_n}{(x^2 + px + q)^n}.$$
Do this for each distinct quadratic factor of $g(x)$ that cannot be factored into linear factors with real coefficients.
3. Set the original fraction $f(x)/g(x)$ equal to the sum of all these partial fractions. Clear the resulting equation of fractions and arrange the terms in decreasing powers of x.
4. Equate the coefficients of corresponding powers of x and solve the resulting equations for the undetermined coefficients.

You Try It

EXAMPLE 1 Distinct Linear Factors

Evaluate
$$\int \frac{x^2 + 4x + 1}{(x - 1)(x + 1)(x + 3)}\, dx$$
using partial fractions.

Solution The partial fraction decomposition has the form
$$\frac{x^2 + 4x + 1}{(x - 1)(x + 1)(x + 3)} = \frac{A}{x - 1} + \frac{B}{x + 1} + \frac{C}{x + 3}.$$
To find the values of the undetermined coefficients A, B, and C we clear fractions and get
$$\begin{aligned} x^2 + 4x + 1 &= A(x + 1)(x + 3) + B(x - 1)(x + 3) + C(x - 1)(x + 1) \\ &= (A + B + C)x^2 + (4A + 2B)x + (3A - 3B - C). \end{aligned}$$
The polynomials on both sides of the above equation are identical, so we equate coefficients of like powers of x obtaining
$$\begin{aligned} &\text{Coefficient of } x^2: && A + B + C = 1 \\ &\text{Coefficient of } x^1: && 4A + 2B = 4 \\ &\text{Coefficient of } x^0: && 3A - 3B - C = 1 \end{aligned}$$

There are several ways for solving such a system of linear equations for the unknowns A, B, and C, including elimination of variables, or the use of a calculator or computer. Whatever method is used, the solution is $A = 3/4$, $B = 1/2$, and $C = -1/4$. Hence we have

$$\int \frac{x^2 + 4x + 1}{(x-1)(x+1)(x+3)}\,dx = \int \left[\frac{3}{4}\frac{1}{x-1} + \frac{1}{2}\frac{1}{x+1} - \frac{1}{4}\frac{1}{x+3}\right] dx$$
$$= \frac{3}{4}\ln|x-1| + \frac{1}{2}\ln|x+1| - \frac{1}{4}\ln|x+3| + K,$$

where K is the arbitrary constant of integration (to avoid confusion with the undetermined coefficient we labeled as C). ■

EXAMPLE 2 A Repeated Linear Factor

Evaluate

$$\int \frac{6x+7}{(x+2)^2}\,dx.$$

Solution First we express the integrand as a sum of partial fractions with undetermined coefficients.

$$\frac{6x+7}{(x+2)^2} = \frac{A}{x+2} + \frac{B}{(x+2)^2}$$
$$6x + 7 = A(x+2) + B \qquad \text{Multiply both sides by } (x+2)^2.$$
$$= Ax + (2A + B)$$

Equating coefficients of corresponding powers of x gives

$$A = 6 \quad \text{and} \quad 2A + B = 12 + B = 7, \quad \text{or} \quad A = 6 \quad \text{and} \quad B = -5.$$

Therefore,

$$\int \frac{6x+7}{(x+2)^2}\,dx = \int \left(\frac{6}{x+2} - \frac{5}{(x+2)^2}\right) dx$$
$$= 6\int \frac{dx}{x+2} - 5\int (x+2)^{-2}\,dx$$
$$= 6\ln|x+2| + 5(x+2)^{-1} + C$$ ■

EXAMPLE 3 Integrating an Improper Fraction

Evaluate

$$\int \frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3}\,dx.$$

Solution First we divide the denominator into the numerator to get a polynomial plus a proper fraction.

$$\begin{array}{r|l} & 2x \\ \hline x^2 - 2x - 3 & 2x^3 - 4x^2 - x - 3 \\ & \underline{2x^3 - 4x^2 - 6x} \\ & \qquad\qquad 5x - 3 \end{array}$$

Then we write the improper fraction as a polynomial plus a proper fraction.

$$\frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3} = 2x + \frac{5x - 3}{x^2 - 2x - 3}$$

We found the partial fraction decomposition of the fraction on the right in the opening example, so

$$\begin{aligned}\int \frac{2x^3 - 4x^2 - x - 3}{x^2 - 2x - 3}\,dx &= \int 2x\,dx + \int \frac{5x - 3}{x^2 - 2x - 3}\,dx \\ &= \int 2x\,dx + \int \frac{2}{x+1}\,dx + \int \frac{3}{x-3}\,dx \\ &= x^2 + 2\ln|x+1| + 3\ln|x-3| + C.\end{aligned}$$

A quadratic polynomial is **irreducible** if it cannot be written as the product of two linear factors with real coefficients.

EXAMPLE 4 Integrating with an Irreducible Quadratic Factor in the Denominator

Evaluate

$$\int \frac{-2x + 4}{(x^2 + 1)(x - 1)^2}\,dx$$

using partial fractions.

Solution The denominator has an irreducible quadratic factor as well as a repeated linear factor, so we write

$$\frac{-2x + 4}{(x^2 + 1)(x - 1)^2} = \frac{Ax + B}{x^2 + 1} + \frac{C}{x - 1} + \frac{D}{(x - 1)^2}. \tag{2}$$

Clearing the equation of fractions gives

$$\begin{aligned}-2x + 4 &= (Ax + B)(x - 1)^2 + C(x - 1)(x^2 + 1) + D(x^2 + 1) \\ &= (A + C)x^3 + (-2A + B - C + D)x^2 \\ &\quad + (A - 2B + C)x + (B - C + D).\end{aligned}$$

Equating coefficients of like terms gives

Coefficients of x^3:	$0 = A + C$
Coefficients of x^2:	$0 = -2A + B - C + D$
Coefficients of x^1:	$-2 = A - 2B + C$
Coefficients of x^0:	$4 = B - C + D$

We solve these equations simultaneously to find the values of A, B, C, and D:

$-4 = -2A, \quad A = 2$	Subtract fourth equation from second.
$C = -A = -2$	From the first equation
$B = 1$	$A = 2$ and $C = -2$ in third equation.
$D = 4 - B + C = 1.$	From the fourth equation

We substitute these values into Equation (2), obtaining

$$\frac{-2x+4}{(x^2+1)(x-1)^2} = \frac{2x+1}{x^2+1} - \frac{2}{x-1} + \frac{1}{(x-1)^2}.$$

Finally, using the expansion above we can integrate:

$$\begin{aligned}\int \frac{-2x+4}{(x^2+1)(x-1)^2}\,dx &= \int \left(\frac{2x+1}{x^2+1} - \frac{2}{x-1} + \frac{1}{(x-1)^2}\right) dx \\ &= \int \left(\frac{2x}{x^2+1} + \frac{1}{x^2+1} - \frac{2}{x-1} + \frac{1}{(x-1)^2}\right) dx \\ &= \ln(x^2+1) + \tan^{-1} x - 2\ln|x-1| - \frac{1}{x-1} + C. \quad \blacksquare\end{aligned}$$

You Try It

EXAMPLE 5 A Repeated Irreducible Quadratic Factor

Evaluate

$$\int \frac{dx}{x(x^2+1)^2}.$$

Solution The form of the partial fraction decomposition is

$$\frac{1}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}$$

Multiplying by $x(x^2+1)^2$, we have

$$\begin{aligned}1 &= A(x^2+1)^2 + (Bx+C)x(x^2+1) + (Dx+E)x \\ &= A(x^4+2x^2+1) + B(x^4+x^2) + C(x^3+x) + Dx^2 + Ex \\ &= (A+B)x^4 + Cx^3 + (2A+B+D)x^2 + (C+E)x + A\end{aligned}$$

If we equate coefficients, we get the system

$$A+B=0, \qquad C=0, \qquad 2A+B+D=0, \qquad C+E=0, \qquad A=1.$$

Solving this system gives $A=1$, $B=-1$, $C=0$, $D=-1$, and $E=0$. Thus,

$$\begin{aligned}\int \frac{dx}{x(x^2+1)^2} &= \int \left[\frac{1}{x} + \frac{-x}{x^2+1} + \frac{-x}{(x^2+1)^2}\right] dx \\ &= \int \frac{dx}{x} - \int \frac{x\,dx}{x^2+1} - \int \frac{x\,dx}{(x^2+1)^2} \\ &= \int \frac{dx}{x} - \frac{1}{2}\int \frac{du}{u} - \frac{1}{2}\int \frac{du}{u^2} \qquad u = x^2+1,\ du = 2x\,dx \\ &= \ln|x| - \frac{1}{2}\ln|u| + \frac{1}{2u} + K \\ &= \ln|x| - \frac{1}{2}\ln(x^2+1) + \frac{1}{2(x^2+1)} + K \\ &= \ln\frac{|x|}{\sqrt{x^2+1}} + \frac{1}{2(x^2+1)} + K. \quad \blacksquare\end{aligned}$$

Historical Biography

Oliver Heaviside
(1850–1925)

The Heaviside "Cover-up" Method for Linear Factors

When the degree of the polynomial $f(x)$ is less than the degree of $g(x)$ and

$$g(x) = (x - r_1)(x - r_2)\cdots(x - r_n)$$

is a product of n distinct linear factors, each raised to the first power, there is a quick way to expand $f(x)/g(x)$ by partial fractions.

EXAMPLE 6 Using the Heaviside Method

Find A, B, and C in the partial-fraction expansion

$$\frac{x^2 + 1}{(x - 1)(x - 2)(x - 3)} = \frac{A}{x - 1} + \frac{B}{x - 2} + \frac{C}{x - 3}. \tag{3}$$

Solution If we multiply both sides of Equation (3) by $(x - 1)$ to get

$$\frac{x^2 + 1}{(x - 2)(x - 3)} = A + \frac{B(x - 1)}{x - 2} + \frac{C(x - 1)}{x - 3}$$

and set $x = 1$, the resulting equation gives the value of A:

$$\frac{(1)^2 + 1}{(1 - 2)(1 - 3)} = A + 0 + 0,$$

$$A = 1.$$

Thus, the value of A is the number we would have obtained if we had covered the factor $(x - 1)$ in the denominator of the original fraction

$$\frac{x^2 + 1}{(x - 1)(x - 2)(x - 3)} \tag{4}$$

and evaluated the rest at $x = 1$:

$$A = \frac{(1)^2 + 1}{\underset{\substack{\Uparrow \\ \text{Cover}}}{\boxed{(x - 1)}}\,(1 - 2)(1 - 3)} = \frac{2}{(-1)(-2)} = 1.$$

Similarly, we find the value of B in Equation (3) by covering the factor $(x - 2)$ in Equation (4) and evaluating the rest at $x = 2$:

$$B = \frac{(2)^2 + 1}{(2 - 1)\underset{\substack{\Uparrow \\ \text{Cover}}}{\boxed{(x - 2)}}(2 - 3)} = \frac{5}{(1)(-1)} = -5.$$

Finally, C is found by covering the $(x - 3)$ in Equation (4) and evaluating the rest at $x = 3$:

$$C = \frac{(3)^2 + 1}{(3 - 1)(3 - 2)\underset{\substack{\Uparrow \\ \text{Cover}}}{\boxed{(x - 3)}}} = \frac{10}{(2)(1)} = 5.$$

■

Heaviside Method

1. *Write the quotient with $g(x)$ factored:*

$$\frac{f(x)}{g(x)} = \frac{f(x)}{(x - r_1)(x - r_2)\cdots(x - r_n)}.$$

2. *Cover the factors $(x - r_i)$ of $g(x)$ one at a time*, each time replacing all the uncovered x's by the number r_i. This gives a number A_i for each root r_i:

$$A_1 = \frac{f(r_1)}{(r_1 - r_2)\cdots(r_1 - r_n)}$$

$$A_2 = \frac{f(r_2)}{(r_2 - r_1)(r_2 - r_3)\cdots(r_2 - r_n)}$$

$$\vdots$$

$$A_n = \frac{f(r_n)}{(r_n - r_1)(r_n - r_2)\cdots(r_n - r_{n-1})}.$$

3. *Write the partial-fraction expansion of $f(x)/g(x)$* as

$$\frac{f(x)}{g(x)} = \frac{A_1}{(x - r_1)} + \frac{A_2}{(x - r_2)} + \cdots + \frac{A_n}{(x - r_n)}.$$

EXAMPLE 7 Integrating with the Heaviside Method

Evaluate

$$\int \frac{x + 4}{x^3 + 3x^2 - 10x}\,dx.$$

Solution The degree of $f(x) = x + 4$ is less than the degree of $g(x) = x^3 + 3x^2 - 10x$, and, with $g(x)$ factored,

$$\frac{x + 4}{x^3 + 3x^2 - 10x} = \frac{x + 4}{x(x - 2)(x + 5)}.$$

The roots of $g(x)$ are $r_1 = 0$, $r_2 = 2$, and $r_3 = -5$. We find

$$A_1 = \frac{0 + 4}{\underset{\substack{\Uparrow \\ \text{Cover}}}{\boxed{x}}\,(0 - 2)(0 + 5)} = \frac{4}{(-2)(5)} = -\frac{2}{5}$$

$$A_2 = \frac{2 + 4}{2\,\underset{\substack{\Uparrow \\ \text{Cover}}}{\boxed{(x - 2)}}\,(2 + 5)} = \frac{6}{(2)(7)} = \frac{3}{7}$$

$$A_3 = \frac{-5 + 4}{(-5)(-5 - 2)\,\underset{\substack{\Uparrow \\ \text{Cover}}}{\boxed{(x + 5)}}} = \frac{-1}{(-5)(-7)} = -\frac{1}{35}.$$

Therefore,

$$\frac{x+4}{x(x-2)(x+5)} = -\frac{2}{5x} + \frac{3}{7(x-2)} - \frac{1}{35(x+5)},$$

and

$$\int \frac{x+4}{x(x-2)(x+5)}\,dx = -\frac{2}{5}\ln|x| + \frac{3}{7}\ln|x-2| - \frac{1}{35}\ln|x+5| + C. \qquad \blacksquare$$

Other Ways to Determine the Coefficients

Another way to determine the constants that appear in partial fractions is to differentiate, as in the next example. Still another is to assign selected numerical values to x.

EXAMPLE 8 Using Differentiation

Find A, B, and C in the equation

$$\frac{x-1}{(x+1)^3} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{C}{(x+1)^3}.$$

Solution We first clear fractions:

$$x - 1 = A(x+1)^2 + B(x+1) + C.$$

Substituting $x = -1$ shows $C = -2$. We then differentiate both sides with respect to x, obtaining

$$1 = 2A(x+1) + B.$$

Substituting $x = -1$ shows $B = 1$. We differentiate again to get $0 = 2A$, which shows $A = 0$. Hence,

$$\frac{x-1}{(x+1)^3} = \frac{1}{(x+1)^2} - \frac{2}{(x+1)^3}. \qquad \blacksquare$$

In some problems, assigning small values to x such as $x = 0, \pm 1, \pm 2$, to get equations in A, B, and C provides a fast alternative to other methods.

EXAMPLE 9 Assigning Numerical Values to x

Find A, B, and C in

$$\frac{x^2+1}{(x-1)(x-2)(x-3)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3}.$$

Solution Clear fractions to get

$$x^2 + 1 = A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2).$$

Then let $x = 1, 2, 3$ successively to find A, B, and C:

$$\begin{aligned}
x = 1: \qquad (1)^2 + 1 &= A(-1)(-2) + B(0) + C(0) \\
2 &= 2A \\
A &= 1 \\
x = 2: \qquad (2)^2 + 1 &= A(0) + B(1)(-1) + C(0) \\
5 &= -B \\
B &= -5 \\
x = 3: \qquad (3)^2 + 1 &= A(0) + B(0) + C(2)(1) \\
10 &= 2C \\
C &= 5.
\end{aligned}$$

Conclusion:

$$\frac{x^2 + 1}{(x-1)(x-2)(x-3)} = \frac{1}{x-1} - \frac{5}{x-2} + \frac{5}{x-3}.$$

■

EXERCISES 8.3

Expanding Quotients into Partial Fractions

Expand the quotients in Exercises 1–8 by partial fractions.

1. $\dfrac{5x - 13}{(x - 3)(x - 2)}$

2. $\dfrac{5x - 7}{x^2 - 3x + 2}$

3. $\dfrac{x + 4}{(x + 1)^2}$

4. $\dfrac{2x + 2}{x^2 - 2x + 1}$

5. $\dfrac{z + 1}{z^2(z - 1)}$

6. $\dfrac{z}{z^3 - z^2 - 6z}$

7. $\dfrac{t^2 + 8}{t^2 - 5t + 6}$

8. $\dfrac{t^4 + 9}{t^4 + 9t^2}$

Nonrepeated Linear Factors

In Exercises 9–16, express the integrands as a sum of partial fractions and evaluate the integrals.

9. $\displaystyle\int \frac{dx}{1 - x^2}$

10. $\displaystyle\int \frac{dx}{x^2 + 2x}$

11. $\displaystyle\int \frac{x + 4}{x^2 + 5x - 6}\,dx$

12. $\displaystyle\int \frac{2x + 1}{x^2 - 7x + 12}\,dx$

13. $\displaystyle\int_4^8 \frac{y\,dy}{y^2 - 2y - 3}$

14. $\displaystyle\int_{1/2}^1 \frac{y + 4}{y^2 + y}\,dy$

15. $\displaystyle\int \frac{dt}{t^3 + t^2 - 2t}$

16. $\displaystyle\int \frac{x + 3}{2x^3 - 8x}\,dx$

Repeated Linear Factors

In Exercises 17–20, express the integrands as a sum of partial fractions and evaluate the integrals.

17. $\displaystyle\int_0^1 \frac{x^3\,dx}{x^2 + 2x + 1}$

18. $\displaystyle\int_{-1}^0 \frac{x^3\,dx}{x^2 - 2x + 1}$

19. $\displaystyle\int \frac{dx}{(x^2 - 1)^2}$

20. $\displaystyle\int \frac{x^2\,dx}{(x - 1)(x^2 + 2x + 1)}$

Irreducible Quadratic Factors

In Exercises 21–28, express the integrands as a sum of partial fractions and evaluate the integrals.

21. $\displaystyle\int_0^1 \frac{dx}{(x + 1)(x^2 + 1)}$

22. $\displaystyle\int_1^{\sqrt{3}} \frac{3t^2 + t + 4}{t^3 + t}\,dt$

23. $\displaystyle\int \frac{y^2 + 2y + 1}{(y^2 + 1)^2}\,dy$

24. $\displaystyle\int \frac{8x^2 + 8x + 2}{(4x^2 + 1)^2}\,dx$

25. $\displaystyle\int \frac{2s + 2}{(s^2 + 1)(s - 1)^3}\,ds$

26. $\displaystyle\int \frac{s^4 + 81}{s(s^2 + 9)^2}\,ds$

27. $\displaystyle\int \frac{2\theta^3 + 5\theta^2 + 8\theta + 4}{(\theta^2 + 2\theta + 2)^2}\,d\theta$

28. $\displaystyle\int \frac{\theta^4 - 4\theta^3 + 2\theta^2 - 3\theta + 1}{(\theta^2 + 1)^3}\,d\theta$

Improper Fractions

In Exercises 29–34, perform long division on the integrand, write the proper fraction as a sum of partial fractions, and then evaluate the integral.

29. $\displaystyle\int \frac{2x^3 - 2x^2 + 1}{x^2 - x}\,dx$

30. $\displaystyle\int \frac{x^4}{x^2 - 1}\,dx$

31. $\displaystyle\int \frac{9x^3 - 3x + 1}{x^3 - x^2}\,dx$

32. $\displaystyle\int \frac{16x^3}{4x^2 - 4x + 1}\,dx$

33. $\displaystyle\int \frac{y^4 + y^2 - 1}{y^3 + y}\,dy$

34. $\displaystyle\int \frac{2y^4}{y^3 - y^2 + y - 1}\,dy$

Evaluating Integrals

Evaluate the integrals in Exercises 35–40.

35. $\displaystyle\int \frac{e^t\,dt}{e^{2t} + 3e^t + 2}$

36. $\displaystyle\int \frac{e^{4t} + 2e^{2t} - e^t}{e^{2t} + 1}\,dt$

37. $\displaystyle\int \frac{\cos y\,dy}{\sin^2 y + \sin y - 6}$

38. $\displaystyle\int \frac{\sin\theta\,d\theta}{\cos^2\theta + \cos\theta - 2}$

39. $\displaystyle\int \frac{(x-2)^2\tan^{-1}(2x) - 12x^3 - 3x}{(4x^2+1)(x-2)^2}\,dx$

40. $\displaystyle\int \frac{(x+1)^2\tan^{-1}(3x) + 9x^3 + x}{(9x^2+1)(x+1)^2}\,dx$

Initial Value Problems

Solve the initial value problems in Exercises 41–44 for x as a function of t.

41. $(t^2 - 3t + 2)\dfrac{dx}{dt} = 1 \quad (t > 2), \quad x(3) = 0$

42. $(3t^4 + 4t^2 + 1)\dfrac{dx}{dt} = 2\sqrt{3}, \quad x(1) = -\pi\sqrt{3}/4$

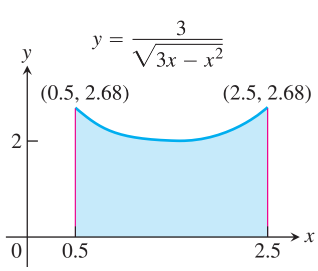
43. $(t^2 + 2t)\dfrac{dx}{dt} = 2x + 2 \quad (t, x > 0), \quad x(1) = 1$

44. $(t + 1)\dfrac{dx}{dt} = x^2 + 1 \quad (t > -1), \quad x(0) = \pi/4$

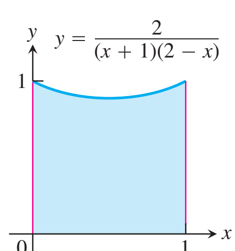
Applications and Examples

In Exercises 45 and 46, find the volume of the solid generated by revolving the shaded region about the indicated axis.

45. The x-axis

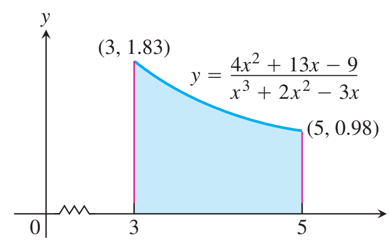


46. The y-axis



47. Find, to two decimal places, the x-coordinate of the centroid of the region in the first quadrant bounded by the x-axis, the curve $y = \tan^{-1} x$, and the line $x = \sqrt{3}$.

48. Find the x-coordinate of the centroid of this region to two decimal places.



49. Social diffusion Sociologists sometimes use the phrase "social diffusion" to describe the way information spreads through a population. The information might be a rumor, a cultural fad, or news about a technical innovation. In a sufficiently large population, the number of people x who have the information is treated as a differentiable function of time t, and the rate of diffusion, dx/dt, is assumed to be proportional to the number of people who have the information times the number of people who do not. This leads to the equation

$$\frac{dx}{dt} = kx(N - x),$$

where N is the number of people in the population.

Suppose t is in days, $k = 1/250$, and two people start a rumor at time $t = 0$ in a population of $N = 1000$ people.

a. Find x as a function of t.

b. When will half the population have heard the rumor? (This is when the rumor will be spreading the fastest.)

50. Second-order chemical reactions Many chemical reactions are the result of the interaction of two molecules that undergo a change to produce a new product. The rate of the reaction typically depends on the concentrations of the two kinds of molecules. If a is the amount of substance A and b is the amount of substance B at time $t = 0$, and if x is the amount of product at time t, then the rate of formation of x may be given by the differential equation

$$\frac{dx}{dt} = k(a - x)(b - x),$$

or

$$\frac{1}{(a - x)(b - x)}\frac{dx}{dt} = k,$$

where k is a constant for the reaction. Integrate both sides of this equation to obtain a relation between x and t **(a)** if $a = b$, and **(b)** if $a \neq b$. Assume in each case that $x = 0$ when $t = 0$.

51. An integral connecting π to the approximation 22/7

a. Evaluate $\displaystyle\int_0^1 \frac{x^4(x-1)^4}{x^2+1}\,dx$.

b. How good is the approximation $\pi \approx 22/7$? Find out by expressing $\left(\dfrac{22}{7} - \pi\right)$ as a percentage of π.

c. Graph the function $y = \dfrac{x^4(x-1)^4}{x^2+1}$ for $0 \le x \le 1$. Experiment with the range on the y-axis set between 0 and 1, then between 0 and 0.5, and then decreasing the range until the graph can be seen. What do you conclude about the area under the curve?

52. Find the second-degree polynomial $P(x)$ such that $P(0) = 1$, $P'(0) = 0$, and

$$\int \frac{P(x)}{x^3(x-1)^2}\,dx$$

is a rational function.

8.4 Trigonometric Integrals

Trigonometric integrals involve algebraic combinations of the six basic trigonometric functions. In principle, we can always express such integrals in terms of sines and cosines, but it is often simpler to work with other functions, as in the integral

$$\int \sec^2 x\, dx = \tan x + C.$$

The general idea is to use identities to transform the integrals we have to find into integrals that are easier to work with.

Products of Powers of Sines and Cosines

We begin with integrals of the form:

$$\int \sin^m x \cos^n x\, dx,$$

where m and n are nonnegative integers (positive or zero). We can divide the work into three cases.

Case 1 If m is odd, we write m as $2k + 1$ and use the identity $\sin^2 x = 1 - \cos^2 x$ to obtain

$$\sin^m x = \sin^{2k+1} x = (\sin^2 x)^k \sin x = (1 - \cos^2 x)^k \sin x. \tag{1}$$

Then we combine the single $\sin x$ with dx in the integral and set $\sin x\, dx$ equal to $-d(\cos x)$.

Case 2 If m is even and n is odd in $\int \sin^m x \cos^n x\, dx$, we write n as $2k + 1$ and use the identity $\cos^2 x = 1 - \sin^2 x$ to obtain

$$\cos^n x = \cos^{2k+1} x = (\cos^2 x)^k \cos x = (1 - \sin^2 x)^k \cos x.$$

We then combine the single $\cos x$ with dx and set $\cos x\, dx$ equal to $d(\sin x)$.

Case 3 If both m and n are even in $\int \sin^m x \cos^n x\, dx$, we substitute

$$\sin^2 x = \frac{1 - \cos 2x}{2}, \qquad \cos^2 x = \frac{1 + \cos 2x}{2} \tag{2}$$

to reduce the integrand to one in lower powers of $\cos 2x$.

Here are some examples illustrating each case.

EXAMPLE 1 *m* is Odd

Evaluate

$$\int \sin^3 x \cos^2 x\, dx.$$

Solution

$$\begin{aligned}
\int \sin^3 x \cos^2 x \, dx &= \int \sin^2 x \cos^2 x \sin x \, dx \\
&= \int (1 - \cos^2 x) \cos^2 x \, (-d\,(\cos x)) \\
&= \int (1 - u^2)(u^2)(-du) && u = \cos x \\
&= \int (u^4 - u^2)\, du \\
&= \frac{u^5}{5} - \frac{u^3}{3} + C \\
&= \frac{\cos^5 x}{5} - \frac{\cos^3 x}{3} + C.
\end{aligned}$$

■

EXAMPLE 2 *m* is Even and *n* is Odd

Evaluate

$$\int \cos^5 x \, dx.$$

Solution

$$\begin{aligned}
\int \cos^5 x \, dx = \int \cos^4 x \cos x \, dx &= \int (1 - \sin^2 x)^2 \, d(\sin x) && m = 0 \\
&= \int (1 - u^2)^2 \, du && u = \sin x \\
&= \int (1 - 2u^2 + u^4)\, du \\
&= u - \frac{2}{3}u^3 + \frac{1}{5}u^5 + C = \sin x - \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x + C.
\end{aligned}$$

■

EXAMPLE 3 *m* and *n* are Both Even

Evaluate

$$\int \sin^2 x \cos^4 x \, dx.$$

Solution

$$\begin{aligned}
\int \sin^2 x \cos^4 x \, dx &= \int \left(\frac{1 - \cos 2x}{2}\right)\left(\frac{1 + \cos 2x}{2}\right)^2 dx \\
&= \frac{1}{8}\int (1 - \cos 2x)(1 + 2\cos 2x + \cos^2 2x)\, dx \\
&= \frac{1}{8}\int (1 + \cos 2x - \cos^2 2x - \cos^3 2x)\, dx \\
&= \frac{1}{8}\left[x + \frac{1}{2}\sin 2x - \int (\cos^2 2x + \cos^3 2x)\, dx\right].
\end{aligned}$$

For the term involving $\cos^2 2x$ we use

$$\int \cos^2 2x\,dx = \frac{1}{2}\int (1 + \cos 4x)\,dx$$

$$= \frac{1}{2}\left(x + \frac{1}{4}\sin 4x\right).$$

Omitting the constant of integration until the final result

For the $\cos^3 2x$ term we have

$$\int \cos^3 2x\,dx = \int (1 - \sin^2 2x)\cos 2x\,dx$$

$u = \sin 2x,$
$du = 2\cos 2x\,dx$

$$= \frac{1}{2}\int (1 - u^2)\,du = \frac{1}{2}\left(\sin 2x - \frac{1}{3}\sin^3 2x\right).$$

Again omitting C

Combining everything and simplifying we get

$$\int \sin^2 x \cos^4 x\,dx = \frac{1}{16}\left(x - \frac{1}{4}\sin 4x + \frac{1}{3}\sin^3 2x\right) + C.$$

■

Eliminating Square Roots

In the next example, we use the identity $\cos^2\theta = (1 + \cos 2\theta)/2$ to eliminate a square root.

EXAMPLE 4 Evaluate

$$\int_0^{\pi/4} \sqrt{1 + \cos 4x}\,dx.$$

Solution To eliminate the square root we use the identity

$$\cos^2\theta = \frac{1 + \cos 2\theta}{2}, \qquad \text{or} \qquad 1 + \cos 2\theta = 2\cos^2\theta.$$

With $\theta = 2x$, this becomes

$$1 + \cos 4x = 2\cos^2 2x.$$

Therefore,

$$\int_0^{\pi/4} \sqrt{1 + \cos 4x}\,dx = \int_0^{\pi/4} \sqrt{2\cos^2 2x}\,dx = \int_0^{\pi/4} \sqrt{2}\sqrt{\cos^2 2x}\,dx$$

$$= \sqrt{2}\int_0^{\pi/4} |\cos 2x|\,dx = \sqrt{2}\int_0^{\pi/4} \cos 2x\,dx$$

$\cos 2x \geq 0$ on $[0, \pi/4]$

$$= \sqrt{2}\left[\frac{\sin 2x}{2}\right]_0^{\pi/4} = \frac{\sqrt{2}}{2}[1 - 0] = \frac{\sqrt{2}}{2}.$$

■

Integrals of Powers of tan x and sec x

We know how to integrate the tangent and secant and their squares. To integrate higher powers we use the identities $\tan^2 x = \sec^2 x - 1$ and $\sec^2 x = \tan^2 x + 1$, and integrate by parts when necessary to reduce the higher powers to lower powers.

EXAMPLE 5 Evaluate

$$\int \tan^4 x\,dx.$$

Solution

$$\begin{aligned}\int \tan^4 x\,dx &= \int \tan^2 x \cdot \tan^2 x\,dx = \int \tan^2 x \cdot (\sec^2 x - 1)\,dx \\ &= \int \tan^2 x \sec^2 x\,dx - \int \tan^2 x\,dx \\ &= \int \tan^2 x \sec^2 x\,dx - \int (\sec^2 x - 1)\,dx \\ &= \int \tan^2 x \sec^2 x\,dx - \int \sec^2 x\,dx + \int dx.\end{aligned}$$

In the first integral, we let

$$u = \tan x, \qquad du = \sec^2 x\,dx$$

and have

$$\int u^2\,du = \frac{1}{3}u^3 + C_1.$$

The remaining integrals are standard forms, so

$$\int \tan^4 x\,dx = \frac{1}{3}\tan^3 x - \tan x + x + C.$$

■

EXAMPLE 6 Evaluate

$$\int \sec^3 x\,dx.$$

Solution We integrate by parts, using

$$u = \sec x, \qquad dv = \sec^2 x\,dx, \qquad v = \tan x, \qquad du = \sec x \tan x\,dx.$$

Then

$$\begin{aligned}\int \sec^3 x\,dx &= \sec x \tan x - \int (\tan x)(\sec x \tan x\,dx) \\ &= \sec x \tan x - \int (\sec^2 x - 1)\sec x\,dx \qquad \tan^2 x = \sec^2 x - 1 \\ &= \sec x \tan x + \int \sec x\,dx - \int \sec^3 x\,dx.\end{aligned}$$

Combining the two secant-cubed integrals gives

$$2\int \sec^3 x\,dx = \sec x \tan x + \int \sec x\,dx$$

and

$$\int \sec^3 x\,dx = \frac{1}{2}\sec x \tan x + \frac{1}{2}\ln|\sec x + \tan x| + C.$$

Products of Sines and Cosines

The integrals

$$\int \sin mx \sin nx\,dx, \qquad \int \sin mx \cos nx\,dx, \qquad \text{and} \qquad \int \cos mx \cos nx\,dx$$

arise in many places where trigonometric functions are applied to problems in mathematics and science. We can evaluate these integrals through integration by parts, but two such integrations are required in each case. It is simpler to use the identities

$$\sin mx \sin nx = \frac{1}{2}[\cos(m-n)x - \cos(m+n)x], \tag{3}$$

$$\sin mx \cos nx = \frac{1}{2}[\sin(m-n)x + \sin(m+n)x], \tag{4}$$

$$\cos mx \cos nx = \frac{1}{2}[\cos(m-n)x + \cos(m+n)x]. \tag{5}$$

These come from the angle sum formulas for the sine and cosine functions (Section 1.6). They give functions whose antiderivatives are easily found.

EXAMPLE 7 Evaluate

$$\int \sin 3x \cos 5x\,dx.$$

Solution From Equation (4) with $m = 3$ and $n = 5$ we get

$$\begin{aligned}\int \sin 3x \cos 5x\,dx &= \frac{1}{2}\int [\sin(-2x) + \sin 8x]\,dx \\ &= \frac{1}{2}\int (\sin 8x - \sin 2x)\,dx \\ &= -\frac{\cos 8x}{16} + \frac{\cos 2x}{4} + C.\end{aligned}$$

EXERCISES 8.4

Products of Powers of Sines and Cosines

Evaluate the integrals in Exercises 1–14.

1. $\displaystyle\int_0^{\pi/2} \sin^5 x\,dx$

2. $\displaystyle\int_0^{\pi} \sin^5 \frac{x}{2}\,dx$

3. $\displaystyle\int_{-\pi/2}^{\pi/2} \cos^3 x\,dx$

4. $\displaystyle\int_0^{\pi/6} 3\cos^5 3x\,dx$

5. $\displaystyle\int_0^{\pi/2} \sin^7 y\,dy$

6. $\displaystyle\int_0^{\pi/2} 7\cos^7 t\,dt$

7. $\int_0^{\pi} 8\sin^4 x\,dx$

8. $\int_0^{1} 8\cos^4 2\pi x\,dx$

9. $\int_{-\pi/4}^{\pi/4} 16\sin^2 x\cos^2 x\,dx$

10. $\int_0^{\pi} 8\sin^4 y\cos^2 y\,dy$

11. $\int_0^{\pi/2} 35\sin^4 x\cos^3 x\,dx$

12. $\int_0^{\pi} \sin 2x\cos^2 2x\,dx$

13. $\int_0^{\pi/4} 8\cos^3 2\theta\sin 2\theta\,d\theta$

14. $\int_0^{\pi/2} \sin^2 2\theta\cos^3 2\theta\,d\theta$

Integrals with Square Roots

Evaluate the integrals in Exercises 15–22.

15. $\int_0^{2\pi} \sqrt{\dfrac{1-\cos x}{2}}\,dx$

16. $\int_0^{\pi} \sqrt{1-\cos 2x}\,dx$

17. $\int_0^{\pi} \sqrt{1-\sin^2 t}\,dt$

18. $\int_0^{\pi} \sqrt{1-\cos^2\theta}\,d\theta$

19. $\int_{-\pi/4}^{\pi/4} \sqrt{1+\tan^2 x}\,dx$

20. $\int_{-\pi/4}^{\pi/4} \sqrt{\sec^2 x-1}\,dx$

21. $\int_0^{\pi/2} \theta\sqrt{1-\cos 2\theta}\,d\theta$

22. $\int_{-\pi}^{\pi} (1-\cos^2 t)^{3/2}\,dt$

Powers of Tan x and Sec x

Evaluate the integrals in Exercises 23–32.

23. $\int_{-\pi/3}^{0} 2\sec^3 x\,dx$

24. $\int e^x\sec^3 e^x\,dx$

25. $\int_0^{\pi/4} \sec^4\theta\,d\theta$

26. $\int_0^{\pi/12} 3\sec^4 3x\,dx$

27. $\int_{\pi/4}^{\pi/2} \csc^4\theta\,d\theta$

28. $\int_{\pi/2}^{\pi} 3\csc^4\dfrac{\theta}{2}\,d\theta$

29. $\int_0^{\pi/4} 4\tan^3 x\,dx$

30. $\int_{-\pi/4}^{\pi/4} 6\tan^4 x\,dx$

31. $\int_{\pi/6}^{\pi/3} \cot^3 x\,dx$

32. $\int_{\pi/4}^{\pi/2} 8\cot^4 t\,dt$

Products of Sines and Cosines

Evaluate the integrals in Exercises 33–38.

33. $\int_{-\pi}^{0} \sin 3x\cos 2x\,dx$

34. $\int_0^{\pi/2} \sin 2x\cos 3x\,dx$

35. $\int_{-\pi}^{\pi} \sin 3x\sin 3x\,dx$

36. $\int_0^{\pi/2} \sin x\cos x\,dx$

37. $\int_0^{\pi} \cos 3x\cos 4x\,dx$

38. $\int_{-\pi/2}^{\pi/2} \cos x\cos 7x\,dx$

Theory and Examples

39. **Surface area** Find the area of the surface generated by revolving the arc

$$x = t^{2/3},\quad y = t^2/2,\quad 0 \le t \le 2,$$

about the x-axis.

40. **Arc length** Find the length of the curve

$$y = \ln(\cos x),\quad 0 \le x \le \pi/3.$$

41. **Arc length** Find the length of the curve

$$y = \ln(\sec x),\quad 0 \le x \le \pi/4.$$

42. **Center of gravity** Find the center of gravity of the region bounded by the x-axis, the curve $y = \sec x$, and the lines $x = -\pi/4$, $x = \pi/4$.

43. **Volume** Find the volume generated by revolving one arch of the curve $y = \sin x$ about the x-axis.

44. **Area** Find the area between the x-axis and the curve $y = \sqrt{1+\cos 4x}$, $0 \le x \le \pi$.

45. **Orthogonal functions** Two functions f and g are said to be **orthogonal** on an interval $a \le x \le b$ if $\int_a^b f(x)g(x)\,dx = 0$.

a. Prove that $\sin mx$ and $\sin nx$ are orthogonal on any interval of length 2π provided m and n are integers such that $m^2 \ne n^2$.

b. Prove the same for $\cos mx$ and $\cos nx$.

c. Prove the same for $\sin mx$ and $\cos nx$ even if $m = n$.

46. **Fourier series** A finite Fourier series is given by the sum

$$f(x) = \sum_{n=1}^{N} a_n \sin nx = a_1\sin x + a_2\sin 2x + \cdots + a_N\sin Nx$$

Show that the mth coefficient a_m is given by the formula

$$a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin mx\,dx.$$

8.5 Trigonometric Substitutions

Trigonometric substitutions can be effective in transforming integrals involving $\sqrt{a^2 - x^2}$, $\sqrt{a^2 + x^2}$, and $\sqrt{x^2 - a^2}$ into integrals we can evaluate directly.

Three Basic Substitutions

The most common substitutions are $x = a\tan\theta$, $x = a\sin\theta$, and $x = a\sec\theta$. They come from the reference right triangles in Figure 8.2.

With $x = a\tan\theta$,

$$a^2 + x^2 = a^2 + a^2\tan^2\theta = a^2(1 + \tan^2\theta) = a^2\sec^2\theta.$$

With $x = a\sin\theta$,

$$a^2 - x^2 = a^2 - a^2\sin^2\theta = a^2(1 - \sin^2\theta) = a^2\cos^2\theta.$$

With $x = a\sec\theta$,

$$x^2 - a^2 = a^2\sec^2\theta - a^2 = a^2(\sec^2\theta - 1) = a^2\tan^2\theta.$$

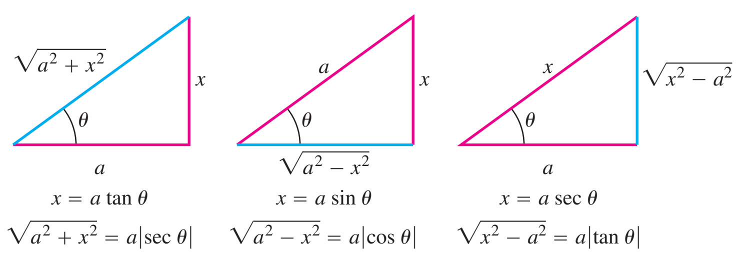


FIGURE 8.2 Reference triangles for the three basic substitutions identifying the sides labeled x and a for each substitution.

We want any substitution we use in an integration to be reversible so that we can change back to the original variable afterward. For example, if $x = a\tan\theta$, we want to be able to set $\theta = \tan^{-1}(x/a)$ after the integration takes place. If $x = a\sin\theta$, we want to be able to set $\theta = \sin^{-1}(x/a)$ when we're done, and similarly for $x = a\sec\theta$.

As we know from Section 7.7, the functions in these substitutions have inverses only for selected values of θ (Figure 8.3). For reversibility,

$$x = a\tan\theta \quad \text{requires} \quad \theta = \tan^{-1}\left(\frac{x}{a}\right) \quad \text{with} \quad -\frac{\pi}{2} < \theta < \frac{\pi}{2},$$

$$x = a\sin\theta \quad \text{requires} \quad \theta = \sin^{-1}\left(\frac{x}{a}\right) \quad \text{with} \quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2},$$

$$x = a\sec\theta \quad \text{requires} \quad \theta = \sec^{-1}\left(\frac{x}{a}\right) \quad \text{with} \quad \begin{cases} 0 \le \theta < \dfrac{\pi}{2} & \text{if } \dfrac{x}{a} \ge 1, \\[2ex] \dfrac{\pi}{2} < \theta \le \pi & \text{if } \dfrac{x}{a} \le -1. \end{cases}$$

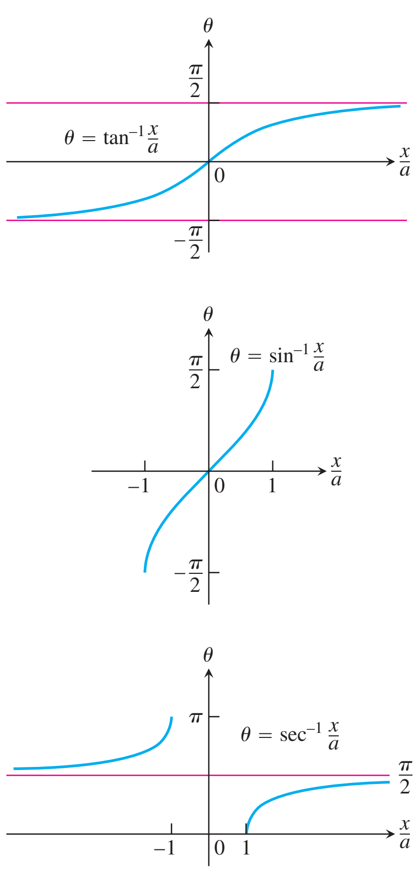


FIGURE 8.3 The arctangent, arcsine, and arcsecant of x/a, graphed as functions of x/a.

To simplify calculations with the substitution $x = a\sec\theta$, we will restrict its use to integrals in which $x/a \ge 1$. This will place θ in $[0, \pi/2)$ and make $\tan\theta \ge 0$. We will then have $\sqrt{x^2 - a^2} = \sqrt{a^2\tan^2\theta} = |a\tan\theta| = a\tan\theta$, free of absolute values, provided $a > 0$.

EXAMPLE 1 Using the Substitution $x = a\tan\theta$

Evaluate

$$\int \frac{dx}{\sqrt{4 + x^2}}.$$

You Try It

Solution We set

$$x = 2\tan\theta, \qquad dx = 2\sec^2\theta\, d\theta, \qquad -\frac{\pi}{2} < \theta < \frac{\pi}{2},$$
$$4 + x^2 = 4 + 4\tan^2\theta = 4(1 + \tan^2\theta) = 4\sec^2\theta.$$

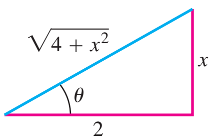


FIGURE 8.4 Reference triangle for $x = 2\tan\theta$ (Example 1):

$$\tan\theta = \frac{x}{2}$$

and

$$\sec\theta = \frac{\sqrt{4+x^2}}{2}.$$

Then

$$\begin{aligned}
\int \frac{dx}{\sqrt{4+x^2}} &= \int \frac{2\sec^2\theta\, d\theta}{\sqrt{4\sec^2\theta}} = \int \frac{\sec^2\theta\, d\theta}{|\sec\theta|} && \sqrt{\sec^2\theta} = |\sec\theta| \\
&= \int \sec\theta\, d\theta && \sec\theta > 0 \text{ for } -\frac{\pi}{2} < \theta < \frac{\pi}{2} \\
&= \ln|\sec\theta + \tan\theta| + C \\
&= \ln\left|\frac{\sqrt{4+x^2}}{2} + \frac{x}{2}\right| + C && \text{From Fig. 8.4} \\
&= \ln\left|\sqrt{4+x^2} + x\right| + C'. && \text{Taking } C' = C - \ln 2
\end{aligned}$$

Notice how we expressed $\ln|\sec\theta + \tan\theta|$ in terms of x: We drew a reference triangle for the original substitution $x = 2\tan\theta$ (Figure 8.4) and read the ratios from the triangle. ■

You Try It

EXAMPLE 2 Using the Substitution $x = a\sin\theta$

Evaluate

$$\int \frac{x^2\, dx}{\sqrt{9-x^2}}.$$

Solution We set

$$x = 3\sin\theta, \qquad dx = 3\cos\theta\, d\theta, \qquad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$$
$$9 - x^2 = 9 - 9\sin^2\theta = 9(1 - \sin^2\theta) = 9\cos^2\theta.$$

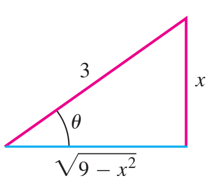


FIGURE 8.5 Reference triangle for $x = 3\sin\theta$ (Example 2):

$$\sin\theta = \frac{x}{3}$$

and

$$\cos\theta = \frac{\sqrt{9-x^2}}{3}.$$

Then

$$\begin{aligned}
\int \frac{x^2\, dx}{\sqrt{9-x^2}} &= \int \frac{9\sin^2\theta \cdot 3\cos\theta\, d\theta}{|3\cos\theta|} \\
&= 9\int \sin^2\theta\, d\theta && \cos\theta > 0 \text{ for } -\frac{\pi}{2} < \theta < \frac{\pi}{2} \\
&= 9\int \frac{1 - \cos 2\theta}{2}\, d\theta \\
&= \frac{9}{2}\left(\theta - \frac{\sin 2\theta}{2}\right) + C \\
&= \frac{9}{2}(\theta - \sin\theta\cos\theta) + C && \sin 2\theta = 2\sin\theta\cos\theta \\
&= \frac{9}{2}\left(\sin^{-1}\frac{x}{3} - \frac{x}{3}\cdot\frac{\sqrt{9-x^2}}{3}\right) + C && \text{Fig. 8.5} \\
&= \frac{9}{2}\sin^{-1}\frac{x}{3} - \frac{x}{2}\sqrt{9-x^2} + C.
\end{aligned}$$
■

EXAMPLE 3 Using the Substitution $x = a \sec \theta$

Evaluate

$$\int \frac{dx}{\sqrt{25x^2 - 4}}, \qquad x > \frac{2}{5}.$$

Solution We first rewrite the radical as

$$\sqrt{25x^2 - 4} = \sqrt{25\left(x^2 - \frac{4}{25}\right)}$$
$$= 5\sqrt{x^2 - \left(\frac{2}{5}\right)^2}$$

to put the radicand in the form $x^2 - a^2$. We then substitute

$$x = \frac{2}{5}\sec\theta, \qquad dx = \frac{2}{5}\sec\theta\tan\theta\, d\theta, \qquad 0 < \theta < \frac{\pi}{2}$$

$$x^2 - \left(\frac{2}{5}\right)^2 = \frac{4}{25}\sec^2\theta - \frac{4}{25}$$
$$= \frac{4}{25}(\sec^2\theta - 1) = \frac{4}{25}\tan^2\theta$$

$$\sqrt{x^2 - \left(\frac{2}{5}\right)^2} = \frac{2}{5}|\tan\theta| = \frac{2}{5}\tan\theta. \qquad \begin{array}{l}\tan\theta > 0 \text{ for} \\ 0 < \theta < \pi/2\end{array}$$

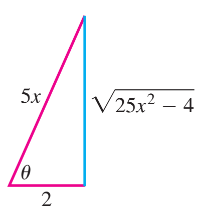


FIGURE 8.6 If $x = (2/5)\sec\theta$, $0 < \theta < \pi/2$, then $\theta = \sec^{-1}(5x/2)$, and we can read the values of the other trigonometric functions of θ from this right triangle (Example 3).

With these substitutions, we have

$$\int \frac{dx}{\sqrt{25x^2 - 4}} = \int \frac{dx}{5\sqrt{x^2 - (4/25)}} = \int \frac{(2/5)\sec\theta\tan\theta\, d\theta}{5 \cdot (2/5)\tan\theta}$$
$$= \frac{1}{5}\int \sec\theta\, d\theta = \frac{1}{5}\ln|\sec\theta + \tan\theta| + C$$
$$= \frac{1}{5}\ln\left|\frac{5x}{2} + \frac{\sqrt{25x^2 - 4}}{2}\right| + C. \qquad \text{Fig. 8.6} \quad ■$$

A trigonometric substitution can sometimes help us to evaluate an integral containing an integer power of a quadratic binomial, as in the next example.

EXAMPLE 4 Finding the Volume of a Solid of Revolution

Find the volume of the solid generated by revolving about the x-axis the region bounded by the curve $y = 4/(x^2 + 4)$, the x-axis, and the lines $x = 0$ and $x = 2$.

Solution We sketch the region (Figure 8.7) and use the disk method:

$$V = \int_0^2 \pi[R(x)]^2\, dx = 16\pi\int_0^2 \frac{dx}{(x^2 + 4)^2}. \qquad R(x) = \frac{4}{x^2 + 4}$$

To evaluate the integral, we set

$$x = 2\tan\theta, \qquad dx = 2\sec^2\theta\, d\theta, \qquad \theta = \tan^{-1}\frac{x}{2},$$
$$x^2 + 4 = 4\tan^2\theta + 4 = 4(\tan^2\theta + 1) = 4\sec^2\theta$$

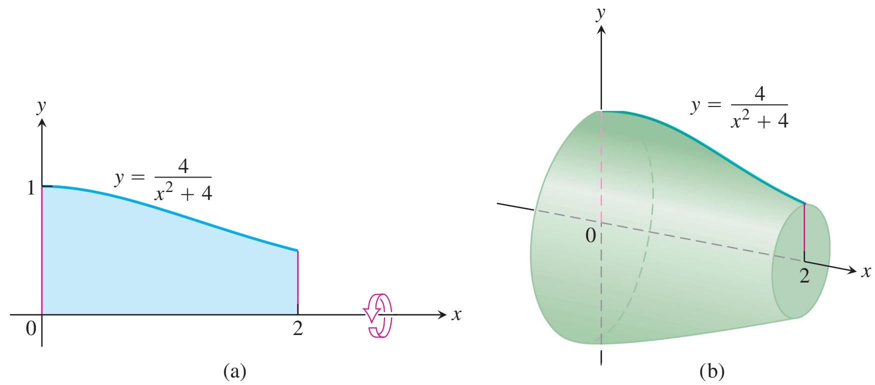


FIGURE 8.7 The region (a) and solid (b) in Example 4.

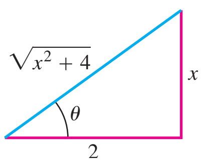


FIGURE 8.8 Reference triangle for $x = 2\tan\theta$ (Example 4).

(Figure 8.8). With these substitutions,

$$
\begin{aligned}
V &= 16\pi \int_0^2 \frac{dx}{(x^2+4)^2} \\
&= 16\pi \int_0^{\pi/4} \frac{2\sec^2\theta\, d\theta}{(4\sec^2\theta)^2} && \theta = 0 \text{ when } x = 0; \\
& && \theta = \pi/4 \text{ when } x = 2 \\
&= 16\pi \int_0^{\pi/4} \frac{2\sec^2\theta\, d\theta}{16\sec^4\theta} = \pi \int_0^{\pi/4} 2\cos^2\theta\, d\theta \\
&= \pi \int_0^{\pi/4} (1 + \cos 2\theta)\, d\theta = \pi\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi/4} && 2\cos^2\theta = 1 + \cos 2\theta \\
&= \pi\left[\frac{\pi}{4} + \frac{1}{2}\right] \approx 4.04.
\end{aligned}
$$

■

You Try It

EXAMPLE 5 Finding the Area of an Ellipse

Find the area enclosed by the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Solution Because the ellipse is symmetric with respect to both axes, the total area A is four times the area in the first quadrant (Figure 8.9). Solving the equation of the ellipse for $y \geq 0$, we get

$$\frac{y^2}{b^2} = 1 - \frac{x^2}{a^2} = \frac{a^2 - x^2}{a^2},$$

or

$$y = \frac{b}{a}\sqrt{a^2 - x^2} \qquad 0 \leq x \leq a$$

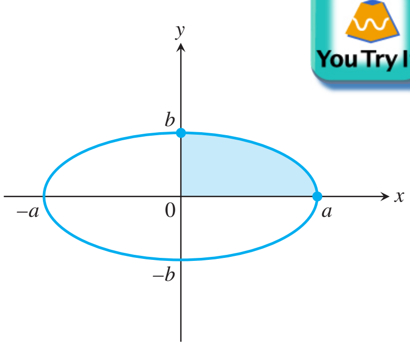


FIGURE 8.9 The ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ in Example 5.

The area of the ellipse is

$$\begin{aligned}
A &= 4\int_0^a \frac{b}{a}\sqrt{a^2 - x^2}\,dx \\
&= 4\frac{b}{a}\int_0^{\pi/2} a\cos\theta \cdot a\cos\theta\,d\theta \\
&= 4ab\int_0^{\pi/2} \cos^2\theta\,d\theta \\
&= 4ab\int_0^{\pi/2} \frac{1 + \cos 2\theta}{2}\,d\theta \\
&= 2ab\left[\theta + \frac{\sin 2\theta}{2}\right]_0^{\pi/2} \\
&= 2ab\left[\frac{\pi}{2} + 0 - 0\right] = \pi ab.
\end{aligned}$$

$x = a\sin\theta$, $dx = a\cos\theta\,d\theta$,
$\theta = 0$ when $x = 0$;
$\theta = \pi/2$ when $x = a$

If $a = b = r$ we get that the area of a circle with radius r is πr^2. ■

EXERCISES 8.5

Basic Trigonometric Substitutions

Evaluate the integrals in Exercises 1–28.

1. $\displaystyle\int \frac{dy}{\sqrt{9+y^2}}$

2. $\displaystyle\int \frac{3\,dy}{\sqrt{1+9y^2}}$

3. $\displaystyle\int_{-2}^{2} \frac{dx}{4+x^2}$

4. $\displaystyle\int_{0}^{2} \frac{dx}{8+2x^2}$

5. $\displaystyle\int_{0}^{3/2} \frac{dx}{\sqrt{9-x^2}}$

6. $\displaystyle\int_{0}^{1/2\sqrt{2}} \frac{2\,dx}{\sqrt{1-4x^2}}$

7. $\displaystyle\int \sqrt{25-t^2}\,dt$

8. $\displaystyle\int \sqrt{1-9t^2}\,dt$

9. $\displaystyle\int \frac{dx}{\sqrt{4x^2-49}}, \quad x > \frac{7}{2}$

10. $\displaystyle\int \frac{5\,dx}{\sqrt{25x^2-9}}, \quad x > \frac{3}{5}$

11. $\displaystyle\int \frac{\sqrt{y^2-49}}{y}\,dy, \quad y > 7$

12. $\displaystyle\int \frac{\sqrt{y^2-25}}{y^3}\,dy, \quad y > 5$

13. $\displaystyle\int \frac{dx}{x^2\sqrt{x^2-1}}, \quad x > 1$

14. $\displaystyle\int \frac{2\,dx}{x^3\sqrt{x^2-1}}, \quad x > 1$

15. $\displaystyle\int \frac{x^3\,dx}{\sqrt{x^2+4}}$

16. $\displaystyle\int \frac{dx}{x^2\sqrt{x^2+1}}$

17. $\displaystyle\int \frac{8\,dw}{w^2\sqrt{4-w^2}}$

18. $\displaystyle\int \frac{\sqrt{9-w^2}}{w^2}\,dw$

19. $\displaystyle\int_{0}^{\sqrt{3}/2} \frac{4x^2\,dx}{(1-x^2)^{3/2}}$

20. $\displaystyle\int_{0}^{1} \frac{dx}{(4-x^2)^{3/2}}$

21. $\displaystyle\int \frac{dx}{(x^2-1)^{3/2}}, \quad x > 1$

22. $\displaystyle\int \frac{x^2\,dx}{(x^2-1)^{5/2}}, \quad x > 1$

23. $\displaystyle\int \frac{(1-x^2)^{3/2}}{x^6}\,dx$

24. $\displaystyle\int \frac{(1-x^2)^{1/2}}{x^4}\,dx$

25. $\displaystyle\int \frac{8\,dx}{(4x^2+1)^2}$

26. $\displaystyle\int \frac{6\,dt}{(9t^2+1)^2}$

27. $\displaystyle\int \frac{v^2\,dv}{(1-v^2)^{5/2}}$

28. $\displaystyle\int \frac{(1-r^2)^{5/2}}{r^8}\,dr$

In Exercises 29–36, use an appropriate substitution and then a trigonometric substitution to evaluate the integrals.

29. $\displaystyle\int_{0}^{\ln 4} \frac{e^t\,dt}{\sqrt{e^{2t}+9}}$

30. $\displaystyle\int_{\ln(3/4)}^{\ln(4/3)} \frac{e^t\,dt}{(1+e^{2t})^{3/2}}$

31. $\displaystyle\int_{1/12}^{1/4} \frac{2\,dt}{\sqrt{t}+4t\sqrt{t}}$

32. $\displaystyle\int_{1}^{e} \frac{dy}{y\sqrt{1+(\ln y)^2}}$

33. $\displaystyle\int \frac{dx}{x\sqrt{x^2-1}}$

34. $\displaystyle\int \frac{dx}{1+x^2}$

35. $\displaystyle\int \frac{x\,dx}{\sqrt{x^2-1}}$

36. $\displaystyle\int \frac{dx}{\sqrt{1-x^2}}$

Initial Value Problems

Solve the initial value problems in Exercises 37–40 for y as a function of x.

37. $x\dfrac{dy}{dx} = \sqrt{x^2-4}, \quad x \ge 2, \quad y(2) = 0$

38. $\sqrt{x^2-9}\,\dfrac{dy}{dx} = 1, \quad x > 3, \quad y(5) = \ln 3$

39. $(x^2+4)\dfrac{dy}{dx} = 3, \quad y(2) = 0$

40. $(x^2+1)^2\dfrac{dy}{dx} = \sqrt{x^2+1}, \quad y(0) = 1$

Applications

41. Find the area of the region in the first quadrant that is enclosed by the coordinate axes and the curve $y = \sqrt{9 - x^2/3}$.

42. Find the volume of the solid generated by revolving about the x-axis the region in the first quadrant enclosed by the coordinate axes, the curve $y = 2/(1 + x^2)$, and the line $x = 1$.

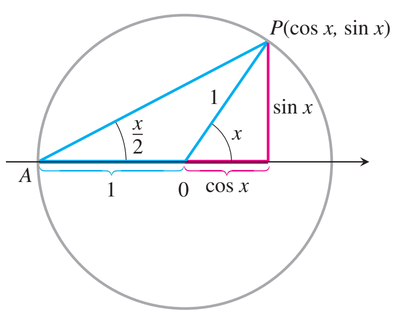
The Substitution $z = \tan(x/2)$

The substitution

$$z = \tan\frac{x}{2} \tag{1}$$

reduces the problem of integrating a rational expression in $\sin x$ and $\cos x$ to a problem of integrating a rational function of z. This in turn can be integrated by partial fractions.

From the accompanying figure



we can read the relation

$$\tan\frac{x}{2} = \frac{\sin x}{1 + \cos x}.$$

To see the effect of the substitution, we calculate

$$\cos x = 2\cos^2\left(\frac{x}{2}\right) - 1 = \frac{2}{\sec^2(x/2)} - 1$$

$$= \frac{2}{1 + \tan^2(x/2)} - 1 = \frac{2}{1 + z^2} - 1$$

$$\cos x = \frac{1 - z^2}{1 + z^2}, \tag{2}$$

and

$$\sin x = 2\sin\frac{x}{2}\cos\frac{x}{2} = 2\,\frac{\sin(x/2)}{\cos(x/2)}\cdot\cos^2\left(\frac{x}{2}\right)$$

$$= 2\tan\frac{x}{2}\cdot\frac{1}{\sec^2(x/2)} = \frac{2\tan(x/2)}{1 + \tan^2(x/2)}$$

$$\sin x = \frac{2z}{1 + z^2}. \tag{3}$$

Finally, $x = 2\tan^{-1} z$, so

$$dx = \frac{2\,dz}{1 + z^2}. \tag{4}$$

Examples

a. $$\int \frac{1}{1 + \cos x}\,dx = \int \frac{1 + z^2}{2}\,\frac{2\,dz}{1 + z^2}$$
$$= \int dz = z + C$$
$$= \tan\left(\frac{x}{2}\right) + C$$

b. $$\int \frac{1}{2 + \sin x}\,dx = \int \frac{1 + z^2}{2 + 2z + 2z^2}\,\frac{2\,dz}{1 + z^2}$$
$$= \int \frac{dz}{z^2 + z + 1} = \int \frac{dz}{(z + (1/2))^2 + 3/4}$$
$$= \int \frac{du}{u^2 + a^2}$$
$$= \frac{1}{a}\tan^{-1}\left(\frac{u}{a}\right) + C$$
$$= \frac{2}{\sqrt{3}}\tan^{-1}\frac{2z + 1}{\sqrt{3}} + C$$
$$= \frac{2}{\sqrt{3}}\tan^{-1}\frac{1 + 2\tan(x/2)}{\sqrt{3}} + C$$

Use the substitutions in Equations (1)–(4) to evaluate the integrals in Exercises 43–50. Integrals like these arise in calculating the average angular velocity of the output shaft of a universal joint when the input and output shafts are not aligned.

43. $\displaystyle\int \frac{dx}{1 - \sin x}$

44. $\displaystyle\int \frac{dx}{1 + \sin x + \cos x}$

45. $\displaystyle\int_0^{\pi/2} \frac{dx}{1 + \sin x}$

46. $\displaystyle\int_{\pi/3}^{\pi/2} \frac{dx}{1 - \cos x}$

47. $\displaystyle\int_0^{\pi/2} \frac{d\theta}{2 + \cos\theta}$

48. $\displaystyle\int_{\pi/2}^{2\pi/3} \frac{\cos\theta\,d\theta}{\sin\theta\cos\theta + \sin\theta}$

49. $\displaystyle\int \frac{dt}{\sin t - \cos t}$

50. $\displaystyle\int \frac{\cos t\,dt}{1 - \cos t}$

Use the substitution $z = \tan(\theta/2)$ to evaluate the integrals in Exercises 51 and 52.

51. $\displaystyle\int \sec\theta\,d\theta$

52. $\displaystyle\int \csc\theta\,d\theta$

8.6 Integral Tables and Computer Algebra Systems

As we have studied, the basic techniques of integration are substitution and integration by parts. We apply these techniques to transform unfamiliar integrals into integrals whose forms we recognize or can find in a table. But where do the integrals in the tables come from? They come from applying substitutions and integration by parts, or by differentiating important functions that arise in practice or applications and tabling the results (as we did in creating Table 8.1). When an integral matches an integral in the table or can be changed into one of the tabulated integrals with some appropriate combination of algebra, trigonometry, substitution, and calculus, the matched result can be used to solve the integration problem at hand.

Computer Algebra Systems (CAS) can also be used to evaluate an integral, if such a system is available. However, beware that there are many relatively simple functions, like $\sin(x^2)$ or $1/\ln x$ for which even the most powerful computer algebra systems cannot find explicit antiderivative formulas because no such formulas exist.

In this section we discuss how to use tables and computer algebra systems to evaluate integrals.

Integral Tables

A Brief Table of Integrals is provided at the back of the book, after the index. (More extensive tables appear in compilations such as *CRC Mathematical Tables*, which contain thousands of integrals.) The integration formulas are stated in terms of constants a, b, c, m, n, and so on. These constants can usually assume any real value and need not be integers. Occasional limitations on their values are stated with the formulas. Formula 5 requires $n \neq -1$, for example, and Formula 11 requires $n \neq 2$.

The formulas also assume that the constants do not take on values that require dividing by zero or taking even roots of negative numbers. For example, Formula 8 assumes that $a \neq 0$, and Formulas 13(a) and (b) cannot be used unless b is positive.

The integrals in Examples 1–5 of this section can be evaluated using algebraic manipulation, substitution, or integration by parts. Here we illustrate how the integrals are found using the Brief Table of Integrals.

EXAMPLE 1 Find

$$\int x(2x+5)^{-1}\,dx.$$

Solution We use Formula 8 (not 7, which requires $n \neq -1$):

$$\int x(ax+b)^{-1}\,dx = \frac{x}{a} - \frac{b}{a^2}\ln|ax+b| + C.$$

With $a = 2$ and $b = 5$, we have

$$\int x(2x+5)^{-1}\,dx = \frac{x}{2} - \frac{5}{4}\ln|2x+5| + C. \qquad \blacksquare$$

You Try It

EXAMPLE 2 Find

$$\int \frac{dx}{x\sqrt{2x+4}}.$$

Solution We use Formula 13(b):

$$\int \frac{dx}{x\sqrt{ax+b}} = \frac{1}{\sqrt{b}} \ln \left| \frac{\sqrt{ax+b} - \sqrt{b}}{\sqrt{ax+b} + \sqrt{b}} \right| + C, \qquad \text{if } b > 0.$$

With $a = 2$ and $b = 4$, we have

$$\int \frac{dx}{x\sqrt{2x+4}} = \frac{1}{\sqrt{4}} \ln \left| \frac{\sqrt{2x+4} - \sqrt{4}}{\sqrt{2x+4} + \sqrt{4}} \right| + C$$

$$= \frac{1}{2} \ln \left| \frac{\sqrt{2x+4} - 2}{\sqrt{2x+4} + 2} \right| + C. \qquad \blacksquare$$

Formula 13(a), which would require $b < 0$ here, is not appropriate in Example 2. It *is* appropriate, however, in the next example.

EXAMPLE 3 Find

$$\int \frac{dx}{x\sqrt{2x-4}}.$$

Solution We use Formula 13(a):

$$\int \frac{dx}{x\sqrt{ax-b}} = \frac{2}{\sqrt{b}} \tan^{-1} \sqrt{\frac{ax-b}{b}} + C.$$

With $a = 2$ and $b = 4$, we have

$$\int \frac{dx}{x\sqrt{2x-4}} = \frac{2}{\sqrt{4}} \tan^{-1} \sqrt{\frac{2x-4}{4}} + C = \tan^{-1} \sqrt{\frac{x-2}{2}} + C. \qquad \blacksquare$$

EXAMPLE 4 Find

$$\int \frac{dx}{x^2\sqrt{2x-4}}.$$

Solution We begin with Formula 15:

$$\int \frac{dx}{x^2\sqrt{ax+b}} = -\frac{\sqrt{ax+b}}{bx} - \frac{a}{2b} \int \frac{dx}{x\sqrt{ax+b}} + C.$$

With $a = 2$ and $b = -4$, we have

$$\int \frac{dx}{x^2\sqrt{2x-4}} = -\frac{\sqrt{2x-4}}{-4x} + \frac{2}{2 \cdot 4} \int \frac{dx}{x\sqrt{2x-4}} + C.$$

We then use Formula 13(a) to evaluate the integral on the right (Example 3) to obtain

$$\int \frac{dx}{x^2\sqrt{2x-4}} = \frac{\sqrt{2x-4}}{4x} + \frac{1}{4} \tan^{-1} \sqrt{\frac{x-2}{2}} + C. \qquad \blacksquare$$

EXAMPLE 5 Find

$$\int x \sin^{-1} x \, dx.$$

Solution We use Formula 99:

$$\int x^n \sin^{-1} ax\,dx = \frac{x^{n+1}}{n+1}\sin^{-1} ax - \frac{a}{n+1}\int \frac{x^{n+1}\,dx}{\sqrt{1-a^2x^2}}, \qquad n \neq -1.$$

With $n = 1$ and $a = 1$, we have

$$\int x \sin^{-1} x\,dx = \frac{x^2}{2}\sin^{-1} x - \frac{1}{2}\int \frac{x^2\,dx}{\sqrt{1-x^2}}.$$

The integral on the right is found in the table as Formula 33:

$$\int \frac{x^2}{\sqrt{a^2-x^2}}\,dx = \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) - \frac{1}{2}x\sqrt{a^2-x^2} + C.$$

With $a = 1$,

$$\int \frac{x^2\,dx}{\sqrt{1-x^2}} = \frac{1}{2}\sin^{-1} x - \frac{1}{2}x\sqrt{1-x^2} + C.$$

The combined result is

$$\begin{aligned}\int x \sin^{-1} x\,dx &= \frac{x^2}{2}\sin^{-1} x - \frac{1}{2}\left(\frac{1}{2}\sin^{-1} x - \frac{1}{2}x\sqrt{1-x^2} + C\right)\\ &= \left(\frac{x^2}{2} - \frac{1}{4}\right)\sin^{-1} x + \frac{1}{4}x\sqrt{1-x^2} + C'.\end{aligned}$$ ■

Reduction Formulas

The time required for repeated integrations by parts can sometimes be shortened by applying formulas like

$$\int \tan^n x\,dx = \frac{1}{n-1}\tan^{n-1} x - \int \tan^{n-2} x\,dx \tag{1}$$

$$\int (\ln x)^n\,dx = x(\ln x)^n - n\int (\ln x)^{n-1}\,dx \tag{2}$$

$$\int \sin^n x \cos^m x\,dx = -\frac{\sin^{n-1} x \cos^{m+1} x}{m+n} + \frac{n-1}{m+n}\int \sin^{n-2} x \cos^m x\,dx \qquad (n \neq -m). \tag{3}$$

Formulas like these are called **reduction formulas** because they replace an integral containing some power of a function with an integral of the same form with the power reduced. By applying such a formula repeatedly, we can eventually express the original integral in terms of a power low enough to be evaluated directly.

You Try It

EXAMPLE 6 Using a Reduction Formula

Find

$$\int \tan^5 x\,dx.$$

Solution We apply Equation (1) with $n = 5$ to get

$$\int \tan^5 x\,dx = \frac{1}{4}\tan^4 x - \int \tan^3 x\,dx.$$

We then apply Equation (1) again, with $n = 3$, to evaluate the remaining integral:

$$\int \tan^3 x\,dx = \frac{1}{2}\tan^2 x - \int \tan x\,dx = \frac{1}{2}\tan^2 x + \ln|\cos x| + C.$$

The combined result is

$$\int \tan^5 x\,dx = \frac{1}{4}\tan^4 x - \frac{1}{2}\tan^2 x - \ln|\cos x| + C'.$$ ■

As their form suggests, reduction formulas are derived by integration by parts.

EXAMPLE 7 Deriving a Reduction Formula

Show that for any positive integer n,

$$\int (\ln x)^n\,dx = x(\ln x)^n - n\int (\ln x)^{n-1}\,dx.$$

Solution We use the integration by parts formula

$$\int u\,dv = uv - \int v\,du$$

with

$$u = (\ln x)^n, \qquad du = n(\ln x)^{n-1}\frac{dx}{x}, \qquad dv = dx, \qquad v = x,$$

to obtain

$$\int (\ln x)^n\,dx = x(\ln x)^n - n\int (\ln x)^{n-1}\,dx.$$ ■

Sometimes two reduction formulas come into play.

EXAMPLE 8 Find

$$\int \sin^2 x \cos^3 x\,dx.$$

Solution 1 We apply Equation (3) with $n = 2$ and $m = 3$ to get

$$\begin{aligned}\int \sin^2 x \cos^3 x\,dx &= -\frac{\sin x\cos^4 x}{2+3} + \frac{1}{2+3}\int \sin^0 x\cos^3 x\,dx\\ &= -\frac{\sin x\cos^4 x}{5} + \frac{1}{5}\int \cos^3 x\,dx.\end{aligned}$$

We can evaluate the remaining integral with Formula 61 (another reduction formula):

$$\int \cos^n ax\,dx = \frac{\cos^{n-1}ax\sin ax}{na} + \frac{n-1}{n}\int \cos^{n-2}ax\,dx.$$

With $n = 3$ and $a = 1$, we have

$$\begin{aligned}\int \cos^3 x\,dx &= \frac{\cos^2 x \sin x}{3} + \frac{2}{3}\int \cos x\,dx \\ &= \frac{\cos^2 x \sin x}{3} + \frac{2}{3}\sin x + C.\end{aligned}$$

The combined result is

$$\begin{aligned}\int \sin^2 x \cos^3 x\,dx &= -\frac{\sin x \cos^4 x}{5} + \frac{1}{5}\left(\frac{\cos^2 x \sin x}{3} + \frac{2}{3}\sin x + C\right) \\ &= -\frac{\sin x \cos^4 x}{5} + \frac{\cos^2 x \sin x}{15} + \frac{2}{15}\sin x + C'.\end{aligned}$$

Solution 2 Equation (3) corresponds to Formula 68 in the table, but there is another formula we might use, namely Formula 69. With $a = 1$, Formula 69 gives

$$\int \sin^n x \cos^m x\,dx = \frac{\sin^{n+1} x \cos^{m-1} x}{m+n} + \frac{m-1}{m+n}\int \sin^n x \cos^{m-2} x\,dx.$$

In our case, $n = 2$ and $m = 3$, so that

$$\begin{aligned}\int \sin^2 x \cos^3 x\,dx &= \frac{\sin^3 x \cos^2 x}{5} + \frac{2}{5}\int \sin^2 x \cos x\,dx \\ &= \frac{\sin^3 x \cos^2 x}{5} + \frac{2}{5}\left(\frac{\sin^3 x}{3}\right) + C \\ &= \frac{\sin^3 x \cos^2 x}{5} + \frac{2}{15}\sin^3 x + C.\end{aligned}$$

As you can see, it is faster to use Formula 69, but we often cannot tell beforehand how things will work out. Do not spend a lot of time looking for the "best" formula. Just find one that will work and forge ahead.

Notice also that Formulas 68 (Solution 1) and 69 (Solution 2) lead to different-looking answers. That is often the case with trigonometric integrals and is no cause for concern. The results are equivalent, and we may use whichever one we please. ■

Nonelementary Integrals

The development of computers and calculators that find antiderivatives by symbolic manipulation has led to a renewed interest in determining which antiderivatives can be expressed as finite combinations of elementary functions (the functions we have been studying) and which cannot. Integrals of functions that do not have elementary antiderivatives are called **nonelementary** integrals. They require infinite series (Chapter 11) or numerical methods for their evaluation. Examples of the latter include the error function (which measures the probability of random errors)

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt$$

and integrals such as

$$\int \sin x^2\,dx \qquad \text{and} \qquad \int \sqrt{1 + x^4}\,dx$$

that arise in engineering and physics. These and a number of others, such as

$$\int \frac{e^x}{x}\,dx, \qquad \int e^{(e^x)}\,dx, \qquad \int \frac{1}{\ln x}\,dx, \qquad \int \ln(\ln x)\,dx, \qquad \int \frac{\sin x}{x}\,dx,$$

$$\int \sqrt{1 - k^2 \sin^2 x}\,dx, \qquad 0 < k < 1,$$

look so easy they tempt us to try them just to see how they turn out. It can be proved, however, that there is no way to express these integrals as finite combinations of elementary functions. The same applies to integrals that can be changed into these by substitution. The integrands all have antiderivatives, as a consequence of the Fundamental Theorem of the Calculus, Part 1, because they are continuous. However, none of the antiderivatives is elementary.

None of the integrals you are asked to evaluate in the present chapter falls into this category, but you may encounter nonelementary integrals in your other work.

Integration with a CAS

A powerful capability of computer algebra systems is their ability to integrate symbolically. This is performed with the **integrate command** specified by the particular system (for example, **int** in Maple, **Integrate** in Mathematica).

EXAMPLE 9 Using a CAS with a Named Function

Suppose that you want to evaluate the indefinite integral of the function

$$f(x) = x^2\sqrt{a^2 + x^2}.$$

Using Maple, you first define or name the function:

$$> f := x\text{^}2 * \text{sqrt}\,(a\text{^}2 + x\text{^}2);$$

Then you use the integrate command on f, identifying the variable of integration:

$$> \text{int}(f, x);$$

Maple returns the answer

$$\frac{1}{4}x(a^2 + x^2)^{3/2} - \frac{1}{8}a^2x\sqrt{a^2 + x^2} - \frac{1}{8}a^4 \ln\left(x + \sqrt{a^2 + x^2}\right).$$

If you want to see if the answer can be simplified, enter

$$> \text{simplify}(\%);$$

Maple returns

$$\frac{1}{8}a^2x\sqrt{a^2 + x^2} + \frac{1}{4}x^3\sqrt{a^2 + x^2} - \frac{1}{8}a^4 \ln\left(x + \sqrt{a^2 + x^2}\right).$$

If you want the definite integral for $0 \le x \le \pi/2$, you can use the format

$$> \text{int}(f, x = 0..\text{Pi}/2);$$

Maple (Version 5.1) will return the expression

$$\frac{1}{64}\pi(4a^2 + \pi^2)^{(3/2)} - \frac{1}{32}a^2\pi\sqrt{4a^2 + \pi^2} + \frac{1}{8}a^4 \ln(2)$$
$$- \frac{1}{8}a^4 \ln\left(\pi + \sqrt{4a^2 + \pi^2}\right) + \frac{1}{16}a^4 \ln(a^2).$$

You can also find the definite integral for a particular value of the constant a:

$$> a := 1;$$

$$> \operatorname{int}(f, x = 0..1);$$

Maple returns the numerical answer

$$\frac{3}{8}\sqrt{2} + \frac{1}{8}\ln\left(\sqrt{2} - 1\right).$$

■

EXAMPLE 10 Using a CAS Without Naming the Function

Use a CAS to find

$$\int \sin^2 x \cos^3 x\, dx.$$

Solution With Maple, we have the entry

$$> \operatorname{int}((\sin\text{^}2)(x) * (\cos\text{^}3)(x), x);$$

with the immediate return

$$-\frac{1}{5}\sin(x)\cos(x)^4 + \frac{1}{15}\cos(x)^2\sin(x) + \frac{2}{15}\sin(x).$$

■

EXAMPLE 11 A CAS May Not Return a Closed Form Solution

Use a CAS to find

$$\int (\cos^{-1} ax)^2\, dx.$$

Solution Using Maple, we enter

$$> \operatorname{int}((\arccos(a * x))\text{^}2, x);$$

and Maple returns the expression

$$\int \arccos(ax)^2\, dx,$$

indicating that it does not have a closed form solution known by Maple. In Chapter 11, you will see how series expansion may help to evaluate such an integral. ■

Computer algebra systems vary in how they process integrations. We used Maple 5.1 in Examples 9–11. Mathematica 4 would have returned somewhat different results:

1. In Example 9, given

$$\mathit{In\ [1]}{:=}\ \text{Integrate}\ [x\text{^}2 * \text{Sqrt}\ [a\text{^}2 + x\text{^}2], x]$$

Mathematica returns

$$\mathit{Out\ [1]}{=}\ \sqrt{a^2 + x^2}\left(\frac{a^2 x}{8} + \frac{x^3}{4}\right) - \frac{1}{8}a^4\,\text{Log}\left[x + \sqrt{a^2 + x^2}\right]$$

without having to simplify an intermediate result. The answer is close to Formula 22 in the integral tables.

2. The Mathematica answer to the integral

$$\textit{In [2]:=}\ \text{Integrate [Sin }[x]\text{^2 * Cos }[x]\text{^3}, x]$$

in Example 10 is

$$\textit{Out [2]=}\ \frac{\text{Sin }[x]}{8} - \frac{1}{48}\text{Sin }[3\,x] - \frac{1}{80}\text{Sin }[5\,x]$$

differing from both the Maple answer and the answers in Example 8.

3. Mathematica does give a result for the integration

$$\textit{In [3]:=}\ \text{Integrate [ArcCos }[a * x]\text{^2}, x]$$

in Example 11, provided $a \neq 0$:

$$\textit{Out [3]=}\ -2x - \frac{2\sqrt{1 - a^2 x^2}\,\text{ArcCos }[a\,x]}{a} + x\,\text{ArcCos }[a\,x]^2$$

Although a CAS is very powerful and can aid us in solving difficult problems, each CAS has its own limitations. There are even situations where a CAS may further complicate a problem (in the sense of producing an answer that is extremely difficult to use or interpret). Note, too, that neither Maple nor Mathematica return an arbitrary constant $+C$. On the other hand, a little mathematical thinking on your part may reduce the problem to one that is quite easy to handle. We provide an example in Exercise 111.

EXERCISES 8.6

Using Integral Tables

Use the table of integrals at the back of the book to evaluate the integrals in Exercises 1–38.

1. $\displaystyle\int \frac{dx}{x\sqrt{x-3}}$

2. $\displaystyle\int \frac{dx}{x\sqrt{x+4}}$

3. $\displaystyle\int \frac{x\,dx}{\sqrt{x-2}}$

4. $\displaystyle\int \frac{x\,dx}{(2x+3)^{3/2}}$

5. $\displaystyle\int x\sqrt{2x-3}\,dx$

6. $\displaystyle\int x(7x+5)^{3/2}\,dx$

7. $\displaystyle\int \frac{\sqrt{9-4x}}{x^2}\,dx$

8. $\displaystyle\int \frac{dx}{x^2\sqrt{4x-9}}$

9. $\displaystyle\int x\sqrt{4x-x^2}\,dx$

10. $\displaystyle\int \frac{\sqrt{x-x^2}}{x}\,dx$

11. $\displaystyle\int \frac{dx}{x\sqrt{7+x^2}}$

12. $\displaystyle\int \frac{dx}{x\sqrt{7-x^2}}$

13. $\displaystyle\int \frac{\sqrt{4-x^2}}{x}\,dx$

14. $\displaystyle\int \frac{\sqrt{x^2-4}}{x}\,dx$

15. $\displaystyle\int \sqrt{25-p^2}\,dp$

16. $\displaystyle\int q^2\sqrt{25-q^2}\,dq$

17. $\displaystyle\int \frac{r^2}{\sqrt{4-r^2}}\,dr$

18. $\displaystyle\int \frac{ds}{\sqrt{s^2-2}}$

19. $\displaystyle\int \frac{d\theta}{5+4\sin 2\theta}$

20. $\displaystyle\int \frac{d\theta}{4+5\sin 2\theta}$

21. $\displaystyle\int e^{2t}\cos 3t\,dt$

22. $\displaystyle\int e^{-3t}\sin 4t\,dt$

23. $\displaystyle\int x\cos^{-1}x\,dx$

24. $\displaystyle\int x\tan^{-1}x\,dx$

25. $\displaystyle\int \frac{ds}{(9-s^2)^2}$

26. $\displaystyle\int \frac{d\theta}{(2-\theta^2)^2}$

27. $\displaystyle\int \frac{\sqrt{4x+9}}{x^2}\,dx$

28. $\displaystyle\int \frac{\sqrt{9x-4}}{x^2}\,dx$

29. $\displaystyle\int \frac{\sqrt{3t-4}}{t}\,dt$

30. $\displaystyle\int \frac{\sqrt{3t+9}}{t}\,dt$

31. $\displaystyle\int x^2\tan^{-1}x\,dx$

32. $\displaystyle\int \frac{\tan^{-1}x}{x^2}\,dx$

33. $\displaystyle\int \sin 3x\cos 2x\,dx$

34. $\displaystyle\int \sin 2x\cos 3x\,dx$

35. $\int 8 \sin 4t \sin \frac{t}{2}\, dt$

36. $\int \sin \frac{t}{3} \sin \frac{t}{6}\, dt$

37. $\int \cos \frac{\theta}{3} \cos \frac{\theta}{4}\, d\theta$

38. $\int \cos \frac{\theta}{2} \cos 7\theta\, d\theta$

Substitution and Integral Tables

In Exercises 39–52, use a substitution to change the integral into one you can find in the table. Then evaluate the integral.

39. $\int \frac{x^3 + x + 1}{(x^2 + 1)^2}\, dx$

40. $\int \frac{x^2 + 6x}{(x^2 + 3)^2}\, dx$

41. $\int \sin^{-1} \sqrt{x}\, dx$

42. $\int \frac{\cos^{-1} \sqrt{x}}{\sqrt{x}}\, dx$

43. $\int \frac{\sqrt{x}}{\sqrt{1 - x}}\, dx$

44. $\int \frac{\sqrt{2 - x}}{\sqrt{x}}\, dx$

45. $\int \cot t \sqrt{1 - \sin^2 t}\, dt, \quad 0 < t < \pi/2$

46. $\int \frac{dt}{\tan t \sqrt{4 - \sin^2 t}}$

47. $\int \frac{dy}{y\sqrt{3 + (\ln y)^2}}$

48. $\int \frac{\cos \theta\, d\theta}{\sqrt{5 + \sin^2 \theta}}$

49. $\int \frac{3\, dr}{\sqrt{9r^2 - 1}}$

50. $\int \frac{3\, dy}{\sqrt{1 + 9y^2}}$

51. $\int \cos^{-1} \sqrt{x}\, dx$

52. $\int \tan^{-1} \sqrt{y}\, dy$

Using Reduction Formulas

Use reduction formulas to evaluate the integrals in Exercises 53–72.

53. $\int \sin^5 2x\, dx$

54. $\int \sin^5 \frac{\theta}{2}\, d\theta$

55. $\int 8 \cos^4 2\pi t\, dt$

56. $\int 3 \cos^5 3y\, dy$

57. $\int \sin^2 2\theta \cos^3 2\theta\, d\theta$

58. $\int 9 \sin^3 \theta \cos^{3/2} \theta\, d\theta$

59. $\int 2 \sin^2 t \sec^4 t\, dt$

60. $\int \csc^2 y \cos^5 y\, dy$

61. $\int 4 \tan^3 2x\, dx$

62. $\int \tan^4 \left(\frac{x}{2}\right) dx$

63. $\int 8 \cot^4 t\, dt$

64. $\int 4 \cot^3 2t\, dt$

65. $\int 2 \sec^3 \pi x\, dx$

66. $\int \frac{1}{2} \csc^3 \frac{x}{2}\, dx$

67. $\int 3 \sec^4 3x\, dx$

68. $\int \csc^4 \frac{\theta}{3}\, d\theta$

69. $\int \csc^5 x\, dx$

70. $\int \sec^5 x\, dx$

71. $\int 16x^3 (\ln x)^2\, dx$

72. $\int (\ln x)^3\, dx$

Powers of x Times Exponentials

Evaluate the integrals in Exercises 73–80 using table Formulas 103–106. These integrals can also be evaluated using tabular integration (Section 8.2).

73. $\int x e^{3x}\, dx$

74. $\int x e^{-2x}\, dx$

75. $\int x^3 e^{x/2}\, dx$

76. $\int x^2 e^{\pi x}\, dx$

77. $\int x^2\, 2^x\, dx$

78. $\int x^2\, 2^{-x}\, dx$

79. $\int x \pi^x\, dx$

80. $\int x 2^{\sqrt{2}x}\, dx$

Substitutions with Reduction Formulas

Evaluate the integrals in Exercises 81–86 by making a substitution (possibly trigonometric) and then applying a reduction formula.

81. $\int e^t \sec^3 (e^t - 1)\, dt$

82. $\int \frac{\csc^3 \sqrt{\theta}}{\sqrt{\theta}}\, d\theta$

83. $\int_0^1 2\sqrt{x^2 + 1}\, dx$

84. $\int_0^{\sqrt{3}/2} \frac{dy}{(1 - y^2)^{5/2}}$

85. $\int_1^2 \frac{(r^2 - 1)^{3/2}}{r}\, dr$

86. $\int_0^{1/\sqrt{3}} \frac{dt}{(t^2 + 1)^{7/2}}$

Hyperbolic Functions

Use the integral tables to evaluate the integrals in Exercises 87–92.

87. $\int \frac{1}{8} \sinh^5 3x\, dx$

88. $\int \frac{\cosh^4 \sqrt{x}}{\sqrt{x}}\, dx$

89. $\int x^2 \cosh 3x\, dx$

90. $\int x \sinh 5x\, dx$

91. $\int \operatorname{sech}^7 x \tanh x\, dx$

92. $\int \operatorname{csch}^3 2x \coth 2x\, dx$

Theory and Examples

Exercises 93–100 refer to formulas in the table of integrals at the back of the book.

93. Derive Formula 9 by using the substitution $u = ax + b$ to evaluate

$$\int \frac{x}{(ax + b)^2}\, dx.$$

94. Derive Formula 17 by using a trigonometric substitution to evaluate

$$\int \frac{dx}{(a^2 + x^2)^2}.$$

95. Derive Formula 29 by using a trigonometric substitution to evaluate

$$\int \sqrt{a^2 - x^2}\, dx.$$

96. Derive Formula 46 by using a trigonometric substitution to evaluate

$$\int \frac{dx}{x^2\sqrt{x^2 - a^2}}.$$

97. Derive Formula 80 by evaluating

$$\int x^n \sin ax\, dx$$

by integration by parts.

98. Derive Formula 110 by evaluating

$$\int x^n (\ln ax)^m\, dx$$

by integration by parts.

99. Derive Formula 99 by evaluating

$$\int x^n \sin^{-1} ax\, dx$$

by integration by parts.

100. Derive Formula 101 by evaluating

$$\int x^n \tan^{-1} ax\, dx$$

by integration by parts.

101. Surface area Find the area of the surface generated by revolving the curve $y = \sqrt{x^2 + 2}$, $0 \le x \le \sqrt{2}$, about the x-axis.

102. Arc length Find the length of the curve $y = x^2$, $0 \le x \le \sqrt{3}/2$.

103. Centroid Find the centroid of the region cut from the first quadrant by the curve $y = 1/\sqrt{x + 1}$ and the line $x = 3$.

104. Moment about y-axis A thin plate of constant density $\delta = 1$ occupies the region enclosed by the curve $y = 36/(2x + 3)$ and the line $x = 3$ in the first quadrant. Find the moment of the plate about the y-axis.

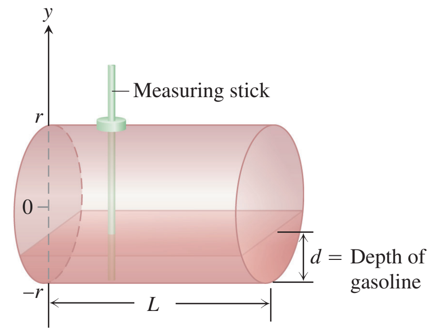
T **105.** Use the integral table and a calculator to find to two decimal places the area of the surface generated by revolving the curve $y = x^2$, $-1 \le x \le 1$, about the x-axis.

106. Volume The head of your firm's accounting department has asked you to find a formula she can use in a computer program to calculate the year-end inventory of gasoline in the company's tanks. A typical tank is shaped like a right circular cylinder of radius r and length L, mounted horizontally, as shown here. The data come to the accounting office as depth measurements taken with a vertical measuring stick marked in centimeters.

a. Show, in the notation of the figure here, that the volume of gasoline that fills the tank to a depth d is

$$V = 2L\int_{-r}^{-r+d} \sqrt{r^2 - y^2}\, dy.$$

b. Evaluate the integral.



107. What is the largest value

$$\int_a^b \sqrt{x - x^2}\, dx$$

can have for any a and b? Give reasons for your answer.

108. What is the largest value

$$\int_a^b x\sqrt{2x - x^2}\, dx$$

can have for any a and b? Give reasons for your answer.

COMPUTER EXPLORATIONS

In Exercises 109 and 110, use a CAS to perform the integrations.

109. Evaluate the integrals

a. $\int x \ln x\, dx$ **b.** $\int x^2 \ln x\, dx$ **c.** $\int x^3 \ln x\, dx.$

d. What pattern do you see? Predict the formula for $\int x^4 \ln x\, dx$ and then see if you are correct by evaluating it with a CAS.

e. What is the formula for $\int x^n \ln x\, dx$, $n \ge 1$? Check your answer using a CAS.

110. Evaluate the integrals

a. $\int \frac{\ln x}{x^2}\, dx$ **b.** $\int \frac{\ln x}{x^3}\, dx$ **c.** $\int \frac{\ln x}{x^4}\, dx.$

d. What pattern do you see? Predict the formula for

$$\int \frac{\ln x}{x^5}\, dx$$

and then see if you are correct by evaluating it with a CAS.

e. What is the formula for

$$\int \frac{\ln x}{x^n}\, dx, \quad n \ge 2?$$

Check your answer using a CAS.

111. **a.** Use a CAS to evaluate

$$\int_0^{\pi/2} \frac{\sin^n x}{\sin^n x + \cos^n x}\, dx$$

where n is an arbitrary positive integer. Does your CAS find the result?

b. In succession, find the integral when $n = 1, 2, 3, 5, 7$. Comment on the complexity of the results.

c. Now substitute $x = (\pi/2) - u$ and add the new and old integrals. What is the value of

$$\int_0^{\pi/2} \frac{\sin^n x}{\sin^n x + \cos^n x}\, dx?$$

This exercise illustrates how a little mathematical ingenuity solves a problem not immediately amenable to solution by a CAS.

8.7 Numerical Integration

As we have seen, the ideal way to evaluate a definite integral $\int_a^b f(x)\,dx$ is to find a formula $F(x)$ for one of the antiderivatives of $f(x)$ and calculate the number $F(b) - F(a)$. But some antiderivatives require considerable work to find, and still others, like the antiderivatives of $\sin(x^2)$, $1/\ln x$, and $\sqrt{1 + x^4}$, have no elementary formulas.

Another situation arises when a function is defined by a table whose entries were obtained experimentally through instrument readings. In this case a formula for the function may not even exist.

Whatever the reason, when we cannot evaluate a definite integral with an antiderivative, we turn to numerical methods such as the *Trapezoidal Rule* and *Simpson's Rule* developed in this section. These rules usually require far fewer subdivisions of the integration interval to get accurate results compared to the various rectangle rules presented in Sections 5.1 and 5.2. We also estimate the error obtained when using these approximation methods.

Trapezoidal Approximations

When we cannot find a workable antiderivative for a function f that we have to integrate, we partition the interval of integration, replace f by a closely fitting polynomial on each subinterval, integrate the polynomials, and add the results to approximate the integral of f. In our presentation we assume that f is positive, but the only requirement is for f to be continuous over the interval of integration $[a, b]$.

The Trapezoidal Rule for the value of a definite integral is based on approximating the region between a curve and the x-axis with trapezoids instead of rectangles, as in Figure 8.10. It is not necessary for the subdivision points $x_0, x_1, x_2, \ldots, x_n$ in the figure to be evenly spaced, but the resulting formula is simpler if they are. We therefore assume that the length of each subinterval is

$$\Delta x = \frac{b - a}{n}.$$

The length $\Delta x = (b - a)/n$ is called the **step size** or **mesh size**. The area of the trapezoid that lies above the ith subinterval is

$$\Delta x\left(\frac{y_{i-1} + y_i}{2}\right) = \frac{\Delta x}{2}(y_{i-1} + y_i),$$

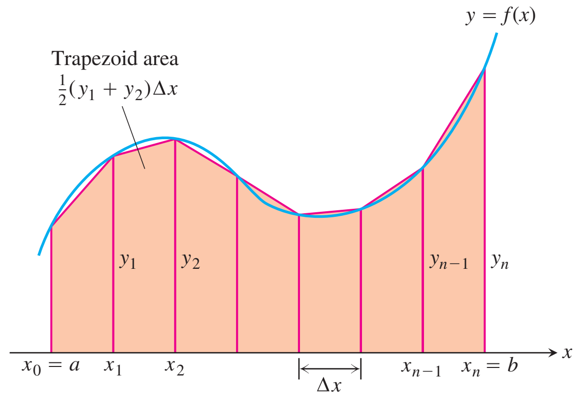


FIGURE 8.10 The Trapezoidal Rule approximates short stretches of the curve $y = f(x)$ with line segments. To approximate the integral of f from a to b, we add the areas of the trapezoids made by joining the ends of the segments to the x-axis.

where $y_{i-1} = f(x_{i-1})$ and $y_i = f(x_i)$. This area is the length Δx of the trapezoid's horizontal "altitude" times the average of its two vertical "bases." (See Figure 8.10.) The area below the curve $y = f(x)$ and above the x-axis is then approximated by adding the areas of all the trapezoids:

$$\begin{aligned} T &= \frac{1}{2}(y_0 + y_1)\Delta x + \frac{1}{2}(y_1 + y_2)\Delta x + \cdots \\ &\quad + \frac{1}{2}(y_{n-2} + y_{n-1})\Delta x + \frac{1}{2}(y_{n-1} + y_n)\Delta x \\ &= \Delta x\left(\frac{1}{2}y_0 + y_1 + y_2 + \cdots + y_{n-1} + \frac{1}{2}y_n\right) \\ &= \frac{\Delta x}{2}(y_0 + 2y_1 + 2y_2 + \cdots + 2y_{n-1} + y_n), \end{aligned}$$

where

$$y_0 = f(a), \qquad y_1 = f(x_1), \qquad \ldots, \qquad y_{n-1} = f(x_{n-1}), \qquad y_n = f(b).$$

The Trapezoidal Rule says: Use T to estimate the integral of f from a to b.

The Trapezoidal Rule

To approximate $\int_a^b f(x)\,dx$, use

$$T = \frac{\Delta x}{2}\left(y_0 + 2y_1 + 2y_2 + \cdots + 2y_{n-1} + y_n\right).$$

The y's are the values of f at the partition points

$$x_0 = a, x_1 = a + \Delta x, x_2 = a + 2\Delta x, \ldots, x_{n-1} = a + (n-1)\Delta x, x_n = b,$$

where $\Delta x = (b - a)/n$.

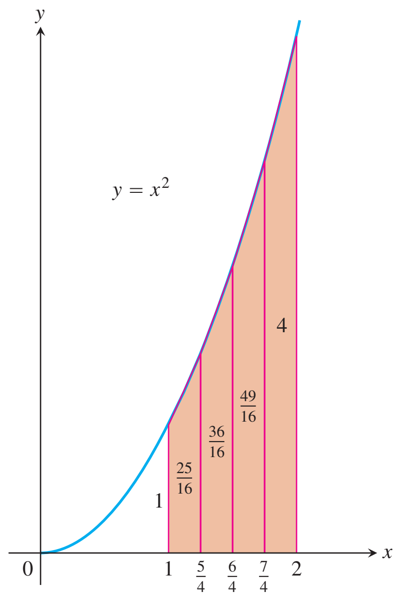


FIGURE 8.11 The trapezoidal approximation of the area under the graph of $y = x^2$ from $x = 1$ to $x = 2$ is a slight overestimate (Example 1).

TABLE 8.3

x	$y = x^2$
1	1
$\frac{5}{4}$	$\frac{25}{16}$
$\frac{6}{4}$	$\frac{36}{16}$
$\frac{7}{4}$	$\frac{49}{16}$
2	4

EXAMPLE 1 Applying the Trapezoidal Rule

Use the Trapezoidal Rule with $n = 4$ to estimate $\int_1^2 x^2\,dx$. Compare the estimate with the exact value.

Solution Partition [1, 2] into four subintervals of equal length (Figure 8.11). Then evaluate $y = x^2$ at each partition point (Table 8.3).

Using these y values, $n = 4$, and $\Delta x = (2 - 1)/4 = 1/4$ in the Trapezoidal Rule, we have

$$\begin{aligned} T &= \frac{\Delta x}{2}\left(y_0 + 2y_1 + 2y_2 + 2y_3 + y_4\right) \\ &= \frac{1}{8}\left(1 + 2\left(\frac{25}{16}\right) + 2\left(\frac{36}{16}\right) + 2\left(\frac{49}{16}\right) + 4\right) \\ &= \frac{75}{32} = 2.34375. \end{aligned}$$

The exact value of the integral is

$$\int_1^2 x^2\,dx = \frac{x^3}{3}\bigg]_1^2 = \frac{8}{3} - \frac{1}{3} = \frac{7}{3}.$$

The T approximation overestimates the integral by about half a percent of its true value of $7/3$. The percentage error is $(2.34375 - 7/3)/(7/3) \approx 0.00446$, or 0.446%. ■

We could have predicted that the Trapezoidal Rule would overestimate the integral in Example 1 by considering the geometry of the graph in Figure 8.11. Since the parabola is concave *up*, the approximating segments lie above the curve, giving each trapezoid slightly more area than the corresponding strip under the curve. In Figure 8.10, we see that the straight segments lie *under* the curve on those intervals where the curve is concave *down*, causing the Trapezoidal Rule to *underestimate* the integral on those intervals.

EXAMPLE 2 Averaging Temperatures

An observer measures the outside temperature every hour from noon until midnight, recording the temperatures in the following table.

Time	N	1	2	3	4	5	6	7	8	9	10	11	M
Temp	63	65	66	68	70	69	68	68	65	64	62	58	55

What was the average temperature for the 12-hour period?

Solution We are looking for the average value of a continuous function (temperature) for which we know values at discrete times that are one unit apart. We need to find

$$\operatorname{av}(f) = \frac{1}{b - a}\int_a^b f(x)\,dx,$$

without having a formula for $f(x)$. The integral, however, can be approximated by the Trapezoidal Rule, using the temperatures in the table as function values at the points of a 12-subinterval partition of the 12-hour interval (making $\Delta x = 1$).

$$\begin{aligned} T &= \frac{\Delta x}{2}\left(y_0 + 2y_1 + 2y_2 + \cdots + 2y_{11} + y_{12}\right) \\ &= \frac{1}{2}\left(63 + 2\cdot 65 + 2\cdot 66 + \cdots + 2\cdot 58 + 55\right) \\ &= 782 \end{aligned}$$

Using T to approximate $\int_a^b f(x)\,dx$, we have

$$\text{av}(f) \approx \frac{1}{b-a} \cdot T = \frac{1}{12} \cdot 782 \approx 65.17.$$

Rounding to the accuracy of the given data, we estimate the average temperature as 65 degrees. ■

Error Estimates for the Trapezoidal Rule

As n increases and the step size $\Delta x = (b-a)/n$ approaches zero, T approaches the exact value of $\int_a^b f(x)\,dx$. To see why, write

$$\begin{aligned} T &= \Delta x \left(\frac{1}{2} y_0 + y_1 + y_2 + \cdots + y_{n-1} + \frac{1}{2} y_n \right) \\ &= (y_1 + y_2 + \cdots + y_n)\Delta x + \frac{1}{2}(y_0 - y_n)\,\Delta x \\ &= \sum_{k=1}^{n} f(x_k)\Delta x + \frac{1}{2}[f(a) - f(b)]\,\Delta x. \end{aligned}$$

As $n \to \infty$ and $\Delta x \to 0$,

$$\sum_{k=1}^{n} f(x_k)\Delta x \to \int_a^b f(x)\,dx \qquad \text{and} \qquad \frac{1}{2}[f(a) - f(b)]\Delta x \to 0.$$

Therefore,

$$\lim_{n\to\infty} T = \int_a^b f(x)\,dx + 0 = \int_a^b f(x)\,dx.$$

This means that in theory we can make the difference between T and the integral as small as we want by taking n large enough, assuming only that f is integrable. In practice, though, how do we tell how large n should be for a given tolerance?

We answer this question with a result from advanced calculus, which says that if f'' is continuous on $[a, b]$, then

$$\int_a^b f(x)\,dx = T - \frac{b-a}{12} \cdot f''(c)(\Delta x)^2$$

for some number c between a and b. Thus, as Δx approaches zero, the error defined by

$$E_T = -\frac{b-a}{12} \cdot f''(c)(\Delta x)^2$$

approaches zero as the *square* of Δx.

The inequality

$$|E_T| \le \frac{b-a}{12} \max|f''(x)|(\Delta x)^2,$$

where max refers to the interval $[a, b]$, gives an upper bound for the magnitude of the error. In practice, we usually cannot find the exact value of $\max|f''(x)|$ and have to estimate an upper bound or "worst case" value for it instead. If M is any upper bound for the values of $|f''(x)|$ on $[a, b]$, so that $|f''(x)| \le M$ on $[a, b]$, then

$$|E_T| \le \frac{b-a}{12} M(\Delta x)^2.$$

If we substitute $(b - a)/n$ for Δx, we get

$$|E_T| \le \frac{M(b-a)^3}{12n^2}.$$

This is the inequality we normally use in estimating $|E_T|$. We find the best M we can and go on to estimate $|E_T|$ from there. This may sound careless, but it works. To make $|E_T|$ small for a given M, we just make n large.

The Error Estimate for the Trapezoidal Rule

If f'' is continuous and M is any upper bound for the values of $|f''|$ on $[a, b]$, then the error E_T in the trapezoidal approximation of the integral of f from a to b for n steps satisfies the inequality

$$|E_T| \le \frac{M(b-a)^3}{12n^2}.$$

EXAMPLE 3 Bounding the Trapezoidal Rule Error

Find an upper bound for the error incurred in estimating

$$\int_0^{\pi} x \sin x \, dx$$

with the Trapezoidal Rule with $n = 10$ steps (Figure 8.12).

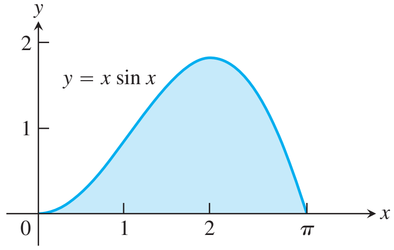


FIGURE 8.12 Graph of the integrand in Example 3.

Solution With $a = 0$, $b = \pi$, and $n = 10$, the error estimate gives

$$|E_T| \le \frac{M(b-a)^3}{12n^2} = \frac{\pi^3}{1200} M.$$

The number M can be any upper bound for the magnitude of the second derivative of $f(x) = x \sin x$ on $[0, \pi]$. A routine calculation gives

$$f''(x) = 2\cos x - x \sin x,$$

so

$$\begin{aligned} |f''(x)| &= |2\cos x - x\sin x| \\ &\le 2|\cos x| + |x||\sin x| \\ &\le 2 \cdot 1 + \pi \cdot 1 = 2 + \pi. \end{aligned}$$

$|\cos x|$ and $|\sin x|$ never exceed 1, and $0 \le x \le \pi$.

We can safely take $M = 2 + \pi$. Therefore,

$$|E_T| \le \frac{\pi^3}{1200} M = \frac{\pi^3(2+\pi)}{1200} < 0.133.$$

Rounded up to be safe

The absolute error is no greater than 0.133.

For greater accuracy, we would not try to improve M but would take more steps. With $n = 100$ steps, for example, we get

$$|E_T| \le \frac{(2+\pi)\pi^3}{120{,}000} < 0.00133 = 1.33 \times 10^{-3}.$$

■

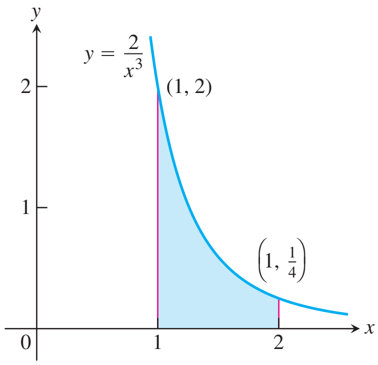


FIGURE 8.13 The continuous function $y = 2/x^3$ has its maximum value on [1, 2] at $x = 1$.

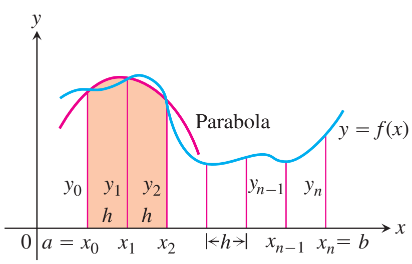


FIGURE 8.14 Simpson's Rule approximates short stretches of the curve with parabolas.

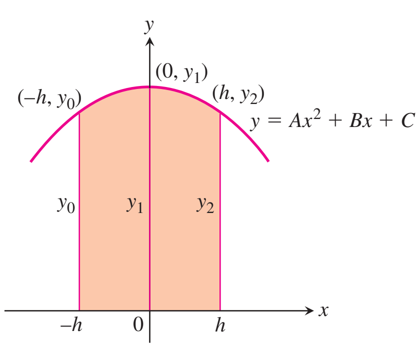


FIGURE 8.15 By integrating from $-h$ to h, we find the shaded area to be

$$\frac{h}{3}(y_0 + 4y_1 + y_2).$$

EXAMPLE 4 Finding How Many Steps Are Needed for a Specific Accuracy

How many subdivisions should be used in the Trapezoidal Rule to approximate

$$\ln 2 = \int_1^2 \frac{1}{x}\,dx$$

with an error whose absolute value is less than 10^{-4}?

Solution With $a = 1$ and $b = 2$, the error estimate is

$$|E_T| \le \frac{M(2-1)^3}{12n^2} = \frac{M}{12n^2}.$$

This is one of the rare cases in which we actually can find $\max|f''|$ rather than having to settle for an upper bound M. With $f(x) = 1/x$, we find

$$f''(x) = \frac{d^2}{dx^2}(x^{-1}) = 2x^{-3} = \frac{2}{x^3}.$$

On [1, 2], $y = 2/x^3$ decreases steadily from a maximum of $y = 2$ to a minimum of $y = 1/4$ (Figure 8.13). Therefore, $M = 2$ and

$$|E_T| \le \frac{2}{12n^2} = \frac{1}{6n^2}.$$

The error's absolute value will therefore be less than 10^{-4} if

$$\frac{1}{6n^2} < 10^{-4}, \qquad \frac{10^4}{6} < n^2, \qquad \frac{100}{\sqrt{6}} < n, \qquad \text{or} \qquad 40.83 < n.$$

The first integer beyond 40.83 is $n = 41$. With $n = 41$ subdivisions we can guarantee calculating ln 2 with an error of magnitude less than 10^{-4}. Any larger n will work, too. ■

Simpson's Rule: Approximations Using Parabolas

Riemann sums and the Trapezoidal Rule both give reasonable approximations to the integral of a continuous function over a closed interval. The Trapezoidal Rule is more efficient, giving a better approximation for small values of n, which makes it a faster algorithm for numerical integration.

Another rule for approximating the definite integral of a continuous function results from using parabolas instead of the straight line segments which produced trapezoids. As before, we partition the interval $[a, b]$ into n subintervals of equal length $h = \Delta x = (b - a)/n$, but this time we require that n be an *even* number. On each consecutive pair of intervals we approximate the curve $y = f(x) \ge 0$ by a parabola, as shown in Figure 8.14. A typical parabola passes through three consecutive points (x_{i-1}, y_{i-1}), (x_i, y_i), and (x_{i+1}, y_{i+1}) on the curve.

Let's calculate the shaded area beneath a parabola passing through three consecutive points. To simplify our calculations, we first take the case where $x_0 = -h$, $x_1 = 0$, and $x_2 = h$ (Figure 8.15), where $h = \Delta x = (b - a)/n$. The area under the parabola will be the same if we shift the y-axis to the left or right. The parabola has an equation of the form

$$y = Ax^2 + Bx + C,$$

so the area under it from $x = -h$ to $x = h$ is

$$\begin{aligned} A_p &= \int_{-h}^{h} (Ax^2 + Bx + C)\,dx \\ &= \frac{Ax^3}{3} + \frac{Bx^2}{2} + Cx \bigg]_{-h}^{h} \\ &= \frac{2Ah^3}{3} + 2Ch = \frac{h}{3}(2Ah^2 + 6C). \end{aligned}$$

Since the curve passes through the three points $(-h, y_0)$, $(0, y_1)$, and (h, y_2), we also have

$$y_0 = Ah^2 - Bh + C, \qquad y_1 = C, \qquad y_2 = Ah^2 + Bh + C,$$

from which we obtain

$$\begin{aligned} C &= y_1, \\ Ah^2 - Bh &= y_0 - y_1, \\ Ah^2 + Bh &= y_2 - y_1, \\ 2Ah^2 &= y_0 + y_2 - 2y_1. \end{aligned}$$

Hence, expressing the area A_p in terms of the ordinates y_0, y_1, and y_2, we have

$$A_p = \frac{h}{3}(2Ah^2 + 6C) = \frac{h}{3}((y_0 + y_2 - 2y_1) + 6y_1) = \frac{h}{3}(y_0 + 4y_1 + y_2).$$

Now shifting the parabola horizontally to its shaded position in Figure 8.14 does not change the area under it. Thus the area under the parabola through (x_0, y_0), (x_1, y_1), and (x_2, y_2) in Figure 8.14 is still

$$\frac{h}{3}(y_0 + 4y_1 + y_2).$$

Similarly, the area under the parabola through the points (x_2, y_2), (x_3, y_3), and (x_4, y_4) is

$$\frac{h}{3}(y_2 + 4y_3 + y_4).$$

Computing the areas under all the parabolas and adding the results gives the approximation

$$\begin{aligned} \int_a^b f(x)\,dx &\approx \frac{h}{3}(y_0 + 4y_1 + y_2) + \frac{h}{3}(y_2 + 4y_3 + y_4) + \cdots \\ &\quad + \frac{h}{3}(y_{n-2} + 4y_{n-1} + y_n) \\ &= \frac{h}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n). \end{aligned}$$

HISTORICAL BIOGRAPHY

Thomas Simpson
(1720–1761)

The result is known as Simpson's Rule, and it is again valid for any continuous function $y = f(x)$ (Exercise 38). The function need not be positive, as in our derivation. The number n of subintervals must be even to apply the rule because each parabolic arc uses two subintervals.

Simpson's Rule

To approximate $\int_a^b f(x)\,dx$, use

$$S = \frac{\Delta x}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + \cdots + 2y_{n-2} + 4y_{n-1} + y_n).$$

The y's are the values of f at the partition points

$$x_0 = a, x_1 = a + \Delta x, x_2 = a + 2\Delta x, \ldots, x_{n-1} = a + (n-1)\Delta x, x_n = b.$$

The number n is even, and $\Delta x = (b - a)/n$.

Note the pattern of the coefficients in the above rule: 1, 4, 2, 4, 2, 4, 2, . . . , 4, 2, 1.

TABLE 8.4

x	$y = 5x^4$
0	0
$\frac{1}{2}$	$\frac{5}{16}$
1	5
$\frac{3}{2}$	$\frac{405}{16}$
2	80

EXAMPLE 5 Applying Simpson's Rule

Use Simpson's Rule with $n = 4$ to approximate $\int_0^2 5x^4\,dx$.

Solution Partition [0, 2] into four subintervals and evaluate $y = 5x^4$ at the partition points (Table 8.4). Then apply Simpson's Rule with $n = 4$ and $\Delta x = 1/2$:

$$\begin{aligned} S &= \frac{\Delta x}{3}\left(y_0 + 4y_1 + 2y_2 + 4y_3 + y_4\right) \\ &= \frac{1}{6}\left(0 + 4\left(\frac{5}{16}\right) + 2(5) + 4\left(\frac{405}{16}\right) + 80\right) \\ &= 32\frac{1}{12}. \end{aligned}$$

This estimate differs from the exact value (32) by only 1/12, a percentage error of less than three-tenths of one percent, and this was with just four subintervals. ■

Error Estimates for Simpson's Rule

To estimate the error in Simpson's rule, we start with a result from advanced calculus that says that if the fourth derivative $f^{(4)}$ is continuous, then

$$\int_a^b f(x)\,dx = S - \frac{b-a}{180} \cdot f^{(4)}(c)(\Delta x)^4$$

for some point c between a and b. Thus, as Δx approaches zero, the error,

$$E_S = -\frac{b-a}{180} \cdot f^{(4)}(c)(\Delta x)^4,$$

approaches zero as the *fourth power* of Δx (This helps to explain why Simpson's Rule is likely to give better results than the Trapezoidal Rule.)

The inequality

$$|E_S| \leq \frac{b-a}{180} \max|f^{(4)}(x)|\,(\Delta x)^4$$

where max refers to the interval $[a, b]$, gives an upper bound for the magnitude of the error. As with $\max|f''|$ in the error formula for the Trapezoidal Rule, we usually cannot find the exact value of $\max|f^{(4)}(x)|$ and have to replace it with an upper bound. If M is any upper bound for the values of $|f^{(4)}|$ on $[a, b]$, then

$$|E_S| \leq \frac{b-a}{180} M(\Delta x)^4.$$

Substituting $(b-a)/n$ for Δx in this last expression gives

$$|E_S| \leq \frac{M(b-a)^5}{180n^4}.$$

This is the formula we usually use in estimating the error in Simpson's Rule. We find a reasonable value for M and go on to estimate $|E_S|$ from there.

The Error Estimate for Simpson's Rule

If $f^{(4)}$ is continuous and M is any upper bound for the values of $|f^{(4)}|$ on $[a, b]$, then the error E_S in the Simpson's Rule approximation of the integral of f from a to b for n steps satisfies the inequality

$$|E_S| \leq \frac{M(b-a)^5}{180n^4}.$$

As with the Trapezoidal Rule, we often cannot find the smallest possible value of M. We just find the best value we can and go on from there.

EXAMPLE 6 Bounding the Error in Simpson's Rule

Find an upper bound for the error in estimating $\int_0^2 5x^4\,dx$ using Simpson's Rule with $n = 4$ (Example 5).

Solution To estimate the error, we first find an upper bound M for the magnitude of the fourth derivative of $f(x) = 5x^4$ on the interval $0 \leq x \leq 2$. Since the fourth derivative has the constant value $f^{(4)}(x) = 120$, we take $M = 120$. With $b - a = 2$ and $n = 4$, the error estimate for Simpson's Rule gives

$$|E_S| \leq \frac{M(b-a)^5}{180n^4} = \frac{120(2)^5}{180 \cdot 4^4} = \frac{1}{12}.$$ ■

EXAMPLE 7 Comparing the Trapezoidal Rule and Simpson's Rule Approximations

As we saw in Chapter 7, the value of ln 2 can be calculated from the integral

$$\ln 2 = \int_1^2 \frac{1}{x}\,dx.$$

Table 8.5 shows T and S values for approximations of $\int_1^2 (1/x)\,dx$ using various values of n. Notice how Simpson's Rule dramatically improves over the Trapezoidal Rule. In particular, notice that when we double the value of n (thereby halving the value of $h = \Delta x$), the T error is divided by 2 *squared*, whereas the S error is divided by 2 *to the fourth*.

TABLE 8.5 Trapezoidal Rule approximations (T_n) and Simpson's Rule approximations (S_n) of $\ln 2 = \int_1^2 (1/x)\,dx$

n	T_n	|Error| less than . . .	S_n	|Error| less than . . .
10	0.6937714032	0.0006242227	0.6931502307	0.0000030502
20	0.6933033818	0.0001562013	0.6931473747	0.0000001942
30	0.6932166154	0.0000694349	0.6931472190	0.0000000385
40	0.6931862400	0.0000390595	0.6931471927	0.0000000122
50	0.6931721793	0.0000249988	0.6931471856	0.0000000050
100	0.6931534305	0.0000062500	0.6931471809	0.0000000004

This has a dramatic effect as $\Delta x = (2 - 1)/n$ gets very small. The Simpson approximation for $n = 50$ rounds accurately to seven places and for $n = 100$ agrees to nine decimal places (billionths)! ■

If $f(x)$ is a polynomial of degree less than four, then its fourth derivative is zero, and

$$E_S = -\frac{b-a}{180} f^{(4)}(c)(\Delta x)^4 = -\frac{b-a}{180}(0)(\Delta x)^4 = 0.$$

Thus, there will be no error in the Simpson approximation of any integral of f. In other words, if f is a constant, a linear function, or a quadratic or cubic polynomial, Simpson's Rule will give the value of any integral of f exactly, whatever the number of subdivisions. Similarly, if f is a constant or a linear function, then its second derivative is zero and

$$E_T = -\frac{b-a}{12} f''(c)(\Delta x)^2 = -\frac{b-a}{12}(0)(\Delta x)^2 = 0.$$

The Trapezoidal Rule will therefore give the exact value of any integral of f. This is no surprise, for the trapezoids fit the graph perfectly. Although decreasing the step size Δx reduces the error in the Simpson and Trapezoidal approximations in theory, it may fail to do so in practice.

When Δx is very small, say $\Delta x = 10^{-5}$, computer or calculator round-off errors in the arithmetic required to evaluate S and T may accumulate to such an extent that the error formulas no longer describe what is going on. Shrinking Δx below a certain size can actually make things worse. Although this is not an issue in this book, you should consult a text on numerical analysis for alternative methods if you are having problems with round-off.

EXAMPLE 8 Estimate

$$\int_0^2 x^3\,dx$$

with Simpson's Rule.

Solution The fourth derivative of $f(x) = x^3$ is zero, so we expect Simpson's Rule to give the integral's exact value with any (even) number of steps. Indeed, with $n = 2$ and $\Delta x = (2 - 0)/2 = 1$,

$$S = \frac{\Delta x}{3}(y_0 + 4y_1 + y_2)$$
$$= \frac{1}{3}((0)^3 + 4(1)^3 + (2)^3) = \frac{12}{3} = 4,$$

while

$$\int_0^2 x^3\,dx = \frac{x^4}{4}\Big]_0^2 = \frac{16}{4} - 0 = 4.$$

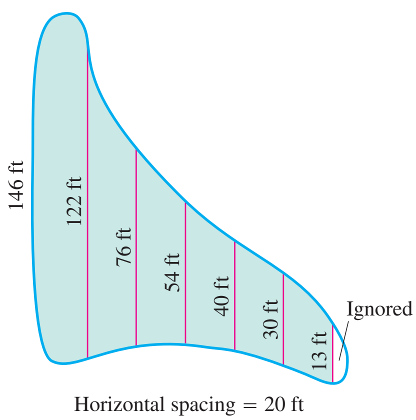


FIGURE 8.16 The dimensions of the swamp in Example 9.

EXAMPLE 9 Draining a Swamp

A town wants to drain and fill a small polluted swamp (Figure 8.16). The swamp averages 5 ft deep. About how many cubic yards of dirt will it take to fill the area after the swamp is drained?

Solution To calculate the volume of the swamp, we estimate the surface area and multiply by 5. To estimate the area, we use Simpson's Rule with $\Delta x = 20$ ft and the y's equal to the distances measured across the swamp, as shown in Figure 8.16.

$$S = \frac{\Delta x}{3}(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + 4y_5 + y_6)$$
$$= \frac{20}{3}(146 + 488 + 152 + 216 + 80 + 120 + 13) = 8100$$

The volume is about $(8100)(5) = 40{,}500$ ft^3 or 1500 yd^3.

EXERCISES 8.7

Estimating Integrals

The instructions for the integrals in Exercises 1–10 have two parts, one for the Trapezoidal Rule and one for Simpson's Rule.

I. Using the Trapezoidal Rule

a. Estimate the integral with $n = 4$ steps and find an upper bound for $|E_T|$.

b. Evaluate the integral directly and find $|E_T|$.

c. Use the formula $(|E_T|/(\text{true value})) \times 100$ to express $|E_T|$ as a percentage of the integral's true value.

II. Using Simpson's Rule

a. Estimate the integral with $n = 4$ steps and find an upper bound for $|E_S|$.

b. Evaluate the integral directly and find $|E_S|$.

c. Use the formula $(|E_S|/(\text{true value})) \times 100$ to express $|E_S|$ as a percentage of the integral's true value.

1. $\displaystyle\int_1^2 x\,dx$

2. $\displaystyle\int_1^3 (2x - 1)\,dx$

3. $\displaystyle\int_{-1}^1 (x^2 + 1)\,dx$

4. $\displaystyle\int_{-2}^0 (x^2 - 1)\,dx$

5. $\displaystyle\int_0^2 (t^3 + t)\,dt$

6. $\displaystyle\int_{-1}^1 (t^3 + 1)\,dt$

7. $\displaystyle\int_1^2 \frac{1}{s^2}\,ds$

8. $\displaystyle\int_2^4 \frac{1}{(s - 1)^2}\,ds$

9. $\displaystyle\int_0^\pi \sin t\,dt$

10. $\displaystyle\int_0^1 \sin \pi t\,dt$

In Exercises 11–14, use the tabulated values of the integrand to estimate the integral with **(a)** the Trapezoidal Rule and **(b)** Simpson's Rule with $n = 8$ steps. Round your answers to five decimal places. Then **(c)** find the integral's exact value and the approximation error E_T or E_S, as appropriate.

11. $\displaystyle\int_0^1 x\sqrt{1 - x^2}\,dx$

x	$x\sqrt{1 - x^2}$
0	0.0
0.125	0.12402
0.25	0.24206
0.375	0.34763
0.5	0.43301
0.625	0.48789
0.75	0.49608
0.875	0.42361
1.0	0

12. $\displaystyle\int_0^3 \frac{\theta}{\sqrt{16 + \theta^2}}\,d\theta$

θ	$\theta/\sqrt{16 + \theta^2}$
0	0.0
0.375	0.09334
0.75	0.18429
1.125	0.27075
1.5	0.35112
1.875	0.42443
2.25	0.49026
2.625	0.58466
3.0	0.6

13. $\displaystyle\int_{-\pi/2}^{\pi/2} \frac{3\cos t}{(2 + \sin t)^2}\,dt$

t	$(3\cos t)/(2 + \sin t)^2$
−1.57080	0.0
−1.17810	0.99138
−0.78540	1.26906
−0.39270	1.05961
0	0.75
0.39270	0.48821
0.78540	0.28946
1.17810	0.13429
1.57080	0

14. $\displaystyle\int_{\pi/4}^{\pi/2} (\csc^2 y)\sqrt{\cot y}\,dy$

y	$(\csc^2 y)\sqrt{\cot y}$
0.78540	2.0
0.88357	1.51606
0.98175	1.18237
1.07992	0.93998
1.17810	0.75402
1.27627	0.60145
1.37445	0.46364
1.47262	0.31688
1.57080	0

The Minimum Number of Subintervals

In Exercises 15–26, estimate the minimum number of subintervals needed to approximate the integrals with an error of magnitude less than 10^{-4} by **(a)** the Trapezoidal Rule and **(b)** Simpson's Rule. (The integrals in Exercises 15–22 are the integrals from Exercises 1–8.)

15. $\displaystyle\int_1^2 x\,dx$

16. $\displaystyle\int_1^3 (2x - 1)\,dx$

17. $\displaystyle\int_{-1}^1 (x^2 + 1)\,dx$

18. $\displaystyle\int_{-2}^0 (x^2 - 1)\,dx$

19. $\displaystyle\int_0^2 (t^3 + t)\,dt$

20. $\displaystyle\int_{-1}^1 (t^3 + 1)\,dt$

21. $\displaystyle\int_1^2 \frac{1}{s^2}\,ds$

22. $\displaystyle\int_2^4 \frac{1}{(s - 1)^2}\,ds$

23. $\displaystyle\int_0^3 \sqrt{x + 1}\,dx$

24. $\displaystyle\int_0^3 \frac{1}{\sqrt{x + 1}}\,dx$

25. $\displaystyle\int_0^2 \sin(x + 1)\,dx$

26. $\displaystyle\int_{-1}^1 \cos(x + \pi)\,dx$

Applications

27. Volume of water in a swimming pool A rectangular swimming pool is 30 ft wide and 50 ft long. The table shows the depth $h(x)$ of the water at 5-ft intervals from one end of the pool to the other. Estimate the volume of water in the pool using the Trapezoidal Rule with $n = 10$, applied to the integral

$$V = \int_0^{50} 30 \cdot h(x)\,dx.$$

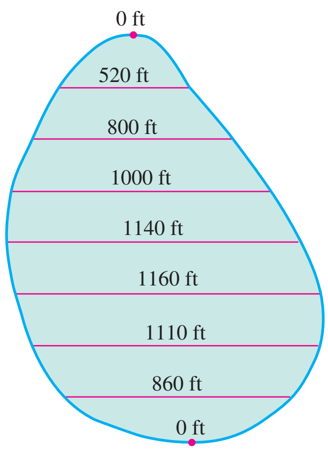
Position (ft) x	Depth (ft) $h(x)$	Position (ft) x	Depth (ft) $h(x)$
0	6.0	30	11.5
5	8.2	35	11.9
10	9.1	40	12.3
15	9.9	45	12.7
20	10.5	50	13.0
25	11.0		

Exercises

28. Stocking a fish pond As the fish and game warden of your township, you are responsible for stocking the town pond with fish before the fishing season. The average depth of the pond is 20 ft. Using a scaled map, you measure distances across the pond at 200-ft intervals, as shown in the accompanying diagram.

a. Use the Trapezoidal Rule to estimate the volume of the pond.

b. You plan to start the season with one fish per 1000 cubic feet. You intend to have at least 25% of the opening day's fish population left at the end of the season. What is the maximum number of licenses the town can sell if the average seasonal catch is 20 fish per license?



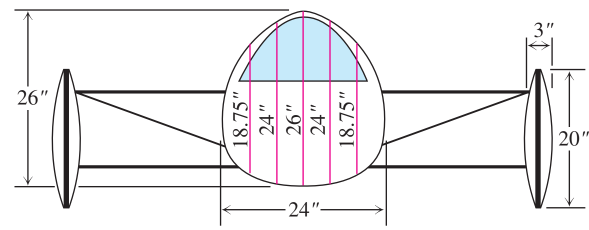
Vertical spacing = 200 ft

29. Ford® Mustang Cobra™ The accompanying table shows time-to-speed data for a 1994 Ford Mustang Cobra accelerating from rest to 130 mph. How far had the Mustang traveled by the time it reached this speed? (Use trapezoids to estimate the area under the velocity curve, but be careful: The time intervals vary in length.)

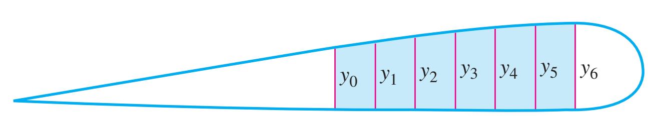
Speed change	Time (sec)
Zero to 30 mph	2.2
40 mph	3.2
50 mph	4.5
60 mph	5.9
70 mph	7.8
80 mph	10.2
90 mph	12.7
100 mph	16.0
110 mph	20.6
120 mph	26.2
130 mph	37.1

Source: Car and Driver, April 1994.

30. Aerodynamic drag A vehicle's aerodynamic drag is determined in part by its cross-sectional area, so, all other things being equal, engineers try to make this area as small as possible. Use Simpson's Rule to estimate the cross-sectional area of the body of James Worden's solar-powered Solectria® automobile at MIT from the diagram.



31. Wing design The design of a new airplane requires a gasoline tank of constant cross-sectional area in each wing. A scale drawing of a cross-section is shown here. The tank must hold 5000 lb of gasoline, which has a density of 42 lb/ft^3. Estimate the length of the tank.



$y_0 = 1.5$ ft, $y_1 = 1.6$ ft, $y_2 = 1.8$ ft, $y_3 = 1.9$ ft,
$y_4 = 2.0$ ft, $y_5 = y_6 = 2.1$ ft Horizontal spacing = 1 ft

32. Oil consumption on Pathfinder Island A diesel generator runs continuously, consuming oil at a gradually increasing rate until it must be temporarily shut down to have the filters replaced.

Use the Trapezoidal Rule to estimate the amount of oil consumed by the generator during that week.

Day	Oil consumption rate (liters/h)
Sun	0.019
Mon	0.020
Tue	0.021
Wed	0.023
Thu	0.025
Fri	0.028
Sat	0.031
Sun	0.035

Theory and Examples

33. Usable values of the sine-integral function *The sine-integral function,*

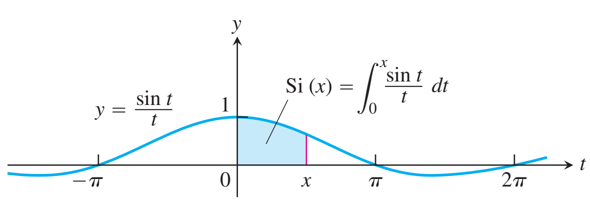
$$\mathrm{Si}(x) = \int_0^x \frac{\sin t}{t}\,dt, \qquad \text{“Sine integral of } x\text{”}$$

is one of the many functions in engineering whose formulas cannot be simplified. There is no elementary formula for the antiderivative of $(\sin t)/t$. The values of $\mathrm{Si}(x)$, however, are readily estimated by numerical integration.

Although the notation does not show it explicitly, the function being integrated is

$$f(t) = \begin{cases} \dfrac{\sin t}{t}, & t \neq 0 \\ 1, & t = 0, \end{cases}$$

the continuous extension of $(\sin t)/t$ to the interval $[0, x]$. The function has derivatives of all orders at every point of its domain. Its graph is smooth, and you can expect good results from Simpson's Rule.



a. Use the fact that $|f^{(4)}| \le 1$ on $[0, \pi/2]$ to give an upper bound for the error that will occur if

$$\mathrm{Si}\left(\frac{\pi}{2}\right) = \int_0^{\pi/2} \frac{\sin t}{t}\,dt$$

is estimated by Simpson's Rule with $n = 4$.

b. Estimate $\mathrm{Si}(\pi/2)$ by Simpson's Rule with $n = 4$.

c. Express the error bound you found in part (a) as a percentage of the value you found in part (b).

34. The error function *The error function,*

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\,dt,$$

important in probability and in the theories of heat flow and signal transmission, must be evaluated numerically because there is no elementary expression for the antiderivative of e^{-t^2}.

a. Use Simpson's Rule with $n = 10$ to estimate $\operatorname{erf}(1)$.

b. In $[0, 1]$,

$$\left|\frac{d^4}{dt^4}\left(e^{-t^2}\right)\right| \le 12.$$

Give an upper bound for the magnitude of the error of the estimate in part (a).

35. (*Continuation of Example 3.*) The error bounds for E_T and E_S are "worst case" estimates, and the Trapezoidal and Simpson Rules are often more accurate than the bounds suggest. The Trapezoidal Rule estimate of

$$\int_0^\pi x \sin x\,dx$$

in Example 3 is a case in point.

a. Use the Trapezoidal Rule with $n = 10$ to approximate the value of the integral. The table to the right gives the necessary y-values.

x	$x \sin x$
0	0
$(0.1)\pi$	0.09708
$(0.2)\pi$	0.36932
$(0.3)\pi$	0.76248
$(0.4)\pi$	1.19513
$(0.5)\pi$	1.57080
$(0.6)\pi$	1.79270
$(0.7)\pi$	1.77912
$(0.8)\pi$	1.47727
$(0.9)\pi$	0.87372
π	0

b. Find the magnitude of the difference between π, the integral's value, and your approximation in part (a). You will find the difference to be considerably less than the upper bound of 0.133 calculated with $n = 10$ in Example 3.

T **c.** The upper bound of 0.133 for $|E_T|$ in Example 3 could have been improved somewhat by having a better bound for

$$|f''(x)| = |2\cos x - x\sin x|$$

on $[0, \pi]$. The upper bound we used was $2 + \pi$. Graph f'' over $[0, \pi]$ and use Trace or Zoom to improve this upper bound.

Use the improved upper bound as M to make an improved estimate of $|E_T|$. Notice that the Trapezoidal Rule approximation in part (a) is also better than this improved estimate would suggest.

T **36.** (*Continuation of Exercise 35.*)

a. Show that the fourth derivative of $f(x) = x \sin x$ is

$$f^{(4)}(x) = -4\cos x + x\sin x.$$

Use Trace or Zoom to find an upper bound M for the values of $|f^{(4)}|$ on $[0, \pi]$.

b. Use the value of M from part (a) to obtain an upper bound for the magnitude of the error in estimating the value of

$$\int_0^{\pi} x \sin x \, dx$$

with Simpson's Rule with $n = 10$ steps.

c. Use the data in the table in Exercise 35 to estimate $\int_0^{\pi} x \sin x \, dx$ with Simpson's Rule with $n = 10$ steps.

d. To six decimal places, find the magnitude of the difference between your estimate in part (c) and the integral's true value, π. You will find the error estimate obtained in part (b) to be quite good.

37. Prove that the sum T in the Trapezoidal Rule for $\int_a^b f(x)\,dx$ is a Riemann sum for f continuous on $[a, b]$. (*Hint:* Use the Intermediate Value Theorem to show the existence of c_k in the subinterval $[x_{k-1}, x_k]$ satisfying $f(c_k) = (f(x_{k-1}) + f(x_k))/2$.)

38. Prove that the sum S in Simpson's Rule for $\int_a^b f(x)\,dx$ is a Riemann sum for f continuous on $[a, b]$. (See Exercise 37.)

T Numerical Integration

As we mentioned at the beginning of the section, the definite integrals of many continuous functions cannot be evaluated with the Fundamental Theorem of Calculus because their antiderivatives lack elementary formulas. Numerical integration offers a practical way to estimate the values of these so-called *nonelementary integrals*. If your calculator or computer has a numerical integration routine, try it on the integrals in Exercises 39–42.

39. $\displaystyle\int_0^1 \sqrt{1 + x^4}\, dx$ — A nonelementary integral that came up in Newton's research

40. $\displaystyle\int_0^{\pi/2} \frac{\sin x}{x}\, dx$ — The integral from Exercise 33. To avoid division by zero, you may have to start the integration at a small positive number like 10^{-6} instead of 0.

41. $\displaystyle\int_0^{\pi/2} \sin(x^2)\, dx$ — An integral associated with the diffraction of light

42. $\displaystyle\int_0^{\pi/2} 40\sqrt{1 - 0.64\cos^2 t}\, dt$ — The length of the ellipse $(x^2/25) + (y^2/9) = 1$

T 43. Consider the integral $\int_0^{\pi} \sin x \, dx$.

a. Find the Trapezoidal Rule approximations for $n = 10, 100$, and 1000.

b. Record the errors with as many decimal places of accuracy as you can.

c. What pattern do you see?

d. Explain how the error bound for E_T accounts for the pattern.

T 44. (*Continuation of Exercise 43.*) Repeat Exercise 43 with Simpson's Rule and E_S.

45. Consider the integral $\int_{-1}^{1} \sin(x^2)\, dx$.

a. Find f'' for $f(x) = \sin(x^2)$.

b. Graph $y = f''(x)$ in the viewing window $[-1, 1]$ by $[-3, 3]$.

c. Explain why the graph in part (b) suggests that $|f''(x)| \le 3$ for $-1 \le x \le 1$.

d. Show that the error estimate for the Trapezoidal Rule in this case becomes

$$|E_T| \le \frac{(\Delta x)^2}{2}.$$

e. Show that the Trapezoidal Rule error will be less than or equal to 0.01 in magnitude if $\Delta x \le 0.1$.

f. How large must n be for $\Delta x \le 0.1$?

46. Consider the integral $\int_{-1}^{1} \sin(x^2)\, dx$.

a. Find $f^{(4)}$ for $f(x) = \sin(x^2)$. (You may want to check your work with a CAS if you have one available.)

b. Graph $y = f^{(4)}(x)$ in the viewing window $[-1, 1]$ by $[-30, 10]$.

c. Explain why the graph in part (b) suggests that $|f^{(4)}(x)| \le 30$ for $-1 \le x \le 1$.

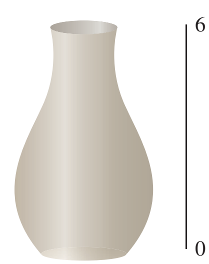
d. Show that the error estimate for Simpson's Rule in this case becomes

$$|E_S| \le \frac{(\Delta x)^4}{3}.$$

e. Show that the Simpson's Rule error will be less than or equal to 0.01 in magnitude if $\Delta x \le 0.4$.

f. How large must n be for $\Delta x \le 0.4$?

T 47. A vase We wish to estimate the volume of a flower vase using only a calculator, a string, and a ruler. We measure the height of the vase to be 6 in. We then use the string and the ruler to find circumferences of the vase (in inches) at half-inch intervals. (We list them from the top down to correspond with the picture of the vase.)



Circumferences	
5.4	10.8
4.5	11.6
4.4	11.6
5.1	10.8
6.3	9.0
7.8	6.3
9.4	

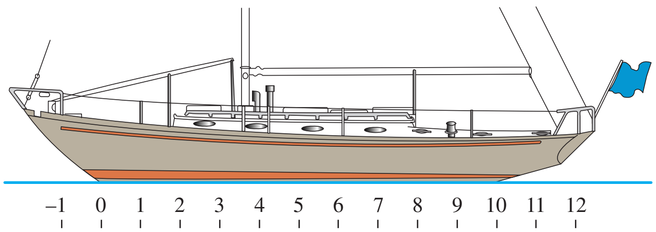
a. Find the areas of the cross-sections that correspond to the given circumferences.

b. Express the volume of the vase as an integral with respect to y over the interval $[0, 6]$.

c. Approximate the integral using the Trapezoidal Rule with $n = 12$.

d. Approximate the integral using Simpson's Rule with $n = 12$. Which result do you think is more accurate? Give reasons for your answer.

T 48. A sailboat's displacement To find the volume of water displaced by a sailboat, the common practice is to partition the waterline into 10 subintervals of equal length, measure the cross-sectional area $A(x)$ of the submerged portion of the hull at each partition point, and then use Simpson's Rule to estimate the integral of $A(x)$ from one end of the waterline to the other. The table here lists the area measurements at "Stations" 0 through 10, as the partition points are called, for the cruising sloop *Pipedream*, shown here. The common subinterval length (distance between consecutive stations) is $\Delta x = 2.54$ ft (about 2 ft 6-1/2 in., chosen for the convenience of the builder).



a. Estimate *Pipedream*'s displacement volume to the nearest cubic foot.

Station	Submerged area (ft^2)
0	0
1	1.07
2	3.84
3	7.82
4	12.20
5	15.18
6	16.14
7	14.00
8	9.21
9	3.24
10	0

b. The figures in the table are for seawater, which weighs 64 lb/ft^3. How many pounds of water does *Pipedream* displace? (Displacement is given in pounds for small craft and in long tons (1 long ton = 2240 lb) for larger vessels.) (Data from *Skene's Elements of Yacht Design* by Francis S. Kinney (Dodd, Mead, 1962.)

c. **Prismatic coefficients** A boat's prismatic coefficient is the ratio of the displacement volume to the volume of a prism whose height equals the boat's waterline length and whose base equals the area of the boat's largest submerged cross-section. The best sailboats have prismatic coefficients between 0.51 and 0.54. Find *Pipedream*'s prismatic coefficient, given a waterline length of 25.4 ft and a largest submerged cross-sectional area of 16.14 ft^2 (at Station 6).

T 49. Elliptic integrals The length of the ellipse

$$x = a\cos t, \quad y = b\sin t, \quad 0 \le t \le 2\pi$$

turns out to be

$$\text{Length} = 4a\int_0^{\pi/2} \sqrt{1 - e^2\cos^2 t}\, dt,$$

where e is the ellipse's eccentricity. The integral in this formula, called an *elliptic integral*, is nonelementary except when $e = 0$ or 1.

a. Use the Trapezoidal Rule with $n = 10$ to estimate the length of the ellipse when $a = 1$ and $e = 1/2$.

b. Use the fact that the absolute value of the second derivative of $f(t) = \sqrt{1 - e^2\cos^2 t}$ is less than 1 to find an upper bound for the error in the estimate you obtained in part (a).

T 50. The length of one arch of the curve $y = \sin x$ is given by

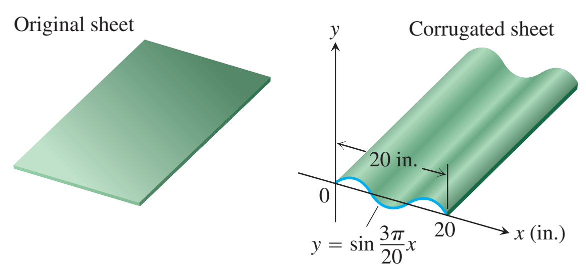
$$L = \int_0^{\pi} \sqrt{1 + \cos^2 x}\, dx.$$

Estimate L by Simpson's Rule with $n = 8$.

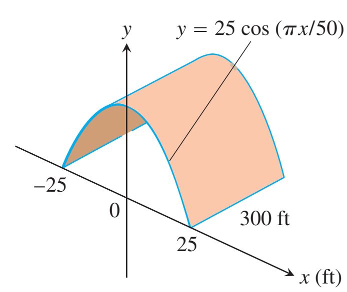
T 51. Your metal fabrication company is bidding for a contract to make sheets of corrugated iron roofing like the one shown here. The cross-sections of the corrugated sheets are to conform to the curve

$$y = \sin\frac{3\pi}{20}x, \quad 0 \le x \le 20 \text{ in}.$$

If the roofing is to be stamped from flat sheets by a process that does not stretch the material, how wide should the original material be? To find out, use numerical integration to approximate the length of the sine curve to two decimal places.



T 52. Your engineering firm is bidding for the contract to construct the tunnel shown here. The tunnel is 300 ft long and 50 ft wide at the base. The cross-section is shaped like one arch of the curve $y = 25\cos(\pi x/50)$. Upon completion, the tunnel's inside surface (excluding the roadway) will be treated with a waterproof sealer that costs \$1.75 per square foot to apply. How much will it cost to apply the sealer? (*Hint:* Use numerical integration to find the length of the cosine curve.)



Surface Area

Find, to two decimal places, the areas of the surfaces generated by revolving the curves in Exercises 53–56 about the x-axis.

53. $y = \sin x, \quad 0 \le x \le \pi$

54. $y = x^2/4, \quad 0 \le x \le 2$

55. $y = x + \sin 2x, \quad -2\pi/3 \le x \le 2\pi/3$ (the curve in Section 4.4, Exercise 5)

56. $y = \frac{x}{12}\sqrt{36 - x^2}, 0 \le x \le 6$ (the surface of the plumb bob in Section 6.1, Exercise 56)

Estimating Function Values

57. Use numerical integration to estimate the value of

$$\sin^{-1} 0.6 = \int_0^{0.6} \frac{dx}{\sqrt{1 - x^2}}.$$

For reference, $\sin^{-1} 0.6 = 0.64350$ to five decimal places.

58. Use numerical integration to estimate the value of

$$\pi = 4\int_0^1 \frac{1}{1 + x^2}\,dx.$$

8.8 Improper Integrals

Up to now, definite integrals have been required to have two properties. First, that the domain of integration $[a, b]$ be finite. Second, that the range of the integrand be finite on this domain. In practice, we may encounter problems that fail to meet one or both of these conditions. The integral for the area under the curve $y = (\ln x)/x^2$ from $x = 1$ to $x = \infty$ is an example for which the domain is infinite (Figure 8.17a). The integral for the area under the curve of $y = 1/\sqrt{x}$ between $x = 0$ and $x = 1$ is an example for which the range of the integrand is infinite (Figure 8.17b). In either case, the integrals are said to be *improper* and are calculated as limits. We will see that improper integrals play an important role when investigating the convergence of certain infinite series in Chapter 11.

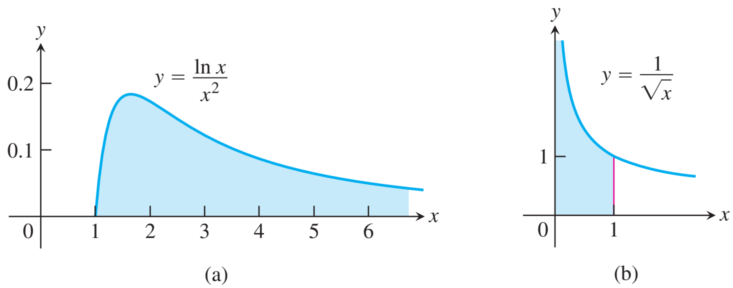


FIGURE 8.17 Are the areas under these infinite curves finite?

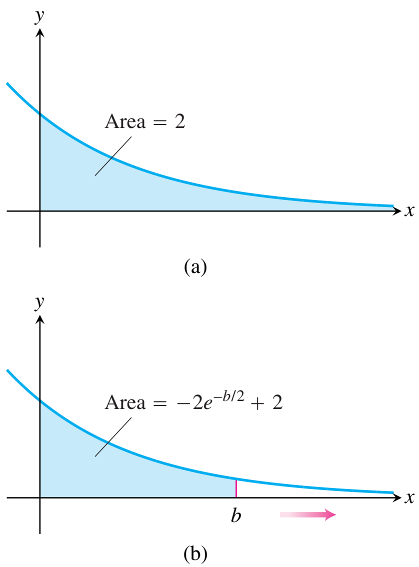


FIGURE 8.18 (a) The area in the first quadrant under the curve $y = e^{-x/2}$ is (b) an improper integral of the first type.

Infinite Limits of Integration

Consider the infinite region that lies under the curve $y = e^{-x/2}$ in the first quadrant (Figure 8.18a). You might think this region has infinite area, but we will see that the natural value to assign is finite. Here is how to assign a value to the area. First find the area $A(b)$ of the portion of the region that is bounded on the right by $x = b$ (Figure 8.18b).

$$A(b) = \int_0^b e^{-x/2}\, dx = -2e^{-x/2}\Big]_0^b = -2e^{-b/2} + 2$$

Then find the limit of $A(b)$ as $b \to \infty$

$$\lim_{b\to\infty} A(b) = \lim_{b\to\infty} (-2e^{-b/2} + 2) = 2.$$

The value we assign to the area under the curve from 0 to ∞ is

$$\int_0^{\infty} e^{-x/2}\, dx = \lim_{b\to\infty} \int_0^b e^{-x/2}\, dx = 2.$$

DEFINITION Type I Improper Integrals

Integrals with infinite limits of integration are **improper integrals of Type I**.

1. If $f(x)$ is continuous on $[a, \infty)$, then

$$\int_a^{\infty} f(x)\, dx = \lim_{b\to\infty} \int_a^b f(x)\, dx.$$

2. If $f(x)$ is continuous on $(-\infty, b]$, then

$$\int_{-\infty}^{b} f(x)\, dx = \lim_{a\to-\infty} \int_a^b f(x)\, dx.$$

3. If $f(x)$ is continuous on $(-\infty, \infty)$, then

$$\int_{-\infty}^{\infty} f(x)\, dx = \int_{-\infty}^{c} f(x)\, dx + \int_c^{\infty} f(x)\, dx,$$

where c is any real number.

In each case, if the limit is finite we say that the improper integral **converges** and that the limit is the **value** of the improper integral. If the limit fails to exist, the improper integral **diverges**.

It can be shown that the choice of c in Part 3 of the definition is unimportant. We can evaluate or determine the convergence or divergence of $\int_{-\infty}^{\infty} f(x)\, dx$ with any convenient choice.

Any of the integrals in the above definition can be interpreted as an area if $f \geq 0$ on the interval of integration. For instance, we interpreted the improper integral in Figure 8.18 as an area. In that case, the area has the finite value 2. If $f \geq 0$ and the improper integral diverges, we say the area under the curve is **infinite**.

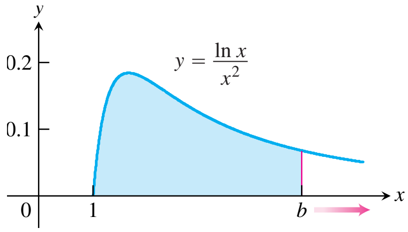


FIGURE 8.19 The area under this curve is an improper integral (Example 1).

EXAMPLE 1 Evaluating an Improper Integral on $[1, \infty)$

Is the area under the curve $y = (\ln x)/x^2$ from $x = 1$ to $x = \infty$ finite? If so, what is it?

Solution We find the area under the curve from $x = 1$ to $x = b$ and examine the limit as $b \to \infty$. If the limit is finite, we take it to be the area under the curve (Figure 8.19). The area from 1 to b is

$$\int_1^b \frac{\ln x}{x^2}\, dx = \left[(\ln x)\left(-\frac{1}{x}\right)\right]_1^b - \int_1^b \left(-\frac{1}{x}\right)\left(\frac{1}{x}\right) dx$$

Integration by parts with $u = \ln x$, $dv = dx/x^2$, $du = dx/x$, $v = -1/x$.

$$= -\frac{\ln b}{b} - \left[\frac{1}{x}\right]_1^b$$

$$= -\frac{\ln b}{b} - \frac{1}{b} + 1.$$

The limit of the area as $b \to \infty$ is

$$\begin{aligned}\int_1^\infty \frac{\ln x}{x^2}\,dx &= \lim_{b\to\infty}\int_1^b \frac{\ln x}{x^2}\,dx\\ &= \lim_{b\to\infty}\left[-\frac{\ln b}{b} - \frac{1}{b} + 1\right]\\ &= -\left[\lim_{b\to\infty}\frac{\ln b}{b}\right] - 0 + 1\\ &= -\left[\lim_{b\to\infty}\frac{1/b}{1}\right] + 1 = 0 + 1 = 1. \qquad \text{l'Hôpital's Rule}\end{aligned}$$

Thus, the improper integral converges and the area has finite value 1. ■

EXAMPLE 2 Evaluating an Integral on $(-\infty, \infty)$

Evaluate

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2}.$$

Solution According to the definition (Part 3), we can write

$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \int_{-\infty}^{0} \frac{dx}{1+x^2} + \int_{0}^{\infty} \frac{dx}{1+x^2}.$$

Next we evaluate each improper integral on the right side of the equation above.

$$\begin{aligned}\int_{-\infty}^{0} \frac{dx}{1+x^2} &= \lim_{a\to-\infty}\int_a^0 \frac{dx}{1+x^2}\\ &= \lim_{a\to-\infty} \tan^{-1}x\Big]_a^0\\ &= \lim_{a\to-\infty}(\tan^{-1}0 - \tan^{-1}a) = 0 - \left(-\frac{\pi}{2}\right) = \frac{\pi}{2}\end{aligned}$$

$$\begin{aligned}\int_{0}^{\infty} \frac{dx}{1+x^2} &= \lim_{b\to\infty}\int_0^b \frac{dx}{1+x^2}\\ &= \lim_{b\to\infty} \tan^{-1}x\Big]_0^b\\ &= \lim_{b\to\infty}(\tan^{-1}b - \tan^{-1}0) = \frac{\pi}{2} - 0 = \frac{\pi}{2}\end{aligned}$$

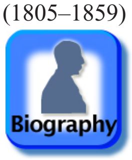
Thus,

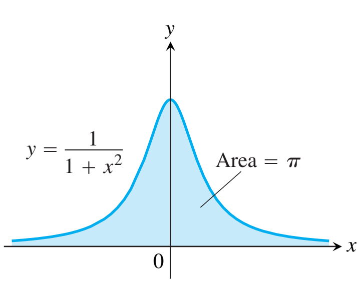
$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \frac{\pi}{2} + \frac{\pi}{2} = \pi.$$

Since $1/(1+x^2) > 0$, the improper integral can be interpreted as the (finite) area beneath the curve and above the x-axis (Figure 8.20). ■

HISTORICAL BIOGRAPHY

Lejeune Dirichlet
(1805–1859)





NOT TO SCALE

FIGURE 8.20 The area under this curve is finite (Example 2).

The Integral $\int_1^{\infty} \frac{dx}{x^p}$

The function $y = 1/x$ is the boundary between the convergent and divergent improper integrals with integrands of the form $y = 1/x^p$. As the next example shows, the improper integral converges if $p > 1$ and diverges if $p \leq 1$.

EXAMPLE 3 Determining Convergence

For what values of p does the integral $\int_1^{\infty} dx/x^p$ converge? When the integral does converge, what is its value?

Solution If $p \neq 1$,

$$\int_1^b \frac{dx}{x^p} = \frac{x^{-p+1}}{-p+1}\bigg]_1^b = \frac{1}{1-p}\left(b^{-p+1} - 1\right) = \frac{1}{1-p}\left(\frac{1}{b^{p-1}} - 1\right).$$

Thus,

$$\begin{aligned}\int_1^{\infty} \frac{dx}{x^p} &= \lim_{b\to\infty} \int_1^b \frac{dx}{x^p} \\ &= \lim_{b\to\infty} \left[\frac{1}{1-p}\left(\frac{1}{b^{p-1}} - 1\right)\right] = \begin{cases} \dfrac{1}{p-1}, & p > 1 \\ \infty, & p < 1 \end{cases}\end{aligned}$$

because

$$\lim_{b\to\infty} \frac{1}{b^{p-1}} = \begin{cases} 0, & p > 1 \\ \infty, & p < 1. \end{cases}$$

Therefore, the integral converges to the value $1/(p-1)$ if $p > 1$ and it diverges if $p < 1$.

If $p = 1$, the integral also diverges:

$$\begin{aligned}\int_1^{\infty} \frac{dx}{x^p} &= \int_1^{\infty} \frac{dx}{x} \\ &= \lim_{b\to\infty} \int_1^b \frac{dx}{x} \\ &= \lim_{b\to\infty} \ln x \bigg]_1^b \\ &= \lim_{b\to\infty} (\ln b - \ln 1) = \infty.\end{aligned}$$

■

Integrands with Vertical Asymptotes

Another type of improper integral arises when the integrand has a vertical asymptote—an infinite discontinuity—at a limit of integration or at some point between the limits of integration. If the integrand f is positive over the interval of integration, we can again interpret the improper integral as the area under the graph of f and above the x-axis between the limits of integration.

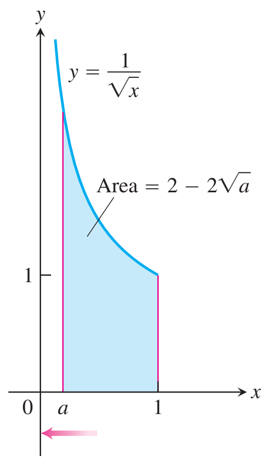


FIGURE 8.21 The area under this curve is

$$\lim_{a\to 0^+}\int_a^1 \left(\frac{1}{\sqrt{x}}\right) dx = 2,$$

an improper integral of the second kind.

Consider the region in the first quadrant that lies under the curve $y = 1/\sqrt{x}$ from $x = 0$ to $x = 1$ (Figure 8.17b). First we find the area of the portion from a to 1 (Figure 8.21).

$$\int_a^1 \frac{dx}{\sqrt{x}} = 2\sqrt{x}\Big]_a^1 = 2 - 2\sqrt{a}$$

Then we find the limit of this area as $a \to 0^+$:

$$\lim_{a\to 0^+}\int_a^1 \frac{dx}{\sqrt{x}} = \lim_{a\to 0^+}\left(2 - 2\sqrt{a}\right) = 2.$$

The area under the curve from 0 to 1 is finite and equals

$$\int_0^1 \frac{dx}{\sqrt{x}} = \lim_{a\to 0^+}\int_a^1 \frac{dx}{\sqrt{x}} = 2.$$

DEFINITION Type II Improper Integrals

Integrals of functions that become infinite at a point within the interval of integration are **improper integrals of Type II**.

1. If $f(x)$ is continuous on $(a, b]$ and is discontinuous at a then
$$\int_a^b f(x)\,dx = \lim_{c\to a^+}\int_c^b f(x)\,dx.$$

2. If $f(x)$ is continuous on $[a, b)$ and is discontinuous at b, then
$$\int_a^b f(x)\,dx = \lim_{c\to b^-}\int_a^c f(x)\,dx.$$

3. If $f(x)$ is discontinuous at c, where $a < c < b$, and continuous on $[a, c) \cup (c, b]$, then
$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx.$$

In each case, if the limit is finite we say the improper integral **converges** and that the limit is the **value** of the improper integral. If the limit does not exist, the integral **diverges**.

In Part 3 of the definition, the integral on the left side of the equation converges if *both* integrals on the right side converge; otherwise it diverges.

EXAMPLE 4 A Divergent Improper Integral

Investigate the convergence of

$$\int_0^1 \frac{1}{1 - x}\,dx.$$

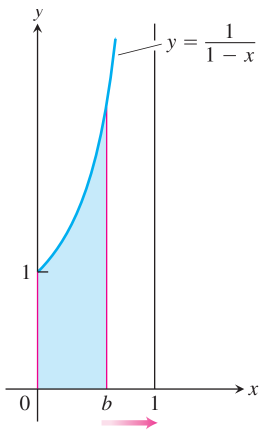


FIGURE 8.22 The limit does not exist:

$$\int_0^1 \left(\frac{1}{1-x}\right) dx = \lim_{b\to 1^-} \int_0^b \frac{1}{1-x}\,dx = \infty.$$

The area beneath the curve and above the x-axis for $[0, 1)$ is not a real number (Example 4).

Solution The integrand $f(x) = 1/(1-x)$ is continuous on $[0, 1)$ but is discontinuous at $x = 1$ and becomes infinite as $x \to 1^-$ (Figure 8.22). We evaluate the integral as

$$\begin{aligned}\lim_{b\to 1^-} \int_0^b \frac{1}{1-x}\,dx &= \lim_{b\to 1^-} \Big[-\ln|1-x|\Big]_0^b \\ &= \lim_{b\to 1^-} [-\ln(1-b) + 0] = \infty.\end{aligned}$$

The limit is infinite, so the integral diverges. ■

EXAMPLE 5 Vertical Asympote at an Interior Point

Evaluate

$$\int_0^3 \frac{dx}{(x-1)^{2/3}}.$$

Solution The integrand has a vertical asymptote at $x = 1$ and is continuous on $[0, 1)$ and $(1, 3]$ (Figure 8.23). Thus, by Part 3 of the definition above,

$$\int_0^3 \frac{dx}{(x-1)^{2/3}} = \int_0^1 \frac{dx}{(x-1)^{2/3}} + \int_1^3 \frac{dx}{(x-1)^{2/3}}.$$

Next, we evaluate each improper integral on the right-hand side of this equation.

$$\begin{aligned}\int_0^1 \frac{dx}{(x-1)^{2/3}} &= \lim_{b\to 1^-} \int_0^b \frac{dx}{(x-1)^{2/3}} \\ &= \lim_{b\to 1^-} 3(x-1)^{1/3}\Big]_0^b \\ &= \lim_{b\to 1^-} [3(b-1)^{1/3} + 3] = 3\end{aligned}$$

$$\begin{aligned}\int_1^3 \frac{dx}{(x-1)^{2/3}} &= \lim_{c\to 1^+} \int_c^3 \frac{dx}{(x-1)^{2/3}} \\ &= \lim_{c\to 1^+} 3(x-1)^{1/3}\Big]_c^3 \\ &= \lim_{c\to 1^+} \left[3(3-1)^{1/3} - 3(c-1)^{1/3}\right] = 3\sqrt[3]{2}\end{aligned}$$

We conclude that

$$\int_0^3 \frac{dx}{(x-1)^{2/3}} = 3 + 3\sqrt[3]{2}.$$ ■

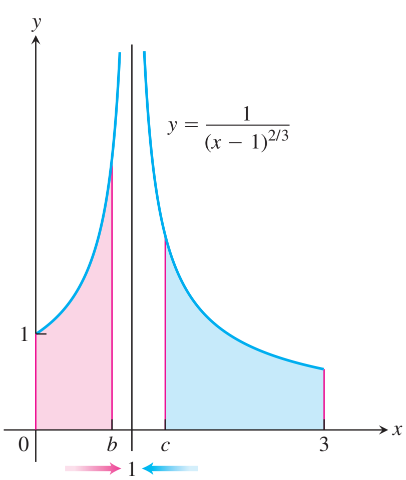


FIGURE 8.23 Example 5 shows the convergence of

$$\int_0^3 \frac{1}{(x-1)^{2/3}}\,dx = 3 + 3\sqrt[3]{2},$$

so the area under the curve exists (so it is a real number).

EXAMPLE 6 A Convergent Improper Integral

Evaluate

$$\int_2^\infty \frac{x+3}{(x-1)(x^2+1)}\,dx.$$

Solution

$$\int_2^{\infty} \frac{x+3}{(x-1)(x^2+1)}\,dx = \lim_{b\to\infty}\int_2^b \frac{x+3}{(x-1)(x^2+1)}\,dx$$

$$= \lim_{b\to\infty}\int_2^b \left(\frac{2}{x-1} - \frac{2x+1}{x^2+1}\right)dx \qquad \text{Partial fractions}$$

$$= \lim_{b\to\infty}\left[2\ln(x-1) - \ln(x^2+1) - \tan^{-1}x\right]_2^b$$

$$= \lim_{b\to\infty}\left[\ln\frac{(x-1)^2}{x^2+1} - \tan^{-1}x\right]_2^b \qquad \text{Combine the logarithms.}$$

$$= \lim_{b\to\infty}\left[\ln\left(\frac{(b-1)^2}{b^2+1}\right) - \tan^{-1}b\right] - \ln\left(\frac{1}{5}\right) + \tan^{-1}2$$

$$= 0 - \frac{\pi}{2} + \ln 5 + \tan^{-1}2 \approx 1.1458$$

Notice that we combined the logarithms in the antiderivative *before* we calculated the limit as $b\to\infty$. Had we not done so, we would have encountered the indeterminate form

$$\lim_{b\to\infty}(2\ln(b-1) - \ln(b^2+1)) = \infty - \infty.$$

The way to evaluate the indeterminate form, of course, is to combine the logarithms, so we would have arrived at the same answer in the end. ■

Computer algebra systems can evaluate many convergent improper integrals. To evaluate the integral in Example 6 using Maple, enter

$$> f{:=}\ (x+3)/((x-1)*(x\verb|^|2+1));$$

Then use the integration command

$$> \text{int}(f, x = 2..\text{infinity});$$

Maple returns the answer

$$-\frac{1}{2}\pi + \ln(5) + \arctan(2).$$

To obtain a numerical result, use the evaluation command **evalf** and specify the number of digits, as follows:

$$> \text{evalf}(\%, 6);$$

The symbol % instructs the computer to evaluate the last expression on the screen, in this case $(-1/2)\pi + \ln(5) + \arctan(2)$. Maple returns 1.14579.

Using Mathematica, entering

$$\textit{In [1]:=}\ \text{Integrate}\,[(x+3)/((x-1)(x\verb|^|2+1)), \{x, 2, \text{Infinity}\}]$$

returns

$$\textit{Out [1]=}\ \frac{-\text{Pi}}{2} + \text{ArcTan}\,[2] + \text{Log}\,[5].$$

To obtain a numerical result with six digits, use the command "N[%, 6]"; it also yields 1.14579.

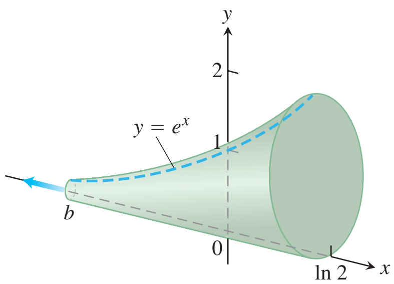


FIGURE 8.24 The calculation in Example 7 shows that this infinite horn has a finite volume.

EXAMPLE 7 Finding the Volume of an Infinite Solid

The cross-sections of the solid horn in Figure 8.24 perpendicular to the x-axis are circular disks with diameters reaching from the x-axis to the curve $y = e^x$, $-\infty < x \leq \ln 2$. Find the volume of the horn.

Solution The area of a typical cross-section is

$$A(x) = \pi(\text{radius})^2 = \pi\left(\frac{1}{2}y\right)^2 = \frac{\pi}{4}e^{2x}.$$

We define the volume of the horn to be the limit as $b \to -\infty$ of the volume of the portion from b to $\ln 2$. As in Section 6.1 (the method of slicing), the volume of this portion is

$$\begin{aligned} V &= \int_b^{\ln 2} A(x)\,dx = \int_b^{\ln 2} \frac{\pi}{4}e^{2x}\,dx = \frac{\pi}{8}e^{2x}\bigg]_b^{\ln 2} \\ &= \frac{\pi}{8}(e^{\ln 4} - e^{2b}) = \frac{\pi}{8}(4 - e^{2b}). \end{aligned}$$

As $b \to -\infty$, $e^{2b} \to 0$ and $V \to (\pi/8)(4 - 0) = \pi/2$. The volume of the horn is $\pi/2$. ■

EXAMPLE 8 An Incorrect Calculation

Evaluate

$$\int_0^3 \frac{dx}{x-1}.$$

Solution Suppose we fail to notice the discontinuity of the integrand at $x = 1$, interior to the interval of integration. If we evaluate the integral as an ordinary integral we get

$$\int_0^3 \frac{dx}{x-1} = \ln|x-1|\bigg]_0^3 = \ln 2 - \ln 1 = \ln 2.$$

This result is *wrong* because the integral is improper. The correct evaluation uses limits:

$$\int_0^3 \frac{dx}{x-1} = \int_0^1 \frac{dx}{x-1} + \int_1^3 \frac{dx}{x-1}$$

where

$$\begin{aligned} \int_0^1 \frac{dx}{x-1} &= \lim_{b\to 1^-}\int_0^b \frac{dx}{x-1} = \lim_{b\to 1^-}\ln|x-1|\bigg]_0^b \\ &= \lim_{b\to 1^-}(\ln|b-1| - \ln|-1|) \\ &= \lim_{b\to 1^-}\ln(1-b) = -\infty. \qquad 1 - b \to 0^+ \text{ as } b \to 1^- \end{aligned}$$

Since $\int_0^1 dx/(x-1)$ is divergent, the original integral $\int_0^3 dx/(x-1)$ is divergent. ■

Example 8 illustrates what can go wrong if you mistake an improper integral for an ordinary integral. Whenever you encounter an integral $\int_a^b f(x)\,dx$ you must examine the function f on $[a, b]$ and then decide if the integral is improper. If f is continuous on $[a, b]$, it will be proper, an ordinary integral.

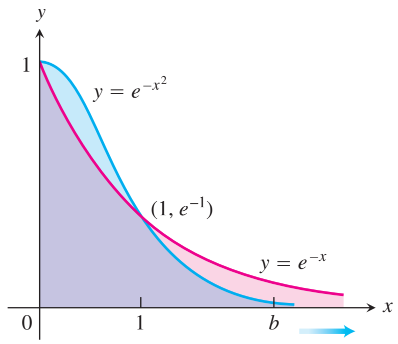


FIGURE 8.25 The graph of e^{-x^2} lies below the graph of e^{-x} for $x > 1$ (Example 9).

Tests for Convergence and Divergence

When we cannot evaluate an improper integral directly, we try to determine whether it converges or diverges. If the integral diverges, that's the end of the story. If it converges, we can use numerical methods to approximate its value. The principal tests for convergence or divergence are the Direct Comparison Test and the Limit Comparison Test.

EXAMPLE 9 Investigating Convergence

Does the integral $\int_1^\infty e^{-x^2}\,dx$ converge?

Solution By definition,

$$\int_1^\infty e^{-x^2}\,dx = \lim_{b\to\infty}\int_1^b e^{-x^2}\,dx.$$

We cannot evaluate the latter integral directly because it is nonelementary. But we *can* show that its limit as $b \to \infty$ is finite. We know that $\int_1^b e^{-x^2}\,dx$ is an increasing function of b. Therefore either it becomes infinite as $b \to \infty$ or it has a finite limit as $b \to \infty$. It does not become infinite: For every value of $x \geq 1$ we have $e^{-x^2} \leq e^{-x}$ (Figure 8.25), so that

$$\int_1^b e^{-x^2}\,dx \leq \int_1^b e^{-x}\,dx = -e^{-b} + e^{-1} < e^{-1} \approx 0.36788.$$

Hence

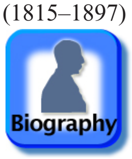
$$\int_1^\infty e^{-x^2}\,dx = \lim_{b\to\infty}\int_1^b e^{-x^2}\,dx$$

converges to some definite finite value. We do not know exactly what the value is except that it is something positive and less than 0.37. Here we are relying on the completeness property of the real numbers, discussed in Appendix 4. ■

The comparison of e^{-x^2} and e^{-x} in Example 9 is a special case of the following test.

HISTORICAL BIOGRAPHY

Karl Weierstrass
(1815–1897)



THEOREM 1 Direct Comparison Test

Let f and g be continuous on $[a, \infty)$ with $0 \leq f(x) \leq g(x)$ for all $x \geq a$. Then

1. $\int_a^\infty f(x)\,dx$ converges if $\int_a^\infty g(x)\,dx$ converges
2. $\int_a^\infty g(x)\,dx$ diverges if $\int_a^\infty f(x)\,dx$ diverges.

The reasoning behind the argument establishing Theorem 1 is similar to that in Example 9.

If $0 \leq f(x) \leq g(x)$ for $x \geq a$, then

$$\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx, \qquad b > a.$$

From this it can be argued, as in Example 9, that

$$\int_a^\infty f(x)\,dx \text{ converges if } \int_a^\infty g(x)\,dx \text{ converges.}$$

Turning this around says that

$$\int_a^\infty g(x)\,dx \text{ diverges if } \int_a^\infty f(x)\,dx \text{ diverges.}$$

EXAMPLE 10 Using the Direct Comparison Test

(a) $\displaystyle\int_1^\infty \frac{\sin^2 x}{x^2}\,dx$ converges because

$$0 \le \frac{\sin^2 x}{x^2} \le \frac{1}{x^2} \quad \text{on} \quad [1, \infty) \quad \text{and} \quad \int_1^\infty \frac{1}{x^2}\,dx \text{ converges.}$$ Example 3

(b) $\displaystyle\int_1^\infty \frac{1}{\sqrt{x^2 - 0.1}}\,dx$ diverges because

$$\frac{1}{\sqrt{x^2 - 0.1}} \ge \frac{1}{x} \quad \text{on} \quad [1, \infty) \quad \text{and} \quad \int_1^\infty \frac{1}{x}\,dx \text{ diverges.}$$ Example 3

■

THEOREM 2 Limit Comparison Test

If the positive functions f and g are continuous on $[a, \infty)$ and if

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = L, \qquad 0 < L < \infty,$$

then

$$\int_a^\infty f(x)\,dx \qquad \text{and} \qquad \int_a^\infty g(x)\,dx$$

both converge or both diverge.

A proof of Theorem 2 is given in advanced calculus.

Although the improper integrals of two functions from a to ∞ may both converge, this does not mean that their integrals necessarily have the same value, as the next example shows.

EXAMPLE 11 Using the Limit Comparison Test

Show that

$$\int_1^{\infty} \frac{dx}{1+x^2}$$

converges by comparison with $\int_1^{\infty} (1/x^2)\, dx$. Find and compare the two integral values.

Solution The functions $f(x) = 1/x^2$ and $g(x) = 1/(1+x^2)$ are positive and continuous on $[1, \infty)$. Also,

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{1/x^2}{1/(1+x^2)} = \lim_{x\to\infty} \frac{1+x^2}{x^2}$$

$$= \lim_{x\to\infty} \left(\frac{1}{x^2} + 1\right) = 0 + 1 = 1,$$

a positive finite limit (Figure 8.26). Therefore, $\int_1^{\infty} \frac{dx}{1+x^2}$ converges because $\int_1^{\infty} \frac{dx}{x^2}$ converges.

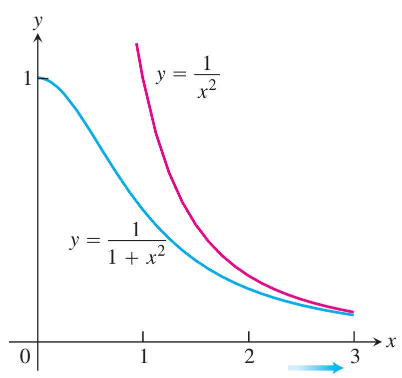


FIGURE 8.26 The functions in Example 11.

The integrals converge to different values, however.

$$\int_1^{\infty} \frac{dx}{x^2} = \frac{1}{2-1} = 1 \qquad \text{Example 3}$$

and

$$\int_1^{\infty} \frac{dx}{1+x^2} = \lim_{b\to\infty} \int_1^{b} \frac{dx}{1+x^2}$$

$$= \lim_{b\to\infty} [\tan^{-1} b - \tan^{-1} 1] = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$ ■

EXAMPLE 12 Using the Limit Comparison Test

Show that

$$\int_1^{\infty} \frac{3}{e^x + 5}\, dx$$

converges.

Solution From Example 9, it is easy to see that $\int_1^{\infty} e^{-x}\, dx = \int_1^{\infty} (1/e^x)\, dx$ converges. Moreover, we have

$$\lim_{x\to\infty} \frac{1/e^x}{3/(e^x+5)} = \lim_{x\to\infty} \frac{e^x+5}{3e^x} = \lim_{x\to\infty} \left(\frac{1}{3} + \frac{5}{3e^x}\right) = \frac{1}{3},$$

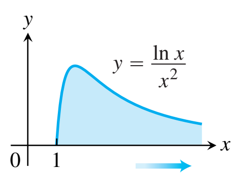
a positive finite limit. As far as the convergence of the improper integral is concerned, $3/(e^x + 5)$ behaves like $1/e^x$. ■

Types of Improper Integrals Discussed in This Section

INFINITE LIMITS OF INTEGRATION: TYPE I

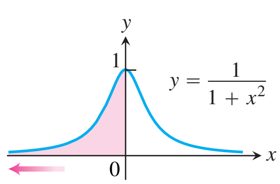
1. Upper limit

$$\int_1^{\infty} \frac{\ln x}{x^2}\,dx = \lim_{b\to\infty}\int_1^{b} \frac{\ln x}{x^2}\,dx$$



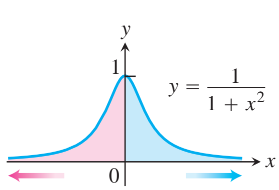
2. Lower limit

$$\int_{-\infty}^{0} \frac{dx}{1+x^2} = \lim_{a\to-\infty}\int_a^{0} \frac{dx}{1+x^2}$$



3. Both limits

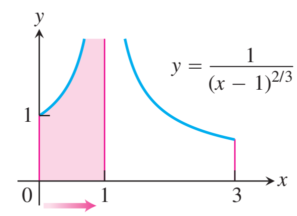
$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \lim_{b\to-\infty}\int_b^{0} \frac{dx}{1+x^2} + \lim_{c\to\infty}\int_0^{c} \frac{dx}{1+x^2}$$



INTEGRAND BECOMES INFINITE: TYPE II

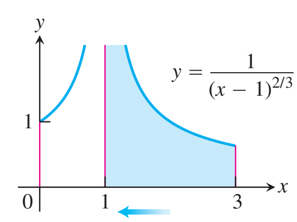
4. Upper endpoint

$$\int_0^{1} \frac{dx}{(x-1)^{2/3}} = \lim_{b\to1^-}\int_0^{b} \frac{dx}{(x-1)^{2/3}}$$



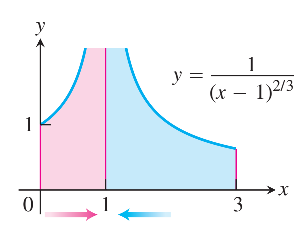
5. Lower endpoint

$$\int_1^{3} \frac{dx}{(x-1)^{2/3}} = \lim_{d\to1^+}\int_d^{3} \frac{dx}{(x-1)^{2/3}}$$



6. Interior point

$$\int_0^{3} \frac{dx}{(x-1)^{2/3}} = \int_0^{1} \frac{dx}{(x-1)^{2/3}} + \int_1^{3} \frac{dx}{(x-1)^{2/3}}$$



EXERCISES 8.8

Evaluating Improper Integrals

Evaluate the integrals in Exercises 1–34 without using tables.

1. $\displaystyle\int_0^\infty \frac{dx}{x^2+1}$

2. $\displaystyle\int_1^\infty \frac{dx}{x^{1.001}}$

3. $\displaystyle\int_0^1 \frac{dx}{\sqrt{x}}$

4. $\displaystyle\int_0^4 \frac{dx}{\sqrt{4-x}}$

5. $\displaystyle\int_{-1}^1 \frac{dx}{x^{2/3}}$

6. $\displaystyle\int_{-8}^1 \frac{dx}{x^{1/3}}$

7. $\displaystyle\int_0^1 \frac{dx}{\sqrt{1-x^2}}$

8. $\displaystyle\int_0^1 \frac{dr}{r^{0.999}}$

9. $\displaystyle\int_{-\infty}^{-2} \frac{2\,dx}{x^2-1}$

10. $\displaystyle\int_{-\infty}^2 \frac{2\,dx}{x^2+4}$

11. $\displaystyle\int_2^\infty \frac{2}{v^2-v}\,dv$

12. $\displaystyle\int_2^\infty \frac{2\,dt}{t^2-1}$

13. $\displaystyle\int_{-\infty}^\infty \frac{2x\,dx}{(x^2+1)^2}$

14. $\displaystyle\int_{-\infty}^\infty \frac{x\,dx}{(x^2+4)^{3/2}}$

15. $\displaystyle\int_0^1 \frac{\theta+1}{\sqrt{\theta^2+2\theta}}\,d\theta$

16. $\displaystyle\int_0^2 \frac{s+1}{\sqrt{4-s^2}}\,ds$

17. $\displaystyle\int_0^\infty \frac{dx}{(1+x)\sqrt{x}}$

18. $\displaystyle\int_1^\infty \frac{1}{x\sqrt{x^2-1}}\,dx$

19. $\displaystyle\int_0^\infty \frac{dv}{(1+v^2)(1+\tan^{-1}v)}$

20. $\displaystyle\int_0^\infty \frac{16\tan^{-1}x}{1+x^2}\,dx$

21. $\displaystyle\int_{-\infty}^0 \theta e^\theta\,d\theta$

22. $\displaystyle\int_0^\infty 2e^{-\theta}\sin\theta\,d\theta$

23. $\displaystyle\int_{-\infty}^0 e^{-|x|}\,dx$

24. $\displaystyle\int_{-\infty}^\infty 2xe^{-x^2}\,dx$

25. $\displaystyle\int_0^1 x\ln x\,dx$

26. $\displaystyle\int_0^1 (-\ln x)\,dx$

27. $\displaystyle\int_0^2 \frac{ds}{\sqrt{4-s^2}}$

28. $\displaystyle\int_0^1 \frac{4r\,dr}{\sqrt{1-r^4}}$

29. $\displaystyle\int_1^2 \frac{ds}{s\sqrt{s^2-1}}$

30. $\displaystyle\int_2^4 \frac{dt}{t\sqrt{t^2-4}}$

31. $\displaystyle\int_{-1}^4 \frac{dx}{\sqrt{|x|}}$

32. $\displaystyle\int_0^2 \frac{dx}{\sqrt{|x-1|}}$

33. $\displaystyle\int_{-1}^\infty \frac{d\theta}{\theta^2+5\theta+6}$

34. $\displaystyle\int_0^\infty \frac{dx}{(x+1)(x^2+1)}$

Testing for Convergence

In Exercises 35–64, use integration, the Direct Comparison Test, or the Limit Comparison Test to test the integrals for convergence. If more than one method applies, use whatever method you prefer.

35. $\displaystyle\int_0^{\pi/2} \tan\theta\,d\theta$

36. $\displaystyle\int_0^{\pi/2} \cot\theta\,d\theta$

37. $\displaystyle\int_0^\pi \frac{\sin\theta\,d\theta}{\sqrt{\pi-\theta}}$

38. $\displaystyle\int_{-\pi/2}^{\pi/2} \frac{\cos\theta\,d\theta}{(\pi-2\theta)^{1/3}}$

39. $\displaystyle\int_0^{\ln 2} x^{-2}e^{-1/x}\,dx$

40. $\displaystyle\int_0^1 \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,dx$

41. $\displaystyle\int_0^\pi \frac{dt}{\sqrt{t}+\sin t}$

42. $\displaystyle\int_0^1 \frac{dt}{t-\sin t}$ (*Hint:* $t \ge \sin t$ for $t \ge 0$)

43. $\displaystyle\int_0^2 \frac{dx}{1-x^2}$

44. $\displaystyle\int_0^2 \frac{dx}{1-x}$

45. $\displaystyle\int_{-1}^1 \ln|x|\,dx$

46. $\displaystyle\int_{-1}^1 -x\ln|x|\,dx$

47. $\displaystyle\int_1^\infty \frac{dx}{x^3+1}$

48. $\displaystyle\int_4^\infty \frac{dx}{\sqrt{x}-1}$

49. $\displaystyle\int_2^\infty \frac{dv}{\sqrt{v-1}}$

50. $\displaystyle\int_0^\infty \frac{d\theta}{1+e^\theta}$

51. $\displaystyle\int_0^\infty \frac{dx}{\sqrt{x^6+1}}$

52. $\displaystyle\int_2^\infty \frac{dx}{\sqrt{x^2-1}}$

53. $\displaystyle\int_1^\infty \frac{\sqrt{x+1}}{x^2}\,dx$

54. $\displaystyle\int_2^\infty \frac{x\,dx}{\sqrt{x^4-1}}$

55. $\displaystyle\int_\pi^\infty \frac{2+\cos x}{x}\,dx$

56. $\displaystyle\int_\pi^\infty \frac{1+\sin x}{x^2}\,dx$

57. $\displaystyle\int_4^\infty \frac{2\,dt}{t^{3/2}-1}$

58. $\displaystyle\int_2^\infty \frac{1}{\ln x}\,dx$

59. $\displaystyle\int_1^\infty \frac{e^x}{x}\,dx$

60. $\displaystyle\int_{e^e}^\infty \ln(\ln x)\,dx$

61. $\displaystyle\int_1^\infty \frac{1}{\sqrt{e^x-x}}\,dx$

62. $\displaystyle\int_1^\infty \frac{1}{e^x-2^x}\,dx$

63. $\displaystyle\int_{-\infty}^\infty \frac{dx}{\sqrt{x^4+1}}$

64. $\displaystyle\int_{-\infty}^\infty \frac{dx}{e^x+e^{-x}}$

Theory and Examples

65. Find the values of p for which each integral converges.

a. $\displaystyle\int_1^2 \frac{dx}{x(\ln x)^p}$ **b.** $\displaystyle\int_2^\infty \frac{dx}{x(\ln x)^p}$

66. $\int_{-\infty}^{\infty} f(x)\,dx$ **may not equal** $\lim_{b\to\infty}\int_{-b}^{b} f(x)\,dx$ Show that

$$\int_0^\infty \frac{2x\,dx}{x^2+1}$$

diverges and hence that

$$\int_{-\infty}^\infty \frac{2x\,dx}{x^2+1}$$

diverges. Then show that

$$\lim_{b\to\infty}\int_{-b}^{b}\frac{2x\,dx}{x^2+1}=0.$$

Exercises 67–70 are about the infinite region in the first quadrant between the curve $y = e^{-x}$ and the x-axis.

67. Find the area of the region.

68. Find the centroid of the region.

69. Find the volume of the solid generated by revolving the region about the y-axis.

70. Find the volume of the solid generated by revolving the region about the x-axis.

71. Find the area of the region that lies between the curves $y = \sec x$ and $y = \tan x$ from $x = 0$ to $x = \pi/2$.

72. The region in Exercise 71 is revolved about the x-axis to generate a solid.

a. Find the volume of the solid.

b. Show that the inner and outer surfaces of the solid have infinite area.

73. Estimating the value of a convergent improper integral whose domain is infinite

a. Show that

$$\int_3^\infty e^{-3x}\,dx=\frac{1}{3}e^{-9}<0.000042,$$

and hence that $\int_3^\infty e^{-x^2}\,dx < 0.000042$. Explain why this means that $\int_0^\infty e^{-x^2}\,dx$ can be replaced by $\int_0^3 e^{-x^2}\,dx$ without introducing an error of magnitude greater than 0.000042.

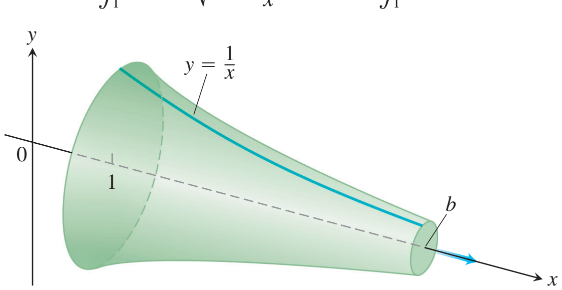
T **b.** Evaluate $\int_0^3 e^{-x^2}\,dx$ numerically.

74. The infinite paint can or Gabriel's horn As Example 3 shows, the integral $\int_1^\infty (dx/x)$ diverges. This means that the integral

$$\int_1^\infty 2\pi\frac{1}{x}\sqrt{1+\frac{1}{x^4}}\,dx,$$

which measures the *surface area* of the solid of revolution traced out by revolving the curve $y = 1/x$, $1 \le x$, about the x-axis, diverges also. By comparing the two integrals, we see that, for every finite value $b > 1$,

$$\int_1^b 2\pi\frac{1}{x}\sqrt{1+\frac{1}{x^4}}\,dx > 2\pi\int_1^b\frac{1}{x}\,dx.$$



However, the integral

$$\int_1^\infty \pi\left(\frac{1}{x}\right)^2 dx$$

for the *volume* of the solid converges. **(a)** Calculate it. **(b)** This solid of revolution is sometimes described as a can that does not hold enough paint to cover its own interior. Think about that for a moment. It is common sense that a finite amount of paint cannot cover an infinite surface. But if we fill the horn with paint (a finite amount), then we *will* have covered an infinite surface. Explain the apparent contradiction.

75. Sine-integral function The integral

$$\mathrm{Si}\,(x)=\int_0^x\frac{\sin t}{t}\,dt,$$

called the *sine-integral function,* has important applications in optics.

T **a.** Plot the integrand $(\sin t)/t$ for $t > 0$. Is the Si function everywhere increasing or decreasing? Do you think $\mathrm{Si}\,(x) = 0$ for $x > 0$? Check your answers by graphing the function $\mathrm{Si}\,(x)$ for $0 \le x \le 25$.

b. Explore the convergence of

$$\int_0^\infty\frac{\sin t}{t}\,dt.$$

If it converges, what is its value?

76. Error function The function

$$\mathrm{erf}\,(x)=\int_0^x\frac{2e^{-t^2}}{\sqrt{\pi}}\,dt,$$

called the *error function*, has important applications in probability and statistics.

T **a.** Plot the error function for $0 \le x \le 25$.

b. Explore the convergence of

$$\int_0^\infty\frac{2e^{-t^2}}{\sqrt{\pi}}\,dt.$$

If it converges, what appears to be its value? You will see how to confirm your estimate in Section 15.3, Exercise 37.

77. Normal probability distribution function The function

$$f(x)=\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

is called the *normal probability density function* with mean μ and standard deviation σ. The number μ tells where the distribution is centered, and σ measures the "scatter" around the mean.

From the theory of probability, it is known that

$$\int_{-\infty}^{\infty}f(x)\,dx=1.$$

In what follows, let $\mu = 0$ and $\sigma = 1$.

T **a.** Draw the graph of f. Find the intervals on which f is increasing, the intervals on which f is decreasing, and any local extreme values and where they occur.

b. Evaluate

$$\int_{-n}^{n} f(x)\,dx$$

for $n = 1, 2, 3$.

c. Give a convincing argument that

$$\int_{-\infty}^{\infty} f(x)\,dx = 1.$$

(*Hint:* Show that $0 < f(x) < e^{-x/2}$ for $x > 1$, and for $b > 1$,

$$\int_{b}^{\infty} e^{-x/2}\,dx \to 0 \quad \text{as} \quad b \to \infty.)$$

78. Here is an argument that $\ln 3$ equals $\infty - \infty$. Where does the argument go wrong? Give reasons for your answer.

$$\begin{aligned}
\ln 3 &= \ln 1 + \ln 3 = \ln 1 - \ln \frac{1}{3} \\
&= \lim_{b\to\infty} \ln\left(\frac{b-2}{b}\right) - \ln\frac{1}{3} \\
&= \lim_{b\to\infty} \left[\ln \frac{x-2}{x}\right]_3^b \\
&= \lim_{b\to\infty} \left[\ln (x-2) - \ln x\right]_3^b \\
&= \lim_{b\to\infty} \int_3^b \left(\frac{1}{x-2} - \frac{1}{x}\right) dx \\
&= \int_3^{\infty} \left(\frac{1}{x-2} - \frac{1}{x}\right) dx \\
&= \int_3^{\infty} \frac{1}{x-2}\,dx - \int_3^{\infty} \frac{1}{x}\,dx \\
&= \lim_{b\to\infty} \left[\ln (x-2)\right]_3^b - \lim_{b\to\infty} \left[\ln x\right]_3^b \\
&= \infty - \infty.
\end{aligned}$$

79. Show that if $f(x)$ is integrable on every interval of real numbers and a and b are real numbers with $a < b$, then

a. $\int_{-\infty}^{a} f(x)\,dx$ and $\int_{a}^{\infty} f(x)\,dx$ both converge if and only if $\int_{-\infty}^{b} f(x)\,dx$ and $\int_{b}^{\infty} f(x)\,dx$ both converge.

b. $\int_{-\infty}^{a} f(x)\,dx + \int_{a}^{\infty} f(x)\,dx = \int_{-\infty}^{b} f(x)\,dx + \int_{b}^{\infty} f(x)\,dx$ when the integrals involved converge.

80. a. Show that if f is even and the necessary integrals exist, then

$$\int_{-\infty}^{\infty} f(x)\,dx = 2\int_{0}^{\infty} f(x)\,dx.$$

b. Show that if f is odd and the necessary integrals exist, then

$$\int_{-\infty}^{\infty} f(x)\,dx = 0.$$

Use direct evaluation, the comparison tests, and the results in Exercise 80, as appropriate, to determine the convergence or divergence of the integrals in Exercises 81–88. If more than one method applies, use whatever method you prefer.

81. $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{\sqrt{x^2+1}}$

82. $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{\sqrt{x^6+1}}$

83. $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{e^x+e^{-x}}$

84. $\displaystyle\int_{-\infty}^{\infty} \frac{e^{-x}\,dx}{x^2+1}$

85. $\displaystyle\int_{-\infty}^{\infty} e^{-|x|}\,dx$

86. $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{(x+1)^2}$

87. $\displaystyle\int_{-\infty}^{\infty} \frac{|\sin x| + |\cos x|}{|x|+1}\,dx$

(*Hint:* $|\sin\theta| + |\cos\theta| \geq \sin^2\theta + \cos^2\theta$.)

88. $\displaystyle\int_{-\infty}^{\infty} \frac{x\,dx}{(x^2+1)(x^2+2)}$

COMPUTER EXPLORATIONS

Exploring Integrals of $x^p \ln x$

In Exercises 89–92, use a CAS to explore the integrals for various values of p (include noninteger values). For what values of p does the integral converge? What is the value of the integral when it does converge? Plot the integrand for various values of p.

89. $\displaystyle\int_{0}^{e} x^p \ln x\,dx$

90. $\displaystyle\int_{e}^{\infty} x^p \ln x\,dx$

91. $\displaystyle\int_{0}^{\infty} x^p \ln x\,dx$

92. $\displaystyle\int_{-\infty}^{\infty} x^p \ln |x|\,dx$

Chapter 8 Additional and Advanced Exercises

Challenging Integrals

Evaluate the integrals in Exercises 1–10.

1. $\int (\sin^{-1} x)^2\, dx$

2. $\int \dfrac{dx}{x(x+1)(x+2)\cdots(x+m)}$

3. $\int x \sin^{-1} x\, dx$

4. $\int \sin^{-1} \sqrt{y}\, dy$

5. $\int \dfrac{d\theta}{1-\tan^2\theta}$

6. $\int \ln\left(\sqrt{x}+\sqrt{1+x}\right) dx$

7. $\int \dfrac{dt}{t-\sqrt{1-t^2}}$

8. $\int \dfrac{(2e^{2x}-e^x)\, dx}{\sqrt{3e^{2x}-6e^x-1}}$

9. $\int \dfrac{dx}{x^4+4}$

10. $\int \dfrac{dx}{x^6-1}$

Limits

Evaluate the limits in Exercises 11 and 12.

11. $\lim\limits_{x\to\infty} \int_{-x}^{x} \sin t\, dt$

12. $\lim\limits_{x\to 0^+} x \int_{x}^{1} \dfrac{\cos t}{t^2}\, dt$

Evaluate the limits in Exercises 13 and 14 by identifying them with definite integrals and evaluating the integrals.

13. $\lim\limits_{n\to\infty} \sum\limits_{k=1}^{n} \ln \sqrt[n]{1+\dfrac{k}{n}}$

14. $\lim\limits_{n\to\infty} \sum\limits_{k=0}^{n-1} \dfrac{1}{\sqrt{n^2-k^2}}$

Theory and Applications

15. **Finding arc length** Find the length of the curve

$$y = \int_0^x \sqrt{\cos 2t}\, dt, \quad 0 \le x \le \pi/4.$$

16. **Finding arc length** Find the length of the curve $y = \ln(1-x^2)$, $0 \le x \le 1/2$.

17. **Finding volume** The region in the first quadrant that is enclosed by the x-axis and the curve $y = 3x\sqrt{1-x}$ is revolved about the y-axis to generate a solid. Find the volume of the solid.

18. **Finding volume** The region in the first quadrant that is enclosed by the x-axis, the curve $y = 5/\left(x\sqrt{5-x}\right)$, and the lines $x = 1$ and $x = 4$ is revolved about the x-axis to generate a solid. Find the volume of the solid.

19. **Finding volume** The region in the first quadrant enclosed by the coordinate axes, the curve $y = e^x$, and the line $x = 1$ is revolved about the y-axis to generate a solid. Find the volume of the solid.

20. **Finding volume** The region in the first quadrant that is bounded above by the curve $y = e^x - 1$, below by the x-axis, and on the right by the line $x = \ln 2$ is revolved about the line $x = \ln 2$ to generate a solid. Find the volume of the solid.

21. **Finding volume** Let R be the "triangular" region in the first quadrant that is bounded above by the line $y = 1$, below by the curve $y = \ln x$, and on the left by the line $x = 1$. Find the volume of the solid generated by revolving R about

a. the x-axis. **b.** the line $y = 1$.

22. **Finding volume** (*Continuation of Exercise 21.*) Find the volume of the solid generated by revolving the region R about

a. the y-axis. **b.** the line $x = 1$.

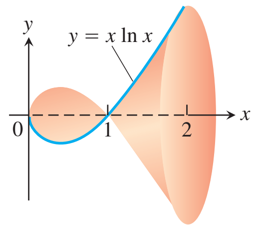
23. **Finding volume** The region between the x-axis and the curve

$$y = f(x) = \begin{cases} 0, & x = 0 \\ x \ln x, & 0 < x \le 2 \end{cases}$$

is revolved about the x-axis to generate the solid shown here.

a. Show that f is continuous at $x = 0$.

b. Find the volume of the solid.



24. Finding volume The infinite region bounded by the coordinate axes and the curve $y = -\ln x$ in the first quadrant is revolved about the x-axis to generate a solid. Find the volume of the solid.

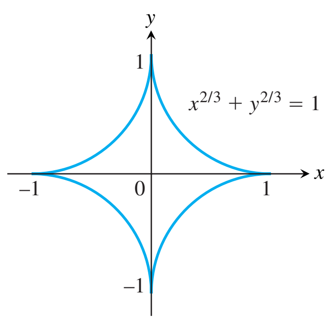
25. Centroid of a region Find the centroid of the region in the first quadrant that is bounded below by the x-axis, above by the curve $y = \ln x$, and on the right by the line $x = e$.

26. Centroid of a region Find the centroid of the region in the plane enclosed by the curves $y = \pm(1 - x^2)^{-1/2}$ and the lines $x = 0$ and $x = 1$.

27. Length of a curve Find the length of the curve $y = \ln x$ from $x = 1$ to $x = e$.

28. Finding surface area Find the area of the surface generated by revolving the curve in Exercise 27 about the y-axis.

29. The length of an astroid The graph of the equation $x^{2/3} + y^{2/3} = 1$ is one of a family of curves called *astroids* (not "asteroids") because of their starlike appearance (see accompanying figure). Find the length of this particular astroid.



30. The surface generated by an astroid Find the area of the surface generated by revolving the curve in Exercise 29 about the x-axis.

31. Find a curve through the origin whose length is

$$\int_0^4 \sqrt{1 + \frac{1}{4x}}\, dx.$$

32. Without evaluating either integral, explain why

$$2\int_{-1}^{1} \sqrt{1 - x^2}\, dx = \int_{-1}^{1} \frac{dx}{\sqrt{1 - x^2}}.$$

T **33. a.** Graph the function $f(x) = e^{(x - e^x)}$, $-5 \le x \le 3$.

b. Show that $\int_{-\infty}^{\infty} f(x)\, dx$ converges and find its value.

34. Find $\lim_{n\to\infty} \int_0^1 \frac{ny^{n-1}}{1 + y}\, dy$.

35. Derive the integral formula

$$\int x\left(\sqrt{x^2 - a^2}\right)^n dx = \frac{\left(\sqrt{x^2 - a^2}\right)^{n+2}}{n + 2} + C, \quad n \ne -2.$$

36. Prove that

$$\frac{\pi}{6} < \int_0^1 \frac{dx}{\sqrt{4 - x^2 - x^3}} < \frac{\pi\sqrt{2}}{8}.$$

(*Hint:* Observe that for $0 < x < 1$, we have $4 - x^2 > 4 - x^2 - x^3 > 4 - 2x^2$, with the left-hand side becoming an equality for $x = 0$ and the right-hand side becoming an equality for $x = 1$.)

37. For what value or values of a does

$$\int_1^{\infty} \left(\frac{ax}{x^2 + 1} - \frac{1}{2x}\right) dx$$

converge? Evaluate the corresponding integral(s).

38. For each $x > 0$, let $G(x) = \int_0^{\infty} e^{-xt}\, dt$. Prove that $xG(x) = 1$ for each $x > 0$.

39. Infinite area and finite volume What values of p have the following property: The area of the region between the curve $y = x^{-p}$, $1 \le x < \infty$, and the x-axis is infinite but the volume of the solid generated by revolving the region about the x-axis is finite.

40. Infinite area and finite volume What values of p have the following property: The area of the region in the first quadrant enclosed by the curve $y = x^{-p}$, the y-axis, the line $x = 1$, and the interval $[0, 1]$ on the x-axis is infinite but the volume of the solid generated by revolving the region about one of the coordinate axes is finite.

Tabular Integration

The technique of tabular integration also applies to integrals of the form $\int f(x)g(x)\, dx$ when neither function can be differentiated repeatedly to become zero. For example, to evaluate

$$\int e^{2x} \cos x\, dx$$

we begin as before with a table listing successive derivatives of e^{2x} and integrals of $\cos x$:

e^{2x} **and its derivatives**		**cos x and its integrals**	
e^{2x}	$(+)$	$\cos x$	
$2e^{2x}$	$(-)$	$\sin x$	
$4e^{2x}$	$(+)$	$-\cos x$	← *Stop here:* Row is same as first row except for multiplicative constants (4 on the left, -1 on the right)

We stop differentiating and integrating as soon as we reach a row that is the same as the first row except for multiplicative constants. We interpret the table as saying

$$\int e^{2x} \cos x\, dx = +(e^{2x} \sin x) - (2e^{2x}(-\cos x)) + \int (4e^{2x})(-\cos x)\, dx.$$

We take signed products from the diagonal arrows and a signed integral for the last horizontal arrow. Transposing the integral on the right-hand side over to the left-hand side now gives

$$5\int e^{2x}\cos x\,dx = e^{2x}\sin x + 2e^{2x}\cos x$$

or

$$\int e^{2x}\cos x\,dx = \frac{e^{2x}\sin x + 2e^{2x}\cos x}{5} + C,$$

after dividing by 5 and adding the constant of integration.

Use tabular integration to evaluate the integrals in Exercises 41–48.

41. $\int e^{2x}\cos 3x\,dx$

42. $\int e^{3x}\sin 4x\,dx$

43. $\int \sin 3x \sin x\,dx$

44. $\int \cos 5x \sin 4x\,dx$

45. $\int e^{ax}\sin bx\,dx$

46. $\int e^{ax}\cos bx\,dx$

47. $\int \ln(ax)\,dx$

48. $\int x^2\ln(ax)\,dx$

The Gamma Function and Stirling's Formula

Euler's gamma function $\Gamma(x)$ ("gamma of x"; Γ is a Greek capital g) uses an integral to extend the factorial function from the nonnegative integers to other real values. The formula is

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}\,dt, \qquad x > 0.$$

For each positive x, the number $\Gamma(x)$ is the integral of $t^{x-1}e^{-t}$ with respect to t from 0 to ∞. Figure 8.27 shows the graph of Γ near the origin. You will see how to calculate $\Gamma(1/2)$ if you do Additional Exercise 31 in Chapter 15.

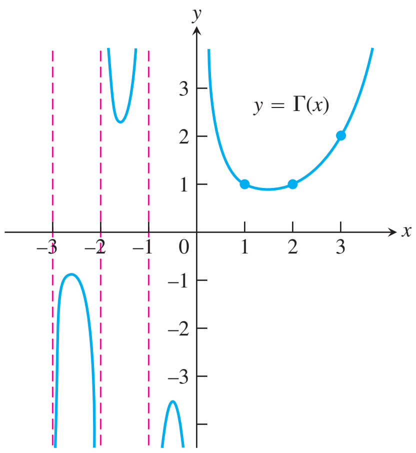


FIGURE 8.27 Euler's gamma function $\Gamma(x)$ is a continuous function of x whose value at each positive integer $n + 1$ is $n!$. The defining integral formula for Γ is valid only for $x > 0$, but we can extend Γ to negative noninteger values of x with the formula $\Gamma(x) = (\Gamma(x+1))/x$, which is the subject of Exercise 49.

49. If n is a nonnegative integer, $\Gamma(n+1) = n!$

a. Show that $\Gamma(1) = 1$.

b. Then apply integration by parts to the integral for $\Gamma(x+1)$ to show that $\Gamma(x+1) = x\Gamma(x)$. This gives

$$\begin{aligned}\Gamma(2) &= 1\Gamma(1) = 1\\ \Gamma(3) &= 2\Gamma(2) = 2\\ \Gamma(4) &= 3\Gamma(3) = 6\\ &\vdots\\ \Gamma(n+1) &= n\,\Gamma(n) = n! \end{aligned} \tag{1}$$

c. Use mathematical induction to verify Equation (1) for every nonnegative integer n.

50. Stirling's formula Scottish mathematician James Stirling (1692–1770) showed that

$$\lim_{x\to\infty}\left(\frac{e}{x}\right)^x\sqrt{\frac{x}{2\pi}}\,\Gamma(x) = 1,$$

so for large x,

$$\Gamma(x) = \left(\frac{x}{e}\right)^x\sqrt{\frac{2\pi}{x}}\,(1 + \epsilon(x)), \qquad \epsilon(x)\to 0 \text{ as } x\to\infty. \tag{2}$$

Dropping $\epsilon(x)$ leads to the approximation

$$\Gamma(x) \approx \left(\frac{x}{e}\right)^x\sqrt{\frac{2\pi}{x}} \qquad \textbf{(Stirling's formula)}. \tag{3}$$

a. Stirling's approximation for $n!$ Use Equation (3) and the fact that $n! = n\Gamma(n)$ to show that

$$n! \approx \left(\frac{n}{e}\right)^n\sqrt{2n\pi} \qquad \textbf{(Stirling's approximation)}. \tag{4}$$

As you will see if you do Exercise 64 in Section 11.1, Equation (4) leads to the approximation

$$\sqrt[n]{n!} \approx \frac{n}{e}. \tag{5}$$

T **b.** Compare your calculator's value for $n!$ with the value given by Stirling's approximation for $n = 10, 20, 30, \ldots$, as far as your calculator can go.

T **c.** A refinement of Equation (2) gives

$$\Gamma(x) = \left(\frac{x}{e}\right)^x\sqrt{\frac{2\pi}{x}}\,e^{1/(12x)}(1 + \epsilon(x)),$$

or

$$\Gamma(x) \approx \left(\frac{x}{e}\right)^x \sqrt{\frac{2\pi}{x}}\, e^{1/(12x)}$$

which tells us that

$$n! \approx \left(\frac{n}{e}\right)^n \sqrt{2n\pi}\, e^{1/(12n)}. \tag{6}$$

Compare the values given for 10! by your calculator, Stirling's approximation, and Equation (6).

Chapter 8 Practice Exercises

Integration Using Substitutions

Evaluate the integrals in Exercises 1–82. To transform each integral into a recognizable basic form, it may be necessary to use one or more of the techniques of algebraic substitution, completing the square, separating fractions, long division, or trigonometric substitution.

1. $\int x\sqrt{4x^2 - 9}\, dx$

2. $\int 6x\sqrt{3x^2 + 5}\, dx$

3. $\int x(2x + 1)^{1/2}\, dx$

4. $\int x(1 - x)^{-1/2}\, dx$

5. $\int \frac{x\, dx}{\sqrt{8x^2 + 1}}$

6. $\int \frac{x\, dx}{\sqrt{9 - 4x^2}}$

7. $\int \frac{y\, dy}{25 + y^2}$

8. $\int \frac{y^3\, dy}{4 + y^4}$

9. $\int \frac{t^3\, dt}{\sqrt{9 - 4t^4}}$

10. $\int \frac{2t\, dt}{t^4 + 1}$

11. $\int z^{2/3}(z^{5/3} + 1)^{2/3}\, dz$

12. $\int z^{-1/5}(1 + z^{4/5})^{-1/2}\, dz$

13. $\int \frac{\sin 2\theta\, d\theta}{(1 - \cos 2\theta)^2}$

14. $\int \frac{\cos\theta\, d\theta}{(1 + \sin\theta)^{1/2}}$

15. $\int \frac{\sin t}{3 + 4\cos t}\, dt$

16. $\int \frac{\cos 2t}{1 + \sin 2t}\, dt$

17. $\int \sin 2x\, e^{\cos 2x}\, dx$

18. $\int \sec x \tan x\, e^{\sec x}\, dx$

19. $\int e^{\theta} \sin(e^{\theta}) \cos^2(e^{\theta})\, d\theta$

20. $\int e^{\theta} \sec^2(e^{\theta})\, d\theta$

21. $\int 2^{x-1}\, dx$

22. $\int 5^{x\sqrt{2}}\, dx$

23. $\int \frac{dv}{v \ln v}$

24. $\int \frac{dv}{v(2 + \ln v)}$

25. $\int \frac{dx}{(x^2 + 1)(2 + \tan^{-1} x)}$

26. $\int \frac{\sin^{-1} x}{\sqrt{1 - x^2}}\, dx$

27. $\int \frac{2\, dx}{\sqrt{1 - 4x^2}}$

28. $\int \frac{dx}{\sqrt{49 - x^2}}$

29. $\int \frac{dt}{\sqrt{16 - 9t^2}}$

30. $\int \frac{dt}{\sqrt{9 - 4t^2}}$

31. $\int \frac{dt}{9 + t^2}$

32. $\int \frac{dt}{1 + 25t^2}$

33. $\int \frac{4\, dx}{5x\sqrt{25x^2 - 16}}$

34. $\int \frac{6\, dx}{x\sqrt{4x^2 - 9}}$

35. $\int \frac{dx}{\sqrt{4x - x^2}}$

36. $\int \frac{dx}{\sqrt{4x - x^2 - 3}}$

37. $\int \frac{dy}{y^2 - 4y + 8}$

38. $\int \frac{dt}{t^2 + 4t + 5}$

39. $\int \frac{dx}{(x - 1)\sqrt{x^2 - 2x}}$

40. $\int \frac{dv}{(v + 1)\sqrt{v^2 + 2v}}$

41. $\int \sin^2 x\, dx$

42. $\int \cos^2 3x\, dx$

43. $\int \sin^3 \frac{\theta}{2}\, d\theta$ **44.** $\int \sin^3 \theta \cos^2 \theta\, d\theta$

45. $\int \tan^3 2t\, dt$ **46.** $\int 6 \sec^4 t\, dt$

47. $\int \frac{dx}{2 \sin x \cos x}$ **48.** $\int \frac{2\, dx}{\cos^2 x - \sin^2 x}$

49. $\int_{\pi/4}^{\pi/2} \sqrt{\csc^2 y - 1}\, dy$ **50.** $\int_{\pi/4}^{3\pi/4} \sqrt{\cot^2 t + 1}\, dt$

51. $\int_0^{\pi} \sqrt{1 - \cos^2 2x}\, dx$ **52.** $\int_0^{2\pi} \sqrt{1 - \sin^2 \frac{x}{2}}\, dx$

53. $\int_{-\pi/2}^{\pi/2} \sqrt{1 - \cos 2t}\, dt$ **54.** $\int_{\pi}^{2\pi} \sqrt{1 + \cos 2t}\, dt$

55. $\int \frac{x^2}{x^2 + 4}\, dx$ **56.** $\int \frac{x^3}{9 + x^2}\, dx$

57. $\int \frac{4x^2 + 3}{2x - 1}\, dx$ **58.** $\int \frac{2x}{x - 4}\, dx$

59. $\int \frac{2y - 1}{y^2 + 4}\, dy$ **60.** $\int \frac{y + 4}{y^2 + 1}\, dy$

61. $\int \frac{t + 2}{\sqrt{4 - t^2}}\, dt$ **62.** $\int \frac{2t^2 + \sqrt{1 - t^2}}{t\sqrt{1 - t^2}}\, dt$

63. $\int \frac{\tan x\, dx}{\tan x + \sec x}$ **64.** $\int \frac{\cot x}{\cot x + \csc x}\, dx$

65. $\int \sec (5 - 3x)\, dx$ **66.** $\int x \csc (x^2 + 3)\, dx$

67. $\int \cot \left(\frac{x}{4}\right) dx$ **68.** $\int \tan (2x - 7)\, dx$

69. $\int x\sqrt{1 - x}\, dx$ **70.** $\int 3x\sqrt{2x + 1}\, dx$

71. $\int \sqrt{z^2 + 1}\, dz$ **72.** $\int (16 + z^2)^{-3/2}\, dz$

73. $\int \frac{dy}{\sqrt{25 + y^2}}$ **74.** $\int \frac{dy}{\sqrt{25 + 9y^2}}$

75. $\int \frac{dx}{x^2\sqrt{1 - x^2}}$ **76.** $\int \frac{x^3\, dx}{\sqrt{1 - x^2}}$

77. $\int \frac{x^2\, dx}{\sqrt{1 - x^2}}$ **78.** $\int \sqrt{4 - x^2}\, dx$

79. $\int \frac{dx}{\sqrt{x^2 - 9}}$ **80.** $\int \frac{12\, dx}{(x^2 - 1)^{3/2}}$

81. $\int \frac{\sqrt{w^2 - 1}}{w}\, dw$ **82.** $\int \frac{\sqrt{z^2 - 16}}{z}\, dz$

Integration by Parts

Evaluate the integrals in Exercises 83–90 using integration by parts.

83. $\int \ln (x + 1)\, dx$ **84.** $\int x^2 \ln x\, dx$

85. $\int \tan^{-1} 3x\, dx$ **86.** $\int \cos^{-1} \left(\frac{x}{2}\right) dx$

87. $\int (x + 1)^2 e^x\, dx$ **88.** $\int x^2 \sin (1 - x)\, dx$

89. $\int e^x \cos 2x\, dx$ **90.** $\int e^{-2x} \sin 3x\, dx$

Partial Fractions

Evaluate the integrals in Exercises 91–110. It may be necessary to use a substitution first.

91. $\int \frac{x\, dx}{x^2 - 3x + 2}$ **92.** $\int \frac{x\, dx}{x^2 + 4x + 3}$

93. $\int \frac{dx}{x(x + 1)^2}$ **94.** $\int \frac{x + 1}{x^2(x - 1)}\, dx$

95. $\int \frac{\sin \theta\, d\theta}{\cos^2 \theta + \cos \theta - 2}$ **96.** $\int \frac{\cos \theta\, d\theta}{\sin^2 \theta + \sin \theta - 6}$

97. $\int \frac{3x^2 + 4x + 4}{x^3 + x}\, dx$ **98.** $\int \frac{4x\, dx}{x^3 + 4x}$

99. $\int \frac{v + 3}{2v^3 - 8v}\, dv$ **100.** $\int \frac{(3v - 7)\, dv}{(v - 1)(v - 2)(v - 3)}$

101. $\int \frac{dt}{t^4 + 4t^2 + 3}$ **102.** $\int \frac{t\, dt}{t^4 - t^2 - 2}$

103. $\int \frac{x^3 + x^2}{x^2 + x - 2}\, dx$ **104.** $\int \frac{x^3 + 1}{x^3 - x}\, dx$

105. $\int \frac{x^3 + 4x^2}{x^2 + 4x + 3}\, dx$ **106.** $\int \frac{2x^3 + x^2 - 21x + 24}{x^2 + 2x - 8}\, dx$

107. $\int \frac{dx}{x\left(3\sqrt{x + 1}\right)}$ **108.** $\int \frac{dx}{x\left(1 + \sqrt[3]{x}\right)}$

109. $\int \frac{ds}{e^s - 1}$ **110.** $\int \frac{ds}{\sqrt{e^s + 1}}$

Trigonometric Substitutions

Evaluate the integrals in Exercises 111–114 **(a)** without using a trigonometric substitution, **(b)** using a trigonometric substitution.

111. $\int \frac{y\, dy}{\sqrt{16 - y^2}}$ **112.** $\int \frac{x\, dx}{\sqrt{4 + x^2}}$

113. $\int \frac{x\, dx}{4 - x^2}$ **114.** $\int \frac{t\, dt}{\sqrt{4t^2 - 1}}$

Quadratic Terms

Evaluate the integrals in Exercises 115–118.

115. $\int \frac{x\, dx}{9 - x^2}$ **116.** $\int \frac{dx}{x(9 - x^2)}$

117. $\int \frac{dx}{9 - x^2}$ **118.** $\int \frac{dx}{\sqrt{9 - x^2}}$

Trigonometric Integrals

Evaluate the integrals in Exercises 119–126.

119. $\int \sin^3 x \cos^4 x\, dx$

120. $\int \cos^5 x \sin^5 x\, dx$

121. $\int \tan^4 x \sec^2 x\, dx$

122. $\int \tan^3 x \sec^3 x\, dx$

123. $\int \sin 5\theta \cos 6\theta\, d\theta$

124. $\int \cos 3\theta \cos 3\theta\, d\theta$

125. $\int \sqrt{1 + \cos(t/2)}\, dt$

126. $\int e^t \sqrt{\tan^2 e^t + 1}\, dt$

Numerical Integration

127. According to the error-bound formula for Simpson's Rule, how many subintervals should you use to be sure of estimating the value of

$$\ln 3 = \int_1^3 \frac{1}{x}\, dx$$

by Simpson's Rule with an error of no more than 10^{-4} in absolute value? (Remember that for Simpson's Rule, the number of subintervals has to be even.)

128. A brief calculation shows that if $0 \le x \le 1$, then the second derivative of $f(x) = \sqrt{1 + x^4}$ lies between 0 and 8. Based on this, about how many subdivisions would you need to estimate the integral of f from 0 to 1 with an error no greater than 10^{-3} in absolute value using the Trapezoidal Rule?

129. A direct calculation shows that

$$\int_0^{\pi} 2 \sin^2 x\, dx = \pi.$$

How close do you come to this value by using the Trapezoidal Rule with $n = 6$? Simpson's Rule with $n = 6$? Try them and find out.

130. You are planning to use Simpson's Rule to estimate the value of the integral

$$\int_1^2 f(x)\, dx$$

with an error magnitude less than 10^{-5}. You have determined that $|f^{(4)}(x)| \le 3$ throughout the interval of integration. How many subintervals should you use to assure the required accuracy? (Remember that for Simpson's Rule the number has to be even.)

131. Mean temperature Compute the average value of the temperature function

$$f(x) = 37 \sin\left(\frac{2\pi}{365}(x - 101)\right) + 25$$

for a 365-day year. This is one way to estimate the annual mean air temperature in Fairbanks, Alaska. The National Weather Service's official figure, a numerical average of the daily normal mean air temperatures for the year, is 25.7°F, which is slightly higher than the average value of $f(x)$.

132. Heat capacity of a gas Heat capacity C_v is the amount of heat required to raise the temperature of a given mass of gas with constant volume by 1°C, measured in units of cal/deg-mol (calories per degree gram molecular weight). The heat capacity of oxygen depends on its temperature T and satisfies the formula

$$C_v = 8.27 + 10^{-5}(26T - 1.87T^2).$$

Find the average value of C_v for $20° \le T \le 675°$C and the temperature at which it is attained.

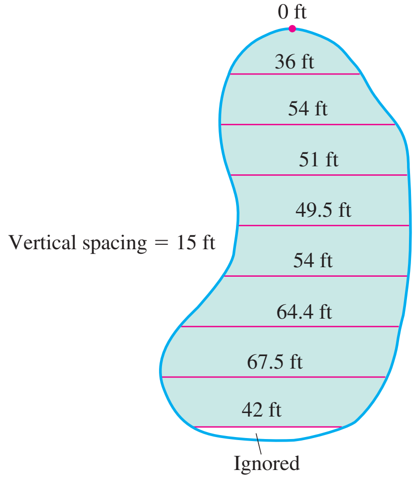
133. Fuel efficiency An automobile computer gives a digital readout of fuel consumption in gallons per hour. During a trip, a passenger recorded the fuel consumption every 5 min for a full hour of travel.

Time	Gal/h	Time	Gal/h
0	2.5	35	2.5
5	2.4	40	2.4
10	2.3	45	2.3
15	2.4	50	2.4
20	2.4	55	2.4
25	2.5	60	2.3
30	2.6		

a. Use the Trapezoidal Rule to approximate the total fuel consumption during the hour.

b. If the automobile covered 60 mi in the hour, what was its fuel efficiency (in miles per gallon) for that portion of the trip?

134. A new parking lot To meet the demand for parking, your town has allocated the area shown here. As the town engineer, you have been asked by the town council to find out if the lot can be built for \$11,000. The cost to clear the land will be \$0.10 a square foot, and the lot will cost \$2.00 a square foot to pave. Use Simpson's Rule to find out if the job can be done for \$11,000.



Improper Integrals

Evaluate the improper integrals in Exercises 135–144.

135. $\displaystyle\int_0^3 \frac{dx}{\sqrt{9-x^2}}$

136. $\displaystyle\int_0^1 \ln x\,dx$

137. $\displaystyle\int_{-1}^1 \frac{dy}{y^{2/3}}$

138. $\displaystyle\int_{-2}^0 \frac{d\theta}{(\theta+1)^{3/5}}$

139. $\displaystyle\int_3^\infty \frac{2\,du}{u^2-2u}$

140. $\displaystyle\int_1^\infty \frac{3v-1}{4v^3-v^2}\,dv$

141. $\displaystyle\int_0^\infty x^2e^{-x}\,dx$

142. $\displaystyle\int_{-\infty}^0 xe^{3x}\,dx$

143. $\displaystyle\int_{-\infty}^\infty \frac{dx}{4x^2+9}$

144. $\displaystyle\int_{-\infty}^\infty \frac{4dx}{x^2+16}$

Convergence or Divergence

Which of the improper integrals in Exercises 145–150 converge and which diverge?

145. $\displaystyle\int_6^\infty \frac{d\theta}{\sqrt{\theta^2+1}}$

146. $\displaystyle\int_0^\infty e^{-u}\cos u\,du$

147. $\displaystyle\int_1^\infty \frac{\ln z}{z}\,dz$

148. $\displaystyle\int_1^\infty \frac{e^{-t}}{\sqrt{t}}\,dt$

149. $\displaystyle\int_{-\infty}^\infty \frac{2\,dx}{e^x+e^{-x}}$

150. $\displaystyle\int_{-\infty}^\infty \frac{dx}{x^2(1+e^x)}$

Assorted Integrations

Evaluate the integrals in Exercises 151–218. The integrals are listed in random order.

151. $\displaystyle\int \frac{x\,dx}{1+\sqrt{x}}$

152. $\displaystyle\int \frac{x^3+2}{4-x^2}\,dx$

153. $\displaystyle\int \frac{dx}{x(x^2+1)^2}$

154. $\displaystyle\int \frac{\cos\sqrt{x}}{\sqrt{x}}\,dx$

155. $\displaystyle\int \frac{dx}{\sqrt{-2x-x^2}}$

156. $\displaystyle\int \frac{(t-1)\,dt}{\sqrt{t^2-2t}}$

157. $\displaystyle\int \frac{du}{\sqrt{1+u^2}}$

158. $\displaystyle\int e^t\cos e^t\,dt$

159. $\displaystyle\int \frac{2-\cos x+\sin x}{\sin^2 x}\,dx$

160. $\displaystyle\int \frac{\sin^2\theta}{\cos^2\theta}\,d\theta$

161. $\displaystyle\int \frac{9\,dv}{81-v^4}$

162. $\displaystyle\int \frac{\cos x\,dx}{1+\sin^2 x}$

163. $\displaystyle\int \theta\cos(2\theta+1)\,d\theta$

164. $\displaystyle\int_2^\infty \frac{dx}{(x-1)^2}$

165. $\displaystyle\int \frac{x^3\,dx}{x^2-2x+1}$

166. $\displaystyle\int \frac{d\theta}{\sqrt{1+\sqrt{\theta}}}$

167. $\displaystyle\int \frac{2\sin\sqrt{x}\,dx}{\sqrt{x}\sec\sqrt{x}}$

168. $\displaystyle\int \frac{x^5\,dx}{x^4-16}$

169. $\displaystyle\int \frac{dy}{\sin y\cos y}$

170. $\displaystyle\int \frac{d\theta}{\theta^2-2\theta+4}$

171. $\displaystyle\int \frac{\tan x}{\cos^2 x}\,dx$

172. $\displaystyle\int \frac{dr}{(r+1)\sqrt{r^2+2r}}$

173. $\displaystyle\int \frac{(r+2)\,dr}{\sqrt{-r^2-4r}}$

174. $\displaystyle\int \frac{y\,dy}{4+y^4}$

175. $\displaystyle\int \frac{\sin 2\theta\,d\theta}{(1+\cos 2\theta)^2}$

176. $\displaystyle\int \frac{dx}{(x^2-1)^2}$

177. $\displaystyle\int_{\pi/4}^{\pi/2} \sqrt{1+\cos 4x}\,dx$

178. $\displaystyle\int (15)^{2x+1}\,dx$

179. $\displaystyle\int \frac{x\,dx}{\sqrt{2-x}}$

180. $\displaystyle\int \frac{\sqrt{1-v^2}}{v^2}\,dv$

181. $\displaystyle\int \frac{dy}{y^2-2y+2}$

182. $\displaystyle\int \ln\sqrt{x-1}\,dx$

183. $\displaystyle\int \theta^2\tan(\theta^3)\,d\theta$

184. $\displaystyle\int \frac{x\,dx}{\sqrt{8-2x^2-x^4}}$

185. $\displaystyle\int \frac{z+1}{z^2(z^2+4)}\,dz$

186. $\displaystyle\int x^3e^{(x^2)}\,dx$

187. $\displaystyle\int \frac{t\,dt}{\sqrt{9-4t^2}}$

188. $\displaystyle\int_0^{\pi/10} \sqrt{1+\cos 5\theta}\,d\theta$

189. $\displaystyle\int \frac{\cot\theta\,d\theta}{1+\sin^2\theta}$

190. $\displaystyle\int \frac{\tan^{-1}x}{x^2}\,dx$

191. $\displaystyle\int \frac{\tan\sqrt{y}}{2\sqrt{y}}\,dy$

192. $\displaystyle\int \frac{e^t\,dt}{e^{2t}+3e^t+2}$

193. $\displaystyle\int \frac{\theta^2 d\theta}{4-\theta^2}$

194. $\displaystyle\int \frac{1-\cos 2x}{1+\cos 2x}\,dx$

195. $\displaystyle\int \frac{\cos(\sin^{-1}x)}{\sqrt{1-x^2}}\,dx$

196. $\displaystyle\int \frac{\cos x\,dx}{\sin^3 x-\sin x}$

197. $\displaystyle\int \sin\frac{x}{2}\cos\frac{x}{2}\,dx$

198. $\displaystyle\int \frac{x^2-x+2}{(x^2+2)^2}\,dx$

199. $\displaystyle\int \frac{e^t\,dt}{1+e^t}$

200. $\displaystyle\int \tan^3 t\,dt$

201. $\displaystyle\int_1^\infty \frac{\ln y}{y^3}\,dy$

202. $\displaystyle\int \frac{3+\sec^2 x+\sin x}{\tan x}\,dx$

203. $\displaystyle\int \frac{\cot v\,dv}{\ln\sin v}$

204. $\displaystyle\int \frac{dx}{(2x-1)\sqrt{x^2-x}}$

205. $\displaystyle\int e^{\ln\sqrt{x}}\,dx$

206. $\displaystyle\int e^\theta\sqrt{3+4e^\theta}\,d\theta$

207. $\displaystyle\int \frac{\sin 5t\,dt}{1+(\cos 5t)^2}$

208. $\displaystyle\int \frac{dv}{\sqrt{e^{2v}-1}}$

209. $\displaystyle\int (27)^{3\theta+1}\,d\theta$

210. $\displaystyle\int x^5\sin x\,dx$

211. $\displaystyle\int \frac{dr}{1+\sqrt{r}}$

212. $\displaystyle\int \frac{4x^3-20x}{x^4-10x^2+9}\,dx$

213. $\displaystyle\int \frac{8\,dy}{y^3(y+2)}$

214. $\displaystyle\int \frac{(t+1)\,dt}{(t^2+2t)^{2/3}}$

215. $\displaystyle\int \frac{8\,dm}{m\sqrt{49m^2-4}}$

216. $\displaystyle\int \frac{dt}{t(1 + \ln t)\sqrt{(\ln t)(2 + \ln t)}}$

217. $\displaystyle\int_0^1 3(x - 1)^2 \left(\int_0^x \sqrt{1 + (t - 1)^4}\, dt \right) dx$

218. $\displaystyle\int_2^{\infty} \frac{4v^3 + v - 1}{v^2(v - 1)(v^2 + 1)}\, dv$

219. Suppose for a certain function f it is known that

$$f'(x) = \frac{\cos x}{x}, \quad f(\pi\ 2) = a, \quad \text{and} \quad f(3\pi\ 2) = b.$$

Use integration by parts to evaluate

$$\int_{\pi/2}^{3\pi/2} f(x)\, dx.$$

220. Find a positive number a satisfying

$$\int_0^a \frac{dx}{1 + x^2} = \int_a^{\infty} \frac{dx}{1 + x^2}.$$

Chapter 8 Questions to Guide Your Review

1. What basic integration formulas do you know?
2. What procedures do you know for matching integrals to basic formulas?
3. What is the formula for integration by parts? Where does it come from? Why might you want to use it?
4. When applying the formula for integration by parts, how do you choose the u and dv? How can you apply integration by parts to an integral of the form $\int f(x)\,dx$?
5. What is tabular integration? Give an example.
6. What is the goal of the method of partial fractions?

7. When the degree of a polynomial $f(x)$ is less than the degree of a polynomial $g(x)$, how do you write $f(x)/g(x)$ as a sum of partial fractions if $g(x)$

a. is a product of distinct linear factors?

b. consists of a repeated linear factor?

c. contains an irreducible quadratic factor?

What do you do if the degree of f is *not* less than the degree of g?

8. If an integrand is a product of the form $\sin^n x \cos^m x$, where m and n are nonnegative integers, how do you evaluate the integral? Give a specific example of each case.

9. What substitutions are made to evaluate integrals of $\sin mx \sin nx$, $\sin mx \cos nx$, and $\cos mx \cos nx$? Give an example of each case.

10. What substitutions are sometimes used to transform integrals involving $\sqrt{a^2 - x^2}$, $\sqrt{a^2 + x^2}$, and $\sqrt{x^2 - a^2}$ into integrals that can be evaluated directly? Give an example of each case.

11. What restrictions can you place on the variables involved in the three basic trigonometric substitutions to make sure the substitutions are reversible (have inverses)?

12. How are integral tables typically used? What do you do if a particular integral you want to evaluate is not listed in the table?

13. What is a reduction formula? How are reduction formulas typically derived? How are reduction formulas used? Give an example.

14. You are collaborating to produce a short "how-to" manual for numerical integration, and you are writing about the Trapezoidal Rule. **(a)** What would you say about the rule itself and how to use it? How to achieve accuracy? **(b)** What would you say if you were writing about Simpson's Rule instead?

15. How would you compare the relative merits of Simpson's Rule and the Trapezoidal Rule?

16. What is an improper integral of Type I? Type II? How are the values of various types of improper integrals defined? Give examples.

17. What tests are available for determining the convergence and divergence of improper integrals that cannot be evaluated directly? Give examples of their use.

Chapter 8 Technology Application Projects

Mathematica/Maple Module

Riemann, Trapezoidal, and Simpson Approximations

Part I: Visualize the error involved in using Riemann sums to approximate the area under a curve.

Part II: Build a table of values and compute the relative magnitude of the error as a function of the step size Δx.

Part III: Investigate the effect of the derivative function on the error.

Parts IV and **V**: Trapezoidal Rule approximations.

Part VI: Simpson's Rule approximations.

Mathematica/Maple Module

Games of Chance: Exploring the Monte Carlo Probabilistic Technique for Numerical Integration

Graphically explore the Monte Carlo method for approximating definite integrals.

Mathematica/Maple Module

Computing Probabilities with Improper Integrals

Graphically explore the Monte Carlo method for approximating definite integrals.