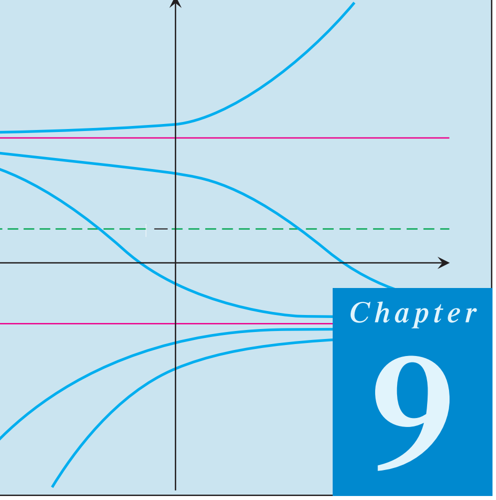


Chapter 9

FURTHER APPLICATIONS OF INTEGRATION

HISTORICAL BIOGRAPHY

Carl Friedrich Gauss
(1777–1855)

Biography History

OVERVIEW In Section 4.8 we introduced differential equations of the form $dy/dx = f(x)$, where y is an unknown function being differentiated. For a continuous function f, we found the general solution $y(x)$ by integration: $y(x) = \int f(x)\,dx$. (Remember that the indefinite integral represents *all* the antiderivatives of f, so it contains an arbitrary constant $+C$ which must be shown once an antiderivative is found.) Many applications in the sciences, engineering, and economics involve a model formulated by even more general differential equations. In Section 7.5, for example, we found that exponential growth and decay is modeled by a differential equation of the form $dy/dx = ky$, for some constant $k \neq 0$. We have not yet considered differential equations such as $dy/dx = y - x$, yet such equations arise frequently in applications. In this chapter, we study several differential equations having the form $dy/dx = f(x, y)$, where f is a function of *both* the independent and dependent variables. We use the theory of indefinite integration to solve these differential equations, and investigate analytic, graphical, and numerical solution methods.

9.1 Slope Fields and Separable Differential Equations

HISTORICAL BIOGRAPHY

Jules Henri Poincaré
(1854–1912)

Biography

In calculating derivatives by implicit differentiation (Section 3.6), we found that the expression for the derivative dy/dx often contained both variables x and y, not just the independent variable x. We begin this section by considering the general differential equation $dy/dx = f(x, y)$ and what is meant by a solution to it. Then we investigate equations having a special form for which the function f can be expressed as a product of a function of x and a function of y.

General First-Order Differential Equations and Solutions

A **first-order differential equation** is an equation

$$\frac{dy}{dx} = f(x, y) \tag{1}$$

in which $f(x, y)$ is a function of two variables defined on a region in the xy-plane. The equation is of *first-order* because it involves only the first derivative dy/dx (and not higher-order derivatives). We point out that the equations

$$y' = f(x, y) \quad \text{and} \quad \frac{d}{dx}y = f(x, y),$$

are equivalent to Equation (1) and all three forms will be used interchangeably in the text.

A **solution** of Equation (1) is a differentiable function $y = y(x)$ defined on an interval I of x-values (perhaps infinite) such that

$$\frac{d}{dx}y(x) = f(x, y(x))$$

on that interval. That is, when $y(x)$ and its derivative $y'(x)$ are substituted into Equation (1), the resulting equation is true for all x over the interval I. The **general solution** to a first-order differential equation is a solution that contains all possible solutions. The general solution always contains an arbitrary constant, but having this property doesn't mean a solution is the general solution. That is, a solution may contain an arbitrary constant without being the general solution. Establishing that a solution *is* the general solution may require deeper results from the theory of differential equations and is best studied in a more advanced course.

EXAMPLE 1 Verifying Solution Functions

Show that every member of the family of functions

$$y = \frac{C}{x} + 2$$

is a solution of the first-order differential equation

$$\frac{dy}{dx} = \frac{1}{x}(2 - y)$$

on the interval $(0, \infty)$, where C is any constant.

Solution Differentiating $y = C/x + 2$ gives

$$\frac{dy}{dx} = C\frac{d}{dx}\left(\frac{1}{x}\right) + 0 = -\frac{C}{x^2}.$$

Thus we need only verify that for all $x \in (0, \infty)$,

$$-\frac{C}{x^2} = \frac{1}{x}\left[2 - \left(\frac{C}{x} + 2\right)\right].$$

This last equation follows immediately by expanding the expression on the right side:

$$\frac{1}{x}\left[2 - \left(\frac{C}{x} + 2\right)\right] = \frac{1}{x}\left(-\frac{C}{x}\right) = -\frac{C}{x^2}.$$

Therefore, for every value of C, the function $y = C/x + 2$ is a solution of the differential equation. ■

As was the case in finding antiderivatives, we often need a *particular* rather than the general solution to a first-order differential equation $y' = f(x, y)$. The **particular solution** satisfying the initial condition $y(x_0) = y_0$ is the solution $y = y(x)$ whose value is y_0 when $x = x_0$. Thus the graph of the particular solution passes through the point (x_0, y_0) in the xy-plane. A **first-order initial value problem** is a differential equation $y' = f(x, y)$ whose solution must satisfy an initial condition $y(x_0) = y_0$.

EXAMPLE 2 Verifying That a Function Is a Particular Solution

Video Video

Show that the function

$$y = (x + 1) - \frac{1}{3}e^x$$

is a solution to the first-order initial value problem

$$\frac{dy}{dx} = y - x, \qquad y(0) = \frac{2}{3}.$$

Solution The equation

$$\frac{dy}{dx} = y - x$$

is a first-order differential equation with $f(x, y) = y - x$.

On the left:

$$\frac{dy}{dx} = \frac{d}{dx}\left(x + 1 - \frac{1}{3}e^x\right) = 1 - \frac{1}{3}e^x.$$

On the right:

$$y - x = (x + 1) - \frac{1}{3}e^x - x = 1 - \frac{1}{3}e^x.$$

The function satisfies the initial condition because

$$y(0) = \left[(x + 1) - \frac{1}{3}e^x\right]_{x=0} = 1 - \frac{1}{3} = \frac{2}{3}.$$

The graph of the function is shown in Figure 9.1. ■

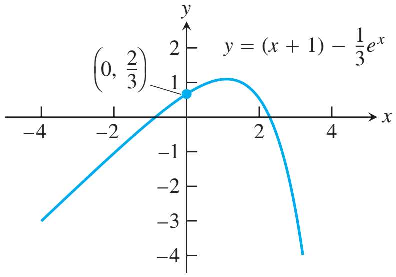


FIGURE 9.1 Graph of the solution $y = (x + 1) - \frac{1}{3}e^x$ to the differential equation $dy/dx = y - x$, with initial condition $y(0) = \frac{2}{3}$ (Example 2).

Slope Fields: Viewing Solution Curves

Each time we specify an initial condition $y(x_0) = y_0$ for the solution of a differential equation $y' = f(x, y)$, the **solution curve** (graph of the solution) is required to pass through the point (x_0, y_0) and to have slope $f(x_0, y_0)$ there. We can picture these slopes graphically by drawing short line segments of slope $f(x, y)$ at selected points (x, y) in the region of the xy-plane that constitutes the domain of f. Each segment has the same slope as the solution curve through (x, y) and so is tangent to the curve there. The resulting picture is called a **slope field** (or **direction field**) and gives a visualization of the general shape of the solution curves. Figure 9.2a shows a slope field, with a particular solution sketched into it in Figure 9.2b. We see how these line segments indicate the direction the solution curve takes at each point it passes through.

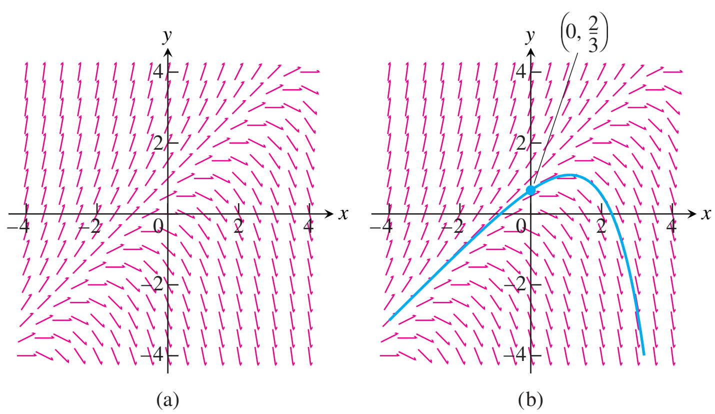


FIGURE 9.2 (a) Slope field for $\frac{dy}{dx} = y - x$. (b) The particular solution curve through the point $\left(0, \frac{2}{3}\right)$ (Example 2).

Figure 9.3 shows three slope fields and we see how the solution curves behave by following the tangent line segments in these fields.

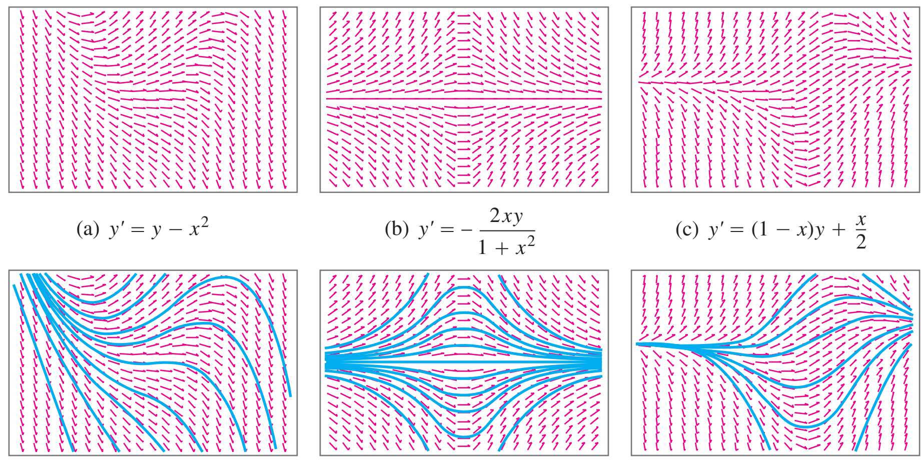


FIGURE 9.3 Slope fields (top row) and selected solution curves (bottom row). In computer renditions, slope segments are sometimes portrayed with arrows, as they are here. This is not to be taken as an indication that slopes have directions, however, for they do not.

Constructing a slope field with pencil and paper can be quite tedious. All our examples were generated by a computer.

While general differential equations are difficult to solve, many important equations that arise in science and applications have special forms that make them solvable by special techniques. One such class is the separable equations.

Separable Equations

The equation $y' = f(x, y)$ is **separable** if f can be expressed as a product of a function of x and a function of y. The differential equation then has the form

$$\frac{dy}{dx} = g(x)H(y).$$

When we rewrite this equation in the form

$$\frac{dy}{dx} = \frac{g(x)}{h(y)}, \qquad H(y) = \frac{1}{h(y)}$$

its differential form allows us to collect all y terms with dy and all x terms with dx:

$$h(y)\,dy = g(x)\,dx.$$

Now we simply integrate both sides of this equation:

$$\int h(y)\,dy = \int g(x)\,dx. \tag{2}$$

After completing the integrations we obtain the solution y defined implicitly as a function of x.

The justification that we can simply integrate both sides in Equation (2) is based on the Substitution Rule (Section 5.5):

$$\begin{aligned} \int h(y)\,dy &= \int h(y(x))\frac{dy}{dx}\,dx \\ &= \int h(y(x))\frac{g(x)}{h(y(x))}\,dx \qquad \frac{dy}{dx}=\frac{g(x)}{h(y)} \\ &= \int g(x)\,dx. \end{aligned}$$

EXAMPLE 3 Solving a Separable Equation

Solve the differential equation

$$\frac{dy}{dx} = (1 + y^2)e^x.$$

Solution Since $1 + y^2$ is never zero, we can solve the equation by separating the variables.

$$\begin{aligned} \frac{dy}{dx} &= (1 + y^2)e^x \\ dy &= (1 + y^2)e^x\,dx && \text{Treat } dy/dx \text{ as a quotient of differentials and multiply both sides by } dx. \\ \frac{dy}{1 + y^2} &= e^x\,dx && \text{Divide by } (1 + y^2). \\ \int \frac{dy}{1 + y^2} &= \int e^x\,dx && \text{Integrate both sides.} \\ \tan^{-1} y &= e^x + C && C \text{ represents the combined constants of integration.} \end{aligned}$$

The equation $\tan^{-1} y = e^x + C$ gives y as an implicit function of x. When $-\pi/2 < e^x + C < \pi/2$, we can solve for y as an explicit function of x by taking the tangent of both sides:

$$\begin{aligned} \tan(\tan^{-1} y) &= \tan(e^x + C) \\ y &= \tan(e^x + C). \end{aligned}$$

■

EXAMPLE 4 Solve the equation

$$(x + 1)\frac{dy}{dx} = x(y^2 + 1).$$

Solution We change to differential form, separate the variables, and integrate:

$$\begin{aligned} (x + 1)\,dy &= x(y^2 + 1)\,dx \\ \frac{dy}{y^2 + 1} &= \frac{x\,dx}{x + 1} && x \neq -1 \\ \int \frac{dy}{1 + y^2} &= \int \left(1 - \frac{1}{x + 1}\right)dx \\ \tan^{-1} y &= x - \ln|x + 1| + C. \end{aligned}$$

■

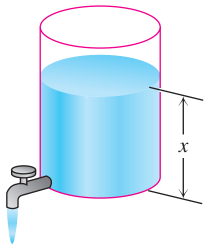


FIGURE 9.4 The rate at which water runs out is $k\sqrt{x}$, where k is a positive constant. In Example 5, $k = 1/2$ and x is measured in feet.

The initial value problem

$$\frac{dy}{dt} = ky, \qquad y(0) = y_0$$

involves a separable differential equation, and the solution $y = y_0 e^{kt}$ gives the Law of Exponential Change (Section 7.5). We found this initial value problem to be a model for such phenomena as population growth, radioactive decay, and heat transfer. We now present an application involving a different separable first-order equation.

Torricelli's Law

Torricelli's Law says that if you drain a tank like the one in Figure 9.4, the rate at which the water runs out is a constant times the square root of the water's depth x. The constant depends on the size of the drainage hole. In Example 5, we assume that the constant is $1/2$.

EXAMPLE 5 Draining a Tank

A right circular cylindrical tank with radius 5 ft and height 16 ft that was initially full of water is being drained at the rate of $0.5\sqrt{x}\ \text{ft}^3/\text{min}$. Find a formula for the depth and the amount of water in the tank at any time t. How long will it take to empty the tank?

HISTORICAL BIOGRAPHY

Evangelista Torricelli (1608–1647)

Solution The volume of a right circular cylinder with radius r and height h is $V = \pi r^2 h$, so the volume of water in the tank (Figure 9.4) is

$$V = \pi r^2 h = \pi(5)^2 x = 25\pi x.$$

Diffentiation leads to

$$\frac{dV}{dt} = 25\pi \frac{dx}{dt} \qquad \text{Negative because } V \text{ is decreasing and } dx/dt < 0$$

$$-0.5\sqrt{x} = 25\pi \frac{dx}{dt} \qquad \text{Torricelli's Law}$$

Thus we have the initial value problem

$$\frac{dx}{dt} = -\frac{\sqrt{x}}{50\pi},$$

$$x(0) = 16 \qquad \text{The water is 16 ft deep when } t = 0.$$

We solve the differential equation by separating the variables.

$$x^{-1/2}\,dx = -\frac{1}{50\pi}\,dt$$

$$\int x^{-1/2}\,dx = -\int \frac{1}{50\pi}\,dt \qquad \text{Integrate both sides.}$$

$$2x^{1/2} = -\frac{1}{50\pi}t + C \qquad \text{Constants combined}$$

The initial condition $x(0) = 16$ determines the value of C.

$$2(16)^{1/2} = -\frac{1}{50\pi}(0) + C$$

$$C = 8$$

With $C = 8$, we have

$$2x^{1/2} = -\frac{1}{50\pi}t + 8 \qquad \text{or} \qquad x^{1/2} = 4 - \frac{t}{100\pi}.$$

The formulas we seek are

$$x = \left(4 - \frac{t}{100\pi}\right)^2 \qquad \text{and} \qquad V = 25\pi x = 25\pi\left(4 - \frac{t}{100\pi}\right)^2.$$

At any time t, the water in the tank is $(4 - t/(100\pi))^2$ ft deep and the amount of water is $25\pi(4 - t/(100\pi))^2$ ft^3. At $t = 0$, we have $x = 16$ ft and $V = 400\pi$ ft^3, as required. The tank will empty $(V = 0)$ in $t = 400\pi$ minutes, which is about 21 hours. ■

EXERCISES 9.1

Verifying Solutions

In Exercises 1 and 2, show that each function $y = f(x)$ is a solution of the accompanying differential equation.

1. $2y' + 3y = e^{-x}$

a. $y = e^{-x}$ **b.** $y = e^{-x} + e^{-(3/2)x}$

c. $y = e^{-x} + Ce^{-(3/2)x}$

2. $y' = y^2$

a. $y = -\dfrac{1}{x}$ **b.** $y = -\dfrac{1}{x+3}$ **c.** $y = -\dfrac{1}{x+C}$

In Exercises 3 and 4, show that the function $y = f(x)$ is a solution of the given differential equation.

3. $y = \dfrac{1}{x}\displaystyle\int_1^x \frac{e^t}{t}\,dt, \quad x^2y' + xy = e^x$

4. $y = \dfrac{1}{\sqrt{1+x^4}}\displaystyle\int_1^x \sqrt{1+t^4}\,dt, \quad y' + \frac{2x^3}{1+x^4}y = 1$

In Exercises 5–8, show that each function is a solution of the given initial value problem.

	Differential equation	Initial condition	Solution candidate
5.	$y' + y = \dfrac{2}{1+4e^{2x}}$	$y(-\ln 2) = \dfrac{\pi}{2}$	$y = e^{-x}\tan^{-1}(2e^x)$
6.	$y' = e^{-x^2} - 2xy$	$y(2) = 0$	$y = (x-2)e^{-x^2}$
7.	$xy' + y = -\sin x,$ $x > 0$	$y\left(\dfrac{\pi}{2}\right) = 0$	$y = \dfrac{\cos x}{x}$
8.	$x^2y' = xy - y^2,$ $x > 1$	$y(e) = e$	$y = \dfrac{x}{\ln x}$

Separable Equations

Solve the differential equation in Exercises 9–18.

9. $2\sqrt{xy}\dfrac{dy}{dx} = 1, \quad x, y > 0$

10. $\dfrac{dy}{dx} = x^2\sqrt{y}, \quad y > 0$

11. $\dfrac{dy}{dx} = e^{x-y}$

12. $\dfrac{dy}{dx} = 3x^2 e^{-y}$

13. $\dfrac{dy}{dx} = \sqrt{y}\cos^2\sqrt{y}$

14. $\sqrt{2xy}\dfrac{dy}{dx} = 1$

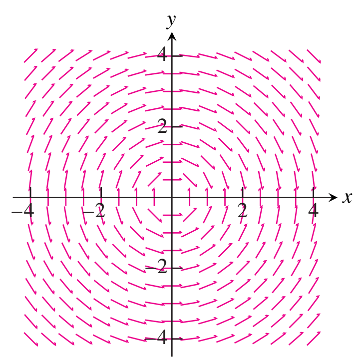
15. $\sqrt{x}\dfrac{dy}{dx} = e^{y+\sqrt{x}}, \quad x > 0$

16. $(\sec x)\dfrac{dy}{dx} = e^{y+\sin x}$

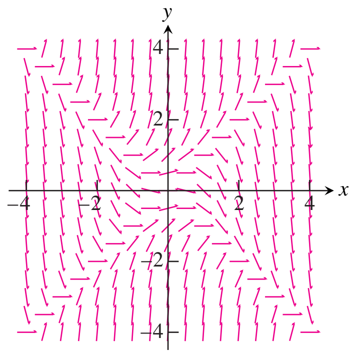
17. $\dfrac{dy}{dx} = 2x\sqrt{1-y^2}, \quad -1 < y < 1$

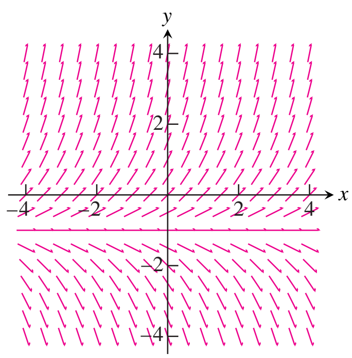
18. $\dfrac{dy}{dx} = \dfrac{e^{2x-y}}{e^{x+y}}$

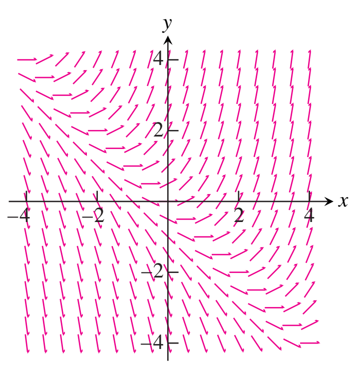
In Exercises 19–22, match the differential equations with their slope fields, graphed here.



(a)


(b)


(c)


(d)

19. $y' = x + y$

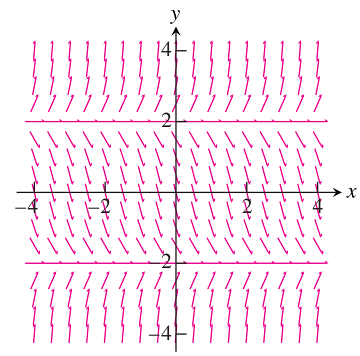
20. $y' = y + 1$

21. $y' = -\frac{x}{y}$

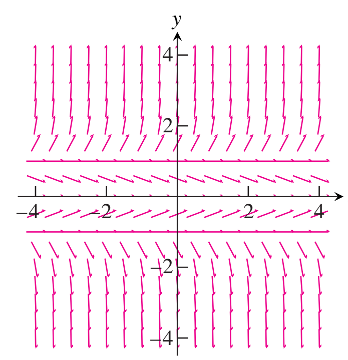
22. $y' = y^2 - x^2$

In Exercises 23 and 24, copy the slope fields and sketch in some of the solution curves.

23. $y' = (y + 2)(y - 2)$



24. $y' = y(y + 1)(y - 1)$



COMPUTER EXPLORATIONS

Slope Fields and Solution Curves

In Exercises 25–30, obtain a slope field and add to it graphs of the solution curves passing through the given points.

25. $y' = y$ with

a. $(0, 1)$ **b.** $(0, 2)$ **c.** $(0, -1)$

26. $y' = 2(y - 4)$ with

a. $(0, 1)$ **b.** $(0, 4)$ **c.** $(0, 5)$

27. $y' = y(x + y)$ with

a. $(0, 1)$ **b.** $(0, -2)$ **c.** $(0, 1/4)$ **d.** $(-1, -1)$

28. $y' = y^2$ with

a. $(0, 1)$ **b.** $(0, 2)$ **c.** $(0, -1)$ **d.** $(0, 0)$

29. $y' = (y - 1)(x + 2)$ with

a. $(0, -1)$ **b.** $(0, 1)$ **c.** $(0, 3)$ **d.** $(1, -1)$

30. $y' = \dfrac{xy}{x^2 + 4}$ with

a. $(0, 2)$ **b.** $(0, -6)$ **c.** $\left(-2\sqrt{3}, -4\right)$

In Exercises 31 and 32, obtain a slope field and graph the particular solution over the specified interval. Use your CAS DE solver to find the general solution of the differential equation.

31. A logistic equation $y' = y(2 - y)$, $y(0) = 1/2$; $0 \le x \le 4$, $0 \le y \le 3$

32. $y' = (\sin x)(\sin y)$, $y(0) = 2$; $-6 \le x \le 6$, $-6 \le y \le 6$

Exercises 33 and 34 have no explicit solution in terms of elementary functions. Use a CAS to explore graphically each of the differential equations.

33. $y' = \cos(2x - y)$, $y(0) = 2$; $0 \le x \le 5$, $0 \le y \le 5$; $y(2)$

34. A Gompertz equation $y' = y(1/2 - \ln y)$, $y(0) = 1/3$; $0 \le x \le 4$, $0 \le y \le 3$; $y(3)$

35. Use a CAS to find the solutions of $y' + y = f(x)$ subject to the initial condition $y(0) = 0$, if $f(x)$ is

a. $2x$ **b.** $\sin 2x$ **c.** $3e^{x/2}$ **d.** $2e^{-x/2}\cos 2x$.

Graph all four solutions over the interval $-2 \le x \le 6$ to compare the results.

36. a. Use a CAS to plot the slope field of the differential equation

$$y' = \frac{3x^2 + 4x + 2}{2(y - 1)}$$

over the region $-3 \le x \le 3$ and $-3 \le y \le 3$.

b. Separate the variables and use a CAS integrator to find the general solution in implicit form.

c. Using a CAS implicit function grapher, plot solution curves for the arbitrary constant values $C = -6, -4, -2, 0, 2, 4, 6$.

d. Find and graph the solution that satisfies the initial condition $y(0) = -1$.

9.2 First-Order Linear Differential Equations

The exponential growth/decay equation $dy/dx = ky$ (Section 7.5) is a separable differential equation. It is also a special case of a differential equation having a *linear form*. Linear differential equations model a number of real-world phenomena, including electrical circuits and chemical mixture problems.

A first-order **linear** differential equation is one that can be written in the form

$$\frac{dy}{dx} + P(x)y = Q(x), \tag{1}$$

where P and Q are continuous functions of x. Equation (1) is the linear equation's **standard form**.

Since the exponential growth/decay equation can be put in the standard form

$$\frac{dy}{dx} - ky = 0,$$

we see it is a linear equation with $P(x) = -k$ and $Q(x) = 0$. Equation (1) is *linear* (in y) because y and its derivative dy/dx occur only to the first power, are not multiplied together, nor do they appear as the argument of a function (such as $\sin y$, e^y, or $\sqrt{dy/dx}$).

EXAMPLE 1 Finding the Standard Form

Put the following equation in standard form:

$$x\frac{dy}{dx} = x^2 + 3y, \qquad x > 0.$$

Solution

$$x\frac{dy}{dx} = x^2 + 3y$$

$$\frac{dy}{dx} = x + \frac{3}{x}y \qquad \text{Divide by } x$$

$$\frac{dy}{dx} - \frac{3}{x}y = x \qquad \text{Standard form with } P(x) = -3/x \text{ and } Q(x) = x$$

Notice that $P(x)$ is $-3/x$, not $+3/x$. The standard form is $y' + P(x)y = Q(x)$, so the minus sign is part of the formula for $P(x)$. ■

Solving Linear Equations

We solve the equation

$$\frac{dy}{dx} + P(x)y = Q(x) \tag{2}$$

by multiplying both sides by a *positive* function $v(x)$ that transforms the left side into the derivative of the product $v(x) \cdot y$. We will show how to find v in a moment, but first we want to show how, once found, it provides the solution we seek.

Here is why multiplying by $v(x)$ works:

$$\frac{dy}{dx} + P(x)y = Q(x)$$ Original equation is in standard form.

$$v(x)\frac{dy}{dx} + P(x)v(x)y = v(x)Q(x)$$ Multiply by positive $v(x)$.

$$\frac{d}{dx}(v(x) \cdot y) = v(x)Q(x)$$ $v(x)$ is chosen to make $v\frac{dy}{dx} + Pvy = \frac{d}{dx}(v \cdot y)$.

$$v(x) \cdot y = \int v(x)Q(x)\,dx$$ Integrate with respect to x.

$$y = \frac{1}{v(x)}\int v(x)Q(x)\,dx \tag{3}$$

Equation (3) expresses the solution of Equation (2) in terms of the function $v(x)$ and $Q(x)$. We call $v(x)$ an **integrating factor** for Equation (2) because its presence makes the equation integrable.

Why doesn't the formula for $P(x)$ appear in the solution as well? It does, but indirectly, in the construction of the positive function $v(x)$. We have

$$\frac{d}{dx}(vy) = v\frac{dy}{dx} + Pvy$$ Condition imposed on v

$$v\frac{dy}{dx} + y\frac{dv}{dx} = v\frac{dy}{dx} + Pvy$$ Product Rule for derivatives

$$y\frac{dv}{dx} = Pvy$$ The terms $v\frac{dy}{dx}$ cancel.

This last equation will hold if

$$\frac{dv}{dx} = Pv$$

$$\frac{dv}{v} = P\,dx$$ Variables separated, $v > 0$

$$\int \frac{dv}{v} = \int P\,dx$$ Integrate both sides.

$$\ln v = \int P\,dx$$ Since $v > 0$, we do not need absolute value signs in $\ln v$.

$$e^{\ln v} = e^{\int P\,dx}$$ Exponentiate both sides to solve for v.

$$v = e^{\int P\,dx} \tag{4}$$

Thus a formula for the general solution to Equation (1) is given by Equation (3), where $v(x)$ is given by Equation (4). However, rather than memorizing the formula, just remember how to find the integrating factor once you have the standard form so $P(x)$ is correctly identified.

> To solve the linear equation $y' + P(x)y = Q(x)$, multiply both sides by the integrating factor $v(x) = e^{\int P(x)\,dx}$ and integrate both sides.

When you integrate the left-side product in this procedure, you always obtain the product $v(x)y$ of the integrating factor and solution function y because of the way v is defined.

EXAMPLE 2 Solving a First-Order Linear Differential Equation

Solve the equation

$$x\frac{dy}{dx} = x^2 + 3y, \qquad x > 0.$$

HISTORICAL BIOGRAPHY

Adrien Marie Legendre (1752–1833)

Solution First we put the equation in standard form (Example 1):

$$\frac{dy}{dx} - \frac{3}{x}y = x,$$

so $P(x) = -3/x$ is identified.

The integrating factor is

$$\begin{aligned} v(x) = e^{\int P(x)\,dx} &= e^{\int(-3/x)\,dx} \\ &= e^{-3\ln|x|} && \text{Constant of integration is 0, so } v \text{ is as simple as possible.} \\ &= e^{-3\ln x} && x > 0 \\ &= e^{\ln x^{-3}} = \frac{1}{x^3}. \end{aligned}$$

Next we multiply both sides of the standard form by $v(x)$ and integrate:

$$\begin{aligned} \frac{1}{x^3}\cdot\left(\frac{dy}{dx} - \frac{3}{x}y\right) &= \frac{1}{x^3}\cdot x \\ \frac{1}{x^3}\frac{dy}{dx} - \frac{3}{x^4}y &= \frac{1}{x^2} \\ \frac{d}{dx}\left(\frac{1}{x^3}y\right) &= \frac{1}{x^2} && \text{Left side is } \tfrac{d}{dx}(v\cdot y). \\ \frac{1}{x^3}y &= \int \frac{1}{x^2}\,dx && \text{Integrate both sides.} \\ \frac{1}{x^3}y &= -\frac{1}{x} + C. \end{aligned}$$

Solving this last equation for y gives the general solution:

$$y = x^3\left(-\frac{1}{x} + C\right) = -x^2 + Cx^3, \qquad x > 0.$$ ■

EXAMPLE 3 Solving a First-Order Linear Initial Value Problem

Solve the equation

$$xy' = x^2 + 3y, \qquad x > 0,$$

given the initial condition $y(1) = 2$.

Solution We first solve the differential equation (Example 2), obtaining

$$y = -x^2 + Cx^3, \qquad x > 0.$$

We then use the initial condition to find C:

$$y = -x^2 + Cx^3$$

$$2 = -(1)^2 + C(1)^3 \qquad y = 2 \text{ when } x = 1$$

$$C = 2 + (1)^2 = 3.$$

The solution of the initial value problem is the function $y = -x^2 + 3x^3$. ■

EXAMPLE 4 Find the particular solution of

$$3xy' - y = \ln x + 1, \qquad x > 0,$$

satisfying $y(1) = -2$.

Solution With $x > 0$, we write the equation in standard form:

$$y' - \frac{1}{3x}y = \frac{\ln x + 1}{3x}.$$

Then the integrating factor is given by

$$v = e^{\int -dx/3x} = e^{(-1/3)\ln x} = x^{-1/3}. \qquad x > 0$$

Thus

$$x^{-1/3}y = \frac{1}{3}\int (\ln x + 1)x^{-4/3}\,dx. \qquad \text{Left side is } vy.$$

Integration by parts of the right side gives

$$x^{-1/3}y = -x^{-1/3}(\ln x + 1) + \int x^{-4/3}\,dx + C.$$

Therefore

$$x^{-1/3}y = -x^{-1/3}(\ln x + 1) - 3x^{-1/3} + C$$

or, solving for y,

$$y = -(\ln x + 4) + Cx^{1/3}.$$

When $x = 1$ and $y = -2$ this last equation becomes

$$-2 = -(0 + 4) + C,$$

so

$$C = 2.$$

Substitution into the equation for y gives the particular solution

$$y = 2x^{1/3} - \ln x - 4.$$

In solving the linear equation in Example 2, we integrated both sides of the equation after multiplying each side by the integrating factor. However, we can shorten the amount of work, as in Example 4, by remembering that the left side *always* integrates into the product $v(x) \cdot y$ of the integrating factor times the solution function. From Equation (3) this means that

$$v(x)y = \int v(x)Q(x)\,dx.$$

We need only integrate the product of the integrating factor $v(x)$ with the right side $Q(x)$ of Equation (1) and then equate the result with $v(x)y$ to obtain the general solution. Nevertheless, to emphasize the role of $v(x)$ in the solution process, we sometimes follow the complete procedure as illustrated in Example 2.

Observe that if the function $Q(x)$ is identically zero in the standard form given by Equation (1), the linear equation is separable:

$$\frac{dy}{dx} + P(x)y = Q(x)$$

$$\frac{dy}{dx} + P(x)y = 0 \qquad Q(x) \equiv 0$$

$$dy = -P(x)\,dx \qquad \text{Separating the variables}$$

We now present two applied problems modeled by a first-order linear differential equation.

RL Circuits

The diagram in Figure 9.5 represents an electrical circuit whose total resistance is a constant R ohms and whose self-inductance, shown as a coil, is L henries, also a constant. There is a switch whose terminals at a and b can be closed to connect a constant electrical source of V volts.

Ohm's Law, $V = RI$, has to be modified for such a circuit. The modified form is

$$L\frac{di}{dt} + Ri = V, \tag{5}$$

where i is the intensity of the current in amperes and t is the time in seconds. By solving this equation, we can predict how the current will flow after the switch is closed.

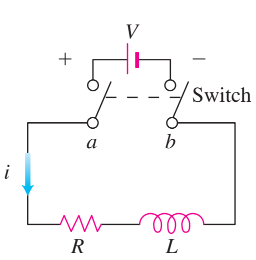


FIGURE 9.5 The *RL* circuit in Example 5.

EXAMPLE 5 Electric Current Flow

The switch in the *RL* circuit in Figure 9.5 is closed at time $t = 0$. How will the current flow as a function of time?

Solution Equation (5) is a first-order linear differential equation for i as a function of t. Its standard form is

$$\frac{di}{dt} + \frac{R}{L}i = \frac{V}{L}, \tag{6}$$

and the corresponding solution, given that $i = 0$ when $t = 0$, is

$$i = \frac{V}{R} - \frac{V}{R}e^{-(R/L)t} \tag{7}$$

(Exercise 32). Since R and L are positive, $-(R/L)$ is negative and $e^{-(R/L)t} \to 0$ as $t \to \infty$. Thus,

$$\lim_{t\to\infty} i = \lim_{t\to\infty}\left(\frac{V}{R} - \frac{V}{R}e^{-(R/L)t}\right) = \frac{V}{R} - \frac{V}{R}\cdot 0 = \frac{V}{R}.$$

At any given time, the current is theoretically less than V/R, but as time passes, the current approaches the **steady-state value** V/R. According to the equation

$$L\frac{di}{dt} + Ri = V,$$

$I = V/R$ is the current that will flow in the circuit if either $L = 0$ (no inductance) or $di/dt = 0$ (steady current, $i =$ constant) (Figure 9.6).

Equation (7) expresses the solution of Equation (6) as the sum of two terms: a **steady-state solution** V/R and a **transient solution** $-(V/R)e^{-(R/L)t}$ that tends to zero as $t \to \infty$. ■

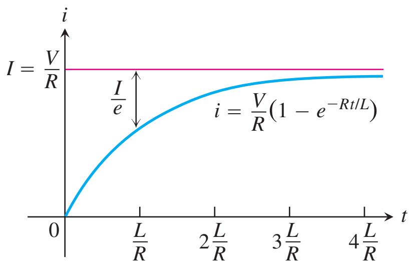


FIGURE 9.6 The growth of the current in the RL circuit in Example 5. I is the current's steady-state value. The number $t = L/R$ is the time constant of the circuit. The current gets to within 5% of its steady-state value in 3 time constants (Exercise 31).

Mixture Problems

A chemical in a liquid solution (or dispersed in a gas) runs into a container holding the liquid (or the gas) with, possibly, a specified amount of the chemical dissolved as well. The mixture is kept uniform by stirring and flows out of the container at a known rate. In this process, it is often important to know the concentration of the chemical in the container at any given time. The differential equation describing the process is based on the formula

$$\begin{matrix}\text{Rate of change}\\ \text{of amount}\\ \text{in container}\end{matrix} = \begin{pmatrix}\text{rate at which}\\ \text{chemical}\\ \text{arrives}\end{pmatrix} - \begin{pmatrix}\text{rate at which}\\ \text{chemical}\\ \text{departs.}\end{pmatrix} \tag{8}$$

If $y(t)$ is the amount of chemical in the container at time t and $V(t)$ is the total volume of liquid in the container at time t, then the departure rate of the chemical at time t is

$$\begin{aligned}\text{Departure rate} &= \frac{y(t)}{V(t)}\cdot(\text{outflow rate})\\ &= \begin{pmatrix}\text{concentration in}\\ \text{container at time } t\end{pmatrix}\cdot(\text{outflow rate}).\end{aligned} \tag{9}$$

Accordingly, Equation (8) becomes

$$\frac{dy}{dt} = (\text{chemical's arrival rate}) - \frac{y(t)}{V(t)}\cdot(\text{outflow rate}). \tag{10}$$

If, say, y is measured in pounds, V in gallons, and t in minutes, the units in Equation (10) are

$$\frac{\text{pounds}}{\text{minutes}} = \frac{\text{pounds}}{\text{minutes}} - \frac{\text{pounds}}{\text{gallons}}\cdot\frac{\text{gallons}}{\text{minutes}}.$$

Video

EXAMPLE 6 Oil Refinery Storage Tank

In an oil refinery, a storage tank contains 2000 gal of gasoline that initially has 100 lb of an additive dissolved in it. In preparation for winter weather, gasoline containing 2 lb of

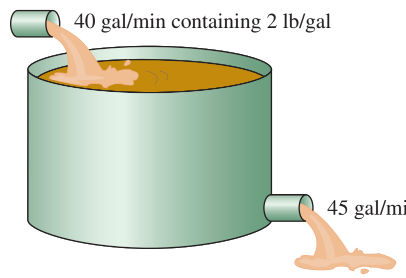


FIGURE 9.7 The storage tank in Example 6 mixes input liquid with stored liquid to produce an output liquid.

additive per gallon is pumped into the tank at a rate of 40 gal/min. The well-mixed solution is pumped out at a rate of 45 gal/min. How much of the additive is in the tank 20 min after the pumping process begins (Figure 9.7)?

Solution Let y be the amount (in pounds) of additive in the tank at time t. We know that $y = 100$ when $t = 0$. The number of gallons of gasoline and additive in solution in the tank at any time t is

$$\begin{aligned} V(t) &= 2000 \text{ gal} + \left(40 \frac{\text{gal}}{\text{min}} - 45 \frac{\text{gal}}{\text{min}}\right)(t \text{ min}) \\ &= (2000 - 5t) \text{ gal}. \end{aligned}$$

Therefore,

$$\begin{aligned} \text{Rate out} &= \frac{y(t)}{V(t)} \cdot \text{outflow rate} && \text{Eq. (9)} \\ &= \left(\frac{y}{2000 - 5t}\right) 45 && \text{Outflow rate is 45 gal/min. and } v = 2000 - 5t. \\ &= \frac{45y}{2000 - 5t} \frac{\text{lb}}{\text{min}}. \end{aligned}$$

Also,

$$\begin{aligned} \text{Rate in} &= \left(2 \frac{\text{lb}}{\text{gal}}\right)\left(40 \frac{\text{gal}}{\text{min}}\right) \\ &= 80 \frac{\text{lb}}{\text{min}}. && \text{Eq. (10)} \end{aligned}$$

The differential equation modeling the mixture process is

$$\frac{dy}{dt} = 80 - \frac{45y}{2000 - 5t}$$

in pounds per minute.

To solve this differential equation, we first write it in standard form:

$$\frac{dy}{dt} + \frac{45}{2000 - 5t} y = 80.$$

Thus, $P(t) = 45/(2000 - 5t)$ and $Q(t) = 80$.

The integrating factor is

$$\begin{aligned} v(t) = e^{\int P\,dt} &= e^{\int \frac{45}{2000-5t}\,dt} \\ &= e^{-9\ln(2000-5t)} \qquad 2000 - 5t > 0 \\ &= (2000 - 5t)^{-9}. \end{aligned}$$

Multiplying both sides of the standard equation by $v(t)$ and integrating both sides gives,

$$\begin{aligned} (2000 - 5t)^{-9} \cdot \left(\frac{dy}{dt} + \frac{45}{2000 - 5t}\,y \right) &= 80(2000 - 5t)^{-9} \\ (2000 - 5t)^{-9} \frac{dy}{dt} + 45(2000 - 5t)^{-10}\,y &= 80(2000 - 5t)^{-9} \\ \frac{d}{dt}\left[(2000 - 5t)^{-9} y \right] &= 80(2000 - 5t)^{-9} \\ (2000 - 5t)^{-9} y &= \int 80(2000 - 5t)^{-9}\,dt \\ (2000 - 5t)^{-9} y &= 80 \cdot \frac{(2000 - 5t)^{-8}}{(-8)(-5)} + C. \end{aligned}$$

The general solution is

$$y = 2(2000 - 5t) + C(2000 - 5t)^9.$$

Because $y = 100$ when $t = 0$, we can determine the value of C:

$$\begin{aligned} 100 &= 2(2000 - 0) + C(2000 - 0)^9 \\ C &= -\frac{3900}{(2000)^9}. \end{aligned}$$

The particular solution of the initial value problem is

$$y = 2(2000 - 5t) - \frac{3900}{(2000)^9}(2000 - 5t)^9.$$

The amount of additive 20 min after the pumping begins is

$$y(20) = 2[2000 - 5(20)] - \frac{3900}{(2000)^9}[2000 - 5(20)]^9 \approx 1342 \text{ lb}.$$ ■

EXERCISES 9.2

First-Order Linear Equations

Solve the differential equations in Exercises 1–14.

Exercises

1. $x\dfrac{dy}{dx} + y = e^x, \quad x > 0$

2. $e^x\dfrac{dy}{dx} + 2e^x y = 1$

3. $xy' + 3y = \dfrac{\sin x}{x^2}, \quad x > 0$

4. $y' + (\tan x)y = \cos^2 x, \quad -\pi/2 < x < \pi/2$

5. $x\dfrac{dy}{dx} + 2y = 1 - \dfrac{1}{x}, \quad x > 0$

6. $(1 + x)y' + y = \sqrt{x}$

7. $2y' = e^{x/2} + y$

8. $e^{2x}y' + 2e^{2x}y = 2x$

9. $xy' - y = 2x\ln x$

10. $x\dfrac{dy}{dx} = \dfrac{\cos x}{x} - 2y, \quad x > 0$

11. $(t - 1)^3\dfrac{ds}{dt} + 4(t - 1)^2 s = t + 1, \quad t > 1$

12. $(t+1)\dfrac{ds}{dt} + 2s = 3(t+1) + \dfrac{1}{(t+1)^2}, \quad t > -1$

13. $\sin\theta \dfrac{dr}{d\theta} + (\cos\theta)r = \tan\theta, \quad 0 < \theta < \pi/2$

14. $\tan\theta \dfrac{dr}{d\theta} + r = \sin^2\theta, \quad 0 < \theta < \pi/2$

Solving Initial Value Problems

Solve the initial value problems in Exercises 15–20.

15. $\dfrac{dy}{dt} + 2y = 3, \quad y(0) = 1$

16. $t\dfrac{dy}{dt} + 2y = t^3, \quad t > 0, \quad y(2) = 1$

17. $\theta\dfrac{dy}{d\theta} + y = \sin\theta, \quad \theta > 0, \quad y(\pi/2) = 1$

18. $\theta\dfrac{dy}{d\theta} - 2y = \theta^3 \sec\theta \tan\theta, \quad \theta > 0, \quad y(\pi/3) = 2$

19. $(x+1)\dfrac{dy}{dx} - 2(x^2+x)y = \dfrac{e^{x^2}}{x+1}, \quad x > -1, \quad y(0) = 5$

20. $\dfrac{dy}{dx} + xy = x, \quad y(0) = -6$

21. Solve the exponential growth/decay initial value problem for y as a function of t thinking of the differential equation as a first-order linear equation with $P(x) = -k$ and $Q(x) = 0$:

$$\frac{dy}{dt} = ky \quad (k \text{ constant}), \quad y(0) = y_0$$

22. Solve the following initial value problem for u as a function of t:

$$\frac{du}{dt} + \frac{k}{m}u = 0 \quad (k \text{ and } m \text{ positive constants}), \quad u(0) = u_0$$

a. as a first-order linear equation.

b. as a separable equation.

Theory and Examples

23. Is either of the following equations correct? Give reasons for your answers.

a. $x\displaystyle\int \frac{1}{x}\,dx = x\ln|x| + C$ **b.** $x\displaystyle\int \frac{1}{x}\,dx = x\ln|x| + Cx$

24. Is either of the following equations correct? Give reasons for your answers.

a. $\dfrac{1}{\cos x}\displaystyle\int \cos x\,dx = \tan x + C$

b. $\dfrac{1}{\cos x}\displaystyle\int \cos x\,dx = \tan x + \dfrac{C}{\cos x}$

25. Salt mixture A tank initially contains 100 gal of brine in which 50 lb of salt are dissolved. A brine containing 2 lb/gal of salt runs into the tank at the rate of 5 gal/min. The mixture is kept uniform by stirring and flows out of the tank at the rate of 4 gal/min.

a. At what rate (pounds per minute) does salt enter the tank at time t?

b. What is the volume of brine in the tank at time t?

c. At what rate (pounds per minute) does salt leave the tank at time t?

d. Write down and solve the initial value problem describing the mixing process.

e. Find the concentration of salt in the tank 25 min after the process starts.

26. Mixture problem A 200-gal tank is half full of distilled water. At time $t = 0$, a solution containing 0.5 lb/gal of concentrate enters the tank at the rate of 5 gal/min, and the well-stirred mixture is withdrawn at the rate of 3 gal/min.

a. At what time will the tank be full?

b. At the time the tank is full, how many pounds of concentrate will it contain?

27. Fertilizer mixture A tank contains 100 gal of fresh water. A solution containing 1 lb/gal of soluble lawn fertilizer runs into the tank at the rate of 1 gal/min, and the mixture is pumped out of the tank at the rate of 3 gal/min. Find the maximum amount of fertilizer in the tank and the time required to reach the maximum.

28. Carbon monoxide pollution An executive conference room of a corporation contains 4500 ft^3 of air initially free of carbon monoxide. Starting at time $t = 0$, cigarette smoke containing 4% carbon monoxide is blown into the room at the rate of 0.3 ft^3/min. A ceiling fan keeps the air in the room well circulated and the air leaves the room at the same rate of 0.3 ft^3/min. Find the time when the concentration of carbon monoxide in the room reaches 0.01%.

29. Current in a closed *RL* circuit How many seconds after the switch in an RL circuit is closed will it take the current i to reach half of its steady state value? Notice that the time depends on R and L and not on how much voltage is applied.

30. Current in an open *RL* circuit If the switch is thrown open after the current in an RL circuit has built up to its steady-state value $I = V/R$, the decaying current (graphed here) obeys the equation

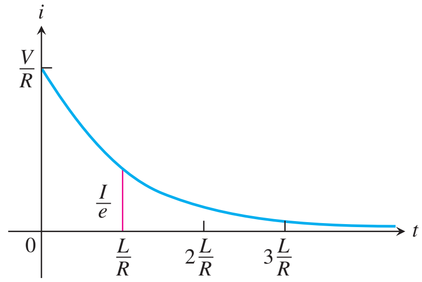
$$L\frac{di}{dt} + Ri = 0,$$

which is Equation (5) with $V = 0$.

a. Solve the equation to express i as a function of t.

b. How long after the switch is thrown will it take the current to fall to half its original value?

c. Show that the value of the current when $t = L/R$ is I/e. (The significance of this time is explained in the next exercise.)



31. Time constants Engineers call the number L/R the *time constant* of the RL circuit in Figure 9.6. The significance of the time constant is that the current will reach 95% of its final value within 3 time constants of the time the switch is closed (Figure 9.6). Thus, the time constant gives a built-in measure of how rapidly an individual circuit will reach equilibrium.

a. Find the value of i in Equation (7) that corresponds to $t = 3L/R$ and show that it is about 95% of the steady-state value $I = V/R$.

b. Approximately what percentage of the steady-state current will be flowing in the circuit 2 time constants after the switch is closed (i.e., when $t = 2L/R$)?

32. Derivation of Equation (7) in Example 5

a. Show that the solution of the equation

$$\frac{di}{dt} + \frac{R}{L}i = \frac{V}{L}$$

is

$$i = \frac{V}{R} + Ce^{-(R/L)t}.$$

b. Then use the initial condition $i(0) = 0$ to determine the value of C. This will complete the derivation of Equation (7).

c. Show that $i = V/R$ is a solution of Equation (6) and that $i = Ce^{-(R/L)t}$ satisfies the equation

$$\frac{di}{dt} + \frac{R}{L}i = 0.$$

HISTORICAL BIOGRAPHY

James Bernoulli
(1654–1705)

A **Bernoulli differential equation** is of the form

$$\frac{dy}{dx} + P(x)y = Q(x)y^n.$$

Observe that, if $n = 0$ or 1, the Bernoulli equation is linear. For other values of n, the substitution $u = y^{1-n}$ transforms the Bernoulli equation into the linear equation

$$\frac{du}{dx} + (1 - n)P(x)u = (1 - n)Q(x).$$

For example, in the equation

$$\frac{dy}{dx} - y = e^{-x}y^2$$

we have $n = 2$, so that $u = y^{1-2} = y^{-1}$ and $du/dx = -y^{-2}\,dy/dx$. Then $dy/dx = -y^2\,du/dx = -u^{-2}\,du/dx$. Substitution into the original equation gives

$$-u^{-2}\frac{du}{dx} - u^{-1} = e^{-x}u^{-2}$$

or, equivalently,

$$\frac{du}{dx} + u = -e^{-x}.$$

This last equation is linear in the (unknown) dependent variable u.

Solve the differential equations in Exercises 33–36.

33. $y' - y = -y^2$

34. $y' - y = xy^2$

35. $xy' + y = y^{-2}$

36. $x^2y' + 2xy = y^3$

9.3 Euler's Method

HISTORICAL BIOGRAPHY

Leonhard Euler
(1703–1783)

If we do not require or cannot immediately find an *exact* solution for an initial value problem $y' = f(x, y)$, $y(x_0) = y_0$ we can often use a computer to generate a table of approximate numerical values of y for values of x in an appropriate interval. Such a table is called a **numerical solution** of the problem, and the method by which we generate the table is called a **numerical method**. Numerical methods are generally fast and accurate, and they are often the methods of choice when exact formulas are unnecessary, unavailable, or overly complicated. In this section, we study one such method, called Euler's method, upon which many other numerical methods are based.

Euler's Method

Given a differential equation $dy/dx = f(x, y)$ and an initial condition $y(x_0) = y_0$, we can approximate the solution $y = y(x)$ by its linearization

$$L(x) = y(x_0) + y'(x_0)(x - x_0) \qquad \text{or} \qquad L(x) = y_0 + f(x_0, y_0)(x - x_0).$$

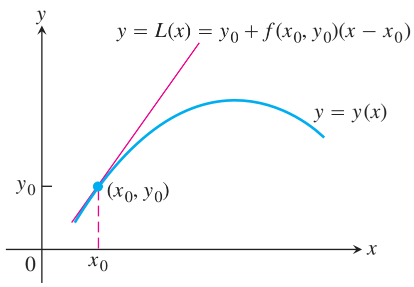


FIGURE 9.8 The linearization $L(x)$ of $y = y(x)$ at $x = x_0$.

The function $L(x)$ gives a good approximation to the solution $y(x)$ in a short interval about x_0 (Figure 9.8). The basis of Euler's method is to patch together a string of linearizations to approximate the curve over a longer stretch. Here is how the method works.

We know the point (x_0, y_0) lies on the solution curve. Suppose that we specify a new value for the independent variable to be $x_1 = x_0 + dx$. (Recall that $dx = \Delta x$ in the definition of differentials.) If the increment dx is small, then

$$y_1 = L(x_1) = y_0 + f(x_0, y_0)\, dx$$

is a good approximation to the exact solution value $y = y(x_1)$. So from the point (x_0, y_0), which lies *exactly* on the solution curve, we have obtained the point (x_1, y_1), which lies very close to the point $(x_1, y(x_1))$ on the solution curve (Figure 9.9).

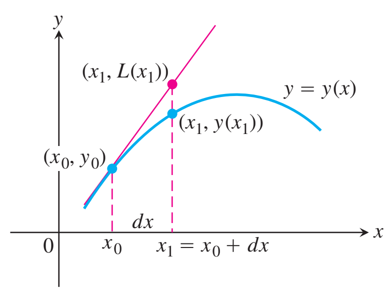


FIGURE 9.9 The first Euler step approximates $y(x_1)$ with $y_1 = L(x_1)$.

Using the point (x_1, y_1) and the slope $f(x_1, y_1)$ of the solution curve through (x_1, y_1), we take a second step. Setting $x_2 = x_1 + dx$, we use the linearization of the solution curve through (x_1, y_1) to calculate

$$y_2 = y_1 + f(x_1, y_1)\, dx.$$

This gives the next approximation (x_2, y_2) to values along the solution curve $y = y(x)$ (Figure 9.10). Continuing in this fashion, we take a third step from the point (x_2, y_2) with slope $f(x_2, y_2)$ to obtain the third approximation

$$y_3 = y_2 + f(x_2, y_2)\, dx,$$

and so on. We are literally building an approximation to one of the solutions by following the direction of the slope field of the differential equation.

The steps in Figure 9.10 are drawn large to illustrate the construction process, so the approximation looks crude. In practice, dx would be small enough to make the red curve hug the blue one and give a good approximation throughout.

EXAMPLE 1 Using Euler's Method

Find the first three approximations y_1, y_2, y_3 using Euler's method for the initial value problem

$$y' = 1 + y, \qquad y(0) = 1,$$

starting at $x_0 = 0$ with $dx = 0.1$.

Solution We have $x_0 = 0, y_0 = 1, x_1 = x_0 + dx = 0.1, x_2 = x_0 + 2dx = 0.2$, and $x_3 = x_0 + 3dx = 0.3$.

$$\begin{aligned}
\textit{First:}\quad y_1 &= y_0 + f(x_0, y_0)\, dx \\
&= y_0 + (1 + y_0)\, dx \\
&= 1 + (1 + 1)(0.1) = 1.2 \\
\textit{Second:}\quad y_2 &= y_1 + f(x_1, y_1)\, dx \\
&= y_1 + (1 + y_1)\, dx \\
&= 1.2 + (1 + 1.2)(0.1) = 1.42 \\
\textit{Third:}\quad y_3 &= y_2 + f(x_2, y_2)\, dx \\
&= y_2 + (1 + y_2)\, dx \\
&= 1.42 + (1 + 1.42)(0.1) = 1.662
\end{aligned}$$

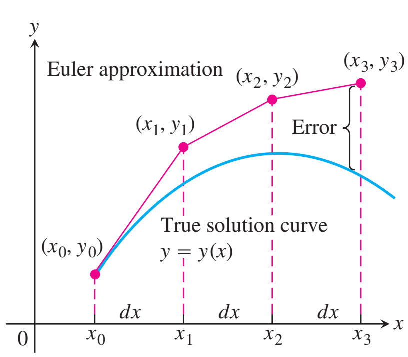


FIGURE 9.10 Three steps in the Euler approximation to the solution of the initial value problem $y' = f(x, y), y(x_0) = y_0$. As we take more steps, the errors involved usually accumulate, but not in the exaggerated way shown here.

The step-by-step process used in Example 1 can be continued easily. Using equally spaced values for the independent variable in the table and generating n of them, set

$$\begin{aligned}
x_1 &= x_0 + dx \\
x_2 &= x_1 + dx \\
&\vdots \\
x_n &= x_{n-1} + dx.
\end{aligned}$$

Then calculate the approximations to the solution,

$$y_1 = y_0 + f(x_0, y_0)\,dx$$
$$y_2 = y_1 + f(x_1, y_1)\,dx$$
$$\vdots$$
$$y_n = y_{n-1} + f(x_{n-1}, y_{n-1})\,dx.$$

The number of steps n can be as large as we like, but errors can accumulate if n is too large.

Euler's method is easy to implement on a computer or calculator. A computer program generates a table of numerical solutions to an initial value problem, allowing us to input x_0 and y_0, the number of steps n, and the step size dx. It then calculates the approximate solution values $y_1, y_2, \ldots, y_n$ in iterative fashion, as just described.

Solving the separable equation in Example 1, we find that the exact solution to the initial value problem is $y = 2e^x - 1$. We use this information in Example 2.

You Try It

EXAMPLE 2 Investigating the Accuracy of Euler's Method

Use Euler's method to solve

$$y' = 1 + y, \qquad y(0) = 1,$$

on the interval $0 \le x \le 1$, starting at $x_0 = 0$ and taking

(a) $dx = 0.1$

(b) $dx = 0.05$.

Compare the approximations with the values of the exact solution $y = 2e^x - 1$.

Solution

(a) We used a computer to generate the approximate values in Table 9.1. The "error" column is obtained by subtracting the unrounded Euler values from the unrounded values found using the exact solution. All entries are then rounded to four decimal places.

TABLE 9.1 Euler solution of $y' = 1 + y$, $y(0) = 1$, step size $dx = 0.1$

x	y (Euler)	y (exact)	Error
0	1	1	0
0.1	1.2	1.2103	0.0103
0.2	1.42	1.4428	0.0228
0.3	1.662	1.6997	0.0377
0.4	1.9282	1.9836	0.0554
0.5	2.2210	2.2974	0.0764
0.6	2.5431	2.6442	0.1011
0.7	2.8974	3.0275	0.1301
0.8	3.2872	3.4511	0.1639
0.9	3.7159	3.9192	0.2033
1.0	4.1875	4.4366	0.2491

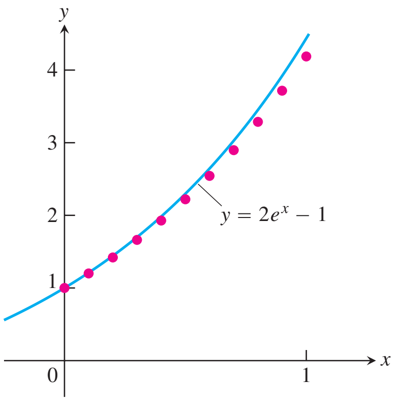


FIGURE 9.11 The graph of $y = 2e^x - 1$ superimposed on a scatterplot of the Euler approximations shown in Table 9.1 (Example 2).

By the time we reach $x = 1$ (after 10 steps), the error is about 5.6% of the exact solution. A plot of the exact solution curve with the scatterplot of Euler solution points from Table 9.1 is shown in Figure 9.11.

(b) One way to try to reduce the error is to decrease the step size. Table 9.2 shows the results and their comparisons with the exact solutions when we decrease the step size to 0.05, doubling the number of steps to 20. As in Table 9.1, all computations are performed before rounding. This time when we reach $x = 1$, the relative error is only about 2.9%.

TABLE 9.2 Euler solution of $y' = 1 + y$, $y(0) = 1$, step size $dx = 0.05$

x	y (Euler)	y (exact)	Error
0	1	1	0
0.05	1.1	1.1025	0.0025
0.10	1.205	1.2103	0.0053
0.15	1.3153	1.3237	0.0084
0.20	1.4310	1.4428	0.0118
0.25	1.5526	1.5681	0.0155
0.30	1.6802	1.6997	0.0195
0.35	1.8142	1.8381	0.0239
0.40	1.9549	1.9836	0.0287
0.45	2.1027	2.1366	0.0340
0.50	2.2578	2.2974	0.0397
0.55	2.4207	2.4665	0.0458
0.60	2.5917	2.6442	0.0525
0.65	2.7713	2.8311	0.0598
0.70	2.9599	3.0275	0.0676
0.75	3.1579	3.2340	0.0761
0.80	3.3657	3.4511	0.0853
0.85	3.5840	3.6793	0.0953
0.90	3.8132	3.9192	0.1060
0.95	4.0539	4.1714	0.1175
1.00	4.3066	4.4366	0.1300

■

It might be tempting to reduce the step size even further in Example 2 to obtain greater accuracy. Each additional calculation, however, not only requires additional computer time but more importantly adds to the buildup of round-off errors due to the approximate representations of numbers inside the computer.

The analysis of error and the investigation of methods to reduce it when making numerical calculations are important but are appropriate for a more advanced course. There are numerical methods more accurate than Euler's method, as you can see in a further study of differential equations. We study one improvement here.

HISTORICAL BIOGRAPHY

Carl Runge
(1856–1927)

Improved Euler's Method

We can improve on Euler's method by taking an average of two slopes. We first estimate y_n as in the original Euler method, but denote it by z_n. We then take the average of $f(x_{n-1}, y_{n-1})$ and $f(x_n, z_n)$ in place of $f(x_{n-1}, y_{n-1})$ in the next step. Thus, we calculate the next approximation y_n using

$$z_n = y_{n-1} + f(x_{n-1}, y_{n-1})\, dx$$
$$y_n = y_{n-1} + \left[\frac{f(x_{n-1}, y_{n-1}) + f(x_n, z_n)}{2}\right] dx.$$

EXAMPLE 3 Investigating the Accuracy of the Improved Euler's Method

Use the improved Euler's method to solve

$$y' = 1 + y, \qquad y(0) = 1,$$

on the interval $0 \le x \le 1$, starting at $x_0 = 0$ and taking $dx = 0.1$. Compare the approximations with the values of the exact solution $y = 2e^x - 1$.

Solution We used a computer to generate the approximate values in Table 9.3. The "error" column is obtained by subtracting the unrounded improved Euler values from the unrounded values found using the exact solution. All entries are then rounded to four decimal places.

TABLE 9.3 Improved Euler solution of $y' = 1 + y$, $y(0) = 1$, step size $dx = 0.1$

x	y (improved Euler)	y (exact)	Error
0	1	1	0
0.1	1.21	1.2103	0.0003
0.2	1.4421	1.4428	0.0008
0.3	1.6985	1.6997	0.0013
0.4	1.9818	1.9836	0.0018
0.5	2.2949	2.2974	0.0025
0.6	2.6409	2.6442	0.0034
0.7	3.0231	3.0275	0.0044
0.8	3.4456	3.4511	0.0055
0.9	3.9124	3.9192	0.0068
1.0	4.4282	4.4366	0.0084

By the time we reach $x = 1$ (after 10 steps), the relative error is about 0.19%. ■

By comparing Tables 9.1 and 9.3, we see that the improved Euler's method is considerably more accurate than the regular Euler's method, at least for the initial value problem $y' = 1 + y, \quad y(0) = 1$.

EXAMPLE 4 Oil Refinery Storage Tank Revisited

In Example 6, Section 9.2, we looked at a problem involving an additive mixture entering a 2000-gallon gasoline tank that was simultaneously being pumped. The analysis gave the initial value problem

$$\frac{dy}{dt} = 80 - \frac{45y}{2000 - 5t}, \qquad y(0) = 100$$

where $y(t)$ is the amount of additive in the tank at time t. The question was to find $y(20)$. Using Euler's method with an increment of $dt = 0.2$ (or 100 steps) gives the approximations

$$y(0.2) \approx 115.55, \qquad y(0.4) \approx 131.0298, \ldots$$

ending with $y(20) \approx 1344.3616$. The relative error from the exact solution $y(20) = 1342$ is about 0.18%. ■

EXERCISES 9.3

Calculating Euler Approximations

In Exercises 1–6, use Euler's method to calculate the first three approximations to the given initial value problem for the specified increment size. Calculate the exact solution and investigate the accuracy of your approximations. Round your results to four decimal places.

1. $y' = 1 - \frac{y}{x}, \quad y(2) = -1, \quad dx = 0.5$
2. $y' = x(1 - y), \quad y(1) = 0, \quad dx = 0.2$
3. $y' = 2xy + 2y, \quad y(0) = 3, \quad dx = 0.2$
4. $y' = y^2(1 + 2x), \quad y(-1) = 1, \quad dx = 0.5$

T 5. $y' = 2xe^{x^2}, \quad y(0) = 2, \quad dx = 0.1$

T 6. $y' = y + e^x - 2, \quad y(0) = 2, \quad dx = 0.5$

7. Use the Euler method with $dx = 0.2$ to estimate $y(1)$ if $y' = y$ and $y(0) = 1$. What is the exact value of $y(1)$?
8. Use the Euler method with $dx = 0.2$ to estimate $y(2)$ if $y' = y/x$ and $y(1) = 2$. What is the exact value of $y(2)$?

T 9. Use the Euler method with $dx = 0.5$ to estimate $y(5)$ if $y' = y^2/\sqrt{x}$ and $y(1) = -1$. What is the exact value of $y(5)$?

T 10. Use the Euler method with $dx = 1/3$ to estimate $y(2)$ if $y' = y - e^{2x}$ and $y(0) = 1$. What is the exact value of $y(2)$?

Improved Euler's Method

In Exercises 11 and 12, use the improved Euler's method to calculate the first three approximations to the given initial value problem. Compare the approximations with the values of the exact solution.

11. $y' = 2y(x + 1), \quad y(0) = 3, \quad dx = 0.2$

    (See Exercise 3 for the exact solution.)

12. $y' = x(1 - y), \quad y(1) = 0, \quad dx = 0.2$

    (See Exercise 2 for the exact solution.)

COMPUTER EXPLORATIONS

Euler's Method

In Exercises 13–16, use Euler's method with the specified step size to estimate the value of the solution at the given point x^*. Find the value of the exact solution at x^*.

13. $y' = 2xe^{x^2}, \quad y(0) = 2, \quad dx = 0.1, \quad x^* = 1$
14. $y' = y + e^x - 2, \quad y(0) = 2, \quad dx = 0.5, \quad x^* = 2$
15. $y' = \sqrt{x}/y, \quad y > 0, \quad y(0) = 1, \quad dx = 0.1, \quad x^* = 1$
16. $y' = 1 + y^2, \quad y(0) = 0, \quad dx = 0.1, \quad x^* = 1$

In Exercises 17 and 18, **(a)** find the exact solution of the initial value problem. Then compare the accuracy of the approximation with $y(x^*)$ using Euler's method starting at x_0 with step size **(b)** 0.2, **(c)** 0.1, and **(d)** 0.05.

17. $y' = 2y^2(x - 1), \quad y(2) = -1/2, \quad x_0 = 2, \quad x^* = 3$
18. $y' = y - 1, \quad y(0) = 3, \quad x_0 = 0, \quad x^* = 1$

Improved Euler's Method

In Exercises 19 and 20, compare the accuracy of the approximation with $y(x^*)$ using the improved Euler's method starting at x_0 with step size

a. 0.2 **b.** 0.1 **c.** 0.05

d. Describe what happens to the error as the step size decreases.

19. $y' = 2y^2(x - 1), \quad y(2) = -1/2, \quad x_0 = 2, \quad x^* = 3$

    (See Exercise 17 for the exact solution.)

20. $y' = y - 1, \quad y(0) = 3, \quad x_0 = 0, \quad x^* = 1$

    (See Exercise 18 for the exact solution.)

Exploring Differential Equations Graphically

Use a CAS to explore graphically each of the differential equations in Exercises 21–24. Perform the following steps to help with your explorations.

a. Plot a slope field for the differential equation in the given xy-window.

b. Find the general solution of the differential equation using your CAS DE solver.

c. Graph the solutions for the values of the arbitrary constant $C = -2, -1, 0, 1, 2$ superimposed on your slope field plot.

d. Find and graph the solution that satisfies the specified initial condition over the interval $[0, b]$.

e. Find the Euler numerical approximation to the solution of the initial value problem with 4 subintervals of the x-interval and plot the Euler approximation superimposed on the graph produced in part (d).

f. Repeat part (e) for 8, 16, and 32 subintervals. Plot these three Euler approximations superimposed on the graph from part (e).

g. Find the error $(y(\text{exact}) - y(\text{Euler}))$ at the specified point $x = b$ for each of your four Euler approximations. Discuss the improvement in the percentage error.

21. $y' = x + y, \quad y(0) = -7/10; \quad -4 \le x \le 4, \quad -4 \le y \le 4; \quad b = 1$

22. $y' = -x/y, \quad y(0) = 2; \quad -3 \le x \le 3, \quad -3 \le y \le 3; \quad b = 2$

23. A logistic equation $y' = y(2 - y), \quad y(0) = 1/2; \quad 0 \le x \le 4, \quad 0 \le y \le 3; \quad b = 3$

24. $y' = (\sin x)(\sin y), \quad y(0) = 2; \quad -6 \le x \le 6, \quad -6 \le y \le 6; \quad b = 3\pi/2$

9.4 Graphical Solutions of Autonomous Differential Equations

In Chapter 4 we learned that the sign of the first derivative tells where the graph of a function is increasing and where it is decreasing. The sign of the second derivative tells the concavity of the graph. We can build on our knowledge of how derivatives determine the shape of a graph to solve differential equations graphically. The starting ideas for doing so are the notions of *phase line* and *equilibrium value*. We arrive at these notions by investigating what happens when the derivative of a differentiable function is zero from a point of view different from that studied in Chapter 4.

Equilibrium Values and Phase Lines

When we differentiate implicitly the equation

$$\frac{1}{5}\ln(5y - 15) = x + 1$$

we obtain

$$\frac{1}{5}\left(\frac{5}{5y - 15}\right)\frac{dy}{dx} = 1.$$

Solving for $y' = dy/dx$ we find $y' = 5y - 15 = 5(y - 3)$. In this case the derivative y' is a function of y only (the dependent variable) and is zero when $y = 3$.

A differential equation for which dy/dx is a function of y only is called an **autonomous** differential equation. Let's investigate what happens when the derivative in an autonomous equation equals zero.

DEFINITION Equilibrium Values

If $dy/dx = g(y)$ is an autonomous differential equation, then the values of y for which $dy/dx = 0$ are called **equilibrium values** or **rest points**.

Thus, equilibrium values are those at which no change occurs in the dependent variable, so y is at *rest*. The emphasis is on the value of y where $dy/dx = 0$, not the value of x, as we studied in Chapter 4.

EXAMPLE 1 Finding Equilibrium Values

The equilibrium values for the autonomous differential equation

$$\frac{dy}{dx} = (y + 1)(y - 2)$$

are $y = -1$ and $y = 2$. ■

To construct a graphical solution to an autonomous differential equation like the one in Example 1, we first make a **phase line** for the equation, a plot on the y-axis that shows the equation's equilibrium values along with the intervals where dy/dx and d^2y/dx^2 are positive and negative. Then we know where the solutions are increasing and decreasing, and the concavity of the solution curves. These are the essential features we found in Section 4.4, so we can determine the shapes of the solution curves without having to find formulas for them.

Video Video You Try It

EXAMPLE 2 Drawing a Phase Line and Sketching Solution Curves

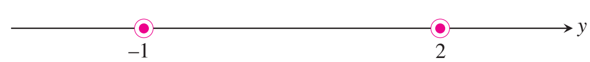
Draw a phase line for the equation

$$\frac{dy}{dx} = (y + 1)(y - 2)$$

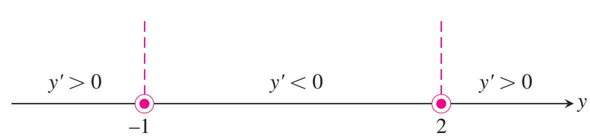
and use it to sketch solutions to the equation.

Solution

1. *Draw a number line for y and mark the equilibrium values $y = -1$ and $y = 2$, where $dy/dx = 0$.*



2. *Identify and label the intervals where $y' > 0$ and $y' < 0$.* This step resembles what we did in Section 4.3, only now we are marking the y-axis instead of the x-axis.



We can encapsulate the information about the sign of y' on the phase line itself. Since $y' > 0$ on the interval to the left of $y = -1$, a solution of the differential equation with a y-value less than -1 will increase from there toward $y = -1$. We display this information by drawing an arrow on the interval pointing to -1.

Similarly, $y' < 0$ between $y = -1$ and $y = 2$, so any solution with a value in this interval will decrease toward $y = -1$.

For $y > 2$, we have $y' > 0$, so a solution with a y-value greater than 2 will increase from there without bound.

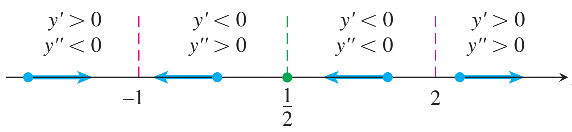
In short, solution curves below the horizontal line $y = -1$ in the xy-plane rise toward $y = -1$. Solution curves between the lines $y = -1$ and $y = 2$ fall away from $y = 2$ toward $y = -1$. Solution curves above $y = 2$ rise away from $y = 2$ and keep going up.

3. *Calculate y'' and mark the intervals where $y'' > 0$ and $y'' < 0$.* To find y'', we differentiate y' *with respect to* x, using implicit differentiation.

$$y' = (y + 1)(y - 2) = y^2 - y - 2 \qquad \text{Formula for } y' \ldots$$

$$\begin{aligned} y'' &= \frac{d}{dx}(y') = \frac{d}{dx}(y^2 - y - 2) \\ &= 2yy' - y' \\ &= (2y - 1)y' \\ &= (2y - 1)(y + 1)(y - 2). \end{aligned} \qquad \text{differentiated implicitly with respect to } x.$$

From this formula, we see that y'' changes sign at $y = -1$, $y = 1/2$, and $y = 2$. We add the sign information to the phase line.



4. *Sketch an assortment of solution curves in the xy-plane.* The horizontal lines $y = -1$, $y = 1/2$, and $y = 2$ partition the plane into horizontal bands in which we know the signs of y' and y''. In each band, this information tells us whether the solution curves rise or fall and how they bend as x increases (Figure 9.12).

The "equilibrium lines" $y = -1$ and $y = 2$ are also solution curves. (The constant functions $y = -1$ and $y = 2$ satisfy the differential equation.) Solution curves that cross the line $y = 1/2$ have an inflection point there. The concavity changes from concave down (above the line) to concave up (below the line).

As predicted in Step 2, solutions in the middle and lower bands approach the equilibrium value $y = -1$ as x increases. Solutions in the upper band rise steadily away from the value $y = 2$. ■

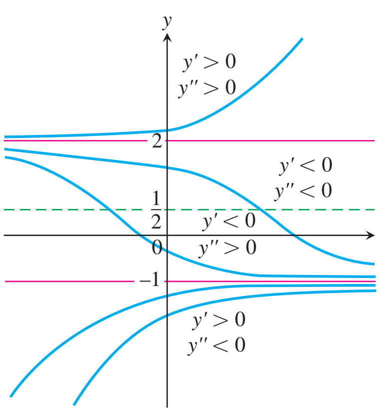


FIGURE 9.12 Graphical solutions from Example 2 include the horizontal lines $y = -1$ and $y = 2$ through the equilibrium values.

Stable and Unstable Equilibria

Look at Figure 9.12 once more, in particular at the behavior of the solution curves near the equilibrium values. Once a solution curve has a value near $y = -1$, it tends steadily toward that value; $y = -1$ is a **stable equilibrium**. The behavior near $y = 2$ is just the opposite: all solutions except the equilibrium solution $y = 2$ itself move *away* from it as x increases. We call $y = 2$ an **unstable equilibrium**. If the solution is *at* that value, it stays, but if it is off by any amount, no matter how small, it moves away. (Sometimes an equilibrium value is unstable because a solution moves away from it only on one side of the point.)

Now that we know what to look for, we can already see this behavior on the initial phase line. The arrows lead away from $y = 2$ and, once to the left of $y = 2$, toward $y = -1$.

We now present several applied examples for which we can sketch a family of solution curves to the differential equation models using the method in Example 2.

In Section 7.5 we solved analytically the differential equation

$$\frac{dH}{dt} = -k(H - H_S), \qquad k > 0$$

modeling Newton's law of cooling. Here H is the temperature (amount of heat) of an object at time t and H_S is the constant temperature of the surrounding medium. Our first example uses a phase line analysis to understand the graphical behavior of this temperature model over time.

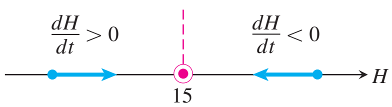


FIGURE 9.13 First step in constructing the phase line for Newton's law of cooling in Example 3. The temperature tends towards the equilibrium (surrounding-medium) value in the long run.

EXAMPLE 3 Cooling Soup

What happens to the temperature of the soup when a cup of hot soup is placed on a table in a room? We know the soup cools down, but what does a typical temperature curve look like as a function of time?

Solution Suppose that the surrounding medium has a constant Celsius temperature of 15°C. We can then express the difference in temperature as $H(t) - 15$. Assuming H is a differentiable function of time t, by Newton's law of cooling, there is a constant of proportionality $k > 0$ such that

$$\frac{dH}{dt} = -k(H - 15) \tag{1}$$

(*minus* k to give a negative derivative when $H > 15$).

Since $dH/dt = 0$ at $H = 15$, the temperature 15°C is an equilibrium value. If $H > 15$, Equation (1) tells us that $(H - 15) > 0$ and $dH/dt < 0$. If the object is hotter than the room, it will get cooler. Similarly, if $H < 15$, then $(H - 15) < 0$ and $dH/dt > 0$. An object cooler than the room will warm up. Thus, the behavior described by Equation (1) agrees with our intuition of how temperature should behave. These observations are captured in the initial phase line diagram in Figure 9.13. The value $H = 15$ is a stable equilibrium.

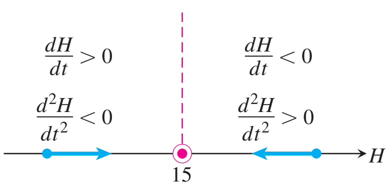


FIGURE 9.14 The complete phase line for Newton's law of cooling (Example 3).

We determine the concavity of the solution curves by differentiating both sides of Equation (1) with respect to t:

$$\frac{d}{dt}\left(\frac{dH}{dt}\right) = \frac{d}{dt}(-k(H - 15))$$

$$\frac{d^2H}{dt^2} = -k\frac{dH}{dt}.$$

Since $-k$ is negative, we see that d^2H/dt^2 is positive when $dH/dt < 0$ and negative when $dH/dt > 0$. Figure 9.14 adds this information to the phase line.

The completed phase line shows that if the temperature of the object is above the equilibrium value of 15°C, the graph of $H(t)$ will be decreasing and concave upward. If the temperature is below 15°C (the temperature of the surrounding medium), the graph of $H(t)$ will be increasing and concave downward. We use this information to sketch typical solution curves (Figure 9.15).

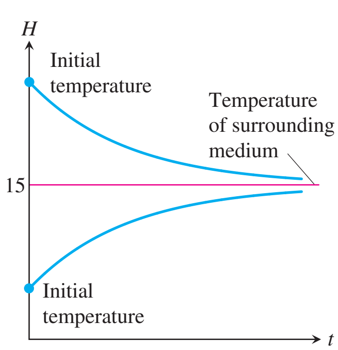


FIGURE 9.15 Temperature versus time. Regardless of initial temperature, the object's temperature $H(t)$ tends toward 15°C, the temperature of the surrounding medium.

From the upper solution curve in Figure 9.15, we see that as the object cools down, the rate at which it cools slows down because dH/dt approaches zero. This observation is implicit in Newton's law of cooling and contained in the differential equation, but the flattening of the graph as time advances gives an immediate visual representation of the phenomenon. The ability to discern physical behavior from graphs is a powerful tool in understanding real-world systems. ■

EXAMPLE 4 Analyzing the Fall of a Body Encountering a Resistive Force

Galileo and Newton both observed that the rate of change in momentum encountered by a moving object is equal to the net force applied to it. In mathematical terms,

$$F = \frac{d}{dt}(mv) \tag{2}$$

where F is the force and m and v the object's mass and velocity. If m varies with time, as it will if the object is a rocket burning fuel, the right-hand side of Equation (2) expands to

$$m\frac{dv}{dt} + v\frac{dm}{dt}$$

using the Product Rule. In many situations, however, m is constant, $dm/dt = 0$, and Equation (2) takes the simpler form

$$F = m\frac{dv}{dt} \qquad \text{or} \qquad F = ma, \tag{3}$$

known as *Newton's second law of motion*.

In free fall, the constant acceleration due to gravity is denoted by g and the one force acting downward on the falling body is

$$F_p = mg,$$

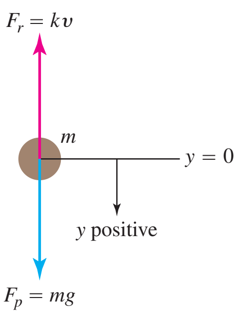


FIGURE 9.16 An object falling under the influence of gravity with a resistive force assumed to be proportional to the velocity.

the propulsion due to gravity. If, however, we think of a real body falling through the air—say, a penny from a great height or a parachutist from an even greater height—we know that at some point air resistance is a factor in the speed of the fall. A more realistic model of free fall would include air resistance, shown as a force F_r in the schematic diagram in Figure 9.16.

For low speeds well below the speed of sound, physical experiments have shown that F_r is approximately proportional to the body's velocity. The net force on the falling body is therefore

$$F = F_p - F_r,$$

giving

$$m\frac{dv}{dt} = mg - kv$$

$$\frac{dv}{dt} = g - \frac{k}{m}v. \tag{4}$$

We can use a phase line to analyze the velocity functions that solve this differential equation.

The equilibrium point, obtained by setting the right-hand side of Equation (4) equal to zero, is

$$v = \frac{mg}{k}.$$

If the body is initially moving faster than this, dv/dt is negative and the body slows down. If the body is moving at a velocity below mg/k, then $dv/dt > 0$ and the body speeds up. These observations are captured in the initial phase line diagram in Figure 9.17.

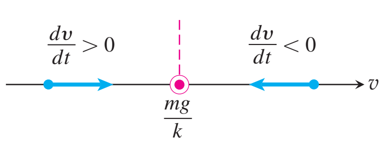


FIGURE 9.17 Initial phase line for Example 4.

We determine the concavity of the solution curves by differentiating both sides of Equation (4) with respect to t:

$$\frac{d^2v}{dt^2} = \frac{d}{dt}\left(g - \frac{k}{m}v\right) = -\frac{k}{m}\frac{dv}{dt}.$$

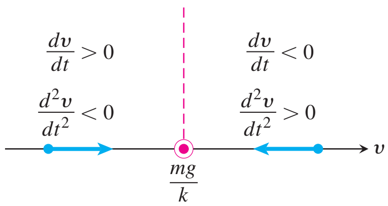


FIGURE 9.18 The completed phase line for Example 4.

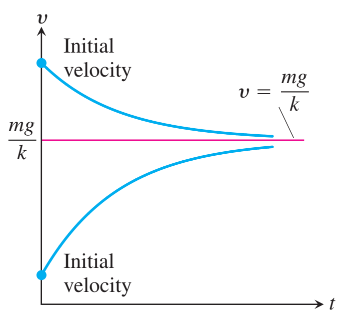


FIGURE 9.19 Typical velocity curves in Example 4. The value $v = mg/k$ is the terminal velocity.

We see that $d^2v/dt^2 < 0$ when $v < mg/k$ and $d^2v/dt^2 > 0$ when $v > mg/k$. Figure 9.18 adds this information to the phase line. Notice the similarity to the phase line for Newton's law of cooling (Figure 9.14). The solution curves are similar as well (Figure 9.19).

Figure 9.19 shows two typical solution curves. Regardless of the initial velocity, we see the body's velocity tending toward the limiting value $v = mg/k$. This value, a stable equilibrium point, is called the body's **terminal velocity**. Skydivers can vary their terminal velocity from 95 mph to 180 mph by changing the amount of body area opposing the fall. ■

EXAMPLE 5 Analyzing Population Growth in a Limiting Environment

You Try It

In Section 7.5 we examined population growth using the model of exponential change. That is, if P represents the number of individuals and we neglect departures and arrivals, then

$$\frac{dP}{dt} = kP, \tag{5}$$

where $k > 0$ is the birthrate minus the death rate per individual per unit time.

Because the natural environment has only a limited number of resources to sustain life, it is reasonable to assume that only a maximum population M can be accommodated. As the population approaches this **limiting population** or **carrying capacity**, resources become less abundant and the growth rate k decreases. A simple relationship exhibiting this behavior is

$$k = r(M - P),$$

where $r > 0$ is a constant. Notice that k decreases as P increases toward M and that k is negative if P is greater than M. Substituting $r(M - P)$ for k in Equation (5) gives the differential equation

$$\frac{dP}{dt} = r(M - P)P = rMP - rP^2. \tag{6}$$

The model given by Equation (6) is referred to as **logistic growth**.

We can forecast the behavior of the population over time by analyzing the phase line for Equation (6). The equilibrium values are $P = M$ and $P = 0$, and we can see that $dP/dt > 0$ if $0 < P < M$ and $dP/dt < 0$ if $P > M$. These observations are recorded on the phase line in Figure 9.20.

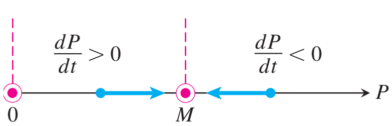


FIGURE 9.20 The initial phase line for Equation 6.

We determine the concavity of the population curves by differentiating both sides of Equation (6) with respect to t:

$$\begin{aligned} \frac{d^2P}{dt^2} &= \frac{d}{dt}(rMP - rP^2) \\ &= rM\frac{dP}{dt} - 2rP\frac{dP}{dt} \\ &= r(M - 2P)\frac{dP}{dt}. \end{aligned} \tag{7}$$

If $P = M/2$, then $d^2P/dt^2 = 0$. If $P < M/2$, then $(M - 2P)$ and dP/dt are positive and $d^2P/dt^2 > 0$. If $M/2 < P < M$, then $(M - 2P) < 0$, $dP/dt > 0$, and $d^2P/dt^2 < 0$. If $P > M$, then $(M - 2P)$ and dP/dt are both negative and $d^2P/dt^2 > 0$. We add this information to the phase line (Figure 9.21).

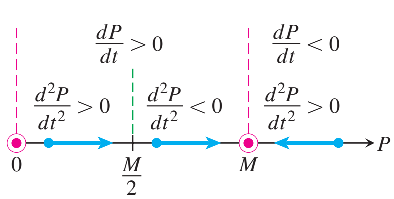


FIGURE 9.21 The completed phase line for logistic growth (Equation 6).

The lines $P = M/2$ and $P = M$ divide the first quadrant of the tP-plane into horizontal bands in which we know the signs of both dP/dt and d^2P/dt^2. In each band, we know how the solution curves rise and fall, and how they bend as time passes. The equilibrium lines $P = 0$ and $P = M$ are both population curves. Population curves crossing the line $P = M/2$ have an inflection point there, giving them a **sigmoid** shape (curved in two directions like a letter S). Figure 9.22 displays typical population curves. ■

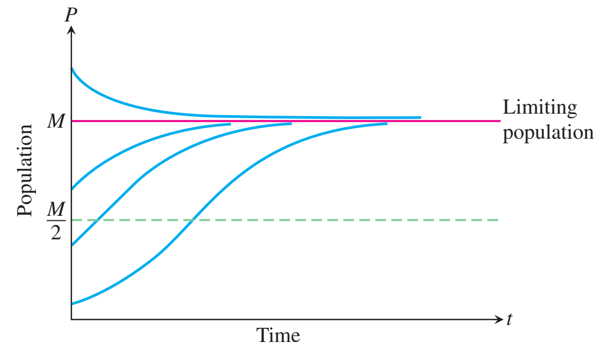


FIGURE 9.22 Population curves in Example 5.

EXERCISES 9.4

Phase Lines and Solution Curves

In Exercises 1–8,

a. Identify the equilibrium values. Which are stable and which are unstable?

b. Construct a phase line. Identify the signs of y' and y''.

c. Sketch several solution curves.

1. $\dfrac{dy}{dx} = (y + 2)(y - 3)$

2. $\dfrac{dy}{dx} = y^2 - 4$

3. $\dfrac{dy}{dx} = y^3 - y$

4. $\dfrac{dy}{dx} = y^2 - 2y$

5. $y' = \sqrt{y}, \quad y > 0$

6. $y' = y - \sqrt{y}, \quad y > 0$

7. $y' = (y - 1)(y - 2)(y - 3)$

8. $y' = y^3 - y^2$

Models of Population Growth

The autonomous differential equations in Exercises 9–12 represent models for population growth. For each exercise, use a phase line analysis to sketch solution curves for $P(t)$, selecting different starting values $P(0)$ (as in Example 5). Which equilibria are stable, and which are unstable?

9. $\dfrac{dP}{dt} = 1 - 2P$

10. $\dfrac{dP}{dt} = P(1 - 2P)$

11. $\dfrac{dP}{dt} = 2P(P - 3)$

12. $\dfrac{dP}{dt} = 3P(1 - P)\left(P - \dfrac{1}{2}\right)$

13. Catastrophic continuation of Example 5 Suppose that a healthy population of some species is growing in a limited environment and that the current population P_0 is fairly close to the carrying capacity M_0. You might imagine a population of fish living in a freshwater lake in a wilderness area. Suddenly a catastrophe such as the Mount St. Helens volcanic eruption contaminates the lake and destroys a significant part of the food and oxygen on which the fish depend. The result is a new environment with a carrying capacity M_1 considerably less than M_0 and, in fact, less than the current population P_0. Starting at some time before the catastrophe, sketch a "before-and-after" curve that shows how the fish population responds to the change in environment.

14. Controlling a population The fish and game department in a certain state is planning to issue hunting permits to control the deer population (one deer per permit). It is known that if the deer population falls below a certain level m, the deer will become extinct. It is also known that if the deer population rises above the carrying capacity M, the population will decrease back to M through disease and malnutrition.

a. Discuss the reasonableness of the following model for the growth rate of the deer population as a function of time:

$$\frac{dP}{dt} = rP(M - P)(P - m),$$

where P is the population of the deer and r is a positive constant of proportionality. Include a phase line.

b. Explain how this model differs from the logistic model $dP/dt = rP(M - P)$. Is it better or worse than the logistic model?

c. Show that if $P > M$ for all t, then $\lim_{t\to\infty} P(t) = M$.

d. What happens if $P < m$ for all t?

e. Discuss the solutions to the differential equation. What are the equilibrium points of the model? Explain the dependence of the steady-state value of P on the initial values of P. About how many permits should be issued?

Applications and Examples

Exercises

15. Skydiving If a body of mass m falling from rest under the action of gravity encounters an air resistance proportional to the square of velocity, then the body's velocity t seconds into the fall satisfies the equation.

$$m\frac{dv}{dt} = mg - kv^2, \qquad k > 0$$

where k is a constant that depends on the body's aerodynamic properties and the density of the air. (We assume that the fall is too short to be affected by changes in the air's density.)

a. Draw a phase line for the equation.

b. Sketch a typical velocity curve.

c. For a 160-lb skydiver ($mg = 160$) and with time in seconds and distance in feet, a typical value of k is 0.005. What is the diver's terminal velocity?

16. Resistance proportional to $\sqrt{v}$ A body of mass m is projected vertically downward with initial velocity v_0. Assume that the resisting force is proportional to the square root of the velocity and find the terminal velocity from a graphical analysis.

17. Sailing A sailboat is running along a straight course with the wind providing a constant forward force of 50 lb. The only other force acting on the boat is resistance as the boat moves through the water. The resisting force is numerically equal to five times the boat's speed, and the initial velocity is 1 ft/sec. What is the maximum velocity in feet per second of the boat under this wind?

18. The spread of information Sociologists recognize a phenomenon called *social diffusion,* which is the spreading of a piece of information, technological innovation, or cultural fad among a population. The members of the population can be divided into two classes: those who have the information and those who do not. In a fixed population whose size is known, it is reasonable to assume that the rate of diffusion is proportional to the number who have the information times the number yet to receive it. If X denotes the number of individuals who have the information in a population of N people, then a mathematical model for social diffusion is given by

$$\frac{dX}{dt} = kX(N - X),$$

where t represents time in days and k is a positive constant.

a. Discuss the reasonableness of the model.

b. Construct a phase line identifying the signs of X' and X''.

c. Sketch representative solution curves.

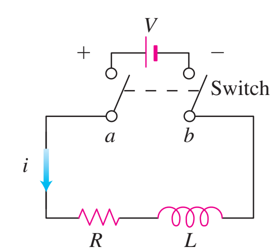
d. Predict the value of X for which the information is spreading most rapidly. How many people eventually receive the information?

19. Current in an *RL*-circuit The accompanying diagram represents an electrical circuit whose total resistance is a constant R ohms and whose self-inductance, shown as a coil, is L henries, also a constant. There is a switch whose terminals at a and b can be closed to connect a constant electrical source of V volts.

Ohm's Law, $V = Ri$, has to be modified for such a circuit. The modified form is

$$L\frac{di}{dt} + Ri = V,$$

where i is the intensity of the current in amperes and t is the time in seconds. By solving this equation, we can predict how the current will flow after the switch is closed.



Use a phase line analysis to sketch the solution curve assuming that the switch in the *RL*-circuit is closed at time $t = 0$. What happens to the current as $t \to \infty$? This value is called the *steady-state solution.*

20. A pearl in shampoo Suppose that a pearl is sinking in a thick fluid, like shampoo, subject to a frictional force opposing its fall and proportional to its velocity. Suppose that there is also a resistive buoyant force exerted by the shampoo. According to *Archimedes' principle,* the buoyant force equals the weight of the fluid displaced by the pearl. Using m for the mass of the pearl and P for the mass of the shampoo displaced by the pearl as it descends, complete the following steps.

a. Draw a schematic diagram showing the forces acting on the pearl as it sinks, as in Figure 9.16.

b. Using $v(t)$ for the pearl's velocity as a function of time t, write a differential equation modeling the velocity of the pearl as a falling body.

c. Construct a phase line displaying the signs of v' and v''.

d. Sketch typical solution curves.

e. What is the terminal velocity of the pearl?

9.5 Applications of First-Order Differential Equations

We now look at three applications of the differential equations we have been studying. The first application analyzes an object moving along a straight line while subject to a force opposing its motion. The second is a model of population growth which takes into account factors in the environment placing limits on growth, such as the availability of food or other vital resources. The last application considers a curve or curves intersecting each curve in a second family of curves *orthogonally* (that is, at right angles).

Resistance Proportional to Velocity

In some cases it is reasonable to assume that the resistance encountered by a moving object, such as a car coasting to a stop, is proportional to the object's velocity. The faster the object moves, the more its forward progress is resisted by the air through which it passes. To describe this in mathematical terms, we picture the object as a mass m moving along a coordinate line with position function s and velocity v at time t. From Newton's second law of motion, the resisting force opposing the motion is

$$\text{Force} = \text{mass} \times \text{acceleration} = m\frac{dv}{dt}.$$

We can express the assumption that the resisting force is proportional to velocity by writing

$$m\frac{dv}{dt} = -kv \qquad \text{or} \qquad \frac{dv}{dt} = -\frac{k}{m}v \qquad (k > 0).$$

This is a separable differential equation representing exponential change. The solution to the equation with initial condition $v = v_0$ at $t = 0$ is (Section 7.5)

$$v = v_0 e^{-(k/m)t}. \tag{1}$$

What can we learn from Equation (1)? For one thing, we can see that if m is something large, like the mass of a 20,000-ton ore boat in Lake Erie, it will take a long time for the velocity to approach zero (because t must be large in the exponent of the equation in order to make kt/m large enough for v to be small). We can learn even more if we integrate Equation (1) to find the position s as a function of time t.

Suppose that a body is coasting to a stop and the only force acting on it is a resistance proportional to its speed. How far will it coast? To find out, we start with Equation (1) and solve the initial value problem

$$\frac{ds}{dt} = v_0 e^{-(k/m)t}, \qquad s(0) = 0.$$

Integrating with respect to t gives

$$s = -\frac{v_0 m}{k}e^{-(k/m)t} + C.$$

Substituting $s = 0$ when $t = 0$ gives

$$0 = -\frac{v_0 m}{k} + C \qquad \text{and} \qquad C = \frac{v_0 m}{k}.$$

The body's position at time t is therefore

$$s(t) = -\frac{v_0 m}{k} e^{-(k/m)t} + \frac{v_0 m}{k} = \frac{v_0 m}{k}\left(1 - e^{-(k/m)t}\right). \tag{2}$$

To find how far the body will coast, we find the limit of $s(t)$ as $t \to \infty$. Since $-(k/m) < 0$, we know that $e^{-(k/m)t} \to 0$ as $t \to \infty$, so that

$$\begin{aligned}\lim_{t\to\infty} s(t) &= \lim_{t\to\infty} \frac{v_0 m}{k}\left(1 - e^{-(k/m)t}\right) \\ &= \frac{v_0 m}{k}(1 - 0) = \frac{v_0 m}{k}.\end{aligned}$$

Thus,

$$\text{Distance coasted} = \frac{v_0 m}{k}. \tag{3}$$

This is an ideal figure, of course. Only in mathematics can time stretch to infinity. The number $v_0 m/k$ is only an upper bound (albeit a useful one). It is true to life in one respect, at least: if m is large, it will take a lot of energy to stop the body. That is why ocean liners have to be docked by tugboats. Any liner of conventional design entering a slip with enough speed to steer would smash into the pier before it could stop.

In the English system, where weight is measured in pounds, mass is measured in **slugs**. Thus,

$$\text{Pounds} = \text{slugs} \times 32,$$

assuming the gravitational constant is 32 ft/sec^2.

EXAMPLE 1 A Coasting Ice Skater

You Try It

For a 192-lb ice skater, the k in Equation (1) is about 1/3 slug/sec and $m = 192/32 = 6$ slugs. How long will it take the skater to coast from 11 ft/sec (7.5 mph) to 1 ft/sec? How far will the skater coast before coming to a complete stop?

Solution We answer the first question by solving Equation (1) for t:

$$\begin{aligned} 11e^{-t/18} &= 1 && \text{Eq. (1) with } k = 1/3,\ m = 6,\ v_0 = 11,\ v = 1 \\ e^{-t/18} &= 1/11 \\ -t/18 &= \ln(1/11) = -\ln 11 \\ t &= 18 \ln 11 \approx 43 \text{ sec}. \end{aligned}$$

We answer the second question with Equation (3):

$$\begin{aligned}\text{Distance coasted} &= \frac{v_0 m}{k} = \frac{11 \cdot 6}{1/3} \\ &= 198 \text{ ft}.\end{aligned}$$

■

Modeling Population Growth

In Section 7.5 we modeled population growth with the Law of Exponential Change:

$$\frac{dP}{dt} = kP, \qquad P(0) = P_0$$

where P is the population at time t, $k > 0$ is a constant growth rate, and P_0 is the size of the population at time $t = 0$. In Section 7.5 we found the solution $P = P_0 e^{kt}$ to this model. However, an issue to be addressed is "how good is the model?"

To begin an assessment of the model, notice that the exponential growth differential equation says that

$$\frac{dP/dt}{P} = k \tag{4}$$

is constant. This rate is called the **relative growth rate**. Now, Table 9.4 gives the world population at midyear for the years 1980 to 1989. Taking $dt = 1$ and $dP \approx \Delta P$, we see from the table that the relative growth rate in Equation (4) is approximately the constant 0.017. Thus, based on the tabled data with $t = 0$ representing 1980, $t = 1$ representing 1981, and so forth, the world population could be modeled by

$$\text{Differential equation:} \quad \frac{dP}{dt} = 0.017P$$

$$\text{Initial condition:} \quad P(0) = 4454.$$

TABLE 9.4 World population (midyear)

Year	Population (millions)	$\Delta P/P$
1980	4454	$76/4454 \approx 0.0171$
1981	4530	$80/4530 \approx 0.0177$
1982	4610	$80/4610 \approx 0.0174$
1983	4690	$80/4690 \approx 0.0171$
1984	4770	$81/4770 \approx 0.0170$
1985	4851	$82/4851 \approx 0.0169$
1986	4933	$85/4933 \approx 0.0172$
1987	5018	$87/5018 \approx 0.0173$
1988	5105	$85/5105 \approx 0.0167$
1989	5190	

Source: U.S. Bureau of the Census (Sept., 1999): www.census.gov/ipc/www/worldpop.html.

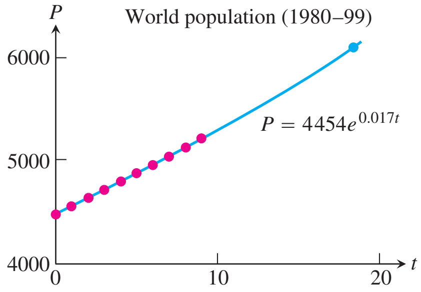


FIGURE 9.23 Notice that the value of the solution $P = 4454e^{0.017t}$ is 6152.16 when $t = 19$, which is slightly higher than the actual population in 1999.

The solution to this initial value problem gives the population function $P = 4454e^{0.017t}$. In year 1999 (so $t = 19$), the solution predicts the world population in midyear to be about 6152 million, or 6.15 billion (Figure 9.23), which is more than the actual population of 6001 million given by the U.S. Bureau of the Census (Table 9.5). Let's examine more recent data to see if there is a change in the growth rate.

Table 9.5 shows the world population for the years 1990 to 2002. From the table we see that the relative growth rate is positive but decreases as the population increases due to environmental, economic, and other factors. On average, the growth rate decreases by about 0.0003 per year over the years 1990 to 2002. That is, the graph of k in Equation (4) is closer to being a line with a negative slope $-r = -0.0003$. In Example 5 of Section 9.4 we proposed the more realistic **logistic growth model**

$$\frac{dP}{dt} = r(M - P)P, \tag{5}$$

where M is the maximum population, or **carrying capacity**, that the environment is capable of sustaining in the long run. Comparing Equation (5) with the exponential model, we see that $k = r(M - P)$ is a linearly decreasing function of the population rather than a constant. The graphical solution curves to the logistic model of Equation (5) were obtained in Section 9.4 and are displayed (again) in Figure 9.24. Notice from the graphs that if $P < M$, the population grows toward M; if $P > M$, the growth rate will be negative (as $r > 0$, $M > 0$) and the population decreasing.

TABLE 9.5 Recent world population

Year	Population (millions)	$\Delta P/P$
1990	5275	$84/5275 \approx 0.0159$
1991	5359	$84/5359 \approx 0.0157$
1992	5443	$81/5443 \approx 0.0149$
1993	5524	$81/5524 \approx 0.0147$
1994	5605	$80/5605 \approx 0.0143$
1995	5685	$79/5685 \approx 0.0139$
1996	5764	$80/5764 \approx 0.0139$
1997	5844	$79/5844 \approx 0.0135$
1998	5923	$78/5923 \approx 0.0132$
1999	6001	$78/6001 \approx 0.0130$
2000	6079	$73/6079 \approx 0.0120$
2001	6152	$76/6152 \approx 0.0124$
2002	6228	?
2003	?	

Source: U.S. Bureau of the Census (Sept., 2003): www.census.gov/ipc/www/worldpop.html.

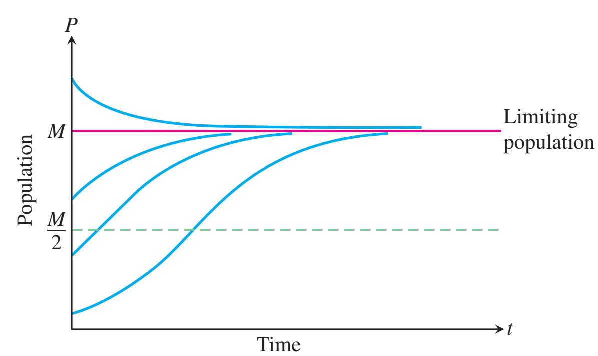


FIGURE 9.24 Solution curves to the logistic population model $dP/dt = r(M - P)P$.

EXAMPLE 2 Modeling a Bear Population

A national park is known to be capable of supporting 100 grizzly bears, but no more. Ten bears are in the park at present. We model the population with a logistic differential equation with $r = 0.001$ (although the model may not give reliable results for very small population levels).

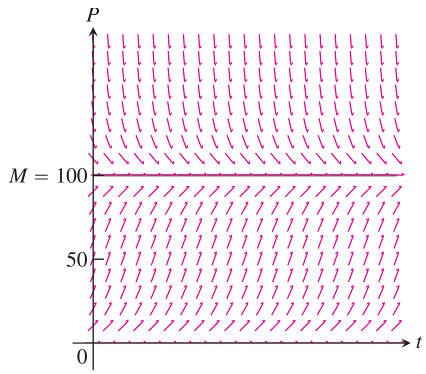


FIGURE 9.25 A slope field for the logistic differential equation $dP/dt = 0.001(100 - P)P$ (Example 2).

(a) Draw and describe a slope field for the differential equation.

(b) Use Euler's method with step size $dt = 1$ to estimate the population size in 20 years.

(c) Find a logistic growth analytic solution $P(t)$ for the population and draw its graph.

(d) When will the bear population reach 50?

Solution

(a) *Slope field.* The carrying capacity is 100, so $M = 100$. The solution we seek is a solution to the following differential equation.

$$\frac{dP}{dt} = 0.001(100 - P)P$$

Figure 9.25 shows a slope field for this differential equation. There appears to be a horizontal asymptote at $P = 100$. The solution curves fall toward this level from above and rise toward it from below.

(b) *Euler's method.* With step size $dt = 1$, $t_0 = 0$, $P(0) = 10$, and

$$\frac{dP}{dt} = f(t, P) = 0.001(100 - P)P,$$

we obtain the approximations in Table 9.6, using the iteration formula

$$P_n = P_{n-1} + 0.001(100 - P_{n-1})P_{n-1}.$$

TABLE 9.6 Euler solution of $dP/dt = 0.001(100 - P)P$, $P(0) = 10$, step size $dt = 1$

t	*P* (Euler)	*t*	*P* (Euler)
0	10		
1	10.9	11	24.3629
2	11.8712	12	26.2056
3	12.9174	13	28.1395
4	14.0423	14	30.1616
5	15.2493	15	32.2680
6	16.5417	16	34.4536
7	17.9222	17	36.7119
8	19.3933	18	39.0353
9	20.9565	19	41.4151
10	22.6130	20	43.8414

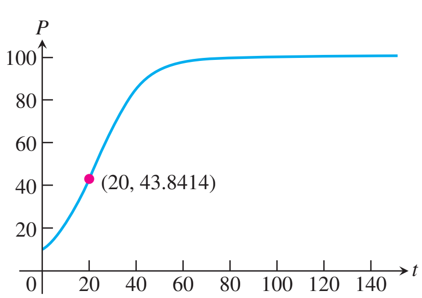


FIGURE 9.26 Euler approximations of the solution to $dP/dt = 0.001(100 - P)P$, $P(0) = 10$, step size $dt = 1$.

There are approximately 44 grizzly bears after 20 years. Figure 9.26 shows a graph of the Euler approximation over the interval $0 \le t \le 150$ with step size $dt = 1$. It looks like the lower curves we sketched in Figure 9.24.

(c) *Analytic solution*. We can assume that $t = 0$ when the bear population is 10, so $P(0) = 10$. The logistic growth model we seek is the solution to the following initial value problem.

Differential equation: $\frac{dP}{dt} = 0.001(100 - P)P$

Initial condition: $P(0) = 10$

To prepare for integration, we rewrite the differential equation in the form

$$\frac{1}{P(100 - P)} \frac{dP}{dt} = 0.001.$$

Using partial fraction decomposition on the left-hand side and multiplying both sides by 100, we get

$$\left(\frac{1}{P} + \frac{1}{100 - P}\right) \frac{dP}{dt} = 0.1$$

$$\ln|P| - \ln|100 - P| = 0.1t + C \quad \text{Integrate with respect to } t.$$

$$\ln\left|\frac{P}{100 - P}\right| = 0.1t + C$$

$$\ln\left|\frac{100 - P}{P}\right| = -0.1t - C \quad \ln\frac{a}{b} = -\ln\frac{b}{a}$$

$$\left|\frac{100 - P}{P}\right| = e^{-0.1t - C} \quad \text{Exponentiate.}$$

$$\frac{100 - P}{P} = (\pm e^{-C})e^{-0.1t}$$

$$\frac{100}{P} - 1 = Ae^{-0.1t} \quad \text{Let } A = \pm e^{-c}.$$

$$P = \frac{100}{1 + Ae^{-0.1t}}. \quad \text{Solve for } P.$$

This is the general solution to the differential equation. When $t = 0$, $P = 10$, and we obtain

$$10 = \frac{100}{1 + Ae^0}$$

$$1 + A = 10$$

$$A = 9.$$

Thus, the logistic growth model is

$$P = \frac{100}{1 + 9e^{-0.1t}}.$$

Its graph (Figure 9.27) is superimposed on the slope field from Figure 9.25.

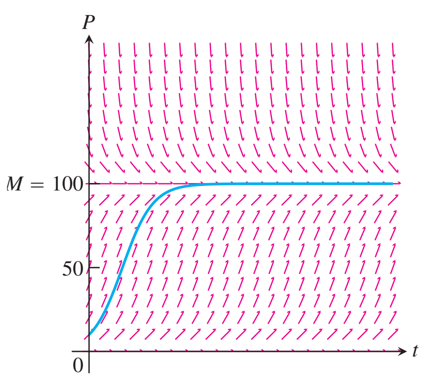


FIGURE 9.27 The graph of

$$P = \frac{100}{1 + 9e^{-0.1t}}$$

superimposed on the slope field in Figure 9.25 (Example 2).

(d) When will the bear population reach 50? For this model,

$$\begin{aligned} 50 &= \frac{100}{1 + 9e^{-0.1t}} \\ 1 + 9e^{-0.1t} &= 2 \\ e^{-0.1t} &= \frac{1}{9} \\ e^{0.1t} &= 9 \\ t &= \frac{\ln 9}{0.1} \approx 22 \text{ years}. \end{aligned}$$

The solution of the general logistic differential equation

$$\frac{dP}{dt} = r(M - P)P$$

can be obtained as in Example 2. In Exercise 10, we ask you to show that the solution is

$$P = \frac{M}{1 + Ae^{-rMt}}.$$

The value of A is determined by an appropriate initial condition.

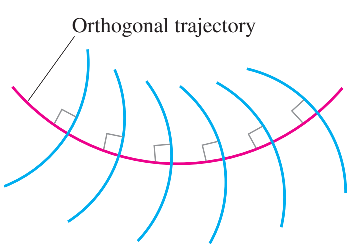


FIGURE 9.28 An orthogonal trajectory intersects the family of curves at right angles, or orthogonally.

Orthogonal Trajectories

An **orthogonal trajectory** of a family of curves is a curve that intersects each curve of the family at right angles, or *orthogonally* (Figure 9.28). For instance, each straight line through the origin is an orthogonal trajectory of the family of circles $x^2 + y^2 = a^2$, centered at the origin (Figure 9.29). Such mutually orthogonal systems of curves are of particular importance in physical problems related to electrical potential, where the curves in one family correspond to flow of electric current and those in the other family correspond to curves of constant potential. They also occur in hydrodynamics and heat-flow problems.

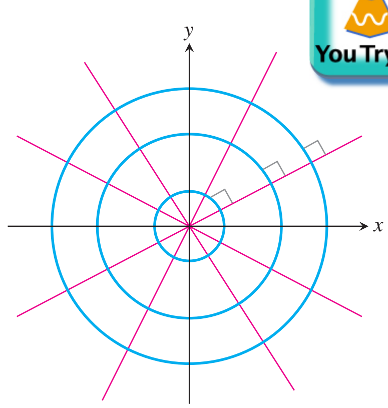


FIGURE 9.29 Every straight line through the origin is orthogonal to the family of circles centered at the origin.

EXAMPLE 3 Finding Orthogonal Trajectories

You Try It

Find the orthogonal trajectories of the family of curves $xy = a$, where $a \neq 0$ is an arbitrary constant.

Solution The curves $xy = a$ form a family of hyperbolas with asymptotes $y = \pm x$. First we find the slopes of each curve in this family, or their dy/dx values. Differentiating $xy = a$ implicitly gives

$$x\frac{dy}{dx} + y = 0 \qquad \text{or} \qquad \frac{dy}{dx} = -\frac{y}{x}.$$

Thus the slope of the tangent line at any point (x, y) on one of the hyperbolas $xy = a$ is $y' = -y/x$. On an orthogonal trajectory the slope of the tangent line at this same point must be the negative reciprocal, or x/y. Therefore, the orthogonal trajectories must satisfy the differential equation

$$\frac{dy}{dx} = \frac{x}{y}.$$

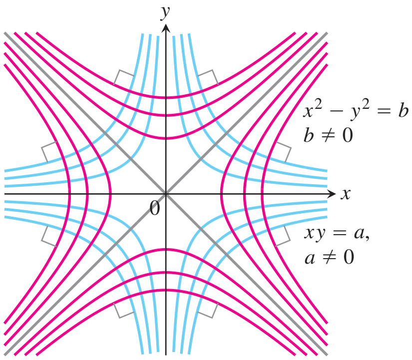


FIGURE 9.30 Each curve is orthogonal to every curve it meets in the other family (Example 3).

This differential equation is separable and we solve it as in Section 9.1:

$$y\,dy = x\,dx \qquad \text{Separate variables.}$$

$$\int y\,dy = \int x\,dx \qquad \text{Integrate both sides.}$$

$$\frac{1}{2}y^2 = \frac{1}{2}x^2 + C$$

$$y^2 - x^2 = b, \tag{6}$$

where $b = 2C$ is an arbitrary constant. The orthogonal trajectories are the family of hyperbolas given by Equation (6) and sketched in Figure 9.30. ■

EXERCISES 9.5

1. **Coasting bicycle** A 66-kg cyclist on a 7-kg bicycle starts coasting on level ground at 9 m/sec. The k in Equation (1) is about 3.9 kg/sec.
   a. About how far will the cyclist coast before reaching a complete stop?
   b. How long will it take the cyclist's speed to drop to 1 m/sec?

2. **Coasting battleship** Suppose that an Iowa class battleship has mass around 51,000 metric tons (51,000,000 kg) and a k value in Equation (1) of about 59,000 kg/sec. Assume that the ship loses power when it is moving at a speed of 9 m/sec.
   a. About how far will the ship coast before it is dead in the water?
   b. About how long will it take the ship's speed to drop to 1 m/sec?

3. The data in Table 9.7 were collected with a motion detector and a CBL™ by Valerie Sharritts, a mathematics teacher at St. Francis DeSales High School in Columbus, Ohio. The table shows the distance s (meters) coasted on in-line skates in t sec by her daughter Ashley when she was 10 years old. Find a model for Ashley's position given by the data in Table 9.7 in the form of Equation (2). Her initial velocity was $v_0 = 2.75$ m/sec, her mass $m = 39.92$ kg (she weighed 88 lb), and her total coasting distance was 4.91 m.

TABLE 9.7 Ashley Sharritts skating data

t (sec)	s (m)	t (sec)	s (m)	t (sec)	s (m)
0	0	2.24	3.05	4.48	4.77
0.16	0.31	2.40	3.22	4.64	4.82
0.32	0.57	2.56	3.38	4.80	4.84
0.48	0.80	2.72	3.52	4.96	4.86
0.64	1.05	2.88	3.67	5.12	4.88
0.80	1.28	3.04	3.82	5.28	4.89
0.96	1.50	3.20	3.96	5.44	4.90
1.12	1.72	3.36	4.08	5.60	4.90
1.28	1.93	3.52	4.18	5.76	4.91
1.44	2.09	3.68	4.31	5.92	4.90
1.60	2.30	3.84	4.41	6.08	4.91
1.76	2.53	4.00	4.52	6.24	4.90
1.92	2.73	4.16	4.63	6.40	4.91
2.08	2.89	4.32	4.69	6.56	4.91

4. **Coasting to a stop** Table 9.8 shows the distance s (meters) coasted on in-line skates in terms of time t (seconds) by Kelly Schmitzer. Find a model for her position in the form of Equation (2). Her initial velocity was $v_0 = 0.80$ m/sec, her mass $m = 49.90$ kg (110 lb), and her total coasting distance was 1.32 m.

5. **Guppy population** A 2000-gal tank can support no more than 150 guppies. Six guppies are introduced into the tank. Assume that the rate of growth of the population is

$$\frac{dP}{dt} = 0.0015(150 - P)P,$$

where time t is in weeks.

   a. Find a formula for the guppy population in terms of t.
   b. How long will it take for the guppy population to be 100? 125?

6. **Gorilla population** A certain wild animal preserve can support no more than 250 lowland gorillas. Twenty-eight gorillas were known to be in the preserve in 1970. Assume that the rate of growth of the population is

$$\frac{dP}{dt} = 0.0004(250 - P)P,$$

where time t is in years.

TABLE 9.8 Kelly Schmitzer skating data

t (sec)	*s* (m)	*t* (sec)	*s* (m)	*t* (sec)	*s* (m)
0	0	1.5	0.89	3.1	1.30
0.1	0.07	1.7	0.97	3.3	1.31
0.3	0.22	1.9	1.05	3.5	1.32
0.5	0.36	2.1	1.11	3.7	1.32
0.7	0.49	2.3	1.17	3.9	1.32
0.9	0.60	2.5	1.22	4.1	1.32
1.1	0.71	2.7	1.25	4.3	1.32
1.3	0.81	2.9	1.28	4.5	1.32

a. Find a formula for the gorilla population in terms of t.

b. How long will it take for the gorilla population to reach the carrying capacity of the preserve?

7. Pacific halibut fishery The Pacific halibut fishery has been modeled by the logistic equation

$$\frac{dy}{dt} = r(M - y)y$$

where $y(t)$ is the total weight of the halibut population in kilograms at time t (measured in years), the carrying capacity is estimated to be $M = 8 \times 10^7$ kg, and $r = 0.08875 \times 10^{-7}$ per year.

a. If $y(0) = 1.6 \times 10^7$ kg, what is the total weight of the halibut population after 1 year?

b. When will the total weight in the halibut fishery reach 4×10^7 kg?

8. Modified logistic model Suppose that the logistic differential equation in Example 2 is modified to

$$\frac{dP}{dt} = 0.001(100 - P)P - c$$

for some constant c.

a. Explain the meaning of the constant c. What values for c might be realistic for the grizzly bear population?

T **b.** Draw a direction field for the differential equation when $c = 1$. What are the equilibrium solutions (Section 9.4)?

c. Sketch several solution curves in your direction field from part (a). Describe what happens to the grizzly bear population for various initial populations.

9. Exact solutions Find the exact solutions to the following initial value problems.

a. $y' = 1 + y, \quad y(0) = 1$

b. $y' = 0.5(400 - y)y, \quad y(0) = 2$

10. Logistic differential equation Show that the solution of the differential equation

$$\frac{dP}{dt} = r(M - P)P$$

is

$$P = \frac{M}{1 + Ae^{-rMt}},$$

where A is an arbitrary constant.

11. Catastrophic solution Let k and P_0 be positive constants.

a. Solve the initial value problem?

$$\frac{dP}{dt} = kP^2, \quad P(0) = P_0$$

T **b.** Show that the graph of the solution in part (a) has a vertical asymptote at a positive value of t. What is that value of t?

12. Extinct populations Consider the population model

$$\frac{dP}{dt} = r(M - P)(P - m),$$

where $r > 0$, M is the maximum sustainable population, and m is the minimum population below which the species becomes extinct.

a. Let $m = 100$, and $M = 1200$, and assume that $m < P < M$. Show that the differential equation can be rewritten in the form

$$\left[\frac{1}{1200 - P} + \frac{1}{P - 100}\right]\frac{dP}{dt} = 1100r$$

and solve this separable equation.

b. Find the solution to part (a) that satisfies $P(0) = 300$.

c. Solve the differential equation with the restriction $m < P < M$.

Orthogonal Trajectories

In Exercises 13–18, find the orthogonal trajectories of the family of curves. Sketch several members of each family.

13. $y = mx$

14. $y = cx^2$

15. $kx^2 + y^2 = 1$

16. $2x^2 + y^2 = c^2$

17. $y = ce^{-x}$

18. $y = e^{kx}$

19. Show that the curves $2x^2 + 3y^2 = 5$ and $y^2 = x^3$ are orthogonal.

20. Find the family of solutions of the given differential equation and the family of orthogonal trajectories. Sketch both families.

a. $x\,dx + y\,dy = 0$ **b.** $x\,dy - 2y\,dx = 0$

21. Suppose a and b are positive numbers. Sketch the parabolas

$$y^2 = 4a^2 - 4ax \quad \text{and} \quad y^2 = 4b^2 + 4bx$$

in the same diagram. Show that they intersect at $\left(a - b, \pm 2\sqrt{ab}\right)$, and that each "$a$-parabola" is orthogonal to every "b-parabola."

Chapter 9 Additional and Advanced Exercises

Theory and Applications

1. **Transport through a cell membrane** Under some conditions, the result of the movement of a dissolved substance across a cell's membrane is described by the equation

$$\frac{dy}{dt} = k\frac{A}{V}(c - y).$$

In this equation, y is the concentration of the substance inside the cell and dy/dt is the rate at which y changes over time. The letters

k, A, V, and c stand for constants, k being the *permeability coefficient* (a property of the membrane), A the surface area of the membrane, V the cell's volume, and c the concentration of the substance outside the cell. The equation says that the rate at which the concentration changes within the cell is proportional to the difference between it and the outside concentration.

a. Solve the equation for $y(t)$, using y_0 to denote $y(0)$.

b. Find the steady-state concentration, $\lim_{t\to\infty} y(t)$. (Based on *Some Mathematical Models in Biology*, edited by R. M. Thrall, J. A. Mortimer, K. R. Rebman, and R. F. Baum, rev. ed., Dec. 1967, PB-202 364, pp. 101–103; distributed by N.T.I.S., U.S. Department of Commerce.)

2. Oxygen flow mixture Oxygen flows through one tube into a liter flask filled with air, and the mixture of oxygen and air (considered well stirred) escapes through another tube. Assuming that air contains 21% oxygen, what percentage of oxygen will the flask contain after 5 L have passed through the intake tube?

3. Carbon dioxide in a classroom If the average person breathes 20 times per minute, exhaling each time 100 in^3 of air containing 4% carbon dioxide, find the percentage of carbon dioxide in the air of a 10,000 ft^3 closed room 1 hour after a class of 30 students enters. Assume that the air is fresh at the start, that the ventilators admit 1000 ft^3 of fresh air per minute, and that the fresh air contains 0.04% carbon dioxide.

4. Height of a rocket If an external force F acts upon a system whose mass varies with time, Newton's law of motion is

$$\frac{d(mv)}{dt} = F + (v + u)\frac{dm}{dt}.$$

In this equation, m is the mass of the system at time t, v its velocity, and $v + u$ is the velocity of the mass that is entering (or leaving) the system at the rate dm/dt. Suppose that a rocket of initial mass m_0 starts from rest, but is driven upward by firing some of its mass directly backward at the constant rate of $dm/dt = -b$ units per second and at constant speed relative to the rocket $u = -c$. The only external force acting on the rocket is $F = -mg$ due to gravity. Under these assumptions, show that the height of the rocket above the ground at the end of t seconds (t small compared to m_0/b) is

$$y = c\left[t + \frac{m_0 - bt}{b}\ln\frac{m_0 - bt}{m_0}\right] - \frac{1}{2}gt^2.$$

5. a. Assume that $P(x)$ and $Q(x)$ are continuous over the interval $[a, b]$. Use the Fundamental Theorem of Calculus, Part 1 to show that any function y satisfying the equation

$$v(x)y = \int v(x)Q(x)\,dx + C$$

for $v(x) = e^{\int P(x)\,dx}$ is a solution to the first-order linear equation

$$\frac{dy}{dx} + P(x)y = Q(x).$$

b. If $C = y_0 v(x_0) - \int_{x_0}^{x} v(t)Q(t)\,dt$, then show that any solution y in part (a) satisfies the initial condition $y(x_0) = y_0$.

6. (*Continuation of Exercise 5.*) Assume the hypotheses of Exercise 5, and assume that $y_1(x)$ and $y_2(x)$ are both solutions to the first-order linear equation satisfying the initial condition $y(x_0) = y_0$.

a. Verify that $y(x) = y_1(x) - y_2(x)$ satisfies the initial value problem

$$y' + P(x)y = 0, \quad y(x_0) = 0.$$

b. For the integrating factor $v(x) = e^{\int P(x)\,dx}$, show that

$$\frac{d}{dx}\left(v(x)[y_1(x) - y_2(x)]\right) = 0.$$

Conclude that $v(x)[y_1(x) - y_2(x)] \equiv \text{constant}$.

c. From part (a), we have $y_1(x_0) - y_2(x_0) = 0$. Since $v(x) > 0$ for $a < x < b$, use part (b) to establish that $y_1(x) - y_2(x) \equiv 0$ on the interval (a, b). Thus $y_1(x) = y_2(x)$ for all $a < x < b$.

Chapter 9 Practice Exercises

In Exercises 1–20 solve the differential equation.

1. $\frac{dy}{dx} = \sqrt{y}\cos^2\sqrt{y}$
2. $y' = \frac{3y(x+1)^2}{y-1}$
3. $yy' = \sec y^2 \sec^2 x$
4. $y\cos^2 x\, dy + \sin x\, dx = 0$
5. $y' = xe^y\sqrt{x-2}$
6. $y' = xye^{x^2}$
7. $\sec x\, dy + x\cos^2 y\, dx = 0$
8. $2x^2\, dx - 3\sqrt{y}\csc x\, dy = 0$
9. $y' = \frac{e^y}{xy}$
10. $y' = xe^{x-y}\csc y$
11. $x(x-1)\, dy - y\, dx = 0$
12. $y' = (y^2-1)x^{-1}$
13. $2y' - y = xe^{x/2}$
14. $\frac{y'}{2} + y = e^{-x}\sin x$
15. $xy' + 2y = 1 - x^{-1}$
16. $xy' - y = 2x\ln x$
17. $(1+e^x)\, dy + (ye^x + e^{-x})\, dx = 0$
18. $e^{-x}\, dy + (e^{-x}y - 4x)\, dx = 0$
19. $(x + 3y^2)\, dy + y\, dx = 0$ (*Hint:* $d(xy) = y\, dx + x\, dy$)
20. $x\, dy + (3y - x^{-2}\cos x)\, dx = 0, \quad x > 0$

Initial Value Problems

In Exercises 21–30 solve the initial value problem.

21. $\frac{dy}{dx} = e^{-x-y-2}, \quad y(0) = -2$
22. $\frac{dy}{dx} = \frac{y\ln y}{1+x^2}, \quad y(0) = e^2$
23. $(x+1)\frac{dy}{dx} + 2y = x, \quad x > -1, \quad y(0) = 1$
24. $x\frac{dy}{dx} + 2y = x^2 + 1, \quad x > 0, \quad y(1) = 1$
25. $\frac{dy}{dx} + 3x^2y = x^2, \quad y(0) = -1$
26. $x\, dy + (y - \cos x)\, dx = 0, \quad y\left(\frac{\pi}{2}\right) = 0$
27. $x\, dy - \left(y + \sqrt{y}\right) dx = 0, \quad y(1) = 1$
28. $y^{-2}\frac{dx}{dy} = \frac{e^x}{e^{2x}+1}, \quad y(0) = 1$
29. $xy' + (x-2)y = 3x^3e^{-x}, \quad y(1) = 0$
30. $y\, dx + (3x - xy + 2)\, dy = 0, \quad y(2) = -1, \quad y < 0$

Euler's Method

In Exercises 31 and 32, use the stated method to solve the initial value problem on the given interval starting at x_0 with $dx = 0.1$.

T 31. **Euler:** $y' = y + \cos x, \quad y(0) = 0; \quad 0 \le x \le 2; \quad x_0 = 0$

T 32. **Improved Euler:** $y' = (2-y)(2x+3), \quad y(-3) = 1; \quad -3 \le x \le -1; \quad x_0 = -3$

In Exercises 33 and 34, use the stated method with $dx = 0.05$ to estimate $y(c)$ where y is the solution to the given initial value problem.

T 33. **Improved Euler:**

$$c = 3; \quad \frac{dy}{dx} = \frac{x-2y}{x+1}, \quad y(0) = 1$$

T **34. Euler:**

$$c = 4; \quad \frac{dy}{dx} = \frac{x^2 - 2y + 1}{x}, \quad y(1) = 1$$

In Exercises 35 and 36, use the stated method to solve the initial value problem graphically, starting at $x_0 = 0$ with

a. $dx = 0.1$. **b.** $dx = -0.1$.

T **35. Euler:**

$$\frac{dy}{dx} = \frac{1}{e^{x+y+2}}, \quad y(0) = -2$$

T **36. Improved Euler:**

$$\frac{dy}{dx} = -\frac{x^2 + y}{e^y + x}, \quad y(0) = 0$$

Slope Fields

In Exercises 37–40, sketch part of the equation's slope field. Then add to your sketch the solution curve that passes through the point $P(1, -1)$. Use Euler's method with $x_0 = 1$ and $dx = 0.2$ to estimate $y(2)$. Round your answers to four decimal places. Find the exact value of $y(2)$ for comparison.

37. $y' = x$ **38.** $y' = 1/x$

39. $y' = xy$ **40.** $y' = 1/y$

Autonomous Differential Equations and Phase Lines

In Exercises 41 and 42.

a. Identify the equilibrium values. Which are stable and which are unstable?

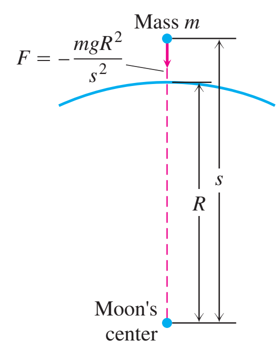
b. Construct a phase line. Identify the signs of y' and y''.

c. Sketch a representative selection of solution curves.

41. $\frac{dy}{dx} = y^2 - 1$ **42.** $\frac{dy}{dx} = y - y^2$

Applications

43. Escape velocity The gravitational attraction F exerted by an airless moon on a body of mass m at a distance s from the moon's center is given by the equation $F = -mg\,R^2 s^{-2}$, where g is the acceleration of gravity at the moon's surface and R is the moon's radius (see accompanying figure). The force F is negative because it acts in the direction of decreasing s.



a. If the body is projected vertically upward from the moon's surface with an initial velocity v_0 at time $t = 0$, use Newton's second law, $F = ma$, to show that the body's velocity at position s is given by the equation

$$v^2 = \frac{2gR^2}{s} + {v_0}^2 - 2gR.$$

Thus, the velocity remains positive as long as $v_0 \geq \sqrt{2gR}$. The velocity $v_0 = \sqrt{2gR}$ is the moon's **escape velocity**. A body projected upward with this velocity or a greater one will escape from the moon's gravitational pull.

b. Show that if $v_0 = \sqrt{2gR}$, then

$$s = R\left(1 + \frac{3v_0}{2R}t\right)^{2/3}.$$

44. Coasting to a stop Table 9.9 shows the distance s (meters) coasted on in-line skates in t sec by Johnathon Krueger. Find a model for his position in the form of Equation (2) of Section 9.5. His initial velocity was $v_0 = 0.86$ m/sec, his mass $m = 30.84$ kg (he weighed 68 lb), and his total coasting distance 0.97 m.

TABLE 9.9 Johnathon Krueger skating data

t (sec)	s (m)	t (sec)	s (m)	t (sec)	s (m)
0	0	0.93	0.61	1.86	0.93
0.13	0.08	1.06	0.68	2.00	0.94
0.27	0.19	1.20	0.74	2.13	0.95
0.40	0.28	1.33	0.79	2.26	0.96
0.53	0.36	1.46	0.83	2.39	0.96
0.67	0.45	1.60	0.87	2.53	0.97
0.80	0.53	1.73	0.90	2.66	0.97

Chapter 9 Questions to Guide Your Review

1. What is a first-order differential equation? When is a function a solution of such an equation?
2. How do you solve separable first-order differential equations?
3. What is the law of exponential change? How can it be derived from an initial value problem? What are some of the applications of the law?
4. What is the slope field of a differential equation $y' = f(x, y)$? What can we learn from such fields?
5. How do you solve linear first-order differential equations?
6. Describe Euler's method for solving the initial value problem $y' = f(x, y), y(x_0) = y_0$ numerically. Give an example. Comment on the method's accuracy. Why might you want to solve an initial value problem numerically?
7. Describe the improved Euler's method for solving the initial value problem $y' = f(x, y), y(x_0) = y_0$ numerically. How does it compare with Euler's method?
8. What is an autonomous differential equation? What are its equilibrium values? How do they differ from critical points? What is a stable equilibrium value? Unstable?
9. How do you construct the phase line for an autonomous differential equation? How does the phase line help you produce a graph which qualitatively depicts a solution to the differential equation?
10. Why is the exponential model unrealistic for predicting long-term population growth? How does the logistic model correct for the deficiency in the exponential model for population growth? What is the logistic differential equation? What is the form of its solution? Describe the graph of the logistic solution.

Chapter 9 Technology Application Projects

Mathematica/Maple Module

Drug Dosages: Are They Effective? Are They Safe?

Formulate and solve an initial value model for the absorption of blood in the bloodstream.

Mathematica/Maple Module

First-Order Differential Equations and Slope Fields

Plot slope fields and solution curves for various initial conditions to selected first-order differential equations.