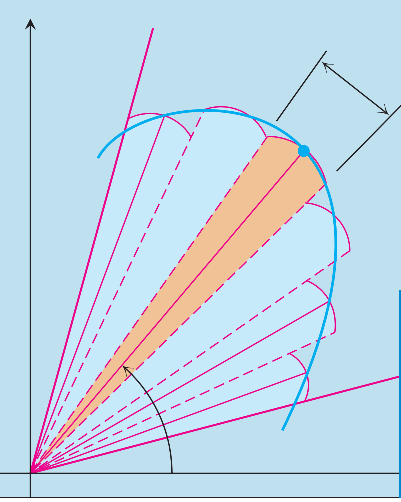


Chapter 10

Conic Sections and Polar Coordinates

OVERVIEW In this chapter we give geometric definitions of parabolas, ellipses, and hyperbolas and derive their standard equations. These curves are called *conic sections*, or *conics*, and model the paths traveled by planets, satellites, and other bodies whose motions are driven by inverse square forces. In Chapter 13 we will see that once the path of a moving body is known to be a conic, we immediately have information about the body's velocity and the force that drives it. Planetary motion is best described with the help of polar coordinates, so we also investigate curves, derivatives, and integrals in this new coordinate system.

10.1 Conic Sections and Quadratic Equations

In Chapter 1 we defined a **circle** as the set of points in a plane whose distance from some fixed center point is a constant radius value. If the center is (h, k) and the radius is a, the standard equation for the circle is $(x - h)^2 + (y - k)^2 = a^2$. It is an example of a conic section, which are the curves formed by cutting a double cone with a plane (Figure 10.1); hence the name *conic section.*

We now describe parabolas, ellipses, and hyperbolas as the graphs of quadratic equations in the coordinate plane.

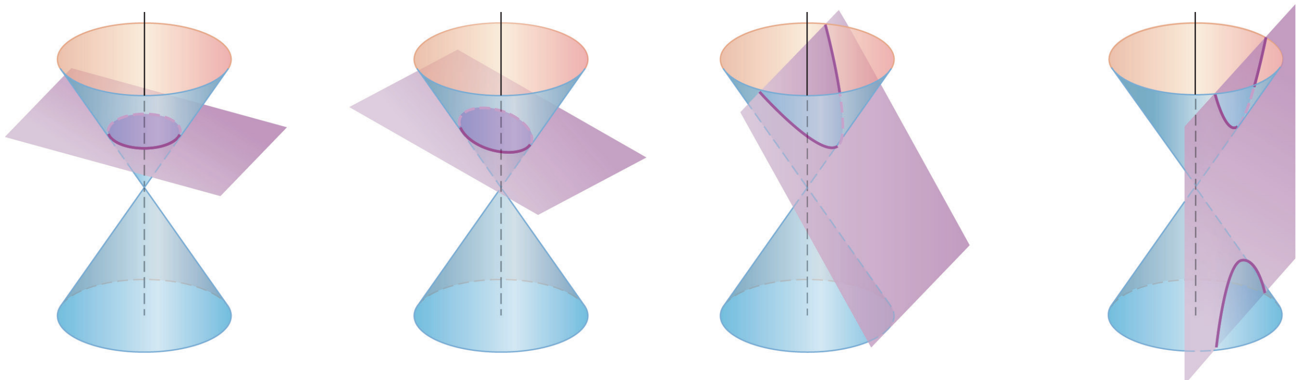
Parabolas

DEFINITIONS Parabola, Focus, Directrix

A set that consists of all the points in a plane equidistant from a given fixed point and a given fixed line in the plane is a **parabola**. The fixed point is the **focus** of the parabola. The fixed line is the **directrix**.

If the focus F lies on the directrix L, the parabola is the line through F perpendicular to L. We consider this to be a degenerate case and assume henceforth that F does not lie on L.

A parabola has its simplest equation when its focus and directrix straddle one of the coordinate axes. For example, suppose that the focus lies at the point $F(0, p)$ on the positive y-axis and that the directrix is the line $y = -p$ (Figure 10.2). In the notation of the figure,



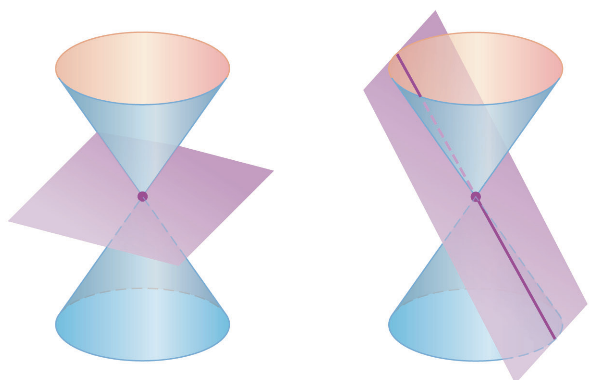
Circle: plane perpendicular to cone axis

Ellipse: plane oblique to cone axis

Parabola: plane parallel to side of cone

Hyperbola: plane cuts both halves of cone

(a)



Point: plane through cone vertex only

Single line: plane tangent to cone

Pair of intersecting lines

(b)

FIGURE 10.1 The standard conic sections (a) are the curves in which a plane cuts a double cone. Hyperbolas come in two parts, called *branches*. The point and lines obtained by passing the plane through the cone's vertex (b) are *degenerate* conic sections.

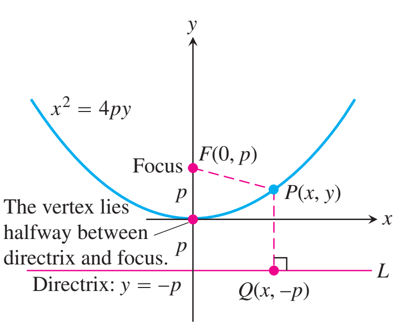


FIGURE 10.2 The standard form of the parabola $x^2 = 4py, p > 0$.

a point $P(x, y)$ lies on the parabola if and only if $PF = PQ$. From the distance formula,

$$PF = \sqrt{(x-0)^2 + (y-p)^2} = \sqrt{x^2 + (y-p)^2}$$

$$PQ = \sqrt{(x-x)^2 + (y-(-p))^2} = \sqrt{(y+p)^2}.$$

When we equate these expressions, square, and simplify, we get

$$y = \frac{x^2}{4p} \quad \text{or} \quad x^2 = 4py. \qquad \text{Standard form} \qquad (1)$$

These equations reveal the parabola's symmetry about the y-axis. We call the y-axis the **axis** of the parabola (short for "axis of symmetry").

The point where a parabola crosses its axis is the **vertex**. The vertex of the parabola $x^2 = 4py$ lies at the origin (Figure 10.2). The positive number p is the parabola's **focal length**.

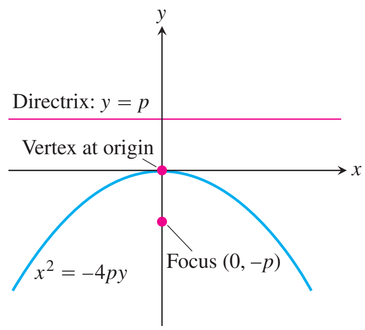


FIGURE 10.3 The parabola $x^2 = -4py, p > 0$.

If the parabola opens downward, with its focus at $(0, -p)$ and its directrix the line $y = p$, then Equations (1) become

$$y = -\frac{x^2}{4p} \qquad \text{and} \qquad x^2 = -4py$$

(Figure 10.3). We obtain similar equations for parabolas opening to the right or to the left (Figure 10.4 and Table 10.1).

TABLE 10.1 Standard-form equations for parabolas with vertices at the origin $(p > 0)$

Equation	Focus	Directrix	Axis	Opens
$x^2 = 4py$	$(0, p)$	$y = -p$	y-axis	Up
$x^2 = -4py$	$(0, -p)$	$y = p$	y-axis	Down
$y^2 = 4px$	$(p, 0)$	$x = -p$	x-axis	To the right
$y^2 = -4px$	$(-p, 0)$	$x = p$	x-axis	To the left

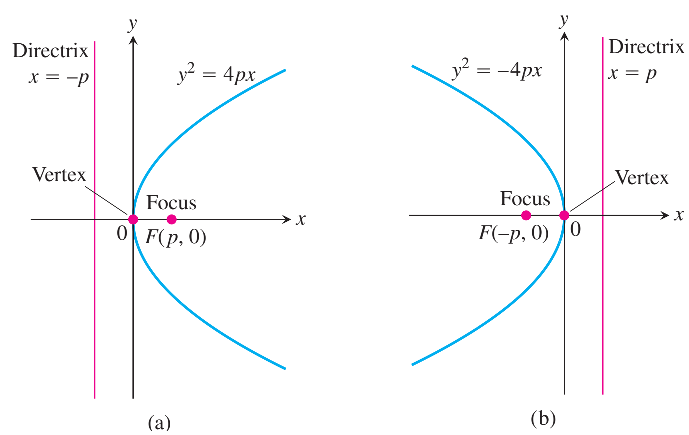


FIGURE 10.4 (a) The parabola $y^2 = 4px$. (b) The parabola $y^2 = -4px$.

You Try It

EXAMPLE 1 Find the focus and directrix of the parabola $y^2 = 10x$.

Solution We find the value of p in the standard equation $y^2 = 4px$:

$$4p = 10, \qquad \text{so} \qquad p = \frac{10}{4} = \frac{5}{2}.$$

Then we find the focus and directrix for this value of p:

$$\text{Focus:} \qquad (p, 0) = \left(\frac{5}{2}, 0\right)$$

$$\text{Directrix:} \qquad x = -p \qquad \text{or} \qquad x = -\frac{5}{2}.$$

■

The horizontal and vertical shift formulas in Section 1.5, can be applied to the equations in Table 10.1 to give equations for a variety of parabolas in other locations (see Exercises 39, 40, and 45–48).

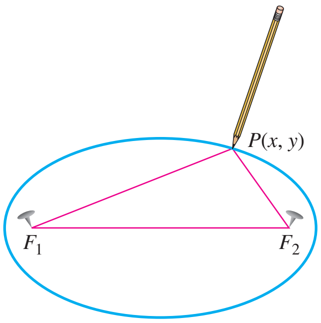


FIGURE 10.5 One way to draw an ellipse uses two tacks and a loop of string to guide the pencil.

Ellipses

> **DEFINITIONS Ellipse, Foci**
> An **ellipse** is the set of points in a plane whose distances from two fixed points in the plane have a constant sum. The two fixed points are the **foci** of the ellipse.

The quickest way to construct an ellipse uses the definition. Put a loop of string around two tacks F_1 and F_2, pull the string taut with a pencil point P, and move the pencil around to trace a closed curve (Figure 10.5). The curve is an ellipse because the sum $PF_1 + PF_2$, being the length of the loop minus the distance between the tacks, remains constant. The ellipse's foci lie at F_1 and F_2.

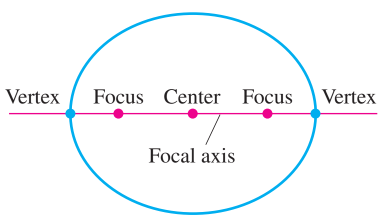


FIGURE 10.6 Points on the focal axis of an ellipse.

> **DEFINITIONS Focal Axis, Center, Vertices**
> The line through the foci of an ellipse is the ellipse's **focal axis**. The point on the axis halfway between the foci is the **center**. The points where the focal axis and ellipse cross are the ellipse's **vertices** (Figure 10.6).

If the foci are $F_1(-c, 0)$ and $F_2(c, 0)$ (Figure 10.7), and $PF_1 + PF_2$ is denoted by $2a$, then the coordinates of a point P on the ellipse satisfy the equation

$$\sqrt{(x+c)^2 + y^2} + \sqrt{(x-c)^2 + y^2} = 2a.$$

To simplify this equation, we move the second radical to the right-hand side, square, isolate the remaining radical, and square again, obtaining

$$\frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} = 1. \tag{2}$$

Since $PF_1 + PF_2$ is greater than the length F_1F_2 (triangle inequality for triangle PF_1F_2), the number $2a$ is greater than $2c$. Accordingly, $a > c$ and the number $a^2 - c^2$ in Equation (2) is positive.

The algebraic steps leading to Equation (2) can be reversed to show that every point P whose coordinates satisfy an equation of this form with $0 < c < a$ also satisfies the equation $PF_1 + PF_2 = 2a$. A point therefore lies on the ellipse if and only if its coordinates satisfy Equation (2).

If

$$b = \sqrt{a^2 - c^2}, \tag{3}$$

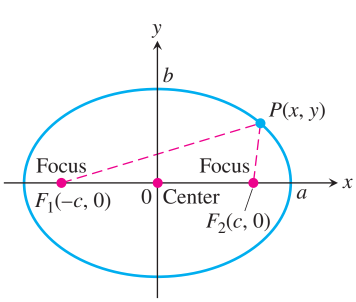


FIGURE 10.7 The ellipse defined by the equation $PF_1 + PF_2 = 2a$ is the graph of the equation $(x^2/a^2) + (y^2/b^2) = 1$, where $b^2 = a^2 - c^2$.

then $a^2 - c^2 = b^2$ and Equation (2) takes the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \tag{4}$$

Equation (4) reveals that this ellipse is symmetric with respect to the origin and both coordinate axes. It lies inside the rectangle bounded by the lines $x = \pm a$ and $y = \pm b$. It crosses the axes at the points $(\pm a, 0)$ and $(0, \pm b)$. The tangents at these points are perpendicular to the axes because

$$\frac{dy}{dx} = -\frac{b^2 x}{a^2 y}$$ Obtained from Equation (4) by implicit differentiation

is zero if $x = 0$ and infinite if $y = 0$.

The **major axis** of the ellipse in Equation (4) is the line segment of length $2a$ joining the points $(\pm a, 0)$. The **minor axis** is the line segment of length $2b$ joining the points $(0, \pm b)$. The number a itself is the **semimajor axis**, the number b the **semiminor axis**. The number c, found from Equation (3) as

$$c = \sqrt{a^2 - b^2},$$

is the **center-to-focus distance** of the ellipse.

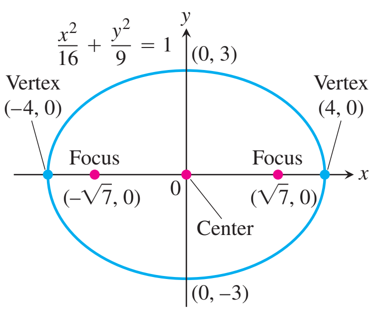


FIGURE 10.8 An ellipse with its major axis horizontal (Example 2).

EXAMPLE 2 Major Axis Horizontal

The ellipse

$$\frac{x^2}{16} + \frac{y^2}{9} = 1 \tag{5}$$

(Figure 10.8) has

Semimajor axis: $a = \sqrt{16} = 4$, Semiminor axis: $b = \sqrt{9} = 3$

Center-to-focus distance: $c = \sqrt{16 - 9} = \sqrt{7}$

Foci: $(\pm c, 0) = \left(\pm\sqrt{7}, 0\right)$

Vertices: $(\pm a, 0) = (\pm 4, 0)$

Center: $(0, 0)$. ■

EXAMPLE 3 Major Axis Vertical

The ellipse

$$\frac{x^2}{9} + \frac{y^2}{16} = 1, \tag{6}$$

obtained by interchanging x and y in Equation (5), has its major axis vertical instead of horizontal (Figure 10.9). With a^2 still equal to 16 and b^2 equal to 9, we have

Semimajor axis: $a = \sqrt{16} = 4$, Semiminor axis: $b = \sqrt{9} = 3$

Center-to-focus distance: $c = \sqrt{16 - 9} = \sqrt{7}$

Foci: $(0, \pm c) = \left(0, \pm\sqrt{7}\right)$

Vertices: $(0, \pm a) = (0, \pm 4)$

Center: $(0, 0)$. ■

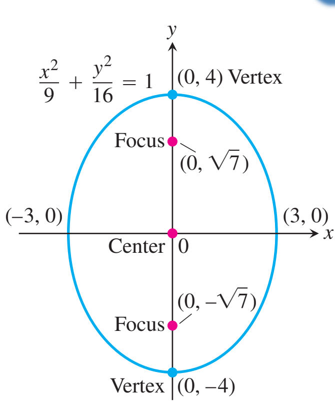


FIGURE 10.9 An ellipse with its major axis vertical (Example 3).

There is never any cause for confusion in analyzing Equations (5) and (6). We simply find the intercepts on the coordinate axes; then we know which way the major axis runs because it is the longer of the two axes. The center always lies at the origin and the foci and vertices lie on the major axis.

Standard-Form Equations for Ellipses Centered at the Origin

Foci on the x-axis: $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad (a > b)$

Center-to-focus distance: $c = \sqrt{a^2 - b^2}$

Foci: $(\pm c, 0)$

Vertices: $(\pm a, 0)$

Foci on the y-axis: $\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1 \quad (a > b)$

Center-to-focus distance: $c = \sqrt{a^2 - b^2}$

Foci: $(0, \pm c)$

Vertices: $(0, \pm a)$

In each case, a is the semimajor axis and b is the semiminor axis.

Hyperbolas

DEFINITIONS Hyperbola, Foci

A **hyperbola** is the set of points in a plane whose distances from two fixed points in the plane have a constant difference. The two fixed points are the **foci** of the hyperbola.

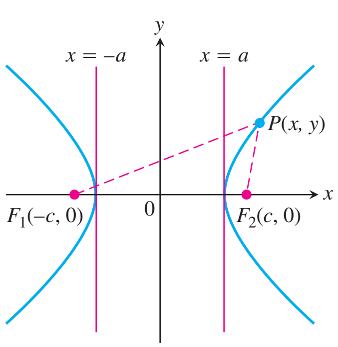


FIGURE 10.10 Hyperbolas have two branches. For points on the right-hand branch of the hyperbola shown here, $PF_1 - PF_2 = 2a$. For points on the left-hand branch, $PF_2 - PF_1 = 2a$. We then let $b = \sqrt{c^2 - a^2}$.

If the foci are $F_1(-c, 0)$ and $F_2(c, 0)$ (Figure 10.10) and the constant difference is $2a$, then a point (x, y) lies on the hyperbola if and only if

$$\sqrt{(x + c)^2 + y^2} - \sqrt{(x - c)^2 + y^2} = \pm 2a. \tag{7}$$

To simplify this equation, we move the second radical to the right-hand side, square, isolate the remaining radical, and square again, obtaining

$$\frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} = 1. \tag{8}$$

So far, this looks just like the equation for an ellipse. But now $a^2 - c^2$ is negative because $2a$, being the difference of two sides of triangle PF_1F_2, is less than $2c$, the third side.

The algebraic steps leading to Equation (8) can be reversed to show that every point P whose coordinates satisfy an equation of this form with $0 < a < c$ also satisfies Equation (7). A point therefore lies on the hyperbola if and only if its coordinates satisfy Equation (8).

If we let b denote the positive square root of $c^2 - a^2$,

$$b = \sqrt{c^2 - a^2}, \tag{9}$$

then $a^2 - c^2 = -b^2$ and Equation (8) takes the more compact form

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1. \tag{10}$$

The differences between Equation (10) and the equation for an ellipse (Equation 4) are the minus sign and the new relation

$$c^2 = a^2 + b^2. \qquad \text{From Equation (9)}$$

Like the ellipse, the hyperbola is symmetric with respect to the origin and coordinate axes. It crosses the x-axis at the points $(\pm a, 0)$. The tangents at these points are vertical because

$$\frac{dy}{dx} = \frac{b^2 x}{a^2 y} \qquad \text{Obtained from Equation (10) by implicit differentiation}$$

is infinite when $y = 0$. The hyperbola has no y-intercepts; in fact, no part of the curve lies between the lines $x = -a$ and $x = a$.

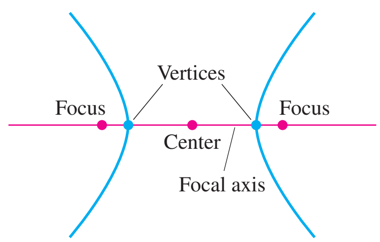


FIGURE 10.11 Points on the focal axis of a hyperbola.

DEFINITIONS Focal Axis, Center, Vertices

The line through the foci of a hyperbola is the **focal axis**. The point on the axis halfway between the foci is the hyperbola's **center**. The points where the focal axis and hyperbola cross are the **vertices** (Figure 10.11).

Asymptotes of Hyperbolas and Graphing

If we solve Equation (10) for y we obtain

$$y^2 = b^2\left(\frac{x^2}{a^2} - 1\right)$$
$$= \frac{b^2}{a^2}x^2\left(1 - \frac{a^2}{x^2}\right)$$

or, taking square roots,

$$y = \pm\frac{b}{a}x\sqrt{1 - \frac{a^2}{x^2}}.$$

As $x \to \pm\infty$, the factor $\sqrt{1 - a^2/x^2}$ approaches 1, and the factor $\pm(b/a)x$ is dominant. Thus the lines

$$y = \pm\frac{b}{a}x$$

are the two **asymptotes** of the hyperbola defined by Equation (10). The asymptotes give the guidance we need to graph hyperbolas quickly. The fastest way to find the equations of the asymptotes is to replace the 1 in Equation (10) by 0 and solve the new equation for y:

$$\underbrace{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1}_{\text{hyperbola}} \rightarrow \underbrace{\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0}_{\text{0 for 1}} \rightarrow \underbrace{y = \pm\frac{b}{a}x}_{\text{asymptotes}}.$$

Standard-Form Equations for Hyperbolas Centered at the Origin

Foci on the x-axis: $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$

Center-to-focus distance: $c = \sqrt{a^2 + b^2}$

Foci: $(\pm c, 0)$

Vertices: $(\pm a, 0)$

Asymptotes: $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 0$ or $y = \pm \dfrac{b}{a} x$

Foci on the y-axis: $\dfrac{y^2}{a^2} - \dfrac{x^2}{b^2} = 1$

Center-to-focus distance: $c = \sqrt{a^2 + b^2}$

Foci: $(0, \pm c)$

Vertices: $(0, \pm a)$

Asymptotes: $\dfrac{y^2}{a^2} - \dfrac{x^2}{b^2} = 0$ or $y = \pm \dfrac{a}{b} x$

Notice the difference in the asymptote equations (b/a in the first, a/b in the second).

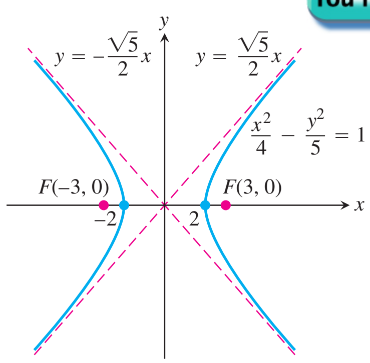


FIGURE 10.12 The hyperbola and its asymptotes in Example 4.

EXAMPLE 4 Foci on the x-axis

The equation

$$\frac{x^2}{4} - \frac{y^2}{5} = 1 \tag{11}$$

is Equation (10) with $a^2 = 4$ and $b^2 = 5$ (Figure 10.12). We have

Center-to-focus distance: $c = \sqrt{a^2 + b^2} = \sqrt{4 + 5} = 3$

Foci: $(\pm c, 0) = (\pm 3, 0)$, Vertices: $(\pm a, 0) = (\pm 2, 0)$

Center: $(0, 0)$

Asymptotes: $\dfrac{x^2}{4} - \dfrac{y^2}{5} = 0$ or $y = \pm \dfrac{\sqrt{5}}{2} x.$ ■

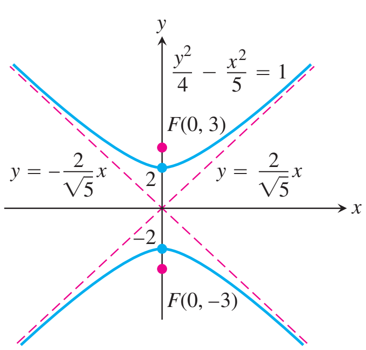


FIGURE 10.13 The hyperbola and its asymptotes in Example 5.

EXAMPLE 5 Foci on the y-axis

The hyperbola

$$\frac{y^2}{4} - \frac{x^2}{5} = 1,$$

obtained by interchanging x and y in Equation (11), has its vertices on the y-axis instead of the x-axis (Figure 10.13). With a^2 still equal to 4 and b^2 equal to 5, we have

Center-to-focus distance: $c = \sqrt{a^2 + b^2} = \sqrt{4 + 5} = 3$

Foci: $(0, \pm c) = (0, \pm 3)$, Vertices: $(0, \pm a) = (0, \pm 2)$

Center: $(0, 0)$

Asymptotes: $\dfrac{y^2}{4} - \dfrac{x^2}{5} = 0$ or $y = \pm \dfrac{2}{\sqrt{5}} x.$ ■

Reflective Properties

The chief applications of parabolas involve their use as reflectors of light and radio waves. Rays originating at a parabola's focus are reflected out of the parabola parallel to the parabola's axis (Figure 10.14 and Exercise 90). Moreover, the time any ray takes from the focus to a line parallel to the parabola's directrix (thus perpendicular to its axis) is the same for each of the rays. These properties are used by flashlight, headlight, and spotlight reflectors and by microwave broadcast antennas.

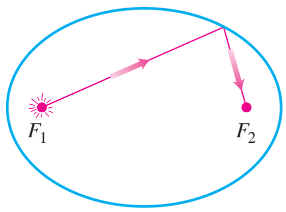


FIGURE 10.15 An elliptical mirror (shown here in profile) reflects light from one focus to the other.

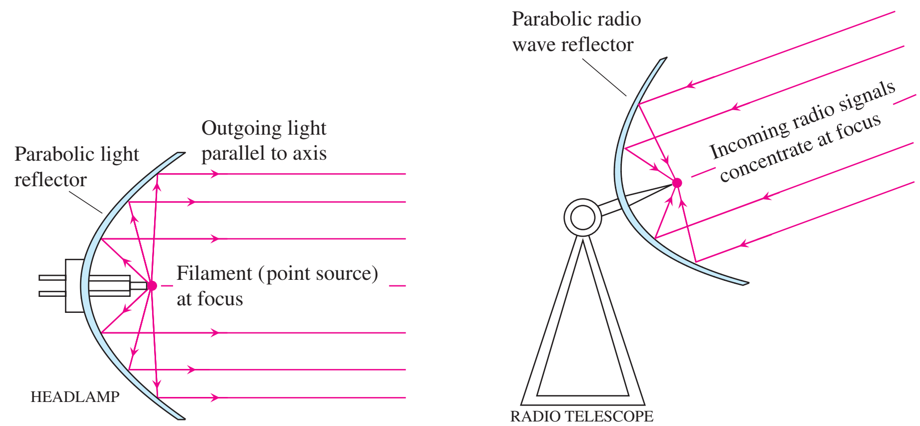


FIGURE 10.14 Parabolic reflectors can generate a beam of light parallel to the parabola's axis from a source at the focus; or they can receive rays parallel to the axis and concentrate them at the focus.

If an ellipse is revolved about its major axis to generate a surface (the surface is called an *ellipsoid*) and the interior is silvered to produce a mirror, light from one focus will be reflected to the other focus (Figure 10.15). Ellipsoids reflect sound the same way, and this property is used to construct *whispering galleries*, rooms in which a person standing at one focus can hear a whisper from the other focus. (Statuary Hall in the U.S. Capitol building is a whispering gallery.)

Light directed toward one focus of a hyperbolic mirror is reflected toward the other focus. This property of hyperbolas is combined with the reflective properties of parabolas and ellipses in designing some modern telescopes. In Figure 10.16 starlight reflects off a primary parabolic mirror toward the mirror's focus F_P. It is then reflected by a small hyperbolic mirror, whose focus is $F_H = F_P$, toward the second focus of the hyperbola, $F_E = F_H$. Since this focus is shared by an ellipse, the light is reflected by the elliptical mirror to the ellipse's second focus to be seen by an observer.

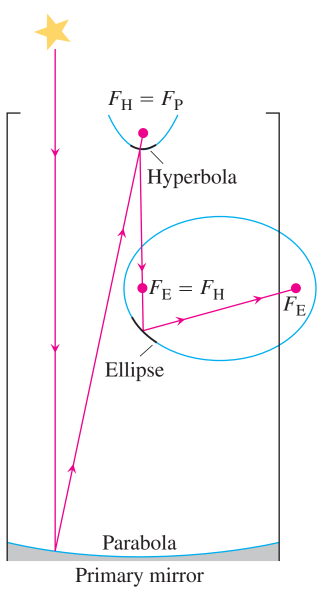


FIGURE 10.16 Schematic drawing of a reflecting telescope.

EXERCISES 10.1

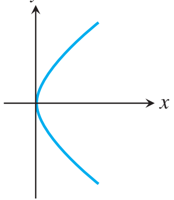
Identifying Graphs

Match the parabolas in Exercises 1–4 with the following equations:

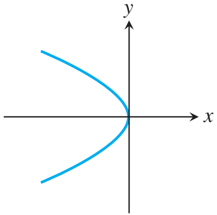
$$x^2 = 2y, \quad x^2 = -6y, \quad y^2 = 8x, \quad y^2 = -4x.$$

Then find the parabola's focus and directrix.

1.



2.



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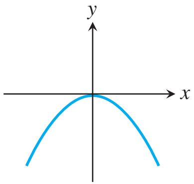

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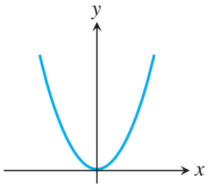


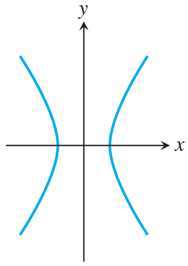
Match each conic section in Exercises 5–8 with one of these equations:

$$\frac{x^2}{4} + \frac{y^2}{9} = 1, \qquad \frac{x^2}{2} + y^2 = 1,$$

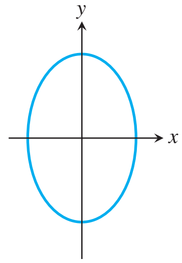
$$\frac{y^2}{4} - x^2 = 1, \qquad \frac{x^2}{4} - \frac{y^2}{9} = 1.$$

Then find the conic section's foci and vertices. If the conic section is a hyperbola, find its asymptotes as well.

5.



6.



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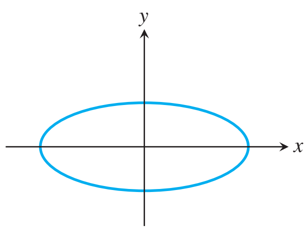
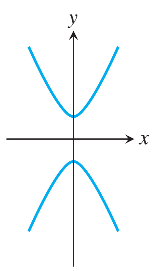


8.



Parabolas

Exercises 9–16 give equations of parabolas. Find each parabola's focus and directrix. Then sketch the parabola. Include the focus and directrix in your sketch.

9. $y^2 = 12x$ **10.** $x^2 = 6y$ **11.** $x^2 = -8y$

12. $y^2 = -2x$ **13.** $y = 4x^2$ **14.** $y = -8x^2$

15. $x = -3y^2$ **16.** $x = 2y^2$

Ellipses

Exercises 17–24 give equations for ellipses. Put each equation in standard form. Then sketch the ellipse. Include the foci in your sketch.

17. $16x^2 + 25y^2 = 400$ **18.** $7x^2 + 16y^2 = 112$

19. $2x^2 + y^2 = 2$ **20.** $2x^2 + y^2 = 4$

21. $3x^2 + 2y^2 = 6$ **22.** $9x^2 + 10y^2 = 90$

23. $6x^2 + 9y^2 = 54$ **24.** $169x^2 + 25y^2 = 4225$

Exercises 25 and 26 give information about the foci and vertices of ellipses centered at the origin of the xy-plane. In each case, find the ellipse's standard-form equation from the given information.

25. Foci: $\left(\pm\sqrt{2}, 0\right)$
Vertices: $(\pm 2, 0)$

26. Foci: $(0, \pm 4)$
Vertices: $(0, \pm 5)$

Hyperbolas

Exercises 27–34 give equations for hyperbolas. Put each equation in standard form and find the hyperbola's asymptotes. Then sketch the hyperbola. Include the asymptotes and foci in your sketch.

27. $x^2 - y^2 = 1$ **28.** $9x^2 - 16y^2 = 144$

29. $y^2 - x^2 = 8$ **30.** $y^2 - x^2 = 4$

31. $8x^2 - 2y^2 = 16$ **32.** $y^2 - 3x^2 = 3$

33. $8y^2 - 2x^2 = 16$ **34.** $64x^2 - 36y^2 = 2304$

Exercises 35–38 give information about the foci, vertices, and asymptotes of hyperbolas centered at the origin of the xy-plane. In each case, find the hyperbola's standard-form equation from the information given.

35. Foci: $\left(0, \pm\sqrt{2}\right)$
Asymptotes: $y = \pm x$

36. Foci: $(\pm 2, 0)$
Asymptotes: $y = \pm \frac{1}{\sqrt{3}} x$

37. Vertices: $(\pm 3, 0)$
Asymptotes: $y = \pm \frac{4}{3} x$

38. Vertices: $(0, \pm 2)$
Asymptotes: $y = \pm \frac{1}{2} x$

Shifting Conic Sections

39. The parabola $y^2 = 8x$ is shifted down 2 units and right 1 unit to generate the parabola $(y + 2)^2 = 8(x - 1)$.

a. Find the new parabola's vertex, focus, and directrix.

b. Plot the new vertex, focus, and directrix, and sketch in the parabola.

40. The parabola $x^2 = -4y$ is shifted left 1 unit and up 3 units to generate the parabola $(x + 1)^2 = -4(y - 3)$.

a. Find the new parabola's vertex, focus, and directrix.

b. Plot the new vertex, focus, and directrix, and sketch in the parabola.

41. The ellipse $(x^2/16) + (y^2/9) = 1$ is shifted 4 units to the right and 3 units up to generate the ellipse

$$\frac{(x - 4)^2}{16} + \frac{(y - 3)^2}{9} = 1.$$

a. Find the foci, vertices, and center of the new ellipse.

b. Plot the new foci, vertices, and center, and sketch in the new ellipse.

42. The ellipse $(x^2/9) + (y^2/25) = 1$ is shifted 3 units to the left and 2 units down to generate the ellipse

$$\frac{(x + 3)^2}{9} + \frac{(y + 2)^2}{25} = 1.$$

a. Find the foci, vertices, and center of the new ellipse.

b. Plot the new foci, vertices, and center, and sketch in the new ellipse.

43. The hyperbola $(x^2/16) - (y^2/9) = 1$ is shifted 2 units to the right to generate the hyperbola

$$\frac{(x - 2)^2}{16} - \frac{y^2}{9} = 1.$$

a. Find the center, foci, vertices, and asymptotes of the new hyperbola.

b. Plot the new center, foci, vertices, and asymptotes, and sketch in the hyperbola.

44. The hyperbola $(y^2/4) - (x^2/5) = 1$ is shifted 2 units down to generate the hyperbola

$$\frac{(y+2)^2}{4} - \frac{x^2}{5} = 1.$$

a. Find the center, foci, vertices, and asymptotes of the new hyperbola.

b. Plot the new center, foci, vertices, and asymptotes, and sketch in the hyperbola.

Exercises 45–48 give equations for parabolas and tell how many units up or down and to the right or left each parabola is to be shifted. Find an equation for the new parabola, and find the new vertex, focus, and directrix.

45. $y^2 = 4x$, left 2, down 3

46. $y^2 = -12x$, right 4, up 3

47. $x^2 = 8y$, right 1, down 7

48. $x^2 = 6y$, left 3, down 2

Exercises 49–52 give equations for ellipses and tell how many units up or down and to the right or left each ellipse is to be shifted. Find an equation for the new ellipse, and find the new foci, vertices, and center.

49. $\frac{x^2}{6} + \frac{y^2}{9} = 1$, left 2, down 1

50. $\frac{x^2}{2} + y^2 = 1$, right 3, up 4

51. $\frac{x^2}{3} + \frac{y^2}{2} = 1$, right 2, up 3

52. $\frac{x^2}{16} + \frac{y^2}{25} = 1$, left 4, down 5

Exercises 53–56 give equations for hyperbolas and tell how many units up or down and to the right or left each hyperbola is to be shifted. Find an equation for the new hyperbola, and find the new center, foci, vertices, and asymptotes.

53. $\frac{x^2}{4} - \frac{y^2}{5} = 1$, right 2, up 2

54. $\frac{x^2}{16} - \frac{y^2}{9} = 1$, left 2, down 1

55. $y^2 - x^2 = 1$, left 1, down 1

56. $\frac{y^2}{3} - x^2 = 1$, right 1, up 3

Find the center, foci, vertices, asymptotes, and radius, as appropriate, of the conic sections in Exercises 57–68.

57. $x^2 + 4x + y^2 = 12$

58. $2x^2 + 2y^2 - 28x + 12y + 114 = 0$

59. $x^2 + 2x + 4y - 3 = 0$

60. $y^2 - 4y - 8x - 12 = 0$

61. $x^2 + 5y^2 + 4x = 1$

62. $9x^2 + 6y^2 + 36y = 0$

63. $x^2 + 2y^2 - 2x - 4y = -1$

64. $4x^2 + y^2 + 8x - 2y = -1$

65. $x^2 - y^2 - 2x + 4y = 4$

66. $x^2 - y^2 + 4x - 6y = 6$

67. $2x^2 - y^2 + 6y = 3$

68. $y^2 - 4x^2 + 16x = 24$

Inequalities

Sketch the regions in the xy-plane whose coordinates satisfy the inequalities or pairs of inequalities in Exercises 69–74.

69. $9x^2 + 16y^2 \leq 144$

70. $x^2 + y^2 \geq 1$ and $4x^2 + y^2 \leq 4$

71. $x^2 + 4y^2 \geq 4$ and $4x^2 + 9y^2 \leq 36$

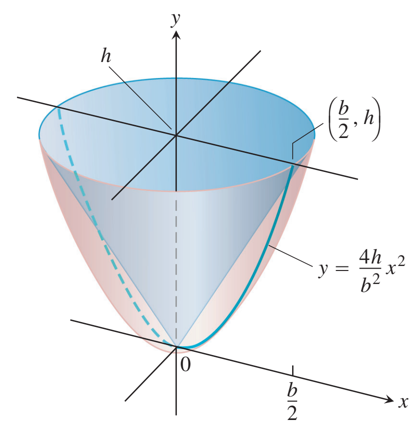
72. $(x^2 + y^2 - 4)(x^2 + 9y^2 - 9) \leq 0$

73. $4y^2 - x^2 \geq 4$

74. $|x^2 - y^2| \leq 1$

Theory and Examples

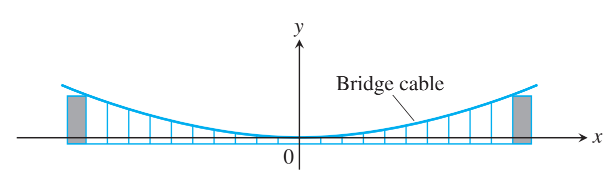
75. Archimedes' formula for the volume of a parabolic solid The region enclosed by the parabola $y = (4h/b^2)x^2$ and the line $y = h$ is revolved about the y-axis to generate the solid shown here. Show that the volume of the solid is $3/2$ the volume of the corresponding cone.



76. Suspension bridge cables hang in parabolas The suspension bridge cable shown here supports a uniform load of w pounds per horizontal foot. It can be shown that if H is the horizontal tension of the cable at the origin, then the curve of the cable satisfies the equation

$$\frac{dy}{dx} = \frac{w}{H}x.$$

Show that the cable hangs in a parabola by solving this differential equation subject to the initial condition that $y = 0$ when $x = 0$.



77. Find an equation for the circle through the points (1, 0), (0, 1), and (2, 2).

78. Find an equation for the circle through the points (2, 3), (3, 2), and $(-4, 3)$.

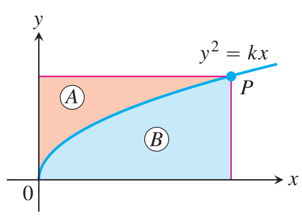
79. Find an equation for the circle centered at $(-2, 1)$ that passes through the point (1, 3). Is the point (1.1, 2.8) inside, outside, or on the circle?

80. Find equations for the tangents to the circle $(x - 2)^2 + (y - 1)^2 = 5$ at the points where the circle crosses the coordinate axes. (*Hint:* Use implicit differentiation.)

81. If lines are drawn parallel to the coordinate axes through a point P on the parabola $y^2 = kx$, $k > 0$, the parabola partitions the rectangular region bounded by these lines and the coordinate axes into two smaller regions, A and B.

a. If the two smaller regions are revolved about the y-axis, show that they generate solids whose volumes have the ratio 4:1.

b. What is the ratio of the volumes generated by revolving the regions about the x-axis?



82. Show that the tangents to the curve $y^2 = 4px$ from any point on the line $x = -p$ are perpendicular.

83. Find the dimensions of the rectangle of largest area that can be inscribed in the ellipse $x^2 + 4y^2 = 4$ with its sides parallel to the coordinate axes. What is the area of the rectangle?

84. Find the volume of the solid generated by revolving the region enclosed by the ellipse $9x^2 + 4y^2 = 36$ about the **(a)** x-axis, **(b)** y-axis.

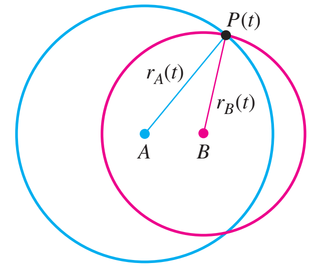
85. The "triangular" region in the first quadrant bounded by the x-axis, the line $x = 4$, and the hyperbola $9x^2 - 4y^2 = 36$ is revolved about the x-axis to generate a solid. Find the volume of the solid.

86. The region bounded on the left by the y-axis, on the right by the hyperbola $x^2 - y^2 = 1$, and above and below by the lines $y = \pm 3$ is revolved about the y-axis to generate a solid. Find the volume of the solid.

87. Find the centroid of the region that is bounded below by the x-axis and above by the ellipse $(x^2/9) + (y^2/16) = 1$.

88. The curve $y = \sqrt{x^2 + 1}, 0 \le x \le \sqrt{2}$, which is part of the upper branch of the hyperbola $y^2 - x^2 = 1$, is revolved about the x-axis to generate a surface. Find the area of the surface.

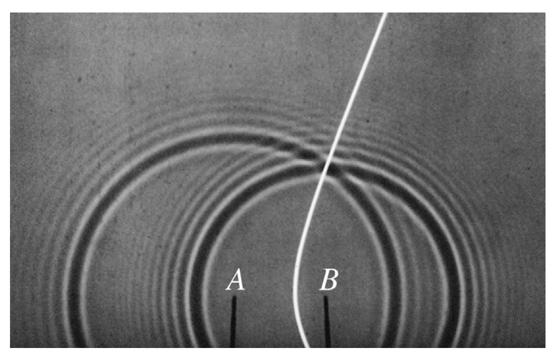
89. The circular waves in the photograph here were made by touching the surface of a ripple tank, first at A and then at B. As the waves expanded, their point of intersection appeared to trace a hyperbola. Did it really do that? To find out, we can model the waves with circles centered at A and B.



At time t, the point P is $r_A(t)$ units from A and $r_B(t)$ units from B. Since the radii of the circles increase at a constant rate, the rate at which the waves are traveling is

$$\frac{dr_A}{dt} = \frac{dr_B}{dt}.$$

Conclude from this equation that $r_A - r_B$ has a constant value, so that P must lie on a hyperbola with foci at A and B.



90. The reflective property of parabolas The figure here shows a typical point $P(x_0, y_0)$ on the parabola $y^2 = 4px$. The line L is tangent to the parabola at P. The parabola's focus lies at $F(p, 0)$. The ray L' extending from P to the right is parallel to the x-axis. We show that light from F to P will be reflected out along L' by showing that β equals α. Establish this equality by taking the following steps.

a. Show that $\tan \beta = 2p/y_0$.

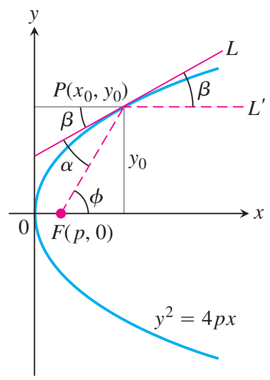
b. Show that $\tan \phi = y_0/(x_0 - p)$.

c. Use the identity

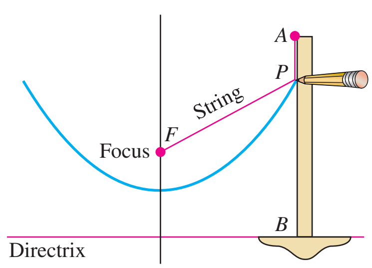
$$\tan \alpha = \frac{\tan \phi - \tan \beta}{1 + \tan \phi \tan \beta}$$

to show that $\tan \alpha = 2p/y_0$.

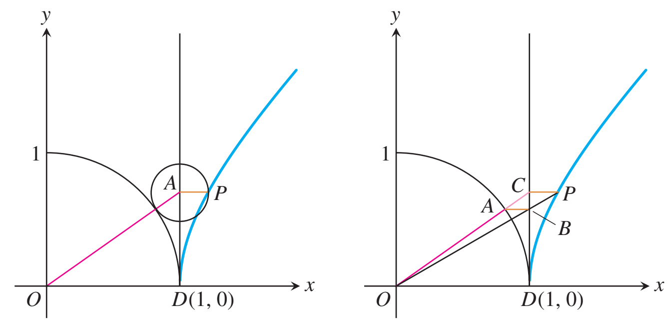
Since α and β are both acute, $\tan \beta = \tan \alpha$ implies $\beta = \alpha$.



91. How the astronomer Kepler used string to draw parabolas Kepler's method for drawing a parabola (with more modern tools) requires a string the length of a T square and a table whose edge can serve as the parabola's directrix. Pin one end of the string to the point where you want the focus to be and the other end to the upper end of the T square. Then, holding the string taut against the T square with a pencil, slide the T square along the table's edge. As the T square moves, the pencil will trace a parabola. Why?



92. Construction of a hyperbola The following diagrams appeared (unlabeled) in Ernest J. Eckert, "Constructions Without Words," *Mathematics Magazine*, Vol. 66, No. 2, Apr. 1993, p. 113. Explain the constructions by finding the coordinates of the point P.



(a) (b)

93. The width of a parabola at the focus Show that the number $4p$ is the *width* of the parabola $x^2 = 4py$ $(p > 0)$ at the focus by showing that the line $y = p$ cuts the parabola at points that are $4p$ units apart.

94. The asymptotes of $(x^2/a^2) - (y^2/b^2) = 1$ Show that the vertical distance between the line $y = (b/a)x$ and the upper half of the right-hand branch $y = (b/a)\sqrt{x^2 - a^2}$ of the hyperbola $(x^2/a^2) - (y^2/b^2) = 1$ approaches 0 by showing that

$$\lim_{x\to\infty}\left(\frac{b}{a}x - \frac{b}{a}\sqrt{x^2 - a^2}\right) = \frac{b}{a}\lim_{x\to\infty}\left(x - \sqrt{x^2 - a^2}\right) = 0.$$

Similar results hold for the remaining portions of the hyperbola and the lines $y = \pm(b/a)x$.

10.2 Classifying Conic Sections by Eccentricity

We now show how to associate with each conic section a number called the conic section's *eccentricity*. The eccentricity reveals the conic section's type (circle, ellipse, parabola, or hyperbola) and, in the case of ellipses and hyperbolas, describes the conic section's general proportions.

Eccentricity

Although the center-to-focus distance c does not appear in the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \qquad (a > b)$$

for an ellipse, we can still determine c from the equation $c = \sqrt{a^2 - b^2}$. If we fix a and vary c over the interval $0 \leq c \leq a$, the resulting ellipses will vary in shape (Figure 10.17). They are circles if $c = 0$ (so that $a = b$) and flatten as c increases. If $c = a$, the foci and vertices overlap and the ellipse degenerates into a line segment.

We use the ratio of c to a to describe the various shapes the ellipse can take. We call this ratio the ellipse's eccentricity.

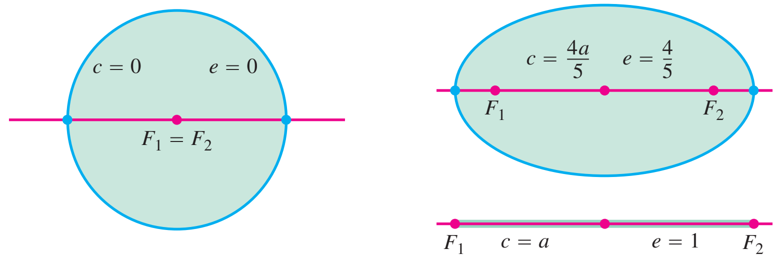


FIGURE 10.17 The ellipse changes from a circle to a line segment as c increases from 0 to a.

> **DEFINITION Eccentricity of an Ellipse**
> The **eccentricity** of the ellipse $(x^2/a^2) + (y^2/b^2) = 1$ $(a > b)$ is
> $$e = \frac{c}{a} = \frac{\sqrt{a^2 - b^2}}{a}.$$

TABLE 10.2 Eccentricities of planetary orbits

Mercury	0.21	Saturn	0.06
Venus	0.01	Uranus	0.05
Earth	0.02	Neptune	0.01
Mars	0.09	Pluto	0.25
Jupiter	0.05		

The planets in the solar system revolve around the sun in (approximate) elliptical orbits with the sun at one focus. Most of the orbits are nearly circular, as can be seen from the eccentricities in Table 10.2. Pluto has a fairly eccentric orbit, with $e = 0.25$, as does Mercury, with $e = 0.21$. Other members of the solar system have orbits that are even more eccentric. Icarus, an asteroid about 1 mile wide that revolves around the sun every 409 Earth days, has an orbital eccentricity of 0.83 (Figure 10.18).

EXAMPLE 1 Halley's Comet

The orbit of Halley's comet is an ellipse 36.18 astronomical units long by 9.12 astronomical units wide. (One *astronomical unit* [AU] is 149,597,870 km, the semimajor axis of Earth's orbit.) Its eccentricity is

$$e = \frac{\sqrt{a^2 - b^2}}{a} = \frac{\sqrt{(36.18/2)^2 - (9.12/2)^2}}{(1/2)(36.18)} = \frac{\sqrt{(18.09)^2 - (4.56)^2}}{18.09} \approx 0.97. \quad ■$$

HISTORICAL BIOGRAPHY

Edmund Halley
(1656–1742)

Whereas a parabola has one focus and one directrix, each **ellipse** has two foci and two **directrices**. These are the lines perpendicular to the major axis at distances $\pm a/e$ from the center. The parabola has the property that

$$PF = 1 \cdot PD \tag{1}$$

for any point P on it, where F is the focus and D is the point nearest P on the directrix. For an ellipse, it can be shown that the equations that replace Equation (1) are

$$PF_1 = e \cdot PD_1, \qquad PF_2 = e \cdot PD_2. \tag{2}$$

Here, e is the eccentricity, P is any point on the ellipse, F_1 and F_2 are the foci, and D_1 and D_2 are the points on the directrices nearest P (Figure 10.19).

In both Equations (2) the directrix and focus must correspond; that is, if we use the distance from P to F_1, we must also use the distance from P to the directrix at the same end of the ellipse. The directrix $x = -a/e$ corresponds to $F_1(-c, 0)$, and the directrix $x = a/e$ corresponds to $F_2(c, 0)$.

The eccentricity of a hyperbola is also $e = c/a$, only in this case c equals $\sqrt{a^2 + b^2}$ instead of $\sqrt{a^2 - b^2}$. In contrast to the eccentricity of an ellipse, the eccentricity of a hyperbola is always greater than 1.

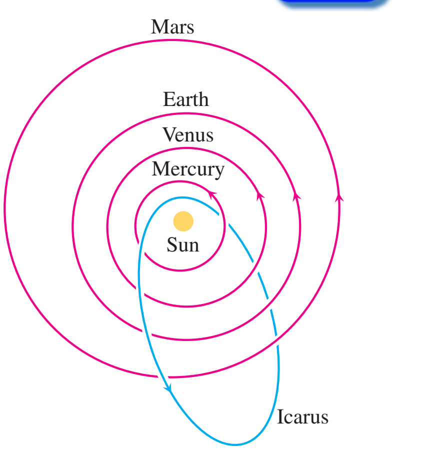


FIGURE 10.18 The orbit of the asteroid Icarus is highly eccentric. Earth's orbit is so nearly circular that its foci lie inside the sun.

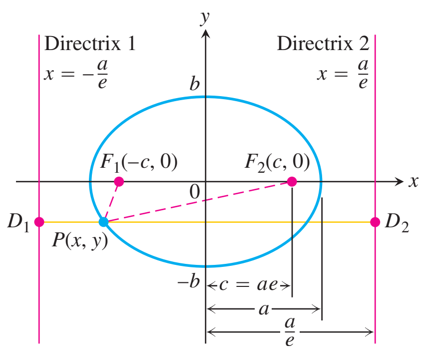


FIGURE 10.19 The foci and directrices of the ellipse $(x^2/a^2) + (y^2/b^2) = 1$. Directrix 1 corresponds to focus F_1, and directrix 2 to focus F_2.

> **DEFINITION Eccentricity of a Hyperbola**
> The **eccentricity** of the hyperbola $(x^2/a^2) - (y^2/b^2) = 1$ is
> $$e = \frac{c}{a} = \frac{\sqrt{a^2 + b^2}}{a}.$$

In both ellipse and hyperbola, the eccentricity is the ratio of the distance between the foci to the distance between the vertices (because $c/a = 2c/2a$).

$$\text{Eccentricity} = \frac{\text{distance between foci}}{\text{distance between vertices}}$$

In an ellipse, the foci are closer together than the vertices and the ratio is less than 1. In a hyperbola, the foci are farther apart than the vertices and the ratio is greater than 1.

EXAMPLE 2 Finding the Vertices of an Ellipse

Locate the vertices of an ellipse of eccentricity 0.8 whose foci lie at the points $(0, \pm 7)$.

Solution Since $e = c/a$, the vertices are the points $(0, \pm a)$ where

$$a = \frac{c}{e} = \frac{7}{0.8} = 8.75,$$

or $(0, \pm 8.75)$. ■

EXAMPLE 3 Eccentricity of a Hyperbola

Find the eccentricity of the hyperbola $9x^2 - 16y^2 = 144$.

Solution We divide both sides of the hyperbola's equation by 144 to put it in standard form, obtaining

$$\frac{9x^2}{144} - \frac{16y^2}{144} = 1 \quad \text{and} \quad \frac{x^2}{16} - \frac{y^2}{9} = 1.$$

With $a^2 = 16$ and $b^2 = 9$, we find that $c = \sqrt{a^2 + b^2} = \sqrt{16 + 9} = 5$, so

$$e = \frac{c}{a} = \frac{5}{4}.$$ ■

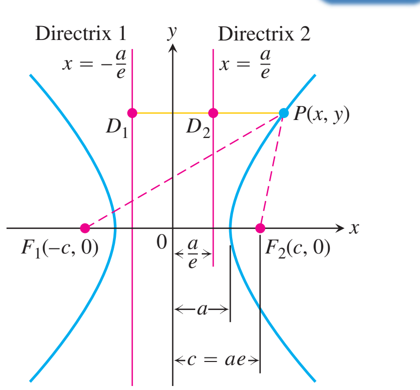


FIGURE 10.20 The foci and directrices of the hyperbola $(x^2/a^2) - (y^2/b^2) = 1$. No matter where P lies on the hyperbola, $PF_1 = e \cdot PD_1$ and $PF_2 = e \cdot PD_2$.

As with the ellipse, it can be shown that the lines $x = \pm a/e$ act as **directrices** for the **hyperbola** and that

$$PF_1 = e \cdot PD_1 \quad \text{and} \quad PF_2 = e \cdot PD_2. \tag{3}$$

Here P is any point on the hyperbola, F_1 and F_2 are the foci, and D_1 and D_2 are the points nearest P on the directrices (Figure 10.20).

To complete the picture, we define the eccentricity of a parabola to be $e = 1$. Equations (1) to (3) then have the common form $PF = e \cdot PD$.

DEFINITION Eccentricity of a Parabola

The **eccentricity** of a parabola is $e = 1$.

The "focus–directrix" equation $PF = e \cdot PD$ unites the parabola, ellipse, and hyperbola in the following way. Suppose that the distance PF of a point P from a fixed point F (the focus) is a constant multiple of its distance from a fixed line (the directrix). That is, suppose

$$PF = e \cdot PD, \tag{4}$$

where e is the constant of proportionality. Then the path traced by P is

(a) a *parabola* if $e = 1$,

(b) an *ellipse* of eccentricity e if $e < 1$, and

(c) a *hyperbola* of eccentricity e if $e > 1$.

There are no coordinates in Equation (4) and when we try to translate it into coordinate form it translates in different ways, depending on the size of e. At least, that is what happens in Cartesian coordinates. However, in polar coordinates, as we will see in Section 10.8, the equation $PF = e \cdot PD$ translates into a single equation regardless of the value of e, an equation so simple that it has been the equation of choice of astronomers and space scientists for nearly 300 years.

Given the focus and corresponding directrix of a hyperbola centered at the origin and with foci on the x-axis, we can use the dimensions shown in Figure 10.20 to find e. Knowing e, we can derive a Cartesian equation for the hyperbola from the equation $PF = e \cdot PD$, as in the next example. We can find equations for ellipses centered at the origin and with foci on the x-axis in a similar way, using the dimensions shown in Figure 10.19.

EXAMPLE 4 Cartesian Equation for a Hyperbola

Find a Cartesian equation for the hyperbola centered at the origin that has a focus at (3, 0) and the line $x = 1$ as the corresponding directrix.

Solution We first use the dimensions shown in Figure 10.20 to find the hyperbola's eccentricity. The focus is

$$(c, 0) = (3, 0) \qquad \text{so} \qquad c = 3.$$

The directrix is the line

$$x = \frac{a}{e} = 1, \qquad \text{so} \qquad a = e.$$

When combined with the equation $e = c/a$ that defines eccentricity, these results give

$$e = \frac{c}{a} = \frac{3}{e}, \qquad \text{so} \qquad e^2 = 3 \quad \text{and} \quad e = \sqrt{3}.$$

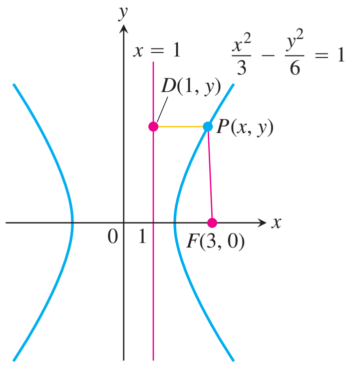


FIGURE 10.21 The hyperbola and directrix in Example 4.

Knowing e, we can now derive the equation we want from the equation $PF = e \cdot PD$. In the notation of Figure 10.21, we have

$$PF = e \cdot PD \qquad \text{Equation (4)}$$

$$\sqrt{(x-3)^2 + (y-0)^2} = \sqrt{3}\,|x-1| \qquad e = \sqrt{3}$$

$$x^2 - 6x + 9 + y^2 = 3(x^2 - 2x + 1)$$

$$2x^2 - y^2 = 6$$

$$\frac{x^2}{3} - \frac{y^2}{6} = 1.$$

■

EXERCISES 10.2

Ellipses

In Exercises 1–8, find the eccentricity of the ellipse. Then find and graph the ellipse's foci and directrices.

1. $16x^2 + 25y^2 = 400$

2. $7x^2 + 16y^2 = 112$

3. $2x^2 + y^2 = 2$

4. $2x^2 + y^2 = 4$

5. $3x^2 + 2y^2 = 6$

6. $9x^2 + 10y^2 = 90$

7. $6x^2 + 9y^2 = 54$

8. $169x^2 + 25y^2 = 4225$

Exercises 9–12 give the foci or vertices and the eccentricities of ellipses centered at the origin of the *xy*-plane. In each case, find the ellipse's standard-form equation.

9. Foci: $(0, \pm 3)$
Eccentricity: 0.5

10. Foci: $(\pm 8, 0)$
Eccentricity: 0.2

11. Vertices: $(0, \pm 70)$
Eccentricity: 0.1

12. Vertices: $(\pm 10, 0)$
Eccentricity: 0.24

Exercises 13–16 give foci and corresponding directrices of ellipses centered at the origin of the *xy*-plane. In each case, use the dimensions in Figure 10.19 to find the eccentricity of the ellipse. Then find the ellipse's standard-form equation.

13. Focus: $\left(\sqrt{5}, 0\right)$
Directrix: $x = \dfrac{9}{\sqrt{5}}$

14. Focus: $(4, 0)$
Directrix: $x = \dfrac{16}{3}$

15. Focus: $(-4, 0)$
Directrix: $x = -16$

16. Focus: $\left(-\sqrt{2}, 0\right)$
Directrix: $x = -2\sqrt{2}$

17. Draw an ellipse of eccentricity $4/5$. Explain your procedure.

18. Draw the orbit of Pluto (eccentricity 0.25) to scale. Explain your procedure.

19. The endpoints of the major and minor axes of an ellipse are $(1, 1)$, $(3, 4)$, $(1, 7)$, and $(-1, 4)$. Sketch the ellipse, give its equation in standard form, and find its foci, eccentricity, and directrices.

20. Find an equation for the ellipse of eccentricity $2/3$ that has the line $x = 9$ as a directrix and the point $(4, 0)$ as the corresponding focus.

21. What values of the constants a, b, and c make the ellipse

$$4x^2 + y^2 + ax + by + c = 0$$

lie tangent to the x-axis at the origin and pass through the point $(-1, 2)$? What is the eccentricity of the ellipse?

22. The reflective property of ellipses An ellipse is revolved about its major axis to generate an ellipsoid. The inner surface of the ellipsoid is silvered to make a mirror. Show that a ray of light emanating from one focus will be reflected to the other focus. Sound waves also follow such paths, and this property is used in constructing "whispering galleries." (*Hint:* Place the ellipse in standard position in the *xy*-plane and show that the lines from a point P on the ellipse to the two foci make congruent angles with the tangent to the ellipse at P.)

Hyperbolas

In Exercises 23–30, find the eccentricity of the hyperbola. Then find and graph the hyperbola's foci and directrices.

23. $x^2 - y^2 = 1$

24. $9x^2 - 16y^2 = 144$

25. $y^2 - x^2 = 8$

26. $y^2 - x^2 = 4$

27. $8x^2 - 2y^2 = 16$

28. $y^2 - 3x^2 = 3$

29. $8y^2 - 2x^2 = 16$

30. $64x^2 - 36y^2 = 2304$

Exercises 31–34 give the eccentricities and the vertices or foci of hyperbolas centered at the origin of the *xy*-plane. In each case, find the hyperbola's standard-form equation.

31. Eccentricity: 3
Vertices: $(0, \pm 1)$

32. Eccentricity: 2
Vertices: $(\pm 2, 0)$

33. Eccentricity: 3
Foci: $(\pm 3, 0)$

34. Eccentricity: 1.25
Foci: $(0, \pm 5)$

Exercises 35–38 give foci and corresponding directrices of hyperbolas centered at the origin of the xy-plane. In each case, find the hyperbola's eccentricity. Then find the hyperbola's standard-form equation.

35. Focus: $(4, 0)$
Directrix: $x = 2$

36. Focus: $\left(\sqrt{10}, 0\right)$
Directrix: $x = \sqrt{2}$

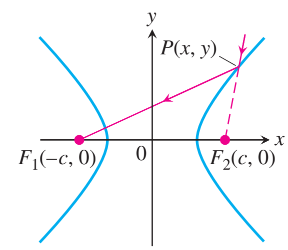
37. Focus: $(-2, 0)$
Directrix: $x = -\frac{1}{2}$

38. Focus: $(-6, 0)$
Directrix: $x = -2$

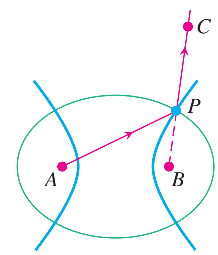
39. A hyperbola of eccentricity $3/2$ has one focus at $(1, -3)$. The corresponding directrix is the line $y = 2$. Find an equation for the hyperbola.

T **40. The effect of eccentricity on a hyperbola's shape** What happens to the graph of a hyperbola as its eccentricity increases? To find out, rewrite the equation $(x^2/a^2) - (y^2/b^2) = 1$ in terms of a and e instead of a and b. Graph the hyperbola for various values of e and describe what you find.

41. The reflective property of hyperbolas Show that a ray of light directed toward one focus of a hyperbolic mirror, as in the accompanying figure, is reflected toward the other focus. (*Hint:* Show that the tangent to the hyperbola at P bisects the angle made by segments PF_1 and PF_2.)



42. A confocal ellipse and hyperbola Show that an ellipse and a hyperbola that have the same foci A and B, as in the accompanying figure, cross at right angles at their point of intersection. (*Hint:* A ray of light from focus A that met the hyperbola at P would be reflected from the hyperbola as if it came directly from B (Exercise 41). The same ray would be reflected off the ellipse to pass through B (Exercise 22).)



10.3 Quadratic Equations and Rotations

In this section, we examine the Cartesian graph of any equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0, \tag{1}$$

in which A, B, and C are not all zero, and show that it is nearly always a conic section. The exceptions are the cases in which there is no graph at all or the graph consists of two parallel lines. It is conventional to call all graphs of Equation (1), curved or not, **quadratic curves**.

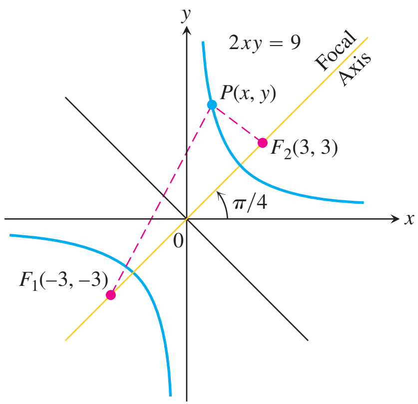


FIGURE 10.22 The focal axis of the hyperbola $2xy = 9$ makes an angle of $\pi/4$ radians with the positive x-axis.

The Cross Product Term

You may have noticed that the term Bxy did not appear in the equations for the conic sections in Section 10.1. This happened because the axes of the conic sections ran parallel to (in fact, coincided with) the coordinate axes.

To see what happens when the parallelism is absent, let us write an equation for a hyperbola with $a = 3$ and foci at $F_1(-3, -3)$ and $F_2(3, 3)$ (Figure 10.22). The equation $|PF_1 - PF_2| = 2a$ becomes $|PF_1 - PF_2| = 2(3) = 6$ and

$$\sqrt{(x + 3)^2 + (y + 3)^2} - \sqrt{(x - 3)^2 + (y - 3)^2} = \pm 6.$$

When we transpose one radical, square, solve for the radical that still appears, and square again, the equation reduces to

$$2xy = 9, \tag{2}$$

a case of Equation (1) in which the cross product term is present. The asymptotes of the hyperbola in Equation (2) are the x- and y-axes, and the focal axis makes an angle of $\pi/4$

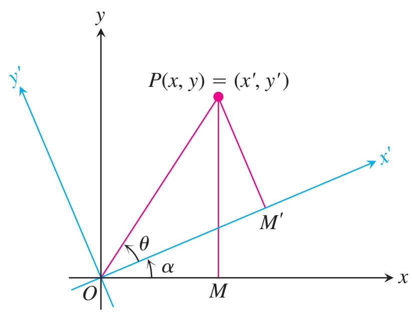


FIGURE 10.23 A counterclockwise rotation through angle α about the origin.

radians with the positive x-axis. As in this example, the cross product term is present in Equation (1) only when the axes of the conic are tilted.

To eliminate the xy-term from the equation of a conic, we rotate the coordinate axes to eliminate the "tilt" in the axes of the conic. The equations for the rotations we use are derived in the following way. In the notation of Figure 10.23, which shows a counterclockwise rotation about the origin through an angle α,

$$\begin{aligned} x &= OM = OP\cos(\theta + \alpha) = OP\cos\theta\cos\alpha - OP\sin\theta\sin\alpha \\ y &= MP = OP\sin(\theta + \alpha) = OP\cos\theta\sin\alpha + OP\sin\theta\cos\alpha. \end{aligned} \quad (3)$$

Since

$$OP\cos\theta = OM' = x'$$

and

$$OP\sin\theta = M'P = y',$$

Equations (3) reduce to the following.

Equations for Rotating Coordinate Axes

$$\begin{aligned} x &= x'\cos\alpha - y'\sin\alpha \\ y &= x'\sin\alpha + y'\cos\alpha \end{aligned} \quad (4)$$

EXAMPLE 1 Finding an Equation for a Hyperbola

The x- and y-axes are rotated through an angle of $\pi/4$ radians about the origin. Find an equation for the hyperbola $2xy = 9$ in the new coordinates.

Solution Since $\cos\pi/4 = \sin\pi/4 = 1/\sqrt{2}$, we substitute

$$x = \frac{x' - y'}{\sqrt{2}}, \qquad y = \frac{x' + y'}{\sqrt{2}}$$

from Equations (4) into the equation $2xy = 9$ and obtain

$$\begin{aligned} 2\left(\frac{x' - y'}{\sqrt{2}}\right)\left(\frac{x' + y'}{\sqrt{2}}\right) &= 9 \\ x'^2 - y'^2 &= 9 \\ \frac{x'^2}{9} - \frac{y'^2}{9} &= 1. \end{aligned}$$

See Figure 10.24. ■

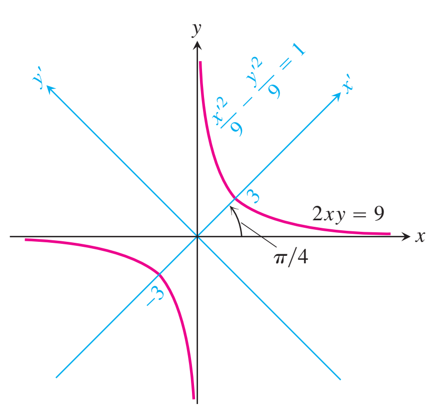


FIGURE 10.24 The hyperbola in Example 1 (x' and y' are the coordinates).

If we apply Equations (4) to the quadratic equation (1), we obtain a new quadratic equation

$$A'x'^2 + B'x'y' + C'y'^2 + D'x' + E'y' + F' = 0. \quad (5)$$

The new and old coefficients are related by the equations

$$\begin{aligned} A' &= A\cos^2\alpha + B\cos\alpha\sin\alpha + C\sin^2\alpha \\ B' &= B\cos 2\alpha + (C - A)\sin 2\alpha \\ C' &= A\sin^2\alpha - B\sin\alpha\cos\alpha + C\cos^2\alpha \\ D' &= D\cos\alpha + E\sin\alpha \\ E' &= -D\sin\alpha + E\cos\alpha \\ F' &= F. \end{aligned} \tag{6}$$

These equations show, among other things, that if we start with an equation for a curve in which the cross product term is present $(B \neq 0)$, we can find a rotation angle α that produces an equation in which no cross product term appears $(B' = 0)$. To find α, we set $B' = 0$ in the second equation in (6) and solve the resulting equation,

$$B\cos 2\alpha + (C - A)\sin 2\alpha = 0,$$

for α. In practice, this means determining α from one of the two equations

Angle of Rotation

$$\cot 2\alpha = \frac{A - C}{B} \qquad \text{or} \qquad \tan 2\alpha = \frac{B}{A - C}. \tag{7}$$

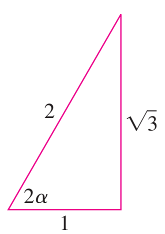


FIGURE 10.25 This triangle identifies $2\alpha = \cot^{-1}(1/\sqrt{3})$ as $\pi/3$ (Example 2).

EXAMPLE 2 Finding the Angle of Rotation

The coordinate axes are to be rotated through an angle α to produce an equation for the curve

$$2x^2 + \sqrt{3}\,xy + y^2 - 10 = 0$$

that has no cross product term. Find α and the new equation. Identify the curve.

Solution The equation $2x^2 + \sqrt{3}\,xy + y^2 - 10 = 0$ has $A = 2, B = \sqrt{3}$, and $C = 1$. We substitute these values into Equation (7) to find α:

$$\cot 2\alpha = \frac{A - C}{B} = \frac{2 - 1}{\sqrt{3}} = \frac{1}{\sqrt{3}}.$$

From the right triangle in Figure 10.25, we see that one appropriate choice of angle is $2\alpha = \pi/3$, so we take $\alpha = \pi/6$. Substituting $\alpha = \pi/6$, $A = 2$, $B = \sqrt{3}$, $C = 1$, $D = E = 0$, and $F = -10$ into Equations (6) gives

$$A' = \frac{5}{2}, \qquad B' = 0, \qquad C' = \frac{1}{2}, \qquad D' = E' = 0, \qquad F' = -10.$$

Equation (5) then gives

$$\frac{5}{2}x'^2 + \frac{1}{2}y'^2 - 10 = 0, \qquad \text{or} \qquad \frac{x'^2}{4} + \frac{y'^2}{20} = 1.$$

The curve is an ellipse with foci on the new y'-axis (Figure 10.26). ■

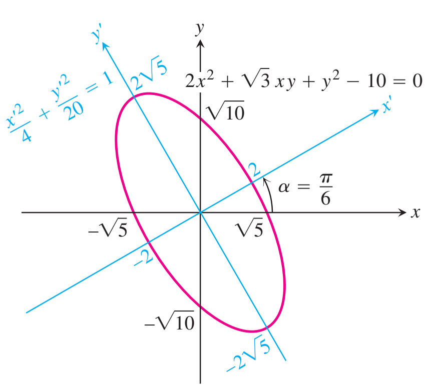


FIGURE 10.26 The conic section in Example 2.

Possible Graphs of Quadratic Equations

We now return to the graph of the general quadratic equation.

Since axes can always be rotated to eliminate the cross product term, there is no loss of generality in assuming that this has been done and that our equation has the form

$$Ax^2 + Cy^2 + Dx + Ey + F = 0. \tag{8}$$

Equation (8) represents

(a) a *circle* if $A = C \neq 0$ (special cases: the graph is a point or there is no graph at all);

(b) a *parabola* if Equation (8) is quadratic in one variable and linear in the other;

(c) an *ellipse* if A and C are both positive or both negative (special cases: circles, a single point, or no graph at all);

(d) a *hyperbola* if A and C have opposite signs (special case: a pair of intersecting lines);

(e) a *straight line* if A and C are zero and at least one of D and E is different from zero;

(f) *one or two straight lines* if the left-hand side of Equation (8) can be factored into the product of two linear factors.

See Table 10.3 for examples.

TABLE 10.3 Examples of quadratic curves $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$

	A	*B*	*C*	*D*	*E*	*F*	**Equation**	**Remarks**
Circle	1		1			−4	$x^2 + y^2 = 4$	$A = C; F < 0$
Parabola			1	−9			$y^2 = 9x$	Quadratic in y, linear in x
Ellipse	4		9			−36	$4x^2 + 9y^2 = 36$	A, C have same sign, $A \neq C; F < 0$
Hyperbola	1		−1			−1	$x^2 - y^2 = 1$	A, C have opposite signs
One line (still a conic section)	1						$x^2 = 0$	y-axis
Intersecting lines (still a conic section)		1		1	−1	−1	$xy + x - y - 1 = 0$	Factors to $(x - 1)(y + 1) = 0$, so $x = 1, y = -1$
Parallel lines (not a conic section)	1			−3		2	$x^2 - 3x + 2 = 0$	Factors to $(x - 1)(x - 2) = 0$, so $x = 1, x = 2$
Point	1		1				$x^2 + y^2 = 0$	The origin
No graph	1					1	$x^2 = -1$	No graph

The Discriminant Test

We do not need to eliminate the xy-term from the equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \tag{9}$$

to tell what kind of conic section the equation represents. If this is the only information we want, we can apply the following test instead.

As we have seen, if $B \neq 0$, then rotating the coordinate axes through an angle α that satisfies the equation

$$\cot 2\alpha = \frac{A - C}{B} \tag{10}$$

will change Equation (9) into an equivalent form

$$A'x'^2 + C'y'^2 + D'x' + E'y' + F' = 0 \tag{11}$$

without a cross product term.

Now, the graph of Equation (11) is a (real or degenerate)

(a) *parabola* if A' or $C' = 0$; that is, if $A'C' = 0$;

(b) *ellipse* if A' and C' have the same sign; that is, if $A'C' > 0$;

(c) *hyperbola* if A' and C' have opposite signs; that is, if $A'C' < 0$.

It can also be verified from Equations (6) that for any rotation of axes,

$$B^2 - 4AC = B'^2 - 4A'C'. \tag{12}$$

This means that the quantity $B^2 - 4AC$ is not changed by a rotation. But when we rotate through the angle α given by Equation (10), B' becomes zero, so

$$B^2 - 4AC = -4A'C'.$$

Since the curve is a parabola if $A'C' = 0$, an ellipse if $A'C' > 0$, and a hyperbola if $A'C' < 0$, the curve must be a parabola if $B^2 - 4AC = 0$, an ellipse if $B^2 - 4AC < 0$, and a hyperbola if $B^2 - 4AC > 0$. The number $B^2 - 4AC$ is called the **discriminant** of Equation (9).

The Discriminant Test

With the understanding that occasional degenerate cases may arise, the quadratic curve $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ is

(a) a **parabola** if $B^2 - 4AC = 0$,
(b) an **ellipse** if $B^2 - 4AC < 0$,
(c) a **hyperbola** if $B^2 - 4AC > 0$.

EXAMPLE 3 Applying the Discriminant Test

(a) $3x^2 - 6xy + 3y^2 + 2x - 7 = 0$ represents a parabola because

$$B^2 - 4AC = (-6)^2 - 4 \cdot 3 \cdot 3 = 36 - 36 = 0.$$

(b) $x^2 + xy + y^2 - 1 = 0$ represents an ellipse because

$$B^2 - 4AC = (1)^2 - 4 \cdot 1 \cdot 1 = -3 < 0.$$

(c) $xy - y^2 - 5y + 1 = 0$ represents a hyperbola because

$$B^2 - 4AC = (1)^2 - 4(0)(-1) = 1 > 0.$$ ■

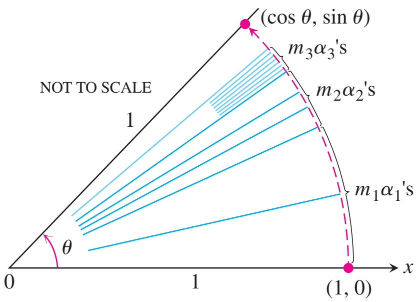


FIGURE 10.27 To calculate the sine and cosine of an angle θ between 0 and 2π, the calculator rotates the point (1, 0) to an appropriate location on the unit circle and displays the resulting coordinates.

USING TECHNOLOGY How Calculators Use Rotations to Evaluate Sines and Cosines

Some calculators use rotations to calculate sines and cosines of arbitrary angles. The procedure goes something like this: The calculator has, stored,

1. ten angles or so, say

$$\alpha_1 = \sin^{-1}(10^{-1}), \qquad \alpha_2 = \sin^{-1}(10^{-2}), \qquad \dots, \qquad \alpha_{10} = \sin^{-1}(10^{-10}),$$

and

2. twenty numbers, the sines and cosines of the angles $\alpha_1, \alpha_2, \dots, \alpha_{10}$.

To calculate the sine and cosine of an arbitrary angle θ, we enter θ (in radians) into the calculator. The calculator subtracts or adds multiples of 2π to θ to replace θ by the angle between 0 and 2π that has the same sine and cosine as θ (we continue to call the angle θ). The calculator then "writes" θ as a sum of multiples of α_1 (as many as possible without overshooting) plus multiples of α_2 (again, as many as possible), and so on, working its way to α_{10}. This gives

$$\theta \approx m_1\alpha_1 + m_2\alpha_2 + \cdots + m_{10}\alpha_{10}.$$

The calculator then rotates the point (1, 0) through m_1 copies of α_1 (through α_1, m_1 times in succession), plus m_2 copies of α_2, and so on, finishing off with m_{10} copies of α_{10} (Figure 10.27). The coordinates of the final position of (1, 0) on the unit circle are the values the calculator gives for $(\cos\theta, \sin\theta)$.

EXERCISES 10.3

Using the Discriminant

Use the discriminant $B^2 - 4AC$ to decide whether the equations in Exercises 1–16 represent parabolas, ellipses, or hyperbolas.

Exercises

1. $x^2 - 3xy + y^2 - x = 0$

2. $3x^2 - 18xy + 27y^2 - 5x + 7y = -4$

3. $3x^2 - 7xy + \sqrt{17}\, y^2 = 1$

4. $2x^2 - \sqrt{15}\, xy + 2y^2 + x + y = 0$

5. $x^2 + 2xy + y^2 + 2x - y + 2 = 0$

6. $2x^2 - y^2 + 4xy - 2x + 3y = 6$

7. $x^2 + 4xy + 4y^2 - 3x = 6$

8. $x^2 + y^2 + 3x - 2y = 10$

9. $xy + y^2 - 3x = 5$

10. $3x^2 + 6xy + 3y^2 - 4x + 5y = 12$

11. $3x^2 - 5xy + 2y^2 - 7x - 14y = -1$

12. $2x^2 - 4.9xy + 3y^2 - 4x = 7$

13. $x^2 - 3xy + 3y^2 + 6y = 7$

14. $25x^2 + 21xy + 4y^2 - 350x = 0$

15. $6x^2 + 3xy + 2y^2 + 17y + 2 = 0$

16. $3x^2 + 12xy + 12y^2 + 435x - 9y + 72 = 0$

Rotating Coordinate Axes

In Exercises 17–26, rotate the coordinate axes to change the given equation into an equation that has no cross product (xy) term. Then identify the graph of the equation. (The new equations will vary with the size and direction of the rotation you use.)

Exercises

17. $xy = 2$

18. $x^2 + xy + y^2 = 1$

19. $3x^2 + 2\sqrt{3}\, xy + y^2 - 8x + 8\sqrt{3}\, y = 0$

20. $x^2 - \sqrt{3}\, xy + 2y^2 = 1$

21. $x^2 - 2xy + y^2 = 2$

22. $3x^2 - 2\sqrt{3}\, xy + y^2 = 1$

23. $\sqrt{2}\, x^2 + 2\sqrt{2}\, xy + \sqrt{2}\, y^2 - 8x + 8y = 0$

24. $xy - y - x + 1 = 0$

25. $3x^2 + 2xy + 3y^2 = 19$

26. $3x^2 + 4\sqrt{3}\, xy - y^2 = 7$

27. Find the sine and cosine of an angle in Quadrant I through which the coordinate axes can be rotated to eliminate the cross product term from the equation

$$14x^2 + 16xy + 2y^2 - 10x + 26{,}370y - 17 = 0.$$

Do not carry out the rotation.

28. Find the sine and cosine of an angle in Quadrant II through which the coordinate axes can be rotated to eliminate the cross product term from the equation

$$4x^2 - 4xy + y^2 - 8\sqrt{5}\,x - 16\sqrt{5}\,y = 0.$$

Do not carry out the rotation.

T The conic sections in Exercises 17–26 were chosen to have rotation angles that were "nice" in the sense that once we knew $\cot 2\alpha$ or $\tan 2\alpha$ we could identify 2α and find $\sin\alpha$ and $\cos\alpha$ from familiar triangles.

In Exercises 29–34, use a calculator to find an angle α through which the coordinate axes can be rotated to change the given equation into a quadratic equation that has no cross product term. Then find $\sin\alpha$ and $\cos\alpha$ to two decimal places and use Equations (6) to find the coefficients of the new equation to the nearest decimal place. In each case, say whether the conic section is an ellipse, a hyperbola, or a parabola.

29. $x^2 - xy + 3y^2 + x - y - 3 = 0$

30. $2x^2 + xy - 3y^2 + 3x - 7 = 0$

31. $x^2 - 4xy + 4y^2 - 5 = 0$

32. $2x^2 - 12xy + 18y^2 - 49 = 0$

33. $3x^2 + 5xy + 2y^2 - 8y - 1 = 0$

34. $2x^2 + 7xy + 9y^2 + 20x - 86 = 0$

Theory and Examples

35. What effect does a 90° rotation about the origin have on the equations of the following conic sections? Give the new equation in each case.

a. The ellipse $(x^2/a^2) + (y^2/b^2) = 1 \quad (a > b)$

b. The hyperbola $(x^2/a^2) = (y^2/b^2) = 1$

c. The circle $x^2 + y^2 = a^2$

d. The line $y = mx$ **e.** The line $y = mx + b$

36. What effect does a 180° rotation about the origin have on the equations of the following conic sections? Give the new equation in each case.

a. The ellipse $(x^2/a^2) + (y^2/b^2) = 1 \quad (a > b)$

b. The hyperbola $(x^2/a^2) = (y^2/b^2) = 1$

c. The circle $x^2 + y^2 = a^2$

d. The line $y = mx$ **e.** The line $y = mx + b$

37. The Hyperbola $xy = a$ The hyperbola $xy = 1$ is one of many hyperbolas of the form $xy = a$ that appear in science and mathematics.

a. Rotate the coordinate axes through an angle of 45° to change the equation $xy = 1$ into an equation with no xy-term. What is the new equation?

b. Do the same for the equation $xy = a$.

38. Find the eccentricity of the hyperbola $xy = 2$.

39. Can anything be said about the graph of the equation $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ if $AC < 0$? Give reasons for your answer.

40. Degenerate conics Does any nondegenerate conic section $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$ have all of the following properties?

a. It is symmetric with respect to the origin.

b. It passes through the point $(1, 0)$.

c. It is tangent to the line $y = 1$ at the point $(-2, 1)$.

Give reasons for your answer.

41. Show that the equation $x^2 + y^2 = a^2$ becomes $x'^2 + y'^2 = a^2$ for every choice of the angle α in the rotation equations (4).

42. Show that rotating the axes through an angle of $\pi/4$ radians will eliminate the xy-term from Equation (1) whenever $A = C$.

43. a. Decide whether the equation

$$x^2 + 4xy + 4y^2 + 6x + 12y + 9 = 0$$

represents an ellipse, a parabola, or a hyperbola.

b. Show that the graph of the equation in part (a) is the line $2y = -x - 3$.

44. a. Decide whether the conic section with equation

$$9x^2 + 6xy + y^2 - 12x - 4y + 4 = 0$$

represents a parabola, an ellipse, or a hyperbola.

b. Show that the graph of the equation in part (a) is the line $y = -3x + 2$.

45. a. What kind of conic section is the curve $xy + 2x - y = 0$?

b. Solve the equation $xy + 2x - y = 0$ for y and sketch the curve as the graph of a rational function of x.

c. Find equations for the lines parallel to the line $y = -2x$ that are normal to the curve. Add the lines to your sketch.

46. Prove or find counterexamples to the following statements about the graph of $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$.

a. If $AC > 0$, the graph is an ellipse.

b. If $AC > 0$, the graph is a hyperbola.

c. If $AC < 0$, the graph is a hyperbola.

47. A nice area formula for ellipses When $B^2 - 4AC$ is negative, the equation

$$Ax^2 + Bxy + Cy^2 = 1$$

represents an ellipse. If the ellipse's semi-axes are a and b, its area is πab (a standard formula). Show that the area is also given by the formula $2\pi/\sqrt{4AC - B^2}$. (*Hint:* Rotate the coordinate axes to eliminate the xy-term and apply Equation (12) to the new equation.)

48. Other invariants We describe the fact that $B'^2 - 4A'C'$ equals $B^2 - 4AC$ after a rotation about the origin by saying that the discriminant of a quadratic equation is an *invariant* of the equation.

Use Equations (6) to show that the numbers **(a)** $A + C$ and **(b)** $D^2 + E^2$ are also invariants, in the sense that

$$A' + C' = A + C \quad \text{and} \quad D'^2 + E'^2 = D^2 + E^2.$$

We can use these equalities to check against numerical errors when we rotate axes.

49. A proof that $B'^2 - 4A'C' = B^2 - 4AC$ Use Equations (6) to show that $B'^2 - 4A'C' = B^2 - 4AC$ for any rotation of axes about the origin.

10.4 Conics and Parametric Equations; The Cycloid

Curves in the Cartesian plane defined by parametric equations, and the calculation of their derivatives, were introduced in Section 3.5. There we studied parametrizations of lines, circles, and ellipses. In this section we discuss parametrization of parabolas, hyperbolas, cycloids, brachistocrones, and tautocrones.

Parabolas and Hyperbolas

In Section 3.5 we used the parametrization

$$x = \sqrt{t}, \qquad y = t, \qquad t > 0$$

to describe the motion of a particle moving along the right branch of the parabola $y = x^2$. In the following example we obtain a parametrization of the entire parabola, not just its right branch.

You Try It

EXAMPLE 1 An Entire Parabola

The position $P(x, y)$ of a particle moving in the xy-plane is given by the equations and parameter interval

$$x = t, \qquad y = t^2, \qquad -\infty < t < \infty .$$

Identify the particle's path and describe the motion.

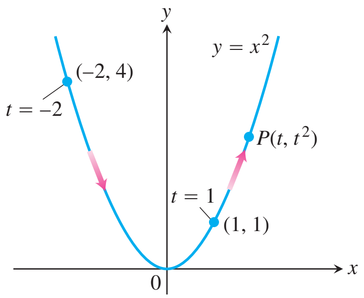


FIGURE 10.28 The path defined by $x = t, y = t^2, -\infty < t < \infty$ is the entire parabola $y = x^2$ (Example 1).

Solution We identify the path by eliminating t between the equations $x = t$ and $y = t^2$, obtaining

$$y = (t)^2 = x^2.$$

The particle's position coordinates satisfy the equation $y = x^2$, so the particle moves along this curve.

In contrast to Example 10 in Section 3.5, the particle now traverses the entire parabola. As t increases from $-\infty$ to ∞, the particle comes down the left-hand side, passes through the origin, and moves up the right-hand side (Figure 10.28). ■

As Example 1 illustrates, any curve $y = f(x)$ has the parametrization $x = t$, $y = f(t)$. This is so simple we usually do not use it, but the point of view is occasionally helpful.

You Try It

EXAMPLE 2 A Parametrization of the Right-hand Branch of the Hyperbola $x^2 - y^2 = 1$

Describe the motion of the particle whose position $P(x, y)$ at time t is given by

$$x = \sec t, \qquad y = \tan t, \qquad -\frac{\pi}{2} < t < \frac{\pi}{2}.$$

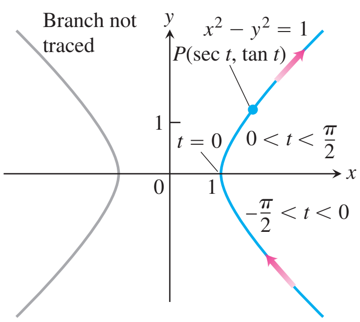


FIGURE 10.29 The equations $x = \sec t$, $y = \tan t$ and interval $-\pi/2 < t < \pi/2$ describe the right-hand branch of the hyperbola $x^2 - y^2 = 1$ (Example 2).

Solution We find a Cartesian equation for the coordinates of P by eliminating t between the equations

$$\sec t = x, \qquad \tan t = y.$$

We accomplish this with the identity $\sec^2 t - \tan^2 t = 1$, which yields

$$x^2 - y^2 = 1.$$

Since the particle's coordinates (x, y) satisfy the equation $x^2 - y^2 = 1$, the motion takes place somewhere on this hyperbola. As t runs between $-\pi/2$ and $\pi/2$, $x = \sec t$ remains positive and $y = \tan t$ runs between $-\infty$ and ∞, so P traverses the hyperbola's right-hand branch. It comes in along the branch's lower half as $t \to 0^-$, reaches $(1, 0)$ at $t = 0$, and moves out into the first quadrant as t increases toward $\pi/2$ (Figure 10.29). ■

Cycloids

The problem with a pendulum clock whose bob swings in a circular arc is that the frequency of the swing depends on the amplitude of the swing. The wider the swing, the longer it takes the bob to return to center (its lowest position).

This does not happen if the bob can be made to swing in a *cycloid*. In 1673, Christiaan Huygens designed a pendulum clock whose bob would swing in a cycloid, a curve we define in Example 3. He hung the bob from a fine wire constrained by guards that caused it to draw up as it swung away from center (Figure 10.30).

HISTORICAL BIOGRAPHY

Christiaan Huygens (1629–1695)

Biography

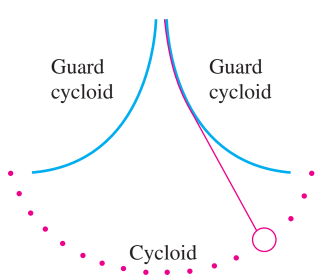


FIGURE 10.30 In Huygens' pendulum clock, the bob swings in a cycloid, so the frequency is independent of the amplitude.

EXAMPLE 3 Parametrizing a Cycloid

A wheel of radius a rolls along a horizontal straight line. Find parametric equations for the path traced by a point P on the wheel's circumference. The path is called a **cycloid**.

Solution We take the line to be the x-axis, mark a point P on the wheel, start the wheel with P at the origin, and roll the wheel to the right. As parameter, we use the angle t through which the wheel turns, measured in radians. Figure 10.31 shows the wheel a short while later, when its base lies at units from the origin. The wheel's center C lies at (at, a) and the coordinates of P are

$$x = at + a\cos\theta, \qquad y = a + a\sin\theta.$$

To express θ in terms of t, we observe that $t + \theta = 3\pi/2$ in the figure, so that

$$\theta = \frac{3\pi}{2} - t.$$

This makes

$$\cos\theta = \cos\left(\frac{3\pi}{2} - t\right) = -\sin t, \qquad \sin\theta = \sin\left(\frac{3\pi}{2} - t\right) = -\cos t.$$

The equations we seek are

$$x = at - a\sin t, \qquad y = a - a\cos t.$$

These are usually written with the a factored out:

$$x = a(t - \sin t), \qquad y = a(1 - \cos t). \tag{1}$$

Figure 10.32 shows the first arch of the cycloid and part of the next. ■

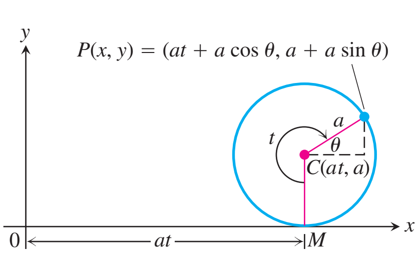


FIGURE 10.31 The position of $P(x, y)$ on the rolling wheel at angle t (Example 3).

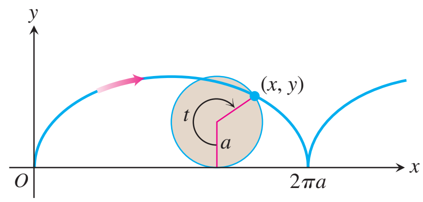


FIGURE 10.32 The cycloid $x = a(t - \sin t), y = a(1 - \cos t)$, for $t \geq 0$.

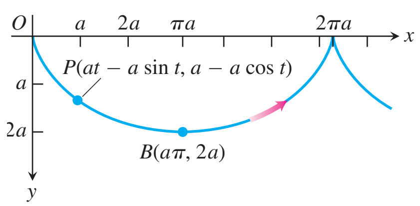


FIGURE 10.33 To study motion along an upside-down cycloid under the influence of gravity, we turn Figure 10.32 upside down. This points the y-axis in the direction of the gravitational force and makes the downward y-coordinates positive. The equations and parameter interval for the cycloid are still

$$x = a(t - \sin t),$$
$$y = a(1 - \cos t), \quad t \geq 0.$$

The arrow shows the direction of increasing t.

Brachistochrones and Tautochrones

If we turn Figure 10.32 upside down, Equations (1) still apply and the resulting curve (Figure 10.33) has two interesting physical properties. The first relates to the origin O and the point B at the bottom of the first arch. Among all smooth curves joining these points, the cycloid is the curve along which a frictionless bead, subject only to the force of gravity, will slide from O to B the fastest. This makes the cycloid a **brachistochrone** ("brah-*kiss*-toe-krone"), or shortest time curve for these points. The second property is that even if you start the bead partway down the curve toward B, it will still take the bead the same amount of time to reach B. This makes the cycloid a **tautochrone** ("*taw*-toe-krone"), or same-time curve for O and B.

Are there any other brachistochrones joining O and B, or is the cycloid the only one? We can formulate this as a mathematical question in the following way. At the start, the kinetic energy of the bead is zero, since its velocity is zero. The work done by gravity in moving the bead from (0, 0) to any other point (x, y) in the plane is mgy, and this must equal the change in kinetic energy. That is,

$$mgy = \frac{1}{2}mv^2 - \frac{1}{2}m(0)^2.$$

Thus, the velocity of the bead when it reaches (x, y) has to be

$$v = \sqrt{2gy}.$$

That is,

$$\frac{ds}{dt} = \sqrt{2gy}$$

ds is the arc length differential along the bead's path.

or

$$dt = \frac{ds}{\sqrt{2gy}} = \frac{\sqrt{1 + (dy/dx)^2}\,dx}{\sqrt{2gy}}.$$

The time T_f it takes the bead to slide along a particular path $y = f(x)$ from O to $B(a\pi, 2a)$ is

$$T_f = \int_{x=0}^{x=a\pi} \sqrt{\frac{1 + (dy/dx)^2}{2gy}}\,dx. \tag{2}$$

What curves $y = f(x)$, if any, minimize the value of this integral?

At first sight, we might guess that the straight line joining O and B would give the shortest time, but perhaps not. There might be some advantage in having the bead fall vertically at first to build up its velocity faster. With a higher velocity, the bead could travel a longer path and still reach B first. Indeed, this is the right idea. The solution, from a branch of mathematics known as the *calculus of variations,* is that the original cycloid from O to B is the one and only brachistochrone for O and B.

While the solution of the brachistrochrone problem is beyond our present reach, we can still show why the cycloid is a tautochrone. For the cycloid, Equation (2) takes the form

$$\begin{aligned} T_{\text{cycloid}} &= \int_{x=0}^{x=a\pi} \sqrt{\frac{dx^2 + dy^2}{2gy}} \\ &= \int_{t=0}^{t=\pi} \sqrt{\frac{a^2(2 - 2\cos t)}{2ga(1 - \cos t)}}\,dt \\ &= \int_0^{\pi} \sqrt{\frac{a}{g}}\,dt = \pi\sqrt{\frac{a}{g}}. \end{aligned}$$

From Equations (1), $dx = a(1 - \cos t)\,dt$, $dy = a\sin t\,dt$, and $y = a(1 - \cos t)$

Thus, the amount of time it takes the frictionless bead to slide down the cycloid to B after it is released from rest at O is $\pi\sqrt{a/g}$.

Suppose that instead of starting the bead at O we start it at some lower point on the cycloid, a point (x_0, y_0) corresponding to the parameter value $t_0 > 0$. The bead's velocity at any later point (x, y) on the cycloid is

$$v = \sqrt{2g(y - y_0)} = \sqrt{2ga(\cos t_0 - \cos t)}. \qquad y = a(1 - \cos t)$$

Accordingly, the time required for the bead to slide from (x_0, y_0) down to B is

$$\begin{aligned}
T &= \int_{t_0}^{\pi} \sqrt{\frac{a^2(2 - 2\cos t)}{2ga(\cos t_0 - \cos t)}}\,dt = \sqrt{\frac{a}{g}} \int_{t_0}^{\pi} \sqrt{\frac{1 - \cos t}{\cos t_0 - \cos t}}\,dt \\
&= \sqrt{\frac{a}{g}} \int_{t_0}^{\pi} \sqrt{\frac{2\sin^2(t/2)}{(2\cos^2(t_0/2) - 1) - (2\cos^2(t/2) - 1)}}\,dt \\
&= \sqrt{\frac{a}{g}} \int_{t_0}^{\pi} \frac{\sin(t/2)\,dt}{\sqrt{\cos^2(t_0/2) - \cos^2(t/2)}} \\
&= \sqrt{\frac{a}{g}} \int_{t=t_0}^{t=\pi} \frac{-2\,du}{\sqrt{a^2 - u^2}} \qquad \begin{aligned} u &= \cos(t/2) \\ -2\,du &= \sin(t/2)\,dt \\ c &= \cos(t_0/2) \end{aligned} \\
&= 2\sqrt{\frac{a}{g}} \left[-\sin^{-1}\frac{u}{c}\right]_{t=t_0}^{t=\pi} \\
&= 2\sqrt{\frac{a}{g}} \left[-\sin^{-1}\frac{\cos(t/2)}{\cos(t_0/2)}\right]_{t_0}^{\pi} \\
&= 2\sqrt{\frac{a}{g}}(-\sin^{-1} 0 + \sin^{-1} 1) = \pi\sqrt{\frac{a}{g}}.
\end{aligned}$$

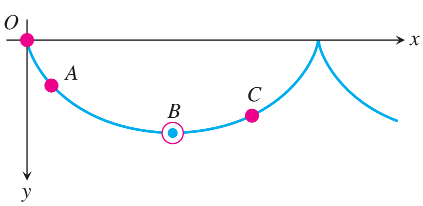


FIGURE 10.34 Beads released simultaneously on the cycloid at O, A, and C will reach B at the same time.

This is precisely the time it takes the bead to slide to B from O. It takes the bead the same amount of time to reach B no matter where it starts. Beads starting simultaneously from O, A, and C in Figure 10.34, for instance, will all reach B at the same time. This is the reason that Huygens' pendulum clock is independent of the amplitude of the swing.

EXERCISES 10.4

Parametric Equations for Conics

Exercises 1–12 give parametric equations and parameter intervals for the motion of a particle in the xy-plane. Identify the particle's path by finding a Cartesian equation for it. Graph the Cartesian equation. (The graphs will vary with the equation used.) Indicate the portion of the graph traced by the particle and the direction of motion.

Exercises

1. $x = \cos t, \quad y = \sin t, \quad 0 \leq t \leq \pi$

2. $x = \sin(2\pi(1 - t)), \quad y = \cos(2\pi(1 - t)); \quad 0 \leq t \leq 1$

3. $x = 4\cos t, \quad y = 5\sin t; \quad 0 \leq t \leq \pi$

4. $x = 4\sin t, \quad y = 5\cos t; \quad 0 \leq t \leq 2\pi$

5. $x = t, \quad y = \sqrt{t}; \quad t \geq 0$

6. $x = \sec^2 t - 1, \quad y = \tan t; \quad -\pi/2 < t < \pi/2$

7. $x = -\sec t, \quad y = \tan t; \quad -\pi/2 < t < \pi/2$

8. $x = \csc t, \quad y = \cot t; \quad 0 < t < \pi$

9. $x = t, \quad y = \sqrt{4 - t^2}; \quad 0 \leq t \leq 2$

10. $x = t^2, \quad y = \sqrt{t^4 + 1}; \quad t \geq 0$

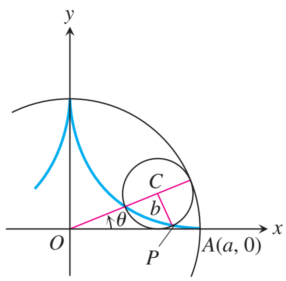
11. $x = -\cosh t, \quad y = \sinh t; \quad -\infty < t < \infty$

12. $x = 2\sinh t, \quad y = 2\cosh t; \quad -\infty < t < \infty$

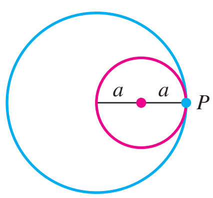
13. Hypocycloids When a circle rolls on the inside of a fixed circle, any point P on the circumference of the rolling circle describes a *hypocycloid*. Let the fixed circle be $x^2 + y^2 = a^2$, let the radius of the rolling circle be b, and let the initial position of the tracing point P be $A(a, 0)$. Find parametric equations for the hypocycloid, using as the parameter the angle θ from the positive x-axis to the line joining the circles' centers. In particular, if

$b = a/4$, as in the accompanying figure, show that the hypocycloid is the astroid

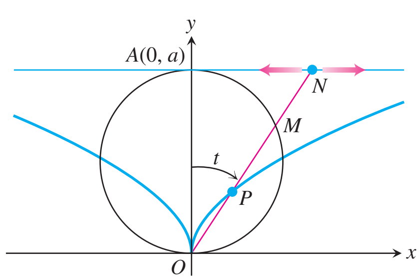
$$x = a\cos^3\theta, \quad y = a\sin^3\theta.$$



14. **More about hypocycloids** The accompanying figure shows a circle of radius a tangent to the inside of a circle of radius $2a$. The point P, shown as the point of tangency in the figure, is attached to the smaller circle. What path does P trace as the smaller circle rolls around the inside of the larger circle?



15. As the point N moves along the line $y = a$ in the accompanying figure, P moves in such a way that $OP = MN$. Find parametric equations for the coordinates of P as functions of the angle t that the line ON makes with the positive y-axis.



16. **Trochoids** A wheel of radius a rolls along a horizontal straight line without slipping. Find parametric equations for the curve traced out by a point P on a spoke of the wheel b units from its center. As parameter, use the angle θ through which the wheel turns. The curve is called a *trochoid*, which is a cycloid when $b = a$.

Distance Using Parametric Equations

17. Find the point on the parabola $x = t, y = t^2, -\infty < t < \infty$, closest to the point $(2, 1/2)$. (*Hint:* Minimize the square of the distance as a function of t.)

18. Find the point on the ellipse $x = 2\cos t, y = \sin t, 0 \le t \le 2\pi$ closest to the point $(3/4, 0)$. (*Hint:* Minimize the square of the distance as a function of t.)

T GRAPHER EXPLORATIONS

If you have a parametric equation grapher, graph the following equations over the given intervals.

19. **Ellipse** $x = 4\cos t, \quad y = 2\sin t$, over
 - **a.** $0 \le t \le 2\pi$
 - **b.** $0 \le t \le \pi$
 - **c.** $-\pi/2 \le t \le \pi/2$.

20. **Hyperbola branch** $x = \sec t$ (enter as $1/\cos(t)$), $y = \tan t$ (enter as $\sin(t)/\cos(t)$), over
 - **a.** $-1.5 \le t \le 1.5$
 - **b.** $-0.5 \le t \le 0.5$
 - **c.** $-0.1 \le t \le 0.1$.

21. **Parabola** $x = 2t + 3, \quad y = t^2 - 1, \quad -2 \le t \le 2$

22. **Cycloid** $x = t - \sin t, \quad y = 1 - \cos t$, over
 - **a.** $0 \le t \le 2\pi$
 - **b.** $0 \le t \le 4\pi$
 - **c.** $\pi \le t \le 3\pi$.

23. **A nice curve (a deltoid)**

$$x = 2\cos t + \cos 2t, \quad y = 2\sin t - \sin 2t; \quad 0 \le t \le 2\pi$$

What happens if you replace 2 with -2 in the equations for x and y? Graph the new equations and find out.

24. **An even nicer curve**

$$x = 3\cos t + \cos 3t, \quad y = 3\sin t - \sin 3t; \quad 0 \le t \le 2\pi$$

What happens if you replace 3 with -3 in the equations for x and y? Graph the new equations and find out.

25. **Three beautiful curves**
 - **a.** *Epicycloid:*

$$x = 9\cos t - \cos 9t, \quad y = 9\sin t - \sin 9t; \quad 0 \le t \le 2\pi$$

 - **b.** *Hypocycloid:*

$$x = 8\cos t + 2\cos 4t, \quad y = 8\sin t - 2\sin 4t; \quad 0 \le t \le 2\pi$$

 - **c.** *Hypotrochoid:*

$$x = \cos t + 5\cos 3t, \quad y = 6\cos t - 5\sin 3t; \quad 0 \le t \le 2\pi$$

26. **More beautiful curves**
 - **a.** $x = 6\cos t + 5\cos 3t, \quad y = 6\sin t - 5\sin 3t; \quad 0 \le t \le 2\pi$
 - **b.** $x = 6\cos 2t + 5\cos 6t, \quad y = 6\sin 2t - 5\sin 6t; \quad 0 \le t \le \pi$
 - **c.** $x = 6\cos t + 5\cos 3t, \quad y = 6\sin 2t - 5\sin 3t; \quad 0 \le t \le 2\pi$
 - **d.** $x = 6\cos 2t + 5\cos 6t, \quad y = 6\sin 4t - 5\sin 6t; \quad 0 \le t \le \pi$

10.5 Polar Coordinates

In this section, we study polar coordinates and their relation to Cartesian coordinates. While a point in the plane has just one pair of Cartesian coordinates, it has infinitely many pairs of polar coordinates. This has interesting consequences for graphing, as we will see in the next section.

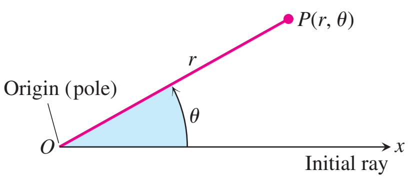


FIGURE 10.35 To define polar coordinates for the plane, we start with an origin, called the pole, and an initial ray.

Definition of Polar Coordinates

To define polar coordinates, we first fix an **origin** O (called the **pole**) and an **initial ray** from O (Figure 10.35). Then each point P can be located by assigning to it a **polar coordinate pair** (r, θ) in which r gives the directed distance from O to P and θ gives the directed angle from the initial ray to ray OP.

> **Polar Coordinates**
>
> $P(r, \theta)$
>
> Directed distance from O to P — Directed angle from initial ray to OP

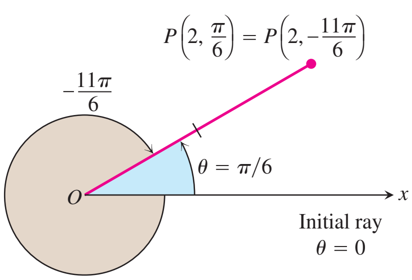


FIGURE 10.36 Polar coordinates are not unique.

As in trigonometry, θ is positive when measured counterclockwise and negative when measured clockwise. The angle associated with a given point is not unique. For instance, the point 2 units from the origin along the ray $\theta = \pi/6$ has polar coordinates $r = 2$, $\theta = \pi/6$. It also has coordinates $r = 2$, $\theta = -11\pi/6$ (Figure 10.36). There are occasions when we wish to allow r to be negative. That is why we use directed distance in defining $P(r, \theta)$. The point $P(2, 7\pi/6)$ can be reached by turning $7\pi/6$ radians counterclockwise from the initial ray and going forward 2 units (Figure 10.37). It can also be reached by turning $\pi/6$ radians counterclockwise from the initial ray and going *backward* 2 units. So the point also has polar coordinates $r = -2$, $\theta = \pi/6$.

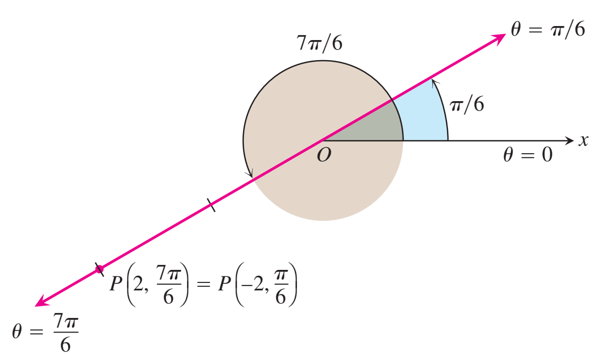


FIGURE 10.37 Polar coordinates can have negative r-values.

EXAMPLE 1 Finding Polar Coordinates

Find all the polar coordinates of the point $P(2, \pi/6)$.

Solution We sketch the initial ray of the coordinate system, draw the ray from the origin that makes an angle of $\pi/6$ radians with the initial ray, and mark the point $(2, \pi/6)$ (Figure 10.38). We then find the angles for the other coordinate pairs of P in which $r = 2$ and $r = -2$.

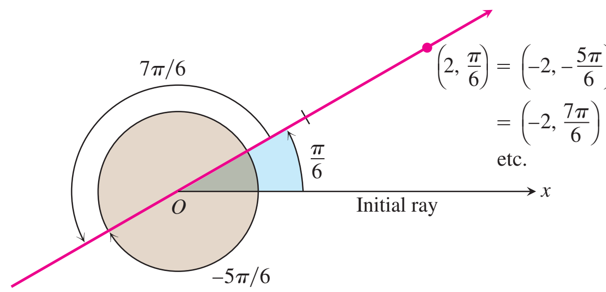


FIGURE 10.38 The point $P(2, \pi/6)$ has infinitely many polar coordinate pairs (Example 1).

For $r = 2$, the complete list of angles is

$$\frac{\pi}{6}, \quad \frac{\pi}{6} \pm 2\pi, \quad \frac{\pi}{6} \pm 4\pi, \quad \frac{\pi}{6} \pm 6\pi, \quad \ldots .$$

For $r = -2$, the angles are

$$-\frac{5\pi}{6}, \quad -\frac{5\pi}{6} \pm 2\pi, \quad -\frac{5\pi}{6} \pm 4\pi, \quad -\frac{5\pi}{6} \pm 6\pi, \quad \ldots .$$

The corresponding coordinate pairs of P are

$$\left(2, \frac{\pi}{6} + 2n\pi\right), \qquad n = 0, \pm 1, \pm 2, \ldots$$

and

$$\left(-2, -\frac{5\pi}{6} + 2n\pi\right), \qquad n = 0, \pm 1, \pm 2, \ldots .$$

When $n = 0$, the formulas give $(2, \pi/6)$ and $(-2, -5\pi/6)$. When $n = 1$, they give $(2, 13\pi/6)$ and $(-2, 7\pi/6)$, and so on. ■

Polar Equations and Graphs

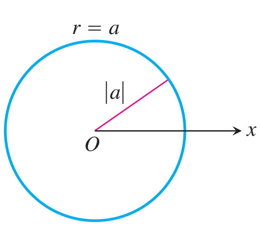


FIGURE 10.39 The polar equation for a circle is $r = a$.

If we hold r fixed at a constant value $r = a \neq 0$, the point $P(r, \theta)$ will lie $|a|$ units from the origin O. As θ varies over any interval of length 2π, P then traces a circle of radius $|a|$ centered at O (Figure 10.39).

If we hold θ fixed at a constant value $\theta = \theta_0$ and let r vary between $-\infty$ and ∞, the point $P(r, \theta)$ traces the line through O that makes an angle of measure θ_0 with the initial ray.

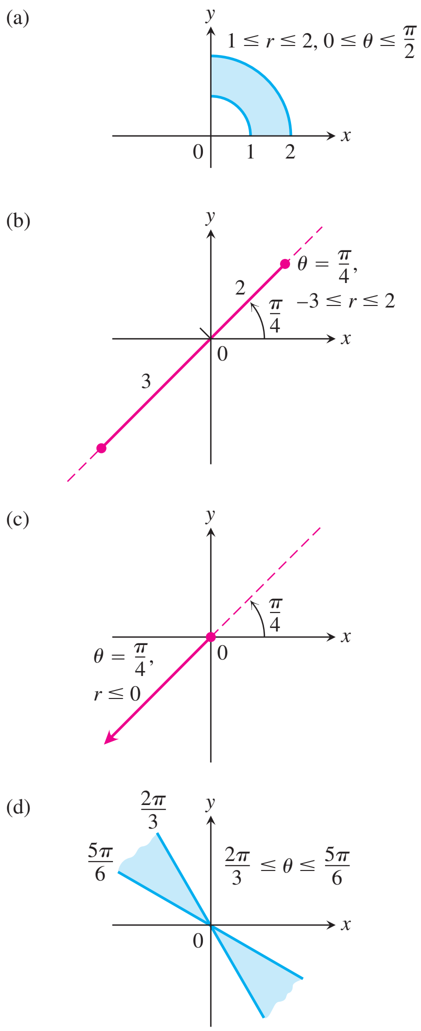


FIGURE 10.40 The graphs of typical inequalities in r and θ (Example 3).

Equation	Graph
$r = a$	Circle radius $\lvert a \rvert$ centered at O
$\theta = \theta_0$	Line through O making an angle θ_0 with the initial ray

EXAMPLE 2 Finding Polar Equations for Graphs

(a) $r = 1$ and $r = -1$ are equations for the circle of radius 1 centered at O.

(b) $\theta = \pi/6$, $\theta = 7\pi/6$, and $\theta = -5\pi/6$ are equations for the line in Figure 10.38. ■

Equations of the form $r = a$ and $\theta = \theta_0$ can be combined to define regions, segments, and rays.

EXAMPLE 3 Identifying Graphs

Graph the sets of points whose polar coordinates satisfy the following conditions.

(a) $1 \le r \le 2$ and $0 \le \theta \le \frac{\pi}{2}$

(b) $-3 \le r \le 2$ and $\theta = \frac{\pi}{4}$

(c) $r \le 0$ and $\theta = \frac{\pi}{4}$

(d) $\frac{2\pi}{3} \le \theta \le \frac{5\pi}{6}$ (no restriction on r)

Solution The graphs are shown in Figure 10.40. ■

Relating Polar and Cartesian Coordinates

When we use both polar and Cartesian coordinates in a plane, we place the two origins together and take the initial polar ray as the positive x-axis. The ray $\theta = \pi/2$, $r > 0$, becomes the positive y-axis (Figure 10.41). The two coordinate systems are then related by the following equations.

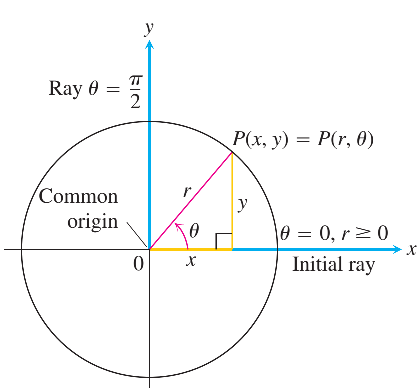


FIGURE 10.41 The usual way to relate polar and Cartesian coordinates.

Equations Relating Polar and Cartesian Coordinates

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad x^2 + y^2 = r^2$$

The first two of these equations uniquely determine the Cartesian coordinates x and y given the polar coordinates r and θ. On the other hand, if x and y are given, the third equation gives two possible choices for r (a positive and a negative value). For each selection, there is a unique $\theta \in [0, 2\pi)$ satisfying the first two equations, each then giving a polar coordinate representation of the Cartesian point (x, y). The other polar coordinate representations for the point can be determined from these two, as in Example 1.

EXAMPLE 4 Equivalent Equations

Polar equation	Cartesian equivalent
$r\cos\theta = 2$	$x = 2$
$r^2\cos\theta\sin\theta = 4$	$xy = 4$
$r^2\cos^2\theta - r^2\sin^2\theta = 1$	$x^2 - y^2 = 1$
$r = 1 + 2r\cos\theta$	$y^2 - 3x^2 - 4x - 1 = 0$
$r = 1 - \cos\theta$	$x^4 + y^4 + 2x^2y^2 + 2x^3 + 2xy^2 - y^2 = 0$

With some curves, we are better off with polar coordinates; with others, we aren't. ■

EXAMPLE 5 Converting Cartesian to Polar

Find a polar equation for the circle $x^2 + (y - 3)^2 = 9$ (Figure 10.42).

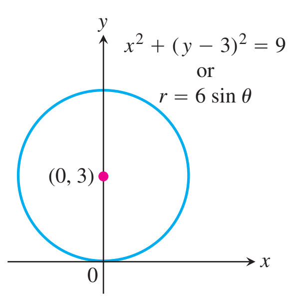


FIGURE 10.42 The circle in Example 5.

Solution

$$\begin{aligned} x^2 + y^2 - 6y + 9 &= 9 && \text{Expand } (y-3)^2. \\ x^2 + y^2 - 6y &= 0 && \text{The 9's cancel.} \\ r^2 - 6r\sin\theta &= 0 && x^2 + y^2 = r^2 \\ r = 0 \quad \text{or} \quad r - 6\sin\theta &= 0 \\ r &= 6\sin\theta && \text{Includes both possibilities} \end{aligned}$$

We will say more about polar equations of conic sections in Section 10.8. ■

EXAMPLE 6 Converting Polar to Cartesian

Replace the following polar equations by equivalent Cartesian equations, and identify their graphs.

(a) $r\cos\theta = -4$

(b) $r^2 = 4r\cos\theta$

(c) $r = \dfrac{4}{2\cos\theta - \sin\theta}$

Solution We use the substitutions $r\cos\theta = x$, $r\sin\theta = y$, $r^2 = x^2 + y^2$.

(a) $r\cos\theta = -4$

The Cartesian equation:
$$\begin{aligned} r\cos\theta &= -4 \\ x &= -4 \end{aligned}$$

The graph: Vertical line through $x = -4$ on the x-axis

(b) $r^2 = 4r\cos\theta$

The Cartesian equation:
$$\begin{aligned} r^2 &= 4r\cos\theta \\ x^2 + y^2 &= 4x \\ x^2 - 4x + y^2 &= 0 \\ x^2 - 4x + 4 + y^2 &= 4 && \text{Completing the square} \\ (x-2)^2 + y^2 &= 4 \end{aligned}$$

The graph: Circle, radius 2, center $(h, k) = (2, 0)$

(c) $r = \dfrac{4}{2\cos\theta - \sin\theta}$

The Cartesian equation:

$$\begin{aligned} r(2\cos\theta - \sin\theta) &= 4 \\ 2r\cos\theta - r\sin\theta &= 4 \\ 2x - y &= 4 \\ y &= 2x - 4 \end{aligned}$$

The graph: Line, slope $m = 2$, y-intercept $b = -4$ ■

EXERCISES 10.5

Polar Coordinate Pairs

1. Which polar coordinate pairs label the same point?
   a. $(3, 0)$ b. $(-3, 0)$ c. $(2, 2\pi/3)$
   d. $(2, 7\pi/3)$ e. $(-3, \pi)$ f. $(2, \pi/3)$
   g. $(-3, 2\pi)$ h. $(-2, -\pi/3)$
2. Which polar coordinate pairs label the same point?
   a. $(-2, \pi/3)$ b. $(2, -\pi/3)$ c. (r, θ)
   d. $(r, \theta + \pi)$ e. $(-r, \theta)$ f. $(2, -2\pi/3)$
   g. $(-r, \theta + \pi)$ h. $(-2, 2\pi/3)$
3. Plot the following points (given in polar coordinates). Then find all the polar coordinates of each point.
   a. $(2, \pi/2)$ b. $(2, 0)$
   c. $(-2, \pi/2)$ d. $(-2, 0)$
4. Plot the following points (given in polar coordinates). Then find all the polar coordinates of each point.
   a. $(3, \pi/4)$ b. $(-3, \pi/4)$
   c. $(3, -\pi/4)$ d. $(-3, -\pi/4)$

Polar to Cartesian Coordinates

5. Find the Cartesian coordinates of the points in Exercise 1.
6. Find the Cartesian coordinates of the following points (given in polar coordinates).
   a. $\left(\sqrt{2}, \pi/4\right)$ b. $(1, 0)$
   c. $(0, \pi/2)$ d. $\left(-\sqrt{2}, \pi/4\right)$
   e. $(-3, 5\pi/6)$ f. $(5, \tan^{-1}(4/3))$
   g. $(-1, 7\pi)$ h. $\left(2\sqrt{3}, 2\pi/3\right)$

Graphing Polar Equations and Inequalities

Graph the sets of points whose polar coordinates satisfy the equations and inequalities in Exercises 7–22.

7. $r = 2$
8. $0 \le r \le 2$
9. $r \ge 1$
10. $1 \le r \le 2$
11. $0 \le \theta \le \pi/6, \quad r \ge 0$
12. $\theta = 2\pi/3, \quad r \le -2$
13. $\theta = \pi/3, \quad -1 \le r \le 3$
14. $\theta = 11\pi/4, \quad r \ge -1$
15. $\theta = \pi/2, \quad r \ge 0$
16. $\theta = \pi/2, \quad r \le 0$
17. $0 \le \theta \le \pi, \quad r = 1$
18. $0 \le \theta \le \pi, \quad r = -1$
19. $\pi/4 \le \theta \le 3\pi/4, \quad 0 \le r \le 1$
20. $-\pi/4 \le \theta \le \pi/4, \quad -1 \le r \le 1$
21. $-\pi/2 \le \theta \le \pi/2, \quad 1 \le r \le 2$
22. $0 \le \theta \le \pi/2, \quad 1 \le |r| \le 2$

Polar to Cartesian Equations

Replace the polar equations in Exercises 23–48 by equivalent Cartesian equations. Then describe or identify the graph.

23. $r\cos\theta = 2$
24. $r\sin\theta = -1$
25. $r\sin\theta = 0$
26. $r\cos\theta = 0$
27. $r = 4\csc\theta$
28. $r = -3\sec\theta$
29. $r\cos\theta + r\sin\theta = 1$
30. $r\sin\theta = r\cos\theta$
31. $r^2 = 1$
32. $r^2 = 4r\sin\theta$
33. $r = \dfrac{5}{\sin\theta - 2\cos\theta}$
34. $r^2\sin 2\theta = 2$
35. $r = \cot\theta\csc\theta$
36. $r = 4\tan\theta\sec\theta$
37. $r = \csc\theta\, e^{r\cos\theta}$
38. $r\sin\theta = \ln r + \ln\cos\theta$
39. $r^2 + 2r^2\cos\theta\sin\theta = 1$
40. $\cos^2\theta = \sin^2\theta$
41. $r^2 = -4r\cos\theta$
42. $r^2 = -6r\sin\theta$
43. $r = 8\sin\theta$
44. $r = 3\cos\theta$
45. $r = 2\cos\theta + 2\sin\theta$
46. $r = 2\cos\theta - \sin\theta$
47. $r\sin\left(\theta + \dfrac{\pi}{6}\right) = 2$
48. $r\sin\left(\dfrac{2\pi}{3} - \theta\right) = 5$

Cartesian to Polar Equations

Replace the Cartesian equations in Exercises 49–62 by equivalent polar equations.

49. $x = 7$
50. $y = 1$
51. $x = y$
52. $x - y = 3$
53. $x^2 + y^2 = 4$
54. $x^2 - y^2 = 1$
55. $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$
56. $xy = 2$

57. $y^2 = 4x$

58. $x^2 + xy + y^2 = 1$

59. $x^2 + (y - 2)^2 = 4$

60. $(x - 5)^2 + y^2 = 25$

61. $(x - 3)^2 + (y + 1)^2 = 4$

62. $(x + 2)^2 + (y - 5)^2 = 16$

Theory and Examples

63. Find all polar coordinates of the origin.

64. Vertical and horizontal lines

a. Show that every vertical line in the xy-plane has a polar equation of the form $r = a \sec \theta$.

b. Find the analogous polar equation for horizontal lines in the xy-plane.

10.6 Graphing in Polar Coordinates

This section describes techniques for graphing equations in polar coordinates.

Symmetry

Figure 10.43 illustrates the standard polar coordinate tests for symmetry.

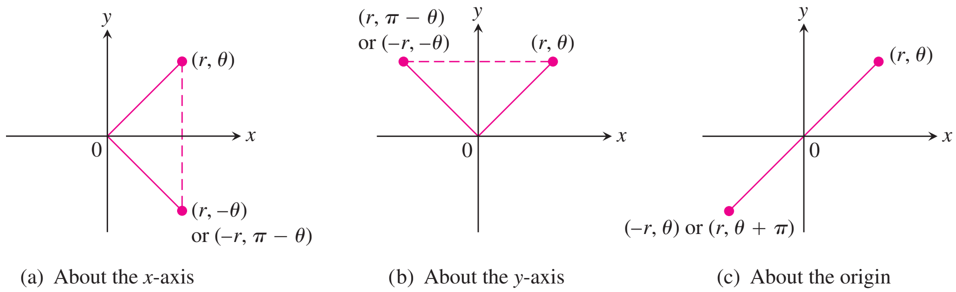


FIGURE 10.43 Three tests for symmetry in polar coordinates.

> **Symmetry Tests for Polar Graphs**
>
> 1. *Symmetry about the x-axis:* If the point (r, θ) lies on the graph, the point $(r, -\theta)$ or $(-r, \pi - \theta)$ lies on the graph (Figure 10.43a).
> 2. *Symmetry about the y-axis:* If the point (r, θ) lies on the graph, the point $(r, \pi - \theta)$ or $(-r, -\theta)$ lies on the graph (Figure 10.43b).
> 3. *Symmetry about the origin:* If the point (r, θ) lies on the graph, the point $(-r, \theta)$ or $(r, \theta + \pi)$ lies on the graph (Figure 10.43c).

Slope

The slope of a polar curve $r = f(\theta)$ is given by dy/dx, not by $r' = df/d\theta$. To see why, think of the graph of f as the graph of the parametric equations

$$x = r\cos\theta = f(\theta)\cos\theta, \qquad y = r\sin\theta = f(\theta)\sin\theta.$$

If f is a differentiable function of θ, then so are x and y and, when $dx/d\theta \neq 0$, we can calculate dy/dx from the parametric formula

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} \qquad \text{Section 3.5, Equation (2) with } t = \theta$$

$$= \frac{\dfrac{d}{d\theta}(f(\theta)\cdot \sin\theta)}{\dfrac{d}{d\theta}(f(\theta)\cdot \cos\theta)}$$

$$= \frac{\dfrac{df}{d\theta}\sin\theta + f(\theta)\cos\theta}{\dfrac{df}{d\theta}\cos\theta - f(\theta)\sin\theta} \qquad \text{Product Rule for derivatives}$$

Slope of the Curve $r = f(\theta)$

$$\left.\frac{dy}{dx}\right|_{(r,\theta)} = \frac{f'(\theta)\sin\theta + f(\theta)\cos\theta}{f'(\theta)\cos\theta - f(\theta)\sin\theta},$$

provided $dx/d\theta \neq 0$ at (r, θ).

If the curve $r = f(\theta)$ passes through the origin at $\theta = \theta_0$, then $f(\theta_0) = 0$, and the slope equation gives

$$\left.\frac{dy}{dx}\right|_{(0,\theta_0)} = \frac{f'(\theta_0)\sin\theta_0}{f'(\theta_0)\cos\theta_0} = \tan\theta_0.$$

If the graph of $r = f(\theta)$ passes through the origin at the value $\theta = \theta_0$, the slope of the curve there is $\tan\theta_0$. The reason we say "slope at $(0, \theta_0)$" and not just "slope at the origin" is that a polar curve may pass through the origin (or any point) more than once, with different slopes at different θ-values. This is not the case in our first example, however.

EXAMPLE 1 A Cardioid

Graph the curve $r = 1 - \cos\theta$.

Solution The curve is symmetric about the x-axis because

$$(r, \theta) \text{ on the graph} \Rightarrow r = 1 - \cos\theta$$
$$\Rightarrow r = 1 - \cos(-\theta) \qquad \cos\theta = \cos(-\theta)$$
$$\Rightarrow (r, -\theta) \text{ on the graph}.$$

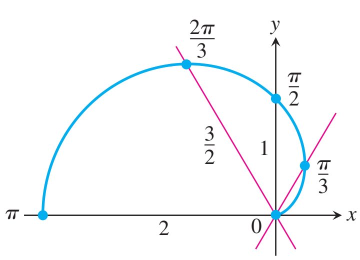
As θ increases from 0 to π, $\cos\theta$ decreases from 1 to -1, and $r = 1 - \cos\theta$ increases from a minimum value of 0 to a maximum value of 2. As θ continues on from π to 2π, $\cos\theta$ increases from -1 back to 1 and r decreases from 2 back to 0. The curve starts to repeat when $\theta = 2\pi$ because the cosine has period 2π.

The curve leaves the origin with slope $\tan(0) = 0$ and returns to the origin with slope $\tan(2\pi) = 0$.

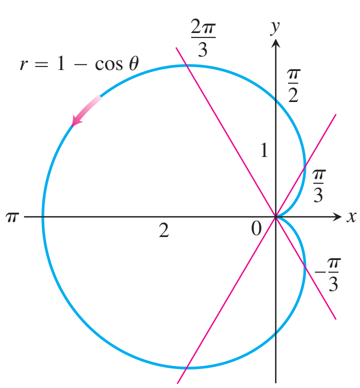
We make a table of values from $\theta = 0$ to $\theta = \pi$, plot the points, draw a smooth curve through them with a horizontal tangent at the origin, and reflect the curve across the x-axis to complete the graph (Figure 10.44). The curve is called a *cardioid* because of its heart shape. Cardioid shapes appear in the cams that direct the even layering of thread on bobbins and reels, and in the signal-strength pattern of certain radio antennas. ■

θ	$r = 1 - \cos\theta$
0	0
$\frac{\pi}{3}$	$\frac{1}{2}$
$\frac{\pi}{2}$	1
$\frac{2\pi}{3}$	$\frac{3}{2}$
π	2

(a)



(b)



(c)

FIGURE 10.44 The steps in graphing the cardioid $r = 1 - \cos\theta$ (Example 1). The arrow shows the direction of increasing θ.

EXAMPLE 2 Graph the Curve $r^2 = 4\cos\theta$.

Solution The equation $r^2 = 4\cos\theta$ requires $\cos\theta \geq 0$, so we get the entire graph by running θ from $-\pi/2$ to $\pi/2$. The curve is symmetric about the x-axis because

$$
\begin{aligned}
(r, \theta) \text{ on the graph} &\Rightarrow r^2 = 4\cos\theta \\
&\Rightarrow r^2 = 4\cos(-\theta) \qquad \cos\theta = \cos(-\theta) \\
&\Rightarrow (r, -\theta) \text{ on the graph.}
\end{aligned}
$$

The curve is also symmetric about the origin because

$$
\begin{aligned}
(r, \theta) \text{ on the graph} &\Rightarrow r^2 = 4\cos\theta \\
&\Rightarrow (-r)^2 = 4\cos\theta \\
&\Rightarrow (-r, \theta) \text{ on the graph.}
\end{aligned}
$$

Together, these two symmetries imply symmetry about the y-axis.

The curve passes through the origin when $\theta = -\pi/2$ and $\theta = \pi/2$. It has a vertical tangent both times because $\tan\theta$ is infinite.

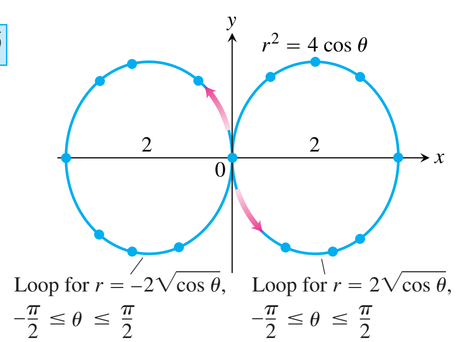
For each value of θ in the interval between $-\pi/2$ and $\pi/2$, the formula $r^2 = 4\cos\theta$ gives two values of r:

$$r = \pm 2\sqrt{\cos\theta}.$$

We make a short table of values, plot the corresponding points, and use information about symmetry and tangents to guide us in connecting the points with a smooth curve (Figure 10.45). ■

θ	$\cos\theta$	$r = \pm 2\sqrt{\cos\theta}$
0	1	± 2
$\pm\frac{\pi}{6}$	$\frac{\sqrt{3}}{2}$	$\approx \pm 1.9$
$\pm\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\approx \pm 1.7$
$\pm\frac{\pi}{3}$	$\frac{1}{2}$	$\approx \pm 1.4$
$\pm\frac{\pi}{2}$	0	0

(a)



(b)

FIGURE 10.45 The graph of $r^2 = 4\cos\theta$. The arrows show the direction of increasing θ. The values of r in the table are rounded (Example 2).

A Technique for Graphing

One way to graph a polar equation $r = f(\theta)$ is to make a table of (r, θ)-values, plot the corresponding points, and connect them in order of increasing θ. This can work well if enough points have been plotted to reveal all the loops and dimples in the graph. Another method of graphing that is usually quicker and more reliable is to

1. first graph $r = f(\theta)$ in the *Cartesian* $r\theta$-plane,
2. then use the Cartesian graph as a "table" and guide to sketch the *polar* coordinate graph.

This method is better than simple point plotting because the first Cartesian graph, even when hastily drawn, shows at a glance where r is positive, negative, and nonexistent, as well as where r is increasing and decreasing. Here's an example.

EXAMPLE 3 A Lemniscate

Graph the curve

$$r^2 = \sin 2\theta .$$

Solution Here we begin by plotting r^2 (not r) as a function of θ in the Cartesian $r^2\theta$-plane. See Figure 10.46a. We pass from there to the graph of $r = \pm\sqrt{\sin 2\theta}$ in the $r\theta$-plane (Figure 10.46b), and then draw the polar graph (Figure 10.46c). The graph in Figure 10.46b "covers" the final polar graph in Figure 10.46c twice. We could have managed with either loop alone, with the two upper halves, or with the two lower halves. The double covering does no harm, however, and we actually learn a little more about the behavior of the function this way. ■

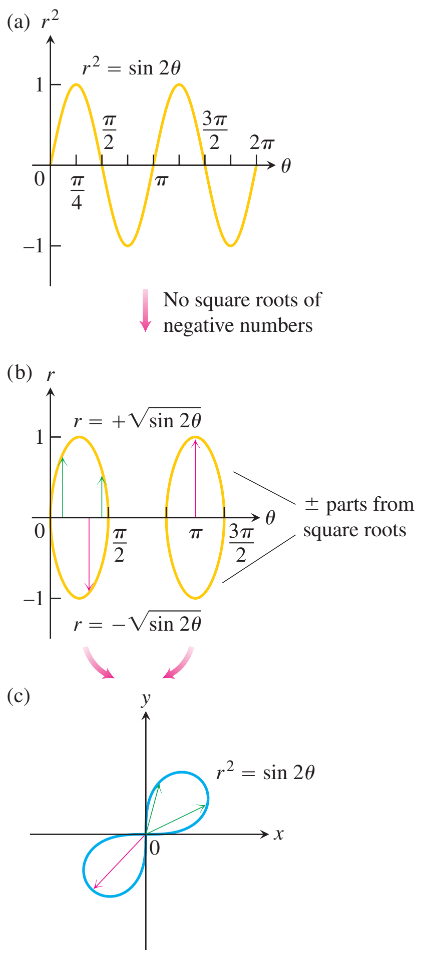


FIGURE 10.46 To plot $r = f(\theta)$ in the Cartesian $r\theta$-plane in (b), we first plot $r^2 = \sin 2\theta$ in the $r^2\theta$-plane in (a) and then ignore the values of θ for which $\sin 2\theta$ is negative. The radii from the sketch in (b) cover the polar graph of the lemniscate in (c) twice (Example 3).

Finding Points Where Polar Graphs Intersect

The fact that we can represent a point in different ways in polar coordinates makes extra care necessary in deciding when a point lies on the graph of a polar equation and in determining the points in which polar graphs intersect. The problem is that a point of intersection may satisfy the equation of one curve with polar coordinates that are different from the ones with which it satisfies the equation of another curve. Thus, solving the equations of two curves simultaneously may not identify all their points of intersection. One sure way to identify all the points of intersection is to graph the equations.

EXAMPLE 4 Deceptive Polar Coordinates

Show that the point $(2, \pi/2)$ lies on the curve $r = 2\cos 2\theta$.

Solution It may seem at first that the point $(2, \pi/2)$ does not lie on the curve because substituting the given coordinates into the equation gives

$$2 = 2\cos 2\left(\frac{\pi}{2}\right) = 2\cos \pi = -2,$$

which is not a true equality. The magnitude is right, but the sign is wrong. This suggests looking for a pair of coordinates for the same given point in which r is negative, for example, $(-2, -(\pi/2))$. If we try these in the equation $r = 2\cos 2\theta$, we find

$$-2 = 2\cos 2\left(-\frac{\pi}{2}\right) = 2(-1) = -2,$$

and the equation is satisfied. The point $(2, \pi/2)$ does lie on the curve. ■

EXAMPLE 5 Elusive Intersection Points

Find the points of intersection of the curves

$$r^2 = 4\cos\theta \qquad \text{and} \qquad r = 1 - \cos\theta .$$

HISTORICAL BIOGRAPHY

Johannes Kepler
(1571–1630)

Solution In Cartesian coordinates, we can always find the points where two curves cross by solving their equations simultaneously. In polar coordinates, the story is different. Simultaneous solution may reveal some intersection points without revealing others. In this example, simultaneous solution reveals only two of the four intersection points. The others are found by graphing. (Also, see Exercise 49.)

If we substitute $\cos\theta = r^2/4$ in the equation $r = 1 - \cos\theta$, we get

$$r = 1 - \cos\theta = 1 - \frac{r^2}{4}$$

$$4r = 4 - r^2$$

$$r^2 + 4r - 4 = 0$$

$$r = -2 \pm 2\sqrt{2}. \qquad \text{Quadratic formula}$$

The value $r = -2 - 2\sqrt{2}$ has too large an absolute value to belong to either curve. The values of θ corresponding to $r = -2 + 2\sqrt{2}$ are

$$\theta = \cos^{-1}(1 - r) \qquad \text{From } r = 1 - \cos\theta$$

$$= \cos^{-1}\left(1 - \left(2\sqrt{2} - 2\right)\right) \qquad \text{Set } r = 2\sqrt{2} - 2.$$

$$= \cos^{-1}\left(3 - 2\sqrt{2}\right)$$

$$= \pm 80°. \qquad \text{Rounded to the nearest degree}$$

We have thus identified two intersection points: $(r, \theta) = (2\sqrt{2} - 2, \pm 80°)$.

If we graph the equations $r^2 = 4\cos\theta$ and $r = 1 - \cos\theta$ together (Figure 10.47), as we can now do by combining the graphs in Figures 10.44 and 10.45, we see that the curves also intersect at the point $(2, \pi)$ and the origin. Why weren't the r-values of these points revealed by the simultaneous solution? The answer is that the points $(0, 0)$ and $(2, \pi)$ are not on the curves "simultaneously." They are not reached at the same value of θ. On the curve $r = 1 - \cos\theta$, the point $(2, \pi)$ is reached when $\theta = \pi$. On the curve $r^2 = 4\cos\theta$, it is reached when $\theta = 0$, where it is identified not by the coordinates $(2, \pi)$, which do not satisfy the equation, but by the coordinates $(-2, 0)$, which do. Similarly, the cardioid reaches the origin when $\theta = 0$, but the curve $r^2 = 4\cos\theta$ reaches the origin when $\theta = \pi/2$. ■

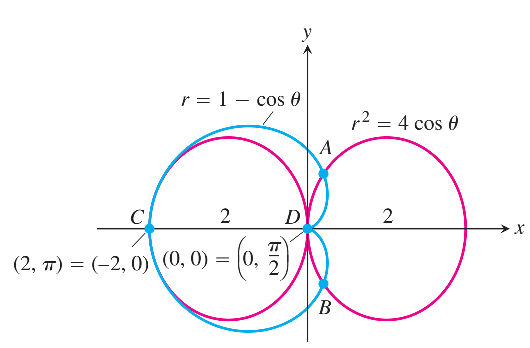


FIGURE 10.47 The four points of intersection of the curves $r = 1 - \cos\theta$ and $r^2 = 4\cos\theta$ (Example 5). Only A and B were found by simultaneous solution. The other two were disclosed by graphing.

USING TECHNOLOGY **Graphing Polar Curves Parametrically**

For complicated polar curves we may need to use a graphing calculator or computer to graph the curve. If the device does not plot polar graphs directly, we can convert $r = f(\theta)$ into parametric form using the equations

$$x = r\cos\theta = f(\theta)\cos\theta, \qquad y = r\sin\theta = f(\theta)\sin\theta.$$

Then we use the device to draw a parametrized curve in the Cartesian xy-plane. It may be required to use the parameter t rather than θ for the graphing device.

EXERCISES 10.6

Symmetries and Polar Graphs

Identify the symmetries of the curves in Exercises 1–12. Then sketch the curves.

1. $r = 1 + \cos\theta$
2. $r = 2 - 2\cos\theta$
3. $r = 1 - \sin\theta$
4. $r = 1 + \sin\theta$
5. $r = 2 + \sin\theta$
6. $r = 1 + 2\sin\theta$
7. $r = \sin(\theta/2)$
8. $r = \cos(\theta/2)$
9. $r^2 = \cos\theta$
10. $r^2 = \sin\theta$
11. $r^2 = -\sin\theta$
12. $r^2 = -\cos\theta$

Graph the lemniscates in Exercises 13–16. What symmetries do these curves have?

13. $r^2 = 4\cos 2\theta$
14. $r^2 = 4\sin 2\theta$
15. $r^2 = -\sin 2\theta$
16. $r^2 = -\cos 2\theta$

Slopes of Polar Curves

Find the slopes of the curves in Exercises 17–20 at the given points. Sketch the curves along with their tangents at these points.

17. Cardioid $r = -1 + \cos\theta; \quad \theta = \pm\pi/2$
18. Cardioid $r = -1 + \sin\theta; \quad \theta = 0, \pi$
19. Four-leaved rose $r = \sin 2\theta; \quad \theta = \pm\pi/4, \pm 3\pi/4$
20. Four-leaved rose $r = \cos 2\theta; \quad \theta = 0, \pm\pi/2, \pi$

Limaçons

Graph the limaçons in Exercises 21–24. Limaçon ("*lee*-ma-sahn") is Old French for "snail." You will understand the name when you graph the limaçons in Exercise 21. Equations for limaçons have the form $r = a \pm b\cos\theta$ or $r = a \pm b\sin\theta$. There are four basic shapes.

21. Limaçons with an inner loop

a. $r = \dfrac{1}{2} + \cos\theta$ **b.** $r = \dfrac{1}{2} + \sin\theta$

22. Cardioids

a. $r = 1 - \cos\theta$ **b.** $r = -1 + \sin\theta$

23. Dimpled limaçons

a. $r = \dfrac{3}{2} + \cos\theta$ **b.** $r = \dfrac{3}{2} - \sin\theta$

24. Oval limaçons

a. $r = 2 + \cos\theta$ **b.** $r = -2 + \sin\theta$

Graphing Polar Inequalities

25. Sketch the region defined by the inequalities $-1 \le r \le 2$ and $-\pi/2 \le \theta \le \pi/2$.

26. Sketch the region defined by the inequalities $0 \le r \le 2\sec\theta$ and $-\pi/4 \le \theta \le \pi/4$.

In Exercises 27 and 28, sketch the region defined by the inequality.

27. $0 \le r \le 2 - 2\cos\theta$
28. $0 \le r^2 \le \cos\theta$

Intersections

29. Show that the point $(2, 3\pi/4)$ lies on the curve $r = 2\sin 2\theta$.

30. Show that $(1/2, 3\pi/2)$ lies on the curve $r = -\sin(\theta/3)$.

Find the points of intersection of the pairs of curves in Exercises 31–38.

31. $r = 1 + \cos\theta, \quad r = 1 - \cos\theta$
32. $r = 1 + \sin\theta, \quad r = 1 - \sin\theta$
33. $r = 2\sin\theta, \quad r = 2\sin 2\theta$
34. $r = \cos\theta, \quad r = 1 - \cos\theta$
35. $r = \sqrt{2}, \quad r^2 = 4\sin\theta$
36. $r^2 = \sqrt{2}\sin\theta, \quad r^2 = \sqrt{2}\cos\theta$
37. $r = 1, \quad r^2 = 2\sin 2\theta$
38. $r^2 = \sqrt{2}\cos 2\theta, \quad r^2 = \sqrt{2}\sin 2\theta$

T Find the points of intersection of the pairs of curves in Exercises 39–42.

39. $r^2 = \sin 2\theta, \quad r^2 = \cos 2\theta$
40. $r = 1 + \cos\dfrac{\theta}{2}, \quad r = 1 - \sin\dfrac{\theta}{2}$
41. $r = 1, \quad r = 2\sin 2\theta$
42. $r = 1, \quad r^2 = 2\sin 2\theta$

T Grapher Explorations

43. Which of the following has the same graph as $r = 1 - \cos\theta$?

a. $r = -1 - \cos\theta$ **b.** $r = 1 + \cos\theta$

Confirm your answer with algebra.

44. Which of the following has the same graph as $r = \cos 2\theta$?

a. $r = -\sin(2\theta + \pi/2)$ **b.** $r = -\cos(\theta/2)$

Confirm your answer with algebra.

45. A rose within a rose Graph the equation $r = 1 - 2\sin 3\theta$.

46. The nephroid of Freeth Graph the nephroid of Freeth:

$$r = 1 + 2\sin\frac{\theta}{2}.$$

47. Roses Graph the roses $r = \cos m\theta$ for $m = 1/3, 2, 3$, and 7.

48. Spirals Polar coordinates are just the thing for defining spirals. Graph the following spirals.

a. $r = \theta$ **b.** $r = -\theta$

c. *A logarithmic spiral:* $r = e^{\theta/10}$

d. *A hyperbolic spiral:* $r = 8/\theta$

e. *An equilateral hyperbola:* $r = \pm 10/\sqrt{\theta}$

(Use different colors for the two branches.)

Theory and Examples

49. (*Continuation of Example 5.*) The simultaneous solution of the equations

$$r^2 = 4\cos\theta \qquad (1)$$

$$r = 1 - \cos\theta \qquad (2)$$

in the text did not reveal the points (0, 0) and $(2, \pi)$ in which their graphs intersected.

a. We could have found the point $(2, \pi)$, however, by replacing the (r, θ) in Equation (1) by the equivalent $(-r, \theta + \pi)$ to obtain

$$\begin{aligned} r^2 &= 4\cos\theta \\ (-r)^2 &= 4\cos(\theta + \pi) \qquad (3) \\ r^2 &= -4\cos\theta. \end{aligned}$$

Solve Equations (2) and (3) simultaneously to show that $(2, \pi)$ is a common solution. (This will still not reveal that the graphs intersect at (0, 0).)

b. The origin is still a special case. (It often is.) Here is one way to handle it: Set $r = 0$ in Equations (1) and (2) and solve each equation for a corresponding value of θ. Since $(0, \theta)$ is the origin for *any* θ, this will show that both curves pass through the origin even if they do so for different θ-values.

50. If a curve has any two of the symmetries listed at the beginning of the section, can anything be said about its having or not having the third symmetry? Give reasons for your answer.

***51.** Find the maximum width of the petal of the four-leaved rose $r = \cos 2\theta$, which lies along the x-axis.

***52.** Find the maximum height above the x-axis of the cardioid $r = 2(1 + \cos\theta)$.

10.7 Areas and Lengths in Polar Coordinates

This section shows how to calculate areas of plane regions, lengths of curves, and areas of surfaces of revolution in polar coordinates.

Area in the Plane

The region *OTS* in Figure 10.48 is bounded by the rays $\theta = \alpha$ and $\theta = \beta$ and the curve $r = f(\theta)$. We approximate the region with n nonoverlapping fan-shaped circular sectors based on a partition P of angle *TOS*. The typical sector has radius $r_k = f(\theta_k)$ and central angle of radian measure $\Delta\theta_k$. Its area is $\Delta\theta_k/2\pi$ times the area of a circle of radius r_k, or

$$A_k = \frac{1}{2} r_k^{\,2}\, \Delta\theta_k = \frac{1}{2}\left(f(\theta_k)\right)^2 \Delta\theta_k\,.$$

The area of region *OTS* is approximately

$$\sum_{k=1}^{n} A_k = \sum_{k=1}^{n} \frac{1}{2}\left(f(\theta_k)\right)^2 \Delta\theta_k\,.$$

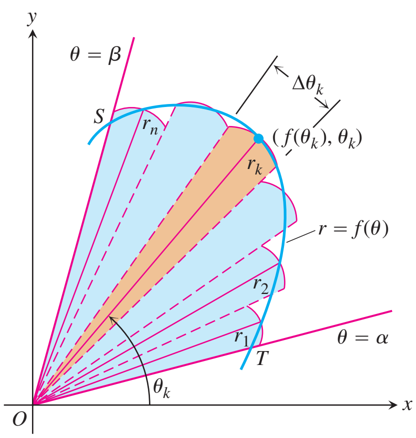


FIGURE 10.48 To derive a formula for the area of region *OTS*, we approximate the region with fan-shaped circular sectors.

If f is continuous, we expect the approximations to improve as the norm of the partition $\|P\| \to 0$, and we are led to the following formula for the region's area:

$$A = \lim_{\|P\| \to 0} \sum_{k=1}^{n} \frac{1}{2}\left(f(\theta_k)\right)^2 \Delta\theta_k$$
$$= \int_{\alpha}^{\beta} \frac{1}{2}\left(f(\theta)\right)^2 d\theta.$$

Area of the Fan-Shaped Region Between the Origin and the Curve $r = f(\theta), \alpha \le \theta \le \beta$

$$A = \int_{\alpha}^{\beta} \frac{1}{2} r^2 \, d\theta.$$

This is the integral of the **area differential** (Figure 10.49)

$$dA = \frac{1}{2} r^2 \, d\theta = \frac{1}{2}(f(\theta))^2 \, d\theta.$$

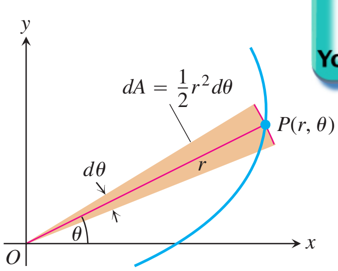


FIGURE 10.49 The area differential dA for the curve $n = f(\theta)$.

EXAMPLE 1 Finding Area

Find the area of the region in the plane enclosed by the cardioid $r = 2(1 + \cos\theta)$.

Solution We graph the cardioid (Figure 10.50) and determine that the radius OP sweeps out the region exactly once as θ runs from 0 to 2π. The area is therefore

$$\int_{\theta=0}^{\theta=2\pi} \frac{1}{2} r^2 \, d\theta = \int_0^{2\pi} \frac{1}{2} \cdot 4(1 + \cos\theta)^2 \, d\theta$$
$$= \int_0^{2\pi} 2(1 + 2\cos\theta + \cos^2\theta)\, d\theta$$
$$= \int_0^{2\pi} \left(2 + 4\cos\theta + 2\,\frac{1 + \cos 2\theta}{2}\right) d\theta$$
$$= \int_0^{2\pi} (3 + 4\cos\theta + \cos 2\theta)\, d\theta$$
$$= \left[3\theta + 4\sin\theta + \frac{\sin 2\theta}{2}\right]_0^{2\pi} = 6\pi - 0 = 6\pi.$$

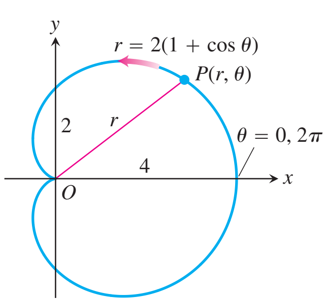


FIGURE 10.50 The cardioid in Example 1.

EXAMPLE 2 Finding Area

Find the area inside the smaller loop of the limaçon

$$r = 2\cos\theta + 1.$$

Solution After sketching the curve (Figure 10.51), we see that the smaller loop is traced out by the point (r, θ) as θ increases from $\theta = 2\pi/3$ to $\theta = 4\pi/3$. Since the curve is symmetric about the x-axis (the equation is unaltered when we replace θ by $-\theta$), we may calculate the area of the shaded half of the inner loop by integrating from $\theta = 2\pi/3$ to $\theta = \pi$. The area we seek will be twice the resulting integral:

$$A = 2\int_{2\pi/3}^{\pi} \frac{1}{2} r^2 \, d\theta = \int_{2\pi/3}^{\pi} r^2 \, d\theta.$$

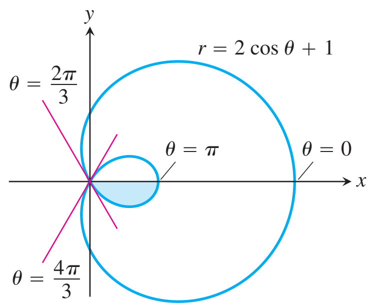


FIGURE 10.51 The limaçon in Example 2. Limaçon (pronounced LEE-ma-sahn) is an old French word for *snail*.

Since

$$
\begin{aligned}
r^2 &= (2\cos\theta + 1)^2 = 4\cos^2\theta + 4\cos\theta + 1 \\
&= 4\cdot\frac{1+\cos 2\theta}{2} + 4\cos\theta + 1 \\
&= 2 + 2\cos 2\theta + 4\cos\theta + 1 \\
&= 3 + 2\cos 2\theta + 4\cos\theta,
\end{aligned}
$$

we have

$$
\begin{aligned}
A &= \int_{2\pi/3}^{\pi} (3 + 2\cos 2\theta + 4\cos\theta)\, d\theta \\
&= \Big[3\theta + \sin 2\theta + 4\sin\theta\Big]_{2\pi/3}^{\pi} \\
&= (3\pi) - \left(2\pi - \frac{\sqrt{3}}{2} + 4\cdot\frac{\sqrt{3}}{2}\right) \\
&= \pi - \frac{3\sqrt{3}}{2}.
\end{aligned}
$$
■

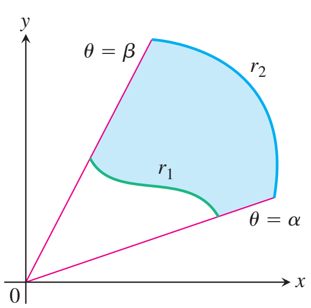


FIGURE 10.52 The area of the shaded region is calculated by subtracting the area of the region between r_1 and the origin from the area of the region between r_2 and the origin.

You Try It

To find the area of a region like the one in Figure 10.52, which lies between two polar curves $r_1 = r_1(\theta)$ and $r_2 = r_2(\theta)$ from $\theta = \alpha$ to $\theta = \beta$, we subtract the integral of $(1/2){r_1}^2\, d\theta$ from the integral of $(1/2){r_2}^2\, d\theta$. This leads to the following formula.

Area of the Region $0 \le r_1(\theta) \le r \le r_2(\theta)$, $\alpha \le \theta \le \beta$

$$A = \int_\alpha^\beta \frac{1}{2}{r_2}^2\, d\theta - \int_\alpha^\beta \frac{1}{2}{r_1}^2\, d\theta = \int_\alpha^\beta \frac{1}{2}\left({r_2}^2 - {r_1}^2\right) d\theta \qquad (1)$$

EXAMPLE 3 Finding Area Between Polar Curves

Find the area of the region that lies inside the circle $r = 1$ and outside the cardioid $r = 1 - \cos\theta$.

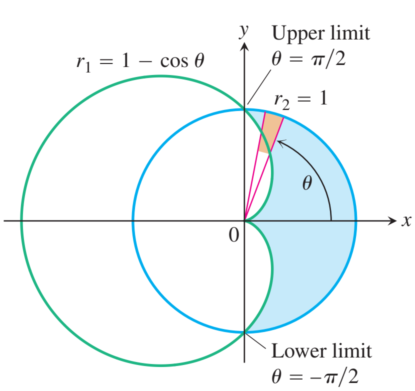


FIGURE 10.53 The region and limits of integration in Example 3.

Solution We sketch the region to determine its boundaries and find the limits of integration (Figure 10.53). The outer curve is $r_2 = 1$, the inner curve is $r_1 = 1 - \cos\theta$, and θ runs from $-\pi/2$ to $\pi/2$. The area, from Equation (1), is

$$
\begin{aligned}
A &= \int_{-\pi/2}^{\pi/2} \frac{1}{2}\left({r_2}^2 - {r_1}^2\right) d\theta \\
&= 2\int_0^{\pi/2} \frac{1}{2}\left({r_2}^2 - {r_1}^2\right) d\theta \qquad \text{Symmetry} \\
&= \int_0^{\pi/2} (1 - (1 - 2\cos\theta + \cos^2\theta))\, d\theta \\
&= \int_0^{\pi/2} (2\cos\theta - \cos^2\theta)\, d\theta = \int_0^{\pi/2}\left(2\cos\theta - \frac{1+\cos 2\theta}{2}\right) d\theta \\
&= \left[2\sin\theta - \frac{\theta}{2} - \frac{\sin 2\theta}{4}\right]_0^{\pi/2} = 2 - \frac{\pi}{4}.
\end{aligned}
$$
■

Length of a Polar Curve

We can obtain a polar coordinate formula for the length of a curve $r = f(\theta)$, $\alpha \le \theta \le \beta$, by parametrizing the curve as

$$x = r\cos\theta = f(\theta)\cos\theta, \qquad y = r\sin\theta = f(\theta)\sin\theta, \qquad \alpha \le \theta \le \beta. \tag{2}$$

The parametric length formula, Equation (1) from Section 6.3, then gives the length as

$$L = \int_\alpha^\beta \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta.$$

This equation becomes

$$L = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

when Equations (2) are substituted for x and y (Exercise 33).

Length of a Polar Curve

If $r = f(\theta)$ has a continuous first derivative for $\alpha \le \theta \le \beta$ and if the point $P(r, \theta)$ traces the curve $r = f(\theta)$ exactly once as θ runs from α to β, then the length of the curve is

$$L = \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta. \tag{3}$$

You Try It

EXAMPLE 4 Finding the Length of a Cardioid

Find the length of the cardioid $r = 1 - \cos\theta$.

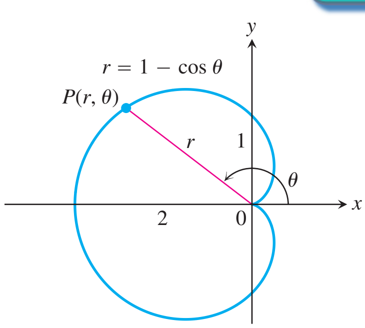


FIGURE 10.54 Calculating the length of a cardioid (Example 4).

Solution We sketch the cardioid to determine the limits of integration (Figure 10.54). The point $P(r, \theta)$ traces the curve once, counterclockwise as θ runs from 0 to 2π, so these are the values we take for α and β.

With

$$r = 1 - \cos\theta, \qquad \frac{dr}{d\theta} = \sin\theta,$$

we have

$$\begin{aligned} r^2 + \left(\frac{dr}{d\theta}\right)^2 &= (1 - \cos\theta)^2 + (\sin\theta)^2 \\ &= 1 - 2\cos\theta + \underbrace{\cos^2\theta + \sin^2\theta}_{1} = 2 - 2\cos\theta \end{aligned}$$

and

$$\begin{aligned} L &= \int_\alpha^\beta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta = \int_0^{2\pi} \sqrt{2 - 2\cos\theta}\, d\theta \\ &= \int_0^{2\pi} \sqrt{4\sin^2\frac{\theta}{2}}\, d\theta \qquad 1 - \cos\theta = 2\sin^2\frac{\theta}{2} \end{aligned}$$

$$= \int_0^{2\pi} 2\left|\sin\frac{\theta}{2}\right| d\theta$$

$$= \int_0^{2\pi} 2\sin\frac{\theta}{2}\, d\theta \qquad \sin\frac{\theta}{2} \ge 0 \quad \text{for} \quad 0 \le \theta \le 2\pi$$

$$= \left[-4\cos\frac{\theta}{2}\right]_0^{2\pi} = 4 + 4 = 8.$$ ■

Area of a Surface of Revolution

To derive polar coordinate formulas for the area of a surface of revolution, we parametrize the curve $r = f(\theta)$, $\alpha \le \theta \le \beta$, with Equations (2) and apply the surface area equations for parametrized curves in Section 6.5.

Area of a Surface of Revolution of a Polar Curve

If $r = f(\theta)$ has a continuous first derivative for $\alpha \le \theta \le \beta$ and if the point $P(r, \theta)$ traces the curve $r = f(\theta)$ exactly once as θ runs from α to β, then the areas of the surfaces generated by revolving the curve about the x- and y-axes are given by the following formulas:

1. Revolution about the x-axis $(y \ge 0)$:

$$S = \int_\alpha^\beta 2\pi r \sin\theta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta \qquad (4)$$

2. Revolution about the y-axis $(x \ge 0)$:

$$S = \int_\alpha^\beta 2\pi r \cos\theta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta \qquad (5)$$

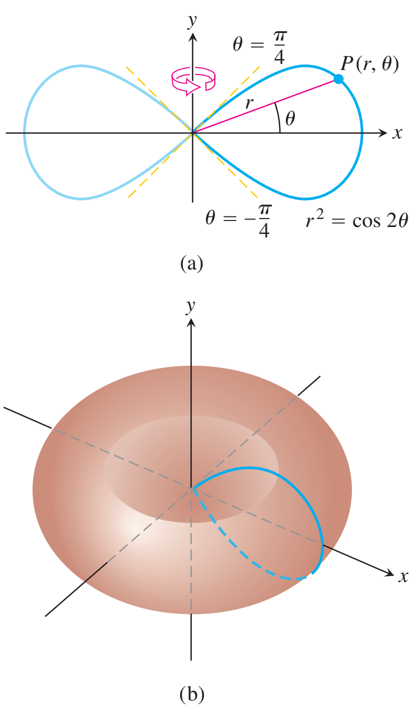


FIGURE 10.55 The right-hand half of a lemniscate (a) is revolved about the y-axis to generate a surface (b), whose area is calculated in Example 5.

EXAMPLE 5 Finding Surface Area

Find the area of the surface generated by revolving the right-hand loop of the lemniscate $r^2 = \cos 2\theta$ about the y-axis.

Solution We sketch the loop to determine the limits of integration (Figure 10.55a). The point $P(r, \theta)$ traces the curve once, counterclockwise as θ runs from $-\pi/4$ to $\pi/4$, so these are the values we take for α and β.

We evaluate the area integrand in Equation (5) in stages. First,

$$2\pi r \cos\theta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} = 2\pi\cos\theta\sqrt{r^4 + \left(r\frac{dr}{d\theta}\right)^2}. \qquad (6)$$

Next, $r^2 = \cos 2\theta$, so

$$2r\frac{dr}{d\theta} = -2\sin 2\theta$$

$$r\frac{dr}{d\theta} = -\sin 2\theta$$

$$\left(r\frac{dr}{d\theta}\right)^2 = \sin^2 2\theta.$$

Finally, $r^4 = (r^2)^2 = \cos^2 2\theta$, so the square root on the right-hand side of Equation (6) simplifies to

$$\sqrt{r^4 + \left(r\frac{dr}{d\theta}\right)^2} = \sqrt{\cos^2 2\theta + \sin^2 2\theta} = 1.$$

All together, we have

$$\begin{aligned} S &= \int_{\alpha}^{\beta} 2\pi r \cos\theta \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta \qquad \text{Equation (5)} \\ &= \int_{-\pi/4}^{\pi/4} 2\pi \cos\theta \cdot (1)\, d\theta \\ &= 2\pi \Big[\sin\theta\Big]_{-\pi/4}^{\pi/4} \\ &= 2\pi\left[\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}\right] = 2\pi\sqrt{2}. \end{aligned}$$

■

EXERCISES 10.7

Areas Inside Polar Curves

Find the areas of the regions in Exercises 1–6.

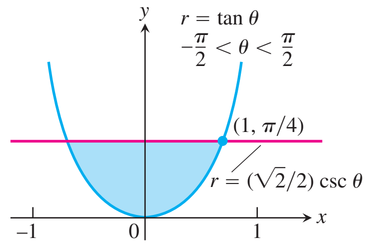
1. Inside the oval limaçon $r = 4 + 2\cos\theta$
2. Inside the cardioid $r = a(1 + \cos\theta), \quad a > 0$
3. Inside one leaf of the four-leaved rose $r = \cos 2\theta$
4. Inside the lemniscate $r^2 = 2a^2 \cos 2\theta, \quad a > 0$
5. Inside one loop of the lemniscate $r^2 = 4\sin 2\theta$
6. Inside the six-leaved rose $r^2 = 2\sin 3\theta$

Areas Shared by Polar Regions

Find the areas of the regions in Exercises 7–16.

7. Shared by the circles $r = 2\cos\theta$ and $r = 2\sin\theta$
8. Shared by the circles $r = 1$ and $r = 2\sin\theta$
9. Shared by the circle $r = 2$ and the cardioid $r = 2(1 - \cos\theta)$
10. Shared by the cardioids $r = 2(1 + \cos\theta)$ and $r = 2(1 - \cos\theta)$
11. Inside the lemniscate $r^2 = 6\cos 2\theta$ and outside the circle $r = \sqrt{3}$
12. Inside the circle $r = 3a\cos\theta$ and outside the cardioid $r = a(1 + \cos\theta), a > 0$
13. Inside the circle $r = -2\cos\theta$ and outside the circle $r = 1$
14. **a.** Inside the outer loop of the limaçon $r = 2\cos\theta + 1$ (See Figure 10.51.)

    b. Inside the outer loop and outside the inner loop of the limaçon $r = 2\cos\theta + 1$
15. Inside the circle $r = 6$ above the line $r = 3\csc\theta$
16. Inside the lemniscate $r^2 = 6\cos 2\theta$ to the right of the line $r = (3/2)\sec\theta$
17. **a.** Find the area of the shaded region in the accompanying figure.



    b. It looks as if the graph of $r = \tan\theta, -\pi/2 < \theta < \pi/2$, could be asymptotic to the lines $x = 1$ and $x = -1$. Is it? Give reasons for your answer.
18. The area of the region that lies inside the cardioid curve $r = \cos\theta + 1$ and outside the circle $r = \cos\theta$ is not

$$\frac{1}{2}\int_0^{2\pi} [(\cos\theta + 1)^2 - \cos^2\theta]\, d\theta = \pi.$$

Why not? What *is* the area? Give reasons for your answers.

Lengths of Polar Curves

Find the lengths of the curves in Exercises 19–27.

19. The spiral $r = \theta^2, \quad 0 \le \theta \le \sqrt{5}$

20. The spiral $r = e^{\theta}/\sqrt{2}, \quad 0 \le \theta \le \pi$

21. The cardioid $r = 1 + \cos\theta$

22. The curve $r = a\sin^2(\theta/2), \quad 0 \le \theta \le \pi, \quad a > 0$

23. The parabolic segment $r = 6/(1 + \cos\theta), \quad 0 \le \theta \le \pi/2$

24. The parabolic segment $r = 2/(1 - \cos\theta), \quad \pi/2 \le \theta \le \pi$

25. The curve $r = \cos^3(\theta/3), \quad 0 \le \theta \le \pi/4$

26. The curve $r = \sqrt{1 + \sin 2\theta}, \quad 0 \le \theta \le \pi\sqrt{2}$

27. The curve $r = \sqrt{1 + \cos 2\theta}, \quad 0 \le \theta \le \pi\sqrt{2}$

28. Circumferences of circles As usual, when faced with a new formula, it is a good idea to try it on familiar objects to be sure it gives results consistent with past experience. Use the length formula in Equation (3) to calculate the circumferences of the following circles $(a > 0)$:

a. $r = a$ **b.** $r = a\cos\theta$ **c.** $r = a\sin\theta$

Surface Area

Find the areas of the surfaces generated by revolving the curves in Exercises 29–32 about the indicated axes.

29. $r = \sqrt{\cos 2\theta}, \quad 0 \le \theta \le \pi/4, \quad y\text{-axis}$

30. $r = \sqrt{2}e^{\theta/2}, \quad 0 \le \theta \le \pi/2, \quad x\text{-axis}$

31. $r^2 = \cos 2\theta, \quad x\text{-axis}$

32. $r = 2a\cos\theta, \quad a > 0, \quad y\text{-axis}$

Theory and Examples

33. The length of the curve $r = f(\theta), \alpha \le \theta \le \beta$ Assuming that the necessary derivatives are continuous, show how the substitutions

$$x = f(\theta)\cos\theta, \quad y = f(\theta)\sin\theta$$

(Equations 2 in the text) transform

$$L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$$

into

$$L = \int_{\alpha}^{\beta} \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta .$$

34. Average value If f is continuous, the average value of the polar coordinate r over the curve $r = f(\theta), \alpha \le \theta \le \beta$, with respect to θ is given by the formula

$$r_{av} = \frac{1}{\beta - \alpha}\int_{\alpha}^{\beta} f(\theta)\, d\theta .$$

Use this formula to find the average value of r with respect to θ over the following curves $(a > 0)$.

a. The cardioid $r = a(1 - \cos\theta)$

b. The circle $r = a$

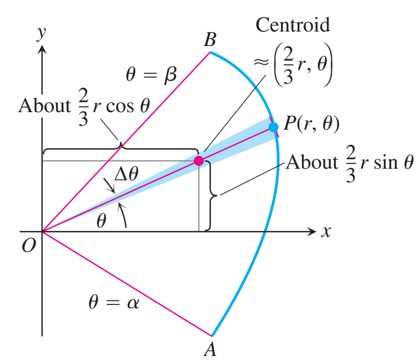
c. The circle $r = a\cos\theta, \quad -\pi/2 \le \theta \le \pi/2$

35. $r = f(\theta)$ vs. $r = 2f(\theta)$ Can anything be said about the relative lengths of the curves $r = f(\theta), \alpha \le \theta \le \beta$, and $r = 2f(\theta)$, $\alpha \le \theta \le \beta$? Give reasons for your answer.

36. $r = f(\theta)$ vs. $r = 2f(\theta)$ The curves $r = f(\theta), \alpha \le \theta \le \beta$, and $r = 2f(\theta), \alpha \le \theta \le \beta$, are revolved about the x-axis to generate surfaces. Can anything be said about the relative areas of these surfaces? Give reasons for your answer.

Centroids of Fan-Shaped Regions

Since the centroid of a triangle is located on each median, two-thirds of the way from the vertex to the opposite base, the lever arm for the moment about the x-axis of the thin triangular region in the accompanying figure is about $(2/3)r\sin\theta$. Similarly, the lever arm for the moment of the triangular region about the y-axis is about $(2/3)r\cos\theta$. These approximations improve as $\Delta\theta \to 0$ and lead to the following formulas for the coordinates of the centroid of region AOB:



$$\bar{x} = \frac{\displaystyle\int \frac{2}{3}r\cos\theta \cdot \frac{1}{2}r^2\, d\theta}{\displaystyle\int \frac{1}{2}r^2\, d\theta} = \frac{\displaystyle\frac{2}{3}\int r^3\cos\theta\, d\theta}{\displaystyle\int r^2\, d\theta},$$

$$\bar{y} = \frac{\displaystyle\int \frac{2}{3}r\sin\theta \cdot \frac{1}{2}r^2\, d\theta}{\displaystyle\int \frac{1}{2}r^2\, d\theta} = \frac{\displaystyle\frac{2}{3}\int r^3\sin\theta\, d\theta}{\displaystyle\int r^2\, d\theta},$$

with limits $\theta = \alpha$ to $\theta = \beta$ on all integrals.

37. Find the centroid of the region enclosed by the cardioid $r = a(1 + \cos\theta)$.

38. Find the centroid of the semicircular region $0 \le r \le a$, $0 \le \theta \le \pi$.

10.8 Conic Sections in Polar Coordinates

Polar coordinates are important in astronomy and astronautical engineering because the ellipses, parabolas, and hyperbolas along which satellites, moons, planets, and comets approximately move can all be described with a single relatively simple coordinate equation. We develop that equation here.

Lines

Suppose the perpendicular from the origin to line L meets L at the point $P_0(r_0, \theta_0)$, with $r_0 \geq 0$ (Figure 10.56). Then, if $P(r, \theta)$ is any other point on L, the points P, P_0, and O are the vertices of a right triangle, from which we can read the relation

$$r_0 = r\cos(\theta - \theta_0).$$

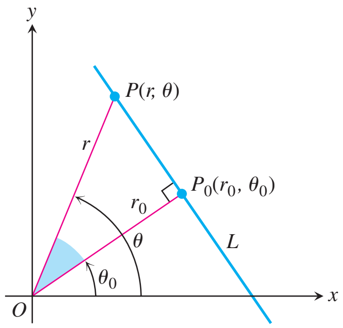


FIGURE 10.56 We can obtain a polar equation for line L by reading the relation $r_0 = r\cos(\theta - \theta_0)$ from the right triangle OP_0P.

The Standard Polar Equation for Lines

If the point $P_0(r_0, \theta_0)$ is the foot of the perpendicular from the origin to the line L, and $r_0 \geq 0$, then an equation for L is

$$r\cos(\theta - \theta_0) = r_0. \qquad (1)$$

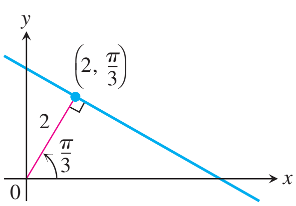


FIGURE 10.57 The standard polar equation of this line converts to the Cartesian equation $x + \sqrt{3}\,y = 4$ (Example 1).

EXAMPLE 1 Converting a Line's Polar Equation to Cartesian Form

Use the identity $\cos(A - B) = \cos A\cos B + \sin A\sin B$ to find a Cartesian equation for the line in Figure 10.57.

Solution

$$r\cos\left(\theta - \frac{\pi}{3}\right) = 2$$

$$r\left(\cos\theta\cos\frac{\pi}{3} + \sin\theta\sin\frac{\pi}{3}\right) = 2$$

$$\frac{1}{2}r\cos\theta + \frac{\sqrt{3}}{2}r\sin\theta = 2$$

$$\frac{1}{2}x + \frac{\sqrt{3}}{2}y = 2$$

$$x + \sqrt{3}\,y = 4$$

■

Circles

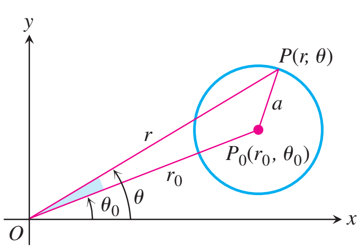


FIGURE 10.58 We can get a polar equation for this circle by applying the Law of Cosines to triangle OP_0P.

To find a polar equation for the circle of radius a centered at $P_0(r_0, \theta_0)$, we let $P(r, \theta)$ be a point on the circle and apply the Law of Cosines to triangle OP_0P (Figure 10.58). This gives

$$a^2 = r_0^2 + r^2 - 2r_0r\cos(\theta - \theta_0).$$

If the circle passes through the origin, then $r_0 = a$ and this equation simplifies to

$$a^2 = a^2 + r^2 - 2ar\cos(\theta - \theta_0)$$
$$r^2 = 2ar\cos(\theta - \theta_0)$$
$$r = 2a\cos(\theta - \theta_0).$$

If the circle's center lies on the positive x-axis, $\theta_0 = 0$ and we get the further simplification

$$r = 2a\cos\theta$$

(see Figure 10.59a).

If the center lies on the positive y-axis, $\theta = \pi/2$, $\cos(\theta - \pi/2) = \sin\theta$, and the equation $r = 2a\cos(\theta - \theta_0)$ becomes

$$r = 2a\sin\theta$$

(see Figure 10.59b).

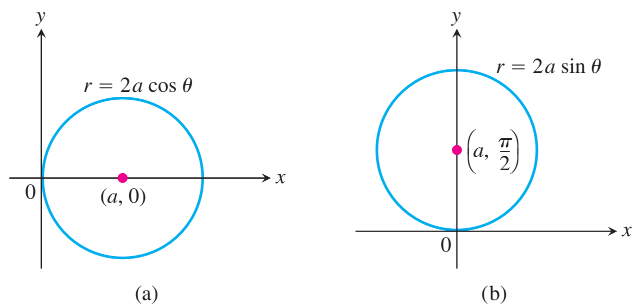


FIGURE 10.59 Polar equation of a circle of radius a through the origin with center on (a) the positive x-axis, and (b) the positive y-axis.

Equations for circles through the origin centered on the negative x- and y-axes can be obtained by replacing r with $-r$ in the above equations (Figure 10.60).

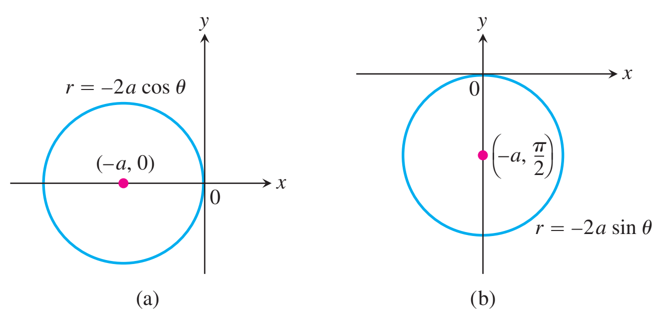


FIGURE 10.60 Polar equation of a circle of radius a through the origin with center on (a) the negative x-axis, and (b) the negative y-axis.

EXAMPLE 2 Circles Through the Origin

Radius	Center (polar coordinates)	Polar equation
3	$(3, 0)$	$r = 6\cos\theta$
2	$(2, \pi/2)$	$r = 4\sin\theta$
1/2	$(-1/2, 0)$	$r = -\cos\theta$
1	$(-1, \pi/2)$	$r = -2\sin\theta$

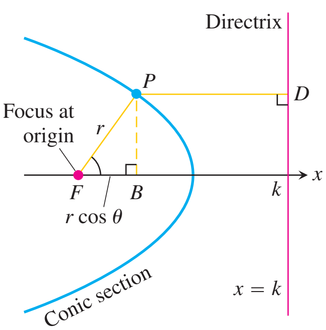


FIGURE 10.61 If a conic section is put in the position with its focus placed at the origin and a directrix perpendicular to the initial ray and right of the origin, we can find its polar equation from the conic's focus–directrix equation.

Ellipses, Parabolas, and Hyperbolas

To find polar equations for ellipses, parabolas, and hyperbolas, we place one focus at the origin and the corresponding directrix to the right of the origin along the vertical line $x = k$ (Figure 10.61). This makes

$$PF = r$$

and

$$PD = k - FB = k - r\cos\theta.$$

The conic's focus–directrix equation $PF = e \cdot PD$ then becomes

$$r = e(k - r\cos\theta),$$

which can be solved for r to obtain

Polar Equation for a Conic with Eccentricity e

$$r = \frac{ke}{1 + e\cos\theta}, \qquad (2)$$

where $x = k > 0$ is the vertical directrix.

This equation represents an ellipse if $0 < e < 1$, a parabola if $e = 1$, and a hyperbola if $e > 1$. That is, ellipses, parabolas, and hyperbolas all have the same basic equation expressed in terms of eccentricity and location of the directrix.

EXAMPLE 3 Polar Equations of Some Conics

$e = \frac{1}{2}$: ellipse $\quad r = \frac{k}{2 + \cos\theta}$

$e = 1$: parabola $\quad r = \frac{k}{1 + \cos\theta}$

$e = 2$: hyperbola $\quad r = \frac{2k}{1 + 2\cos\theta}$

You may see variations of Equation (2) from time to time, depending on the location of the directrix. If the directrix is the line $x = -k$ to the left of the origin (the origin is still a focus), we replace Equation (2) by

$$r = \frac{ke}{1 - e\cos\theta}.$$

The denominator now has a $(-)$ instead of a $(+)$. If the directrix is either of the lines $y = k$ or $y = -k$, the equations have sines in them instead of cosines, as shown in Figure 10.62.

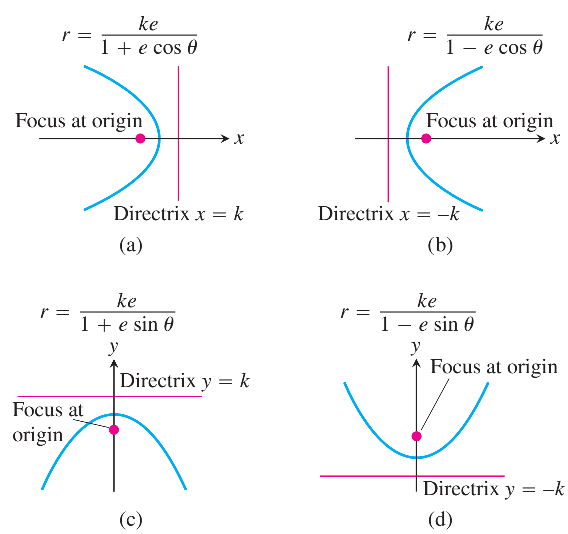


FIGURE 10.62 Equations for conic sections with eccentricity $e > 0$, but different locations of the directrix. The graphs here show a parabola, so $e = 1$.

You Try It

EXAMPLE 4 Polar Equation of a Hyperbola

Find an equation for the hyperbola with eccentricity $3/2$ and directrix $x = 2$.

Solution We use Equation (2) with $k = 2$ and $e = 3/2$:

$$r = \frac{2(3/2)}{1 + (3/2)\cos\theta} \quad \text{or} \quad r = \frac{6}{2 + 3\cos\theta}.$$

■

EXAMPLE 5 Finding a Directrix

Find the directrix of the parabola

$$r = \frac{25}{10 + 10\cos\theta}.$$

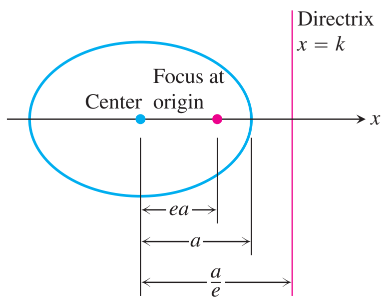


FIGURE 10.63 In an ellipse with semimajor axis a, the focus–directrix distance is $k = (a/e) - ea$, so $ke = a(1 - e^2)$.

Solution We divide the numerator and denominator by 10 to put the equation in standard form:

$$r = \frac{5/2}{1 + \cos\theta}.$$

This is the equation

$$r = \frac{ke}{1 + e\cos\theta}$$

with $k = 5/2$ and $e = 1$. The equation of the directrix is $x = 5/2$. ■

From the ellipse diagram in Figure 10.63, we see that k is related to the eccentricity e and the semimajor axis a by the equation

$$k = \frac{a}{e} - ea.$$

From this, we find that $ke = a(1 - e^2)$. Replacing ke in Equation (2) by $a(1 - e^2)$ gives the standard polar equation for an ellipse.

Polar Equation for the Ellipse with Eccentricity e and Semimajor Axis a

$$r = \frac{a(1 - e^2)}{1 + e\cos\theta} \qquad (3)$$

Notice that when $e = 0$, Equation (3) becomes $r = a$, which represents a circle.

Equation (3) is the starting point for calculating planetary orbits.

EXAMPLE 6 The Planet Pluto's Orbit

Find a polar equation for an ellipse with semimajor axis 39.44 AU (astronomical units) and eccentricity 0.25. This is the approximate size of Pluto's orbit around the sun.

Solution We use Equation (3) with $a = 39.44$ and $e = 0.25$ to find

$$r = \frac{39.44(1 - (0.25)^2)}{1 + 0.25\cos\theta} = \frac{147.9}{4 + \cos\theta}.$$

At its point of closest approach (perihelion) where $\theta = 0$, Pluto is

$$r = \frac{147.9}{4 + 1} = 29.58 \text{ AU}$$

from the sun. At its most distant point (aphelion) where $\theta = \pi$, Pluto is

$$r = \frac{147.9}{4 - 1} = 49.3 \text{ AU}$$

from the sun (Figure 10.64). ■

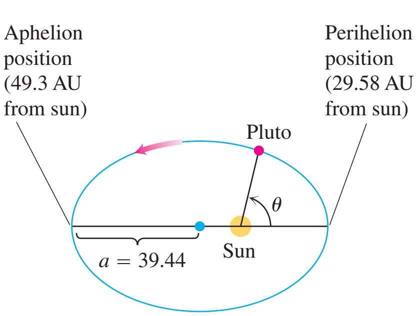


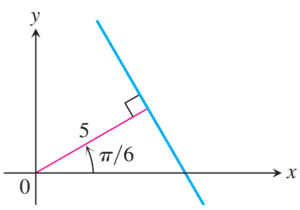
FIGURE 10.64 The orbit of Pluto (Example 6).

EXERCISES 10.8

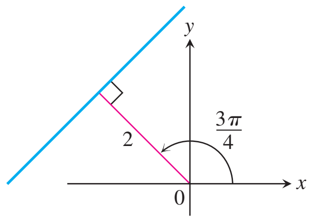
Lines

Find polar and Cartesian equations for the lines in Exercises 1–4.

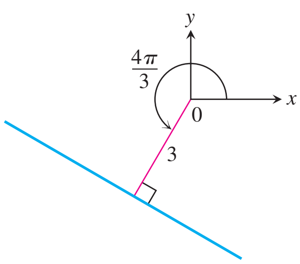
1.



2.



3.



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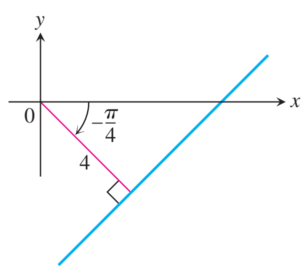


Sketch the lines in Exercises 5–8 and find Cartesian equations for them.

5. $r\cos\left(\theta - \frac{\pi}{4}\right) = \sqrt{2}$

6. $r\cos\left(\theta + \frac{3\pi}{4}\right) = 1$

7. $r\cos\left(\theta - \frac{2\pi}{3}\right) = 3$

8. $r\cos\left(\theta + \frac{\pi}{3}\right) = 2$

Find a polar equation in the form $r\cos(\theta - \theta_0) = r_0$ for each of the lines in Exercises 9–12.

9. $\sqrt{2}\,x + \sqrt{2}\,y = 6$

10. $\sqrt{3}\,x - y = 1$

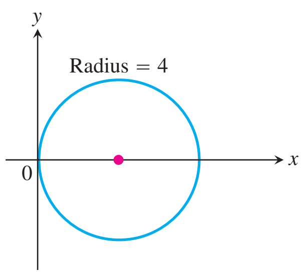
11. $y = -5$

12. $x = -4$

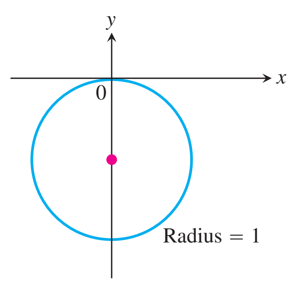
Circles

Find polar equations for the circles in Exercises 13–16.

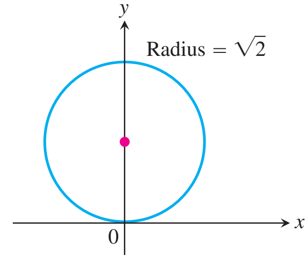
13.



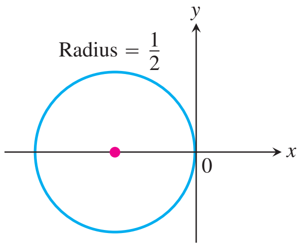
14.



15.



16.



Sketch the circles in Exercises 17–20. Give polar coordinates for their centers and identify their radii.

17. $r = 4\cos\theta$

18. $r = 6\sin\theta$

19. $r = -2\cos\theta$

20. $r = -8\sin\theta$

Find polar equations for the circles in Exercises 21–28. Sketch each circle in the coordinate plane and label it with both its Cartesian and polar equations.

21. $(x - 6)^2 + y^2 = 36$

22. $(x + 2)^2 + y^2 = 4$

23. $x^2 + (y - 5)^2 = 25$

24. $x^2 + (y + 7)^2 = 49$

25. $x^2 + 2x + y^2 = 0$

26. $x^2 - 16x + y^2 = 0$

27. $x^2 + y^2 + y = 0$

28. $x^2 + y^2 - \frac{4}{3}y = 0$

Conic Sections from Eccentricities and Directrices

Exercises 29–36 give the eccentricities of conic sections with one focus at the origin, along with the directrix corresponding to that focus. Find a polar equation for each conic section.

29. $e = 1, \quad x = 2$

30. $e = 1, \quad y = 2$

31. $e = 5, \quad y = -6$

32. $e = 2, \quad x = 4$

33. $e = 1/2, \quad x = 1$

34. $e = 1/4, \quad x = -2$

35. $e = 1/5, \quad y = -10$

36. $e = 1/3, \quad y = 6$

Parabolas and Ellipses

Sketch the parabolas and ellipses in Exercises 37–44. Include the directrix that corresponds to the focus at the origin. Label the vertices with appropriate polar coordinates. Label the centers of the ellipses as well.

37. $r = \dfrac{1}{1 + \cos\theta}$

38. $r = \dfrac{6}{2 + \cos\theta}$

39. $r = \dfrac{25}{10 - 5\cos\theta}$

40. $r = \dfrac{4}{2 - 2\cos\theta}$

41. $r = \dfrac{400}{16 + 8\sin\theta}$

42. $r = \dfrac{12}{3 + 3\sin\theta}$

43. $r = \dfrac{8}{2 - 2\sin\theta}$

44. $r = \dfrac{4}{2 - \sin\theta}$

Graphing Inequalities

Sketch the regions defined by the inequalities in Exercises 45 and 46.

45. $0 \le r \le 2\cos\theta$

46. $-3\cos\theta \le r \le 0$

T Grapher Explorations

Graph the lines and conic sections in Exercises 47–56.

47. $r = 3\sec(\theta - \pi/3)$

48. $r = 4\sec(\theta + \pi/6)$

49. $r = 4\sin\theta$

50. $r = -2\cos\theta$

51. $r = 8/(4 + \cos\theta)$

52. $r = 8/(4 + \sin\theta)$

53. $r = 1/(1 - \sin\theta)$

54. $r = 1/(1 + \cos\theta)$

55. $r = 1/(1 + 2\sin\theta)$

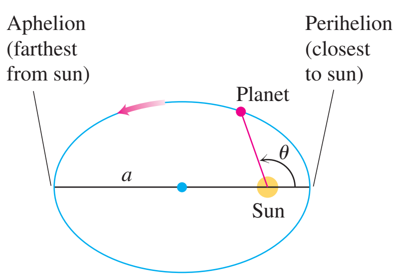
56. $r = 1/(1 + 2\cos\theta)$

Theory and Examples

57. Perihelion and aphelion A planet travels about its sun in an ellipse whose semimajor axis has length a. (See accompanying figure.)

a. Show that $r = a(1 - e)$ when the planet is closest to the sun and that $r = a(1 + e)$ when the planet is farthest from the sun.

b. Use the data in the table in Exercise 58 to find how close each planet in our solar system comes to the sun and how far away each planet gets from the sun.



58. Planetary orbits In Example 6, we found a polar equation for the orbit of Pluto. Use the data in the table below to find polar equations for the orbits of the other planets.

Planet	Semimajor axis (astronomical units)	Eccentricity
Mercury	0.3871	0.2056
Venus	0.7233	0.0068
Earth	1.000	0.0167
Mars	1.524	0.0934
Jupiter	5.203	0.0484
Saturn	9.539	0.0543
Uranus	19.18	0.0460
Neptune	30.06	0.0082
Pluto	39.44	0.2481

59. a. Find Cartesian equations for the curves $r = 4\sin\theta$ and $r = \sqrt{3}\sec\theta$.

b. Sketch the curves together and label their points of intersection in both Cartesian and polar coordinates.

60. Repeat Exercise 59 for $r = 8\cos\theta$ and $r = 2\sec\theta$.

61. Find a polar equation for the parabola with focus (0, 0) and directrix $r\cos\theta = 4$.

62. Find a polar equation for the parabola with focus (0, 0) and directrix $r\cos(\theta - \pi/2) = 2$.

63. a. The space engineer's formula for eccentricity The space engineer's formula for the eccentricity of an elliptical orbit is

$$e = \frac{r_{max} - r_{min}}{r_{max} + r_{min}},$$

where r is the distance from the space vehicle to the attracting focus of the ellipse along which it travels. Why does the formula work?

b. Drawing ellipses with string You have a string with a knot in each end that can be pinned to a drawing board. The string is 10 in. long from the center of one knot to the center of the other. How far apart should the pins be to use the method illustrated in Figure 10.5 (Section 10.1) to draw an ellipse of eccentricity 0.2? The resulting ellipse would resemble the orbit of Mercury.

64. Halley's comet (See Section 10.2, Example 1.)

a. Write an equation for the orbit of Halley's comet in a coordinate system in which the sun lies at the origin and the other focus lies on the negative x-axis, scaled in astronomical units.

b. How close does the comet come to the sun in astronomical units? In kilometers?

c. What is the farthest the comet gets from the sun in astronomical units? In kilometers?

In Exercises 65–68, find a polar equation for the given curve. In each case, sketch a typical curve.

65. $x^2 + y^2 - 2ay = 0$

66. $y^2 = 4ax + 4a^2$

67. $x\cos\alpha + y\sin\alpha = p$ (α, p constant)

68. $(x^2 + y^2)^2 + 2ax(x^2 + y^2) - a^2y^2 = 0$

COMPUTER EXPLORATIONS

69. Use a CAS to plot the polar equation

$$r = \frac{ke}{1 + e\cos\theta}$$

for various values of k and e, $-\pi \le \theta \le \pi$. Answer the following questions.

a. Take $k = -2$. Describe what happens to the plots as you take e to be 3/4, 1, and 5/4. Repeat for $k = 2$.

b. Take $k = -1$. Describe what happens to the plots as you take e to be $7/6, 5/4, 4/3, 3/2, 2, 3, 5, 10$, and 20. Repeat for $e = 1/2, 1/3, 1/4, 1/10$, and $1/20$.

c. Now keep $e > 0$ fixed and describe what happens as you take k to be $-1, -2, -3, -4$, and -5. Be sure to look at graphs for parabolas, ellipses, and hyperbolas.

70. Use a CAS to plot the polar ellipse

$$r = \frac{a(1 - e^2)}{1 + e\cos\theta}$$

for various values of $a > 0$ and $0 < e < 1, -\pi \leq \theta \leq \pi$.

a. Take $e = 9/10$. Describe what happens to the plots as you let a equal $1, 3/2, 2, 3, 5$, and 10. Repeat with $e = 1/4$.

b. Take $a = 2$. Describe what happens as you take e to be $9/10, 8/10, 7/10, \ldots, 1/10, 1/20$, and $1/50$.

Chapter 10 Additional and Advanced Exercises

Finding Conic Sections

1. Find an equation for the parabola with focus (4, 0) and directrix $x = 3$. Sketch the parabola together with its vertex, focus, and directrix.
2. Find the vertex, focus, and directrix of the parabola
$$x^2 - 6x - 12y + 9 = 0.$$
3. Find an equation for the curve traced by the point $P(x, y)$ if the distance from P to the vertex of the parabola $x^2 = 4y$ is twice the distance from P to the focus. Identify the curve.
4. A line segment of length $a + b$ runs from the x-axis to the y-axis. The point P on the segment lies a units from one end and b units from the other end. Show that P traces an ellipse as the ends of the segment slide along the axes.
5. The vertices of an ellipse of eccentricity 0.5 lie at the points $(0, \pm 2)$. Where do the foci lie?
6. Find an equation for the ellipse of eccentricity $2/3$ that has the line $x = 2$ as a directrix and the point (4, 0) as the corresponding focus.
7. One focus of a hyperbola lies at the point $(0, -7)$ and the corresponding directrix is the line $y = -1$. Find an equation for the hyperbola if its eccentricity is **(a)** 2, **(b)** 5.
8. Find an equation for the hyperbola with foci $(0, -2)$ and $(0, 2)$ that passes through the point (12, 7).
9. **a.** Show that the line
$$b^2xx_1 + a^2yy_1 - a^2b^2 = 0$$
is tangent to the ellipse $b^2x^2 + a^2y^2 - a^2b^2 = 0$ at the point (x_1, y_1) on the ellipse.

   b. Show that the line
$$b^2xx_1 - a^2yy_1 - a^2b^2 = 0$$
is tangent to the hyperbola $b^2x^2 - a^2y^2 - a^2b^2 = 0$ at the point (x_1, y_1) on the hyperbola.
10. Show that the tangent to the conic section
$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$
at a point (x_1, y_1) on it has an equation that may be written in the form
$$Axx_1 + B\left(\frac{x_1y + xy_1}{2}\right) + Cyy_1 + D\left(\frac{x + x_1}{2}\right) + E\left(\frac{y + y_1}{2}\right) + F = 0.$$

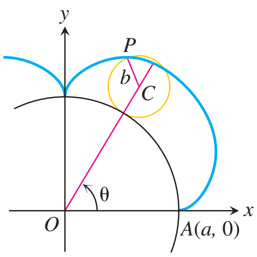
Equations and Inequalities

What points in the xy-plane satisfy the equations and inequalities in Exercises 11–18? Draw a figure for each exercise.

11. $(x^2 - y^2 - 1)(x^2 + y^2 - 25)(x^2 + 4y^2 - 4) = 0$
12. $(x + y)(x^2 + y^2 - 1) = 0$
13. $(x^2/9) + (y^2/16) \le 1$
14. $(x^2/9) - (y^2/16) \le 1$
15. $(9x^2 + 4y^2 - 36)(4x^2 + 9y^2 - 16) \le 0$
16. $(9x^2 + 4y^2 - 36)(4x^2 + 9y^2 - 16) > 0$
17. $x^4 - (y^2 - 9)^2 = 0$
18. $x^2 + xy + y^2 < 3$

Parametric Equations and Cycloids

19. Epicycloids When a circle rolls externally along the circumference of a second, fixed circle, any point P on the circumference of the rolling circle describes an *epicycloid*, as shown here. Let the fixed circle have its center at the origin O and have radius a.



Let the radius of the rolling circle be b and let the initial position of the tracing point P be $A(a, 0)$. Find parametric equations for the epicycloid, using as the parameter the angle θ from the positive x-axis to the line through the circles' centers.

20. **a.** Find the centroid of the region enclosed by the x-axis and the cycloid arch

$$x = a(t - \sin t), \quad y = a(1 - \cos t); \quad 0 \le t \le 2\pi.$$

b. Find the first moments about the coordinate axes of the curve

$$x = (2/3)t^{3/2}, \quad y = 2\sqrt{t}; \quad 0 \le t \le \sqrt{3}.$$

Polar Coordinates

21. **a.** Find an equation in polar coordinates for the curve

$$x = e^{2t}\cos t, \quad y = e^{2t}\sin t; \quad -\infty < t < \infty.$$

b. Find the length of the curve from $t = 0$ to $t = 2\pi$.

22. Find the length of the curve $r = 2\sin^3(\theta/3), 0 \le \theta \le 3\pi$, in the polar coordinate plane.

23. Find the area of the surface generated by revolving the first-quadrant portion of the cardioid $r = 1 + \cos\theta$ about the x-axis. (*Hint*: Use the identities $1 + \cos\theta = 2\cos^2(\theta/2)$ and $\sin\theta = 2\sin(\theta/2)\cos(\theta/2)$ to simplify the integral.)

24. Sketch the regions enclosed by the curves $r = 2a\cos^2(\theta/2)$ and $r = 2a\sin^2(\theta/2), a > 0$, in the polar coordinate plane and find the area of the portion of the plane they have in common.

Exercises 25–28 give the eccentricities of conic sections with one focus at the origin of the polar coordinate plane, along with the directrix for that focus. Find a polar equation for each conic section.

25. $e = 2, \quad r\cos\theta = 2$

26. $e = 1, \quad r\cos\theta = -4$

27. $e = 1/2, \quad r\sin\theta = 2$

28. $e = 1/3, \quad r\sin\theta = -6$

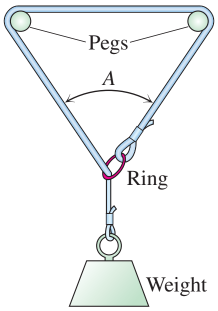
Theory and Examples

29. A rope with a ring in one end is looped over two pegs in a horizontal line. The free end, after being passed through the ring, has a weight suspended from it to make the rope hang taut. If the rope slips freely over the pegs and through the ring, the weight will descend as far as possible. Assume that the length of the rope is at least four times as great as the distance between the pegs and that the configuration of the rope is symmetric with respect to the line of the vertical part of the rope.

a. Find the angle A formed at the bottom of the loop in the accompanying figure.

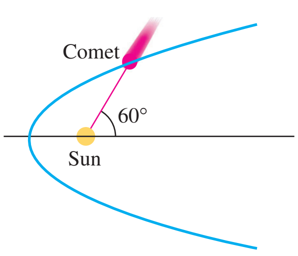
b. Show that for each fixed position of the ring on the rope, the possible locations of the ring in space lie on an ellipse with foci at the pegs.

c. Justify the original symmetry assumption by combining the result in part (b) with the assumption that the rope and weight will take a rest position of minimal potential energy.



30. Two radar stations lie 20 km apart along an east–west line. A low-flying plane traveling from west to east is known to have a speed of v_0 km/sec. At $t = 0$ a signal is sent from the station at $(-10, 0)$, bounces off the plane, and is received at $(10, 0)$ $30/c$ seconds later (c is the velocity of the signal). When $t = 10/v_0$, another signal is sent out from the station at $(-10, 0)$, reflects off the plane, and is once again received $30/c$ seconds later by the other station. Find the position of the plane when it reflects the second signal under the assumption that v_0 is much less than c.

31. A comet moves in a parabolic orbit with the sun at the focus. When the comet is 4×10^7 miles from the sun, the line from the comet to the sun makes a 60° angle with the orbit's axis, as shown here. How close will the comet come to the sun?



32. Find the points on the parabola $x = 2t, y = t^2, -\infty < t < \infty$, closest to the point (0, 3).

33. Find the eccentricity of the ellipse $x^2 + xy + y^2 = 1$ to the nearest hundredth.

34. Find the eccentricity of the hyperbola $xy = 1$.

35. Is the curve $\sqrt{x} + \sqrt{y} = 1$ part of a conic section? If so, what kind of conic section? If not, why not?

36. Show that the curve $2xy - \sqrt{2}\,y + 2 = 0$ is a hyperbola. Find the hyperbola's center, vertices, foci, axes, and asymptotes.

37. Find a polar coordinate equation for

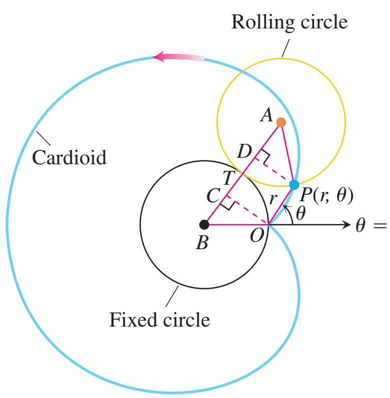
a. the parabola with focus at the origin and vertex at $(a, \pi/4)$;

b. the ellipse with foci at the origin and (2, 0) and one vertex at (4, 0);

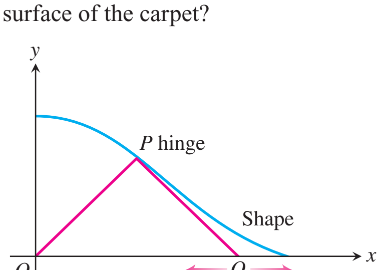
c. the hyperbola with one focus at the origin, center at $(2, \pi/2)$, and a vertex at $(1, \pi/2)$.

38. Any line through the origin will intersect the ellipse $r = 3/(2 + \cos\theta)$ in two points P_1 and P_2. Let d_1 be the distance between P_1 and the origin and let d_2 be the distance between P_2 and the origin. Compute $(1/d_1) + (1/d_2)$.

39. Generating a cardioid with circles Cardioids are special epicycloids (Exercise 18). Show that if you roll a circle of radius a about another circle of radius a in the polar coordinate plane, as in the accompanying figure, the original point of contact P will trace a cardioid. (*Hint:* Start by showing that angles OBC and PAD both have measure θ.)



40. A bifold closet door A bifold closet door consists of two 1-ft-wide panels, hinged at point P. The outside bottom corner of one panel rests on a pivot at O (see the accompanying figure). The outside bottom corner of the other panel, denoted by Q, slides along a straight track, shown in the figure as a portion of the x-axis. Assume that as Q moves back and forth, the bottom of the door rubs against a thick carpet. What shape will the door sweep out on the surface of the carpet?

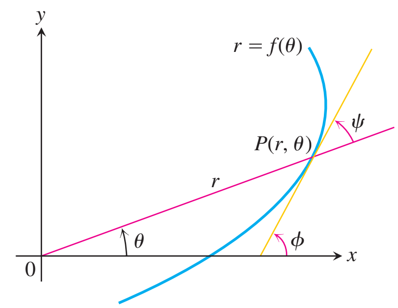


The Angle Between the Radius Vector and the Tangent Line to a Polar Coordinate Curve

In Cartesian coordinates, when we want to discuss the direction of a curve at a point, we use the angle ϕ measured counterclockwise from the positive x-axis to the tangent line. In polar coordinates, it is more convenient to calculate the angle ψ from the *radius vector* to the tangent line (see the accompanying figure). The angle ϕ can then be calculated from the relation

$$\phi = \theta + \psi, \tag{1}$$

which comes from applying the Exterior Angle Theorem to the triangle in the accompanying figure.



Suppose the equation of the curve is given in the form $r = f(\theta)$, where $f(\theta)$ is a differentiable function of θ. Then

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta \tag{2}$$

are differentiable functions of θ with

$$\begin{aligned} \frac{dx}{d\theta} &= -r\sin\theta + \cos\theta\,\frac{dr}{d\theta}, \\ \frac{dy}{d\theta} &= r\cos\theta + \sin\theta\,\frac{dr}{d\theta}. \end{aligned} \tag{3}$$

Since $\psi = \phi - \theta$ from (1),

$$\tan\psi = \tan(\phi - \theta) = \frac{\tan\phi - \tan\theta}{1 + \tan\phi\tan\theta}.$$

Furthermore,

$$\tan\phi = \frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}$$

because $\tan\phi$ is the slope of the curve at P. Also,

$$\tan\theta = \frac{y}{x}.$$

Hence

$$\tan\psi = \frac{\dfrac{dy/d\theta}{dx/d\theta} - \dfrac{y}{x}}{1 + \dfrac{y}{x}\dfrac{dy/d\theta}{dx/d\theta}} = \frac{x\dfrac{dy}{d\theta} - y\dfrac{dx}{d\theta}}{x\dfrac{dx}{d\theta} + y\dfrac{dy}{d\theta}}. \tag{4}$$

The numerator in the last expression in Equation (4) is found from Equations (2) and (3) to be

$$x\frac{dy}{d\theta} - y\frac{dx}{d\theta} = r^2.$$

Similarly, the denominator is

$$x\frac{dx}{d\theta} + y\frac{dy}{d\theta} = r\frac{dr}{d\theta}.$$

When we substitute these into Equation (4), we obtain

$$\tan\psi = \frac{r}{dr/d\theta}. \qquad (5)$$

This is the equation we use for finding ψ as a function of θ.

41. Show, by reference to a figure, that the angle β between the tangents to two curves at a point of intersection may be found from the formula

$$\tan\beta = \frac{\tan\psi_2 - \tan\psi_1}{1 + \tan\psi_2\tan\psi_1}. \qquad (6)$$

When will the two curves intersect at right angles?

42. Find the value of $\tan\psi$ for the curve $r = \sin^4(\theta/4)$.

43. Find the angle between the radius vector to the curve $r = 2a\sin 3\theta$ and its tangent when $\theta = \pi/6$.

T **44.** **a.** Graph the hyperbolic spiral $r\theta = 1$. What appears to happen to ψ as the spiral winds in around the origin?

b. Confirm your finding in part (a) analytically.

45. The circles $r = \sqrt{3}\cos\theta$ and $r = \sin\theta$ intersect at the point $(\sqrt{3}/2, \pi/3)$. Show that their tangents are perpendicular there.

46. Sketch the cardioid $r = a(1 + \cos\theta)$ and circle $r = 3a\cos\theta$ in one diagram and find the angle between their tangents at the point of intersection that lies in the first quadrant.

47. Find the points of intersection of the parabolas

$$r = \frac{1}{1 - \cos\theta} \quad \text{and} \quad r = \frac{3}{1 + \cos\theta}$$

and the angles between their tangents at these points.

48. Find points on the cardioid $r = a(1 + \cos\theta)$ where the tangent line is **(a)** horizontal, **(b)** vertical.

49. Show that parabolas $r = a/(1 + \cos\theta)$ and $r = b/(1 - \cos\theta)$ are orthogonal at each point of intersection $(ab \neq 0)$.

50. Find the angle at which the cardioid $r = a(1 - \cos\theta)$ crosses the ray $\theta = \pi/2$.

51. Find the angle between the line $r = 3\sec\theta$ and the cardioid $r = 4(1 + \cos\theta)$ at one of their intersections.

52. Find the slope of the tangent line to the curve $r = a\tan(\theta/2)$ at $\theta = \pi/2$.

53. Find the angle at which the parabolas $r = 1/(1 - \cos\theta)$ and $r = 1/(1 - \sin\theta)$ intersect in the first quadrant.

54. The equation $r^2 = 2\csc 2\theta$ represents a curve in polar coordinates.

a. Sketch the curve.

b. Find an equivalent Cartesian equation for the curve.

c. Find the angle at which the curve intersects the ray $\theta = \pi/4$.

55. Suppose that the angle ψ from the radius vector to the tangent line of the curve $r = f(\theta)$ has the constant value α.

a. Show that the area bounded by the curve and two rays $\theta = \theta_1$, $\theta = \theta_2$, is proportional to $r_2{}^2 - r_1{}^2$, where (r_1, θ_1) and (r_2, θ_2) are polar coordinates of the ends of the arc of the curve between these rays. Find the factor of proportionality.

b. Show that the length of the arc of the curve in part (a) is proportional to $r_2 - r_1$, and find the proportionality constant.

56. Let P be a point on the hyperbola $r^2\sin 2\theta = 2a^2$. Show that the triangle formed by OP, the tangent at P, and the initial line is isosceles.

Chapter 10 Practice Exercises

Graphing Conic Sections

Sketch the parabolas in Exercises 1–4. Include the focus and directrix in each sketch.

1. $x^2 = -4y$

2. $x^2 = 2y$

3. $y^2 = 3x$

4. $y^2 = -(8/3)x$

Find the eccentricities of the ellipses and hyperbolas in Exercises 5–8. Sketch each conic section. Include the foci, vertices, and asymptotes (as appropriate) in your sketch.

5. $16x^2 + 7y^2 = 112$

6. $x^2 + 2y^2 = 4$

7. $3x^2 - y^2 = 3$

8. $5y^2 - 4x^2 = 20$

Shifting Conic Sections

Exercises 9–14 give equations for conic sections and tell how many units up or down and to the right or left each curve is to be shifted. Find an equation for the new conic section and find the new foci, vertices, centers, and asymptotes, as appropriate. If the curve is a parabola, find the new directrix as well.

9. $x^2 = -12y$, right 2, up 3

10. $y^2 = 10x$, left 1/2, down 1

11. $\frac{x^2}{9} + \frac{y^2}{25} = 1$, left 3, down 5

12. $\frac{x^2}{169} + \frac{y^2}{144} = 1$, right 5, up 12

13. $\frac{y^2}{8} - \frac{x^2}{2} = 1$, right 2, up $2\sqrt{2}$

14. $\frac{x^2}{36} - \frac{y^2}{64} = 1$, left 10, down 3

Identifying Conic Sections

Identify the conic sections in Exercises 15–22 and find their foci, vertices, centers, and asymptotes (as appropriate). If the curve is a parabola, find its directrix as well.

15. $x^2 - 4x - 4y^2 = 0$

16. $4x^2 - y^2 + 4y = 8$

17. $y^2 - 2y + 16x = -49$

18. $x^2 - 2x + 8y = -17$

19. $9x^2 + 16y^2 + 54x - 64y = -1$

20. $25x^2 + 9y^2 - 100x + 54y = 44$

21. $x^2 + y^2 - 2x - 2y = 0$

22. $x^2 + y^2 + 4x + 2y = 1$

Using the Discriminant

What conic sections or degenerate cases do the equations in Exercises 23–28 represent? Give a reason for your answer in each case.

23. $x^2 + xy + y^2 + x + y + 1 = 0$

24. $x^2 + 4xy + 4y^2 + x + y + 1 = 0$

25. $x^2 + 3xy + 2y^2 + x + y + 1 = 0$

26. $x^2 + 2xy - 2y^2 + x + y + 1 = 0$

27. $x^2 - 2xy + y^2 = 0$

28. $x^2 - 3xy + 4y^2 = 0$

Rotating Conic Sections

Identify the conic sections in Exercises 29–32. Then rotate the coordinate axes to find a new equation for the conic section that has no cross product term. (The new equations will vary with the size and direction of the rotations used.)

29. $2x^2 + xy + 2y^2 - 15 = 0$

30. $3x^2 + 2xy + 3y^2 = 19$

31. $x^2 + 2\sqrt{3}\,xy - y^2 + 4 = 0$

32. $x^2 - 3xy + y^2 = 5$

Identifying Parametric Equations in the Plane

Exercises 33–36 give parametric equations and parameter intervals for the motion of a particle in the xy-plane. Identify the particle's path by finding a Cartesian equation for it. Graph the Cartesian equation and indicate the direction of motion and the portion traced by the particle.

33. $x = (1/2)\tan t$, $y = (1/2)\sec t$; $-\pi/2 < t < \pi/2$

34. $x = -2\cos t$, $y = 2\sin t$; $0 \le t \le \pi$

35. $x = -\cos t$, $y = \cos^2 t$; $0 \le t \le \pi$

36. $x = 4\cos t$, $y = 9\sin t$; $0 \le t \le 2\pi$

Graphs in the Polar Plane

Sketch the regions defined by the polar coordinate inequalities in Exercises 37 and 38.

37. $0 \le r \le 6\cos\theta$

38. $-4\sin\theta \le r \le 0$

Match each graph in Exercises 39–46 with the appropriate equation (a)–(l). There are more equations than graphs, so some equations will not be matched.

a. $r = \cos 2\theta$

b. $r\cos\theta = 1$

c. $r = \frac{6}{1 - 2\cos\theta}$

d. $r = \sin 2\theta$

e. $r = \theta$

f. $r^2 = \cos 2\theta$

g. $r = 1 + \cos\theta$

h. $r = 1 - \sin\theta$

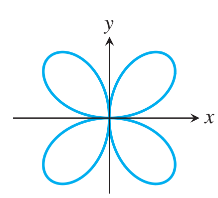
i. $r = \frac{2}{1 - \cos\theta}$

j. $r^2 = \sin 2\theta$

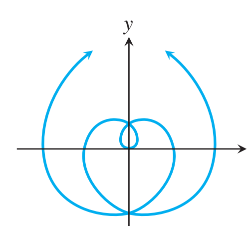
k. $r = -\sin\theta$

l. $r = 2\cos\theta + 1$

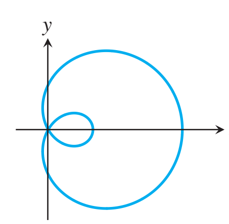
39. Four-leaved rose



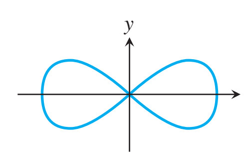
40. Spiral



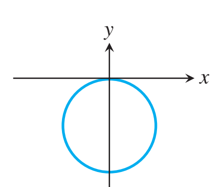
41. Limaçon



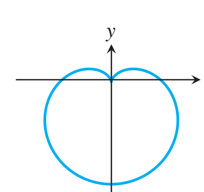
42. Lemniscate



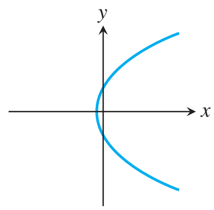
43. Circle



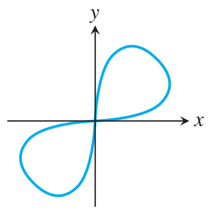
44. Cardioid



45. Parabola



46. Lemniscate



Intersections of Graphs in the Polar Plane

Find the points of intersection of the curves given by the polar coordinate equations in Exercises 47–54.

47. $r = \sin\theta, \quad r = 1 + \sin\theta$ **48.** $r = \cos\theta, \quad r = 1 - \cos\theta$

49. $r = 1 + \cos\theta, \quad r = 1 - \cos\theta$

50. $r = 1 + \sin\theta, \quad r = 1 - \sin\theta$

51. $r = 1 + \sin\theta, \quad r = -1 + \sin\theta$

52. $r = 1 + \cos\theta, \quad r = -1 + \cos\theta$

53. $r = \sec\theta, \quad r = 2\sin\theta$ **54.** $r = -2\csc\theta, \quad r = -4\cos\theta$

Polar to Cartesian Equations

Sketch the lines in Exercises 55–60. Also, find a Cartesian equation for each line.

55. $r\cos\left(\theta + \frac{\pi}{3}\right) = 2\sqrt{3}$ **56.** $r\cos\left(\theta - \frac{3\pi}{4}\right) = \frac{\sqrt{2}}{2}$

57. $r = 2\sec\theta$ **58.** $r = -\sqrt{2}\sec\theta$

59. $r = -(3/2)\csc\theta$ **60.** $r = \left(3\sqrt{3}\right)\csc\theta$

Find Cartesian equations for the circles in Exercises 61–64. Sketch each circle in the coordinate plane and label it with both its Cartesian and polar equations.

61. $r = -4\sin\theta$ **62.** $r = 3\sqrt{3}\sin\theta$

63. $r = 2\sqrt{2}\cos\theta$ **64.** $r = -6\cos\theta$

Cartesian to Polar Equations

Find polar equations for the circles in Exercises 65–68. Sketch each circle in the coordinate plane and label it with both its Cartesian and polar equations.

65. $x^2 + y^2 + 5y = 0$ **66.** $x^2 + y^2 - 2y = 0$

67. $x^2 + y^2 - 3x = 0$ **68.** $x^2 + y^2 + 4x = 0$

Conic Sections in Polar Coordinates

Sketch the conic sections whose polar coordinate equations are given in Exercises 69–72. Give polar coordinates for the vertices and, in the case of ellipses, for the centers as well.

69. $r = \frac{2}{1 + \cos\theta}$ **70.** $r = \frac{8}{2 + \cos\theta}$

71. $r = \frac{6}{1 - 2\cos\theta}$ **72.** $r = \frac{12}{3 + \sin\theta}$

Exercises 73–76 give the eccentricities of conic sections with one focus at the origin of the polar coordinate plane, along with the directrix for that focus. Find a polar equation for each conic section.

73. $e = 2, \quad r\cos\theta = 2$ **74.** $e = 1, \quad r\cos\theta = -4$

75. $e = 1/2, \quad r\sin\theta = 2$ **76.** $e = 1/3, \quad r\sin\theta = -6$

Area, Length, and Surface Area in the Polar Plane

Find the areas of the regions in the polar coordinate plane described in Exercises 77–80.

77. Enclosed by the limaçon $r = 2 - \cos\theta$

78. Enclosed by one leaf of the three-leaved rose $r = \sin 3\theta$

79. Inside the "figure eight" $r = 1 + \cos 2\theta$ and outside the circle $r = 1$

80. Inside the cardioid $r = 2(1 + \sin\theta)$ and outside the circle $r = 2\sin\theta$

Find the lengths of the curves given by the polar coordinate equations in Exercises 81–84.

81. $r = -1 + \cos\theta$

82. $r = 2\sin\theta + 2\cos\theta, \quad 0 \le \theta \le \pi/2$

83. $r = 8\sin^3(\theta/3), \quad 0 \le \theta \le \pi/4$

84. $r = \sqrt{1 + \cos 2\theta}, \quad -\pi/2 \le \theta \le \pi/2$

Find the areas of the surfaces generated by revolving the polar coordinate curves in Exercises 85 and 86 about the indicated axes.

85. $r = \sqrt{\cos 2\theta}, \quad 0 \le \theta \le \pi/4, \quad x$-axis

86. $r^2 = \sin 2\theta, \quad y$-axis

Theory and Examples

87. Find the volume of the solid generated by revolving the region enclosed by the ellipse $9x^2 + 4y^2 = 36$ about **(a)** the x-axis, **(b)** the y-axis.

88. The "triangular" region in the first quadrant bounded by the x-axis, the line $x = 4$, and the hyperbola $9x^2 - 4y^2 = 36$ is revolved about the x-axis to generate a solid. Find the volume of the solid.

89. A ripple tank is made by bending a strip of tin around the perimeter of an ellipse for the wall of the tank and soldering a flat bottom onto this. An inch or two of water is put in the tank and you drop a marble into it, right at one focus of the ellipse. Ripples radiate outward through the water, reflect from the strip around the edge of the tank, and a few seconds later a drop of water spurts up at the second focus. Why?

90. LORAN A radio signal was sent simultaneously from towers A and B, located several hundred miles apart on the northern California coast. A ship offshore received the signal from A 1400 microseconds before receiving the signal from B. Assuming that the signals traveled at the rate of 980 ft/microsecond, what can be said about the location of the ship relative to the two towers?

91. On a level plane, at the same instant, you hear the sound of a rifle and that of the bullet hitting the target. What can be said about your location relative to the rifle and target?

92. Archimedes spirals The graph of an equation of the form $r = a\theta$, where a is a nonzero constant, is called an *Archimedes spiral*. Is there anything special about the widths between the successive turns of such a spiral?

93. a. Show that the equations $x = r\cos\theta$, $y = r\sin\theta$ transform the polar equation

$$r = \frac{k}{1 + e\cos\theta}$$

into the Cartesian equation

$$(1 - e^2)x^2 + y^2 + 2kex - k^2 = 0.$$

b. Then apply the criteria of Section 10.3 to show that

$$\begin{aligned} e = 0 &\Rightarrow \text{circle.} \\ 0 < e < 1 &\Rightarrow \text{ellipse.} \\ e = 1 &\Rightarrow \text{parabola.} \\ e > 1 &\Rightarrow \text{hyperbola.} \end{aligned}$$

94. A satellite orbit A satellite is in an orbit that passes over the North and South Poles of the earth. When it is over the South Pole it is at the highest point of its orbit, 1000 miles above the earth's surface. Above the North Pole it is at the lowest point of its orbit, 300 miles above the earth's surface.

a. Assuming that the orbit is an ellipse with one focus at the center of the earth, find its eccentricity. (Take the diameter of the earth to be 8000 miles.)

b. Using the north–south axis of the earth as the x-axis and the center of the earth as origin, find a polar equation for the orbit.

Chapter 10 Questions to Guide Your Review

1. What is a parabola? What are the Cartesian equations for parabolas whose vertices lie at the origin and whose foci lie on the coordinate axes? How can you find the focus and directrix of such a parabola from its equation?
2. What is an ellipse? What are the Cartesian equations for ellipses centered at the origin with foci on one of the coordinate axes? How can you find the foci, vertices, and directrices of such an ellipse from its equation?
3. What is a hyperbola? What are the Cartesian equations for hyperbolas centered at the origin with foci on one of the coordinate axes? How can you find the foci, vertices, and directrices of such an ellipse from its equation?
4. What is the eccentricity of a conic section? How can you classify conic sections by eccentricity? How are an ellipse's shape and eccentricity related?
5. Explain the equation $PF = e \cdot PD$.
6. What is a quadratic curve in the xy-plane? Give examples of degenerate and nondegenerate quadratic curves.
7. How can you find a Cartesian coordinate system in which the new equation for a conic section in the plane has no xy-term? Give an example.
8. How can you tell what kind of graph to expect from a quadratic equation in x and y? Give examples.
9. What are some typical parametrizations for conic sections?
10. What is a cycloid? What are typical parametric equations for cycloids? What physical properties account for the importance of cycloids?
11. What are polar coordinates? What equations relate polar coordinates to Cartesian coordinates? Why might you want to change from one coordinate system to the other?
12. What consequence does the lack of uniqueness of polar coordinates have for graphing? Give an example.
13. How do you graph equations in polar coordinates? Include in your discussion symmetry, slope, behavior at the origin, and the use of Cartesian graphs. Give examples.
14. How do you find the area of a region $0 \le r_1(\theta) \le r \le r_2(\theta)$, $\alpha \le \theta \le \beta$, in the polar coordinate plane? Give examples.
15. Under what conditions can you find the length of a curve $r = f(\theta)$, $\alpha \le \theta \le \beta$, in the polar coordinate plane? Give an example of a typical calculation.
16. Under what conditions can you find the area of the surface generated by revolving a curve $r = f(\theta)$, $\alpha \le \theta \le \beta$, about the x-axis? The y-axis? Give examples of typical calculations.
17. What are the standard equations for lines and conic sections in polar coordinates? Give examples.

Chapter 10 Technology Application Projects

Mathematica/Maple Module

Radar Tracking of a Moving Object

Part I: Convert from polar to Cartesian coordinates.

Mathematica/Maple Module

Parametric and Polar Equations with a Figure Skater

Part I: Visualize position, velocity, and acceleration to analyze motion defined by parametric equations.

Part II: Find and analyze the equations of motion for a figure skater tracing a polar plot.