$\boxed{11.2\text{-}1}$

$$d(KE) = \frac{1}{2} v^2 dm, \quad KE = \int \frac{1}{2} v^2 dm$$

disp. $= \underset{\sim}{N}\,\underset{\sim}{d}$, $\quad v = \underset{\sim}{N}\,\underset{\sim}{\dot{d}}$, $\quad dm = \rho\, dV$

$$v^2 = v^T v = \underset{\sim}{\dot{d}}^T \underset{\sim}{N}^T \underset{\sim}{N}\,\underset{\sim}{\dot{d}}$$

$$KE = \frac{1}{2} \underset{\sim}{\dot{d}}^T \underbrace{\int \underset{\sim}{N}^T \underset{\sim}{N} \rho\, dV}_{\underset{\sim}{m}} \underset{\sim}{\dot{d}} = \frac{1}{2} \underset{\sim}{\dot{d}}^T \underset{\sim}{m}\, \underset{\sim}{\dot{d}}$$

More generally, with displacement components u, v, and w,

$\begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = \underset{\sim}{N}\,\underset{\sim}{d}$, velocities are $\begin{Bmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{Bmatrix} = \underset{\sim}{N}\,\underset{\sim}{\dot{d}}$

The resultant velocity squared is

$$\begin{Bmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{Bmatrix}^T \begin{Bmatrix} \dot{u} \\ \dot{v} \\ \dot{w} \end{Bmatrix} = \underset{\sim}{\dot{d}}\, \underset{\sim}{N}^T \underset{\sim}{N}\, \underset{\sim}{\dot{d}} \quad \text{(as above)}$$

$\boxed{11.2\text{-}2}$

$$\{\underset{\sim}{u}\} = [\underset{\sim}{N}]\{\underset{\sim}{d}\}, \text{ so } \{\ddot{\underset{\sim}{u}}\} = [\underset{\sim}{N}]\{\ddot{\underset{\sim}{d}}\}.$$

$$\{\underset{\sim}{r}\} = [\underset{\sim}{m}]\{\ddot{\underset{\sim}{d}}\} = \int [\underset{\sim}{N}]^T[\underset{\sim}{N}]\rho\{\ddot{\underset{\sim}{d}}\}dV = \int [\underset{\sim}{N}]^T\rho\{\ddot{\underset{\sim}{u}}\}dV$$

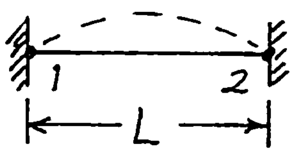
But $\rho\{\ddot{\underset{\sim}{u}}\}$ are "effective" body forces, so

$\{\underset{\sim}{r}\} = \int [\underset{\sim}{N}]^T\{\underset{\sim}{F}\}\,dV$ as in Chapters 3 & 4.

$\boxed{11.3\text{-}1}$

(a) No. $m_{ii} = \int \rho N_i^2 \, dV$, and since ρ, N_i^2, and dV are all positive, $m_{ii} > 0$.

(b) The half-wave of a typical mode is spanned by one element:



Nodes have only rotational d.o.f.

$$\left(\frac{EI}{L^3} \begin{bmatrix} 4L^2 & 2L^2 \\ 2L^2 & 4L^2 \end{bmatrix} - \omega^2 \frac{mL^2}{24} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \begin{Bmatrix} \theta_{z1} \\ \theta_{z2} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

$\theta_{z1} = -\theta_{z2}$, so 1^{st} eq. becomes

$$\left(\frac{2EI}{L} - \omega^2 \frac{mL^2}{24} \right) \theta_{z1} = 0$$

$$\omega^2 = \frac{48EI}{mL^3}, \qquad \omega = 6.93\sqrt{\frac{EI}{mL^3}}$$

$\boxed{11.3\text{-}2}$

For the three respective motions, correct kinetic energies are

$$KE_1 = \frac{1}{2} m v^2, \quad KE_2 = \frac{1}{2}\frac{mL^2}{12}\Omega^2, \quad KE_3 = \frac{1}{2}\frac{mL^2}{3}\Omega^2$$

where $m = \rho A L$. In terms of nodal d.o.f. and mass matrix $\underset{\sim}{m}$, $KE = \frac{1}{2}\dot{\underset{\sim}{d}}^T \underset{\sim}{m}\, \dot{\underset{\sim}{d}}$.

(a) $\underset{\sim}{m} = \frac{m}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\dot{\underset{\sim}{d}}_1 = v[1 \quad 1]^T, \quad KE_1 = \frac{mv^2}{2}$ ✓

$\dot{\underset{\sim}{d}}_2 = \frac{\Omega L}{2}[-1 \quad 1]^T, \quad KE_2 = \frac{mL^2\Omega^2}{8}$ ×

$\dot{\underset{\sim}{d}}_3 = \Omega L[0 \quad 1]^T, \quad KE_3 = \frac{mL^2\Omega^2}{2}$ ×

(b) $\underset{\sim}{m} = \frac{m}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

$\dot{\underset{\sim}{d}}_1 = v[1 \quad 1]^T, \quad KE_1 = \frac{mv^2}{2}$ ✓

$\dot{\underset{\sim}{d}}_2 = \frac{\Omega L}{2}[-1 \quad 1]^T, \quad KE_2 = \frac{mL^2\Omega^2}{24}$ ✓

$\dot{\underset{\sim}{d}} = \Omega[0 \quad 1]^T, \quad KE_3 = \frac{mL^2\Omega^2}{6}$ ✓

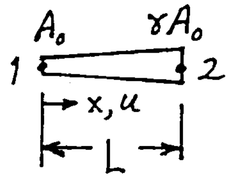
(c) $\underset{\sim}{m} = \frac{m}{420}\Big[$(as in Eq. 11.3-5)$\Big]$

$\dot{\underset{\sim}{d}}_1 = v[1 \quad 0 \quad 1 \quad 0]^T, \quad KE_1 = \frac{mv^2}{2}$ ✓

$\dot{\underset{\sim}{d}}_2$ [illegible] $\frac{L}{2} \quad 1]^T, \quad KE_2 = \frac{mL^2\Omega^2}{24}$ ✓

$\dot{\underset{\sim}{d}}_3 = \Omega[0 \quad 1 \quad L \quad 1]^T, \quad KE_3 = \frac{mL^2\Omega^2}{6}$ ✓

$\boxed{11.3\text{-}3}$



$u = \lfloor \underset{\sim}{N} \rfloor \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$, $A = \lfloor \underset{\sim}{N} \rfloor \begin{Bmatrix} A_0 \\ \gamma A_0 \end{Bmatrix}$

where $\lfloor \underset{\sim}{N} \rfloor = \begin{bmatrix} \frac{L-x}{L} & \frac{x}{L} \end{bmatrix}$

$$[\underset{\sim}{m}] = \int \rho \lfloor \underset{\sim}{N} \rfloor^T \lfloor \underset{\sim}{N} \rfloor A \, dx, \quad A = \frac{A_0}{L}\left[L + (\gamma - 1)x\right]$$

$$[\underset{\sim}{m}] = \frac{\rho A_0}{L^3} \int_0^L \begin{bmatrix} (L-x)^2 & x(L-x) \\ x(L-x) & x^2 \end{bmatrix} \left[L + (\gamma-1)x\right] dx$$

$$\int_0^L (L-x)^2 dx = \int_0^L x^2 dx = \frac{L^3}{3}, \quad \int_0^L x(L-x) dx = \frac{L^3}{6}$$

$$\int_0^L (L-x)^2 x \, dx = \frac{L^4}{12}, \quad \int_0^L x^2 (L-x) dx = \frac{L^4}{12}, \quad \int_0^L x^3 dx = \frac{L^4}{4}$$

$$[\underset{\sim}{m}] = \rho A_0 L \left(\begin{bmatrix} 1/3 & 1/6 \\ 1/6 & 1/3 \end{bmatrix} + (\gamma - 1) \begin{bmatrix} 1/12 & 1/12 \\ 1/12 & 1/4 \end{bmatrix} \right)$$

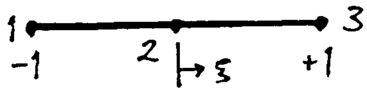
$$[\underset{\sim}{m}] = \frac{\rho A_0 L}{12} \begin{bmatrix} 4 + (\gamma-1) & 2 + (\gamma-1) \\ 2 + (\gamma-1) & 4 + 3(\gamma-1) \end{bmatrix}$$

(c) $\{\ddot{\underset{\sim}{d}}\} = \begin{Bmatrix} a \\ 0 \\ a \\ 0 \end{Bmatrix}$, $[\underset{\sim}{m}]\{\ddot{\underset{\sim}{d}}\} = \dfrac{ma}{6} \begin{Bmatrix} 3 \\ 0 \\ 3 \\ 0 \end{Bmatrix} = \{\underset{\sim}{r}\}$

$\sum F_x = \lfloor 1 \;\; 0 \;\; 1 \;\; 0 \rfloor \{\underset{\sim}{r}\} = \dfrac{ma}{6} 6 = ma$ ✓

$[\underset{\sim}{m}]\{\ddot{\underset{\sim}{d}}\} = \dfrac{ma}{2} \begin{Bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{Bmatrix} = \{\underset{\sim}{r}\}$, $\sum F_x = \dfrac{ma}{2} 2 = ma$ ✓

11.3-4



$$[\underset{\sim}{m}] = \int_{-1}^{1} \lfloor \underset{\sim}{N} \rfloor^T \lfloor \underset{\sim}{N} \rfloor \rho A \frac{L}{2} d\xi, \quad \lfloor \underset{\sim}{N} \rfloor = \left[\frac{-\xi+\xi^2}{2}, 1-\xi^2, \frac{\xi+\xi^2}{2} \right]$$

$$\int_{-1}^{1} \left(\frac{-\xi+\xi^2}{2} \right)^2 d\xi = \frac{8}{30}, \quad \int_{-1}^{1} \frac{-\xi+\xi^2}{2} (1-\xi^2) d\xi = \frac{4}{30}$$

$$\int_{-1}^{1} \frac{-\xi+\xi^2}{2} \left(\frac{\xi+\xi^2}{2} \right) d\xi = -\frac{2}{30}, \quad \int_{-1}^{1} (1-\xi^2)^2 d\xi = \frac{32}{30}$$

$$\int_{-1}^{1} (1-\xi^2) \frac{\xi+\xi^2}{2} d\xi = \frac{4}{30}, \quad \int_{-1}^{1} \left(\frac{\xi+\xi^2}{2} \right)^2 d\xi = \frac{8}{30}$$

with $m = \rho A L$,

$$[\underset{\sim}{m}] = \frac{m}{30} \begin{bmatrix} 4 & 2 & -1 \\ 2 & 16 & 2 \\ -1 & 2 & 4 \end{bmatrix}$$

$\boxed{11.3\text{-}5}$

Exact: $\frac{1}{2} I \Omega^2 = \frac{1}{2} \frac{mL^2}{12} \Omega^2 = \frac{1}{24} mL^2\Omega^2 = 0.04167\, mL^2\Omega^2$

FEA: $KE = \frac{1}{2}\{\dot{\underset{\sim}{d}}\}[\underset{\sim}{m}]\{\dot{\underset{\sim}{d}}\}$

Here $\{\dot{\underset{\sim}{d}}\} = \begin{Bmatrix} -\Omega L/2 \\ \Omega \\ \Omega L/2 \\ \Omega \end{Bmatrix}$

(a) HRZ:

$$[\underset{\sim}{m}]\{\dot{\underset{\sim}{d}}\} = \frac{m}{78}\begin{bmatrix} 39 & & & \\ & L^2 & & \\ & & 39 & \\ & & & L^2 \end{bmatrix}\{\dot{\underset{\sim}{d}}\} = \frac{m\Omega}{78}\begin{Bmatrix} -39L/2 \\ L^2 \\ 39L/2 \\ L^2 \end{Bmatrix}$$

$$KE = \frac{1}{2}\{\dot{\underset{\sim}{d}}\}^T\left([\underset{\sim}{m}]\{\dot{\underset{\sim}{d}}\}\right) = \frac{1}{2}\frac{m\Omega^2L^2}{78}(21.5) = 0.1378\, m\Omega^2L^2$$

Error is 231% (high)

(b) Particle-lumped:

$$[\underset{\sim}{m}]\{\dot{\underset{\sim}{d}}\} = \frac{m}{2}\begin{bmatrix} 1 & & & \\ & 0 & & \\ & & 1 & \\ & & & 0 \end{bmatrix}\{\dot{\underset{\sim}{d}}\} = \frac{m\Omega}{2}\begin{Bmatrix} -L/2 \\ 0 \\ L/2 \\ 0 \end{Bmatrix}$$

$$KE = \frac{1}{2}\{\dot{\underset{\sim}{d}}\}^T\left([\underset{\sim}{m}]\{\dot{\underset{\sim}{d}}\}\right) = \frac{1}{2} m\Omega^2L^2\left(\frac{1}{4}\right) = 0.1250\, m\Omega^2L^2$$

Error is 200% (high)

$\boxed{11.3-6}$

By FEA:

$$KE = \frac{1}{2}\{\dot{\underset{\sim}{d}}\}^T[\underset{\sim}{m}]\{\dot{\underset{\sim}{d}}\} = \frac{1}{2}\left\lfloor -\frac{\Omega L}{2} \quad \Omega \quad \frac{\Omega L}{2} \quad \Omega \right\rfloor \begin{bmatrix} m/2 & & & \\ & m\alpha L^2 & & \\ & & m/2 & \\ & & & m\alpha L^2 \end{bmatrix} \begin{Bmatrix} -\Omega L/2 \\ \Omega \\ \Omega L/2 \\ \Omega \end{Bmatrix}$$

$$KE = \frac{m\Omega^2}{2}\left\lfloor -\frac{L}{2} \quad 1 \quad \frac{L}{2} \quad 1 \right\rfloor \begin{Bmatrix} -L/4 \\ \alpha L^2 \\ L/4 \\ \alpha L^2 \end{Bmatrix} = \frac{m\Omega^2 L^2}{2}\left(\frac{1}{4} + 2\alpha\right)$$

Exact KE: $\frac{1}{2} I \Omega^2 = \frac{1}{2}\frac{mL^2}{12}\Omega^2$

Equate KE's: $\frac{1}{12} = \frac{1}{4} + 2\alpha \quad$ so $\alpha = -\frac{1}{12}$

For a one-element s.s. beam, only the (negative) rotational masses remain. Hence ω^2 is negative and natural frequencies ω are imaginary. We conclude that negative masses are dangerous.

11.3-7

(a) Consider x-direction motion first.

$\{\text{nodal forces}\} = [\underset{\sim}{m}_x]\lfloor \ddot{u}_1 \quad \ddot{u}_2 \quad \ddot{u}_3 \rfloor^T$

$u = \lfloor \underset{\sim}{N} \rfloor \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$ where $\lfloor \underset{\sim}{N} \rfloor = \lfloor \xi_1 \quad \xi_2 \quad \xi_3 \rfloor$ and

ξ_1, ξ_2, ξ_3 are area coordinates (Chap. 5).

$$[\underset{\sim}{m}_x] = \rho t \int_A \lfloor \underset{\sim}{N} \rfloor^T \lfloor \underset{\sim}{N} \rfloor \, dA = \rho t \int_A \begin{bmatrix} \xi_1^2 & \xi_1\xi_2 & \xi_1\xi_3 \\ \xi_1\xi_2 & \xi_2^2 & \xi_2\xi_3 \\ \xi_1\xi_3 & \xi_2\xi_3 & \xi_3^2 \end{bmatrix} dA$$

Use Eq. to integrate; result is $[\underset{\sim}{m}_x] = \frac{\rho t A}{12} \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$

Since the y-dir. field has the same form, the $[\underset{\sim}{m}]$ that operates on $\lfloor u_1 \; v_1 \; u_2 \; v_2 \; u_3 \; v_3 \rfloor^T$ is

(b)

$$[\underset{\sim}{m}] = \frac{\rho t A}{12} \begin{bmatrix} 2 & 0 & 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 & 0 & 1 \\ 1 & 0 & 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 & 0 & 1 \\ 1 & 0 & 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 1 & 0 & 2 \end{bmatrix}$$

$\boxed{11.3\text{-}8}$ For the entire element, $J = \frac{A}{4} = \frac{ab}{4}$

$$m_{ij} = \iint \rho N_i N_j t \, dx \, dy = \rho t J \int_{-1}^{1} \int_{-1}^{1} N_i N_j \, d\xi \, d\eta$$

$$N_1 = \frac{1}{4}(1-\xi)(1-\eta), \quad N_2 = \frac{1}{4}(1+\xi)(1-\eta), \quad N_3 = \frac{1}{4}(1+\xi)(1+\eta), \quad N_4 = \frac{1}{4}(1-\xi)(1+\eta)$$

$$N_1^2 = \frac{1}{4^2}(1-\xi)^2(1-\eta)^2 = \frac{1}{4^2}(1-2\xi+\xi^2)(1-2\eta+\eta^2)$$

$$N_1 N_2 = \frac{1}{4^2}(1-\xi^2)(1-\eta)^2 = \frac{1}{4^2}(1-\xi^2)(1-2\eta+\eta^2)$$

$$N_1 N_3 = \frac{1}{4^2}(1-\xi^2)(1-\eta^2)$$

$$N_1 N_4 = \frac{1}{4^2}(1-\xi)^2(1-\eta^2) = \frac{1}{4^2}(1-2\xi+\xi^2)(1-\eta^2)$$

$$N_2 N_4 = \frac{1}{4^2}(1-\xi^2)(1-\eta^2)$$

$$N_2 N_3 = \frac{1}{4^2}(1+\xi)^2(1-\eta^2) = \frac{1}{4^2}(1+2\xi+\xi^2)(1-\eta^2)$$

$$N_2^2 = \frac{1}{4^2}(1+\xi)^2(1-\eta)^2 = \frac{1}{4^2}(1+2\xi+\xi^2)(1-2\xi+\eta^2)$$

$$N_3^2 = \frac{1}{4^2}(1+\xi)^2(1+\eta)^2 = \frac{1}{4^2}(1+2\xi+\xi^2)(1+2\eta+\eta^2)$$

$$N_3 N_4 = \frac{1}{4^2}(1-\xi^2)(1+\eta)^2 = \frac{1}{4^2}(1-\xi^2)(1+2\eta+\eta^2)$$

$$N_4^2 = \frac{1}{4^2}(1-\xi)^2(1+\eta)^2 = \frac{1}{4}(1-2\xi+\xi^2)(1+2\eta+\eta^2)$$

Note that first powers of ξ and η integrate to zero. Therefore

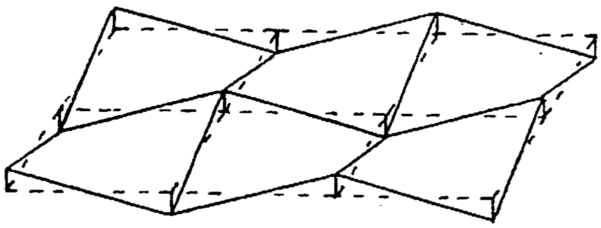
$$m_{11} = m_{22} = m_{33} = m_{44}, \quad m_{12} = m_{14} = m_{23} = m_{34}, \quad m_{13} = m_{24}$$

$$m_{11} = \frac{\rho abt}{4^3} \int_{-1}^{1} \int_{-1}^{1} (1+\xi^2)(1+\eta^2) \, d\xi \, d\eta = \frac{\rho abt}{4^3} \left[\xi + \frac{\xi^3}{3}\right]_{-1}^{1} \left[\eta + \frac{\eta^3}{3}\right]_{-1}^{1}$$

$$m_{11} = \frac{\rho abt}{4^3}\left(2+\frac{2}{3}\right)\left(2+\frac{2}{3}\right) = \frac{\rho abt}{9} = 4\frac{\rho abt}{36}$$

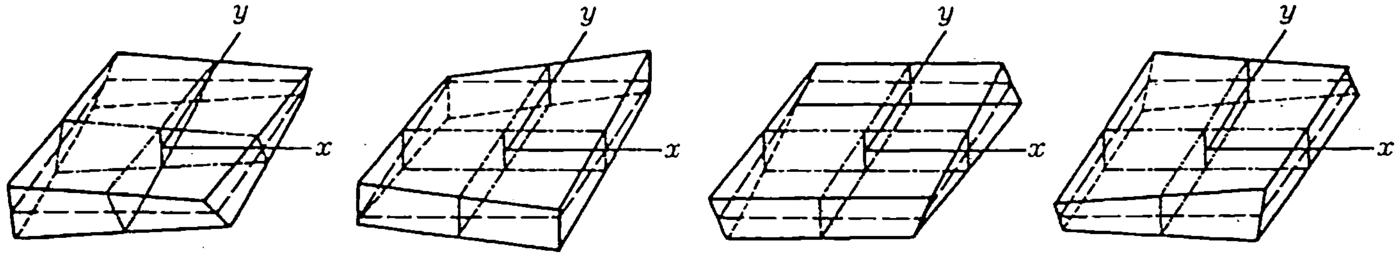
(continues)

11.3-7



$\boxed{11.3-8}$

$$\{\underset{\sim}{\kappa}_M\} = \begin{Bmatrix} \psi_{x,x} \\ \psi_{y,y} \\ \psi_{x,y} + \psi_{y,x} \\ \psi_x - w_{,x} \\ \psi_y - w_{,y} \end{Bmatrix} \begin{matrix} \text{curvature} \\ \text{curvature} \\ \text{twist} \\ \gamma_{zx} \\ \gamma_{yz} \end{matrix}$$



In-plane twist mode $w=0, \Psi_x=-y, \Psi_y=x$ | w-hourglass mode $w=xy, \Psi_x=\Psi_y=0$ | Ψ_x-hourglass mode $w=0, \Psi_x=xy, \Psi_y=0$ | Ψ_y-hourglass mode $w=0, \Psi_x=0, \Psi_y=xy$

For the above four modes, $\{\underset{\sim}{\kappa}_M\}$ is

$$\begin{array}{ccccl} \begin{Bmatrix} 0 \\ 0 \\ -1+1=0 \\ x \\ -y \end{Bmatrix} & \begin{Bmatrix} 0 \\ 0 \\ 0 \\ -x \\ -y \end{Bmatrix} & \begin{Bmatrix} y \\ 0 \\ x \\ 0 \\ xy \end{Bmatrix} & \begin{Bmatrix} 0 \\ x \\ y \\ xy \\ 0 \end{Bmatrix} & \begin{matrix} \text{bending} \\ \text{\& twist} \\ \text{transverse} \\ \text{shear} \end{matrix} \\ \text{in-plane twist} & w\text{-hourglass} & \psi_x\text{-hourglass} & \psi_y\text{-hourglass} & \end{array}$$

Reduced integration: one Gauss pt. at $x=y=0$, so all 4 above vectors are zero there.

Selective integration: one point for shear terms (at $x=y=0$), four points for bending terms. The four points have nonzero x and y, and so detect the curvature and twist [illegible] with the ψ_x and ψ_y hourglass modes; they cease to be mechanisms.

$\boxed{11.3-8}$ (concluded)

$$m_{12} = \frac{\rho abt}{4^3}\int_{-1}^{1}\int_{-1}^{1}(1-\xi^2)(1-2\eta+\eta^2)\,d\xi\,d\eta = \frac{\rho abt}{4^3}\left[\xi-\frac{\xi^3}{3}\right]_{-1}^{1}\left[\eta+\frac{\eta^3}{3}\right]_{-1}^{1}$$

$$m_{12} = \frac{\rho abt}{4^3}\left(2-\frac{2}{3}\right)\left(2+\frac{2}{3}\right) = \frac{\rho abt}{18} = 2\,\frac{\rho abt}{36}$$

$$m_{13} = \frac{\rho abt}{4^3}\int_{-1}^{1}\int_{-1}^{1}(1-\xi^2)(1-\eta^2)\,d\xi\,d\eta = \frac{\rho abt}{4^3}\left[\xi-\frac{\xi^3}{3}\right]_{-1}^{1}\left[\eta-\frac{\eta^3}{3}\right]_{-1}^{1}$$

$$m_{13} = \frac{\rho abt}{4^3}\left(2-\frac{2}{3}\right)\left(2-\frac{2}{3}\right) = \frac{\rho abt}{36}$$

$$[\underset{\sim}{m}] = \frac{\rho abt}{36}\begin{bmatrix} 4 & 2 & 1 & 2 \\ & 4 & 2 & 1 \\ & & 4 & 2 \\ \text{symm.} & & & 4 \end{bmatrix}$$

$\boxed{11.3\text{-}9}$

$\underset{\sim}{d}_A = \underset{\sim}{T}\,\underset{\sim}{d}_B$ where the d.o.f. are

$\underset{\sim}{d}_A = [u_A \quad v_A \quad \theta_A]^T$ at A

$\underset{\sim}{d}_B = [u_B \quad v_B \quad \theta_B]^T$ at node B

The relational (transformation) matrix is

$$\underset{\sim}{T} = \begin{bmatrix} 1 & 0 & -L \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Mass matrix for d.o.f. $\underset{\sim}{d}_A$ is $\underset{\sim}{m}_A = \begin{bmatrix} m & & \\ & m & \\ & & 0 \end{bmatrix}$

For d.o.f. $\underset{\sim}{d}_B$ it is

$$\underset{\sim}{m}_B = \underset{\sim}{T}^T \underset{\sim}{m}_A \underset{\sim}{T} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -L & 0 & 1 \end{bmatrix} \begin{bmatrix} m & 0 & -mL \\ 0 & m & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\underset{\sim}{m}_B = \begin{bmatrix} m & 0 & -mL \\ 0 & m & 0 \\ -mL & 0 & mL^2 \end{bmatrix}$$

A diagonal form of $\underset{\sim}{m}_B$, call it $\underset{\sim}{m}_{BD}$, would yield from $\underset{\sim}{m}_{BD}\ddot{\underset{\sim}{d}}_B$ —

- No torque associated with θ_B if there is translational accel. $\ddot{u}_B$
- No force associated with u_B if there is angular acceleration $\ddot{\theta}_B$.

11.3-10

(a) Take node 5 as typical side node.
From Eqs. 6.4-1, $N_5 = \frac{1}{2}(1-\xi^2)(1-\eta)$

$$\int_{-1}^{1}\int_{-1}^{1} N_5^2 \, d\xi \, d\eta = \frac{1}{4}\int_{-1}^{1}\int_{-1}^{1}(1-\xi^2)^2(1-\eta)^2 d\xi \, d\eta =$$

$$\frac{1}{4}\left[2\left(1-\frac{2}{3}+\frac{1}{5}\right)2\left(1+\frac{1}{3}\right)\right] = \frac{8}{15}\,\frac{4}{3} = \frac{32}{45}$$

Take node 3 as typical corner node. From Eqs. 6.4-1, $N_3 = \frac{1}{4}(1+\xi+\eta+\xi\eta) - \frac{1}{4}(1+\xi-\eta^2-\xi\eta^2)$
$- \frac{1}{4}(1-\xi^2+\eta-\xi^2\eta)$

$$N_3 = \frac{1}{4}(-1+\xi^2+\eta^2+\xi\eta+\xi\eta^2+\xi^2\eta)$$

Since odd powers will integrate to zero, let's discard them in N_3^2. Thus, what's left is

$$N_3^2 = \frac{1}{16}(1+\xi^4+\eta^4+3\xi^2\eta^2+\xi^2\eta^4+\xi^4\eta^2-2\xi^2-2\eta^2)$$

$$\int_{-1}^{1} N_3^2 \, d\xi = \frac{1}{8}\left(\frac{8}{15}-\frac{4}{5}\eta^2+\frac{4}{3}\eta^4\right)$$

$$\int_{-1}^{1}\int_{-1}^{1} N_3^2 \, d\xi \, d\eta = \frac{1}{4}\left(\frac{8}{15}-\frac{4}{15}+\frac{4}{15}\right) = \frac{2}{15}$$

$$s = \sum_{1}^{8} m_{ii} = \rho t \frac{A}{4}\left(4\,\frac{32}{45}+4\,\frac{2}{15}\right) = \rho t A \frac{38}{45}$$

$$m = \rho t A \,; \quad \frac{m}{s} = \frac{45}{38}$$

$$\text{side node } m_{ii} = \frac{45}{38}\left(\rho t \frac{A}{4}\,\frac{32}{45}\right) = \frac{8}{38}\rho A t = \frac{16}{76} m$$

$$\text{corner node } m_{ii} = \frac{45}{38}\left(\rho t \frac{A}{4}\,\frac{2}{15}\right) = \frac{1.5}{38}\rho A t = \frac{3}{76} m$$

(b) Take node 5 as typical side node.
From Table 6.6-1, $N_5 = \frac{1}{2}(1-\xi^2)(\eta^2-\eta)$

$$\int_{-1}^{1}\int_{-1}^{1} N_5^2 \, d\xi \, d\eta = \frac{1}{4}\left[2\left(1-\frac{2}{3}+\frac{1}{5}\right)2\left(\frac{1}{5}+\frac{1}{3}\right)\right] = \left(\frac{8}{15}\right)^2$$

Take node 1 as typical corner node. From Table 6.4-1, $N_1 = \frac{\xi\eta}{4}(1-\xi)(1-\eta)$. Dropping odd powers, which will integrate to zero,

$$N_1^2 = \frac{1}{16}(\xi^2\eta^2+\xi^4\eta^2+\xi^2\eta^4+\xi^4\eta^4)$$

$$\int_{-1}^{1}\int_{-1}^{1} N_1^2 \, d\xi \, d\eta = \frac{1}{4}\left(\frac{1}{3}\,\frac{1}{3}+\frac{1}{5}\,\frac{1}{3}+\frac{1}{3}\,\frac{1}{5}+\frac{1}{5}\,\frac{1}{5}\right) = \frac{16}{225}$$

Center node: $\int_{-1}^{1}\int_{-1}^{1}(1-\xi^2)^2(1-\eta^2)^2 d\xi \, d\eta$

$$= \left[2\left(1-\frac{2}{3}+\frac{1}{5}\right)\right]^2 = \left(\frac{16}{15}\right)^2$$

$$s = \sum_{1}^{9} m_{ii} = \rho t \frac{A}{4}\left[\left(\frac{8}{15}\right)^2 4 + \frac{16}{225}4 + \left(\frac{16}{15}\right)^2\right]$$

$$s = \rho t A (0.64), \quad m = \rho t A, \quad \frac{m}{s} = \frac{1}{0.64}$$

$$\text{side node } m_{ii} = \frac{1}{0.64}\left[\rho t \frac{A}{4}\left(\frac{8}{15}\right)^2\right] = \rho t A (0.1111) = \frac{m}{9} = \frac{4m}{36}$$

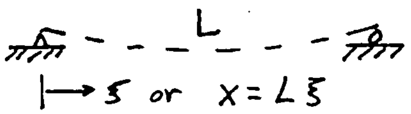
$$\text{corner node } m_{ii} = \frac{1}{0.64}\left[\rho t \frac{A}{4}\left(\frac{16}{225}\right)\right] = \rho t A (0.0278) = \frac{m}{36}$$

$$\text{center node } m_{ii} = \frac{1}{0.64}\left[\rho t \frac{A}{4}\left(\frac{16}{15}\right)^2\right] = \rho t A (0.444) = \frac{16 m}{36}$$

$\boxed{11.3\text{-}11}$

(a) For $\theta_1 = \theta_2$, $v = (1-\xi)v_1 + \xi v_2$, which is a linear function

Next consider a simply supported beam ($v_1 = v_2 = 0$) with $\theta_{z1} = -\theta_{z2}$.



$$v = \frac{L}{2}(\xi - \xi^2)(-\theta_{z2}) + \frac{L}{2}(-\xi + \xi^2)\theta_{z2}$$

$$v = L(\xi^2 - \xi)\theta_{z2} = L\left(\frac{x^2}{L^2} - \frac{x}{L}\right)\theta_{z2}$$

$$\frac{dv}{dx} = L\left(\frac{2x}{L^2} - \frac{1}{L}\right)\theta_{z2} \qquad \frac{d^2v}{dx^2} = \frac{2}{L}\theta_{z2} \qquad \text{(A)}$$

Beam theory: consider constant curvature, $d^2v/dx^2 = c_1$

Then $dv/dx = c_1 x + c_2$ and $v = c_1 x^2/2 + c_2 x + c_3$

Boundary conditions: $v = 0$ at $x = 0$, so $c_3 = 0$

$v = 0$ at $x = L$, so $c_2 = -c_1 L/2$

$$\frac{(dv/dx)_{x=L}}{d^2v/dx^2} = \frac{c_1 L + c_2}{c_1} = \frac{L}{2}$$

agree

And from Eqs. (A),

$$\frac{(dv/dx)_{x=L}}{d^2v/dx^2} = L\left(\frac{2}{L} - \frac{1}{L}\right)\theta_{z2} \Big/ (2/L)\theta_{z2} = \frac{L}{2}$$

(b) $$\lfloor \tilde{N} \rfloor = \left\lfloor \left(1 - \frac{x}{L}\right) \quad \left(\frac{x}{2} - \frac{x^2}{2L}\right) \quad \frac{x}{L} \quad \left(-\frac{x}{2} + \frac{x^2}{2L}\right) \right\rfloor$$

$$[\tilde{m}] = \rho A \int_0^L \lfloor \tilde{N} \rfloor^T \lfloor \tilde{N} \rfloor \, dx$$

By tedious expansion and integration, and with $m = \rho A L$,

$$[\tilde{m}] = \frac{m}{120}\begin{bmatrix} 40 & 5L & 20 & -5L \\ 5L & L^2 & 5L & -L^2 \\ 20 & 5L & 40 & -5L \\ -5L & -L^2 & -5L & L^2 \end{bmatrix}$$

(c) $$s = 2\left(40\frac{m}{120}\right) = \frac{2m}{3}, \qquad \frac{m}{s} = \frac{3}{2}$$

$$[\tilde{m}] = \frac{m}{120}\left\lceil 60 \quad \frac{3L^2}{2} \quad 60 \quad \frac{3L^2}{2} \right\rfloor = \frac{m}{80}\left\lceil 40 \quad L^2 \quad 40 \quad L^2 \right\rfloor$$

$\boxed{11.4-1}$

Let λ, $\Delta\lambda$, and amplitude $\mathbf{D}$ correspond to mode i.

$$(\mathbf{D}+\Delta\mathbf{D})^T(\mathbf{K}+\Delta\mathbf{K})(\mathbf{D}+\Delta\mathbf{D}) = (\lambda+\Delta\lambda)(\mathbf{D}+\Delta\mathbf{D})^T(\mathbf{M}+\Delta\mathbf{M})(\mathbf{D}+\Delta\mathbf{D})$$

$$\mathbf{D}^T\mathbf{K}\mathbf{D} + \mathbf{D}^T\mathbf{K}\Delta\mathbf{D} + \Delta\mathbf{D}^T\mathbf{K}\mathbf{D} + \mathbf{D}^T\Delta\mathbf{K}\mathbf{D} + (\text{higher terms}) = (\lambda\mathbf{D}^T\mathbf{M}\mathbf{D} + \Delta\lambda\mathbf{D}^T\mathbf{M}\mathbf{D}) + \lambda(\mathbf{D}^T\mathbf{M}\Delta\mathbf{D} + \Delta\mathbf{D}^T\mathbf{M}\mathbf{D} + \mathbf{D}^T\Delta\mathbf{M}\mathbf{D} + \text{higher terms})$$

But $\mathbf{D}^T\mathbf{K}\mathbf{D} = \lambda\mathbf{D}^T\mathbf{M}\mathbf{D}$ from Eq. 11.4-13.
Also $\mathbf{D}^T\mathbf{K}\Delta\mathbf{D} = \Delta\mathbf{D}^T\mathbf{K}\mathbf{D}$, $\mathbf{D}^T\mathbf{M}\Delta\mathbf{D} = \Delta\mathbf{D}^T\mathbf{M}\mathbf{D}$.
Thus, and dropping higher-order terms,

$$2\Delta\mathbf{D}^T\mathbf{K}\mathbf{D} + \mathbf{D}^T\Delta\mathbf{K}\mathbf{D} = \Delta\lambda\mathbf{D}^T\mathbf{M}\mathbf{D} + \lambda(2\Delta\mathbf{D}^T\mathbf{M}\mathbf{D} + \mathbf{D}^T\Delta\mathbf{M}\mathbf{D})$$

or

$$2\Delta\mathbf{D}^T\underbrace{(\mathbf{K}-\lambda\mathbf{M})\mathbf{D}}_{\text{zero}} + \mathbf{D}^T(\Delta\mathbf{K}-\lambda\Delta\mathbf{M})\mathbf{D} = \Delta\lambda\mathbf{D}^T\mathbf{M}\mathbf{D}$$

Finally $$\Delta\lambda = \frac{\mathbf{D}^T(\Delta\mathbf{K}-\lambda\Delta\mathbf{M})\mathbf{D}}{\mathbf{D}^T\mathbf{M}\mathbf{D}}$$

11.4-2

(a) Exact result: $\begin{vmatrix} 2-\lambda & -2 \\ -2 & 5-\lambda \end{vmatrix} = 0$ satisfied by $\lambda = 1$ & $\lambda = 6$.

Approximate, first mode:

$$\begin{bmatrix} 1.7 & 1.0 \end{bmatrix} \begin{bmatrix} 2 & -2 \\ -2 & 5 \end{bmatrix} \begin{Bmatrix} 1.7 \\ 1.0 \end{Bmatrix} = 3.98$$

$$\begin{bmatrix} 1.7 & 1.0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} 1.7 \\ 1.0 \end{Bmatrix} = 3.89$$

$\lambda_1 \approx \dfrac{3.98}{3.89} = 1.023$ (high, as expected)

Approximate, second mode:

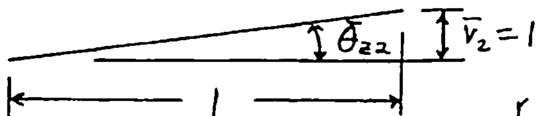
$$\begin{bmatrix} 1.2 & -2.0 \end{bmatrix} \begin{bmatrix} 2 & -2 \\ -2 & 5 \end{bmatrix} \begin{Bmatrix} 1.2 \\ -2.0 \end{Bmatrix} = 32.48$$

$$\begin{bmatrix} 1.2 & -2.0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} 1.2 \\ -2.0 \end{Bmatrix} = 5.44$$

$\lambda_2 \approx \dfrac{32.48}{5.44} = 5.97$ (low, as expected)

Approx. evals. more accurate than approx. evecs.

(b) $\underset{\sim}{K} = \dfrac{EI}{L^3}\begin{bmatrix} 12 & -6L \\ -6L & 4L^2 \end{bmatrix}$, $\underset{\sim}{M} = \dfrac{\rho AL}{420}\begin{bmatrix} 156 & -22L \\ -22L & 4L^2 \end{bmatrix}$



$\bar{\theta}_{z2} = \dfrac{1}{L}$, $\bar{\underset{\sim}{D}}_1 = \begin{Bmatrix} 1 \\ 1/L \end{Bmatrix}$

$\underset{\sim}{K}\bar{\underset{\sim}{D}}_1 = \dfrac{EI}{L^3}\begin{Bmatrix} 6 \\ -2L \end{Bmatrix}$, $\bar{\underset{\sim}{D}}_1^T \underset{\sim}{K} \bar{\underset{\sim}{D}}_1 = \dfrac{4EI}{L^3}$

$\underset{\sim}{M}\bar{\underset{\sim}{D}}_1 = \dfrac{\rho AL}{420}\begin{Bmatrix} 134 \\ -18L \end{Bmatrix}$, $\bar{\underset{\sim}{D}}_1^T \underset{\sim}{M} \bar{\underset{\sim}{D}}_1 = 116\dfrac{\rho AL}{420}$

$$\omega_1^2 \approx \frac{4}{116/420}\frac{EI}{\rho AL^4} = 14.48\frac{EI}{\rho AL^4}$$

$\omega_1 \approx 3.81\left(\dfrac{EI}{\rho AL^4}\right)^{1/2}$ $\quad$ $\left[\text{exact } \omega_1 = 3.516(EI/\rho AL^4)^{1/2}\right]$

Better: use shape of cantilever beam under transverse tip load P:

$v = \dfrac{PL^3}{3EI}$, $\theta = \dfrac{PL^2}{2EI} = \dfrac{3v}{2L}$ $\quad$ so use $\bar{\underset{\sim}{D}}_1 = \begin{Bmatrix} 1 \\ 3/2L \end{Bmatrix}$

With foregoing $\underset{\sim}{K}$ and $\underset{\sim}{M}$,

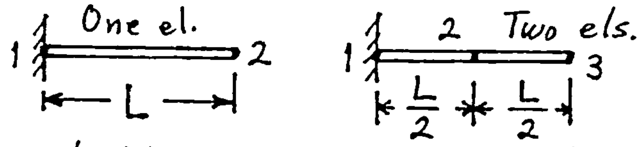
$\underset{\sim}{K}\underset{\sim}{D}_1 = \dfrac{EI}{L^3}\begin{Bmatrix} 3 \\ 0 \end{Bmatrix}$, $\bar{\underset{\sim}{D}}_1^T \underset{\sim}{K} \bar{\underset{\sim}{D}}_1 = \dfrac{3EI}{L^3}$

$\underset{\sim}{M}\underset{\sim}{D}_1 = \dfrac{\rho AL}{420}\begin{Bmatrix} 123 \\ -16L \end{Bmatrix}$, $\bar{\underset{\sim}{D}}_1^T \underset{\sim}{M} \bar{\underset{\sim}{D}}_1 = \dfrac{99\rho AL}{420}$

$\omega_1^2 \approx \dfrac{3(420)}{99}\dfrac{EI}{\rho AL^4}$

$\omega_1^2 \approx 12.73\dfrac{EI}{\rho AL^4}$, $\omega_1 \approx 3.568\left(\dfrac{EI}{\rho AL^4}\right)^{1/2}$

11.4-3



(a) One el. $\left(\frac{AE}{L} - \omega_1^2 \frac{m}{3}\right)\bar{u}_2 = 0, \quad \omega_1^2 = \frac{3AE}{mL}$

Two els. $\left(\frac{AE}{L/2}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} - \frac{\frac{m}{2}\omega^2}{6}\begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix}\right)\begin{Bmatrix} \bar{u}_2 \\ \bar{u}_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$

$7\lambda^2 - 10\lambda + 1 = 0$, where $\lambda = \frac{m\omega^2 L}{24AE}$

$\lambda_1 = 0.1082, \quad \omega_1^2 = 2.597\,(AE/mL)$

(b) One el. $\left(\frac{AE}{L} - \omega_1^2 \frac{m}{2}\right)\bar{u}_2 = 0, \quad \omega_1^2 = \frac{2AE}{mL}$

Two els. $\left(\frac{AE}{L/2}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} - \frac{\frac{m}{2}\omega^2}{2}\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{Bmatrix} \bar{u}_2 \\ \bar{u}_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$

$2\lambda^2 - 4\lambda + 1 = 0$, where $\lambda = \frac{m\omega^2 L}{8AE}$

$\lambda_1 = 0.2929, \quad \omega_1^2 = 2.343\,(AE/mL)$

(c)

$[\underset{\sim}{m}] = \frac{1}{2}\left(\frac{m}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} + \frac{m}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = \frac{m}{12}\begin{bmatrix} 5 & 1 \\ 1 & 5 \end{bmatrix}$

One el. $\left(\frac{AE}{L} - \omega_1^2 \frac{5m}{12}\right)\bar{u}_2 = 0, \quad \omega_1^2 = \frac{2.4AE}{mL}$

Two els. $\left(\frac{AE}{L/2}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} - \frac{\frac{m}{2}\omega^2}{12}\begin{bmatrix} 10 & 1 \\ 1 & 5 \end{bmatrix}\right)\begin{Bmatrix} \bar{u}_2 \\ \bar{u}_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$

$49\lambda^2 - 22\lambda + 1 = 0$, where $\lambda = \frac{m\omega^2 L}{48AE}$

[illegible]2, $\omega_1^2 = 2.463\,(AE/mL)$

(d) $\left(\frac{AE}{L} - \omega_1^2 m \frac{3-\beta}{6}\right)\bar{u}_2 = 0, \quad \omega_1^2 = \frac{6AE}{(3-\beta)mL}$

Exact: $\omega_1^2 = \left(\frac{\pi}{2}\right)^2 \frac{AE}{mL}$

Equate these ω_1^2 values; thus

$\frac{6}{3-\beta} = \left(\frac{\pi}{2}\right)^2$ and $\beta = 0.568$

$\boxed{11.4\text{-}4}$

Exact $\omega_1 = \frac{\pi}{2}\left(\frac{AE}{mL}\right)^{1/2} = 1.5708\left(\frac{AE}{mL}\right)^{1/2}$

frequencies $\div (AE/mL)^{1/2}$ (from Prob. 11.4-3)

N	consis.	lumped	ave.
1	1.732	1.414	1.549
2	1.612	1.531	1.569

% errors

N	consis.	lumped	ave.
1	10.3	-10.0	-1.4
2	2.6	-2.5	-0.1
F	3.96	4.00	14

where F is the factor of reduction of magnitude of % error in going from N=1 to N=2 elements

These F factors agree with the respective error estimates $O(h^2)$, $O(h^2)$, and $O(h^4)$.

$\boxed{11.4\text{-}5}$

(a) $m = \rho AL$. Let $\lambda = \frac{3L}{AE}\frac{m\omega^2}{30} = \frac{mL\omega^2}{10AE}$

The eigenvalue problem becomes

$$\begin{vmatrix} 7-4\lambda & -8-2\lambda & 1+\lambda \\ -8-2\lambda & 16(1-\lambda) & -8-2\lambda \\ 1+\lambda & -8-2\lambda & 7-4\lambda \end{vmatrix} = 0, \text{ from which}$$

$5\lambda(5\lambda^2 - 36\lambda + 36) = 0$ $\quad \lambda_1 = 0$, $\omega_1 = 0$

$\lambda = \frac{36 \pm \sqrt{36^2 - 4(5)36}}{10}$ $\quad \lambda_2 = 1.2$, $\omega_2 = 3.464\sqrt{\frac{AE}{mL}}$

$\lambda_3 = 6.0$, $\omega_3 = 7.746\sqrt{\frac{AE}{mL}}$

(b) $m = \rho AL$. Let $\lambda = \frac{3L}{AE}\frac{m\omega^2}{6} = \frac{mL\omega^2}{2AE}$

The eigenvalue problem becomes

$$\begin{vmatrix} 7-\lambda & -8 & 1 \\ -8 & 16-4\lambda & -8 \\ 1 & -8 & 7-\lambda \end{vmatrix} = 0, \text{ from which}$$

$4\lambda(\lambda^2 - 18\lambda + 72) = 0$ $\quad \lambda_1 = 0$, $\omega_1 = 0$

$\lambda = \frac{18 \pm \sqrt{18^2 - 4(72)}}{2}$ $\quad \lambda_2 = 6$, $\omega_2 = 3.464\sqrt{\frac{AE}{mL}}$

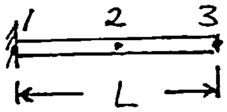
$\lambda_3 = 12$, $\omega_3 = 4.899\sqrt{\frac{AE}{mL}}$

(c) Rigid body translation

(d) Exact: $\omega_1 = 0$, $\omega_2 = 3.142\sqrt{\frac{AE}{mL}}$, $\omega_3 = 6.283\sqrt{\frac{AE}{mL}}$.

Errors of (a): 0%, +10.3%, +23.3% $\qquad$ Errors of (b): 0%, +10.3%, −22.0%

11.4-6



Exact: $\omega_1 = 1.571\sqrt{\frac{AE}{mL}}$

$\omega_2 = 4.712\sqrt{\frac{AE}{mL}}$

(a) $$\left(\frac{AE}{3L}\begin{bmatrix}16 & -8\\ -8 & 7\end{bmatrix} - \frac{m\omega^2}{30}\begin{bmatrix}16 & 2\\ 2 & 4\end{bmatrix}\right)\begin{Bmatrix}\bar{u}_2\\ \bar{u}_3\end{Bmatrix} = \begin{Bmatrix}0\\ 0\end{Bmatrix}$$

Let $\lambda = \frac{mL\omega^2}{10AE}$; then $\begin{vmatrix}16(1-\lambda) & -8-2\lambda\\ -8-2\lambda & 7-4\lambda\end{vmatrix} = 0$

$15\lambda^2 - 52\lambda + 12 = 0$ $\quad \omega_1 = 1.577\sqrt{\frac{AE}{mL}}$

$\lambda_1 = 0.248596$

$\lambda_2 = 3.21807$ $\quad \omega_2 = 5.673\sqrt{\frac{AE}{mL}}$

(b) $$\left(\frac{AE}{3L}\begin{bmatrix}16 & -8\\ -8 & 7\end{bmatrix} - \frac{m\omega^2}{6}\begin{bmatrix}4 & 0\\ 0 & 1\end{bmatrix}\right)\begin{Bmatrix}\bar{u}_2\\ \bar{u}_3\end{Bmatrix} = \begin{Bmatrix}0\\ 0\end{Bmatrix}$$

Let $\lambda = \frac{mL\omega^2}{2AE}$; then $\begin{vmatrix}16-4\lambda & -8\\ -8 & 7-\lambda\end{vmatrix} = 0$

$\lambda^2 - 11\lambda + 12 = 0$ $\quad \omega_1 = 1.523\sqrt{\frac{AE}{mL}}$

$\lambda_1 = 1.22800$

$\lambda_2 = 9.77200$ $\quad \omega_2 = 4.421\sqrt{\frac{AE}{mL}}$

(c) $$\left(\frac{AE}{3L}\begin{bmatrix}16 & -8\\ -8 & 7\end{bmatrix} - \frac{m\omega^2}{3}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\right)\begin{Bmatrix}\bar{u}_2\\ \bar{u}_3\end{Bmatrix} = \begin{Bmatrix}0\\ 0\end{Bmatrix}$$

Let $\lambda = \frac{mL\omega^2}{AE}$; then $\begin{vmatrix}16-\lambda & -8\\ -8 & 7-\lambda\end{vmatrix} = 0$

$\lambda^2 - 23\lambda + 48 = 0$ $\quad \omega_1 = 1.524\sqrt{\frac{AE}{mL}}$

$\lambda_1 = 2.32122$

$\lambda_2 = 20.6788$ $\quad \omega_2 = 4.547\sqrt{\frac{AE}{mL}}$

(d) $\{d\}^T[k]\{d\} = \begin{Bmatrix}1\\ 2\end{Bmatrix}^T\begin{bmatrix}16 & -8\\ -8 & 7\end{bmatrix}\begin{Bmatrix}1\\ 2\end{Bmatrix} = 12$ $\quad \left(*\frac{AE}{3L}\right)$

Consistent:

$\{d\}^T[m]\{d\} = \begin{Bmatrix}1\\ 2\end{Bmatrix}^T\begin{bmatrix}16 & 2\\ 2 & 4\end{bmatrix}\begin{Bmatrix}1\\ 2\end{Bmatrix} = 40$ $\quad \left(*\frac{m}{30}\right)$

$\omega_1^2 = \frac{12/3}{40/30}\frac{AE}{mL}$, $\quad \omega_1 = 1.732\sqrt{\frac{AE}{mL}}$

HRZ-lumped:

$\{d\}^T[m]\{d\} = \begin{Bmatrix}1\\ 2\end{Bmatrix}^T\begin{bmatrix}4 & 0\\ 0 & 1\end{bmatrix}\begin{Bmatrix}1\\ 2\end{Bmatrix} = 8$ $\quad \left(*\frac{m}{6}\right)$

$\omega_1^2 = \frac{12/3}{8/6}\frac{AE}{mL}$, $\quad \omega_1 = 1.732\sqrt{\frac{AE}{mL}}$

Ad-hoc lumped:

$\{d\}^T[m]\{d\} = \begin{Bmatrix}1\\ 2\end{Bmatrix}^T\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}\begin{Bmatrix}1\\ 2\end{Bmatrix} = 5$ $\quad \left(*\frac{m}{3}\right)$

$\omega_1^2 = \frac{12/3}{5/3}\frac{AE}{mL}$, $\quad \omega_1 = 1.549\sqrt{\frac{AE}{mL}}$

11.4-7

Exact $\omega_1 = 2.4674\,[EI/\rho A L^4]^{1/2}$, where L is the half-length. In each part of this problem,

$$[\underset{\sim}{k}] = \frac{2EI}{2L}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}, \text{ and } \left([\underset{\sim}{k}] - \omega^2[\underset{\sim}{m}]\right)\begin{Bmatrix}\bar{\theta}_1\\ \bar{\theta}_2\end{Bmatrix} = \begin{Bmatrix}0\\ 0\end{Bmatrix}$$

(a) $[\underset{\sim}{m}] = [\underset{\sim}{0}]$, as no mass is associated with d.o.f. θ_1 and θ_2. No solution ($\omega \to \infty$).

(a) $$[\underset{\sim}{m}] = \frac{\rho A(2L)}{420}\begin{bmatrix}4(2L)^2 & -3(2L)^2\\ -3(2L)^2 & 4(2L)^2\end{bmatrix}$$

Let $\lambda = \dfrac{2\rho A L^4\omega^2}{105EI}$, then $\begin{vmatrix}2-4\lambda & 1+3\lambda\\ 1+3\lambda & 2-4\lambda\end{vmatrix} = 0$

$7\lambda^2 - 22\lambda + 3 = 0$

$\lambda_1 = 1/7$ $\qquad \omega_1 = 2.739\,[EI/\rho A L^4]^{1/2}$

$\lambda_2 = 21/7$ $\qquad \omega_2 = 12.55\,[EI/\rho A L^4]^{1/2}$

(b) $[\underset{\sim}{m}] = [0\quad 0]$. No solution (or ω's $= \infty$)

(c) $$[\underset{\sim}{m}] = \frac{17.5(2\rho A L)(2L)^2}{2(210)}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

Let $\lambda = \dfrac{\rho A L^4\omega^2}{3EI}$, then $\begin{vmatrix}2-\lambda & 1\\ 1 & 2-\lambda\end{vmatrix} = 0$

$\lambda^2 - 4\lambda + 3 = 0$ $\qquad \omega_1 = 1.732\,[EI/\rho A L^4]^{1/2}$

$\lambda_1 = 1$

$\lambda_2 = 3$ $\qquad \omega_2 = 3.000\,[EI/\rho A L^4]^{1/2}$

(d) $$[\underset{\sim}{m}] = \frac{\rho A(2L)}{78}\begin{bmatrix}(2L)^2 & 0\\ 0 & (2L)^2\end{bmatrix} = \frac{4\rho A L^3}{39}\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$$

Let $\lambda = \dfrac{4\rho A L^4\omega^2}{39EI}$, then $\begin{vmatrix}2-\lambda & 1\\ 1 & 2-\lambda\end{vmatrix} = 0$

$\lambda^2 - 4\lambda + 3 = 0$ $\qquad \omega_1 = 3.122\,[EI/\rho A L^4]^{1/2}$

$\lambda_1 = 1$

$\lambda_3 = 3$ $\qquad \omega_2 = 5.408\,[EI/\rho A L^4]^{1/2}$

(e) $$[\underset{\sim}{m}] = \frac{\rho A(2L)}{120}(2L)^2\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} = \frac{\rho A L^3}{15}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$$

Let $\lambda = \dfrac{\cdots\,\omega^2}{15}$, then $\begin{vmatrix}2-\lambda & 1+\lambda\\ 1+\lambda & 2-\lambda\end{vmatrix} = 0$

$-6\lambda + 3 = 0$, $\lambda = \frac{1}{2}$, $\omega_1 = 2.739\,[EI/\rho A L^4]^{1/2}$

(Same as ω_1 of part (a) -- const. $v_{,xx}$ mode)

11.4-8

Exact $\omega_1 = 3.516\,[EI/\rho A L^4]^{1/2}$

$$[\underset{\sim}{k}] = \frac{2EI}{L^3}\begin{bmatrix} 6 & -3L \\ -3L & 2L^2 \end{bmatrix}, \quad \left([\underset{\sim}{k}] - \omega^2[\underset{\sim}{m}]\right)\begin{Bmatrix} \bar{v}_2 \\ \bar{\theta}_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Lumped $[\underset{\sim}{m}]$ solution:

$$[\underset{\sim}{m}] = \frac{\rho A L}{2}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

Let $\lambda = \dfrac{\rho A L^4 \omega^2}{4EI}$, then $\begin{vmatrix} 6-\lambda & -3L \\ -3L & 2L^2 \end{vmatrix} = 0$

$\lambda = \frac{3}{2}$, $\quad \omega_1 = 2.449\,[EI/\rho A L^4]^{1/2}$

(ω_2 is not provided)

Using $[\underset{\sim}{m}]$ from Prob. 11.3-11(b):

$$[\underset{\sim}{m}] = \frac{\rho A L}{120}\begin{bmatrix} 40 & -5L \\ -5L & L^2 \end{bmatrix}$$

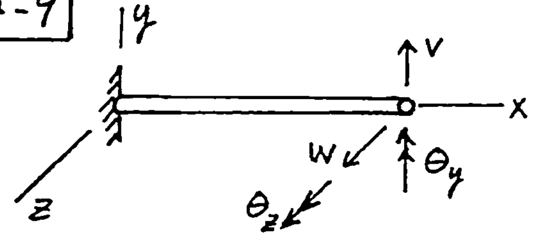
Let $\lambda = \dfrac{\rho A L^4 \omega^2}{240 EI}$, then $\begin{vmatrix} 6-40\lambda & -3L+5\lambda L \\ -3L+5\lambda L & (2-\lambda)L^2 \end{vmatrix} = 0$

$15\lambda^2 - 56\lambda + 3 = 0$ $\quad \omega_1 = 3.612\,[EI/\rho A L^4]^{1/2}$

$\lambda_1 = 0.054363$

$\lambda_2 = 3.6790$ $\quad \omega_2 = 29.714\,[EI/\rho A L^4]^{1/2}$

$\boxed{11.4\text{-}9}$



$$\{\bar{D}\} = \begin{Bmatrix} v \\ \theta_z \\ w \\ \theta_z \end{Bmatrix}$$

$$[M] = \begin{bmatrix} m_{11} & m_{12} & 0 & 0 \\ m_{12} & m_{22} & 0 & 0 \\ 0 & 0 & m_{11} & m_{12} \\ 0 & 0 & m_{12} & m_{22} \end{bmatrix}$$

Activate y-direction motion: $\{\bar{D}_y\} = \begin{Bmatrix} v \\ \theta_z \\ 0 \\ 0 \end{Bmatrix}$

$$[M]\{\bar{D}_y\} = \begin{Bmatrix} m_{11}v + m_{12}\theta_z \\ m_{12}v + m_{22}\theta_z \\ 0 \\ 0 \end{Bmatrix}$$

Premultiply by z-direction motion, $\{\bar{D}_z\} = \begin{Bmatrix} 0 \\ 0 \\ w \\ \theta_y \end{Bmatrix}$

Get $\{D_z\}^T([M]\{\bar{D}_y\}) = 0$

$\boxed{11.4\text{-}10}$

Exact: $c = \pi^2/4 = 2.4674$

(a) $p=2$ $\quad c = \dfrac{3.000(1) - 2.597(4)}{1-4} = 2.463$

(b) $p=2$ $\quad c = \dfrac{2.000(1) - 2.343(4)}{1-4} = 2.457$

(c) $p=4$ $\quad c = \dfrac{2.400(1) - 2.463(16)}{1-16} = 2.467$

$\boxed{11.5\text{-}1}$

(a) $\xi_1 = 0.03 \qquad \omega_1 = 2\pi(5) = 10$

$\xi_2 = 0.20 \qquad \omega_2 = 2\pi(15) = 30$

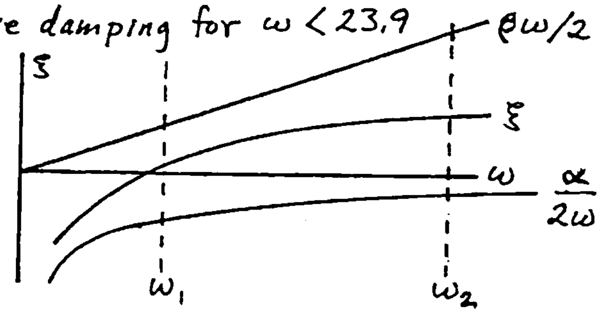
Apply Eqs. 11.5-3:

$$\beta = 2\,\frac{0.2(30\pi) - 0.03(10\pi)}{(30\pi)^2 - (10\pi)^2} = 0.004536$$

$$\alpha = 2(10\pi)(30\pi)\,\frac{0.03(30\pi) - 0.2(10\pi)}{(30\pi)^2 - (10\pi)^2} = -2.592$$

(b) $\xi = \dfrac{\alpha}{2\omega} + \dfrac{\beta\omega}{2} = -\dfrac{1.296}{\omega} + 0.002268\omega$

Negative damping for $\omega < 23.9$



$\boxed{11.6\text{-}1}$

(a) Work out $[T]^T([K][T])$.

$$\begin{bmatrix} K_{mm} & K_{ms} \\ K_{ms}^T & K_{ss} \end{bmatrix} \begin{bmatrix} I \\ -K_{ss}^{-1} K_{ms}^T \end{bmatrix} = \underbrace{\begin{bmatrix} K_{mm} - K_{ms} K_{ss}^{-1} K_{ms}^T \\ 0 \end{bmatrix}}$$

$$\begin{bmatrix} I & -K_{ms} K_{ss}^{-1} \end{bmatrix} \begin{bmatrix} \quad \end{bmatrix} = K_{mm} - K_{ms} K_{ss}^{-1} K_{ms}^T$$

$K_{ss}^{-T} = K_{ss}^{-1}$ by symmetry

Same result as given by "static condensation."

(b)

$$\begin{bmatrix} I & -K_{ms} K_{ss}^{-T} \end{bmatrix} \begin{bmatrix} M_{mm} & M_{ms} \\ M_{ms}^T & M_{ss} \end{bmatrix} \begin{bmatrix} I \\ -K_{ss}^{-1} K_{ms}^T \end{bmatrix} =$$

same

$$\begin{bmatrix} I & -K_{ms} K_{ss}^{-1} \end{bmatrix} \begin{bmatrix} M_{mm} - M_{ms} K_{ss}^{-1} K_{ms}^T \\ M_{ms}^T - M_{ss} K_{ss}^{-1} K_{ms}^T \end{bmatrix} =$$

$$M_{mm} - M_{ms} K_{ss}^{-1} K_{ms}^T - K_{ms} K_{ss}^{-1} M_{ms}^T + K_{ms} K_{ss}^{-1} M_{ss} K_{ss}^{-1} K_{ms}^T$$

11.6-2

(a)
$$\begin{bmatrix} \mathbf{F}_{mm} & \mathbf{F}_{ms} \\ \mathbf{F}_{ms}^T & \mathbf{F}_{ss} \end{bmatrix} \begin{Bmatrix} \mathbf{R}_m \\ \mathbf{R}_s \end{Bmatrix} = \begin{Bmatrix} \mathbf{D}_m \\ \mathbf{D}_s \end{Bmatrix}$$

With $\mathbf{R}_s = \mathbf{0}$, the upper partition yields $\mathbf{R}_m = \mathbf{F}_{mm}^{-1} \mathbf{D}_m$. Therefore

$$\begin{Bmatrix} \mathbf{D}_m \\ \mathbf{D}_s \end{Bmatrix} = \begin{bmatrix} \mathbf{F}_{mm} \\ \mathbf{F}_{ms}^T \end{bmatrix} \{\mathbf{R}_m\} = \begin{bmatrix} \mathbf{F}_{mm} \\ \mathbf{F}_{ms}^T \end{bmatrix} [\mathbf{F}_{mm}]^{-1} \{\mathbf{D}_m\}$$

$$\begin{Bmatrix} \mathbf{D}_m \\ \mathbf{D}_s \end{Bmatrix} = [\mathbf{T}_F]\{\mathbf{D}_m\}, \text{ where } [\mathbf{T}_F] = \begin{bmatrix} \mathbf{I} \\ \mathbf{F}_{ms}^T \mathbf{F}_{mm}^{-1} \end{bmatrix}$$

(b)
We must show that $\mathbf{F}_{ms}^T \mathbf{F}_{mm}^{-1} = -\mathbf{K}_{ss}^{-1} \mathbf{K}_{ms}^T$

The product of stiffness and flexibility matrices must be a unit matrix.

$$\begin{bmatrix} \mathbf{K}_{mm} & \mathbf{K}_{ms} \\ \mathbf{K}_{ms}^T & \mathbf{K}_{ss} \end{bmatrix} \begin{bmatrix} \mathbf{F}_{mm} & \mathbf{F}_{ms} \\ \mathbf{F}_{ms}^T & \mathbf{F}_{ss} \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}$$

Row 2 times column 1 is $\mathbf{K}_{ms}^T \mathbf{F}_{mm} + \mathbf{K}_{ss} \mathbf{F}_{ms}^T = 0$

Premult. by $\mathbf{K}_{ss}^{-1}$: $\mathbf{K}_{ss}^{-1} \mathbf{K}_{ms}^T \mathbf{F}_{mm} = -\mathbf{F}_{ms}^T$

Postmult. by $\mathbf{F}_{mm}^{-1}$: $\mathbf{K}_{ss}^{-1} \mathbf{K}_{ms}^T = -\mathbf{F}_{ms}^T \mathbf{F}_{mm}^{-1}$

(c) Consider as many load vectors as there are master d.o.f., & let each load vector consist of a single unit load on a master d.o.f. Also set $\mathbf{R}_s = \mathbf{0}$.

$$\begin{bmatrix} \mathbf{K}_{..} & \mathbf{K}_{..} \\ \mathbf{K}_{ms}^T & \mathbf{K}_{ss} \end{bmatrix} \begin{bmatrix} \mathbf{D}_m \\ \mathbf{D}_s \end{bmatrix} = \begin{bmatrix} \mathbf{R}_m \\ \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathbf{I} \\ \mathbf{0} \end{bmatrix}$$

Solve; note that $[\mathbf{F}] = [\mathbf{K}]^{-1}$

$$\begin{bmatrix} \mathbf{D}_m \\ \mathbf{D}_s \end{bmatrix} = \begin{bmatrix} \mathbf{F}_{mm} & \mathbf{F}_{ms} \\ \mathbf{F}_{ms}^T & \mathbf{F}_{ss} \end{bmatrix} \begin{bmatrix} \mathbf{I} \\ \mathbf{0} \end{bmatrix} = \begin{bmatrix} \mathbf{F}_{mm} \\ \mathbf{F}_{ms}^T \end{bmatrix}$$

The physical meaning of $[\mathbf{F}_{mm}]$ is that each of its columns represents the displacements of master d.o.f. produced by a unit load on one master d.o.f.

(d) Usually $m \ll s$, so $[\mathbf{F}_{mm}]$ is much smaller than $[\mathbf{K}_{ss}]$ -- cheaper to invert.

$\boxed{11.6\text{-}3}$

$$[\underset{\sim}{K}] = \begin{bmatrix} k & -k \\ -k & k \end{bmatrix}, \quad [\underset{\sim}{M}] = \begin{bmatrix} m & 0 \\ 0 & m \end{bmatrix}$$

$K_{ss} = k, \; K_{ms} = -k, \; [\underset{\sim}{T}] = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$ $\quad [\underset{\sim}{T}]^T[\underset{\sim}{K}][\underset{\sim}{T}] = 0$

$\qquad [\underset{\sim}{T}]^T[\underset{\sim}{M}][\underset{\sim}{T}] = 2m$

$$[0 - \omega^2(2m)]\bar{u}_1 = 0$$

Hence $\omega = 0$; the rigid body mode; OK.

11.6-4

(a) With c a constant, $\dfrac{M_{11}}{K_{11}} = c\,\dfrac{156}{12}$

and $\dfrac{M_{22}}{K_{22}} = c\,\dfrac{4L^2}{4L^2} = c$. $\dfrac{M_{11}}{K_{11}} > \dfrac{M_{22}}{K_{22}}$, therefore the choice is proper.

(b)
$$[\underset{\sim}{T}] = \begin{bmatrix} -\dfrac{L^3}{12EI}\,\dfrac{6EI}{L^2} \\ 1 \end{bmatrix} = \begin{Bmatrix} -\dfrac{L}{2} \\ 1 \end{Bmatrix}$$

$$[\underset{\sim}{T}]^T[\underset{\sim}{K}][\underset{\sim}{T}] = \left[-\frac{L}{2} \quad 1\right]\frac{EI}{L^3}\begin{Bmatrix} -6L+6L \\ -3L^2+4L^2 \end{Bmatrix} = \frac{EI}{L}$$

$$[\underset{\sim}{T}]^T[\underset{\sim}{M}][\underset{\sim}{T}] = \left[-\frac{L}{2} \quad 1\right]\frac{m}{420}\begin{Bmatrix} -78L-13L \\ 6.5L^2+4L^2 \end{Bmatrix} = \frac{14mL^2}{105}$$

$$\left(\frac{EI}{L} - \omega^2\frac{14mL^2}{105}\right)\bar{\theta}_2 = 0, \quad \omega^2 = 7.5\frac{EI}{mL^3}$$

(c) Using the first recovery method (Eq. 11.6-3)

$$\begin{Bmatrix} \bar{v}_1 \\ \bar{\theta}_2 \end{Bmatrix} = [\underset{\sim}{T}]\bar{\theta}_2, \quad \bar{v}_1 = -\frac{L}{2}\bar{\theta}_2$$

$$\omega_1^2 = \frac{\dfrac{EI}{L^3}\left[-\dfrac{L}{2} \quad 1\right]\begin{bmatrix} 12 & 6L \\ 6L & 4L^2 \end{bmatrix}\begin{Bmatrix} -L/2 \\ 1 \end{Bmatrix}}{\dfrac{m}{420}\left[-\dfrac{L}{2} \quad 1\right]\begin{bmatrix} 156 & -13L \\ -13L & 4L^2 \end{bmatrix}\begin{Bmatrix} -L/2 \\ 1 \end{Bmatrix}}$$

$$\omega_1^2 = \frac{420EI}{mL^3}\,\frac{L^2}{56L} = 7.5\frac{EI}{mL^3} \quad \text{(no improvement)}$$

Using the 2^{nd} recovery method (Eq. 11.6-6),

$$\bar{v}_1 = -\left[\frac{12EI}{L^3} - 7.5\frac{EI}{mL^3}\frac{156m}{420}\right]^{-1}\left[\frac{6EI}{L^2} - 7.5\frac{EI}{mL^3}\left(-\frac{13mL}{420}\right)\right]\bar{\theta}_2$$

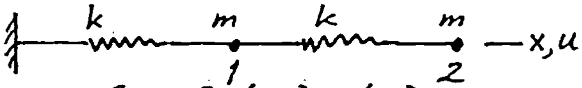
$$\bar{v}_1 = -\left[\frac{EI}{L^3}(12-2.186)\right]^{-1}\left[\frac{EI}{L^2}(6+0.232)\right]\bar{\theta}_2$$

$$\bar{v}_1 = -0.67636L \quad \text{for } \bar{\theta}_2 = 1$$

$$\omega_1^2 = \frac{EI\,[-.6764L, 1]\begin{bmatrix} 12 & 6L \\ 6L & 4L^2 \end{bmatrix}\begin{Bmatrix} -.6764L \\ 1 \end{Bmatrix}}{\dfrac{m}{420}[-.6764L, 1]\begin{bmatrix} 156 & -13 \\ -13 & 4L^2 \end{bmatrix}\begin{Bmatrix} -.6764L \\ 1 \end{Bmatrix}}$$

$$\omega_1^2 = \frac{420EI}{mL^2}\,\frac{1.373L^2}{92.95} = 6.205\frac{EI}{mL^3}$$

11.6-5



$$\left(k\begin{bmatrix}1 & -1\\ -1 & 2\end{bmatrix} - \omega^2\begin{bmatrix}m & 0\\ 0 & m\end{bmatrix}\right)\begin{Bmatrix}\bar{u}_2\\ \bar{u}_1\end{Bmatrix} = \begin{Bmatrix}0\\ 0\end{Bmatrix}$$

where, for convenience, d.o.f. are ordered so form of Eq. 11.6-3 need not be changed. D.o.f. $\bar{u}_1$ has the smaller M_{ii}/K_{ii}, so it is slave. From Eq. 11.6-3, $[\underset{\sim}{T}] = \begin{Bmatrix}1.0\\ 0.5\end{Bmatrix}$

With $k=1$ and $m=2$, Eqs. 11.6-4 yield

$$\lfloor 1.0, 0.5\rfloor\begin{bmatrix}1 & -1\\ -1 & 2\end{bmatrix}\begin{Bmatrix}1.0\\ 0.5\end{Bmatrix} = 0.5$$

$$\lfloor 1.0, 0.5\rfloor\begin{bmatrix}2 & 0\\ 0 & 2\end{bmatrix}\begin{Bmatrix}1.0\\ 0.5\end{Bmatrix} = 2.5$$

$(0.5 - 2.5\omega_1^2)\bar{u}_2 = 0$, $\omega_1^2 = 0.200$, $\omega_1 = 0.4472$

(The full 2 by 2 system yields $\omega_1 = 0.4370$)

For (say) $\bar{u}_2 = 1$, Eq. 11.6-3 yields $\bar{u}_1 = 0.5$. Then Rayleigh quotient yields

$$\omega_1^2 = \frac{\lfloor 1.0, 0.5\rfloor\begin{bmatrix}1 & -1\\ -1 & 2\end{bmatrix}\begin{Bmatrix}1.0\\ 0.5\end{Bmatrix}}{\lfloor 1.0, 0.5\rfloor\begin{bmatrix}2 & 0\\ 0 & 2\end{bmatrix}\begin{Bmatrix}1.0\\ 0.5\end{Bmatrix}} = 0.200 \text{ as before}$$

For (say) $\bar{u}_2 = 1$, Eq. 11.6-6 yields

$$\bar{u}_1 = -\left[2.0 - 0.2(2)\right]^{-1}\left[-1.0 - 0.2(0)\right](1.0) = 0.625$$

Then Rayleigh quotient yields

$$\omega_1^2 = \frac{\cdots\begin{bmatrix}1 & -1\\ -1 & 2\end{bmatrix}\begin{Bmatrix}1.0\\ 0.625\end{Bmatrix}}{\lfloor 1.0, 0.625\rfloor\begin{bmatrix}2 & 0\\ 0 & 2\end{bmatrix}\begin{Bmatrix}1.0\\ 0.625\end{Bmatrix}} \qquad \begin{matrix}\omega_1^2 = 0.1910\\ \omega_1 = 0.4370\end{matrix}$$

11.7-1

With $\underset{\sim}{C}$ from Eq. 11.2-12, the operations $\underset{\sim}{\phi}^T \underset{\sim}{C} \underset{\sim}{\phi}$ yield the damping in Eq. 11.7-6, i.e.

$$\underset{\sim}{\phi}^T \underset{\sim}{C} \underset{\sim}{\phi} = \begin{bmatrix} 2\xi_1\omega_1 & & & \\ & 2\xi_2\omega_2 & & \\ & & \ddots & \\ & & & 2\xi_n\omega_n \end{bmatrix}$$

Premultiply by $\underset{\sim}{\phi}^{-T}$; postmultiply by $\underset{\sim}{\phi}^{-1}$

$$\underset{\sim}{C} = \underset{\sim}{\phi}^{-T} \begin{bmatrix} 2\xi_1\omega_1 & & & \\ & 2\xi_2\omega_2 & & \\ & & \ddots & \\ & & & 2\xi_n\omega_n \end{bmatrix} \underset{\sim}{\phi}^{-1}$$

$\boxed{11.7\text{-}2}$

Let $\bar{D}_i^*$ be a vector before normalization.

Evaluate c in $(\bar{D}_i^*)^T M \bar{D}_i^* = c$

Scaled vector $\bar{D}_i = \frac{1}{\sqrt{c}} \bar{D}_i^*$ will yield $\bar{D}_i^T M \bar{D}_i = 1$

From Prob. 11.4-2a, $\omega_1^2 = 1$, $\omega_2^2 = 6$, and $c = \sqrt{5}$

Now use Eqs. 11.7-5 and 11.7-6

$$[\phi] = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix}$$

$$\begin{Bmatrix} P_1 \\ P_2 \end{Bmatrix} = \frac{1}{\sqrt{5}} \begin{bmatrix} 2 & 1 \\ 1 & -2 \end{bmatrix} \begin{Bmatrix} R_1 \\ R_2 \end{Bmatrix} = \frac{1}{\sqrt{5}} \begin{Bmatrix} 2R_1 + R_2 \\ R_1 - 2R_2 \end{Bmatrix}$$

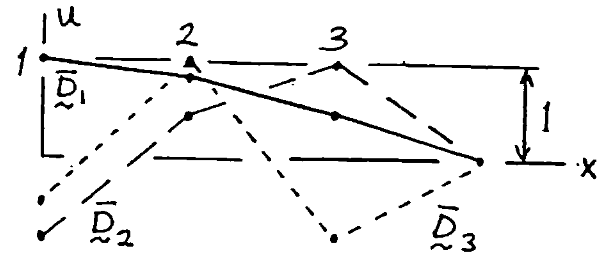
Eq. 13.6-5 yields

$$\ddot{z}_1 + z_1 = (2R_1 + R_2)/\sqrt{5}$$

$$\ddot{z}_2 + 6z_2 = (R_1 - 2R_2)/\sqrt{5}$$

11.7-3

(a)



$M = mI$, so for M-matrix orthogonality,

$\bar{D}_1^T \bar{D}_2 = -0.802 + 0.802(0.445) + 0.445(1) = 0$

$\bar{D}_1^T \bar{D}_3 = -0.445 + 0.802(1) + 0.445(-0.802) = 0$

$\bar{D}_2^T \bar{D}_3 = -0.802(-0.445) + 0.445(1) - 0.802 = 0$

$$K = k\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}, \quad K\bar{D}_1 = k\begin{Bmatrix} 0.198 \\ 0.159 \\ 0.088 \end{Bmatrix},$$

$$K\bar{D}_2 = k\begin{Bmatrix} -1.247 \\ 0.692 \\ 1.555 \end{Bmatrix}, \quad K\bar{D}_3 = k\begin{Bmatrix} -1.445 \\ 3.247 \\ -2.604 \end{Bmatrix}, \quad \bar{D}_i^T K \bar{D}_j = 0 \text{ if } i \neq j$$

(b) Must scale $\bar{D}_1$ and $\bar{D}_2$ according to Eq. 11.7-1, where M is a unit matrix in this problem.

$c_1^2 \bar{D}_1^T I \bar{D}_1 = c_1^2(1.841) = 1; \quad c_1 = 0.737$

$c_2^2 \bar{D}_2^T I \bar{D}_2 = c_2^2(1.841) = 1; \quad c_2 = 0.737$

The rescaled eigenvectors are

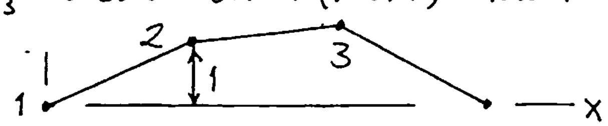
$$\bar{D}_1 = \begin{Bmatrix} 0.737 \\ 0.591 \\ 0.328 \end{Bmatrix} \qquad \bar{D}_2 = \begin{Bmatrix} -0.591 \\ 0.328 \\ 0.737 \end{Bmatrix}$$

Eq. 11.7-4: $\begin{Bmatrix} 0 \\ u_2 \\ u_3 \end{Bmatrix} = \bar{D}_1(1) + \bar{D}_2 z_2$

First d.o.f. yields $0 = 0.737 - 0.591 z_2$

$z_2 = 1.247$

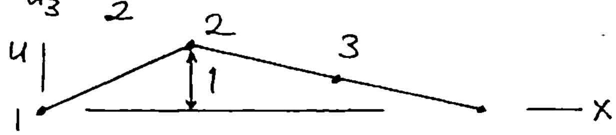
$\therefore \; u_3 = 0.328 + 0.737(1.247) = 1.247$



(c) For $u_1 = 0$ and $u_2 = 1$, from Eq. 11.6-2,

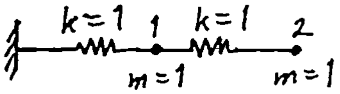
$\bar{D}_s = u_3 = -K_{ss}^{-1} K_{ms}^T \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = -\left(\frac{1}{2}\right)\begin{bmatrix} 0 & -1 \end{bmatrix}\begin{Bmatrix} 0 \\ 1 \end{Bmatrix}$

$u_3 = \frac{1}{2}$



$\boxed{11.7\text{-}4}$

(a)



$$\left(\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} - \omega^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{Bmatrix} \bar{u}_1 \\ \bar{u}_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Solution of this eigenproblem yields

$\lambda_1 = 0.381966$, $\omega_1 = 0.618034$

$\lambda_2 = 2.61803$, $\omega_2 = 1.618034$

Eigenvectors, respectively un-normalized and normalized, are

$$\begin{Bmatrix} 1 \\ 1.61803 \end{Bmatrix} \rightarrow \begin{Bmatrix} 0.52573 \\ 0.85065 \end{Bmatrix}, \quad \begin{Bmatrix} 1 \\ -0.618034 \end{Bmatrix} \rightarrow \begin{Bmatrix} 0.85065 \\ -0.52573 \end{Bmatrix}$$

Hence $[\underset{\sim}{\phi}] = \begin{bmatrix} 0.52573 & 0.85065 \\ 0.85065 & -0.52573 \end{bmatrix}$.

Initial conditions, using Eq. 11.7-4, are

$$\begin{Bmatrix} z_1 \\ z_2 \end{Bmatrix}_0 = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}, \quad \begin{Bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{Bmatrix}_0 = [\underset{\sim}{\phi}]^T \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{Bmatrix} 0 \\ 1 \end{Bmatrix} = \begin{Bmatrix} 0.85065 \\ -0.52573 \end{Bmatrix}$$

With $\xi_i = p_i = 0$, Eq. 11.7-6 has the solution

$$z_i = A_i \sin \omega_i t + B_i \cos \omega_i t$$

$B_i = 0$ because $z_i = 0$ at $t = 0$.

Next, $\dot{z}_i = A_i \omega_i \cos \omega_i t$, and at $t = 0$

$$\left.\begin{aligned} \dot{z}_1 &= 0.85065 = A_1(0.618) \\ \dot{z}_2 &= -0.52573 = A_2(1.618) \end{aligned}\right\} \begin{aligned} A_1 &= 1.3764 \\ A_2 &= -0.3249 \end{aligned}$$

$z_1 = 1.3764 \sin 0.618t$

$z_2 = -0.3249 \sin 1.618t$

By Eq. 11.7-4, $\{\underset{\sim}{D}\} = [\underset{\sim}{\phi}]^T\{\underset{\sim}{z}\}$ yields

$$\begin{Bmatrix} \text{[illegible]} \\ u_2 \end{Bmatrix} = \begin{Bmatrix} \text{[illegible]}\text{in } 0.618t - 0.2764 \sin 1.618t \\ 1.171 \sin 0.618t + 0.1708 \sin 1.618t \end{Bmatrix}$$

	t=0	t=1	t=2	t=3	t=4	t=5
u_1	0	.1432	.7095	.9684	.3972	-.2315
[illegible]	[illegible]	[illegible]	[illegible]	.9552	.7589	.2262

(b) $\ddot{u}_1 = -0.276 \sin 0.618t + 0.7236 \sin 1.618t$

$\ddot{u}_2 = -0.447 \sin 0.618t - 0.447 \sin 1.618t$

$$[\underset{\sim}{M}]\{\ddot{\underset{\sim}{D}}\} = \begin{Bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{Bmatrix}$$

$$[\underset{\sim}{K}]\{\underset{\sim}{D}\} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} 0.276 \sin 0.618t - 0.7236 \sin 1.618t \\ 0.447 \sin 0.618t + 0.447 \sin 1.618t \end{Bmatrix}$$

$$[\underset{\sim}{M}]\{\ddot{\underset{\sim}{D}}\} + [\underset{\sim}{K}]\{\underset{\sim}{D}\} = \{\underset{\sim}{0}\} \checkmark$$

(c) Compare greatest magnitudes, regardless of the times at which they appear.

$$\frac{0.7236}{0.7236 + 0.2764} = 0.7236 \quad (27.6\,\%\text{ error in } u_1)$$

$$\frac{1.171}{1.171 + 0.1708} = 0.8727 \quad (12.7\,\%\text{ error in } u_2)$$

$\boxed{11.7\text{-}5}$

From Eq. 11.7-6, with constant loads, $\ddot{z}_i + \omega_i^2 z_i = P_i$. Hence

$z_i = A_i \sin\omega_i t + B_i \cos\omega_i t + \dfrac{P_i}{\omega_i^2}$, $\dot{z}_i = A_i\omega_i \cos\omega_i t - B_i \omega_i \sin\omega_i t$

At rest and not deformed at $t=0$, so $A_i = 0$, $B_i = -\dfrac{P_i}{\omega_i^2}$

Thus $z_i = \dfrac{P_i}{\omega_i^2}(1-\cos\omega_i t)$ and $\ddot{z}_i = P_i \cos\omega_i t$

From Prob. 11.7-4, $[\underset{\sim}{\phi}] = \begin{bmatrix} 0.5257 & 0.8507 \\ 0.8507 & -0.5257 \end{bmatrix}$ and $\begin{matrix}\omega_1 = 0.618 \\ \omega_2 = 1.618\end{matrix}$

For $\{\underset{\sim}{R}\} = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}$, $\begin{Bmatrix} P_1 \\ P_2 \end{Bmatrix} = [\underset{\sim}{\phi}]^T \{\underset{\sim}{R}\} = \begin{Bmatrix} 0.8507 \\ -0.5257 \end{Bmatrix}$

(a) First mode: $\dfrac{P_1}{\omega_1^2} = 2.227$, $\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} 0.5257 \\ 0.8507 \end{Bmatrix} 2.227(1-\cos 0.618t)$

$$\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} 1.171 \\ 1.895 \end{Bmatrix}(1-\cos 0.618t)$$

	$t=2$	$t=4$	$t=6$	$t=8$	$t=10$
u_1	0.786	2.089	2.159	0.902	0.006
u_2	1.272	3.379	3.492	1.459	0.010

(b) Second mode: $\dfrac{P_2}{\omega_2^2} = -0.2008$, $\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} 0.8507 \\ -0.5257 \end{Bmatrix}(-0.2008)(1-\cos 1.618t)$

$$\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} -0.1708 \\ 0.1056 \end{Bmatrix}(1-\cos 1.618t)$$

	$t=2$	$t=4$	$t=6$	$t=8$	$t=10$
u_1	-0.341	-0.003	-0.335	-0.012	-0.323
u_2	0.211	0.002	0.207	0.011	0.200

Combine with first mode results to get final results:

	$t=2$	$t=4$	$t=6$	$t=8$	$t=10$
u_1	0.445	2.086	1.824	0.890	-0.317
u_2	..483	3.381	3.699	1.470	0.210

(continues)

$\boxed{11.7\text{-}5}$ (concluded)

(c) For mode 1, $[\underset{\sim}{K}]\{\underset{\sim}{D}\} = \begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} 0.447 \\ 0.724 \end{Bmatrix}(1-\cos 0.618t)$

$$\{\underset{\sim}{R}\}-[\underset{\sim}{M}]\{\ddot{\underset{\sim}{D}}\}-[\underset{\sim}{K}]\{\underset{\sim}{D}\} = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} - \begin{Bmatrix} 0.447 \\ 0.724 \end{Bmatrix}\cos 0.618t - \begin{Bmatrix} 0.447 \\ 0.724 \end{Bmatrix}(1-\cos 0.618t)$$

$$= \begin{Bmatrix} -0.447 \\ 0.276 \end{Bmatrix}$$

Eq. 11.7-9: $e(t) = 0.526$ (not small - indicates significant error)

For mode 2 (alone), $[\underset{\sim}{K}]\{\underset{\sim}{D}\} = \begin{Bmatrix} -0.447 \\ 0.276 \end{Bmatrix}(1-\cos 1.618t)$

$$[\underset{\sim}{M}]\{\ddot{\underset{\sim}{D}}\} = \begin{Bmatrix} 0.447 \\ -0.276 \end{Bmatrix}\cos 1.618t$$

Using modes 1 and 2, $\{\underset{\sim}{R}\}-[\underset{\sim}{M}]\{\ddot{\underset{\sim}{D}}\}-[\underset{\sim}{K}]\{\underset{\sim}{D}\} = 0$ and $e(t) = 0$

(d) $[\underset{\sim}{M}][\underset{\sim}{\Phi}][\underset{\sim}{\Phi}]^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{Bmatrix} 0.5257 \\ 0.8507 \end{Bmatrix}\lfloor 0.5257 \quad 0.8507 \rfloor = \begin{bmatrix} 0.2764 & 0.4472 \\ 0.4472 & 0.7273 \end{bmatrix}$

$\{\underset{\sim}{R}\}^{ext}_{approx} = \begin{Bmatrix} 0.2764 \\ 0.4472 \end{Bmatrix}$. From Eq. 11.7-11,

$\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\{\Delta\underset{\sim}{D}\} = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix} - \{\underset{\sim}{R}\}^{ext}_{approx}$, $\{\Delta\underset{\sim}{D}\} = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}\begin{Bmatrix} -0.4472 \\ 0.2727 \end{Bmatrix} = \begin{Bmatrix} -0.1745 \\ 0.0982 \end{Bmatrix}$

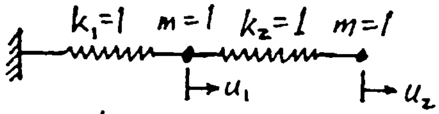
Add to results of part (a), to obtain:

	$t=2$	$t=4$	$t=6$	$t=8$	$t=10$
u_1	0.612	1.914	1.984	0.728	-0.169
u_2	1.370	3.477	3.590	1.557	0.108

Using the static correction, $[\underset{\sim}{K}]\{\Delta\underset{\sim}{D}\} = \begin{Bmatrix} -0.447 \\ 0.273 \end{Bmatrix}$

Including these terms in the numerator of Eq. 11.7-9, for mode 1 alone (as in the first part of part (c) above) gives $e(t) = 0$ (except for small rounding error).

$\boxed{11.8\text{-}1}$



Initial conditions: $u_1 = u_2 = \dot{u}_1 = 0,\ \dot{u}_2 = 1$

(a) Say $\{\underset{\sim}{w}_1\} = \begin{Bmatrix}1\\0\end{Bmatrix}$ in Eq. 11.8-4. Then Eq. 11.8-5 gives

$$\begin{bmatrix}2 & -1\\-1 & 1\end{bmatrix}\begin{Bmatrix}u_1^*\\u_2^*\end{Bmatrix} = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{Bmatrix}1\\0\end{Bmatrix} = \begin{Bmatrix}1\\0\end{Bmatrix},\quad \begin{Bmatrix}u_1^*\\u_2^*\end{Bmatrix} = \begin{Bmatrix}1\\1\end{Bmatrix}$$

$$\begin{Bmatrix}u_1^{**}\\u_2^{**}\end{Bmatrix} = \begin{Bmatrix}1\\1\end{Bmatrix} - \lfloor 1\ \ 0\rfloor\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{Bmatrix}1\\1\end{Bmatrix}\begin{Bmatrix}1\\0\end{Bmatrix} = \begin{Bmatrix}1-1\\1-0\end{Bmatrix} = \begin{Bmatrix}0\\1\end{Bmatrix}$$

$[\underset{\sim}{W}] = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$ (yields original system)

(b) Say $\{\underset{\sim}{w}_1\} = \begin{Bmatrix}0\\1\end{Bmatrix}$ in Eq. 11.8-4. Then Eq. 11.8-5 gives

$$\begin{bmatrix}2 & -1\\-1 & 1\end{bmatrix}\begin{Bmatrix}u_1^*\\u_2^*\end{Bmatrix} = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{Bmatrix}0\\1\end{Bmatrix} = \begin{Bmatrix}0\\1\end{Bmatrix},\quad \begin{Bmatrix}u_1^*\\u_2^*\end{Bmatrix} = \begin{Bmatrix}1\\2\end{Bmatrix}$$

$$\begin{Bmatrix}u_1^{**}\\u_2^{**}\end{Bmatrix} = \begin{Bmatrix}1\\2\end{Bmatrix} - \lfloor 0\ \ 1\rfloor\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{Bmatrix}1\\2\end{Bmatrix}\begin{Bmatrix}0\\1\end{Bmatrix} = \begin{Bmatrix}1-0\\2-2\end{Bmatrix} = \begin{Bmatrix}1\\0\end{Bmatrix}$$

$[\underset{\sim}{W}] = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$ (yields original system but with d.o.f. labels interchanged)

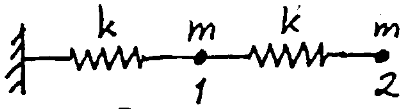
(c) $[\underset{\sim}{W}^*] = \begin{Bmatrix}1\\2\end{Bmatrix}$; normalized $[\underset{\sim}{W}] = \frac{1}{\sqrt{5}}\begin{Bmatrix}1\\2\end{Bmatrix}$

Eq. 11.8-6 becomes

$$\ddot{y} + \frac{1}{5}\lfloor 1\ \ 2\rfloor\begin{bmatrix}2 & -1\\-1 & 1\end{bmatrix}\begin{Bmatrix}1\\2\end{Bmatrix}y = 0, \text{ or } \ddot{y} + \frac{2}{5}y = 0$$

Let $y = \bar{y}\sin\omega t$, this yields $\omega = \sqrt{\frac{2}{5}} = .6325$ (full system gives $\omega = .618$). The approximate solution amounts to use of an assumed mode shape in the Rayleigh quotient.

$\boxed{11.9\text{-}1}$



is substructure 1:

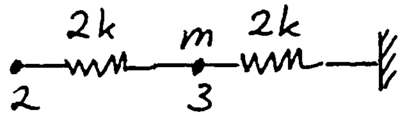
$$[\underset{\sim}{K}]_1 = \left[\begin{array}{c|c} 2 & -1 \\ \hline -1 & 1 \end{array}\right] \begin{array}{l} u_1 \\ u_2 \end{array} \qquad \psi = -\frac{1}{2}(-1) = \frac{1}{2}\ , \quad \begin{bmatrix} \underset{\sim}{\psi} \\ \underset{\sim}{I} \end{bmatrix} = \begin{Bmatrix} 1/2 \\ 1 \end{Bmatrix}$$

Mode 1, with node 2 fixed, is $(2-\omega^2)u_1 = 0$, $\omega^2 = \sqrt{2}$, $u_1 = 1$ (say)

$$[\underset{\sim}{W}]_1 = \begin{bmatrix} 1 & 1/2 \\ 0 & 1 \end{bmatrix}, \qquad [\underset{\sim}{W}]_1^T[\underset{\sim}{K}]_1[\underset{\sim}{W}]_1 = \begin{bmatrix} 2 & 0 \\ 0 & 1/2 \end{bmatrix} \begin{array}{l} a_1 \\ u_2 \end{array}$$

Elect to associate m at node 2 with substructure 1.

$$[\underset{\sim}{W}]_1^T[\underset{\sim}{M}]_1[\underset{\sim}{W}]_1 = \begin{bmatrix} 1 & 1/2 \\ 1/2 & 5/4 \end{bmatrix} \begin{array}{l} a_1 \\ u_2 \end{array}$$



is substructure 2:

From substructure 1, double stiffness, reorder $[\underset{\sim}{W}]_1$ to get

$$[\underset{\sim}{W}]_2 = \begin{bmatrix} 1 & 0 \\ 1/2 & 1 \end{bmatrix}, \quad [\underset{\sim}{W}]_2^T \begin{bmatrix} 2 & -2 \\ -2 & 4 \end{bmatrix} [\underset{\sim}{W}]_2 = \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix} \begin{array}{l} u_2 \\ a_2 \end{array}$$

$$[\underset{\sim}{W}]_2^T \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} [\underset{\sim}{W}]_2 = \begin{bmatrix} 1/4 & 1/2 \\ 1/2 & 1 \end{bmatrix} \begin{array}{l} u_2 \\ a_2 \end{array}$$

Synthesized structure, vibration problem:

$$\left(\begin{bmatrix} 2 & 0 & 0 \\ 0 & \frac{1}{2}+1 & 0 \\ 0 & 0 & 4 \end{bmatrix} - \omega^2 \begin{bmatrix} 1 & 1/2 & 0 \\ 1/2 & \frac{5}{4}+\frac{1}{2} & 1/2 \\ 0 & 1/2 & 1 \end{bmatrix} \right) \begin{Bmatrix} a_1 \\ u_2 \\ a_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

Multiply by 2, substitute $\mu = 1/\omega^2$, switch signs:

$$\left(\begin{bmatrix} 2 & 1 & 0 \\ 1 & 3 & 1 \\ 0 & 1 & 2 \end{bmatrix} - \mu \begin{bmatrix} 4 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 8 \end{bmatrix} \right) \begin{Bmatrix} a_1 \\ u_2 \\ a_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

$$\begin{vmatrix} 2-4\mu & 1 & 0 \\ 1 & 3-3\mu & 1 \\ 0 & 1 & 2-8\mu \end{vmatrix} = 0$$

By substituting $\mu_1 = \left(\frac{1}{0.9246}\right)^2$, $\mu_2 = \left(\frac{1}{1.574}\right)^2$, and $\mu_3 = \left(\frac{1}{2.381}\right)^2$, we check that the determinant is indeed zero in each case.

$\boxed{11.10-1}$

Eq. 11.10-1, with $\xi = 0$: $\quad \bar{u} = \dfrac{F_o/k}{\pm(1-\beta^2)}$, where $\beta = \dfrac{\Omega}{\omega}$

Want $\dfrac{\bar{u}}{F_o/k} = 1.10$

Positive root: $\quad 1.10 = \dfrac{1}{1-\beta^2}$, $\quad \beta = 0.3015$

Negative root: $\quad 1.10 = -\dfrac{1}{1-\beta^2}$, $\quad \beta = 1.3817$

$$0.3015 < \frac{\Omega}{\omega} < 1.3817$$

$\boxed{11.12\text{-}1}$

Forward: see Eq. 11.12-2a. Terms that contain Δt to second and higher powers discarded. This implies second-order accuracy according to the argument that follows Eq. 11.12-2b.

Backward: Write Eq. 11.12-2b for the next time step.

$$\underset{\sim}{D}_n = \underset{\sim}{D}_{n+1} - \Delta t\, \dot{\underset{\sim}{D}}_{n+1} + \frac{\Delta t^2}{2}\, \ddot{\underset{\sim}{D}}_{n+1} - \cdots$$

Solve for $\underset{\sim}{D}_{n+1}$ and discard Δt^2 and higher terms.

$$\underset{\sim}{D}_{n+1} = \underset{\sim}{D}_n + \Delta t\, \dot{\underset{\sim}{D}}_{n+1}$$

Second-order accurate according to the same argument.

$\boxed{11.12\text{-}2}$

$\mathbf{C} = \mathbf{0}$ in Eq. 11.12-6: $\frac{1}{\Delta t^2}\mathbf{M}\mathbf{D}_{n+1} = \mathbf{R}_n^{ext} - \mathbf{R}_n^{int} + \frac{1}{\Delta t^2}\mathbf{M}\left(\mathbf{D}_n + \Delta t\,\dot{\mathbf{D}}_{n-\frac{1}{2}}\right)$

Write Eq. 11.2-5a for the previous time step: $\mathbf{D}_n = \mathbf{D}_{n-1} + \Delta t\,\dot{\mathbf{D}}_{n-\frac{1}{2}}$

Solve the latter equation for $\Delta t\,\dot{\mathbf{D}}_{n-\frac{1}{2}}$ and subs. into former equation:

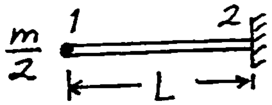
$$\frac{1}{\Delta t^2}\mathbf{M}\mathbf{D}_{n+1} = \mathbf{R}_n^{ext} - \mathbf{R}_n^{int} + \frac{1}{\Delta t^2}\mathbf{M}\left(\underbrace{\mathbf{D}_n + \mathbf{D}_n}_{2\mathbf{D}_n} - \mathbf{D}_{n-1}\right)$$

agree

Eq. 11.12-3 with $\mathbf{C} = \mathbf{0}$ is

$$\frac{1}{\Delta t^2}\mathbf{M}\mathbf{D}_{n+1} = \mathbf{R}_n^{ext} - \mathbf{R}_n^{int} + \frac{2}{\Delta t^2}\mathbf{M}\mathbf{D}_n - \frac{1}{\Delta t^2}\mathbf{M}\mathbf{D}_{n-1}$$

11.12-3



Exact ω^2 is

$$\omega^2 = \frac{k}{m/2} = \frac{2AE}{mL}$$

Element bound, from Eq. 11.12-15:

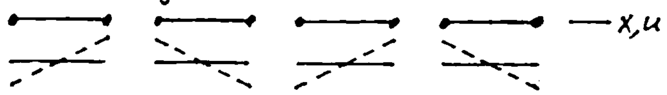
$$(\omega_{max})_e^2 = \frac{4E}{\rho L^2} = \frac{4EA}{L(\rho L A)} = \frac{4EA}{mL} \quad \text{(100\% high)}$$

Gershgorin bound: $[\underset{\sim}{K}] = \frac{AE}{L}$, $[\underset{\sim}{M}] = \frac{m}{2}$

$$\omega_{max}^2 \leq \frac{k}{m/2} \quad \text{(exact)}$$

11.12-4

(a) Imagine identical but unconnected two-node bar elements, each unsupported, vibrating axially, and 180° out of phase with its neighbors on either side.



We can connect the elements without changing anything.

(b) If we fix one end of the model sketched in the solution of part (a), the vibration mode is perturbed there, but the disturbance is not much felt at the other end, particularly if there are a great many elements. Agreement improves.

$\boxed{11.12-5}$

$$\Delta t_{cr} = 2/\omega = 2.0$$

From Eq. 11.12-14, $u_{-1} = -\Delta t$

From Eq. 11.12-3

$$\frac{1}{\Delta t^2} u_{n+1} = -u_n + \frac{1}{\Delta t^2}(2u_n - u_{n-1})$$

$$u_{n+1} = (2 - \Delta t^2) u_n - u_{n-1}$$

(a) $\Delta t = 1$, $u_{n+1} = u_n - u_{n-1}$

n	-1	0	1	2	3	4	5
u_n	-1	0	1	1	0	-1	-1

(b) $\Delta t = \sqrt{2}$, $u_{n+1} = -u_{n-1}$

n	-1	0	1	2	3	4	5
u_n	$-\sqrt{2}$	0	$\sqrt{2}$	0	$-\sqrt{2}$	0	$\sqrt{2}$

(c) $\Delta t = 2$ (critical), $u_{n+1} = -2u_n - u_{n-1}$

n	-1	0	1	2	3	4	5
u_n	-2	0	2	-4	6	-8	10

Blows up in arithmetic fashion.

(d) $\Delta t = 3$ (> critical), $u_{n+1} = -7u_n - u_{n-1}$

n	-1	0	1	2	3	4	5
u_n	-3	0	3	-21	144	-1008	6912

Blows up in geometric fashion.

11.12-6

Note that $\Delta t_{cr} = \frac{2}{\omega} = \frac{2}{1} = 2.0$

Eq. 11.12-3: $\frac{1}{\Delta t^2} D_{n+1} = 1 - \left(1 - \frac{2}{\Delta t^2}\right) D_n - \frac{1}{\Delta t^2} D_{n-1}$

or $D_{n+1} = \Delta t^2 - (\Delta t^2 - 2) D_n - D_{n-1}$

Also $D_{-1} = 0 - 0 + \frac{\Delta t^2}{2}(1) = \frac{\Delta t^2}{2}$

(a) $\Delta t = 0.5$, $D_{-1} = 0.125$

$D_{n+1} = 0.250 + 1.750 D_n - D_{n-1}$

t	0	0.5	1.0	1.5	2.0
D	0	0.125	0.469	0.945	1.436

t	2.5	3.0	3.5	4.0	4.5
D	1.817	1.994	1.923	1.621	1.163

t	5.0	5.5	6.0	6.5	7.0
D	0.665	0.251	0.024	0.041	0.298

(b) $\Delta t = 1.0$, $D_{-1} = 0.50$

$D_{n+1} = 1.0 + D_n - D_{n-1}$

t	0	1	2	3	4	5	6	7
D	0	0.5	1.5	2.0	1.5	0.5	0	0.5

(c) $\Delta t = 2.0$, $D_{-1} = 2.0$

$D_{n+1} = 4 - 2D_n - D_{n-1}$

t	0	2	4	6	8	10
D	0	2	0	2	0	2

(d) $\Delta t = 3.0$, $D_{-1} = 4.5$

$D_{n+1} = 9 - 7D_n - D_{n-1}$

t	0	3	6	9	12	15
D	0	4.5	-22.5	162	-1103	7565

(e) $D + \ddot{D} = 1$, $D = A\sin t + B\cos t + 1$

At $t = 0$, $D = 0$, $\therefore B = -1$

At $t = 0$, $\dot{D} = 0$, $\therefore A = 0$ $\quad D = 1 - \cos t$

t	0	0.5	1	1.5	2	2.5
D	0	0.122	0.460	0.929	1.416	1.801

t	3	3.5	4	4.5	5	5.5
D	1.990	1.937	1.654	1.211	0.716	0.291

t	6	6.5	7	8	9	10
D	0.040	0.023	0.246	1.146	1.911	1.839

$\boxed{11.12\text{-}7}$

Eq. 11.2-12, with proposed representation of viscous terms, is

$$\mathbf{M}\ddot{\mathbf{D}}_n + \alpha\mathbf{M}\dot{\mathbf{D}}_n + \beta\mathbf{K}\dot{\mathbf{D}}_{n-\frac{1}{2}} + \mathbf{K}\mathbf{D}_n = \mathbf{R}_n^{ext}$$

Approximate $\dot{\mathbf{D}}_n$ and $\ddot{\mathbf{D}}_n$ by Eqs. 11.12-1. Leave $\dot{\mathbf{D}}_{n-\frac{1}{2}}$ as-is so it can be computed with historical information. Thus

$$\frac{1}{\Delta t^2}\mathbf{M}(\mathbf{D}_{n+1} - 2\mathbf{D}_n + \mathbf{D}_{n-1}) + \frac{\alpha}{2\Delta t}\mathbf{M}(\mathbf{D}_{n+1} - \mathbf{D}_{n-1})$$
$$+\beta\mathbf{K}\dot{\mathbf{D}}_{n-\frac{1}{2}} + \mathbf{K}\mathbf{D}_n = \mathbf{R}_n^{ext}$$

Which is, after rearrangement,

$$\left(\frac{1}{\Delta t^2} + \frac{\alpha}{2\Delta t}\right)\mathbf{M}\mathbf{D}_{n+1} = \mathbf{R}_n^{ext} - \mathbf{K}\mathbf{D}_n + \frac{1}{\Delta t^2}\mathbf{M}\left[2\mathbf{D}_n - \left(1 - \frac{\alpha\Delta t}{2}\right)\mathbf{D}_{n-1}\right]$$
$$-\beta\mathbf{K}\dot{\mathbf{D}}_{n-\frac{1}{2}}$$

Compute $\beta\mathbf{K}\dot{\mathbf{D}}_{n-\frac{1}{2}}$ in element-by-element fashion.

The contribution of one element to this is

$$\beta\mathbf{k}\dot{\mathbf{d}}_{n-\frac{1}{2}} = \beta\int\mathbf{B}^T\mathbf{E}\mathbf{B}\,dV\,\dot{\mathbf{d}}_{n-\frac{1}{2}} = \beta\int\mathbf{B}^T\mathbf{E}\dot{\boldsymbol{\epsilon}}_{n-\frac{1}{2}}\,dV = \beta\int\mathbf{B}^T\dot{\boldsymbol{\sigma}}_{n-\frac{1}{2}}\,dV$$

We replace $\mathbf{K}\mathbf{D}_n$ by $\mathbf{R}_n^{int} = \sum\mathbf{r}_n^{int} = \sum\int\mathbf{B}^T(\boldsymbol{\sigma}_n + \beta\dot{\boldsymbol{\sigma}}_{n-\frac{1}{2}})\,dV$

$\boxed{11.13\text{-}1}$

$$D_{n+1} = D_n + \Delta t\,\dot{D}_n + \frac{\Delta t^2}{2}\ddot{D}_n + \frac{\Delta t^3}{6}\dddot{D}_n + \cdots \qquad (A)$$

$$D_n = D_{n+1} - \Delta t\,\dot{D}_{n+1} + \frac{\Delta t^2}{2}\ddot{D}_{n+1} - \frac{\Delta t^3}{6}\dddot{D}_{n+1} + \cdots \qquad (B)$$

Solve (B) for $\underset{\sim}{D}_{n+1}$

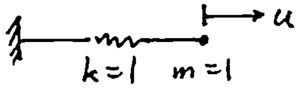
$$D_{n+1} = D_n + \Delta t\,\dot{D}_{n+1} - \frac{\Delta t^2}{2}\ddot{D}_{n+1} + \frac{\Delta t^3}{6}\dddot{D}_{n+1} - \cdots \qquad (C)$$

Add (A) & (C); divide result by 2.

$$\underbrace{D_{n+1} \doteq D_n + \frac{\Delta t}{2}(\dot{D}_n + \dot{D}_{n+1})}_{(D)\ -\ \text{result requested}} + \frac{\Delta t^2}{4}(\ddot{D}_n - \ddot{D}_{n+1}) + \cdots$$

Terms omitted from Eq. (D) depend on Δt^2, so error is $O(\Delta t^2)$.

$\boxed{11.13-2}$



$\beta = \frac{1}{4}, \quad \gamma = \frac{1}{2}$

(a) $\Delta t = 2$

Eq. 11.13-5: $2u_{n+1} = u_n + 2\dot{u}_n + \ddot{u}_n$

Eq. 11.13-4b: $\dot{u}_{n+1} = u_{n+1} - u_n - \dot{u}_n$

Eq. 11.13-4a: $\ddot{u}_{n+1} = u_{n+1} - u_n - 2\dot{u}_n - \ddot{u}_n$

n	u_n	$\dot{u}_n$	$\ddot{u}_n$
0	0	1	0
1	1	0	-1
2	0	-1	0
3	-1	0	1
4	0	1	0

(b) $\Delta t = 1$

Eq. 11.13-5: $(4+1)u_{n+1} = 4(u_n + \dot{u}_n) + \ddot{u}_n$

or $u_{n+1} = 0.8(u_n + \dot{u}_n) + 0.2\ddot{u}_n$

Eq. 11.13-4b: $\dot{u}_{n+1} = 2(u_{n+1} - u_n) - \dot{u}_n$

Eq. 11.13-4a: $\ddot{u}_{n+1} = 4(u_{n+1} - u_n - \dot{u}_n) - \ddot{u}_n$

n	u_n	$\dot{u}_n$	$\ddot{u}_n$
0	0	1	0
1	0.80	0.6	-0.8
2	0.96	-0.28	-0.96
3	0.352	-0.963	-0.352
4	-0.5592		

$\boxed{11.14\text{-}1}$

(a) $$2\xi\omega\,\frac{z_{n+1}-z_{n-1}}{2\Delta t} + \omega^2 z_n = 0$$

or $$z_{n+1} + \frac{\omega\Delta t}{\xi} - z_{n-1} = 0 \qquad (A)$$

From Eq. 13.13-5: $z_n = C\lambda^n$, $z_{n+1} = C\lambda^{n+1}$, $z_{n-1} = C\lambda^{n-1}$

So (A) becomes

$$C\lambda^{n+1} + \frac{\omega\Delta t}{\xi}C\lambda^n - C\lambda^{n-1} = 0$$

Divide by $C\lambda^{n-1}$; $\lambda^2 + \frac{\omega\Delta t}{\xi}\lambda - 1 = 0$

$$\lambda_{1,2} = \frac{1}{2}\left[-\frac{\omega\Delta t}{2} \pm \sqrt{\frac{\omega^2\Delta t^2}{\xi^2} - 4(1)(-1)}\right]$$

always real

Since $\lambda_1\lambda_2 = c/a = -1$, & since λ_1 & λ_2 are real & distinct, then one $|\lambda| < 1$ while other $|\lambda| > 1$ and <u>method is unstable.</u>

(b) $$2\xi\omega\underbrace{(\dot z_{n+1} + \dot z_n)}_{\frac{2}{\Delta t}(z_{n+1}-z_n)} + \omega^2(z_{n+1}+z_n) = 0$$

$$z_{n+1} - z_n + \frac{\omega\Delta t}{4\xi}(z_{n+1}+z_n) = 0$$

$$\left(1+\frac{\omega\Delta t}{4\xi}\right)z_{n+1} + \left(\frac{\omega\Delta t}{4\xi} - 1\right)z_n = 0.$$ Subs. $z_n = C\lambda^n$, $z_{n+1} = C\lambda^{n+1}$

$$\left(1+\frac{\omega\Delta t}{4\xi}\right)\lambda + \frac{\omega\Delta t}{4\xi} - 1 = 0$$

$$\lambda = \left(1-\frac{\omega\Delta t}{4\xi}\right)\frac{1}{1+\frac{\omega\Delta t}{4\xi}}$$ Let $h = \frac{\omega\Delta t}{4\xi}$ always positive

$\left|\lambda\right| = \left|\frac{1-h}{1+h}\right| < 1$ for all h (i.e. for all Δt): <u>unconditionally stable</u>

(c) $$2\xi\omega\dot z_n + \omega^2 z_n = 0$$

$$2\xi\omega\frac{1}{\Delta t}(z_{n+1}-z_n) + \omega^2 z_n = 0$$

$$z_{n+1} + \left(\frac{\omega\Delta t}{2\xi} - 1\right)z_n = 0$$ Subs. $z_n = C\lambda^n$, $z_{n+1} = C\lambda^{n+1}$

$$\lambda + \left(\frac{\omega\Delta t}{2\xi} - 1\right) = 0 \quad \text{or} \quad \lambda = 1 - \frac{\omega\Delta t}{2\xi}$$

always positive

For $|\lambda| \le 1$, must have

$$-1 \le 1 - \frac{\omega\Delta t}{2\xi} \le 1 \quad \text{or} \quad -2 \le \underbrace{-\frac{\omega\Delta t}{2\xi} \le 0}_{\text{always satisfied}}$$

Then $-2 \le -\frac{\omega\Delta t}{2\xi}$, $\frac{\omega\Delta t}{2\xi} \le 2$, i.e. must have $\Delta t \le \frac{4\xi}{\omega}$ for stability (conditionally stable)

(d) $$2\xi\omega\dot z_{n+1} + \omega^2 z_{n+1} = 0$$

$$2\xi\omega\frac{1}{\Delta t}(z_{n+1}-z_n) + \omega^2 z_{n+1} = 0$$

$$z_{n+1} - z_n + \frac{\omega\Delta t}{2\xi}z_{n+1} = 0$$ Subs. $z_n = C\lambda^n$, etc.

$$\left(1+\frac{\omega\Delta t}{\xi}\right)\lambda - 1 = 0, \quad \lambda = \frac{1}{1+\frac{\omega\Delta t}{2\xi}}$$

always pos.

$|\lambda| < 1$ for all Δt <u>unconditionally stable</u>

11.14-2

In Prob. 11.12-5, $k=1$ and $=1$, so
$\omega_{exact}=1$ and $P_{exact}=2\pi$.

(a) $\Delta t=1$, period $=6\Delta t=6$

$$P=\frac{2\pi/b}{2\pi/\omega}=\frac{6}{2\pi}=\frac{3}{\pi}=0.9549$$

Eq. 11.14-20:

$$P=\omega\Delta t\left(\tan^{-1}\frac{(1)\sqrt{4-1}}{2-1}\right)^{-1}=(1)\frac{1}{\tan^{-1}\sqrt{3}}=0.9549$$

(b) $\Delta t=\sqrt{2}$, period $=4\Delta t=4\sqrt{2}$

$$P=\frac{2\pi/b}{2\pi/\omega}=\frac{4\sqrt{2}}{2\pi}=\frac{2\sqrt{2}}{\pi}$$

Eq. 11.14-20:

$$P=\omega\Delta t\left(\tan^{-1}\frac{\sqrt{2}}{2-2}\right)^{-1}=\frac{\sqrt{2}}{\pi/2}=\frac{2\sqrt{2}}{\pi}$$

(c) $\Delta t=2$, period $=2\Delta t=4$, $P=\frac{4}{2\pi}=\frac{2}{\pi}$

Eq. 11.14-20:

$$P=\omega\Delta t\left(\tan^{-1}\frac{0}{-2}\right)^{-1}=\frac{2}{\pi}$$

11.14-3

In Prob. 11.13-2, $k=1$ and $m=1$, so $\omega_{exact}=1$ and $P_{exact}=2\pi$.

(a) $\text{Period} = 4\Delta t = 4(2) = 8$

$$P = \frac{2\pi/b}{2\pi/\omega} = \frac{8}{2\pi} = \frac{4}{\pi}$$

Eq. 11.14-21:

$$P = 2\left(\tan^{-1}\frac{4(2)}{4-4}\right)^{-1} = \frac{2}{\pi/2} = \frac{4}{\pi}$$

(b) Determine when $u_n = 0$ by a linear interpolation approximation.

$$n \approx 3 + \frac{0.352}{0.352+0.559} = 3.39$$

$\text{period} \approx 3.39\,\Delta t = 3.39$

$$P = \frac{2\pi/b}{2\pi/\omega} = \frac{2(3.39)}{2\pi} = \frac{3.39}{\pi} = 1.08$$

Eq. 11.14-21:

$$P = (1)\left(\tan^{-1}\frac{4}{4-1}\right)^{-1} = \frac{1}{\tan\frac{4}{3}} = 1.08$$

11.14-4

(a) Eq. 11.14-20:

$\omega \Delta t = 0: \quad P = 0$

$\omega \Delta t = 1: \quad P = (1)\left[\tan^{-1}\frac{\sqrt{3}}{1}\right]^{-1} = 0.955$

$\omega \Delta t = \sqrt{2}: \quad P = \sqrt{2}\left[\tan^{-1}\frac{\sqrt{2}\sqrt{2}}{0}\right]^{-1} = \frac{\sqrt{2}}{\pi/2} = 0.900$

$\omega \Delta t = 2: \quad P = 2\left[\tan^{-1}\frac{2(0)}{-2}\right]^{-1} = \frac{2}{\pi} = 0.637$

(b) Eq. 11.14-21: $\quad \omega \Delta t = 0: \quad P = 0$

$\omega \Delta t = 1: \quad P = (1)\left[\tan^{-1}\frac{4}{3}\right]^{-1} = \frac{1}{0.9273} = 1.078$

$\omega \Delta t = 2: \quad P = (2)\left[\tan^{-1}\frac{8}{0}\right]^{-1} = \frac{2}{\pi/2} = \frac{4}{\pi} = 1.273$

$\omega \Delta t = 4: \quad P = (4)\left[\tan^{-1}\frac{16}{-12}\right]^{-1} = \frac{4}{2.214} = 1.806$

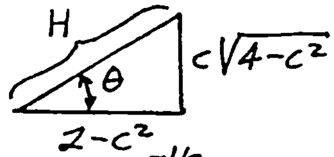
11.14-5

(a) Substitute $b\,\Delta t$ from Eq. 11.14-19 into Eq. 11.14-22:

$$2-\omega^2\Delta t^2 = 2e^{a\Delta t}\cos\left[\arctan\frac{\omega\Delta t\sqrt{4-\omega^2\Delta t^2}}{2-\omega^2\Delta t^2}\right]$$ Solve for $a\Delta t$:

$$a\,\Delta t = \ln\left(\frac{1-\omega^2\Delta t^2/2}{\cos\left[\arctan\frac{\omega\Delta t\sqrt{4-\omega^2\Delta t^2}}{2-\omega^2\Delta t^2}\right]}\right)$$ Let $c=\omega\Delta t$;

$$a\,\Delta t = \ln\left(\frac{1-(c^2/2)}{\cos\left[\arctan\frac{c\sqrt{4-c^2}}{2-c^2}\right]}\right)$$



$$H=\left[(2-c^2)^2+(c\sqrt{4-c^2})^2\right]^{1/2}=\left[4-4c^2+c^4+4c^2-c^4\right]^{1/2}=4^{1/2}=2$$

Hence $\cos\theta = \frac{2-c^2}{2} = 1-(c^2/2)$ and $a\Delta t = \ln 1 = 0$

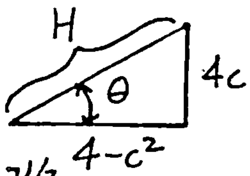
(b) Obtain $\lambda_1+\lambda_2$ from Eq. 11.14-13 and 11.14-16; thus

$$e^{a\Delta t}\cos b\Delta t = \frac{4-\omega^2\Delta t^2}{4+\omega^2\Delta t^2}$$ Then from Eq. 11.14-21, where $h=\omega^2\Delta t^2/4$,

$$\tan bt = \frac{4\omega\Delta t}{4-\omega^2\Delta t^2}$$ Combine these two eqs. to obtain

$$e^{a\Delta t}\cos\left[\arctan\frac{4\omega\Delta t}{4-\omega^2\Delta t^2}\right] = \frac{4-\omega^2\Delta t^2}{4+\omega^2\Delta t^2}.$$ With $c=\omega\Delta t$, we get

$$a\Delta t = \ln\left(\frac{4-c^2}{(4+c^2)\cos\left[\arctan\frac{4c}{4-c^2}\right]}\right)$$



$$H=\left[(4-c^2)^2+(4c)^2\right]^{1/2}=\left[16-8c^2+c^4+16c^2\right]^{1/2}=4+c^2$$

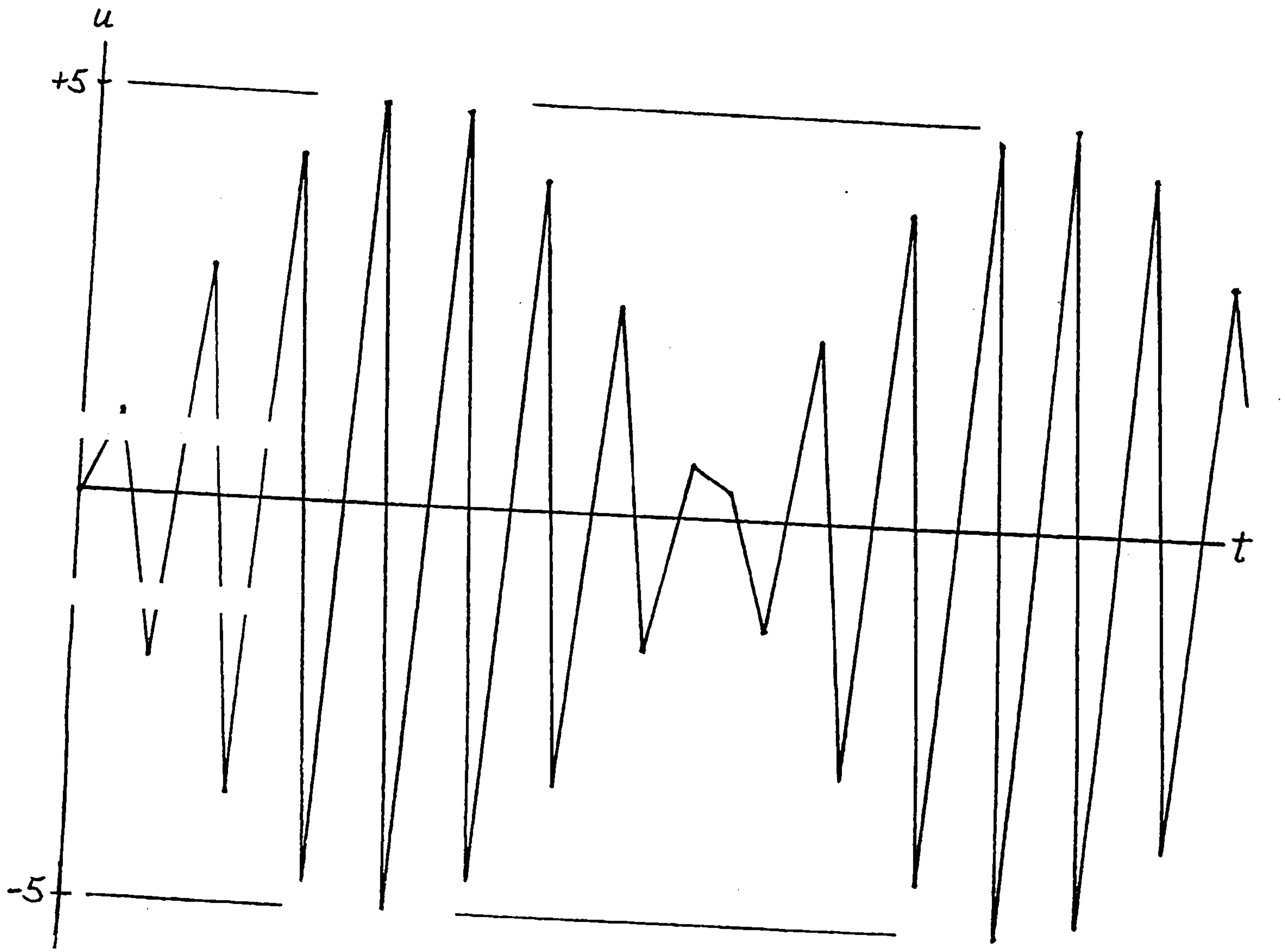
Hence $\cos\theta = \frac{4-c^2}{4+c^2}$, and

$$a\Delta t = \ln\frac{4-c^2}{(4+c^2)\frac{4-c^2}{4+c^2}} = \ln 1 = 0$$

$\boxed{11.14\text{-}6}$ Note that $\Delta t_{cr} = 2$, so we operate just under the stability limit.

$$u_{n+1} = (2-\Delta t^2)u_n - u_{n-1} = -1.96\,u_n - u_{n-1}$$

n	u_n	n	u_n	n	u_n	n	u_n
-1	-1	9	4.89	19	-3.10	29	-2.29
0	0	10	-4.56	20	3.83	30	1.36
1	1	11	4.05	21	-4.40	31	-0.37
2	-1.96	12	-3.38	22	4.79	32	-0.64
3	2.84	13	2.57	23	-5.00	33	1.62
4	-3.61	14	-1.66	24	5.00	34	-2.53
5	4.23	15	0.68	25	-4.81	35	3.35
6	-4.69	16	0.32	26	4.42	36	-4.02
7	4.95	17	-1.32	27	-3.85	37	4.52
8	-5.02	18	2.25	28	3.14	38	-4.88



11.17-1

Node 51 amplitude, mode 3, from Table 11.17-1, is 0.0034.

Modal load, mode 3:

$$P_3 = 0.0034(3000) = 10.2$$

From Fig. 11.17-2, $f_3 = 70.77$ Hz

hence $\omega_3 = 2\pi f_3 = 444.7$ /s

With $\beta_3 = 1$, Eq. 11.10-1 yields ($\xi_3 = 0.02$)

$$z_3 = \frac{P_3/\omega_3^2}{2\xi_3} = 1.28(10)^{-3}$$

Node 16 amplitude, mode 3, from Table 11.17-1, is 0.0238. Hence

$$\bar{u}_{16} = 0.0238\, z_3 = 30.7(10)^{-6}\ \mathrm{m}$$

which agrees with value plotted in Fig. 11.17-3b.