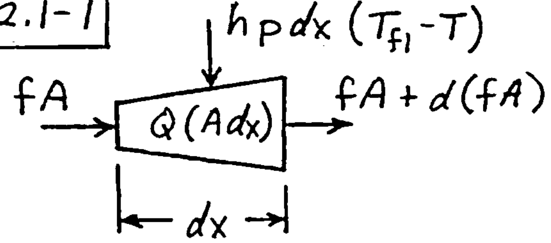
$\boxed{12.1\text{-}1}$



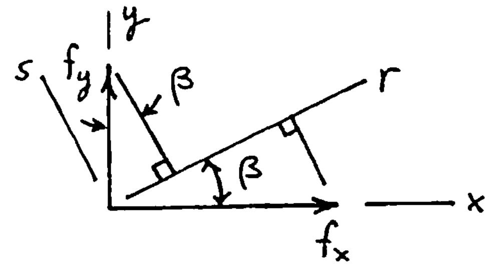
$$fA + hp\,dx\,(T_{f1}-T) + QA\,dx - \left[fA + d(fA)\right] = c\rho\dot{T}A\,dx$$

$$-\frac{d}{dx}(fA) + hp(T_{f1}-T) + QA - c\rho\dot{T}A = 0$$

Substitute $f = -kT_{,x}$

$$\frac{d}{dx}(AkT_{,x}) + AQ + hp(T_{f1}-T) - Ac\rho\dot{T} = 0$$

$\boxed{12.1\text{-}2}$



$$f_r = f_x \cos\beta + f_y \sin\beta$$
$$f_s = -f_x \sin\beta + f_y \cos\beta$$

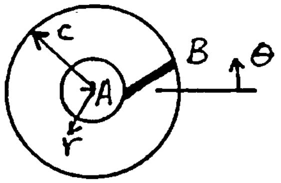
or

$$\begin{Bmatrix} f_r \\ f_s \end{Bmatrix} = \underbrace{\begin{bmatrix} \cos\beta & \sin\beta \\ -\sin\beta & \cos\beta \end{bmatrix}}_{[\underset{\sim}{\Lambda}]} \begin{Bmatrix} f_x \\ f_y \end{Bmatrix}$$

But $[\underset{\sim}{\Lambda}]$ is orthogonal; $[\underset{\sim}{\Lambda}]^{-1} = [\underset{\sim}{\Lambda}]^T = \begin{bmatrix} \cos\beta & -\sin\beta \\ \sin\beta & \cos\beta \end{bmatrix}$

Therefore $\begin{Bmatrix} f_x \\ f_y \end{Bmatrix} = [\underset{\sim}{\Lambda}]^T \begin{Bmatrix} f_r \\ f_s \end{Bmatrix}$

12.1-3



Consider a disk, viewed along its axis. Let AB be a path that is much more conductive than the remainder. If A is heated, B will be much hotter than other parts of $r=c$. Thus, T on $r=c$ is not symmetric w.r.t. $\theta=0$, but would become so if AB is radial.

$\boxed{12.1-4}$

(a) Use $\{\underset{\sim}{\partial}\}$ and $\{\underset{\sim}{T}_\partial\}$ from Eqs. 12.1-15

$$[\underset{\sim}{\kappa}]\{\underset{\sim}{T}_\partial\} = k\lfloor T_{,r} \quad \tfrac{1}{r}T_{,\theta} \quad T_{,z}\rfloor^T$$

Eq. 12.1-14a becomes

$$k\left(\tfrac{1}{r}T_{,r} + T_{,rr} + \tfrac{1}{r^2}T_{,\theta\theta} + T_{,zz}\right) + Q - c\rho\dot{T} = 0$$

Eq. 12.1-14b becomes

$$f_B = k(\ell T_{,r} + n T_{,z})$$

(b) $\{\underset{\sim}{\partial}\} = \begin{Bmatrix} \frac{1}{r} + \frac{\partial}{\partial r} \\ \frac{1}{r}\frac{\partial}{\partial\theta} \end{Bmatrix}$, $\{\underset{\sim}{T}_\partial\} = \begin{Bmatrix} T_{,r} \\ \frac{1}{r}T_{,\theta} \end{Bmatrix}$

Eq. 12.1-14a:

$$\{\underset{\sim}{\partial}\}^T \begin{Bmatrix} \kappa_{11}T_{,r} + \kappa_{12}\frac{1}{r}T_{,\theta} \\ \kappa_{21}T_{,r} + \kappa_{22}\frac{1}{r}T_{,\theta} \end{Bmatrix} + Q = c\rho\dot{T} \qquad (\kappa_{12}=\kappa_{21})$$

$$\kappa_{11}T_{,rr} + \kappa_{11}\tfrac{1}{r}T_{,r} + \kappa_{12}\tfrac{1}{r^2}T_{,\theta} - \kappa_{12}\tfrac{1}{r^2}T_{,\theta} + \kappa_{12}\tfrac{1}{r}T_{,r\theta}$$

$+\kappa_{21}\frac{1}{r}T_{,r\theta} + \kappa_{22}\frac{1}{r^2}T_{,\theta\theta} + Q = \rho c\dot{T}$. Gather terms.

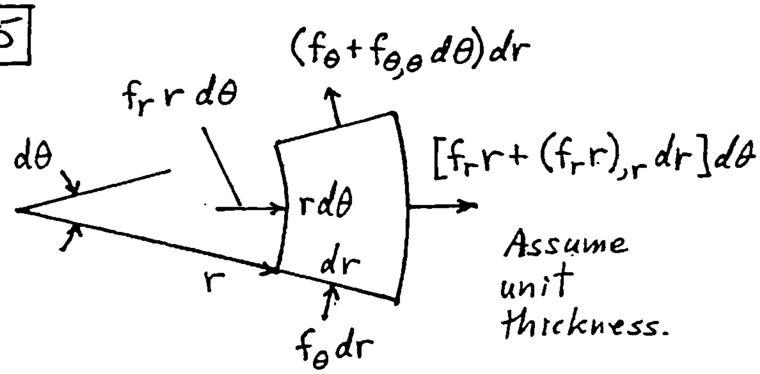
$$\kappa_{11}\left(T_{,rr} + \tfrac{1}{r}T_{,r}\right) + 2\kappa_{12}\tfrac{1}{r}T_{,r\theta} + \kappa_{22}\tfrac{1}{r^2}T_{,\theta\theta} + Q = c\rho\dot{T}$$

In Eq. 12.1-14b, $l_B = 1$ & $m_B = 0$.

$$f_B = \lfloor 1 \quad 0\rfloor \begin{Bmatrix} \kappa_{11}T_{,r} + \kappa_{12}\frac{1}{r}T_{,\theta} \\ \kappa_{21}T_{,r} + \kappa_{22}\frac{1}{r}T_{,\theta} \end{Bmatrix}$$

$$f_B = \kappa_{11}T_{,r} + \kappa_{12}\tfrac{1}{r}T_{,\theta}$$

12.1-5



Assume unit thickness.

Net inward flux from the above is

$$-[f_r r + (f_r r)_{,r}\, dr]\, d\theta + f_r r\, d\theta - (f_\theta + f_{\theta,\theta}\, d\theta)\, dr + f_\theta\, dr$$

or $-\left[(f_r r)_{,r} + f_{\theta,\theta}\right] dr\, d\theta$

Net inward heat flow per unit volume is

$$-(r f_{r,r} + f_r + f_{\theta,\theta})\, dr\, d\theta + Q r\, dr\, d\theta$$

Set equal to $\rho c \dot{T} r\, dr\, d\theta$ and divide by $r\, dr\, d\theta$

$$-f_{r,r} - \frac{1}{r} f_r - \frac{1}{r} f_{\theta,\theta} + Q = \rho c \dot{T} \qquad (A)$$

If orthotropic, $f_r = -k_{11} T_{,r} - k_{12} \frac{1}{r} T_{,\theta}$

$f_\theta = -k_{21} T_{,r} - k_{22} \frac{1}{r} T_{,\theta}$

$(k_{12} = k_{21})$

Eq. (A) becomes

$$k_{11} T_{,rr} - k_{12} \frac{1}{r^2} T_{,\theta} + k_{12} \frac{1}{r} T_{,r\theta} + k_{11} \frac{1}{r} T_{,r}$$
$$+ k_{12} \frac{1}{r^2} T_{,\theta} + k_{21} \frac{1}{r} T_{,r\theta} + k_{22} \frac{1}{r^2} T_{,\theta\theta} + Q = c\rho \dot{T}$$

Gather terms

$$k_{11}\left(T_{,rr} + \frac{1}{r} T_{,r}\right) + 2 k_{12} \frac{1}{r} T_{,r\theta} + k_{22} \frac{1}{r^2} T_{,\theta\theta} + Q - c\rho \dot{T} = 0$$

$\boxed{12.2\text{-}1}$

Eq. 4.7-6 is $\dfrac{\partial F}{\partial T} - \dfrac{\partial}{\partial x}\dfrac{\partial F}{\partial T_{,x}} - \dfrac{\partial}{\partial y}\dfrac{\partial F}{\partial T_{,y}} = 0$ (A)

where, from Eq. 12.2-1, $F = \frac{1}{2}\{\underset{\sim}{T}_\partial\}^T[\underset{\sim}{k}]\{\underset{\sim}{T}_\partial\} - QT + c\rho\dot{T}T$ (B)

in which, for a plane problem, $\{\underset{\sim}{T}_\partial\} = \begin{Bmatrix} T_{,x} \\ T_{,y} \end{Bmatrix}$ (C)

Rewriting (A), with $\{\underset{\sim}{\partial}\} = \begin{Bmatrix} \partial/\partial x \\ \partial/\partial y \end{Bmatrix}$,

$$\frac{\partial F}{\partial T} - \{\underset{\sim}{\partial}\}^T \begin{Bmatrix} \partial F/\partial T_{,x} \\ \partial F/\partial T_{,y} \end{Bmatrix} = 0 \qquad \text{(D)}$$

Substitute (B) into (D)

$$-Q + c\rho\dot{T} - \{\underset{\sim}{\partial}\}^T\left([\underset{\sim}{k}]\{\underset{\sim}{T}_\partial\}\right) = 0$$

12.2-2

$$\Pi = \iint\left(\frac{1}{2}k_x T_{,x}^2 + k_{xy}T_{,x}T_{,y} + \frac{1}{2}k_y T_{,y}^2\right.$$

$$\left. - QT - 2hT_f T + hT^2 + \rho c \dot{T} T\right) dx\,dy$$

$$- \int h\left(T_f T - \frac{1}{2}T^2\right) dS - \int f_B T\, dS$$

$$\delta\Pi = 0 = \iint\left(kT_{,x}\,\delta T_{,x} + k_{xy}T_{,x}\,\delta T_{,y} + k_{xy}T_{,y}\,\delta T_{,x}\right.$$

$$+ k_y T_{,y}\,\delta T_{,y} - Q\,\delta T - 2hT_f\,\delta T + 2hT\,\delta T$$

$$\left. + \rho c \dot{T}\,\delta T\right) dx\,dy - \int h\left(T_f\,\delta T - T\,\delta T\right) dS - \int f_B\,\delta T\, dS$$

Integrations by parts:

$$\iint k_x T_{,x}\,\delta T_{,x}\, dx\,dy = -\iint (k_x T_{,x})_{,x}\,\delta T\, dx\,dy + \int k_x T_{,x}\,\delta T\, l_B\, dS$$

$$\iint k_{xy} T_{,x}\,\delta T_{,y}\, dx\,dy = -\iint (k_{xy} T_{,x})_{,y}\,\delta T\, dx\,dy + \int k_{xy} T_{,x}\,\delta T\, m_B\, dS$$

Similar for next 2 terms.

Thus

$$\delta\Pi = 0 = \iint\Big[-(k_x T_{,x} + k_{xy}T_{,y})_{,x} - (k_{xy}T_{,x} + k_y T_{,y})_{,y}$$

$$- Q - 2h(T_f - T) + \rho c \dot{T}\Big]\delta T\, dx\,dy + \int\Big[(k_x T_{,x} + k_{xy}T_{,y})\, l_B$$

$$+ (k_{xy}T_{,x} + k_y T_{,y})\, m_B - h(T_f - T) - f_B\Big]\delta T\, dS$$

Vanishing of [---] in double integral yields Eq. 12.1-7. Vanishing of [---] in surface integral yields Eq. 12.1-12b. Lateral sur- 2 added to both.

$\boxed{12.2\text{-}3}$

Assume $\tilde{T} = \underset{\sim}{N}\, \underset{\sim}{T}_e$; use Eq. 12.1-10

$$\int_0^L \underset{\sim}{N}^T \left[(Ak\tilde{T}_{,x})_{,x} + QA + h(T_f - \tilde{T})p \right] dx = 0$$

Integrate by parts:

$$\int_0^L \underset{\sim}{N}^T \left[(Ak\tilde{T}_{,x})_{,x} \right] dx = -\int_0^L \underset{\sim}{N}_{,x}^T Ak\tilde{T}_{,x}\, dx + \left(\underset{\sim}{N}^T Ak\tilde{T}_{,x} \right)_0^L$$

But $\left(\underset{\sim}{N}^T Ak\tilde{T}_{,x} \right)_0^L = -\left(\underset{\sim}{N}^T Af \right)_0^L = \begin{Bmatrix} Af_0 \\ -Af_L \end{Bmatrix}$

Also subs. $\tilde{T} = \underset{\sim}{N}\, \underset{\sim}{T}_e$ & $\tilde{T}_{,x} = \underset{\sim}{N}_{,x} \underset{\sim}{T}_e$ into the first (residual) equation. Thus

$$\underbrace{-\int_0^L \underset{\sim}{N}_{,x} Ak\, \underset{\sim}{N}_{,x}\, dx\, \underset{\sim}{T}_e}_{[\underset{\sim}{k}]} + \underbrace{\int_0^L \underset{\sim}{N}^T QA\, dx}_{\{\underset{\sim}{r}_Q\}}$$

$$\underbrace{-\int_0^L \underset{\sim}{N}^T hp\, \underset{\sim}{N}\, dx\, \underset{\sim}{T}_e}_{[\underset{\sim}{h}_{ls}]} + \underbrace{\int_0^L \underset{\sim}{N}^T hp T_f\, dx}_{\{\underset{\sim}{r}_{ls}\}} + \begin{Bmatrix} Af_0 \\ -Af_L \end{Bmatrix} = \underset{\sim}{0}$$

$$\left([\underset{\sim}{k}] + [\underset{\sim}{h}_{ls}] \right) \{\underset{\sim}{T}_e\} = \{\underset{\sim}{r}_Q\} + \{\underset{\sim}{r}_{ls}\} + \begin{Bmatrix} Af_0 \\ -Af_L \end{Bmatrix}$$

Say Af is positive when directed <u>into</u> el.; thus $Af_0 = Af_1$ & $-Af_L = Af_2$

$$\left([\underset{\sim}{k}] + [\underset{\sim}{h}_{ls}] \right) \{\underset{\sim}{T}_e\} = \{\underset{\sim}{r}_Q\} + \{\underset{\sim}{r}_{ls}\} + \begin{Bmatrix} Af_1 \\ Af_2 \end{Bmatrix}$$

$\boxed{12.2\text{-}4}$

(a) $\begin{Bmatrix} T_{,x} \\ T_{,y} \end{Bmatrix} = [\underset{\sim}{B}] \begin{Bmatrix} T_1 \\ T_2 \\ T_3 \end{Bmatrix}$ where, from Eqs.

7.2-4 and 7.2-6,

$[\underset{\sim}{B}] = \frac{1}{2A} \begin{bmatrix} y_{23} & y_{31} & y_{12} \\ x_{32} & x_{13} & x_{21} \end{bmatrix}$, $2A = x_{21}y_{31} - x_{31}y_{21}$

$[\underset{\sim}{\kappa}] = k \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, so $[\underset{\sim}{k}] = Ak [\underset{\sim}{B}]^T [\underset{\sim}{B}]$

(b) On side 1-3, $\xi_2 = 0$, & $\lfloor \underset{\sim}{N} \rfloor = \lfloor \xi_1 \;\; 0 \;\; \xi_3 \rfloor$
Use Eq. 7.3-5 to integrate along 1-3.

$$[\underset{\sim}{h}] = h \int \begin{bmatrix} \xi_1^2 & 0 & \xi_1\xi_3 \\ 0 & 0 & 0 \\ \xi_1\xi_3 & 0 & \xi_3^2 \end{bmatrix} dL_{13} = \frac{hL_{13}}{6} \begin{bmatrix} 2 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$

where $L_{13} = \left[(x_3 - x_1)^2 + (y_3 - y_1)^2\right]^{1/2}$.

(c) $[\underset{\sim}{h}] = \frac{hL_{13}}{2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $[\underset{\sim}{c}] = \frac{\rho c A}{3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(d) $\lfloor \underset{\sim}{N} \rfloor = \lfloor \xi_1 \;\; \xi_2 \;\; \xi_3 \rfloor$. By Eq. 7.3-7,

$$\int_A \xi_i \, dA = 2A \frac{1}{3!} = \frac{A}{3} \quad \text{for} \quad i = 1, 2, 3$$

Hence, with Q constant, & unit thickness,

$$\{\underset{\sim}{r}_Q\} = Q \int \lfloor \underset{\sim}{N} \rfloor^T dA = \frac{QA}{3} \begin{Bmatrix} 1 \\ 1 \\ 1 \end{Bmatrix}$$

$\boxed{12.2\text{-}5}$

(a) As in Prob. 12.2-4,

$$[\underset{\sim}{B}] = \frac{1}{2A}\begin{bmatrix} y_{23} & y_{31} & y_{12} \\ x_{32} & x_{13} & x_{21} \end{bmatrix}, \quad 2A = x_{21}y_{31} - x_{31}y_{21}$$

$$\text{Then } [\underset{\sim}{k}] = \int [\underset{\sim}{B}]^T[\underset{\sim}{B}]k\,dV = k[\underset{\sim}{B}]^T[\underset{\sim}{B}]\int 2\pi r\,dA$$

$r = \lfloor \xi_1 \ \ \xi_2 \ \ \xi_3 \rfloor \begin{Bmatrix} r_1 \\ r_2 \\ r_3 \end{Bmatrix}$ and $\int \xi_i\,dA = \frac{A}{3}$ for $i = 1, 2, 3$ from Eq. 7.3-7. Thus

$$\int 2\pi r\,dA = 2\pi \frac{A}{3}(r_1 + r_2 + r_3)$$

(b) Integral for $[\underset{\sim}{h}]$ in Prob. 12.2-4, applies, except dL_{13} becomes $2\pi r\,dL_{13}$ where $r = \xi_1 r_1 + \xi_3 r_3$ along side 1-3.

By Eq. 7.3-5,

$$\int \xi_1^3\,dL_{13} = \int \xi_3^3\,dL_{13} = \frac{L_{13}}{4}, \quad \int \xi_1^2\xi_3\,dL_{13} = \int \xi_1\xi_3^2\,dL_{13} = \frac{L_{13}}{12}$$

Hence

$$[\underset{\sim}{h}] = h(2\pi L_{13})\begin{bmatrix} \frac{r_1}{4} + \frac{r_3}{12}, & 0, & \frac{r_1+r_3}{12} \\ 0 & 0 & 0 \\ \frac{r_1+r_3}{12}, & 0, & \frac{r_1}{12} + \frac{r_3}{4} \end{bmatrix}$$

(c)

$$[\underset{\sim}{h}] = 2\pi \frac{r_1+r_3}{2} L_{12} h \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$[\underset{\sim}{c}] = 2\pi \frac{r_1+r_2+r_3}{3} \frac{\rho c A}{3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(d)

$$\{\underset{\sim}{r}_Q\} = Q\int \begin{Bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{Bmatrix} 2\pi \underbrace{(\xi_1 r_1 + \xi_2 r_2 + \xi_3 r_3)}_{r} dA$$

By Eq. 7.3-7, $\int \xi_i^2\,dA = \frac{A}{6}$ & $\int \xi_i \xi_j\,dA = \frac{A}{12}$

$$\{\underset{\sim}{r}_Q\} = 2\pi Q \frac{A}{12}\begin{Bmatrix} 2r_1 + r_2 + r_3 \\ r_1 + 2r_2 + r_3 \\ r_1 + r_2 + 2r_3 \end{Bmatrix}$$

$\boxed{12.2\text{-}6}$

$$T = \underbrace{\left\lfloor \frac{r_2-r}{r_2-r_1} \quad \frac{r-r_1}{r_2-r_1} \right\rfloor}_{\lfloor N \rfloor} \begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix}, \quad T_{,r} = \underbrace{\frac{1}{r_2-r_1} \lfloor -1 \quad 1 \rfloor}_{\lfloor B \rfloor} \begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix}$$

$$[k] = \int_{-\pi}^{\pi} \int_{r_1}^{r_2} \lfloor B \rfloor^T \lfloor B \rfloor k\, r\, dr\, d\theta = 2\pi \frac{r_2^2 - r_1^2}{2(r_2-r_1)^2} k \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$$[k] = \frac{r_2+r_1}{r_2-r_1} \pi k \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$[h]$ and $\{r_h\}$ are not present on lateral surfaces

For possible convection on edge $r = r_1$,

$$[h] = \int_{-\pi}^{\pi} h \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} \lfloor 1 \quad 0 \rfloor (1)\, r_1\, d\theta = 2\pi h r_1 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\{r_h\} = \int_{-\pi}^{\pi} \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} h T_{f1} (1)\, r_1\, d\theta = 2\pi h T_{f1} r_1 \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$$

or, for convection on edge $r = r_2$,

$$[h] = 2\pi h r_2 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad \{r_h\} = 2\pi h T_{f1} r_2 \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}$$

$\{r_B\}$ replace $h T_{f1}$ by f_B in $\{r_h\}$ expressions

$$[c] = \int_{-\pi}^{\pi} \int_{r_1}^{r_2} \lfloor N \rfloor^T \lfloor N \rfloor \rho c\, r\, dr\, d\theta$$

(algebraic expressions do not simplify)

$$\{r_Q\} = \frac{2\pi Q}{r_2-r_1} \int_{r_1}^{r_2} \begin{Bmatrix} r_2 - r \\ r - r_1 \end{Bmatrix} r\, dr = \frac{2\pi Q}{r_2-r_1} \begin{Bmatrix} r_2 \frac{r^2}{2} - \frac{r^3}{3} \\ \frac{r^3}{3} - r_1 \frac{r^2}{2} \end{Bmatrix}_{r_1}^{r_2}$$

$$\{r_Q\} = \frac{\pi Q}{3(r_2-r_1)} \begin{Bmatrix} r_2^3 - 3r_1^2 r_2 + 2r_1^3 \\ 2r_1^3 - 3r_1 r_2^2 + r_1^3 \end{Bmatrix}$$

$\boxed{12.2\text{-}7}$

$$\frac{Ak}{L}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} T_1 \\ T_2 \\ T_3 \end{Bmatrix} = \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix}$$

Only relative temperatures matter in this problem, so impose $T_3 = 0$ (we will add T_3 to T_1 & T_2 after solving); also set $q_2 = 0$.

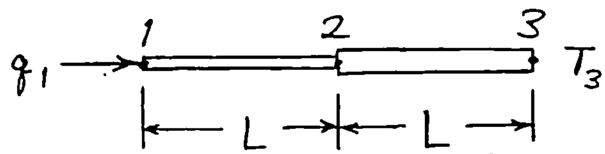
$$\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}\begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix} = \frac{Lq_1}{Ak}\begin{Bmatrix} 1 \\ 0 \end{Bmatrix}, \quad \begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix} = \frac{Lq_1}{Ak}\begin{Bmatrix} 2 \\ 1 \end{Bmatrix}$$

Re-introduce T_3 for final temps.

$$\begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix} = \begin{Bmatrix} T_3 \\ T_3 \end{Bmatrix} + \frac{Lq_1}{Ak}\begin{Bmatrix} 2 \\ 1 \end{Bmatrix}$$

$$q_3 = A\left[-k\,\frac{T_3 - T_2}{L}\right] = -\frac{Ak}{L}\left(-\frac{Lq_1}{Ak}\right) = q_1 \quad \checkmark$$

$\boxed{12.2\text{-}8}$



$$\frac{A_0 k}{L}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 1+2 & -2 \\ 0 & -2 & 2 \end{bmatrix}\begin{Bmatrix} T_1 \\ T_2 \\ T_3 \end{Bmatrix} = \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \end{Bmatrix}$$

As in Problem 8.3, use $q_2 = 0$ & $T_3 = 0$.

$$\begin{bmatrix} 1 & -1 \\ -1 & 3 \end{bmatrix}\begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix} = \frac{Lq_1}{A_0 k}\begin{Bmatrix} 1 \\ 0 \end{Bmatrix}$$

Solve for T_1 and T_2 ; reintroduce T_3.

$$\begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix} = \begin{Bmatrix} T_3 \\ T_3 \end{Bmatrix} + \frac{Lq_1}{2A_0 k}\begin{Bmatrix} 3 \\ 1 \end{Bmatrix}$$

$$q_3 = 2A_0\left[-k\,\frac{T_3 - T_2}{L}\right] = -\frac{2A_0 k}{L}\left(-\frac{Lq_1}{2A_0 k}\right) = q_1 \checkmark$$

$\boxed{12.2\text{-}9}$

$$\frac{Ak}{L}\begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 1+1 & -1 & 0 \\ 0 & -1 & 1+1 & -1 \\ 0 & 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{Bmatrix} = \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix}$$

Impose $q_2 = q_3 = 0$, $T_1 = 0$, $T_4 = 300$
(the latter adds $300Ak/L$ to the r.h.s.)

$$\frac{Ak}{L}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{Bmatrix} T_2 \\ T_3 \end{Bmatrix} = \frac{Ak}{L}\begin{Bmatrix} 0 \\ 300 \end{Bmatrix}$$

$$\begin{Bmatrix} T_2 \\ T_3 \end{Bmatrix} = \frac{1}{3}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{Bmatrix} 0 \\ 300 \end{Bmatrix} = \begin{Bmatrix} 100 \\ 200 \end{Bmatrix}$$

(as expected)

12.2-10

$$\frac{A_0 k}{L}\begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 1+2 & -2 & 0 \\ 0 & -2 & 2+3 & -3 \\ 0 & 0 & -3 & 3 \end{bmatrix}\begin{Bmatrix} T_1 \\ T_2 \\ T_3 \\ T_4 \end{Bmatrix} = \begin{Bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \end{Bmatrix}$$

Impose $q_2 = q_3 = 0$, $T_1 = 0$, $T_4 = 300$ (the latter adds $300(3A_0 k/L)$ to the r.h.s.)

$$\frac{A_0 k}{L}\begin{bmatrix} 3 & -2 \\ -2 & 5 \end{bmatrix}\begin{Bmatrix} T_2 \\ T_3 \end{Bmatrix} = \frac{A_0 k}{L}\begin{Bmatrix} 0 \\ 900 \end{Bmatrix}$$

$$\begin{Bmatrix} T_2 \\ T_3 \end{Bmatrix} = \frac{1}{11}\begin{bmatrix} 5 & 2 \\ 2 & 3 \end{bmatrix}\begin{Bmatrix} 0 \\ 900 \end{Bmatrix} = \begin{Bmatrix} 163.6 \\ 245.5 \end{Bmatrix}$$

$\boxed{12.2\text{-}11}$

(a) With $\underset{\sim}{N} = \left[\frac{L-x}{L} \quad \frac{x}{L}\right]$, $dS = p\,dx$ where p = perimeter of criss section,

$$\int \underset{\sim}{N}^T \underset{\sim}{N}\, h\, dS = ph \int_0^L \begin{bmatrix} \left(\frac{L-x}{L}\right)^2 & \frac{L-x}{L}\frac{x}{L} \\ \frac{L-x}{L}\frac{x}{L} & \frac{x^2}{L^2} \end{bmatrix} dx$$

$$= \frac{phL}{3}\begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix} = \frac{hS_e}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$\int_0^L \underset{\sim}{N}^T h T_f\, dS = \int_0^L \begin{Bmatrix} \frac{L-x}{L} \\ \frac{x}{L} \end{Bmatrix} h T_f\, p\,dx = \frac{hT_f S_e}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

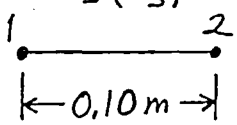
(b) $S_e = 0.010$

$$\left.\begin{aligned} \frac{hS_e}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} &= \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \\ \frac{Ak}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} &= \begin{bmatrix} 0.4 & -0.4 \\ -0.4 & 0.4 \end{bmatrix} \end{aligned}\right\} \text{add: } \begin{bmatrix} 2.4 & 0.6 \\ 0.6 & 2.4 \end{bmatrix}$$

Also $\frac{hT_f S_e}{2} = 600$.

Combine elements and set $T_1 = 0$.

$$\begin{bmatrix} 4.8 & 0.6 \\ 0.6 & 2.4 \end{bmatrix}\begin{Bmatrix} T_2 \\ T_3 \end{Bmatrix} = 600\begin{Bmatrix} 2 \\ 1 \end{Bmatrix}, \quad \begin{Bmatrix} T_2 \\ T_3 \end{Bmatrix} = \begin{Bmatrix} 226 \\ 194 \end{Bmatrix}$$

(c) 

$S_e = \frac{1}{3}(0.020)$ for the left element

$$\frac{Ak}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = 0.6\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$ Half this for the right el.

$$\frac{hS_e}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} = 0.667\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$ Twice this for the other el.

$$\frac{hT_f S_e}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix} = 400\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$ Twice this for the other el.

Assemble and set $T_1 = 0$.

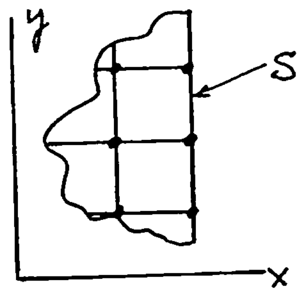
$$\begin{bmatrix} 0.6+0.3+2(0.667)+2(1.333) & 1.333-0.3 \\ 1.333-0.3 & 2(1.333)+0.3 \end{bmatrix} * \begin{Bmatrix} T_2 \\ T_3 \end{Bmatrix} = \begin{Bmatrix} 1200 \\ 800 \end{Bmatrix}$$

$$\begin{bmatrix} 4.900 & 1.033 \\ 1.033 & 2.967 \end{bmatrix}\begin{Bmatrix} T_2 \\ T_3 \end{Bmatrix} = \begin{Bmatrix} 1200 \\ 800 \end{Bmatrix}$$

$$\begin{Bmatrix} T_2 \\ T_3 \end{Bmatrix} = \begin{Bmatrix} 203 \\ 199 \end{Bmatrix}$$

(d) The model change from (b) to (c) has made T_2 and T_3 closer to ambient and decreased the gradient from 2 to 3. The actual gradient is so close to node 1 that these coarse models cannot represent it (analytical equations for fins show as much).

12.2-12



Eq. 12.1-2: $\begin{Bmatrix} f_x \\ f_y \end{Bmatrix} - k \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{Bmatrix} T_{,x} \\ T_{,y} \end{Bmatrix}$

	Known	Unknown
(a)	$T_{,x} = 0$	$T \quad T_{,y}$
(b)	$T \quad T_{,y}$	$T_{,x}$
(c)	$T_{,x}$	$T \quad T_{,y}$
(d)	———	$T \quad T_{,x} \quad T_{,y}$

12.3-1

$$\frac{Ak}{L}\begin{bmatrix}1&-1&0&0\\-1&2&-1&0\\0&-1&2&-1\\0&0&-1&1\end{bmatrix}\begin{Bmatrix}T_1\\T_2\\T_3\\T_4\end{Bmatrix}=\begin{Bmatrix}-\\0\\0\\-\end{Bmatrix}$$

Impose $T_1=0$, $T_4=500$, let $k=73$.

$$73\begin{bmatrix}2&-1\\-1&2\end{bmatrix}\begin{Bmatrix}T_2\\T_3\end{Bmatrix}=\begin{Bmatrix}0\\73(500)\end{Bmatrix}$$

from which $T_2=166.7\,°C$, $T_3=333.3\,°C$

In all elements,

$$f=-73\,\frac{500-0}{3(0.04)}=-304{,}200\,\frac{W}{m^2}$$

Compute el. ave. T's & associated k's.

el. 1, $T_{ave}=83.3$, $k=73-5=68$

el. 2, $T_{ave}=250$, $k=73-15=58$

el. 3, $T_{ave}=416.7$, $k=73-25=48$

$$\frac{A}{L}\begin{bmatrix}68&-68&0&0\\-68&68+58&-58&0\\0&-58&58+48&-48\\0&0&-48&48\end{bmatrix}\begin{Bmatrix}T_1\\T_2\\T_3\\T_4\end{Bmatrix}=\begin{Bmatrix}-\\0\\0\\-\end{Bmatrix}$$

Impose $T_1=0$, $T_4=500$

$$\begin{bmatrix}126&-58\\-58&106\end{bmatrix}\begin{Bmatrix}T_2\\T_3\end{Bmatrix}=\begin{Bmatrix}0\\48(500)\end{Bmatrix}$$

from which $T_2=139.3\,°C$, $T_3=302.6\,°C$

Element fluxes are

$$f_1=-68\,\frac{139.3-0}{0.04}=-236{,}800\,\frac{W}{m^2}$$

$$f_2=-58\,\frac{302.6-139.3}{0.04}=-236{,}800\,\frac{W}{m^2}$$

$$f_3=-48\,\frac{500-302.6}{0.04}=-236{,}800\,\frac{W}{m^2}$$

The fluxes agree (a check).

Compute el. ave. T's & associated k's.

el. 1, $T_{ave}=69.65$, $k=68.8$

el. 2, $T_{ave}=221.0$, $k=59.7$

el. 3, $T_{ave}=401.3$, $k=48.9$

$$\frac{A}{L}\begin{bmatrix}68.8&-68.8&0&0\\-68.8&68.8+59.7&-59.7&0\\0&-59.7&59.7+48.9&-48.9\\0&0&-48.9&48.9\end{bmatrix}\begin{Bmatrix}T_1\\T_2\\T_3\\T_4\end{Bmatrix}=\begin{Bmatrix}-\\0\\0\\-\end{Bmatrix}$$

Impose $T_1=0$, $T_4=500$.

$$\begin{bmatrix}128.5&-59.7\\-59.7&108.6\end{bmatrix}\begin{Bmatrix}T_2\\T_3\end{Bmatrix}=\begin{Bmatrix}0\\48.9(500)\end{Bmatrix}$$

from which $T_2=140.5\,°C$, $T_3=302.4\,°C$

Element fluxes are

$$f_1=-68.8\,\frac{140.5-0}{0.04}=-241{,}600\,\frac{W}{m^2}$$

$$f_2=-59.7\,\frac{302.4-140.5}{0.04}=-241{,}600\,\frac{W}{m^2}$$

$$f_3=-48.9\,\frac{500-302.4}{0.04}=-241{,}600\,\frac{W}{m^2}$$

The fluxes agree (a check).

$\boxed{12.3\text{-}2}$

$$q = \frac{k}{t}(T_b - T_a) = h_r(T_c - T_b)$$

hence $\left(\frac{k}{t} + h_r\right)T_b = \frac{k}{t}T_a + h_r T_c$ (a)

where $\frac{k}{t} = \frac{0.7}{0.01} = 70$ and, from Eqs. 8.3-2 and 8.3-4,

$$h_r = \frac{\sigma}{\frac{1}{0.6} + \frac{1}{0.6} - 1}(T_c^2 + T_b^2)(T_c + T_b)$$

Using absolute temperatures, $T_a = 293\,°K$ and $T_c = 873\,°K$, h_r & Eq. (a) become

$h_r = 2.43(10)^{-8}(873^2 + T_b^2)(873 + T_b)$ &

$(70 + h_r)T_b = 20{,}510 + 873\,h_r$ (b)

First cycle: $T_b = 100°C = 373\,°K$; use (b):

$(70 + 27.3)T_b = 20{,}510 + 873(27.3)$

$T_b = 455.7\,°K$ Use this in (b); second cycle:

$(70 + 31.3)T_b = 20{,}510 + 873(31.3)$

$T_b = 472.2\,°K$ Use this in (c); third cycle:

$(70 + 32.2)T_b = 20{,}510 + 873(32.2)$

$T_b = 475.8\,°K = 202.8°C$
(close enough)

$\boxed{12.4\text{-}1}$

All symbols but λ are arrays

Given $\bar{T}_i^T C \bar{T}_i = 1$ (A)

Prove $\bar{T}_i^T C \bar{T}_j = 0$ for $i \neq j$ (B)

Premultiply Eq. 12.4-2 by $\bar{T}^T$:

(a) $\bar{T}_i^T K_T \bar{T}_j = \lambda_j \bar{T}_i^T C \bar{T}_j$ prem. jth by $\bar{T}_i^T$

(b) $\bar{T}_j^T K_T \bar{T}_i = \lambda_i \bar{T}_j^T C \bar{T}_i$ prem. ith by $\bar{T}_j^T$

(c) $\bar{T}_i^T K_T \bar{T}_j = \lambda_i \bar{T}_i^T C \bar{T}_j$ transpose of (b)

$0 = (\lambda_j - \lambda_i) \bar{T}_i^T C \bar{T}_j$ (a) − (c)

But $\lambda_i \neq \lambda_j$, so $\bar{T}_i^T C \bar{T}_j = 0$ Q.E.D.

If we repeat foregoing argument with Eq. 12.4-2 in form $(C - \frac{1}{\lambda} K_T)\bar{T} = 0$, get $\bar{T}_i^T K_T \bar{T}_j = 0$

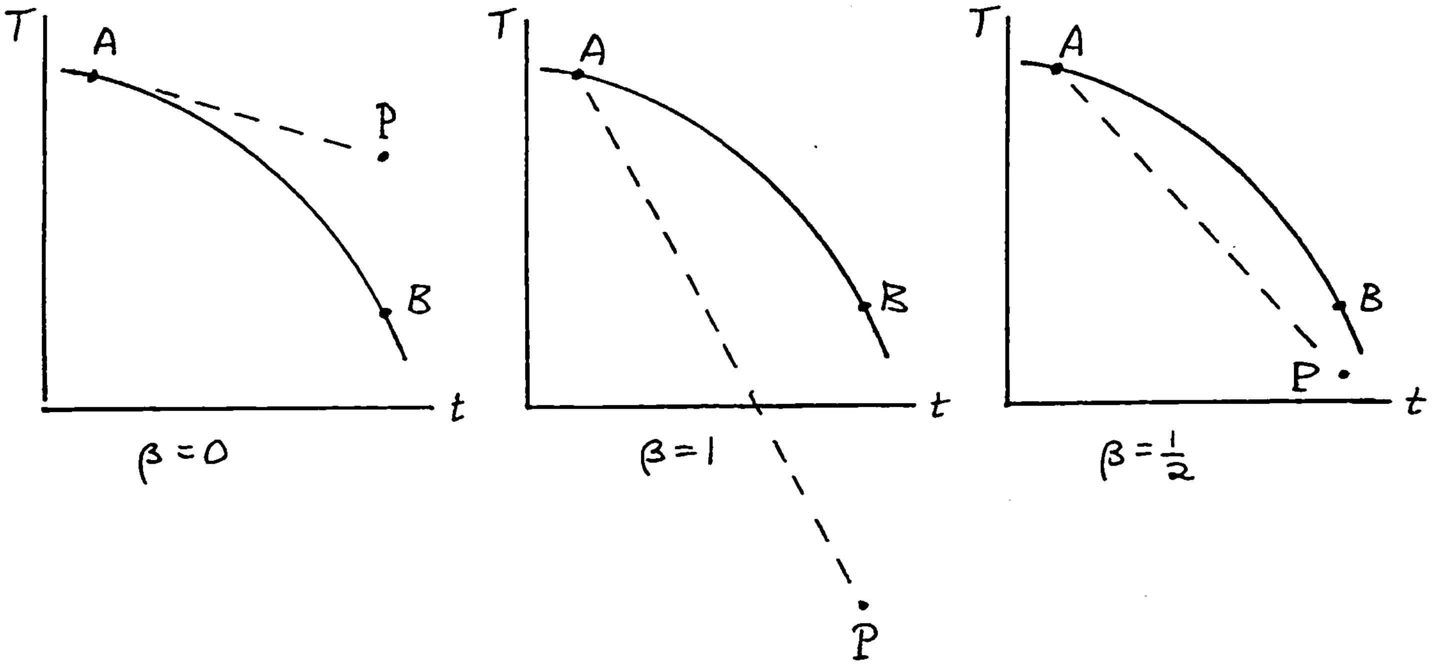
To show second of Eqs. 12.4-3: we know $\phi^T C \phi = I$ from (A) & (B). Hence

$$\bar{T}_i^T (K_T - \lambda_j C)\bar{T}_j = 0$$

$$\bar{T}_i^T K_T \bar{T}_j = \lambda_j \bar{T}_i^T C \bar{T}_j \begin{cases} = 0 & \text{for } i \neq j \\ = \lambda_j & \text{for } i = j \end{cases}$$

Hence $\phi^T K_T \phi = \lceil \lambda \rfloor$

$\boxed{12.4\text{-}2}$ P is the predicted T



$\boxed{12.4\text{-}3}$

(a) When $\dot{T}=0$, $T = 3/6 = 0.500$

(b) Assume solution of form $T = a(1-e^{-bt})$

Must have $a = 0.5$ to check part (a).

Thus $\dot{T} = 0.5be^{-bt}$ and diff. eq. becomes

$3(1-e^{-bt}) + 1.06e^{-bt} = 3$. Therefore $b=3$,

and exact solution is $T = 0.5(1-e^{-3t})$

(c) $(6-2\lambda)\bar{T} = 0$, $\lambda = 3$, $\Delta t_{cr} = \frac{2}{\lambda} = \frac{2}{3}$

(d)–(k): Eq. 12.4-8 becomes

$$\left(\frac{1}{\Delta t}2 + 6\beta\right)T_{n+1} = \left(\frac{1}{\Delta t}2 - 6(1-\beta)\right)T_n + 3$$

(d)–(g): $\Delta t = 0.1$, $(20+6\beta)T_{n+1} = (14+6\beta)T_n + 3$

(h)–(k): $\Delta t = 1.0$, $(2+6\beta)T_{n+1} = (-4+6\beta)T_n + 3$

(d) $20T_{n+1} = 14T_n + 3$ (h) $2T_{n+1} = -4T_n + 3$

(e) $23T_{n+1} = 17T_n + 3$ (i) $5T_{n+1} = -T_n + 3$

(f) $24T_{n+1} = 18T_n + 3$ (j) $6T_{n+1} = 3$

(g) $26T_{n+1} = 20T_n + 3$ (k) $8T_{n+1} = 2T_n + 3$

Collected numerical results:

time t	T_{exact}	$T_{(d)}$	$T_{(e)}$	$T_{(f)}$	$T_{(g)}$	$T_{(h)}$	$T_{(i)}$	$T_{(j)}$	$T_{(k)}$
0	0	0	0	0	0				
0.1	0.1296	.150	.130	.125	.115				
0.2	0.2256	.255	.227	.219	.204				
0.3	0.2967	.329	.298	.289	.272				
0.4	0.3494	.380	.351	.342	.325				
0.5	0.3884	.416	.390	.381	.365				
1.0	0.4751					1.5	.600	.500	.375
2.0	0.4988					-1.5	.480	.500	.469
3.0	0.4999					4.5	.504	.500	.492
4.0	0.5000					-7.5	.499	.500	.498
5.0	0.5000					16.5	.500	.500	.500

12.4-4

(a) From a handbook formula,

$T = 4(1 - e^{-t/3})$. Check by substitution:

$$2\left[4(1-e^{-t/3})\right] + 6\left[4\left(\tfrac{1}{3}e^{-t/3}\right)\right] \stackrel{?}{=} 8$$

$$8 - 8e^{-t/3} + 8e^{-t/3} \stackrel{?}{=} 8 \quad \text{Yes; OK}$$

(b) $\Delta t = 1$; Eq. 12.4-8 becomes $(\beta = \frac{1}{2})$

$$\left[\tfrac{1}{2}2 + 6\right]T_{n+1} = 8 - \left[\tfrac{1}{2}2 - 6\right]T_n$$

$$7T_{n+1} = 8 + 5T_n \quad \text{or} \quad T_{n+1} = \tfrac{1}{7}(8 + 5T_n)$$

t	n	T_n	T_{exact}
0	0	0	0
1	1	1.143	1.134
2	2	1.959	1.946
3	3	2.542	2.528
4	4	2.959	2.946
5	5	3.256	3.245
6	6	3.469	3.459
7	7	3.621	3.612
8	8	3.729	3.722

(c) $\Delta t = 10$; Eq. 12.4-8 becomes $(\beta = \frac{1}{2})$

$$\left[\tfrac{1}{2}2 + \tfrac{1}{10}6\right]T_{n+1} = 8 - \left[\tfrac{1}{2}2 - \tfrac{1}{10}6\right]T_n$$

$$1.6T_{n+1} = 8 - 0.4T_n \quad \text{or} \quad T_{n+1} = 5 - \frac{T_n}{4}$$

t	n	T_n	T_{exact}
0	0	0	0
10	1	5.00	3.875
20	2	3.75	3.995
-	-	4.06	4.000
40	4	3.98	4.000
50	5	4.004	4.000
60	6	3.999	4.000
70	7	4.000	4.000
80	8	4.000	4.000

$\boxed{12.7\text{-}1}$

(a) Consider a cube one unit on a side, so $V = 1 \cdot 1 \cdot 1 = 1$

After strains appear in all three coordinate directions, volume is

$$V + dV = (1+\epsilon_x)(1+\epsilon_y)(1+\epsilon_z) = 1 + \epsilon_x + \epsilon_y + \epsilon_z + \underbrace{\text{higher order terms}}_{\text{omit}}$$

$$dV = (V+dV) - V = \epsilon_x + \epsilon_y + \epsilon_z = dV/V$$

(b) With Π from Eq. 12.7-6,

$$\delta\Pi = \int_V \left(p_{,x}\,\delta p_{,x} + p_{,y}\,\delta p_{,y} + p_{,z}\,\delta p_{,z} + \frac{\rho}{B}\ddot{p}\,\delta p\right) dV + \int_{S_s} \rho \ddot{u}_n\,\delta p\, dS + \int_{S_f} \frac{1}{g}\ddot{p}\,\delta p\, dS$$

← (The last integral is a term for surface waves)

Integrate by parts: e.g. for the $p_{,x}\,\delta p_{,x}$ term,

$$\int_V p_{,x}\,\delta p_{,x}\, dV = -\int_V p_{,xx}\,\delta p\, dV + \int_S p_{,x}\, l\, \delta p\, dS$$

After all three integ. by parts, the integrand of the surface integral thus generated is $p_{,x} l + p_{,y} m + p_{,z} n$ which is $p_{,n}$. Also, $S = S_s + S_f$. Therefore

$$\delta\Pi = -\int_V \left(p_{,xx} + p_{,yy} + p_{,zz} - \frac{\rho}{B}\ddot{p}\right)\delta p\, dV + \int_{S_s} (p_{,n} + \rho \ddot{u}_n)\,\delta p\, dS + \int_{S_f} \left(p_{,n} + \frac{1}{g}\ddot{p}\right)\delta p\, dS$$

For $\delta\Pi = 0$, integrands of the three integrals must vanish separately. Thus

$$p_{,xx} + p_{,yy} + p_{,zz} - \frac{\rho}{B}\ddot{p} = 0 \quad \text{in } V \qquad \text{as in Eq. 12.7-4}$$

$$p_{,n} + \rho \ddot{u}_n = 0 \quad \text{on } S_s \qquad \text{as in Eq. 12.7-5}$$

$$p_{,n} + \frac{1}{g}\ddot{p} = 0 \quad \text{on } S_f \qquad \text{as noted above Eq. 12.8-7}$$

(c)
$$\Pi = \int \left[\bar{p}_{,x}^2 + \bar{p}_{,y}^2 + \bar{p}_{,z}^2 - \left(\frac{\omega \bar{p}}{c}\right)^2\right] dV$$

$$\delta\Pi = 2\int \left[\bar{p}_{,x}\,\delta\bar{p}_{,x} + \bar{p}_{,y}\,\delta\bar{p}_{,y} + \bar{p}_{,z}\,\delta\bar{p}_{,z} - \frac{\omega^2}{c^2}\bar{p}\,\delta\bar{p}\right] dV$$

Integrate by parts: e.g. the 1st of 3 is

$$\int \bar{p}_{,x}\,\delta\bar{p}_{,x}\, dV = -\int \bar{p}_{,xx}\,\delta\bar{p}\, dV + \int \bar{p}_{,x}\,\delta\bar{p}\, l\, dS$$

Thus $\delta\Pi = 2\int \left[-\nabla^2\bar{p} - \frac{\omega^2}{c^2}\bar{p}\right]\delta p\, dV + \int \bar{p}_{,n}\,\delta p\, dS$

From Eq. 12.7-5, surface integral vanishes if $\ddot{u}_n = 0$. Hence, $\delta\Pi = 0$ implies

$$\nabla^2\bar{p} + \frac{\omega^2}{c^2}\bar{p} = 0$$

$\boxed{12.7\text{-}2}$

(a) Matrices are like $[\underset{\sim}{k}]$ and $[\underset{\sim}{m}]$ of a truss el. $\left(\frac{A}{L}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix} - \frac{\omega^2 AL}{6c^2}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}\right)\begin{Bmatrix}p_1\\ p_2\end{Bmatrix} = \begin{Bmatrix}0\\ 0\end{Bmatrix}$

Let $\lambda = \frac{\omega^2 L^2}{6c^2}$; $\begin{vmatrix}1-2\lambda & -1-\lambda\\ -1-\lambda & 1-2\lambda\end{vmatrix} = 3\lambda^2 - 6\lambda = 0$

$\lambda = 0$ or $\lambda = 2$; for latter, $\omega^2 = \frac{12c^2}{L^2}$, $\omega = 3.46\frac{c}{L}$

(b) Matrices are like stress stiffness matrix $[\underset{\sim}{k}_\sigma]$ & mass matrix $[\underset{\sim}{m}]$ of standard beam el.

Set $p_{1,x}=0$ & $p_{2,x}=0$ $\left(\frac{A}{30L}\begin{bmatrix}36 & -36\\ -36 & 36\end{bmatrix} - \frac{\omega^2 A}{c^2}\frac{L}{420}\begin{bmatrix}156 & 54\\ 54 & 156\end{bmatrix}\right)\begin{Bmatrix}p_1\\ p_2\end{Bmatrix} = \begin{Bmatrix}0\\ 0\end{Bmatrix}$

Let $\lambda = \frac{\omega^2 L^2}{14c^2}$; $\begin{vmatrix}6-26\lambda & -6-9\lambda\\ -6-9\lambda & 6-26\lambda\end{vmatrix} = 0$

$595\lambda^2 - 420\lambda = 0$. $\lambda = 0$ or $\lambda = 0.7059$.

For the latter, $\omega^2 = 9.8824\frac{c^2}{L^2}$, $\omega = 3.1436\frac{c}{L}$

12.7-3

Fig. 12.7-2, with $p_1 = 0$ and $p_5 = 0$. Thus, after combining two of the elements whose matrices appear in Eqs. 12.7-14,

$$\left(\frac{A}{3L}\begin{bmatrix} 16 & -8 & 0 \\ -8 & 14 & -8 \\ 0 & -8 & 16 \end{bmatrix} - \left(\frac{\omega}{c}\right)^2 \frac{AL}{30}\begin{bmatrix} 16 & 2 & 0 \\ 2 & 8 & 2 \\ 0 & 2 & 16 \end{bmatrix} \right) \begin{Bmatrix} p_2 \\ p_3 \\ p_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix}$$

For the first mode, $p_2 = p_4$, so

$$\left(\frac{A}{3L}\begin{bmatrix} 16 & -8 \\ -16 & 14 \end{bmatrix} - \frac{\omega^2}{c^2}\left(\frac{AL}{30}\right)\begin{bmatrix} 16 & 2 \\ 4 & 8 \end{bmatrix} \right) \begin{Bmatrix} p_2 \\ p_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Let $\lambda = \dfrac{\omega^2 L^2}{10c^2}$, then $\begin{vmatrix} 16(1-\lambda) & -8-2\lambda \\ -16-4\lambda & 14-8\lambda \end{vmatrix} = 0$

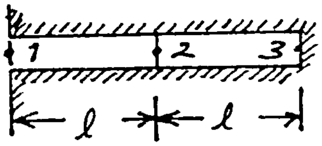
from which $120\lambda^2 - 416\lambda + 96 = 0$

$$\lambda_1 = \frac{416 - 356.34}{240} = 0.248596$$

$$\omega_1 = 1.5767 \frac{c}{L}$$

12.7-4

(a) A is uniform and cancels out.



$[\underset{\sim}{k}] = \dfrac{1}{\ell}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

$[\underset{\sim}{c}] = \dfrac{\ell}{6c^2}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

Set $p_1 = 0$. Thus

$$\left(\frac{1}{\ell}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix} - \frac{\omega^2 \ell}{6c^2}\begin{bmatrix} 4 & 1 \\ 1 & 2 \end{bmatrix} \right) \begin{Bmatrix} \bar{p}_2 \\ \bar{p}_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Let $\lambda = \dfrac{\omega^2 \ell^2}{6c^2}$, $\begin{vmatrix} 2-4\lambda & -1-\lambda \\ -1-\lambda & 1-2\lambda \end{vmatrix} = 0$

Yields $7\lambda^2 - 10\lambda + 1 = 0$, $\lambda_1 = 0.10819$

$$\omega_1 = \frac{0.8057c}{\ell} = \frac{1.611c}{L} \qquad \left(\ell = \frac{L}{2}\right)$$

(b) Matrices from Eq. 12.7-14.
A is uniform and cancels out.

$$[\underset{\sim}{k}] = \frac{1}{3L}\begin{bmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{bmatrix}, \quad [\underset{\sim}{c}] = \frac{L}{30c^2}\begin{bmatrix} 4 & 2 & -1 \\ 2 & 16 & 2 \\ -1 & 2 & 4 \end{bmatrix}$$

where, in getting $[\underset{\sim}{c}]$, we use $\dfrac{\rho}{B} = \dfrac{1}{c^2}$.
For this problem set $p_1 = 0$. Thus

$$\left(\frac{1}{3L}\begin{bmatrix} 16 & -8 \\ -8 & 7 \end{bmatrix} - \frac{\omega^2 L}{30c^2}\begin{bmatrix} 16 & 2 \\ 2 & 4 \end{bmatrix} \right) \begin{Bmatrix} \bar{p}_2 \\ \bar{p}_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

Let $\lambda = \dfrac{L^2 \omega^2}{10c^2}$, $\begin{vmatrix} 16-16\lambda & -8-2\lambda \\ -8-2\lambda & 7-4\lambda \end{vmatrix} = 0$

Yields $15\lambda^2 - 52\lambda + 12 = 0$, $\lambda_1 = 0.2486$,

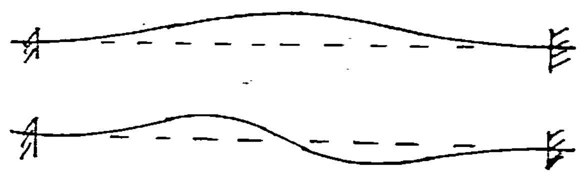
[illegible]$/L^2$; $\omega_1 = \dfrac{1.5767c}{L}$

$$\omega_{exact} = \frac{\pi c}{2L} = \frac{1.5707c}{L}$$

12.8-1

No fluid:

With fluid:



12.8-2

In the lower partition of Eq. 12.8-6, $\mathbf{M}_F$ and $\mathbf{C}_F$ are zero, but $\mathbf{W}_F$ appears in place of $\mathbf{M}_F$. Thus in place of Eq. 12.8-8 we obtain

$$\mathbf{K}_F \mathbf{P} + \rho \mathbf{S} \ddot{\mathbf{D}} + \mathbf{W}_F \ddot{\mathbf{P}} = \mathbf{0}$$

Substitute $\mathbf{D} = \bar{\mathbf{D}} \sin \omega t$; thus

$$\mathbf{K}_F \bar{\mathbf{P}} - \omega^2 \rho \mathbf{S} \bar{\mathbf{D}} - \omega^2 \mathbf{W}_F \bar{\mathbf{P}} = \mathbf{0}$$

$$\mathbf{P} = \left[\mathbf{K}_F - \omega^2 \mathbf{W}_F \right]^{-1} \omega^2 \rho \mathbf{S} \bar{\mathbf{D}} \qquad (A)$$

From the upper partition of Eq. 12.8-6, with $\mathbf{D} = \bar{\mathbf{D}} \sin \omega t$,

$$\mathbf{K} \bar{\mathbf{D}} - \mathbf{S}^T \bar{\mathbf{P}} - \omega^2 \mathbf{M} \bar{\mathbf{D}} = \mathbf{0} \qquad (B)$$

Substitute for $\mathbf{P}$ from (A); thus (B) yields

$$\left[\mathbf{K}_F - \omega^2 \Big(\underbrace{\rho \mathbf{S}^T \left[\mathbf{K}_F - \omega^2 \mathbf{W}_F \right]^{-1} \mathbf{S}}_{\text{added mass}} + \mathbf{M} \Big) \right] \bar{\mathbf{D}} = \mathbf{0}$$