$\boxed{15.1\text{-}1}$

(a) Let $\sigma_x = kz$, where k = constant (σ_x linear in z). Hence

$$M_x = \int \sigma_x z\,dz = k\int_{-t/2}^{t/2} z^2 dz = \frac{kt^3}{12} = \frac{\sigma_x t^3}{12z},$$

$$\sigma_x = \frac{M_x z}{t^3/12}. \quad \text{At } z = \pm\frac{t}{2},\ \sigma_x = \pm\frac{6M_x}{t^2}$$

(b) If τ_{yz} parabolic, $\tau_{yz} = k\left(\frac{t^2}{4} - z^2\right)$ where k = constant.

$$Q_y = \int_{-t/2}^{t/2} \tau_{yz}\,dz = k\left(\frac{t^2}{4}z - \frac{z^3}{3}\right)_{-t/2}^{t/2} = k\frac{t^3}{3},$$

$$k = 6Q_y/t^3,\quad \tau_{yz} = \frac{6Q_y}{t^3}\left(\frac{t^2}{4} - z^2\right).$$

$$\text{At } z = 0,\ \tau_{yz} = \frac{6Q_y}{t^3}\frac{t^2}{4} = 1.5\frac{Q_y}{t}$$

15.1-2

As shown in Prob. 15.1-1,

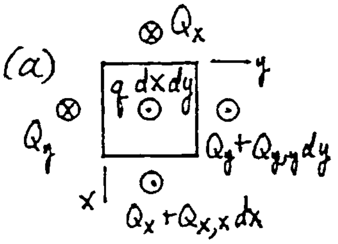
$$\sigma_x = \frac{M_x z}{t^3/12} \qquad \sigma_y = \frac{M_y z}{t^3/12} \qquad \tau_{xy} = \frac{M_{xy} z}{t^3/12} \qquad \sigma_n = \frac{M_n z}{t^3/12}$$

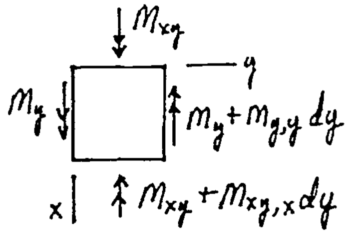
Substitute these stresses into the σ_n expression given in the problem statement and cancel the common factor $\frac{z}{t^3/12}$. Thus

$$M_n = \frac{1}{2}(M_x + M_y) + \frac{1}{2}(M_x - M_y)\cos 2\theta + M_{xy} \sin 2\theta$$

$\boxed{15.1\text{-}3}$

(a)





(moments with x-comp.)

Sum z forces:

$$-Q_y dx - Q_x dy + (Q_x + Q_{x,x}dx)dy + (Q_y + Q_{y,y}dy)dx + q\,dx\,dy = 0 \text{ ; yields } Q_{x,x} + Q_{y,y} + q = 0$$

Sum moments about line $y = dy/2$:

$$0 = -M_y dx - M_{xy}dy + (M_y + M_{y,y}dy)dx + (M_{xy} + M_{xy,x}dx)dy - Q_y dx\frac{dy}{2} - (Q_y + Q_{y,y}dy)dx\frac{dy}{2}$$

Neglect higher-order term; get

$$M_{y,y} + M_{xy,x} = Q_y$$

Similarly, moments about line $x = dx/2$ yield

$$M_{x,x} + M_{xy,y} = Q_x$$

(b) $\left.\begin{array}{l} Q_{x,x} = M_{x,xx} + M_{xy,xy} \\ Q_{y,y} = M_{y,yy} + M_{xy,xy} \end{array}\right\}$ into $Q_{x,x} + Q_{y,y} + q = 0$

Hence $M_{x,xx} + 2M_{xy,xy} + M_{y,yy} + q = 0$

(c) $M_x = -D(w_{,xx} + \nu w_{,xy})$

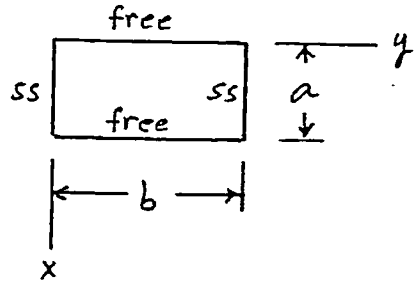
$M_y = -D(w_{,yy} + \nu w_{,xx})$

$M_{xy} = -D(1-\nu)w_{,xy}$

$$-D(w_{,xxxx} + 2w_{,xxyy} + w_{,yyyy}) + q = 0$$

or $\nabla^4 w = q/D$

$\boxed{15.1\text{-}4}$



Bending to a cylindrical surface. Apply beam theory, with uniformly distributed load $q(1)$ per unit length.

$$w = \frac{5qb^4}{384D}, \quad D = \frac{Et^3}{12(1-\nu^2)}, \quad w = \frac{qb^4(1-\nu^2)}{6.4Et^3}$$

Bending moments at center are

$$M_y = \frac{qb^2}{8}, \quad M_x = \nu\frac{qb^2}{8}, \quad M_{xy} = 0$$

$$\sigma_1 = \sigma_y = \frac{6M_y}{t^2} = \frac{3qb^2}{4t^2}$$

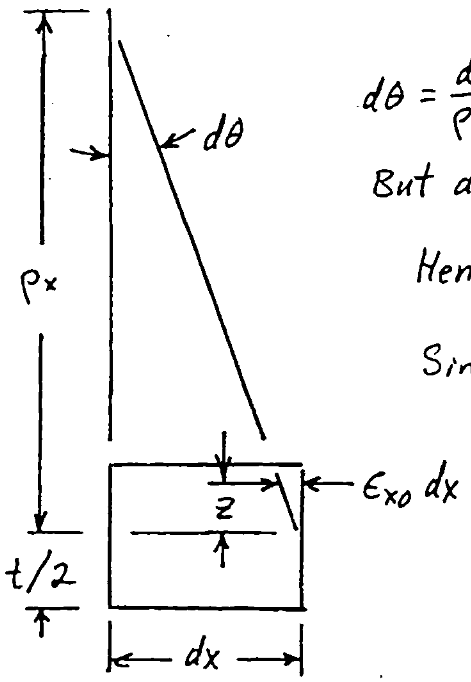
$$\sigma_2 = \sigma_x = \frac{6M_x}{t^2} = \nu\frac{3qb^2}{4t^2}$$

$$\sigma_3 = \sigma_z = 0$$

15.1-5

The given temperature field is $T = -2T_0 \frac{z}{t}$

so initial strains are $\epsilon_{x0} = \epsilon_{y0} = \alpha T = -2\alpha T_0 \frac{z}{t}$



$$d\theta = \frac{dx}{\rho_x}$$

But also $d\theta = \dfrac{\epsilon_{x0}\, dx}{z} = -\dfrac{2\alpha T_0}{t}\, dx$

Hence $\kappa_{0x} = \dfrac{1}{\rho_x} = -\dfrac{2\alpha T_0}{t}$

Similarly $\kappa_{0y} = -\dfrac{2\alpha T_0}{t}$

$$\{\underset{\sim}{\kappa_0}\} = -\frac{2\alpha T_0}{t}\begin{Bmatrix} 1 \\ 1 \\ 0 \end{Bmatrix}$$

$\boxed{15.1\text{-}6}$

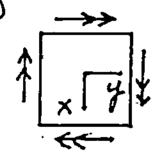
From Eqs. 15.1-4, with $\{\underset{\sim}{\kappa_0}\} = \{\underset{\sim}{0}\}$,

$$\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = -D \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix} \begin{Bmatrix} w_{,xx} \\ w_{,yy} \\ 2w_{,xy} \end{Bmatrix}$$

(a) $w_{,xx} = 2c_1 \quad w_{,yy} = 2c_1 \quad w_{,xy} = 0$

$M_x = M_y = -2(1+\nu)Dc_1$

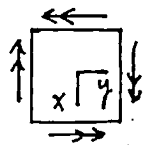
$M_{xy} = 0$



(b) $w_{,xx} = -2c_2, \; w_{,yy} = 2c_2, \; w_{,xy} = 0$

$M_x = 2Dc_2(1-\nu)$ $\quad M_{xy} = 0$

$M_y = -2Dc_2(1-\nu)$



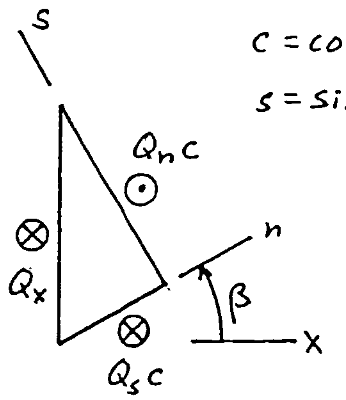
15.1-7

Error proportional to $\frac{1}{1-\nu^2}$, so error usually less than 10% (deflections). In bending to a cylindrical surface with largest stress σ_a, transverse stress is $\sigma_t = 0$ in beam theory but $\sigma_t = \nu\sigma_a$ in plate theory.

Must ask how much error is allowed. If unable to decide, could use 3-D elements (expensive), which should be between beam and plate analyses.

Theoretical studies offer correction factor based on width-to-thickness ratio of beam; see W. C. Young, Roark's Formulas for Stress and Strain.

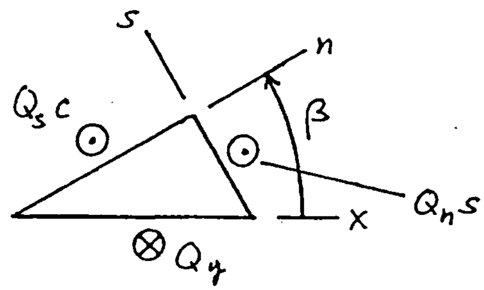
15.1-8



$c = \cos\beta$

$s = \sin\beta$

$$-Q_x - Q_s c + Q_n c = 0 \quad (a)$$



$$-Q_y + Q_s c + Q_n s = 0 \quad (b)$$

Together, (a) and (b) are $\begin{Bmatrix} Q_x \\ Q_y \end{Bmatrix} = \begin{bmatrix} c & -s \\ s & c \end{bmatrix} \begin{Bmatrix} Q_n \\ Q_s \end{Bmatrix}$

or $\{\underset{\sim}{Q}\} = [\underset{\sim}{T}]^T \{\underset{\sim}{Q}'\}$

$$[\underset{\sim}{G}] = \begin{bmatrix} c & -s \\ s & c \end{bmatrix} kt \begin{bmatrix} G_n & 0 \\ 0 & G_s \end{bmatrix} \begin{bmatrix} c & s \\ -s & c \end{bmatrix}$$

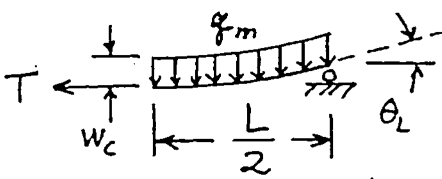
$$[\underset{\sim}{G}] = kt \begin{bmatrix} c & -s \\ s & c \end{bmatrix} \begin{bmatrix} cG_n & sG_n \\ -sG_s & cG_s \end{bmatrix} = kt \begin{bmatrix} c^2 G_n + s^2 G_s & cs(G_n - G_s) \\ cs(G_n - G_s) & s^2 G_n + c^2 G_s \end{bmatrix}$$

(Assumes that k is direction-independent.)

If $G_n = G_s$, reduces to $[\underset{\sim}{G}] = ktG \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

15.1-9

(a) Relate tension T to load q_m.



$$w = \frac{4w_c}{L^2}(Lx - x^2)$$

Moments about right end:

$$T w_c - \left(q_m \frac{L}{2}\right)\frac{L}{4} = 0 \;; \quad T = \frac{q_m L^2}{8 w_c} \quad \text{(a)}$$

Or, $q_m \frac{L}{2} = T\theta_L$ where θ_L is dw/dx at $x = L$. Thus $q_m \frac{L}{2} = T\frac{4w_c}{L}$; $T = \frac{q_m L^2}{8w_c}$

Length change $= \delta$

$$\delta = \frac{1}{2}\int_0^L \left(\frac{dw}{dx}\right)^2 dx = \frac{1}{2}\,\frac{16 w_c^2}{L^4}\int_0^L (L^2 - 4Lx + 4x^2)\,dx$$

$$\delta = \frac{8 w_c^2}{L^4}\frac{L^3}{3} = \frac{8w_c^2}{3L} \;, \quad \text{strain} = \epsilon = \frac{\delta}{L}$$

$$T = AE\epsilon = btE\frac{8w_c^2}{3L^2}$$

Substitute from (a) to eliminate T

Hence $\dfrac{q_m L^2}{8 w_c} = \dfrac{8Ebt w_c^2}{3L^2}$

$$q_m = \frac{64 Ebt w_c^3}{3L^4} = \frac{64Ebt^4}{3L^4}\left(\frac{w_c}{t}\right)^3$$

(b) Center deflection, bending alone, is

$$\frac{5q_b L^4}{384EI} \quad \text{so} \quad q_b = \frac{384\,EI}{5L^4} w_c$$

$$q_b + q_s = \frac{384E(bt^4/12)}{5L^4}\left(\frac{w_c}{t}\right) + \frac{64Ebt^4}{3L^4}\left(\frac{w_c}{t}\right)^3$$

$$q_b + q_s = \frac{Ebt^4}{L^4}\left[6.40\left(\frac{w_c}{t}\right) + 21.3\left(\frac{w_c}{t}\right)^3\right]$$

For $\dfrac{w_c}{t} = 0.5$,

$$q_b + q_s = \frac{Ebt^4}{L^4}\left[3.20 + 2.67\right]$$

roughly equal

15.2-1

(a) Should have $w_{,y}$ linear in x if it is to depend on only $w_{,y}$ nodal values at nodes 3 and 4. But, from Eq. 15.2-4,

$w_{,y} = \lfloor 0,0,1,0,x,2y,0,x^2,2xy,3y^2,x^3,3xy^2 \rfloor \{\underset{\sim}{a}\}$

$(w_{,y})_{y=b} = \lfloor 0,0,1,0,x,2b,0,x^2,2bx,3b^2,x^3,3b^2x \rfloor \{\underset{\sim}{a}\}$

This edge slope is a cubic in x — needs 4 d.o.f. to define. Hence it must depend on some nodal d.o.f. other than $w_{,y}$ at nodes 3 & 4. It would be only fortuitous (or constant curvature case) if the "other" d.o.f. are such that $w_{,y}$ matches between adjacent els.

(b) On $y = b$, $w = \lfloor \text{cubic in } x \rfloor \{\underset{\sim}{a}\}$
Requires 4 d.o.f. to define; they are (presumably) w_3, w_4, $w_{,x3}$, $w_{,x4}$. These same d.o.f. used in adjacent el. to define w of same edge. Hence, same w; compatible. And, if w same, so is $w_{,x}$.
(c) Then w would be quartic on all edges. Five d.o.f. needed to define quartic; these are more than available as nodal d.o.f. at ends of each edge. Expect that w and both slopes will be incompatible.

15.2-2

(a) Let $w_{,n}$ = edge-normal slope.
First arrangement of d.o.f:
w, $w_{,x}$, $w_{,y}$, $w_{,xx}$, $w_{,xy}$, $w_{,yy}$ at 1, 2, 3
$w_{,n}$ at 4, 5, 6

Fifth degree terms: $x^5, x^4y, x^3y^2, x^2y^3, xy^4, y^5$.
On $x=0$, w is 5th degree in y. Requires 6 d.o.f. to define and 6 are available (w, $w_{,y}$, and $w_{,yy}$ at nodes 1 & 3). And, on $x=0$, $w_{,x}$ is 4th degree in y. Requires 5 d.o.f. to define and 5 are available ($w_{,x}$ and $w_{,xy}$ at nodes 1 & 3, $w_{,x}$ at node 6). Compatibility expected.

Second arrangement of d.o.f.:
w, $w_{,x}$, $w_{,y}$ at all nodes, also $w_{,xy}$ at
On $x=0$, 6 d.o.f. needed & available to define w (w & $w_{,y}$ at 1, 3, 6). Also, 5 d.o.f. needed & available to define $w_{,x}$ ($w_{,x}$ & $w_{,xy}$ at 1 & 3, $w_{,x}$ at 6). Compat. expected.

(b) D.o.f. w, $w_{,x}$, $w_{,y}$ at corner nodes, w and $w_{,n}$ at side nodes.
On $x=0$, 6 d.o.f. needed & available to define w (w at nodes 1, 3, 8, 9 and $w_{,y}$ at nodes 1 & 3). 5 d.o.f. needed to define $w_{,x}$ but only 4 available on $x=0$ ($w_{,x}$ at 1, 3, 8, 9). Compatibility expected for w and $w_{,y}$ but not for $w_{,x}$.

$\boxed{15.2\text{-}3}$

From standard cubic beam functions, written i.t.o. y rather than x,

$$w_{,y} = \lfloor \tilde{N}_{,y} \rfloor \{\tilde{d}\}$$

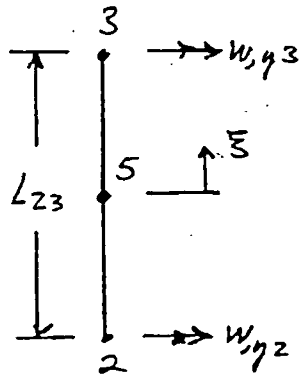
$$w_{,y} = \left\lfloor -\frac{6y}{L_{23}^2} + \frac{6y^2}{L_{23}^3} \quad 1 - \frac{4y}{L_{23}} + \frac{3y^2}{L_{23}^2} \quad \frac{6y}{L_{23}^2} - \frac{6y^2}{L_{23}^3} \quad -\frac{2y}{L_{23}} + \frac{3y^2}{L_{23}^2} \right\rfloor \begin{Bmatrix} w_2 \\ w_{,y2} \\ w_3 \\ w_{,y3} \end{Bmatrix}$$

$$\text{At } y = \frac{L_{23}}{2}, \quad w_{,y} = w_{,y5} = \left\lfloor -\frac{3}{2L_{23}} \quad -\frac{1}{4} \quad \frac{3}{2L_{23}} \quad -\frac{1}{4} \right\rfloor \begin{Bmatrix} w_2 \\ w_{,y2} \\ w_3 \\ w_{,y3} \end{Bmatrix} \qquad (A)$$

Quadratic variation of ψ_y: can use shape functions from Sec. 6.1.

$$\psi_y = \frac{1}{2}(-\xi + \xi^2) w_{,y2} + (1 - \xi^2) w_{,y5} + \frac{1}{2}(\xi + \xi^2) w_{,y3}$$

Substitute from (A):



$$\psi_y = \frac{3}{2L_{23}}(1-\xi^2)(w_3 - w_2) + \left(-\frac{1}{2}\xi + \frac{1}{2}\xi^2 - \frac{1}{4} + \frac{1}{4}\xi^2\right) w_{,y2} + \left(\frac{1}{2}\xi + \frac{1}{2}\xi^2 - \frac{1}{4} + \frac{1}{4}\xi^2\right) w_{,y3}$$

$$\psi_y = \frac{3}{2L_{23}}(1-\xi^2)(w_3 - w_2) + \frac{1}{4}(-1 - 2\xi + 3\xi^2) w_{,y2} + \frac{1}{4}(-1 + 2\xi + 3\xi^2) w_{,y3}$$

Shear constraint: will enforce $w_{,y2} = \psi_2$ and $w_{,y3} = \psi_3$.

15.2-4

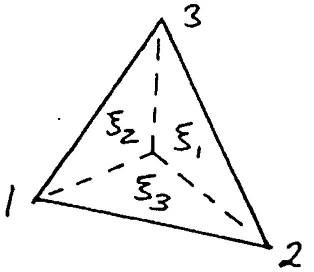
(a) Not changed: all concentrated center load cases, since the formula would create nodal moment loads only from distributed lood. Also, the $N=1$ case of distributed load on a clamped plate, since w at the plate center is the only d.o.f. not restrained.

(b) Changed very little: other uniform-load clamped-edge cases, since nodal moment loads whose vectors are parallel to mesh boundaries are discarded at boundary nodes when edge-tangent rotations are suppressed. Other nodal moment loads are probably small, as contributions from connected elements may nearly cancel.

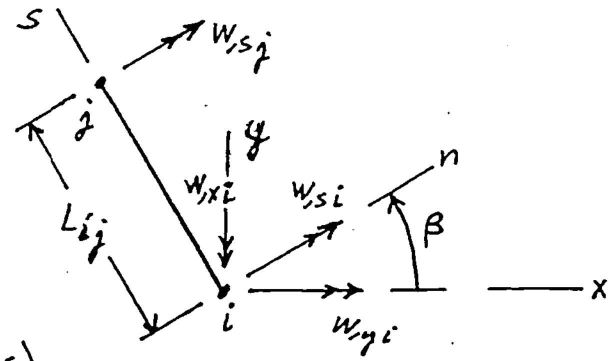
(c) Noticeably changed: uniformly loaded, simply-supported cases. Edge-tangent moment vectors will appear at boundary nodes and associated with d.o.f. that remain active.

15.2-5

Area coordinates:
$\xi_1 \quad \xi_2 \quad \xi_3$



Typical side ij:



Along side ij, $\quad w_{ij} = \dfrac{s(L_{ij}-s)}{2L_{ij}}(w_{,si} - w_{,sj})$

which gives, at midside, $\quad w_{ij\,max} = \dfrac{L_{ij}}{8}(w_{,si} - w_{,sj})$

But $\quad w_{,s} = w_{,y}\cos\beta - w_{,x}\sin\beta = w_{,y}\dfrac{y_j - y_i}{L_{ij}} - w_{,x}\dfrac{x_i - x_j}{L_{ij}}$

Hence $\quad w_{ij\,max} = \dfrac{1}{8}\left[(y_j - y_i)(w_{,yi} - w_{,yj}) + (x_i - x_j)(w_{,xj} - w_{,xi})\right]$

Use w's at vertices and midsides to interpolate quadratically over the plate:

$$w = \xi_1 w_1 + \xi_2 w_2 + \xi_3 w_3 + 4\left(\xi_1\xi_2 w_{12\,max} + \xi_2\xi_3 w_{23\,max} + \xi_3\xi_1 w_{31\,max}\right)$$

$$\begin{aligned} w = {} & \xi_1 w_1 + \xi_2 w_2 + \xi_3 w_3 \\ & + \frac{\xi_1\xi_2}{2}\left[(y_2 - y_1)(w_{,y1} - w_{,y2}) + (x_1 - x_2)(w_{,x2} - w_{,x1})\right] \\ & \frac{\xi_2\xi_3}{2}\left[(y_3 - y_2)(w_{,y2} - w_{,y3}) + (x_2 - x_3)(w_{,x3} - w_{,x2})\right] \\ & + \frac{\xi_3\xi_1}{2}\left[(y_1 - y_3)(w_{,y3} - w_{,y1}) + (x_3 - x_1)(w_{,x1} - w_{,x3})\right] \end{aligned}$$

Can gather terms and revise form if desired.

$\boxed{15.2\text{-}6}$

From Eqs. 15.2-5 and 15.2-12 we see that strains $\{\underset{\sim}{\epsilon}\}$ will contain only linear terms. Therefore terms in the stiffness matrix integrand will be no higher than quadratic, and either 3-point formula in Table 7.4-1 will integrate $[\underset{\sim}{k}]$ exactly.

$\boxed{15.2\text{-}7}$

(a) Incompatible elements – note that center deflection is overestimated even for concentrated center load, which would not happen with compatible elements.

(b) Uniform load, simply supported, M_c:

Halve h, cut error by factor $\approx \frac{1}{4}$; $2^m = 4$, $m = 2$

Uniform load, clamped, w_c:

Halve h, cut error by factor $\approx \frac{1}{3}$; $2^m = 3$, $m = 1.6$

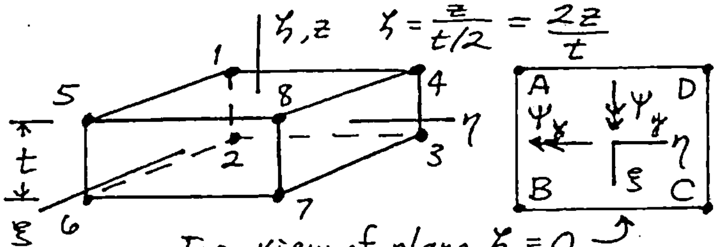
(c) w_c: halve h, cut error by factor $\approx \frac{1}{4}$; $2^m = 4$, $m = 2$

M_c: " " " " " " $\approx \frac{1}{4}$; $2^m = 4$, $m = 2$

M_{xy0}: " " " " " " ≈ 0.3; $2^m = \frac{1}{0.3}$, $m = 1.7$

15.3-1

$\zeta = \frac{z}{t/2} = \frac{2z}{t}$



Top view of plane $\zeta = 0$

(The sketch is simplified; the element need not be rectangular; Ψ_x need not be η-parallel; Ψ_y need not be ξ-parallel.

$$w_1 = w_2 = w_A \qquad u_2 = -u_1 = (t/2)\Psi_{xA}$$
$$w_5 = w_6 = w_B \qquad u_6 = -u_5 = (t/2)\Psi_{xB}$$
$$w_7 = w_8 = w_C \qquad u_8 = -u_7 = (t/2)\Psi_{xC}$$
$$w_3 = w_4 = w_D \qquad u_4 = -u_3 = (t/2)\Psi_{xD}$$

(Formulas for V's similar.)

Consider e.g. nodes 1 and 2: with $w_1 = w_2$,

$$\frac{1}{8}(1-\xi)(1-\eta)(1+\zeta)w_1 + \frac{1}{8}(1-\xi)(1-\eta)(1-\zeta)w_2 = \frac{1}{4}(1-\xi)(1-\eta)w_A$$

And with u_1 and u_2 as defined above,

$$u = \frac{1}{8}(1-\xi)(1-\eta)\left(1+\frac{2z}{t}\right)\left(-\frac{t}{2}\Psi_{xA}\right) + \frac{1}{8}(1-\xi)(1-\eta)\left(1-\frac{2z}{t}\right)\left(+\frac{t}{2}\Psi_{xA}\right) + \cdots$$

$$u = -\frac{z}{4}(1-\xi)(1-\eta)\Psi_{xA} + \cdots, \quad \text{etc.}$$

$\boxed{15.3\text{-}2}$

$$\underset{\sim}{k} = \int \underset{N\times5}{\underset{\sim}{B}_M^T}\, \underset{5\times5}{\underset{\sim}{D}_m}\, \underset{5\times N}{\underset{\sim}{B}_M}\, dA = \int (\underset{\sim}{B}_b + \underset{\sim}{B}_s)^T \underset{\sim}{D}_m (\underset{\sim}{B}_b + \underset{\sim}{B}_s)\, dA$$

$$\underset{\sim}{k} = \int \underset{\sim}{B}_b^T \underset{\sim}{D}_m \underset{\sim}{B}_b\, dA + \int \underset{\sim}{B}_s^T \underset{\sim}{D}_m \underset{\sim}{B}_s\, dA + \int \underset{\sim}{B}_s^T \underset{\sim}{D}_m \underset{\sim}{B}_b\, dA + \int \underset{\sim}{B}_b^T \underset{\sim}{D}_m \underset{\sim}{B}_s\, dA \qquad (A)$$

$$\underset{\sim}{D}_m = \begin{bmatrix} [3\times3] & \text{zero} \\ \text{zero} & [2\times2] \end{bmatrix} \qquad \underset{\sim}{B} = \underset{\sim}{B}_b + \underset{\sim}{B}_s$$

rows 5 & 6 zero

first 3 rows zero

$\underset{\sim}{D}_M \underset{\sim}{B}_b$: rows 5 & 6 zero, so $\underset{\sim}{B}_s^T(\underset{\sim}{D}_M \underset{\sim}{B}_b) = \underset{\sim}{0}$

$\underset{\sim}{D}_M \underset{\sim}{B}_s$: rows 1, 2, 3 zero, so $\underset{\sim}{B}_b^T(\underset{\sim}{D}_M \underset{\sim}{B}_s) = \underset{\sim}{0}$

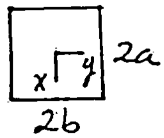
So Eq. (A) reduces to Eq. 15.3-6.

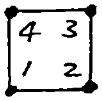
15.3-3

Nodal moments would appear if lateral displacement were created by nodal rotation, but this is not the case for C^0 elements; w depends only on nodal w_i.

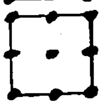
Let $A = 4ab$

F_i = nodal force

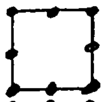




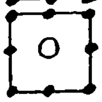
$F_i = qA/4$ (see Fig. 3.11-3)



$F_i = qA/36$ corners
$F_i = qA/9$ midsides
$F_i = 4qA/9$ center
(see Problem 6.9-1)



$F_i = -qA/12$ corners
$F_i = qA/3$ midsides
(see Fig. 3.11-3)



Same as preceding (center node has no w d.o.f.)

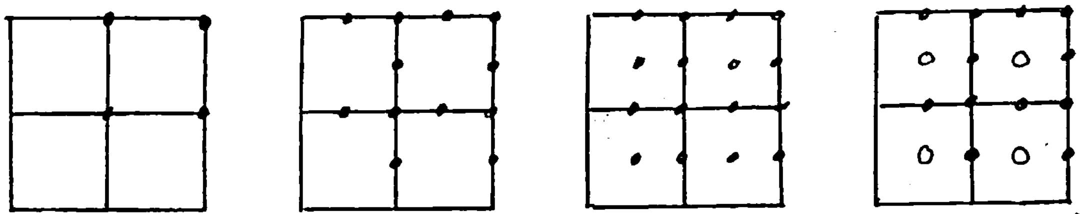
15.3-4

Evaluate the number of "free" d.o.f. remaining after boundary conditions have been imposed, then subtract (no. of els.)* (no. of shear constraints per element).
Consider elements in the following order: 4-node, 8-node, 9-node, heterosis.

(a) 1 el./quadrant

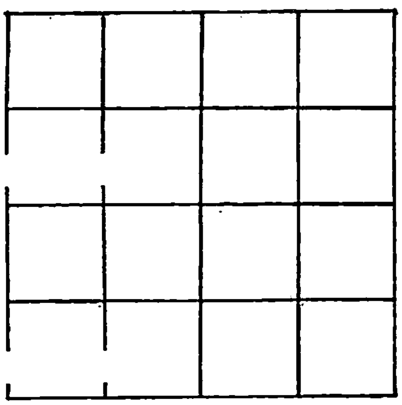
4-node	8-node	9-node	heterosis
$1-2=-1$	$5-8=-3$	$8-8=0$	$7-8=-1$
no d.o.f. left	no d.o.f. left	no d.o.f. left	no d.o.f. left

(b)



	4-node	8-node	9-node	heterosis
D.o.f.:	$3+2(2)+1=8$	$5(3)+6(2)+1=28$	$9(3)+6(2)+1=40$	$5(3)+10(2)+1=36$
	$8-4(2)=0$	$28-4(8)=-4$	$40-4(8)=8$	$36-4(8)=4$
	no d.o.f. left	no d.o.f. left	8 d.o.f. left	4 d.o.f. left

(c)



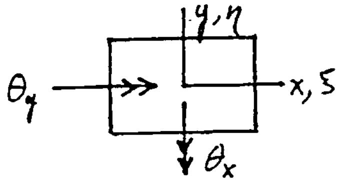
4-node el.:
$9(3)+6(2)+1=40$ free d.o.f.
$40-16(2)=8$ d.o.f. left

8-node el.:
$33(3)+14(2)+1=128$ free d.o.f.
$128-16(8)=0$ (no d.o.f. left)

9-node el.:
$128+16(3)=176$ free d.o.f.
$176-16(8)=48$ d.o.f. left

Heterosis
$128+16(2)=160$ free d.o.f.
$160-16(8)=32$ d.o.f. left

15.3-5



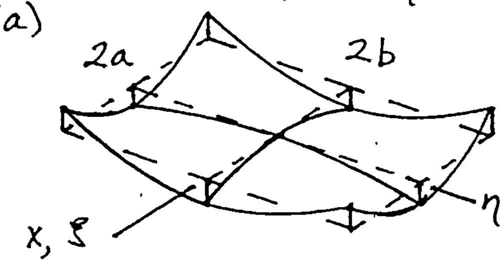
Shear strains incorrect except as follows:

Both γ_{yz} and γ_{zx} correctly evaluated (as zero) when there is rigid body motion ($w = a_0 + a_1 x + a_2 y$), bending to a cylindrical surface, or constant twist, as then M's do not vary with x or y. Also correct if nodal rotations are zero and nodal w's create a state of pure transverse shear strain, uniform over the element.

$\boxed{11.3\text{-}6}$

$w = 3\xi^2\eta^2 - \xi^2 - \eta^2$

(a)



ξ	η	w
0	0	0
±1	0	-1
0	±1	-1
±1	±1	1

(b) With $\psi_x = \psi_y = 0$, transverse shear strains are produced by w alone.

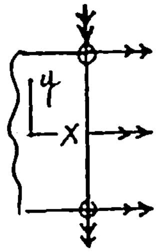
$$\gamma_{zx} = w_{,x} = \frac{w_{,\xi}}{a} = \frac{1}{a}\left(6\xi\eta^2 - 2\xi\right) = \frac{2\xi}{a}\left(3\eta^2 - 1\right)$$

$\gamma_{zx} = 0$ at $\eta = \pm\frac{1}{\sqrt{3}}$. In similar fashion we can show that $\gamma_{yz} = 0$ at $\xi = \pm\frac{1}{\sqrt{3}}$.

(c) Supports only at midsides or only at corners; the latter is more likely for a mesh. Any loads that tend to bend solid lines in sketch into the shape shown.

$\boxed{15.3-9}$

Two transverse shear strains (e.g. γ_{yz} and γ_{zx}) are set to zero at each of 4 Gauss points. Thus we can eliminate $2*4=8$ d.o.f. such as lateral deflection and normal rotation at side nodes. This leaves, e.g. along a y-parallel as shown



Thus, w and edge-normal rotation are linear in y but edge-tangent rotation is quadratic in y, so a moment M_y linear in y can be represented.

$\boxed{15.4\text{-}1}$

$$w = \frac{L-x}{L} w_1 + \frac{x}{L} w_2 \qquad \psi = \frac{L-x}{L}\psi_1 + \frac{x}{L}\psi_2$$

$$\psi_{,x} = -\frac{1}{L}\psi_1 + \frac{1}{L}\psi_2 = \frac{1}{L}\begin{bmatrix} -1 & 1 \end{bmatrix}\begin{Bmatrix} \psi_1 \\ \psi_2 \end{Bmatrix}$$

$$U_b = \frac{EI}{2L^2}\psi_{,x}^T \psi_{,x} L = \frac{EI}{L}\begin{Bmatrix} \psi_1 \\ \psi_2 \end{Bmatrix}^T \begin{Bmatrix} -1 \\ 1 \end{Bmatrix}\begin{bmatrix} -1 & 1 \end{bmatrix}\begin{Bmatrix} \psi_1 \\ \psi_2 \end{Bmatrix} = \begin{Bmatrix} \psi_1 \\ \psi_2 \end{Bmatrix}^T \underbrace{\frac{EI}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}}_{[k_b]}\begin{Bmatrix} \psi_1 \\ \psi_2 \end{Bmatrix}$$

(Expand $[k_b]$ to 4×4 by adding zeros corresponding to w_1 and w_2 d.o.f.)

$$\gamma_{zx} = w_{,x} - \psi = \begin{bmatrix} -\frac{1}{L} & -\frac{L-x}{L} & \frac{1}{L} & -\frac{x}{L} \end{bmatrix} \underbrace{\begin{Bmatrix} w_1 \\ \psi_1 \\ w_2 \\ \psi_2 \end{Bmatrix}}_{\{d\}}$$

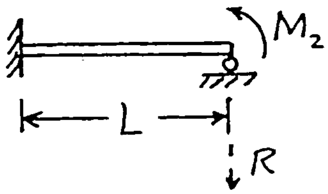
Evaluate γ_{zx} at $x = \frac{L}{2}$ for one-point integration; then

$$U_s = \frac{GA_s}{2}\{d\}^T \begin{Bmatrix} -1/L \\ -1/2 \\ 1/L \\ -1/2 \end{Bmatrix}\begin{bmatrix} -\frac{1}{L} & -\frac{1}{2} & \frac{1}{L} & -\frac{1}{2} \end{bmatrix}\{d\} L$$

$$U_s = \frac{1}{2}\{d\}^T GA_s \begin{bmatrix} 1/L & 1/2 & -1/L & 1/2 \\ 1/2 & L/4 & -1/2 & L/4 \\ -1/L & -1/2 & 1/L & -1/2 \\ 1/2 & L/4 & -1/2 & L/4 \end{bmatrix}\{d\} = \frac{1}{2}\{d\}^T[k_s]\{d\}$$

$\boxed{15.4\text{-}2}$

(a)



Beam theory: zero tip deflection, so

$$\frac{M_2 L^2}{2EI} - \frac{RL^3}{3EI} = 0 \; ; \quad R = \frac{3M_2}{2L}$$

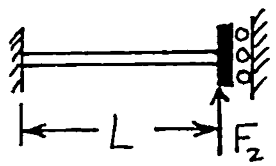
$$\psi_2 = \frac{M_2 L}{EI} - \frac{RL^2}{2EI} = \frac{M_2 L}{4EI}$$

From last d.o.f. in Eq. 15.4-4:

$$\left[\frac{EI}{L} + \frac{L/4}{\left(\frac{L^2}{12EI} + \frac{1}{GA_s}\right)}\right]\psi_2 = M_2$$

Reduces to $\dfrac{12EI + 4GA_sL^2}{12L + \dfrac{GA_sL^3}{EI}}\psi_2 = M_2$ $\begin{cases} \text{Small } GA_s, \; \dfrac{EI}{L}\psi_2 = M_2 \\ \text{Large } GA_s, \; \dfrac{4EI}{L}\psi_2 = M_2 \end{cases}$

(b)



Beam theory: $w_2 = 2\dfrac{F_2(L/2)^3}{3EI} = \dfrac{F_2L^3}{12EI}$

From next-to-last d.o.f. in Eq. 15.4-4:

$$\frac{1}{L}\,\frac{1}{\left(\frac{L^2}{12EI} + \frac{1}{GA_s}\right)}w_2 = F_2$$

$$\frac{1}{L}\,\frac{12EIGA_s}{GA_sL^2 + 12EI}w_2 = F_2$$

Small GA_s, $w_2 = \dfrac{F_2L}{GA_s}$

Large GA_s, $w_2 = \dfrac{F_2L^3}{12EI}$

15.4-3 $\quad x = \frac{L}{2}\xi$

The only nonzero d.o.f. is w_2, so $\psi_{,x} = 0$ throughout. With $\frac{d}{dx} = \frac{2}{L}\frac{d}{d\xi}$,

$$\gamma_{zx} = w_{,x} - \psi = w_{,x} = \frac{d}{dx}(1-\xi^2)w_2 = \frac{2}{L}(-2\xi)w_2 = -\frac{4\xi}{L}w_2$$

There is only one stiffness coefficient. Gauss point locations of an order 2 rule are at $\xi = \pm 1/\sqrt{3}$, and weights are each unity.

$$k = GA_s\int_{-1}^{1}\left(-\frac{4\xi}{L}\right)^2 \frac{L}{2}\,d\xi = \frac{8GA_s}{L}\left[\left(-\frac{1}{\sqrt{3}}\right)^2 + \left(\frac{1}{\sqrt{3}}\right)^2\right] = \frac{16GA_s}{3L}$$

The consistent load at node 2 is $\frac{2qL}{3}$, so

$$\frac{16GA_s}{3L}w_2 = \frac{2qL}{3}, \qquad w_2 = \frac{qL^2}{8GA_s}$$

We want to include the exact bending deflection $\frac{qL^4}{384EI}$

$$\frac{qL^2}{8GA^*} = \frac{qL^2}{8GA_s} + \frac{qL^4}{384EI}, \quad \text{hence} \quad \frac{1}{GA^*} = \frac{1}{GA} + \frac{L^2}{48EI}$$

15.6-1

Assume that all of the plate is in contact; solve as usual. Where the solution shows an upward deflection (with downward load), eliminate the foundation there in the next solution. Repeat until convergence. It may be necessary to re-introduce foundation in some places where it was previously removed, when a reanalysis shows renewed contact.