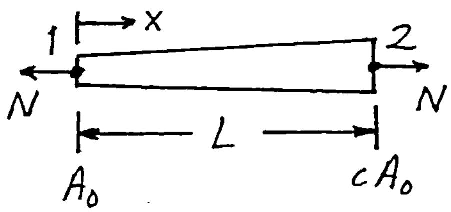
2.2-1



(a) Eq. 2.2-1:

$$[\underset{\sim}{k}] = \frac{A_{ave}E}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{(1+c)A_0}{2}\frac{E}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Exact: let δ = elongation

$$\delta = \int_0^L \frac{N\,dx}{AE} \quad \text{where } A = A_0 + (c-1)\frac{x}{L}A_0$$

$$\delta = \frac{N}{A_0E}\,\frac{L}{c-1}\left(1+(c-1)\frac{x}{L}\right)\Big|_0^L = \frac{NL\ln c}{A_0E(c-1)}$$

For $\delta = 1$, $N = \dfrac{A_0E(c-1)}{L\ln c}$

$$[\underset{\sim}{k}] = \frac{A_0E(c-1)}{L\ln c}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$ Now let $c=2$:

Conventional:

$$[\underset{\sim}{k}] = 1.5\frac{A_0E}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Exact:

$$[\underset{\sim}{k}] = 1.443\frac{A_0E}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

(b) Exact: $\delta = \dfrac{PL_T}{2A_0E}\ln 3 = 0.5493\dfrac{PL_T}{A_0E}$

One el.: $\delta = \dfrac{PL_T}{(2A_0)E} = 0.5\dfrac{PL_T}{A_0E}$ 8.98% low

Two els.:

$$\delta = \frac{P(L_T/2)}{A_0E}\left(\frac{1}{1.5}+\frac{1}{2.5}\right) = 0.5333\frac{PL_T}{A_0E}$$

2.91% low

Three els.:

$$\delta = \frac{P(L_T/3)}{A_0E}\left(\frac{1}{1.333}+\frac{1}{2.000}+\frac{1}{2.667}\right)$$

$$\delta = 0.5417\frac{PL_T}{A_0E}$$ 1.39% low

Four els.:

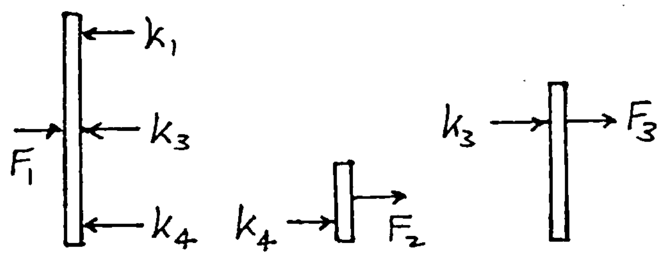
$$\delta = \frac{P(L_T/4)}{A_0E}\left(\frac{1}{1.25}+\frac{1}{1.75}+\frac{1}{2.25}+\frac{1}{2.75}\right)$$

$$\delta = 0.5449\frac{PL_T}{A_0E}$$ 0.81% low

2.2-2

(a) $$\begin{bmatrix} k_1+k_3+k_4 & -k_4 & -k_3 \\ -k_4 & k_4 & 0 \\ -k_3 & 0 & k_2+k_3 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix}$$

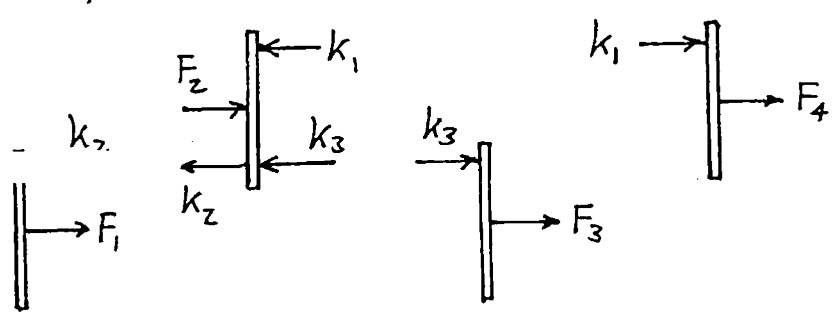
Example — column 1:



(b)

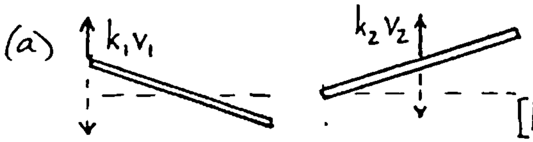
$$\begin{bmatrix} k_2+k_5+k_6 & -k_2 & -k_6 & 0 \\ -k_2 & k_1+k_2+k_3 & -k_3 & -k_1 \\ -k_6 & -k_3 & k_3+k_4+k_6 & -k_4 \\ 0 & -k_1 & -k_4 & k_1+k_4 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{Bmatrix}$$

Example — column 2:



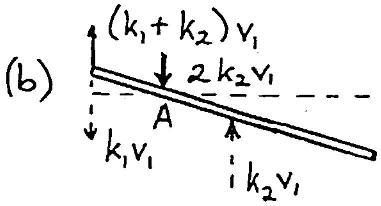
2.2-3 Activate d.o.f. in turn to unit value; each time calculate loads that therefore must be applied to the two d.o.f.

(a)



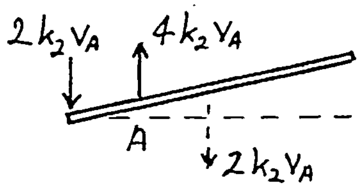
$$\begin{array}{ccc} & v_1 & v_2 \\ [\underset{\sim}{k}] = & \left[\begin{array}{c} k_1 \\ 0 \end{array}\right. & \left.\begin{array}{c} 0 \\ k_2 \end{array}\right] \end{array}$$

(b)

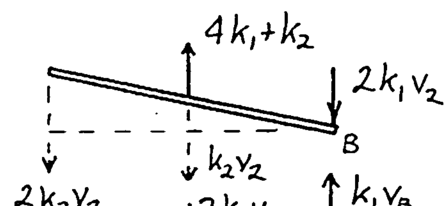


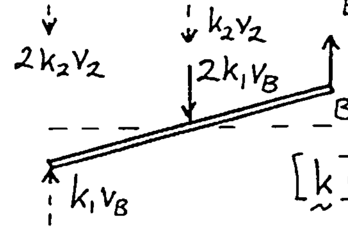
$$\begin{array}{cc} v_1 & v_A \end{array}$$

$$[\underset{\sim}{k}] = \begin{bmatrix} k_1 + k_2 & -2k_2 \\ -2k_2 & 4k_2 \end{bmatrix}$$



(c)

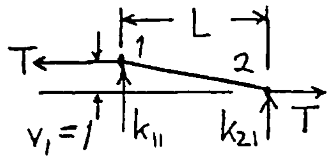




$$\begin{array}{cc} v_2 & v_B \end{array}$$

$$[\underset{\sim}{k}] = \begin{bmatrix} 4k_1 + k_2 & -2k_1 \\ -2k_1 & k_1 \end{bmatrix}$$

$\boxed{2.2\text{-}4}$

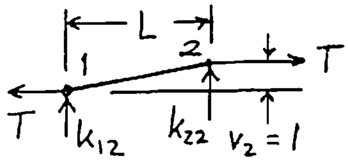


$\sum M_{node\ 2} = 0$

$k_{11}L - T(1) = 0$

$k_{11} = \frac{T}{L}$

$\sum F_y = 0;\ k_{21} = -\frac{T}{L}$



$\sum M_{node\ 1} = 0$

$k_{22}L - T(1) = 0$

$k_{22} = \frac{T}{L}$

$\sum F_y = 0;\ k_{12} = -\frac{T}{L}$

$$[\underset{\sim}{k}] = \frac{T}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$\boxed{2.2\text{-}5}$ Apply Eq. 2.2-9.

(a)
$$\frac{k}{t}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} 0 \\ T_2 \\ 200 \end{Bmatrix} = \begin{Bmatrix} f_1 \\ 0 \\ f_3 \end{Bmatrix}$$

2^{nd} eq. gives $T_2 = 100°C$

1^{st} eq. gives $f_1 = -100k/t$

3^{rd} eq. gives $f_3 = 100k/t$

(b)
$$\frac{k}{t}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}\begin{Bmatrix} 400 \\ T_2 \\ T_3 \end{Bmatrix} = \begin{Bmatrix} f_1 \\ 0 \\ \bar{f} \end{Bmatrix}$$

Unknowns are T_2, T_3, f_1

2^{nd} eq. gives $T_2 = 200 + \frac{T_3}{2}$

Substitute into 3^{rd} equation:

$$\frac{k}{t}\left(-200 - \frac{T_3}{2} + T_3\right) = \bar{f}$$

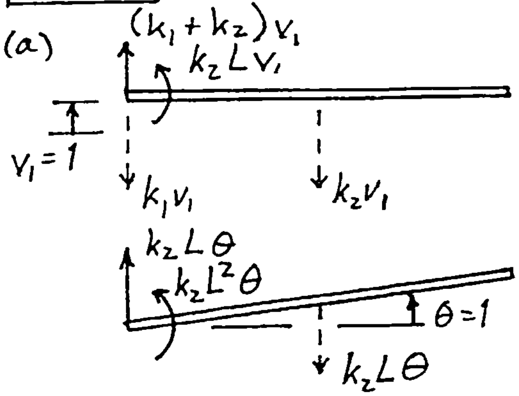
$$\therefore\ T_3 = 400 + 2\frac{\bar{f}t}{k}$$

Hence $T_2 = 400 + \frac{\bar{f}t}{k}$

1^{st} eq. gives $f_1 = -\bar{f}$

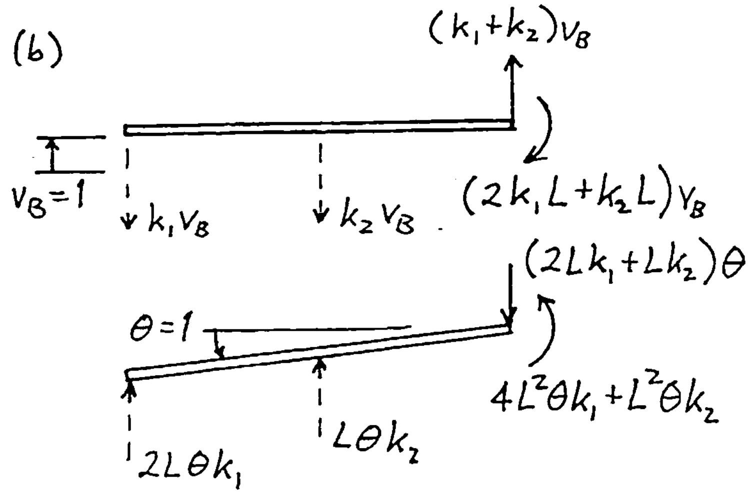
$\boxed{2.3-1}$

(a)



$$[\underset{\sim}{k}] = \begin{bmatrix} k_1+k_2 & k_2 L \\ k_2 L & k_2 L^2 \end{bmatrix}$$

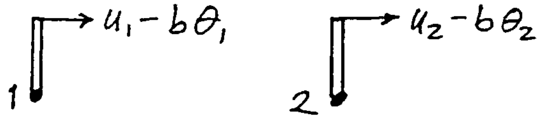
(b)



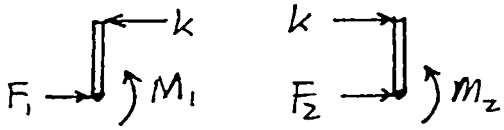
$$[\underset{\sim}{k}] = \begin{bmatrix} k_1+k_2 & -(2k_1+k_2)L \\ -(2k_1+k_2)L & (4k_1+k_2)L^2 \end{bmatrix}$$

2.3-2

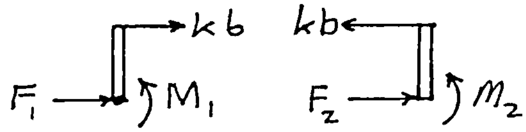
(a) Displacements at spring ends:



Col. 1 of $[\underset{\sim}{k}]$: $u_1 = 1, \theta_1 = u_2 = \theta_2 = 0$



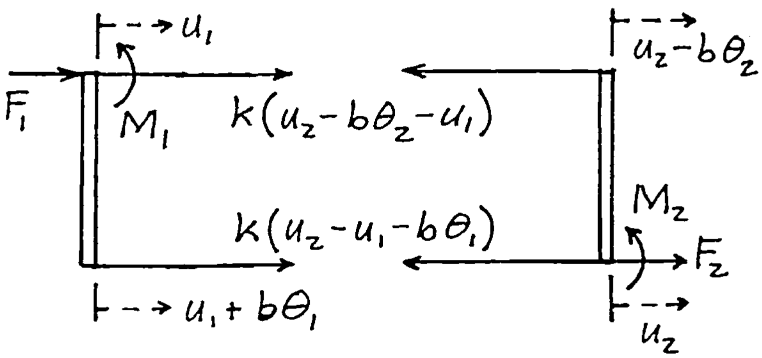
Col. 2 of $[\underset{\sim}{k}]$: $u_1 = 0, \theta_1 = 1, u_2 = \theta_2 = 0$



and so on.

$$[\underset{\sim}{k}] = \begin{bmatrix} k & -kb & -k & kb \\ -kb & kb^2 & kb & -kb^2 \\ -k & kb & k & -kb \\ kb & -kb^2 & -kb & kb^2 \end{bmatrix}$$

(b)



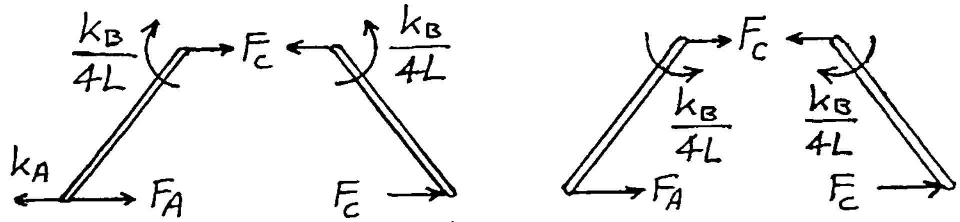
.... $u_1, \theta_1, u_2, \theta_2$ one unit, in turn.

$$[\underset{\sim}{k}] = \begin{bmatrix} 2k & bk & -2k & bk \\ bk & b^2k & -bk & 0 \\ -2k & -bk & 2k & -bk \\ bk & 0 & -bk & b^2k \end{bmatrix}$$

2.3-3

(a) Change of angle at B is $\dfrac{u_C - u_A}{4L}$

Activate u_A one unit, then u_C one unit.

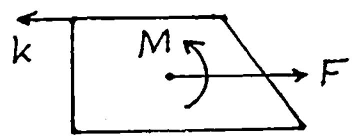


$$4LF_C + \frac{k_B}{4L} = 0;\quad F_C = -\frac{k_B}{16L^2} \qquad\qquad F_C = \frac{k_B}{16L^2}$$

$$F_A = k_A - F_C = k_A + \frac{k_B}{16L^2} \qquad\qquad F_A = -F_C$$

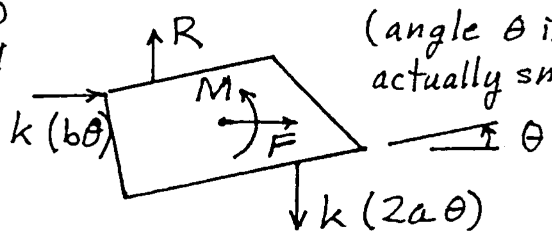
$$[\underset{\sim}{k}] = \begin{bmatrix} k_A + \dfrac{k_B}{16L^2} & -\dfrac{k_B}{16L^2} \\ -\dfrac{k_B}{16L^2} & \dfrac{k_B}{16L^2} \end{bmatrix}$$

(b) $u = 1$
$\theta = 0$



$F = k$
$M = -kb$

$u = 0$
$\theta = 1$



(angle θ is actually small)

$$R = k\,(2a\theta)$$

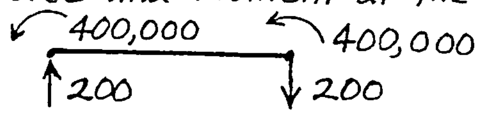
$$M = b\left[k\,(b\theta)\right] + a\left[R + k(2a\theta)\right]$$

$$\left[kb^2 + k\,(2a^2) + k\,(2a^2)\right]\theta$$

$$[\underset{\sim}{k}] = \begin{bmatrix} k & -kb \\ -kb & k(4a^2 + b^2) \end{bmatrix}$$

$\boxed{2.3-4}$

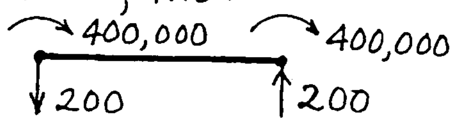
With force units N and length units mm, equilibrium considerations provide the force and moment at the right end.



Thus $k_{11} = 200$, $k_{21} = 400{,}000$,
$k_{31} = -200$, $k_{41} = 400{,}000$.

By the symmetry of $\underset{\sim}{k}$, we also have the first row of $\underset{\sim}{k}$.

On physical grounds, we also know that if we lift not the left end but the right end by 1 mm, then



This gives row and colum 3 of $\underset{\sim}{k}$.
So altogether we know

$$\underset{\sim}{k} = \begin{bmatrix} 200 & 400{,}000 & -200 & 400{,}000 \\ 400{,}000 & ? & -400{,}000 & ? \\ -200 & -400{,}000 & 200 & -400{,}000 \\ 400{,}000 & ? & -400{,}000 & ? \end{bmatrix}$$

2.3-5

(a) Column 2: use Fig. 2.3-1d (beam cantilevered from node 2 with unit rotation at node 1).

$v_1 = 0$ at node 1 $\quad 0 = \dfrac{k_{12}L^3}{3EI} - \dfrac{k_{22}L^2}{2EI}$

$\theta_{z1} = 1$ at node 1 $\quad 1 = -\dfrac{k_{12}L^2}{2EI} + \dfrac{k_{22}L}{EI}$

From which $k_{12} = \dfrac{6EI}{L^2}$, $k_{22} = \dfrac{4EI}{L}$

$\Sigma F_y = 0 = k_{12} + k_{32}$, $k_{32} = -\dfrac{6EI}{L^2}$

$\Sigma M_{node\,2} = 0 = k_{22} + k_{42} - k_{12}L$

$k_{42} = k_{12}L - k_{22} = \dfrac{2EI}{L}$

(b) Column 3: use Fig. 2.3-1e (beam cantilevered from node 1 with unit displacement at node 2).

$v_2 = 1$ at node 2 $\quad 1 = \dfrac{k_{33}L^3}{3EI} + \dfrac{k_{43}L^2}{2EI}$

$\theta_{z2} = 0$ at node 2 $\quad 0 = \dfrac{k_{33}L^2}{2EI} + \dfrac{k_{43}L}{EI}$

From which $k_{33} = \dfrac{12EI}{L^3}$, $k_{43} = -\dfrac{6EI}{L^2}$

$\Sigma F_y = 0 = k_{13} + k_{33}$, $k_{13} = -\dfrac{12EI}{L^3}$

$\Sigma M_{node\,1} = 0 = k_{23} + k_{43} + k_{33}L$

$k_{23} = -k_{33}L - k_{43} = -\dfrac{6EI}{L^2}$

(c) Column 4: use Fig. 2.3-1f (beam cantilevered from node 1 with unit rotation at node 2).

$v_2 = 0$ at node 2 $\quad 0 = \dfrac{k_{34}L^3}{3EI} + \dfrac{k_{44}L^2}{2EI}$

$\theta_{z2} = 1$ at node 2 $\quad 1 = \dfrac{k_{34}L^2}{2EI} + \dfrac{k_{44}L}{EI}$

From which $k_{34} = -\dfrac{6EI}{L^2}$, $k_{44} = \dfrac{4EI}{L}$

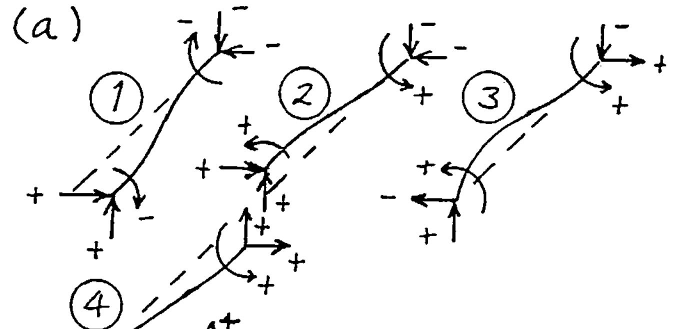
$\Sigma F_y = 0 = k_{14} + k_{34}$, $k_{14} = \dfrac{6EI}{L^2}$

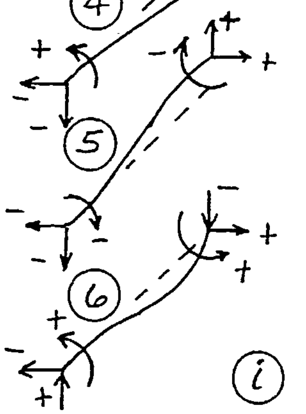
$\Sigma M_{node\,1} = 0 = k_{24} + k_{44} + k_{34}L$

$k_{24} = -k_{34}L - k_{44} = \dfrac{2EI}{L}$

2.3-6

(a)

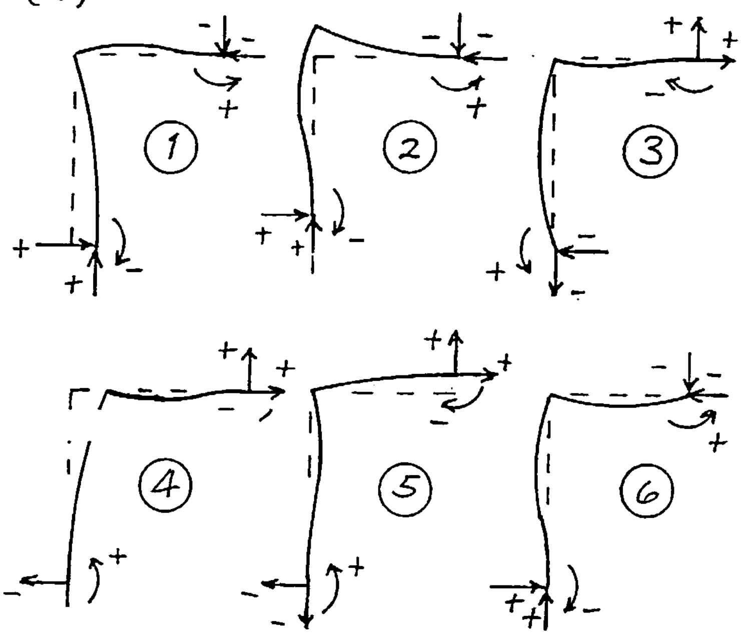




$$\begin{bmatrix} + & + & - & - & - & - \\ + & + & + & - & - & + \\ - & + & + & + & - & + \\ - & - & + & + & + & + \\ - & - & - & + & + & - \\ - & + & + & + & - & + \end{bmatrix}$$

(i) = for column i

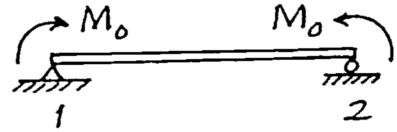
(b)



$$\begin{bmatrix} + & + & - & - & - & + \\ + & + & - & - & - & + \\ - & - & + & + & + & - \\ - & - & + & + & + & - \\ - & - & + & + & + & - \\ + & + & - & - & - & + \end{bmatrix}$$

(i) = for column i

$\boxed{2.3\text{-}7}$



(a)

From Eq. 2.3-6, with θ_{z1} and θ_{z2} the active d.o.f.,

$$\frac{EI_z}{(1+\phi_y)L}\begin{bmatrix} 4+\phi_y & 2-\phi_y \\ 2-\phi_y & 4+\phi_y \end{bmatrix}\begin{Bmatrix} \theta_{z1} \\ \theta_{z2} \end{Bmatrix} = \begin{Bmatrix} -M_o \\ M_o \end{Bmatrix}$$

From which $\theta_{z1} = -\theta_{z2} = -\dfrac{M_o L}{2EI_z}$

(b) $$v_2 = \frac{P}{Y_1} = \frac{(1+\phi_y)PL^3}{12EI_z}$$

$$\phi_y = \frac{12EI_z k_y}{AGL^2} = \frac{12E[3(4^3)/12]1.2}{3(4)(E/2)L^2} = \frac{38.4}{L^2}$$

$$v_2 = \frac{(1+38.4/L^2)PL^3}{12E\,[3(4^3)/12]} = \frac{1+38.4/L^2}{192}\,\frac{PL^3}{E}$$

No trans. shear deformation: $v_2 = \dfrac{PL^3}{192E} = \bar{v}$

$L=8$: $v_2 = 1.6\bar{v}$; 37.5% from shear

$L=16$: $v_2 = 1.15\bar{v}$; 13.0% from shear

$L=32$: $v_2 = 1.0375\bar{v}$; 3.6% from shear

(c) $$v_2 = \frac{(1+\phi_y)PL^3}{12EI_z} = \frac{PL^3}{12EI_z}\left[\frac{AGL^2+12EI_zk_y}{AGL^2}\right]$$

$$v_2 = \frac{--}{AG}\left[\frac{AGL^2}{12EI_z} + k_y\right]$$

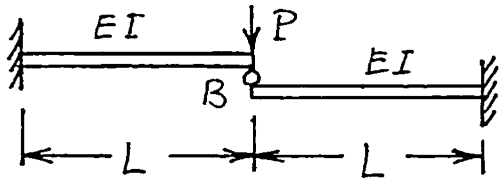
$L \to \infty$: $v_2 \to \dfrac{PL^3}{12EI_z}$

(as in elementary beam theory)

$L \to 0$: $v_2 \to \dfrac{PL}{AG}k_y$

2.3-8

But for torsional stiffness of members, the problem would be the same as the following plane problem:



for which $v = \frac{PL^3}{6EI}$ at B.

For the given problem, note that $\theta_x = \theta_y$ at B. Stiffnesses of the two members add. Including torsional stiffness and letting $\theta_B = \theta_x = \theta_y$ at B, we have

$$\begin{bmatrix} 2\frac{12EI}{L^3} & 2\frac{6EI}{L^2} \\ 2\frac{6EI}{L^2} & 2\frac{4EI}{L} + 2\frac{GK}{L} \end{bmatrix} \begin{Bmatrix} v_B \\ \theta_B \end{Bmatrix} = \begin{Bmatrix} -P \\ 0 \end{Bmatrix}$$

Second eq. gives $\theta_B = -\frac{6EI}{(4EI+GK)L} v_B$

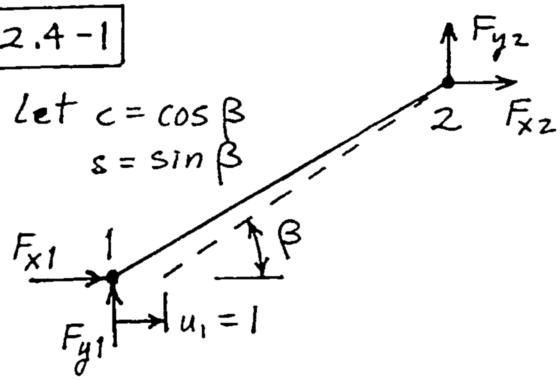
Then first eq. gives

$$v_B = -\frac{PL^3}{24EI}\left[1 - \frac{3}{4 + \frac{GK}{EI}}\right]^{-1}$$

I section: partial restraint of cross-section warping will be present at A, B, and C. Restraint will increase torsional stiffness as compared with foregoing equation and therefore reduce $|v_B|$.

$\boxed{2.4-1}$

Let $c = \cos\beta$
$s = \sin\beta$



Shortening $= (1)\cos\beta = c$

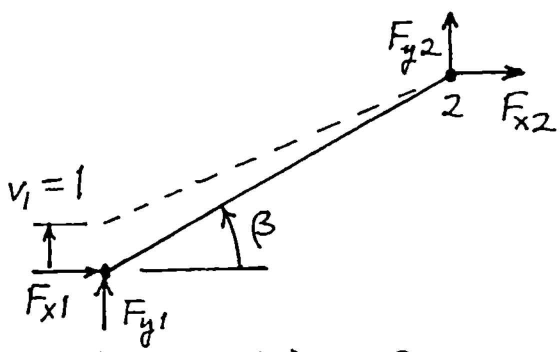
Axial force $= N = \frac{AE}{L}c$

$$F_{x1} = -F_{x2} = cN = \frac{AE}{L}c^2$$

$$F_{y1} = -F_{y2} = sN = \frac{AE}{L}cs$$

So column 1 of $[\underset{\sim}{k}]$ is

$$\frac{AE}{L}\lfloor c^2 \quad cs \quad -c^2 \quad -cs \rfloor^T$$



Shortening $= (1)\sin\beta = s$

$= N = \frac{AE}{L}s$

$$F_{x1} = -F_{x2} = cN = \frac{AE}{L}cs$$

$$F_{u1} = -F_{u2} = sN = \frac{AE}{L}s^2$$

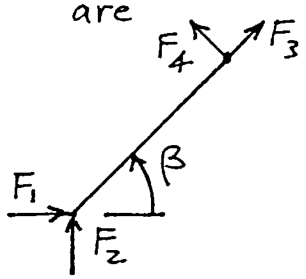
So column 2 of $[\underset{\sim}{k}]$ is

$$\frac{AE}{L}\lfloor cs \quad s^2 \quad -cs \quad -s^2 \rfloor^T$$

Similarly for columns 3 and 4 of $[\underset{\sim}{k}]$.

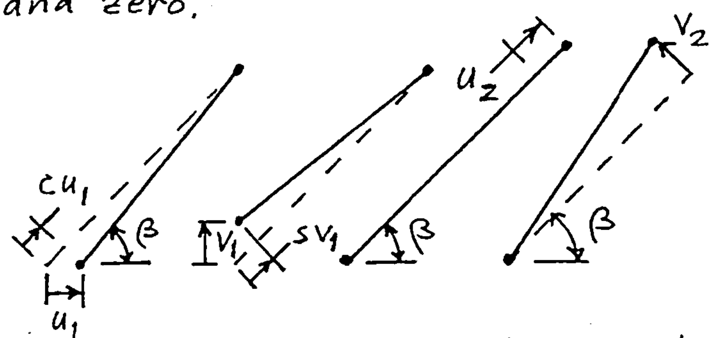
$\boxed{2.4\text{-}2}$

(a) Axial force $= N = \frac{AE}{L}\delta$, where δ is the axial deformation. Nodal forces are



$$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \end{Bmatrix} = \begin{Bmatrix} Nc \\ Ns \\ -N \\ 0 \end{Bmatrix} \qquad \begin{matrix} c = \cos\beta \\ s = \sin\beta \end{matrix}$$

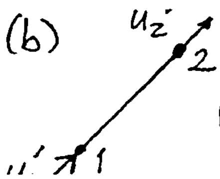
Write this column for each of the following cases, whose axial deformations are respectively cu_1, sv_1, u_2, and zero.



With u_1, v_1, u_2, v_2 each unity, we get

$$[\underset{\sim}{k}] = \frac{AE}{L}\begin{bmatrix} c^2 & cs & -c & 0 \\ cs & s^2 & -s & 0 \\ -c & -s & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(b)



$$[\underset{\sim}{k}'] = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \qquad \begin{Bmatrix} u_1' \\ u_2' \end{Bmatrix} = \underbrace{\begin{bmatrix} c & s & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}}_{[\underset{\sim}{T}]} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{Bmatrix}$$

$$[\underset{\sim}{k}'][\underset{\sim}{T}] = \frac{AE}{L}\begin{bmatrix} c & s & -1 & 0 \\ -c & -s & 1 & 0 \end{bmatrix}$$

$$[\underset{\sim}{T}]^T[\underset{\sim}{k}'][\underset{\sim}{T}] = \frac{AE}{L}\begin{bmatrix} c^2 & cs & -c & 0 \\ cs & s^2 & -s & 0 \\ -c & -s & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$\boxed{2.4\text{-}3}$ Let $k = AE/L$

$i = 1,2$

(a)

D.O.F. :	u_i	v_i	w_i	
Axial comp. :	$l_1 u_i$	$m_1 v_i$	$n_1 w_i$	for bar on x' axis
Axial force :	$k l_1 u_i$	$k m_1 v_i$	$k n_1 w_i$	

x, y, z force comps. at node i; $i = 1,2$

$$\left\{\begin{matrix} k l_1^2 u_i & k l_1 m_1 v_i & k l_1 n_1 w_i \\ k l_1 m_1 u_i & k m_1^2 v_i & k m_1 n_1 w_i \\ k l_1 n_1 u_i & k m_1 n_1 v_i & k n_1^2 w_i \end{matrix}\right\}\text{magnitudes}$$

Hence

$$[\underset{\sim}{k}] = \frac{AE}{L}\begin{bmatrix} l_1^2 & l_1 m_1 & l_1 n_1 & -l_1^2 & -l_1 m_1 & -l_1 n_1 \\ l_1 m_1 & m_1^2 & m_1 n_1 & -l_1 m_1 & -m_1^2 & -m_1 n_1 \\ l_1 n_1 & m_1 n_1 & n_1^2 & -l_1 n_1 & -m_1 n_1 & -n_1^2 \\ -l_1^2 & -l_1 m_1 & -l_1 n_1 & -l_1^2 & l_1 m_1 & l_1 n_1 \\ -l_1 m_1 & -m_1^2 & -m_1 n_1 & -l_1 m_1 & m_1^2 & m_1 n_1 \\ -l_1 n_1 & -m_1 n_1 & -n_1^2 & l_1 n_1 & m_1 n_1 & n_1^2 \end{bmatrix}$$

(b)

$$[\underset{\sim}{k}'][\underset{\sim}{T}] = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} l_1 & m_1 & n_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & l_1 & m_1 & n_1 \end{bmatrix}$$

$$= \frac{AE}{L}\begin{bmatrix} l_1 & m_1 & n_1 & -l_1 & -m_1 & -n_1 \\ -l_1 & -m_1 & -n_1 & l_1 & m_1 & n_1 \end{bmatrix}$$

$[\underset{\sim}{k}] = [\underset{\sim}{T}]^T([\underset{\sim}{k}'][T])$

Gives same $[\underset{\sim}{k}]$ as in part (a).

$\boxed{2.4\text{-}4}$

matrix to transform is $[\underset{\sim}{k}'] = \begin{bmatrix} k_1 & 0 \\ 0 & k_2 \end{bmatrix}$

(a) $$\begin{Bmatrix} v_1 \\ v_2 \end{Bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix} \begin{Bmatrix} v_1 \\ v_A \end{Bmatrix} = [\underset{\sim}{T}] \begin{Bmatrix} v_1 \\ v_A \end{Bmatrix}$$

$$[\underset{\sim}{T}]^T([\underset{\sim}{k}'][\underset{\sim}{T}]) = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} k_1 & 0 \\ -k_2 & 2k_2 \end{bmatrix} = \begin{bmatrix} k_1+k_2 & -2k_2 \\ -2k_2 & 4k_2 \end{bmatrix}$$

(b) $$\begin{Bmatrix} v_1 \\ v_2 \end{Bmatrix} = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix} \begin{Bmatrix} v_2 \\ v_B \end{Bmatrix} = [\underset{\sim}{T}] \begin{Bmatrix} v_2 \\ v_B \end{Bmatrix}$$

$$[\underset{\sim}{T}]^T([\underset{\sim}{k}'][\underset{\sim}{T}]) = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 2k_1 & -k_1 \\ k_2 & 0 \end{bmatrix} = \begin{bmatrix} 4k_1+k_2 & -2k_1 \\ -2k_1 & k_1 \end{bmatrix}$$

(c) $$\begin{Bmatrix} v_1 \\ v_2 \end{Bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & L \end{bmatrix} \begin{Bmatrix} v_1 \\ \theta \end{Bmatrix} = [\underset{\sim}{T}] \begin{Bmatrix} v_1 \\ \theta \end{Bmatrix}$$

$$[\underset{\sim}{T}]^T([\underset{\sim}{k}'][\underset{\sim}{T}]) = \begin{bmatrix} 1 & 1 \\ 0 & L \end{bmatrix} \begin{bmatrix} k_1 & 0 \\ k_2 & k_2 L \end{bmatrix} = \begin{bmatrix} k_1+k_2 & k_2 L \\ k_2 L & k_2 L^2 \end{bmatrix}$$

(d) $$\begin{Bmatrix} v_1 \\ v_2 \end{Bmatrix} = \begin{bmatrix} 1 & -2L \\ 1 & -L \end{bmatrix} \begin{Bmatrix} v_B \\ \theta \end{Bmatrix} = [\underset{\sim}{T}] \begin{Bmatrix} v_B \\ \theta \end{Bmatrix}$$

$$[\underset{\sim}{T}]^T([\underset{\sim}{k}'][\underset{\sim}{T}]) = \begin{bmatrix} 1 & 1 \\ -2L & -L \end{bmatrix} \begin{bmatrix} k_1 & -2Lk_1 \\ k_2 & -Lk_2 \end{bmatrix} = \begin{bmatrix} k_1+k_2 & -(2k_1+k_2)L \\ -(2k_1+k_2)L & (4k_1+k_2)L^2 \end{bmatrix}$$

$\boxed{2.5-1}$

$$[\underset{\sim}{k}]_1\{\underset{\sim}{d}\}_1 = \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \begin{Bmatrix} d_2 \\ d_3 \\ d_1 \end{Bmatrix} \qquad \begin{matrix} d_2 \rightarrow D_1 \\ d_3 \rightarrow D_4 \\ d_1 \rightarrow D_2 \end{matrix}$$

$$[\underset{\sim}{k}]_2\{\underset{\sim}{d}\}_2 = \begin{bmatrix} b_1 & b_2 & b_3 \\ b_4 & b_5 & b_6 \\ b_7 & b_8 & b_9 \end{bmatrix} \begin{Bmatrix} d_2 \\ d_3 \\ d_1 \end{Bmatrix} \qquad \begin{matrix} d_2 \rightarrow D_4 \\ d_3 \rightarrow D_3 \\ d_1 \rightarrow D_2 \end{matrix}$$

Reorder, expand, and add.

$$\underbrace{\left(\begin{bmatrix} a_1 & a_3 & 0 & a_4 \\ a_7 & a_9 & 0 & a_8 \\ 0 & 0 & 0 & 0 \\ a_4 & a_6 & 0 & a_5 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & b_9 & b_8 & b_7 \\ 0 & b_6 & b_5 & b_4 \\ 0 & b_3 & b_2 & b_1 \end{bmatrix} \right)}_{[\underset{\sim}{K}]} \begin{Bmatrix} D_1 \\ D_2 \\ D_3 \\ D_4 \end{Bmatrix}$$

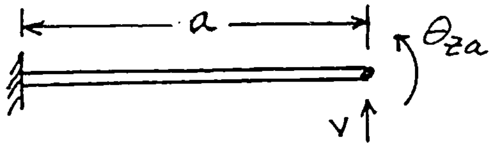
$\boxed{2.5-2}$

$$[\underset{\sim}{K}]: \begin{bmatrix} A,C & C & A & C & C \\ C & C & & C & C \\ A & & A,B & B & B \\ C & C & B & B,C & B,C \\ C & C & B & B,C & B,C \end{bmatrix}$$

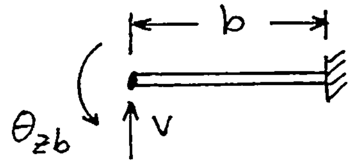
$$\{\underset{\sim}{R}\}: \begin{Bmatrix} A,C \\ C \\ A,B \\ B,C \\ B,C \end{Bmatrix}$$

$\boxed{2.5\text{-}3}$

No rotational connection at the hinge — retain two θ_z d.o.f. there, so $\{\underset{\sim}{D}\} = \lfloor v \quad \theta_{za} \quad \theta_{zb} \rfloor^T$



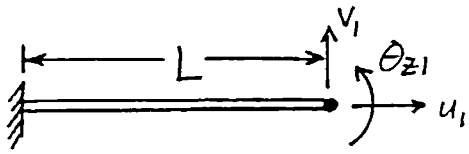
$$[\underset{\sim}{k}]_a = EI_z \begin{array}{c} \begin{bmatrix} 12/a^3 & -6/a^2 & 0 \\ -6/a^2 & 4/a & 0 \\ 0 & 0 & 0 \end{bmatrix} \end{array} \begin{array}{l} v \\ \theta_{za} \\ \theta_{zb} \end{array}$$



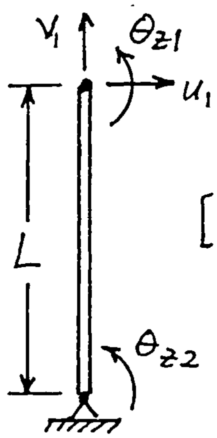
$$[\underset{\sim}{k}]_b = EI_z \begin{bmatrix} 12/b^3 & 0 & -6/b^2 \\ 0 & 0 & 0 \\ -6/b^2 & 0 & 4/b \end{bmatrix} \begin{array}{l} v \\ \theta_{za} \\ \theta_{zb} \end{array}$$

$$[K] = EI_z \begin{bmatrix} 12/a^3 + 12/b^3 & -6/a^2 & -6/b^2 \\ -6/a^2 & 4/a & 0 \\ -6/b^2 & 0 & 4/b \end{bmatrix} \begin{array}{l} v \\ \theta_{za} \\ \theta_{zb} \end{array}$$

$\boxed{2.5\text{-}4}$ $a = AE/L$ $b = EI/L^3$



$$[\underset{\sim}{k}] = \begin{array}{c} \left[\begin{array}{cccc} a & 0 & 0 & 0 \\ 0 & 12b & -6bL & 0 \\ 0 & -6bL & 4bL^2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \end{array} \begin{array}{l} u_1 \\ v_1 \\ \theta_{z1} \\ \theta_{z2} \end{array}$$

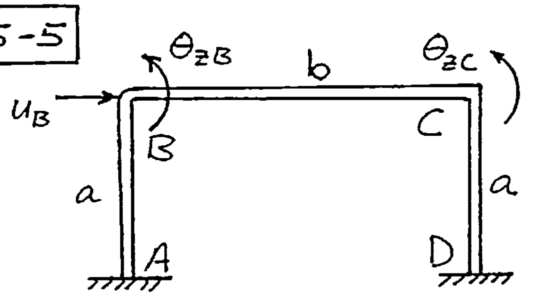


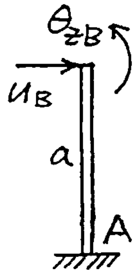
$$[\underset{\sim}{k}] = \left[\begin{array}{cccc} 12b & 0 & 6bL & 6bL \\ 0 & a & 0 & 0 \\ 6bL & 0 & 4bL^2 & 2bL^2 \\ 6bL & 0 & 2bL^2 & 4bL^2 \end{array}\right] \begin{array}{l} u_1 \\ v_1 \\ \theta_{z1} \\ \theta_{z2} \end{array}$$

Add ; get

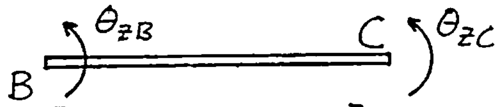
$$[\underset{\sim}{K}] = \left[\begin{array}{cccc} a+12b & 0 & 0 & 0 \\ 0 & a+12b & -6bL & 0 \\ 6bL & -6bL & 8bL^2 & 2bL^2 \\ 6bL & 0 & 2bL^2 & 4bL^2 \end{array}\right]$$

2.5-5

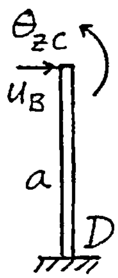




$$\frac{EI_z}{a^3}\begin{array}{c}\left[\begin{array}{ccc} 12 & 6a & 0 \\ 6a & 4a^2 & 0 \\ 0 & 0 & 0 \end{array}\right]\end{array}\begin{array}{l} u_B \\ \theta_{zB} \\ \theta_{zC} \end{array}$$



$$\frac{EI_z}{b^3}\left[\begin{array}{ccc} 0 & 0 & 0 \\ 0 & 4b^2 & 2b^2 \\ 0 & 2b^2 & 4b^2 \end{array}\right]\begin{array}{l} u_B \\ \theta_{zB} \\ \theta_{zC} \end{array}$$



$$\frac{EI_z}{a^3}\left[\begin{array}{ccc} 12 & 0 & 6a \\ 0 & 0 & 0 \\ 6a & 0 & 4a^2 \end{array}\right]\begin{array}{l} u_B \\ \theta_{zB} \\ \theta_{zC} \end{array}$$

$[\underset{\sim}{K}]$ = sum of the above

$$[\underset{\sim}{K}] = EI_z\left[\begin{array}{ccc} 24/a^3 & 6/a^2 & 6/a^2 \\ 6/a^2 & 4/a+4/b & 2/b \\ 6/a^2 & 2/b & 4/a+4/b \end{array}\right]\begin{array}{l} u_B \\ \theta_{zB} \\ \theta_{zC} \end{array}$$

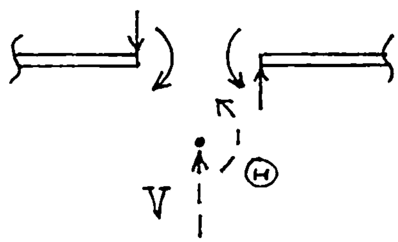
$\boxed{2.5\text{-}6}$

Sign changes in $[\underset{\sim}{k}]$ of Eq. 2.3-5:

- In columns 3 and 4 (because directions of d.o.f. are reversed)
- In rows 3 and 4 (because directions of loads are reversed)

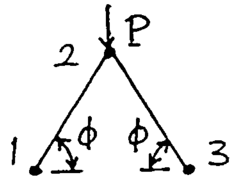
Thus k_{33} and k_{44} end up positive.

Awkward because d.o.f. don't match when elements are assembled.



If V and Θ are global d.o.f., must change signs again in left-hand element to make its $\{\underset{\sim}{d}\}$ match $\{\underset{\sim}{D}\}$.

$\boxed{2.5\text{-}7}$



$c = \cos\phi$

$s = \sin\phi$

Bar 1-2: Eq. 2.4-6, with $\beta = \phi$

$$[\underset{\sim}{k}] = \frac{AE}{L}\left[\begin{array}{cccc} c^2 & cs & -c^2 & -cs \\ cs & s^2 & -cs & -s^2 \\ -c^2 & -cs & c^2 & cs \\ -cs & -s^2 & cs & s^2 \end{array}\right]\begin{array}{c} u_1 \\ v_1 \\ u_2 \\ v_2 \end{array}$$

Bar 2-3: Eq. 2.4-6, with $\beta = -\phi$

$$[\underset{\sim}{k}] = \frac{AE}{L}\left[\begin{array}{cccc} c^2 & -cs & -c^2 & cs \\ -cs & s^2 & cs & -s^2 \\ -c^2 & cs & c^2 & -cs \\ cs & -s^2 & -cs & s^2 \end{array}\right]\begin{array}{c} u_2 \\ v_2 \\ u_3 \\ v_3 \end{array}$$

Omit rows and columns corresponding to fixed d.o.f. u_1, v_1, u_3, v_3 and sum overlapping terms at node 2.

$$\frac{AE}{L}\begin{bmatrix} 2c^2 & 0 \\ 0 & 2s^2 \end{bmatrix}\begin{Bmatrix} u_2 \\ v_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -P \end{Bmatrix} \quad \therefore \quad \begin{array}{l} u_2 = 0 \\ v_2 = -\dfrac{PL}{2AEs^2} \end{array}$$

Member elongation: $e = v_2 \sin\phi = v_2 s$

$$\epsilon = \frac{e}{L} = -\frac{P}{2AEs} \qquad \sigma = E\epsilon = -\frac{P}{2As}$$

as $\phi \to 0$

(according to linear small-displacement theory)

$\boxed{2.6\text{-}1}$

(a) $\lfloor u_1 \;\; v_1 \;\; \theta_{z1} \mid u_2 \;\; v_2 \;\; \theta_{z2} \rfloor =$

$\lfloor c_1 \;\; c_2 \;\; c_3 \mid c_1 \;\; c_2+Lc_3 \;\; c_3 \rfloor$

where c_1, c_2, c_3 are constants, and c_3 must be small.

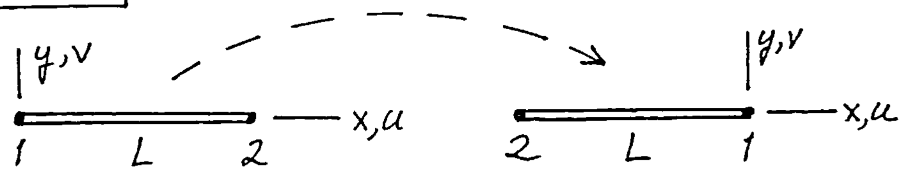
(b)

$\lfloor u_1 \;\; v_1 \;\; w_1 \;\; \theta_{x1} \;\; \theta_{y1} \;\; \theta_{z1} \mid u_2 \;\; v_2 \;\; w_2 \;\; \theta_{x2} \;\; \theta_{y2} \;\; \theta_{z2} \rfloor =$

$\lfloor c_1 \;\; c_2 \;\; c_3 \;\; c_4 \;\; c_5 \;\; c_6 \mid c_1 \;\; c_2+Lc_6 \;\; c_3-Lc_5 \;\; c_4 \;\; c_5 \;\; c_6 \rfloor$

where constants c_5 and c_6 must be small.

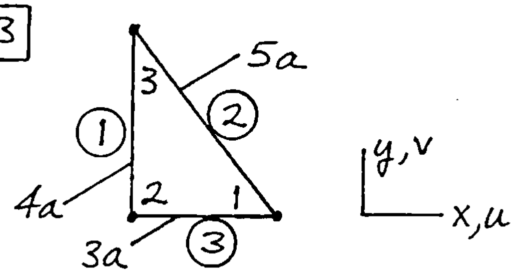
2.6-2



$$\lfloor u_1 \quad v_1 \quad \theta_{z1} \quad u_2 \quad v_2 \quad \theta_{z2} \rfloor = \lfloor 0 \quad 0 \quad \pi \quad -2L \quad 0 \quad \pi \rfloor$$

$[\underset{\sim}{k}]\{\underset{\sim}{d}\} \neq \{\underset{\sim}{0}\}$ because $[\underset{\sim}{k}]$ is based on the original (undeformed, undisplaced) geometry, and so is a good approximation only if deformations and rotational displacements are small.

$\boxed{2.6\text{-}3}$



(a) $\{\underset{\sim}{D}\}_1 = c_1 \lfloor -3 \;\; 4 \;\; -3 \;\; 4 \;\; -3 \;\; 4 \rfloor^T$

where c_1 is a constant

(b) $\{\underset{\sim}{D}\}_2 = c_2 \lfloor 4a \;\; 3a \;\; 4a \;\; 0 \;\; 0 \;\; 0 \rfloor^T$

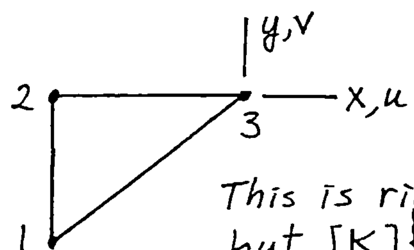
where c_2 is a small constant

(c) $\{\underset{\sim}{D}\}_3 = c_3 \lfloor 4a \;\; 0 \;\; 4a \;\; -3a \;\; 0 \;\; -3a \rfloor^T$

where c_3 is a small constant

(d) Yes: there are no values of the c_i such that the $\{\underset{\sim}{D}\}_i$ sum to zero.

(e) $\{\underset{\sim}{D}\} = \lfloor -7 \;\; -3 \;\; -4 \;\; 0 \;\; 0 \;\; -4 \rfloor^T$ gives the displaced structure as



This is rigid-body displacement, but $[\underset{\sim}{K}]\{\underset{\sim}{D}\} \neq \{\underset{\sim}{0}\}$ for the reason stated in Problem 2.6-2.

$\boxed{2.6\text{-}4}$

(a) We get row-sums (or column-sums, due to symmetry of $[\underset{\sim}{k}]$) from $[\underset{\sim}{k}]\{\underset{\sim}{d}\}$ with $\{\underset{\sim}{d}\} = \lfloor 1 \; 1 \; 1 \; 1 \; 1 \; 1 \rfloor^T$, but this $\{\underset{\sim}{d}\}$ is not rigid-body motion.

(b) $\{\underset{\sim}{d}\} = \lfloor c_1 \quad c_2 \quad c_3 \quad c_1 - Rc_3 \quad c_2 + Rc_3 \quad c_3 \rfloor^T$

where the c_i are constants and c_3 must be small.

(c) $$c_1 \begin{Bmatrix} 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{Bmatrix} \qquad c_2 \begin{Bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \end{Bmatrix} \qquad c_3 \begin{Bmatrix} 0 \\ 0 \\ 1 \\ -R \\ R \\ 1 \end{Bmatrix}$$

where c_3 is a small constant. The latter vector is rigid-body rotation about the node at $x = y = 0$.

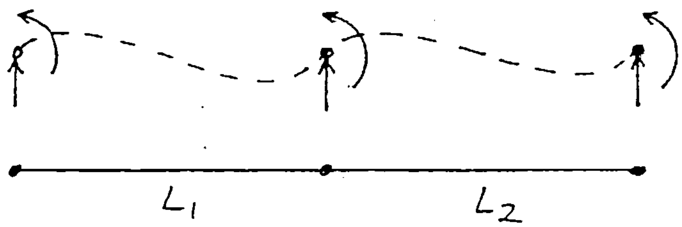
$\boxed{2.6\text{-}5}$

(a) We get row-sums (or column-sums, due to symmetry of $[\underset{\sim}{K}]$) from $[\underset{\sim}{K}]\{\underset{\sim}{D}\}$ with $\{\underset{\sim}{D}\} = \lfloor 1 \;\; 1 \;\; 1 \;\; 1 \;\; 1 \;\; 1 \rfloor^T$, but this $\{D\}$ is not rigid-body motion.

(b)
$$\{\underset{\sim}{D}\} = \lfloor c_1 \quad c_2 \quad c_1 + L_1 c_2 \quad c_2 \quad c_1 + (L_1 + L_2) c_2 \quad c_2 \rfloor^T$$

where the c_i are constants and c_2 is small.

(c)

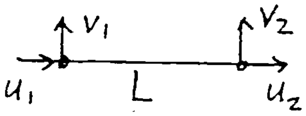


Direction of middle force arrow is upward, as shown, if $L_1 > L_2$.

$\boxed{2.6\text{-}6}$ $\{\underset{\sim}{d}\} = \lfloor u_1 \;\; v_1 \;\; u_2 \;\; v_2 \rfloor^T$

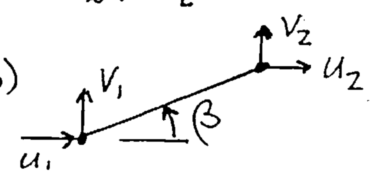
Let c_1, c_2, c_3 be constants, with c_3 a small rotation.

(a)



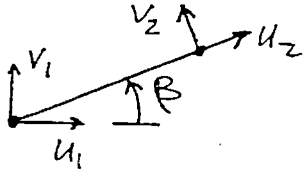
$$\{\underset{\sim}{d}\} = \lfloor c_1 \quad c_2 \quad c_1 \quad c_2 + L c_3 \rfloor^T$$

(b)



$$\{\underset{\sim}{d}\} = \lfloor c_1 \quad c_2 \quad c_1 - c_3 L \sin\beta \quad c_2 + c_3 L \cos\beta \rfloor^T$$

(c)

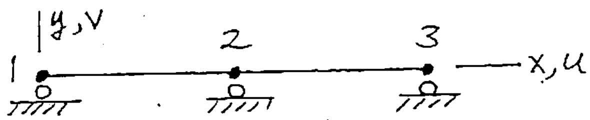


$$\{\underset{\sim}{d}\} = \lfloor c_1 \quad c_2 \quad c_1 \cos\beta + c_2 \sin\beta \quad -c_1 \sin\beta + c_2 \cos\beta + c_3 L \rfloor^T$$

2.6-7

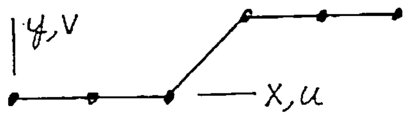
(a) With n_R restraints it is possible to overconstrain one motion while at the same time underconstraining another. Examples, using beam elements:

2D: $n_R = 3$, d.o.f. u, v, θ_z at each node



Only v_1, v_2, v_3 constrained

3D:

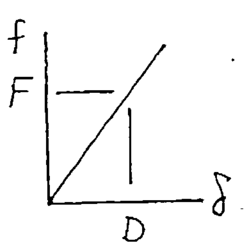


$n_R = 6$, d.o.f. $u, v, w, \theta_x, \theta_y, \theta_z$ at each node.
Only θ_x (or θ_y, θ_z) restrained at each node.

(b) Always restrain u, v, θ_z at node 1.
Also restrain:
(1) w, θ_x, θ_y at node 1, or
(2) w, θ_x, θ_y at node 2, or
(3) w, θ_x, θ_y at node 3, or
(4) w at nodes 1 and 2, θ_x at node 2, or
(5) w at nodes 1 and 2, θ_x at node 3, or
(6) w at node 2, θ_x at node 1, θ_y at node 3,
Etc.

2.6-8

Strain energy in a linearly elastic structure due to gradually applied loads $\{\underset{\sim}{R}\}$ is $\{\underset{\sim}{D}\}^T\{\underset{\sim}{R}\}/2$, e.g. for a single load F,



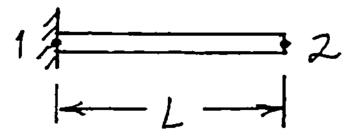
$$U = \text{work} = \int_0^D f \, d\delta = \frac{1}{2} F D$$

But $\{\underset{\sim}{R}\} = [\underset{\sim}{K}]\{\underset{\sim}{D}\}$, so

$$U = \frac{1}{2}\{\underset{\sim}{D}\}^T[\underset{\sim}{K}]\{\underset{\sim}{D}\}$$

$\boxed{2.7\text{-}1}$

(a)



From Eq. 2.3-5,

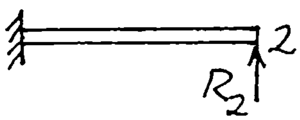
$$EI_z\begin{bmatrix}12/L^3 & -6/L^2\\ -6/L^2 & 4/L\end{bmatrix}\begin{Bmatrix}v_2\\ \theta_{z2}\end{Bmatrix}=\begin{Bmatrix}R_2\\ M_2\end{Bmatrix} \qquad (A)$$

Set $v_2 = \bar{v}_2$

$$EI_z\begin{bmatrix}1 & 0\\ 0 & 4/L\end{bmatrix}\begin{Bmatrix}v_2\\ \theta_{z2}\end{Bmatrix}=\begin{Bmatrix}EI_z\bar{v}_2\\ 6EI_z\bar{v}_2/L^2\end{Bmatrix} \text{ gives } \theta_{z2}=\frac{3\bar{v}_2}{2L},\ v_2=\bar{v}_2$$

Eq. (A) then gives $R_2 = EI_z\left[\frac{12}{L^3}\bar{v}_2-\frac{6}{L^2}\frac{3\bar{v}_2}{2L}\right]=\frac{3EI_z\bar{v}_2}{L^3}$

Beam theory:

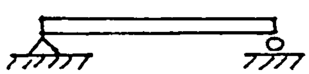


$$\underbrace{v_2=\frac{R_2L^3}{3EI_z},\quad \theta_{z2}=\frac{R_2L^2}{2EI_z}}$$

$$\theta_{z2}=\frac{L^2}{2EI_z}\,\frac{3EI_z}{L^3}v_2=\frac{3v_2}{2L}$$

$$R_2=\frac{2EI_z}{L^2}\theta_{z2}=\frac{3EI_z}{L^3}v_2$$

(b)



From Eq. 2.3-5,

$$\frac{2EI_z}{L}\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}\begin{Bmatrix}\theta_{z1}\\ \theta_{z2}\end{Bmatrix}=\begin{Bmatrix}M_1\\ M_2\end{Bmatrix} \qquad (B)$$

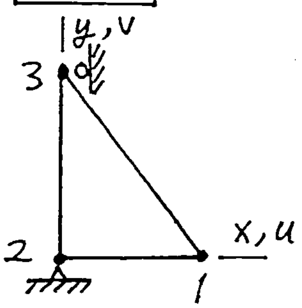
Set $\theta_{z1}=\bar{\theta}_{z1}$

$$\frac{2EI_z}{L}\begin{bmatrix}1 & 0\\ 0 & 2\end{bmatrix}\begin{Bmatrix}\theta_{z1}\\ \theta_{z2}\end{Bmatrix}=\begin{Bmatrix}2EI_z\bar{\theta}_{z1}/L\\ -2EI_z\bar{\theta}_{z1}/L\end{Bmatrix} \text{ gives } \theta_{z1}=\bar{\theta}_{z1},\ \theta_{z2}=-\frac{\bar{\theta}_{z1}}{2}$$

Eq. (B) then gives $M_1=\frac{2EI_z}{L}\left[2\bar{\theta}_{z1}-\frac{\bar{\theta}_{z1}}{2}\right]=\frac{3EI_z}{L}\bar{\theta}_{z1}$

These results appear in tables of beam deflections.

2.7-2



For $k_1 = k_2 = k_3 = k$, $P = 0$, Eq. 2.7-8 is

$$k\begin{bmatrix} 1.36 & -0.48 & 0.48 \\ -0.48 & 0.64 & -0.64 \\ 0.48 & -0.64 & 1.64 \end{bmatrix}\begin{Bmatrix} u_1 \\ v_1 \\ v_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \qquad (A)$$

Impose $u_1 = c, v_1 = 0$ by method of Eq. 2.7-6

$$k\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1.64 \end{bmatrix}\begin{Bmatrix} u_1 \\ v_1 \\ v_3 \end{Bmatrix} = \begin{Bmatrix} ck \\ 0 \\ -0.48ck \end{Bmatrix} \quad \therefore \quad \begin{Bmatrix} u_1 \\ v_1 \\ v_3 \end{Bmatrix} = \begin{Bmatrix} c \\ 0 \\ -0.2927c \end{Bmatrix}$$

Use the latter vector on the left-hand side of Eq. (A).

$$k\begin{bmatrix} 1.36 & -0.48 & 0.48 \\ -0.48 & 0.64 & -0.64 \\ 0.48 & -0.64 & 1.64 \end{bmatrix}\begin{Bmatrix} c \\ 0 \\ -0.2927c \end{Bmatrix} = \begin{Bmatrix} 1.22ck \\ -0.2927ck \\ 0 \end{Bmatrix} \begin{matrix} \leftarrow F_{x1} \\ \leftarrow F_{y1} \\ \end{matrix}$$

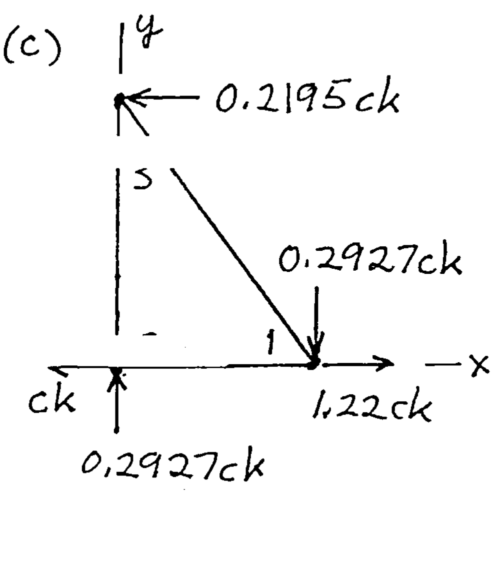
(b) Return to Eq. 2.5-10 with $k_1 = k_2 = k_3 = k$ and

$$\{\underset{\sim}{D}\} = \lfloor c \quad 0 \quad 0 \quad 0 \quad 0 \quad -0.2927c \rfloor^T$$

Hence

$$[\underset{\sim}{K}]\{\underset{\sim}{D}\} = \{\underset{\sim}{R}\} = ck\lfloor 1.22 \quad -0.2927 \quad -1 \quad 0.2927 \quad -0.2195 \quad 0 \rfloor^T$$

(c)



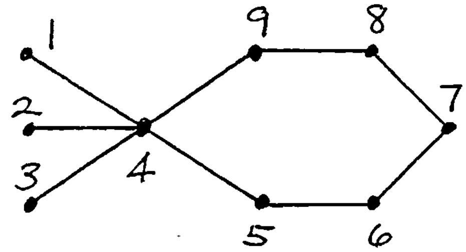
$\Sigma F_x = 0$ ✓

$\Sigma F_y = 0$ ✓

$\Sigma M_2 = (0.2195ck)4 - (0.2927ck)3$

$= 0$ ✓

$\boxed{2.8-1}$

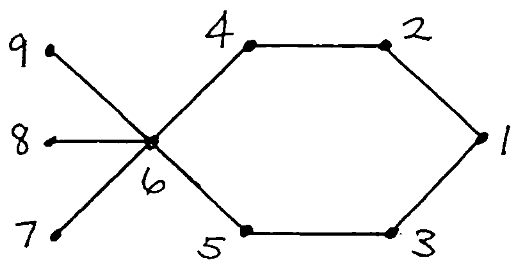


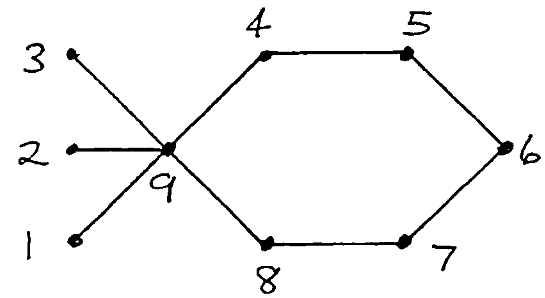
$$
\begin{array}{c}
\left[\begin{array}{ccccccccc}
x & & & x & & & & & \\
 & x & & x & & & & & \\
 & & x & x & & & & & \\
 & & & x & x & & & & x \\
 & & & & x & x & & & \\
 & & & & & x & x & & \\
 & \text{symmetric} & & & & & x & x & \\
 & & & & & & & x & x \\
 & & & & & & & & x
\end{array}\right]
\begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \end{array} \\
\begin{array}{ccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{array}
\end{array}
$$

$b_{max} = 6$ (in row 4)

fills $= 3$ (in column 9)

$\boxed{2.8\text{-}2}$





$$
\begin{array}{c}
\begin{array}{ccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{array} \\
\left[\begin{array}{ccccccccc}
\times & \times & \times & & & & & & \\
 & \times & & \times & & & & & \\
 & & \times & & \times & & & & \\
 & & & \times & & \times & & & \\
 & & & & \times & \times & & & \\
 & & & & & \times & \times & \times & \times \\
 & & & & & & \times & & \\
 \text{symm.} & & & & & & & \times & \\
 & & & & & & & & \times
\end{array}\right]
\end{array}
\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \end{array}
$$

$p = 24$

$b_{max} = 4$ (row 6)

fills = 6

$$
\begin{array}{c}
\begin{array}{ccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \end{array} \\
\left[\begin{array}{ccccccccc}
\times & & & & & & & & \times \\
 & \times & & & & & & & \times \\
 & & \times & & & & & & \times \\
 & & & \times & \times & & & & \times \\
 & & & & \times & \times & & & \\
 & & & & & \times & \times & & \\
 & & & & & & \times & \times & \\
 \text{symm.} & & & & & & & \times & \times \\
 & & & & & & & & \times
\end{array}\right]
\end{array}
\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \end{array}
$$

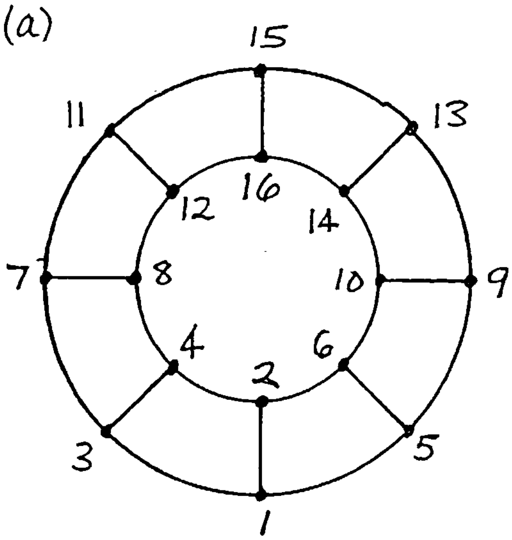
$p = 21$

$b_{max} = 9$ (row 1)

fills = 3

$\boxed{2.8\text{-}3}$

(a)



(candidate numbering)

$b_{max} = 5$

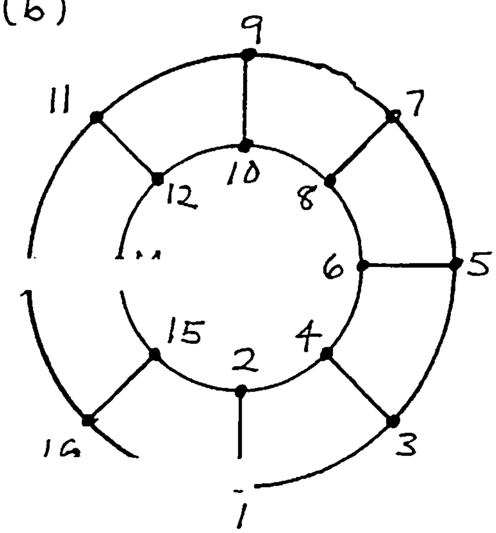
$p = 12(5) + 3 + 3 + 2 + 1 = 69$

fills $= 29$

$$
\begin{array}{cccccccccccccccc|c}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & & 12 & & 14 & & 16 & \\
\hline
\times & \times & \times & & \times & & & & & & & & & & & & 1 \\
 & \times & & \times & & \times & & & & & & & & & & & 2 \\
 & & \times & \times & & & \times & & & & & & & & & & 3 \\
 & & & \times & & & & \times & & & & & & & & & 4 \\
 & & & & \times & \times & & & \times & & & & & & & & 5 \\
 & & & & & \times & & & & \times & & & & & & & 6 \\
 & & & & & & \times & \times & & & \times & & & & & & 7 \\
 & & & & & & & \times & & & & \times & & & & & 8 \\
 & & & & & & & & \times & \times & & & \times & & & & 9 \\
 & & & & & & & & & \times & & & & \times & & & 10 \\
 & & & & & & & & & & \times & \times & & & \times & & 11 \\
 & & & & & & & & & & & \times & & & & \times & 12 \\
 & & & & & & & & & & & & \times & \times & \times & & 13 \\
 & & & & & & & & & & & & & \times & & \times & 14 \\
 & & & & & & & & & & & & & & \times & \times & 15 \\
 & & & & & & & & & & & & & & & \times & 16
\end{array}
$$

symmetric

(b)



(candidate numbering)

$b_{max} = 16$ (row 1)

$p = 12(3) + 14 + 16 = 66$

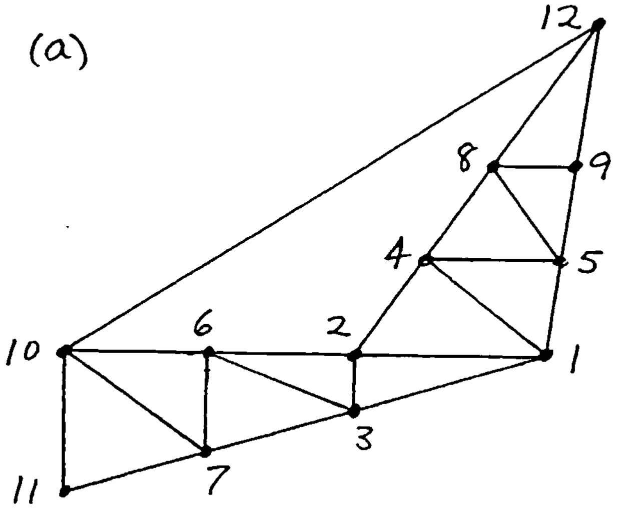
fills $= 6 + 11 + 12 = 29$

$$
\begin{array}{cccccccccccccccc|c}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & & 12 & & 14 & & 16 & \\
\hline
\times & \times & \times & & & & & & & & & & & & & \times & 1 \\
 & \times & & \times & & & & & & & & & & & \times & & 2 \\
 & & \times & \times & \times & & & & & & & & & & & & 3 \\
 & & & \times & & \times & & & & & & & & & & & 4 \\
 & & & & \times & \times & \times & & & & & & & & & & 5 \\
 & & & & & \times & & \times & & & & & & & & & 6 \\
 & & & & & & \times & \times & \times & & & & & & & & 7 \\
 & & & & & & & \times & & \times & & & & & & & 8 \\
 & & & & & & & & \times & \times & \times & & & & & & 9 \\
 & & & & & & & & & \times & & \times & & & & & 10 \\
 & & & & & & & & & & \times & \times & \times & & & & 11 \\
 & & & & & & & & & & & \times & & \times & & & 12 \\
 & & & & & & & & & & & & \times & \times & & \times & 13 \\
 & & & & & & & & & & & & & \times & \times & & 14 \\
 & & & & & & & & & & & & & & \times & \times & 15 \\
 & & & & & & & & & & & & & & & \times & 16
\end{array}
$$

symmetric

$\boxed{2.8-4}$

(a)



(candidate numbering)

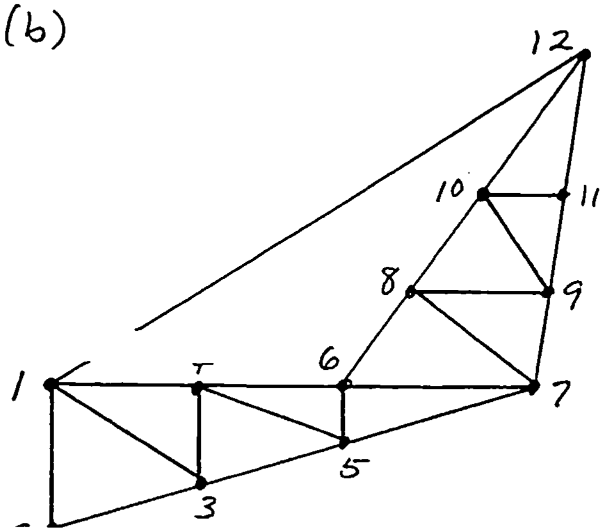
$$
\begin{array}{cccccccccccc|c}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & & 12 & \\
\hline
X & X & X & X & X & & & & & & & & 1 \\
 & X & X & X & & X & & & & & & & 2 \\
 & & X & & & X & X & & & & & & 3 \\
 & & & X & X & & & X & & & & & 4 \\
 & & & & X & & & X & X & & & & 5 \\
 & & & & & X & X & & & X & & & 6 \\
 & & & & & & X & & & X & X & & 7 \\
 & & & & & & & X & X & & & X & 8 \\
 & & & & & & & & X & & & X & 9 \\
\multicolumn{4}{l}{\text{symmetric}} & & & & & & X & X & X & 10 \\
 & & & & & & & & & & X & & 11 \\
 & & & & & & & & & & & X & 12
\end{array}
$$

$b_{max} = 5$

$p = 8(5) + 4 + 3 + 2 + 1 = 50$

fills $= 16$

(b)



(candidate numbering)

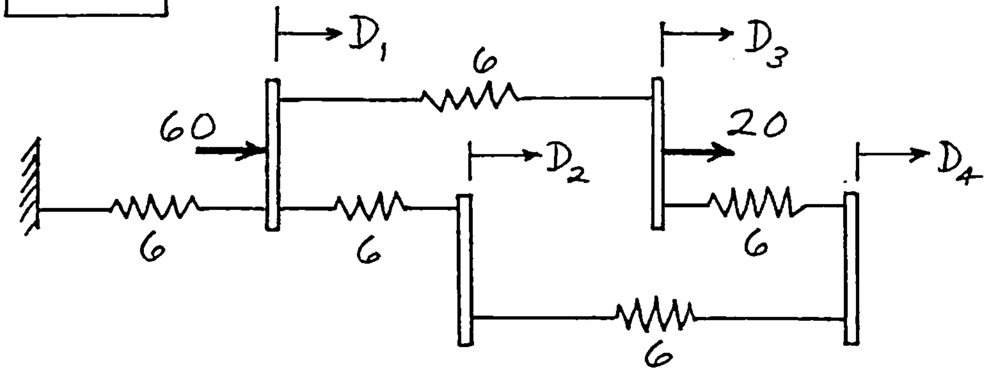
$$
\begin{array}{cccccccccccc|c}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & & 12 & \\
\hline
X & X & X & X & & & & & & & & X & 1 \\
 & X & X & & & & & & & & & & 2 \\
 & & X & X & X & & & & & & & & 3 \\
 & & & X & X & X & & & & & & & 4 \\
 & & & & X & X & X & & & & & & 5 \\
 & & & & & X & X & X & & & & & 6 \\
 & & & & & & X & X & X & & & & 7 \\
 & & & & & & & X & X & X & & & 8 \\
 & & & & & & & & X & X & X & & 9 \\
\multicolumn{4}{l}{\text{symmetric}} & & & & & & X & X & X & 10 \\
 & & & & & & & & & & X & X & 11 \\
 & & & & & & & & & & & X & 12
\end{array}
$$

$b_{max} = 12$

$p = 8(3) + 4 + 2 + 1 + 12 = 43$

fills $= 9$

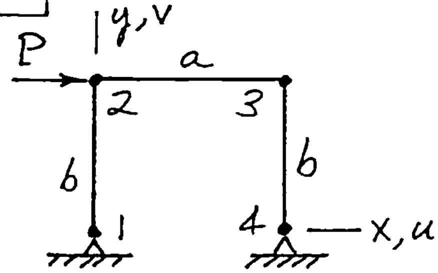
2.8-5



Spring k's and nodal loads shown by numbers.

$\boxed{2.8\text{-}6}$

(a)



$$AE\begin{bmatrix} 1/a & 0 & -1/a & 0 \\ 0 & 1/b & 0 & 0 \\ -1/a & 0 & 1/a & 0 \\ 0 & 0 & 0 & 1/b \end{bmatrix}\begin{Bmatrix} u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix} = \begin{Bmatrix} P \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

After the first elimination,

$$AE\begin{bmatrix} 1/a & 0 & -1/a & 0 \\ 0 & 1/b & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/b \end{bmatrix}\begin{Bmatrix} u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix} = \begin{Bmatrix} P \\ 0 \\ P \\ 0 \end{Bmatrix}$$

Trouble will appear in the third elimination (the u_3 equation).

(b)

$$\frac{EI_z}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} v_1 \\ \theta_{z1} \\ v_2 \\ \theta_{z2} \end{Bmatrix} = \begin{Bmatrix} P \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

After the first elimination,

$$\frac{EI_z}{L^3}\begin{bmatrix} \cdots & 6L & -12 & 6L \\ 0 & L^2 & 0 & -L^2 \\ 0 & 0 & 0 & 0 \\ 0 & -L^2 & 0 & L^2 \end{bmatrix}\begin{Bmatrix} v_1 \\ \theta_{z1} \\ v_2 \\ \theta_{z2} \end{Bmatrix} = \begin{Bmatrix} P \\ -PL/2 \\ P \\ -PL/2 \end{Bmatrix}$$

Trouble will appear in the third elimination (the v_2 equation).

In both cases, trouble appears in the first equation for which restraint becomes necessary to prevent rigid body motion or a mechanism. If no mechanism is possible and d.o.f. include rotations as well as displacements, the offending equation involves the last-numbered node, since full restraint at this node would suffice to prevent rigid body motion.

2.8-7

(a) Given system

$$\begin{bmatrix} 12 & -6 & 0 \\ -6 & 12 & -6 \\ 0 & -6 & 6 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} 24 \\ 24 \\ 0 \end{Bmatrix}$$

1st elimination

$$\begin{bmatrix} 12 & -6 & 0 \\ 0 & 9 & -6 \\ 0 & -6 & 6 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} 24 \\ 36 \\ 0 \end{Bmatrix}$$

2nd elimination

$$\begin{bmatrix} 12 & -6 & 0 \\ 0 & 9 & -6 \\ 0 & 0 & 2 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} 24 \\ 36 \\ 24 \end{Bmatrix}$$

Back-substitution

$u_4 = 24/2 = 12$

$u_3 = (36 + 6u_4)/9 = (36+72)/9 = 12$

$u_2 = (24 + 6u_3)/12 = (24+72)/12 = 8$

(b) After the first elimination, $K_{22} = 9$. This is the stiffness seen by u_3 when node 2 is free to move, so members 1 and 2 are in series (for which $k = 3$). Stiffness $k = 3$ adds to stiffness $k = 6$ of member 3. After the second elimination, $K_{33} = 2$. This is the stiffness seen by node 4 when nodes 2 and 3 are freed, so that node 4 sees all three members in series, for which $\frac{1}{k} = \frac{1}{6} + \frac{1}{6} + \frac{1}{6}$ and hence $k = 2$.

(c)

1st iteration	2nd iteration	3rd iteration	4th iteration
$u_2 = 2$	$u_2 = 3.50$	$u_2 = 4.625$	$u_2 = 5.469$
$u_3 = 3$	$u_3 = 5.25$	$u_3 = 6.9375$	$u_3 = 8.203$
$u_4 = 3$	$u_4 = 5.25$	$u_4 = 6.9375$	$u_4 = 8.203$

$\boxed{2.8-8}$

(a) Given eqs.

$$k\begin{bmatrix} 3 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 2 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ F_2 \\ 0 \end{Bmatrix}$$

1^{st} elimination

$$k\begin{bmatrix} 3 & -1 & -1 \\ 0 & 2/3 & -1/3 \\ 0 & -1/3 & 5/3 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ F_2 \\ 0 \end{Bmatrix}$$

2^{nd} elimination

$$k\begin{bmatrix} 3 & -1 & -1 \\ 0 & 2/3 & -1/3 \\ 0 & 0 & 3/2 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ F_2 \\ F_2/2 \end{Bmatrix}$$

Back substitution

$$ku_3 = \frac{1}{3}F_2 \quad , \quad u_3 = \frac{F_2}{3k}$$

$$ku_2 = \frac{3}{2}\left[F_2 + \frac{1}{3}\frac{F_2}{3}\right] \quad , \quad u_2 = \frac{5F_2}{3k}$$

$$ku_1 = \frac{1}{3}\left[\frac{5F_2}{3} + \frac{F_2}{3}\right] \quad , \quad u_1 = \frac{2F_2}{3k}$$

(b) Given $\{\underset{\sim}{R}\}$ 1^{st} elimination 2^{nd} elimination

$$\begin{Bmatrix} F_1 \\ 0 \\ 0 \end{Bmatrix} \longrightarrow \begin{Bmatrix} F_1 \\ F_1/3 \\ F_1/3 \end{Bmatrix} \longrightarrow \begin{Bmatrix} F_1 \\ F_1/3 \\ F_1/3 + F_1/6 \end{Bmatrix} = F_1\begin{Bmatrix} 1 \\ 1/3 \\ 1/2 \end{Bmatrix}$$

Back substitution

$$ku_3 = \frac{2}{3}\frac{F_1}{2} = \frac{F_1}{3}$$

$$ku_2 = \frac{3}{2}\left[\frac{F_1}{3} + \frac{1}{3}\frac{F_1}{3}\right] = \frac{2F_1}{3}$$

$$ku_1 = \frac{1}{3}\left[F_1 + \frac{2F_1}{3} + \frac{F_1}{3}\right] = \frac{2F_1}{3}$$

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \frac{F_1}{k}\begin{Bmatrix} 2/3 \\ 2/3 \\ 1/3 \end{Bmatrix}$$

$\boxed{2.8-9}$ Given eqs. $\qquad$ 1st elimination

(a)

$$k\begin{bmatrix}3 & -1 & -1 & 0\\ -1 & 3 & -1 & -1\\ -1 & -1 & 3 & -1\\ 0 & -1 & -1 & 2\end{bmatrix}\begin{Bmatrix}u_1\\u_2\\u_3\\u_4\end{Bmatrix}=\begin{Bmatrix}0\\F_2\\0\\0\end{Bmatrix},\quad k\begin{bmatrix}3 & -1 & -1 & 0\\ 0 & 8/3 & -4/3 & -1\\ 0 & -4/3 & 8/3 & -1\\ 0 & -1 & -1 & 2\end{bmatrix}\begin{Bmatrix}u_1\\u_2\\u_3\\u_4\end{Bmatrix}=\begin{Bmatrix}0\\F_2\\0\\0\end{Bmatrix}$$

2nd elimination $\qquad$ 3rd elimination

$$k\begin{bmatrix}3 & -1 & -1 & 0\\ 0 & 8/3 & -4/3 & -1\\ 0 & 0 & 2 & -3/2\\ 0 & 0 & -3/2 & 13/8\end{bmatrix}\begin{Bmatrix}u_1\\u_2\\u_3\\u_4\end{Bmatrix}=\begin{Bmatrix}0\\F_2\\F_2/2\\3F_2/8\end{Bmatrix},\quad k\begin{bmatrix}3 & -1 & -1 & 0\\ 0 & 8/3 & -4/3 & -1\\ 0 & 0 & 2 & -3/2\\ 0 & 0 & 0 & 1/2\end{bmatrix}\begin{Bmatrix}u_1\\u_2\\u_3\\u_4\end{Bmatrix}=\begin{Bmatrix}0\\F_2\\F_2/2\\3F_2/4\end{Bmatrix}$$

Back substitution

$$ku_4=\frac{3F_2}{2}$$
$$ku_3=\frac{1}{2}\left[\frac{F_2}{2}+\frac{3}{2}\,\frac{3F_2}{2}\right]=\frac{11F_2}{8}$$
$$ku_2=\frac{3}{8}\left[F_2+\frac{4}{3}\,\frac{11F_2}{8}+\frac{3F_2}{2}\right]=\frac{13F_2}{8}$$
$$ku_1=\frac{1}{3}\left[\frac{13F_2}{8}+\frac{11F_2}{8}\right]=F_2$$

$$\begin{Bmatrix}u_1\\u_2\\u_3\\u_4\end{Bmatrix}=\frac{F_3}{k}\begin{Bmatrix}1.000\\1.625\\1.375\\1.500\end{Bmatrix}$$

(b) Given $\{\underset{\sim}{R}\}$ $\quad$ 1st elim. $\quad$ 2nd elim. $\quad$ 3rd elim.

$$\begin{Bmatrix}F_1\\0\\0\\0\end{Bmatrix}\longrightarrow\begin{Bmatrix}F_1\\F_1/3\\F_1/3\\0\end{Bmatrix}\longrightarrow\begin{Bmatrix}F_1\\F_1/3\\F_1/3+F_1/6\\F_1/8\end{Bmatrix}\longrightarrow\begin{Bmatrix}F_1\\F_1/3\\F_1/2\\F_1/8+3F_1/8\end{Bmatrix}=F_1\begin{Bmatrix}1\\1/3\\1/2\\1/2\end{Bmatrix}$$

Back substitution

$$ku_4=F_1$$
$$ku_3=\frac{1}{2}\left[\frac{F_1}{2}+\frac{3F_1}{2}\right]=F_1$$
$$ku_2=\frac{3}{8}\left[\frac{F_1}{3}+\frac{4}{3}F_1+F_1\right]=F_1$$
$$ku_1=\frac{1}{3}\left[F_1+F_1+F_1\right]=F_1$$

$$\begin{Bmatrix}u_1\\u_2\\u_3\\u_4\end{Bmatrix}=\frac{F_1}{k}\begin{Bmatrix}1\\1\\1\\1\end{Bmatrix}$$

$\boxed{2.8\text{-}10}$

(a)

$$\begin{bmatrix} 1.36 & -0.48 & 0.48 \\ -0.48 & 0.64 & -0.64 \\ 0.48 & -0.64 & 1.64 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ v_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -P/k \\ 0 \end{Bmatrix}$$

Divide row 1 by 1.36, then mult. by 0.48 & −0.48 resp.; add to rows 2 & 3.

$$\begin{bmatrix} 1 & -0.3529 & 0.3529 \\ 0 & 0.4706 & -0.4706 \\ 0 & -0.4706 & 1.4706 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ v_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -P/k \\ 0 \end{Bmatrix}$$

Add row 2 to row 3, then divide row 2 by 0.4706.

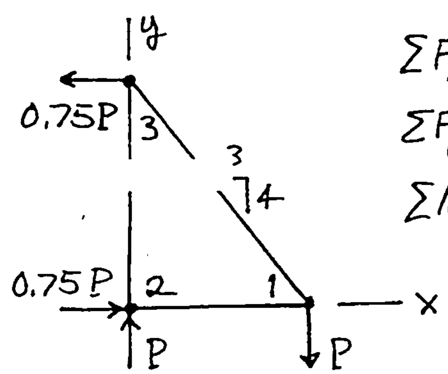
$$\begin{bmatrix} 1 & -0.3529 & 0.3529 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ v_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -2.125P/k \\ -P/k \end{Bmatrix}$$

Back-substitute.

$$v_3 = -P/k$$
$$v_1 = -2.125P/k - P/k = -3.125P/k$$
$$u_1 = 0.3529(-3.125P/k) - 0.3529(-P/k)$$
$$= -0.75P/k$$

(b) Use Eq. 2.5-10

$$p_2 = k\left[-\left(-0.75\frac{P}{k}\right) + 0\right] = 0.75P$$
$$q_2 = k\left[0 - \left(-\frac{P}{k}\right)\right] = P$$
$$p_3 = k\left[-0.36\left(-0.75\frac{P}{k}\right) + 0.48\left(-3.125\frac{P}{k}\right) + 0.36(0) - 0.48\left(-\frac{P}{k}\right)\right] = -0.75P$$



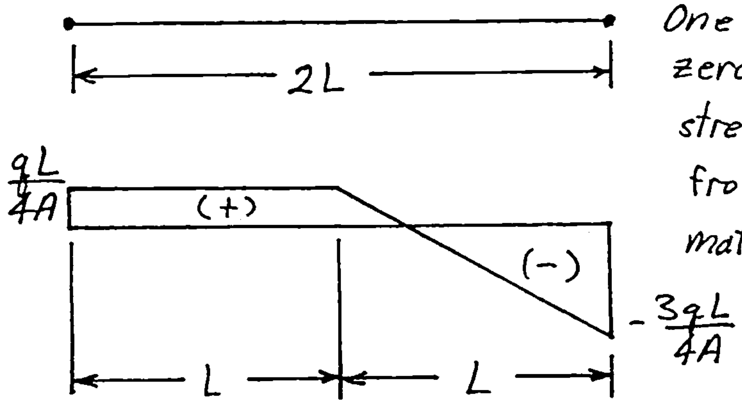
$\Sigma F_x = 0$ ✓

$\Sigma F_y = 0$ ✓

$\Sigma M_2 = 0.75P(4) - P(3) = 0$ ✓

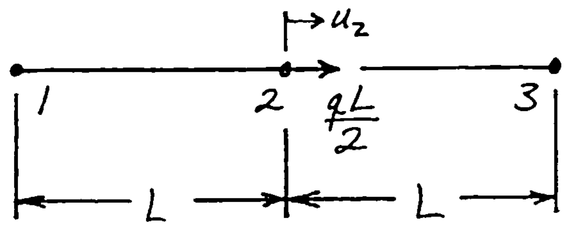
2.9-1

(a)



One element – nodal d.o.f. are zero, so the only contribution to stress is element stress, which from elementary mechanics of materials is as shown.

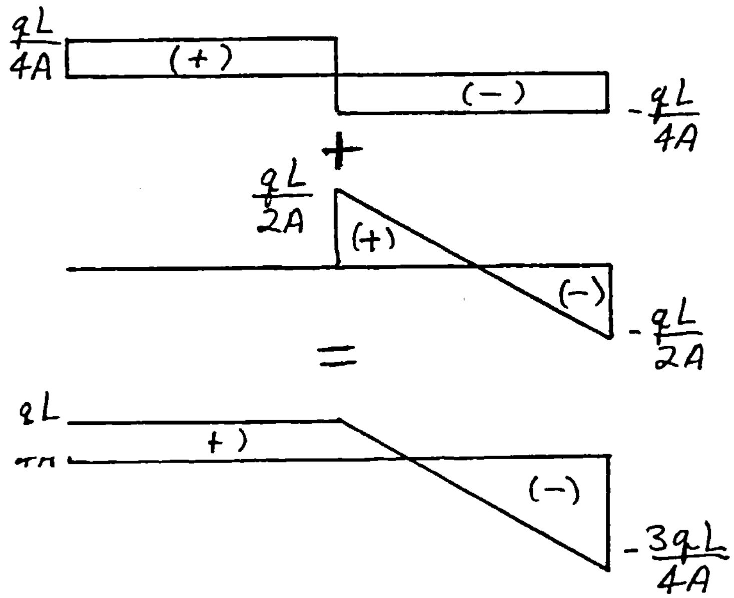
(b)



$$u_2 = \frac{qL/2}{2AE/L} = \frac{qL^2}{4AE}$$

$$\epsilon_{1\text{-}2} = \frac{u_2}{L}, \quad \sigma_{1\text{-}2} = E\epsilon_{1\text{-}2} = \frac{qL}{4A}$$

$$\epsilon_{2\text{-}3} = -\frac{u_2}{L}, \quad \sigma_{2\text{-}3} = E\epsilon_{2\text{-}3} = -\frac{qL}{4A}$$



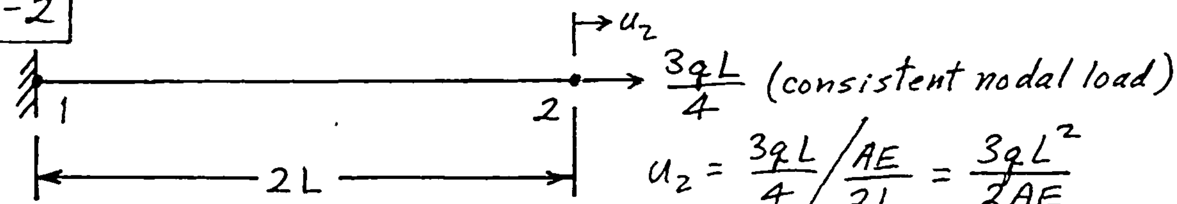
Stress due to u_2

Element stress

Sum of the two

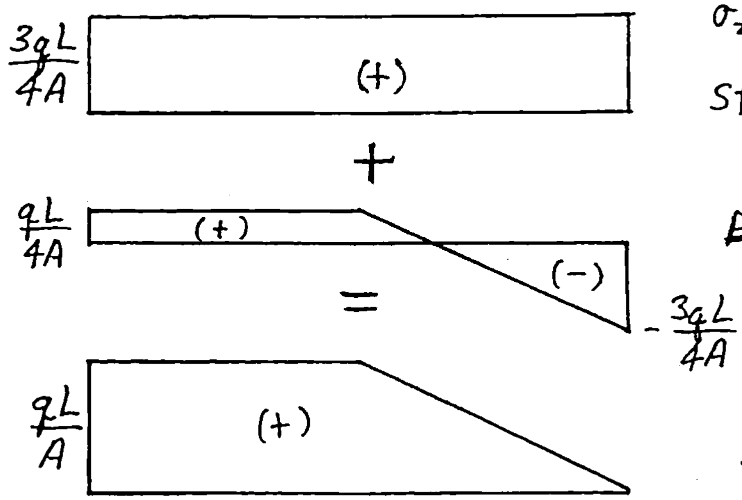
$\boxed{2.9\text{-}2}$

(a)



$$u_2 = \frac{3qL}{4} \Big/ \frac{AE}{2L} = \frac{3qL^2}{2AE}$$

$$\sigma_2 = E\,\frac{u_2}{2L} = \frac{3qL}{4A}$$

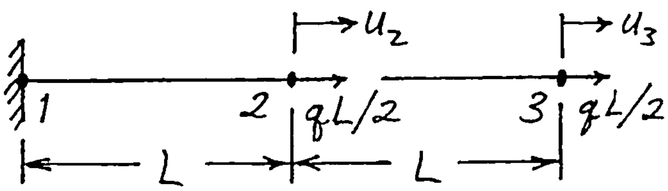


Stress due to u_2

Element stress

Sum of the two

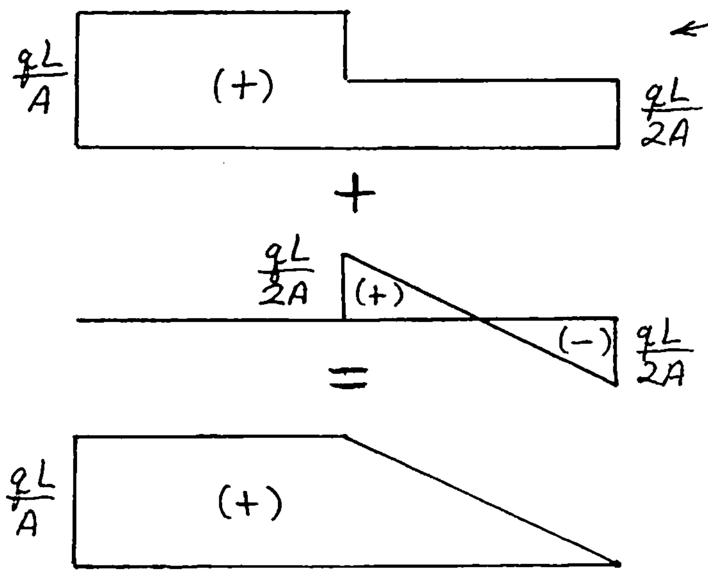
(b)



$$u_2 = \frac{qL}{AE/L} = \frac{qL^2}{AE}$$

$$u_3 = u_2 + \frac{qL/2}{AE/L} = \frac{3qL^2}{2AE}$$

$$\left\{ \begin{array}{l} \sigma_{1\text{-}2} = E\,\frac{u_2}{L} = \frac{qL}{A} \\ \sigma_{2\text{-}3} = E\,\frac{u_3 - u_2}{L} = \frac{qL}{2A} \end{array} \right.$$

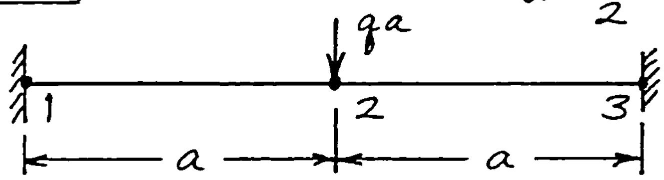


Element stress

Sum of the two

$\boxed{2.9\text{-}3}$

$a = \frac{L}{2}$

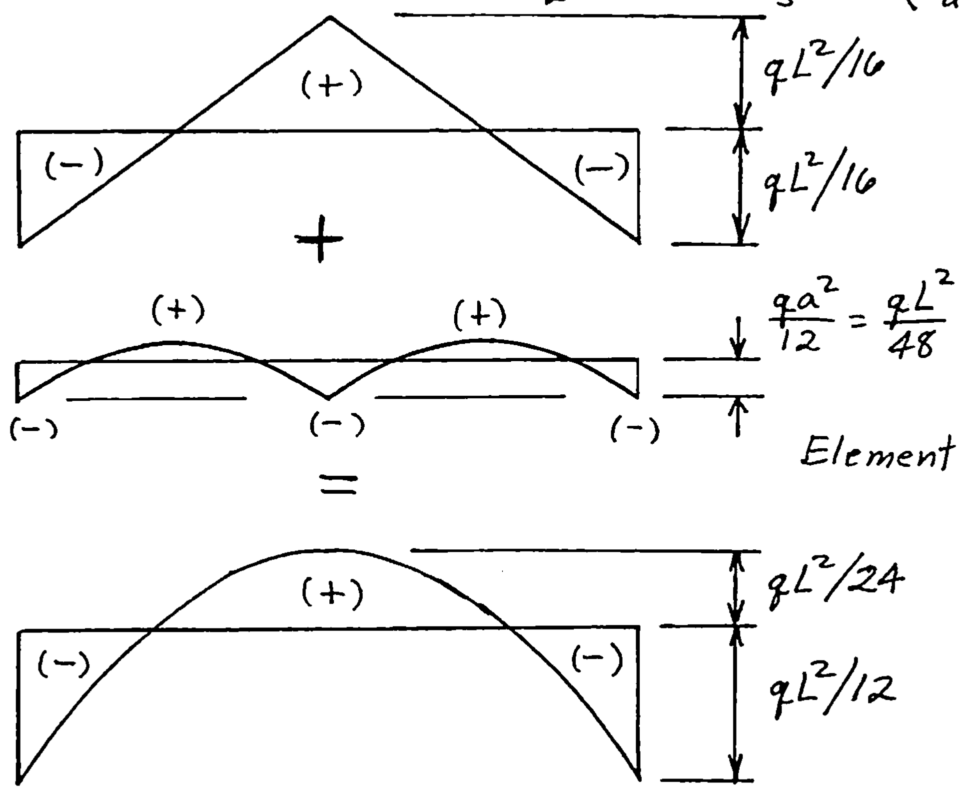


Vertical deflection at center node is the only nonzero d.o.f.

$$2\,\frac{12EI}{a^3}\,v_2 = -qa, \quad v_2 = -\frac{qa^4}{24EI}$$

$$\text{or } v_2 = -\frac{qL^4}{384EI}$$

$$M_2 = -M_1 = -M_3 = EI\left(-\frac{6}{a^2}\right)v_2 = \frac{qa^2}{4} = \frac{qL^2}{16}$$

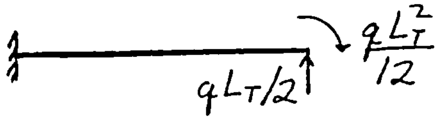


Element moments

Sum of the two (exact)

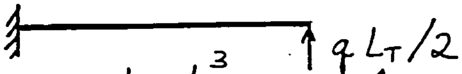
$\boxed{2.9\text{-}4}$

(a) Exact: $v = \frac{qL_T^4}{8EI}$, $M = \frac{qL_T^2}{2}$



$$v = \frac{qL_T}{2}\frac{L_T^3}{3EI} - \frac{qL_T^2}{12}\frac{L_T^2}{2EI} = \frac{qL_T^4}{8EI} \quad \text{exact}$$

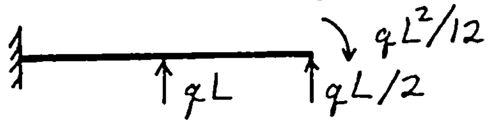
$$M = \frac{qL_T}{2}L_T - \frac{qL_T^2}{12} = 0.417qL_T^2 \quad -16.7\%$$



$$v = \frac{qL_T}{2}\frac{L_T^3}{3EI} = \frac{qL_T^4}{6EI} \quad \text{———} \quad +33.3\%$$

$$M = \frac{qL_T}{2}L_T = \frac{qL_T^2}{2} \quad \text{———} \quad \text{exact}$$

(b)



$L = \frac{L_T}{2}$

$$v = v_{\text{center force}} + v_{\text{end force}} + v_{\text{end moment}}$$

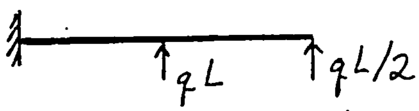
$$v = \left(qL\frac{L^3}{3EI} + qL\frac{L^2}{2EI}L\right) + \left(\frac{qL}{2}\frac{(2L)^3}{3EI}\right)$$

$$-\left(\frac{qL^2}{12}\frac{(2L)^2}{2EI}\right) = \frac{qL^4}{EI}\left(\frac{5}{6} + \frac{4}{3} - \frac{1}{6}\right)$$

$$v = \frac{2qL^4}{EI} = \frac{qL_T^4}{8EI} \quad \text{———} \quad \text{exact}$$

$$M = qL(L) + \frac{qL}{2}(2L) - \frac{qL^2}{12}$$

$$M = \frac{23qL^2}{12} = \frac{23qL_T^2}{48} = 0.479qL_T^2 \quad -4.2\%$$



$L = \frac{L_T}{2}$

Omit $v_{\text{end moment}}$ from foregoing

$$v = \frac{qL^4}{EI}\left(\frac{5}{6} + \frac{4}{3}\right) = \frac{13qL^4}{6EI} = 0.135\frac{qL_T^4}{EI}$$

+8.3%

$$M = qL(L) + \frac{qL}{2}2L = 2qL^2 = \frac{qL_T^2}{2} \quad \text{exact}$$

$\boxed{2.9\text{-}5}$

(a) $\begin{Bmatrix} v_2 \\ \theta_{z2} \end{Bmatrix} = [\underset{\sim}{K}]^{-1}\{\underset{\sim}{R}\} = \dfrac{L}{6EI_z}\begin{bmatrix} 2L^2 & 3L \\ 3L & 6 \end{bmatrix}\begin{Bmatrix} F_2 \\ M_2 \end{Bmatrix}$ where $F_2 = P\left(\dfrac{3a^2}{L^2} - \dfrac{2a^3}{L^3}\right)$, $M_2 = P\left(\dfrac{a^3}{L^2} - \dfrac{a^2}{L}\right)$

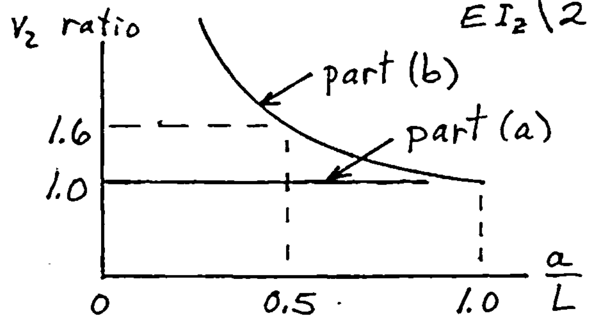
from which $v_2 = \dfrac{Pa^2}{EI_z}\left(\dfrac{L}{2} - \dfrac{a}{6}\right)$, $\theta_{z2} = \dfrac{Pa^2}{2EI_z}$

(b) $\begin{Bmatrix} v_2 \\ \theta_{z2} \end{Bmatrix} = [\underset{\sim}{K}]^{-1}\{\underset{\sim}{R}\} = \dfrac{L}{6EI_z}\begin{bmatrix} 2L^2 & 3L \\ 3L & 6 \end{bmatrix}\begin{Bmatrix} Pa/L \\ 0 \end{Bmatrix} = \dfrac{PaL}{EI_z}\begin{Bmatrix} L/3 \\ 1/2 \end{Bmatrix}$

(c) Beam theory: $v_2 = \dfrac{Pa^3}{3EI_z} + \dfrac{Pa^2}{2EI_z}(L-a) = \dfrac{Pa^2}{EI_z}\left(\dfrac{L}{2} - \dfrac{a}{6}\right)$

Deflection ratios: in part (a), unity [part (a) exact]

In part (b), ratio $= \dfrac{PaL^2/3EI_z}{\dfrac{Pa^2}{EI_z}\left(\dfrac{L}{2} - \dfrac{a}{6}\right)} = \dfrac{2}{\dfrac{a}{L}\left(3 - \dfrac{a}{L}\right)}$



(d) Beam theory: $M_1 = Pa$

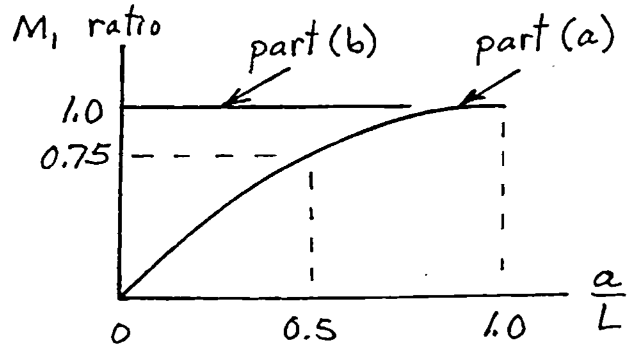
Moment ratio, part (a): from FEA,

$M_1 = EI_z\left[\left(\dfrac{6}{L^2}\right)\dfrac{Pa^2}{EI_z}\left(\dfrac{L}{2} - \dfrac{a}{6}\right) + \left(-\dfrac{2}{L}\right)\dfrac{Pa^2}{2EI_z}\right] = \dfrac{Pa^2}{L}\left(2 - \dfrac{a}{L}\right)$

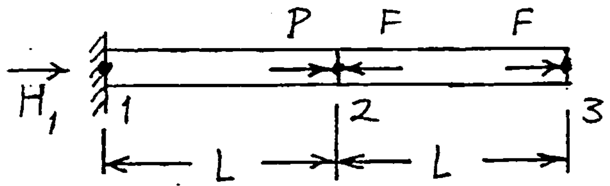
moment ratio $= \dfrac{a}{L}\left(2 - \dfrac{a}{L}\right)$

And in part (b): from FEA,

$M_1 = EI_z\left[\left(\dfrac{6}{L^2}\right)\dfrac{PaL^2}{3EI_z} + \left(-\dfrac{2}{L}\right)\dfrac{PaL}{2EI_z}\right] = Pa$, moment ratio $= 1$ (exact)



2.10-1



$$\sigma_0 = -E\alpha \frac{T_2 + T_3}{2} \quad \text{(right element only)}$$

$$F = |A\sigma_0| = AE\alpha \frac{T_2 + T_3}{2}$$

In Eq. 2.10-3, set $u_1 = 0$ and $H_3 = 0$. Thus

$$\frac{AE}{L}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} P - EA\alpha(T_2+T_3)/2 \\ EA\alpha(T_2+T_3)/2 \end{Bmatrix}$$

from which $u_2 = \dfrac{PL}{AE}$, $u_3 = \dfrac{PL}{AE} + \alpha L \dfrac{T_2+T_3}{2}$

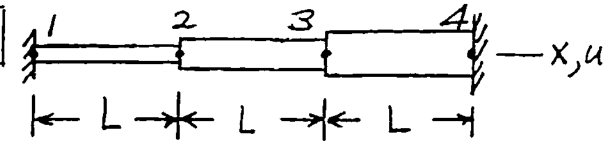
1st of Eqs. 2.10-3 then gives $H_1 = -P$

Stresses:

$$\sigma_{1\text{-}2} = E\frac{u_2}{L} + (\text{zero}) = \frac{P}{A} \quad \checkmark$$

$$\sigma_{2\text{-}3} = E\frac{u_3 - u_2}{L} + \sigma_0 = E\alpha\frac{T_2+T_3}{2} + \left(-E\alpha\frac{T_2+T_3}{2}\right) = 0 \quad \checkmark$$

2.10-2



$L = \frac{L_T}{3}$

Thermal loads at nodes 2 and 3 are

Node 2: $\alpha E(1.1A_0)\Delta T \rightarrow \; \leftarrow \alpha E(1.3A_0)\Delta T \quad = \quad \leftarrow 0.2\alpha E A_0 \Delta T$

Node 3: $\alpha E(1.3A_0)\Delta T \rightarrow \; \leftarrow \alpha E(1.5A_0)\Delta T \quad = \quad \leftarrow 0.2\alpha E A_0 \Delta T$

$\underset{\sim}{k}$ is given by Eq. 2.2-7, with A being $1.1A_0$, $1.3A_0$, and $1.5A_0$ for the respective elements.

$$\underset{\sim}{K} = \frac{A_0E}{L}\left(\begin{bmatrix} 1.1 & -1.1 & 0 & 0 \\ -1.1 & 1.1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1.3 & -1.3 & 0 \\ 0 & -1.3 & 1.3 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1.5 & -1.5 \\ 0 & 0 & -1.5 & 1.5 \end{bmatrix}\right) = \frac{A_0E}{L}\begin{bmatrix} \cdot & \cdot & \cdot & \cdot \\ \cdot & 2.4 & -1.3 & \cdot \\ \cdot & -1.3 & 2.8 & \cdot \\ \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

For the nonzero d.o.f. u_2 and u_3

$$\frac{A_0E}{L}\begin{bmatrix} 2.4 & -1.3 \\ -1.3 & 2.8 \end{bmatrix}\begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} -0.2 \\ -0.2 \end{Bmatrix}\alpha E A_0 \Delta T$$

Solve: $\begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \alpha L \Delta T \begin{Bmatrix} -0.163 \\ -0.147 \end{Bmatrix}$

Initial stress is $-E\alpha\Delta T$ in each el.

each element:

$$\sigma_1 = E\frac{u_2}{L} + \sigma_0 = -1.163\, E\alpha\Delta T$$

$$\sigma_2 = E\frac{u_3 - u_2}{L} + \sigma_0 = -0.984\, E\alpha\Delta T$$

$$\sigma_3 = E\frac{-u_3}{L} + \sigma_0 = -0.853\, E\alpha\Delta T$$

Nodal average stresses:

Node 2, $\frac{\sigma_1 + \sigma_2}{2} = -1.074\, E\alpha\Delta T$

Node 3, $\frac{\sigma_2 + \sigma_3}{2} = -0.919\, E\alpha\Delta T$

Exact solution: compute axial force P that negates free thermal expansion.

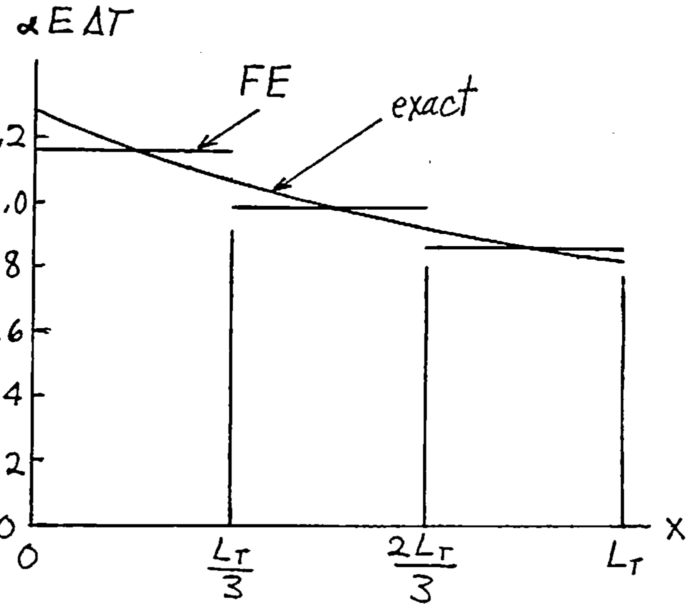
$$L_T\alpha\Delta T = \int_0^{L_T}\frac{P\,dx}{AE} = \frac{P}{E}\int_0^{L_T}\frac{dx}{A_0\left(1+0.6\frac{x}{L_T}\right)}$$

$$L_T\alpha\Delta T = \frac{PL_T}{EA_0(0.6)}\ln\left(1+0.6\frac{x}{L_T}\right)\Big|_0^{L_T}$$

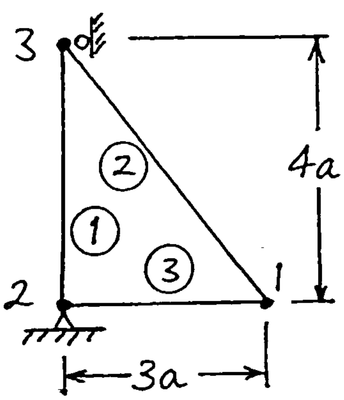
$$L_T\alpha\Delta T = \frac{PL_T}{0.6EA_0}\ln 1.6 = 0.783\frac{PL_T}{EA_0}$$

$P = 1.277\,\alpha E A_0 \Delta T$ (compressive)

location	A	$\sigma = P/A$
node 1	A_0	$-1.277\,\alpha E\Delta T$
el. 1	$1.1A_0$	$-1.161\,\alpha E\Delta T$
node 2	$1.2A_0$	$-1.064\,\alpha E\Delta T$
el. 2	$1.3A_0$	$-0.982\,\alpha E\Delta T$
node 3	$1.4A_0$	$-0.912\,\alpha E\Delta T$
el. 3	$1.5A_0$	$-0.851\,\alpha E\Delta T$
node 4	$1.6A_0$	$-0.798\,\alpha E\Delta T$



2.10-3



Heat bar 2 only. In that bar,

$$\sigma_0 = -E\alpha T, \quad F = |A\sigma_0| = EA\alpha T$$

Eq. 2.7-8 becomes

$$\begin{bmatrix} k_3 + 0.36k_2 & -0.48k_2 & 0.48k_2 \\ -0.48k_2 & 0.64k_2 & -0.64k_2 \\ 0.48k_2 & -0.64k_2 & k_1 + 0.64k_2 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ v_3 \end{Bmatrix} = \begin{Bmatrix} 0.6F \\ -0.8F \\ 0.8F \end{Bmatrix}$$

where $k_2 = \dfrac{AE}{5a}$

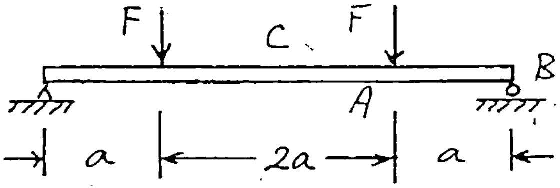
Solution is $\begin{Bmatrix} u_1 \\ v_1 \\ v_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -6.25a\alpha T \\ 0 \end{Bmatrix}$

Bar stresses: $\sigma_1 = 0$, $\sigma_3 = 0$, and

$$\sigma_2 = \frac{E}{5a}(0.8)(6.25a\alpha T) + (-E\alpha T) = 0$$

$\boxed{2.11\text{-}1}$

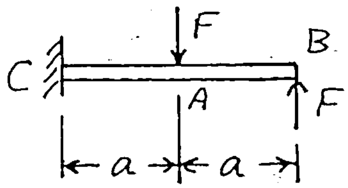
First case (symmetric loads)



$F = \frac{P}{2}$

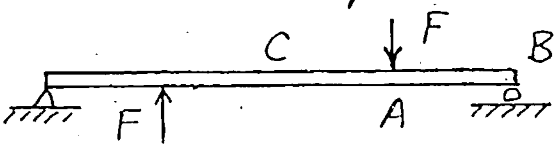
$a = \frac{L}{4}$

Consider right half to get deflection of A relative to B

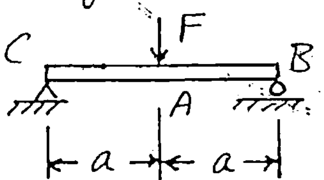


$$\Delta = \underbrace{\frac{(Fa)a}{EI}}_{\theta_A} a + \frac{Fa^3}{3EI} = \frac{4Fa^3}{3EI}$$

Second case (antisymmetric loads)



Again consider right half



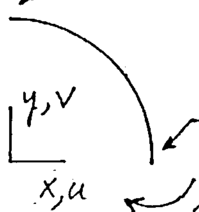
$$\Delta = \frac{F(2a)^3}{48EI} = \frac{Fa^3}{6EI}$$

$$\text{Sum} = \frac{Fa^3}{EI}\left(\frac{4}{3} + \frac{1}{6}\right) = \frac{3Fa^3}{2EI} = \frac{3}{2EI}\frac{P}{2}\left(\frac{L}{4}\right)^3 = \frac{3PL^3}{256EI}$$

Handbook formula: $\frac{P(3L/4)^2(L/4)^2}{3EIL} = \frac{3PL^3}{256EI}$

$\boxed{2.11\text{-}2}$

At top node, $u = 0$ and $\theta_z = 0$



At bottom node, $u = 0$ and $v = 0$

$M_o/2$ (the only load term)

Also prevent out-of-plane motion: at (say) top node, set $w = \theta_x = \theta_y = 0$.

$\boxed{2.11-2}$

(a) Symmetric case:

Restrain at $x=0$: u, θ_y, θ_z

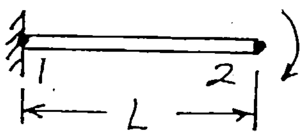
Zero loads at $x=0$: y-direction transverse shear force
z-direction transverse shear force
torque

(b) Antisymmetric case:

Restrain at $x=0$: v, w, θ_x

Zero loads at $x=0$: x-direction force
moment M_y
moment M_z

$\boxed{2.10-4}$ With no deformation, stress is $-E\alpha T$ on top surface, $+E\alpha T$ on bottom surface. The associated bending moment is $M_z = \frac{\sigma I}{c} = E\alpha T \frac{t\,(2c)^3}{12c} = \frac{2}{3} E\alpha T t c^2$

(a) 

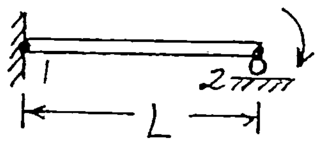
M_z for which $v_2 = -\frac{M_z L^2}{2EI}$, $\theta_2 = -\frac{M_z L}{EI}$

With $v_1 = \theta_{z1} = 0$, Eq. 2.9-4 gives

$$M = EI\left[\left(\frac{6}{L^2} - \frac{12x}{L^3}\right)\left(-\frac{M_z L^2}{2EI}\right) + \left(-\frac{2}{L} + \frac{6x}{L^2}\right)\left(-\frac{M_z L}{EI}\right)\right] = -M_z$$

for all x

Net M, from nodal d.o.f. + temperature change, is zero, so zero stress is predicted.

(b) 

M_z for which $\frac{4EI}{L}\theta_2 = -M_z$, $\theta_2 = -\frac{M_z L}{4EI}$

With θ_2 the only nonzero d.o.f., Eq. 2.9-4 gives

$$M = EI\left(-\frac{2}{L} + \frac{6x}{L^2}\right)\left(-\frac{M_z L}{4EI}\right) = \frac{M_z}{2}\left(1 - \frac{3x}{L}\right)$$

Net M, including M from temperature change, is

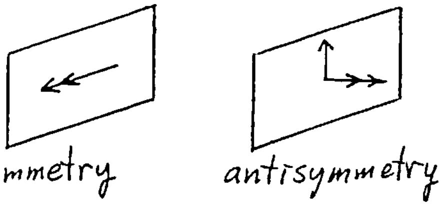
$$\frac{M_z}{2}\left(1 - \frac{3x}{L}\right) + M_z = \frac{3M_z}{2}\left(1 - \frac{x}{L}\right) = E\alpha T t c^2\left(1 - \frac{x}{L}\right)$$

On the top surface,

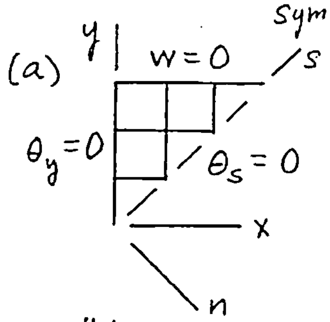
$$\sigma = -\frac{Mc}{I} = -\frac{c}{t(2c)^3/12} E\alpha T t c^2\left(1 - \frac{x}{L}\right)$$

$\boxed{2.11-4}$

D.o.f. restrained:



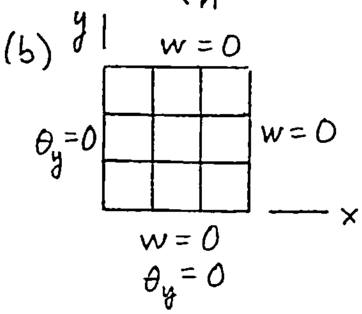
(a)



Symmetry w.r.t. $x=0$ and $n=0$ planes

Apply loads $P/2$ to nodes on symm. planes.

(b)



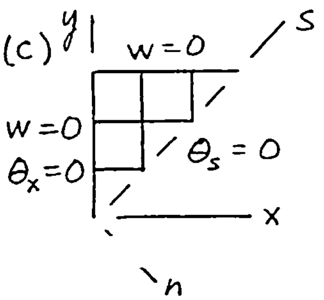
Symmetry w.r.t. $x=0$ plane

Antisymmetry w.r.t. $y=0$ plane

Apply loads $P/2$ to nodes on $x=0$.

Could go further: cut the quadrant in half by an x-parallel line; impose symmetry ($\theta_x=0$) along this cut.

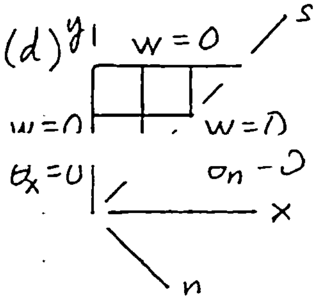
(c)



Symmetry w.r.t. $n=0$ plane

Antisymmetry w.r.t. $x=0$ plane

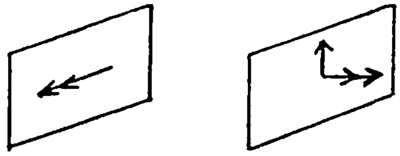
(d)



Antisymmetry w.r.t. $x=0$ and $n=0$ planes

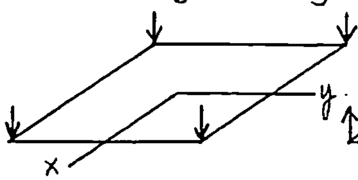
2.11-5

D. o.f. restrained:



symmetry . antisymmetry

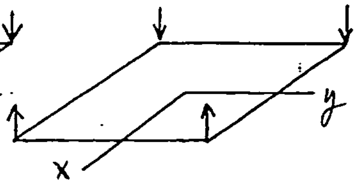
Analyze the following four cases, with each load = $P/4$. Only one quadrant (e.g. the first) need be treated. Deflections in quadrants 2, 3, 4 can can be obtained from deflections in quadrant 1 by use of the symmetry and antisymmetry condtions noted.



$\theta_y = 0$ on $x=0$ plane

$\theta_x = 0$ on $y=0$ plane

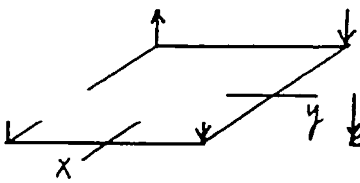
$w(x) = w(-x)$
$w(y) = w(-y)$



$\left.\begin{matrix} w=0 \\ \theta_x=0 \end{matrix}\right\}$ on $x=0$ plane

$\theta_x = 0$ on $y=0$ plane

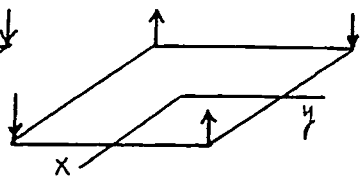
$w(x) = -w(-x)$
$w(y) = w(-y)$



$\theta_y = 0$ on $x=0$ plane

$\left.\begin{matrix} w=0 \\ \theta_y=0 \end{matrix}\right\}$ . . $=0$ plane

$w(x) = w(-x)$
$w(y) = -w(-y)$



$\left.\begin{matrix} w=0 \\ \theta_x=0 \end{matrix}\right\}$ on $x=0$ plane

$\left.\begin{matrix} w=0 \\ \theta_y=0 \end{matrix}\right\}$ on $y=0$ plane

$w(x) = -w(-x)$
$w(y) = -w(-y)$