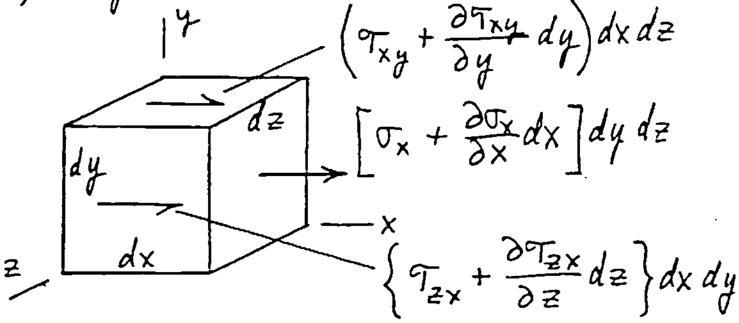
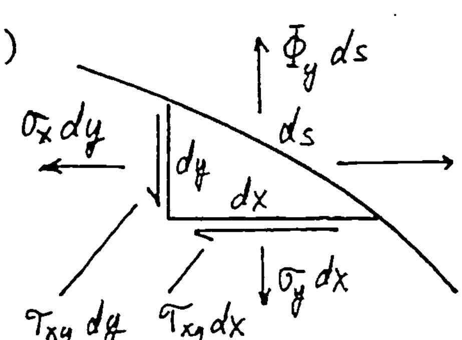
3.1-1

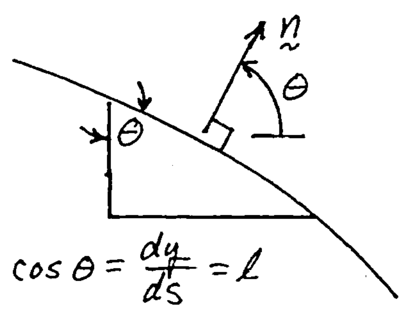
(a) E.g. in the x direction,



$$\sum F_{x\ \text{axis}} = 0 = -\sigma_x\, dy\, dz - \tau_{xy}\, dx\, dz$$
$$- \tau_{zx}\, dx\, dy + (---)\, dx\, dz + [---]\, dy \cdot dz$$
$$+ \{---\}\, dx\, dy + F_x\, dx\, dy\, dz \qquad \text{yields}$$
$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + F_x = 0$$

(b)





$$\cos\theta = \frac{dy}{ds} = \ell$$
$$\sin\theta = \frac{dx}{ds} = m$$

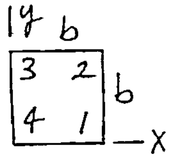
Thickness immaterial; can take as unity.

$$\sum F_x = 0 = -\sigma_x\, dy - \tau_{xy}\, dx + \Phi_x\, ds$$
$$\sum F_y = 0 = -\sigma_y\, dx - \tau_{xy}\, dy + \Phi_y\, ds$$
$$\Phi_x = \sigma_x \frac{dy}{ds} + \tau_{xy} \frac{dx}{ds}$$
$$\Phi_y = \tau_{xy} \frac{dy}{ds} + \sigma_y \frac{dx}{ds}$$
$$\Phi_x = \ell \sigma_x + m \tau_{xy}$$
$$\Phi_y = \ell \tau_{xy} + m \sigma_y$$

$\boxed{3.1\text{-}2}$

(a)

$$\sigma_x = -6a_1x^2,\ \sigma_y = 12a_1x^2,\ \tau_{xy} = 12a_1y^2$$



<u>On 1–2:</u> $l=1,\ m=0,\ x=b$

$$\Phi_x = \sigma_x = -6a_1b^2$$
$$\Phi_y = \tau_{xy} = 12a_1y^2$$

<u>On 2-3:</u> $l=0,\ m=1;\ y=b$

$$\Phi_x = \tau_{xy} = 12a_1b^2,\quad \Phi_y = \sigma_y = 12a_1x^2$$

<u>On 3–4:</u> $l=-1,\ m=0,\ x=0$

$$\Phi_x = 0,\quad \Phi_y = -\tau_{xy} = -12a_1y^2$$

<u>On 4–1:</u> $l=0,\ m=-1,\ y=0$

$$\Phi_x = -\tau_{xy} = 0,\quad \Phi_y = -\sigma_y = -12a_1x^2$$

(b) Check equilibrium.

x-dir. $\sigma_{x,x} + \tau_{xy,y} = -12a_1x + 24a_1y \neq 0$

y-dir. $\tau_{xy,x} + \sigma_{y,y} = 0+0 = 0$ ✓

Equil. not satisfied; field not possible.

$\boxed{3.1\text{-}3}$

$$\sigma_x = 3a_1x^2y,\ \sigma_y = a_1y^3,\ \tau_{xy} = -3a_1xy^2$$

Equilibrium:

$$\sigma_{x,x} + \tau_{xy,y} = 6a_1xy - 6a_1xy = 0 \quad ✓$$
$$\tau_{xy,x} + \sigma_{y,y} = -3a_1y^2 + 3a_1y^2 = 0 \quad ✓$$

Check compatibility:

$$\epsilon_x = \frac{1}{E}\left(3a_1x^2y - \nu a_1y^3\right)$$
$$\epsilon_y = \frac{1}{E}\left(a_1y^3 - 3\nu a_1x^2y\right)$$
$$\gamma_{xy} = \frac{2(1+\nu)}{E}\left(-3a_1xy^2\right)$$

$$\epsilon_{x,yy} + \epsilon_{y,xx} \stackrel{?}{=} \gamma_{xy,xy}$$

$$\frac{1}{E}\left(-6\nu a_1y - 6\nu a_1y\right) \stackrel{?}{=} \frac{2(1+\nu)}{E}\left(-6a_1y\right)$$

$\nu \stackrel{?}{=} 1+\nu$ Not possible.

3.1-5

$$\epsilon_x = u_{,x} = a_2 + 2a_4 x + a_5 y$$
$$\epsilon_y = v_{,y} = a_9 + a_{11} x + 2a_{12} y$$
$$\gamma_{xy} = u_{,y} + v_{,x} = a_3 + a_5 x + 2a_6 y + a_8 + 2a_{10} x + a_{11} y$$

Let $A = \frac{E}{1-\nu^2}$. Then

$$\sigma_x = A(\epsilon_x + \nu\epsilon_y), \quad \sigma_y = A(\epsilon_y + \nu\epsilon_x), \quad \tau_{xy} = G\gamma_{xy}$$

Equilibrium eqs. become

$$\sigma_{x,x} + \tau_{xy,y} = A(2a_4 + \nu a_{11}) + G(2a_6 + a_{11})$$
$$\tau_{xy,x} + \sigma_{y,y} = G(a_5 + 2a_{10}) + A(2a_{12} + \nu a_5)$$

When are these 2 eqs. satisfied?
With $G = (1-\nu)A/2$,

$$2a_4 + \nu a_{11} + \frac{1-\nu}{2}(2a_6 + a_{11}) = 0$$
$$\frac{1-\nu}{2}(a_5 + 2a_{10}) + 2a_{12} + \nu a_5 = 0$$

or

$$2a_4 + (1-\nu)a_6 + \frac{1+\nu}{2}a_{11} = 0$$
$$2a_{12} + (1-\nu)a_{10} + \frac{1+\nu}{2}a_5 = 0$$

3.1-4

(a) $\Delta V = V_{final} - V_{original}$

$$\Delta V = [(1+\epsilon_x)dx\,(1+\epsilon_y)dy\,(1+\epsilon_z)dz] - [dx\,dy\,dz]$$
$$\Delta V = (\epsilon_x + \epsilon_y + \epsilon_z)\,dx\,dy\,dz + \text{higher order tems}$$
$$\Delta V = (\epsilon_x + \epsilon_y + \epsilon_z)\,dx\,dy\,dz \quad \text{for small strains}$$
$$\frac{\Delta V}{V} = \frac{\Delta V}{dx\,dy\,dz} = \epsilon_x + \epsilon_y + \epsilon_z$$

(b) Use 3D form of Eqs. 3.1-4, with $\sigma_x = \sigma_y = \sigma_z = -p$:

$$\epsilon_x = \frac{-p}{E}(1-2\nu) \qquad \epsilon_y = \epsilon_x \qquad \epsilon_z = \epsilon_x$$

Hence $\frac{\Delta V}{V} = -\frac{3p}{E}(1-2\nu)$ and $\frac{\Delta V/V}{p} = -\frac{3(1-2\nu)}{E}$

(c) As $\nu \to 0.5$, $\frac{\Delta V}{V} \to 0$

$\boxed{3.1\text{-}4}$

(a) $\Delta V = V_{final} - V_{original}$

$$\Delta V = \left[(1+\epsilon_x)dx\,(1+\epsilon_y)dy\,(1+\epsilon_z)dz\right] - \left[dx\,dy\,dz\right]$$

$$\Delta V = (\epsilon_x + \epsilon_y + \epsilon_z)\,dx\,dy\,dz + \text{higher order tems}$$

$$\Delta V = (\epsilon_x + \epsilon_y + \epsilon_z)\,dx\,dy\,dz \quad \text{for small strains}$$

$$\frac{\Delta V}{V} = \frac{\Delta V}{dx\,dy\,dz} = \epsilon_x + \epsilon_y + \epsilon_z$$

(b) Use 3D form of Eqs. 3.1-4, with $\sigma_x = \sigma_y = \sigma_z = -p$:

$$\epsilon_x = \frac{-p}{E}(1-2\nu) \qquad \epsilon_y = \epsilon_x \qquad \epsilon_z = \epsilon_x$$

Hence $\dfrac{\Delta V}{V} = -\dfrac{3p}{E}(1-2\nu)$ and $\dfrac{\Delta V/V}{p} = -\dfrac{3(1-2\nu)}{E}$

(c) As $\nu \to 0.5$, $\dfrac{\Delta V}{V} \to 0$

$\boxed{3.2\text{-}1}$

$$N_1 = \frac{(2-x)(3-x)}{(2)(3)} = \frac{1}{6}(6-5x+x^2)$$

$$N_2 = \frac{-x(3-x)}{(-2)(1)} = \frac{1}{2}(3x-x^2)$$

$$N_3 = \frac{-x(2-x)}{(-3)(-1)} = \frac{1}{3}(-2x+x^2)$$

(a) $\sum_1^3 N_i = \frac{1}{6}(6-5x+x^2+9x-3x^2-4x+2x^2) = 1$

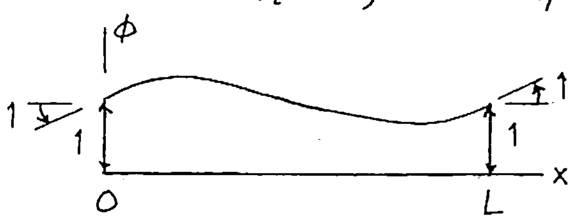
(b) $N_{1,x} = \frac{1}{6}(-5+2x)$ $\quad \sum_1^3 N_{i,x} = \frac{1}{6}(-5+2x$

$N_{2,x} = \frac{1}{2}(3-2x)$ $\quad +9-6x-4+4x)$

$N_{3,x} = \frac{1}{3}(-2+2x)$ $\quad = 0$

$\boxed{3.2\text{-}2}$

If all 4 $\phi_i = 1$, the shape is



Clearly $\phi \neq 1$ throughout ($\Sigma N_i = 1$ if $\Sigma N_i \phi_i = \phi$ gives $1 = 1$ for $\phi = \phi_i = 1$).
Also, not all N_i have the same units.

$\boxed{3.1\text{-}5}$

$$\epsilon_x = u_{,x} = a_2 + 2a_4 x + a_5 y$$
$$\epsilon_y = v_{,y} = a_9 + a_{11} x + 2a_{12} y$$
$$\gamma_{xy} = u_{,y} + v_{,x} = a_3 + a_5 x + 2a_6 y + a_8 + 2a_{10} x + a_{11} y$$

Let $A = \dfrac{E}{1-\nu^2}$. Then

$$\sigma_x = A(\epsilon_x + \nu\epsilon_y), \quad \sigma_y = A(\epsilon_y + \nu\epsilon_x), \quad \tau_{xy} = G\gamma_{xy}$$

Equilibrium eqs. become

$$\sigma_{x,x} + \tau_{xy,y} = A(2a_4 + \nu a_{11}) + G(2a_6 + a_{11})$$
$$\tau_{xy,x} + \sigma_{y,y} = G(a_5 + 2a_{10}) + A(2a_{12} + \nu a_5)$$

When are these 2 eqs. satisfied?
With $G = (1-\nu)A/2$,

$$2a_4 + \nu a_{11} + \frac{1-\nu}{2}(2a_6 + a_{11}) = 0$$
$$\frac{1-\nu}{2}(a_5 + 2a_{10}) + 2a_{12} + \nu a_5 = 0$$

or

$$2a_4 + (1-\nu)a_6 + \frac{1+\nu}{2}a_{11} = 0$$
$$2a_{12} + (1-\nu)a_{10} + \frac{1+\nu}{2}a_5 = 0$$

$\boxed{3.2\text{-}3}$

(a) $\phi = \sum N_i \phi_i$; use Eqs. 3.2-7

$$\phi = \frac{(3-x)(5-x)(8-x)}{(2)(4)(7)}2 + \frac{(1-x)(5-x)(8-x)}{(-2)(2)(5)}2 + \frac{(1-x)(3-x)(8-x)}{(-4)(-2)(3)}2 + \frac{(1-x)(3-x)(5-x)}{(-7)(-5)(-3)}5$$

(b)
$$\phi_0 = \frac{3\cdot5\cdot8}{56}2 + \frac{1\cdot5\cdot8}{-20}2 + \frac{1\cdot3\cdot8}{24}2 + \frac{1\cdot3\cdot5}{-105}5$$
$$= \frac{120}{28} - \frac{40}{10} + \frac{24}{12} - \frac{15}{21} = 1.57$$

Similarly

$$\phi_2 = \frac{1\cdot3\cdot6}{28} - \frac{(-1)\cdot3\cdot6}{10} + \frac{(-1)\cdot1\cdot6}{12} - \frac{(-1)\cdot1\cdot3}{21} = 2.086$$

$$\phi_4 = \frac{(-1)\cdot1\cdot4}{28} - \frac{(-3)\cdot1\cdot4}{10} + \frac{(-3)\cdot(-1)\cdot4}{12} - \frac{(-3)\cdot(-1)\cdot1}{21} = 1.914$$

$$\phi_7 = \frac{(-4)\cdot(-2)\cdot1}{28} - \frac{(-6)\cdot(-2)\cdot1}{10} + \frac{(-6)\cdot(-4)\cdot1}{12} - \frac{(-6)\cdot(-4)\cdot(-2)}{21} = 3.371$$

$\boxed{3.2\text{-}4}$

First two of Eqs. 3.2-8 yield $a_1 = \phi_1$, $a_2 = \phi_{,x1}$

The second two equations become

$$\begin{bmatrix} L^2 & L^3 \\ 2L & 3L^2 \end{bmatrix} \begin{Bmatrix} a_3 \\ a_4 \end{Bmatrix} = \begin{Bmatrix} \phi_2 - \phi_1 - L\phi_{,x1} \\ \phi_{,x2} - \phi_{,x1} \end{Bmatrix},$$

$$\begin{Bmatrix} a_3 \\ a_4 \end{Bmatrix} = \frac{1}{L^4} \begin{bmatrix} 3L^2 & -L^3 \\ -2L & L^2 \end{bmatrix} \begin{Bmatrix} \phi_2 - \phi_1 - L\phi_{,x1} \\ \phi_{,x2} - \phi_{,x1} \end{Bmatrix} = \begin{Bmatrix} (-3\phi_1 - 2L\phi_{,x1} + 3\phi_2 - L\phi_{,x2})/L^2 \\ (2\phi_1 + L\phi_{,x1} - 2\phi_2 + L\phi_{,x2})/L^3 \end{Bmatrix}$$

$$[\underset{\sim}{A}]^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -3/L^2 & -2/L & 3/L^2 & -1/L \\ 2/L^3 & 1/L^2 & -2/L^3 & 1/L^2 \end{bmatrix}$$

with $\lfloor \underset{\sim}{X} \rfloor = \lfloor 1 \quad x \quad x^2 \quad x^3 \rfloor$,

$$\lfloor \underset{\sim}{X} \rfloor [\underset{\sim}{A}]^{-1} = \Big\lfloor \underbrace{1 - \frac{3x^2}{L^2} + \frac{2x^3}{L^3}}_{N_1},\ \underbrace{x - \frac{2x^2}{L} + \frac{x^3}{L^2}}_{N_2},\ \underbrace{\frac{3x^2}{L^2} - \frac{2x^3}{L^3}}_{N_3},\ \underbrace{-\frac{x^2}{L} + \frac{x^3}{L^2}}_{N_4} \Big\rfloor$$

$\boxed{3.2\text{-}5}$

$$\phi = \lfloor \underset{\sim}{X} \rfloor \{\underset{\sim}{a}\} = \lfloor 1 \quad x \quad x^2 \rfloor \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix}$$

$$\phi_{,x} = \lfloor 0 \quad 1 \quad 2x \rfloor \{\underset{\sim}{a}\}$$

Evaluate given conditions (let $\theta_1 = \phi_{,x1}$)

$$\begin{Bmatrix} \phi_1 \\ \theta_1 \\ \phi_2 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & L & L^2 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix}$$

$a_1 = \phi_1$, $a_2 = \theta_1$

3^{rd} eq. is then

$$\phi_2 - \phi_1 - L\theta_1 = L^2 a_3$$

$$\therefore \; a_3 = \frac{\phi_2 - \phi_1}{L^2} - \frac{\theta_1}{L}$$

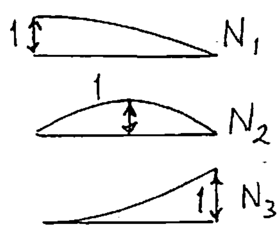
$$\begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{1}{L^2} & -\frac{1}{L} & \frac{1}{L^2} \end{bmatrix} \begin{Bmatrix} \phi_1 \\ \theta_1 \\ \phi_2 \end{Bmatrix} = [\underset{\sim}{A}]^{-1} \{\underset{\sim}{d}\}$$

$\phi = \lfloor \underset{\sim}{X} \rfloor [\underset{\sim}{A}]^{-1} \{\underset{\sim}{d}\} = \lfloor \underset{\sim}{N} \rfloor \{\underset{\sim}{d}\}$ where

$$\lfloor \underset{\sim}{N} \rfloor = \left\lfloor 1 - \frac{x^2}{L^2} \quad x\left(1 - \frac{x}{L}\right) \quad \frac{x^2}{L^2} \right\rfloor . \text{ Also}$$

$$\lfloor \underset{\sim}{N}_{,x} \rfloor = \left\lfloor -\frac{2x}{L^2} \quad 1 - \frac{2x}{L} \quad \frac{2x}{L^2} \right\rfloor$$

$i \downarrow$	At $x=0$		At $x=L$	
	N_i	$N_{i,x}$	N_i	$N_{i,x}$
1	1	0	0	$-2/L$
2	0	1	0	-1
3	0	0	1	$2/L$



$\boxed{3.2\text{-}6}$

Lagrange: 3 pts. define parabola.
Piecewise C^0: two straight lines.
Piecewise C^1: two cubics.

(a) (b) (c)

$\boxed{3.3\text{-}1}$

(a) $u = \lfloor 1 \quad x \rfloor \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix}$, $\lfloor \underset{\sim}{B}_a \rfloor = \dfrac{\partial}{\partial x} \lfloor 1 \quad x \rfloor = \lfloor 0 \quad 1 \rfloor$

$$[\underset{\sim}{k}_a] = \int_0^L \lfloor \underset{\sim}{B}_a \rfloor^T \lfloor \underset{\sim}{B}_a \rfloor AE\,dx = AEL \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Let $x_1 = 0$, $x_2 = L$ in Eq. 3.2-4. Then $[\underset{\sim}{A}] = \begin{bmatrix} 1 & 0 \\ 1 & L \end{bmatrix}$, $[\underset{\sim}{A}]^{-1} = \begin{bmatrix} 1 & 0 \\ -1/L & 1/L \end{bmatrix}$

$$[\underset{\sim}{k}] = [\underset{\sim}{A}]^{-T}[\underset{\sim}{k}_a][\underset{\sim}{A}]^{-1} = \begin{bmatrix} 1 & -1/L \\ 0 & 1/L \end{bmatrix} \left(AEL \begin{bmatrix} 0 & 0 \\ -1/L & 1/L \end{bmatrix} \right) = \frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \checkmark$$

(b) $v = \lfloor 1 \quad x \quad x^2 \quad x^3 \rfloor \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{Bmatrix}$, $\dfrac{d^2 v}{dx^2} = \lfloor 0 \quad 0 \quad 2 \quad 6x \rfloor \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{Bmatrix} = \lfloor \underset{\sim}{B}_a \rfloor \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{Bmatrix}$

$$[\underset{\sim}{k}_a] = \int_0^L \lfloor \underset{\sim}{B}_a \rfloor^T \lfloor \underset{\sim}{B}_a \rfloor EI\,dx = EI \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 4L & 6L^2 \\ 0 & 0 & 6L^2 & 12L^3 \end{bmatrix}$$

$[\underset{\sim}{A}]^{-1}$ is calculated in Problem 3.2-4

$$[\underset{\sim}{k}] = [\underset{\sim}{A}]^{-T}[\underset{\sim}{k}_a][\underset{\sim}{A}]^{-1} = \begin{bmatrix} 1 & 0 & -3/L^2 & 2/L^3 \\ 0 & 1 & -2/L & 1/L^2 \\ 0 & 0 & 3/L^2 & -2/L^3 \\ 0 & 0 & -1/L & 1/L^2 \end{bmatrix} \left(EI \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 2 \\ 6 & 0 & -6 & 6L \end{bmatrix} \right)$$

$$[\underset{\sim}{k}] = EI \begin{bmatrix} 12/L^3 & 6/L^2 & -12/L^3 & 6/L^2 \\ 6/L^2 & 4/L & -6/L^2 & 2/L \\ -12/L^3 & -6/L^2 & 12/L^3 & -6/L^2 \\ 6/L^2 & 2/L & -6/L^2 & 4/L \end{bmatrix} \checkmark$$

$\boxed{3.3\text{-}2}$

(a)
Follow the method of Prob. 3.3-1. That is, use the "a-basis", $u = \lfloor 1 \; x \; x^2 \; x^3 \rfloor \{\underset{\sim}{a}\}$,
$[\underset{\sim}{k}_a] = \int_0^L \lfloor \underset{\sim}{B}_a \rfloor^T \lfloor \underset{\sim}{B}_a \rfloor AE\,dx$, where $\lfloor \underset{\sim}{B}_a \rfloor = \lfloor 0 \;\; 1 \;\; 2x \;\; 3x^2 \rfloor$, $[\underset{\sim}{k}] = [\underset{\sim}{A}]^{-T}[\underset{\sim}{k}_a][\underset{\sim}{A}]^{-1}$,
and $[\underset{\sim}{A}]^{-1}$ is computed in Prob. 3.2-4.

$$\underbrace{AE\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & L & L^2 & L^3 \\ 0 & L^2 & \frac{4}{3}L^3 & \frac{3}{2}L^4 \\ 0 & L^3 & \frac{3}{2}L^4 & \frac{9}{5}L^5 \end{bmatrix}}_{[\underset{\sim}{k}_a]}, \quad \underbrace{AE\begin{bmatrix} 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ -L & -\frac{L^2}{6} & L & \frac{L^2}{6} \\ -.9L^2 & -.2L^3 & .9L^2 & .3L^3 \end{bmatrix}}_{[\underset{\sim}{k}_a][\underset{\sim}{A}]^{-1}}$$

$$\underbrace{\begin{bmatrix} 1 & 0 & -3/L^2 & 2/L^3 \\ 0 & 1 & -2/L & 1/L^2 \\ 0 & 0 & 3/L^2 & -2/L^3 \\ 0 & 0 & -1/L & 1/L^2 \end{bmatrix}}_{[\underset{\sim}{A}]^{-T}}, \quad \underbrace{\frac{AE}{30L}\begin{bmatrix} 36 & 3L & -36 & 3L \\ 3L & 4L^2 & -3L & -L^2 \\ -36 & -3L & 36 & -3L \\ 3L & -L^2 & -3L & 4L^2 \end{bmatrix}}_{[\underset{\sim}{k}]}$$

(b) Must not enforce interelement continuity of ϵ_x at these locations. Where two elements meet, can condense the ϵ_x d.o.f. in one of them before assembly.

$\boxed{3.3\text{-}3}$ Eqs. 3.2-7: $x_1 = 0,\ x_2 = \frac{L}{2},\ x_3 = L$

$$N_1 = \frac{\left(\frac{L}{2}-x\right)(L-x)}{\frac{L}{2}L} = \frac{1}{L^2}\left(L^2-3Lx+2x^2\right)$$

$$N_2 = \frac{(-x)(L-x)}{-\frac{L}{2}\left(\frac{L}{2}\right)} = \frac{4}{L^2}\left(Lx-x^2\right)$$

$$N_3 = \frac{-x\left(\frac{L}{2}-x\right)}{-L\left(-\frac{L}{2}\right)} = \frac{1}{L^2}\left(2x^2-Lx\right)$$

$$\lfloor \underset{\sim}{B} \rfloor = \frac{d}{dx}\lfloor \underset{\sim}{N} \rfloor = \frac{1}{L^2}\lfloor -3L+4x \quad 4(L-2x) \quad 4x-L \rfloor$$

$$[\underset{\sim}{k}] = \int_0^L \lfloor \underset{\sim}{B} \rfloor^T \lfloor \underset{\sim}{B} \rfloor AE\,dx$$

$$[\underset{\sim}{k}] = \frac{AE}{L^4}\int_0^L \begin{bmatrix} 9L-24Lx+16x^2 & -12L^2+40Lx-32x^2 & 3L^2-16Lx+16x^2 \\ & 16\left(L^2-4Lx+4x^2\right) & 4\left(-L^2+6Lx-8x^2\right) \\ \text{symmetric} & & L^2-8Lx+16x^2 \end{bmatrix} dx$$

$$[\underset{\sim}{k}] = \frac{AE}{L^4}\begin{bmatrix} 9L^2x-12Lx^2+\frac{16}{3}x^3 & -12L^2x+20Lx^2-\frac{32}{3}x^3 & 3L^2x-8Lx^2+\frac{16}{3}x^3 \\ & 16\left(L^2x-2Lx^2+\frac{4}{3}x^3\right) & 4\left(-L^2x+3Lx^2-\frac{8}{3}x^3\right) \\ \text{symmetric} & & L^2x-4Lx^2+\frac{16}{3}x^3 \end{bmatrix}_0^L$$

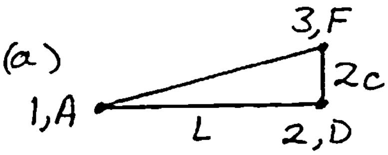
$$[\underset{\sim}{k}] = \frac{AE}{3L}\begin{bmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{bmatrix}$$

$\boxed{3.4\text{-}1}$ Beam theory: $I = \frac{t}{12}(2c)^3 = \frac{2tc^3}{3}$

$$v_D = v_F = -\frac{ML^2}{2EI} = -\frac{3ML^2}{4Etc^3}$$

$$u_F = -u_D = c\,\frac{ML}{EI} = \frac{3ML}{2Etc^2}$$

(a)



Obtain $[\underset{\sim}{B}]$ from Eq. 3.4-10

$$\{\underset{\sim}{\epsilon}\} = [\underset{\sim}{B}]\{\underset{\sim}{d}\} = \begin{bmatrix} -1/L & 0 & 1/L & 0 & 0 & 0 \\ 0 & 0 & 0 & -1/2c & 0 & 1/2c \\ 0 & -1/L & -1/2c & 1/L & 1/2c & 0 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ u_D \\ v_D \\ u_F \\ v_F \end{Bmatrix}$$

$$\epsilon_x = \frac{u_D}{L} = -\frac{3M}{2Etc^2}, \qquad \epsilon_y = \frac{v_F - v_D}{2c} = 0,$$

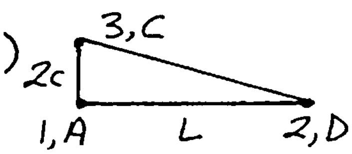
$$\gamma_{xy} = -\frac{u_D}{2c} + \frac{v_D}{L} + \frac{u_F}{2c} = \frac{u_F}{c} + \frac{v_D}{L} = \frac{3ML}{4Etc^3}$$

With $\nu = 0$, $\sigma_x = E\epsilon_x = \frac{3M}{2tc^2}$, $\sigma_y = E\epsilon_y = 0$, $\tau_{xy} = \frac{E}{2}\gamma_{xy} = \frac{3ML}{4tc^3}$

Beam theory, top & bottom: $|\sigma_x| = \frac{Mc}{8tc^3/12} = \frac{3M}{2tc^2}$

With σ_x & τ_{xy} from FEA, $\frac{\tau_{xy}}{\sigma_x} = \frac{L}{2c} \to \infty$ as $\frac{L}{c} \to \infty$

(b)



Obtain $[\underset{\sim}{B}]$ from Eq. 3.4-10

$$\{\underset{\sim}{\epsilon}\} = [\underset{\sim}{B}]\{\underset{\sim}{d}\} = \begin{bmatrix} -1/L & 0 & 1/L & 0 & 0 & 0 \\ 0 & -1/2c & 0 & 0 & 0 & 1/2c \\ -1/2c & -1/L & 0 & 1/L & 1/2c & 0 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ u_D \\ v_D \\ 0 \\ 0 \end{Bmatrix}$$

$$\epsilon_x = \frac{u_D}{L} = -\frac{3M}{2Etc^2}, \qquad \epsilon_y = 0, \qquad \gamma_{xy} = \frac{v_D}{L} = -\frac{3ML}{4Etc^3}$$

With $\nu = 0$, $\sigma_x = E\epsilon_x = -\frac{3M}{2tc^2}$, $\sigma_y = E\epsilon_y = 0$

$$\tau_{xy} = \frac{E}{2}\gamma_{xy} = -\frac{3ML}{8tc^3}$$

Beam theory: as in part (a)

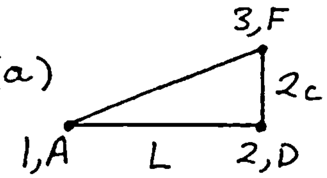
With σ_x & τ_{xy} from FEA, $\frac{\tau_{xy}}{\sigma_x} = \frac{L}{4c} \to \infty$ as $\frac{L}{c} \to \infty$

$\boxed{3.4\text{-}2}$ Beam theory: $I = \frac{t}{12}(2c)^3 = \frac{2tc^3}{3}$

$$v_D = v_F = \frac{PL^3}{3EI} = \frac{PL^3}{2Etc^3}$$

$$u_F = -u_D = -c\frac{PL^2}{2EI} = -\frac{3PL^2}{4Etc^2}$$

(a)



Obtain $[\underset{\sim}{B}]$ from Eq. 3.4-10

$$\{\underset{\sim}{\epsilon}\} = [\underset{\sim}{B}]\{\underset{\sim}{d}\} = \begin{bmatrix} -1/L & 0 & 1/L & 0 & 0 & 0 \\ 0 & 0 & 0 & -1/2c & 0 & 1/2c \\ 0 & -1/L & -1/2c & 1/L & 1/2c & 0 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ u_D \\ v_D \\ u_F \\ v_F \end{Bmatrix}$$

$$\epsilon_x = \frac{u_D}{L} = \frac{3PL}{4Etc^2}, \quad \epsilon_y = \frac{v_F - v_D}{2c} = 0,$$

$$\gamma_{xy} = -\frac{u_D}{2c} + \frac{v_D}{L} + \frac{u_F}{2c} = \frac{u_F}{c} + \frac{v_D}{L} = -\frac{PL^2}{4Etc^3}$$

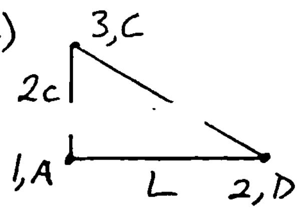
With $\nu = 0$, $\sigma_x = E\epsilon_x = \frac{3PL}{4tc^2}$, $\sigma_y = E\epsilon_y = 0$, $\tau_{xy} = \frac{E}{2}\gamma_{xy} = -\frac{PL^2}{8tc^3}$

Beam theory, top & bottom: $|\sigma_x| = \frac{P(L-x)c}{I} = \frac{3P(L-x)}{2tc^2}$

Beam theory, neutral axis: $\tau_{xy} = \frac{3}{2}\frac{P}{2tc} = \frac{3P}{4tc}$

With σ_x and τ_{xy} from FEA, $\frac{\tau_{xy}}{\sigma_x} = -\frac{L}{6c} \longrightarrow -\infty$ as $\frac{L}{c} \to \infty$

(b)



Obtain $[\underset{\sim}{B}]$ from Eq. 3.4-10

$$\{\underset{\sim}{\epsilon}\} = [\underset{\sim}{B}]\{\underset{\sim}{d}\} = \begin{bmatrix} -1/L & 0 & 1/L & 0 & 0 & 0 \\ 0 & -1/2c & 0 & 0 & 0 & 1/2c \\ -1/2c & -1/L & 0 & 1/L & 1/2c & 0 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ u_D \\ v_D \\ 0 \\ 0 \end{Bmatrix}$$

$$\epsilon_x = \frac{u_D}{L} = \frac{3PL}{4Etc^2}, \quad \epsilon_y = 0, \quad \gamma_{xy} = \frac{v_D}{L} = \frac{PL^2}{2Etc^3}$$

With $\nu = 0$, $\sigma_x = E\epsilon_x = \frac{3PL}{4tc^2}$, $\sigma_y = E\epsilon_y = 0$, $\tau_{xy} = \frac{E}{2}\gamma_{xy} = \frac{PL^2}{4tc^3}$

Beam theory: as in part (a)

With σ_x and τ_{xy} from FEA, $\frac{\tau_{xy}}{\sigma_x} = \frac{L}{3c} \longrightarrow \infty$ as $\frac{L}{c} \to \infty$

$\boxed{3.4\text{-}3}$

(a) The only nonzero d.o.f. are u_3 and v_3. From Eq. 3.4-10,

$$\begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix} = [\underset{\sim}{B}]\{\underset{\sim}{d}\} = \begin{bmatrix} 0 & 0 \\ 0 & 1/a \\ 1/a & 0 \end{bmatrix} \begin{Bmatrix} u_3 \\ v_3 \end{Bmatrix} , \quad \text{For } \nu = 0,$$

$$[\underset{\sim}{k}] = Et \frac{a^2}{2} \begin{bmatrix} 0 & 0 & 1/a \\ 0 & 1/a & 0 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & \\ & & 1/2 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$[\underset{\sim}{k}] = \frac{Et}{2} \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1/2 & 0 \end{bmatrix} = \frac{Et}{2} \begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix}$$

(b) 4 triangles each resist P with ϵ_y stiffness $Et/4$
4 triangles each resist P with γ_{xy} stiffness $Et/2$
Net stiffness $= 4\left(\frac{Et}{4} + \frac{Et}{2}\right) = 3Et$, so $v = \frac{P}{3Et}$
at center node (independent of a).

$\boxed{3.4\text{-}4}$ With u_1 and v_1 the only nonzero d.o.f., $\epsilon_x = 0$, $\epsilon_y = \frac{v_1}{b}$, $\gamma_{xy} = \frac{u_1}{b}$

$$\{\underset{\sim}{\epsilon}\} = [\underset{\sim}{B}]\{\underset{\sim}{d}\} = \frac{1}{b}\begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{Bmatrix} u_1 \\ v_1 \end{Bmatrix}$$

For $\nu = 0$,

$$[\underset{\sim}{k}] = E(abt)\frac{1}{b^2}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 1 & & \\ & 1 & \\ & & 1/2 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$[\underset{\sim}{k}] = \frac{Eat}{b}\begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 1/2 & 0 \end{bmatrix} = \frac{Eat}{b}\begin{bmatrix} 1/2 & 0 \\ 0 & 1 \end{bmatrix}$$

$\boxed{3.5-1}$

(a) Let $\phi = u$ or v. ϕ depends only on y. Assume

$$\phi = c_1 + c_2 y + c_3 y^2 \qquad (a)$$

Substitute:

$\phi = 0$ at $y = 0$, $\therefore c_1 = 0$

$\phi = 0$ at $y = b$, $\quad 0 = c_2 b + c_3 b^2$

$\phi = 1$ at $y = 2b$, $\quad 1 = c_2(2b) + c_3(4b^2)$

Solve: $\quad c_2 = -\dfrac{1}{2b}, \quad c_3 = \dfrac{1}{2b^2}$

Eq. (a) becomes $\quad \phi = \underbrace{-\dfrac{y}{2b} + \dfrac{y^2}{2b^2}}_{N_3}$

(b) On $y = 0$, $N_4 = 1 - \left(\dfrac{x}{a}\right)^2$

Zero at $x = \pm a$, unity at $x = 0$

On $y = b$, $N_4 = -\left(\dfrac{x}{a}\right)^2 + \dfrac{1}{4}$

Zero at $x = \pm(a/2)$

At $y = 2b$, $N_4 = 1 - 2 + 1 = 0$

(c) Here $u = N_3 u_3 + N_4 u_4$, $v = N_3 v_3 + N_4 v_4$

$$\epsilon_x = \frac{\partial N_3}{\partial x} u_3 + \frac{\partial N_4}{\partial x} u_4 = -\frac{2x}{a^2} u_4$$

$$\epsilon_y = \frac{\partial N_3}{\partial y} v_3 + \frac{\partial N_4}{\partial y} v_4$$

$$= \left(-\frac{1}{2b} + \frac{y}{b^2}\right) v_3 + \left[-\frac{1}{b} + \frac{y}{2b^2}\right] v_4$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \left(- - -\right) u_3 + \left[- - -\right] u_4 - \frac{2x}{a^2} v_4$$

$\boxed{3.6\text{-}1}$

$$\epsilon_x = a_2 + a_4 y \,, \quad \epsilon_y = a_7 + a_8 x$$

$$\gamma_{xy} = a_3 + a_4 x + a_6 + a_8 y$$

$$\sigma_{x,x} = \frac{E}{1-\nu^2}(\epsilon_x + \nu\epsilon_y)_{,x} = \frac{E}{1-\nu^2}(\nu a_8)$$

$$\sigma_{y,y} = \frac{E}{1-\nu^2}(\epsilon_y + \nu\epsilon_x)_{,y} = \frac{E}{1-\nu^2}(\nu a_4)$$

$$\tau_{xy} = \frac{E}{2(1+\nu)}\gamma_{xy} \qquad \tau_{xy,x} = \frac{E}{2(1+\nu)} a_4$$

Subs. into $\qquad \tau_{xy,y} = \dfrac{E}{2(1+\nu)} a_8$

$$\left.\begin{aligned} \sigma_{x,x} + \tau_{xy,y} + F_x &= 0 \\ \sigma_{y,y} + \tau_{xy,x} + F_y &= 0 \end{aligned}\right\} \text{yields} \left\{\begin{aligned} C E a_8 + F_x &= 0 \\ C E a_4 + F_y &= 0 \end{aligned}\right.$$

where $C = \dfrac{\nu}{1-\nu^2} + \dfrac{1}{2(1+\nu)} = \dfrac{1}{2(1-\nu)}$

Eqs. satisfied if $a_4 = a_8 = 0$ when $F_x = F_y = 0$ or if particular values for F_x & F_y const.

3.6-2

(a)
$$\{\underset{\sim}{d}\} = \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix} = \begin{Bmatrix} a_1 - a_2 a - a_3 b \\ a_4 - a_5 a - a_6 b \\ a_1 + a_2 a - a_3 b \\ a_4 + a_5 a - a_6 b \\ a_1 + a_2 a + a_3 b \\ a_4 + a_5 a + a_6 b \\ a_1 - a_2 a + a_3 b \\ a_4 - a_5 a + a_6 b \end{Bmatrix}$$

Eqs. 3.6-4 and 3.6-5 show that

$$[\underset{\sim}{N}]\{\underset{\sim}{d}\} = \begin{Bmatrix} a_1 + a_2 x + a_3 y \\ a_4 + a_5 x + a_6 y \end{Bmatrix}$$

(b)
$$\epsilon_x = \frac{1}{4ab}\begin{bmatrix} -(b-y) & 0 & b-y & 0 & b+y & 0 & -(b+y) & 0 \end{bmatrix}\{\underset{\sim}{d}\}$$

$$= a_2$$

$$\epsilon_y = \frac{1}{4ab}\begin{bmatrix} 0 & -(a-x) & 0 & -(a+x) & 0 & a+x & 0 & a-x \end{bmatrix}\{\underset{\sim}{d}\}$$

$$= a_6$$

$$\gamma_{xy} = \frac{1}{4ab}\begin{bmatrix} -(a-x) & -(b-y) & -(a+x) & b-y \\ a+x & b+y & a-x & -(b+y) \end{bmatrix}\{\underset{\sim}{d}\}$$

$$= a_3 + a_5$$

$\boxed{3.6\text{-}3}$ Beam theory will give $u_F = -u_D$, $v_F = v_D$

(a) From Eq. 3.6-6,

$$\begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix} = \frac{1}{4ab} \begin{bmatrix} \cdots & b-y & 0 & b+y & 0 & \cdots \\ \cdots & 0 & -(a+x) & 0 & a+x & \cdots \\ \cdots & -(a+x) & b-y & a+x & b+y & \cdots \end{bmatrix} \begin{Bmatrix} \vdots \\ u_D \\ v_D \\ -u_D \\ v_D \\ \vdots \end{Bmatrix}$$

From beam theory, Prob. 3.4-1, $u_D = -\dfrac{3ML}{2Etc^2}$, $v_D = -\dfrac{3ML^2}{4Etc^3}$

Hence

$$\epsilon_x = \frac{3MLy}{4Etabc^2}, \quad \epsilon_y = 0, \quad \gamma_{xy} = \frac{3ML(a+x)}{4abEtc^2} - \frac{3ML^2}{8aEtc^3}$$

For $\nu = 0$, $\sigma_x = E\epsilon_x = \dfrac{3MLy}{4tabc^2}$, $\sigma_y = E\epsilon_y = 0$,

$$\tau_{xy} = \frac{E}{2}\gamma_{xy} = \frac{3ML(a+x)}{8abtc^2} - \frac{3ML^2}{16atc^3}$$

But $L = 2a$ and $c = b$, so $\sigma_x = \dfrac{3My}{2tb^3}$, $\sigma_y = 0$, $\tau_{xy} = \dfrac{3Mx}{4b^3t}$
(in the coordinate system of Fig. 3.6-1). In this system,
$\sigma_x = \dfrac{My}{I} = \dfrac{3My}{2tb^3}$, $\sigma_y = \tau_{xy} = 0$ according to beam theory.

(b) From beam theory, Prob. 3.4-2, $u_D = \dfrac{3PL^2}{4Etc^2}$, $v_D = \dfrac{PL^3}{2Etc^3}$

Hence

$$\epsilon_x = -\frac{3PL^2y}{8abEtc^2}, \quad \epsilon_y = 0, \quad \gamma_{xy} = \frac{PL^3}{4aEtc^3} - \frac{3PL^2(a+x)}{8abEtc^2}$$

For $\nu = 0$, $\sigma_x = E\epsilon_x = -\dfrac{3PL^2y}{8abtc^2}$, $\sigma_y = E\epsilon_y = 0$,

$$\tau_{xy} = \frac{E}{2}\gamma_{xy} = \frac{PL^3}{8atc^3} - \frac{3PL^2(a+x)}{16abtc^2}$$

But $L = 2a$ and $c = b$, so $\sigma_x = \dfrac{3Pay}{2tb^3}$, $\sigma_y = 0$, $\tau_{xy} = \dfrac{Pa}{4tb^3}(a-3x)$
(in the coordinate system of Fig. 3.6-1). In this system,

$$\sigma_x = \frac{My}{I} = \frac{P(a-x)y}{t(2b)^3/12} = \frac{3P(a-x)y}{2tb^3}, \quad \sigma_y = 0, \text{ and}$$

$$\tau_{xy} = \frac{3}{2}\frac{P}{2tb} = \frac{3P}{4tb} \quad \text{according to beam theory.}$$

$\boxed{3.6-4}$

(a) $K_{48,39}$ is associated with v_{24} and u_{20}. Since node 24 is not in element j, el. j does not contribute to $K_{48,39}$

(b) $K_{37,37} \leftarrow k_{1,1}$ of el. j. $[\underset{\sim}{E}]$ is diagonal (since $\nu = 0$), $[\underset{\sim}{E}] = 10^7 \lceil 1 \quad 1 \quad \tfrac{1}{2} \rfloor$. We need only col. 1 of $[\underset{\sim}{B}]$, Eq. 3.6-6

$$k_{1,1} = \frac{1}{16} \int_{-1}^{1}\int_{-1}^{1} \begin{Bmatrix} -(1-y) \\ 0 \\ -(1-x) \end{Bmatrix}^T \begin{Bmatrix} -(1-y) \\ 0 \\ -\frac{1}{2}(1-x) \end{Bmatrix} 10^7 (1)\, dx\, dy$$

After integration, $k_{1,1} = 10^7/2$.

(c) $K_{59,61} \leftarrow k_{3,5}$ of el. j. Need cols. 3 & 5 of $[\underset{\sim}{B}]$, Eq. 3.6-6. As in (b),

$$k_{3,5} = \frac{1}{16} \int_{-1}^{1}\int_{-1}^{1} \begin{Bmatrix} 1-y \\ 0 \\ -(1+x) \end{Bmatrix}^T \begin{Bmatrix} 1+y \\ 0 \\ \frac{1}{2}(1+x) \end{Bmatrix} 10^7 (1)\, dx\, dy$$

After integration, $k_{3,5} = 0$

$\boxed{3.6\text{-}5}$

Method 1: Evaluate $u = a_1 + a_2 x + a_3 y + a_4 xy$ at nodes, solve for a's, gather coefficients of u_1, u_2, u_3, u_4.

$u_1 = a_1$ Node 1

$u_2 = u_1 + a_2(2a)$; $a_2 = \dfrac{u_2 - u_1}{2a}$ Node 2

$u_4 = u_1 + a_3(2b)$; $a_3 = \dfrac{u_4 - u_1}{2b}$ Node 4

$$u_3 = u_1 + \frac{u_2 - u_1}{2a} 2a + \frac{u_4 - u_1}{2b} 2b + a_4 (2a)(2b)$$

from which $a_4 = \dfrac{u_1 - u_2 + u_3 - u_4}{4ab}$

$$u = u_1 + \frac{u_2 - u_1}{2a} x + \frac{u_4 - u_1}{2b} y + \frac{u_1 - u_2 + u_3 - u_4}{4ab} xy$$

$$u = \left(1 - \frac{x}{2a} - \frac{y}{2b} + \frac{xy}{4ab}\right) u_1 + \left(\frac{x}{2a} - \frac{xy}{4ab}\right) u_2 + \left(\frac{xy}{4ab}\right) u_3 + \left(\frac{y}{2b} - \frac{xy}{4ab}\right) u_4$$

Coefficients of the u_i are the N_i.

Method 2: Take the product of one-dimensional N_i (as for a bar element).

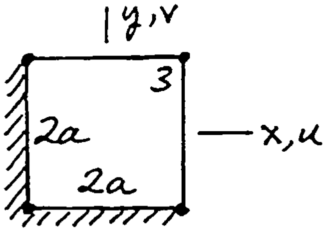
$$N_1 = \frac{2a - x}{2a} \frac{2b - y}{2b} \qquad N_2 = \frac{x}{2a} \frac{2b - y}{2b}$$

$$N_3 = \frac{x}{2a} \frac{y}{2b} \qquad N_4 = \frac{2a - x}{2a} \frac{y}{2b}$$

Method 3: In Eqs. 3.6-4, replace x by $x-a$ and y by $y-b$.

$\boxed{3.6-6}$

Consider the lower-left element of the four-element structure.



In the assembled structure, v_3 is the only nonzero d.o.f., so we need only this one stiffness coefficient.

$$\begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix} = [\underset{\sim}{B}]\{\underset{\sim}{d}\} = \frac{1}{4a^2}\begin{Bmatrix} 0 \\ a+x \\ a+y \end{Bmatrix} v_3 \quad \text{(from Eq. 3.6-6)}$$

$$k = \frac{1}{16a^4}\int_{-a}^{a}\int_{-a}^{a} \lfloor 0 \quad a+x \quad a+y \rfloor \begin{bmatrix} E & & \\ & E & \\ & & E/2 \end{bmatrix} \begin{Bmatrix} 0 \\ a+x \\ a+y \end{Bmatrix} t \, dx \, dy$$

$$k = \frac{Et}{16a^4}\int_{-a}^{a}\int_{-a}^{a}\left[(a+x)^2 + \frac{(a+y)^2}{2}\right] dx\, dy = \frac{Et}{16a^4}\left[\frac{16a^4}{3} + \frac{8a^4}{3}\right] = \frac{Et}{2}$$

For the four-element structure, $v_{center} = \dfrac{P/4}{Et/2} = \dfrac{P}{2Et}$

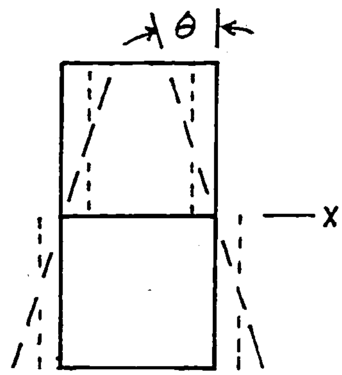
3.6-7

Assume that beam theory would provide almost the exact tip deflection, and recall Eq. 3.6-11:

$$\frac{\theta_{el}}{\theta_b} = \frac{1-\nu^2}{1+\frac{1-\nu}{2}\left(\frac{a}{b}\right)^2}$$

(a) Expect the 2^{nd} mesh to be better, since a/b is half as large in the 2^{nd} mesh

(b) Two elements of the first mesh, with bending deformation:



Smaller dashed lines represent deformation due to ϵ_x in elements. Assume this is exact. Remainder comes from rotation of element faces and suffers from parasitic shear: with $\nu = 0$,

$$\frac{\theta_{el}}{\theta_b} = \frac{1}{1+\frac{1}{2}(1)} = \frac{2}{3}$$

Assume that γ_{xy} and ϵ_x contribute equally, therefore estimate error as the average:

$$e \simeq \frac{1}{2}\left[0 + \left(1-\frac{2}{3}\right)\right] \simeq 17\%$$

For the second mesh, $\frac{a}{b} = \frac{1}{2}$, so with $\nu = 0$

$$\frac{\theta_{el}}{\theta_b} = \frac{1}{1+\frac{1}{2}\left(\frac{1}{2}\right)^2} = 0.89 \qquad \text{i.e. } e \simeq 11\%$$

$\boxed{3.7\text{-}1}$

$$\phi_{152} = \frac{-x(a-x)}{2a^2}\phi_1 + \frac{(a-x)(a+x)}{a^2}\phi_5 + \frac{x(a+x)}{2a^2}\phi_2$$

$$\phi_{43} = \frac{a-x}{2a}\phi_4 + \frac{a+x}{2a}\phi_3$$

$$\phi = \frac{b-y}{2b}\phi_{152} + \frac{b+y}{2b}\phi_{43} = \sum_{i=1}^{5} N_i \phi_i$$

$$N_1 = \frac{-x(a-x)(b-y)}{4a^2b}, \quad N_2 = \frac{x(a+x)(b-y)}{4a^2b}$$

$$N_3 = \frac{(a+x)(b+y)}{4ab}, \quad N_4 = \frac{(a-x)(b+y)}{4ab}$$

$$N_5 = \frac{(a^2-x^2)(b-y)}{2a^2b}$$

3.7-2

(a)
$$\phi_{184} = \sum_{i=1}^{3} \bar{N}_i * (\phi_i \text{ on } x=-a)$$
$$\phi_{597} = \sum_{i=1}^{3} \bar{N}_i * (\phi_i \text{ on } x=0)$$
$$\phi_{263} = \sum_{i=1}^{3} \bar{N}_i * (\phi_i \text{ on } x=+a)$$

for which $\bar{N}_1, \bar{N}_2, \bar{N}_3$ come from Eq. 3.2-7 with $x \to y$, $x_1 \to -b$, $x_2=0$, $x_3 \to b$

$$\bar{N}_1 = \frac{-y(b-y)}{2b^2}, \quad \bar{N}_2 = \frac{(-b-y)(b-y)}{-b(b)} = \frac{b^2-y^2}{b^2}$$

$$\bar{N}_3 = \frac{(-b-y)(-y)}{-2b(-b)} = \frac{y(b+y)}{2b^2}.$$ In 9-node el.,

$$\phi = \frac{-x(a-x)}{2a^2}\phi_{184} + \frac{a^2-x^2}{a^2}\phi_{597} + \frac{x(a+x)}{2a^2}\phi_{263}$$

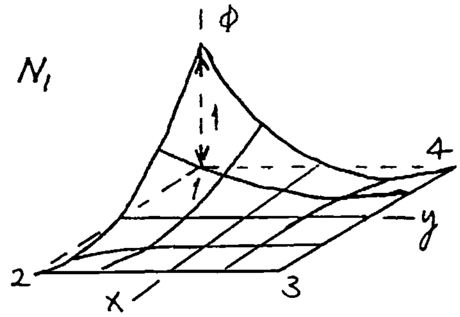
$$\phi = N_a\phi_{184} + N_b\phi_{597} + N_c\phi_{263}$$

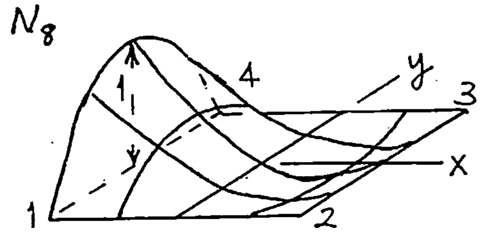
$$\phi = N_a(\bar{N}_1\phi_1 + \bar{N}_2\phi_8 + \bar{N}_3\phi_4) + N_b(\bar{N}_1\phi_5 + \bar{N}_2\phi_9 + \bar{N}_3\phi_7) + N_c(\bar{N}_1\phi_2 + \bar{N}_2\phi_6 + \bar{N}_3\phi_3)$$

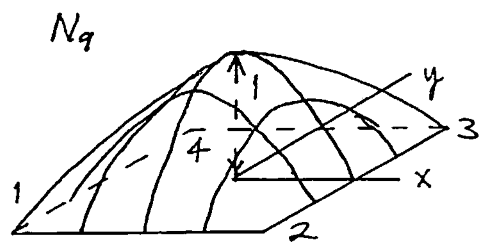
$$\phi = \sum_{i=1}^{9} N_i\phi_i \quad \text{where}$$

$$N_1 = N_a\bar{N}_1 = xy(a-x)(b-y)/4a^2b^2$$
$$N_2 = N_c\bar{N}_1 = -xy(a+x)(b-y)/4a^2b^2$$
$$N_3 = N_c\bar{N}_3 = xy(a+x)(b+y)/4a^2b^2$$
$$N_4 = N_a\bar{N}_3 = -xy(a-x)(b+y)/4a^2b^2$$
$$N_5 = N_b\bar{N}_1 = -(a^2-x^2)y(b-y)/2a^2b^2$$
$$N_6 = N_c\bar{N}_2 = x(a+x)(b^2-y^2)/2a^2b^2$$
$$N_7 = N_b\bar{N}_3 = (a^2-x^2)y(b+y)/2a^2b^2$$
$$N_8 = N_a\bar{N}_2 = -x(a-x)(b^2-y^2)/2a^2b^2$$
$$N_9 = N_b\bar{N}_2 = (a^2-x^2)(b^2-y^2)/a^2b^2$$

(b)



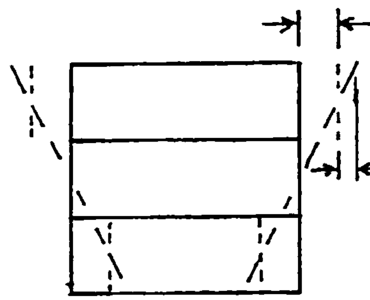




$\boxed{3.8\text{-}1}$ Similar to Problem 3.6-7. Again use Eq. 3.6-11 with $\nu = 0$ for simplicity. Best mesh to worst, with estimated error e :

(c) Aspect ratio $\frac{1}{3}$, $\dfrac{1}{1+\frac{1}{2}\left(\frac{1}{3}\right)^2} = 0.947$, $e \simeq 1 - 0.947 = 5.3\%$

(b) Aspect ratio 3



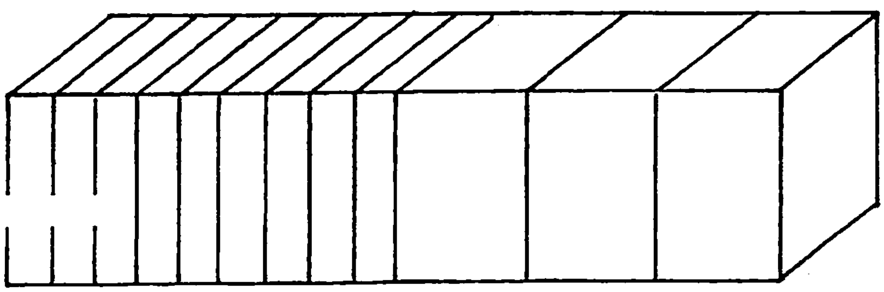
← 2 parts (due to ϵ_x in top & bottom layers – assume this is exact)

← 1 part (due to bending of each element; influenced by parasitic shear)

$$\frac{2}{3} + \frac{1}{3}\left(\frac{1}{1+\frac{1}{2}(3)^2}\right) = 0.727, \quad e \simeq 1 - 0.727 = 27\%$$

(c) Aspect ratio 1, $\dfrac{1}{1+\frac{1}{2}(1)^2} = 0.667$, $e \simeq 1 - 0.667 = 33\%$

A better 12-element mesh (for the given loading):



$\boxed{3.9\text{-}1}$

Rigid-body lateral translation $\bar{u}$: $\{\underset{\sim}{d}\} = \lfloor \bar{u} \quad 0 \quad \bar{u} \quad 0 \rfloor^T$

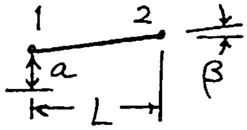
Rigid-body rotation through small angle θ about node 1:

$$\{\underset{\sim}{d}\} = \lfloor 0 \quad \theta \quad L\theta \quad \theta \rfloor^T$$

In both cases, straightforward multiplcation shows that $[\underset{\sim}{B}]\{\underset{\sim}{d}\}$ and $[\underset{\sim}{k}]\{\underset{\sim}{d}\}$ are both zero.

$\boxed{3.9\text{-}2}$

(a)



$w = a + \beta x$ gives

$w_1 = a\ ,\ \theta_1 = \beta$

$w_2 = a + \beta L,\ \theta_2 = \beta$

Subs. into given field; get $w = a + \beta x$

OK

(b) From given field, $w_{,xx} = (\theta_2 - \theta_1)/L$

By beam theory: let M_0 = const. moment

$EI\, w_{,xx} = M_0$ (a)

$\int_{\theta_1}^{\theta_2} EI\, dw_{,x} = \int_0^L M_0\, dx$

$EI(\theta_2 - \theta_1) = M_0 L$ (b)

Eliminate M_0 between (a) & (b); get

$w_{,xx} = (\theta_2 - \theta_1)/L$

(c) $w_{,xx} = \lfloor B \rfloor \{d\} = \left\lfloor 0, -\frac{1}{L}, 0, \frac{1}{L} \right\rfloor \lfloor w_1\ \theta_1\ w_2\ \theta_2 \rfloor^T$

$$[k] = \int_0^L \lfloor B \rfloor^T \lfloor B \rfloor EI\, dx = \frac{EI}{L}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{bmatrix}$$

This $[k]$ offers no resistance to w_1 & w_2 and develops no nodal forces in response to θ_1 & θ_2. $[k]$ has rank 1.

$\boxed{3.9\text{-}3}$

(a)

Evaluate $u = \lfloor 1, x^2 \rfloor \{\underset{\sim}{a}\}$ at nodes.

$$\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & L^2 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix}, \quad \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} = \frac{1}{L^2} \begin{bmatrix} L^2 & 0 \\ -1 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = [\underset{\sim}{A}]^{-1} \{\underset{\sim}{u}\}$$

$$u = \lfloor 1, x^2 \rfloor [\underset{\sim}{A}]^{-1} \{\underset{\sim}{u}\} = \left\lfloor \frac{L^2 - x^2}{L^2} \quad \frac{x^2}{L^2} \right\rfloor \{\underset{\sim}{u}\}$$

$$\epsilon_x = \left\lfloor -\frac{2x}{L^2} \quad \frac{2x}{L^2} \right\rfloor \{\underset{\sim}{u}\} = \lfloor \underset{\sim}{B} \rfloor \{\underset{\sim}{u}\}$$

$$[\underset{\sim}{k}] = \int_0^L \lfloor \underset{\sim}{B} \rfloor^T \lfloor \underset{\sim}{B} \rfloor AE \, dx = \frac{4AE}{3L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Too large by factor $\frac{4}{3}$. Source is defective u: no x^1 term present, so state of constant ϵ_x is not possible.

(b)

Evaluate $u = \lfloor x, x^2 \rfloor \{\underset{\sim}{a}\}$ at nodes.

$$\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{bmatrix} -\frac{L}{2} & \frac{L^2}{4} \\ \frac{L}{2} & \frac{L^2}{4} \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix}, \quad \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} = \frac{1}{L^2} \begin{bmatrix} -L & L \\ 2 & 2 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = [\underset{\sim}{A}]^{-1} \{\underset{\sim}{u}\}$$

$$u = \lfloor x, x^2 \rfloor [\underset{\sim}{A}]^{-1} \{\underset{\sim}{u}\} = \left\lfloor \frac{2x^2 - Lx}{L^2} \quad \frac{2x^2 + Lx}{L^2} \right\rfloor \{\underset{\sim}{u}\}$$

$$\epsilon_x = \left\lfloor \frac{4x - L}{L^2} \quad \frac{4x + L}{L^2} \right\rfloor \{\underset{\sim}{u}\} = \lfloor \underset{\sim}{B} \rfloor \{\underset{\sim}{u}\}$$

$$[\underset{\sim}{k}] = \int_{-L/2}^{L/2} \lfloor \underset{\sim}{B} \rfloor^T \lfloor \underset{\sim}{B} \rfloor AE \, dx = \frac{AE}{3L} \begin{bmatrix} 7 & 1 \\ 1 & 7 \end{bmatrix}$$

[illegible] $,2\}$ when $\{\underset{\sim}{d}\} = \{1, 1\}$ (rigid body translation). Source of trouble is defective u: no constant term present.

3.9-4

Exact ans.: $u_2 = \frac{PL}{AE}$, $u_3 = 2\frac{PL}{AE}$

(a) $\frac{4AE}{3L}\begin{bmatrix} 2 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ P \end{Bmatrix}$, $\begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \frac{PL}{AE}\begin{Bmatrix} 3/4 \\ 3/2 \end{Bmatrix}$

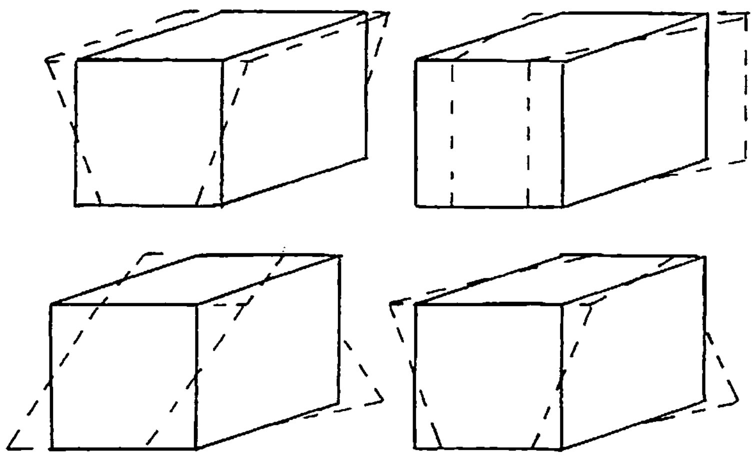
$\sigma_{x=0} = E\lfloor 0, 0 \rfloor\{\underset{\sim}{d}\} = 0$ (Exact σ_x is P/A for all x)

(b) $\frac{AE}{3L}\begin{bmatrix} 14 & 1 \\ 1 & 7 \end{bmatrix}\begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ P \end{Bmatrix}$, $\begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \frac{PL}{AE}\begin{Bmatrix} -3/97 \\ 42/97 \end{Bmatrix}$

$\sigma_{x=0} = E\left\lfloor \frac{-3L}{L^2}, \frac{-L}{L^2} \right\rfloor\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = -\frac{1}{L}\frac{-3PL}{97AE} = \frac{3P}{97A}$

at $x=0$ in structure; at $x=-\frac{L}{2}$ in element.

3.9-5



With the node numbering of Fig. 3.8 1a, x-direction nodal d.o.f. of the foregoing states are, with c a constant,

$$\begin{Bmatrix} c \\ c \\ -c \\ -c \\ -c \\ -c \\ c \\ c \end{Bmatrix} \qquad \begin{Bmatrix} c \\ -c \\ -c \\ c \\ -c \\ c \\ c \\ -c \end{Bmatrix}$$

$$\begin{Bmatrix} -c \\ \\ -c \\ c \\ -c \\ c \\ \\ c \end{Bmatrix} \qquad \begin{Bmatrix} c \\ -c \\ c \\ -c \\ -c \\ c \\ -c \\ c \end{Bmatrix}$$

$\boxed{3.10\text{-}1}$

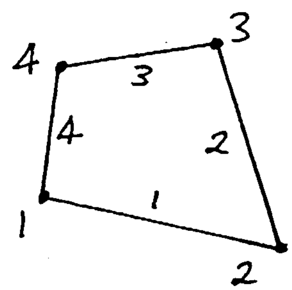
Let $a_1 = \frac{1}{8}(x_2 - x_3)$ $\quad a_2 = \frac{1}{8}(x_3 - x_1)$ $\quad a_3 = \frac{1}{8}(x_1 - x_2)$

$b_1 = \frac{1}{8}(y_3 - y_2)$ $\quad b_2 = \frac{1}{8}(y_1 - y_3)$ $\quad b_3 = \frac{1}{8}(y_2 - y_1)$

$$\begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \\ u_5 \\ v_5 \\ u_6 \\ v_6 \end{Bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ \frac{1}{2} & 0 & -b_3 & \frac{1}{2} & 0 & b_3 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & -a_3 & 0 & \frac{1}{2} & a_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 0 & -b_1 & \frac{1}{2} & 0 & b_1 \\ 0 & 0 & 0 & 0 & \frac{1}{2} & -a_1 & 0 & \frac{1}{2} & a_1 \\ \frac{1}{2} & 0 & b_2 & 0 & 0 & 0 & \frac{1}{2} & 0 & -b_2 \\ 0 & \frac{1}{2} & a_2 & 0 & 0 & 0 & 0 & \frac{1}{2} & -a_2 \end{bmatrix}}_{[\underset{\sim}{T}]} \begin{Bmatrix} u_1 \\ v_1 \\ \omega_1 \\ u_2 \\ v_2 \\ \omega_2 \\ u_3 \\ v_3 \\ \omega_3 \end{Bmatrix}$$

3.10-2

Consider e.g. a quadrilateral element, and apply Eq. 3.10-1 to each side.



$$\frac{\delta_{m1}}{L_1} = \frac{1}{8}(\omega_2 - \omega_1)$$

$$\frac{\delta_{m2}}{L_2} = \frac{1}{8}(\omega_3 - \omega_2)$$

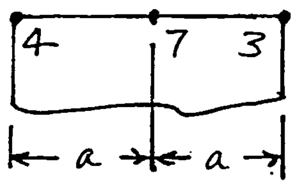
$$\frac{\delta_{m3}}{L_3} = \frac{1}{8}(\omega_4 - \omega_3)$$

$$\frac{\delta_{m4}}{L_4} = \frac{1}{8}(\omega_1 - \omega_4)$$

Add; thus $$\sum_{i=1}^{4} \frac{\delta_{mi}}{L_i} = 0$$

$\boxed{3.11\text{-}1}$

Eq. 3.11-6 : $\begin{Bmatrix} F_4 \\ F_7 \\ F_3 \end{Bmatrix} = \underbrace{\frac{a}{15}\begin{bmatrix} 4 & 2 & -1 \\ 2 & 16 & 2 \\ -1 & 2 & 4 \end{bmatrix}}_{[\underset{\sim}{H}]} \begin{Bmatrix} q_4 \\ q_7 \\ q_3 \end{Bmatrix}$



Unit thickness

(a) $\begin{Bmatrix} F_4 \\ F_7 \\ F_3 \end{Bmatrix} = [\underset{\sim}{H}] \begin{Bmatrix} \sigma \\ 0 \\ -\sigma \end{Bmatrix} = \frac{\sigma a}{3} \begin{Bmatrix} 1 \\ 0 \\ -1 \end{Bmatrix}$

Couple-moment: $M = F_4(2a) = \frac{2\sigma a^2}{3}$

Flexure formula:

$$M = \frac{\sigma I}{c} = \frac{\sigma (2a)^3/12}{a} = \frac{2\sigma a^2}{3}$$

(b) $\begin{Bmatrix} F_4 \\ F_7 \\ F_3 \end{Bmatrix} = [\underset{\sim}{H}] \begin{Bmatrix} 0 \\ \sigma/2 \\ \sigma \end{Bmatrix} = \frac{\sigma a}{3} \begin{Bmatrix} 0 \\ 2 \\ 1 \end{Bmatrix}$

Moment: $M = 2aF_3 + aF_7 = \frac{4\sigma a^2}{3}$

Flexure formula, for section $4a$ units deep:

$$M = \frac{\sigma I}{c} = \frac{\sigma (4a)^3/12}{2a} = \frac{8\sigma a^2}{3}$$

Contribution of half the section is

$$\frac{M}{2} = \frac{4\sigma a^2}{3}$$

(c) $\begin{Bmatrix} F_4 \\ F_7 \\ F_3 \end{Bmatrix} = [\underset{\sim}{H}] \begin{Bmatrix} 0 \\ \tau \\ 0 \end{Bmatrix} = \frac{\tau a}{15} \begin{Bmatrix} 2 \\ 16 \\ 2 \end{Bmatrix}$

Shear force: $F_4 + F_7 + F_3 = \frac{4\tau a}{3}$

Beam theory (parabolic distribution):

$$\tau = \frac{3}{2}\frac{V}{2a}, \quad V = \frac{4\tau a}{3}$$

$\boxed{3.11-2}$

With $[\underset{\sim}{N}]$ from Eq. 3.11-5, and constant q,

$$\begin{Bmatrix} F_1 \\ F_2 \\ F_3 \end{Bmatrix} = \int_{-a}^{a} [\underset{\sim}{N}]\, q\, dx = \int_{-a}^{a} \frac{1}{2a^2} \begin{Bmatrix} x(x-a) \\ 2(a^2-x^2) \\ x(x+a) \end{Bmatrix} q\, dx$$

$$= \frac{q}{2a^2} \begin{Bmatrix} \frac{x^3}{3} - \frac{ax^2}{2} \\ 2a^2x - \frac{2x^3}{3} \\ \frac{x^3}{3} + \frac{ax^2}{2} \end{Bmatrix}_{-a}^{a} = \frac{q}{2a^2} \begin{Bmatrix} 2a^3/3 \\ 8a^3/3 \\ 2a^3/3 \end{Bmatrix} = q \begin{Bmatrix} a/3 \\ 4a/3 \\ a/3 \end{Bmatrix} = F \begin{Bmatrix} 1/6 \\ 2/3 \\ 1/6 \end{Bmatrix}$$

where $F = 2qa$

$\boxed{3.11-3}$

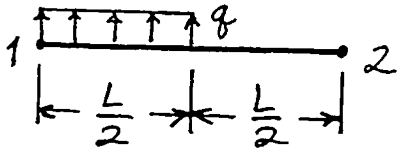
$M_1 \, \delta\theta_{z1} = \int_0^L v \, q \, dx$ *where* $v = N_2 \, \delta\theta_{z1}$ *and* q *is constant*

Hence $M_1 = q \int_0^L \left(x - \frac{2x^2}{L} + \frac{x^3}{L^2} \right) dx = q L^2 \left(\frac{1}{2} - \frac{2}{3} + \frac{1}{4} \right) = \frac{qL^2}{12}$

$\boxed{3.11\text{-}4}$

(a)

$$\{\underset{\sim}{r}_e\} = \int_0^{L/2} \lfloor \underset{\sim}{N} \rfloor^T dx = q \begin{Bmatrix} x - \dfrac{x^3}{L^2} + \dfrac{x^4}{2L^3} \\ \dfrac{x^2}{2} - \dfrac{2x^3}{3L} + \dfrac{x^4}{4L^2} \\ \dfrac{x^3}{L^2} - \dfrac{x^4}{2L^3} \\ -\dfrac{x^3}{3L} + \dfrac{x^4}{4L^2} \end{Bmatrix}_0^{L/2}$$

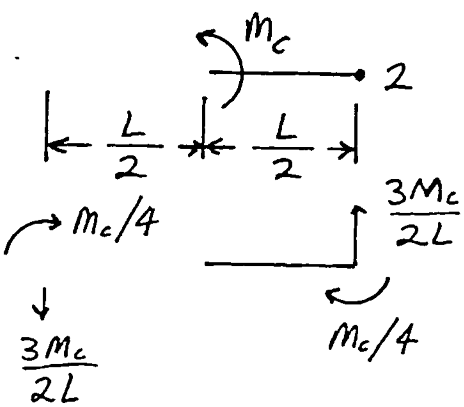


$$\{\underset{\sim}{r}_e\} = q \begin{Bmatrix} 0.40625L \\ 0.05729L^2 \\ 0.09375L \\ -0.02604L^2 \end{Bmatrix} \qquad \begin{aligned} \Sigma F_z &= \frac{qL}{2} \\ \Sigma M_0 &= r_{e3} L + r_{e2} + r_{e4} \\ &= \frac{qL^2}{8} \end{aligned}$$

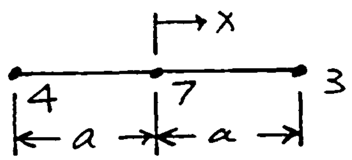
(b) $\{\underset{\sim}{r}_e\} = \left[\dfrac{d\underset{\sim}{N}}{dx}\right]_*^T M_c$ where $\left[\dfrac{d\underset{\sim}{N}}{dx}\right]_*^T$ is evaluated at $x = \dfrac{L}{2}$

From N_i in Fig. 3.2-4 we obtain

$$\begin{Bmatrix} -\dfrac{6x}{L^2} + \dfrac{6x^2}{L^3} \\ 1 - \dfrac{4x}{L} + \dfrac{3x^2}{L^2} \\ \dfrac{6x}{L^2} - \dfrac{6x^2}{L^3} \\ -\dfrac{2x}{L} + \dfrac{3x^2}{L^2} \end{Bmatrix}_{x=\frac{L}{2}} = \begin{Bmatrix} -\dfrac{6}{2L} + \dfrac{6}{4L} \\ 1 - 2 + \dfrac{3}{4} \\ \dfrac{6}{2L} - \dfrac{6}{4L} \\ -1 + \dfrac{3}{4} \end{Bmatrix} = \begin{Bmatrix} -\dfrac{3}{2L} \\ -\dfrac{1}{4} \\ \dfrac{3}{2L} \\ -\dfrac{1}{4} \end{Bmatrix}$$



$\boxed{3.11-5}$



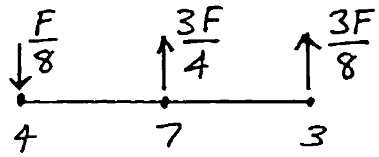
$$v = \lfloor N_4 \quad N_7 \quad N_3 \rfloor^T \begin{Bmatrix} v_4 \\ v_7 \\ v_3 \end{Bmatrix}$$

Apply Eq. 3.2-7: $N_4 = \dfrac{-x(a-x)}{a(2a)} = -\dfrac{x(a-x)}{2a^2}$

$$N_7 = \frac{(-a-x)(a-x)}{-a(a)} = \frac{a^2-x^2}{a^2}$$

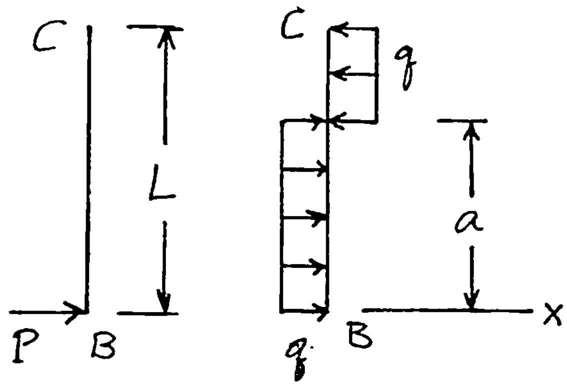
$$N_3 = \frac{(-a-x)(-x)}{-2a(-a)} = \frac{x(a+x)}{2a^2}$$

$$\begin{Bmatrix} F_4 \\ F_7 \\ F_3 \end{Bmatrix} = \begin{Bmatrix} N_4 \\ N_7 \\ N_3 \end{Bmatrix}_{x=\frac{a}{2}} F = \begin{Bmatrix} -\dfrac{(a/2)^2}{2a^2} \\ \dfrac{a^2-(a/2)^2}{a^2} \\ \dfrac{a/2\,(3a/2)}{2a^2} \end{Bmatrix} F = \begin{Bmatrix} -1/8 \\ 3/4 \\ 3/8 \end{Bmatrix} F$$



$F_4 + F_7 + F_3 = F$ ✓ $\qquad \Sigma M_4 = \dfrac{3F}{4}a + \dfrac{3F}{8}(2a) = \dfrac{3Fa}{4}$ ✓

$\boxed{3.11\text{-}6}$



where $q = \dfrac{(1+\sqrt{2})P}{L}$

$a = \dfrac{L}{\sqrt{2}}$

x-direction force: $qa - q(L-a) = q(2a-L)$

$$= \frac{(1+\sqrt{2})P}{L}\left(\frac{2L}{\sqrt{2}} - L\right)$$

$$= \frac{(1+\sqrt{2})(2-\sqrt{2})}{\sqrt{2}}P$$

$$= \frac{2+\sqrt{2}-2}{\sqrt{2}}P = P \checkmark$$

Moment about B: $qa\dfrac{a}{2} - q(L-a)\dfrac{L+a}{2} = \dfrac{q}{2}(2a^2-L^2)$

$$= \frac{q}{2}(L^2-L^2) = 0 \checkmark$$

$\boxed{3.11\text{-}7}$

Evaluate N_i of Eqs. 3.6-4 at Q $(x = \frac{a}{2},\ y = 0)$:

$$\{\underset{\sim}{r}_e\} = [\underset{\sim}{N}]_Q^T \begin{Bmatrix} P_x \\ P_y \end{Bmatrix} = \frac{1}{4ab} \begin{bmatrix} ab/2 & 0 & 3ab/2 & 0 & 3ab/2 & 0 & ab/2 & 0 \\ 0 & ab/2 & 0 & 3ab/2 & 0 & 3ab/2 & 0 & ab/2 \end{bmatrix}^T \begin{Bmatrix} P_x \\ P_y \end{Bmatrix}$$

$$\{\underset{\sim}{r}_e\} = \frac{1}{8} \begin{bmatrix} 1 & 0 & 3 & 0 & 3 & 0 & 1 & 0 \\ 0 & 1 & 0 & 3 & 0 & 3 & 0 & 1 \end{bmatrix}^T \begin{Bmatrix} 8 \\ 6 \end{Bmatrix}$$

$$\{\underset{\sim}{r}_e\} = \left\lfloor 1 \quad \frac{3}{4} \quad 3 \quad \frac{9}{4} \quad 3 \quad \frac{9}{4} \quad 1 \quad \frac{3}{4} \right\rfloor^T$$

$\boxed{3.11\text{-}8}$ Get N_i from Eq. 3.2-7. Let $a = \frac{L}{2}$. $B_i = \frac{d}{dx} N_i$

$$N_1 = -\frac{x(a-x)}{2a^2} \qquad N_2 = \frac{a^2-x^2}{a^2} \qquad N_3 = \frac{x(a+x)}{2a^2}$$

$$B_1 = -\frac{a-2x}{2a^2} \qquad B_2 = -\frac{2x}{a^2} \qquad B_3 = \frac{a+2x}{2a^2}$$

$$\{\underset{\sim}{r}_e\} = \int [\underset{\sim}{B}]^T [\underset{\sim}{E}] \{\underset{\sim}{\epsilon}_0\} dV = \int_{-a}^{a} \begin{Bmatrix} B_1 \\ B_2 \\ B_3 \end{Bmatrix} E \left(\alpha T_3 \frac{a+x}{2a} \right) A\,dx$$

$$\{\underset{\sim}{r}_e\} = \frac{EA\alpha T_3}{4a^3} \int_{-a}^{a} \begin{Bmatrix} -(a-2x) \\ -4x \\ a+2x \end{Bmatrix} (a+x)\,dx = \frac{EA\alpha T_3}{4a^3} \int_{-a}^{a} \begin{Bmatrix} -a^2+x+2x^2 \\ -4ax-4x^2 \\ a^2+3x+2x^2 \end{Bmatrix} dx$$

$$\{\underset{\sim}{r}_e\} = \frac{EA\alpha T_3}{4a^3} \begin{Bmatrix} -2a^3+4a^3/3 \\ -8a^3/3 \\ 2a^3+4a^3/3 \end{Bmatrix} = \frac{EA\alpha T_3}{4a^3} \begin{Bmatrix} -2a^3/3 \\ -8a^3/3 \\ 10a^3/3 \end{Bmatrix} = \frac{EA\alpha T_3}{6} \begin{Bmatrix} -1 \\ -4 \\ 5 \end{Bmatrix}$$

$\boxed{3.11\text{-}9}$

$$\{\underset{\sim}{r}_e\} = -\int [\underset{\sim}{B}]^T \{\underset{\sim}{\sigma}_0\}\, dV$$

To show $\Sigma r_{ei} = 0$, we ask if $\lfloor \underset{\sim}{I} \rfloor \{\underset{\sim}{r}_e\} = 0$, where $\lfloor \underset{\sim}{I} \rfloor$ is a row vector filled with 1's. We get $\Sigma r_{ei} = 0$ if $\lfloor \underset{\sim}{I} \rfloor \{\underset{\sim}{B}_j\} = 0$, where $\{\underset{\sim}{B}_j\}$ is a column of $[\underset{\sim}{B}]^T$, for all j in $[\underset{\sim}{B}]^T$. Now $[\underset{\sim}{B}] = [\underset{\sim}{\partial}][\underset{\sim}{N}]$, so each row of $[\underset{\sim}{B}]$ (and each column of $[\underset{\sim}{B}]^T$) is a derivative of $\lfloor \underset{\sim}{N} \rfloor$. But $\Sigma N_i = 1$ for C^0 elements, so $\Sigma N_{i,x} = 0$.

$\boxed{3.11\text{-}10}$

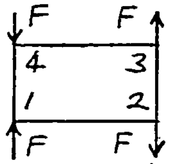
$$[\underset{\sim}{E}]\{\underset{\sim}{\epsilon}_0\} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}\begin{Bmatrix} \alpha T_0 x \\ \alpha T_0 x \\ 0 \end{Bmatrix} = \frac{E\alpha T_0 x}{1+\nu}\begin{Bmatrix} 1 \\ 1 \\ 0 \end{Bmatrix}$$

$$\{\underset{\sim}{r}_e\} = \iint [\underset{\sim}{B}]^T[\underset{\sim}{E}]\{\underset{\sim}{\epsilon}_0\}\, t\, dx\, dy, \quad N_i = \frac{(a \pm x)(b \pm y)}{4ab}$$

$N_{i,x}$ indep. of x but $\{\underset{\sim}{\epsilon}_0\}$ linear in x, so integration w.r.t. x with limits $-a$ to $+a$ yields zero x-direction nodal loads.

$$\underset{y\text{-dir.}}{\{\underset{\sim}{r}_e\}} = \frac{E\alpha T_0}{(1+\nu)4ab}\int_{-b}^{b}\int_{-a}^{a}\begin{Bmatrix} -(a-x) \\ -(a+x) \\ (a+x) \\ (a-x) \end{Bmatrix} x\, dx\, dy$$

$$\underset{y\text{-dir.}}{\{\underset{\sim}{r}_e\}} = \frac{E a^2 \alpha T_0}{3(1+\nu)}\begin{Bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{Bmatrix} = \begin{Bmatrix} F \\ -F \\ F \\ -F \end{Bmatrix}$$



(as applied to structure nodes)

$\boxed{3.11\text{-}11}$

(a) First the nodal loads, all 3 cases.

$$\lfloor \underset{\sim}{N} \rfloor = \left\lfloor \frac{L-s}{L} \quad \frac{s}{L} \right\rfloor, \quad F = \frac{c}{A}(L_T - x)$$

$$\{\underset{\sim}{r}_e\} = \int_0^L \lfloor \underset{\sim}{N} \rfloor^T F \, dV \quad \text{where } dV = A\,ds$$

One element: $s = x$, $L_T = L$, $\{\underset{\sim}{r}_e\} = \frac{cL^2}{6}\begin{Bmatrix}2\\1\end{Bmatrix}$

Two elements: in the first, $s = x$, $L_T = 2L$

in the second, $x = L + s$, $L_T = 2L$

$$\{\underset{\sim}{r}_e\}_1 = \frac{cL^2}{6}\begin{Bmatrix}5\\4\end{Bmatrix}, \quad \{\underset{\sim}{r}_e\}_2 = \frac{cL^2}{6}\begin{Bmatrix}2\\1\end{Bmatrix}$$

Three elements: $L_T = 3L$; in the respective elements, $x = s$, $x = L+s$, $x = 2L+s$.

$$\{\underset{\sim}{r}_e\}_1 = \frac{cL^2}{6}\begin{Bmatrix}8\\7\end{Bmatrix}, \quad \{\underset{\sim}{r}_e\}_2 = \frac{cL^2}{6}\begin{Bmatrix}5\\4\end{Bmatrix}, \quad \{\underset{\sim}{r}_e\}_3 = \frac{cL^2}{6}\begin{Bmatrix}2\\1\end{Bmatrix}$$

(b) • One el. $(L = L_T)$ $\quad \frac{AE}{L} u_2 = \frac{cL^2}{6}$, $\quad u_2 = \frac{cL^3}{6AE}$

$$\sigma_x = E\frac{u_2}{L} = \frac{cL^2}{6A}$$

• Two els. $(L_T = 2L)$ $\quad \frac{AE}{L}\begin{bmatrix}2 & -1\\-1 & 1\end{bmatrix}\begin{Bmatrix}u_2\\u_3\end{Bmatrix} = \frac{cL^2}{6}\begin{Bmatrix}4+2\\1\end{Bmatrix}$

$$\begin{Bmatrix}u_2\\u_3\end{Bmatrix} = \frac{cL^3}{6AE}\begin{Bmatrix}7\\8\end{Bmatrix} = \frac{cL_T^3}{6EA}\begin{Bmatrix}7/8\\1\end{Bmatrix} \quad \text{(exact)}$$

$$\sigma_{x1} = E\frac{u_2}{L} = \frac{7cL^2}{6A} = \frac{7}{24}\frac{cL_T^2}{A} = 0.2917\frac{cL_T^2}{A}$$

$$\sigma_{x2} = E\frac{u_3 - u_2}{L} = \frac{1}{7}\sigma_{x1} = 0.0417\frac{cL_T^2}{A}$$

•

$$\frac{AE}{L}\begin{bmatrix} & & 0\\-1 & 2 & -1\\0 & -1 & 1\end{bmatrix}\begin{Bmatrix}u_2\\u_3\\u_4\end{Bmatrix} = \frac{cL^2}{6}\begin{Bmatrix}7+5\\4+2\\1\end{Bmatrix} \quad \text{3 els. } (L_T = 3L)$$

$$\begin{Bmatrix}u_2\\ \\u_4\end{Bmatrix} \quad \frac{cL^3}{6AE}\begin{Bmatrix}19\\26\\27\end{Bmatrix}$$

(exact)

$$\sigma_{x1} = E\frac{u_2}{L} = 0.3519\frac{cL_T^2}{A}$$

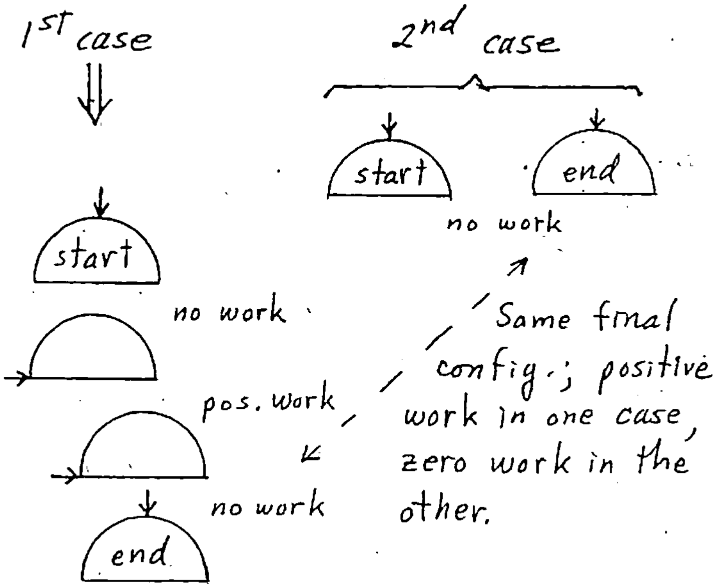
$$\sigma_{x2} = E\frac{u_3 - u_2}{L} = 0.1296\frac{cL_T^2}{A}$$

$$\sigma_{x3} = E\frac{u_4 - u_3}{L} = 0.0185\frac{cL_T^2}{A}$$

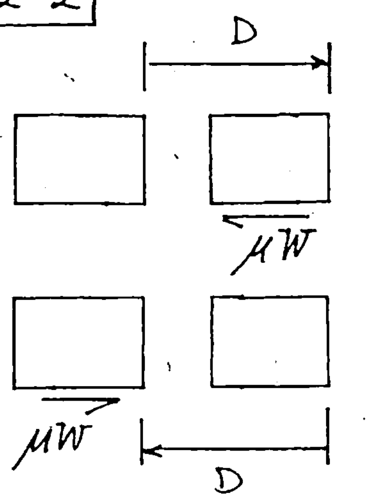
Exact stresses: multipliers of cL_T^2/A are:

$\frac{x}{L_T}$	0	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	$\frac{5}{6}$	1
factor	.5	.3472	.2813	.1250	.0313	.0139	0
	left end			middle			right end

$\boxed{4.2\text{-}1}$



$\boxed{4.2\text{-}2}$



Work of friction force:

$(-\mu W)D$

$\mu W(-D)$

Net work of friction force is $-2\mu WD$
Not zero, so not conservative.