$\boxed{5.1\text{-}1}$

$$\Pi_p = \int_0^{L_T} \left(\frac{1}{2} AE\, \tilde{u}_{,x}^{\,2} - cx\tilde{u} \right) dx - Pu_L$$

Assume $\tilde{u} = a_1 x + a_2 x^2$, then $\tilde{u}_{,x} = a_1 + 2a_2 x$

$$\tilde{u}_{,x}^{\,2} = a_1^2 + 4a_1 a_2 x + 4a_2 x^2$$

$$\Pi_p = \frac{AE}{2}\left(a_1^2 L_T + 2a_1 a_2 L_T^2 + \frac{4}{3} a_2^2 L_T^3\right) - c\left(a_1 \frac{L_T^3}{3} + a_2 \frac{L_T^4}{4}\right) - P\left(a_1 L_T + a_2 L_T^2\right)$$

$$\frac{\partial \Pi_p}{\partial a_1} = 0 = AE\left(a_1 L_T + a_2 L_T^2\right) - \frac{cL_T^3}{3} - PL_T$$

$$\frac{\partial \Pi_p}{\partial a_2} = 0 = AE\left(a_1 L_T^2 + \frac{4}{3} a_2 L_T^3\right) - \frac{cL_T^4}{4} - PL_T^2$$

$$\begin{bmatrix} 1 & L_T \\ L_T & \frac{4}{3} L_T^2 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} = \begin{Bmatrix} \dfrac{cL_T^2}{3AE} + \dfrac{P}{AE} \\ \dfrac{cL_T^3}{4AE} + \dfrac{PL_T}{AE} \end{Bmatrix}$$

$$\begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} = \frac{3}{L_T^2} \begin{bmatrix} \frac{4}{3} L_T^2 & -L_T \\ -L_T & 1 \end{bmatrix} \begin{Bmatrix} \dfrac{cL_T^2}{3AE} + \dfrac{P}{AE} \\ \dfrac{cL_T^3}{4AE} + \dfrac{PL^T}{AE} \end{Bmatrix} = \begin{Bmatrix} \dfrac{7cL_T^2}{12AE} + \dfrac{P}{AE} \\ -\dfrac{cL_T}{4AE} \end{Bmatrix}$$

$$\tilde{u} = \frac{P}{AE} x + \frac{7cL_T^2}{12AE} x - \frac{cL_T}{4AE} x^2$$ same as Eq. 5.1-10

$\boxed{5.1\text{-}2}$

$$\int_0^L W[EI\,\tilde{v}_{,xxxx} - q]\,dx = 0, \quad \tilde{v} = a\underbrace{x(L-x)}_{W}$$

Integrate by parts twice:

$$\int_0^L W_{,x}[-EI\,\tilde{v}_{,xxx}]\,dx - \int_0^L Wq\,dx + EI\underbrace{\left[W\,\tilde{v}_{,xxx}\right]_0^L}_{\downarrow} = 0$$

But $W = 0$ at $x = 0$ and at $x = L$: $[\downarrow] = 0$

$$\int_0^L W_{,xx}[EI\,\tilde{v}_{,xx}]\,dx - \int_0^L Wq\,dx - EI\left[W_{,x}\tilde{v}_{,xx}\right]_0^L = 0$$

But, as nonessential boundary condition, ends are simply supported; $\tilde{v}_{,xx} = 0$ at ends.

Also $\tilde{v}_{,xx} = -2a$ and $W_{,xx} = -2$, so

$$\int_0^L -2\,EI(-2a)\,dx - \int_0^L x(L-x)\,q\,dx = 0 \quad \text{and}$$

$$a = \frac{1}{4EIL}\int_0^L x(L-x)\,q\,dx \ ; \quad \text{center } \tilde{v} = a\frac{L^2}{4}$$

(a) $q = q_0 \sin\frac{\pi x}{L}$. Let $\theta = \frac{\pi x}{L}$; then

$$a = \frac{1}{4EIL}\int_0^L x(L-x)\,q_0 \sin\frac{\pi x}{L}\,dx \quad \text{becomes}$$

$$a = \frac{L^2 q_0}{EI\pi^2}\int_0^\pi \theta\left(1 - \frac{\theta}{\pi}\right)\sin\theta\,d\theta = 0.03225\frac{q_0 L^2}{EI}$$

At center, $\tilde{v} = a\frac{L^2}{4} = 0.00806\frac{q_0 L^4}{EI}$

Exact $v = v_c \sin\frac{\pi x}{L}$ where v_c = center v.

$$EI\,v_{,xxxx} = q, \quad EI\frac{\pi^4}{L^4}v_c \sin\frac{\pi x}{L} = q_0 \sin\frac{\pi x}{L}$$

$$\frac{q_0 L^4}{\pi^4 EI} \qquad 0.01027\frac{q_0 L^4}{EI}$$

Approx. center deflection is 21.5% low.

$M = EI\,v_{,xx}$: at center,

$$\text{exact } M_c = EI\frac{q_0 L^4}{\pi^4 EI}\left(-\frac{\pi^2}{L^2}\right) = 0.1013\,q_0 L^2$$

Approx. $M = EI(-2a) = 0.0645\,q_0 L^2$ (36% low)

(b)

$$a = \frac{q_0}{4EIL}\int_0^L x(L-x)\,dx = \frac{q_0}{4EIL}\frac{L^2}{4}\frac{2L}{3} = \frac{q_0 L^2}{24EI}$$

At center, $\tilde{v} = a\frac{L^2}{4} = \frac{q_0 L^4}{96EI} = 0.01042\frac{q_0 L^4}{EI}$

Exact is $\frac{5q_0 L^4}{384EI} = 0.01302\frac{q_0 L^4}{EI}$

Approx. center deflection is 20.0% low.

At center, exact $M_c = \frac{q_0 L^2}{8}$

Approx. $M = EI(-2a) = \frac{q_0 L^2}{12}$ (33% low)

$\boxed{5.2-1}$

At $x = \frac{L_T}{3}$, from collocation:

$$\tilde{u} = \left(\frac{P}{AE} + \frac{cL_T^2}{2AE}\right)\frac{L_T}{3} - \frac{cL_T}{6AE}\frac{L_T^2}{9} = \frac{PL_T}{3AE} - \frac{4cL_T^3}{27AE}$$

At $x = \frac{L_T}{3}$, exact solution (Eq. 5.1-4):

$$u = \frac{PL_T}{3AE} + \frac{cL_T^2}{2AE}\frac{L_T}{3} - \frac{c}{6AE}\frac{L_T^3}{27} = \frac{PL_T}{3AE} - \frac{4.33cL_T^3}{27AE}$$

That $\tilde{u} \neq u$ at the collocation point does not imply an error. The residual depends on derivatives of $\tilde{u}$. Making the residual vanish at a certain point does not imply that the dependent variable is exact at that point.

$\boxed{5.2\text{-}2}$ $\frac{d^2u}{dx^2} + \frac{c}{AE} = 0$ in V Assume $\tilde{u} = a_1 x + a_2 x^2$

$\frac{du}{dx} = \frac{P}{AE}$ at $x = L_T$ then $R = 2a_2 + \frac{c}{AE}$

$R_B = a_1 + 2a_2 L_T - \frac{P}{AE}$

Collocation Point doesn't matter; $a_2 = -\frac{c}{2AE}$ from $R = 0$

$a_1 - \frac{cL_T}{AE} - \frac{P}{AE} = 0$; $a_1 = \frac{P}{AE} + \frac{cL_T}{AE}$ from $R_B = 0$

$u = \frac{Px}{AE} + \frac{c}{AE}\left(L_T x - \frac{x^2}{2}\right)$ [exact]

Subdomain $\int_0^{L_T}\left(2a_2 + \frac{c}{AE}\right)dx = 0$; $a_2 = -\frac{c}{2AE}$ Same results as above.

Least Squares $\left(\alpha = \frac{1}{L_T}\right)$

$$I = \int_0^{L_T}\left(2a_2 + \frac{c}{AE}\right)^2 dx + \frac{1}{L_T}\left[a_1 + 2a_2 L_T - \frac{P}{AE}\right]^2$$

$$I = \left(2a_2 + \frac{c}{AE}\right)^2 L_T + \frac{1}{L_T}\left[a_1 + 2a_2 L_T - \frac{P}{AE}\right]^2$$

$$\frac{\partial I}{\partial a_1} = \frac{2}{L_T}\left(a_1 + 2a_2 L_T - \frac{P}{AE}\right) = 0, \quad \frac{\partial I}{\partial a_2} = 2L_T\left(2a_2 + \frac{c}{AE}\right)2 + \frac{2}{L_T}[\cdots]2L_T = 0$$

$$\frac{2}{L_T}a_1 + 4a_2 = \frac{2P}{AEL_T}$$

$$4a_1 + 16L_T a_2 = \frac{4P}{AE} - \frac{4cL_T}{AE}$$

Same results as above.

Least Squares Collocation (at $x = \frac{L_T}{3}$ and $x = L_T$ for R, at $x = L$ for R_B. Also set $\alpha = 1/L_T$.)

$$\begin{Bmatrix} R_1 \\ R_2 \\ R_3 \end{Bmatrix} = \begin{bmatrix} 0 & 2 \\ 0 & 2 \\ 1/L_T & 2 \end{bmatrix}\begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} - \begin{Bmatrix} -c/AE \\ -c/AE \\ P/AEL_T \end{Bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 1/L_T \\ 2 & 2 & 2 \end{bmatrix}\begin{bmatrix} 0 & 2 \\ 0 & 2 \\ 1/L_T & 2 \end{bmatrix}\begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} = \begin{bmatrix} 0 & 0 & 1/L_T \\ 2 & 2 & 2 \end{bmatrix}\begin{Bmatrix} -c/AE \\ -c/AE \\ P/AEL_T \end{Bmatrix}$$

$$\begin{bmatrix} 1/L_T^2 & 2/L_T \\ 2/L_T & 12 \end{bmatrix}\begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} = \begin{Bmatrix} P/AEL_T^2 \\ -4c/AE + 2P/AEL_T \end{Bmatrix}$$ Same results as above

[continued, next page]

$\boxed{5.2\text{-}2}$ (continued) <u>Galerkin</u>

$$R_i = \int_0^{L_T} \left(-\frac{dW_i}{dx}\frac{d\tilde{u}}{dx} + W_i \frac{c}{AE} \right) dx + W_i \frac{P}{AE}\bigg|_{L_T} \quad \text{where} \quad \begin{matrix} W_1 = x \\ W_2 = x^2 \end{matrix}$$

$$0 = \int_0^{L_T} \left[(-1)(a_1 + 2a_2 x) + x\frac{c}{AE} \right] dx + \frac{PL_T}{AE}$$

$$0 = \int_0^{L_T} \left[(-2x)(a_1 + 2a_2 x) + x^2\frac{c}{AE} \right] dx + \frac{PL_T^2}{AE}$$

$$0 = -L_T a_1 - L_T^2 a_2 + \frac{cL_T^2}{2AE} + \frac{PL_T}{AE}$$

$$0 = -L_T^2 a_1 - \frac{4}{3} L_T^3 a_2 + \frac{cL_T^3}{3} + \frac{PL_T^2}{AE}$$

$$\begin{bmatrix} 1 & L_T \\ L_T & 4L_T^2/3 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} = \begin{Bmatrix} cL_T/2AE + P/AE \\ cL_T^2/3AE + PL_T/AE \end{Bmatrix} \quad \text{Same results as before}$$

$\boxed{5.2\text{-}3}$

Exact values: At $x=0.5$, $u = 1.1492$
At $x=0.7$, $u = 0.8325$

Residual methods:

$\tilde{u} = 3 - 2x + a(x^2 - x)$, $\tilde{u}_{,x} = -2 + a(2x-1)$,

$\tilde{u}_{,xx} = 2a$, $R = 4ax^2 - (4a+8)x + 2a$

(a) $R = 0$ at $x = \frac{1}{2}$ yields $a = 4$
and $\tilde{u} = 4x^2 - 6x + 3$

At $x=0.5$, $\tilde{u} = 1.00$; -13.0% error
At $x=0.7$, $\tilde{u} = 0.76$; -8.7% error

(b) $\int_0^1 R\,dx = \frac{4a}{3} - (2a+4) + 2a = 0$ yields
$a = 3$ and $\tilde{u} = 3x^2 - 5x + 3$

At $x = 0.5$, $\tilde{u} = 1.25$; $+8.8\%$ error
At $x = 0.7$, $\tilde{u} = 0.97$; $+16.5\%$ error

(c) $\frac{\partial}{\partial a}\int_0^1 R^2\,dx = 0$, $\int_0^1 R\frac{\partial R}{\partial a}\,dx = 0$

$$\int_0^1 \left[4ax^2 - (4a+8)x + 2a\right](4x^2 - 4x + 2)\,dx = 0$$

yields $a = \frac{20}{7}$, $\tilde{u} = \frac{20}{7}x^2 - \frac{34}{7}x + 3$

At $x=0.5$, $\tilde{u} = 1.2857$; $+11.9\%$ error
At $x = 0.7$, $\tilde{u} = 1.0000$; $+20.1\%$ error

(d) $I = \left[\frac{4a}{9} - \frac{4a}{3} - \frac{8}{3} + 2a\right]^2 + \left[\frac{16a}{9} - \frac{8a}{3} - \frac{16}{3} + 2a\right]^2$

$I = \frac{200a^2}{81} - \frac{160a}{9} + \frac{320}{9}$, $\frac{\partial I}{\partial a} = \frac{400a}{81} - \frac{160}{9}$

$\frac{\partial I}{\partial a} = 0$ yields $a = \frac{18}{5}$, $\tilde{u} = \frac{18}{5}x^2 - \frac{28}{5}x + 3$

At $x = 0.5$, $\tilde{u} = 1.100$; -4.3% error
At $x = 0.7$, $\tilde{u} = 0.844$; $+1.4\%$ error

(e) $\frac{\partial \tilde{u}}{\partial a} = x^2 - x$, $\int_0^1 (x^2 - x)R\,dx = 0$

yields $a = \frac{10}{3}$, $\tilde{u} = \frac{10}{3}x^2 - \frac{16}{3}x + 3$

At $x = 0.5$, $\tilde{u} = 1.1667$; $+1.5\%$ error
At $x = 0.7$, $\tilde{u} = 0.9000$; $+8.1\%$ error

$\boxed{5.2\text{-}4}$

Exact values: At $x=0.5$, $u=1.4715$
At $x=0.7$, $u=2.5864$

Residual methods:

$\tilde{u} = a_1 x + a_2 x^2$, $\tilde{u}_{,x} = a_1 + 2a_2 x$

$R = \tilde{u}_{,x} + 2\tilde{u} - 16x = (1+2x)a_1 + 2x(1+x)a_2 - 16x$

(a)

$$\begin{Bmatrix} R_1 \\ R_2 \\ R_3 \end{Bmatrix} = \begin{bmatrix} 1.5 & 0.625 \\ 2.0 & 1.500 \\ 2.5 & 2.625 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} - \begin{Bmatrix} 4 \\ 8 \\ 12 \end{Bmatrix} = [\underset{\sim}{Q}]\{\underset{\sim}{a}\} - \{\underset{\sim}{c}\}$$

Apply Eq. 5.2-13c: $\begin{bmatrix} 12.5 & 10.5 \\ 10.5 & 9.5313 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} = \begin{Bmatrix} 52 \\ 46 \end{Bmatrix}$

$\begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} = \begin{Bmatrix} 1.4202 \\ 3.2616 \end{Bmatrix}$, $\tilde{u} = 1.4202x + 3.2616x^2$

At $x=0.5$, $\tilde{u} = 1.5255$; $+3.7\%$ error

At $x=0.7$, $\tilde{u} = 2.5923$; $+0.2\%$ error

(b) $\frac{\partial \tilde{u}}{\partial a_1} = x$, $\frac{\partial \tilde{u}}{\partial a_2} = x^2$; set $\int_0^1 xR\,dx = \int_0^1 x^2 R\,dx = 0$

Thus

$$\left.\begin{aligned} \left(\tfrac{1}{2}+\tfrac{2}{3}\right)a_1 + \left(\tfrac{2}{3}+\tfrac{1}{2}\right)a_2 - \tfrac{16}{3} &= 0 \\ \left(\tfrac{1}{3}+\tfrac{1}{2}\right)a_1 + \left(\tfrac{1}{2}+\tfrac{2}{5}\right)a_2 - 4 &= 0 \end{aligned}\right\} \begin{aligned} a_1 &= 1.7143 \\ a_2 &= 2.8571 \end{aligned}$$

At $x=0.5$, $\tilde{u} = 1.5714$; $+6.8\%$ error

At $x=0.7$, $\tilde{u} = 2.5429$; -1.7% error

$\boxed{5.2\text{-}5}$ $\tilde{u} = a_1 x + a_2 x^2 \qquad R = 2a_2 + cx$

$$R_S = (a_1 + 2a_2 L_T) - b$$

(a) $R = 0 = 2a_2 + c\dfrac{L_T}{2}$,

$R_S = 0$ yields $a_1 = b + \dfrac{cL_T^2}{2}$

For $b=0$ & $c = L_T = 1$, $\tilde{u} = \dfrac{x}{2} - \dfrac{x^2}{4}$, $\tilde{u}_{,x} = \dfrac{1}{2} - \dfrac{x}{2}$

(b) $\displaystyle\int_0^{L_T/2} (2a_2 + cx)\,dx = 2a_2\frac{L_T}{2} + \frac{c}{2}\left(\frac{L_T}{2}\right)^2 = 0,\ a_2 = -\frac{cL_T}{8}$

$R_B = 0$ yields $a_1 = b + \dfrac{cL_T^2}{4}$

For $b = 0$ & $c = L_T = 1$, $\tilde{u} = \dfrac{x}{4} - \dfrac{x^2}{8}$, $\tilde{u}_{,x} = \dfrac{1}{4} - \dfrac{x}{4}$

(c) $$\begin{Bmatrix} R_1 \\ R_2 \\ R_3 \\ R_S \end{Bmatrix} = \begin{bmatrix} 0 & 2 \\ 0 & 2 \\ 0 & 2 \\ 1/L_T & 2 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} - \begin{Bmatrix} 0 \\ -cL_T/2 \\ -cL_T \\ b/L_T \end{Bmatrix}$$

$$\begin{bmatrix} \dfrac{1}{L_T^2} & \dfrac{2}{L_T} \\ \dfrac{2}{L_T} & 16 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} = \begin{Bmatrix} \dfrac{b}{L_T^2} \\ -\dfrac{3c}{L_T} + \dfrac{2b}{L_T} \end{Bmatrix}, \quad \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} = \begin{Bmatrix} b + \dfrac{cL_T^2}{2} \\ -\dfrac{cL_T}{4} \end{Bmatrix}$$

For $b=0$ & $c = L_T = 1$, $\tilde{u} = \dfrac{x}{2} - \dfrac{x^2}{4}$, $\tilde{u}_{,x} = \dfrac{1}{2} - \dfrac{x}{2}$

(d) Becomes simple collocation

$$\begin{bmatrix} 0 & 2 \\ 1/L_T & 2 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} - \begin{Bmatrix} -cL_T/3 \\ b/L_T \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}, \quad a_1 = b + \frac{cL_T}{3},\ a_2 = -\frac{cL_T^2}{6}$$

For $b=0$ & $c = L_T = 1$, $\tilde{u} = \dfrac{x}{3} - \dfrac{x^2}{6}$, $\tilde{u}_{,x} = \dfrac{1}{3} - \dfrac{x}{3}$

Summary of foregoing results:

	(a)	(b)	(c)	(d)	(exact)
u @ $L_T/2$	.1875	.0938	.1875	.1250	.2292
u @ L_T	.2500	.1250	.2500	.1667	.3333
$u_{,x}$ @ 0	.5000	.2500	.5000	.3333	.5000
$u_{,x}$ @ $L_T/2$	.2500	.1250	.2500	.1667	.3750
$u_{,x}$ @ L_T	0	0	0	0	0

(e) Consider n residuals R_i. Let c_i = constants. Form of Eq. 15.3-7 becomes

$$\{\underset{\sim}{R}\} = \begin{Bmatrix} R_1 \\ R_2 \\ \vdots \\ R_S \end{Bmatrix} = \begin{bmatrix} 0 & 2 \\ 0 & 2 \\ \vdots & \vdots \\ \alpha & 2\alpha \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} - \begin{Bmatrix} c_1 \\ c_2 \\ \vdots \\ \alpha b \end{Bmatrix}$$

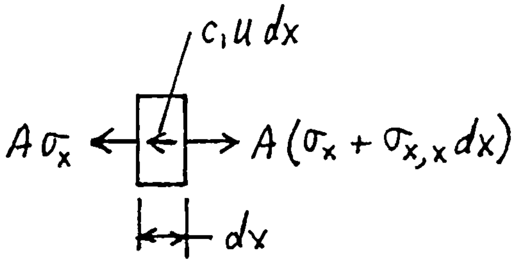
Form of Eq. 15.3-8c, & its solution, are

$$\begin{bmatrix} \alpha^2 & 2\alpha^2 \\ 2\alpha^2 & 4n + 4\alpha^2 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} = \underbrace{\begin{Bmatrix} \alpha^2 b \\ 2\sum c_i + 2\alpha^2 b \end{Bmatrix}}$$

$$\begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} = \frac{1}{4n\alpha^2} \begin{bmatrix} 4n + 4\alpha^2 & -2\alpha^2 \\ -2\alpha^2 & \alpha^2 \end{bmatrix} \begin{Bmatrix} \ \\ \ \end{Bmatrix} = \begin{Bmatrix} b - \dfrac{1}{n}\sum c_i \\ \dfrac{1}{2n}\sum c_i \end{Bmatrix}$$

This result is independent of α due to the fortuitous absence of α from interior residuals.

$\boxed{5.3\text{-}1}$



Axial equilibrium:

$$\left.\begin{aligned} A\sigma_{x,x} - c_1 u &= 0 \\ \sigma_x &= E u_{,x} \end{aligned}\right\} AE\,u_{,xx} - c_1 u = 0$$

Assume $\tilde{u} = \lfloor \underset{\sim}{N} \rfloor \{\underset{\sim}{d}\} = \left\lfloor \frac{L-x}{L} \quad \frac{x}{L} \right\rfloor \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$ over one element of length L

$$R = 0 = \int_0^L \lfloor \underset{\sim}{N} \rfloor^T \left(AE\,\tilde{u}_{,xx} - c_1 \tilde{u}\right) dx$$

Substitute for $\tilde{u}$ and integrate by parts

$$0 = \int_0^L \lfloor \underset{\sim}{N}_{,x} \rfloor^T \lfloor \underset{\sim}{N}_{,x} \rfloor AE\,dx\,\{\underset{\sim}{d}\} + \int_0^L \lfloor \underset{\sim}{N} \rfloor^T \lfloor \underset{\sim}{N} \rfloor c_1\,dx\,\{\underset{\sim}{d}\} - \Big[\lfloor \underset{\sim}{N} \rfloor^T AE\,\tilde{u}_{,x}\Big]_0^L$$

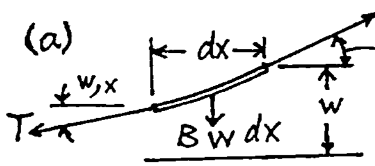
$$\left(\frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + c_1 L \begin{bmatrix} 1/3 & 1/6 \\ 1/6 & 1/3 \end{bmatrix}\right)\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} (-AE\,\tilde{u}_{,x})_0 \\ (AE\,\tilde{u}_{,x})_L \end{Bmatrix}$$

$$\left(\frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \frac{c_1 L}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\right)\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} P \\ -F \end{Bmatrix}$$

where $F = c_2 u_L$

$\boxed{5.3\text{-}2}$

(a)



Sum vertical forces:

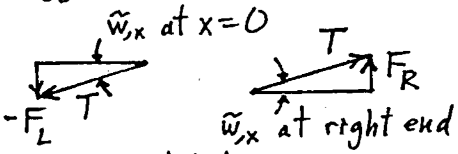
$$-Tw_{,x} + T\left[w_{,x} + d(w_{,x})\right] - Bw\,dx = 0$$

$$T\,d(w_{,x}) = Bw\,dx\,, \qquad Tw_{,xx} - Bw = 0$$

(b) $\tilde{w} = \lfloor \underset{\sim}{N} \rfloor \{\underset{\sim}{d}\}$ and $\sum \int_0^L \lfloor \underset{\sim}{N} \rfloor^T (T\tilde{w}_{,xx} - B\tilde{w})\,dx = 0$

Integrate by parts

$$\sum \int_0^L \left(-\lfloor \underset{\sim}{N}_{,x} \rfloor^T T\,\tilde{w}_{,x} - \lfloor \underset{\sim}{N} \rfloor^T B\tilde{w}\right) dx + \sum \Big[\lfloor \underset{\sim}{N} \rfloor^T T\tilde{w}_{,x}\Big]_0^L = 0$$



F_L & F_R enter the last term

Also substitute $\tilde{w} = \lfloor \underset{\sim}{N} \rfloor \{\underset{\sim}{d}\}$ & $\tilde{w}_{,x} = \lfloor \underset{\sim}{N}_{,x} \rfloor \{\underset{\sim}{d}\}$

$$\sum \Big(\underbrace{\int_0^L \lfloor \underset{\sim}{N}_{,x} \rfloor^T T \lfloor \underset{\sim}{N}_{,x} \rfloor\,dx}_{[\underset{\sim}{k}]} + \underbrace{\int_0^L \lfloor \underset{\sim}{N} \rfloor^T B \lfloor \underset{\sim}{N} \rfloor\,dx}_{[\underset{\sim}{k}_f]} \Big) \{\underset{\sim}{d}\} = \begin{Bmatrix} F_L \\ \vdots \\ F_R \end{Bmatrix}$$

$\boxed{5.3\text{-}3}$

(a) $$\int_0^L \lfloor \mathbf{N} \rfloor^T (T\tilde{w}_{,xx} - \rho_L \ddot{w})\,dx = 0$$

Also, $\tilde{w} = \lfloor \mathbf{N} \rfloor \{\mathbf{d}\}$. Integrate by parts.

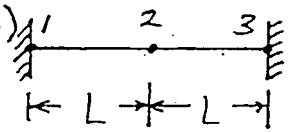
$$-\int_0^L \lfloor \mathbf{N}_{,x} \rfloor^T T\,\tilde{w}_{,x}\,dx - \int_0^L \lfloor \mathbf{N} \rfloor^T \rho_L \ddot{w}\,dx + \Big[\lfloor \mathbf{N} \rfloor^T T\,\tilde{w}_{,x}\Big]_0^L = 0$$

Last term associated with transverse support forces, whose d.o.f. are discarded for simply supported ends. Subs: $\tilde{w} = \lfloor \mathbf{N} \rfloor \{\mathbf{d}\}$, $\tilde{w}_{,x} = \lfloor \mathbf{N}_{,x} \rfloor \{\mathbf{d}\}$

$$\underbrace{\int_0^L \lfloor \mathbf{N}_{,x} \rfloor^T T \lfloor \mathbf{N}_{,x} \rfloor\,dx}_{[\mathbf{k}]}\{\mathbf{d}\} + \underbrace{\int_0^L \lfloor \mathbf{N} \rfloor^T \rho_L \lfloor \mathbf{N} \rfloor\,dx}_{[\mathbf{m}]}\{\ddot{\mathbf{d}}\} = 0$$

Here $\lfloor \mathbf{N} \rfloor = \left\lfloor \frac{L-x}{L} \quad \frac{x}{L} \right\rfloor$ and $\lfloor \mathbf{N}_{,x} \rfloor = \frac{1}{L}\lfloor -1 \quad 1 \rfloor$

$$[\mathbf{k}] = \frac{T}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}, \quad [\mathbf{m}] = \frac{\rho_L L}{6}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

(b) 

Assemble two elements & suppress d.o.f. w_1 & w_3. Thus

$$2\frac{T}{L}w_2 + 4\frac{\rho_L L}{6}\ddot{w}_2 = 0$$

Set $w_2 = \bar{w}_2 \sin\omega t$

$\ddot{w}_2 = -\omega^2 \bar{w}_2 \sin\omega t$ $\qquad \left(2\frac{T}{L} + 4\frac{\rho_L L}{6}\omega^2\right)\bar{w}_2 = 0$

$$\omega^2 = 3\frac{T}{\rho_L L^2}$$

$\boxed{5.3\text{-}4}$

Left hand side of Eq. 5.3-18, after integration by parts of the term that contains $F\tilde{v}_{,xx}$, contains the additional terms

$$+\int_0^L \lfloor \mathbf{N}_{,x} \rfloor^T F\,\tilde{v}_{,x}\,dx + \int_0^L \lfloor \mathbf{N} \rfloor^T B\,\tilde{v}\,dx$$

Substitute $\tilde{v} = \lfloor \mathbf{N} \rfloor \{\mathbf{d}\}$ and $\tilde{v}_{,x} = \lfloor \mathbf{N}_{,x} \rfloor \{\mathbf{d}\}$

The additional terms become

$$+\underbrace{\int_0^L \lfloor \mathbf{N}_{,x} \rfloor^T F \lfloor \mathbf{N}_{,x} \rfloor\,dx}_{[\mathbf{k}_\sigma]}\{\mathbf{d}\} + \underbrace{\int_0^L \lfloor \mathbf{N} \rfloor^T B \lfloor \mathbf{N} \rfloor\,dx}_{[\mathbf{k}_f]}\{\mathbf{d}\}$$

$\boxed{5.3\text{-}5}$

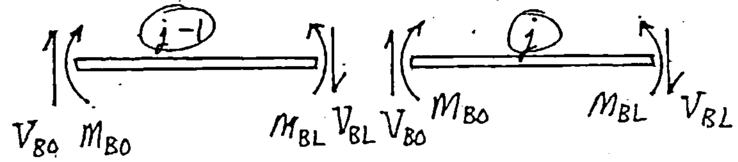
We want to show that the load terms are $\{\underset{\sim}{R}\} = \left(\lfloor \underset{\sim}{N}_{,x} \rfloor^T m_B - \lfloor \underset{\sim}{N} \rfloor^T V_B\right)_0^L$

(with assembly of elements implied).

Use cubic shape functions and insert limits 0 to L:

$$\begin{Bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{Bmatrix} M_{BL} - \begin{Bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{Bmatrix} M_{BO} - \begin{Bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{Bmatrix} V_{BL} + \begin{Bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{Bmatrix} V_{BO}$$

Consider adjacent elements $j-1$ and j.



Consider e.g. assembly of vertical forces where elements $j-1$ and j meet. The dashed line connects forces having the same global d.o.f. number. These add, to yield the net force $(V_{BO})_j - (V_{BL})_{j-1}$ at the shared node.

$$\begin{array}{cc} (j-1) & (j) \\ -V_{BL}\begin{Bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{Bmatrix} & +V_{BO}\begin{Bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \end{array}$$

$\boxed{5.3\text{-}6}$

$A = \frac{L-x}{L} A_1 + \frac{x}{L} A_2$. Use Eq. 5.3-23.

$\lfloor \underset{\sim}{N}_{,x} \rfloor^T \lfloor \underset{\sim}{N}_{,x} \rfloor = \frac{1}{L^2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$, so integral is

$$\frac{k}{L^2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\int_0^L \left(\frac{L-x}{L} A_1 + \frac{x}{L} A_2\right) dx =$$

$$\frac{k}{L^2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\left(\frac{A_1 L}{2} + \frac{A_2 L}{2}\right) = \frac{k(A_1 + A_2)}{2L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

Eq. 5.3-24 becomes: $\frac{k(A_1+A_2)}{2L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} T_1 \\ T_2 \end{Bmatrix} = \begin{Bmatrix} A_1 q_1 \\ A_2 q_2 \end{Bmatrix}$

$\boxed{5.3-7}$

Consider adjacent els. $j-1$ and j.

$\left[A_{j-1}q_{r(j-1)}\right]_a \quad \left[A_{j-1}q_{r(j-1)}\right]_b \quad \left[A_j q_{rj}\right]_b \quad \left[A_j q_{rj}\right]_c$

Connect elements at node b. Right-hand side of Eq. 5.3-23 becomes

$$\begin{array}{l} \text{node } a \text{ --} \\ \text{node } b \text{ --} \\ \text{node } c \text{ --------} \end{array} \begin{Bmatrix} A_{j-1}q_{r(j-1)} \\ -A_{j-1}q_{r(j-1)} \\ \end{Bmatrix} + \begin{Bmatrix} \\ A_j q_{rj} \\ -A_j q_{rj} \end{Bmatrix} = \begin{Bmatrix} A_{j-1}q_{r(j-1)} \\ \Delta \\ -A_j q_{rj} \end{Bmatrix}$$

where $\Delta = \left[A_j q_{rj}\right]_b - \left[A_{j-1}q_{r(j-1)}\right]_b = 0$

because of interelement continuity. The other two entries are complete if there are no additional elements.

$\boxed{5.5\text{-}1}$

Write $\phi_{,x}^2 = \phi_{,x}^T \phi_{,x}$ and $\phi_{,y}^2 = \phi_{,y}^T \phi_{,y}$

Substitute $\phi = \underset{\sim}{N}\,\underset{\sim}{\phi}_e$, $\phi_{,x} = \underset{\sim}{N}_{,x}\underset{\sim}{\phi}_e$, $\phi_{,y} = \underset{\sim}{N}_{,y}\underset{\sim}{\phi}_e$

$$\Pi = \frac{1}{2}\underset{\sim}{\phi}_e^T \iint \left(\underset{\sim}{N}_{,x}^T k_x \underset{\sim}{N}_{,x} + \underset{\sim}{N}_{,y}^T k_y \underset{\sim}{N}_{,y}\right) dx\,dy\, \underset{\sim}{\phi}_e$$

$$- \underset{\sim}{\phi}_e^T \iint \underset{\sim}{N}^T Q\, dx\,dy - \underset{\sim}{\phi}_e^T \int \underset{\sim}{N}^T f_B\, dS$$

$$\left\{\frac{\partial \Pi}{\partial \underset{\sim}{\phi}_e}\right\} = 0 = \iint \left(\underset{\sim}{N}_{,x}^T k_x \underset{\sim}{N}_{,x} + \underset{\sim}{N}_{,y}^T k_y \underset{\sim}{N}_{,y}\right) dx\,dy\, \underset{\sim}{\phi}_e$$

$$- \iint \underset{\sim}{N}^T Q\, dx\,dy - \int \underset{\sim}{N}^T f_B\, dS$$

$\boxed{5.5\text{-}2}$

Residual equation is

$$\int \underset{\sim}{N}^T \big(\underbrace{\tilde{p}_{,xx} + \tilde{p}_{,yy} + \tilde{p}_{,zz}}_{\nabla^2 \tilde{p}} + \frac{\omega^2}{c^2}\tilde{p}\big)\, dV = 0 \qquad (A)$$

Integrate by parts (see Eq. 5.5-5, with $k_x = k_y = k_z = 1$).

$$\int \underset{\sim}{N}^T \nabla^2 \tilde{p}\, dV = \int \underset{\sim}{N}^T \big(\overbrace{\tilde{p}_{,x}\ell + \tilde{p}_{,y}m + \tilde{p}_{,z}n}^{=\tilde{p}_{,n} = 0}\big)\, dS$$

$$- \int \big(\underset{\sim}{N}_{,x}^T \tilde{p}_{,x} + \underset{\sim}{N}_{,y}^T \tilde{p}_{,y} + \underset{\sim}{N}_{,z}^T \tilde{p}_{,z}\big)\, dV$$

Hence, with $\tilde{p}_{,x} = \underset{\sim}{N}_{,x}^T \underset{\sim}{p}_e$ etc., Eq. (A) becomes

$$\int \big(\underset{\sim}{N}_{,x}^T \underset{\sim}{N}_{,x} + \underset{\sim}{N}_{,y}^T \underset{\sim}{N}_{,y} + \underset{\sim}{N}_{,z}^T \underset{\sim}{N}_{,z}\big)\, dV\, \underset{\sim}{p}_e$$

$$- \omega^2 \int \frac{1}{c^2} \underset{\sim}{N}^T \underset{\sim}{N}\, dV\, \underset{\sim}{p}_e = 0$$

5.5-3

Apply integ. by parts. First, use Eq. 5.4-7 on first term of –

$$\int \underset{\sim}{N}^T\left(\frac{1}{r}(r\tilde{\phi}_{,r})_{,r}+\frac{1}{r^2}\tilde{\phi}_{,\theta\theta}+\tilde{\phi}_{,zz}+\frac{Q}{k}\right)dV=0$$

Thus

$$\int \underset{\sim}{N}^T\left(\frac{1}{r}(r\tilde{\phi}_{,r})_{,r}\right)dV=-\int \underset{\sim}{N}^T_{,r}\tilde{\phi}_{,r}\,dV+\int \underset{\sim}{N}^T\tilde{\phi}_{,r}\,l\,dS$$

Integration of second term by parts produces no surface integral because $m=0$ for a normal to the boundary.

$$\int \underset{\sim}{N}^T\frac{1}{r^2}\tilde{\phi}_{,\theta\theta}\,dV=-\int \underset{\sim}{N}^T_{,\theta}\frac{1}{r^2}\tilde{\phi}_{,\theta}\,dV.$$ Finally

$$\int \underset{\sim}{N}^T\tilde{\phi}_{,zz}\,dV=-\int \underset{\sim}{N}^T_{,z}\tilde{\phi}_{,z}\,dV+\int \underset{\sim}{N}^T\tilde{\phi}_{,z}\,n\,dS$$

With the given boundary condition, we have

$$\int\left(\underset{\sim}{N}^T_{,r}\tilde{\phi}_{,r}+\underset{\sim}{N}^T_{,\theta}\frac{1}{r^2}\tilde{\phi}_{,\theta}+\underset{\sim}{N}^T_{,z}\tilde{\phi}_{,z}\right)k\,dV$$

$$\int \underset{\sim}{N}^T Q\,dV+\int \underset{\sim}{N}^T f_B\,dS$$

Substitute $\tilde{\phi}_{,r}=\underset{\sim}{N}_{,r}\underset{\sim}{\phi}_e$, etc. Thus

$$\int\left(\underset{\sim}{N}^T_{,r}\underset{\sim}{N}_{,r}+\frac{1}{r^2}\underset{\sim}{N}^T_{,\theta}\underset{\sim}{N}_{,\theta}+\underset{\sim}{N}^T_{,z}\underset{\sim}{N}_{,z}\right)k\,dV\,\underset{\sim}{\phi}_e=\underset{\sim}{r}$$

where $\underset{\sim}{r}=\int \underset{\sim}{N}^T Q\,dV+\int \underset{\sim}{N}^T f_B\,dS$

Note: $dV = r\,dr\,dz$ for a 1-radian segment.

$\boxed{5.5\text{-}4}$

Multiply by weighting function $\underset{\sim}{N}^T$:

$$\iint \underset{\sim}{N}^T \left(\tilde{\Psi}_{,xx} + \tilde{\Psi}_{,yy} + A\tilde{\Psi}_{,x} + B\tilde{\Psi}_{,y} + C \right) dx\,dy = 0 \quad (A)$$

Integrate first two terms by parts.

$$\iint \underset{\sim}{N}^T \tilde{\Psi}_{,xx}\, dx\,dy = -\iint \underset{\sim}{N}_{,x}^T \tilde{\Psi}_{,x}\, dx\,dy + \int \underset{\sim}{N}^T \tilde{\Psi}_{,x}\, l\, dS$$

$$\iint \underset{\sim}{N}^T \tilde{\Psi}_{,yy}\, dx\,dy = -\iint \underset{\sim}{N}_{,y}^T \tilde{\Psi}_{,y}\, dx\,dy + \int \underset{\sim}{N}^T \tilde{\Psi}_{,y}\, m\, dS$$

Boundary terms: $\tilde{\Psi}_{,x} l + \tilde{\Psi}_{,y} m = \tilde{\Psi}_{,n} = 0$.

Subs. $\tilde{\Psi}_{,x} = \underset{\sim}{N}_{,x}^T \underset{\sim}{\Psi}_e$ etc. into what remains of Eq. (A).

$$\iint \left(-\underset{\sim}{N}_{,x}^T \underset{\sim}{N}_{,x} - \underset{\sim}{N}_{,y}^T \underset{\sim}{N}_{,y} + A\underset{\sim}{N}^T \underset{\sim}{N}_{,x} + B\underset{\sim}{N}^T \underset{\sim}{N}_{,y} \right) dx\,dy\ \underset{\sim}{\Psi}_e = -\iint \underset{\sim}{N}^T C\, dx\,dy$$

Here the coefficient matrix of nodal d.o.f. $\underset{\sim}{\Psi}_e$ is not symmetric.

$\boxed{5.5\text{-}5}$

Companions to Eq. 5.5-11 are

$$\iint \underset{\sim}{N}^T \tilde{\tau}_{xy,y}\, dx\, dy = -\iint \underset{\sim}{N}_{,y}^T \tilde{\tau}_{xy}\, dx\, dy + \int \underset{\sim}{N}^T \tilde{\tau}_{xy}\, m\, dS$$

$$\iint \underset{\sim}{N}^T \tilde{\tau}_{xy,x}\, dx\, dy = -\iint \underset{\sim}{N}_{,x}^T \tilde{\tau}_{xy}\, dx\, dy + \int \underset{\sim}{N}^T \tilde{\tau}_{xy}\, l\, dS$$

$$\iint \underset{\sim}{N}^T \tilde{\sigma}_{y,y}\, dx\, dy = -\iint \underset{\sim}{N}_{,y}^T \tilde{\sigma}_{y}\, dx\, dy + \int \underset{\sim}{N}^T \tilde{\sigma}_{y}\, m\, dS$$

In view of surface-traction Eqs. 5.5-8, Eqs. 5.5-10 now read

$$-\iint \begin{bmatrix} \underset{\sim}{N}_{,x}^T & \underset{\sim}{0}^T & \underset{\sim}{N}_{,y}^T \\ \underset{\sim}{0}^T & \underset{\sim}{N}_{,y}^T & \underset{\sim}{N}_{,x}^T \end{bmatrix} \begin{Bmatrix} \tilde{\sigma}_x \\ \tilde{\sigma}_y \\ \tilde{\tau}_{xy} \end{Bmatrix} dx\, dy +$$

$$\iint \begin{bmatrix} \underset{\sim}{N}^T & \underset{\sim}{0}^T \\ \underset{\sim}{0}^T & \underset{\sim}{N}^T \end{bmatrix} \begin{Bmatrix} F_x \\ F_y \end{Bmatrix} dx\, dy + \int \begin{bmatrix} \underset{\sim}{N}^T & \underset{\sim}{0}^T \\ \underset{\sim}{0}^T & \underset{\sim}{N}^T \end{bmatrix} \begin{Bmatrix} \Phi_x \\ \Phi_y \end{Bmatrix} dS = \underset{\sim}{0}$$

Or, using conventional notation,

$$-\iint [\underset{\sim}{B}]^T \{\tilde{\underset{\sim}{\sigma}}\}\, dx\, dy + \iint [\underset{\sim}{N}]^T \{\underset{\sim}{F}\}\, dx\, dy$$

$$+ \int [\underset{\sim}{N}]^T \{\underset{\sim}{\Phi}\}\, dS \qquad (A)$$

But $\{\tilde{\underset{\sim}{\sigma}}\} = [\underset{\sim}{E}](\{\tilde{\underset{\sim}{\epsilon}}\} - \{\underset{\sim}{\epsilon}_0\}) + \{\underset{\sim}{\sigma}_0\}$

and $\{\tilde{\underset{\sim}{\epsilon}}\} = [\underset{\sim}{B}]\{\underset{\sim}{d}\}$

so $\{\tilde{\underset{\sim}{\sigma}}\} = [\underset{\sim}{E}][\underset{\sim}{B}]\{\underset{\sim}{d}\} - [\underset{\sim}{E}]\{\underset{\sim}{\epsilon}_0\} + \{\underset{\sim}{\sigma}_0\}$ (B)

Eqs. (A) and (B) yield the standard eqs.

$\boxed{5.5\text{-}6}$

Consider unit thickness, as usual. There are only radial displacements, so we need only $\tilde{u} = \underset{\sim}{N}\,\underset{\sim}{d}$. $dV = 2\pi r\,dr$.

$$\int \underset{\sim}{N}^T\left(\frac{1}{r}\frac{d}{dr}(r\tilde{\sigma}_r) - \frac{\tilde{\sigma}_\theta}{r} + \rho\omega^2 r\right)dV = 0 \qquad (A)$$

Apply Eq. 5.4-7 with $l = 1$.

$$\int \underset{\sim}{N}^T\left[\frac{1}{r}\frac{d}{dr}(r\tilde{\sigma}_r)\right]dV = -\int \underset{\sim}{N}_{,r}^T\tilde{\sigma}_r\,dV + \int \underset{\sim}{N}^T\tilde{\sigma}_r\,dS$$

Last term vanishes, as $\sigma_r = 0$ @ $r = r_i$ & $r = r_o$.

(A) becomes

$$\int_{r_i}^{r_o}\left[\underset{\sim}{N}_{,r}^T \quad \frac{1}{r}\underset{\sim}{N}^T\right]\begin{Bmatrix}\tilde{\sigma}_r\\ \tilde{\sigma}_\theta\end{Bmatrix}2\pi r\,dr = \overbrace{\int_{r_i}^{r_o}\underset{\sim}{N}^T\rho\omega^2 r\,(2\pi r)\,dr}^{\{\underset{\sim}{r}\}}$$

$$\begin{Bmatrix}\tilde{\sigma}_r\\ \tilde{\sigma}_\theta\end{Bmatrix} = \frac{E}{1-\nu^2}\begin{bmatrix}1 & \nu\\ \nu & 1\end{bmatrix}\begin{Bmatrix}\tilde{\epsilon}_r\\ \tilde{\epsilon}_\theta\end{Bmatrix} = [\underset{\sim}{E}]\begin{Bmatrix}\tilde{u}_{,r}\\ \frac{\tilde{u}}{r}\end{Bmatrix} = [\underset{\sim}{E}]\begin{bmatrix}\underset{\sim}{N}_{,r}\\ \frac{1}{r}\underset{\sim}{N}\end{bmatrix}\{\underset{\sim}{d}\}$$

$$\int_{r_i}^{r_o}\left[\underset{\sim}{N}_{,r}^T \;\; \frac{1}{r}\underset{\sim}{N}^T\right][\underset{\sim}{E}]\begin{bmatrix}\underset{\sim}{N}_{,r}\\ \frac{1}{r}\underset{\sim}{N}\end{bmatrix}2\pi r\,dr\,\{\underset{\sim}{d}\} = \{\underset{\sim}{r}\}$$

$\boxed{5.5\text{-}7}$

Multiply equilibrium eqs. by weight function $\tilde{N}^T$ and integrate.

$$\int \tilde{N}^T\left(\frac{1}{r}(r\tilde{\sigma}_r)_{,r} + \tilde{\tau}_{rz,z} - \frac{\tilde{\sigma}_\theta}{r}\right)dV = 0 \qquad (A)$$

$$\int \tilde{N}^T\left(\frac{1}{r}(r\tilde{\tau}_{rz})_{,r} + \tilde{\sigma}_{z,z}\right)dV = 0 \qquad (B)$$

Integrations by parts (e.g. Eq. 5.4-7):

$$\int \tilde{N}^T\left(\frac{1}{r}(r\tilde{\sigma}_r)_{,r}\right)dV = -\int \tilde{N}_{,r}^T\tilde{\sigma}_r\,dV + \int \tilde{N}^T\tilde{\sigma}_r\,l\,dS$$

$$\int \tilde{N}^T\tilde{\tau}_{rz,z}\,dV = -\int \tilde{N}_{,z}^T\tilde{\tau}_{rz}\,dV + \int \tilde{N}^T\tilde{\tau}_{rz}\,n\,dS$$

$$\int \tilde{N}^T\left(\frac{1}{r}(r\tilde{\tau}_{rz})_{,r}\right)dV = -\int \tilde{N}_{,r}^T\tilde{\tau}_{rz}\,dV + \int \tilde{N}^T\tilde{\tau}_{rz}\,l\,dS$$

$$\int \tilde{N}^T\tilde{\sigma}_{z,z}\,dV = -\int \tilde{N}_{,z}^T\tilde{\sigma}_z\,dV + \int \tilde{N}^T\tilde{\sigma}_z\,n\,dS$$

Now $l\tilde{\sigma}_r + n\tilde{\tau}_{rz} = \Phi_r$ & $l\tilde{\tau}_{rz} + n\tilde{\sigma}_z = \Phi_z$, so the two residual eqs. (A) & (B) become

$$\int \begin{bmatrix} \tilde{N}_{,r}^T & \tilde{0}^T & \frac{1}{r}\tilde{N}^T & \tilde{N}_{,z}^T \\ \tilde{0}^T & \tilde{N}_{,z}^T & \tilde{0}^T & \tilde{N}_{,r}^T \end{bmatrix} \begin{Bmatrix} \tilde{\sigma}_r \\ \tilde{\sigma}_z \\ \tilde{\sigma}_\theta \\ \tilde{\tau}_{rz} \end{Bmatrix} dV =$$

$$\int \begin{bmatrix} \tilde{N}^T & \tilde{0}^T \\ \tilde{0}^T & \tilde{N}^T \end{bmatrix} \begin{Bmatrix} \Phi_r \\ \Phi_z \end{Bmatrix} dS$$

Using conventional notation, this eq. is

$$\int [\tilde{B}]^T\{\tilde{\sigma}\}\,dV = \int [\tilde{N}]^T\{\tilde{\Phi}\}\,dS \qquad (C)$$

But $\{\tilde{\sigma}\} = [\tilde{E}]\{\tilde{\epsilon}\} = [\tilde{E}][\tilde{B}]\{\tilde{d}\}$ (D)

where, with $\{\tilde{d}\} = \lfloor u_1\ u_2 \cdots w_1\ w_2 \cdots \rfloor^T$,

$$\{\tilde{\epsilon}\} = \begin{Bmatrix} \tilde{\epsilon}_r \\ \tilde{\epsilon}_z \\ \tilde{\epsilon}_\theta \\ \gamma_{rz} \end{Bmatrix} = \begin{bmatrix} \tilde{N}_{,r} & \tilde{0} \\ \tilde{0} & \tilde{N}_{,z} \\ \frac{1}{r}\tilde{N} & \tilde{0} \\ \tilde{N}_{,z} & \tilde{N}_{,r} \end{bmatrix} \{\tilde{d}\}$$

Eqs. (C) & (D) yield the standard result:

$$\int [\tilde{B}]^T[\tilde{E}][\tilde{B}]\,dV\,\{\tilde{d}\} = \int [\tilde{N}]^T\{\tilde{\Phi}\}\,dS$$

$\boxed{5.6-1}$ Apply Eq. 5.6-9. Including all six d.o.f. of the two-el. model,

$$A\begin{bmatrix} -L/3E & -1/2 & -L/6E & 1/2 & 0 & 0 \\ -1/2 & 0 & -1/2 & 0 & 0 & 0 \\ -L/6E & -1/2 & -L/3E & 1/2 & 0 & 0 \\ 1/2 & 0 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}\begin{Bmatrix} \sigma_1 \\ u_1 \\ \sigma_2 \\ u_2 \\ \sigma_3 \\ u_3 \end{Bmatrix} +$$

$$A\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -L/3E & -1/2 & -L/6E & 1/2 \\ 0 & 0 & -1/2 & 0 & -1/2 & 0 \\ 0 & 0 & -L/6E & -1/2 & -L/3E & 1/2 \\ 0 & 0 & 1/2 & 0 & 1/2 & 0 \end{bmatrix}\begin{Bmatrix} \sigma_1 \\ u_1 \\ \sigma_2 \\ u_2 \\ \sigma_3 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} \text{load} \\ \text{terms} \end{Bmatrix}$$

Combine. Discard rows & colums 2 and 5 to impose boundary conditions $u_1 = 0$ and $\sigma_3 = 0$. Include load terms $F_{q1} = cL/2$ and $F_{q2} = cL/2$ (as in Eq. 5.6-8). Thus

$$A\begin{bmatrix} -L/3E & -L/6E & 1/2 & 0 \\ -L/6E & -2L/3E & 0 & 1/2 \\ 1/2 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & 0 \end{bmatrix}\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ cL/2 \\ cL/2 \end{Bmatrix}$$

Solutron is

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} cL/A \\ cL/A \\ cL^2/AE \\ 5cL^2/3AE \end{Bmatrix} \begin{matrix} \checkmark \\ \checkmark \\ \checkmark \\ \longleftarrow \text{ exact is } \dfrac{3cL^2}{2AE} \end{matrix}$$

$\boxed{5.6\text{-}2}$ Governing eqs. are $v_{,xx} - \dfrac{M}{EI} = 0$ and $M_{,xx} - q = 0$

Assume, for element fields, $\tilde{v} = \mathbf{N}\tilde{\mathbf{v}}_e$ and $\tilde{M} = \mathbf{N}\mathbf{M}_e$

First eq.: $\displaystyle\int_0^L \mathbf{N}^T\left(\tilde{v}_{,xx} - \frac{\tilde{M}}{EI}\right)dx = 0$. Integrate 1st term by parts:

$$\int_0^L \mathbf{N}^T \tilde{v}_{,xx}\,dx = \left[\mathbf{N}^T \tilde{v}_{,x}\right]_0^L - \int_0^L \mathbf{N}_{,x}^T \tilde{v}_{,x}\,dx$$ Hence 1st eq. becomes

$$-\underbrace{\int_0^L \mathbf{N}_{,x}^T \mathbf{N}_{,x}\,dx}_{\mathbf{H}_{12}}\ \mathbf{v}_e - \underbrace{\int_0^L \frac{1}{EI}\mathbf{N}^T\mathbf{N}\,dx}_{\mathbf{H}_{11}}\ \mathbf{M}_e = -\left[\mathbf{N}^T \tilde{v}_{,x}\right]_0^L$$

← Cancels upon assembly of elements

Second eq.: $\displaystyle\int_0^L \mathbf{N}^T(\tilde{M}_{,xx} - q)\,dx = 0$. Integrate 1st term by parts:

$$\int_0^L \mathbf{N}^T \tilde{M}_{,xx}\,dx = \left[\mathbf{N}^T \tilde{M}_{,x}\right]_0^L - \int_0^L \mathbf{N}_{,x}^T \tilde{M}_{,x}\,dx$$ Hence 2nd eq. becomes

$$-\underbrace{\int_0^L \mathbf{N}_{,x}^T \mathbf{N}_{,x}\,dx}_{\mathbf{H}_{12}}\ \mathbf{M}_e = \underbrace{\int_0^L \mathbf{N}^T q\,dx}_{\mathbf{r}_q} - \left[\mathbf{N}^T \tilde{M}_{,x}\right]_0^L$$

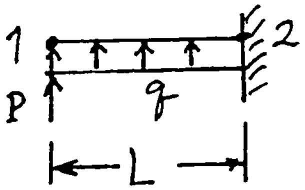
← Cancels upon assembly of els.

Put together: $\begin{bmatrix} \mathbf{H}_{11} & \mathbf{H}_{12} \\ \mathbf{H}_{12} & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \mathbf{M}_e \\ \mathbf{v}_e \end{Bmatrix} = \begin{Bmatrix} 0 \\ -\mathbf{r}_q \end{Bmatrix}$ in which, if $\lfloor \mathbf{N} \rfloor = \left\lfloor \dfrac{L-x}{L} \quad \dfrac{x}{L} \right\rfloor$,

$$[\mathbf{H}_{11}] = \frac{L}{6EI}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \quad [\mathbf{H}_{12}] = \frac{1}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

hence
$$\begin{bmatrix} L/3EI & L/6EI & 1/L & -1/L \\ L/6EI & L/3EI & -1/L & 1/L \\ 1/L & -1/L & 0 & 0 \\ -1/L & 1/L & 0 & 0 \end{bmatrix} \begin{Bmatrix} M_1 \\ M_2 \\ v_1 \\ v_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ -r_{q1} \\ -r_{q2} \end{Bmatrix}$$

Example: $M_1 = 0$, $v_2 = 0$, so



$$\begin{bmatrix} L/3EI & -1/L \\ -1/L & 0 \end{bmatrix} \begin{Bmatrix} M_2 \\ v_1 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -P - qL/2 \end{Bmatrix}$$

Gives $M_2 = PL + \dfrac{qL^2}{2}$, $v_1 = \dfrac{PL^3}{3EI} + \dfrac{qL^4}{6EI}$

Should be $qL^4/8EI$; other terms in M_2 and v_1 are exact.

$\boxed{5.6\text{-}3}$

$$\Pi = \int \left(\sigma_x \frac{du}{dx} - \frac{\sigma_x^2}{2E} - \frac{q}{A} u \right) A\, dx$$

In an element, $\sigma_x = \underset{\sim}{N}\, \underset{\sim}{\sigma}_e$, $u = \underset{\sim}{N}\, \underset{\sim}{u}_e$ hence $\sigma_x^T = \underset{\sim}{\sigma}_e^T \underset{\sim}{N}^T$, $u_{,x} = \underset{\sim}{N}_{,x} \underset{\sim}{u}_e$

$$\Pi = \int \left(\underset{\sim}{\sigma}_e^T \underset{\sim}{N}^T \underset{\sim}{N}_{,x} \underset{\sim}{u}_e - \frac{1}{2E} \underset{\sim}{\sigma}_e^T \underset{\sim}{N}^T \underset{\sim}{N}\, \underset{\sim}{\sigma}_e - \underset{\sim}{u}_e^T \underset{\sim}{N}^T \frac{q}{A} \right) A\, dx$$

$$\frac{\partial \Pi}{\partial \underset{\sim}{\sigma}_e} = \underbrace{\int \underset{\sim}{N}^T \underset{\sim}{N}_{,x} A\, dx}_{[\underset{\sim}{k}_{\sigma u}]}\, \underset{\sim}{u}_e - \underbrace{\int \underset{\sim}{N}^T \underset{\sim}{N} \frac{A}{E}\, dx}_{[\underset{\sim}{k}_{\sigma\sigma}]}\, \underset{\sim}{\sigma}_e$$

$$\frac{\partial \Pi}{\partial \underset{\sim}{u}_e} = \underbrace{\int \underset{\sim}{N}_{,x}^T \underset{\sim}{N} A\, dx}_{[\underset{\sim}{k}_{u\sigma}]}\, \underset{\sim}{\sigma}_e - \underbrace{\int \underset{\sim}{N}^T q\, dx}_{\{\underset{\sim}{r}_q\}}$$

Thus $\begin{bmatrix} -\underset{\sim}{k}_{\sigma\sigma} & \underset{\sim}{k}_{\sigma u} \\ \underset{\sim}{k}_{u\sigma} & \underset{\sim}{0} \end{bmatrix} \begin{Bmatrix} \underset{\sim}{\sigma}_e \\ \underset{\sim}{u}_e \end{Bmatrix} = \begin{Bmatrix} \underset{\sim}{0} \\ \underset{\sim}{r}_q \end{Bmatrix}$ ✓