$\boxed{6.1\text{-}1}$

(a) Solve first of Eqs. 6.1-2 for the a_i:

2^{nd} row gives $a_1 = x_2$, so 1^{st} and 3^{rd} rows become

$$x_1 - x_2 = -a_2 + a_3 \qquad\qquad a_2 = \frac{-x_1 + x_3}{2}$$
$$x_3 - x_2 = a_2 + a_3 \quad \text{from which}$$
$$a_3 = \frac{x_1 - 2x_2 + x_3}{2}$$

$$\begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ -1/2 & 0 & 1/2 \\ 1/2 & -1 & 1/2 \end{bmatrix}}_{[\underset{\sim}{A}]^{-1}} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}$$

$$\lfloor \underset{\sim}{N} \rfloor = \lfloor 1 \quad \xi \quad \xi^2 \rfloor [\underset{\sim}{A}]^{-1} = \left\lfloor \frac{-\xi + \xi^2}{2} \quad 1 - \xi^2 \quad \frac{\xi + \xi^2}{2} \right\rfloor$$

(b) With ξ in place of x, Eq. 3.2-7 is

$$\lfloor \underset{\sim}{N} \rfloor = \left\lfloor \frac{(\xi_2 - \xi)(\xi_3 - \xi)}{(\xi_2 - \xi_1)(\xi_3 - \xi_1)} \quad \frac{(\xi_1 - \xi)(\xi_3 - \xi)}{(\xi_1 - \xi_2)(\xi_3 - \xi_2)} \quad \frac{(\xi_1 - \xi)(\xi_2 - \xi)}{(\xi_1 - \xi_3)(\xi_2 - \xi_3)} \right\rfloor$$

Set $\xi_1 = -1$, $\xi_2 = 0$, $\xi_3 = 1$. Thus

$$\lfloor \underset{\sim}{N} \rfloor = \left\lfloor \frac{-\xi(1-\xi)}{(1)(2)} \quad \frac{(-1-\xi)(1-\xi)}{(-1)(1)} \quad \frac{(-1-\xi)(-\xi)}{(-2)(-1)} \right\rfloor$$

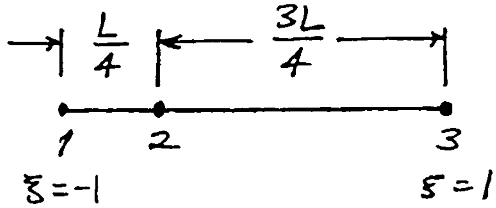
$$= \left\lfloor \frac{-\xi + \xi^2}{2} \quad 1 - \xi^2 \quad \frac{\xi + \xi^2}{2} \right\rfloor$$

$\boxed{6.1\text{-}2}$

(a) $J = \begin{bmatrix} \frac{-1+2\xi}{2} & -2\xi & \frac{1+2\xi}{2} \end{bmatrix} \begin{Bmatrix} x_1 \\ x_1 + L/2 \\ x_1 + L \end{Bmatrix} = -2\xi \frac{L}{2} + \frac{1+2\xi}{2} L = \frac{L}{2}$

(b) ϵ_x becomes infinite at node 1 if $J = 0$ in Eq. 6.1-7 at $\xi = -1$
From Eq. 6.1-6, with $\xi = -1$,

$$0 = \begin{bmatrix} -\frac{3}{2} & 2 & -\frac{1}{2} \end{bmatrix} \begin{Bmatrix} 0 \\ x_2 \\ L \end{Bmatrix} = 2x_2 - \frac{L}{2} \qquad \text{so } x_2 = \frac{L}{4}$$



$\boxed{6.1\text{-}3}$

From Problem 6.1-2a, $J = \frac{L}{2}$. Eq. 6.1-8 becomes

$$[\underset{\sim}{k}] = \int_{-1}^{1} \lfloor \underset{\sim}{B} \rfloor^T \lfloor \underset{\sim}{B} \rfloor AE \frac{L}{2} d\xi = \frac{2AE}{L} \int_{-1}^{1} \begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix} d\xi$$

where $a = \frac{1}{4}(-1+2\xi)^2 = \frac{1}{4}(1-4\xi+4\xi^2)$

$b = -\xi(-1+2\xi) = \xi - 2\xi^2$

$c = \frac{1}{4}(-1+2\xi)(1+2\xi) = \frac{1}{4}(-1+4\xi^2)$

$d = 4\xi^2$

$e = -\xi(1+2\xi) = -\xi - 2\xi^2$

$f = \frac{1}{4}(1+2\xi)^2 = \frac{1}{4}(1+4\xi+4\xi^2)$

$\int_{-1}^{1} a\, d\xi = \frac{14}{4(3)}$ $\qquad \int_{-1}^{1} b\, d\xi = -\frac{4}{3}$ $\qquad \int_{-1}^{1} c\, d\xi = \frac{2}{4(3)}$

$\int_{-1}^{1} d\, d\xi = \frac{8}{3}$ $\qquad \int_{-1}^{1} e\, d\xi = -\frac{4}{3}$ $\qquad \int_{-1}^{1} f\, d\xi = \frac{14}{4(3)}$

$$[\underset{\sim}{k}] = \frac{AE}{3L} \begin{bmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{bmatrix}$$

$\boxed{6.1-4}$

$$x = \lfloor \underset{\sim}{N} \rfloor \begin{Bmatrix} x_1 \\ x_2 \end{Bmatrix}, \quad u = \lfloor \underset{\sim}{N} \rfloor \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}, \quad \textit{where}$$

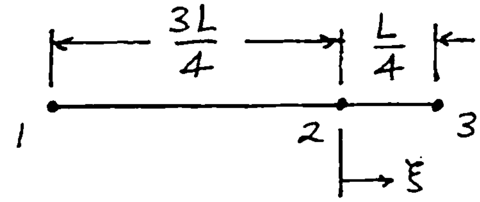
$$\lfloor \underset{\sim}{N} \rfloor = \frac{1}{2} \lfloor 1-\xi, \; 1+\xi \rfloor, \quad J = \frac{dx}{d\xi} = \frac{x_2 - x_1}{2} = \frac{L}{2}$$

$$\epsilon_x = \frac{du}{dx} = \frac{du}{d\xi}\frac{d\xi}{dx} = \frac{1}{J}\frac{du}{d\xi} = \frac{2}{L}\left\lfloor -\frac{1}{2} \quad \frac{1}{2} \right\rfloor \{\underset{\sim}{d}\} = \underbrace{\frac{1}{L}\lfloor -1 \quad 1 \rfloor}_{\lfloor \underset{\sim}{B} \rfloor} \{\underset{\sim}{d}\}$$

$$[\underset{\sim}{k}] = \int_{-1}^{1} \lfloor \underset{\sim}{B} \rfloor^T \lfloor \underset{\sim}{B} \rfloor AE J \, d\xi$$

$$[\underset{\sim}{k}] = \frac{AE}{2L} \int_{-1}^{1} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} d\xi = \frac{AE}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$\boxed{6.1\text{-}5}$



Eq. 6.1-6: $J = \left\lfloor \frac{1}{2}(-1+2\xi) \quad -2\xi \quad \frac{1}{2}(1+2\xi) \right\rfloor \begin{Bmatrix} 0 \\ 3L/4 \\ L \end{Bmatrix}$

$$J = \frac{L}{2}(1-\xi)$$

If only u_3 is nonzero, Eqs. 6.1-5 and 6.1-7 yield

$$\epsilon_x = \frac{(1+2\xi)/2}{L(1-\xi)/2}\, u_3 = \frac{1+2\xi}{1-\xi}\,\frac{u_3}{L}$$

Node 1: $\xi = -1$, $\epsilon_x = -\frac{1}{2}\frac{u_3}{L}$

Node 2: $\xi = 0$, $\epsilon_x = \frac{u_3}{L}$

Node 3: $\xi = 1$, $\epsilon_x = \frac{1}{0}\frac{u_3}{L} = \infty$

$\boxed{6.2\text{-}1}$

(a) $x = \lfloor 1, \xi, \eta, \xi\eta \rfloor \{\underset{\sim}{a}\}$. Subs. ξ & η coords. of nodes.

$$\lfloor x_1\ x_2\ x_3\ x_4 \rfloor^T = [\underset{\sim}{A}]\{\underset{\sim}{a}\}, \quad [\underset{\sim}{A}] = \begin{bmatrix} 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix}$$

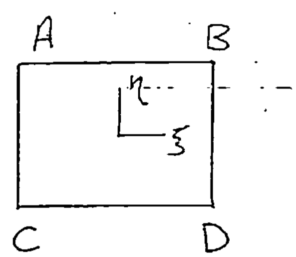
(b) $\lfloor N_1\ N_2\ N_3\ N_4 \rfloor = \lfloor 1\ \xi\ \eta\ \xi\eta \rfloor [\underset{\sim}{A}]^{-1}$. By comparing this with Eqs. 6.2-3,

$$[\underset{\sim}{A}]^{-1} = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & -1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix}$$

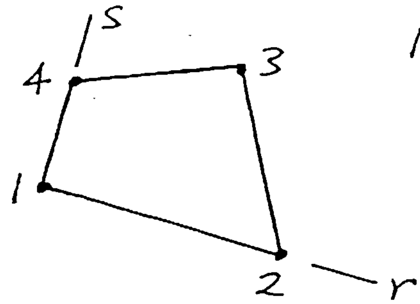
(c) Indeed $[\underset{\sim}{A}][\underset{\sim}{A}]^{-1} = [\underset{\sim}{I}]$.

$\boxed{6.2\text{-}2}$

We answer by noting which of the N_i become unity when ξ & η define a corner coordinate.



6.2-3

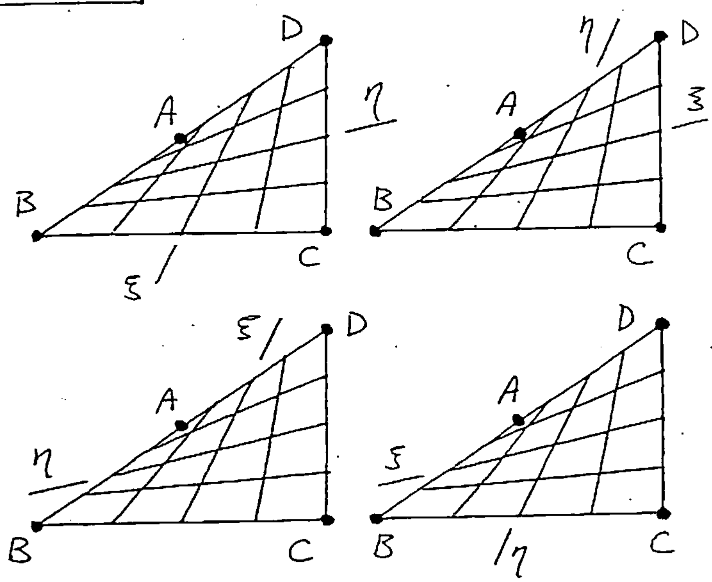


By simple inspection and trial,

$$N_1 = (1-r)(1-s)$$
$$N_2 = r(1-s)$$
$$N_3 = rs$$
$$N_4 = (1-r)s$$

Each N_i is unity at node i and zero at node j, where $j \neq i$.

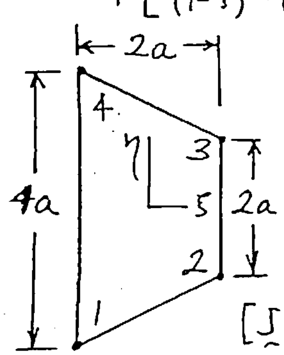
6.2-4



6.2-5

Apply Eq. 6.2-6 to the element shown.

$$[\underset{\sim}{J}] = \frac{1}{4}\begin{bmatrix} -(1-\eta) & (1-\eta) & (1+\eta) & -(1+\eta) \\ -(1-\xi) & -(1+\xi) & (1+\xi) & (1-\xi) \end{bmatrix}\begin{bmatrix} -a & -2a \\ a & -a \\ a & a \\ -a & 2a \end{bmatrix}$$



$$[\underset{\sim}{J}] = \begin{bmatrix} a & \frac{a}{2}\eta \\ 0 & \frac{a}{2}(3-\xi) \end{bmatrix}$$

$$J = \det[\underset{\sim}{J}] = \frac{a^2}{2}(3-\xi)$$

$[\underset{\sim}{J}] = f(\xi, \eta)$ but $J = f(\xi)$

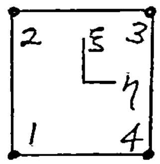
$\boxed{6.2\text{-}6}$

From Eq. 6.2-6, let

$$[\underset{\sim}{D}_N] = \frac{1}{4}\begin{bmatrix} -(1-\eta) & (1-\eta) & (1+\eta) & -(1+\eta) \\ -(1-\xi) & -(1+\xi) & (1+\xi) & (1-\xi) \end{bmatrix}$$

Then

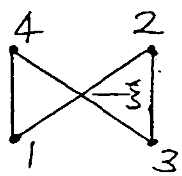
(a) $[\underset{\sim}{J}] = [\underset{\sim}{D}_N]\begin{bmatrix} -1 & -1 \\ -1 & 1 \\ 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$



$J = |\underset{\sim}{J}| = -1$

Implies left-handed $\xi\eta$ axes.

(b) $[\underset{\sim}{J}] = [\underset{\sim}{D}_N]\begin{bmatrix} -1 & -1 \\ 1 & 1 \\ 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -\eta \\ 0 & -\xi \end{bmatrix}$



$J = |\underset{\sim}{J}| = -\xi$

Implies "bow-tie" element.

6.2-7

Use Eq. 6.2-6. Define the 2 by 8 matrix in Eq. 6.2-6 as $[\underset{\sim}{D}_N]$ and write it in the form

$$[\underset{\sim}{D}_N] = \frac{1}{4}\begin{bmatrix} -1 & 1 & 1 & -1 \\ -1 & -1 & 1 & 1 \end{bmatrix} + \frac{1}{4}\begin{bmatrix} \eta & -\eta & \eta & -\eta \\ \xi & -\xi & \xi & -\xi \end{bmatrix}$$

(a) $[\underset{\sim}{D}_N]\begin{bmatrix} -3 & -2 \\ 3 & -2 \\ 3 & 2 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}, J = 6$

(b) $[\underset{\sim}{D}_N]\begin{bmatrix} 3 & -2 \\ 3 & 2 \\ -3 & 2 \\ -3 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ -3 & 0 \end{bmatrix}, J = 6$

(c) $[\underset{\sim}{D}_N]\begin{bmatrix} 0 & -2 \\ 3 & 0 \\ 3 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 3/2 & -\eta \\ 0 & 1-\xi \end{bmatrix}, J = \frac{3}{2}(1-\xi)$

(d) $[\underset{\sim}{D}_N]\begin{bmatrix} -3 & -2 \\ 0 & -2 \\ 3 & 2 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 3/2 & 0 \\ 3/2 & 2 \end{bmatrix}, J = 3$

Area ratios & J	(a)	(b)	(c)	(d)
$A_{el} \div A_{2\times2}$	6	6	3/2	3
J	6	6	$\frac{3}{2}(1-\xi)$	3

J is ratio $\frac{dx\,dy}{d\xi\,d\eta}$ & is unity if $\xi = x$ & $\eta = y$, which is the case of 2x2 square.

$\boxed{6.3-1}$

$$\text{exact } I = \int_{-1}^{1} \phi \, dx = 2a_1 + \frac{2}{3}a_3$$

Let W = weights, $\pm p$ = location.

$$W(a_1 - a_2p + a_3p^2 - a_4p^3) + W(a_1 + a_2p + a_3p^2 + a_4p^3) = 2a_1 + \frac{2}{3}a_3$$

Reduces to

$$Wa_1 + Wa_3p^2 = a_1 + \frac{1}{3}a_3$$

Must be true for any a_1 & a_3, so

$a_1(W-1) = 0$ (a)

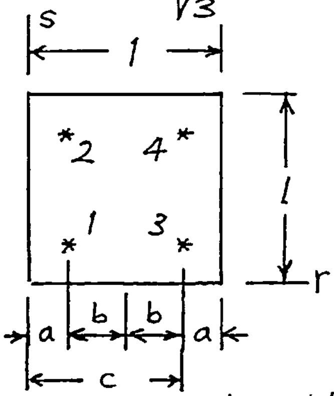
$a_3(Wp^2 - \frac{1}{3}) = 0$ (b)

(a) yields $W = 1$, hence (b) yields

$p = \pm \frac{1}{\sqrt{3}} = \pm 0.57735\ldots$

$\boxed{6.3\text{-}2}$

$b = \dfrac{0.5}{\sqrt{3}}$, $a = 0.5 - b$, $c = 0.5 + b$



$a = 0.21132\ldots$

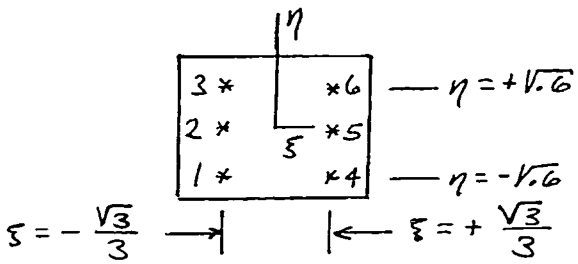
$c = 0.78868\ldots$

point	r_i	s_i
1	a	a
2	a	c
3	c	a
4	c	c

Weights : $\frac{1}{2}$ each, since

$$\int_0^1 \int_0^1 dr\, ds = \sum_{i=1}^{2} \sum_{j=1}^{2} W_i W_j = 1$$

$$\boxed{6.3\text{-}3}$$



$$I = (1)\left(\frac{5}{9}\phi_1 + \frac{8}{9}\phi_2 + \frac{5}{9}\phi_3\right) + (1)\left(\frac{5}{9}\phi_4 + \frac{8}{9}\phi_5 + \frac{5}{9}\phi_6\right)$$

$$I = \frac{5}{9}\left(\phi_1 + \phi_3 + \phi_4 + \phi_6\right) + \frac{8}{9}\left(\phi_2 + \phi_5\right)$$

$\boxed{6.3\text{-}4}$

	$W_i W_j W_k$
8 points nearest corners	$\left(\frac{5}{9}\right)^3$
6 points nearest middle of faces	$\left(\frac{5}{9}\right)\left(\frac{8}{9}\right)^2$
12 points nearest middle of edges	$\left(\frac{5}{9}\right)^2\left(\frac{8}{9}\right)$
1 point at center $(\xi = \eta = \zeta = 0)$	$\left(\frac{8}{9}\right)^3$

27 points $(= 3 \times 3 \times 3)$

Sum of weight products:

$$8\left(\frac{5}{9}\right)^3 + 6\left(\frac{5}{9}\right)\left(\frac{8}{9}\right)^2 + 12\left(\frac{5}{9}\right)^2\left(\frac{8}{9}\right) + \left(\frac{8}{9}\right)^3 = 8 \quad \checkmark$$

$\boxed{6.3\text{-}5}$

$$I_{exact} = \int_0^{12} y\,dx = 6\,\frac{2+4}{2} + 6(4) = 18+24 = 42$$

$$x = 6 + 6\xi, \qquad J = \frac{dx}{d\xi} = 6$$

	error
$I_1 = 6\left[2(4)\right] = 48$	$+14.3\%$
$I_2 = 6\left[\left(4 - 2\frac{1}{\sqrt{3}}\right) + 4\right] = 41.072$	-2.2%
$I_3 = 6\left[\left(4 - 2\sqrt{0.6}\right)\frac{5}{9} + 4\left(\frac{8}{9}\right) + 4\left(\frac{5}{9}\right)\right] = 42.836$	$+2.0\%$

Convergence is not monotonic

$\boxed{6.3-6}$

(a) Exact: $\int_{-1}^{1}(\xi^2+\xi^3)\,d\xi = \frac{2}{3} = I$

1 pt. $I_1 = 2(0+0) = 0$ 100% low

2 pts. Let $a = \sqrt{3}/3$, then

$I_2 = (a^2-a^3)+(a^2+a^3) = 2a^2 = \frac{2}{3}$ exact

3 pts. Let $b = \sqrt{0.6}$, then

$I_3 = \frac{5}{9}(b^2-b^3)+\frac{8}{9}(0)+\frac{5}{9}(b^2+b^3)$

$I_3 = \frac{10}{9}b^2 = \frac{2}{3}$ exact

(b) $\int_{-1}^{1}\cos 1.5\xi\,d\xi = \left.\frac{\sin 1.5\xi}{1.5}\right|_{-1}^{1} = 1.3300 = I$

1 pt. $I_1 = 2(1) = 2$ +50.4%

2 pts. $I_2 = \cos\left(-\frac{1.5}{\sqrt{3}}\right)+\cos\left(\frac{1.5}{\sqrt{3}}\right) = 1.2957$ −2.58%

3 pts. $I_3 = \frac{5}{9}\cos\left(-1.5\sqrt{0.6}\right)+\frac{8}{9}\cos(0)$

$+\frac{5}{9}\cos\left(1.5\sqrt{0.6}\right) = 1.3307$ +0.05%

(c) $\int_{-1}^{1}\frac{1-\xi}{2+\xi}\,d\xi = \int_{-1}^{1}\frac{d\xi}{2+\xi} - \int_{-1}^{1}\frac{\xi\,d\xi}{2+\xi}$

$= \ln(2+\xi)\Big|_{-1}^{1} - \Big[2+\xi-2\ln(2+\xi)\Big]_{-1}^{1}$

$= \ln 3 - 2 + 2\ln 3 = 3\ln 3 - 2 = 1.2958 = I$

1 pt. $I_1 = 2\,\frac{1}{2} = 1$ −22.8%

2 pts. $I_2 = \frac{1+\frac{1}{\sqrt{3}}}{2-\frac{1}{\sqrt{3}}} + \frac{1-\frac{1}{\sqrt{3}}}{2+\frac{1}{\sqrt{3}}} = 1.2727$ −1.8%

3 pts. $I_3 = \frac{5}{9}\,\frac{1+\sqrt{0.6}}{2-\sqrt{0.6}} + \frac{8}{9}\,\frac{1}{2} + \frac{5}{9}\,\frac{1-\sqrt{0.6}}{2+\sqrt{0.6}}$

$= 1.2941$ −0.13%

(d) $x = \begin{bmatrix}\frac{1-\xi}{2} & \frac{1+\xi}{2}\end{bmatrix}\begin{Bmatrix}1\\7\end{Bmatrix} = \frac{8+6\xi}{2} = 4+3\xi$

$I = \int_{1}^{7}\frac{dx}{x} = \ln 7 = 1.94591 = \int_{-1}^{1}\frac{3\,d\xi}{4+3\xi}$

$I_1 = 2\,\frac{3}{4} = 1.5$ −22.9%

$I_2 = \frac{3}{4-\sqrt{3}} + \frac{3}{4+\sqrt{3}} = 1.84615$ −5.1%

$I_3 = \frac{5}{9}\left(\frac{3}{4-3\sqrt{.6}}+\frac{3}{4+3\sqrt{.6}}\right)+\frac{8}{9}\,\frac{3}{4} = 1.92453$ −1.1%

$\boxed{6.3\text{-}7}$

$$\int_{-1}^{1}(3+\xi^2)\,d\xi = \left(3\xi + \frac{\xi^2}{3}\right)_{-1}^{1} = \frac{20}{3}$$

$$I = \frac{20}{3}\int_{-1}^{1}\frac{d\eta}{2+\eta^2} = \frac{20}{3}\,\frac{1}{\sqrt{2}}\arctan\frac{\eta}{\sqrt{2}}\Big|_{-1}^{1}$$

$$I = 5.8028$$

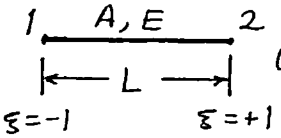
1 pt. $I_1 = (2)(2)\left(\frac{3}{2}\right) = 6$ $\qquad +3.4\%$

2 pts. $I_2 = 4\,\dfrac{3+\frac{1}{3}}{2+\frac{1}{3}} = 5.7143$ $\qquad -1.53\%$

3 pts. $I_3 = 4\left(\dfrac{25}{81}\,\dfrac{3+0.6}{2+0.6}\right) + 2\left(\dfrac{40}{81}\,\dfrac{3+0.6}{2}\right)$

$\qquad + 2\left(\dfrac{40}{81}\,\dfrac{3}{2+0.6}\right) + \dfrac{64}{81}\,\dfrac{3}{2} = 5.8120$ $\qquad +0.16\%$

$\boxed{6.3\text{-}8}$



$$u = \begin{bmatrix} \frac{1-\xi}{2} & \frac{1+\xi}{2} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$J = L/2$$

$$\epsilon_x = \frac{1}{J}\frac{du}{d\xi} = \frac{2}{L}\begin{bmatrix} -\frac{1}{2} & \frac{1}{2} \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \frac{1}{L}\begin{bmatrix} -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$[\underset{\sim}{k}] = \int_{-1}^{1} \lfloor \underset{\sim}{B} \rfloor^T \lfloor \underset{\sim}{B} \rfloor A E J\, d\xi = \frac{E}{2L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\int_{-1}^{1} A\, d\xi$$

(a) A const.; $\int_{-1}^{1} A\, d\xi = A\int_{-1}^{1} d\xi = A(1+1) = 2A$

$$[\underset{\sim}{k}] = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

(b) $\int_{-1}^{1} A\, d\xi = \int_{-1}^{1}\left(\frac{1-\xi}{2}A_1 + \frac{1+\xi}{2}A_2\right) d\xi$

$$\left.\begin{aligned} &= (1)\left(\frac{1+1/\sqrt{3}}{2}A_1 + \frac{1-1/\sqrt{3}}{2}A_2\right) \\ &+ (1)\left(\frac{1-1/\sqrt{3}}{2}A_1 + \frac{1+1/\sqrt{3}}{2}A_2\right) \end{aligned}\right\} \text{2-pt. Gauss rule}$$

$= A_1 + A_2$, so $[\underset{\sim}{k}] = \frac{A_1 + A_2}{2}\frac{E}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

(c) Evaluate integral of (b) at $\xi = 0$ with weight 2. Thus

$$\int_{-1}^{1} A\, d\xi = 2\left(\frac{A_1}{2} + \frac{A_2}{2}\right) = A_1 + A_2$$

Same result as in part (b).

$\boxed{6.3-9}$

From Eq. 6.1-7, with $J = L/2$,

$$\epsilon_x = \frac{1}{L}\underbrace{\lfloor (-1+2\xi) \quad -4\xi \quad (1+2\xi) \rfloor}_{\lfloor B_1 \rfloor}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \frac{1}{L}\lfloor B_1 \rfloor \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$[k] = \int_{-1}^{1} AE \lfloor B \rfloor^T \lfloor B \rfloor \, J \, d\xi = AE \frac{L}{2}\frac{1}{L^2}\int_{-1}^{1} \lfloor B_1 \rfloor^T \lfloor B_1 \rfloor \, d\xi$$

$$[k] = \frac{AE}{2L}\underbrace{\lfloor B_1 \rfloor^T \lfloor B_1 \rfloor}_{\text{at } \xi = -\frac{1}{\sqrt{3}}} + \frac{AE}{2L}\underbrace{\lfloor B_1 \rfloor^T \lfloor B_1 \rfloor}_{\text{at } \xi = +\frac{1}{\sqrt{3}}}$$

$$[k] = \frac{AE}{2L}\begin{Bmatrix} -2.1547 \\ 2.3094 \\ -0.1547 \end{Bmatrix}\lfloor --- \rfloor + \frac{AE}{2L}\begin{Bmatrix} 0.1547 \\ -2.3094 \\ 2.1547 \end{Bmatrix}\lfloor --- \rfloor$$

$$[k] = \frac{AE}{2L}\begin{bmatrix} 4.6427 & -4.9761 & 0.3333 \\ & 5.3333 & -0.3573 \\ \text{symm.} & & 0.0239 \end{bmatrix}$$

$$+ \frac{AE}{2L}\begin{bmatrix} 0.0239 & -0.3573 & 0.3333 \\ & 5.3333 & -4.9761 \\ \text{symm.} & & 4.6427 \end{bmatrix}$$

$$[k] = \frac{AE}{2L}\begin{bmatrix} 4\frac{2}{3} & -5\frac{1}{3} & \frac{2}{3} \\ & 10\frac{2}{3} & -5\frac{1}{3} \\ \text{symm.} & & 4\frac{2}{3} \end{bmatrix} = \frac{AE}{3L}\begin{bmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{bmatrix}$$

$\boxed{6.3\text{-}10}$

$$\lfloor \underset{\sim}{B} \rfloor = \lfloor B_1 \quad B_2 \quad -B_1 \quad B_4 \rfloor \quad \text{where, with } x = \frac{L}{2}(1+\xi),$$

$$B_1 = -\frac{6}{L^2} + \frac{12x}{L^3} = -\frac{6}{L} + \frac{12}{L^3}\frac{L}{2}(1+\xi) = \frac{6\xi}{L^2}$$

$$B_2 = -\frac{4}{L} + \frac{6x}{L^2} = -\frac{4}{L} + \frac{6}{L^2}\frac{L}{2}(1+\xi) = \frac{1}{L}(-1+3\xi)$$

$$B_4 = -\frac{2}{L} + \frac{6x}{L^2} = -\frac{2}{L} + \frac{6}{L^2}\frac{L}{2}(1+\xi) = \frac{1}{L}(1+3\xi)$$

$$k_{11} = EI\int_{-1}^{1} B_1^2 \frac{L}{2}\,d\xi = \frac{EIL}{2}\int_{-1}^{1}\frac{36\xi^2}{L^4}\,d\xi = \frac{18EI}{L^3}\left(\frac{1}{3}+\frac{1}{3}\right) = \frac{12EI}{L^3}$$

$$k_{12} = EI\int_{-1}^{1} B_1 B_2 \frac{L}{2}\,d\xi = \frac{3EI}{L^2}\int_{-1}^{1}(-\xi+3\xi^2)\,d\xi = \frac{3EI}{L^2}\left[\left(\frac{1}{\sqrt{3}}+\frac{3}{3}\right)+\left(-\frac{1}{\sqrt{3}}+\frac{3}{3}\right)\right] = \frac{6EI}{L^2}$$

$k_{13} = -k_{11}$ by inspection

$$k_{14} = EI\int_{-1}^{1} B_1 B_4 \frac{L}{2}\,d\xi = \frac{3EI}{L^2}\int_{-1}^{1}(\xi+3\xi^2)\,d\xi = \frac{3EI}{L^2}\left[\left(-\frac{1}{\sqrt{3}}+\frac{3}{3}\right)+\left(\frac{1}{\sqrt{3}}+\frac{3}{3}\right)\right] = \frac{6EI}{L^2}$$

$$k_{22} = EI\int_{-1}^{1} B_2^2 \frac{L}{2}\,d\xi = \frac{EI}{2L}\int_{-1}^{1}(-1+3\xi)^2\,d\xi = \frac{EI}{2L}\left[\left(-1-\frac{3}{\sqrt{3}}\right)^2+\left(-1+\frac{3}{\sqrt{3}}\right)^2\right]$$

$$= \frac{EI}{2L}\left[7.464+0.536\right] = \frac{4EI}{L}$$

$k_{23} = -k_{12}$ by inspection

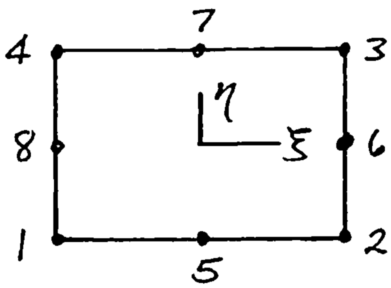
$$k_{24} = \frac{EI}{2L}\int_{-1}^{1} B_2 B_4 \frac{L}{2}\,d\xi = \frac{EI}{2L}\int_{-1}^{1}(-1+9\xi^2)\,d\xi = \frac{EI}{2L}\left[\left(-1+\frac{9}{3}\right)+\left(-1+\frac{9}{3}\right)\right] = \frac{2EI}{L}$$

$k_{--} = k_{..}$ by inspection

$k_{34} = -k_{14}$ by inspection

$$\text{L} \quad - EI\int_{-1}^{1} B_4^2 \frac{L}{2}\,d\xi = \frac{EI}{2L}\left[\left(1-\frac{6}{\sqrt{3}}+\frac{9}{3}\right)+\left(1+\frac{6}{\sqrt{3}}+\frac{9}{3}\right)\right] = \frac{4EI}{L}$$

$\boxed{6.4\text{-}1}$



Left edge

$$N_4 = \frac{1}{2}(1+\eta) - \frac{1}{2}(1-\eta^2) = \frac{\eta+\eta^2}{2}$$

$$N_8 = 1-\eta^2$$

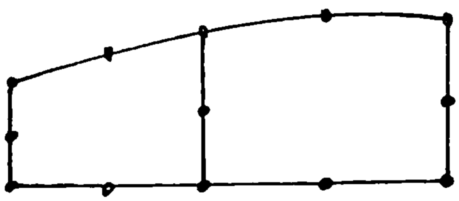
$$N_1 = \frac{1}{2}(1-\eta) - \frac{1}{2}(1-\eta^2) = \frac{-\eta+\eta^2}{2}$$

Right edge

$$N_3 = \frac{1}{2}(1+\eta) - \frac{1}{2}(1-\eta^2) = \frac{\eta+\eta^2}{2}$$

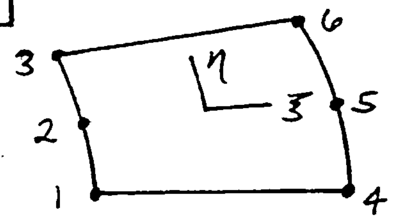
$$N_6 = 1-\eta^2$$

$$N_2 = \frac{1}{2}(1-\eta) - \frac{1}{2}(1-\eta^2) = \frac{-\eta+\eta^2}{2}$$



Hence on a common edge such as this, the field quantity is defined by the same three nodal d.o.f. and uses the same N_i whether viewed from the left element or the right, so the identical curve $\phi = \phi(\eta)$ is produced.

$\boxed{6.4\text{-}2}$



Can use Eq. 6.1-4, but apply to sides 1-2-3 and 4-5-6:

On $\xi = -1$, $\phi = \frac{-\eta+\eta^2}{2}\phi_1 + (1-\eta^2)\phi_2 + \frac{\eta+\eta^2}{2}\phi_3$

On $\xi = +1$, $\phi = \frac{-\eta+\eta^2}{2}\phi_4 + (1-\eta^2)\phi_5 + \frac{\eta+\eta^2}{2}\phi_6$

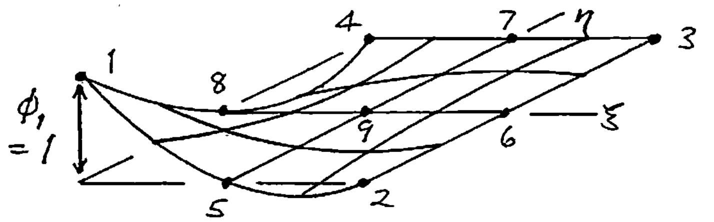
Sweep using shape function $\frac{1-\xi}{2}$ for left edge, $\frac{1+\xi}{2}$ for right. Thus for the six-node element,

$$N_1 = \frac{1-\xi}{2}\left(\frac{-\eta+\eta^2}{2}\right) \qquad N_4 = \frac{1+\xi}{2}\left(\frac{-\eta+\eta^2}{2}\right)$$

$$N_2 = \frac{1-\xi}{2}(1-\eta^2) \qquad N_5 = \frac{1+\xi}{2}(1-\eta^2)$$

$$N_3 = \frac{1-\xi}{2}\left(\frac{\eta+\eta^2}{2}\right) \qquad N_6 = \frac{1+\xi}{2}\left(\frac{\eta+\eta^2}{2}\right)$$

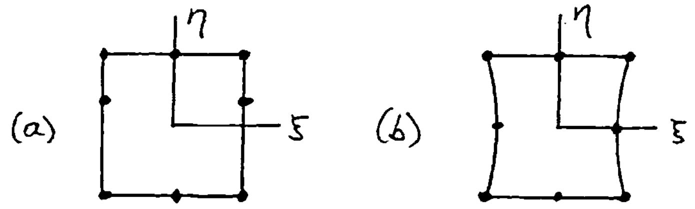
6.4-3



$$N_1 = \frac{1}{4}(1-\xi)(1-\eta) - \frac{1}{4}(1-\xi^2)(1-\eta) - \frac{1}{4}(1-\xi)(1-\eta^2) + \frac{1}{4}(1-\xi^2)(1-\eta^2)$$

ξ	η	$N_1/64$
$-1/2$	$-1/2$	$36-18-18+9 = 9$ $\uparrow$
$1/2$	$-1/2$	$12-18-6+9 = -3$ $\downarrow$
$1/2$	$1/2$	$4-6-6+9 = 1$ $\uparrow$
$-1/2$	$1/2$	$12-6-18+9 = -3$ $\downarrow$

$\boxed{6.4-4}$



$\boxed{6.4\text{-}5}$

On $\xi = -1$, $\phi = \eta\phi_4 + (1-\eta)\phi_8$ (A)

on $\eta = -1$, $\phi = \xi\phi_2 + (1-\xi)\phi_5$ (B)

At node 1, $\xi = \eta = -1$, (A) & (B) give

$$\phi_1 = -\phi_4 - 2\phi_8 = -\phi_2 - 2\phi_5 \qquad (C)$$

But ϕ_2, ϕ_4, ϕ_5 & ϕ_8 are independent, which contradicts (C).

Also, if N_1 of Table 6.6-1 omitted, ϕ along $\xi = -1$ is quadratic in η (from N_8), which contradicts (A). Similar for ϕ along $\eta = -1$ & (C).

$\boxed{6.4\text{-}6}$

For a beam that extends from $-\xi$ to $+\xi$ in natural coordinates, consider

$$v = a_1(1 + \cos \pi\xi)$$

where a_1 is a generalized d.o.f. Hence

$$\frac{dv}{dx} = \frac{2}{L}\frac{dv}{d\xi} = -a_1 \frac{2}{L}(\pi \sin \pi\xi)$$

At ends $\pm L/2$, v and dv/dx both vanish; OK.

6.6-1

Consider Eqs. 3.6-10, which are for a Q4 element in pure bending:

$$\epsilon_x = -\frac{\theta_{e1} y}{2a} \qquad \epsilon_y = 0 \qquad \gamma_{xy} = -\frac{\theta_{e1} x}{2a}$$

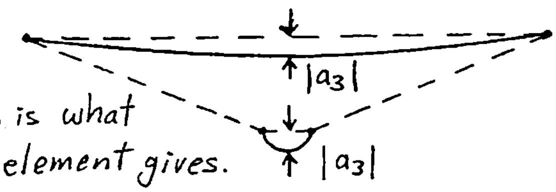
These equations pertain to a four-node element in which bending dominates and internal d.o.f. are omitted. If nodal d.o.f. (and hence θ_{e1}) are rather accurate, then ϵ_x is rather accurate but ϵ_y and γ_{xy} are not. Thus for stresses:

$$\sigma_x = \frac{E}{1-\nu^2}(\epsilon_x + \nu\epsilon_y) \qquad \text{some error}$$

$$\sigma_y = \frac{E}{1-\nu^2}(\epsilon_y + \nu\epsilon_x) \qquad \text{larger error}$$

$$\tau_{xy} = G\gamma_{xy} \qquad \text{very large error}$$

6.6-2



This is what the element gives.

But, for a correct model of pure bending, both arcs should have the same center (the center of curv. of the beam).

$\boxed{6.7-1}$

$$\begin{bmatrix} 12 & -6 & 0 \\ -6 & 12 & -6 \\ 0 & -6 & 6 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \\ u_4 \end{Bmatrix} = \begin{Bmatrix} 24 \\ 24 \\ 0 \end{Bmatrix}, \text{or} \begin{bmatrix} 6 & -6 & 0 \\ -6 & 12 & -6 \\ 0 & -6 & 12 \end{bmatrix} \begin{Bmatrix} u_4 \\ u_3 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 24 \\ 24 \end{Bmatrix}$$

$$[\underset{\sim}{k}_{cc}] = \begin{bmatrix} 12 & -6 \\ -6 & 12 \end{bmatrix}, \quad [\underset{\sim}{k}_{cc}]^{-1} = \frac{1}{18} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

$$[\underset{\sim}{k}]_{cond.} = 6 - \lfloor -6 \quad 0 \rfloor \frac{1}{18} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{Bmatrix} -6 \\ 0 \end{Bmatrix} = 6 - 4 = 2$$

$$\{\underset{\sim}{r}\}_{cond.} = 0 - \lfloor -6 \quad 0 \rfloor \frac{1}{18} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{Bmatrix} 24 \\ 24 \end{Bmatrix} = \frac{6}{18} 72 = 24$$

i.e. $2u_4 = 24$ so $u_4 = 12$

Recover $\{\underset{\sim}{d}_c\}$:

$$\begin{Bmatrix} u_3 \\ u_2 \end{Bmatrix} = -\frac{1}{18} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \left(\begin{Bmatrix} -6 \\ 0 \end{Bmatrix} 12 - \begin{Bmatrix} 24 \\ 24 \end{Bmatrix} \right) = \begin{Bmatrix} 12 \\ 8 \end{Bmatrix}$$

Check: (order of d.o.f. here as originally written)

$$\begin{bmatrix} 12 & -6 & 0 \\ -6 & 12 & -6 \\ 0 & -6 & 6 \end{bmatrix} \begin{Bmatrix} 8 \\ 12 \\ 12 \end{Bmatrix} = \begin{Bmatrix} 24 \\ 24 \\ 0 \end{Bmatrix} \quad \checkmark$$

$\boxed{6.7\text{-}2}$

(a) Fill in column 1 of $[\underset{\sim}{k}]$ by symmetry : $[\underset{\sim}{k}] = \dfrac{AE}{3L}\begin{bmatrix} 7 & -8 & 1 \\ -8 & & \\ 1 & & \end{bmatrix}$

D.o.f. u_1 and u_3 create the same nodal forces: using this, and symmetry to fill in the last row, $[\underset{\sim}{k}] = \dfrac{AE}{3L}\begin{bmatrix} 7 & -8 & 1 \\ -8 & & -8 \\ 1 & -8 & 7 \end{bmatrix}$

Finally, there is no force at node 2 if $u_1 = u_2 = u_3$, so $[\underset{\sim}{k}] = \dfrac{AE}{3L}\begin{bmatrix} 7 & -8 & 1 \\ -8 & 16 & -8 \\ 1 & -8 & 7 \end{bmatrix}$

(b) To suit explanation in Section 6.7, reorder d.o.f.

$$[\underset{\sim}{k}] = \frac{AE}{3L}\begin{array}{c} \begin{bmatrix} 7 & 1 & -8 \\ 1 & 7 & -8 \\ -8 & -8 & 16 \end{bmatrix} \end{array}\begin{array}{c} u_1 \\ u_3 \\ u_2 \end{array}$$

To condense u_2, apply Eq. 6.7-3:
$[\underset{\sim}{k}_{cc}] = \dfrac{AE}{3L}16$, $[\underset{\sim}{k}_{cc}]^{-1} = \dfrac{3L}{16AE}$
$[\underset{\sim}{k}_{rc}] = \dfrac{AE}{3L}\begin{Bmatrix} -8 \\ -8 \end{Bmatrix}$

$$[\underset{\sim}{k}_{cond}] = \frac{AE}{3L}\begin{bmatrix} 7 & 1 \\ 1 & 7 \end{bmatrix} - \frac{AE}{3L}\begin{Bmatrix} -8 \\ -8 \end{Bmatrix}\frac{3L}{16AE}\frac{AE}{3L}\lfloor -8 \quad -8 \rfloor$$

$$= \frac{AE}{3L}\begin{bmatrix} 7 & 1 \\ 1 & 7 \end{bmatrix} - \frac{AE}{3L}\begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix} = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{array}{c} u_1 \\ u_3 \end{array}$$

(c) With d.o.f. in the order $u_1 \; u_3 \; u_2$, the load vector is $\{\underset{\sim}{r}_e\} = \dfrac{qL}{6}\begin{Bmatrix} 1 \\ 1 \\ 4 \end{Bmatrix}$

$$\{\underset{\sim}{r}_{cond}\} = \frac{qL}{6}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix} - \frac{AE}{3L}\begin{Bmatrix} -8 \\ -8 \end{Bmatrix}\frac{3L}{16AE}\frac{4qL}{6} = \frac{qL}{6}\begin{Bmatrix} 1+2 \\ 1+2 \end{Bmatrix} = \frac{qL}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$$

[illegible]ed system: $\dfrac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_3 \end{Bmatrix} = \dfrac{qL}{2}\begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$ Gives $u_3 = \dfrac{qL^2}{2AE}$ if $u_1 = 0$

Then from Eq. 6.7-2:

$$u_2 = \{\underset{\sim}{d}_c\} = -\frac{3L}{16AE}\left(\frac{AE}{3L}\lfloor -8 \quad -8 \rfloor\begin{Bmatrix} 0 \\ qL^2/2AE \end{Bmatrix} - \frac{4qL}{6}\right) = \frac{3qL^2}{8AE}$$

Check nodal loads using computed d.o.f. :

$$\frac{AE}{3L}\begin{bmatrix} 7 & 1 & -8 \\ 1 & 7 & -8 \\ -8 & -8 & 16 \end{bmatrix}\begin{Bmatrix} 0 \\ qL^2/2AE \\ 3qL^2/8AE \end{Bmatrix} = \frac{qL}{3}\begin{Bmatrix} \frac{1}{2}-3 \\ \frac{7}{2}-3 \\ -4+6 \end{Bmatrix} = qL\begin{Bmatrix} -5/6 \\ 1/6 \\ 2/3 \end{Bmatrix}$$ Satisfies equil.

$\boxed{6.7\text{-}3}$

$$[\underset{\sim}{k}] = \frac{EI}{L^3}\begin{array}{c}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}\end{array}\begin{array}{l} v_1 \\ \theta_{z1} \\ v_2 \\ \theta_{z2} \end{array}$$

Condense θ_{z2}: apply Eq. 6.7-3

$$[k_{cond}] = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 \\ 6L & 4L^2 & -6L \\ -12 & -6L & 12 \end{bmatrix} - \frac{EI}{L^3}\begin{Bmatrix} 6L \\ 2L^2 \\ -6L \end{Bmatrix}\frac{L}{4EI}\frac{EI}{L^3}\lfloor 6L \quad 2L^2 \quad -6L \rfloor$$

$$= \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 \\ 6L & 4L^2 & -6L \\ -12 & -6L & 12 \end{bmatrix} - \frac{EI}{L^3}\begin{bmatrix} 9 & 3L & -9 \\ 3L & L^2 & -3L \\ -9 & -3L & 9 \end{bmatrix}$$

$$= \frac{EI}{L^3}\begin{bmatrix} 3 & 3L & -3 \\ 3L & 3L^2 & -3L \\ -3 & -3L & 3 \end{bmatrix}$$

$$= \frac{3EI}{L^3}\begin{bmatrix} 1 & L & -1 \\ L & L^2 & -L \\ -1 & -L & 1 \end{bmatrix}$$

6.7-4

Let $[\underset{\sim}{k}_I] = \begin{bmatrix} \underset{6\times 6}{[\underset{\sim}{k}]} & [\underset{\sim}{0}] \\ [\underset{\sim}{0}] & [\underset{\sim}{0}] \end{bmatrix}$ in which $\underset{6\times 6}{[\underset{\sim}{k}]}$ appears in Eq. 2.3-6

Let $[\underset{\sim}{k}_{II}] = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & k_1 & 0 & 0 & 0 & -k_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & k_2 & 0 & -k_2 \\ 0 & 0 & -k_1 & 0 & 0 & 0 & k_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -k_2 & 0 & k_2 \end{bmatrix}$

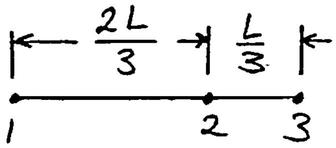
Both $[\underset{\sim}{k}_I]$ and $[\underset{\sim}{k}_{II}]$ operate on the nodal d.o.f. $\{\underset{\sim}{d}\} = \begin{Bmatrix} u_1 \\ v_1 \\ \theta_{z1} \\ u_2 \\ v_2 \\ \theta_{z2} \\ \beta_{z1} \\ \beta_{z2} \end{Bmatrix}$

Form $[\underset{\sim}{k}_I] + [\underset{\sim}{k}_{II}]$, condense out d.o.f. θ_{z1} and θ_{z2}, discard the rows and columns that correspond to θ_{z1} and θ_{z2}; what remains is a 6 by 6 matrix that operates on nodal d.o.f. $\{\underset{\sim}{d}_c\} = \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ \beta_{z1} \\ \beta_{z2} \end{Bmatrix}$

(Rearrange as may be convenient or necessary.)

$\boxed{6.8-1}$

$$\epsilon_x = \frac{1}{J}\left\lfloor \frac{-1+2\xi}{2} \quad -2\xi \quad \frac{1+2\xi}{2} \right\rfloor \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$



$$J = \left\lfloor \frac{-1+2\xi}{2} \quad -2\xi \quad \frac{1+2\xi}{2} \right\rfloor \begin{Bmatrix} 0 \\ 2L/3 \\ L \end{Bmatrix} \qquad \text{For } \xi = 0,\ J = J_0 = \frac{L}{2}$$

For $\xi = 0$, $$\epsilon_x = \underbrace{\frac{2}{L}\left\lfloor -\frac{1}{2} \quad 0 \quad \frac{1}{2} \right\rfloor}_{\lfloor \underset{\sim}{B}_0 \rfloor} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$[\underset{\sim}{k}] = \int_{-1}^{1} \lfloor \underset{\sim}{B} \rfloor^T \lfloor \underset{\sim}{B} \rfloor A E J \, d\xi$$

For one-point quadrature, with weight factor $W = 2$,

$$[\underset{\sim}{k}] = W \lfloor \underset{\sim}{B}_0 \rfloor^T \lfloor \underset{\sim}{B}_0 \rfloor A E J_0 = 2AE\frac{L}{2}\left(\frac{2}{L}\right)^2 \begin{Bmatrix} -1/2 \\ 0 \\ 1/2 \end{Bmatrix} \left\lfloor -1/2 \quad 0 \quad 1/2 \right\rfloor$$

$$[\underset{\sim}{k}] = \frac{AE}{L}\begin{bmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$\boxed{6.8-2}$$

$$V = \int_{-1}^{1}\int_{-1}^{1} Jt \, d\xi \, d\eta$$

(a) From Eq. 6.2-6, powers of ξ & η in $[\underset{\sim}{J}]$ go to $\begin{bmatrix} \eta^1 & \eta^1 \\ \xi^1 & \xi^1 \end{bmatrix}$. Also, t is bilinear. Hence Jt contains terms to $\xi^2\eta$, $\xi\eta^2$, and $\xi^2\eta^2$. Second powers: need 2×2.

(b) In similar fashion, powers in $[\underset{\sim}{J}]$ and t go to $\begin{bmatrix} \eta^2 & \eta^2 \\ \xi^2 & \xi^2 \end{bmatrix}$ and $\xi\eta^2$, $\xi^2\eta$ in t.

Jt contains ξ^4 & η^4. Need 3×3 rule: $(2 \cdot 3 - 1) = 5 > 4$.

$\boxed{6.8\text{-}3}$

$$V = \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} J \, d\xi \, d\eta \, d\zeta .$$

Eq. 6.5-2 shows that in $[\underset{\sim}{J}]$ we will find powers of ξ, η, ζ up to: →

$$\begin{bmatrix} \eta\zeta & \eta\zeta & \eta\zeta \\ \xi\zeta & \xi\zeta & \xi\zeta \\ \xi\eta & \xi\eta & \xi\eta \end{bmatrix}$$

Determinant(J) contains ξ^2, η^2, & ζ^2.

Need $2\times2\times2$ rule: $(2\cdot2-1)=3>2$.

$\boxed{6.8\text{-}4}$

Jacobian J is constant. Look for highest powers of ξ and η in the product $[\underset{\sim}{B}]^T[\underset{\sim}{B}]t$, where $t = \Sigma N_i t_i$

(a) Q4 element:
From Eqs. 6.2-3, t displays ξ^1 and η^1
From Eq. 6.2-11, $[\underset{\sim}{B}]$ displays ξ^1 and η^1 (J is constant)

Hence $[\underset{\sim}{B}]^T[\underset{\sim}{B}]t$ displays ξ^3 and η^3; need a 2 by 2 rule

(b) Q8 element:
From Eqs. 6.4-2, t displays ξ^2 and η^2
From Table 6.4-1, $[\underset{\sim}{B}]$ will contain ξ^2 and η^2 (J is constant)

Hence $[\underset{\sim}{B}]^T[\underset{\sim}{B}]t$ displays ξ^6 and η^6; need a 4 by 4 rule

6.8-5

For rectangular elements the 2 by 2 rule is exact, so changing the rule to 3 by 3 will make no difference.

For nonrectangular elements no rule is exact, but 3 by 3 is more nearly exact than 2 by 2; it stiffens the elements and therefore reduces computed deflection.

$\boxed{6.8\text{-}6}$

Consider square el. 2a units/side.

$$[\underset{\sim}{J}] = \begin{bmatrix} a & 0 \\ 0 & a \end{bmatrix}, \quad [\underset{\sim}{J}]^{-1} = \frac{1}{a}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Eqs. 6.12-3:

$$u = 3\xi\eta^2 - \xi, \quad u_{,\xi} = 3\eta^2 - 1, \quad u_{,\eta} = 6\xi\eta$$

$$v = \eta - 3\xi^2\eta, \quad v_{,\xi} = -6\xi\eta, \quad v_{,\eta} = 1 - 3\xi^2$$

$$\epsilon_x = \frac{1}{a}u_{,\xi} \qquad \epsilon_y = \frac{1}{a}v_{,\eta} \qquad \gamma_{xy} = \frac{1}{a}(u_{,\eta} + v_{,\xi})$$

$\gamma_{xy} = 0$ for all ξ, η

$u_{,\xi} = v_{,\eta} = 0$ for $\xi, \eta = \pm \frac{1}{\sqrt{3}}$; i.e.

$\epsilon_x = \epsilon_y = 0$ at Gauss points of 2×2 rule.

$\boxed{6.8\text{-}7}$

(a) At $\xi = 0$, Eq. 6.1-7 gives $\lfloor \underset{\sim}{B} \rfloor = \frac{1}{J}\left\lfloor -\frac{1}{2} \quad 0 \quad \frac{1}{2} \right\rfloor$. Therefore $\{\underset{\sim}{d}\} = \begin{Bmatrix} 0 \\ u_2 \\ 0 \end{Bmatrix}$ is a spurious mode.

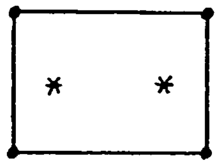
(b) 

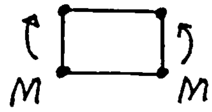
An inflection point (zero curvature) appears at midspan ($\xi = 0$).

At midspan, $x = \frac{L}{2}$, Eq. 3.3-13 yields $\lfloor \underset{\sim}{B} \rfloor = \left\lfloor 0 \quad -\frac{1}{L} \quad 0 \quad \frac{1}{L} \right\rfloor$

$\lfloor \underset{\sim}{B} \rfloor \{\underset{\sim}{d}\} = 0$ when $\{\underset{\sim}{d}\} = \begin{Bmatrix} b \\ c \\ b \\ c \end{Bmatrix}$, so this $\{\underset{\sim}{d}\}$ is a spurious mode.

$\boxed{6.8\text{-}8}$

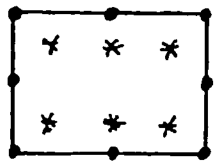
(a) 

8 d.o.f.
3 rigid body modes
no spurious modes:  M not resisted by ϵ_x (but resisted by γ_{xy})
Rank 5

Disadvantages: not frame invariant

(b) Like part (a), except that $\gamma_{xy}=0$ on $\eta=0$, so M is not resisted.
Rank 4

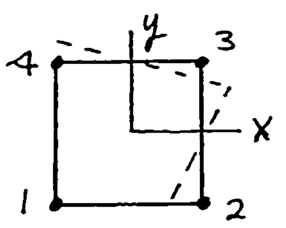
Disadvantages: one spurious mode, not frame invariant

(c) 

See roblem 6.8-6: ϵ_y is now nonzero for the mode of Fig. 6.8-3d, so no spurious mode
Rank 13

Disadvantages: not frame invariant

$\boxed{6.8\text{-}9}$



$x = \xi$
$y = \eta$
$N = \frac{1}{4}(1 \pm x)(1 \pm y)$

For the element shown,

$$\{\underset{\sim}{d}\} = \lfloor 0, 0, -c, 0, c, -c, 0, c \rfloor^T$$

$$u = \sum N_i u_i = c\,(N_3 - N_2)$$

$$u = \frac{c}{4}(1+x)[(1+y)-(1-y)] = \frac{c}{4}(1+x)(2y)$$

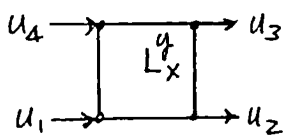
$$v = \sum N_i v_i = c\,(N_4 - N_3)$$

$$v = \frac{c}{4}(1+y)[(1-x)-(1+x)] = \frac{c}{4}(1+y)(-2x)$$

$$\left.\begin{array}{l} \epsilon_x = cy/2\,, \quad \epsilon_y = -cx/2 \\ \gamma_{xy} = \frac{c}{2}[(1+x)-(1+y)] = \frac{c}{2}(x-y) \end{array}\right\} \begin{array}{l} \text{All zero} \\ \text{@ center} \\ (x=y=0). \end{array}$$

(b) Consider u first. $u =$ (mode 7) $+$ rigid-body rotation; i.e. at nodes

$$\begin{Bmatrix} 0 \\ -c \\ +c \\ 0 \end{Bmatrix} = \begin{Bmatrix} +b \\ -b \\ +b \\ -b \end{Bmatrix} + \theta \begin{Bmatrix} +1 \\ +1 \\ -1 \\ -1 \end{Bmatrix}$$



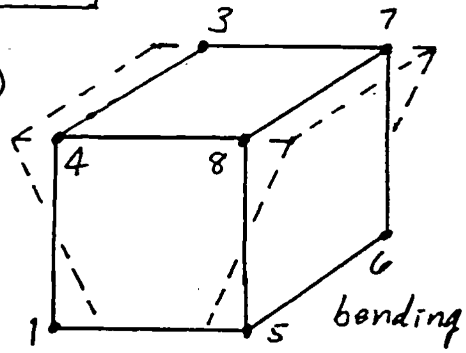
Works if $b = \frac{c}{2}$ and $\theta = -\frac{c}{2}$. Now check that these b & θ values also work for v_i.

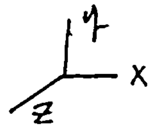
$$\begin{Bmatrix} 0 \\ 0 \\ -c \\ +c \end{Bmatrix} = \underbrace{\begin{Bmatrix} -c/2 \\ +c/2 \\ -c/2 \\ +c/2 \end{Bmatrix}}_{\text{mode 8}} \underbrace{- \frac{c}{2} \begin{Bmatrix} -1 \\ +1 \\ +1 \\ -1 \end{Bmatrix}}_{\text{rotation}}$$

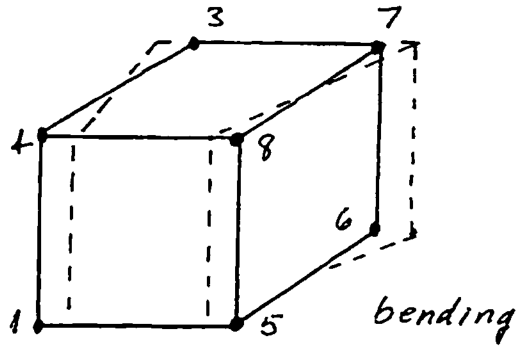
Checks.

6.8-10

(a)

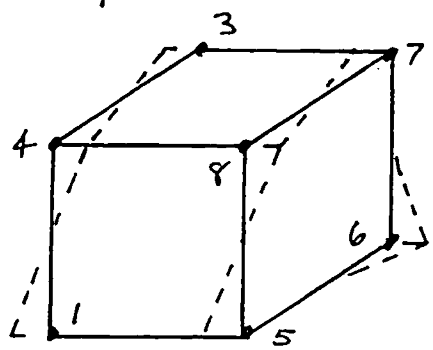


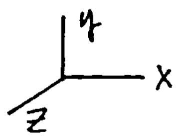


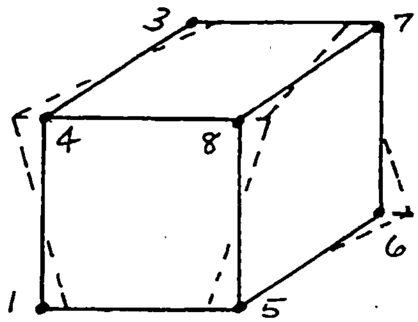


$\{\underset{\sim}{d}_x\}_1 = \lfloor 1 \ 1 \ -1 \ -1 \ -1 \ -1 \ 1 \ 1 \rfloor^T$

$\{\underset{\sim}{d}_x\}_2 = \lfloor 1 \ -1 \ -1 \ 1 \ -1 \ 1 \ 1 \ -1 \rfloor^T$



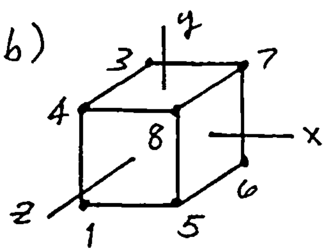




$\{\underset{\sim}{d}_x\}_3 = \lfloor -1 \ 1 \ -1 \ 1 \ -1 \ 1 \ -1 \ 1 \rfloor^T$

$\{\underset{\sim}{d}_x\}_4 = \lfloor 1 \ -1 \ 1 \ -1 \ -1 \ 1 \ -1 \ 1 \rfloor^T$

(b)



Translation: $\{\underset{\sim}{d}_x\}_5 = \lfloor 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \ 1 \rfloor^T$

Const. ϵ_x: $\{\underset{\sim}{d}_x\}_6 = \lfloor 0 \ 0 \ 0 \ 0 \ 1 \ 1 \ 1 \ 1 \rfloor^T$

Const. γ_{xy}: $\{\underset{\sim}{d}_x\}_7 = \lfloor -1 \ -1 \ 1 \ 1 \ -1 \ -1 \ 1 \ 1 \rfloor^T$

Const. γ_{zx}: $\{\underset{\sim}{d}_x\}_8 = \lfloor 1 \ -1 \ -1 \ 1 \ 1 \ -1 \ -1 \ 1 \rfloor^T$

(c) Want to show that $\{\underset{\sim}{d}_x\}_i^T \{\underset{\sim}{d}_x\}_j = 0$ for $i = 1,2,3,4$ & $j = 5,6,7,8$.
Straightforward calculation shows that this is so.

$\boxed{6.8-11}$

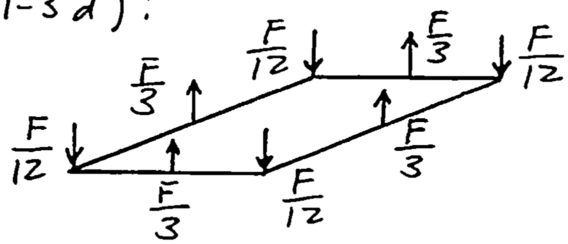
Weight factors: all must be equal, and to obtain volume = 8 for a 2 by 2 by 2 cube, $6W_i = 8$, so $W_i = 4/3$ $(i = 1, 2, \ldots, 6)$.

Spurious modes: any deformation state for which in-plane strains are zero at the middle of each face. Mode 4 of Problem 6.8-10 is such a state (for example, $\epsilon_x = \epsilon_y = \gamma_{xy} = 0$ at the middle of face 1-5-4-8). There are two more such states, involving y- and z-direction nodal displacements respectively. Thus we expect three spurious modes, and a 24 by 24 matrix $[\underset{\sim}{k}]$ of rank 15. There are 6 rigid body modes, 6 constant strain modes, and three bending modes that store strain energy.

Yes, the spurious modes are communicable.

6.9-1

With F the total force applied, uniform pressure on an 8-node rectangular surface gives the following nodal loads (from Fig. 3.11-3 d):



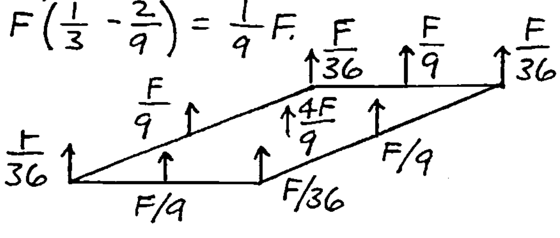
For a nine-node rectangular surface,

$$\int N_9\, dA = \int_{-1}^{1}\int_{-1}^{1}(1-\xi^2)(1-\eta^2)\, d\xi\, d\eta$$

$$\int_{-1}^{1}(1-\eta^2)\, d\eta = 2\left(\eta - \frac{\eta^3}{3}\right)_0^1 = \frac{4}{3}, \quad \int N_9\, dA = \frac{16}{9}$$

If pressure p_0 uniform on 2×2 element, $F_9 = \frac{16}{9} p_0$. Let F = total force = $(2)(2)p_0$. Thus $F_9 = \frac{4}{9}F$. In Table 6.4-1, contribution of N_9 to N_1 thru N_4 is $-\frac{1}{2}\left(-\frac{1}{2}N_9 - \frac{1}{2}N_9\right) - \frac{1}{4}N_9 = \frac{1}{4}N_9$. So, to corner loads in above Fig., add $\frac{1}{4}F_9 = \frac{1}{9}F$, for net result $F\left(-\frac{1}{12} + \frac{1}{9}\right) = \frac{1}{36}F$

In Table 6.4-1, contribution of N_9 to N_5 [illegible] $-\frac{1}{2}N_9$. So, to midside loads in above Fig., add $-\frac{1}{2}F_9 = -\frac{2}{9}F$, for net result $F\left(\frac{1}{3} - \frac{2}{9}\right) = \frac{1}{9}F$.



$\boxed{6.9\text{-}2}$

(a) $$\{\underset{\sim}{r}_e\} = \int_0^L \lfloor \underset{\sim}{N} \rfloor^T q\, dx = q \int_{-1}^{1} \begin{Bmatrix} \frac{1}{2}(-\xi+\xi^2) \\ 1-\xi^2 \\ \frac{1}{2}(\xi+\xi^2) \end{Bmatrix} \frac{L}{2} d\xi = \frac{qL}{2} \begin{Bmatrix} \frac{1}{2}\left(-\frac{\xi^2}{2}+\frac{\xi^3}{3}\right) \\ \xi - \frac{\xi^3}{3} \\ \frac{1}{2}\left(\frac{\xi^2}{2}+\frac{\xi^3}{3}\right) \end{Bmatrix}_{-1}^{1}$$

$$\{\underset{\sim}{r}_e\} = \frac{qL}{2} \begin{Bmatrix} \frac{1}{2}\frac{2}{3} \\ \frac{4}{3} \\ \frac{1}{2}\frac{2}{3} \end{Bmatrix} = \frac{qL}{6} \begin{Bmatrix} 1 \\ 4 \\ 1 \end{Bmatrix}$$

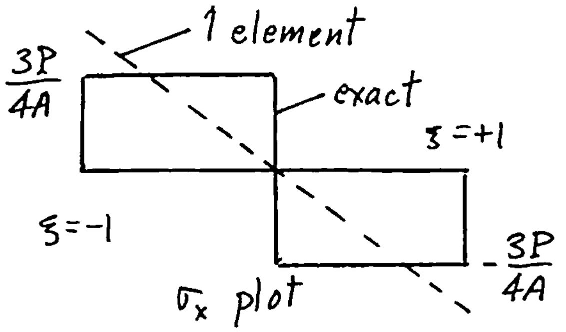
(b) $$r_a = \int_0^L N_a\, q\, dx = q \int_{-1}^{1} (1+\cos \pi\xi) \frac{L}{2} d\xi = \frac{qL}{2}\left(\xi + \frac{\sin \pi\xi}{\pi}\right)_{-1}^{1} = qL$$

6.10-1

(a) $\frac{16AE}{3L}u_2 = P$, $u_2 = \frac{3PL}{16AE}$ $\quad \left(\text{exact}: u_2 = \frac{(P/2)(L/2)}{AE} = \frac{PL}{4E}\right)$

$$\epsilon_x = \frac{dN_2}{dx}u_2 = \frac{1}{J}\frac{dN_2}{d\xi}u_2 = \frac{1}{L/2}(-2\xi)u_2 = -\frac{4\xi}{L}\frac{3PL}{16AE} = -\frac{3P}{4AE}\xi$$

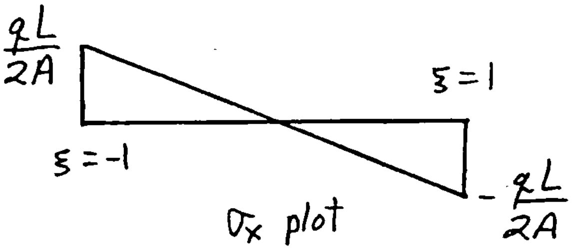
$$\sigma_x = E\epsilon_x = -\frac{3P}{4A}\xi \quad \left(\text{exact}: \sigma_x = \frac{P}{2A} \text{ for } -1<\xi<0\right.$$
$$\left.\sigma_x = -\frac{P}{2A} \text{ for } 0<\xi<1\right)$$



(b) From Problem 6.9-2a, load at node 2 is $\frac{2qL}{3}$

Part (a) repeats with P replaced by $\frac{2qL}{3}$. Thus

$\sigma_x = -\frac{qL}{2A}\xi$ (this is exact)



$\boxed{6.10\text{-}2}$

(a) $B_a = \frac{d^2}{dx^2} = \left[\frac{1}{(L/2)}\right]^2 \frac{d^2}{d\xi^2}$ $\qquad v = a_1(1+\cos\pi\xi)$

$$k_a = EI\int_0^L B_a^{\,2}\,dx = EI\int_{-1}^{1}\left(\frac{-\pi^2\cos\pi\xi}{(L/2)^2}\right)^2 \frac{L}{2}\,d\xi$$

$$k_a = \frac{8EI\pi^4}{L^3}\left(\frac{\xi}{2} + \frac{1}{4\pi}\sin 2\pi\xi\right)\Big|_{-1}^{1} = \frac{8EI\pi^4}{L^3}$$

(b) As in Problem 6.9-2b,

$$r_a = \int_0^L N_a q\,dx = q\int_{-1}^{1}(1+\cos\pi\xi)\frac{L}{2}\,d\xi = \frac{qL}{2}\left(\xi + \frac{\sin\pi\xi}{\pi}\right)\Big|_{-1}^{1} = qL$$

$$k_a a_1 = r_a, \quad a_1 = \frac{r_a}{k_a} = \frac{qL^4}{8\pi^4 EI}$$

Exact center deflection: $\frac{qL^4}{384EI}$

Approx. center deflection: $v_0 = 2a_1 = \frac{qL^4}{4\pi^4 EI}$ (1.45 % low)

(c) $M = EI\frac{d^2v}{dx^2} = EI\left(\frac{-\pi^2\cos\pi\xi}{(L/2)^2}\right)a_1 = -\left(\frac{4\pi^2 EI}{L^2}\cos\pi\xi\right)a_1$

$$|M_{ends}| = |M_{center}| = \frac{4\pi^2 EI}{L^2}a_1 = \frac{4\pi^2 EI}{L^2}\frac{qL^4}{8\pi^4 EI} = \frac{qL^2}{2\pi^2}$$

Exact: $M_{ends} = \frac{qL^2}{12}$, $M_{center} = -\frac{qL^2}{24}$ (for upward load)

Approx. M_{end} is 39.2% low

Approx. $|M_{center}|$ is 21.6% high

$\boxed{6.10\text{-}3}$

(a) As in Problem 6.10-2a, $k_a = \dfrac{8EI\pi^4}{L^3}$

(b) $v = N_a a_1$ where $N_a = 1 + \cos\pi\xi$. At center, $N_a = 2$

$$r_a = (N_a)_{center} P = 2P$$

$$k_a a_1 = r_a, \quad a_1 = \frac{r_a}{k_a} = \frac{PL^3}{4\pi^4 EI}$$

Approx. center deflection: $v_0 = 2a_1 = \dfrac{PL^3}{2\pi^4 EI} = 0.005133\dfrac{PL^3}{EI}$

Exact center deflection: $\dfrac{PL^3}{192EI} = 0.005208\dfrac{PL^3}{EI}$

1.45% low

(c) As in Problem 6.10-2c,

$$|M_{ends}| = |M_{center}| = \frac{4\pi^2 EI}{L^2} a_1 = \frac{PL}{\pi^2}$$

Exact magnitudes are $\dfrac{PL}{8}$

18.9 % low

$\boxed{6.10\text{-}4}$

$$\gamma_{xy} = u_{,y} + v_{,x} = u_{,y} \qquad (\text{here } v = 0)$$

For an el. 2a units on a side, $\xi = x/a$ and $\eta = y/a$, and $N = \frac{1}{4a^2}(a \pm x)(a \pm y)$

$$u_{,y} = \frac{1}{4a^2}\Big[-(a-x)(-\bar{u}) - (a+x)(\bar{u})$$
$$+ (a+x)(-\bar{u}) + (a-x)(\bar{u})\Big]$$

where u = magnitude of corner d.o.f.

$$u_{,y} = \frac{1}{4a^2}(-4\bar{u}x) = -\frac{\bar{u}x}{a^2} = -\frac{\bar{u}}{a}\xi$$

$\boxed{6.10\text{-}5}$

(a,b) $N_1 = \frac{1}{4}(1-r)(1-s) \qquad N_2 = \frac{1}{4}(1+r)(1-s)$

$N_3 = \frac{1}{4}(1+r)(1+s) \qquad N_4 = \frac{1}{4}(1-r)(1+s)$

	N_1	N_2	N_3	N_4
$r=-\sqrt{3},\ s=-\sqrt{3}$	1.866	-0.5	0.134	-0.5
$r=+\sqrt{3},\ s=-\sqrt{3}$	-0.5	1.866	-0.5	0.134
$r=+\sqrt{3},\ s=+\sqrt{3}$	0.134	-0.5	1.866	-0.5
$r=-\sqrt{3},\ s=+\sqrt{3}$	-0.5	0.134	-0.5	1.866

$$\begin{Bmatrix} \sigma_A \\ \sigma_B \\ \sigma_C \\ \sigma_D \end{Bmatrix} = \Big[\uparrow \Big]_{4\times 4} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \end{Bmatrix}$$

This is result of evaluating $\sigma_P = \sum N_i \sigma_i$ at corners $P=A$, $P=B$, $P=C$, and $P=D$.

(c) Evaluate $\sigma_P = \sum N_i \sigma_i$ at midsides $P=E$, $P=F$, $P=G$, $P=H$:

	N_1	N_2	N_3	N_4
$r=0,\ s=-\sqrt{3}$	a	a	b	b
$r=+\sqrt{3},\ s=0$	b	a	a	b
$r=0,\ s=+\sqrt{3}$	b	b	a	a
$r=-\sqrt{3},\ s=0$	a	b	b	a

$a = 0.683$

$b = -0.183$

$$\begin{Bmatrix} \sigma_E \\ \sigma_F \\ \sigma_G \\ \sigma_H \end{Bmatrix} = \Big[\uparrow \Big]_{4\times 4} \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \end{Bmatrix}$$

$\boxed{6.10-6}$

For extrapolation, Eq. 6.5-5 becomes $N = \frac{1}{8}(1 \pm r)(1 \pm s)(1 \pm t)$.

At a corner, with $r, s, t = \pm\sqrt{3}$,

$\frac{1}{8}(1+\sqrt{3})^3 = 2.549$, $\frac{1}{8}(1+\sqrt{3})^2(1-\sqrt{3}) = -0.683$

$\frac{1}{8}(1-\sqrt{3})^3 = -0.049$, $\frac{1}{8}(1+\sqrt{3})(1-\sqrt{3})^2 = 0.183$

At node 8 in Fig. 6.5-1a, with Gauss points given the number of the nearest corner node,

$$\sigma_{node\ 8} = 2.549\sigma_8 - 0.683(\sigma_4 + \sigma_5 + \sigma_7) + 0.183(\sigma_1 + \sigma_3 + \sigma_6) - 0.049\sigma_2$$

Check: $\sigma_{node\ 8} = \bar{\sigma}$ if $\sigma_i = \bar{\sigma}$ for all i.

(b) $N_i = \frac{1}{8}(1-\sqrt{3})(1)^2 = -0.092 \qquad i = 1,2,3,4$

$N_i = \frac{1}{8}(1+\sqrt{3})(1)^2 = 0.341 \qquad i = 5,6,7,8$

$$\sigma_{point} = -0.092(\sigma_1 + \sigma_2 + \sigma_3 + \sigma_4) + 0.341(\sigma_5 + \sigma_6 + \sigma_7 + \sigma_8)$$

Check: $\sigma_{point} = \bar{\sigma}$ if $\sigma_i = \bar{\sigma}$ for all i.

$\boxed{6.10\text{-}7}$

Because J isn't constant in a general element. Let's take an example; the 3-node bar of Fig. 6.1-1, with $u_1 = u_3 = 0$, $u_2 \neq 0$. Compute ϵ_x at node 1. By Eq. 6.1-7,

$$\epsilon_{x1} = \frac{1}{J}\lfloor N_{,\xi} \rfloor \begin{Bmatrix} 0 \\ 0 \\ u_3 \end{Bmatrix} = \frac{1}{J}\lfloor -\tfrac{3}{2} \quad 2 \quad -\tfrac{1}{2} \rfloor \begin{Bmatrix} 0 \\ 0 \\ u_3 \end{Bmatrix} = \frac{-u_3}{2J}$$

At Gauss pts. of 2-pt. rule,

$$\left.\begin{aligned} \epsilon_{x*1} &= \frac{1}{J_1}\,\frac{1 - 2\sqrt{3}/3}{2}\,u_3 \\ \epsilon_{x*2} &= \frac{1}{J_2}\,\frac{1 + 2\sqrt{3}/3}{2}\,u_3 \end{aligned}\right\} \text{for } u_1 = u_2 = 0$$

Extrapolate to node 1:

$$\epsilon_{x1} = \frac{1+\sqrt{3}}{2}\epsilon_{x*1} + \frac{1-\sqrt{3}}{2}\epsilon_{x*2}$$

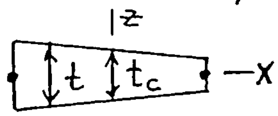
$$\epsilon_{x1} = \left(\frac{-0.105}{J_1} - \frac{0.395}{J_2}\right) u_2$$

Only for $J_1 = J_2 = J$ do the two ϵ_{x1} expressions yield the same result, viz.

$$\epsilon_{x1} = -0.5\frac{u_3}{J} = -0.5\frac{u_3}{L/2} = -\frac{u_3}{L}$$

$\boxed{6.10-8}$

For a bilinear element, t plays no role in Eq. 6.10-1; for the element



shown, ϵ_x is independent of x. An ad hoc adjustment that may yield better accuracy is $\{\underset{\sim}{\epsilon}\} = \frac{t_c}{t}[\underset{\sim}{B}]\{\underset{\sim}{d}\}$, where t_c is thickness at the center (or apply t_c/t to adjust $\{\underset{\sim}{\sigma}\}$ from Eq. 6.10-1).

For a quadratic element, this adjustment should matter less, as side node

can displace relative to corners, thus automatically providing a strain variation.

$\boxed{6.10\text{-}9}$

(a) $J = \left\lfloor \frac{1}{2}(-1+2\xi) \quad -2\xi \quad \frac{1}{2}(1+2\xi) \right\rfloor \begin{Bmatrix} 0 \\ 0.6L \\ L \end{Bmatrix} = -1.2L\xi + \frac{L}{2} + L\xi$

$J = \frac{L}{2}(1-0.4\xi)$

$\epsilon_x = \frac{1}{J}\left\lfloor - \quad - \quad \frac{1}{2}(1+2\xi) \right\rfloor \begin{Bmatrix} 0 \\ 0 \\ L/1000 \end{Bmatrix} = 0.001\,\frac{1+2\xi}{1-0.4\xi}$

Node 1, $\xi = -1$: $\epsilon_x = -0.000714$

Node 2, $\xi = 0$: $\epsilon_x = 0.00100$

Node 3, $\xi = 1$: $\epsilon_x = 0.00500$

(b) 1st Gauss pt., $\xi = -1/\sqrt{3}$: $\epsilon_x = -0.0001257$

2nd Gauss pt., $\xi = 1/\sqrt{3}$: $\epsilon_x = 0.002802$

(c) Extrapolation from Gauss points: let $r = \sqrt{3}\,\xi$

$\epsilon_x = \frac{1}{2}\left\lfloor 1-r \quad 1+r \right\rfloor \begin{Bmatrix} -0.0001257 \\ 0.002802 \end{Bmatrix}$

Node 1, $r = -\sqrt{3}$: $\epsilon_x = -0.00120$

Node 2, $r = 0$: $\epsilon_x = 0.00134$

Node 3, $r = \sqrt{3}$: $\epsilon_x = 0.00387$

(d)

$$\epsilon_x = \frac{0.00500 + 0.00387}{3 - \frac{0.00500}{0.00387}} = 0.00519$$

$\boxed{6.10-10}$

(a) $[\underset{\sim}{k}]\{\underset{\sim}{d}\} = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{Bmatrix} 0 \\ u_2 \end{Bmatrix} = \frac{AE}{L}\begin{Bmatrix} -1 \\ 1 \end{Bmatrix} u_3$

Eq. 6.10-7: $\sigma = \beta$

Eq. 6.10-9: $[\underset{\sim}{Q}] = \int_0^L \begin{Bmatrix} -1/L \\ 1/L \end{Bmatrix} (1) A\,dx = A\begin{Bmatrix} -1 \\ 1 \end{Bmatrix}$

$$[\underset{\sim}{Q}]^T[\underset{\sim}{Q}] = 2A^2$$

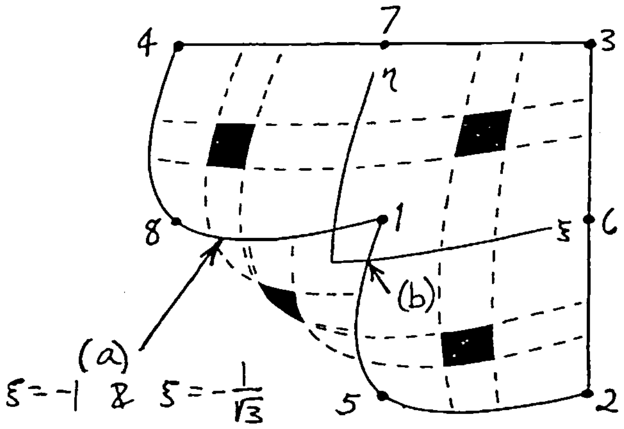
Eq. 6.10-10: $(2A^2)\beta = A\lfloor -1 \quad 1 \rfloor \frac{AE}{L}\begin{Bmatrix} -1 \\ 1 \end{Bmatrix} u_2$

$$2A^2\beta = \frac{A^2E}{L} 2u_2$$

$$\beta = \frac{Eu_2}{L} \qquad \text{so} \quad \sigma = \beta = E\frac{u_2}{L}$$

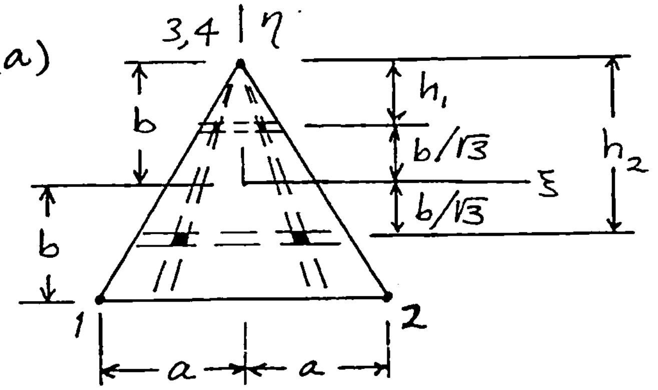
(b) In our development, we require equilibrium, which for the bar requires $\frac{d\sigma_x}{dx} = 0$. Not satisfied by $\sigma_x = \beta_1 + \beta_2 x$.

$\boxed{6.11\text{-}1}$



$\boxed{6.11\text{-}2}$

(a)



(b) Eq. 6.2-6:

$$[\underset{\sim}{J}] = \frac{1}{4}\begin{bmatrix} -(1-\eta) & (1-\eta) & (1+\eta) & -(1+\eta) \\ -(1-\xi) & -(1+\xi) & (1+\xi) & (1-\xi) \end{bmatrix}\begin{bmatrix} -a & -b \\ a & -b \\ 0 & b \\ 0 & b \end{bmatrix}$$

$$[\underset{\sim}{J}] = \frac{1}{4}\begin{bmatrix} 2(1-\eta)a & 0 \\ -2\xi a & 4b \end{bmatrix}, \qquad J = \det[\underset{\sim}{J}] = 2ab(1-\eta)$$

Smallest J in element is $J=0$ at nodes 3,4 (as is obvious by inspection).

J at Gauss points:

At $\eta = -\frac{1}{\sqrt{3}}$, $J = 3.1547ab$ (call it J_{max})

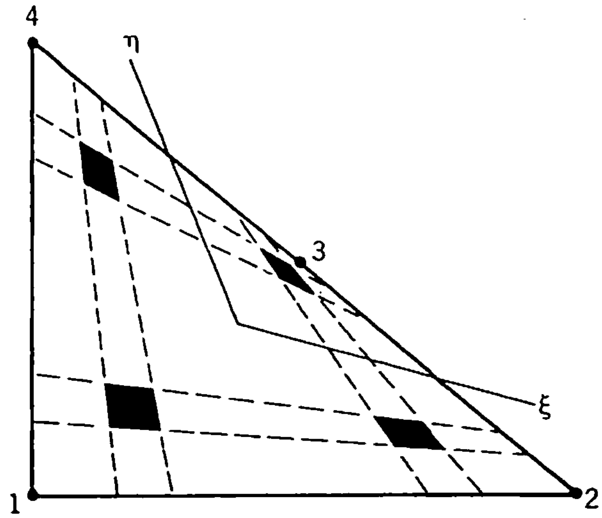
At $\eta = \frac{1}{\sqrt{3}}$, $J = 0.8453ab$ (call it J_{min})

$$\therefore \quad \frac{J_{max}}{J_{min}} = \frac{3.1547}{0.8453} = 3.732$$

Or, more directly, $\dfrac{J_{max}}{J_{min}} = \dfrac{h_2}{h_1} = \dfrac{b + b/\sqrt{3}}{b - b/\sqrt{3}} = \dfrac{\sqrt{3}+1}{\sqrt{3}-1} = 3.732$

6.11-3

(a)



(b)

$$[\underset{\sim}{J}] = \frac{1}{4}\begin{bmatrix} -(1-\eta) & (1-\eta) & (1+\eta) & -(1+\eta) \\ -(1-\xi) & -(1+\xi) & (1+\xi) & (1-\xi) \end{bmatrix}\begin{bmatrix} -1 & -1 \\ 1 & -1 \\ 0 & 0 \\ -1 & 1 \end{bmatrix}$$

$$[\underset{\sim}{J}] = \frac{1}{4}\begin{bmatrix} 3-\eta & -1-\eta \\ -1-\xi & 3-\xi \end{bmatrix}, \qquad J = \det[\underset{\sim}{J}] = 2-\xi-\eta$$

Node 1: $\xi=-1,\ \eta=-1$: $J=4$

Node 2: $\xi=1,\ \eta=-1$: $J=2$

Node 3: $\xi=1,\ \eta=1$: $J=0$

Node 4: $\xi=-1,\ \eta=1$: $J=2$

$\boxed{6.12-1}$

(a) 4-node element:

$$\Sigma N_i = \frac{1}{4}(1+\xi)\left[(1-\eta)+(1+\eta)\right]+\frac{1}{4}(1-\xi)\left[(1-\eta)+(1+\eta)\right]$$

$$= \frac{1}{2}(1+\xi)+\frac{1}{2}(1-\xi) = 1$$

8-node element:

From the above we know that bilinear portions of N_1 through N_4 in the N_i of Eqs. 6.4-1 sum to unity. Also, by inspection, the higher-order terms cancel in the sum of all eight N_i

(b) $\Sigma N_{i,\xi} = \frac{1}{4}\left[-(1-\eta)+(1-\eta)+(1+\eta)-(1+\eta)\right] = 0$

$\Sigma N_{i,\eta} = \frac{1}{4}\left[-(1-\xi)-(1+\xi)+(1+\xi)+(1-\xi)\right] = 0$

$\boxed{6.12\text{-}2}$

$$J = \left\lfloor \tfrac{1}{2}(-1+2\xi) \quad -2\xi \quad \tfrac{1}{2}(1+2\xi) \right\rfloor \begin{Bmatrix} 0 \\ x_2 \\ L \end{Bmatrix} = \frac{L}{2}(1+2\xi) - 2\xi x_2$$

For linear $u = u(x)$,

$$u = \left\lfloor \frac{1-\xi}{2} \quad \frac{1+\xi}{2} \right\rfloor \begin{Bmatrix} u_1 \\ u_3 \end{Bmatrix}, \quad \frac{du}{d\xi} = \left\lfloor -\frac{1}{2} \quad \frac{1}{2} \right\rfloor \begin{Bmatrix} u_1 \\ u_3 \end{Bmatrix} = \frac{u_3 - u_1}{2}$$

$$\epsilon_x = \frac{du}{d\xi}\frac{d\xi}{dx} = \frac{du/d\xi}{dx/d\xi} = \frac{1}{J}\frac{du}{d\xi} = \frac{u_3 - u_1}{L(1+2\xi) - 4\xi x_2}$$

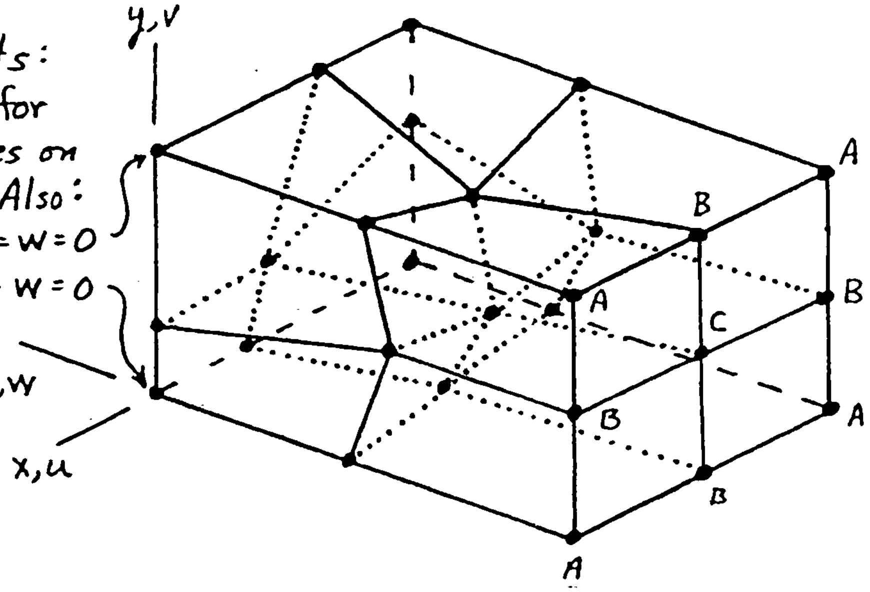
We obtain the expected $\epsilon_x = \frac{u_3 - u_1}{L}$ only if $x_2 = \frac{L}{2}$ (the midpoint)

$\boxed{6.13\text{-}1}$

Zero. In "standard" patch test, only boundary nodes loaded, & $\{\underset{\sim}{D}\} = [\underset{\sim}{K}]^{-1}\{\underset{\sim}{R}\}$ gives $\{\underset{\sim}{D}\}$ consistent with a const. strain state. Therefore, using this $\{\underset{\sim}{D}\}$, $[\underset{\sim}{K}]\{\underset{\sim}{D}\}$ gives an $\{\underset{\sim}{R}\}$ with loads on boundary nodes only.

6.13-2

Restraints:
w = 0 for
all nodes on
z = 0. Also:
u = w = 0
u = v = w = 0



Eight arbitrarily shaped
hexahedra fill a rectangular box, with one
completely internal node. Rectangular faces
on right end suggest z-direction loads of
1 @ A, 2 @ B, 4 @ C.