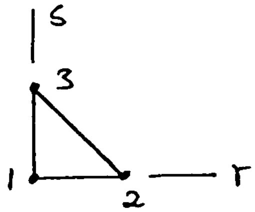
$\boxed{7.1\text{-}1}$



$$\phi = a_1 + a_2 r + a_3 s$$

$$\begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{Bmatrix} = [\underset{\sim}{A}]\{\underset{\sim}{a}\} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix}$$

Solve for the a_i:

$a_1 = \phi_1$
$a_2 = \phi_2 - \phi_1$
$a_3 = \phi_3 - \phi_1$

hence $$\begin{Bmatrix} a_1 \\ a_2 \\ a_3 \end{Bmatrix} = [\underset{\sim}{A}]^{-1}\{\underset{\sim}{\phi}_e\} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{Bmatrix}$$

$$\phi = \lfloor 1 \quad r \quad s \rfloor [\underset{\sim}{A}]^{-1}\{\underset{\sim}{\phi}_e\} = \lfloor 1-r-s \quad r \quad s \rfloor \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{Bmatrix}$$

$\boxed{7.1\text{-}2}$

$$x = \begin{bmatrix} N_1 & N_2 & \cdots & N_6 \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \\ (x_1+x_2)/2 \\ (x_2+x_3)/2 \\ (x_3+x_1)/2 \end{Bmatrix}$$

$$x = \begin{bmatrix} \left(N_1 + \frac{N_4}{2} + \frac{N_6}{2}\right) & \left(N_2 + \frac{N_4}{2} + \frac{N_5}{2}\right) & \left(N_3 + \frac{N_5}{2} + \frac{N_6}{2}\right) \end{bmatrix} \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}$$

$$N_1 + \frac{N_4}{2} + \frac{N_6}{2} = (1-r-s)(1-2r-2s) + 2r(1-r-s) + 2s(1-r-s)$$
$$= (1-r-s)(1-2r-2s+2r+2s) = 1-r-s \quad \checkmark$$

$$N_2 + \frac{N_4}{2} + \frac{N_5}{2} = r(2r-1) + 2r(1-r-s) + 2rs$$
$$= 2r^2 - r + 2r - 2r^2 - 2rs + 2rs = r \quad \checkmark$$

$$N_3 + \frac{N_5}{2} + \frac{N_6}{2} = s(2s-1) + 2rs + 2s(1-r-s)$$
$$= 2s^2 - s + 2rs + 2s - 2rs - 2s^2 = s \quad \checkmark$$

7.1-3

Obtain ith shape function by taking products of functions which, if equated to zero, are equations of lines that do not pass through the ith node. Multiply each such product by a constant c_i such that it becomes unity at node i.

$c_1 N_1 = \left(\frac{1}{3} - r - s\right)\left(\frac{2}{3} - r - s\right)(1 - r - s)$; $\quad c_1 N_1 = \frac{2}{9}$ for $r = s = 0$, so $c_1 = \frac{2}{9}$

$$N_1 = (1 - 3r - 3s)\left(1 - \frac{3}{2}r - \frac{3}{2}s\right)(1 - r - s)$$

$c_2 N_2 = r\left(\frac{1}{3} - r\right)\left(\frac{2}{3} - r\right)$; $\quad c_2 N_2 = \frac{2}{9}$ for $r = 1$, $s = 0$, so $c_2 = \frac{2}{9}$

$$N_2 = r(1 - 3r)\left(1 - \frac{3r}{2}\right)$$

$c_4 N_4 = r(1 - r - s)\left(\frac{2}{3} - r - s\right)$; $\quad c_4 N_4 = \frac{2}{27}$ for $r = \frac{1}{3}$, $s = 0$, so $c_4 = \frac{2}{27}$

$$N_4 = 9r(1 - r - s)\left(1 - \frac{3r}{2} - \frac{3s}{2}\right)$$

$c_5 N_5 = r(1 - r - s)\left(\frac{1}{3} - r\right)$; $\quad c_5 N_5 = -\frac{2}{27}$ for $r = \frac{2}{3}$, $s = 0$, so $c_5 = -\frac{2}{27}$

$$N_5 = 9r(1 - r - s)\left(\frac{3r}{2} - \frac{1}{2}\right)$$

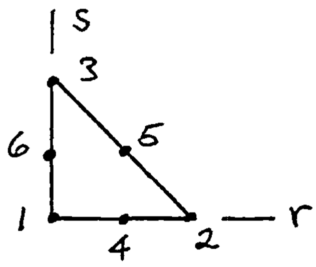
$c_6 N_6 = rs\left(\frac{1}{3} - r\right)$; $\quad c_6 N_6 = \frac{2}{3}\,\frac{1}{3}\left(\frac{1}{3} - \frac{2}{3}\right) = -\frac{2}{27}$ for $r = \frac{2}{3}$, $s = \frac{1}{3}$; $c_6 = -\frac{2}{27}$

$$N_6 = \frac{9}{2} rs(3r - 1)$$

$c_{10} N_{10} = rs(1 - r - s)$; $\quad c_{10} N_{10} = \frac{1}{27}$ for $r = s = \frac{1}{3}$, so $c_{10} = \frac{1}{27}$

$$N_{10} = 27rs(1 - r - s)$$

7.1-4



The Table:

Include only if node i is present in the element

	$i=4$	$i=5$	$i=6$
$N_1 = 1-r-s$	$-\frac{1}{2}N_4$		$-\frac{1}{2}N_6$
$N_2 = r$	$-\frac{1}{2}N_4$	$-\frac{1}{2}N_5$	
$N_3 = s$		$-\frac{1}{2}N_5$	$-\frac{1}{2}N_6$

$$\left.\begin{aligned} N_4 &= 4r(1-r-s) \\ N_5 &= 4rs \\ N_6 &= 4s(1-r-s) \end{aligned}\right\} \text{[from Eqs. 7.1-2]}$$

If nodes 4, 5, 6 are all present, then

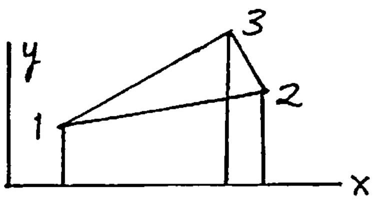
$$N_1 = 1-r-s-2r(1-r-s)-2s(1-r-s) = (1-r-s)(1-2r-2s)$$

$$N_2 = r-2r(1-r-s)-2rs = -r+2r^2 = r(2r-1)$$

$$s-2s(1-r-s) = -s+2s^2 = s(2s-1)$$

Checks N_1, N_2, N_3 in Eqs. 7.1-2.

$\boxed{7.2\text{-}1}$



A = area of triangle
= (2 "tall" trapezoids) minus
(1 "short" trapezoid)

$$A = \frac{y_1+y_3}{2}(x_3-x_1) + \frac{y_2+y_3}{2}(x_2-x_3) - \frac{y_1+y_2}{2}(x_2-x_1)$$

$$2A = x_3y_1 + x_3y_3 - x_1y_1 - x_1y_3 + x_2y_2 + x_2y_3 - x_3y_2 - x_3y_3 + x_1y_1 + x_1y_2 - x_2y_1 - x_2y_2$$

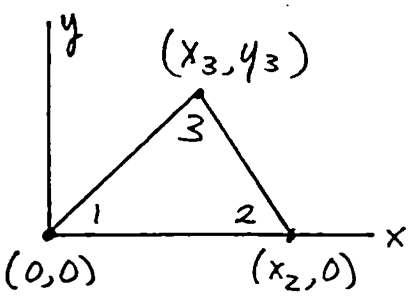
$$2A = x_2(y_3-y_1) - x_1(y_3-y_2) - x_3(y_2-y_1)$$

$$2A = x_2(y_3-y_1) - x_1(y_3-y_1) - x_1y_1 + x_1(y_2-y_1) + x_1y_1 - x_3(y_2-y_1)$$

$$2A = (x_2-x_1)(y_3-y_1) + (x_1-x_3)(y_2-y_1)$$

$2A = x_{21}y_{31} - x_{31}y_{21}$ which is det$[\underset{\sim}{J}]$ from Eq. 7.2-3

$\boxed{7.2-2}$



Eq. 7.2-4:

$$J = x_{21} y_{31} - x_{31} y_{21}$$

$$x_{21} = x_2 - x_1 = x_2$$

$$y_{31} = y_3 - y_1 = y_3$$

$$x_{31} = x_3 - x_1 = x_3$$

$$y_{21} = y_2 - y_1 = 0$$

Hence $J = 2A = x_2 y_3$

Eq. 7.2-6:

$$[\underset{\sim}{B}] = \frac{1}{2A}\begin{bmatrix} y_{23} & y_{31} & y_{12} \\ x_{32} & x_{13} & x_{21} \end{bmatrix} = \frac{1}{x_2 y_3}\begin{bmatrix} -y_3 & y_3 & 0 \\ x_3 - x_2 & -x_3 & x_2 \end{bmatrix}$$

$$[\underset{\sim}{B}] = \begin{bmatrix} -\frac{1}{x_2} & \frac{1}{x_2} & 0 \\ \frac{x_3 - x_2}{x_2 y_3} & -\frac{x_3}{x_2 y_3} & \frac{1}{y_3} \end{bmatrix}$$

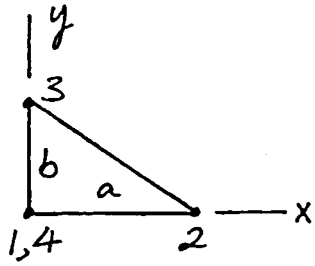
Checks Eq. 3.4-6

$\boxed{7.2\text{-}3}$

$$\phi = N_1\phi_1 + N_2\phi_2 + N_3\phi_3 + N_4\phi_1 = (N_1+N_4)\phi_1 + N_2\phi_2 + N_3\phi_3$$

(for a quadrilateral), $N_1+N_4 = \frac{1}{2}(1-\xi)$, so

$$\phi = \frac{1}{2}(1-\xi)\phi_1 + \frac{1}{4}(1+\xi)(1-\eta)\phi_2 + \frac{1}{4}(1+\xi)(1+\eta)\phi_3$$



Derivatives of shape functions: define

$$[\underset{\sim}{D}_N] = \begin{bmatrix} \phi_{1,\xi} & \phi_{2,\xi} & \phi_{3,\xi} \\ \phi_{1,\eta} & \phi_{2,\eta} & \phi_{3,\eta} \end{bmatrix} = \begin{bmatrix} -1/2 & (1-\eta)/4 & (1+\eta)/4 \\ 0 & -(1+\xi)/4 & (1+\xi)/4 \end{bmatrix}$$

$$[\underset{\sim}{J}] = [\underset{\sim}{D}_N]\begin{bmatrix} 0 & 0 \\ a & 0 \\ 0 & b \end{bmatrix} = \frac{1}{4}\begin{bmatrix} (1-\eta)a & (1+\eta)b \\ -(1+\xi)a & (1+\xi)b \end{bmatrix}$$

$$[\underset{\sim}{\Gamma}] = [\underset{\sim}{J}]^{-1} = \frac{2}{(1+\xi)ab}\begin{bmatrix} (1+\xi)b & -(1+\eta)b \\ (1+\xi)a & (1-\eta)a \end{bmatrix}$$

$$\begin{Bmatrix} \phi_{,x} \\ \phi_{,y} \end{Bmatrix} = [\underset{\sim}{\Gamma}][\underset{\sim}{D}_N]\begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{Bmatrix} = [\underset{\sim}{B}]\begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{Bmatrix} \quad \text{where}$$

$$[\underset{\sim}{B}] = \frac{1}{(1+\xi)ab}\begin{bmatrix} -(1+\xi)b & \frac{b}{2}(1+\xi)(1-\eta)+\frac{b}{2}(1+\xi)(1+\eta) & \frac{b}{2}(1+\xi)(1+\eta)-\frac{b}{2}(1+\xi)(1+\eta) \\ -(1+\xi)a & \frac{a}{2}(1+\xi)(1-\eta)-\frac{a}{2}(1+\xi)(1-\eta) & \frac{a}{2}(1+\xi)(1+\eta)+\frac{a}{2}(1+\xi)(1-\eta) \end{bmatrix}$$

$$[\underset{\sim}{B}] = \frac{1}{(1+\xi)ab}\begin{bmatrix} -(1+\xi)b & (1+\xi)b & 0 \\ -(1+\xi)a & 0 & (1+\xi)a \end{bmatrix} = \begin{bmatrix} -1/a & 1/a & 0 \\ -1/b & 0 & 1/b \end{bmatrix}$$

Eq. 7.2-6:

$$[\underset{\sim}{B}] = \frac{1}{ab}\begin{bmatrix} -b & b & 0 \\ -a & 0 & a \end{bmatrix} = \begin{bmatrix} -1/a & 1/a & 0 \\ -1/b & 0 & 1/b \end{bmatrix}$$

$\boxed{7.3\text{-}1}$

$$\sum_1^6 N_i = 2(\xi_1^2+\xi_2^2+\xi_3^2) - (\xi_1+\xi_2+\xi_3) + 4(\xi_1\xi_2+\xi_2\xi_3+\xi_3\xi_1)$$

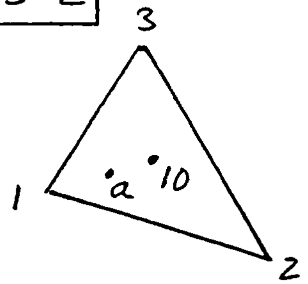
Call the above Eq. (a). Now

$$(\xi_1+\xi_2+\xi_3)^2 = (\xi_1^2+\xi_2^2+\xi_3^2) + 2(\xi_1\xi_2+\xi_2\xi_3+\xi_3\xi_1)$$

If we multiply this equation by 2 and subtract $(\xi_1+\xi_2+\xi_3)$, we get the right hand side of Eq. (a). But since $\xi_1+\xi_2+\xi_3=1$, this means that

$$\sum_1^6 N_i = 2(1)-1 \quad \text{so} \quad \sum_1^6 N_i = 1$$

$\boxed{7.3\text{-}2}$



Point a has coordinates $\xi_1 = \frac{2}{3}$, $\xi_2 = \xi_3 = \frac{1}{6}$

$$\phi_a = 27\,\frac{2}{3}\,\frac{1}{6}\,\frac{1}{6}\,\phi_{10}, \quad \text{so} \quad \phi_{10} = 2\phi_a$$

$$\text{and} \quad \phi = 54\,\xi_1 \xi_2 \xi_3 \phi_a$$

$$V = \int_A 54\,\xi_1 \xi_2 \xi_3\, dA = 54\,(2A)\frac{\phi_a}{(2+1+1+1)!} = \frac{108}{5!} A\phi_a = 0.9 A\phi_a$$

$\boxed{7.3\text{-}3}$

.9 $\quad x = x_1\xi_1 + x_2\xi_2 + x_3\xi_3$

$$\int x^2 dA = \int \left(x_1^2\xi_1^2 + x_2^2\xi_2^2 + x_3^2\xi_3^2 + 2x_1x_2\xi_1\xi_2 + 2x_2x_3\xi_2\xi_3 + 2x_3x_1\xi_3\xi_1 \right) dA$$

Eq. 5.2-8 yields

$\int \xi_i^2 dA = \frac{A}{6}$ & $\int \xi_i \xi_j dA = \frac{A}{12}$. Hence

$$\int x^2 dA = \frac{A}{6}\left(x_1^2 + x_2^2 + x_3^2 + x_1x_2 + x_2x_3 + x_3x_1\right) \quad (a)$$

Add $x_1x_2 - x_1x_2$ to right side & factor

$$\int x^2 dA = \frac{A}{6}\left[x_1(x_1+x_2+x_3) + x_2(x_1+x_2+x_3) - x_1x_2 + x_3^2\right]$$

But $x_1+x_2+x_3 = 0$ because of centroidal coordinates, so

$$\int x^2 dA = \frac{A}{6}\left(x_3^2 - x_1x_2\right) = \frac{A}{6}\left[x_3(-x_1-x_2) - x_1x_2\right] = \frac{A}{6}\left(-x_1x_2 - x_2x_3 - x_3x_1\right) \quad (b)$$

From (a) and (b),

$$\int x^2 dA = \frac{A}{6}\left(x_1^2 + x_2^2 + x_3^2\right) - \int x^2 dA \quad \text{hence} \quad \int x^2 dA = \frac{A}{12}\left(x_1^2 + x_2^2 + x_3^2\right)$$

$\boxed{7.3\text{-}4}$

Evaluate the given ϕ equation at nodes.

$$\begin{Bmatrix}\phi_1\\ \phi_2\\ \phi_3\\ \phi_4\\ \phi_5\\ \phi_6\end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 1 & 0 & 0 & 0\\ 1/4 & 1/4 & 0 & 1/4 & 0 & 0\\ 0 & 1/4 & 1/4 & 0 & 1/4 & 0\\ 1/4 & 0 & 1/4 & 0 & 0 & 1/4\end{bmatrix}\begin{Bmatrix}a_1\\ a_2\\ a_3\\ a_4\\ a_5\\ a_6\end{Bmatrix}$$

Hence $a_1 = \phi_1$, $a_2 = \phi_2$, $a_3 = \phi_3$.
Subs. these into the last 3 eqs.

$$\left.\begin{aligned} a_4 &= 4\phi_4 - \phi_1 - \phi_2\\ a_5 &= 4\phi_5 - \phi_2 - \phi_3\\ a_6 &= 4\phi_6 - \phi_3 - \phi_1 \end{aligned}\right\} \quad \begin{aligned} \phi = {} & \phi_1\xi_1^2 + \phi_2\xi_2^2 + \phi_3\xi_3^2\\ & + (4\phi_4 - \phi_1 - \phi_2)\xi_1\xi_2\\ & + (4\phi_5 - \phi_2 - \phi_3)\xi_2\xi_3\\ & + (4\phi_6 - \phi_3 - \phi_1)\xi_3\xi_1 \end{aligned}$$

$$\begin{aligned} \phi = {} & (\xi_1^2 - \xi_1\xi_2 - \xi_3\xi_1)\phi_1 + (\xi_2^2 - \xi_1\xi_2 - \xi_2\xi_3)\phi_2\\ & + (\xi_3^2 - \xi_2\xi_3 - \xi_3\xi_1)\phi_3 + 4(\xi_1\xi_2\phi_4 + \xi_2\xi_3\phi_5 + \xi_3\xi_1\phi_6) \end{aligned}$$

Now use relation $\xi_1 + \xi_2 + \xi_3 = 1$.

$$\begin{aligned} \xi_1^2 - \xi_1\xi_2 - \xi_3\xi_1 = \xi_1(\xi_1 - \xi_2 - \xi_3) &= \xi_1[\xi_1 - (1 - \xi_1)]\\ &= \xi_1(2\xi_1 - 1) \end{aligned}$$

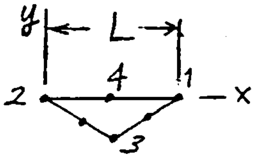
Similarly,

$$\xi_2^2 - \xi_1\xi_2 - \xi_2\xi_3 = \xi_2(2\xi_2 - 1)$$

$$\xi_3^2 - \xi_2\xi_3 - \xi_3\xi_1 = \xi_3(2\xi_3 - 1).$$ Finally,

$$\begin{aligned} \phi = {} & \xi_1(2\xi_1 - 1)\phi_1 + \xi_2(2\xi_2 - 1)\phi_2 + \xi_3(2\xi_3 - 1)\phi_3\\ & + 4\xi_1\xi_2\phi_4 + 4\xi_2\xi_3\phi_5 + 4\xi_3\xi_1\phi_6 \end{aligned}$$

$\boxed{7.3\text{-}5}$



$\{\underset{\sim}{r}_e\} = \int [\underset{\sim}{N}]^T \{\underset{\sim}{\Phi}\}\, dS$

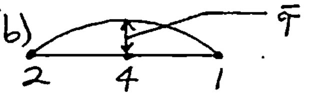
(a) $\begin{Bmatrix} r_{y1} \\ r_{y4} \\ r_{y2} \end{Bmatrix} = \int \begin{Bmatrix} \xi_1(2\xi_1 - 1) \\ 4\xi_1\xi_2 \\ \xi_2(2\xi_2-1) \end{Bmatrix} p\,t\,dL$ ← From Eq. 7.3-4

Use Eq. 7.3-5 to integrate

$\int \xi_i^2 dL = \frac{L}{3}\ , \quad \int \xi_i\, dL = \frac{L}{2}\ , \quad \int \xi_i \xi_j\, dL = \frac{L}{6}$

$$\begin{Bmatrix} r_{y1} \\ r_{y4} \\ r_{y2} \end{Bmatrix} = ptL \begin{Bmatrix} \frac{2}{3} - \frac{1}{2} \\ \frac{4}{6} \\ \frac{2}{3} - \frac{1}{2} \end{Bmatrix} = \frac{ptL}{6} \begin{Bmatrix} 1 \\ 4 \\ 1 \end{Bmatrix}$$

(b)

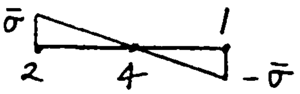


$\tau = 4\xi_1\xi_2\bar{\tau}$

$\int \xi_1^2 \xi_2\, dL = \frac{L}{12}\ , \quad \int \xi_1^3 \xi_2\, dL = \frac{L}{20}\ , \quad \int \xi_1^2 \xi_2^2\, dL = \frac{L}{30}$

$$\begin{Bmatrix} r_{x1} \\ r_{x4} \\ r_{x2} \end{Bmatrix} = \int \begin{Bmatrix} \xi_1(2\xi_1 - 1) \\ 4\xi_1\xi_2 \\ \xi_2(2\xi_2-1) \end{Bmatrix} 4\xi_1\xi_2\bar{\tau}\, t\, dL = \frac{\bar{\tau} t L}{15} \begin{Bmatrix} 1 \\ 8 \\ 1 \end{Bmatrix}$$

(c)



$\sigma = (\xi_1 - \xi_2)\bar{\sigma}$

$$\begin{Bmatrix} r_{y1} \\ r_{y4} \\ r_{y2} \end{Bmatrix} = \int \begin{Bmatrix} \xi_1(2\xi_1 - 1) \\ 4\xi_1\xi_2 \\ \xi_2(2\xi_2-1) \end{Bmatrix} (\xi_1 - \xi_2)\bar{\sigma}\, t\, dL$$

$\int \xi^3\, dL = \frac{L}{4} \qquad \int \xi_1^2 \xi_2^2\, dL = \frac{L}{6}$

$$\begin{Bmatrix} r_{y1} \\ r_{y4} \\ r_{y2} \end{Bmatrix} = \bar{\sigma} t \begin{Bmatrix} 2\frac{L}{4} - \frac{L}{3} - 2\frac{L}{12} + \frac{L}{6} \\ 4\left(\frac{L}{12} - \frac{L}{12}\right) \\ 2\frac{L}{12} - \frac{L}{6} - 2\frac{L}{4} + \frac{L}{3} \end{Bmatrix} = \bar{\sigma} t L \begin{Bmatrix} \frac{1}{6} \\ 0 \\ -\frac{1}{6} \end{Bmatrix}$$

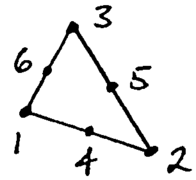
Yields $M = 2\frac{L}{2}\left(\frac{\bar{\sigma} t L}{6}\right) = \frac{\bar{\sigma} t L^2}{6}$

Check by flexure formula:

$$M = \frac{\sigma I}{c} = \frac{\bar{\sigma}\frac{1}{12} t L^3}{L/2} = \frac{\bar{\sigma} t L^2}{6} \checkmark$$

7.3-6

For one face, we deal with the triangle



For constant p, on face of area A,

$$\{\underset{\sim}{r}_e\} = \int [\underset{\sim}{N}]^T p\, dA = p \int [\underset{\sim}{N}]^T dA$$

where the N_i come from Eq. 7.3-4.
Use Eq. 7.3-7 for integration:

$$\int \xi_i\, dA = \frac{A}{3} \qquad \int \xi_i^2\, dA = \frac{A}{6} \qquad \int \xi_i \xi_j\, dA = \frac{A}{12}$$

$$\{\underset{\sim}{r}_e\} = p\frac{A}{12}\begin{Bmatrix} 2(2)-4 \\ 2(2)-4 \\ 2(2)-4 \\ 4 \\ 4 \\ 4 \end{Bmatrix} = \frac{pA}{3}\begin{Bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{Bmatrix}$$

$\boxed{7.4\text{-}1}$ $\phi = a_1 r + a_2 r^2 + a_3 rs + a_4 r^3 + a_5 rs^2$, straight sides, $A = \frac{1}{2}$

(a) In area coordinates, $\phi = a_1 \xi_2 + a_2 \xi_2^2 + a_3 \xi_2 \xi_3 + a_4 \xi_2^3 + a_5 \xi_2 \xi_3^2$

Using Eq. 7.3-7,

$$\int \phi \, dA = a_1 \frac{1}{6} + a_2 \frac{2}{4!} + a_3 \frac{1}{4!} + a_4 \frac{3!}{5!} + a_5 \frac{2}{5!}$$

$$= \frac{a_1}{6} + \frac{a_2}{12} + \frac{a_3}{24} + \frac{a_4}{20} + \frac{a_5}{60}$$

$$= 0.1667 a_1 + 0.08333 a_2 + 0.04167 a_3 + 0.05000 a_4 + 0.01667 a_5$$

(b) Formula 1, Table 7.4-1, $|\underset{\sim}{J}| = 2A = 1$

$$\int \phi \, dA \approx \frac{1}{2}\left(\frac{a_1}{3} + \frac{a_2}{3^2} + \frac{a_3}{3^2} + \frac{a_4}{3^3} + \frac{a_5}{3^3}\right) = \frac{a_1}{6} + \underbrace{\frac{a_2}{18} + \frac{a_3}{18} + \frac{a_4}{54} + \frac{a_5}{54}}_{\text{approx.}}$$

(c) Formula 2, Table 7.4-1, $|\underset{\sim}{J}| = 2A = 1$

$$\int \phi \, dA \approx \frac{1}{2}\frac{1}{3}\left[a_1 \frac{2}{3} + a_2\left(\frac{2}{3}\right)^2 + a_3\left(\frac{2}{3}\right)\left(\frac{1}{6}\right) + a_4\left(\frac{2}{3}\right)^3 + a_5\left(\frac{2}{3}\right)\left(\frac{1}{6}\right)^2\right]$$

$$+ \frac{1}{2}\frac{1}{3}\left[a_1 \frac{1}{6} + a_2\left(\frac{1}{6}\right)^2 + a_3\left(\frac{1}{6}\right)^2 + a_4\left(\frac{1}{6}\right)^3 + a_5\left(\frac{1}{6}\right)^3\right]$$

$$+ \frac{1}{2}\frac{1}{3}\left[a_1 \frac{1}{6} + a_2\left(\frac{1}{6}\right)^2 + a_3\left(\frac{1}{6}\right)\left(\frac{2}{3}\right) + a_4\left(\frac{1}{6}\right)^3 + a_5\left(\frac{1}{6}\right)\left(\frac{2}{3}\right)^2\right]$$

$$= \frac{1}{6}\left[a_1\left(\frac{2}{3} + \frac{1}{6} + \frac{1}{6}\right) + a_2\left(\frac{4}{9} + \frac{1}{36} + \frac{1}{36}\right) + a_3\left(\frac{2}{18} + \frac{1}{36} + \frac{2}{18}\right)\right.$$

$$\left. + a_4\left(\frac{8}{27} + \frac{1}{216} + \frac{1}{216}\right) + a_5\left(\frac{2}{108} + \frac{1}{216} + \frac{4}{54}\right)\right]$$

$$= 0.1667 a_1 + 0.08333 a_2 + 0.04167 a_3 + \underbrace{0.05093 a_4 + 0.01620 a_5}_{\text{approx.}}$$

(d) Formula 3, Table 7.4-1, $|\underset{\sim}{J}| = 2A = 1$

$$\int \phi \, dA \approx \frac{1}{2}\frac{1}{3}\left[a_1 \frac{1}{2} + a_2\left(\frac{1}{2}\right)^2 + a_3(0) + a_4\left(\frac{1}{2}\right)^3 + a_5(0)\right]$$

$$+ \frac{1}{2}\frac{1}{3}\left[a_1(0) + a_2(0) + a_3(0) + a_4(0) + a_5(0)\right]$$

$$+ \frac{1}{2}\frac{1}{3}\left[a_1 \frac{1}{2} + a_2\left(\frac{1}{2}\right)^2 + a_3\left(\frac{1}{2}\right)^2 + a_4\left(\frac{1}{2}\right)^3 + a_5\left(\frac{1}{2}\right)^3\right]$$

$$= \frac{1}{6}\left[a_1 + a_2\left(\frac{1}{2}\right) + a_3\left(\frac{1}{4}\right) + a_4\left(\frac{1}{4}\right) + a_5\left(\frac{1}{8}\right)\right]$$

$$= 0.1667 a_1 + 0.08333 a_2 + 0.04167 a_3 + \underbrace{0.04167 a_4 + 0.02083 a_5}_{\text{approx.}},$$

(continues)

$\boxed{7.4\text{-}1}$ (concluded)

(e) Formula 4, Table 7.4-1, $|\underset{\sim}{J}| = 2A = 1$

$$\int \phi \, dA \approx \frac{1}{2}\left(-\frac{27}{48}\right)\left[a_1\left(\frac{1}{3}\right) + a_2\left(\frac{1}{3}\right)^2 + a_3\left(\frac{1}{3}\right)^2 + a_4\left(\frac{1}{3}\right)^3 + a_5\left(\frac{1}{3}\right)^3\right]$$

$$+ \frac{1}{2}\left(\frac{25}{48}\right)\left[a_1\left(\frac{3}{5}\right) + a_2\left(\frac{3}{5}\right)^2 + a_3\left(\frac{3}{5}\right)\left(\frac{1}{5}\right) + a_4\left(\frac{3}{5}\right)^3 + a_5\left(\frac{3}{5}\right)\left(\frac{1}{5}\right)^2\right]$$

$$+ \frac{1}{2}\left(\frac{25}{48}\right)\left[a_1\left(\frac{1}{5}\right) + a_2\left(\frac{1}{5}\right)^2 + a_3\left(\frac{1}{5}\right)^2 + a_4\left(\frac{1}{5}\right)^3 + a_5\left(\frac{1}{5}\right)^3\right]$$

$$+ \frac{1}{2}\left(\frac{25}{48}\right)\left[a_1\left(\frac{1}{5}\right) + a_2\left(\frac{1}{5}\right)^2 + a_3\left(\frac{1}{5}\right)\left(\frac{3}{5}\right) + a_4\left(\frac{1}{5}\right)^3 + a_5\left(\frac{1}{5}\right)\left(\frac{3}{5}\right)^2\right]$$

$$= \frac{1}{96}\left[a_1(-9+15+5+5) + a_2(-3+9+1+1) + a_3(-3+3+1+3) + a_4\left(-1+\frac{27}{5}+\frac{1}{5}+\frac{1}{5}\right) + a_5\left(-1+\frac{3}{5}+\frac{1}{5}+\frac{9}{5}\right)\right]$$

$$= 0.1667a_1 + 0.08333a_2 + 0.04167a_3 + 0.0500a_4 + 0.01667a_5$$

✓ ✓ ✓ ✓ ✓

$\boxed{7.4-2}$

$$\text{1-pt.} \quad I \approx (1)\frac{1}{1+\frac{1}{3}\frac{1}{3}} = \frac{1}{\frac{10}{9}} = 0.9000$$

$$1^{st}\ \text{3-pt.} \quad I \approx \frac{1}{3}\left[\frac{1}{1+\frac{2}{3}\frac{1}{6}} + \frac{1}{1+\frac{1}{6}\frac{1}{6}} + \frac{1}{1+\frac{1}{6}\frac{2}{3}}\right] = 0.924324$$

$$2^{nd}\ \text{3-pt.} \quad I \approx \frac{1}{3}\left[\frac{1}{1+0} + \frac{1}{1+0} + \frac{1}{1+\frac{1}{4}}\right] = 0.933333$$

$$\text{4-pt.} \quad I \approx -0.5625\,\frac{1}{1+\frac{1}{9}} + 0.5208333\left[\frac{1}{1+.12} + \frac{1}{1+.04} + \frac{1}{1+.12}\right] = 0.924611$$

$\boxed{7.4\text{-}3}$

Eq. 7.2-2: $[\underset{\sim}{J}] = \begin{bmatrix} x_{,r} & y_{,r} \\ x_{,s} & y_{,s} \end{bmatrix}$ $\qquad x = \sum N_i x_i$, $\quad y = \sum N_i y_i$

Contents of $[\underset{\sim}{J}]$ are linear in r and s, so $|\underset{\sim}{J}|$ is quadratic in r and s. By Eq. 7.4-1

$$\text{Volume} = \int t\, dA = \sum_{i=1}^{n} \frac{1}{2} |\underset{\sim}{J}|_i t_i W_i$$

where t is quadratic in r and s. Hence integrand contains 4^{th} powers of r and s. Need degree of precision = 4.

$\boxed{7.4\text{-}4}$ $\phi = a_1 r + a_2 r^2 + a_3 rs + a_4 r^3 + a_5 rs^2 + a_6 rst$ Flat faces, $V = \frac{1}{6}$

(a) In volume coords., $\phi = a_1 \xi_2 + a_2 \xi_2^2 + a_3 \xi_2 \xi_3 + a_4 \xi_2^3 + a_5 \xi_2 \xi_3^2 + a_6 \xi_2 \xi_3 \xi_4$

Using Eq. 7.3-9,

$$\int \phi \, dV = a_1 \frac{1}{4!} + a_2 \frac{2}{5!} + a_3 \frac{1}{5!} + a_4 \frac{3!}{6!} + a_5 \frac{2}{6!} + a_6 \frac{1}{6!}$$

$$= 0.04167 a_1 + 0.01667 a_2 + 0.008333 a_3$$
$$+ 0.008333 a_4 + 0.002778 a_5 + 0.001389 a_6$$

(b) Formula 1, Table 7.4-2, $|\underset{\sim}{J}| = 6V = 1$

$$\int \phi \, dV \approx \frac{1}{6}\left[\frac{a_1}{4} + \frac{a_2}{16} + \frac{a_3}{16} + \frac{a_4}{4^3} + \frac{a_5}{4^3} + \frac{a_6}{4^3}\right]$$

$$= 0.04167 a_1 + 0.01042 (a_2 + a_3) + 0.002604 (a_4 + a_5 + a_6)$$

(c) Formula 2, Table 7.4-2, $|\underset{\sim}{J}| = 6V = 1$

$$a = \frac{5 + 3\sqrt{5}}{20} = 0.58541, \quad b = \frac{5 - \sqrt{5}}{20} = 0.13820$$

$$\int \phi \, dV \simeq \frac{1}{4}\frac{1}{6}\left[a_1 a + a_2 a^2 + a_3 b^2 + a_4 b^3 + a_5 b^3 + a_6 a b^2\right]$$
$$+ \frac{1}{4}\frac{1}{6}\left[a_1 b + a_2 b^2 + a_3 b^2 + a_4 b^3 + a_5 b^3 + a_6 b^3\right]$$
$$+ \frac{1}{4}\frac{1}{6}\left[a_1 b + a_2 b^2 + a_3 ab + a_4 b^3 + a_5 a^2 b + a_6 ab^2\right]$$
$$+ \frac{1}{4}\frac{1}{6}\left[a_1 b + a_2 b^2 + a_3 ab + a_4 a^3 + a_5 ab^2 + a_6 ab^2\right]$$

$$= \frac{1}{24}\left[a_1 (a + 3b) + a_2 (a^2 + 3b^2) + a_3 (2ab + 2b^2)\right.$$
$$\left. + a_4 (a^3 + 3b^3) + a_5 (a^2 b + ab^2 + 2b^3) + a_6 (3ab^2 + b^3)\right]$$

$$= 0.04167 a_1 + 0.01667 a_2 + 0.008333 a_3$$
$$+ 0.008689 a_4 + 0.002659 a_5 + 0.001508 a_6$$

(continues)

$\boxed{7.4\text{-}4}$ (concluded)

(d) Formula 3, Table 7.4-2, $|\underset{\sim}{J}| = 6V = 1$

$$\int \phi \, dV \approx \left(-\frac{4}{5}\right)\frac{1}{6}\left[\frac{a_1}{4} + \frac{a_2}{4^2} + \frac{a_3}{4^2} + \frac{a_4}{4^3} + \frac{a_5}{4^3} + \frac{a_6}{4^3}\right]$$

$$+ \frac{9}{20}\,\frac{1}{6}\left[a_1\frac{1}{2} + a_2\frac{1}{4} + a_3\left(\frac{1}{2}\right)\left(\frac{1}{6}\right) + a_4\left(\frac{1}{2}\right)^3 + a_5\left(\frac{1}{2}\right)\left(\frac{1}{6}\right)^2 + a_6\left(\frac{1}{2}\right)\left(\frac{1}{6}\right)^2\right]$$

$$+ \frac{9}{20}\,\frac{1}{6}\left[a_1\frac{1}{6} + a_2\left(\frac{1}{6}\right)^2 + a_3\left(\frac{1}{6}\right)^2 + a_4\left(\frac{1}{6}\right)^3 + a_5\left(\frac{1}{6}\right)^3 + a_6\left(\frac{1}{6}\right)^3\right]$$

$$+ \frac{9}{20}\,\frac{1}{6}\left[a_1\frac{1}{6} + a_2\left(\frac{1}{6}\right)^2 + a_3\left(\frac{1}{6}\right)^2 + a_4\left(\frac{1}{6}\right)^3 + a_5\left(\frac{1}{6}\right)^3 + a_6\left(\frac{1}{6}\right)^2\left(\frac{1}{2}\right)\right]$$

$$+ \frac{9}{20}\,\frac{1}{6}\left[a_1\frac{1}{6} + a_2\left(\frac{1}{6}\right)^2 + a_3\left(\frac{1}{6}\right)\left(\frac{1}{2}\right) + a_4\left(\frac{1}{6}\right)^3 + a_5\left(\frac{1}{6}\right)\left(\frac{1}{2}\right)^2 + a_6\left(\frac{1}{6}\right)^2\left(\frac{1}{2}\right)\right]$$

$$= a_1\left(-\frac{1}{30} + \frac{9}{240} + 3\,\frac{9}{20(36)}\right) + a_2\left(-\frac{1}{120} + \frac{9}{480} + 3\,\frac{9}{20(36)6}\right)$$

$$+ a_3\left(-\frac{1}{120} + 2\,\frac{9}{20(2)6^2} + 2\,\frac{9}{20(6^3)}\right)$$

$$+ a_4\left(-\frac{1}{5(6)16} + \frac{9}{20(6)8} + 3\,\frac{9}{20(6)6^3}\right)$$

$$+ a_5\left(-\frac{1}{5(6)16} + \frac{9}{20(6)2(36)} + 2\,\frac{9}{20(6^4)} + \frac{9}{20(36)4}\right)$$

$$+ a_6\left(-\frac{1}{5(6)16} + 3\,\frac{9}{20(6)2(36)} + \frac{9}{20(6)6^3}\right)$$

$$= 0.04167a_1 + 0.01667a_2 + 0.008333a_3$$

$$\cdot\, ^{\frown}.008333a_4 + 0.002778a_5 + 0.001389a_6$$