$\boxed{8.1\text{-}1}$

$$L^2 = \underset{\sim}{V} \cdot \underset{\sim}{V} = \lfloor u' \; v' \; w' \rfloor \begin{Bmatrix} u' \\ v' \\ w' \end{Bmatrix} = \{\underset{\sim}{d}'\}^T \{\underset{\sim}{d}'\}$$

$L^2 = ([\underset{\sim}{\Lambda}]\{\underset{\sim}{d}\})^T([\underset{\sim}{\Lambda}]\{\underset{\sim}{d}\})$. Also, $L^2 = \{\underset{\sim}{d}\}^T\{\underset{\sim}{d}\}$.

$\{\underset{\sim}{d}\}^T[\underset{\sim}{\Lambda}]^T[\underset{\sim}{\Lambda}]\{\underset{\sim}{d}\} = \{\underset{\sim}{d}\}^T\{\underset{\sim}{d}\}$, or

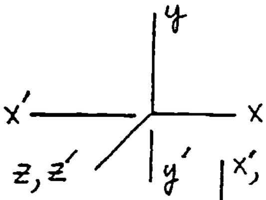
$\{\underset{\sim}{d}\}^T([\underset{\sim}{\Lambda}]^T[\underset{\sim}{\Lambda}] - [\underset{\sim}{I}])\{\underset{\sim}{d}\} = 0$.

Must be true for any $\{\underset{\sim}{d}\}$, so

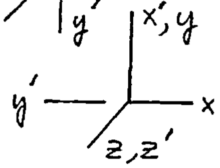
$[\underset{\sim}{\Lambda}]^T[\underset{\sim}{\Lambda}]^T = [\underset{\sim}{I}]$. From Eq. 8.1-1, this means $\sum l_i^2 = \sum m_i^2 = \sum n_i^2 = 1$

and $\sum l_i m_i = \sum m_i n_i = \sum n_i l_i = 0$

$\boxed{8.1-2}$



$$[\underset{\sim}{\Lambda}] = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$



$$[\underset{\sim}{\Lambda}] = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\boxed{8.2\text{-}1}$

$$[\underset{\sim}{E}'] = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix}$$

We can ignore the common multiplier $E/(1-\nu^2)$ in the following. The product $[\underset{\sim}{T}_\epsilon]^T[\underset{\sim}{E}'][\underset{\sim}{T}_\epsilon]$ is found to be symmetric, with the following terms in its upper triangle. Let $c = \cos\theta$, $s = \sin\theta$.

$$(1,1) = c^4 + \nu c^2 s^2 + \nu c^2 s^2 + s^4 + 2c^2s^2 - 2\nu c^2 s^2$$
$$= (c^2+s^2)^2 = 1$$

$$(1,2) = c^2s^2 + \nu c^4 + \nu s^4 + c^2s^2 - 2c^2s^2 + 2\nu c^2 s^2$$
$$= \nu(c^2+s^2)^2 = \nu$$

$$(1,3) = c^3 s(1-\nu) + cs^3(\nu-1) - (1-\nu)(c^3s - cs^3)$$
$$= 0$$

$$(2,2) = s^4 + \nu c^2s^2 + \nu c^2 s^2 + c^4 + 2c^2s^2(1-\nu)$$
$$= (c^2+s^2)^2 = 1$$

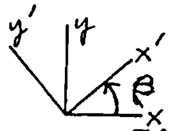
$$(2,3) = cs^3(1-\nu) + c^3s(\nu-1) + (1-\nu)(c^3s - cs^3)$$
$$= 0$$

$$(3,3) = c^2s^2(1-\nu) + c^2s^2(1-\nu) + \frac{1-\nu}{2}(c^2-s^2)^2$$
$$= \frac{1-\nu}{2}(c^2+s^2)^2 = \frac{1-\nu}{2}$$

So, after restoring the multiplier $E/(1-\nu^2)$, we obtain $[\underset{\sim}{E}] = [E']$.

$\boxed{8.2\text{-}2}$

$$[\underset{\sim}{E}'] = \begin{bmatrix} E_a & 0 & 0 \\ 0 & E_b & 0 \\ 0 & 0 & G \end{bmatrix}$$



$c = \cos\beta$
$s = \sin\beta$

$$[\underset{\sim}{T}_\epsilon]^T\left([\underset{\sim}{E}'][\underset{\sim}{T}_\epsilon]\right) =$$

$$\begin{bmatrix} c^2 & s^2 & -2cs \\ s^2 & c^2 & 2cs \\ cs & -cs & c^2-s^2 \end{bmatrix}\begin{bmatrix} c^2E_a & s^2E_a & csE_a \\ s^2E_b & c^2E_b & -cs\,E_b \\ -2csG & 2csG & G(c^2-s^2) \end{bmatrix} = [\underset{\sim}{E}]$$

$E_{11} = c^4E_a + s^4E_b + 4c^2s^2G$

$E_{12} = c^2s^2(E_a+E_b) - 4c^2s^2G \qquad = E_{21}$

$E_{13} = c^3s\,E_a - cs^3E_b - 2cs(c^2-s^2)G = E_{31}$

$E_{22} = s^4E_a + c^4E_b + 4c^2s^2G$

$E_{23} = cs^3E_a - c^3sE_b + 2cs(c^2-s^2)G = E_{32}$

$E_{33} = c^2s^2(E_a+E_b) + G(c^2-s^2)^2$

For $\beta = 0$, $[\underset{\sim}{E}] = [\underset{\sim}{E}']$ ✓

For $\beta = \pi/2$, $E_{11} = E_b$, $E_{22} = E_a$, $E_{33} = G$,
remainder of $[\underset{\sim}{E}]$ null. ✓

$\boxed{8.2\text{-}3}$

$$[\underset{\sim}{T}_\epsilon]^T[\underset{\sim}{T}_\epsilon] = \begin{bmatrix} c^4+s^4+4c^2s^2 & \cdots \\ \vdots & \end{bmatrix}$$

$$c^4+s^4+4c^2s^2 = (c^2+s^2)^2+2c^2s^2 = 1+c^2s^2$$

Unity only if $c^2s^2=0$; not so for all β.

Hence $[\underset{\sim}{T}_\epsilon]^T[\underset{\sim}{T}_\epsilon] \neq [\underset{\sim}{I}]$; $[\underset{\sim}{T}_\epsilon]$ not orthogonal.

$\boxed{8.3\text{-}1}$

For d.o.f. u_1 & u_2, $[\underset{\sim}{k}'] = \frac{AE}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

$u_r = u_2 - u_1$, $u_2 = u_1 + u_r$

$$\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}}_{[\underset{\sim}{T}]} \begin{Bmatrix} u_1 \\ u_r \end{Bmatrix} \qquad [\underset{\sim}{T}][\underset{\sim}{k}'] = \frac{AE}{L}\begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix}$$

$$[\underset{\sim}{T}]^T([\underset{\sim}{k}'][\underset{\sim}{T}]) = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \frac{AE}{L}\begin{bmatrix} 0 & -1 \\ 0 & 1 \end{bmatrix} = \frac{AE}{L}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$\boxed{8.3-2}$

$$u = \lfloor \underset{\sim}{N}' \rfloor \{\underset{\sim}{d}'\} = \left\lfloor \frac{-\xi+\xi^2}{2} \quad 1-\xi^2 \quad \frac{\xi+\xi^2}{2} \right\rfloor \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

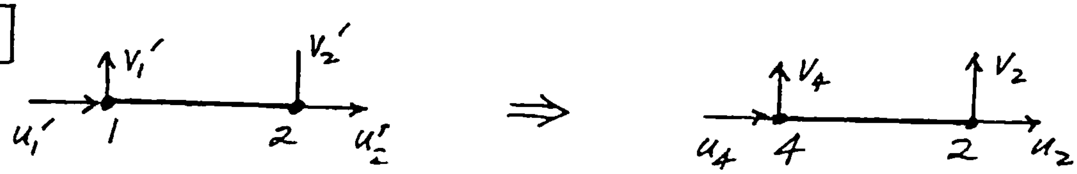
$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 & 1 & 1/2 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_r \\ u_3 \end{Bmatrix} \qquad \text{or} \quad \{\underset{\sim}{d}'\} = [\underset{\sim}{T}]\{\underset{\sim}{d}\}$$

$u = \lfloor \underset{\sim}{N}' \rfloor [\underset{\sim}{T}] \{\underset{\sim}{d}\}$ or $u = \lfloor \underset{\sim}{N} \rfloor \{\underset{\sim}{d}\}$ *where*

$$\lfloor \underset{\sim}{N} \rfloor = \lfloor \underset{\sim}{N}' \rfloor [\underset{\sim}{T}] = \left\lfloor \frac{-\xi+\xi^2+1-\xi^2}{2} \quad 1-\xi^2 \quad \frac{1-\xi^2+\xi+\xi^2}{2} \right\rfloor$$

$$= \left\lfloor \frac{1-\xi}{2} \quad 1-\xi^2 \quad \frac{1+\xi}{2} \right\rfloor$$

$\boxed{8.3\text{-}3}$



$$\begin{Bmatrix} u_1' \\ v_1' \\ u_2' \\ v_2' \end{Bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}}_{[\underset{\sim}{T}]} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ \vdots \\ u_4 \\ v_4 \end{Bmatrix}$$

8.4-1

Set $u_1 = v_1 = u_2 = v_2 = 0$, but temporarily retain u_4. $[\underset{\sim}{K}']\{\underset{\sim}{D}'\} = \{\underset{\sim}{R}'\}$ is

$$k\begin{bmatrix} \frac{3}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & -\frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{3}{2} \end{bmatrix}\begin{Bmatrix} u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix} = \begin{Bmatrix} R_{x3} \\ R_{y3} \\ R_{x4} \\ -P \end{Bmatrix}$$

Now set $u_4 = 0$: discard row & col. 3

$$\frac{k}{2}\begin{bmatrix} 3 & 1 & -1 \\ 1 & 1 & -1 \\ -1 & -1 & 3 \end{bmatrix}\begin{Bmatrix} u_3 \\ v_3 \\ v_4 \end{Bmatrix} = \begin{Bmatrix} R_{x3} \\ R_{y3} \\ -P \end{Bmatrix}, \quad [\underset{\sim}{T}] = \begin{bmatrix} 0.8 & 0.6 & 0 \\ -0.6 & 0.8 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\underbrace{[\underset{\sim}{T}]^T([\underset{\sim}{K}'][\underset{\sim}{T}])}_{} = \begin{bmatrix} 0.8 & -0.6 & 0 \\ 0.6 & 0.8 & 0 \\ 0 & 0 & 1 \end{bmatrix}\frac{k}{2}\begin{bmatrix} 1.8 & 2.6 & -1 \\ 0.2 & 1.4 & -1 \\ -0.2 & -1.4 & 3 \end{bmatrix}$$

$$\frac{k}{2}\begin{bmatrix} 1.32 & 1.24 & -0.2 \\ 1.24 & 2.68 & -1.4 \\ -0.2 & -1.4 & 3.0 \end{bmatrix}\begin{Bmatrix} U_3 \\ V_3 \\ v_4 \end{Bmatrix} = \begin{Bmatrix} 0 \\ R_3 \\ -P \end{Bmatrix}$$

Set $V_3 = 0$, solve for U_3 & v_4.

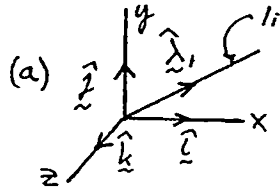
$$\begin{aligned} 1.32\,U_3 - 0.2\,v_4 &= 0 \\ -0.2\,U_3 + 3.0\,v_4 &= -2P/k \end{aligned}, \quad \begin{aligned} U_3 &= -0.102\,P/k \\ v_4 &= -0.673\,P/k \end{aligned}$$

$$F_{1\text{-}3} = \left|k\,(U_3\cos\beta)\right| = k\left(0.102\frac{P}{k}\right)(0.8) = 0.0816P$$

(tension)

$\boxed{8.4\text{-}2}$

(a)



line, $\hat{\lambda}_1 = \ell_1 \hat{i} + \ell_2 \hat{j} + \ell_3 \hat{k}$

Set up orthogonal vectors V_2 & V_3. Arbitrarily define $V_2 = \hat{\lambda}_1 \times \hat{i}$ (make another choice if $\hat{\lambda}_1 = \hat{i}$).

$$V_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \ell_1 & \ell_2 & \ell_3 \\ 1 & 0 & 0 \end{vmatrix} = \ell_3 \hat{j} - \ell_2 \hat{k}$$

Length of V_2 is L_2, $L_2 = (\ell_2^2 + \ell_3^2)^{1/2}$

$$V_3 = \hat{\lambda}_1 \times V_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \ell_1 & \ell_2 & \ell_3 \\ 0 & \ell_3 & -\ell_2 \end{vmatrix} = (-\ell_2^2 - \ell_3^2)\hat{i} + \ell_1 \ell_2 \hat{j} + \ell_1 \ell_3 \hat{k}$$

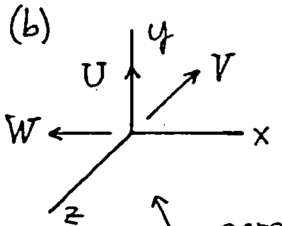
$$L_3 = \left[(-\ell_2^2 - \ell_3^2)^2 + (\ell_1 \ell_2)^2 + (\ell_1 \ell_3)^2 \right]^{1/2}$$

Orthog. unit vectors are $\hat{\lambda}_1$, V_2/L_2, V_3/L_3.
At affected node n, like $[\Lambda]$ in Eq. 8.1-1,

$$[T_n] = \begin{bmatrix} \ell_1 & 0 & (-\ell_2^2 - \ell_3^2)/L_3 \\ \ell_2 & \ell_3/L_2 & \ell_1 \ell_2 / L_3 \\ \ell_3 & -\ell_2/L_2 & \ell_1 \ell_3 / L_3 \end{bmatrix}$$

Put this in matrix $[T]$ similar to Eq. 8.4-3.

(b)



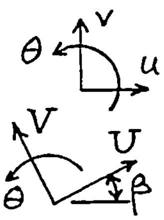
$\ell_2 = 1, \quad \ell_1 = \ell_3 = 0$

$$[T_n] = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix}$$

agree

8.4-3

(a) $$\begin{bmatrix} \frac{AE}{L} & 0 & 0 \\ 0 & \frac{12EI}{L^3} & -\frac{6EI}{L^2} \\ 0 & -\frac{6EI}{L^2} & \frac{4EI}{L} \end{bmatrix} \begin{Bmatrix} u \\ v \\ \theta \end{Bmatrix} = \begin{Bmatrix} 0 \\ -P \\ 0 \end{Bmatrix}$$



$$\begin{Bmatrix} u \\ v \\ \theta \end{Bmatrix} = \underbrace{\begin{bmatrix} c & -s & 0 \\ s & c & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{[\underset{\sim}{T}]} \begin{Bmatrix} U \\ V \\ \theta \end{Bmatrix} \qquad \begin{aligned} c &= \cos\beta \\ s &= \sin\beta \end{aligned}$$

$$\underbrace{\begin{bmatrix} c & s & 0 \\ -s & c & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{[\underset{\sim}{T}]^T} \underbrace{\begin{bmatrix} c\frac{AE}{L} & -s\frac{AE}{L} & 0 \\ s\frac{12EI}{L^3} & c\frac{12EI}{L^3} & -\frac{6EI}{L^2} \\ -s\frac{6EI}{L^2} & -c\frac{6EI}{L^2} & \frac{4EI}{L} \end{bmatrix}}_{[\underset{\sim}{K}'][\underset{\sim}{T}]}$$

"Omit middle" in product (i.e. set V=0)

$$\begin{bmatrix} c^2\frac{AE}{L} + s^2\frac{12EI}{L^3} & -s\frac{6EI}{L^2} \\ -s\frac{6EI}{L^2} & \frac{4EI}{L} \end{bmatrix} \begin{Bmatrix} U \\ \theta \end{Bmatrix} = \begin{Bmatrix} -sP \\ 0 \end{Bmatrix}$$

(b) $$U = -sP\left[c^2\frac{AE}{L} + s^2\frac{12EI}{L^3}\right]^{-1}$$

$\boxed{8.5-1}$

Substitute $\xi, \eta = \pm\frac{1}{2}$ in bilinear shape functions, Eqs. 6.2-3, e.g.

$$u_A = \frac{1}{4}\left(\frac{3}{2}\right)\left(\frac{3}{2}\right)u_1 + \frac{1}{4}\left(\frac{1}{2}\right)\left(\frac{3}{2}\right)u_2 + \frac{1}{4}\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)u_3 + \frac{1}{4}\left(\frac{3}{2}\right)\left(\frac{1}{2}\right)u_4$$

$$u_4 = \frac{1}{16}(9u_1 + 3u_2 + u_3 + 3u_4)$$

$$\begin{Bmatrix} u_A \\ u_B \\ u_C \\ v_A \\ v_B \\ v_C \end{Bmatrix} = \begin{bmatrix} \underset{\sim}{I}_1 & 0 \\ \underset{\sim}{0} & \underset{\sim}{I}_1 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ v_1 \\ \vdots \\ v_4 \end{Bmatrix},$$

where $[I_1]$ is

$$\frac{1}{16}\begin{bmatrix} 9 & 3 & 1 & 3 \\ 3 & 9 & 3 & 1 \\ 1 & 3 & 9 & 3 \end{bmatrix}$$

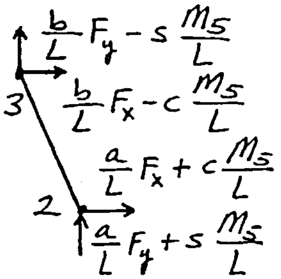
$\boxed{8.5\text{-}2}$

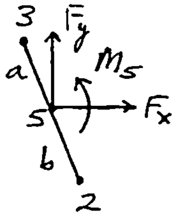
$$\{\underset{\sim}{r}\} = [\underset{\sim}{T}]^T\{\underset{\sim}{r}'\} = \frac{1}{L}\begin{bmatrix} a & 0 & c \\ 0 & a & s \\ b & 0 & -c \\ 0 & b & -s \end{bmatrix}\begin{Bmatrix} F_x \\ F_y \\ M_5 \end{Bmatrix}$$

$c = \cos\beta$

$s = \sin\beta$

(result)





$L = a + b$

Now want to show that these two are stat. equiv.

$$F_x = \frac{b}{L}F_x - c\frac{M_5}{L} + \frac{a}{L}F_x + c\frac{M_5}{L} = \frac{a+b}{L}F_x = F_x$$

$$F_y = \frac{b}{L}F_y - s\frac{M_5}{L} + \frac{a}{L}F_y + s\frac{M_5}{L} = \frac{a+b}{L}F_y = F_y$$

$$M_5 = bc\left(\frac{a}{L}F_x + c\frac{M_5}{L}\right) - ac\left(\frac{b}{L}F_x - c\frac{M_5}{L}\right)$$

$$+ bs\left(\frac{a}{L}F_y + s\frac{M_5}{L}\right) - as\left(\frac{b}{L}F_y - s\frac{M_5}{L}\right)$$

$$= \left[(a+b)c^2 + (a+b)s^2\right]\frac{M_5}{L} = (a+b)\frac{M_5}{L} = M_5$$

$\boxed{8.5\text{-}3}$

$$u_5 = \frac{c}{L_2}u_1 + \frac{d}{L_2}u_4 \qquad u_6 = \frac{a}{L_1}u_2 + \frac{b}{L_1}u_3$$

$$v_5 = \frac{c}{L_2}v_1 + \frac{d}{L_2}v_4 \qquad v_6 = \frac{a}{L_1}v_2 + \frac{b}{L_1}v_3$$

$$\lfloor u_5 \ v_5 \ u_6 \ v_6 \rfloor^T = [\underset{\sim}{T}]\lfloor u_1 \ v_1 \ u_2 \ v_2 \ u_3 \ v_3 \ u_4 \ v_4 \rfloor^T$$

where $[\underset{\sim}{T}]$ is the 4 by 8 matrix

$$\begin{bmatrix} c/L_2 & 0 & 0 & 0 & 0 & 0 & d/L_2 & 0 \\ 0 & c/L_2 & 0 & 0 & 0 & 0 & 0 & d/L_2 \\ 0 & 0 & a/L_1 & 0 & b/L_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & a/L_1 & 0 & b/L_1 & 0 & 0 \end{bmatrix}$$

$\boxed{8.5-4}$

(a) $\underset{8\times1}{\{d_1\}} = \underset{8\times6}{[T_1]}\underset{6\times1}{\{d_2\}}$, where $[T_1]$ is

$$\begin{bmatrix} 1 & 0 & H/2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & H/2 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & -H/2 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & -H/2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

This transformation sets $v_1 = v_4$ and $v_2 = v_3$.

(b) $\underset{6\times1}{\{d_2\}} = \underset{6\times8}{[T_2]}\underset{8\times1}{\{d_1\}}$, where $[T_2]$ is

$$\begin{bmatrix} 1/2 & 0 & 0 & 0 & 0 & 0 & 1/2 & 0 \\ 0 & 1/2 & 0 & 0 & 0 & 0 & 0 & 1/2 \\ 1/H & 0 & 0 & 0 & 0 & 0 & -1/H & 0 \\ 0 & 0 & 1/2 & 0 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1/2 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 1/H & 0 & -1/H & 0 & 0 & 0 \end{bmatrix}$$

This transformation averages the d.o.f. at each end of el. 1 to get the translational d.o.f. of el. 2, and says $\theta_1 = \frac{v_1 - v_4}{H}$, $\theta_2 = \frac{v_2 - v_3}{H}$.

(c) $\{d_1\} = [T_1]\{d_2\} = [T_1][T_2]\{d_1\}$

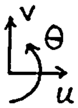
$\{d_2\} = [T_2]\{d_1\} = [T_2][T_1]\{d_2\}$

Hence we expect that $[T_1][T_2]$ and $[T_2][T_1]$ are both unit matrices, but:

$$\underset{8\times6}{[T_1]}\underset{6\times8}{[T_2]} \neq \underset{8\times8}{[I]}, \quad \underset{6\times8}{[T_2]}\underset{8\times6}{[T_1]} = \underset{6\times6}{[I]}$$

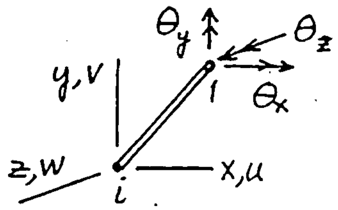
The operation $[T_1]\{d_2\}$ expands 6 pieces of information to 8, and $[T_2][T_1]\{d_2\}$ back again; nothing is lost. The operation $[T_1][T_2]\{d_1\}$ condenses, then expands, but in expanding does not separate the v_i from their combination in the definition of the θ_i.

$\boxed{8.5\text{-}5}$



$$\begin{Bmatrix} u_i \\ w_i \\ \theta_i \\ u_j \\ w_j \\ \theta_j \end{Bmatrix} = \begin{bmatrix} 1 & 0 & b_1 & & & \\ 0 & 1 & -a_1 & & & \\ 0 & 0 & 1 & & & \\ & & & 1 & 0 & b_2 \\ & & & 0 & 1 & -a_2 \\ & & & 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} u_1 \\ w_1 \\ \theta_1 \\ u_2 \\ w_2 \\ \theta_2 \end{Bmatrix}$$

8.5-6



Link $j2$ is similar.

$$\lfloor u_i \; v_i \; w_i \; \theta_{xi} \; \theta_{yi} \; \theta_{zi} \; u_j \; v_j \; w_j \; \theta_{xj} \; \theta_{yj} \; \theta_{zj} \rfloor^T =$$

$$\underset{12\times 12}{\begin{bmatrix} \underset{\sim}{T_1} & \underset{\sim}{0} \\ \underset{\sim}{0} & \underset{\sim}{T_2} \end{bmatrix}} \lfloor u_1 \; v_1 \; w_1 \; \theta_{x1} \; \theta_{y1} \; \theta_{z1} \; u_2 \; v_2 \; w_2 \; \theta_{x2} \; \theta_{y2} \; \theta_{z2} \rfloor^T$$

$$\underset{6\times 6}{[\underset{\sim}{T_1}]} = \begin{bmatrix} 1 & 0 & 0 & 0 & -(z_1-z_i) & (y_1-y_i) \\ 0 & 1 & 0 & (z_1-z_i) & 0 & -(x_1-x_i) \\ 0 & 0 & 1 & -(y_1-y_i) & (x_1-x_i) & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Obtain $[\underset{\sim}{T_2}]$ from $[\underset{\sim}{T_1}]$ by changing i to j and 1 to 2.

$\boxed{8.5\text{-}7}$

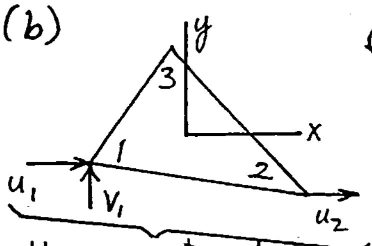
(a) $\circlearrowleft + \quad \theta = \dfrac{u_3 - u_2}{b}$ $\qquad v_1 = v_3 + \theta a$

$v_2 = v_3$

$u_1 = u_3$

$$\begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ -a/b & a/b & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \\ v_3 \end{Bmatrix}$$

(b)



three master d.o.f.

$\circlearrowleft + \quad \theta = \dfrac{u_2 - u_1}{y_1 - y_2}$

(OK if $y_1 \neq y_2$)

Relative to node 1,

$v_2 = \theta(x_2 - x_1)$

$u_3 = -\theta(y_3 - y_1)$

$v_3 = \theta(x_3 - x_1)$

Let $\alpha = \dfrac{1}{y_1 - y_2}$

$$\begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ -(x_2 - x_1)\alpha & 1 & (x_2 - x_1)\alpha \\ 1 + (y_3 - y_1)\alpha & 0 & -(y_3 - y_1)\alpha \\ -(x_3 - x_1)\alpha & 1 & (x_3 - x_1)\alpha \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \end{Bmatrix}$$

$\boxed{8.6\text{-}1}$

<u>Along AB</u> Apply $\{\underset{\sim}{\sigma}\} = [\underset{\sim}{E}]\{\epsilon\}$ with $\epsilon_x = u_{,x}$, etc.

$\sigma_x = -p$, so $\quad -p = \frac{E}{1-\nu^2}(u_{,x} + \nu v_{,y})$

$\tau_{xy} = 0$, so $\quad 0 = u_{,y} + v_{,x}$

These are 2 constraint eqs. on 4 d.o.f.

<u>Along CD</u> Same as AB, except $p = 0$.

<u>Along AD</u> Can transform d.o.f. to the st axes by use of Eqs. 8.1-3 & 8.2-3; then treat like edge CD.

<u>Along BC</u> $u = v = u_{,x} = v_{,x} = 0$. But $u_{,y}$ & $v_{,y}$ are unknown, as are corresp. terms in $\{\underset{\sim}{R}\}$. Can perhaps set these load terms to zero with little consequence

<u>Anisotropy</u> E.g. along AB,

$$\begin{Bmatrix} -p \\ \sigma_y \\ 0 \end{Bmatrix} = [\underset{\sim}{E}]\begin{Bmatrix} \epsilon_x \\ \epsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

From the 1st & 3rd of these equations,

$$\left.\begin{aligned} -p &= E_{11}u_{,x} + E_{12}v_{,y} + E_{13}(u_{,y} + v_{,x}) \\ 0 &= E_{31}u_{,x} + E_{32}v_{,y} + E_{33}(u_{,y} + v_{,x}) \end{aligned}\right\} \text{2 constraint eqs.}$$

$\boxed{8.6\text{-}2}$

(a) Use the beam N_i of Fig. 3.2-4:

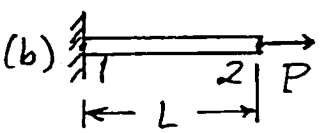
$$u = \lfloor N_1 \; N_2 \; N_3 \; N_4 \rfloor \lfloor u_1 \; \epsilon_{x1} \; u_2 \; \epsilon_{x2} \rfloor^T$$

$\lfloor \underset{\sim}{B} \rfloor = \frac{d}{dx} \lfloor \underset{\sim}{N} \rfloor$, etc. However, it is easier to use the "a-basis", $u = \lfloor 1 \; x \; x^2 \; x^3 \rfloor \{\underset{\sim}{a}\}$

$[\underset{\sim}{k}_a] = \int_0^L \lfloor \underset{\sim}{B}_a \rfloor^T \lfloor \underset{\sim}{B}_a \rfloor AE\,dx$, where $\lfloor \underset{\sim}{B}_a \rfloor = \lfloor 0 \;\; 1 \;\; 2x \;\; 3x^2 \rfloor$, $[\underset{\sim}{k}] = [\underset{\sim}{A}]^{-T} [\underset{\sim}{k}_a] [\underset{\sim}{A}]^{-1}$, and $[\underset{\sim}{A}]^{-1}$ is computed in Prob. 3.2-4

$$\underbrace{AE \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & L & L^2 & L^3 \\ 0 & L^2 & \frac{4}{3}L^3 & \frac{3}{2}L^4 \\ 0 & L^3 & \frac{3}{2}L^4 & \frac{9}{5}L^5 \end{bmatrix}}_{[\underset{\sim}{k}_a]}, \quad \underbrace{AE \begin{bmatrix} 0 & 0 & 0 & 0 \\ -1 & 0_2 & 1 & 0_2 \\ -L & -\frac{L^2}{6} & L & \frac{L^2}{6} \\ -.9L^2 & -.2L^3 & .9L^2 & .3L^3 \end{bmatrix}}_{[\underset{\sim}{k}_a][\underset{\sim}{A}]^{-1}}$$

$$\underbrace{\begin{bmatrix} 1 & 0 & -3/L^2 & 2/L^3 \\ 0 & 1 & -2/L & 1/L^2 \\ 0 & 0 & 3/L^2 & -2/L^3 \\ 0 & 0 & -1/L & 1/L^2 \end{bmatrix}}_{[\underset{\sim}{A}]^{-T}}, \quad \underbrace{\frac{AE}{30L} \begin{bmatrix} 36 & 3L & -36 & 3L \\ 3L & 4L^2 & -3L & -L^2 \\ -36 & -3L & 36 & -3L \\ 3L & -L^2 & -3L & 4L^2 \end{bmatrix}}_{[\underset{\sim}{k}]}$$

(b) 

$u_1 = 0$, so $\frac{AE}{30L} \begin{bmatrix} 4L^2 & -3L & -L^2 \\ -3L & 36 & -3L \\ -L^2 & -3L & 4L^2 \end{bmatrix} \begin{Bmatrix} \epsilon_{x1} \\ u_2 \\ \epsilon_{x2} \end{Bmatrix} = \begin{Bmatrix} 0 \\ P \\ F_{x2} \end{Bmatrix}$

Now impose $\epsilon_{x2} = \frac{P}{AE}$: $\frac{AE}{30L} \begin{bmatrix} 4L^2 & -3L & 0 \\ -3L & 36 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{Bmatrix} \epsilon_{x1} \\ u_2 \\ \epsilon_{x2} \end{Bmatrix} = \begin{Bmatrix} (AE/30L)L^2(P/AE) \\ P + (AE/30L)3L(P/AE) \\ P/AE \end{Bmatrix}$

From the first two equations, inverting the 2 by 2 matrix,

$$\begin{Bmatrix} \epsilon_{x1} \\ u_2 \end{Bmatrix} = \frac{30L}{AE} \frac{1}{135L^2} \begin{bmatrix} 36 & 3L \\ 3L & 4L^2 \end{bmatrix} \begin{Bmatrix} PL/30 \\ 1.1P \end{Bmatrix} = \frac{1}{4.5AEL} \begin{Bmatrix} 1.2PL + 3.3PL \\ 0.1PL^2 + 4.4PL^2 \end{Bmatrix}$$

$$\begin{Bmatrix} \epsilon_{x1} \\ u_2 \end{Bmatrix} = \begin{Bmatrix} P/AE \\ PL/AE \end{Bmatrix} \quad \checkmark$$

$\boxed{8.7\text{-}1}$

$J = \lfloor \underset{\sim}{N}_{,\xi} \rfloor \{ \underset{\sim}{x} \}$

$$J = \left\lfloor \frac{-1+2\xi}{2} \quad -2\xi \quad \frac{1+2\xi}{2} \right\rfloor \begin{Bmatrix} 0 \\ L/3 \\ L \end{Bmatrix} = \frac{L}{3}\left(\frac{3}{2} + \xi\right)$$

$J = 0$ at $\xi = -\frac{3}{2}$

$$x = \lfloor \underset{\sim}{N} \rfloor \{ \underset{\sim}{x} \} = \left\lfloor \frac{-\xi+\xi^2}{2} \quad 1-\xi^2 \quad \frac{\xi+\xi^2}{2} \right\rfloor \begin{Bmatrix} 0 \\ L/3 \\ L \end{Bmatrix} = \frac{L}{3}\left(1 + \frac{3}{2}\xi + \frac{1}{2}\xi^2\right)$$

For $\xi = -\frac{3}{2}$, $x = \frac{L}{3}\left(1 - \frac{9}{4} + \frac{9}{8}\right) = -\frac{L}{24}$, so $\frac{x}{L} = -0.04167$

$\boxed{8.8\text{-}1}$

(a) From Eqs. 3.2-3 and 3.2-8,

$$v = \lfloor \underset{\sim}{X} \rfloor [\underset{\sim}{A}]^{-1} \{\underset{\sim}{d}\} \qquad \text{Hence}$$

$$[\underset{\sim}{k}_f] = \beta [\underset{\sim}{A}]^{-T} \underbrace{\int_0^L \begin{Bmatrix} 1 \\ x \\ x^2 \\ x^3 \end{Bmatrix} \begin{Bmatrix} 1 \\ x \\ x^2 \\ x^3 \end{Bmatrix}^T b\, dx}\, [\underset{\sim}{A}]^{-1}$$

b = width of the beam

$$\begin{bmatrix} L & L^2/2 & L^3/3 & L^4/4 \\ & L^3/3 & L^4/4 & L^5/5 \\ & & L^5/5 & L^6/6 \\ \text{symm.} & & & L^7/7 \end{bmatrix} b = [\underset{\sim}{g}]. \text{ Hence,}$$

$$[\underset{\sim}{k}_f] = \beta [\underset{\sim}{A}]^{-T} [\underset{\sim}{g}] [\underset{\sim}{A}]^{-1} =$$

$$\begin{array}{cc} \dfrac{\beta b L}{420} \begin{bmatrix} 156 & 22L & 54 & -13L \\ & 4L^2 & 13L & -3L^2 \\ & & 156 & -22L \\ \text{symm.} & & & 4L^2 \end{bmatrix} & \begin{matrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{matrix} \end{array}$$

(b) $$v = \underbrace{\left\lfloor \frac{L-x}{L} \quad \frac{(L-x)x}{2L} \quad \frac{x}{L} \quad -\frac{(L-x)x}{2L} \right\rfloor}_{\lfloor \underset{\sim}{N} \rfloor} \begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{Bmatrix}$$

$$[\underset{\sim}{k}_f] = \beta \int_0^L \lfloor \underset{\sim}{N} \rfloor^T \lfloor \underset{\sim}{N} \rfloor b\, dx$$

After tedious expansion & integration,

$$\begin{array}{cc} [\underset{\sim}{k}_f] = \dfrac{\beta b L}{120} \begin{bmatrix} 40 & 5L & 20 & -5L \\ & L^2 & 5L & -L^2 \\ & & 40 & -5L \\ \text{symm.} & & & L^2 \end{bmatrix} & \begin{matrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{matrix} \end{array}$$

(c) $$v = \left\lfloor \frac{L-x}{L} \quad \frac{x}{L} \right\rfloor \begin{Bmatrix} w_1 \\ w_2 \end{Bmatrix} = \lfloor \underset{\sim}{N} \rfloor \begin{Bmatrix} v_1 \\ v_2 \end{Bmatrix}$$

$$\begin{array}{cc} [\underset{\sim}{k}_f] = \beta \displaystyle\int_0^L \lfloor \underset{\sim}{N} \rfloor^T \lfloor \underset{\sim}{N} \rfloor b\, dx = \dfrac{\beta b L}{6} \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} & \begin{matrix} v_1 \\ v_2 \end{matrix} \end{array}$$

(Can include θ_1 & θ_2 by adding zeros to $[\underset{\sim}{k}_f]$)

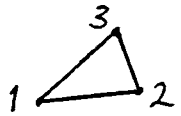
(d) Rigid body motion; gives diagonal $[\underset{\sim}{k}_f]$.

For $\{\underset{\sim}{d}\} = \lfloor v_1 \ \ \theta_1 \ \ v_2 \ \ \theta_2 \rfloor^T$,

$$[\underset{\sim}{k}_f] = \frac{\beta b L}{2} \lceil 1 \quad 0 \quad 1 \quad 0 \rfloor$$

All these formulations are valid; that is, all provide correct convergence with mesh refinement

8.8-2



In area coords.,

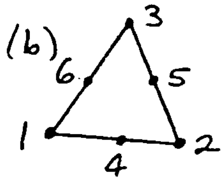
$$w = \lfloor \xi_1 \;\; \xi_2 \;\; \xi_3 \rfloor \begin{Bmatrix} W_1 \\ W_2 \\ W_3 \end{Bmatrix}$$

$$[k_f] = \beta \iint \begin{bmatrix} \xi_1^2 & \xi_1\xi_2 & \xi_1\xi_3 \\ \xi_1\xi_2 & \xi_2^2 & \xi_2\xi_3 \\ \xi_1\xi_3 & \xi_2\xi_3 & \xi_3^2 \end{bmatrix} dA$$

Integrate by use of Eq. 7.3-7.

$$[k_f] = \frac{\beta A}{12}\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

(b)



$$w = \lfloor N \rfloor \begin{Bmatrix} W_1 \\ W_2 \\ W_3 \\ W_4 \\ W_5 \\ W_6 \end{Bmatrix}$$

N_i given by Eq. 7.3-4.
Integrate by use of Eq. 7.3-7.

For $i=1,2,3$, $\int_A N_i^2\, dA = \frac{A}{30}$

For i or $j=1,2,3$ but $i \neq j$, $\int_A N_i N_j\, dA = -\frac{A}{180}$

For $i=4,5,6$, $\int_A N_i^2\, dA = \frac{8A}{45}$

For i or $j=4,5,6$ but $i \neq j$, $\int_A N_i N_j\, dA = \frac{4A}{45}$

$$\int_A N_1 N_5\, dA = \int_A N_2 N_6\, dA = \int_A N_3 N_4\, dA = -\frac{A}{45}$$

Integrals of N_1N_4, N_1N_6, N_2N_4, N_2N_5, N_3N_5, and N_3N_6 are zero. Hence

$$[k_f] = \frac{\text{[illegible]}}{180}\begin{bmatrix} 6 & -1 & -1 & 0 & -4 & 0 \\ -1 & 6 & -1 & 0 & 0 & -4 \\ 1 & -1 & 6 & -4 & 0 & 0 \\ 0 & 0 & -4 & 32 & 16 & 16 \\ -4 & 0 & 0 & 16 & 32 & 16 \\ 0 & -4 & 0 & 16 & 16 & 32 \end{bmatrix}$$

$\boxed{8.8-3}$

Strain energy per unit of area A is

$$dU = \frac{\beta}{2} w\, dA\,(w) + \frac{\alpha}{2} w_{,x}\, dA\,(w_{,x}) + \frac{\alpha}{2} w_{,y}\, dA\,(w_{,y})$$

where $\beta w\, dA$ = force $\qquad \alpha w_{,x}\, dA$ = moment

w = deflection $\qquad w_{,x}$ = rotation

Hence

$$U = \frac{1}{2}\int_A \lfloor w \;\; w_{,x} \;\; w_{,y} \rfloor \begin{bmatrix} \beta & & \\ & \alpha & \\ & & \alpha \end{bmatrix} \begin{Bmatrix} w \\ w_{,x} \\ w_{,y} \end{Bmatrix} dA$$

Let $\lfloor w \;\; w_{,x} \;\; w_{,y} \rfloor^T = \underset{3\times n}{[\underset{\sim}{Q}]}\underset{n\times 1}{\{\underset{\sim}{d}\}}$. Then

$$\underset{n\times n}{[\underset{\sim}{k}_f]} = \int_A [\underset{\sim}{Q}]^T \lceil \beta \;\; \alpha \;\; \alpha \rfloor [\underset{\sim}{Q}]\, dA$$

$\boxed{8.8-4}$

(a) Subs. $\xi = 1 - \frac{2a}{r}$ into $\phi = \frac{1}{2}(1-\xi)\phi_1 + \frac{1}{2}(1+\xi)\phi_3$

$$\phi = \left(\frac{1}{2} - \frac{1}{2} + \frac{a}{r}\right)\phi_1 + \left(\frac{1}{2} + \frac{1}{2} - \frac{a}{r}\right)\phi_3 = \frac{a}{r}\phi_1 + \left(1 - \frac{a}{r}\right)\phi_3$$

$\phi = c$ if $\phi_1 = \phi_3 = c$, $\phi \rightarrow \phi_3$ as $r \rightarrow \infty$

(b) From Eq. 8.8-3,

$$J = \frac{\partial x}{\partial \xi} = \frac{(1-\xi)(-2) - (-2\xi)(-1)}{(1-\xi)^2} x_1 + \frac{(1-\xi)(1) - (1+\xi)(-1)}{(1-\xi)^2} x_2$$

$$J = \frac{-2}{(1-\xi)^2} x_1 + \frac{2}{(1-\xi)^2} x_2 = \frac{2(x_2 - x_1)}{(1-\xi)^2} = \frac{2a}{(1-\xi)^2}$$

(c) $$\lfloor \underset{\sim}{B} \rfloor = \frac{1}{J} \frac{d}{d\xi} \left\lfloor \frac{1-\xi}{2} \quad \frac{1+\xi}{2} \right\rfloor = \frac{(1-\xi)^2}{4a} \lfloor -1 \quad 1 \rfloor$$

$$[\underset{\sim}{k}] = \int_{-1}^{1} \lfloor \underset{\sim}{B} \rfloor^T E \lfloor \underset{\sim}{B} \rfloor J A \, d\xi = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \frac{EA}{16a^2} 2a \int_{-1}^{1} (1-\xi)^2 d\xi$$

$$[\underset{\sim}{k}] = \frac{EA}{8a} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \frac{8}{3} = \frac{EA}{3a} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

$\boxed{8.8-5}$

(a)

$$\phi = \lfloor \underset{\sim}{N} \rfloor \begin{Bmatrix} \phi_1 \\ \phi_2 \\ \phi_5 \\ \phi_6 \end{Bmatrix}$$

$N_1 = \frac{1}{4}(1-\xi)(1-\eta)$

$N_2 = \frac{1}{4}(1-\xi)(1+\eta)$

$N_3 = \frac{1}{4}(1+\xi)(1-\eta)$

$N_4 = \frac{1}{4}(1+\xi)(1+\eta)$

Mapping functions M_i same as in Fig. 8.8-4.

(b) Using derivatives of M_i in Fig. 8.8-4,

$$[\underset{\sim}{J}] = \begin{bmatrix} \frac{-2}{(1-\xi)^2}\frac{1-\eta}{2} & \frac{-2}{(1-\xi)^2}\frac{1+\eta}{2} & \frac{2}{(1-\xi)^2}\frac{1-\eta}{2} & \frac{2}{(1-\xi)^2}\frac{1+\eta}{2} \\ \frac{\xi}{1-\xi} & \frac{-\xi}{1-\xi} & -\frac{1+\xi}{2(1-\xi)} & \frac{1+\xi}{2(1-\xi)} \end{bmatrix} \begin{bmatrix} a & 0 \\ a & 2b \\ 2a & 0 \\ 2a & 2b \end{bmatrix}$$

$$[\underset{\sim}{J}] = \begin{bmatrix} \frac{a}{(1-\xi)^2} & 0 \\ 0 & b \end{bmatrix}, \quad J = \frac{ab}{(1-\xi)^2}$$

(c)

$$[\underset{\sim}{B}] = \begin{bmatrix} \underset{\sim}{N}_{,x} \\ \underset{\sim}{N}_{,y} \end{bmatrix} = [\underset{\sim}{J}]^{-1} \begin{bmatrix} \underset{\sim}{N}_{,\xi} \\ \underset{\sim}{N}_{,\eta} \end{bmatrix}$$

with only ϕ_1 & ϕ_2 active.

$$[\underset{\sim}{B}] = \frac{(1-\xi)^2}{ab} \begin{bmatrix} b & 0 \\ 0 & \frac{a}{(1-\xi)^2} \end{bmatrix} \frac{1}{4} \begin{bmatrix} -(1-\eta) & -(1+\eta) \\ -(1-\xi) & (1-\xi) \end{bmatrix}$$

$$[\underset{\sim}{B}] = \frac{(1-\xi)^2}{4ab} \begin{bmatrix} -b(1-\eta) & -b(1+\eta) \\ -\frac{a}{1-\xi} & \frac{a}{1-\xi} \end{bmatrix}$$

$$[\underset{\sim}{k}] = \int_{-1}^{1}\int_{-1}^{1} [\underset{\sim}{B}]^T k [\underset{\sim}{B}]\, t\, J\, d\xi\, d\eta$$

$$[\underset{\sim}{k}] = \frac{kt}{16ab} \int_{-1}^{1}\int_{-1}^{1} \left[\begin{array}{c|c} b^2(1-\xi)^2(1-\eta)^2 + a^2 & b^2(1-\xi)^2(1-\eta^2) - a^2 \\ b^2(1-\xi)^2(1-\eta^2) - a^2 & b^2(1-\xi)^2(1+\eta)^2 + a^2 \end{array}\right] d\xi\, d\eta$$

$\int_{-1}^{1}(1+\eta)^2 d\eta = \frac{8}{3}$

$\int_{-1}^{1}(1-\eta^2) d\eta = \frac{4}{3}$

$$[\underset{\sim}{k}] = \frac{kt}{16Ab} \begin{bmatrix} \frac{64}{9}b^2 + 2a^2 & \frac{32}{9}b^2 - 2a^2 \\ \frac{32}{9}b^2 - 2a^2 & \frac{64}{9}b^2 + 2a^2 \end{bmatrix}$$

$\boxed{8.8-6}$

$$M_1 = -\frac{2\xi}{1-\xi}\,\frac{-\eta+\eta^2}{2} \qquad M_4 = \frac{1+\xi}{1-\xi}\,\frac{-\eta+\eta^2}{2}$$

$$M_2 = -\frac{2\xi}{1-\xi}(1-\eta^2) \qquad M_5 = \frac{1+\xi}{1-\xi}(1-\eta^2)$$

$$M_3 = -\frac{2\xi}{1-\xi}\,\frac{\eta+\eta^2}{2} \qquad M_6 = \frac{1+\xi}{1-\xi}\,\frac{\eta+\eta^2}{2}$$

$\boxed{8.9\text{-}1}$

(a) $\underset{\sim}{K}^* = \underset{\sim}{K} + \Delta\underset{\sim}{K}$, $\Delta\underset{\sim}{K} = K^* - K = 0.3$

Iterative eq. is $0.5 D_{i+1}^* = 2 - 0.3 D_i^*$

i.e. $D_{i+1}^* = 4 - 0.6 D_i^*$

$D_2^* = 4 - 0.6(4) = 1.6$

$D_3^* = 4 - 0.6(1.6) = 3.04$

$D_4^* = 4 - 0.6(3.04) = 2.176$

$D_5^* = 4 - 0.6(2.176) = 2.6944$

$D_6^* = 4 - 0.6(2.6944) = 2.38336$

Converges to $D^* = 2.50$, i.e. to exact value.

(b) $D_{i+1}^* = \frac{1}{k}(R - \Delta k \, D_i^*) = D - \frac{\Delta k}{k} D_i^*$

$$D_2^* = D - \frac{\Delta k}{k} D$$

$$D_3^* = D - \frac{\Delta k}{k} D_1^* = D - \frac{\Delta k}{k} D + \left(\frac{\Delta k}{k}\right)^2 D$$

$$D_4^* = D - \frac{\Delta k}{k} D_2^* = D\left[1 - \frac{\Delta k}{k} + \left(\frac{\Delta k}{k}\right)^2 - \left(\frac{\Delta k}{k}\right)^3\right]$$

etc., so the series in brackets extends.
The series converges if $\left|\frac{\Delta k}{k}\right| < 1$.

Hence, magnitude of increase or decrease in k must be less than 100%.