$\boxed{9.2\text{-}1}$

Exact solution:

$$\begin{bmatrix} 1.00 & 1.00 \\ 1.00 & 1.01 \end{bmatrix}\begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{Bmatrix} 2.00 \\ 2.01 \end{Bmatrix}, \quad \begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{bmatrix} 101 & -100 \\ -100 & 100 \end{bmatrix}\begin{Bmatrix} 2.00 \\ 2.01 \end{Bmatrix}$$

from which $x = y = 1$.

Small change in matrix: e.g.,

$$\begin{bmatrix} 1.00 & 1.00 \\ 1.00 & 1.02 \end{bmatrix}\begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{Bmatrix} 2.00 \\ 2.01 \end{Bmatrix}, \quad \begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{bmatrix} 51 & -50 \\ -50 & 50 \end{bmatrix}\begin{Bmatrix} 2.00 \\ 2.01 \end{Bmatrix}$$

from which $x = 1.5$, $y = 0.5$.

(Or, if 1.01 changed to 1.00, the matrix is singular & the equations contradictory.)

Small change in vector: e.g.,

$$\begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{bmatrix} 101 & -100 \\ -100 & 100 \end{bmatrix}\begin{Bmatrix} 2.00 \\ 2.02 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 2 \end{Bmatrix}$$

$\boxed{9.2\text{-}2}$

multiply 1^{st} eq. by $\dfrac{\alpha c s}{1+\alpha c^2}$.

$$\begin{bmatrix} k\alpha cs & \dfrac{k(\alpha cs)^2}{1+\alpha c^2} \\ k\alpha cs & k(1+\alpha s^2) \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \end{Bmatrix} = \begin{Bmatrix} \dfrac{\alpha cs}{1+\alpha c^2} P \\ 0 \end{Bmatrix}$$

Subtract 1^{st} eq. from 2^{nd}. 2^{nd} eq. becomes

$$k\left[(1+\alpha s^2) - \frac{(\alpha cs)^2}{1+\alpha c^2}\right] v_1 = -\frac{\alpha cs}{1+\alpha c^2} P = -\frac{cs\,P}{\left(\frac{1}{\alpha}+c^2\right)}$$

As α becomes large, we approach

$$k\left[\alpha s^2 - \alpha s^2\right] v_1 = -\frac{s}{c} P$$

severe cancellation error

$\boxed{9.2\text{-}3}$

$$EI\begin{bmatrix} \frac{12}{L^3}+\frac{12}{(\alpha L)^3} & -\frac{6}{L^2}+\frac{6}{(\alpha L)^2} & \frac{6}{(\alpha L)^2} \\ -\frac{6}{L^2}+\frac{6}{(\alpha L)^2} & \frac{4}{L}+\frac{4}{\alpha L} & \frac{2}{\alpha L} \\ \frac{6}{(\alpha L)^2} & \frac{2}{\alpha L} & \frac{4}{\alpha L} \end{bmatrix}\begin{Bmatrix} v_2 \\ \theta_2 \\ \theta_3 \end{Bmatrix}=\begin{Bmatrix} 0 \\ 0 \\ M_0 \end{Bmatrix}$$

As α becomes small, we approach

$$EI\begin{bmatrix} \frac{12}{(\alpha L)^3} & \frac{6}{(\alpha L)^2} & \frac{6}{(\alpha L)^2} \\ \frac{6}{(\alpha L)^2} & \frac{4}{\alpha L} & \frac{2}{\alpha L} \\ \frac{6}{(\alpha L)^2} & \frac{2}{\alpha L} & \frac{4}{\alpha L} \end{bmatrix}\begin{Bmatrix} v_2 \\ \theta_2 \\ \theta_3 \end{Bmatrix}=\begin{Bmatrix} 0 \\ 0 \\ M_0 \end{Bmatrix}$$

(αL times row 1) − (row 2) = (row 3), i.e. the matrix becomes singular.

9.2-4

(a)
$$\frac{EI}{L^3}\begin{bmatrix} 12+\frac{kL^3}{EI} & -12 & 6L \\ -12 & 12 & -6L \\ 6L & -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} w_1 \\ w_2 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -P \\ 0 \end{Bmatrix}$$

Add $\frac{3}{2L}$ times last eq. to 2^{nd} eq.; subtract $\frac{3}{2L}$ times last eq. from 1^{st} eq.; finally divide last eq. by its pivot. Thus

$$\frac{EI}{L^3}\begin{bmatrix} 3+\frac{kL^3}{EI} & -3 & 0 \\ -3 & 3 & 0 \\ \frac{3}{2L} & -\frac{3}{2L} & 1 \end{bmatrix}\begin{Bmatrix} w_1 \\ w_2 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -P \\ 0 \end{Bmatrix}$$

Add 2^{nd} eq. to 1^{st}, then divide 2^{nd} by 3.

$$\frac{EI}{L^3}\begin{bmatrix} kL^3/EI & 0 & 0 \\ -1 & 1 & 0 \\ \frac{3}{2L} & -\frac{3}{2L} & 1 \end{bmatrix}\begin{Bmatrix} w_1 \\ w_2 \\ \theta_2 \end{Bmatrix} = \begin{Bmatrix} -P \\ -P/3 \\ 0 \end{Bmatrix}$$

$w_1 = -P/k$

$w_2 = w_1 - \frac{PL^3}{3EI} = -P\left(\frac{1}{k}+\frac{L^3}{3EI}\right)$

$\theta_2 = \frac{3}{2L}(-w_1+w_2) = -\frac{PL^2}{2EI}$

But lead coef. in last matrix eq. obtained as $\left(3+\frac{kL^3}{EI}\right)-3$; severe cancellation error if k is small.

(b) 

Set $\theta_2 = \theta_1 + \theta_{21}$ & $w_2 = w_1 + L\theta_1 + w_{21}$

$$\begin{Bmatrix} w_1 \\ \theta_1 \\ w_2 \\ \cdot \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & L & 1 & 0 \\ \cdot & \cdot & \cdot & 1 \end{bmatrix}\begin{Bmatrix} w_1 \\ \theta_1 \\ w_{21} \\ \theta_{21} \end{Bmatrix} = [T]\{d\}$$

Applied to $[k]$ of standard beam el., $[T]^T[k][T]$ yields

$$\frac{EI}{L^3}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 12 & -6L \\ 0 & 0 & -6L & 4L^2 \end{bmatrix}$$

Add stiffness k to w_1 & set $\theta_1 = 0$. Thus

$$\frac{EI}{L^3}\begin{bmatrix} kL^3/EI & 0 & 0 \\ 0 & 12 & -6L \\ 0 & -6L & 4L^2 \end{bmatrix}\begin{Bmatrix} w_1 \\ w_{21} \\ \theta_{21} \end{Bmatrix} = \begin{Bmatrix} -P \\ -P \\ 0 \end{Bmatrix}$$

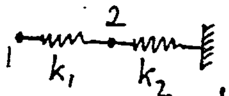
where the load vector is $[T]^T \lfloor 0,0,-P,0 \rfloor^T$ with the second term discarded for $\theta_1 = 0$. The condition of the matrix is independent of k. The solution is

$$\begin{Bmatrix} w_1 \\ w_{21} \\ \theta_{21} \end{Bmatrix} = \begin{Bmatrix} -P/k \\ -PL^3/3EI \\ -PL^2/2EI \end{Bmatrix}, \text{ hence}$$

$w_2 = -\frac{P}{k} + L(0) + w_{21}$
$= -\frac{P}{k} - \frac{PL^3}{3EI}$

$\theta_2 = (0) + \theta_{21} = -\frac{PL^2}{2EI}$

$\boxed{9.3\text{-}1}$



$$\begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1+k_2 \end{bmatrix}\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{Bmatrix} P \\ 0 \end{Bmatrix}$$

(a) Unscaled: $\begin{vmatrix} 10-\lambda & -10 \\ -10 & 11-\lambda \end{vmatrix} = \lambda^2 - 21\lambda + 10 = 0$

$$C(\underset{\sim}{K}) = \frac{20.51}{0.487} = 42.07$$

Scaled: $\begin{vmatrix} 10-10\lambda & -10 \\ -10 & 11-11\lambda \end{vmatrix} = 110\lambda^2 - 220\lambda + 10 = 0$

$$C(\underset{\sim}{K}) = \frac{1.953}{0.0465} = 41.98$$

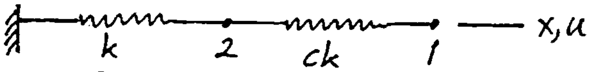
(b)
Unscaled: $\begin{vmatrix} 1-\lambda & -1 \\ -1 & 11-\lambda \end{vmatrix} = \lambda^2 - 12\lambda + 10 = 0$

$$C(\underset{\sim}{K}) = \frac{11.10}{0.901} = 12.32$$

Scaled: $\begin{vmatrix} 1-\lambda & -1 \\ -1 & 11-11\lambda \end{vmatrix} = 11\lambda^2 - 22\lambda + 10 = 0$

$$C(\underset{\sim}{K}) = \frac{1.302}{0.698} = 1.863$$

$\boxed{9.3\text{-}2}$



(a) $[\underset{\sim}{K}] = k\begin{bmatrix} c & -c \\ -c & c+1 \end{bmatrix}$, $\begin{vmatrix} c-\lambda & -c \\ -c & c+1-\lambda \end{vmatrix} = 0$

$\lambda^2 - (2c+1)\lambda + c = 0$, $\lambda = \dfrac{2c+1 \pm \sqrt{4c^2+1}}{2}$

$C(\underset{\sim}{K}) = \dfrac{2c+1+(4c^2+1)^{1/2}}{2c+1-(4c^2+1)^{1/2}}$

By trial, $C(\underset{\sim}{K})_{min} = 5.8284$ for $c = 0.50$

(b) $\begin{vmatrix} c(1-\lambda) & -c \\ -c & (c+1)(1-\lambda) \end{vmatrix} = c(c+1)(1-\lambda)^2 - c^2 = 0$

$(1-\lambda)^2 = \dfrac{c}{c+1}$, $\lambda = 1 \pm \left(\dfrac{c}{c+1}\right)^{1/2}$

$C(\underset{\sim}{K}) = \dfrac{1+\left(\frac{c}{c+1}\right)^{1/2}}{1-\left(\frac{c}{c+1}\right)^{1/2}}$, $C(\underset{\sim}{K})_{min} = 1.0$ for $c=0$

$\boxed{9.3\text{-}3}$

$\cos\beta = \sin\beta = \frac{\sqrt{2}}{2}$. D.o.f. are u_1, v_1.

$$[\underset{\sim}{K}] = \alpha k \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix} + k \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + k \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

$$[\underset{\sim}{K}] = \frac{k}{2} \begin{bmatrix} 2+\alpha & \alpha \\ \alpha & 2+\alpha \end{bmatrix}$$

(a) $\begin{vmatrix} 2+\alpha-\lambda & \alpha \\ \alpha & 2+\alpha-\lambda \end{vmatrix} = 0$, $(2+\alpha-\lambda)^2 - \alpha^2 = 0$

$2+\alpha-\lambda = \pm\alpha$ $\quad \lambda_{max} = 2+2\alpha \quad C(\underset{\sim}{K}) = 1+\alpha$

$\lambda_{min} = 2$

(b) $\begin{vmatrix} (2+\alpha)-\lambda(2+\alpha) & \alpha \\ \alpha & (2+\alpha)-\lambda(2+\alpha) \end{vmatrix} = 0$

$[(1-\lambda)(2+\alpha)]^2 - \alpha^2 = 0$, $\quad 1-\lambda = \pm\frac{\alpha}{2+\alpha}$

$$C(\underset{\sim}{K}) = \frac{1+\frac{\alpha}{2+\alpha}}{1-\frac{\alpha}{2+\alpha}} = \frac{2+2\alpha}{2} = 1+\alpha$$

9.3-4

$$[\underset{\sim}{k}] = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ & 4L^2 & -6L & 2L^2 \\ & & 12 & -6L \\ \text{symm.} & & & 4L^2 \end{bmatrix} \begin{matrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{matrix}$$

(a)  (simply supported)

$$\begin{vmatrix} 4L^2(1-\lambda) & 2L^2 \\ 2L^2 & 4L^2(1-\lambda) \end{vmatrix} = 0, \quad \begin{vmatrix} 2(1-\lambda) & 1 \\ 1 & 2(1-\lambda) \end{vmatrix} = 0$$

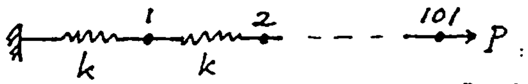
$$2(1-\lambda) = \pm 1, \quad C(\underset{\sim}{K}) = \frac{3/2}{1/2} = 3$$

(b)  (cantilever)

$$\begin{vmatrix} 12(1-\lambda) & -6L \\ -6L & 4L^2(1-\lambda) \end{vmatrix} = 0, \quad \begin{vmatrix} 6(1-\lambda) & -3 \\ -3 & 2(1-\lambda) \end{vmatrix} = 0$$

$$\sqrt{12}\,(1-\lambda) = \pm 3, \quad C(\underset{\sim}{K}) = \frac{1 + 3/\sqrt{12}}{1 - 3/\sqrt{12}} = 13.93$$

$\boxed{9.4\text{-}1}$

(a) 

$$\begin{bmatrix} 200 & -100 & 0 \cdots \\ -100 & 200 & -100 \cdots \\ 0 & -100 & 200 \cdots \\ \vdots & \vdots & \vdots \end{bmatrix}$$ ← NW corner of $[\underset{\sim}{K}]$. Add 0.5 * row 1 to row 2 & normalize row 1.

$$\begin{bmatrix} 1 & -0.5 & 0 \cdots \\ 0 & 150 & -100 \cdots \\ 0 & -100 & 200 \cdots \\ \vdots & \vdots & \vdots \end{bmatrix}$$ Next add $\frac{100}{150}$ * row 2 to row 3 & normalize row 2.

$$\begin{bmatrix} 1 & 0.5 & 0 & 0 \cdots \\ 0 & 1 & -.667 & 0 \cdots \\ 0 & 0 & 133.3 & -100 \cdots \\ \vdots & \vdots & \vdots & \vdots \end{bmatrix}$$

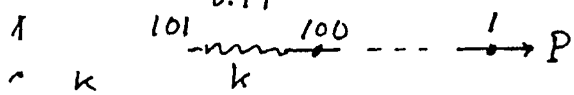
In general, after n-1 eliminations, the n^{th} diagonal coef. becomes $100 + \frac{100}{n}$. Thus, after $n-1 = 99$ eliminations,

$$\begin{bmatrix} 101 & -100 \\ -100 & 100 \end{bmatrix} \begin{Bmatrix} u_{100} \\ u_{101} \end{Bmatrix} = \begin{Bmatrix} 0 \\ P \end{Bmatrix}$$ Decay ratio is $\frac{200}{101} = 1.98$

$$\begin{bmatrix} 100 & \frac{-100}{101} 100 \\ -100 & 100 \end{bmatrix} \begin{Bmatrix} u_{100} \\ u_{101} \end{Bmatrix} = \begin{Bmatrix} 0 \\ P \end{Bmatrix}$$

Last elimination gives $100 + \frac{-100}{101} 100 = 0.99$

Decay ratio is $\frac{100}{0.99} = 101$

(b) 

$$\begin{bmatrix} 100 & -100 & 0 \cdots \\ -100 & 200 & -100 \cdots \\ 0 & -100 & 200 \cdots \end{bmatrix}$$ ← NW corner of $[\underset{\sim}{K}]$. Elimination changes 200's to 100's.

After 99 eliminations,

$$\begin{bmatrix} 100 & -100 \\ -100 & 200 \end{bmatrix} \begin{Bmatrix} u_{100} \\ u_{101} \end{Bmatrix} = \begin{Bmatrix} P \\ 0 \end{Bmatrix}$$ Decay ratio is $\frac{200}{100} = 2$

Last elimination yields $100 u_{101} = P$

Again, decay ratio $= \frac{200}{100} = 2$.

$\boxed{9.4\text{-}2}$

(a) One rigid body motion possible: last eq. (regardless of numbering sequence for the nodes).

(b) Two rigid body motions possible: next to last eq. (a v d.o.f., in usual ordering).

(c) Three rigid body motions possible: third from last eq. (a u d.o.f., in usual ordering).

(d) One rigid body motion possible (axial translation: last equation.

(e) Three rigid body motions possible. Let $\{\underset{\sim}{D}\} = \lfloor u_1 \ v_1 \ u_2 \cdots u_n \ v_n \rfloor^T$. Trouble in third from last (v_{n-1}) provided that v_{n-1} and v_n are not collinear. If they are collinear, we need u_{n-1} to define rotation; then the trouble is detected in 4th from last eq.

$\boxed{9.4\text{-}3}$

From Prob. 9.2-2, original K_{22} is $k(1+\alpha s^2)$ and reduced K_{22} is

$$k(1+\alpha s^2) - \frac{k(\alpha c s)^2}{1+\alpha c^2} = k\frac{1+\alpha s^2+\alpha c^2}{1+\alpha c^2} = k\frac{1+\alpha}{1+\alpha c^2}$$

Decay ratio is

$$\frac{(1+\alpha s^2)(1+\alpha c^2)}{1+\alpha} = \frac{1+\alpha+\alpha^2 c^2 s^2}{1+\alpha} = \frac{\frac{1}{\alpha}+1+\alpha c^2 s^2}{\frac{1}{\alpha}+1}$$

Becomes large if α is large unless $c=0$ (i.e. $\beta = \frac{\pi}{2}$) or $s=0$ (i.e. $\beta = 0$).

$\boxed{9.4\text{-}4}$

Exact: $\begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 0.30303030 \\ 0.30315530 \end{Bmatrix}$

$\begin{bmatrix} 8007 & -8000 \\ -8000 & 8000 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 1 \\ 1 \end{Bmatrix}$ Solve by Gauss elim., 4 digits

$\begin{bmatrix} 1 & -0.9991 \\ 0 & 7.000 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 0.0001249 \\ 1.999 \end{Bmatrix}$

$8000 - 8000(0.9991) = 8000 - 7993 = 7$

Diag. decay ratio $= \dfrac{8000}{7} \approx 10^3$

$\begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 0.2854 \\ 0.2856 \end{Bmatrix}$ lost accuracy in 3 places

$\begin{vmatrix} 8007(1-\lambda) & -8000 \\ -8000 & 8000(1-\lambda) \end{vmatrix} = 0$

$8007(8000)(1-\lambda)^2 = 8000^2$

$8007(1-\lambda) = \pm 8000$

$C(\underset{\sim}{K}) = \dfrac{\lambda_{max}}{\lambda_{min}} = \dfrac{1 + \frac{8000}{8007}}{1 - \frac{8000}{8007}} = \dfrac{16{,}007}{7} = 2287$

$\log_{10} C(\underset{\sim}{K}) = 3.36 \approx$ digits lost

$\boxed{9.5-1}$

$$\{\Delta \underset{\sim}{R}\} = \begin{Bmatrix} 2.88 \\ 1.52 \end{Bmatrix} - \begin{bmatrix} 1.78 & 1.06 \\ 0.94 & 0.56 \end{bmatrix} \begin{Bmatrix} 1.88 \\ -0.44 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -0.0008 \end{Bmatrix}$$

$$e = \frac{-0.44(-0.0008)}{1.88(2.88) + (-0.44)(1.52)} = 7.42(10)^{-5}$$

Exact:

$$\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \frac{1}{0.0004} \begin{bmatrix} 0.56 & -1.06 \\ -0.94 & 1.78 \end{bmatrix} \begin{Bmatrix} 2.88 \\ 1.52 \end{Bmatrix} = \begin{Bmatrix} 4.0000 \\ -4.0000 \end{Bmatrix}$$

$\{\Delta \underset{\sim}{R}\}$ and e are small, but error in solution is large. Equations are ill conditioned; they are the same as

$$1.78 u_1 + 1.06 u_2 = 2.88$$
$$1.78 u_1 + 1.060426 u_2 = 2.87830$$

$\boxed{9.5\text{-}2}$

Exact solution is $u_1 = u_2 = 1.0000$.

1^{st} approx.:

$$\{\Delta \underset{\sim}{R}\} = \begin{Bmatrix} 2.0000 \\ 2.0001 \end{Bmatrix} - \begin{bmatrix} 1 & 1 \\ 1 & 1.0001 \end{bmatrix} \begin{Bmatrix} 2.0 \\ 0.0 \end{Bmatrix}$$

$$\{\Delta \underset{\sim}{R}\} = \begin{Bmatrix} 0.0000 \\ 0.0001 \end{Bmatrix}, \quad e = \frac{2(0) + 0(0.0001)}{2(2) + 2.0001(0)} = 0$$

2^{nd} approx.:

$$\{\Delta \underset{\sim}{R}\} = \begin{Bmatrix} 2.0000 \\ 2.0001 \end{Bmatrix} - \begin{bmatrix} 1 & 1 \\ 1 & 1.0001 \end{bmatrix} \begin{Bmatrix} 1.1 \\ 1.1 \end{Bmatrix} = \begin{Bmatrix} -0.20 \\ -0.20 \end{Bmatrix},$$

$$e = \frac{1.1(-0.20) + 1.1(-0.20)}{1.1(2) + 1.1(2.0001)} = \frac{-0.4}{4.0001} = -0.100$$

The more exact approximation has the larger residual.

$\boxed{9.5-3}$

$$\Delta u_{i+1} = k^{-1}(R - k u_i) = 0.04\overbrace{(0.5 - 28u_i)}^{\Delta R}$$

$u_{i+1} = u_i + \Delta u_{i+1}$ Start with $u_0 = 0.02$

$\Delta R_1 = -0.060$ $\Delta R_2 = 0.0072$

$\Delta u_1 = -0.0024$ $\Delta u_2 = 0.000288$

$u_1 = 0.0176$ $u_2 = 0.01789$

$\Delta R_3 = -0.000864$ Exact:

$\Delta u_3 = -0.0000346$ $u = \dfrac{0.5}{28} = 0.017857$

$u_3 = 0.017853$

$\boxed{9.5\text{-}4}$

Exact: $u_1 = 7$, $u_2 = 4$

(a) First get initial soln. $\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}_1 = \begin{bmatrix} 8 & 5 \\ 5 & 5 \end{bmatrix} \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 8 \\ 5 \end{Bmatrix}$

Now iterate:

$$\begin{Bmatrix} \Delta R_1 \\ \Delta R_2 \end{Bmatrix}_1 = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} - \begin{bmatrix} .3 & -.3 \\ -.3 & .5 \end{bmatrix} \begin{Bmatrix} 8 \\ 5 \end{Bmatrix} = \begin{Bmatrix} .1 \\ -.1 \end{Bmatrix}$$

$$\begin{Bmatrix} \Delta u_1 \\ \Delta u_2 \end{Bmatrix}_2 = \begin{bmatrix} 8 & 5 \\ 5 & 5 \end{bmatrix} \begin{Bmatrix} .1 \\ -.1 \end{Bmatrix} = \begin{Bmatrix} .3 \\ 0 \end{Bmatrix}, \quad \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}_2 = \begin{Bmatrix} 8.3 \\ 5.0 \end{Bmatrix}$$

$$\begin{Bmatrix} \Delta R_1 \\ \Delta R_2 \end{Bmatrix}_2 = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} - \begin{bmatrix} .3 & -.3 \\ -.3 & .5 \end{bmatrix} \begin{Bmatrix} 8.3 \\ 5.0 \end{Bmatrix} = \begin{Bmatrix} .01 \\ -.01 \end{Bmatrix}$$

$$\begin{Bmatrix} \Delta u_1 \\ \Delta u_2 \end{Bmatrix}_3 = \begin{bmatrix} 8 & 5 \\ 5 & 5 \end{bmatrix} \begin{Bmatrix} .01 \\ -.01 \end{Bmatrix} = \begin{Bmatrix} .03 \\ 0 \end{Bmatrix}, \quad \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}_3 = \begin{Bmatrix} 8.33 \\ 5.00 \end{Bmatrix}$$

Next cycle gives $\begin{Bmatrix} 8.333 \\ 5.000 \end{Bmatrix}$ Conv. toward wrong answer.

(b) Iterate from same initial solution.

$$\begin{Bmatrix} \Delta R_1 \\ \Delta R_2 \end{Bmatrix}_1 = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} - \begin{bmatrix} 1/3 & -1/3 \\ -1/3 & 7/12 \end{bmatrix} \begin{Bmatrix} 8 \\ 5 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -.25 \end{Bmatrix}$$

$$\begin{Bmatrix} \Delta u_1 \\ \Delta u_2 \end{Bmatrix}_2 = \begin{bmatrix} 8 & 5 \\ 5 & 5 \end{bmatrix} \begin{Bmatrix} 0 \\ -.25 \end{Bmatrix} = \begin{Bmatrix} -1.25 \\ -1.25 \end{Bmatrix}, \quad \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}_2 = \begin{Bmatrix} 6.75 \\ 3.75 \end{Bmatrix}$$

$$\begin{Bmatrix} \Delta R_1 \\ \Delta R_2 \end{Bmatrix}_2 = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} - \begin{bmatrix} 1/3 & -1/3 \\ -1/3 & 7/12 \end{bmatrix} \begin{Bmatrix} 6.75 \\ 3.75 \end{Bmatrix} = \begin{Bmatrix} 0 \\ .0625 \end{Bmatrix}$$

$$\begin{Bmatrix} \Delta u_1 \\ \Delta u_2 \end{Bmatrix}_3 = \begin{bmatrix} 8 & 5 \\ 5 & 5 \end{bmatrix} \begin{Bmatrix} 0 \\ .0625 \end{Bmatrix} = \begin{Bmatrix} .3125 \\ .3125 \end{Bmatrix}, \quad \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}_3 = \begin{Bmatrix} 7.0625 \\ 4.0625 \end{Bmatrix}$$

$$\begin{Bmatrix} \Delta R_1 \\ \Delta R_2 \end{Bmatrix}_3 = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix} - \begin{bmatrix} 1/3 & -1/3 \\ -1/3 & 7/12 \end{bmatrix} \begin{Bmatrix} 7.0625 \\ 4.0625 \end{Bmatrix} = \begin{Bmatrix} 0 \\ -.015625 \end{Bmatrix}$$

$$\begin{Bmatrix} \Delta u_1 \\ \Delta u_2 \end{Bmatrix}_4 = \begin{bmatrix} 8 & 5 \\ 5 & 5 \end{bmatrix} \begin{Bmatrix} 0 \\ -.015625 \end{Bmatrix} = \begin{Bmatrix} -.078125 \\ -.078125 \end{Bmatrix}$$

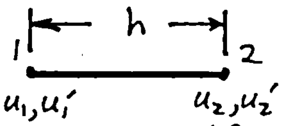
$$\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}_4 = \begin{Bmatrix} 6.984375 \\ 3.984375 \end{Bmatrix}$$ Conv. toward correct answer.

$\boxed{9.6-1}$

At $x = L/2$, $N_1 = N_3 = \frac{1}{2}$

$(L = h)$ $\quad N_2 = \frac{L}{8}$, $N_4 = -\frac{L}{8}$

$u = \frac{1}{2}(u_1 + u_2) + \frac{h}{8}(u_1' - u_2')$. Series:



Expand in series, using u, u', u'', etc. at middle.

$$u_1 = u - \frac{h}{2}u' + \frac{h^2}{8}u'' - \frac{h^3}{48}u''' + \frac{h^4}{384}u'''' - \dots$$

$$u_2 = u + \frac{h}{2}u' + \frac{h^2}{8}u'' + \frac{h^3}{48}u''' + \frac{h^4}{384}u'''' - \dots$$

$$u_1' = u' - \frac{h}{2}u'' + \frac{h^2}{8}u''' - \frac{h^3}{48}u'''' + \dots$$

$$u_2' = u' + \frac{h}{2}u'' + \frac{h^2}{8}u''' + \frac{h^3}{48}u'''' + \dots$$

Hence

$$u = \frac{1}{2}\left(2u + \frac{h^2}{4}u'' + \frac{h^4}{192}u'''' + \dots\right)$$

$$+ \frac{h}{8}\left(-hu'' - \frac{h^3}{24}u'''' - \dots\right) = u - \frac{h^4}{384}u'''' + \dots$$

i.e. error term of order h^4.

Similarly, error in u' is of order h^3.

$\boxed{9.7\text{-}1}$

Full:

$$\frac{4.562(0.1768)-7.124(0.3536)}{0.1768-0.3536} = 9.686$$

$$\frac{7.124(0.0884)-8.302(0.1768)}{0.0884-0.1768} = 9.480$$

Reduced:

$$\frac{11.440(0.1768)-9.323(0.3536)}{0.1768-0.3536} = 7.206$$

$$\frac{9.323(0.0884)-8.922(0.1768)}{0.0884-0.1768} = 8.521$$

Hourglass Controlled:

$$\frac{8.572(0.1768)-8.719(0.3536)}{0.1768-0.3536} = 8.866$$

$$\frac{8.719(0.0884)-8.771(0.1768)}{0.0884-0.1768} = 8.823$$

$\boxed{9.7-2}$

$a + bx = y$. Substitute data values and solve for a and b.

$$\begin{bmatrix} 1 & 1/\sqrt{52} \\ 1 & 1/\sqrt{94} \\ 1 & 1/\sqrt{130} \end{bmatrix} \begin{Bmatrix} a \\ b \end{Bmatrix} = \begin{Bmatrix} 51.37 \\ 61.15 \\ 67.18 \end{Bmatrix} \text{ or } [\underset{\sim}{Q}]\{\underset{\sim}{a}\} = \{\underset{\sim}{c}\}$$

Form $[\underset{\sim}{Q}]^T[\underset{\sim}{Q}]\{\underset{\sim}{a}\} = [\underset{\sim}{Q}]^T\{\underset{\sim}{c}\}$; solve for $\{\underset{\sim}{a}\}$.

$$\begin{bmatrix} 3 & 0.3295229756 \\ \text{symm.} & 0.0375613748 \end{bmatrix} \begin{Bmatrix} a \\ b \end{Bmatrix} = \begin{Bmatrix} 179.7000000 \\ 19.32295396 \end{Bmatrix}$$

we obtain $a = 93.3$, $b = -304.1$

$\boxed{9.7\text{-}3}$

$$h_1 = \frac{1}{\sqrt{52}} = 0.1387, \quad \sigma_1 = 51.37$$
$$h_2 = \frac{1}{\sqrt{94}} = 0.1031, \quad \sigma_2 = 61.15$$
$$h_3 = \frac{1}{\sqrt{130}} = 0.0877, \quad \sigma_3 = 67.18$$

Use Eq. 9.7-1:

$$\sigma_A = \frac{51.37(0.1031) - 61.15(0.1387)}{0.1031 - 0.1387} = 89.62$$

$$\sigma_B = \frac{61.15(0.0877) - 67.18(0.1031)}{0.0877 - 0.1031} = 101.52$$

$$\sigma_C = \frac{51.37(0.0877) - 67.18(0.1387)}{0.0877 - 0.1387} = 94.37$$

$$\frac{1}{3}(\sigma_A + \sigma_B + \sigma_C) = \frac{285.5}{3} = 95.17$$

$\boxed{9.7\text{-}4}$

$h_i = \dfrac{1}{N_i^{1/2}} \qquad h_1 = \dfrac{1}{10}, \; h_2 = \dfrac{1}{10\sqrt{2}}, \; h_3 = \dfrac{1}{20}$

By trial, we discover that convergence rate is $O(h^4)$, e.g.

$\dfrac{4.64 - 4.16}{h_1^4 - h_2^4} = 6400$ and $\dfrac{4.76 - 4.64}{h_2^4 - h_3^4} = 6400$

Eq. 9.7-1: $\phi^0 = \dfrac{4.76 h_2^4 - 4.64 h_3^4}{h_2^4 - h_3^4} = 4.80$

$\boxed{9.7\text{-}5}$

(a) Fig. 3.5-2: $e = O(h^2)$ for displacement, $e = O(h)$ for stress

$$v_A = \frac{0.859(1^2) - 0.961(2^2)}{1^2 - 2^2} = 0.995, \quad \sigma_{xB} = \frac{0.854(1) - 0.956(2)}{1-2} = 1.058$$

(b) CST results in Fig. 3.10-2: $e = O(h^2)$ for displacement

$$N=2,\ N=4: \quad v_A = \frac{0.502(1^2) - 0.765(2^2)}{1^2 - 2^2} = 0.853$$

$$N=4,\ N=8: \quad v_A = \frac{0.765(1^2) - 0.921(2^2)}{1^2 - 2^2} = 0.973$$

(c) Ref. 3.9 results in Fig. 3.10-2: $e = O(h^2)$ for displacement

$$N=2,\ N=4: \quad v_A = \frac{0.852(1^2) - 0.954(2^2)}{1^2 - 2^2} = 0.988$$

$$N=4,\ N=8: \quad v_A = \frac{0.954(1^2) - 0.989(2^2)}{1^2 - 2^2} = 1.001$$

(d) QM6 results in Fig. 6.6-1: $O(h^2)$ for disp., $O(h)$ for stress

$$N=2,\ N=4: \quad v_C = \frac{0.884(1^2) - 0.967(2^2)}{1^2 - 2^2} = 0.995$$

$$\sigma_A = \frac{0.840(1) - 0.978(2)}{1-2} = 1.116$$

$$\sigma_B = \frac{0.788(1) - 0.926(2)}{1-2} = 1.064$$

But if we assume $O(h^3)$ for disp. and $O(h^2)$ for stress, then

$$N=2,\ N=4: \quad v_C = \frac{0.884(1^3) - 0.967(2^3)}{1^3 - 2^3} = 0.979$$

$$\sigma_A = \frac{0.840(1^2) - 0.978(2^2)}{1^2 - 2^2} = 1.024$$

$$\sigma_B = \frac{0.788(1^2) - 0.962(2^2)}{1^2 - 2^2} = 1.020$$

(continues)

$\boxed{9.7\text{-}5}$ (concluded)

(e) Q4 results in Fig. 6.6-1: $O(h^2)$ for disp., $O(h)$ for stress

$N=2$, $N=4$: $$v_c = \frac{0.498(1^2)-0.769(2^2)}{1^2-2^2} = 0.859$$

$$\sigma_A = \frac{0.558(1)-0.830(2)}{1-2} = 1.102$$

$$\sigma_B = \frac{0.457(1)-0.753(2)}{1-2} = 1.049$$

$\boxed{9.7\text{-}6}$

Displacement: error is $O(h^3)$. Apply Eq. 9.7-1:

$$\frac{0.0035(1^3)-0.0041(2^3)}{1^3-2^3} = 0.00419$$

Estimated error of mesh 2:

$$\frac{0.00410-0.00419}{0.00419}100\% = -2.05\%$$

Stress: error is $O(h^2)$. Apply Eq. 9.7-1:

$$\frac{74.23(1^2)-89.03(2^2)}{1^2-2^2} = 93.96$$

Estimated error of mesh 2:

$$\frac{89.03-93.96}{93.96}100\% = -5.25\%$$

$\boxed{9.7\text{-}7}$

Refinement is not regular. Count the number of elements:

$N = 9$ in the coarse mesh

$N = 62$ in the finer mesh

Number of elements per side $\approx \sqrt{N}$. Thus

$N_{es} \approx 3$ coarse mesh

$N_{es} \approx 7.874$ finer mesh

Stress error is $O(h)$

Element side lengths $\approx$ proportional to $\frac{1}{N_{es}}$. Apply Eq. 9.7-1:

$$\phi_\infty = \frac{\phi_1 h_2^q - \phi_2 h_1^q}{h_2^q - h_1^q} = \frac{\phi_1 - \phi_2 (h_1/h_2)^q}{1 - (h_1/h_2)^q}$$

$$\sigma_E = \frac{2.68 - 3.16(7.874/3)^2}{1 - (7.874/3)^2} = 3.24$$

(if stress error is $O(h^2)$)

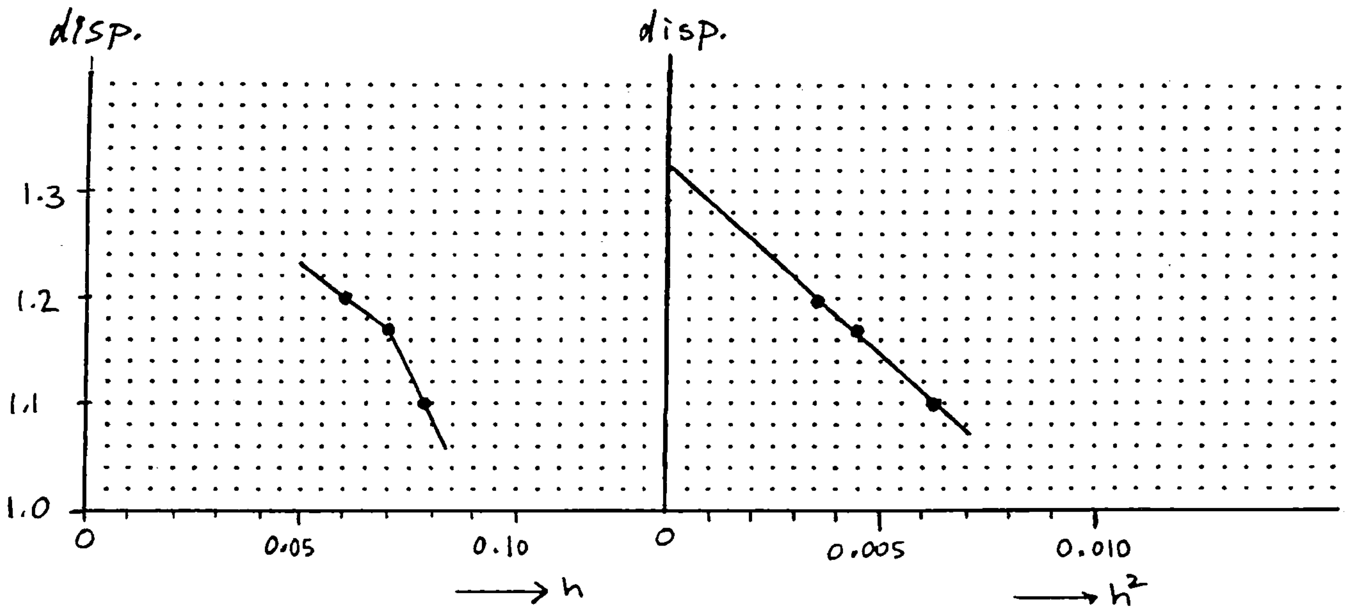
Est. error of finer mesh: $\frac{3.16 - 3.24}{3.24} 100\% = -2.5\%$

9.7-8

(a) No - regular refinement requires $h_{i+1} = h_i/m$, where m is an integer.

(b) With $h = \dfrac{1}{N^{1/3}}$,

mesh	no. of d.o.f., N	disp.	h	h^2
1	2014	1.10	0.0792	0.00627
2	3342	1.17	0.0669	0.00447
3	4560	1.20	0.0603	0.00364



Convergence appears to be $O(h^2)$. This is reasonable: eight-node bricks contain a complete linear polynomial for displacement, so quadratic convergence is expected.

(c) Extrapolated displacement:

$$\phi = \frac{0.00364(1.17) - 0.00447(1.20)}{0.00364 - 0.00447} = 1.332$$

Est. error of finest mesh:

$$e = \frac{1.20 - 1.332}{1.332} 100\% = -9.9\%$$

$\boxed{9.9\text{-}1}$ (a)

$$F_G = \sum \int (\sigma^* - \sigma)^2 \, dV \qquad \text{with} \quad \sigma^* = \mathbf{N}\, \boldsymbol{\sigma}_n^* \quad \text{yields}$$

$$F_G = \sum \int (\mathbf{N}\boldsymbol{\sigma}_n^* - \sigma)^T (\mathbf{N}\boldsymbol{\sigma}^* - \sigma)\, dV = \sum \int (\boldsymbol{\sigma}_n^T \mathbf{N}^T - \sigma)(\mathbf{N}\boldsymbol{\sigma}_n^* - \sigma)\, dV$$

$$F_G = \sum \int \left(\boldsymbol{\sigma}_n^{*T} \mathbf{N}^T \mathbf{N} \boldsymbol{\sigma}_n^* - \boldsymbol{\sigma}_n^{*T} \mathbf{N}^T \sigma - \sigma \mathbf{N} \boldsymbol{\sigma}_n^* + \sigma^2 \right) dV$$

$$F_G = \sum \int \left(\boldsymbol{\sigma}_n^{*T} \mathbf{N}^T \mathbf{N} \boldsymbol{\sigma}_n^* - 2 \boldsymbol{\sigma}_n^{*T} \mathbf{N}^T \sigma + \sigma^2 \right) dV$$

$$\frac{\partial F_G}{\partial \boldsymbol{\sigma}_n^*} = 0 = \sum \int \left(2\mathbf{N}^T \mathbf{N} \boldsymbol{\sigma}_n^* - 2\mathbf{N}^T \sigma \right) dV$$

$$\sum \int \mathbf{N}^T \mathbf{N}\, dV\, \boldsymbol{\sigma}_n^* = \sum \int \mathbf{N}^T \sigma\, dV$$

$$\left(\sum \int \mathbf{N}^T \mathbf{N}\, dV \right) \{\boldsymbol{\sigma}_n^*\}_G = \sum \int \mathbf{N}^T \sigma\, dV$$

(b) $$F_P = \sum (\sigma^* - \sigma)^2 \qquad \text{with} \quad \sigma^* = \mathbf{P}\mathbf{a} \quad \text{yields}$$

$$F_P = \sum (\mathbf{P}\mathbf{a} - \sigma)^T (\mathbf{P}\mathbf{a} - \sigma) = \sum (\mathbf{a}^T \mathbf{P}^T - \sigma)(\mathbf{P}\mathbf{a} - \sigma)$$

$$F_P = \sum \left(\mathbf{a}^T \mathbf{P}^T \mathbf{P} \mathbf{a} - \mathbf{a}^T \mathbf{P}^T \sigma - \sigma \mathbf{P} \mathbf{a} + \sigma^2 \right)$$

$$F_P = \sum \left(\mathbf{a}^T \mathbf{P}^T \mathbf{P} \mathbf{a} - 2\mathbf{a}^T \mathbf{P}^T \sigma + \sigma^2 \right)$$

$$\frac{\partial F_P}{\partial \mathbf{a}} = 0 = \sum \left(2\mathbf{P}^T \mathbf{P} \mathbf{a} - 2\mathbf{P}^T \sigma \right)$$

$$\sum [\mathbf{P}^T \mathbf{P}]\, \mathbf{a} = \sum \mathbf{P}^T \sigma$$

$\boxed{9.9\text{-}2}$

(a) $[\underset{\sim}{A}] = [\underset{\sim}{P}]_A^T [\underset{\sim}{P}]_A + [\underset{\sim}{P}]_B^T [\underset{\sim}{P}]_B$ with $\sigma = \lfloor \underset{\sim}{P} \rfloor \{\underset{\sim}{a}\} = \lfloor 1 \quad x \rfloor \begin{Bmatrix} a_0 \\ a_1 \end{Bmatrix}$

$$[\underset{\sim}{A}] = \begin{Bmatrix} 1 \\ L/2 \end{Bmatrix} \lfloor 1 \quad L/2 \rfloor + \begin{Bmatrix} 1 \\ 3L/2 \end{Bmatrix} \lfloor 1 \quad 3L/2 \rfloor$$

$$[\underset{\sim}{A}] = \begin{bmatrix} 1 & L/2 \\ L/2 & L^2/4 \end{bmatrix} + \begin{bmatrix} 1 & 3L/2 \\ 3L/2 & 9L^2/4 \end{bmatrix} = \begin{bmatrix} 2 & 2L \\ 2L & 2.5L^2 \end{bmatrix}$$

$$\{\underset{\sim}{b}\} = [\underset{\sim}{P}]_A^T \sigma_{xA} + [\underset{\sim}{P}]_B^T \sigma_{xB} = \begin{Bmatrix} 1 \\ L/2 \end{Bmatrix} (1) + \begin{Bmatrix} 1 \\ 3L/2 \end{Bmatrix} (3) = \begin{Bmatrix} 4 \\ 7L/2 \end{Bmatrix}$$

(b) $\begin{bmatrix} 2 & 2L \\ 2L & 2.5L^2 \end{bmatrix} \begin{Bmatrix} a_0 \\ a_1 \end{Bmatrix} = \begin{Bmatrix} 4 \\ 7L/2 \end{Bmatrix}$

$$\begin{Bmatrix} a_0 \\ a_1 \end{Bmatrix} = \frac{1}{L^2} \begin{bmatrix} 2.5L^2 & -2L \\ -2L & 2 \end{bmatrix} \begin{Bmatrix} 4 \\ 7L/2 \end{Bmatrix} = \frac{1}{L^2} \begin{Bmatrix} 3L^2 \\ -L \end{Bmatrix} = \begin{Bmatrix} 3 \\ -1/L \end{Bmatrix}$$

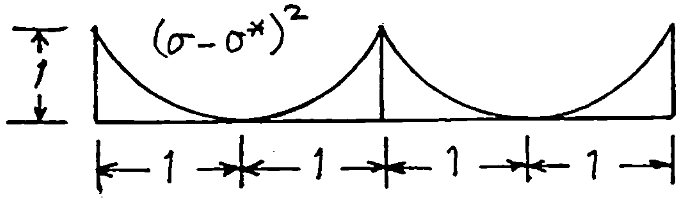
At $x = L/2$, $\sigma = \lfloor 1 \quad L/2 \rfloor \begin{Bmatrix} 3 \\ -1/L \end{Bmatrix} = 2$ ✓

$\boxed{9.10-1}$

For convenience, assume $E=1$.

$$U^2 = 2^2(2) + 4^2(2) = 40$$

The difference between average stress and element stress is represented by four triangles, each of span 1 and altitude (stress) 1. The plot of stress difference squared therefore consists of four parabolas:



Area under each is $\frac{1}{3}$

$$\int (\sigma-\sigma^*)^2 dx = 4\left(\tfrac{1}{3}\right) = 1.333 = e^2$$

$$U^2 + e^2 = 40 + 1.333 = 41.33$$

close!

From the average stress plot,

$$(U^*)^2 = \int_0^4 (1+x)^2 dx = \left[\tfrac{1}{3}(1+x)^3\right]_0^4 = \frac{125}{3} = 41.67$$

$$\eta = \left[\frac{1.333}{41.33}\right]^{1/2} = 0.18$$

$\boxed{9.10\text{-}2}$

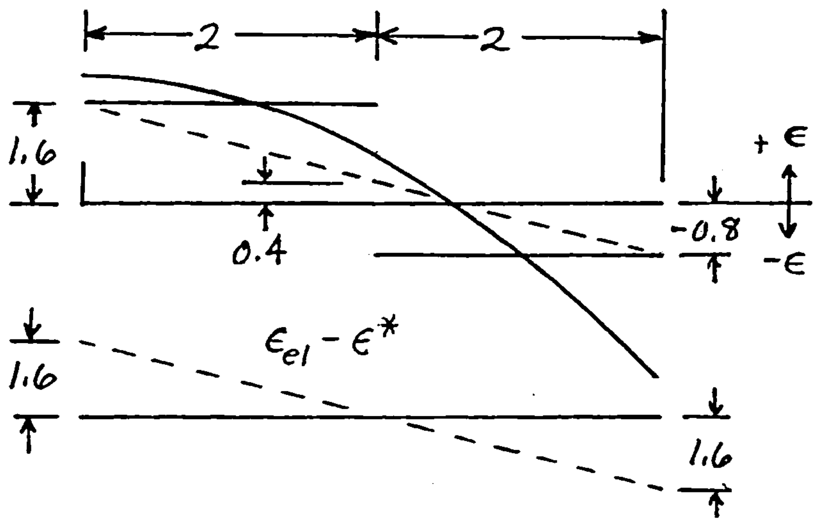
(a) $u = 2x - 0.1x^3$

$\epsilon_x = 2 - 0.3x^2$

A and E do not matter because the bar is uniform.

$$\epsilon_{el} = \frac{u_{i+1} - u_i}{L_i} = \frac{u_{i+1} - u_i}{2}$$

x	u	ϵ_{el}	ϵ^* (nodal ave.)
0	0		1.6
		1.6	
2	3.2		0.4
		-0.8	
4	1.6		-0.8



$\|U\|^2 = 2(1.6)^2 + 2(-0.8)^2 = 6.4 \qquad \|U\| = 2.530$

$\|e\|^2 = \frac{1}{3}2(1.2)^2 + \frac{1}{3}2(1.2)^2 = 1.92 \qquad \|e\| = 1.386$

Exact U: $\int_0^4 \epsilon_x^2\, dx = \int_0^4 (4 - 1.2x^2 + 0.09x^4)\, dx$

$= \left(4x - 0.4x^3 + \frac{0.09}{5}x^5\right)_0^4 = 8.832$

which compares with $\|U\|^2 + \|e\|^2 = 6.4 + 1.92 = 8.32$

$$\eta = \left[\frac{1.92}{8.32}\right]^{1/2} = 0.480$$

(b) Four elements, $L=1$ for each.

x	u	ϵ_{el}	ϵ^* (nodal ave.)
0	0		1.9
		1.9	
1	1.9		1.6
		1.3	
2	3.2		0.7
		0.1	
3	3.3		-0.8
		-1.7	
4	1.6		-1.7

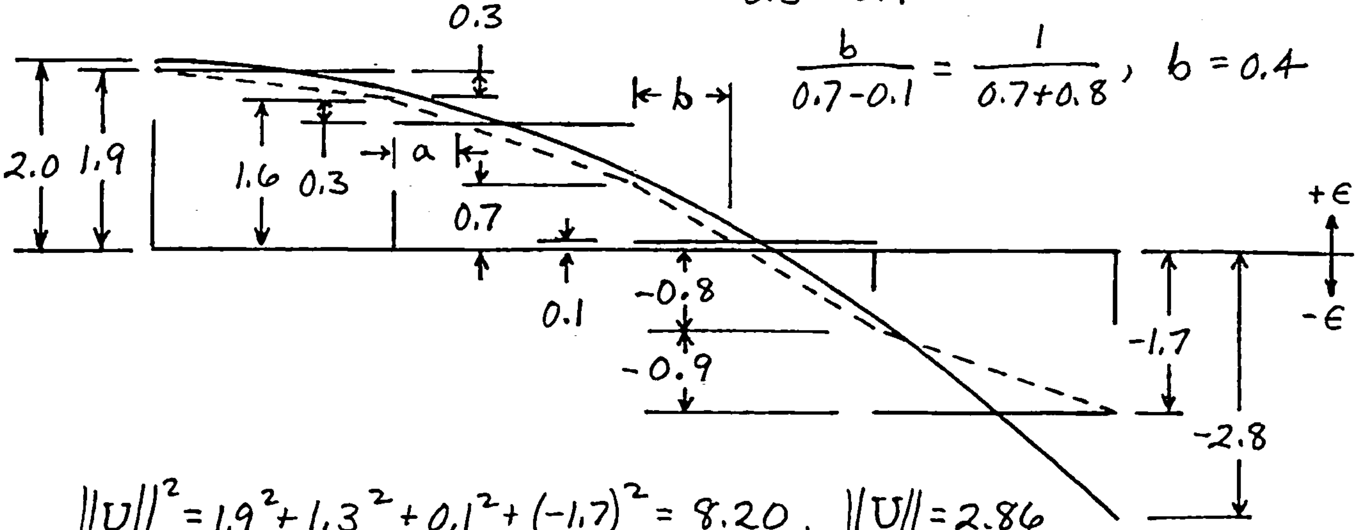
(continues)

9.10-2 (concluded)

$L = 1$ for each element

$\frac{a}{0.3} = \frac{1}{0.9}$, $a = \frac{1}{3}$

$\frac{b}{0.7-0.1} = \frac{1}{0.7+0.8}$, $b = 0.4$



$\|U\|^2 = 1.9^2 + 1.3^2 + 0.1^2 + (-1.7)^2 = 8.20$, $\|U\| = 2.86$

For $\|e\|^2$, we square a number of triangles, obtaining parabolas, whose area is (base)(height)/3, where here (base) $= 1, a, 1-a, b, 1-b, 1$.

$$\|e\|^2 = \frac{1}{3}\left[0.3^2 + 0.3^2\left(\frac{1}{3}\right) + 0.6^2\left(\frac{2}{3}\right) + 0.6^2(0.4) + 0.9^2(0.6) + 0.9^2\right] = 0.60$$

$\|e\| = 0.775$

$\|U\|^2 + \|e\|^2 = 8.20 + 0.60 = 8.80$

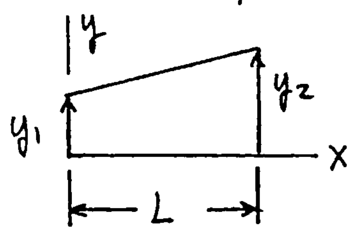
(close to exact $U = 8.832$ on previous page)

$$\eta = \left[\frac{0.60}{8.80}\right]^{1/2} = 0.261$$

(c) $\|e\|$ is roughly halved, so convergence rate of strains and stresses appears to be about $O(h)$, as should be expected.

However the first mesh is so coarse that the true rate may not yet have appeared.

$\boxed{9.10\text{-}3}$ For a trapezoid,



$$\int_0^L y^2\,dx = \frac{L}{3}\left(y_1^2 + y_1 y_2 + y_2^2\right)$$

For a triangle, say $y_2 = 0$, $\int_0^L y^2\,dx = \frac{L}{3} y_1^2$

For the coarse mesh, Prob. 9.10-2a,

$$\|U^*\|^2 = \frac{2}{3}\left[1.6^2 + 1.6(0.4) + 0.4^2\right] + \frac{2/3}{3}0.4^2 + \frac{4/3}{3}(-0.8)^2 = 2.56$$

$$\|U^*\| = 1.60$$

For the finer mesh, Prob. 9.10-2b,

$$\|U^*\|^2 = \frac{1}{3}\left[1.9^2 + 1.9(1.6) + 1.6^2\right] + \frac{1}{3}\left[1.6^2 + 1.6(0.7) + 0.7^2\right]$$

$$+ \frac{0.4}{3}0.7^2 + \frac{0.6}{3}(-0.8)^2 + \frac{1}{3}\left[(-0.8)^2 + (-0.8)(-1.7) + (-1.7)^2\right]$$

$$= 6.283$$

$$\|U^*\| = 2.51$$

Values of $\|U\|^2 + \|e\|^2$, from Prob. 9.10-2:

part (a), 8.32
part (b), 8.80

These values do not agree well with foregoing values of $\|U^*\|^2$ because smoothing by linear interpolation from nodal average values is not very accurate.