Instructor's Solutions Manual

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to accompany

Elementary Differential Equations with Boundary Value Problems

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Chapter 1
Introduction to Differential Equations

Section 1.1

1. This D.E. is of order two because the highest derivative in the equation is y''.

2. Order is 1.

3. This D.E. is of order one because the highest derivative in the equation is y'. (Note: $(y')^3 \neq y'''$)

4. Order is 3.

5 (a). $y = Ce^{t^2}$. Differentiating gives us $y' = Ce^{t^2} \cdot 2t = 2ty$. Therefore, $y' - 2ty = 0$ for any value of C.

5 (b). Substituting into the differential equation yields $y(1) = Ce^{1^2} = Ce$. Using the initial condition, $y(1) = 2 = Ce$. Solving for C, we find $C = 2e^{-1}$.

6. $y''' = 2.\quad y'' = 2t + c_1,\ y^1 = t^2 + c_1 t + c_2,\ y = \dfrac{t^3}{3} + c_1 \dfrac{t^2}{2} + c_2 t + c_3.$

   Order = 3 3 arbitrary constants

7 (a). $y = C_1 \sin 2t + C_2 \cos 2t$. Differentiating gives us $y' = 2C_1 \cos 2t - 2C_2 \sin 2t$ and $y'' = -4C_1 \sin 2t - 4C_2 \cos 2t = -4(C_1 \sin 2t + C_2 \cos 2t) = -4y$. Therefore, $y'' + 4y = -4y + 4y = 0$ and thus $y(t) = C_1 \sin 2t + C_2 \cos 2t$ is a solution of the D.E. $y'' + 4y = 0$.

7 (b). $y\left(\frac{\pi}{4}\right) = C_1(1) + C_2(0) = C_1 = 3$ and $y'\left(\frac{\pi}{4}\right) = 2C_1(0) - 2C_2(1) = -2C_2 = -2 \Rightarrow C_2 = 1$.

8. $y = 2e^{-4t}.\quad y' + ky = -8e^{-4t} + 2ke^{-4t} = 2(k-4)e^{-4t} = 0$

   $\therefore\ k = 4.\quad y(0) = 2 = y_0.\quad \therefore\ k = 4, y_0 = 2.$

9. $y = ct^{-1}$. Differentiating gives us $y' = -ct^{-2}$. Thus $y' + y^2 = -ct^{-2} + c^2 t^{-2} = (c^2 - c)t^{-2} = 0$. Solving this for c, we find that $c^2 - c = c(c-1) = 0$. Therefore, $c = 0,1$.

10. $y = -e^{-t} + \sin t \qquad y' + y = g(t),\ y(0) = y_0. \qquad y' = e^{-t} + \cos t$

    $y' + y = e^{-t} + \cos t - e^{-t} + \sin t = g \quad \therefore \quad g(t) = \cos t + \sin t,\ y(0) = -1 = y_0$

11. $y = t^r$. Differentiating gives us $y' = rt^{r-1}$ and $y'' = r(r-1)t^{r-2}$. Thus $t^2 y'' - 2ty' + 2y = r(r-1)t^r - 2rt^r + 2t^r = [r(r-1) - 2r + 2]t^r = 0$. Solving this for r, we find that $r(r-1) - 2r + 2 = r^2 - 3r + 2 = (r-2)(r-1) = 0$. Therefore, $r = 1,2$.

12. $y = c_1 e^{2t} + c_2 e^{-2t}$. $y' = 2c_1 e^{2t} - 2c_2 e^{-2t}$, $y'' = 4c_1 e^{2t} + 4c_2 e^{-2t} = 4y$
$\therefore\ y'' - 4y = 0$.

13. From (12), $y = C_1 e^{2t} + C_2 e^{-2t}$, which we differentiate to get $y' = 2C_1 e^{2t} - 2C_2 e^{-2t}$. Using the initial conditions, $y(0) = 2$ and $y'(0) = 0$, we have two equations containing C_1 and C_2: $C_1 + C_2 = 2$ and $2C_1 - 2C_2 = 0$. Solving these simultaneous equations gives us $C_1 = C_2 = 1$. Thus, the solution to the initial value problem is $y = e^{2t} + e^{-2t} = 2\cosh(2t)$.

14. $y(0) = c_1 + c_2 = 1,\ 2c_1 - 2c_2 = 2\ \ \therefore\ c_1 = 1,\ c_2 = 0\quad y(t) = e^{2t}$.

15. From (12), $y(t) = C_1 e^{2t} + C_2 e^{-2t}$. Using the initial condition $y(0) = 3$, we find that $C_1 + C_2 = 3$. From the initial condition $\lim_{t\to\infty} y(t) = 0$ and the equation for $y(t)$ given to us in (12), we can conclude that $C_1 = 0$ (if $C_1 \neq 0$, then $\lim_{t\to\infty} = \pm\infty$). Therefore, $C_2 = 3$ and $y(t) = 3e^{-2t}$.

16. $c_1 + c_2 = 10\quad \lim_{t\to-\infty} y(t) = 0\ \Rightarrow\ c_2 = 0\ \ \therefore\ c_1 = 10\ \&\ y(t) = 10e^{2t}$.

17. From the graph, we can see that $y' = -1$ and that $y(1) = 1$. Thus $m = y' - 1 = -1 - 1 = -2$ and $y_0 = y(1) = 1$.

18. $y' = mt\ \Rightarrow\ y = \frac{m}{2}t^2 + c$. From graph, $y = -1$ only at $t = 0\ \ \therefore\ t_0 = 0$.

Also $c = -1$. From graph $y(1) = -0.5\ \ \therefore\ -\frac{1}{2} = \frac{m}{2} - 1\ \Rightarrow\ m = 1$.

19. We know that this is a freefall problem, so we can begin with the generic equation for freefall situations: $y(t) = -\frac{g}{2}t^2 + v_0 t + y_0$. The object is released from rest, so $v_0 = 0$. The impact time corresponds to the time at which $y = 0$, so we are left with the following equation for the impact time t: $0 = -\frac{g}{2}t^2 + y_0$. Solving this for t yields $t = \sqrt{\frac{2y_0}{g}}$. For the velocity at the time of impact: $v = y' = -gt + v_0 = -gt = -\sqrt{2gy_0}$.

20. $x'' = a\quad x' = at + v_0,\ v_0 = x_0 = 0 \Rightarrow x = \frac{at^2}{2} + 0$.

$88 = a(8) \Rightarrow a = 11\ \ ft/\sec^2$. At $t = 8,\ x = 11\left(\frac{64}{2}\right) = 352\,ft$.

21. $a = y'' = 32 - \varepsilon \sin\left(\frac{\pi t}{4}\right)$. Integrating gives us $y' = -32t - \frac{4}{\pi}\varepsilon \cos\left(\frac{\pi t}{4}\right) + C$. The object is dropped from rest, so $y'(0) = 0 = -\frac{4}{\pi}\varepsilon + C$. Solving for C yields $C = \frac{4}{\pi}\varepsilon$, and putting this value back into the equation for y' and simplifying gives us $y' = -32t + \frac{4}{\pi}\varepsilon\left(1 - \cos\left(\frac{\pi t}{4}\right)\right)$. Integrating again gives us $y = -16t^2 + \frac{4}{\pi}\varepsilon t - \left(\frac{4}{\pi}\right)^2 \varepsilon \sin\left(\frac{\pi t}{4}\right) + C'$. Since the object is dropped from a height of 252 ft. (at $t = 0$), $y(0) = C' = 252$ and thus

$y = -16t^2 + \frac{4}{\pi}\varepsilon t - \left(\frac{4}{\pi}\right)^2 \varepsilon \sin\left(\frac{\pi t}{4}\right) + 252$. Finally, since $y(4) = 0$,

$y(4) = 0 = -16 \cdot 4^2 + \frac{4\varepsilon}{\pi} \cdot 4 - \left(\frac{4}{\pi}\right)^2 \varepsilon \sin(\pi) + 252$. Solving for ε yields $\varepsilon = \frac{\pi}{4}$.

Section 1.2

1 (a). The equation is autonomous because y' depends only on y.

1 (b). Setting $y' = 0$, we have $0 = -y + 1$. Solving this for y yields the equilibrium solution: $y = 1$.

2 (a). not autonomous

2 (b). no equilibrium solutions, isoclines are $t = \text{constant}$.

3 (a). The equation is autonomous because y' depends only on y.

3 (b). Setting $y' = 0$, we have $0 = \sin y$. Solving this for y yields the equilibrium solutions: $y = \pm n\pi$.

4 (a). autonomous

4 (b). $y(y-1) = 0, \quad y = 0,\ 1$.

5 (a). The equation is autonomous because y' does not depend explicitly on t.

5 (b). There are no equilibrium solutions because there are no points at which $y' = 0$.

6 (a). not autonomous

6 (b). $y = 0$ is equilibrium solution, isoclines are hyperbolas.

7 (a). $c = -1$: Setting $c = -1$ gives us $-y + 1 = -1$ which, solved for y, reads $y = 2$. This is the isocline for $c = -1$.

$c = 0$: Setting $c = 0$ gives us $-y + 1 = 0$ which, solved for y, reads $y = 1$. This is the isocline for $c = 0$.

$c = 1$: Setting $c = 1$ gives us $-y + 1 = 1$ which, solved for y, reads $y = 0$, the isocline for $c = 1$.

8 (a). $-y+t=-1 \Rightarrow y=t+1$

$-y+t=0 \Rightarrow y=t$

$-y+t=1 \Rightarrow y=t-1$

9 (a). $c=-1$: Setting $c=-1$ gives us $y^2-t^2=-1$ which can be simplified to $t^2-y^2=1$ (a hyperbola). This is the isocline for $c=-1$.

$c=0$: Setting $c=0$ gives us $y^2-t^2=0$ which can be simplified to $y=\pm t$. This is the isocline for $c=0$.

$c=1$: Setting $c=1$ gives us $y^2-t^2=1$ (a hyperbola). This is the isocline for $c=1$.

10. $f(0)=f(2)=0 \quad y'=y(2-y)$

$y'>0$ for $0<y<2$, $y'<0$ for $-\infty<y<0$ and $2<y<\infty$.

11. One example that would fit these criteria is $y'=-(y-1)^2$. For this autonomous D.E., $y'=0$ at $y=1$ and $y'<0$ for $-\infty<y<1$ and $1<y<\infty$.

12. $y'=1$.

13. One example that would fit these criteria is $y'=\sin(2\pi y)$. For this autonomous D.E., $y'=0$ at $y=\dfrac{n}{2}$.

14. c.

15. f.

16. a.

17. b.

18. d.

19. e.

Chapter 2
First Order Linear Differential Equations

Section 2.1

1. This equation is linear because it can be written in the form $y' + p(t)y = g(t)$. It is nonhomogeneous because when it is put in this form, $g(t) \neq 0$.
2. nonlinear
3. This equation is nonlinear because it cannot be written in the form $y' + p(t)y = g(t)$.
4. nonlinear
5. This equation is nonlinear because it cannot be written in the form $y' + p(t)y = g(t)$.
6. linear, homogeneous
7. This equation is nonlinear because it can be written in the form $y' + p(t)y = g(t)$.
8. nonlinear
9. This equation is linear because it cannot be written in the form $y' + p(t)y = g(t)$. It is nonhomogeneous because when it is put in this form, $g(t) \neq 0$.
10. linear, homogeneous

11 (a). Theorem 2.1 guarantees a unique solution for the interval $(-\infty,\infty)$, since $\frac{t}{t^2+1}$ and $\sin(t)$ are both continuous for all t and -2 is on this interval.

11 (b). Theorem 2.1 guarantees a unique solution for the interval $(-\infty,\infty)$, since $\frac{t}{t^2+1}$ and $\sin(t)$ are both continuous for all t and 0 is on this interval.

11 (c). Theorem 2.1 guarantees a unique solution for the interval $(-\infty,\infty)$, since $\frac{t}{t^2+1}$ and $\sin(t)$ are both continuous for all t and π is on this interval.

12 (a). $2 < t < \infty$

12 (b). $-2 < t < 2$

12 (c). $-2 < t < 2$

12 (d). $-\infty < t < -2$

13 (a). For this equation, $p(t)$ is continuous for all $t \neq 2,-2$ and $g(t)$ is continuous for all $t \neq 3$. Therefore, Theorem 2.1 guarantees a unique solution for $(3,\infty)$, the largest interval that includes $t = 5$.

13 (b). For this equation, $p(t)$ is continuous for all $t \neq 2,-2$ and $g(t)$ is continuous for all $t \neq 3$. Therefore, Theorem 2.1 guarantees a unique solution for $(-2,2)$, the largest interval that includes $t = -\frac{3}{2}$.

13 (c). For this equation, $p(t)$ is continuous for all $t \neq 2,-2$ and $g(t)$ is continuous for all $t \neq 3$. Therefore, Theorem 2.1 guarantees a unique solution for $(-2,2)$, the largest interval that includes $t = 0$.

13 (d). For this equation, $p(t)$ is continuous for all $t \neq 2,-2$ and $g(t)$ is continuous for all $t \neq 3$. Therefore, Theorem 2.1 guarantees a unique solution for $(-\infty,-2)$, the largest interval that includes $t = -5$.

13 (e). For this equation, $p(t)$ is continuous for all $t \neq 2,-2$ and $g(t)$ is continuous for all $t \neq 3$. Therefore, Theorem 2.1 guarantees a unique solution for $(-2,2)$, the largest interval that includes $t = \frac{3}{2}$.

14. $\dfrac{\ln\left|t + t^{-1}\right|}{t-2} = \dfrac{\ln\left|\frac{t^2+1}{t}\right|}{t-2}$ undefined at $t = 0,2$.

14 (a). $2 < t < \infty$.

14 (b). $0 < t < 2$.

14 (c). $-\infty < t < 0$.

14 (d). $-\infty < t < 0$.

15. $y(t) = 3e^{t^2}$. Differentiating gives us $y' = 3e^{t^2}(2t) = 2ty$. Substituting these values into the given equation yields $2ty + p(t)y = 0$. Solving this for $p(t)$, we find that $p(t) = -2t$. Putting $t = 0$ into the equation for y gives us $y_0 = 3$.

16(a). $y = Ct^r \qquad y' = Crt^{r-1} \qquad 2ty' - 6y = 0$

$\therefore\ 2Crt^r - 6Ct^r = (2r-6)Ct^r = 0 \ \Rightarrow\ (2r-6)y = 0 \ \Rightarrow\ 2r - 6 = 0 \ \Rightarrow\ r = 3$

$y(-2) = C(-2)^r = 8 \ \Rightarrow\ C \neq 0 \quad \therefore\ C(-2)^3 = 8 \Rightarrow C = -1$

16 (b). $-\infty < t < 0$ since $p(t) = \dfrac{-3}{t}$

16 (c). $y(t) = -t^3,\ -\infty < t < \infty$.

17. $y(t)=0$ satisfies all of these conditions.

Section 2.2

1 (a). First, we will integrate $p(t)=3$ to find $P(t)=3t$. The general solution, then, is $y(t)=Ce^{-P(t)}=Ce^{-3t}$.

1 (b). $y(0)=C=-3$. Therefore, the solution to the initial value problem is $y=-3e^{-3t}$.

2 (a). $y'-\frac{1}{2}y=0 \qquad (e^{-t/2}y)'=0,\ y=Ce^{t/2}$.

2 (b). $y(-1)=Ce^{-1/2}=2,\ C=2e^{1/2} \qquad y(t)=2e^{(t+1)/2}$

3 (a). We can rewrite this equation into the conventional form: $y'-2ty=0$. Then we will integrate $p(t)=-2t$ to find $P(t)=-t^2$. The general solution, then, is $y(t)=Ce^{-P(t)}=Ce^{t^2}$.

3 (b). $y(1)=Ce=3$. Solving for C yields $C=3e^{-1}$. Therefore, the solution to the initial value problem is $y(t)=3e^{-1}e^{t^2}=3e^{(t^2-1)}$.

4 (a). $ty'-4y=0 \Rightarrow y'-\frac{4}{t}y=0. \qquad \int -\frac{4}{t}dt=-4\ln|t|=-\ln(t^4) \quad \therefore \quad \mu=\frac{1}{t^4}$

$\frac{1}{t^4}y'-\frac{4}{t^5}y=(t^{-4}y)'=0 \qquad y=Ct^4$.

4 (b). $y(1)=C=1 \quad \therefore \quad y(t)=t^4$.

5 (a). We can rewrite this equation into the conventional form: $y'+\frac{4}{t}y=0$. Then we will integrate $p(t)=\frac{4}{t}$ to find $P(t)=4\ln|t|=\ln t^4$. The general solution, then, is $y(t)=Ce^{-P(t)}=Ce^{-\ln t^4}=Ce^{\ln t^{-4}}=Ct^{-4}$.

5 (b). $y(1)=C=1$. Therefore, the solution to the initial value problem is $y(t)=t^{-4}$.

6 (a). $\mu=\exp(t-\cos t) \quad \therefore \quad y(t)=Ce^{-(t-\cos t)}$.

6 (b). $y\left(\frac{\pi}{2}\right)=Ce^{-\pi/2}=1 \qquad C=e^{\pi/2} \qquad y=e^{\pi/2}e^{-(t-\cos t)}=e^{\pi/2-t+\cos t}$.

7 (a). First, we will integrate $p(t)=-2\cos(2t)$ to find $P(t)=-\sin(2t)$. The general solution, then, is $y(t)=Ce^{-P(t)}=Ce^{\sin(2t)}$.

7 (b). $y(\pi)=C=-2$. Therefore, the solution to the initial value problem is $y(t)=-2e^{\sin(2t)}$.

8 (a). $((t^2+1)y)'=0 \qquad y=\frac{C}{t^2+1}$.

8 (b). $y(0) = C = 3 \quad \therefore \quad y(t) = \dfrac{3}{t^2+1}$.

9 (a). We can rewrite this equation into the conventional form: $y' - 3(t^2+1)y = 0$. Then we will integrate $p(t) = -3(t^2+1)$ to find $P(t) = -t^3 - 3t$. The general solution, then, is $y(t) = Ce^{-P(t)} = Ce^{t^3+3t}$.

9 (b). $y(1) = Ce^4 = 4$. Solving for C yields $C = 4e^{-4}$. Therefore, the solution to the initial value problem is $y(t) = 4e^{t^3+3t-4}$.

10 (a). $y' + e^{-t}y = 0 \quad \therefore \quad \int e^{-t}dt = -e^{-t} \qquad (-e^{e^{-t}}y)' = 0 \qquad y = Ce^{e^{-t}}$.

10 (b). $y(0) = Ce^1 = 2 \qquad C = 2e^{-1} \qquad y(t) = 2e^{e^{-t}-1}$.

11 (a). #2

11 (b). #3

11 (c). #1

12. $y(t) = y_0 e^{-\alpha t} \qquad 4 = y_0 e^{-\alpha},\ 1 = y_0 e^{-3\alpha}$ Divide: $4 = e^{2\alpha} \ \Rightarrow\ \alpha = \frac{1}{2}\ln 4 = \ln 2$ and $y_0 = e^{3\alpha} = e^{\frac{3}{2}\ln 4} = e^{\ln(8)} = 8. \quad \therefore \quad y(t) = 8e^{-(\ln 2)t}$.

13. First, we should put the equation into our conventional form: $y' - \dfrac{\alpha}{t}y = 0$. Integrating $p(t) = -\dfrac{\alpha}{t}$ gives us $P(t) = -\alpha\ln|t| = \ln|t^{-\alpha}|$. The general solution, then, is $y(t) = Ce^{-P(t)} = Ce^{-\ln|t^{-\alpha}|} = Ce^{\ln|t^{\alpha}|} = Ct^{\alpha}$. Using the general solution and the point $(2,1)$, we can solve for C in terms of α: $y(2) = 1 = C\cdot 2^{\alpha}$; $C = 2^{-\alpha}$. We can then substitute this value for C into the general solution at the point $(4,4)$: $y(4) = 4 = 2^{-\alpha}\cdot 4^{\alpha} = 4^{-\alpha/2}\cdot 4^{\alpha} = 4^{\alpha/2}$. Setting the exponents equal to each other yields $1 = \dfrac{\alpha}{2}; \alpha = 2$. Finally, solving for y_0, $y_0 = y(1) = 2^{-2}\cdot 1^2 = \dfrac{1}{4}$.

14. $z' = 2z,\ z = y+2 \quad \therefore \quad z(0) = -1+2 = 1 \ \Rightarrow\ z = e^{2t} = y+2 \quad \therefore \quad y = -2 + e^{2t}$

15. Putting this equation into a form more like #14, we have $y' = -2ty + 6t = -2t(y-3)$. We will then let $z = y - 3$ (and $z' = y'$, accordingly). Substituting into our modified original equation yields an equation for $z(t)$: $z' = -2tz$, or put in a more conventional form, $z' + 2tz = 0$. Using the same substitution for the initial condition yields $z(0) = 4 - 3 = 1$. Integrating $p(t) = 2t$ gives us $P(t) = t^2$. The general solution is then $z(t) = Ce^{-t^2}$. Our initial condition requires that $C = 1$,

so the solution for $z(t)$ is $z(t) = e^{-t^2}$. In terms of $y(t)$, this solution reads $y - 3 = e^{-t^2}$. Solved for $y(t)$, this solution is $y(t) = e^{-t^2} + 3$.

16 (a). $\dfrac{dB}{dc} = -kB,\ B(0) = -A^*$

16 (b). $B(c) = -A^* e^{-kc} = A(c) - A^* \quad \therefore \quad A(c) = A^*(1 - e^{-kc})$ No. $A(c) \uparrow A^*$ as $c \uparrow \infty$

16 (c). $0.95A^* = A^*(1 - e^{-kc}) \;\Rightarrow\; -0.05 = -e^{-kc} \;\Rightarrow\; -kc = \ln(1/20) = -\ln(20)$

$\therefore \; c_{0.95} = \dfrac{1}{k}\ln(20)$.

17. Solving the equation $y' + cy = 0$ with our method yields the general solution $y(t) = y_0 e^{-ct}$. Looking at the graph, we can see that $y(0) = 2 = y_0$ and $y(-0.4) = 3 = y_0 e^{-c(-0.4)} = 2e^{0.4c}$. Solving for c gives us $c = \dfrac{5}{2}\ln\left(\dfrac{3}{2}\right) \approx 1.01$.

18. $y' = Ce^{-ct} \qquad y(1) = Ce^{-c} = y_0 \;\Rightarrow\; C = y_0 e^{c} \quad \therefore \quad y = y_0 e^{-c(t-1)}$

$y(1) = y_0 = -1 \qquad y(0.3) \approx -\dfrac{1}{2} \quad \therefore \quad -\dfrac{1}{2} = -e^{-c(-0.7)} = 0.7c \approx \ln\left(\dfrac{1}{2}\right)$

$c \approx -\dfrac{1}{0.7}\ln(2) = -0.990 \quad \therefore \quad c = -1$.

19 (a). The general solution to this D.E. is $y(t) = y_0 e^{-t}$, which can be rewritten as $\ln(y) = -t + c$. Thus, this D.E. corresponds to graph #2 with $y_0 = y(0) = e^{\ln(y(0))} = e^2$.

19 (b). The general solution to this D.E. is $y(t) = y_0 e^{t\sin 4t}$, which can be rewritten as $\ln(y) = t\sin 4t + c$. Thus, this D.E. corresponds to graph #1 with $y_0 = y(0) = e^{\ln(y(0))} = 1$.

19 (c). The general solution to this D.E. is $y(t) = y_0 e^{-t^2/2}$, which can be rewritten as $\ln(y) = -\dfrac{t^2}{2} + c$. Thus, this D.E. corresponds to graph #4 with $y_0 = y(0) = e^{\ln(y(0))} = e$.

19 (d). The general solution to this D.E. is $y(t) = y_0 e^{t - \sin 4t}$, which can be rewritten as $\ln(y) = t - \sin 4t + c$. Thus, this D.E. corresponds to graph #3 with $y_0 = y(0) = e^{\ln(y(0))} = 1$.

20. $\ln y(t) = \dfrac{3-1}{4-0}t + 1 = \dfrac{t}{2} + 1 \quad \therefore \quad p(t) = \dfrac{d}{dt}\ \ln(y(t)) = \dfrac{1}{2} \qquad y_0 = e$.

21 (a). Integrating $p(t) = t^n$ gives us $P(t) = \dfrac{t^{n+1}}{n+1}$. Thus the solution to this initial value problem is $y(t) = y_0 e^{-t^{n+1}/n+1}$ which can be rewritten as $\ln y = \ln y_0 - \dfrac{t^{n+1}}{n+1}$.

Substituting values from the table gives us the necessary equations to solve for y_0 and n. First,

$-\frac{1}{4} = \ln y_0 - \frac{1}{n+1}$ and $-4 = \ln y_0 - \frac{2^{n+1}}{n+1}$ can be combined to solve for n:

$4 - \frac{1}{4} = \frac{15}{4} = \frac{2^{n+1}-1}{n+1}$, so $n = 3$. $-\frac{1}{4} = \ln y_0 - \frac{1}{4}$ by substitution, and therefore $y_0 = 1$.

21 (b). $y(t) = y_0 e^{-t^{n+1}/_{n+1}} = 1 \cdot e^{-t^4/_4} \Rightarrow y(-1) = e^{-\frac{1}{4}}$.

Section 2.3

1. For this D.E., $p(t) = 2$. Integrating gives us $P(t) = 2t$. An integrating factor is, then, $\mu(t) = e^{2t}$. Multiplying the D.E. by $\mu(t)$, we obtain $e^{2t}y' + 2e^{2t}y = (e^{2t}y)' = e^{2t}$. Integrating both sides yields $e^{2t}y = \frac{1}{2}e^{2t} + C$. Therefore, the general solution is $y(t) = \frac{1}{2} + Ce^{-2t}$.

2. $y' + 2y = e^{-t} \;\Rightarrow\; (e^{2t}y)' = e^{t} \;\Rightarrow\; e^{2t}y = e^{t} + C \;\Rightarrow\; y = e^{-t} + Ce^{-2t}$.

3. For this D.E., $p(t) = 2$. Integrating gives us $P(t) = 2t$. An integrating factor is, then, $\mu(t) = e^{2t}$. Multiplying the D.E. by $\mu(t)$, we obtain $e^{2t}y' + 2e^{2t}y = (e^{2t}y)' = 1$. Integrating both sides yields $e^{2t}y = t + C$. Therefore, the general solution is $y(t) = te^{-2t} + Ce^{-2t}$.

4. $y' + 2ty = t \;\Rightarrow\; (e^{t^2}y)' = te^{t^2} \;\Rightarrow\; e^{t^2}y = \frac{1}{2}e^{t^2} + C \;\Rightarrow\; y = \frac{1}{2} + Ce^{-t^2}$.

5. Putting this equation into the conventional form gives us $y' + \frac{2}{t}y = t$. For this D.E., $p(t) = \frac{2}{t}$. Integrating gives us $P(t) = 2\ln t$. An integrating factor is, then, $\mu(t) = e^{\ln t^2} = t^2$. Multiplying the D.E. by $\mu(t)$, we obtain $t^2y' + 2ty = (t^2y)' = t^3$. Integrating both sides yields $t^2y = \frac{1}{4}t^4 + C$. Therefore, the general solution is $y(t) = \frac{1}{4}t^2 + Ct^{-2}$.

6. $(t^2+4)y' + 2ty = t^2(t^2+4) \;\Rightarrow\; y' + \frac{2t}{t^2+4}y = t^2,\; \mu = e^{\ln(t^2+4)} = t^2 + 4$

$\therefore\; ((t^2+4)y)' = t^2(t^2+4) = t^4 + 4t^2 \;\Rightarrow\; (t^2+4)y = \frac{t^5}{5} + \frac{4t^3}{3} + C \qquad y = \frac{t^5/_5 + 4t^3/_3 + C}{(t^2+4)}$.

7. For this D.E., $p(t) = 1$. Integrating gives us $P(t) = t$. An integrating factor is, then, $\mu(t) = e^{t}$. Multiplying the D.E. by $\mu(t)$, we obtain $e^{t}y' + e^{t}y = (e^{t}y)' = te^{t}$. Integrating both sides yields $e^{t}y = te^{t} - e^{t} + C$. Therefore, the general solution is $y(t) = t - 1 + Ce^{-t}$.

8. $y' + 2y = \cos 3t \;\Rightarrow\; (e^{2t}y)' = e^{2t}\cos 3t$

$u = e^{2t} \qquad dv = \cos 3t dt$

$du = 2e^{2t}dt \qquad v = \frac{1}{3}\sin 3t \qquad \int e^{2t}\cos 3t dt = \frac{e^{2t}}{3}\sin 3t - \frac{2}{3}\int e^{2t}\sin 3t dt$

$u = e^{2t} \qquad dv = \sin 3t dt$

$du = 2e^{2t}dt \qquad v = -\frac{1}{3}\cos 3t \qquad \int e^{2t}\sin 3t dt = -\frac{e^{2t}}{3}\cos 3t + \frac{2}{3}\int e^{2t}\cos 3t dt$

$\therefore\; I = \frac{e^{2t}}{3}\sin 3t - \frac{2}{3}\left\{-\frac{e^{2t}}{3}\cos 3t + \frac{2}{3}I\right\} \;\Rightarrow\; I(1+\frac{4}{9}) = \frac{e^{2t}}{3}(\sin 3t + 2\cos 3t)$

$\therefore\; I = \frac{3}{13}e^{2t}(\sin 3t + 2\cos 3t)$

$\therefore\; e^{2t}y = \frac{3}{13}e^{2t}(\sin 3t + 2\cos 3t) + C \;\Rightarrow\; y = \frac{3}{13}(\sin 3t + 2\cos 3t) + Ce^{-2t}$

9. For this D.E., $p(t) = -3$. Integrating gives us $P(t) = -3t$. An integrating factor is, then, $\mu(t) = e^{-3t}$. Multiplying the D.E. by $\mu(t)$, we obtain $e^{-3t}y' - 3e^{-3t}y = (e^{-3t}y)' = 6e^{-3t}$. Integrating both sides yields $e^{-3t}y = -2e^{-3t} + C$. Solving for y gives us $y = -2 + Ce^{3t}$, and with our initial condition, $y(0) = 1 = -2 + C$. Solving for C yields $C = 3$, and thus our final solution is $y = -2 + 3e^{3t}$.

10. $y' - 2y = e^{3t},\; y(0) = 3. \qquad (e^{-2t}y)' = e^{t} \;\Rightarrow\; e^{-2t}y = e^{t} + C \;\Rightarrow\; y = e^{3t} + Ce^{2t}$

$y(0) = 1 + C = 3 \;\Rightarrow\; C = 2,\; y = e^{3t} + 2e^{2t}$.

11. Putting this D.E. in the conventional form, we have $y' + \frac{3}{2}y = \frac{1}{2}e^{t}$. For this D.E., $p(t) = \frac{3}{2}$. Integrating gives us $P(t) = \frac{3}{2}t$. An integrating factor is, then, $\mu(t) = e^{\frac{3}{2}t}$. Multiplying the D.E. by $\mu(t)$, we obtain $e^{\frac{3}{2}t}y' + \frac{3}{2}e^{\frac{3}{2}t}y = (e^{\frac{3}{2}t}y)' = \frac{1}{2}e^{\frac{5}{2}t}$. Integrating both sides yields $e^{\frac{3}{2}t}y = \frac{1}{5}e^{\frac{5}{2}t} + C$. Solving for y gives us $y = \frac{1}{5}e^{t} + Ce^{-\frac{3}{2}t}$, and with our initial condition, $y(0) = 0 = \frac{1}{5} + C$. Solving for C yields $C = -\frac{1}{5}$, and thus our final solution is $y = \frac{1}{5}e^{t} - \frac{1}{5}e^{-\frac{3}{2}t}$.

12. $y' + y = 1 + 2e^{-t}\cos(2t),\ y(\pi/2) = 0 \quad \therefore\ (e^t y)' = e^t + 2\cos 2t$

$e^t y = e^t + \sin 2t + C \ \Rightarrow\ y = 1 + e^{-t}\sin 2t + Ce^{-t}$

$y(\pi/2) = 1 + Ce^{-\pi/2} = 0 \ \Rightarrow\ C = -e^{\pi/2};\ y = 1 + e^{-t}\sin 2t - e^{-(t-\pi/2)}$.

13. Putting this D.E. in the conventional form, we have $y' + \frac{\cos(t)}{2}y = -\frac{3}{2}\cos(t)$. For this D.E., $p(t) = \frac{\cos(t)}{2}$. Integrating gives us $P(t) = \frac{\sin(t)}{2}$. An integrating factor is, then, $\mu(t) = e^{\frac{\sin(t)}{2}}$. Multiplying the D.E. by $\mu(t)$, we obtain $e^{\frac{\sin(t)}{2}}y' + \frac{\cos(t)}{2}e^{\frac{\sin(t)}{2}}y = (e^{\frac{\sin(t)}{2}}y)' = -\frac{3\cos(t)}{2}e^{\frac{\sin(t)}{2}}$. Integrating both sides yields $e^{\frac{\sin(t)}{2}}y = -3e^{\frac{\sin(t)}{2}} + C$. Solving for y gives us $y = -3 + Ce^{-\frac{\sin(t)}{2}}$, and with our initial condition, $y(0) = -4 = -3 + C$. Solving for C yields $C = -1$, and thus our final solution is $y = -3 - e^{-\frac{\sin(t)}{2}}$.

14. $y' + 2y = e^{-t} + t + 1,\ y(-1) = e,\ (e^{2t}y)' = e^t + te^{2t} + e^{2t}$

$ye^{2t} = e^t + \frac{1}{2}te^{2t} - \frac{1}{4}e^{2t} + \frac{1}{2}e^{2t} + C \ \Rightarrow\ y = e^{-t} + \frac{t}{2} + \frac{1}{4} + Ce^{-2t}$

$y(-1) = e - \frac{1}{2} + \frac{1}{4} + Ce^2 = e \ \Rightarrow\ C = \frac{1}{4}e^{-2}$

$\therefore\ y = e^{-t} + \frac{t}{2} + \frac{1}{4} + \frac{1}{4}e^{-2(t+1)}$.

15. Putting this D.E. in the conventional form, we have $y' + \frac{3}{t}y = 1 + \frac{1}{t}$. For this D.E., $p(t) = \frac{3}{t}$. Integrating gives us $P(t) = 3\ln(t)$. An integrating factor is, then, $\mu(t) = e^{3\ln(t)} = e^{\ln(t^3)} = t^3$. Multiplying the D.E. by $\mu(t)$, we obtain $t^3y' + 3t^2y = (t^3y)' = t^3 + t^2$. Integrating both sides yields $t^3y = \frac{1}{4}t^4 + \frac{1}{3}t^3 + C$. Solving for y gives us $y = \frac{t}{4} + \frac{1}{3} + Ct^{-3}$, and with our initial condition, $y(-1) = \frac{1}{3} = -\frac{1}{4} + \frac{1}{3} - C$. Solving for C yields $C = -\frac{1}{4}$, and thus our final solution is $y = \frac{t}{4} + \frac{1}{3} - \frac{1}{4}t^{-3}$. The t-interval on which this solution exists is $-\infty < t < 0$.

16. $y' + \frac{4}{t} y = \alpha t,\ \mu = t^4$

$$t^4 y' + 4t^3 y = \alpha t^5 = (t^4 y)' \Rightarrow t^4 y = \alpha \frac{t^6}{6} + C \Rightarrow y = \frac{\alpha t^2}{6} + Ct^{-4}$$

$$y(1) = -\frac{1}{3} = \frac{\alpha}{6} + C \Rightarrow C = -\frac{1}{3} - \frac{\alpha}{6} \equiv 0 \Rightarrow \alpha = -2,\ y = -\frac{t^2}{3}.$$

17. Multiplying both sides of the equation by the integrating factor, $\mu(t) = e^{2t}$, we have $e^{2t} y = e^{2t}(Ce^{-2t} + t + 1) = e^{2t}(t+1) + C$. Differentiating gives us $(e^{2t} y)' = e^{2t}(1) + 2e^{2t}(t+1) = e^{2t}(2t+3)$. Therefore, $(e^{2t} y)' = (\mu(t) y)' = \mu(t) \cdot g(t) = e^{2t}(2t+3) \Rightarrow g(t) = 2t+3$ and $\mu(t) = e^{2t} = e^{P(t)} \Rightarrow P(t) = 2t \Rightarrow p(t) = 2$.

18. $2tCe^{t^2} + pCe^{t^2} = 0 \Rightarrow p(t) = -2t$. Substituting, $(Ce^{t^2} + 2)' - 2t(Ce^{t^2} + 2) = -4t \Rightarrow g(t) = -4t$.

19. Multiplying both sides of the equation by the integrating factor, $\mu(t) = t$, we have $ty = t(Ct^{-1} + 1) = t + C$. Differentiating gives us $(ty)' = 1$. Therefore, $(ty)' = (\mu(t) y)' = \mu(t) \cdot g(t) = 1 = (t)(t^{-1}) \Rightarrow g(t) = t^{-1}$ and $\mu(t) = t = e^{P(t)} \Rightarrow P(t) = \ln t \Rightarrow p(t) = \frac{1}{t} = t^{-1}$.

20. $(e^{-t} + t - 1)' + (e^{-t} + t - 1) = t \Rightarrow g(t) = t,\ y_0 = 0$.

21. $y(t) = -2e^{-t} + e^t + \sin t \Rightarrow y_0 = y(0) = -2 + 1 + 0 = -1$.

If $y(t) = -2e^{-t} + e^t + \sin t$, then $y' = 2e^{-t} + e^t + \cos t$.

Substituting in $y' + y = g(t)$, $(2e^{-t} + e^t + \cos t) + (-2e^{-t} + e^t + \sin t) = 2e^t + \cos t + \sin t = g(t)$.

22. $y' + (1 + \cos t) y = 1 + \cos t,\ y(0) = 3,\ \mu = e^{t + \sin t}$.

$$(e^{t+\sin t} y)' = (1 + \cos t) e^{t + \sin t} = (e^{t + \sin t})' \Rightarrow e^{t+\sin t} y = e^{t + \sin t} + C \Rightarrow y = 1 + Ce^{-(t + \sin t)}.$$

$$y(0) = 1 + C = 3 \Rightarrow C = 2\ \therefore y = 1 + 2e^{-(t+\sin t)} \text{ and } \lim_{t\to\infty} y(t) = 1.$$

23. Putting this D.E. in the conventional form, we have $y' + 2y = e^{-t} - 2$. For this D.E., $p(t) = 2$. An integrating factor is, then, $\mu(t) = e^{2t}$. Multiplying the D.E. by $\mu(t)$, we obtain $e^{2t} y' + 2e^{2t} y = (e^{2t} y)' = e^t - 2e^{2t}$. Integrating both sides yields $e^{2t} y = e^t - e^{2t} + C$. Solving for y gives us $y = e^{-t} - 1 + Ce^{-2t}$, and with our initial condition, $y(0) = -2 = 1 - 1 + C$. Solving for C yields $C = -2$, and thus our final solution is $y = e^{-t} - 1 - 2e^{-2t}$. Therefore, $\lim_{t\to\infty} y(t) = -1$.

24. On [1,2]:

$y' + \frac{1}{t}y = 3t,\ y(1) = 1$. An integrating factor is $\mu(t) = t$. Multiplying the D.E. by $\mu(t)$, we obtain $(ty)' = 3t^2 \Rightarrow ty = t^3 + C \Rightarrow y = t^2 + Ct^{-1},\ y(1) = 1 + C = 1 \Rightarrow C = 0$. Therefore, the solution for $1 \le t \le 2$ is $y = t^2$ and $y(2) = 4$.

On [2,3]:

$y' + \frac{1}{t}y = 0,\ y(2) = 4$. An integrating factor is $\mu(t) = t$. Multiplying the D.E. by $\mu(t)$, we obtain $(ty)' = 0 \Rightarrow ty = C \Rightarrow y = Ct^{-1},\ y(2) = \frac{C}{2} = 4 \Rightarrow C = 8$. Therefore, the solution for $2 \le t \le 3$ is $y = \frac{8}{t}$.

25. On $[0,\pi]$:

$y' + (\sin t)y = \sin t,\ y(0) = 3$. An integrating factor is $\mu(t) = e^{-\cos t}$. Multiplying the D.E. by $\mu(t)$, we obtain $e^{-\cos t}y' + e^{-\cos t}(\sin t)y = (e^{-\cos t}y)' = (\sin t)e^{-\cos t}$. Integrating both sides yields $e^{-\cos t}y = e^{-\cos t} + C$. Solving for y gives us $y = 1 + Ce^{\cos t}$, and with our initial condition, $y(0) = 3 = 1 + Ce \Rightarrow C = 2e^{-1}$. Therefore, the solution for $0 \le t \le \pi$ is $y = 1 + 2e^{\cos t - 1}$ and $y(\pi) = 1 + 2e^{-2}$.

On $[\pi,2\pi]$:

$y' + (\sin t)y = -\sin t,\ y(\pi) = 1 + 2e^{-2}$. Multiplying the D.E. by $\mu(t) = e^{-\cos t}$, we obtain $e^{-\cos t}y' + e^{-\cos t}(\sin t)y = (e^{-\cos t}y)' = (-\sin t)e^{-\cos t}$. Integrating both sides yields $e^{-\cos t}y = -e^{-\cos t} + C$. Solving for y gives us $y = -1 + Ce^{\cos t}$, and with our initial condition, $y(\pi) = 1 + 2e^{-2} = -1 + Ce^{-1} \Rightarrow C = 2e^{1} + 2e^{-1}$. Therefore, the solution for $\pi \le t \le 2\pi$ is $y = -1 + 2e^{\cos t + 1} + 2e^{\cos t - 1}$.

26. On [0,1]: $y' = 2,\ y(0) = 1$.

$y = 2t + C,\ y(0) = C = 1 \Rightarrow C = 1$.

Therefore, the solution for $0 \le t \le 1$ is $y = 2t + 1$ and $y(1) = 3$.

On [1,2]: $y' + \frac{1}{t}y = 2,\ y(1) = 3$. An integrating factor is $\mu(t) = t$. Multiplying the D.E. by $\mu(t)$, we obtain $(ty)' = 2t \Rightarrow ty = t^2 + C \Rightarrow y = t + Ct^{-1},\ y(1) = 1 + C = 3 \Rightarrow C = 2$. Therefore, the solution for $1 \le t \le 2$ is $y = t + \frac{2}{t}$.

27. On [0,1]:

$y' + (2t-1)y = 0,\ y(0) = 3$. An integrating factor is $\mu(t) = e^{t^2-t}$. Multiplying the D.E. by $\mu(t)$, we obtain $e^{t^2-t}y' + e^{t^2-t}(2t-1)y = (e^{t^2-t}y)' = 0$. Integrating both sides yields $e^{t^2-t}y = C$. Solving for y gives us $y = Ce^{t-t^2}$, and with our initial condition, $y(0) = 3 = C$. Therefore, the solution for $0 \le t \le 1$ is $y = 3e^{t-t^2}$ and $y(1) = 3$.

On [1,3]:

$y' + (0)y = y' = 0,\ y(1) = 3$. Integrating gives us $y = C = 3$. Therefore, the solution for $1 \le t \le 3$ is $y = 3$ and $y(3) = 3$.

On [3,4]:

$y' + \left(-\frac{1}{t}\right)y = 0,\ y(3) = 3$. An integrating factor is $\mu(t) = e^{-\ln t} = \frac{1}{t}$. Multiplying the D.E. by $\mu(t)$, we obtain $\frac{1}{t}y' - \frac{1}{t^2}y = (\frac{1}{t}y)' = 0$. Integrating both sides yields $\frac{1}{t}y = C$. Solving for y gives us $y = Ct$, and with our initial condition, $y(3) = 3 = C(3) \Rightarrow C = 1$. Therefore, the solution for $3 \le t \le 4$ is $y = t$.

28. $y(t) = t\{Si(t) - Si(1) + 3\}$

Section 2.4

1. $P(t) = A_0e^{rt} = 5000e^{.05t}$. Thus, $P(30) = 5000e^{.05\cdot 30} = 22408.45$.

2. $P_2(t) = (1+\frac{r}{2})^{2t}A_0 \qquad P_2(30) = (1.025)^{60}\cdot 5000$

$\therefore\ \ln P_2(30) = 60\ln(1.025) + \ln 5000 = 9.999 \qquad P_2(30) \approx 21999$

3 (a). $P_1(t) = (1+r)^tA_0 = (1.06)^tA_0$. Setting $P_1(t) = 2A_0$ yields $2 = 1.06^t$, and solving for t gives us $t \approx 11.9$ years.

3 (b). $P_2(t) = (1+\frac{r}{2})^{2t}A_0 = (1.03)^{2t}A_0$. Setting $P_2(t) = 2A_0$ yields $2 = 1.03^{2t}$, and solving for t gives us $t \approx 11.72$ years.

3 (c). $P(t) = A_0e^{rt} = A_0e^{.06t}$. Setting $P(t) = 2A_0$ yields $2 = e^{.06t}$, and solving for t gives us $t \approx 11.55$ years.

4. With $r = .05 \qquad P(t) = e^{.05t}A_0 \qquad P(10) = e^{0.5}A_0$

With unknown r, $P(8) = e^{r8}A_0 = e^{0.5}A_0$

$\therefore\ 8r = 0.5 \ \Rightarrow\ r = 1/16 \approx 0.0625 \quad (6.25\%)$

5 (a). $P_B{}' = (0.04 + 0.004t)P_B$; $P_B(0) = A_0$.

5 (b). $P_B = A_0 e^{.04t+.002t^2}$. This can be verified easily through differentiation.

5 (c). For Plan A, $P_A(t) = A_0 e^{.06t}$. To find the time t at which Plan B "catches up" with Plan A, let us set $P_A(t) = P_B(t)$: $A_0 e^{.06t} = A_0 e^{.04t+.002t^2}$. Dividing by A_0 and taking the natural logarithm of both sides yields $.06t = .04t + .002t^2$, and solving for t gives us $t = 0$ (the time of the initial investment) and $t = 10$ years (the time at which Plan B "catches up").

6. After 4 yrs, $P(4) = 1000e^{.05(4)}$, $\quad P(10) = 1000e^{.05(4)+.07(6)} = 1000e^{.2+.42} = 1858.93$

7. We can simplify this problem by considering the two deposits separately and then adding the principals of each deposit together at a time of twelve years. We have, then, $1000e^{12r} + 1000e^{6r} = 4000$. Introducing a new variable $x \equiv e^{6r}$, we have $x^2 + x - 4 = 0$. Solving this with the quadratic formula yields one positive value of x: $x \approx 1.5616 = e^{6r}$. Solving for r yields $r \approx 0.0743$.

8. $11{,}000{,}000 = 10{,}000{,}000e^{5k}$. Solving for k yields $k = \frac{1}{5}\ln\left(\frac{11}{10}\right)$.

$P(30) = 10{,}000{,}000e^{\frac{1}{5}\ln\left(\frac{11}{10}\right)(30)} = 10{,}000{,}000e^{\ln\left(\frac{11}{10}\right)^6} = 17{,}715{,}610$.

9. $2 = e^{kt}$, and thus $t = \dfrac{\ln 2}{k} = 5\dfrac{\ln 2}{\ln\frac{11}{10}} \approx 36.36$ days.

10. $1.3 = e^{2k} \;\Rightarrow\; k = \frac{1}{2}\ln(1.3)$. $\quad 3 = e^{kt} \;\Rightarrow\; t = \dfrac{\ln 3}{k} = \dfrac{2\ln(3)}{\ln(1.3)} \approx 8.375$ wks.

11. $80{,}000 = 100{,}000e^{6k}$. Solving for k yields $k = \frac{1}{6}\ln(.8)$. Using this value for k, we have $(80{,}000 + 50{,}000)e^{\ln(0.8)} = 130{,}000 \cdot 0.8 = 104{,}000$.

12 (a). $P' = kP + M,\ P(0) = P_0 \qquad P' - kP = M,\ (e^{-kt}P)' = Me^{-kt}$

$$e^{-kt}P = -\frac{M}{k}e^{-kt} + C \;\Rightarrow\; P = -\frac{M}{k} + Ce^{kt},\ P_0 = -\frac{M}{k} + C$$

$$\therefore\ P(t) = -\frac{M}{k} + \left(P_0 + \frac{M}{k}\right)e^{kt}$$

12 (b). $P_0 = -\dfrac{M}{k}$. $\quad P_0$ and P must be nonnegative $\Rightarrow -\dfrac{M}{k} \geq 0$. If net immigration rate $M > 0$, net growth rate $k < 0$ and vice versa.

12 (c). Set $kP + M = 0 \;\Rightarrow\; P = -\dfrac{M}{k}$. $\quad P(t) = P_0 = -\dfrac{M}{k}$ in this case.

13 (a). For Strategy I, we have $M_I = kP_0$. For Strategy II, we have $M_{II} = P_0(e^k - 1)$.

13 (b). The net profit for each strategy would equal $(M)(\text{profit}/\text{fish})$, and so the profit for Strategy I is, then: $\text{Pr}_I = 500{,}000(.3172)(.75) = 118{,}950$, and the profit for Strategy II is: $\text{Pr}_{II} = 500{,}000(e^{.3172} - 1)(0.6) \approx 111{,}983$. Strategy I would be more profitable for the farm.

14 (a). $P_1(1) = -\dfrac{M}{k} + (P_0 + \dfrac{M}{k})e^k,\ P_1(2) = P_1(1)e^k = -\dfrac{M}{k}e^k + (P_0 + \dfrac{M}{k})e^{2k}$

$P_2(1) = P_0 e^k,\ P_2(2) = -\dfrac{M}{k} + (P_0 e^k + \dfrac{M}{k})e^k$

14 (b). $P_1(2) - P_2(2) = -\dfrac{M}{k}e^k + P_0 e^{2k} + \dfrac{M}{k}e^{2k} + \dfrac{M}{k} - P_0 e^{2k} - \dfrac{M}{k}e^k = \dfrac{M}{k}(e^{2k} - 2e^k + 1)$

$= \dfrac{M}{k}(e^k - 1)^2$. Since $M > 0$, $P_1(2) > P_2(2)$ if $k > 0$ and $P_1(2) < P_2(2)$ if $k < 0$.

14 (c). If $k > 0$, introduce the immigrants as early as possible. If $k < 0$, introduce as late as possible.

15 (a). From the general solution of the radioactive decay equation, $Q(t) = Ce^{-kt}$, we can use the data given to find C and k. $Q(1) = Ce^{-k} = 100$ and $Q(4) = Ce^{-4k} = 30$, so combining these equations, we find that $e^{-3k} = \dfrac{3}{10}$ and therefore, $k = \dfrac{1}{3}\ln\left(\dfrac{10}{3}\right) \approx 0.4013$. Using this value of k with the $t = 1$ data, we find that $C = Q_0 = 149.4\,mg$. $C = Q_0$, since the exponential falls off the expression for Q at $t = 0$.

15 (b). $\tau = \dfrac{\ln 2}{k} \approx 1.727$ months.

15 (c). $0.01 = e^{-kt}$. Solving for t, we have $t = -\dfrac{\ln(0.01)}{k} \approx 11.475$ months.

16 (a). $\tau = \dfrac{\ln 2}{k} = 5730 \ \Rightarrow\ k = \dfrac{\ln 2}{5730}$. $\quad 0.3 = e^{-kt} \ \Rightarrow\ t = \dfrac{-\ln(0.3)}{k}$

$t = \ln(\tfrac{10}{3}) \cdot \dfrac{\tau}{\ln 2} = \left(\dfrac{\ln(\tfrac{10}{3})}{\ln 2}\right)\tau \approx 9953$ yr.

16 (b). From (a) $t = \dfrac{\ln(\tfrac{10}{3})}{\ln 2}\tau \quad \therefore \quad \dfrac{\ln(\tfrac{10}{3})}{\ln 2}(\tau - 30) \le t \le \dfrac{\ln(\tfrac{10}{3})}{\ln 2}(\tau + 30)$

or $9901 \le t \le 10005$ yrs.

16 (c). $\dfrac{Q(60{,}000)}{Q(0)} = e^{-60{,}000k} = e^{-60{,}000\left(\frac{\ln 2}{5730}\right)} \approx 2.83\left(10^{-5}\right)$.

17. $Q' = -kQ + M$. Writing this D.E. in the conventional form, we have $Q' + kQ = M$. For this D.E., $p(t) = k$ and $P(t) = kt$, which yields an integrating factor of $\mu(t) = e^{kt}$. Thus, $e^{kt}Q' + ke^{kt}Q = (e^{kt}Q)' = e^{kt}M$. Integrating both sides gives us $e^{kt}Q = e^{kt}\frac{M}{k} + C$. Solving for Q, we have $Q = \frac{M}{k} + Ce^{-kt}$. $Q_0 = \frac{M}{k} + C$, so our equation for Q in terms of Q_0 now reads $Q(t) = \frac{M}{k} + \left(Q_0 - \frac{M}{k}\right)e^{-kt} = 50e^{-kt} + \frac{M}{k}\left(1 - e^{-kt}\right)$. Setting $Q(2) = 100$ and substituting $k = \frac{\ln 2}{\tau} = \frac{\ln 2}{3} \approx 0.231$, we have $100 = 50e^{-2k} + \frac{M}{k}\left(1 - e^{-2k}\right) = 31.5 + \frac{M}{.231}(0.37)$. Solving for M, we find $M = 42.78$ (mg/yr.).

18. $\tau = \frac{\ln 2}{k} = 8$ days. $\quad Q(t) = Q_0 e^{-kt} = Q_0 e^{-\ln 2 \frac{t}{\tau}}$

$30 = Q_0 e^{-\frac{3}{8}\ln 2} \;\Rightarrow\; Q_0 = 30e^{\frac{3}{8}\ln 2} \approx 38.9\mu\text{g}$

19. $0.99Q_0 = Q_0 e^{-kt}$. Solving for t in terms of k, we have

$t = \frac{1}{k}\ln\left(\frac{100}{99}\right) = \frac{\tau}{\ln 2}\ln\left(\frac{100}{99}\right) = 4 \cdot 10^9 \cdot 0.0145 \approx 0.058 \cdot 10^9$ =58 million years.

20. Contact angle is $180 - 30 + 45 = 195°$ or $\theta_2 - \theta_1 = 3.403$ rad.

$\therefore\; T_2 = e^{0.3(3.403)}(100) \approx 277.6$ lb.

21. The contact angle, $\theta_2 - \theta_1 = 2\pi + 2\pi + \pi = 5\pi$. $T_2 = e^{0.1(\theta_2 - \theta_1)}(100 \cdot 9.8) = e^{0.1 \cdot 5\pi}(980) \approx 4714$ N.

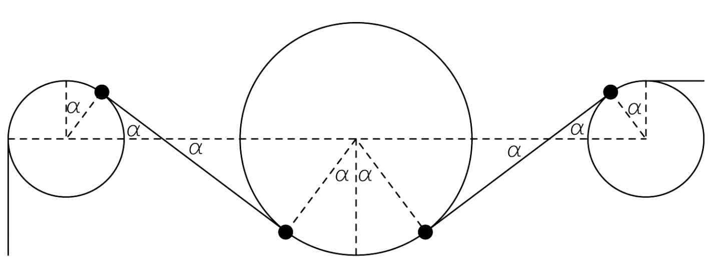
22. Contact $\angle$: $90° + \alpha + \alpha + 90° = 240°$ where $\sin\alpha = \frac{a}{2a} = \frac{1}{2} \;\Rightarrow\; \alpha = 30°$

$\theta_2 - \theta_1 = \frac{4}{3}\pi$ for T_3 and $\frac{2\pi}{3}$ for T_2

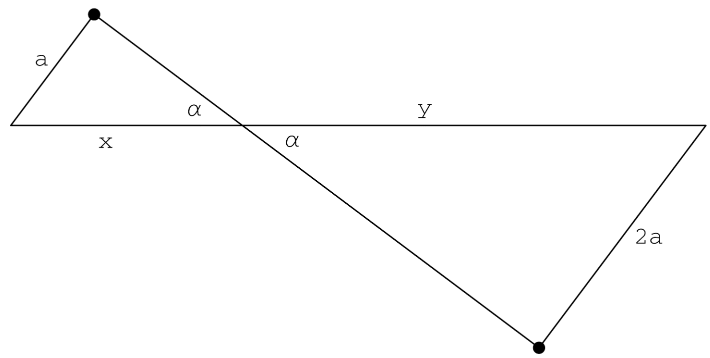
$T_2 = 100e^{.2(2\pi/3)} \approx 152$ lb.

$T_3 = 100e^{.2(4\pi/3)} \approx 231$ lb.

23. The angle, α, is marked at various places on the diagram below. A right angle occurs at each of the dots.



To determine the angle α, part of the diagram is shown here with the radii of the circles marked.



$\sin\alpha = \dfrac{a}{x}$ and $\sin\alpha = \dfrac{2a}{y} \Rightarrow y = 2x$.

In the text, we are given that $x + y = 5a$.

Therefore, $x + 2x = 3x = 5a \Rightarrow x = \dfrac{5a}{3} \therefore \sin\alpha = \dfrac{a}{x} = \dfrac{a}{\frac{5a}{3}} = .6 \Rightarrow \alpha \approx .6435 \ \ radians$.

The corresponding contact angles and belt tensions are:

For T_2: $\frac{\pi}{2} + \alpha \approx 2.214 \ \ radians \Rightarrow T_2 = T_1 e^{\mu(angle)} = 100e^{(.2)(2.214)} \approx 155.7\ lb.$

For T_3: $(\frac{\pi}{2} + \alpha) + 2\alpha = \frac{\pi}{2} + 3\alpha \approx 3.501 \ \ radians$

$\Rightarrow T_3 = T_1 e^{\mu(angle)} = 100e^{(.2)(3.501)} \approx 201.4\ lb.$

For T_4: $(\frac{\pi}{2} + 3\alpha) + \alpha = \frac{\pi}{2} + 4\alpha \approx 4.145 \ \ radians$

$\Rightarrow T_4 = T_1 e^{\mu(angle)} = 100e^{(.2)(4.415)} \approx 229.1\ lb.$

24. Contact $\angle$: $2\pi + 2\pi + 2\pi + \dfrac{\pi}{3} = \dfrac{19\pi}{3}$

$F = \dfrac{1}{3} e^{0.4(19\pi/3)} \approx 953.5$ lb.

Section 2.5

1 (a). To begin, $Q(0) = 0$ and $Q' = (0.2)(3) - \dfrac{Q}{100}(3)$. Putting the second equation in the conventional form, we have $Q' + 0.03Q = 0.6$. Multiplying both sides of this equation by the integrating factor $\mu(t) = e^{0.03t}$ gives us $(e^{0.03t}Q)' = 0.6e^{0.03t}$. Integrating both sides yields $e^{0.03t}Q = 0.6 \cdot \dfrac{100}{3} e^{0.03t} + C = 20e^{0.03t} + C$. Solving for Q, we have $Q = 20 + Ce^{-0.03t}$.

$Q(0)=0=20+C$, so $C=-20$. With this value for C, our final equation for Q is $Q=20(1-e^{-0.03t})$. Thus, $Q(10)=20(1-e^{-0.3})\approx 5.18\,\text{lb}$.

1 (b). $\lim_{t\to\infty} Q(t)=20\,\text{lb}$ and the limiting concentration is $0.2\,\text{lb/gal}$.

2. $V=100(70)(20)=140{,}000m^3$. $\quad Q'=0-\dfrac{Q}{v}r \;\Rightarrow\; Q=Q_0e^{-\frac{r}{v}t}$

$$0.01Q_0=Q_0e^{-\frac{r}{v}30} \;\Rightarrow\; -\frac{r}{v}=\frac{1}{30}\ln(0.01) \;\Rightarrow\; r=\frac{v}{30}\ln(100).$$

$$r=\frac{140{,}000}{30}\ln(100)\approx 21{,}491\ \text{m}^3\!/\text{min}. \qquad \frac{r}{v}=\frac{1}{30}\ln(100)=0.1535 \quad (\approx 15.4\%).$$

3 (a). To begin, $Q(0)=5$ and $Q'=0.25r-\dfrac{Q}{200}r$. Putting the second equation in the conventional form, we have $Q'+0.005rQ=0.25r$. Multiplying both sides of this equation by the integrating factor $\mu(t)=e^{0.005rt}$ gives us $(e^{0.005rt}Q)'=0.25re^{0.005rt}$. Integrating both sides yields $e^{0.005rt}Q=0.25(200)e^{0.005rt}+C=50e^{0.005rt}+C$. Solving for Q, we have $Q=50+Ce^{-0.005rt}$. $Q(0)=5=50+C$, so $C=-45$. With this value for C, our equation for Q now reads $Q=50-45e^{-0.005rt}$. We know that $Q(20)=30=50-45e^{-\frac{20}{200}r}$, and solving for r yields

$$r=\ln\left(\frac{(50-30)}{45}\right)(-10)=10\ln\left(\frac{9}{4}\right)\approx 8.11\,\text{gal/min}.$$

3 (b). This would be impossible, since $Q(t)<50\,\text{lb}$ for all $0\le t<\infty$.

4 (a). $Q'=(10te^{-t/50})(100)-\dfrac{Q}{5000}(100) \qquad Q(0)=0$

$$Q'=-\frac{1}{50}Q+1000te^{-t/50} \;\Rightarrow\; (Qe^{t/50})'=1000t$$

$$Qe^{t/50}=500t^2+C \;\Rightarrow\; Q=500t^2e^{-t/50}+Ce^{-t/50}. \qquad Q(0)=C=0. \qquad \therefore\; Q(t)=500t^2e^{-t/50}\ \text{oz}.$$

4 (b). $Q'=500\left(2t-\frac{t^2}{50}\right)e^{-t/50}=0 \;\Rightarrow\; t^2=100t \;\Rightarrow\; t=100\ \text{min.},$

$$\frac{Q(100)}{5000}=\frac{500(100)^2}{5000}e^{-2}=1000e^{-2}\approx 135.3\ \text{oz}/\text{gal}$$

4 (c). Plot $c(t)$ vs t. Yes.

5 (a). To begin, $Q(0)=10$, $V(0)=100$, and $V(t)=100+t$. Since the tank has a capacity of 700 gallons, $100+t=700$. Solving for t yields $t=600$ minutes.

5 (b). $Q' = (0.5)(3) - \frac{Q}{100+t}(2)$. Putting this in the conventional form, we have $Q' + \frac{2}{100+t}Q = \frac{3}{2}$.

Multiplying both sides of the equation by the integrating factor $\mu(t) = e^{2\ln(100+t)} = (100+t)^2$ gives us $((100+t)^2 Q)' = \frac{3}{2}(100+t)^2$. Integrating both sides yields $(100+t)^2 Q = \frac{(100+t)^3}{2} + C$, and solving for Q, we have $Q = \frac{100+t}{2} + \frac{C}{(100+t)^2}$. $Q(0) = 10 = 50 + \frac{C}{100^2}$, and solving for C yields $C = -40(100)^2 = -400{,}000$.

Substituting this value of C back into our equation for Q gives us our final equation for Q, $Q(t) = \frac{100+t}{2} - \frac{400{,}000}{(100+t)^2}$. $V(t) = 400$ at $t = 300$, so $Q(300) = \frac{400}{2} - \frac{400{,}000}{(400)^2} = 197.5$ lb. The concentration, then, is $\frac{197.5}{400}$ lb/gal.

5 (c). $Q(600) = \frac{700}{2} - \frac{400{,}000}{(700)^2} \approx 349.2$ lb. The concentration, then, is $\frac{349.2}{700} \approx .4988$ lb/gal.

6 (a). $Q' = \alpha\frac{Q}{500}(15) - \frac{Q}{500}(15)$

6 (b). $Q(180) = 0.01Q_0 \qquad Q' = \frac{-(1-\alpha)}{500}(15)Q \qquad Q = Q_0 e^{-.03(1-\alpha)t}$

$.01 = e^{-.03(1-\alpha)(180)} \;\Rightarrow\; e^{-5.4(1-\alpha)} = .01$

$5.4(1-\alpha) = \ln(100) \;\Rightarrow\; 1-\alpha = 0.8528 \;\Rightarrow\; \alpha = 0.1472$.

7 (a). $Q_A(0) = 1000$, $Q_B(0) = 0$, $Q_A{}' = 0 - 1000\left(\frac{Q_A}{500{,}000}\right)$, and

$Q_B{}' = 1000\left(\frac{Q_A}{500{,}000}\right) - 1000\left(\frac{Q_B}{200{,}000}\right)$.

7 (b). Putting the equation for $Q_A{}'$ into the conventional form, we have $Q_A{}' = -\frac{1}{500}Q_A$. Thus, $Q_A = 1000e^{-\frac{t}{500}}$. Putting the equation for $Q_B{}'$ into the conventional form, we have $Q_B{}' + \frac{1}{200}Q_B = 2e^{-\frac{t}{500}}$. Multiplying both sides by the integrating factor $\mu(t) = e^{\frac{t}{200}}$ yields $(Q_B e^{\frac{t}{200}})' = 2e^{t\left(\frac{1}{200}-\frac{1}{500}\right)} = 2e^{\frac{3t}{1000}}$. Integrating both sides gives us $Q_B e^{\frac{t}{200}} = \frac{2000}{3}e^{\frac{3t}{1000}} + C$, and

solving for Q_B, $Q_B = \frac{2000}{3}e^{-\frac{t}{500}} + Ce^{-\frac{t}{200}}$. $Q_B(0) = 0 = \frac{2000}{3} + C$, so $C = -\frac{2000}{3}$. Substituting this value back into our equation, we have $Q_B = \left(\frac{2000}{3}\right)\left(e^{-\frac{t}{500}} - e^{-\frac{t}{200}}\right)$

7 (c). Setting $Q_B{}' = 0$, we have $0 = \left(\frac{2000}{3}\right)\left(-\frac{1}{500}e^{-\frac{t}{500}} + \frac{1}{200}e^{-\frac{t}{200}}\right)$. Since $e^{-\frac{t}{500}+\frac{t}{200}} = \frac{500}{200}$, $\frac{3}{1000}t = \ln\left(\frac{5}{2}\right)$, and thus $t = \frac{1000}{3}\ln\left(\frac{5}{2}\right) \approx 305.4$ hours.

7 (d). Here, we want to determine t_A such that $Q_A(t_A) = \frac{1}{2}$ lb and t_B such that $Q_B(t) \le 0.2$ lb where $t \le t_B$. This can be solved via plotting: $t_A \approx 3800$ hours and $t_B \approx 4056$ hours. Therefore, $t \approx 4056$ hours.

8 (a). $r_i = r_0 = 3 + \sin t \;\Rightarrow\; V = \text{constant}$.

8 (b). Expect $\lim_{t\to\infty} Q(t) = .5(200) = 100$ lb.

The tank is being “flushed out”, albeit in a pulsating manner.

8 (c). $Q' = .5(3+\sin t) - \frac{Q}{200}(3+\sin t),\; Q(0) = 10$

$$Q' + \frac{3+\sin t}{200}Q = \frac{1}{2}(3+\sin t) \;\Rightarrow\; (Qe^{(3t-\cos t)/200})' = \frac{1}{2}(3+\sin t)e^{(3t-\cos t)/200}$$

$$Qe^{(3t-\cos t)/200} = 100e^{3t-\cos t} + C \;\Rightarrow\; Q = 100 + Ce^{-(3t-\cos t)/200}$$

$$Q(0) = 10 = 100 + Ce^{1/200} \;\Rightarrow\; C = -90e^{-1/200} \Rightarrow\; Q(t) = 100 - 90e^{-(3t-\cos t+1)/200}.$$

8(d). $\lim_{t\to\infty} e^{-(3t-\cos t+1)/200} = 0 \;\Rightarrow\; \lim_{t\to\infty} Q(t) = 100$ lb.

9. $f(t) = 3 + \sin t$. Therefore, $\tau = \int_0^t (3+\sin s)ds = [3s - \cos s]_0^t = 3t - \cos t + 1$. Now, $\frac{dQ}{d\tau} = 0.5 - \frac{1}{200}Q$ and $Q(0) = 10$. Putting the first equation into the conventional form, we have $\frac{dQ}{d\tau} + \frac{1}{200}Q = 0.5$, and multiplying both sides by the integrating factor $\mu(t) = e^{\frac{\tau}{200}}$ gives us $\left(e^{\frac{\tau}{200}}Q\right)' = 0.5e^{\frac{\tau}{200}}$. Integrating both sides yields $e^{\frac{\tau}{200}}Q = 100e^{\frac{\tau}{200}} + C$, and solving for Q, $Q = 100 + Ce^{-\frac{\tau}{200}}$. Now, $Q(\tau = 0) = 10 = 100 + C$, and therefore, $C = -90$.

Substituting this back into our equation for Q yields $Q = 100 - 90e^{-\frac{\tau}{200}}$, which in terms of t reads $Q = 100 - 90e^{-\frac{3t-\cos t+1}{200}}$.

10 (a). No limit since we do not expect concentration to stabilize.

10 (b). $Q' = .2(1+\sin t)(3) - \dfrac{Q}{200}(3),\ Q(0) = 10$

10 (c). $Q' + \dfrac{3}{200}Q = 0.6(1+\sin t) \qquad \left(e^{3/200\,t}Q\right)' = 0.6e^{3/200\,t}(1+\sin t).$

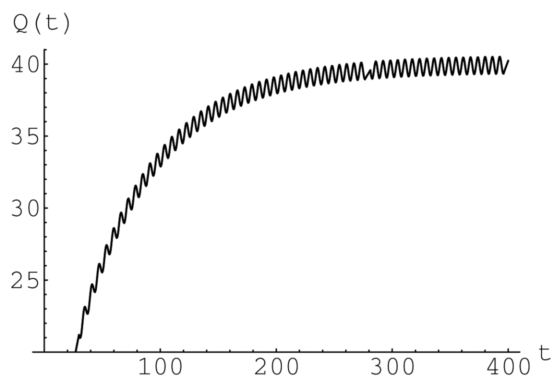
$$\int e^{at}\sin t\,dt = e^{at}\frac{(-\cos t + a\sin t)}{(1+a^2)} \qquad e^{3/200\,t}Q = 0.6\left\{\frac{200}{3}e^{3/200\,t} + \frac{e^{3/200\,t}(-\cos t + \frac{3}{200}\sin t)}{1+(3/200)^2}\right\} + C$$

$$Q(t) = 0.6\left\{\frac{200}{3} + \frac{(-\cos t + \frac{3}{200}\sin t)}{1+(3/200)^2}\right\} + Ce^{-3/200\,t} = 40 + \frac{-0.6\cos t + 0.009\sin t}{1.000225} + Ce^{-3/200\,t}$$

$$Q(0) = 10 = 40 - \frac{0.6}{1.000225} + C \ \Rightarrow\ C = -30 + \frac{0.6}{1.000225}$$

$$Q(t) = 40 - 30e^{-3/200\,t} + \left(\frac{0.6(e^{-3/200\,t} - \cos t) + 0.009\sin t)}{1.000225}\right)$$

10 (d).



11 (a). First, $Q = Q_0e^{-kt}$ for the radioactive material. To find k from the half-life of the material, $\frac{1}{2}Q_0 = Q_0e^{-18k}$. Solving for k, we have $k = \dfrac{\ln 2}{18}$. Thus for the decay of the radioactive material alone, we have $Q(t) = 5e^{-\frac{\ln 2}{18}t}$ with t measured in hours. Now, for the lake, we know that Q varies both with decay and with the water flow. Accordingly, we will begin with the relationship $Q(t+\Delta t) - Q(t) \approx -kQ(t)\Delta t - \dfrac{Q(t)}{V}r\Delta t$.

Using a form of the definition of the derivative and solving for Q', we have

$Q' = -\left(k+\dfrac{r}{V}\right)Q = -\left(\dfrac{\ln 2}{18}+\dfrac{60{,}000}{1{,}200{,}000}\right)Q \approx -0.0885Q$. We know that $Q_0 = Q(0) = 5$ lb, so our final equation for Q reads $Q(t) = 5e^{-0.0885t}$.

11 (b). Here, $(0.0001)(5) = 5e^{-0.0885t}$. Thus, $t = 104.07$ hours.

12. $\theta' = k(S-\theta)$, $S = 72$, $\theta(0) = 350$, $\theta(10) = 290$

$$\theta' + k\theta = kS \ \Rightarrow\ (e^{kt}\theta)' = ke^{kt}S \ \Rightarrow\ e^{kt}\theta = e^{kt}S + C \ \Rightarrow\ \theta = S + Ce^{-kt}$$

$$\theta(0) = \theta_0 = S + C \ \Rightarrow\ C = \theta_0 - S \ \Rightarrow\ \theta = S + (\theta_0 - S)e^{-kt}$$

$$290 = 72 + (350-72)e^{-k(10)} \ \Rightarrow\ 218 = 278e^{-10k},\ 10k = \ln\left(\frac{278}{218}\right)$$

$$k = \frac{1}{10}\ln\left(\frac{278}{218}\right);\ 120 = 72 + (350-72)e^{-kt} \ \Rightarrow\ e^{-kt} = \frac{48}{278}$$

$$t = -\frac{1}{k}\ln\left(\frac{48}{278}\right) = \frac{10\ln\left(\frac{278}{48}\right)}{\ln\left(\frac{278}{218}\right)} = \frac{10(1.756)}{0.243} \approx 72.2 \text{ min.}$$

13. To begin, $\theta = S + (\theta_0 - S)e^{-kt}$. With our substitutions for the time the food was in the oven, this equation reads $120 = 350 + (40-350)e^{-10k}$. Solving for k, we have $k = -\dfrac{1}{10}\ln\left(\dfrac{350-120}{350-40}\right) \approx .02985$. The temperature of the food after 20 minutes in the oven is, then, $\theta(20) = 350 + (40-350)e^{-20k} = 350 - (310)(0.550) \approx 179.5$ degrees. Finally, the food is cooled at room temperature, so $\theta(t) = 110 = 72 + (179.5-72)e^{-0.02985t}$. Solving for t yields

$$t \approx -\frac{1}{0.02985}\ln\left(\frac{110-72}{179.5-72}\right) \approx 34.8 \text{ minutes.}$$

14. $\theta = S + (\theta_0 - S)e^{-kt}$; $170 = 212 + (72-212)e^{-k5}$

$$k = \frac{1}{5}\ln\left(\frac{140}{42}\right) = \frac{1}{5}\ln\left(\frac{10}{3}\right) \text{ min.}^{-1}$$

$$P' = r\left(1 - \frac{Q(t)}{140}\right)P,\ \ P = P_0 \exp\left\{r\left(t - \frac{1}{140}\int_0^t \theta(s)\,ds\right)\right\}$$

$$\theta(t) = 212 + (72-212)e^{-kt} = 212 - 140e^{-kt}$$

$$\int_0^t \theta\,ds = 212t - \frac{140}{k}\left(1 - e^{-kt}\right)$$

$$\therefore\ \ 0.01 = \exp\left\{r\left(10 - \frac{1}{140}\left[2120 - \frac{140}{k}\left(1 - e^{-10k}\right)\right]\right)\right\}$$

$$\frac{1}{100} = \exp\left\{r\left(10 - \frac{212}{14} + \frac{1}{k}\left(1 - e^{-10k}\right)\right)\right\}$$

$$-\ln(100) = r\left(10 - 15.143 + \frac{5}{\ln\left(\frac{10}{3}\right)}(1 - .09)\right) = r(-5.143 + 3.78)$$

$$-4.6052 \approx r(-1.363) \;\Rightarrow\; r \approx 3.379 \text{ min}^{-1}$$

15. For the first cup, $\theta_1 = 72 + (34 - 72)e^{-kt}$. Thus, with the proper substitutions, $53 = 72 - 38e^{-kt_1}$. e^{-kt_1}, then, is equal to $\frac{19}{38}$. For the second cup, $\theta_2 = 34 + (72 - 34)e^{-kt}$. With the proper substitutions, we have $53 = 34 + 38e^{-kt_2}$. e^{-kt_2}, then, is equal to $\frac{19}{38}$. Thus, the two times are equal.

16. $\theta = S + (\theta_0 - S)e^{-kt}$ For casserole, $45 = 72 + (40 - 72)e^{-k2}$

$$-27 = -32e^{-k2}, \quad k = \frac{1}{2}\ln\left(\frac{32}{27}\right)$$

$$S(t) = 72 + 228\left(1 - e^{-\alpha t}\right) \qquad S(2) = 150 = 72 + 228\left(1 - e^{-\alpha 2}\right)$$

$$1 - e^{-2\alpha} = \frac{78}{228} \;\Rightarrow\; e^{-2\alpha} = \frac{150}{228} \;\Rightarrow\; \alpha = \frac{1}{2}\ln\left(\frac{228}{150}\right)$$

$$\theta' = k(S(t) - \theta) \;\Rightarrow\; \theta' + k\theta = kS(t) \;\Rightarrow\; (e^{kt}\theta)' = ke^{kt}S(t)$$

$$= ke^{kt}(72 + 228 - 228e^{-\alpha t}) = ke^{kt}(300 - 228e^{-\alpha t})$$

$$e^{kt}\theta = 300e^{kt} - \frac{228k}{k - \alpha}e^{(k-\alpha)t} + C \;\Rightarrow\; \theta = 300 - \frac{228k}{k - \alpha}e^{-\alpha t} + Ce^{-kt}$$

$$\theta(0) = 45 = 300 - \frac{228k}{k - \alpha} + C \;\Rightarrow\; C = \frac{228k}{k - \alpha} - 255$$

$$\theta(8) = 300 - \frac{228k}{k - \alpha}e^{-8\alpha} + \left(\frac{228k}{k - \alpha} - 255\right)e^{-8k}$$

$$e^{-8\alpha} = \left(e^{-2\alpha}\right)^4 = \left(\frac{150}{228}\right)^4 \approx .1873, \quad e^{-8k} = \left(e^{-2k}\right)^4 = \left(\frac{27}{32}\right)^4 = .5068$$

$$k = \frac{1}{2}\ln\left(\frac{32}{27}\right) \approx 0.08495 \qquad \alpha = \frac{1}{2}\ln\left(\frac{228}{150}\right) \approx 0.2094$$

$$k - \alpha \approx -0.1244 \qquad \frac{k}{k - \alpha} = -0.682843$$

$$\theta(8) = 300 - 228(-0.682843)(.1873379) + (228(-.682843) - 255)(.5068216)$$

$$= 300 + 29.166 - 208.14565 = 121.02°$$

Chapter 3
First Order Nonlinear Differential Equations

Section 3.1

1 (a). Solving for y', we have $y' = \frac{1}{3}(1 - 2t\cos y)$. Thus, $f(t,y) = \frac{1}{3}(1 - 2t\cos y)$.

1 (b). $\frac{\partial f}{\partial y} = \frac{1}{3}(0 + 2t\sin y) = \frac{2}{3}t\sin y$. f and $\frac{\partial f}{\partial y}$ are continuous in the entire ty plane.

1 (c). The largest open rectangle is the entire ty plane, since f and $\frac{\partial f}{\partial y}$ are continuous in the entire ty plane.

2 (a). $f(t,y) = \frac{1}{3t}(1 - 2\cos y)$.

2 (b). $\frac{\partial f}{\partial y} = \frac{2}{3t}\sin y$. f and $\frac{\partial f}{\partial y}$ are continuous when $t < 0,\ t > 0$.

2 (c). $R = \{(t,y) : t > 0, -\infty < y < \infty\}$.

3 (a). Solving for y', we have $y' = -\frac{2t}{1 + y^2}$. Thus, $f(t,y) = -\frac{2t}{1 + y^2}$.

3 (b). $\frac{\partial f}{\partial y} = (-2t)(-1)(1 + y^2)^{-2}(2y) = \frac{4ty}{(1 + y^2)^2}$. f and $\frac{\partial f}{\partial y}$ are continuous in the entire ty plane.

3 (c). The largest open rectangle is the entire ty plane, since f and $\frac{\partial f}{\partial y}$ are continuous in the entire ty plane.

4 (a). $f(t,y) = \frac{-2t}{1 + y^3}$.

4 (b). $\frac{\partial f}{\partial y} = \frac{6ty^2}{(1 + y^3)^2}$. f and $\frac{\partial f}{\partial y}$ are continuous everywhere in the ty-plane except on the line $y = -1$.

4 (c). $R = \{(t,y) : -\infty < t < \infty, y > -1\}$.

5 (a). Solving for y', we have $y' = \tan t - ty^{\frac{1}{3}}$. Thus, $f(t,y) = \tan t - ty^{\frac{1}{3}}$.

5 (b). $\dfrac{\partial f}{\partial y} = -\dfrac{1}{3}ty^{-\frac{2}{3}}$. f and $\dfrac{\partial f}{\partial y}$ are continuous except on the lines $t = \left(n + \dfrac{1}{2}\right)\pi$ (where n is an integer) and $y = 0$.

5 (c). The largest open rectangle is $R = \left\{(t,y): -\dfrac{\pi}{2} < t < \dfrac{\pi}{2}, 0 < y < \infty\right\}$.

6 (a). $f(t,y) = \dfrac{t^2 - e^{-y}}{y^2 - 9}$.

6 (b). $\dfrac{\partial f}{\partial y} = \dfrac{(y^2 + 2y - 9)e^{-y} - 2t^2 y}{(y^2 - 9)^2}$. f and $\dfrac{\partial f}{\partial y}$ are continuous everywhere in the ty-plane except $y = \pm 3$.

6 (c). $R = \{(t,y): -\infty < t < \infty, -3 < y < 3\}$.

7 (a). Solving for y', we have $y' = \dfrac{2 + \tan t}{\cos y}$. Thus, $f(t,y) = \dfrac{2 + \tan t}{\cos y}$.

7 (b). $\dfrac{\partial f}{\partial y} = (2 + \tan t)(-1)(\cos y)^{-2}(-\sin y) = (2 + \tan t)\sec y \tan y$. f and $\dfrac{\partial f}{\partial y}$ are continuous except on the lines $t = \left(n + \dfrac{1}{2}\right)\pi$ (where n is an integer) and $y = \left(m + \dfrac{1}{2}\right)\pi$ (where m is an integer).

7 (c). The largest open rectangle is $R = \left\{(t,y): -\dfrac{\pi}{2} < t < \dfrac{\pi}{2}, -\dfrac{\pi}{2} < y < \dfrac{\pi}{2}\right\}$.

8 (a). $f(t,y) = \dfrac{2 + \tan y}{\cos 2t}$.

8 (b). $\dfrac{\partial f}{\partial y} = \dfrac{\sec^2 y}{\cos 2t}$. f and $\dfrac{\partial f}{\partial y}$ are continuous except where $\tan y$ is not defined and $\cos 2t = 0$, or where $y = (n + \frac{1}{2})\pi$, $n = \ldots,-2,-1,0,1,2,\ldots$, and $t = (m + \frac{1}{2})\frac{\pi}{2}$, $m = \ldots,-2,-1,0,1,2,\ldots$

8 (c). $R = \left\{(t,y): \dfrac{3\pi}{4} < t < \dfrac{5\pi}{4}, -\dfrac{\pi}{2} < y < \dfrac{\pi}{2}\right\}$.

9. One possible example is $y' = \dfrac{1}{t(t-4)(y+1)(y-2)}$ with $(t_0, y_0) = (2,0)$.

10 (a). $f(t,y) = \dfrac{y^2}{t^2}$, $\dfrac{\partial f}{\partial y} = \dfrac{2y}{t^2}$. f and $\dfrac{\partial f}{\partial y}$ are continuous except where $t = 0$.

$R = \{(t,y): 0 < t < \infty, -\infty < y < \infty\}$.

10 (b). No contradiction. If the hypotheses are not satisfied, "bad things need not happen".

11. $\bar{y}(t) = \dfrac{2}{\sqrt{1-(t-1)}}$, so $\bar{y}(0) = \dfrac{2}{\sqrt{2}} = \sqrt{2}$.

12. $\bar{y}(t) = (4 + (t - t_0))^{\frac{3}{2}}$, so $\bar{y}(0) = (4 - t_0)^{\frac{3}{2}} = 1 \Rightarrow t_0 = 3$.

13 (a). $z_1(t) = y(t+2)$, so $z_1(-5) = y(-3) = 2$.

13 (b). $z_2(t) = y(t-2)$, so $z_2(3) = y(1) = 0$.

Section 3.2

1 (a). Antidifferentiation gives us $\frac{y^2}{2} + \cos t = C$. From the initial condition, we have $\frac{(-2)^2}{2} + \cos\frac{\pi}{2} = C = 2$. Then we have $y^2 = 4 - 2\cos t$, $y = -\sqrt{4 - 2\cos t}$.

1 (b). $-\infty < t < \infty$

2 (a). $y^2 y' = 1$, so $\frac{y^3}{3} - t = C$. From the initial condition, we have $\frac{8}{3} - 1 = \frac{5}{3} = C$. Then we have $y^3 = 3t + 5 \Rightarrow y = (3t+5)^{\frac{1}{3}}$.

2 (b). $-\infty < t < \infty$

3 (a). $(y+1)y' + 1 = 0$, so $\frac{y^2}{2} + y + t = C$. From the initial condition, we have $0 + 0 + 1 = C$. Then we have $\frac{y^2}{2} + y + t = 1 \Rightarrow y^2 + 2y + 2(t-1) = 0$, $y = \frac{-2 \pm \sqrt{4 - 8(t-1)}}{2}$. Since $y(1) = 0$, we only want the plus sign. Finally, $y = \frac{-2 + \sqrt{4 - 8(t-1)}}{2} = -1 + \sqrt{3 - 2t}$.

3 (b). $-\infty < t \le \frac{3}{2}$

4 (a). $y^{-2}y' - 2t = 0$, so $-y^{-1} - t^2 = C$. From the initial condition, we have $1 - 0 = C$. Then we have $-y^{-1} = t^2 + 1 \Rightarrow y = \frac{-1}{1 + t^2}$.

4 (b). $-\infty < t < \infty$

5 (a). $y^{-3}y' - t = 0$, so $\frac{y^{-2}}{-2} - \frac{t^2}{2} = C$. From the initial condition, we have $C = -\frac{1}{8}$. Then we have $y^{-2} + t^2 = \frac{1}{4}$, $y = \frac{1}{\sqrt{1/4 - t^2}} = \frac{2}{\sqrt{1 - 4t^2}}$.

5 (b). $-\frac{1}{2} < t < \frac{1}{2}$

6 (a). $e^{-y}y' + (t - \sin t) = 0$, so $-e^{-y} + \left(\frac{t^2}{2} + \cos t\right) = C$. From the initial condition, we have $-1 + 1 = 0 = C$. Then we have $e^{-y} = \frac{t^2}{2} + \cos t \Rightarrow y = -\ln\left(\frac{t^2}{2} + \cos t\right)$.

6 (b). $-\infty < t < \infty$

7 (a). $\frac{1}{1+y^2}y' - 1 = 0$, so $\tan^{-1} y - t = C$. From the initial condition, we have $C = -\frac{\pi}{2}$. Then we have $\tan^{-1} y = t - \frac{\pi}{2}$, $y = \tan\left(t - \frac{\pi}{2}\right)$.

7 (b). $0 < t < \pi$

8 (a). $(\cos y)y' + t^{-2} = 0$, so $\sin y - t^{-1} = C$. From the initial condition, we have $0 - (-1) = 1 = C$. Then we have $\sin y = 1 + t^{-1} \Rightarrow y = \sin^{-1}\left(1 + t^{-1}\right)$.

8 (b). $-\infty < t < -\frac{1}{2}$

9 (a). $\frac{1}{1-y^2}y' - t = 0$.

By partial fractions, $\frac{1}{1-y^2} = \frac{-1}{y^2-1} = \frac{-1}{(y-1)(y+1)} = \frac{-\frac{1}{2}}{y-1} + \frac{\frac{1}{2}}{y+1}$, and so $\frac{1}{2}\ln\left|\frac{y+1}{y-1}\right| - \frac{t^2}{2} = C$.

From the initial condition, we have $\frac{1}{2}\ln 3 = C$. Then we have

$\ln\left|\frac{y+1}{y-1}\right| - t^2 = \ln 3 \Rightarrow \ln\left|\frac{1}{3}\left(\frac{y+1}{y-1}\right)\right| = t^2$, and solving for y yields $y = \frac{3e^{t^2} - 1}{3e^{t^2} + 1}$.

9 (b). $-\infty < t < \infty$

10 (a). $3y^2y' + 2t - 1 = 0$, so $y^3 + t^2 - t = C$. From the initial condition, we have $-1 + 1 - (-1) = 1 = C$. Then we have $y^3 = 1 + t - t^2 \Rightarrow y = \left(1 + t - t^2\right)^{\frac{1}{3}}$.

10 (b). $-\infty < t < \infty$

11 (a). $e^y y' - e^t = 0$, so $e^y - e^t = C$. From the initial condition, we have $C = e - 1$. Then we have $e^y - e^t = e - 1$, $y = \ln\left(e^t + e - 1\right)$.

11 (b). $-\infty < t < \infty$

12 (a). $yy' - t = 0$, so $\frac{y^2}{2} - \frac{t^2}{2} = C$. From the initial condition, we have $2 - 0 = C$. Then we have $\frac{y^2}{2} - \frac{t^2}{2} = 2 \Rightarrow y = -\sqrt{4 + t^2}$.

12 (b). $-\infty < t < \infty$

13 (a). $\sec^2 y(y') + e^{-t} = 0$, so $\tan y - e^{-t} = C$. From the initial condition, we have $C = 1 - 1 = 0$. Then we have $\tan y = e^{-t}$, $y = \tan^{-1}(e^{-t})$.

13 (b). $-\infty < t < \infty$

14 (a). $(2y - \sin y)(y') + (t - \sin t) = 0$, so $y^2 + \cos y + \frac{t^2}{2} + \cos t = C$. From the initial condition, we have $0 + 1 + 0 + 1 = 2 = C$. Then we have $y^2 + \cos y = 2 - \frac{t^2}{2} - \cos t$. There is no explicit solution.

15 (a). $(y+1)e^y y' + (t-2) = 0$, so $ye^y + \frac{(t-2)^2}{2} = C$. From the initial condition, we have $C = 2e^2 + \frac{1}{2}$. Then we have $ye^y = 2e^2 + \frac{1}{2} - \frac{(t-2)^2}{2}$. There is no explicit solution.

16. $y = (4+t)^{-\frac{1}{2}}$, so $y' = -\frac{1}{2}(4+t)^{-\frac{3}{2}} = -\frac{1}{2}y^3 \Rightarrow y' + \frac{1}{2}y^3 = 0$, $y(0) = 4^{-\frac{1}{2}} = \frac{1}{2}$. Therefore, $\alpha = \frac{1}{2}$, $n = 3$, $y_0 = \frac{1}{2}$.

17. $y = \frac{6}{(5+t^4)}$, so $y' = 6(-1)(5+t^4)^{-2}(4t^3) = \frac{-24t^3}{(5+t^4)^2} = -24t^3\left(\frac{y}{6}\right)^2 = -\frac{2}{3}t^3y^2$. Then we have $y' + \frac{2}{3}t^3y^2 = 0$, so $\alpha = \frac{2}{3}$, $n = 3$, $y_0 = \frac{6}{5+1} = 1$.

18. $y^3 + t^2 + \sin y = 4 \Rightarrow 3y^2y' + 2t + (\cos y)y' = 0 \Rightarrow (3y^2 + \cos y)y' + 2t = 0$.
When $t = 2$, $y_0^3 + 4 + \sin y_0 = 4 \Rightarrow y_0^3 + \sin y_0 = 0 \Rightarrow y_0 = 0 \Rightarrow y(2) = 0$.

19. First, $y'e^y + ye^y y' + 2t = \cos t$. Then $(1+y)e^y y' + (2t - \cos t) = 0$. At $t_0 = 0$, we have $y_0 e^{y_0} + 0 = 0$, so $y_0 = 0$, and thus $y(0) = 0$.

20. $y^{-2}y' = 2 \Rightarrow -y^{-1} = 2t + C$, $-y_0^{-1} = C \Rightarrow -y^{-1} = 2t - y_0^{-1} \Rightarrow y^{-1} = y_0^{-1} - 2t \Rightarrow y = \frac{1}{y_0^{-1} - 2t}$.

Require $y_0^{-1} - 2(4) = 0 \Rightarrow y_0 = \frac{1}{8}$.

21 (a). $\left(\frac{K}{S} + 1\right)S' + \alpha = 0$, so $K\ln S + S + \alpha t = C$. From the initial condition, we have $K\ln S_0 + S_0 = C$, so $K\ln S + S = -\alpha t + K\ln S_0 + S_0$.

21 (b). When $t=0$, $S(0)=S_0=1$, so $C=K\cdot 0+1=1$. Then we have $K\ln S+S=-\alpha t+1$. From the other conditions, we have $K\ln\left(\frac{3}{4}\right)+\frac{3}{4}=-\alpha+1\Rightarrow\left(\ln\frac{3}{4}\right)K+\alpha=\frac{1}{4}$ and $K\ln\left(\frac{1}{8}\right)+\frac{1}{8}=-6\alpha+1\Rightarrow\left(\ln\frac{1}{8}\right)K+6\alpha=\frac{7}{8}$. Solving these simultaneous equations yields $K\approx 1.769$ and $\alpha\approx 0.759$.

21 (c). $K\ln\left(\frac{1}{50}\right)+\frac{1}{50}=-\alpha t+1$, so $1.769(-3.912)+0.02=-0.759t+1$. Solving for t yields $t\approx 10.41$.

22. $y'=1+(y+1)^2$. Let $u=y+1$, $u'=1+u^2$, $\frac{1}{(1+u^2)}u'=1\Rightarrow\tan^{-1}(u)=t+C$.

Then, $y(0)=0\Rightarrow u(0)=1$, $\frac{\pi}{4}=0+C\Rightarrow\tan^{-1}(u)=t+\frac{\pi}{4}\Rightarrow u=y+1=\tan\left(t+\frac{\pi}{4}\right)$.

Therefore, $y=\tan\left(t+\frac{\pi}{4}\right)-1$, $-\frac{3\pi}{4}<t<\frac{\pi}{4}$.

23. $y'=t\left((y+2)^2+1\right)$. Letting $u=y+2$, we have $u'=t\left(u^2+1\right)$, so $\frac{1}{u^2+1}u'=t$. Then $\tan^{-1}u=\frac{t^2}{2}+C$. From the initial condition, we have $y(0)=-3$ and $u(0)=-1$, so $-\frac{\pi}{4}=0+C$, $C=-\frac{\pi}{4}$, and $\tan^{-1}u=\frac{t^2}{2}-\frac{\pi}{4}$. In terms of y, this reads $y=-2+\tan\left(\frac{t^2}{2}-\frac{\pi}{4}\right)$.

Setting $-\frac{\pi}{2}<\frac{t^2}{2}-\frac{\pi}{4}<\frac{\pi}{2}$ and simplifying, we have

$-\frac{\pi}{2}<t^2<\frac{3\pi}{2}\Rightarrow|t|<\sqrt{\frac{3\pi}{2}}\Rightarrow-\sqrt{\frac{3\pi}{2}}<t<\sqrt{\frac{3\pi}{2}}$.

24. $y'=(y+1)^2\sin t$. $\frac{y'}{(y+1)^2}=\sin t\Rightarrow\frac{-1}{y+1}=-\cos t+C$.

Then, $y(0)=0\Rightarrow -1=-1+C\Rightarrow C=0\Rightarrow\frac{-1}{y+1}=-\cos t$.

Therefore, $y+1=\sec t\Rightarrow y=\sec t-1$.

25. $Q^{-3}Q'+k=0$, so $\frac{Q^{-2}}{-2}+kt=C'$ and $Q^{-2}=2kt-C$. From the implicit initial condition, we have $Q_0^{-2}=-C$, so $Q^{-2}=2kt+Q_0^{-2}$. Solved for Q, we have $Q(t)=\frac{1}{\sqrt{2kt+Q_0^{-2}}}=\frac{Q_0}{\sqrt{1+2kQ_0^{2}t}}$.

Thus $\frac{1}{2}Q_0 = \frac{Q_0}{\sqrt{1+2kQ_0^2\tau}}$, where τ is the half-life of the reactant. Therefore,

$2 = \sqrt{1+2kQ_0^2\tau}$, which, solved for τ, gives $\tau = \frac{3}{2kQ_0^2}$. Thus the half-life depends upon Q_0.

26. $Q' = -kQ^2,\ Q(0) = Q_0;\ Q^{-2}Q' = -k \Rightarrow -Q^{-1} = -kt + C,\ C = -Q_0^{-1}$. Therefore,

$Q^{-1} = kt + Q_0^{-1} \Rightarrow Q = \frac{1}{kt + Q_0^{-1}} = \frac{Q_0}{1+kQ_0 t},\ Q(10) = 0.4Q_0$. Then,

$0.4Q_0 = \frac{Q_0}{1+kQ_0(10)} \Rightarrow 0.4 + 4kQ_0 = 1 \Rightarrow kQ_0 = 0.15$ and $Q = \frac{Q_0}{1+.15t}$.

Set $Q = 0.25Q_0$. Then, $0.25 = \frac{1}{1+.15t} \Rightarrow t = 20$ min.

27 (a). The equation is nonlinear and separable. $\frac{1}{|y|}y' - 1 = 0$.

27 (b). $|y| = \begin{cases} y, & y \ge 0 \\ -y, & y < 0 \end{cases}$. Thus $\int \frac{dy}{|y|} = \begin{cases} \ln y, & y > 0 \\ -\ln y, & y < 0 \end{cases} \Rightarrow y(t) = \begin{cases} y(0)e^t, & y > 0 \\ y(0)e^{-t}, & y < 0 \end{cases}$.

Since $y(0) = 1 > 0$, the solution $y(t) = e^t$ of $y' = |y|,\ y(0) = 1$ will be identical to that of $y' = y,\ y(0) = 1$ as long as $y(t) = e^t \ge 0$. This is true for all t, however, and so the two solution curves agree.

27 (c). If $y(0) = -1 < 0$, then the solution of $y' = |y|,\ y(0) = -1$, is $y(t) = -e^{-t}$, but the solution of $y' = y,\ y(0) = -1$, is $y(t) = -e^t$.

28. $y' = -y^2$ is graph c. $y' = y^3$ is graph a. $y' = y(4-y)$ is graph b.

29. This is a translation three units to the right of graph (a) in problem 28.

30. Yes. $\frac{1}{f(y)}y' - 1 = 0$.

31. $y'\sin y + y(\cos y)y' - 3 = 0 \Rightarrow y' = \frac{3}{\sin y + y\cos y} = f(y)$. The solution of

$y' = f(y)$ is $H(y) = t + C$, where $H(y) = \int \frac{1}{f(y)}dy$. The solution of $y' = t^2 f(y)$ is therefore

$H(y) = \frac{t^3}{3} + C_2$. From the initial condition, we know that $H(0) = 1 + C \Rightarrow C = H(0) - 1$, and

$H(0) = \frac{1}{3} + C_2$. Thus $C_2 = C + \frac{2}{3}$. Now we have $H(y) = t + C$ and $y\sin y - 3t + 3 = 0$, so

$y\sin y = 3(t-1) \Rightarrow \frac{1}{3}y\sin y = t + (-1) \Rightarrow H(y) = \frac{1}{3}y\sin y$, and $C = -1$.

Therefore, $C_2 = C + \frac{2}{3} = -1 + \frac{2}{3} = -\frac{1}{3}$ and the implicit solution of the initial value problem is

$\frac{y\sin y}{3} = \frac{t^3}{3} - \frac{1}{3} \Rightarrow y\sin y - t^3 + 1 = 0$.

Section 3.3

1. $M = 3t^2 - 2,\ N = y,\ M_y = N_t = 0$, so the equation is exact. $\frac{\partial H}{\partial y} = y \Rightarrow H = \frac{y^2}{2} + g(t)$ and

$\frac{\partial H}{\partial t} = g' = 3t^2 - 2 \Rightarrow g(t) = t^3 - 2t + C \Rightarrow \frac{y^2}{2} + t^3 - 2t = C$.

From the initial condition, we have $\frac{(-2)^2}{2} - 1 - 2(-1) = 2 - 1 + 2 = 3 = C$, and thus

$\frac{y^2}{2} + t^3 - 2t = 3 \Rightarrow y = -\sqrt{6 + 4t - 2t^3}$ (the minus sign can be checked by the initial condition).

2. $M = y + t^3,\ N = t + y^3,\ M_y = N_t = 1$, so the equation is exact.

$\frac{\partial H}{\partial t} = M = y + t^3 \Rightarrow H = yt + \frac{t^4}{4} + h(y)$ and $\frac{\partial H}{\partial y} = t + \frac{dh}{dy} = N = t + y^3 \Rightarrow \frac{dh}{dy} = y^3 \Rightarrow h = \frac{y^4}{4}$.

Therefore, $yt + \frac{t^4}{4} + \frac{y^4}{4} = C,\ y(0) = -2 \Rightarrow C = 4$ and $\frac{y^4}{4} + yt + \frac{t^4}{4} = 4 \Rightarrow y^4 + 4yt + t^4 = 16$.

3. The equation is separable, and therefore it is exact. $\left(y^2 + 1\right)^{-1} y' = 3t^2 + 1$ gives us

$\tan^{-1} y = t^3 + t + C$, and from the initial condition we have $\frac{\pi}{4} = C$. Thus $y = \tan\left(t^3 + t + \frac{\pi}{4}\right)$.

4. $M = 3t^2 y,\ N = 6t + y^3,\ M_y = 3t^2,\ Nt = 6$. Therefore, the differential equation is not exact.

5. $M = e^{t+y} + 3t^2,\ N = e^{t+y} + 2y,\ M_y = N_t = e^{t+y}$, so the equation is exact.

$\frac{\partial H}{\partial t} = M = e^{t+y} + 3t^2 \Rightarrow H = e^{t+y} + t^3 + h(y)$ and $\frac{\partial H}{\partial y} = e^{t+y} + \frac{dh}{dy} = N = e^{t+y} + 2y \Rightarrow \frac{dh}{dy} = 2y$,

and so $h = y^2 + C$. From the initial condition, we have $1 + 0 + 0 = C$, and thus

$e^{t+y} + t^3 + y^2 = 1$.

6. $M = y\cos(ty) + 1,\ N = t\cos(ty) + 2ye^{y^2},\ M_y = N_t = \cos(ty) - ty\sin(ty)$, so the equation is

exact. $\frac{\partial H}{\partial t} = M = y\cos(ty) + 1 \Rightarrow H = \sin(ty) + t + h(y)$

and $\frac{\partial H}{\partial y} = t\cos(ty) + \frac{dh}{dy} = N = t\cos(ty) + 2ye^{y^2} \Rightarrow \frac{dh}{dy} = 2ye^{y^2}$, and so $h = e^{y^2}$. From the initial condition, we have $0 + \pi + 1 = C$, and thus $\sin(ty) + t + e^{y^2} = \pi + 1$.

7. $M_y = \cos(t+y) - y\sin(t+y) + 1,\ N_t = \cos(t+y) - y\sin(t+y) + 1$, so the equation is exact.

$\frac{\partial H}{\partial t} = M = y\cos(t+y) + y + t \Rightarrow H = y\sin(t+y) + yt + \frac{t^2}{2} + h(y)$ and

$\frac{\partial H}{\partial y} = \sin(t+y) + y\cos(t+y) + t + \frac{dh}{dy} = \sin(t+y) + y\cos(t+y) + t + y$. Thus

$\frac{dh}{dy} = y \Rightarrow h = \frac{y^2}{2} + C$, and from the initial condition, we have

$(-1)\sin(1-1) + (-1)(1) + \frac{1}{2} + \frac{1}{2} = 0 = C$, and thus $y\sin(t+y) + yt + \frac{t^2}{2} + \frac{y^2}{2} = 0$.

8. $M = \alpha t^3 y^n,\ N = t^m y^2,\ M_y = n\alpha t^3 y^{n-1},\ N_t = mt^{m-1}y^2$. Therefore,

$m - 1 = 3 \Rightarrow m = 4,\ n - 1 = 2 \Rightarrow n = 3,\ 3\alpha = 4 \Rightarrow \alpha = \frac{4}{3}$.

9. $N = t^2 + y^2\sin t,\ N_t = M_y = 2t + y^2\cos t$. Thus $M = 2ty + \frac{y^3}{3}\cos t + m(t)$.

10. $M = t^2 + y^2\sin t,\ M_y = 2y\sin t = N_t \Rightarrow N = -2y\cos t + n(y)$.

11. $0 + 1 + {y_0}^2 = 5 \Rightarrow y_0 = \pm 2$. $3t^2y + t^3y' + e^t + 2yy' = 0$, so $(t^3 + 2y)y' + (3t^2y + e^t) = 0$ and thus $M = 3t^2y + e^t$ and $N = t^3 + 2y$.

12. $y = -t - \sqrt{4 - t^2} \Rightarrow y(0) = -2 = y_0$. Also, $N_t = a = M_y \Rightarrow$ exact.

Then, $H_y = y + at \Rightarrow H = \frac{y^2}{2} + aty + g(t),\ H_t = at + g' = ay + bt \Rightarrow g' = bt \Rightarrow g = \frac{bt^2}{2}$.

Therefore,

$$H = \frac{y^2}{2} + aty + \frac{bt^2}{2} = C \Rightarrow y^2 + 2aty + (bt^2 - 2C) = 0$$

$$\Rightarrow y = \frac{-2at \pm \sqrt{4a^2t^2 - 4(bt^2 - 2C)}}{2} = -at \pm \sqrt{a^2t^2 - bt^2 + 2C} = -at \pm \sqrt{2C - (b - a^2)t^2}$$

Choose the negative and $a = 1,\ 2C = 4,\ b - a^2 = 1 \Rightarrow b = 2$.

13. $\frac{2y}{y^2+1}y' + \frac{1}{t+1} = 0 \Rightarrow \ln(y^2+1) + \ln(t+1) = C$. From the initial condition, we have $\ln 2 = C$.

Thus $(y^2+1)(t+1) = 2$, and solving for y yields $y = \sqrt{\frac{1-t}{1+t}}$.

14. One example: $(y+2t)+(2y+t)y'=0,\ M=y+2t,\ N=2y+t$

$\dfrac{\partial H}{\partial t}=M=y+2t \Rightarrow H=yt+t^2+h(y),\ \dfrac{\partial H}{\partial y}=t+\dfrac{dh}{dy}=2y+t$. Therefore,

$\dfrac{dh}{dy}=2y \Rightarrow h=y^2 \Rightarrow yt+t^2+y^2=C$.

Section 3.4

1 (a). The equation is both separable and exact.

1 (b). (i) $y'=y(2-y) \Rightarrow y'-2y=-y^2 \Rightarrow\ 1-n=-1=m,\ v=y^{-1} \Rightarrow y=v^{-1}$, thus $y'=-v^2v'$ and $-v^{-2}v'=2v^{-1}-v^{-2}$ or $v'+2v=1,\ v(0)=1$.

(ii) $\left(e^{2t}v\right)'=e^{2t} \Rightarrow e^{2t}v=\dfrac{1}{2}e^{2t}+C$ or $v=\dfrac{1}{2}+Ce^{-2t}$. From the initial condition,

$\dfrac{1}{2}+C=1 \Rightarrow C=\dfrac{1}{2}$, and so $v=\dfrac{1}{2}\left(1+e^{-2t}\right)$.

(iii) $y=v^{-1}=\dfrac{2}{1+e^{-2t}}$.

1 (c). $-\infty<t<\infty$

2 (a). The equation is both separable and exact.

2 (b). (i) $y'=2ty-2ty^2 \Rightarrow\ 1-2=-1=m,\ v=y^{-1} \Rightarrow y=v^{-1}$, thus $-v^{-2}v'=2tv^{-1}-2tv^{-2}$ or $v'+2tv=2t,\ v(0)=-1$.

(ii) $\left(e^{t^2}v\right)'=2te^{t^2} \Rightarrow e^{t^2}v=e^{t^2}+C$ or $v=1+Ce^{-t^2}$. From the initial condition,

$1+C=-1 \Rightarrow C=-2$, and so $v=1-2e^{-t^2}$.

(iii) $y=v^{-1}=\dfrac{1}{1-2e^{-t^2}}$.

2 (c). $-\sqrt{\ln 2}<t<\sqrt{\ln 2}$

3 (a). The equation is neither separable nor exact.

3 (b). (i) $m=1-n=-1,\ v=y^{-1} \Rightarrow y=v^{-1}$, thus $y'=-v^{-2}v'=-v^{-1}+e^tv^{-2} \Rightarrow v'=v-e^t$ or $v'-v=-e^t,\ v(-1)=-1$.

(ii) $\left(e^{-t}v\right)'=-1 \Rightarrow e^{-t}v=-t+C$ or $v=-te^t+Ce^t$. From the initial condition,

$e^{-1}+Ce^{-1}=-1 \Rightarrow C=-(1+e)$, and so $v=e^t(-t-1-e)=-(t+1)e^t-e^{t+1}$.

(iii) $y=v^{-1}=\dfrac{-1}{(t+1)e^t+e^{t+1}}$.

3 (c). $-(1+e) < t < \infty$

4 (a). The equation is both separable and exact.

4 (b). (i) $1-n=2=m,\ v=y^2 \Rightarrow y=v^{\frac{1}{2}}$, thus $y'=\frac{1}{2}v^{-\frac{1}{2}}v'$ and $\frac{1}{2}v^{-\frac{1}{2}}v'=v^{\frac{1}{2}}+v^{-\frac{1}{2}}$ or

$v'=2v+2,\ v(0)=1$.

(ii) $\left(e^{-2t}v\right)'=2e^{-2t} \Rightarrow e^{-2t}v=-e^{-2t}+C$ or $v=-1+Ce^{2t}$. From the initial condition,

$-1+C=1 \Rightarrow C=2$, and so $v=-1+2e^{2t}$.

(iii) $y=-\sqrt{-1+2e^{2t}}$.

4 (c). $-\frac{1}{2}\ln 2 < t < \infty$

5 (a). The equation is neither separable nor exact.

5 (b). (i) $m=1-n=3,\ v=y^3 \Rightarrow y=v^{\frac{1}{3}}$, thus $y'=\frac{1}{3}v^{-\frac{2}{3}}v'$. Then $t\cdot\frac{1}{3}v^{-\frac{2}{3}}v'+v^{\frac{1}{3}}=t^3v^{-\frac{2}{3}}$, and so

$tv'+3v=3t^3,\ v(1)=1$.

(ii) $\left(t^3v\right)'=3t^5 \Rightarrow t^3v=\frac{t^6}{2}+C$ or $v=\frac{1}{2}t^3+\frac{C}{t^3}$. From the initial condition, $\frac{1}{2}+C=1 \Rightarrow C=\frac{1}{2}$,

and so $v=\frac{1}{2}\left(t^3+t^{-3}\right)$.

(iii) $y=v^{\frac{1}{3}}=\left(\frac{1}{2}\left(t^3+t^{-3}\right)\right)^{\frac{1}{3}}$.

5 (c). $0 < t < \infty$

6 (a). The equation is neither separable nor exact.

6 (b). (i) $m=1-n=\frac{2}{3},\ v=y^{\frac{2}{3}} \Rightarrow y=v^{\frac{3}{2}}$, thus $y'=\frac{3}{2}v^{\frac{1}{2}}v'$. Then $\frac{3}{2}v^{\frac{1}{2}}v'-v^{\frac{3}{2}}=tv^{\frac{1}{2}}$, and so

$v'-\frac{2}{3}v=\frac{2}{3}t,\ v(0)=4$.

(ii) $\left(e^{-\frac{2}{3}t}v\right)'=\frac{2}{3}te^{-\frac{2}{3}t} \Rightarrow e^{-\frac{2}{3}t}v=\frac{2}{3}\left(-\frac{3}{2}te^{-\frac{2}{3}t}-\frac{9}{4}e^{-\frac{2}{3}t}\right)+C$ or $v=-t-\frac{3}{2}+Ce^{\frac{2}{3}t}$. From the initial

condition, $-\frac{3}{2}+C=4 \Rightarrow C=\frac{11}{2}$, and so $v=-\left(t+\frac{3}{2}\right)+\frac{11}{2}e^{\frac{2}{3}t}$.

(iii) $y=-\left(\frac{11}{2}e^{\frac{2}{3}t}-\left(t+\frac{3}{2}\right)\right)^{\frac{3}{2}}$.

6 (c). $-\infty < t < \infty$

7. First, let $u = e^y$. Then $y = \ln u$ and $y' = \frac{u'}{u}$. Therefore, $\frac{u'}{u} = 2t^{-1} + \frac{1}{u} \Rightarrow u' - \frac{2}{t}u = 1$ which gives us $\frac{1}{t^2}u' - \frac{2}{t^3}u = \frac{1}{t^2}$. Then we have $\left(t^{-2}u\right)' = t^{-2} \Rightarrow t^{-2}u = -t^{-1} + C$. Solving for u gives us $u = -t + Ct^2$. From the initial condition, we have $y(1) = 0 \Rightarrow u(1) = 1$, and so $u = -t + 2t^2 \Rightarrow y = \ln\left(2t^2 - t\right),\ t > \frac{1}{2}$.

8. First, let $u = y + 1,\ u' = -u + tu^{-1},\ 1 - n = 3$. Therefore,

$v = u^3,\ u = v^{\frac{1}{3}},\ u' = \frac{1}{3}v^{-\frac{2}{3}}v' \Rightarrow \frac{1}{3}v^{-\frac{2}{3}}v' + v^{\frac{1}{3}} = tv^{-\frac{2}{3}}$. Then,

$v' + 3v = 3t,\ y(0) = 1 \Rightarrow u(0) = 2 \Rightarrow v(0) = 8$ and

$v = Ce^{-3t} + at + b,\ a + 3(at + b) = 3t \Rightarrow a = 1,\ b = -\frac{1}{3}$. Therefore,

$v = Ce^{-3t} + t - \frac{1}{3},\ v(0) = C - \frac{1}{3} = 8 \Rightarrow C = \frac{25}{3}$. Then,

$v = \frac{25}{3}e^{-3t} + t - \frac{1}{3},\ y = u - 1 = v^{\frac{1}{3}} - 1 = \left(\frac{25}{3}e^{-3t} + t - \frac{1}{3}\right)^{\frac{1}{3}} - 1,\ -\infty < t < \infty.$

9. $y_0 = 3$ by substitution. Differentiating yields

$$y' = \frac{-3e^{-t}}{1 - 3t} + 3e^{-t}\left(\frac{-1}{(1 - 3t)^2}\right)(-3) = -\frac{3}{(1 - 3t)e^t} + e^t\left(\frac{9}{(1 - 3t)^2 e^{2t}}\right) = -y + e^t y^2.$$

Thus $q(t) = e^t$.

Section 3.5

1. $1 - n = -1,\ v = P^{-1}, P = v^{-1}$. Thus $-v^{-2}v' - rv^{-1} = -\frac{r}{P_e}v^{-2}$, or $v' + rv = \frac{r}{P_e},\ v(0) = P_0^{-1}$. Then $v = Ce^{-rt} + \frac{1}{P_e},\ v(0) = \frac{1}{P_0} = C + \frac{1}{P_e}$. Solving for C yields $C = P_0^{-1} - P_e^{-1}$, so we have $v = P^{-1} = \left(P_0^{-1} - P_e^{-1}\right)e^{-rt} + P_e^{-1}$. Thus $P = \frac{1}{P_0^{-1}e^{-rt} - P_e^{-1}\left(e^{-rt} - 1\right)} = \frac{P_0 P_e}{P_0 - \left(P_0 - P_e\right)e^{-rt}}$.

2. Since P is measured in millions, $P_0 = 0.1,\ r = 0.1,\ P_e = 3$. Therefore,

$P = \frac{0.1(3)}{0.1 - (0.1 - 3)e^{-0.1t}},\ 0.9P_e = 2.7 \Rightarrow 2.7 = \frac{0.3}{0.1 + 2.9e^{-0.1t}} \Rightarrow e^{-.1t} \approx .003831417$

$\Rightarrow t \approx 55.65\ \ years.$

3 (a). Setting $r\left(1-\dfrac{P}{P_e}\right)P+M=0$, we have $-\dfrac{P^2}{P_e}+P+\dfrac{M}{r}=0$. Then

$P^2-P_eP-P_e\dfrac{M}{r}=0 \Rightarrow P=\dfrac{P_e\pm\sqrt{P_e^{\,2}+4P_e{M}/{r}}}{2}$. This makes sense; migration would alter the equilibrium state.

3 (b). $(2P-1)^2=1+4x \Rightarrow \left(P-\dfrac{1}{2}\right)^2=x+\dfrac{1}{4}$, where $x=\dfrac{M}{r}$. This is a parabola with vertex $\left(-\dfrac{1}{4},\dfrac{1}{2}\right)$.

For $x>0$, there is one nonnegative equilibrium solution. Two such solutions exist for $-\dfrac{1}{4}<x\le 0$.

3 (c). When $x=-\dfrac{1}{4}$, the two nonnegative equilibrium solutions coalesce into a single equilibrium value. There are no equilibrium solutions for $x<-\dfrac{1}{4}$. This makes sense, since if the migration out of the colony is too large relative to reproduction, equilibrium could not be achieved.

4. Equilibrium values at 4 and -2. $P=\dfrac{P_e}{2}\left(1\pm\sqrt{1+\dfrac{4M}{P_er}}\right)\Rightarrow 4+(-2)=P_e \Rightarrow P_e=2$.

$4-(-2)=6=P_e\sqrt{1+\dfrac{4M}{P_er}}$ or $3\sqrt{1+\dfrac{2M}{r}}=8=\dfrac{2M}{r}\Rightarrow \dfrac{M}{r}=4$.

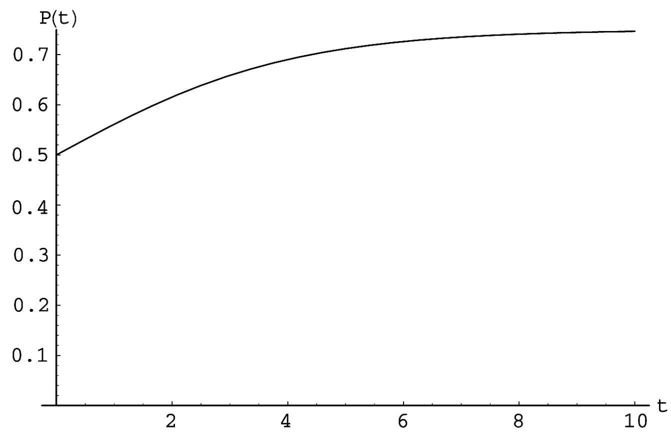
5. $P'=(1-P)P-\dfrac{3}{16},\ P(0)=\dfrac{1}{2}$. Then $P'=-P^2+P-\dfrac{3}{16}=-\left(P-\dfrac{3}{4}\right)\left(P-\dfrac{1}{4}\right)$. Then we have

$\dfrac{1}{\left(P-\frac{1}{4}\right)\left(P-\frac{3}{4}\right)}P'+1=0$, and by partial fractions, we have $\left(\dfrac{-2}{P-\frac{1}{4}}+\dfrac{2}{P-\frac{3}{4}}\right)P'+1=0$. Then

$\displaystyle\int\left[\dfrac{-2}{\left(P-\frac{1}{4}\right)}+\dfrac{2}{\left(P-\frac{3}{4}\right)}\right]dP+t=C$, and so $2\ln\left|\dfrac{P-\frac{3}{4}}{P-\frac{1}{4}}\right|+t=C$.

From the initial condition, we have $C=0$, so

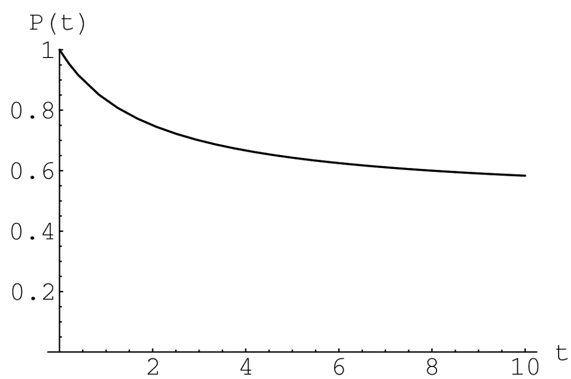
$\left|\dfrac{P-\frac{3}{4}}{P-\frac{1}{4}}\right|=e^{-\frac{t}{2}}$. But $\dfrac{1}{4}<P<\dfrac{3}{4}\Rightarrow \left|P-\tfrac{3}{4}\right|=-\left(P-\tfrac{3}{4}\right)\Rightarrow \dfrac{\frac{3}{4}-P}{P-\frac{1}{4}}=e^{-\frac{t}{2}}$, and then $P(t)=\dfrac{\frac{3}{4}+\frac{1}{4}e^{-\frac{t}{2}}}{1+e^{-\frac{t}{2}}}$.



6. $P' = (1-P)P - \frac{1}{4},\ P_0 = 1 \Rightarrow P' = -\left(P^2 - P + \frac{1}{4}\right) = -\left(P - \frac{1}{2}\right)^2$ which is separable.

$$\frac{1}{\left(P-\frac{1}{2}\right)^2}P' + 1 = 0 \Rightarrow -\left(P - \frac{1}{2}\right)^{-1} + t = C,\ -\left(1 - \frac{1}{2}\right)^{-1} + 0 = -2 = C$$

Therefore, $\left(P - \frac{1}{2}\right)^{-1} = 2 + t \Rightarrow P = \frac{1}{2} + \frac{1}{2+t} = \frac{2 + \frac{t}{2}}{2+t}$.

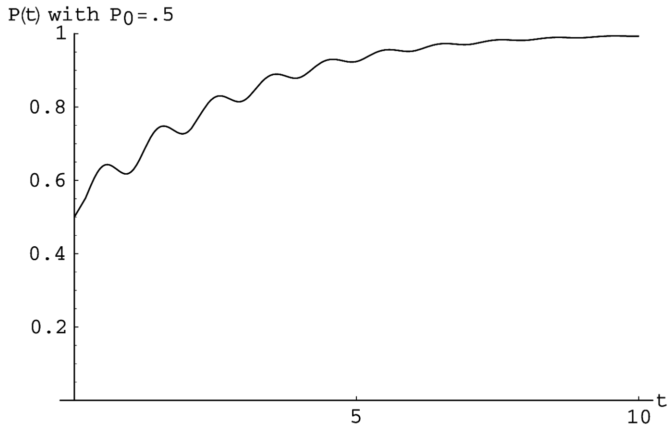


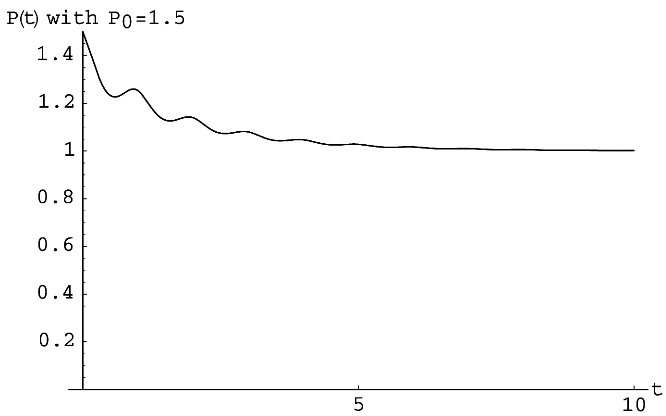
7. $P' = .5\left(1 + 2\sin(2\pi t)\right)(1-P)P,\ P(0) = P_0$. Following the derivation in the chapter with $r(t)$,

we have $R(t) = 0.5\int_0^t \left(1 + 2\sin(2\pi s)\right)ds = \frac{1}{2}\left(s - \frac{1}{\pi}\cos(2\pi s)\right)\Big|_0^t$

$= \frac{1}{2}\left(t - \frac{1}{\pi}\cos(2\pi t) + \frac{1}{\pi}\right) = \frac{1}{2}\left(t + \frac{1}{\pi}\left[1 - \cos(2\pi t)\right]\right)$. Therefore $P = \frac{P_0}{P_0 - \left(P_0 - 1\right)e^{-R(t)}}$ with

$R(t) = \frac{1}{2}\left(t + \frac{1}{\pi}\left[1 - \cos(2\pi t)\right]\right)$.





8. $\tau = \int_0^t r(s)ds \Rightarrow \dfrac{dP}{d\tau} = (1-P)P$. The solution procedure in the text leads to

$P(\tau) = \dfrac{P_0}{P_0 - (P_0 - 1)e^{-\tau}}$. Substitute $\tau = \dfrac{1}{2}\left(t + \dfrac{1}{\pi}[1-\cos(2\pi t)]\right)$.

9. $P' = r(1-P)P = rP - rP^2 \Rightarrow v = P^{-1},\ P = v^{-1},\ -v^{-2}v' = rv^{-1} - rv^{-2} \Rightarrow v' + rv = r,\ v(0) = P_0^{-1}$.

Letting $R(t) = \int_0^t r(s)ds$, we have $\left(e^R v\right)' = re^R \Rightarrow e^R v = e^R + C \Rightarrow v = 1 + Ce^{-R}$. $v(0) = P_0^{-1}$, so

$C = P_0^{-1} - 1$ and thus $v = 1 + \left(P_0^{-1} - 1\right)e^{-R}$. Finally, $P = v^{-1} = \dfrac{1}{1 + \left(P_0^{-1} - 1\right)e^{-R}} = \dfrac{P_0}{P_0 - \left(P_0 - 1\right)e^{-R}}$

with $R(t) = \tau = \dfrac{1}{2}\left(t + \dfrac{1}{\pi}[1-\cos(2\pi t)]\right)$.

10 (a). $P' = k(N-P)P$ with N and P in units of 100,000 and t in months. $N = 5,\ k = 2e^{-t} - 1$.

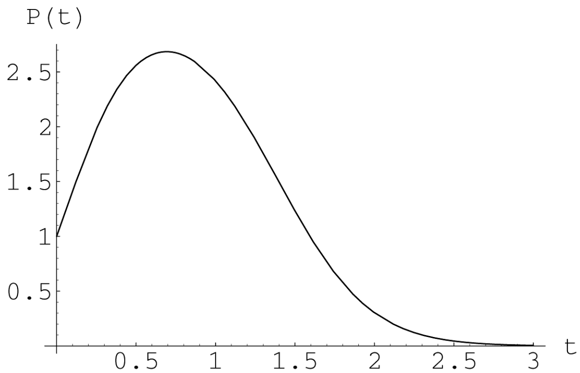
$\tau = \int_0^t (2e^{-s} - 1)ds = \left(-2e^{-s} - s\right)\Big|_0^t = \left(-2e^{-t} - t + 2\right) = -t + 2(1 - e^{-t})$.

$\dfrac{dP}{d\tau} = (5-P)P$ which is separable.

$\dfrac{1}{(5-P)P} = \dfrac{A}{P} + \dfrac{B}{5-P},\ A = \dfrac{1}{5},\ B = \dfrac{1}{5} \Rightarrow \dfrac{1}{5}(\ln P - \ln|5-P|) = \tau + C = \dfrac{1}{5}\ln\left|\dfrac{P}{5-P}\right|$

From the initial condition, $\frac{1}{5}\ln\left(\frac{1}{4}\right) = 0 + C \Rightarrow \ln\left|\frac{P}{5-P}\right| = 5\tau + \ln\left(\frac{1}{4}\right) \Rightarrow \frac{P}{5-P} = \frac{1}{4}e^{5\tau}$

Therefore, $P = \frac{5e^{5\tau}}{4+e^{5\tau}}$, $\tau = -t + 2(1-e^{-t})$.



10 (b). From the plot, $P_{\max} \approx 2.7$ (270,000).

10 (c). From the plot, $t \approx 1.8$ *months*.

11 (a). $(A-B)' = -kAB + kAB = 0$, $A(t) - B(t) = A(0) - B(0) = 5 - 2 = 3$ moles.

11 (b). $B = A - 3$, $A' = -kA(A-3) = k(3-A)A$, $A(0) = 5$.

11 (c). $A(1) = 4$, $A' = 3k\left(1 - \frac{A}{3}\right)A$. Using equation (5), $A(t) = \frac{5\cdot 3}{5-(5-3)e^{-3kt}}$. Thus $A(t) = \frac{15}{5-2e^{-3kt}}$.

We know that $A(1) = 4$, so $\frac{15}{5-2e^{-3k}} = 4$. Solving for e^{-3k} yields $e^{-3k} = \frac{5}{8}$. Thus

$A(4) = \frac{15}{5-2\left(\frac{5}{8}\right)^4} = 3.195$ moles. $B = A - 3 = 0.195$ moles.

Section 3.6

1. With $v_0 = 0$, $v = -\frac{mg}{k}\left(1 - e^{-\frac{k}{m}t}\right)$. Setting $v = -\frac{1}{2}\frac{mg}{k}$ gives us $1 - e^{-\frac{k}{m}t} = \frac{1}{2}$. Thus $e^{-\frac{k}{m}t} = \frac{1}{2}$,

$\frac{k}{m}t = \ln 2$, $t = \frac{m}{k}\ln 2$.

2. $mv' = -mg + \kappa v^2$, $v(0) = 0 \Rightarrow v' = -g + \frac{\kappa}{m}v^2 = \frac{\kappa}{m}\left(v^2 - \frac{mg}{\kappa}\right) \frac{v'}{v^2 - \frac{mg}{\kappa}} = \frac{\kappa}{m}$

$$\frac{1}{v^2-\frac{mg}{\kappa}}=\frac{A}{v-\sqrt{\frac{mg}{\kappa}}}+\frac{B}{v+\sqrt{\frac{mg}{\kappa}}}\Rightarrow A=\frac{1}{2\sqrt{\frac{mg}{\kappa}}},\ B=-\frac{1}{2\sqrt{\frac{mg}{\kappa}}}.\text{ Therefore,}$$

$$\frac{1}{2\sqrt{\frac{mg}{\kappa}}}\ln\left|\frac{v-\sqrt{\frac{mg}{\kappa}}}{v+\sqrt{\frac{mg}{\kappa}}}\right|=\frac{\kappa}{m}t+C,\ v(0)=0\Rightarrow C=0\text{ and } -\sqrt{\frac{mg}{\kappa}}<v\le 0.\text{ Then,}$$

$$\left|\frac{v-\sqrt{\frac{mg}{\kappa}}}{v+\sqrt{\frac{mg}{\kappa}}}\right|=\frac{\sqrt{\frac{mg}{\kappa}}-v}{\sqrt{\frac{mg}{\kappa}}+v}=e^{2\sqrt{\frac{\kappa g}{m}}t}\Rightarrow v=-\sqrt{\frac{mg}{\kappa}}\left(\frac{1-e^{-2\sqrt{\frac{\kappa g}{m}}t}}{1+e^{-2\sqrt{\frac{\kappa g}{m}}t}}\right)=-\sqrt{\frac{mg}{\kappa}}\tanh\left(\sqrt{\frac{\kappa g}{m}}t\right).$$

3. $10\ mi/hr=10\left(\frac{5280}{3600}\right)=14.67$ ft/sec. Then $14.67=\sqrt{\frac{200}{\kappa}}\Rightarrow\kappa\approx .929\frac{\text{lb}\cdot\text{sec}^2}{\text{ft}^2}$.

4 (a). $m\frac{dv}{dt}+kv=0\Rightarrow v(t)=v_0e^{-\frac{k}{m}t},\ m=\frac{3000}{32}slug$

$$\frac{v(4)}{v_0}=\frac{50}{220}=e^{-k\cdot\frac{32}{3000}\cdot 4}\Rightarrow\ln\left(\frac{22}{5}\right)=\frac{128}{3000}k.\text{ Then, }k=\frac{3000}{128}\ln\left(\frac{22}{5}\right)=34.725\ \ lb\cdot\sec/ft.$$

4 (b). $d=\int_0^4 v(t)dt=v_0\int_0^4 e^{-\frac{k}{m}t}dt=v_0\left(-\frac{m}{k}e^{-\frac{k}{m}t}\right)\Big|_0^4=\frac{mv_0}{k}\left(1-e^{-\frac{4k}{m}}\right)$

$$=\frac{3000}{32}\left(220\cdot\frac{5280}{3600}\right)\left(\frac{1}{34.725}\right)\left(\frac{170}{220}\right)\approx 673\ \ ft.$$

5. $mv'+\kappa v^2=0\Rightarrow\frac{v'}{v^2}=-\frac{\kappa}{m}\Rightarrow -v^{-1}=-\frac{\kappa}{m}t+C,\ C=-v_0^{-1}$. Then we have

$v^{-1}=\frac{\kappa}{m}t+v_0^{-1}\Rightarrow v=\frac{v_0}{1+\frac{\kappa}{m}v_0t}$. From the condition provided, we have

$\frac{v(4)}{v_0}=\frac{50}{220}=\frac{1}{1+4\frac{\kappa}{m}v_0}\Rightarrow 4\frac{\kappa}{m}v_0=\frac{1-\frac{5}{22}}{\frac{5}{22}}=\frac{17}{5}$. Solving for κ yields

$$\kappa=\frac{17}{5}\frac{m}{4v_0}=\frac{17}{5}\frac{3000}{32}\cdot\frac{1}{4}\div\left(220\left(\frac{5280}{3600}\right)\right)\approx .247\frac{\text{lb}\cdot\text{sec}^2}{\text{ft}^2}.$$

For the distance traveled, $d=\int_0^4 v(t)dt=v_0\int_0^4\frac{dt}{1+\frac{\kappa v_0}{m}t}=v_0\int_0^4\frac{dt}{1+\frac{17}{20}t}=v_0\left(\frac{20}{17}\right)\ln\left(1+\frac{17}{20}t\right)\Big|_0^4$

$$=220\left(\frac{5280}{3600}\right)\left(\frac{20}{17}\right)\ln\left(1+\frac{17}{5}\right)=562.4\ \text{ft.}$$

6. $mv' + kv = -mg,\ v(0) = v_0 \Rightarrow v(t) = -\frac{mg}{k} + \left(v_0 + \frac{mg}{k}\right)e^{-\frac{k}{m}t}$. Set

$v = 0$: $\frac{mg}{k} = \frac{mg}{k}\left(1 + \frac{kv_0}{mg}\right)e^{-\frac{k}{m}t_m} \Rightarrow \frac{k}{m}t_m = \ln\left(1 + \frac{kv_0}{mg}\right) \Rightarrow t_m = \frac{m}{k}\ln\left(1 + \frac{kv_0}{mg}\right)$.

7. $h = \int_0^{t_m} v(t)dt = \int_0^{t_m}\left[-\frac{mg}{k} + \left(v_0 + \frac{mg}{k}\right)e^{-\frac{k}{m}t}\right]dt = \left[-\frac{mg}{k}t - \frac{m}{k}\left(v_0 + \frac{mg}{k}\right)e^{-\frac{k}{m}t}\right]_0^{t_m}$

$= -\frac{mg}{k}t_m + \frac{m}{k}\left(v_0 + \frac{mg}{k}\right)\left(1 - e^{-\frac{k}{m}t_m}\right)$.

8. $mv' = -mg \Rightarrow v' = -g,\ v(0) = v_0$. Therefore, $v(t) = v_0 - gt$ and $y = -g\frac{t^2}{2} + v_0 t,\ t_m = \frac{v_0}{g}$. The impact time is given by $-g\frac{t_i^2}{2} + v_0 t_i = 0 \Rightarrow -\frac{g}{2}t_i + v_0 = 0 \Rightarrow t_i = \frac{2v_0}{g} = 2t_m$.

9 (a). $v' = -g,\ v_0 = 0 \Rightarrow v = -gt = y' \Rightarrow y = -\frac{1}{2}gt^2 + y_0$. We want to find the time t at which y=7. Thus $7 = -\frac{32}{2}t^2 + 555$, and solving for t yields $t \approx 5.852$ sec. At that time, $v = -32(5.852) \approx -187.3$ft/sec.

9 (b). $mv' + kv = -mg \Rightarrow v' + \frac{kv}{m} = -g,\ v_0 = 0$. Thus $\left(ve^{\frac{k}{m}t}\right)' = -ge^{\frac{k}{m}t} \Rightarrow ve^{\frac{k}{m}t} = -\frac{mg}{k}e^{\frac{k}{m}t} + C$. From the initial condition, we have $C = \frac{mg}{k}$, and so

$$v = -\frac{mg}{k}\left(1 - e^{-\frac{k}{m}t}\right) \Rightarrow y = y_0 + \int_0^t v(s)ds = y_0 - \frac{mg}{k}\left(s + \frac{m}{k}e^{-\frac{k}{m}s}\right)\Big|_0^t = y_0 - \frac{mg}{k}\left(t + \frac{m}{k}\left(e^{-\frac{k}{m}t} - 1\right)\right).$$

$m = \frac{5\frac{1}{8}/16}{32} = \frac{41}{8 \cdot 16 \cdot 32}$ slug, so $\frac{m}{k} = \frac{41}{8(16)(32)(0.0018)} \approx 5.56098\ \text{sec}^{-1}$.

$mg = \frac{41}{8(16)} \approx 0.3203125$ lb, and so solving for t yields

$7 = 555 - 177.95139\left(t - 5.56098\left[1 - e^{\frac{-t}{5.56098}}\right]\right) \Rightarrow t = 7.08513$ sec. Substitution gives us

$$v = \frac{-0.3203125}{0.0018}\left[1 - e^{\frac{-7.08513}{5.56098}}\right] \approx -128.18\ \text{ft/sec}.$$

10. $mg = 180\ \ lb$. For $0 \le t \le 10,\ v' = -g,\ v(0) = 0$.

For $10 < t \le 14,\ mv' + kv = -mg,\ y(14) = 0$.

For $mg = 200,\ \dfrac{200}{\text{k}} = 10\dfrac{5280}{3600} \Rightarrow k = \dfrac{3600(200)}{5280(10)} = 13.63636364$.

10 (a). $v = -gt$ At $t = 10,\ v = -320\ \ ft/\sec$.

10 (b). Solve $v' + \dfrac{k}{m}v = -g,\ v(0) = -320$, for $v(4)$.

$$v(t) = -\frac{mg}{k} + \left(v_0 + \frac{mg}{k}\right)e^{-\frac{k}{m}t} \Rightarrow v(4) = -\frac{180}{13.\overline{63}} + \left(-320 + \frac{180}{13.\overline{63}}\right)e^{-\frac{13.\overline{63}(32)}{180}(4)}$$

$= -13.2 - 306.8(.000061469) = -13.219\ \ ft/\sec$ (basically the terminal velocity).

10 (c). $$h = -\int_0^4 v(t)dt = \left(\frac{mg}{k}t - \left[v_0 + \frac{mg}{k}\right]\left(-\frac{m}{k}\right)e^{-\frac{k}{m}t}\right)\Bigg|_0^4 = \frac{mg}{k}(4) + \frac{m}{k}\left(v_0 + \frac{mg}{k}\right)\left(e^{-\frac{4k}{m}} - 1\right)$$

$$= \frac{4mg}{k} - \frac{m}{k}\left(v_0 + \frac{mg}{k}\right)\left(1 - e^{-\frac{4k}{m}}\right) = \frac{4(180)}{13.\overline{63}} - \frac{180}{32(13.\overline{63})}\left(-320 + \frac{180}{13.\overline{63}}\right)\left(1 - e^{-\frac{4(13.\overline{63})32}{180}}\right)$$

$= 52.8 - 0.4125(-306.8)(0.99994) = 179.347\ \ ft$.

10 (d). $h_{balloon} = h + \dfrac{1}{2}g(10)^2 = 179.347 + 1600 = 1779.347\ \ ft$.

11. For the first situation, $mv_1' + kv_1 = 0,\ v_1 = v_0 e^{-\frac{k}{m}t},\ m = \dfrac{3000}{32},\ k = 25$. Then

$$\frac{50}{220} = e^{\frac{-25\cdot 32}{3000}t_1} \Rightarrow t_1 = \frac{3000}{25(32)}\ln\frac{22}{5} \approx 5.556\ \text{sec}.$$

For the second situation, $mv_2' + k(\tanh t)v_2 = 0,\ v_2' + \dfrac{k}{m}\tanh t(v_2) = 0$. This is a first order linear equation. Letting $\mu = e^{\frac{k}{m}\ln(\cosh t)} = (\cosh t)^{\frac{k}{m}}$, we have

$$\left(v_2(\cosh t)^{\frac{k}{m}}\right)' = 0 \Rightarrow v_2 = C(\cosh t)^{-\frac{k}{m}}.$$

From the initial condition, we have $\cosh(0) = 1 \Rightarrow C = v_0$. Then

$$\frac{v_2}{v_0} = (\cosh t)^{-\frac{k}{m}} \Rightarrow \cosh t_2 = \left(\frac{v_0}{v_2}\right)^{\frac{m}{k}} = \left(\frac{220}{50}\right)^{\frac{3000}{32\cdot 25}}.$$

$\ln(\cosh t_2) = 3.75\ln\left(\dfrac{22}{5}\right) \approx 5.55602 \Rightarrow \cosh t_2 \approx 258.79$, so $t_2 \approx \cosh^{-1}(258.79) \approx 6.249$ sec.

This would be expected, since the size of the drag coefficient would be less for the second situation. Comparing the two values gives us $t_1 \approx 0.89 t_2$. These values do not seem appreciably different. However, it can be shown that this difference in stopping time leads to a difference in stopping distance of approximately 110 ft. If this distance is important for a certain situation, then the idealization is not reasonable.

Section 3.7

1. $\frac{dv}{dt} = -\frac{k}{m}x^2 v \Rightarrow v\frac{dv}{dx} = -\frac{k}{m}x^2 v \Rightarrow \frac{dv}{dx} = -\frac{k}{m}x^2 \Rightarrow v = -\frac{k}{m}\frac{x^3}{3} + C$. When $x = 0$, $v = v_0$.

   Therefore, $v_0 = C$, and so $v = -\frac{k}{m}\frac{x^3}{3} + v_0$ and $x_f^{\ 3} = 3\frac{m}{k}v_0 \Rightarrow x_f = \left(3\frac{m}{k}v_0\right)^{\frac{1}{3}}$.

2. $mv\frac{dv}{dx} = -kxv^2 \Rightarrow \frac{dv}{dx} = -\frac{k}{m}xv \Rightarrow \frac{dv}{dx} + \frac{k}{m}xv = 0$ (first order linear).

   $\frac{d}{dx}\left(e^{\frac{k}{m}\frac{x^2}{2}}v\right) = 0 \Rightarrow v = Ce^{-\frac{kx^2}{2m}},\ C = v_0 \Rightarrow v = v_0 e^{-\frac{kx^2}{2m}}$. Since $v > 0,\ 0 \le x < \infty,\ x_f = \infty$.

3. $mv\frac{dv}{dx} = -ke^{-x} \Rightarrow v\frac{dv}{dx} + \frac{k}{m}e^{-x} = 0 \Rightarrow \frac{v^2}{2} - \frac{k}{m}e^{-x} = C$. Then $C = \frac{v_0^{\ 2}}{2} - \frac{k}{m}$, and so

   $v^2 = 2\left[\frac{v_0^{\ 2}}{2} - \frac{k}{m} + \frac{k}{m}e^{-x}\right] \Rightarrow v = \left[v_0^{\ 2} - 2\frac{k}{m}\left(1 - e^{-x}\right)\right]^{\frac{1}{2}}$. If $v_0^{\ 2} \ge \frac{2k}{m}$, then v>0 for all nonnegative x and $x_f = \infty$. If $v_0^{\ 2} < \frac{2k}{m}$, then we have $v_0^{\ 2} = \frac{2k}{m}\left(1 - e^{-x_f}\right)$, which, solved for x_f, yields

   $x_f = -\ln\left(1 - \frac{mv_0^{\ 2}}{2k}\right)$.

4. $mv\frac{dv}{dx} = -\frac{kv}{1+x} \Rightarrow \frac{dv}{dx} = -\frac{k}{m}\left(\frac{1}{1+x}\right) \Rightarrow v = -\frac{k}{m}\ln(1+x) + C,\ v_0 = C$. Therefore,

   $v = v_0 - \frac{k}{m}\ln(1+x)$ and $\frac{mv_0}{k} = \ln(1+x_f) \Rightarrow x_f = e^{mv_0/k} - 1$.

5. $m\frac{dv}{dt} + kv^2 = 0,\ v(0) = v_0,\ x(0) = 0$. We want to find v when x=d.

   $mv\frac{dv}{dx} + kv^2 = 0 \Rightarrow \frac{dv}{dx} + \frac{k}{m}v = 0 \Rightarrow v = Ce^{-\frac{k}{m}x}$. From the initial condition, $v = v_0 e^{-\frac{k}{m}x}$, and so at x=d, $v = v_0 e^{-\frac{k}{m}d}$.

6. $m\frac{dv}{dt}=-mg-kv^2,\ v(0)=v_0,\ x(0)=0 \Rightarrow mv\frac{dv}{dy}=-mg-kv^2 \Rightarrow \frac{dv}{dy}=-\frac{k}{m}v-gv^{-1}$

$\Rightarrow \frac{dv}{dy}+\frac{k}{m}v=-gv^{-1}$ (Bernoulli).

$1-n=2,\ u=v^2 \Rightarrow v=u^{\frac{1}{2}},\ \frac{dv}{dy}=\frac{1}{2}u^{-\frac{1}{2}}\frac{du}{dy} \Rightarrow \frac{1}{2}u^{-\frac{1}{2}}\frac{du}{dy}+\frac{k}{m}u^{\frac{1}{2}}=-gu^{-\frac{1}{2}}$. Therefore,

$\frac{du}{dy}+\frac{2k}{m}u=-2g,\ u=v_0^2$ when $y=0$. $\frac{d}{dy}\left(e^{\frac{2ky}{m}}u\right)=-2ge^{\frac{2ky}{m}} \Rightarrow e^{\frac{2ky}{m}}u=-\frac{mg}{k}e^{\frac{2ky}{m}}+C,\ C=v_0^2+\frac{mg}{k}$

Therefore, $u=-\frac{mg}{k}+\left(v_0^2+\frac{mg}{k}\right)e^{-\frac{2ky}{m}}=v^2 \Rightarrow v=\left[-\frac{mg}{k}+\left(v_0^2+\frac{mg}{k}\right)e^{-\frac{2ky}{m}}\right]^{\frac{1}{2}}$. This equation is valid for $0\le y\le h$, where $h=$ maximum height.

$-\frac{mg}{k}+\left(v_0^2+\frac{mg}{k}\right)e^{-\frac{2k}{m}h}=0 \Rightarrow -\frac{2k}{m}h=\ln\left[\frac{\frac{mg}{k}}{v_0^2+\frac{mg}{k}}\right] \Rightarrow h=\frac{m}{2k}\ln\left[1+\frac{kv_0^2}{mg}\right].$

7. With x measured as shown and $v=\frac{dx}{dt}$, we have $-m\frac{dv}{dt}=F\cos\theta$. Defining $\cos\theta=\frac{x}{\left(x^2+h^2\right)^{\frac{1}{2}}}$, we have $-mv\frac{dv}{dx}=\frac{Fx}{\left(x^2+h^2\right)^{\frac{1}{2}}} \Rightarrow -m\frac{v^2}{2}=F\left(x^2+h^2\right)^{\frac{1}{2}}+C$. We know that $v=0$ when $x=D$, so $C=-F\left(D^2+h^2\right)^{\frac{1}{2}}$. Then we have $v^2=\frac{2}{m}\left(F\left(D^2+h^2\right)^{\frac{1}{2}}-F\left(x^2+h^2\right)^{\frac{1}{2}}\right)$.

When $x=\frac{D}{3},\ v=-\left(\frac{2F}{m}\left(\sqrt{D^2+h^2}-\sqrt{\frac{D^2}{9}+h^2}\right)\right)^{\frac{1}{2}}$.

8. $P=Fv,\ m\frac{dv}{dt}=F=\frac{P}{v} \Rightarrow mv\frac{dv}{dx}=\frac{P}{v} \Rightarrow v^2\frac{dv}{dx}=\frac{P}{m} \Rightarrow \frac{v^3}{3}=\frac{P}{m}x+C$

$\frac{v_1^3}{3}=\frac{P}{m}x_1+C \Rightarrow C=\frac{v_1^3}{3}-\frac{P}{m}x_1,\ \frac{v_2^3}{3}=\frac{P}{m}x_2+\frac{v_1^3}{3}-\frac{P}{m}x_1$. Therefore,

$x_2-x_1=\frac{m}{P}\left(\frac{v_2^3}{3}-\frac{v_1^3}{3}\right)=\frac{m}{3P}\left(v_2^3-v_1^3\right),\ m=\frac{3000}{32},\ P=200(550)\,ft\cdot lb/\sec$

$v_2=50\frac{5280}{3600}\,ft/\sec,\ v_1=\frac{2}{5}v_2 \Rightarrow \Delta x=\frac{3000}{32}\cdot\frac{1}{3}\cdot\frac{1}{200(550)}\left(\frac{50(5280)}{3600}\right)^3\left(1-\left(\frac{2}{3}\right)^3\right)$

$\Rightarrow \Delta x=112.04(.936)\approx 104.87\,ft.$

9 (a). $mv\frac{dv}{dx}+\kappa_0 xv^2=0,\ v=v_0$ when $x=0$.

9 (b). $\frac{dv}{dx}+\frac{\kappa_0}{m}xv=0 \Rightarrow \left(e^{\frac{\kappa_0 x^2}{2m}}v\right)'=0 \Rightarrow v=v_0 e^{-\frac{\kappa_0 x^2}{2m}}$. Setting $x=d$ and $v=0.01v_0$, we have

$0.01v_0=v_0 e^{-\frac{\kappa_0 d^2}{2m}} \Rightarrow \frac{\kappa_0 d^2}{2m}=\ln 100$. Solving for κ_0 yields $\kappa_0=\frac{2m}{d^2}\ln 100$.

10 (a). $mv\frac{dv}{dr}=-\frac{GmM_e}{r^2}+\kappa v^2 \Rightarrow \frac{dv}{dr}=\frac{\kappa}{m}v-\frac{GM_e}{r^2}v^{-1},\ v=0$ when $r=R_e+h$.

10 (b). Bernoulli equation: $1-n=-1 \Rightarrow n=2,\ u=v^2 \Rightarrow v=u^{\frac{1}{2}} \Rightarrow \frac{dv}{dr}=\frac{1}{2}u^{-\frac{1}{2}}\frac{du}{dr}=\frac{\kappa}{m}u^{\frac{1}{2}}-\frac{GM_e}{r^2}u^{-\frac{1}{2}}$

$\Rightarrow \frac{du}{dr}=\frac{2\kappa}{m}u-\frac{2GM_e}{r^2}$. Therefore,

$$\left(e^{-\frac{2\kappa}{m}r}u\right)'=2GM_e\frac{e^{-\frac{2\kappa}{m}r}}{r^2} \Rightarrow e^{-\frac{2\kappa}{m}(R_e+h)}u\Big|_{r=R_e+h}-e^{-\frac{2\kappa}{m}(R_e)}u\Big|_{r=R_e}-2GM_e\int_{R_e}^{R_e+h}\frac{e^{-\frac{2\kappa}{m}r}}{r^2}dr$$

$$=0-e^{-\frac{2\kappa}{m}(R_e)}u\Big|_{r=R_e}-2GM_e\int_{R_e}^{R_e+h}\frac{e^{-\frac{2\kappa}{m}r}}{r^2}dr.$$

Since $u=v^2,\ v=\frac{dr}{dt}<0,\ v_{impact}=-e^{\frac{\kappa}{m}(R_e)}\left[2GM_e\int_{R_e}^{R_e+h}\frac{e^{-\frac{2\kappa}{m}r}}{r^2}dr\right]^{\frac{1}{2}}$. Let

$r=R_e+s$. Then $v_{impact}=-\left[2GM_e\int_0^h\frac{e^{-\frac{2\kappa}{m}s}}{(R_e+s)^2}ds\right]^{\frac{1}{2}}$.

11. $m\frac{dv}{dt}=-\frac{GM_e m}{r^2} \Rightarrow v\frac{dv}{dr}=-\frac{GM_e}{r^2},\ v=v_0$ when $r=R_e$. Thus $\frac{v^2}{2}=\frac{G}{r}M_e+C$, and from our initial condition, $\frac{{v_0}^2}{2}=\frac{G}{R_e}M_e+C \Rightarrow \frac{v^2}{2}=\frac{{v_0}^2}{2}+GM_e\left(\frac{1}{r}-\frac{1}{R_e}\right)$. Since $v=0$ when $r=R_e+h$,

${v_0}^2=2GM_e\left(\frac{1}{R_e}-\frac{1}{R_e+h}\right)$. Thus

$$v_0=\left[2GM_e\left(\frac{1}{R_e}-\frac{1}{R_e+h}\right)\right]^{\frac{1}{2}}=\left[\frac{2(6.673)(10^{-11})(5.976)(10^{24})}{10^6}\left(\frac{1}{6.371}-\frac{1}{6.591}\right)\right]^{\frac{1}{2}}\approx 2044\text{ m/sec}.$$

12 (a). $m\ell^2\theta''=-mg\ell\sin\theta=m\ell^2\frac{d\omega}{dt} \Rightarrow m\ell^2\omega\frac{d\omega}{d\theta}=-mg\ell\sin\theta$

$m\ell^2\omega\frac{d\omega}{d\theta}=-mg\ell\sin\theta$ and $\omega=-\omega_0$ when $\theta=\pi$.

12 (b). $m\ell^2\frac{\omega^2}{2}=mg\ell\cos\theta+C,\ m\ell^2\frac{{\omega_0}^2}{2}=-mg\ell+C \Rightarrow m\ell^2\frac{\omega^2}{2}-mg\ell\cos\theta=m\ell^2\frac{{\omega_0}^2}{2}+mg\ell$.

12 (c). When $\theta = 0$, $m\ell^2 \dfrac{\omega^2}{2} - mg\ell = m\ell^2 \dfrac{{\omega_0}^2}{2} + mg\ell \Rightarrow \omega^2 = \omega_0^2 + 2mg\ell\left(\dfrac{2}{m\ell^2}\right) = \omega_0^2 + \dfrac{4g}{\ell}$

$\Rightarrow \omega = \sqrt{\omega_0^2 + \dfrac{4g}{\ell}}$.

13. $m\ell^2 \dfrac{\omega^2}{2} = mg\ell\cos\theta + C$, $\omega = \omega_0$ when $\theta = 0$. Therefore, $C = m\ell^2 \dfrac{{\omega_0}^2}{2} - mg\ell$, and so

$m\ell^2 \dfrac{\omega^2}{2} = mg\ell\cos\theta + m\ell^2 \dfrac{{\omega_0}^2}{2} - mg\ell$. We know that $\omega = 0$ when $\theta = \dfrac{3\pi}{4}$, so

$-\dfrac{mg\ell}{\sqrt{2}} + \dfrac{m\ell^2{\omega_0}^2}{2} - mg\ell = 0 \Rightarrow {\omega_0}^2 = \dfrac{2}{m\ell^2} mg\ell\left(1 + \dfrac{1}{\sqrt{2}}\right) = \dfrac{g}{\ell}\left(2 + \sqrt{2}\right)$. Thus

$\omega_0 = \sqrt{\dfrac{g}{\ell}\left(2 + \sqrt{2}\right)} = \sqrt{16\left(2 + \sqrt{2}\right)} \approx 7.391$ rad/sec.

Section 3.8

Note: for exercises 1-5, h=0.1

1 (a). $y = t^2 - t + C$, $y(1) = C = 0 \Rightarrow y = t^2 - t$

1 (b). $y_{k+1} = y_k + h\left(2t_k - 1\right)$

1 (c). $y_1 = 0.1$, $y_2 = 0.22$, $y_3 = 0.36$

1 (d). $y(1.1) = 0.11$, $y(1.2) = 0.24$, $y(1.3) = 0.39$

2 (a). $y = Ce^{-t}$, $y(0) = C = 1 \Rightarrow y = e^{-t}$

2 (b). $y_{k+1} = y_k - hy_k$

2 (c). $y_1 = 0.9$, $y_2 = 0.81$, $y_3 = 0.729$

2 (d). $y(0.1) = 0.90484$, $y(0.2) = 0.81873$, $y(0.3) = 0.74082$

3 (a). $y = Ce^{-\frac{t^2}{2}}$, $y(0) = C = 1 \Rightarrow y = e^{-\frac{t^2}{2}}$

3 (b). $y_{k+1} = y_k - h\left(t_k y_k\right)$

3 (c). $y_1 = 1$, $y_2 = 0.99$, $y_3 = 0.9702$

3 (d). $y(0.1) = 0.99501$, $y(0.2) = 0.98020$, $y(0.3) = 0.955997$

4 (a). $y = Ce^{-t} + t - 1$, $y(0) = C - 1 = 0 \Rightarrow y = e^{-t} + t - 1$

4 (b). $y_{k+1} = y_k + h\left(-y_k + t_k\right)$

4 (c). $y_1 = 0$, $y_2 = 0.01$, $y_3 = 0.029$

4 (d). $y(0.1) = 0.0048374$, $y(0.2) = 0.01873075$, $y(0.3) = 0.040818$

5 (a). $y^{-2}y' = 1,\ -y^{-1} = t + C,\ C = -1 \Rightarrow y = \dfrac{1}{1-t}$

5 (b). $y_{k+1} = y_k + h\left(y_k^{\ 2}\right)$

5 (c). $y_1 = 1.1,\ y_2 = 1.221,\ y_3 = 1.3700841$

5 (d). $y(0.1) = 1.111111,\ y(0.2) = 1.25,\ y(0.3) = 1.4285714$

6. $y_{k+1} = y_k + 0.1\left(\alpha t_k + \beta\right)$. From k=0, $t_0 = 0,\ y_0 = -1$.

For $k = 0,\ y_1 = y_0 + 0.1\left(\alpha t_0 + \beta\right) \Rightarrow -0.9 = -1 + .1(0 + \beta) \Rightarrow 0.1 = .1\beta \Rightarrow \beta = 1$.

For $k = 1,\ y_2 = y_1 + 0.1\left(\alpha t_1 + \beta\right) \Rightarrow -0.81 = -0.9 + .1(\alpha(0.1) + 1) \Rightarrow -0.01 = .01\alpha \Rightarrow \alpha = -1$.

7. $y_{k+1} = y_k + 0.1\left(y_k^{\ n} + \alpha\right)$. From k=0, $t_0 = 1,\ y_0 = 1$.

For $k = 0,\ y_1 = y_0 + .1\left(y_0^{\ n} + \alpha\right) \Rightarrow 0.9 = 1 + .1\left(1^n + \alpha\right) \Rightarrow \left(1^n + \alpha\right) = -1 \Rightarrow \alpha = -2$.

For $k = 1,\ y_2 = y_1 + .1\left(y_1^{\ n} - 2\right) \Rightarrow 0.781 = 0.9 + .1\left(.9^n - 2\right) \Rightarrow \left(.9^n - 2\right) = -1.19$

$\Rightarrow .9^n = .81 \Rightarrow n = 2$.

8 (a). (i)Euler's method will underestimate the exact solution.

(ii)Euler's method will overestimate the exact solution.

(iii)Euler's method will underestimate the exact solution.

(iv)Euler's method will overestimate the exact solution.

8 (b). Exercise 2: decreasing, concave up, underestimates

Exercise 3: decreasing, concave down, overestimates

Exercise 5: increasing, concave up, underestimates

8 (c). Euler's method should initially underestimate (when solution curves are concave up) and then tend to "catch up" (when solution curves become concave down).

9. $y_{k+1} = y_k + h\left(t_k y_k + \sin(2\pi t_k)\right),\ y_0 = 1,\ h = 0.01,\ k = 0,1,\ldots,99$.

10. $V(0) = 90,\ V(t) = 90 + 5t,\ V(T) = 100$ when $T = 2 \Rightarrow 0 \le t \le 2$

$$\frac{dQ}{dt} = 6\left(2 - \cos(\pi t)\right) - 1\cdot\frac{Q}{90 + 5t},\ Q(0) = 0$$

$$Q_{k+1} = Q_k + h\left[6\left(2 - \cos(\pi t_k)\right) - \frac{Q_k}{90 + 5t_k}\right],\ Q_0 = 0,\ h = 0.01,\ k = 0,1,2,,\ldots,199$$

Result: $Q(2) = 23.7556\ldots lb$.

11. $P' = 0.1\left(1 - \dfrac{P}{3}\right)P + e^{-t},\ P(0) = \dfrac{1}{2}$. $P_{k+1} = P_k + h\left[0.1\left(1 - \dfrac{1}{3}P_k\right)P_k + e^{-t_k}\right],\ P_0 = 0.5$. With $h = 0.01,\ k = 0,1,\ldots,199,\ t_k = 0.01k$, $P(2) = 1.502477$ million.

12 (a). $y = Ce^t - 1,\ C = 1 \Rightarrow y = e^t - 1$.

12 (b). $y_{k+1} = y_k + h(y_k + 1),\ y_0 = 0$. For $y_k^{(1)},\ h = 0.02,\ k = 0,1,...49$

For $y_k^{(2)},\ h = 0.01,\ k = 0,1,...99$.

13 (a). $y' - \lambda y = 0,\ \left(e^{-\lambda t} y\right)' = C,\ y = Ce^{\lambda t},\ y(0) = C = y_0$. Thus $y = e^{\lambda t} y_0$.

13 (b). $y_{k+1} = y_k + h\lambda y_k = (1 + \lambda h) y_k$. Therefore

$y_1 = (1 + \lambda h) y_0,\ y_2 = (1 + \lambda h) y_1 = (1 + \lambda h)^2 y_0,\ y_n = (1 + \lambda h)^n y_0,$

13 (c). $y_n = \left(1 + \frac{\lambda t}{n}\right)^n y_0$. Since $\lim_{n\to\infty}\left(1 + \frac{a}{n}\right)^n = e^a$, the result follows.

14 (a). $y^{-2} y' = 1,\ -y^{-1} = t + C,\ C = -1,\ y = \frac{1}{1 - t},\ -\infty < t < 1$

14 (b). $y_{k+1} = y_k + hy_k^2,\ y_0 = 1,\ h = 0.1,\ k = 0,1,...11$

14 (c). Numerical solution becomes worse as $t_k \uparrow 1$. The numerical solution gives the mistaken impression that the interval of existence extends to $t \geq 1$.

Chapter 4
Second Order Linear Differential Equations

Introduction

1 (a). The block's weight is $W_b = \pi(0.5)^2(2)(50) = 25\pi$ lb. Therefore, we have $25\pi = \pi(0.5)^2(Y)(62.4)$. Solving for Y yields $Y = 1.60256$ feet.

1 (b). $y_0 = -\frac{1}{4}$, $y_0' = 0$, and $\omega^2 = \frac{\rho_\ell g}{\rho L} = \frac{W}{W_b}\frac{g}{L} = \frac{62.4}{50}\frac{32}{2} = 19.968$. Therefore, $\omega = 4.469$, and thus our equation for $y(t)$ is $y(t) = -\frac{1}{4}\cos(4.469t)$.

1 (c). The maximum depth to which the block sinks would be $Y + |y_0| = 1.603 + .0.25 = 1.853$ feet.

2. $y(t) = \frac{y_0'}{\omega}\sin\omega t.\ Y + \frac{y_0'}{\omega} = 2 \Rightarrow y_0' = \omega(2 - Y) \approx 1.77596\ \ ft.$

3 (a). We know that the frequency of oscillation is given by $\omega^2 = \frac{\rho_\ell g}{\rho L}$. We can see that the density for the second drum is greater than that of the first. Since all the other variables determining ω would be the same for both drums, $\rho_2 > \rho_1$ means that $\omega_2 < \omega_1$, which in turn means that $T_2 > T_1$. Thus the first drum bobs more rapidly.

3 (b). By the same rationale as in the first part, $L_2 > L_1$ means that $\omega_2 < \omega_1$, which in turn means that $T_2 > T_1$. Thus the first drum bobs more rapidly.

4 (a). $y_0 = 0,\ T = 2$

4 (b). $y_0' = \frac{1}{2}\ \ ft.\ y(t) = \frac{1}{2}\sin\omega t$, with $\omega \cdot 2 = 2\pi \Rightarrow \omega = \pi.\ \therefore\ \ y(t) = \frac{1}{2}\sin\pi t$,

4 (c). $\omega^2 = \pi^2 = \frac{W_l}{W_b}\frac{g}{L} \Rightarrow W_b = \frac{W_l}{\omega^2}\frac{g}{L};\ V = \pi\left(\frac{3}{2}\right)^2 \cdot 5 = \frac{45\pi}{4}\,ft^3$

$$W_b = 62.4\left(\frac{45\pi}{4}\right)\frac{1}{\pi^2}\frac{32}{5} = 62.4 \cdot \frac{9}{\pi} \cdot 8 \approx 1430.1\ \ lb.$$

5 (a). Using the model provided, $my'' = mg - \rho_\ell Vg$. We can rewrite this equation with m in terms of V and ρ: $\rho Vy'' = \rho Vg - \rho_\ell Vg$. Simplifying this equation and solving for y'', we have $y'' = \left(\frac{\rho - \rho_\ell}{\rho}\right)g$. We need not restrict this equation to the motion of cylindrical objects.

5 (b). Using these initial conditions, antidifferentiation yields the general solution

$$y = \left(\frac{\rho - \rho_\ell}{\rho}\right) g\frac{t^2}{2} + y_0{}' t + y_0.$$

6. $$\frac{\rho - \rho_\ell}{\rho} = \frac{30 - 62.4}{62.4} = -.51923.$$

$$\therefore\ 0 = -.51923 \cdot 32 \cdot \frac{t^2}{2} + 0 + 99 \Rightarrow 8.30769t^2 = 99 \Rightarrow t = 11.9166\ldots \text{ sec}.$$

Section 4.1

1. All the relevant functions are continuous everywhere, so Theorem 4.1 guarantees a unique solution for the interval $(-\infty, \infty)$.

2. $t_0 = \pi$. Since $g(t) = \tan t$, $\frac{\pi}{2} < t < \frac{3\pi}{2}$.

3. Dividing the equation by e^t yields the functions $p(t) = 0$, $q(t) = \frac{1}{e^t(t^2 - 1)}$, and $g(t) = \frac{4}{te^t}$.

   These functions are discontinuous at the points ± 1 and 0, and since $t_0 = -2$, the largest t-interval on which Theorem 4.1 guarantees a unique solution is $(-\infty, -1)$.

4. $p(t) = \frac{\sin 2t}{t(t^2 - 9)}$, $q(t) = \frac{2}{t}$, $0 < t < 3$.

5. $y'' + y = 0$, $y(t_0) = y_0$, $y'(t_0) = y_0{}'$ for any $t_0, y_0, y_0{}'$.

6. $y'' + \frac{1}{t-3} y = 0$, $y(t_0) = y_0$, $y'(t_0) = y_0'$, $t_0 > 3$, any y_0, y_0' (not both 0).

7. $y'' + \frac{y'}{t-5} + \frac{y}{t+1} = 0$, $-1 < t_0 < 5$, $y(t_0) = y_0$, $y'(t_0) = y_0{}'$. y_0 and $y_0{}'$ cannot both be zero.

8 (a). $y'' - \frac{y'}{t} + \frac{y}{t^2} = 0$, $t_0 = 1$. $0 < t < \infty$.

8 (b). $t^2(0) - t(1) + t = 0$.

8 (c). No.

9. Theorem 4.1 guarantees a unique solution on the interval $(-4, 4)$, so it is not possible for the limit to hold.

10. Theorem 4.1 guarantees a unique solution on the interval $(-\infty, 3)$, so it is not possible for the limit to hold.

11 (a). B

11 (b). D

11 (c). A

11 (d). C

Section 4.2

1 (a). $y_1'' = 4e^{2t} = 4y_1$ and $y_2'' = 8e^{-2t} = 4y_2$, so both equations are solutions.

1 (b). $W = \begin{vmatrix} e^{2t} & 2e^{-2t} \\ 2e^{2t} & -4e^{-2t} \end{vmatrix} = -8 \neq 0$, so yes, the functions do form a fundamental set of solutions.

1 (c). The general solution would be $y = c_1e^{2t} + c_2 \cdot 2e^{-2t}$. Differentiating yields $y' = 2c_1e^{2t} - 4c_2e^{-2t}$. $y(0) = c_1 + 2c_2 = 1$ and $y'(0) = 2c_1 - 4c_2 = -2$, and solving these two simultaneous equations yields $c_1 = 0$ and $c_2 = \frac{1}{2}$. Thus the unique solution for this initial value problem is $y(t) = e^{-2t}$.

2 (a). $y_1'' = 2e^t = y_1$ and $y_2'' = e^{-t+3} = y_2$, so both equations are solutions.

2 (b). $W = \begin{vmatrix} 2e^t & e^{-t+3} \\ 2e^t & -e^{-t+3} \end{vmatrix} = -4e^3 \neq 0$, so yes, the functions do form a fundamental set of solutions.

2 (c). The general solution would be $y = c_1 \cdot 2e^t + c_2 \cdot e^{-t+3}$. Differentiating yields $y' = 2c_1e^t - c_2e^{-t+3}$. $y(-1) = 2c_1e^{-1} + c_2e^4 = 1$ and $y'(-1) = 2c_1e^{-1} - c_2e^4 = 0$, and solving these two simultaneous equations yields $c_1 = \frac{e}{4}$ and $c_2 = \frac{1}{2e^4}$. Thus the unique solution for this initial value problem is $y(t) = \frac{e^{t+1}}{2} + \frac{e^{-t-1}}{2}$.

3 (a). y_1 is not a solution. y_2 is a solution.

4 (a). Both equations are solutions.

4 (b). $W = \begin{vmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{vmatrix} = 1 \neq 0$, so yes, the functions do form a fundamental set of solutions.

4 (c). The general solution would be $y = c_1 \cos t + c_2 \sin t$. Differentiating yields $y' = -c_1 \sin t + c_2 \cos t$. $y\left(\frac{\pi}{2}\right) = c_2 = 1$ and $y'\left(\frac{\pi}{2}\right) = -c_1 = 1 \Rightarrow c_1 = -1$. Thus the unique solution for this initial value problem is $y(t) = -\cos t + \sin t$.

5 (a). $y_1' = 2e^{2t} = 2y_1$ and $y_1'' = 4e^{2t} = 4y_1$. Thus $y_1'' - 4y_1' + 4y_1 = 4y_1 - 8y_1 + 4y_1 = 0$.

$y_2' = 2te^{2t} + e^{2t}$ and $y_2'' = 4te^{2t} + 2e^{2t} + 2e^{2t} = 4te^{2t} + 4e^{2t}$. Thus

$y_2'' - 4y_2' + 4y_2 = 4te^{2t} + 4e^{2t} - 8te^{2t} - 4e^{2t} + 4te^{2t} = 0$, and so both equations are solutions.

5 (b). $W = \begin{vmatrix} e^{2t} & te^{2t} \\ 2e^{2t} & (2t+1)e^{2t} \end{vmatrix} = e^{4t} \neq 0$, so yes, the functions do form a fundamental set of solutions.

5 (c). The general solution would be $y = c_1 e^{2t} + c_2 te^{2t}$. Differentiating yields

$y' = 2c_1 e^{2t} + 2c_2 te^{2t} + c_2 e^{2t}$. $y(0) = c_1 = 2$ and $y'(0) = 2c_1 + c_2 = 0 \Rightarrow c_2 = -4$.

Thus the unique solution for this initial value problem is $y(t) = 2e^{2t} - 4te^{2t}$.

6 (a). Both equations are solutions.

6 (b). $W = \begin{vmatrix} 1 & e^{t/2} \\ 0 & \frac{1}{2}e^{t/2} \end{vmatrix} = \frac{1}{2}e^{t/2} \neq 0$, so yes, the functions do form a fundamental set of solutions.

6 (c). The general solution would be $y = c_1 + c_2 e^{t/2}$. Differentiating yields $y' = \frac{1}{2}c_2 e^{t/2}$.

$y(2) = c_1 + c_2 e = 0$ and $y'(2) = \frac{1}{2}c_2 e = 2 \Rightarrow c_2 = 4\text{e}^{-1}$ and $\text{c}_1 = -4$. Thus the unique solution for this initial value problem is $y(t) = -4 + 4e^{\frac{t-2}{2}}$.

7 (a). $y_1' = \frac{1}{2}\cos\left(\frac{t}{2} + \frac{\pi}{3}\right)$ and $y_1'' = -\frac{1}{4}\sin\left(\frac{t}{2} + \frac{\pi}{3}\right)$. Thus $4y_1'' + y_1 = -\sin\left(\frac{t}{2} + \frac{\pi}{3}\right) + \sin\left(\frac{t}{2} + \frac{\pi}{3}\right) = 0$.

$y_2' = \frac{1}{2}\cos\left(\frac{t}{2} - \frac{\pi}{3}\right)$ and $y_2'' = -\frac{1}{4}\sin\left(\frac{t}{2} - \frac{\pi}{3}\right)$. Thus $4y_2'' + y_2 = -\sin\left(\frac{t}{2} - \frac{\pi}{3}\right) + \sin\left(\frac{t}{2} - \frac{\pi}{3}\right) = 0$,

and so both equations are solutions.

7 (b). $W = \begin{vmatrix} \sin\left(\frac{t}{2} + \frac{\pi}{3}\right) & \sin\left(\frac{t}{2} - \frac{\pi}{3}\right) \\ \frac{1}{2}\cos\left(\frac{t}{2} + \frac{\pi}{3}\right) & \frac{1}{2}\cos\left(\frac{t}{2} - \frac{\pi}{3}\right) \end{vmatrix}$

$= \frac{1}{2}\left[\sin\left(\frac{t}{2} + \frac{\pi}{3}\right)\cos\left(\frac{t}{2} - \frac{\pi}{3}\right) - \sin\left(\frac{t}{2} - \frac{\pi}{3}\right)\cos\left(\frac{t}{2} + \frac{\pi}{3}\right)\right]$

$= \frac{1}{2}\sin\left(\frac{t}{2} + \frac{\pi}{3} - \frac{t}{2} + \frac{\pi}{3}\right) = \frac{1}{2}\sin\left(\frac{2\pi}{3}\right) \neq 0$, so yes, the two equations do form a fundamental set of solutions.

7 (c). The general solution would be $y = c_1 \sin\left(\frac{t}{2} + \frac{\pi}{3}\right) + c_2 \sin\left(\frac{t}{2} - \frac{\pi}{3}\right)$. Differentiating yields

$y' = \frac{1}{2}c_1 \cos\left(\frac{t}{2} + \frac{\pi}{3}\right) + \frac{1}{2}c_2 \cos\left(\frac{t}{2} - \frac{\pi}{3}\right)$.

Using the first initial condition, $y(0) = c_1 \sin\left(\frac{\pi}{3}\right) + c_2 \sin\left(-\frac{\pi}{3}\right) = 0$, and so $c_1 = c_2$. From the second initial condition, $y'(0) = \frac{1}{2}c_1 \cos\left(\frac{\pi}{3}\right) + \frac{1}{2}c_2 \cos\left(-\frac{\pi}{3}\right) = \frac{1}{4}c_1 + \frac{1}{4}c_2 = 1$. Thus $c_1 = c_2 = 2$, and the unique solution is $y = 2\sin\left(\frac{t}{2} + \frac{\pi}{3}\right) + 2\sin\left(\frac{t}{2} - \frac{\pi}{3}\right) = 4\sin\left(\frac{t}{2}\right)\cos\left(\frac{\pi}{3}\right) = 2\sin\left(\frac{t}{2}\right)$.

8 (a). Both equations are solutions.

8 (b). $W = \begin{vmatrix} 2e^t & e^{2t} \\ 2e^t & 2e^{2t} \end{vmatrix} = 2e^{3t} \neq 0$, so yes, the functions do form a fundamental set of solutions.

8 (c). The general solution would be $y = c_1 \cdot 2e^t + c_2 \cdot e^{2t}$. Differentiating yields $y' = 2c_1e^t + 2c_2e^{2t}$. $y(-1) = 2c_1e^{-1} + c_2e^{-2} = 1$ and $y'(-1) = 2c_1e^{-1} + 2c_2e^{-2} = 0$, and solving these two simultaneous equations yields $c_1 = e$ and $c_2 = -e^2$. Thus the unique solution for this initial value problem is $y(t) = 2e^{t+1} - e^{2(t+1)}$.

9 (a). $y_1' = \frac{1}{t}$, $y_1'' = -\frac{1}{t^2}$, $y_2' = \frac{1}{t}$, and $y_2'' = -\frac{1}{t^2}$. Thus $ty_1'' + y_1' = ty_2'' + y_2' = 0$, and so both equations are solutions.

9 (b). $W = \begin{vmatrix} \ln t & \ln 3t \\ \frac{1}{t} & \frac{1}{t} \end{vmatrix} = \frac{1}{t}(\ln t - \ln 3t) = t^{-1}(\ln t - \ln 3 - \ln t) = -t^{-1}\ln 3 \neq 0$ on $(0,\infty)$, so yes, the two equations do form a fundamental set of solutions.

9 (c). The general solution would be $y = c_1 \ln t + c_2 \ln 3t = (c_1 + c_2)\ln t + c_2 \ln 3$. Differentiating yields $y' = \frac{c_1 + c_2}{t}$. From the first initial condition, $y(3) = (c_1 + c_2)\ln 3 + c_2 \ln 3 = 0$, so $c_1 + 2c_2 = 0$. From the second initial condition, $y'(3) = \frac{c_1 + c_2}{3} = 3$. Thus $c_1 = 18$ and $c_2 = -9$, and so the unique solution is $y = 18\ln t - 9\ln 3t,\ 0 < t < \infty$.

10 (a). Both equations are solutions.

10 (b). $W = \begin{vmatrix} \ln t & \ln 3 \\ \frac{1}{t} & 0 \end{vmatrix} = \frac{-\ln 3}{t} \neq 0$, so yes, the functions do form a fundamental set of solutions.

10 (c). The general solution would be $y = c_1 \cdot \ln t + c_2 \cdot \ln 3$. Differentiating yields $y' = \frac{c_1}{t} + 0$. $y(1) = 0 + c_2 \ln 3 = 0$ and $y'(1) = c_1 = 3$, and solving these two simultaneous equations yields $c_1 = 3$ and $c_2 = 0$. Thus the unique solution for this initial value problem is $y(t) = 3\ln t,\ 0 < t < \infty$.

11 (a). $y_1' = 3t^2$, $y_1'' = 6t$, $y_2' = t^{-2}$, and $y_2'' = -2t^{-3}$. Thus $t^2y_1'' - ty_1' - 3y_1 = t^2y_2'' - ty_2' - 3y_2 = 0$, and so both equations are solutions.

11 (b). $W = \begin{vmatrix} t^3 & -t^{-1} \\ 3t^2 & t^{-2} \end{vmatrix} = t + 3t = 4t \neq 0$ on $(-\infty, 0)$, so yes, the two equations do form a fundamental set of solutions.

11 (c). The general solution would be $y = c_1t^3 - c_2t^{-1}$. Differentiating yields $y' = 3c_1t^2 + c_2t^{-2}$. From the first initial condition, $y(-1) = -c_1 + c_2 = 0$. From the second initial condition, $y'(-1) = 3c_1 + c_2 = -2$. Thus $c_1 = c_2 = -\dfrac{1}{2}$, and so the unique solution is

$y = -\dfrac{1}{2}t^3 + \dfrac{1}{2}t^{-1}, \ -\infty < t < 0$.

12 (a). Both equations are solutions.

12 (b). $W = \begin{vmatrix} e^{-t} & 2e^{1-t} \\ -e^{-t} & -2e^{1-t} \end{vmatrix} = 0$, Therefore, the Wronskian calculation does not establish that y_1 and y_2 form a fundamental set.

13 (a). $y_1' = 1$, $y_1'' = 0$, $y_2' = -1$, and $y_2'' = 0$. Thus both equations are solutions.

13 (b). $W = \begin{vmatrix} t+1 & -t+2 \\ 1 & -1 \end{vmatrix} = -t - 1 + t - 2 = -3 \neq 0$, so yes, the two equations do form a fundamental set of solutions.

13 (c). The general solution would be $y = c_1(t+1) + c_2(2-t)$. Differentiating yields $y' = c_1 - c_2$. From the first initial condition, $y(1) = 2c_1 + c_2 = 4$. From the second initial condition, $y'(1) = c_1 - c_2 = -1$. Thus $c_1 = 1$ and $c_2 = 2$, and so the unique solution is

$y = t + 1 + 4 - 2t = -t + 5$.

14 (a). Both equations are solutions.

14 (b). $W = \begin{vmatrix} \sin\pi t + \cos\pi t & \sin\pi t - \cos\pi t \\ \pi\cos\pi t - \pi\sin\pi t & \pi\cos\pi t + \pi\sin\pi t \end{vmatrix}$

$= 2\pi\sin\pi t\cos\pi t + \pi(\sin^2\pi t + \cos^2\pi t) - 2\pi\sin\pi t\cos\pi t + \pi(\sin^2\pi t + \cos^2\pi t)$,

$= 2\pi \neq 0$ so yes, the functions do form a fundamental set of solutions.

14 (c). The general solution would be $y = c_1(\sin\pi t + \cos\pi t) + c_2(\sin\pi t - \cos\pi t)$. Differentiating yields $y' = \pi c_1(\cos\pi t - \sin\pi t) + \pi c_2(\cos\pi t + \sin\pi t)$.

$y\left(\dfrac{1}{2}\right) = c_1 + c_2 = 1$ and $y'\left(\dfrac{1}{2}\right) = -\pi c_1 + \pi c_2 = 0 \Rightarrow c_1 = c_2 = \frac{1}{2}$.

Thus the unique solution for this initial value problem is

$y=\frac{1}{2}(\sin\pi t+\cos\pi t)+\frac{1}{2}(\sin\pi t-\cos\pi t)=\sin\pi t$.

15 (a). y_1 is not a solution. y_2 is not a solution.

16 (b). $\bar{y}=\sin(2t)\cos\left(\frac{\pi}{4}\right)+\cos(2t)\sin\left(\frac{\pi}{4}\right)=\frac{1}{2\sqrt{2}}(2\cos 2t)+\frac{1}{\sqrt{2}}(\sin 2t)$.

Thus $c_1=\frac{1}{2\sqrt{2}}$ and $c_2=\frac{1}{\sqrt{2}}$.

17 (a). $\bar{y}'=2+1+\ln 3t=\ln 3t+3,\ \bar{y}''=\frac{1}{t}$. Thus $t^2\bar{y}''-t\bar{y}'+\bar{y}=t-3t-t\ln 3t+2t+t\ln 3t=0$.

17(b). $\bar{y}=2t+t(\ln t+\ln 3)=t(2+\ln 3)+t\ln t$. Thus $c_1=2+\ln 3$ and $c_2=1$.

18 (b). $\bar{y}=2\cosh\left(\frac{t}{2}\right)=e^{t/2}+e^{-t/2}=1\cdot e^{-t/2}+\left(-\frac{1}{2}\right)\cdot\left(-2e^{t/2}\right)$.

Thus $c_1=1$ and $c_2=-\frac{1}{2}$.

19. Substituting y_1 into the equation yields $9e^{3t}+3\alpha e^{3t}+\beta e^{3t}=0;\ 3\alpha+\beta=-9$. Substituting y_2 into the equation yields $9e^{-3t}-3\alpha e^{-3t}+\beta e^{-3t}=0;\ -3\alpha+\beta=-9$. Solving the two simultaneous equations gives us $\alpha=0$ and $\beta=-9$.

20 (a). $y_1(t)=t, y_1'(t)=1, y_1''(t)=0,\ y_2(t)=e^t, y_2'(t)=y_2''(t)=e^t$.

$0+p(t)\cdot 1+q(t)\cdot t=0,\ e^t+p(t)\cdot e^t+q(t)\cdot e^t=0$

$p+qt=0,\ e^t(1+p+q)=0\Rightarrow p+q=-1$

$\therefore (t-1)q=1\Rightarrow q=\frac{1}{t-1},\ p=\frac{-t}{t-1}$

20 (b). Both p and q continuous on $(-\infty,1)$ *and* $(1,\infty)$.

20 (c). $W=\begin{vmatrix} t & e^t \\ 1 & e^t \end{vmatrix}=(t-1)e^t \qquad W\neq 0\ \ on\ \ (-\infty,1)\ \ and\ \ (1,\infty)$

20 (d). Yes, $W\neq 0$ on the two intervals on which p and q are both continuous.

21. From Abel's Theorem, we have $W(t)=W(1)e^{-\int_1^t s\,ds}\Rightarrow W(2)=4e^{-\int_1^2 s\,ds}=4e^{-(4-1)/2}=4e^{-3/2}$.

22. Substitute, $4e^{2t}+2\alpha e^{2t}+\beta e^{2t}=0\Rightarrow 2\alpha+\beta=-4$.

Also, $W=e^{-t}\Rightarrow p(t)=\alpha=1$. $\therefore \alpha=1, \beta=-6\ (y''+y'-6y=0)$.

23. $p(t)=0$. From Abel's Theorem, $W(t)=W(t_0)$, which is a constant. Therefore, $W(4)=-3$.

24. $y''+py'+3y=0,\ W=W(t_0)e^{-\int_{t_0}^t p\,ds}=e^{-t^2}$.

$\therefore p(t)=\frac{d}{dt}\left(t^2\right)=2t$.

Section 4.3

1 (a). $W(1)=\begin{vmatrix}2 & -1\\ 2 & -1\end{vmatrix}=0$, so the two solutions do not form a fundamental set.

1 (b). Since $-\frac{1}{2}y_1$ and y_2 satisfy identical initial conditions, we can conclude from Theorem 4.1 that $-\frac{1}{2}y_1(t)\equiv y_2(t)$. Therefore, $\frac{1}{2}y_1(t)+y_2(t)=0,\ -\infty<t<\infty$.

2 (a). $W(-2)=\begin{vmatrix}1 & 0\\ 2 & 1\end{vmatrix}=1\neq 0$, so the two solutions do form a fundamental set.

2 (b). Yes (Theorem 4.7)

3 (a). $W(0)=\begin{vmatrix}0 & -1\\ 1 & 0\end{vmatrix}=1\neq 0$, so the two solutions do form a fundamental set.

3 (b). From Theorem 4.7, the two solutions do form a linearly independent set of functions on $-\infty<t<\infty$.

4 (a). $W(3)=\begin{vmatrix}0 & 1\\ 0 & 2\end{vmatrix}=0$, so the two solutions do not form a fundamental set.

4 (b). By Theorem 4.1, $y_1(t)\equiv 0$. Therefore, $1\cdot y_1(t)+0\cdot y_2(t)=0$ and $y_1(t),y_2(t)$ are linearly dependent.

5 (a). $y_1''-4y_1=4e^{2t}-4e^{2t}=0,\ y_2''-4y_2=4e^{-2t}-4e^{-2t}=0$, so yes, both are solutions to the differential equation.

5 (b). $y_1(1)=e^2,\ y_1'(1)=2e^2,\ y_2(1)=e^{-2},\ y_2'(1)=-2e^{-2}$.

5 (c). $W(1)=\begin{vmatrix}e^2 & e^{-2}\\ 2e^2 & -2e^{-2}\end{vmatrix}=-4\neq 0$, so the solutions do form a fundamental set.

6 (a). $4y_1''-y_1=4\left(\frac{1}{4}e^{t/2}\right)-e^{t/2}=0,\ 4y_2''-y_2=4\left(\frac{-1}{2}e^{-t/2}\right)-\left(-2e^{-t/2}\right)=0$,

6 (b). $y_1(-2)=e^{-1},\ y_1'(-2)=\frac{1}{2}\mathrm{e}^{-1},\ y_2(-2)=-2e,\ y_2'(-2)=e$.

6 (c). $W(-2)=\begin{vmatrix}e^{-1} & -2e\\ \frac{e^{-1}}{2} & e\end{vmatrix}=2\neq 0$, so the solutions do form a fundamental set.

7 (a). $y_1''+9y_1=-9\sin(3(t-1))+9\sin(3(t-1))=0$,

$y_2''+9y_2=-9(2\cos(3(t-1)))+9(2\cos(3(t-1)))=0$, so yes, both are solutions to the differential equation.

7 (b). $y_1(1)=0,\ y_1'(1)=3,\ y_2(1)=2,\ y_2'(1)=0$.

7 (c). $W(1)=\begin{vmatrix}0 & 2\\ 3 & 0\end{vmatrix}=-6\neq 0$, so the solutions do form a fundamental set.

8 (a). $y_1'=e^{-2t}(-2\cos t-\sin t), y_1''=e^{-2t}(3\cos t+4\sin t)$

$y_1''+4y_1'+5y_1=e^{-2t}(3\cos t+4\sin t-8\cos t-4\sin t+5\cos t)=0$

$y_2'=e^{-2t}(-2\sin t+\cos t), y_2''=e^{-2t}(3\sin t-4\cos t)$

$y_2''+4y_2'+5y_2=e^{-2t}(3\sin t-4\cos t-8\sin t+4\cos t+5\sin t)=0$

8 (b). $y_1(0)=1,\ y_1'(0)=-2,\ y_2(0)=0,\ y_2'(0)=1$.

8 (c). $W(0)=\begin{vmatrix}1 & 0\\ -2 & 1\end{vmatrix}=1\neq 0$, so the solutions do form a fundamental set.

9 (a). $y_1''+2y_1'-3y_1=9e^{-3t}-6e^{-3t}-3e^{-3t}=0$,

$y_2''+2y_2'-3y_2=9e^{-3(t-2)}-6e^{-3(t-2)}-3e^{-3(t-2)}=0$, so yes, both are solutions to the differential equation.

9 (b). $y_1(2)=e^{-6},\ y_1'(2)=-3e^{-6},\ y_2(2)=1,\ y_2'(2)=-3$.

9 (c). $W(2)=\begin{vmatrix}e^{-6} & 1\\ -3e^{-6} & -3\end{vmatrix}=0$, so the solutions do not form a fundamental set.

10 (a). $y_1''-6y_1'+9y_1=e^{3(t+2)}(9-18+9)=0$

$y_2''-6y_2'+9y_2=e^{3(t+2)}(9t+6-6(3t+1)+9t))=0$

10 (b). $y_1(-2)=1,\ y_1'(-2)=3,\ y_2(-2)=-2,\ y_2'(-2)=-5$.

10 (c). $W(-2)=\begin{vmatrix}1 & -2\\ 3 & -5\end{vmatrix}=1\neq 0$, so the solutions do form a fundamental set.

11. $[\bar{y}_1\quad \bar{y}_2]=[y_1\quad y_2]\begin{bmatrix}2 & 1\\ -1 & 1\end{bmatrix}$; $\begin{vmatrix}2 & 1\\ -1 & 1\end{vmatrix}=3\neq 0$, so $\{\bar{y}_1\quad \bar{y}_2\}$ is a fundamental set.

12. $[\bar{y}_1\quad \bar{y}_2]=[y_1\quad y_2]\begin{bmatrix}2 & 1\\ -2 & -1\end{bmatrix}$; $\begin{vmatrix}2 & 1\\ -2 & -1\end{vmatrix}=0$, so $\{\bar{y}_1\quad \bar{y}_2\}$ is not a fundamental set.

13. $[\bar{y}_1\quad \bar{y}_2]=[y_1\quad y_2]\begin{bmatrix}0 & 2\\ 1 & -1\end{bmatrix}$; $\begin{vmatrix}0 & 2\\ 1 & -1\end{vmatrix}=-2\neq 0$, so $\{\bar{y}_1\quad \bar{y}_2\}$ is a fundamental set.

14. The set is linearly independent since one function is not a constant multiple of the other.

15. $f_2=\ln t^2=2\ln t=2f_1$, so the set is linearly dependent $(2f_1-f_2=0)$.

16. The set is linearly independent since one function is not a constant multiple of the other.

17. The set is linearly independent since one function is not a constant multiple of the other.

18. Set $k_1 \cdot 2 + k_2 \cdot t + k_3 \cdot (-t^2) = 0$ and evaluate at $t = -1, 0, 1$.

$$\begin{bmatrix} 2 & -1 & -1 \\ 2 & 0 & 0 \\ 2 & 1 & -1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}. \quad \begin{vmatrix} 2 & -1 & -1 \\ 2 & 0 & 0 \\ 2 & 1 & -1 \end{vmatrix} = -2 \begin{vmatrix} -1 & -1 \\ 1 & -1 \end{vmatrix} = -4 \neq 0.$$

Therefore, $k_1 = k_2 = k_3 = 0$ and the set is linearly independent.

19. $f_1 = 2, f_2 = \sin^2 t, f_3 = 2\cos^2 t$. Therefore, $2f_2 + f_3 - f_1 = 0$ on $-3 < t < 2$, so the set is linearly dependent.

20. $f_1 = e^t, f_2 = 2e^{-t}, f_3 = \sinh t$. Therefore, $\frac{1}{2} f_1 - \frac{1}{4} f_2 - f_3 = 0$, so the set is linearly dependent.

21 (a). $f_1 = t, f_2 = 2t = 2f_1$, so the functions form a linearly dependent set.

21 (b). $f_1 = t, f_2 = -t = -f_1$, so the functions form a linearly dependent set.

21 (c). $f_1 = t, f_2 = t - 1$. These functions form a linearly independent set since one function is not a constant multiple of the other.

22 (a). If $f_1 = cf_2$, then $f_1 - cf_2 = 0$. Therefore, the set is linearly dependent.

22 (b). Assume $\{f_1, f_2\}$ is a linearly dependent set. Then

$k_1 f_1(t) + k_2 f_2(t) = 0$ with k_1 and k_2 not both zero. Assume that $k_1 \neq 0$. Then $f_1(t) = -\left(\frac{k_2}{k_1}\right) f_2(t)$ on the domain.

23 (a). If $f_2 = 3f_1 - 2f_3$, then $3f_1 - f_2 - 2f_3 = 0$ on the domain. Therefore, the set is linearly dependent.

23 (b). Assume $\{f_1, f_2, f_3\}$ is a linearly dependent set. Then

$k_1 f_1(t) + k_2 f_2(t) + k_3 f_3(t) = 0$ with k_1, k_2, k_3 not all zero. Assume, without loss of generality, that $k_1 \neq 0$. Then $f_1(t) = -\left(\frac{k_2}{k_1}\right) f_2(t) - \left(\frac{k_3}{k_1}\right) f_3(t)$.

24. Any set of functions containing the zero function is linearly dependent.

Consider $\{0, f_2, f_3, ..., f_n\}$. Then $1 \cdot 0 + 0 \cdot f_2 + 0 \cdot f_3 + ... + 0 \cdot f_n = 0$.

25. Suppose that $f_3 = a_1 f_1 + a_2 f_2$ and $f_3 = b_1 f_1 + b_2 f_2$. Then $(a_1 - b_1) f_1 + (a_2 - b_2) f_2 = 0$. Since the functions are linearly independent, $a_1 - b_1 = a_2 - b_2 = 0$; $a_1 = b_1$, $a_2 = b_2$.

26. On $0 < t < \infty$, $f_2 = |t| = t = f_1$. $\therefore f_1 - f_2 = 0$ and $\{f_1, f_2\}$ is linearly dependent.

On $-\infty < t < \infty$, let $k_1 f_1 + k_2 f_2 = k_1 t + k_2 |t| = 0$ and evaluate at

$t = \pm 1$. Then $k_1 + k_2 = 0$, $-k_1 + k_2 = 0 \Rightarrow k_1 = k_2 = 0$ and this is a linearly independent set.

27. $\begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}$

28. $\begin{bmatrix} -1 & -3 \\ 2 & 1 \end{bmatrix}$

Section 4.4

1 (a). $\lambda^2 + \lambda - 2 = (\lambda + 2)(\lambda - 1) = 0$. Thus $y = c_1 e^{-2t} + c_2 e^t$.

1 (b). $y(0) = c_1 + c_2 = 3,\ y'(0) = -2c_1 + c_2 = -3$. Solving these simultaneous equations yields $c_1 = 2$ and $c_2 = 1$. Thus the unique solution to the initial value problem is $y = 2e^{-2t} + e^t$.

1 (c). $\lim_{t\to-\infty} y = \infty$ and $\lim_{t\to\infty} y = \infty$.

2 (a). $\lambda^2 - \frac{1}{4} = 0 \Rightarrow \lambda = \pm\frac{1}{2}$. Thus $y = c_1 e^{-t/2} + c_2 e^{t/2}$.

2 (b). $y(2) = c_1 e^{-1} + c_2 e^1 = 1,\ y'(2) = -\frac{1}{2}c_1 e^{-1} + \frac{1}{2}c_2 e^1 = 0$. Therefore, $c_2 e = c_1 e^{-1} = \frac{1}{2} \Rightarrow c_1 = \frac{e}{2}$ and $c_2 = \frac{e^{-1}}{2}$. Thus the unique solution to the initial value problem is $y = \frac{1}{2}e^{-(t-2)/2} + \frac{1}{2}e^{(t-2)/2}$.

2 (c). $\lim_{t\to-\infty} y = \infty$ and $\lim_{t\to\infty} y = \infty$.

3 (a). $\lambda^2 - 4\lambda + 3 = (\lambda - 3)(\lambda - 1) = 0$. Thus $y = c_1 e^{3t} + c_2 e^t$.

3 (b). $y(0) = c_1 + c_2 = -1,\ y'(0) = 3c_1 + c_2 = 1$. Solving these simultaneous equations yields $c_1 = 1$ and $c_2 = -2$. Thus the unique solution to the initial value problem is $y = e^{3t} - 2e^t$.

3 (c). $\lim_{t\to-\infty} y = 0$ and $\lim_{t\to\infty} y = \infty$.

4 (a). $2\lambda^2 - 5\lambda + 2 = (2\lambda - 1)(\lambda - 2) = 0$. Thus $y = c_1 e^{t/2} + c_2 e^{2t}$.

4 (b). $y(0) = c_1 + c_2 = -1,\ y'(0) = \frac{1}{2}c_1 + 2c_2 = -5$. Solving these simultaneous equations yields $c_1 = 2$ and $c_2 = -3$. Thus the unique solution to the initial value problem is $y = 2e^{t/2} - 3e^{2t}$.

4 (c). $\lim_{t\to-\infty} y = 0$ and $\lim_{t\to\infty} y = -\infty$.

5 (a). $\lambda^2 - 1 = (\lambda + 1)(\lambda - 1) = 0$. Thus $y = c_1 e^{-t} + c_2 e^t$.

5 (b). $y(0) = c_1 + c_2 = 1,\ y'(0) = -c_1 + c_2 = -1$. Solving these simultaneous equations yields $c_1 = 1$ and $c_2 = 0$. Thus the unique solution to the initial value problem is $y = e^{-t}$.

5 (c). $\lim_{t\to-\infty} y = \infty$ and $\lim_{t\to\infty} y = 0$.

6 (a). $\lambda^2 + 2\lambda = \lambda(\lambda + 2) = 0$. Thus $y = c_1 e^{-2t} + c_2$.

6 (b). $y(-1) = c_1 e^2 + c_2 = 0,\ y'(-1) = -2c_1 e^2 = 2$. Therefore, $c_1 = -e^{-2}$ and $c_2 = 1$, and $y = 1 - e^{-2(t-+1)}$.

6 (c). $\lim_{t\to-\infty} y = -\infty$ and $\lim_{t\to\infty} y = 1$.

7 (a). $\lambda^2 + 5\lambda + 6 = (\lambda+2)(\lambda+3) = 0$. Thus $y = c_1 e^{-3t} + c_2 e^{-2t}$.

7 (b). $y(0) = c_1 + c_2 = 1,\ y'(0) = -3c_1 - 2c_2 = -1$. Solving these simultaneous equations yields $c_1 = -1$ and $c_2 = 2$. Thus the unique solution to the initial value problem is $y = -e^{-3t} + 2e^{-2t}$.

7 (c). $\lim_{t\to-\infty} y = -\infty$ and $\lim_{t\to\infty} y = 0$.

8 (a). $\lambda^2 - 5\lambda + 6 = (\lambda-2)(\lambda-3) = 0$. Thus $y = c_1 e^{2t} + c_2 e^{3t}$.

8 (b). $c_1 + c_2 = 1,\ 2c_1 + 3c_2 = -1$. Therefore, $c_1 = 4$ and $c_2 = -3$, and $y = 4e^{2t} - 3e^{3t}$.

8 (c). $\lim_{t\to-\infty} y = 0$ and $\lim_{t\to\infty} y = -\infty$.

9 (a). $\lambda^2 - 4 = (\lambda+2)(\lambda-2) = 0$. Thus $y = c_1 e^{-2t} + c_2 e^{2t}$.

9 (b). $y(3) = c_1 e^{-6} + c_2 e^{6} = 0,\ y'(3) = -2c_1 e^{-6} + 2c_2 e^{6} = 0$. Solving these simultaneous equations yields $c_1 = 0$ and $c_2 = 0$. Thus the unique solution to the initial value problem is $y = 0$.

9 (c). $\lim_{t\to-\infty} y = 0$ and $\lim_{t\to\infty} y = 0$.

10 (a). $8\lambda^2 - 6\lambda + 1 = (4\lambda-1)(2\lambda-1) = 0$. Thus $y = c_1 e^{t/4} + c_2 e^{t/2}$.

10 (b). $c_1 e^{1/4} + c_2 e^{1/2} = 4,\ \frac{1}{4} c_1 e^{1/4} + \frac{1}{2} c_2 e^{1/2} = \frac{3}{2}$. Therefore, $c_1 = 2e^{-1/4}$ and $c_2 = 2e^{-1/2}$, and $y = 2e^{(t-1)/4} + 2e^{(t-1)/2}$.

10 (c). $\lim_{t\to-\infty} y = 0$ and $\lim_{t\to\infty} y = \infty$.

11 (a). $2\lambda^2 - 3\lambda = (\lambda)(2\lambda-3) = 0$. Thus $y = c_1 + c_2 e^{\frac{3}{2}t}$.

11 (b). $y(-2) = c_1 + c_2 e^{-3} = 3,\ y'(-2) = \dfrac{3}{2} c_2 e^{-3} = 0$. Solving these simultaneous equations yields $c_1 = 3$ and $c_2 = 0$. Thus the unique solution to the initial value problem is $y = 3$.

11 (c). $\lim_{t\to-\infty} y = 3$ and $\lim_{t\to\infty} y = 3$.

12 (a). $\lambda^2 - 6\lambda + 8 = (\lambda-2)(\lambda-4) = 0$. Thus $y = c_1 e^{2t} + c_2 e^{4t}$.

12 (b). $c_1 e^2 + c_2 e^4 = 2,\ 2c_1 e^2 + 4c_2 e^4 = -8$. Therefore, $c_1 = 8e^{-2}$ and $c_2 = -6e^{-4}$, and $y = 8e^{2(t-1)} - 6e^{4(t-1)}$.

12 (c). $\lim_{t\to-\infty} y = 0$ and $\lim_{t\to\infty} y = -\infty$.

13 (a). $\lambda^2 + 4\lambda + 2 = 0$. Thus $\lambda = \dfrac{-4 \pm \sqrt{16-8}}{2} = -2 \pm \sqrt{2}$ and $y = c_1 e^{(-2-\sqrt{2})t} + c_2 e^{(-2+\sqrt{2})t}$.

13 (b). $y(0) = c_1 + c_2 = 0$, $y'(0) = (-2-\sqrt{2})c_1 + (-2+\sqrt{2})c_2 = 4$. Solving these simultaneous equations yields $c_1 = -\sqrt{2}$ and $c_2 = \sqrt{2}$. Thus the unique solution to the initial value problem is $y = -\sqrt{2}e^{(-2-\sqrt{2})t} + \sqrt{2}e^{(-2+\sqrt{2})t}$.

13 (c). $\lim_{t\to-\infty} y = -\infty$ and $\lim_{t\to\infty} y = 0$.

14 (a). $\lambda^2 - 4\lambda - 1 = 0 \Rightarrow \lambda = \dfrac{4 \pm \sqrt{16+4}}{2} = 2 \pm \sqrt{5}$. Thus $y = c_1 e^{(2-\sqrt{5})t} + c_2 e^{(2+\sqrt{5})t}$.

14 (b). $c_1 + c_2 = 1$, $(2 - \sqrt{5})c_1 + (2+\sqrt{5})c_2 = 2 + \sqrt{5}$. Therefore, $c_1 = 0$ and $c_2 = 1$, and $y = e^{(2+\sqrt{5})t}$.

14 (c). $\lim_{t\to-\infty} y = 0$ and $\lim_{t\to\infty} y = \infty$.

15 (a). $2\lambda^2 - 1 = 0$. Thus $\lambda = \pm\dfrac{1}{\sqrt{2}}$ and $y = c_1 e^{-t/\sqrt{2}} + c_2 e^{t/\sqrt{2}}$.

15 (b). $y(0) = c_1 + c_2 = -2$, $y'(0) = -\dfrac{1}{\sqrt{2}}c_1 + \dfrac{1}{\sqrt{2}}c_2 = \sqrt{2}$. Solving these simultaneous equations yields $c_1 = -2$ and $c_2 = 0$. Thus the unique solution to the initial value problem is $y = -2e^{-t/\sqrt{2}}$.

15 (c). $\lim_{t\to-\infty} y = -\infty$ and $\lim_{t\to\infty} y = 0$.

16. Since $\lim_{t\to\infty} e^{-3t} = 0$, $y(t) = c_1 e^{-3t} + 2$, $y(0) = c_1 + 2 = 1 \Rightarrow c_1 = -1$. Therefore, $y(t) = 2 - e^{-3t}$, $\lambda^2 + \alpha\lambda + \beta = \lambda(\lambda+3) \Rightarrow \alpha = 3, \beta = 0$ and $y'(0) = y_0' = 3$.

17 (a). $W = \begin{vmatrix} e^{-t} & ce^{\lambda_2 t} \\ -e^{-t} & \lambda_2 ce^{\lambda_2 t} \end{vmatrix} = (\lambda_2 + 1)ce^{(\lambda_2 - 1)t} = 4e^{2t}$. Thus $\lambda_2 = 3$ and $c = 1$. The second member of the fundamental set is then $y_2 = e^{3t}$.

17 (b). $\lambda^2 + \alpha\lambda + \beta = (\lambda+1)(\lambda-3)$. Therefore $\alpha = -2$ and $\beta = -3$.

17 (c). The general solution is $y = c_1 e^{-t} + c_2 e^{3t}$. Using the initial conditions, we have $y(0) = c_1 + c_2 = 3$, $y'(0) = -c_1 + 3c_2 = 5$. Solving these simultaneous equations yields $c_1 = 1$ and $c_2 = 2$. The unique solution to the initial value problem is $y = e^{-t} + 2e^{3t}$.

18 (a). $\lambda^2 + 2\lambda = 0 \Rightarrow \lambda = 0, -2$. Graph (C) since it is the only equation admitting a nonzero limit as $t \to \infty$.

18 (b). $6\lambda^2 - 5\lambda + 1 = (3\lambda - 1)(2\lambda - 1) = 0 \Rightarrow \lambda = \dfrac{1}{3}, \dfrac{1}{2}$. Graph (B)

18 (c). $\lambda^2 - 1 = 0 \Rightarrow \lambda = 1, -1$. Graph (A).

19. Utilizing the hint given, we can make the substitution $u(t) = y'(t)$. The equation then becomes $u'' - 5u' + 6u = 0$.

The characteristic equation of this new differential equation is $\lambda^2 - 5\lambda + 6 = (\lambda-3)(\lambda-2) = 0$.

Thus $u(t) = c_1{}' e^{2t} + c_2{}' e^{3t} = y'(t)$. Antidifferentiation gives us

$y = \frac{1}{2}c_1{}' e^{2t} + \frac{1}{3}c_2{}' e^{3t} + c_3 = c_1 e^{2t} + c_2 e^{3t} + c_3$.

20 (a). $mx'' + kx' = 0,\ m\lambda^2 + k\lambda = m\lambda\left(\lambda + \frac{k}{m}\right) = 0 \Rightarrow \lambda = -\frac{k}{m}, 0$. Thus $x(t) = c_1 e^{-\frac{k}{m}t} + c_2$.

20 (b). $x' = -\frac{k}{m}c_1 e^{-\frac{k}{m}t}\quad \therefore c_1 + c_2 = x_0,\ -\frac{k}{m}c_1 = v_0 \Rightarrow c_1 = -\frac{m}{k}v_0, c_2 = x_0 + \frac{m}{k}v_0$.

$x(t) = -\frac{mv_0}{k}e^{-\frac{k}{m}t} + x_0 + \frac{mv_0}{k} = x_0 + \frac{mv_0}{k}(1 - e^{-\frac{k}{m}t})$.

20 (c). $\lim_{t\to\infty} x(t) = x_0 + \frac{mv_0}{k}$.

21 (a). $\lambda^2 - \Omega^2 = (\lambda+\Omega)(\lambda-\Omega) = 0$. Thus $r(t) = c_1 e^{-\Omega t} + c_2 e^{\Omega t}$. From the initial conditions,

$r(0) = c_1 + c_2 = r_0$ and $r'(0) = -\Omega c_1 + \Omega c_2 = r_0'$. Solving these simultaneous equations yields

$c_1 = \frac{1}{2}\left(r_0 - \Omega^{-1}r_0'\right)$ and $c_2 = \frac{1}{2}\left(r_0 + \Omega^{-1}r_0'\right)$. Thus the unique solution is

$r = \frac{1}{2}\left(r_0 - \Omega^{-1}r_0'\right)e^{-\Omega t} + \frac{1}{2}\left(r_0 + \Omega^{-1}r_0'\right)e^{\Omega t} = r_0\cosh(\Omega t) + \Omega^{-1}r_0'\sinh(\Omega t)$. When $r_0 = r_0' = 0$,

$r(t) = 0$ and the particle remains at rest at the pivot.

21 (b). $r_0 = 0,\ r_0' = \frac{1}{5},\ \ell = 3,\ \Omega = \frac{30(2\pi)}{60} = \pi$. We need to find t when $r(t) = \ell = 3$. With this condition,

we have $r(t) = 3 = \frac{1}{\pi}\cdot\frac{1}{5}\sinh(\pi t)$. Solved for t, $t = 1.44705$ seconds.

22 (a). $\lambda^2 + \frac{k}{m}\lambda - \Omega^2 = 0$. Thus $\lambda_\pm = \dfrac{-\frac{k}{m} \pm \sqrt{\left(k/m\right)^2 + 4\Omega^2}}{2}$ and the general solution to the differential

equation is $r = c_1 e^{\lambda_- t} + c_2 e^{\lambda_+ t}$. From the initial conditions, we have

$c_1 + c_2 = 0$ and $\lambda_- c_1 + \lambda_+ c_2 = r_0'$. Solving these simultaneous equations yields

$c_2 = -c_1 = \dfrac{r_0'}{\sqrt{\left(k/m\right)^2 + 4\Omega^2}}$, and thus $r(t) = \dfrac{r_0' e^{-\frac{k}{2m}t}\cdot 2\sinh\left[\frac{1}{2}\left(\sqrt{\left(k/m\right)^2 + 4\Omega^2}\right)t\right]}{\sqrt{\left(k/m\right)^2 + 4\Omega^2}}$

22 (b). $m \approx R^3, k \approx R^2\quad \therefore \frac{k}{m} \approx \frac{1}{R}$ would decrease.

22 (c). $\Omega = 20^{rev}\!/_{min} = 20(2\pi)/60 = \frac{2\pi}{3}\,{}^{rad}\!/_{sec}$, $r_0' = 1$, $\frac{k}{m} = 4s^{-1}$

$$\sqrt{\left(\frac{k}{m}\right)^2 + 4\Omega^2} = \sqrt{16 + 4\left(\frac{4\pi^2}{9}\right)} = 5.79189,\ \frac{k}{2m} = 2,\ r_0' = 1.$$

$$\therefore r(t) = \frac{1 \cdot e^{-2t} \cdot 2\sinh[2.89594t]}{5.79189},\ r(2) = 1.03605cm.$$

23 (a). $r = c_2\left[e^{\lambda_+ t} - e^{\lambda_- t}\right]$ where $\lambda_+ > 0$ and $\lambda_- < 0$ (from question 22). For the positive limit,

$$\lambda_+ = \frac{\Omega}{2} = \frac{1}{2}\left(-\frac{k}{m} + \sqrt{\left(k/m\right)^2 + 4\Omega^2}\right).$$ Therefore

$$\Omega + \frac{k}{m} = \sqrt{\left(k/m\right)^2 + 4\Omega^2},$$ and $\Omega^2 + 2\frac{k}{m}\Omega + \left(\frac{k}{m}\right)^2 = \left(\frac{k}{m}\right)^2 + 4\Omega^2$. Then

$2\frac{k}{m}\Omega = 3\Omega^2$, and $\frac{k}{m} = \frac{3}{2}\Omega$.

23 (b). Using the relation reached in part (a), $\left(\frac{k}{m}\right)^2 + 4\Omega^2 = \frac{9}{4}\Omega^2 + 4\Omega^2 = \frac{25}{4}\Omega^2$. Therefore

$$\sqrt{\left(k/m\right)^2 + 4\Omega^2} = \frac{5}{2}\Omega \text{ and } \lambda_- = \frac{1}{2}\left[-\frac{k}{m} - \sqrt{\left(k/m\right)^2 + 4\Omega^2}\right] = \frac{1}{2}\left[-\frac{3}{2}\Omega - \frac{5}{2}\Omega\right] = -2\Omega.$$

$$\lambda_+ = \frac{1}{2}\left[-\frac{3}{2}\Omega + \frac{5}{2}\Omega\right] = \frac{\Omega}{2}, \text{ and so } r(t) = \frac{r_0'}{\frac{5}{2}\Omega}\left[e^{\frac{\Omega}{2}t} - e^{-2\Omega t}\right].$$

Section 4.5

1 (a). $\lambda^2 + 2\lambda + 1 = (\lambda + 1)^2 = 0$. Thus $y = c_1 e^{-t} + c_2 t e^{-t}$.

1 (b). $y' = -c_1 e^{-t} + c_2(1 - t)e^{-t}$. From the initial conditions, we have $c_1 e^{-1} + c_2 e^{-1} = 1$ and $-c_1 e^{-1} + c_2 \cdot 0 = 0$, and thus $c_1 = 0$ and $c_2 = e$. The unique solution is then $y(t) = te^{-(t-1)}$.

1 (c). $\lim_{t\to-\infty} y = -\infty$ and $\lim_{t\to\infty} y = 0$.

2 (a). $9\lambda^2 - 6\lambda + 1 = (3\lambda - 1)^2 = 0$. Thus $y = c_1 e^{t/3} + c_2 t e^{t/3}$.

2 (b). $y' = \frac{1}{3}c_1 e^{t/3} + c_2(1 + \frac{t}{3})e^{t/3}$. $c_1 e + 3c_2 e = -2$ and $\frac{1}{3}c_1 e + 2c_2 e = -\frac{5}{3} \Rightarrow c_1 = e^{-1}$ and $c_2 = -e^{-1}$. The unique solution is then $y(t) = e^{(t-3)/3} - te^{(t-3)/3} = (1-t)e^{(t-3)/3}$.

2 (c). $\lim_{t\to-\infty} y = 0$ and $\lim_{t\to\infty} y = -\infty$.

3 (a). $\lambda^2 + 6\lambda + 9 = (\lambda + 3)^2 = 0$. Thus $y = c_1 e^{-3t} + c_2 t e^{-3t}$.

3 (b). $y' = -3c_1 e^{-3t} + c_2(1-3t)e^{-3t}$. From the initial conditions, we have

$c_1 = 2$ and $-3c_1 + c_2 = -2$, thus $c_2 = 4$. The unique solution is then $y(t) = (2+4t)e^{-3t}$.

3 (c). $\lim_{t\to-\infty} y = -\infty$ and $\lim_{t\to\infty} y = 0$.

4 (a). $25\lambda^2 + 20\lambda + 4 = (5\lambda+2)^2 = 0$. Thus $y = c_1 e^{-\frac{2}{5}t} + c_2 t e^{-\frac{2}{5}t}$.

4 (b). $y' = \frac{-2}{5} c_1 e^{-\frac{2}{5}t} + c_2(1 - \frac{2t}{5})e^{-\frac{2}{5}t}$.

$c_1 e^{-2} + 5c_2 e^{-2} = 4e^{-2}$ and $\frac{-2}{5} c_1 e^{-2} + (1-2)c_2 e^{-2} = -\frac{3}{5} e^{-2} \Rightarrow c_1 = -1$ and $c_2 = 1$.

The unique solution is then $y(t) = (t-1)e^{-\frac{2}{5}t}$.

4 (c). $\lim_{t\to-\infty} y = -\infty$ and $\lim_{t\to\infty} y = 0$.

5 (a). $4\lambda^2 - 4\lambda + 1 = (2\lambda - 1)^2 = 0$. Thus $y = c_1 e^{\frac{t}{2}} + c_2 t e^{\frac{t}{2}}$.

5 (b). $y' = \frac{1}{2} c_1 e^{\frac{t}{2}} + c_2(1 + \frac{t}{2})e^{\frac{t}{2}}$. From the initial conditions, we have

$c_1 e^{\frac{1}{2}} + c_2 e^{\frac{1}{2}} = -4$ and $\frac{1}{2} c_1 e^{\frac{1}{2}} + \frac{3}{2} c_2 e^{\frac{1}{2}} = 0$, and thus $c_1 = -6e^{-\frac{1}{2}}$ and $c_2 = 2e^{-\frac{1}{2}}$. The unique

solution is then $y(t) = (-6+2t)e^{\frac{t-1}{2}}$.

5 (c). $\lim_{t\to-\infty} y = 0$ and $\lim_{t\to\infty} y = \infty$.

6 (a). $\lambda^2 - 4\lambda + 4 = (\lambda - 2)^2 = 0$. Thus $y = c_1 e^{2t} + c_2 t e^{2t}$.

6 (b). $c_1 e^{-2} - c_2 e^{-2} = 2$ and $2c_1 e^{-2} + (1-2)c_2 e^{-2} = 1 \Rightarrow c_1 = -e^2$ and $c_2 = -3e^2$

The unique solution is then $y(t) = (-1-3t)e^{2(t+1)}$.

6 (c). $\lim_{t\to-\infty} y = 0$ and $\lim_{t\to\infty} y = -\infty$.

7 (a). $16\lambda^2 - 8\lambda + 1 = (4\lambda - 1)^2 = 0$. Thus $y = c_1 e^{\frac{t}{4}} + c_2 t e^{\frac{t}{4}}$.

7 (b). From the initial conditions, we have $y(0) = c_1 = -4$ and $y_0' = \frac{1}{4} c_1 + c_2 = 3$. Thus $c_2 = 4$, and so

the unique solution is $y(t) = (-4+4t)e^{\frac{t}{4}}$.

7 (c). $\lim_{t\to-\infty} y = 0$ and $\lim_{t\to\infty} y = \infty$.

8 (a). $\lambda^2 + 2\sqrt{2}\lambda + 2 = (\lambda + \sqrt{2})^2 = 0$. Thus $y = c_1 e^{-\sqrt{2}t} + c_2 t e^{-\sqrt{2}t}$.

8 (b). $c_1 + 0 = 1$ and $-\sqrt{2}c_1 + (1-0)c_2 = 0 \Rightarrow c_1 = 1$ and $c_2 = \sqrt{2}$

The unique solution is then $y(t) = (1+\sqrt{2}t)e^{-\sqrt{2}t}$.

8 (c). $\lim_{t\to-\infty} y = -\infty$ and $\lim_{t\to\infty} y = 0$.

9 (a). $\lambda^2 - 5\lambda + 6.25 = \left(\lambda - \frac{5}{2}\right)^2 = 0$. Thus $y = c_1 e^{\frac{5}{2}t} + c_2 t e^{\frac{5}{2}t}$.

9 (b). $y' = \frac{5}{2} c_1 e^{\frac{5}{2}t} + c_2\left(1 + \frac{5}{2}t\right) e^{\frac{5}{2}t}$. From the initial conditions, we have

$c_1 e^{-5} - 2c_2 e^{-5} = 0$ and $\frac{5}{2} c_1 e^{-5} - 4c_2 e^{-5} = 1$, and thus $c_1 = 2e^5$ and $c_2 = e^5$. The unique solution is

then $y(t) = (2+t)e^{\frac{5}{2}(t+2)}$.

9 (c). $\lim_{t\to-\infty} y = 0$ and $\lim_{t\to\infty} y = \infty$.

10 (a). $3\lambda^2 + 2\sqrt{3}\lambda + 1 = (\sqrt{3}\lambda + 1)^2 = 0$. Thus $y = c_1 e^{-\frac{1}{\sqrt{3}}t} + c_2 t e^{-\frac{1}{\sqrt{3}}t}$.

10 (b). $y' = \frac{-1}{\sqrt{3}} c_1 e^{-\frac{1}{\sqrt{3}}t} + c_2 (1 - \frac{t}{\sqrt{3}}) e^{-\frac{1}{\sqrt{3}}t}$. $c_1 + 0 = 2\sqrt{3}$ and $\frac{-1}{\sqrt{3}} c_1 + (1-0)c_2 = 3 \Rightarrow c_1 = 2\sqrt{3}$ and $c_2 = 5$.

The unique solution is then $y(t) = (2\sqrt{3} + 5t)e^{-\frac{1}{\sqrt{3}}t}$.

10 (c). $\lim_{t\to-\infty} y = -\infty$ and $\lim_{t\to\infty} y = 0$.

11. $\lambda^2 - 2\alpha\lambda + \alpha^2 = (\lambda - \alpha)^2 = 0$. Thus $y = c_1 e^{\alpha t} + c_2 t e^{\alpha t}$. From the initial conditions and the graph provided, $y(0) = c_1 = 0$ and at the maximum, $y' = c_2(1 + \alpha t)e^{\alpha t} = 0$. Solving for the *t* coordinate at the maximum gives us $t_{\max} = -\frac{1}{\alpha} = 2,$ and thus $\alpha = -\frac{1}{2}$.

Solving for the *y* coordinate at the maximum gives us $y_{\max} = c_2\left(-\frac{1}{\alpha}\right)e^{-1} = 2c_2 e^{-1} = 8e^{-1}$, and

thus $c_2 = 4$. Finally, the equation for $y(t)$ is $y(t) = 4te^{-\frac{t}{2}}$, and $\alpha = -\frac{1}{2},\ y_0 = 0,\ y_0' = 4$.

12. $y'' = 0 \ \therefore \alpha = 0,\ y = y_0' t + y_0,\ y(0) = y_0 = 2,\ y(4) = (y_0')(4) + 2 = 0 \Rightarrow y_0' = -\frac{1}{2}$.

Therefore, $y(t) = -\frac{1}{2}t + 2,\ \alpha = 0,\ y_0 = 2,\ y_0' = -\frac{1}{2}$.

13. $4\lambda^2 + 4\lambda + 1 = (2\lambda + 1)^2 = 0$. Thus $y = c_1 e^{-\frac{t}{2}} + c_2 t e^{-\frac{t}{2}}$. From the first point given, we have $y(1) = c_1 e^{-\frac{1}{2}} + c_2 e^{-\frac{1}{2}} = e^{-\frac{1}{2}}$. From the second, we have $y(2) = c_1 e^{-1} + 2c_2 e^{-1} = 0$. Solving these two simultaneous equations yields $c_1 = 2$ and $c_2 = -1$. Finally, we have $y(t) = 2e^{-\frac{t}{2}} - te^{-\frac{t}{2}}$, and differentiation gives us $y' = -e^{-\frac{t}{2}} - \frac{1}{2}(2-t)e^{-\frac{t}{2}}$. Thus $y(0) = 2$ and $y'(0) = -2$.

14 (a). $y_2 = e^t \upsilon,\ y_2' = e^t\upsilon + e^t\upsilon',\ \ y_2'' = e^t v + 2e^t v' + e^t\upsilon''$. Therefore

$t(\upsilon'' + 2\upsilon' + v) - (2t+1)(\upsilon' + v) + (t+1)\upsilon = 0 \Rightarrow$

$t\upsilon'' + (2t - 2t - 1)\upsilon' + (t - 2t - 1 + t + 1)v = t\upsilon'' - \upsilon' = 0$;

$u = v',\ u' - t^{-1}u = 0,\ \left(t^{-1}u\right)' = 0,\ u = ct \Rightarrow v = c\dfrac{t^2}{2}\ \ \therefore y_2 = t^2 e^t$

14 (b). $W = \begin{vmatrix} e^t & t^2 e^t \\ e^t & (t^2+2t)e^t \end{vmatrix} = 2te^{2t} \neq 0$ on $(-\infty, 0)$ and $(0, \infty)$.

14 (c). $p(t) = -\dfrac{(2t+1)}{t},\ q(t) = \dfrac{t+1}{t}$, continuous on $(-\infty, 0)$ and $(0, \infty)$.

15 (a). $y_2 = t\upsilon,\ y_2' = \upsilon + t\upsilon',\ \ y_2'' = t\upsilon'' + 2\upsilon'$. Therefore

$t^2(t\upsilon'' + 2\upsilon') - t(\upsilon + t\upsilon') + t\upsilon = t^3\upsilon'' + t^2\upsilon' = t^2\left(tv'' + v'\right) = 0$, and so

$\left(t\upsilon'\right)' = 0$, which means that $\upsilon' = \dfrac{c}{t}$. Antidifferentiation yields $\upsilon = c\ln|t| + c'$, and thus

$y_2 = t\ln|t|$.

15 (b). $W = \begin{vmatrix} t & t\ln|t| \\ 1 & \ln|t| + 1 \end{vmatrix} = t \neq 0$ on $(-\infty, 0)$ and $(0, \infty)$.

15 (c). $p(t) = -\dfrac{1}{t},\ q(t) = \dfrac{1}{t^2}$, continuous on $(-\infty, 0)$ and $(0, \infty)$.

16 (a). $y_2 = \upsilon\sin t,\ y_2' = \upsilon'\sin t + \upsilon\cos t,\ \ y_2'' = v''\sin t + 2v'\cos t - v\sin t$. Therefore

$(\upsilon''\sin t + 2\upsilon'\cos t - v\sin t) - (2\cot t)(v'\sin t + v\cos t) + (1 + 2\cot^2 t)\sin v = 0 \Rightarrow$

$\upsilon''(\sin t) + \upsilon'(2\cos t - 2\cot t\sin t) + v(-\sin t - 2\cot t\cos t + \sin t + 2\cot^2 t\sin t) = 0$

$\Rightarrow v''\sin t = 0 \Rightarrow v'' = 0,\ v = c_1 t + c_2\ \ \therefore y_2 = t\sin t$.

16 (b). $W = \begin{vmatrix} \sin t & t\sin t \\ \cos t & \sin t + t\cos t \end{vmatrix} = \sin^2 t \neq 0$ on $(n\pi, (n+1)\pi), n = \ldots, -2, -1, 0, 1, 2, \ldots$.

16 (c). $p(t) = -2\cot t,\ q(t) = 1 + 2\cot^2 t$, continuous on same intervals.

17 (a). $y_2 = (t+1)^2\upsilon,\ y_2' = 2(t+1)\upsilon + (t+1)^2\upsilon',\ \ y_2'' = (t+1)^2\upsilon'' + 4(t+1)\upsilon' + 2\upsilon$. Therefore

$(t+1)^4\upsilon'' + 4(t+1)^3\upsilon' + 2(t+1)^2\upsilon - 4\left[2(t+1)^2\upsilon + (t+1)^3\upsilon'\right] + 6(t+1)^2 v$

$= (t+1)^4\upsilon'' = 0$, and so $\upsilon'' = 0$. Antidifferentiation yields $\upsilon = c_1(t+1) + c_2$, and thus

$y_2 = (t+1)^3$.

17 (b). $W = \begin{vmatrix} (t+1)^2 & (t+1)^3 \\ 2(t+1) & 3(t+1)^2 \end{vmatrix} = (t+1)^4 \neq 0$ on $(-\infty, -1)$ and $(-1, \infty)$.

17 (c). $p(t)=-\dfrac{4}{t+1},\ q(t)=\dfrac{6}{(t+1)^2}$, continuous on $(-\infty,-1)$ and $(-1,\infty)$.

18 (a). $y_2=e^{-t^2}\upsilon,\ y_2'=-2te^{-t^2}\upsilon+e^{-t^2}\upsilon',$.

$y_2''=-2e^{-t^2}v+4t^2e^{-t^2}v-2te^{-t^2}v'-2te^{-t^2}v'+e^{-t^2}\upsilon''$.

Therefore, $\upsilon''-4t\upsilon'+(-2+4t^2)v+4t(-2tv+\upsilon')+(2+4t^2)\upsilon=0\Rightarrow v''=0$;

$\therefore v=c_1t+c_2,\ y_2=te^{-t^2}$.

18 (b). $W=\begin{vmatrix} e^{-t^2} & te^{-t^2} \\ -2te^{-t^2} & e^{-t^2}-2t^2e^{-t^2}\end{vmatrix}=e^{-2t^2}\neq 0$ on $(-\infty,\infty)$.

18 (c). $p(t)=4t,\ q(t)=2+4t^2$, continuous on $(-\infty,\infty)$.

19 (a). $y_2=(t-2)^2\upsilon,\ y_2'=2(t-2)\upsilon+(t-2)^2\upsilon',\ y_2''=(t-2)^2\upsilon''+4(t-2)\upsilon'+2\upsilon$. Therefore

$(t-2)^4\upsilon''+4(t-2)^3\upsilon'+2(t-2)^2\upsilon+2(t-2)^2\upsilon+(t-2)^3\upsilon'-4(t-2)^2\upsilon=0$, and so

$\upsilon''+\dfrac{5}{(t-2)}\upsilon'=0$. Thus $\left((t-2)^5\upsilon'\right)'=0$, and antidifferentiation yields $\upsilon=-\dfrac{c_1}{4(t-2)^4}+c_2$,

and thus $y_2=(t-2)^{-2}$.

19 (b). $W=\begin{vmatrix}(t-2)^2 & (t-2)^{-2} \\ 2(t-2) & -2(t-2)^{-3}\end{vmatrix}=-\dfrac{4}{(t-2)}\neq 0$ on $(-\infty,2)$ and $(2,\infty)$.

19 (c). $p(t)=\dfrac{1}{t-2},\ q(t)=-\dfrac{4}{(t-2)^2}$, continuous on $(-\infty,2)$ and $(2,\infty)$.

20 (a). $y_2=e^t\upsilon,\ y_2'=e^t\upsilon+e^t\upsilon',\ y_2''=e^tv+2e^tv'+e^t\upsilon''$. Therefore

$\upsilon''+2\upsilon'+v-\left(2+\dfrac{n-1}{t}\right)(\upsilon'+v)+\left(1+\dfrac{n-1}{t}\right)\upsilon=0\Rightarrow$

$\upsilon''-\dfrac{n-1}{t}v'=0\Rightarrow\left(t^{-(n-1)}v'\right)'=0\Rightarrow v'=c_1t^{n-1}\Rightarrow v=\frac{c_1}{n}t^n,\ \therefore y_2=t^ne^t$

20 (b). $W=\begin{vmatrix}e^t & t^ne^t \\ e^t & (nt^{n-1}+t^n)e^t\end{vmatrix}=nt^{n-1}e^{2t}$.

If $n=1$, $W\neq 0$ on $(-\infty,\infty)$. If $n\geq 2$, $W\neq 0$ on $(-\infty,0),(0,\infty)$.

20 (c). $p(t)=-\left(2+\dfrac{n-1}{t}\right),\ q(t)=\left(1+\dfrac{n-1}{t}\right)$, continuous on same intervals.

Section 4.6

1 (a). $2e^{i\pi/3} = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) = 1 + i\sqrt{3}$

1 (b). $-2\sqrt{2}e^{-i\pi/4} = -2\sqrt{2}\left(\cos\frac{-\pi}{4} + i\sin\frac{-\pi}{4}\right) = -2 + i2$

1 (c). $(2-i)e^{i3\pi/2} = (2-i)(-i) = -1 - i2$.

1 (d). $\frac{1}{2\sqrt{2}}e^{i7\pi/6} = \frac{1}{2\sqrt{2}}\left(\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}\right) = -\frac{\sqrt{3}}{4\sqrt{2}} - i\frac{1}{4\sqrt{2}}$

1 (e). $\left(\sqrt{2}e^{i\pi/6}\right)^4 = 4e^{i2\pi/3} = 4\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right) = 4\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = -2 + i2\sqrt{3}$

2 (a). $2\cos(\sqrt{2}t) + i2\sin(\sqrt{2}t)$

2 (b). $\frac{2}{\pi}e^{-2t}\cos(3t) - i\frac{2}{\pi}e^{-2t}\sin(3t)$

2 (c). $-\frac{1}{2}e^{2t}[\cos(t+\pi) + i\sin(t+\pi)] = \frac{1}{2}e^{2t}\cos t + i\frac{1}{2}e^{2t}\sin t$.

2 (d). $3\sqrt{3}e^{3t}\cos(3t) + i3\sqrt{3}e^{3t}\sin(3t)$

2 (e). $-\frac{\sqrt{2}}{4}(\cos 3\pi t + i\sin 3\pi t) = -\frac{\sqrt{2}}{4}\cos(3\pi t) - i\frac{\sqrt{2}}{4}\sin(3\pi t)$

3 (a). $\lambda^2 + 4 = 0$, and thus $\lambda = \pm i2$.

3 (b). $y = c_1\cos(2t) + c_2\sin(2t)$

3 (c). $y' = -2c_1\sin(2t) + 2c_2\cos(2t)$. Using the initial conditions, we have

$y(\pi/4) = c_2 = -2$ and $y'(\pi/4) = -2c_1 = 1$. Thus

$c_1 = -\frac{1}{2}$, $c_2 = -2$, and $y = -\frac{1}{2}\cos(2t) - 2\sin(2t)$.

4 (a). $\lambda^2 + 2\lambda + 2 = 0 \Rightarrow \lambda = \frac{-2 \pm \sqrt{4-8}}{2} = -1 \pm i$.

4 (b). $y = c_1e^{-t}\cos t + c_2e^{-t}\sin t$

4 (c). $y' = -c_1e^{-t}\cos t - c_1e^{-t}\sin t - c_2e^{-t}\sin t + c_2e^{-t}\cos t$. $y(0) = c_1 = 3$ and $y'(0) = -c_1 + c_2 = -1$.

Thus $c_1 = 3$, $c_2 = 2$, and $y = 3e^{-t}\cos t + 2e^{-t}\sin t$.

5 (a). $9\lambda^2 + 1 = 0$, and thus $\lambda = \pm i\frac{1}{3}$.

5 (b). $y = c_1\cos\left(\frac{t}{3}\right) + c_2\sin\left(\frac{t}{3}\right)$

5 (c). $y' = -\dfrac{1}{3}c_1 \sin\left(\dfrac{t}{3}\right) + \dfrac{1}{3}c_2 \cos\left(\dfrac{t}{3}\right)$. Using the initial conditions, we have

$$y(\pi/2) = \frac{\sqrt{3}}{2}c_1 + \frac{1}{2}c_2 = 4 \text{ and } y'(\pi/2) = -\frac{1}{3}c_1\left(\frac{1}{2}\right) + \frac{1}{3}c_2\left(\frac{\sqrt{3}}{2}\right) = 0.$$

Solving these simultaneous equations gives us

$$c_1 = 2\sqrt{3},\ c_2 = 2, \text{ and } y = 2\sqrt{3}\cos\left(\frac{t}{3}\right) + 2\sin\left(\frac{t}{3}\right).$$

6 (a). $2\lambda^2 - 2\lambda + 1 = 0 \Rightarrow \lambda = \dfrac{2 \pm \sqrt{4-8}}{4} = \dfrac{1}{2} \pm i\dfrac{1}{2}$.

6 (b). $y = c_1 e^{\frac{t}{2}} \cos\left(\frac{t}{2}\right) + c_2 e^{\frac{t}{2}} \sin\left(\frac{t}{2}\right)$

6 (c). $y' = \frac{1}{2}c_1 e^{\frac{t}{2}}\cos\left(\frac{t}{2}\right) - \frac{1}{2}c_1 e^{\frac{t}{2}}\sin\left(\frac{t}{2}\right) + \frac{1}{2}c_2 e^{\frac{t}{2}}\sin\left(\frac{t}{2}\right) + \frac{1}{2}c_2 e^{\frac{t}{2}}\cos\left(\frac{t}{2}\right)$.

$y(-\pi) = -c_2 e^{-\frac{\pi}{2}} = 1$ and $y'(-\pi) = -\frac{1}{2}c_1 e^{-\frac{\pi}{2}}(-1) + \frac{1}{2}c_2 e^{-\frac{\pi}{2}}(-1) = -1$. Thus

$c_1 = -3e^{\frac{\pi}{2}}$, $c_2 = -e^{\frac{\pi}{2}}$, and $y = -3e^{\frac{t+\pi}{2}}\cos\left(\frac{t}{2}\right) - e^{\frac{t+\pi}{2}}\sin\left(\frac{t}{2}\right)$.

7 (a). $\lambda^2 + \lambda + 1 = 0$, and thus $\lambda = -\dfrac{1}{2} \pm i\dfrac{\sqrt{3}}{2}$.

7 (b). $y = c_1 e^{-\frac{t}{2}} \cos\left(\dfrac{\sqrt{3}}{2}t\right) + c_2 e^{-\frac{t}{2}} \sin\left(\dfrac{\sqrt{3}}{2}t\right)$

7 (c). $y' = -\dfrac{1}{2}c_1 e^{-\frac{t}{2}}\cos\left(\dfrac{\sqrt{3}}{2}t\right) - \dfrac{\sqrt{3}}{2}c_1 e^{-\frac{t}{2}}\sin\left(\dfrac{\sqrt{3}}{2}t\right)$

$-\dfrac{1}{2}c_2 e^{-\frac{t}{2}}\sin\left(\dfrac{\sqrt{3}}{2}t\right) + \dfrac{\sqrt{3}}{2}c_2 e^{-\frac{t}{2}}\cos\left(\dfrac{\sqrt{3}}{2}t\right)$.

Using the initial conditions, we have $y(0) = c_1 = -2$ and $y'(0) = -\dfrac{1}{2}c_1 + \dfrac{\sqrt{3}}{2}c_2 = -2$.

Solving these simultaneous equations gives us

$$c_1 = -2,\ c_2 = -2\sqrt{3}, \text{ and } y = -2e^{-\frac{t}{2}}\cos\left(\frac{\sqrt{3}}{2}t\right) - 2\sqrt{3}e^{-\frac{t}{2}}\sin\left(\frac{\sqrt{3}}{2}t\right).$$

8 (a). $\lambda^2 + 4\lambda + 5 = 0 \Rightarrow \lambda = \dfrac{-4 \pm \sqrt{16-20}}{2} = -2 \pm i$.

8 (b). $y = c_1 e^{-2t}\cos t + c_2 e^{-2t}\sin t$

8 (c). $y' = -2c_1 e^{-2t}\cos t - c_1 e^{-2t}\sin t - 2c_2 e^{-2t}\sin t + c_2 e^{-2t}\cos t$.

$y\left(\frac{\pi}{2}\right) = c_2 e^{-\pi} = \frac{1}{2}$ and $y'\left(\frac{\pi}{2}\right) = -c_1 e^{-\pi} - 2c_2 e^{-\pi} = -2$. Thus

$c_1 = e^{\pi}$, $c_2 = \frac{1}{2}e^{\pi}$, and $y = e^{-2\left(t-\frac{\pi}{2}\right)}(\cos t + \frac{1}{2}\sin t)$.

9 (a). $9\lambda^2+6\lambda+2=0$, and thus $\lambda=-\frac{1}{3}\pm i\frac{1}{3}$.

9 (b). $y=c_1e^{-\frac{t}{3}}\cos\left(\frac{t}{3}\right)+c_2e^{-\frac{t}{3}}\sin\left(\frac{t}{3}\right)$

9 (c). $y'=-\frac{1}{3}c_1e^{-\frac{t}{3}}\cos\left(\frac{t}{3}\right)-\frac{1}{3}c_1e^{-\frac{t}{3}}\sin\left(\frac{t}{3}\right)-\frac{1}{3}c_2e^{-\frac{t}{3}}\sin\left(\frac{t}{3}\right)+\frac{1}{3}c_2e^{-\frac{t}{3}}\cos\left(\frac{t}{3}\right)$. Using the initial conditions, we have $y(3\pi)=-c_1e^{-\pi}\Rightarrow c_1=0$ and $y'(3\pi)=-\frac{1}{3}c_2e^{-\pi}=\frac{1}{3}$. Solving these simultaneous equations gives us $c_1=0,\ c_2=-e^{\pi}$, and $y=-e^{-\frac{t}{3}}e^{\pi}\sin\left(\frac{t}{3}\right)$.

10 (a). $\lambda^2+4\pi^2=0\Rightarrow\lambda=\pm 2\pi i$.

10 (b). $y=c_1\cos(2\pi t)+c_2\sin(2\pi t)$

10 (c). $y'=-2\pi c_1\sin(2\pi t)+2\pi c_2\cos(2\pi t)$.

$y(1)=c_1=2$ and $y'(1)=2\pi c_2=1\Rightarrow c_2=\frac{1}{2\pi}$.

Thus $y=2\cos(2\pi t)+\frac{1}{2\pi}\sin(2\pi t)$.

11 (a). $\lambda^2-2\sqrt{2}\lambda+3=0$, and thus $\lambda=\sqrt{2}\pm i$.

11 (b). $y=c_1e^{\sqrt{2}t}\cos(t)+c_2e^{\sqrt{2}t}\sin(t)$

11 (c). $y'=\sqrt{2}c_1e^{\sqrt{2}t}\cos(t)-c_1e^{\sqrt{2}t}\sin(t)+\sqrt{2}c_2e^{\sqrt{2}t}\sin(t)+c_2e^{\sqrt{2}t}\cos(t)$. Using the initial conditions, we have $y(0)=c_1=-\frac{1}{2}$ and $y'(0)=\sqrt{2}c_1+c_2=\sqrt{2}$.

Solving these simultaneous equations gives us

$c_1=-\frac{1}{2},\ c_2=\frac{3\sqrt{2}}{2}$, and $y=-\frac{1}{2}e^{\sqrt{2}t}\cos(t)+\frac{3\sqrt{2}}{2}e^{\sqrt{2}t}\sin(t)$.

12 (a). $9\lambda^2+\pi^2=0\Rightarrow\lambda=\pm\frac{\pi}{3}i$.

12 (b). $y=c_1\cos\left(\frac{\pi}{3}t\right)+c_2\sin\left(\frac{\pi}{3}t\right)$

12 (c). $y'=-\frac{\pi}{3}c_1\sin\left(\frac{\pi}{3}t\right)+\frac{\pi}{3}c_2\cos\left(\frac{\pi}{3}t\right)$.

$y(3)=-c_1=2$ and $y'(3)=-\frac{\pi}{3}c_2=-\pi\Rightarrow c_1=-2,\ c_2=3$.

Thus $y=-2\cos\left(\frac{\pi}{3}t\right)+3\sin\left(\frac{\pi}{3}t\right)$

13. $\lambda = \pm i$, so the characteristic equation must be $\lambda^2 + 1 = 0$. Thus $a = 0$ and $b = 1$.

$y_0 = y(\pi/4) = \dfrac{1}{\sqrt{2}} - 1$ and $y_0' = y'(\pi/4) = \dfrac{1}{\sqrt{2}} + 1$.

14. $\lambda = \pm 2i$, so the characteristic equation must be $\lambda^2 + 4 = 0$. Thus $a = 0$ and $b = 4$.

$y_0 = y(\pi/4) = 2$ and $y_0' = y'(\pi/4) = -2$.

15. $\lambda = -2 \pm i$, so the characteristic equation must be $\lambda^2 + 4\lambda + 5 = 0$. Thus $a = 4$ and $b = 5$.

$y_0 = y(0) = 1$ and $y_0' = y'(0) = -2 - 1 = -3$.

16. $\lambda = 1 \pm 2i$, so the characteristic equation must be $(\lambda - 1)^2 + 4 = \lambda^2 - 2\lambda + 5 = 0$. Thus

$a = -2$ and $b = 5$. $y_0 = y(\pi/6) = \dfrac{1}{2} - \dfrac{\sqrt{3}}{2}$ and $y_0' = y'(\pi/6) = -\dfrac{1}{2} - \dfrac{3\sqrt{3}}{2}$.

17. $\lambda = \pm i\pi$, so the characteristic equation must be $\lambda^2 + \pi^2 = 0$. Thus $a = 0$ and $b = \pi^2$.

$y_0 = y(1/2) = -1$ and $y_0' = y'(1/2) = -\sqrt{3}\pi$.

18. $y = \sin t + \cos t$, so $\alpha = 0$ and $\beta = 1$. $R\cos\delta = 1$ and $R\sin\delta = 1$, so $R = \sqrt{2}$ and $\delta = \dfrac{\pi}{4}$. Thus

$y = \sqrt{2}\cos\left(t - \dfrac{\pi}{4}\right)$.

19. $y = \cos\pi t - \sin\pi t$, so $\alpha = 0$ and $\beta = \pi$. $R\cos\delta = 1$ and $R\sin\delta = -1$, so $R = \sqrt{2}$ and $\delta = \dfrac{7\pi}{4}$.

Thus $y = \sqrt{2}\cos\left(\pi t - \dfrac{7\pi}{4}\right)$.

20. $y = e^t\cos t + \sqrt{3}e^t\sin t$, so $\alpha = 1$ and $\beta = 1$. $R\cos\delta = 1$ and $R\sin\delta = \sqrt{3}$, so $R = 2$ and $\delta = \dfrac{\pi}{3}$.

Thus $y = \sqrt{3}e^t\cos\left(t - \dfrac{\pi}{3}\right)$.

21. $y = -e^{-t}\cos t + \sqrt{3}e^{-t}\sin t$, so $\alpha = -1$ and $\beta = 1$. $R\cos\delta = -1$ and $R\sin\delta = \sqrt{3}$, so

$R = 2$ and $\delta = \dfrac{2\pi}{3}$. Thus $y = 2e^{-t}\cos\left(t - \dfrac{2\pi}{3}\right)$.

22. $y = e^{-2t}\cos 2t - e^{-2t}\sin 2t$, so $\alpha = -2$ and $\beta = 2$. $R\cos\delta = 1$ and $R\sin\delta = -1$, so

$R = \sqrt{2}$ and $\delta = \dfrac{7\pi}{4}$. Thus $y = \sqrt{2}e^{-2t}\cos\left(2t - \dfrac{7\pi}{4}\right)$.

23. $y(t) = 2\cos\left(\dfrac{\pi}{2}t\right)$, $a = 0$, $b = \dfrac{\pi^2}{4}$, $y_0 = 2$, $y_0' = 0$.

24. $y(t) = \cos\left(\dfrac{3}{2}t - \dfrac{\pi}{8}\right)$, $a = 0$, $b = \dfrac{9}{4}$, $y_0 = \cos\left(\dfrac{\pi}{8}\right)$, $y_0' = \dfrac{3}{2}\sin\left(\dfrac{\pi}{8}\right)$.

25. $y(t)=\frac{1}{2}\cos\left(2t-\frac{5\pi}{6}\right),\ a=0,\ b=4,\ y_0=\frac{1}{2}\cos\frac{5\pi}{6},\ y_0'=\sin\frac{5\pi}{6}$.

26 (a). $\lambda^2+\mu\lambda+\omega^2=0$. $\lambda=\frac{-\mu\pm\sqrt{\mu^2-4\omega^2}}{2}=-\frac{\mu}{2}\pm\frac{i}{2}\sqrt{4\omega^2-\mu^2}$

$$y=c_1e^{-\frac{\mu}{2}t}\cos\left(\sqrt{\omega^2-\tfrac{\mu^2}{4}}\,t\right)+c_2e^{-\frac{\mu}{2}t}\sin\left(\sqrt{\omega^2-\tfrac{\mu^2}{4}}\,t\right).$$

26 (b). $y(0)=c_1=2,\ y'(0)=-\frac{\mu}{2}c_1+\sqrt{\omega^2-\tfrac{\mu^2}{4}}\,c_2=0\Rightarrow c_2=\frac{\mu}{\sqrt{\omega^2-\frac{\mu^2}{4}}}$.

$$y=e^{-\frac{\mu}{2}t}\left[2\cos\left(\sqrt{\omega^2-\tfrac{\mu^2}{4}}\,t\right)+\frac{\mu}{\sqrt{\omega^2-\frac{\mu^2}{4}}}\sin\left(\sqrt{\omega^2-\tfrac{\mu^2}{4}}\,t\right)\right].$$

26 (c). $\alpha=-\frac{\mu}{2},\ \beta=\sqrt{\omega^2-\tfrac{\mu^2}{4}}$.

$$R^2=4+\frac{\mu^2}{\omega^2-\frac{\mu^2}{4}}=\frac{4\omega^2}{\omega^2-\frac{\mu^2}{4}}\Rightarrow R=\frac{2\omega}{\sqrt{\omega^2-\frac{\mu^2}{4}}},\ \tan\delta=\frac{\mu}{2\sqrt{\omega^2-\frac{\mu^2}{4}}}=\frac{\mu}{\sqrt{4\omega^2-\mu^2}}.$$

27 (a). $e^{\alpha t}\cos\beta t=\frac{1}{2}e^{(\alpha+i\beta)t}+\frac{1}{2}e^{(\alpha-i\beta)t},\ e^{\alpha t}\sin\beta t=\frac{1}{2i}e^{(\alpha+i\beta)t}-\frac{1}{2i}e^{(\alpha-i\beta)t}$.

27 (b). $\left[e^{\alpha t}\cos\beta t\ ,\ e^{\alpha t}\sin\beta t\right]=\left[e^{(\alpha+i\beta)t}\ ,\ e^{(\alpha-i\beta)t}\right]\begin{bmatrix}\frac{1}{2} & \frac{1}{2i}\\ \frac{1}{2} & -\frac{1}{2i}\end{bmatrix}$.

27 (c). $\det A=-\frac{1}{2i}$, so $A^{-1}=-2i\begin{bmatrix}-\frac{1}{2i} & -\frac{1}{2i}\\ -\frac{1}{2} & \frac{1}{2}\end{bmatrix}=\begin{bmatrix}1 & 1\\ i & -i\end{bmatrix}$.

$\left[e^{(\alpha+i\beta)t}\ ,\ e^{(\alpha-i\beta)t}\right]=\left[e^{\alpha t}\cos\beta t\ ,\ e^{\alpha t}\sin\beta t\right]\begin{bmatrix}1 & 1\\ i & -i\end{bmatrix}$, so

$e^{(\alpha+i\beta)t}=e^{\alpha t}\cos\beta t+ie^{\alpha t}\sin\beta t$ and $e^{(\alpha-i\beta)t}=e^{\alpha t}\cos\beta t-ie^{\alpha t}\sin\beta t$.

28. From Abel's Theorem, $a=0\ \therefore y=c_1\cos 3t+c_2\sin 3t$.

29. $\lambda^2+4i\lambda+5=0$, and thus $\lambda=\frac{-4i\pm\sqrt{-16-20}}{2}=i,\ -5i$. Thus $y=c_1e^{it}+c_2e^{-5it}$.

Section 4.7

2 (a). $ky = mg,\ k = \dfrac{9.8(10)}{0.030} = 3266.67\ \ N/m$ or $3.2667\ \ N/mm$.

2 (b). $my'' + ky = 0,\ y'' + 326.67y = 0,\ y(0) = 0.07,\ y'(0) = 0$.

2 (c). $\omega = \sqrt{\frac{k}{m}} = 18.0739...,\ y = c_1 \cos\omega t + c_2 \sin\omega t$.

$y(0) = 0.07,\ c_2\omega = 0 \Rightarrow c_2 = 0\ \ \therefore y = 0.07\cos(18.0739t)$.

3. $my'' + ky = 0,\ y(0) = 0,\ y'(0) = 2$. The general solution to this differential equation is

$y = c_1 \cos\omega t + c_2 \sin\omega t$, where $\omega = \sqrt{\dfrac{k}{m}}$. From the initial conditions, we have

$y(0) = c_1 = 0$ and $y'(0) = \omega c_2 = 2$, and thus $c_2 = \dfrac{2}{\omega}$. The unique solution is then $y = \dfrac{2}{\omega}\sin\omega t$.

We know that $y_{\max} = 0.2 = \dfrac{2}{\omega}$, so $\omega = 10 = \sqrt{\dfrac{k}{20}}$. Solving for k yields $k = 2000$ N/m.

4 (a). $T = 4\left(\frac{5}{4} - \frac{3}{4}\right) = 2\ \ s$.

4 (b). $f = \dfrac{1}{T} = \dfrac{1}{2}\ \ Hz,\ \omega = 2\pi f = \pi\ \ rad/s$.

4 (c). $R = 3$; the first maximum occurs at

$$t = \frac{3}{4} - \left(\frac{5}{4} - \frac{3}{4}\right) = \frac{1}{4}\ \ \therefore \omega\left(\tfrac{1}{4}\right) - \delta = 0 \Rightarrow \delta = \frac{\pi}{4},\ y = 3\cos\left(\pi t - \tfrac{\pi}{4}\right)\ \ cm.$$

4 (d). $y(0) = 3\cos\left(\frac{\pi}{4}\right) = 2.1213..cm,\ y'(0) = -3\pi\sin\left(-\frac{\pi}{4}\right) = \dfrac{3\pi}{\sqrt{2}} = 6.6643..cm/s$.

5 (a). $my'' + \gamma y' + ky = 0$ with $m = 10,\ \gamma = 7,\ k = 100,\ y(0) = 0.5,\ y'(0) = 1$. Thus with numerical constants the initial value problem reads $y'' + 0.7y' + 10y = 0,\ y(0) = 0.5,\ y'(0) = 1$.

5 (b). $\lambda^2 + 0.7\lambda + 10 = 0$, and thus $\lambda = -0.35 \pm i3.14285 = -\alpha \pm i\beta$. The general solution is then $y = c_1 e^{-0.35t}\cos 3.14285t + c_2 e^{-0.35t}\sin 3.14285t$. From the initial conditions, we have $y(0) = c_1 = 0.5$ and $y'(0) = -0.35c_1 + 3.14285c_2 = 1$. Solving these simultaneous equations yields $c_1 = 0.5$ and $c_2 = 0.37386$. Thus the unique solution to the initial value problem is $y = e^{-0.35t}(0.5\cos(3.14285t) + 0.37386\sin(3.14285t))$. $\lim\limits_{t\to\infty} y(t) = 0$, which means that the damping dissipates the energy of the system, causing the motion to decrease.

6 (a). $my'' + \gamma y' + ky = 0,\ m\lambda^2 + \gamma\lambda + k = 0 \Rightarrow \lambda = \dfrac{-\gamma \pm \sqrt{\gamma^2 - 4mk}}{2m}$. Critical damping: $\gamma^2 = 4mk$.

6 (b). $y = c_1 e^{-\frac{\gamma}{2m}t} + c_2 t e^{-\frac{\gamma}{2m}t},\ y(0) = c_1 = 0,\ y'(0) = c_2 = 4.$

$$y'(t) = c_2\left(1 - \tfrac{\gamma}{2m}t\right)e^{-\frac{\gamma}{2m}t},\ y'(t) = 0 \text{ when } t_m = \frac{2m}{\gamma}.$$

$$y(t_m) = c_2 \frac{2m}{\gamma} e^{-1} = \frac{1}{2} \quad \therefore 4\frac{2m}{\gamma} = \frac{e}{2} \Rightarrow \frac{m}{\gamma} = \frac{e}{16}.$$

$$m = 1\ slug,\ \gamma = 16e^{-1} \approx 5.886...lb \cdot \sec/\,ft,\ k = \frac{\gamma^2}{4m} = 8.6614...lb\,/\,ft.$$

7 (a). $my'' + \gamma y' + ky = 0$ with $y(0) = y_0,\ y'(0) = 0$. $m\lambda^2 + \gamma\lambda + k = 0$, and thus $\lambda = \dfrac{-\gamma \pm \sqrt{\gamma^2 - 4mk}}{2m}$.

We can rewrite this as $\lambda_1 = \dfrac{-\gamma - \sqrt{\gamma^2 - 4mk}}{2m}$ and $\lambda_2 = \dfrac{-\gamma + \sqrt{\gamma^2 - 4mk}}{2m}$. The general solution to this initial value problem is $y = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$. From the initial conditions, we have $y(0) = c_1 + c_2 = y_0$ and $y'(0) = \lambda_1 c_1 + \lambda_2 c_2 = 0$. Solving these simultaneous equations for c_1 and c_2 gives us $c_1 = \dfrac{\lambda_2 y_0}{\lambda_2 - \lambda_1}$ and $c_1 = \dfrac{-\lambda_1 y_0}{\lambda_2 - \lambda_1}$, and thus the unique solution is

$$y = \frac{\left(\lambda_2 e^{\lambda_1 t} - \lambda_1 e^{\lambda_2 t}\right)}{\lambda_2 - \lambda_1} y_0.$$

7 (b).

$$\lambda_1\lambda_2 = \left(-\frac{\gamma}{2m} - \frac{\sqrt{\gamma^2 - 4mk}}{2m}\right)\left(-\frac{\gamma}{2m} + \frac{\sqrt{\gamma^2 - 4mk}}{2m}\right) = \frac{\gamma^2}{4m^2} - \frac{\gamma^2 - 4mk}{4m^2}$$

$$\lambda_1\lambda_2 = \frac{4mk}{4m^2} = \frac{k}{m} \Rightarrow \lambda_2 = \frac{k/m}{\lambda_1} = -\frac{2k}{\gamma + \sqrt{\gamma^2 - 4mk}}.$$ Therefore, $\lim_{\gamma\to\infty} \lambda_2 = 0$.

Then, since $\lambda_1 + \lambda_2 = 2\left(\dfrac{-\gamma}{2m}\right) = -\dfrac{\gamma}{m}$, $\lim_{\gamma\to\infty} \lambda_1 = \lim_{\gamma\to\infty}\left(-\dfrac{\gamma}{m} - \lambda_2\right) = -\infty.$

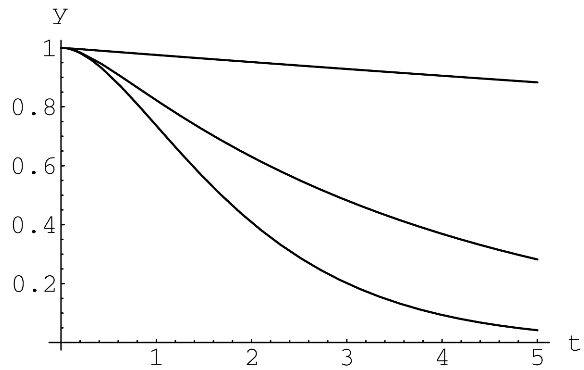
7 (c). $\lim_{\gamma\to\infty} y(t) = \lim_{\gamma\to\infty}\left[\left(\dfrac{\lambda_2 e^{\lambda_1 t} - \lambda_1 e^{\lambda_2 t}}{\lambda_2 - \lambda_1}\right) y_0\right] = y_0 \lim_{\gamma\to\infty}\left[\left(\dfrac{\lambda_2 e^{\lambda_1 t}}{\lambda_2 - \lambda_1} - \dfrac{\lambda_1 e^{\lambda_2 t}}{\lambda_2 - \lambda_1}\right)\right]$

$= y_0\left[0 - \lim_{\gamma\to\infty} \dfrac{e^{\lambda_2 t}}{\frac{\lambda_2}{\lambda_1} - 1}\right] = y_0\left[-\dfrac{1}{0-1}\right] = y_0$. As damping increases, the motion becomes suppressed; the system “locks up” and tends to stay at its initial displacement.

8 (a). $y'' + \gamma y' + y = 0,\ y(0) = 1,\ y'(0) = 0,\ \lambda^2 + \gamma\lambda + 1 = 0 \Rightarrow \lambda = \dfrac{-\gamma \pm \sqrt{\gamma^2 - 4}}{2},$

$$\gamma_{crit}^2 - 4 = 0 \Rightarrow \gamma_{crit} = 2.$$

8 (b).



The plots are consistent. For a fixed t, y is tending toward $y_0 = 1$ as γ increases.

9. For this problem, we must make $\mu = \dfrac{\gamma}{m}$ and $\dfrac{\rho_\ell g}{\rho L} = \dfrac{k}{m}$. The volume of the drum is

$V = \pi\left(\dfrac{5}{2}\right)^2 8 = 50\pi$ cubic feet. Therefore,

$$\frac{\rho_\ell}{\rho} = \frac{\text{weight of equiv. vol. of water}}{\text{weight of drum}} = \frac{50\pi(62.4)}{6000} = 1.634.\ \frac{k}{m} = \frac{\rho_\ell}{\rho}\frac{g}{L} = 1.634\cdot\left(\frac{32}{8}\right) = 6.535\ \text{s}^{-2}$$

and $\dfrac{\gamma}{m} = 0.1\ \text{s}^{-1},\ m = 5\ kg,\ k = 32.675\ {}^{\text{N}}\!/\!_{\text{m}},\ \gamma = 0.5\ {}^{\text{kg}}\!/\!_{\text{s}}$.

10. $my'' + \gamma y' + ky = 0,\ y(0) = 0,\ y'(0) = y_0',\ m\lambda^2 + \gamma\lambda + k = 0 \Rightarrow \lambda = \dfrac{-\gamma \pm \sqrt{\gamma^2 - 4mk}}{2m}$.

10 (a). Underdamped: $\lambda = -\alpha \pm i\beta,\ \alpha = \frac{\gamma}{2m},\ \beta = \dfrac{\sqrt{4mk - \gamma^2}}{2m}$

$$y = c_1 e^{-\alpha t}\cos\beta t + c_2 e^{-\alpha t}\sin\beta t,\ y(0) = c_1 = 0,\ y'(0) = \beta c_2 = y_0' \quad \therefore y(t) = \tfrac{y_0'}{\beta} e^{-\alpha t}\sin\beta t.$$

Critically damped:

$$y = c_1 e^{-\frac{\gamma}{2m}t} + c_2 t e^{-\frac{\gamma}{2m}t},\ y(0) = c_1 = 0,\ y'(0) = c_2 = y_0' \quad \therefore y(t) = y_0' t e^{-\frac{\gamma}{2m}t} = y_0' t e^{-\sqrt{\frac{k}{m}}t}.$$

Overdamped: $\lambda = -\dfrac{\gamma}{2m} \pm \dfrac{1}{2m}\sqrt{\gamma^2 - 4mk} = -\dfrac{\gamma}{2m} \pm \sqrt{\frac{\gamma^2}{4m^2} - \frac{k}{m}}$

$$y = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t},\ c_1 + c_2 = 0,\ \lambda_1 c_1 + \lambda_2 c_2 = y_0' = (\lambda_2 - \lambda_1)c_2 \quad \therefore c_2 = \frac{y_0'}{(\lambda_2 - \lambda_1)} = -c_1.$$

$$y(t) = \frac{1}{\lambda_2 - \lambda_1}\left[e^{\lambda_1 t} - e^{\lambda_2 t}\right]y_0' = \frac{1}{2\sqrt{\frac{\gamma^2}{4m^2} - \frac{k}{m}}} e^{-\frac{\gamma}{2m}t}\left[-e^{-\sqrt{\frac{\gamma^2}{4m^2} - \frac{k}{m}}\ t} + e^{\sqrt{\frac{\gamma^2}{4m^2} - \frac{k}{m}}\ t}\right]y_0'.$$

10 (b). As $\gamma \to \gamma_{crit}$, $e^{-\frac{\gamma}{2m}t} \to e^{-\sqrt{\frac{k}{m}}t}$ and use: $\lim\limits_{x\downarrow 0}\dfrac{\sinh(xt)}{x} = t,\ \lim\limits_{x\uparrow 0}\dfrac{\sin(xt)}{x} = t$.

Section 4.8

1 (a). $y'_P = 3,\ y''_P = 0,\ 0 - 2(3) - 3(3t-1) = -9t - 6 + 3 = -9t - 3$.

1 (b). $\lambda^2 - 2\lambda - 3 = (\lambda - 3)(\lambda + 1) = 0$, and thus $y_C = c_1e^{-t} + c_2e^{3t}$.

1 (c). $y = c_1e^{-t} + c_2e^{3t} + 3t - 1$. From the initial conditions, we have

$y(0) = c_1 + c_2 - 1 = 1$ and $y'(0) = -c_1 + 3c_2 + 3 = 3$. Solving these simultaneous equations yields

$c_1 = \dfrac{3}{2}$ and $c_2 = \dfrac{1}{2}$, and so $y = \dfrac{3}{2}e^{-t} + \dfrac{1}{2}e^{3t} + 3t - 1$.

2 (b). $y_C = c_1e^{-t} + c_2e^{3t}$.

2 (c). $y = c_1e^{-t} + c_2e^{3t} - \dfrac{1}{3}e^{2t}$.

$y(0) = c_1 + c_2 - \frac{1}{3} = 1$ and $y'(0) = -c_1 + 3c_2 - \frac{2}{3} = 0 \Rightarrow c_1 = \dfrac{5}{6}$ and $c_2 = \dfrac{1}{2}$, and so

$y = \frac{5}{6}e^{-t} + \frac{1}{2}e^{3t} - \frac{1}{3}e^{2t}$.

3 (a). $y'_P = 8e^{4t},\ y''_P = 32e^{4t},\ e^{4t}(32 - 8 - 2(2)) = 20e^{4t}$.

3 (b). $\lambda^2 - \lambda - 2 = (\lambda - 2)(\lambda + 1) = 0$, and thus $y_C = c_1e^{-t} + c_2e^{2t}$.

3 (c). $y = c_1e^{-t} + c_2e^{2t} + 2e^{4t}$. From the initial conditions, we have

$y(0) = c_1 + c_2 + 2 = 0$ and $y'(0) = -c_1 + 2c_2 + 8 = 1$. Solving these simultaneous equations yields

$c_1 = 1$ and $c_2 = -3$, and so $y = e^{-t} - 3e^{2t} + 2e^{4t}$.

4 (b). $y_C = c_1e^{-t} + c_2e^{2t}$.

4 (c). $y = c_1e^{-t} + c_2e^{2t} - 5$.

$y(-1) = c_1e^{1} + c_2e^{-2} - 5 = 0$ and $y'(-1) = -c_1e^{1} + 2c_2e^{-2} = 1 \Rightarrow c_1 = 3e^{-1}$ and $c_2 = 2e^{2}$, and so

$y = 3e^{-(t+1)} + 2e^{2(t+1)} - 5$.

5 (a). $2 + (2t - 2) = 2t$.

5 (b). $\lambda^2 + \lambda = \lambda(\lambda + 1) = 0$, and thus $y_C = c_1e^{-t} + c_2$.

5 (c). $y = c_1e^{-t} + c_2 + t^2 - 2t$. From the initial conditions, we have

$y(1) = c_1e^{-1} + c_2 - 1 = 1$ and $y'(1) = -c_1e^{-1} + 2 - 2 = -2$. Solving these simultaneous equations

yields $c_1 = 2e$ and $c_2 = 0$, and so $y = 2e^{-(t-1)} + t^2 - 2t$.

6 (b). $y_C = c_1e^{-t} + c_2$.

6 (c). $y = c_1e^{-t} + c_2 - 2te^{-t}$.

$y(0) = c_1 + c_2 = 2$ and $y'(0) = -c_1 - 2 = 2 \Rightarrow c_1 = -4$ and $c_2 = 6$, and so $y = -4e^{-t} + 6 - 2te^{-t}$.

7 (a). $y'_P = 2 - 2\sin 2t,\ y''_P = -4\cos 2t,\ -4\cos 2t + (2t + \cos 2t) = 2t - 3\cos 2t$.

7 (b). $\lambda^2+1=0$, and thus $y_C=c_1\cos t+c_2\sin t$.

7 (c). $y=c_1\cos t+c_2\sin t+2t+\cos 2t$. From the initial conditions, we have $y(0)=c_1+1=0$ and $y'(0)=c_2+2=0$. Solving these simultaneous equations yields $c_1=-1$ and $c_2=-2$, and so $y=-\cos t-2\sin t+2t+\cos 2t$.

8 (b). $y_C=c_1\cos 2t+c_2\sin 2t$.

8 (c). $y=c_1\cos 2t+c_2\sin 2t+2e^{t-\pi}$.

$y(\pi)=c_1+2=2$ and $y'(\pi)=2c_2+2=0\Rightarrow c_1=0$ and $c_2=-1$, and so $y=-\sin 2t+2e^{t-\pi}$.

9 (a). $10-2(10(t+1))+10(t+1)^2=10-20t-20+10t^2+20t+10=10t^2$.

9 (b). $\lambda^2-2\lambda+2=0$, so $\lambda=1\pm i$, and thus $y_C=c_1e^t\cos t+c_2e^t\sin t$.

9 (c). $y=c_1e^t\cos t+c_2e^t\sin t+5(t+1)^2$. From the initial conditions, we have $y(0)=c_1+5=0$ and $y'(0)=c_1+c_2+10=0$. Solving these simultaneous equations yields $c_1=-5$ and $c_2=-5$, and so $y=-5e^t\cos t-5e^t\sin t+5(t+1)^2$.

10 (b). $y_C=c_1e^t\cos t+c_2e^t\sin t$.

10 (c). $y=c_1e^t\cos t+c_2e^t\sin t+2\cos t+\sin t$.

$y\left(\frac{\pi}{2}\right)=c_2e^{\frac{\pi}{2}}+1=1$ and $y'\left(\frac{\pi}{2}\right)=-c_1e^{\frac{\pi}{2}}+c_2e^{\frac{\pi}{2}}-2=0\Rightarrow c_1=-2e^{-\frac{\pi}{2}}$ and $c_2=0$, and so

$y=-2e^{t-\frac{\pi}{2}}\cos t+2\cos t+\sin t$.

11 (a). $y_P'=\frac{1}{2}(2t+t^2)e^t$, $y_P''=\frac{1}{2}(2+4t+t^2)e^t$, and $\left(\frac{t^2}{2}+2t+1\right)e^t-2\cdot\frac{1}{2}\left(2t+t^2\right)e^t+\frac{t^2}{2}e^t=e^t$.

11 (b). $\lambda^2-2\lambda+1=(\lambda-1)^2=0$, and thus $y_C=c_1e^t+c_2te^t$.

11 (c). $y=c_1e^t+c_2te^t+\frac{t^2}{2}e^t$. From the initial conditions, we have $y(0)=c_1=\text{-}2$ and $y'(0)=c_1+c_2=2$. Solving these simultaneous equations yields $c_1=-2$ and $c_2=4$, and so $y=-2e^t+4te^t+\frac{t^2}{2}e^t$.

12 (b). $y_C=c_1e^t+c_2te^t$.

12 (c). $y=c_1e^t+c_2te^t+t^2+4t+10+\cos t$.

$y(0)=c_1+10+1=1$ and $y'(0)=c_1+c_2+4=3\Rightarrow c_1=-10$ and $c_2=9$, and so

$y=-10e^t+9te^t+t^2+4t+10+\cos t$.

13. First, $y_P=a_1u+a_2v$. Now we have

$$y_P''+p(t)y_P'+q(t)y_P=a_1\left(u''+p(t)u'+q(t)u\right)+a_2\left(v''+p(t)v'+q(t)v\right)=a_1g_1+a_2g_2$$

14. $e^t + 2t + \frac{1}{2} = \frac{1}{2}\left[2e^t + 1\right] + \frac{2}{3}[3t]$. Thus $y_P = \frac{1}{2}u_1 + \frac{2}{3}u_3$.

15. $4e^{-t} - 2 = 2\left[2e^{-t} - t - 1\right] + \frac{2}{3}[3t]$. Thus $y_P = 2u_2 + \frac{2}{3}u_3$.

16. $\cosh t = \frac{1}{2}e^t + \frac{1}{2}e^{-t} = \frac{1}{4}\left[2e^t + 1\right] + \frac{1}{4}\left[2e^{-t} - t - 1\right] + \frac{1}{12}[3t]$. Thus $y_P = \frac{1}{4}u_1 + \frac{1}{4}u_2 + \frac{1}{12}u_3$.

17. Differentiation gives us $y_P' = 2e^{2t} - 2t,\ y_P'' = 4e^{2t} - 2$. From the given differential equation, we have $g(t) = 4e^{2t} - 2 + 2e^{2t} - 2t - e^{2t} + t^2 = 5e^{2t} + t^2 - 2t - 2$.

18. Differentiation gives us $y_P' = 3 + \frac{1}{2}t^{-\frac{1}{2}},\ y_P'' = -\frac{1}{4}t^{-\frac{3}{2}}$. From the given differential equation, we have $g(t) = -\frac{1}{4}t^{-\frac{3}{2}} - 2\left(3 + \frac{1}{2}t^{-\frac{1}{2}}\right) = -6 - t^{-\frac{1}{2}} - \frac{1}{4}t^{-\frac{3}{2}}$.

19. Differentiation gives us $y_P' = 3,\ y_P'' = 0$. From the given differential equation, we have $g(t) = t \cdot 0 + e^t \cdot 3 + 2 \cdot 3t = 3e^t + 6t$.

20. Differentiation gives us $y_P' = \frac{1}{1+t},\ y_P'' = -\frac{1}{(1+t)^2}$. From the given differential equation, we have $g(t) = -\frac{1}{(1+t)^2} + \ln(1+t),\ t > -1$.

21. Differentiation gives us $y_P' = -\sin t,\ y_P'' = -\cos t$. From the given differential equation, we have $g(t) = -\cos t - \sin^2 t + 2|t|\cos t = \left(2|t| - 1\right)\cos t - \sin^2 t$.

22. $(\lambda - 1)(\lambda - 2) = \lambda^2 - 3\lambda + 2 = 0 \Rightarrow\ \alpha = -3,\ \beta = 2$. Differentiation gives us $y_P' = -4e^{-2t},\ y_P'' = 8e^{-2t}$, and so $g(t) = 8e^{-2t} - 3\left(-4e^{-2t}\right) + 2\left(2e^{-2t}\right) = 24e^{-2t}$.

23. $\lambda = -1, 0 \Rightarrow (\lambda + 1)\lambda = 0$, so $\lambda^2 + \lambda = 0$ and $\alpha = 1,\ \beta = 0$. Differentiation gives us $y_P' = 2t,\ y_P'' = 2$, and so $g(t) = 2 + 2t$.

24. $(\lambda - 1)^2 = \lambda^2 - 2\lambda + 1 = 0 \Rightarrow\ \alpha = -2,\ \beta = 1$. Differentiation gives us $y_P' = \left(t^2 + 2t\right)e^t,\ y_P'' = \left(t^2 + 4t + 2\right)e^t$, and so $g(t) = \left(t^2 + 4t + 2\right)e^t - 2\left(t^2 + 2t\right)e^t + t^2e^t = 2e^t$.

25. $\lambda = 1 \pm i \Rightarrow (\lambda - 1)^2 = -1$, so $\lambda^2 - 2\lambda + 2 = 0$ and $\alpha = -2,\ \beta = 2$. Differentiation gives us $y_P' = e^t + \cos t,\ y_P'' = e^t - \sin t$, and so
$g(t) = e^t - \sin t - 2\left[e^t + \cos t\right] + 2\left[e^t + \sin t\right] = e^t - 2\cos t + \sin t$.

26. $\lambda^2 + 4 = 0 \Rightarrow\ \alpha = 0,\ \beta = 4$. Differentiation gives us $y_P' = \cos t,\ y_P'' = -\sin t$, and so $g(t) = -\sin t + 4(-1 + \sin t) = 3\sin t - 4$.

Section 4.9

1 (a). $\lambda^2 - 4 = 0 \Rightarrow y_C = c_1 e^{-2t} + c_2 e^{2t}$

1 (b). $y_P = A_2 t^2 + A_1 t + A_0,\ y_P' = 2A_2 t + A_1,\ y_P'' = 2A_2$.

$y_P'' - 4y_P = 2A_2 - 4(A_2 t^2 + A_1 t + A_0) = 4t^2 \Rightarrow A_0 = -\frac{1}{2}, A_1 = 0, A_2 = -1$.

Therefore, $y_P = -t^2 - \frac{1}{2}$

1 (c). $y = c_1 e^{-2t} + c_2 e^{2t} - t^2 - \frac{1}{2}$

2 (a). $y_C = c_1 e^{-2t} + c_2 e^{2t}$

2 (b). $y_P = -\frac{1}{8}\sin 2t$.

2 (c). $y = c_1 e^{-2t} + c_2 e^{2t} - \frac{1}{8}\sin 2t$.

3 (a). $\lambda^2 + 1 = 0 \Rightarrow y_C = c_1 \cos t + c_2 \sin t$

3 (b). $y_P = Ae^t,\ y_P' = Ae^t,\ y_P'' = Ae^t$.

$y_P'' + y_P = 2Ae^t = 8e^t \Rightarrow A = 4$. Therefore, $y_P = 4e^t$

3 (c). $y = c_1 \cos t + c_2 \sin t + 4e^t$

4 (a). $y_C = c_1 \cos t + c_2 \sin t$

4 (b). $y_P = -\frac{2}{5} e^t \cos t + \frac{1}{5} e^t \sin t$.

4 (c). $y = c_1 \cos t + c_2 \sin t - \frac{2}{5} e^t \cos t + \frac{1}{5} e^t \sin t$.

5 (a). $(\lambda - 2)^2 = 0 \Rightarrow y_C = c_1 e^{2t} + c_2 t e^{2t}$

5 (b). $y_P = At^2 e^{2t},\ y_P' = (2At^2 + 2At)e^{2t},\ y_P'' = (4At^2 + 8At + 2A)e^{2t}$.

$y_P'' - 4y_P' + 4y_P = e^{2t} \Rightarrow A = \frac{1}{2}$. Therefore, $y_P = \frac{t^2}{2} e^{2t}$

5 (c). $y = c_1 e^{2t} + c_2 t e^{2t} + \frac{t^2}{2} e^{2t}$

6 (a). $y_C = c_1 e^{2t} + c_2 t e^{2t}$

6 (b). $y_P = \frac{1}{8}\cos 2t + 2$.

6 (c). $y = c_1 e^{2t} + c_2 t e^{2t} + \frac{1}{8}\cos 2t + 2$.

7 (a). $\lambda^2 + 2\lambda + 2 = 0 \Rightarrow \lambda = -1 \pm i \Rightarrow y_C = c_1 e^{-t}\cos t + c_2 e^{-t}\sin t$

7 (b). $y_P = A_3 t^3 + A_2 t^2 + A_1 t + A_0,\ y_P' = 3A_3 t^2 + 2A_2 t + A_1,\ y_P'' = 6A_3 t + 2A_2$.

$y_P'' + 2y_P' + 2y_P = t^3 \Rightarrow A_0 = 0, A_1 = \frac{3}{2}, A_2 = -\frac{3}{2}, A_3 = \frac{1}{2}$.

Therefore, $y_P = \frac{1}{2}t^3 - \frac{3}{2}t^2 + \frac{3}{2}t$

7 (c). $y = c_1 e^{-t}\cos t + c_2 e^{-t}\sin t + \frac{1}{2}t^3 - \frac{3}{2}t^2 + \frac{3}{2}t$

8 (a). $y_C = c_1 e^{\frac{t}{2}} + c_2 e^{2t}$

8 (b). $y_P = -te^t + e^t$.

8 (c). $y = c_1 e^{\frac{t}{2}} + c_2 e^{2t} - te^t + e^t$.

9 (a). $y_C = c_1 e^{-t}\cos t + c_2 e^{-t}\sin t$

9 (b). $y_P = A_0 e^{-t} + A_1\cos t + A_2 \sin t,\ y_P' = -A_0 e^{-t} - A_1 \sin t + A_2 \cos t,$

$y_P'' = A_0 e^{-t} - A_1 \cos t - A_2 \sin t$. $y_P'' + 2y_P' + 2y_P = e^{-t} + \cos t \Rightarrow A_0 = 1, A_1 = \frac{1}{5}, A_2 = \frac{2}{5}$

Therefore, $y_P = e^{-t} + \frac{1}{5}\cos t + \frac{2}{5}\sin t$

9 (c). $y = c_1 e^{-t}\cos t + c_2 e^{-t}\sin t + e^{-t} + \frac{1}{5}\cos t + \frac{2}{5}\sin t$

10 (a). $y_C = c_1 e^{-t} + c_2$

10 (b). $y_P = 2t^3 - 6t^2 + 12t$.

10 (c). $y = c_1 e^{-t} + c_2 + 2t^3 - 6t^2 + 12t$.

11 (a). $2\lambda^2 - 5\lambda + 2 = (2\lambda - 1)(\lambda - 2) = 0 \Rightarrow \lambda = \frac{1}{2}, 2 \Rightarrow y_C = c_1 e^{\frac{t}{2}} + c_2 e^{2t}$

11 (b). $y_P = A_0 te^{t/2}$. Substituting into the differential equation yields $y_P = 2te^{\frac{t}{2}}$.

11 (c). $y = c_1 e^{\frac{t}{2}} + c_2 e^{2t} + 2te^{\frac{t}{2}}$

12 (a). $y_C = c_1 e^{-t} + c_2$

12 (b). $y_P = -\frac{1}{2}\cos t + \frac{1}{2}\sin t$.

12 (c). $y = c_1 e^{-t} + c_2 - \frac{1}{2}\cos t + \frac{1}{2}\sin t$.

13 (a). $9\lambda^2 - 6\lambda + 1 = (3\lambda - 1)^2 = 0 \Rightarrow y_C = c_1 e^{\frac{t}{3}} + c_2 te^{\frac{t}{3}}$

13 (b). $y_P = (A_1t^3 + A_0t^2)e^{\frac{t}{3}}$. Substituting into the differential equation yields $y_P = \frac{1}{6}t^3e^{\frac{t}{3}}$

13 (c). $y = c_1e^{\frac{t}{3}} + c_2te^{\frac{t}{3}} + \frac{1}{6}t^3e^{\frac{t}{3}}$

14 (a). $y_C = c_1e^{-2t}\cos t + c_2e^{-2t}\sin t$

14 (b). $y_P = t - \frac{5}{4} + \frac{e^{-t}}{2}$.

14 (c). $y = c_1e^{-2t}\cos t + c_2e^{-2t}\sin t + t - \frac{5}{4} + \frac{e^{-t}}{2}$.

15 (a). $\lambda^2 + 4\lambda + 5 = 0 \Rightarrow \lambda = -2 \pm i \Rightarrow y_C = c_1e^{-2t}\cos t + c_2e^{-2t}\sin t$

15 (b). $y_P = Ae^{-2t} + B_1\cos t + B_2\sin t$. Substituting into the differential equation yields

$y_P = 2e^{-2t} + \frac{1}{8}\cos t + \frac{1}{8}\sin t$.

15 (c). $y = c_1e^{-2t}\cos t + c_2e^{-2t}\sin t + 2e^{-2t} + \frac{1}{8}\cos t + \frac{1}{8}\sin t$

16 (a). $y_C = c_1e^{-t} + c_2e^{3t}$

16 (b). $y_P = Ae^{-t}\cos t + Be^{-t}\cos t + C_2t^2 + C_1t + C_0 + t(D_1 + D_0)e^{3t}$

$= Ae^{-t}\cos t + Be^{-t}\cos t + C_2t^2 + C_1t + C_0 + D_1t^2e^{3t} + D_0te^{3t}$

17 (a). $y_C = c_1\cos 3t + c_2\sin 3t$

17 (b). $y_P = t(A_2t^2 + A_1t + A_0)\cos 3t + t(B_2t^2 + B_1t + B_0)\sin 3t + C\cos t + D\sin t$

$= (A_2t^3 + A_1t^2 + A_0t)\cos 3t + (B_2t^3 + B_1t^2 + B_0t)\sin 3t + +C\cos t + D\sin t$

18 (a). $y_C = c_1 + c_2e^t$

18 (b). $y_P = t(A_2t^2 + A_1t + A_0) + t(B_2t^2 + B_1t + B_0)e^t = A_2t^3 + A_1t^2 + A_0t + (B_2t^3 + B_1t^2 + B_0t)e^t$

19 (a). $\lambda^2 - 2\lambda + 2 = (\lambda - 1)^2 + 1 = 0$; $y_C = c_1e^t\cos t + c_2e^t\sin t$

19 (b). $y_P = Ae^{-t}\cos 2t + Be^{-t}\sin 2t + C_1t + C_0 + e^{-t}(D_1t + D_0)\cos t + e^{-t}(E_1t + E_0)\sin t$

20 (a). $y_C = c_1e^t + c_2e^{-t}$

20 (b). $y_P = Ate^t + Bte^{-t} + Ce^{2t} + De^{-2t}$

21 (a). $y_C = c_1\cos 2t + c_2\sin 2t$

21 (b). Using $\sin(2t) = 2\sin t\cos t$ and $\cos(2(2t)) = 2\cos^2 2t - 1$,

$\sin t\cos t + \cos^2 2t = \frac{1}{2}\sin 2t + \frac{1}{2} + \frac{1}{2}\cos 4t$. Therefore,

$y_P = At\cos 2t + Bt\sin 2t + C + D\cos 4t + E\sin 4t$

22 (a). $y_C = c_1 \sin 2t + c_2 \cos 2t$

22 (b). $y_P = Ae^{-2t} + B + Ce^{2t}$

23. $(\lambda+1)(\lambda-2) = \lambda^2 - \lambda - 2 = 0$, so $\alpha = -1$ and $\beta = -2$. $y'' - y' - 2y = 4t$, which leads to the general solution of $y = c_1 e^{-t} + c_2 e^{2t} - 2t + 1$.

24. $\lambda(\lambda+1) = \lambda^2 + \lambda = 0$, so $\alpha = 1$ and $\beta = 0$. $y'' + y' = t$, which leads to the general solution of $y = c_1 + c_2 e^{-t} + \dfrac{t^2}{2} - t$.

25. $(\lambda+2)(\lambda+2) = \lambda^2 + 4\lambda + 4 = 0$, so $\alpha = 4$ and $\beta = 4$. $y'' + 4y' + 4y = 5\sin t$, which leads to the general solution of $y = c_1 e^{-2t} + c_2 t e^{-2t} - \dfrac{4}{5}\cos t + \dfrac{3}{5}\sin t$.

26. $\alpha = 0$ and $\beta = 1$, $y = c_1 \cos t + c_2 \sin t + t - \dfrac{1}{3}\sin 2t$.

27. $\lambda = -1 \pm i2$, so $(\lambda+1)^2 = -4$ and thus $\lambda^2 + 2\lambda + 5 = 0$. Therefore. $\alpha = 2$ and $\beta = 5$. $y'' + 2y' + 5y = 8e^{-t}$, which leads to the general solution of $y = c_1 e^{-t}\cos 2t + c_2 e^{-t}\sin 2t + 2e^{-t}$.

28. Since $y_P = t(A_1 t + A_0) + B_0 t e^{3t}$, we know that 1 and e^{3t} are solutions and $\lambda^2 - 3\lambda = 0$, and thus $\alpha = -3$ and $\beta = 0$.

29. Since $y_P = A_0 t e^{2t} + B_0 t e^{-2t} + C_1 t + C_0$, we know that e^{2t} and e^{-2t} are solutions of the homogeneous differential equation. This means that $\lambda = \pm 2$, so $(\lambda+2)(\lambda-2) = \lambda^2 - 4 = 0$, and thus $\alpha = 0$ and $\beta = -4$.

30. We know that $\cos 2t$ and $\sin 2t$ are solutions and $\lambda^2 + 4 = 0$, and thus $\alpha = 0$ and $\beta = 4$.

31 (a). Graph C

31 (b). Graph E

31 (c). Graph A

31 (d). Graph B

31 (e). Graph D

32. $W_b = 200\ lb$. The weight of an equivalent weight of water is $W_b = 8(62.4) = 499.2\ lb$.
$m_b = \dfrac{200}{32}$. Note: $\dfrac{\rho_\ell}{p} = \dfrac{499.2}{200} = 2.496 \Rightarrow \omega^2 = 2.496\left(\dfrac{32}{2}\right) \Rightarrow \omega = 39.936$.

32 (a). $y'' + \omega^2 y = \dfrac{10}{m_b}\sin \omega t,\ y(0) = 0,\ y'(0) = 0$.

32 (b). $y_c = c_1\cos\omega t + c_2\sin\omega t,\ y_p = At\cos\omega t + Bt\sin\omega t$

$$y_p' = A\cos\omega t - \omega At\sin\omega t + B\sin\omega t + \omega Bt\cos\omega t = (A+\omega Bt)\cos\omega t + (B-\omega At)\sin\omega t$$

$$y_p'' = \omega B\cos\omega t - \omega(A+\omega Bt)\sin\omega t - \omega A\sin\omega t + \omega(B-\omega At)\cos\omega t$$

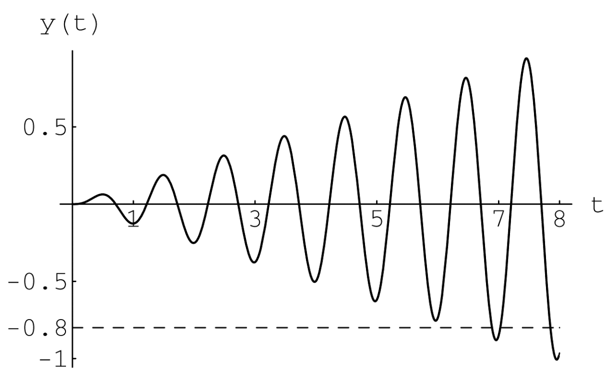
$$y_p'' + \omega^2 y_p = \omega B\cos\omega t - \omega A\sin\omega t - \omega A\sin\omega t + \omega B\cos\omega t = \frac{10}{m_b}\sin\omega t.$$

Therefore, $B=0,\ A=-\dfrac{10}{2\omega m_b}=-\dfrac{5}{\omega m_b}$ and

$$y = c_1\cos\omega t + c_2\sin\omega t - \frac{5}{\omega m_b}t\cos\omega t,\ y(0)=c_1=0,\ y'(0)=\omega c_2 - \frac{5}{\omega m_b}=0 \Rightarrow c_2 = \frac{5}{\omega^2 m_b}.$$

$$y = \frac{5}{\omega^2 m_b}\sin\omega t - \frac{5}{\omega m_b}t\cos\omega t,\ \omega \approx 6.3195\ \sec^{-1},\ m_b = 6.25\ \ slug.$$

32 (c). Before being put into motion, the block is floating with a depth Y submerged, where $62.4(4)Y = 200 \Rightarrow Y = 0.80128...ft$. Therefore, the model is valid if $-0.80128... \le y \le 2-0.80128$. From the graph, $y=-0.80128$ at $t \approx 7$ sec.



33. $y_P'' + p(t)y_P' + q(t)y_P = (u''+iv'') + p(t)(u'+iv') + q(t)(u+iv)$

$= (u'' + p(t)u' + q(t)u) + i(v'' + p(t)v' + q(t)v) = g_1(t) + ig_2(t)$. The real and imaginary parts must be equal on both sides of the equation, so

$u'' + p(t)u' + q(t)u = g_1(t)$ and $v'' + p(t)v' + q(t)v = g_2(t)$.

34 (a). $y'' - y = e^{i2t},\ y_p = Ae^{i2t},\ y_p' = i2Ae^{i2t},\ y_p'' = -4Ae^{i2t}$

$$-4Ae^{i2t} - Ae^{i2t} = e^{i2t} \Rightarrow A = -\frac{1}{5}.$$

34 (b). $Ae^{i2t} = -\dfrac{1}{5}(\cos 2t + i\sin 2t) \Rightarrow u = -\dfrac{1}{5}\cos 2t,\ v = -\dfrac{1}{5}\sin 2t.$

35 (a). $y_P' = iAe^{it},\ y_P'' = -Ae^{it}.\ -Ae^{it} + 2iAe^{it} + Ae^{it} = 2iAe^{it} = e^{it}$, so $A = \dfrac{1}{2i} = -\dfrac{i}{2}$ and $y_P = -\dfrac{i}{2}e^{it}$.

35 (b). $y_P = -\frac{i}{2}(\cos t + i\sin t) = \frac{1}{2}\sin t - \frac{i}{2}\cos t$. Thus $u = \frac{1}{2}\sin t$ and $v = -\frac{1}{2}\cos t$. For the real function,

$u'' + 2u' + u = -\frac{1}{2}\sin t + 2\left(\frac{1}{2}\cos t\right) + \frac{1}{2}\sin t = \cos t$. For the imaginary function,

$v'' + 2v' + v = \frac{1}{2}\cos t + 2\left(\frac{1}{2}\sin t\right) - \frac{1}{2}\cos t = \sin t$.

36 (a). $y_P' = iAe^{it}$, $y_P'' = -Ae^{it}$. $-Ae^{it} + 4Ae^{it} = e^{it} \Rightarrow A = \frac{1}{3}$ and $y_P = \frac{1}{3}e^{it}$.

36 (b). $y_P = \frac{1}{3}(\cos t + i\sin t) \Rightarrow u = \frac{1}{3}\cos t,\ v = \frac{1}{3}\sin t$.

37 (a). $y_P' = A(1+i2t)e^{i2t}$, $y_P'' = (i2+i2-4t)Ae^{i2t} = (-4t+i4)Ae^{i2t}$. $(-4t+i4)Ae^{i2t} + 4Ate^{i2t} = e^{i2t}$, so

$A = -\frac{i}{4}$ and $y_P = -\frac{i}{4}te^{i2t}$.

37 (b). $y_P = -\frac{i}{4}t(\cos 2t + i\sin 2t) = \frac{t}{4}\sin 2t + i\left(-\frac{t}{4}\cos 2t\right)$. Thus $u = \frac{t}{4}\sin 2t$ and $v = -\frac{t}{4}\cos 2t$. For the real function, $u'' + 4u = \cos 2t - t\sin 2t + 4\left(\frac{t}{4}\sin 2t\right) = \cos 2t$. For the imaginary function,

$v'' + 4v = \sin 2t + t\cos 2t - 4\left(\frac{t}{4}\cos 2t\right) = \sin 2t$.

38 (a). $y_P' = -i2Ae^{-i2t}$, $y_P'' = -4Ae^{-i2t}$. $(-4-i2)Ae^{-i2t} = e^{-i2t} \Rightarrow A = \frac{-1}{4+i2} = \frac{-(4-i2)}{20} = -\frac{1}{5} + i\frac{1}{10}$ and

$y_p = \left(-\frac{1}{5} + i\frac{1}{10}\right)e^{-i2t}$.

38 (b). $y_P = \left(-\frac{1}{5} + i\frac{1}{10}\right)(\cos 2t - i\sin 2t) \Rightarrow u = -\frac{1}{5}\cos 2t + \frac{1}{10}\sin 2t,\ v = \frac{1}{5}\sin 2t + \frac{1}{10}\cos 2t$.

39 (a). $y_P' = (1+i)Ae^{(1+i)t}$, $y_P'' = (1+i)^2 Ae^{(1+i)t} = i2Ae^{(1+i)t}$. $i2Ae^{(1+i)t} + Ae^{(1+i)t} = e^{(1+i)t}$, so

$A = \frac{1}{1+i2} = \frac{1-i2}{5}$ and $y_P = \left(\frac{1}{5} - i\frac{2}{5}\right)e^{(1+i)t}$.

39 (b). $y_P = \left(\frac{1}{5} - i\frac{2}{5}\right)e^t(\cos t + i\sin t) = e^t\left(\frac{1}{5}\cos t + \frac{2}{5}\sin t\right) + ie^t\left(\frac{1}{5}\sin t - \frac{2}{5}\cos t\right)$. Thus

$u = e^t\left(\frac{1}{5}\cos t + \frac{2}{5}\sin t\right)$ and $v = e^t\left(\frac{1}{5}\sin t - \frac{2}{5}\cos t\right)$. For the real function,

$u'' + u = e^t\left(\frac{4}{5}\cos t - \frac{2}{5}\sin t\right) + e^t\left(\frac{1}{5}\cos t + \frac{2}{5}\sin t\right) = e^t\cos t$. For the imaginary function,

$v'' + v = e^t\left(\frac{2}{5}\cos t + \frac{4}{5}\sin t\right) + e^t\left(-\frac{2}{5}\cos t + \frac{1}{5}\sin t\right) = e^t\sin t$.

Section 4.10

1 (a). $y_C = c_1\cos 2t + c_2\sin 2t$

1 (b). $\begin{bmatrix}\cos 2t & \sin 2t\\ -2\sin 2t & 2\cos 2t\end{bmatrix}\begin{bmatrix}u_1'\\ u_2'\end{bmatrix}=\begin{bmatrix}0\\ 2\sin 2t\end{bmatrix}$, so $\begin{bmatrix}u_1'\\ u_2'\end{bmatrix}=\frac{1}{2}\begin{bmatrix}2\cos 2t & -\sin 2t\\ 2\sin 2t & \cos 2t\end{bmatrix}\begin{bmatrix}0\\ 2\sin 2t\end{bmatrix}=\begin{bmatrix}-\sin^2 2t\\ \sin 2t\cos 2t\end{bmatrix}$.

Antidifferentiation gives us $u_1 = -\frac{t}{2}+\frac{1}{8}\sin 4t$ and $u_2 = \frac{\sin^2 2t}{4}$. Thus

$y_P = -\frac{t}{2}\cos 2t + \frac{1}{8}\sin 4t\cos 2t + \frac{1}{4}\sin^3 2t$ and

$y = c_1\cos 2t + c_2\sin 2t - \frac{t}{2}\cos 2t + \frac{1}{8}\sin 4t\cos 2t + \frac{1}{4}\sin^3 2t$.

1 (c). $y_P = At\sin 2t + Bt\cos 2t$, $y_P' = (A-2Bt)\sin 2t + (B+2At)\cos 2t$,

$y_P'' = (-4B-4At)\sin 2t + (4A-4Bt)\cos 2t$. $-4B\sin 2t + 4A\cos 2t = 2\sin 2t$, and thus

$A=0,\ B=-\frac{1}{2}$, and $y_P = -\frac{1}{2}t\cos 2t$. Combining the particular solution with the

complementary solution gives us $y = C_1\cos 2t + C_2\sin 2t - \frac{t}{2}\cos 2t$.

To reconcile, $y_P = \frac{1}{8}\sin 4t\cos 2t + \frac{1}{4}\sin^3 2t = \frac{1}{4}\sin 2t\cos^2 2t + \frac{1}{4}\sin^3 2t = \frac{1}{4}\sin 2t$. Therefore, the

solution in (b) can be written $y = c_1\cos 2t + (c_2 + 1/4)\sin 2t - \frac{t}{2}\cos 2t$.

2 (a). $y_C = c_1\cos t + c_2\sin t$

2 (b). $\begin{bmatrix}\cos t & \sin t\\ -\sin t & \cos t\end{bmatrix}\begin{bmatrix}u_1'\\ u_2'\end{bmatrix}=\begin{bmatrix}0\\ \sec t\end{bmatrix}$, so $\begin{bmatrix}u_1'\\ u_2'\end{bmatrix}=\begin{bmatrix}\cos t & -\sin t\\ \sin t & \cos t\end{bmatrix}\begin{bmatrix}0\\ \sec t\end{bmatrix}=\begin{bmatrix}-\frac{\sin t}{\cos t}\\ 1\end{bmatrix}$. Antidifferentiation gives

us $u_1 = \ln|\cos t|$ and $u_2 = t$. Thus $y_P = (\cos t)\ln|\cos t| + t\sin t$ and

$y = c_1\cos t + c_2\sin t + (\cos t)\ln(\cos t) + t\sin t$, since $\cos t > 0$ for $-\frac{\pi}{2} < t < \frac{\pi}{2}$.

2 (c). Method of undetermined coefficients is not applicable.

3 (a). $y_C = c_1e^t + c_2t^2e^t$

3 (b). $\begin{bmatrix}e^t & t^2e^t\\ e^t & (t^2+2t)e^t\end{bmatrix}\begin{bmatrix}u_1'\\ u_2'\end{bmatrix}=\begin{bmatrix}0\\ t\end{bmatrix}$, so $\begin{bmatrix}u_1'\\ u_2'\end{bmatrix}=\frac{1}{2te^{2t}}\begin{bmatrix}(t^2+2t)e^t & -t^2e^t\\ -e^t & e^t\end{bmatrix}\begin{bmatrix}0\\ t\end{bmatrix}=\begin{bmatrix}-\frac{t^2}{2}e^{-t}\\ \frac{1}{2}e^{-t}\end{bmatrix}$.

Antidifferentiation gives us $u_1 = \left(\frac{t^2}{2}+t+1\right)e^{-t}$ and $u_2 = -\frac{1}{2}e^{-t}$. Thus

$y_P = \left(\frac{t^2}{2}+t+1\right)e^{-t}e^t - \frac{1}{2}e^{-t}t^2e^t = t+1$ and $y = c_1e^t + c_2t^2e^t + t + 1$.

3 (c). The method of undetermined coefficients is not applicable.

4 (a). $y_C = c_1e^{-t} + c_2e^t$

4 (b). $\begin{bmatrix} e^{-t} & e^t \\ -e^{-t} & e^t \end{bmatrix}\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{1+e^t} \end{bmatrix}$, so $\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \frac{1}{2}\begin{bmatrix} e^t & -e^t \\ e^{-t} & e^{-t} \end{bmatrix}\begin{bmatrix} 0 \\ \frac{1}{1+e^t} \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}\frac{e^t}{1+e^t} \\ \frac{1}{2}\frac{e^{-t}}{1+e^t} \end{bmatrix}$. Antidifferentiation gives us

$u_1 = -\frac{1}{2}\ln(1+\mathrm{e}^t)$ and $u_2 = -\frac{1}{2}e^{-t} + \frac{1}{2}\ln(1+e^{-t})$. Thus $y_P = -\frac{e^{-t}}{2}\ln(1+\mathrm{e}^t) - \frac{1}{2} + \frac{e^t}{2}\ln(1+e^{-t})$

and $y = c_1e^{-t} + c_2e^t - \frac{e^{-t}}{2}\ln(1+\mathrm{e}^t) - \frac{1}{2} + \frac{e^t}{2}\ln(1+e^{-t})$.

4 (c). The method of undetermined coefficients is not applicable.

5 (a). $y_C = c_1e^{-t} + c_2e^t$

5 (b). $\begin{bmatrix} e^{-t} & e^t \\ -e^{-t} & e^t \end{bmatrix}\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ e^t \end{bmatrix}$, so $\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \frac{1}{2}\begin{bmatrix} e^t & -e^t \\ e^{-t} & e^{-t} \end{bmatrix}\begin{bmatrix} 0 \\ e^t \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}e^{2t} \\ \frac{1}{2} \end{bmatrix}$. Antidifferentiation gives us

$u_1 = -\frac{1}{4}\mathrm{e}^{2t}$ and $u_2 = \frac{t}{2}$. Thus $y_P = -\frac{1}{4}e^t + \frac{t}{2}e^t$ and $y = c_1e^{-t} + c_2e^t - \frac{1}{4}e^t + \frac{t}{2}e^t$.

5 (c). $y_P = Ate^t$, and differentiation gives us $y_P' = A(1+t)e^t$ and $y_P'' = A(2+t)e^t$. Then we have

$A(2+t)e^t - Ate^t = e^t$, and so $A = \frac{1}{2}$, $y_P = \frac{t}{2}e^t$, and $y = C_1e^{-t} + C_2e^t + \frac{t}{2}e^t$.

To reconcile, the solution in (b) can be written $y = c_1e^{-t} + (c_2 - \tfrac{1}{4})e^t + \frac{t}{2}e^t$.

6 (a). $y_1 = t^2$. Use Reduction of Order to obtain $y_2(t)$. $y_2 = t^2v$, $y_2' = 2tv + t^2v'$, $y_2'' = 2v + 4tv' + t^2v''$

Therefore,

$2v + 4tv' + t^2v'' - 4v - 2tv' + 2v = 0 \Rightarrow t^2v'' + 2tv' = (t^2v')' \Rightarrow v' = \frac{k_1}{t^2} \Rightarrow v = -\frac{k_1}{t} + k_2$.

Using $v = t^{-1}$, $y_2 = t$, $y_c = c_1t^2 + c_2t$.

6 (b). $\begin{bmatrix} t^2 & t \\ 2t & 1 \end{bmatrix}\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{t}{1+t^2} \end{bmatrix}$, so $\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = -\frac{1}{t^2}\begin{bmatrix} 1 & -t \\ -2t & t^2 \end{bmatrix}\begin{bmatrix} 0 \\ \frac{t}{1+t^2} \end{bmatrix} = \begin{bmatrix} \frac{1}{1+t^2} \\ \frac{-t}{1+t^2} \end{bmatrix}$. Antidifferentiation gives us

$u_1 = \tan^{-1}\mathrm{t}$ and $u_2 = -\frac{1}{2}\ln(1+t^2)$. Thus $y_P = t^2\tan^{-1}\mathrm{t} - \frac{t}{2}\ln(1+t^2)$ and

$y = c_1t^2 + c_2t + t^2\tan^{-1}\mathrm{t} - \frac{t}{2}\ln(1+t^2)$.

6 (c). The method of undetermined coefficients is not applicable.

7 (a). $y_C = c_1e^t + c_2te^t$

7 (b). $\begin{bmatrix} e^t & te^t \\ e^t & (t+1)e^t \end{bmatrix}\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ e^t \end{bmatrix}$, so $\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \frac{1}{e^{2t}}\begin{bmatrix} (t+1)e^t & -te^t \\ -e^t & e^t \end{bmatrix}\begin{bmatrix} 0 \\ e^t \end{bmatrix} = \begin{bmatrix} -t \\ 1 \end{bmatrix}$. Integrating gives us

$y_P = -\frac{t^2}{2}e^t + t^2e^t = \frac{t^2}{2}e^t$ and $y = c_1e^t + c_2te^t + \frac{t^2}{2}e^t$.

7 (c). $y_P = At^2e^t$, and differentiation gives us $y_P' = A\left(t^2+2t\right)e^t$ and $y_P'' = A\left(t^2+4t+2\right)e^t$. Then we have $A\left(t^2+4t+2\right)e^t - 2A\left(t^2+2t\right)e^t + At^2e^t = e^t$, and so

$A = \frac{1}{2}$, $y_P = \frac{t^2}{2}e^t$, and $y = c_1e^t + c_2te^t + \frac{t^2}{2}e^t$.

8 (a). $y_C = c_1\cos 6t + c_2\sin 6t$

8 (b). $\begin{bmatrix} \cos 6t & \sin 6t \\ -6\sin 6t & 6\cos 6t \end{bmatrix}\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ \csc^3 6t \end{bmatrix}$, so $\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 6\cos 6t & -\sin 6t \\ 6\sin 6t & \cos 6t \end{bmatrix}\begin{bmatrix} 0 \\ \csc^3 6t \end{bmatrix} = \begin{bmatrix} -\frac{\csc^3 6t}{6} \\ \frac{\cos 6t}{6\sin^3 6t} \end{bmatrix}$.

Antidifferentiation gives us $u_1 = \frac{1}{36}\cot(6t)$ and $u_2 = -\frac{1}{72}\csc^2(6t)$. Thus

$y_P = \frac{1}{36}\cos(6t)\cot(6t) - \frac{1}{72}\csc(6t)$ and $y = c_1\cos 6t + c_2\sin 6t + \frac{1}{36}\cos(6t)\cot(6t) - \frac{1}{72}\csc(6t)$.

8 (c). The method of undetermined coefficients is not applicable.

9 (a). $y_C = c_1\sin t + c_2t\sin t$

9 (b). $\begin{bmatrix} \sin t & t\sin t \\ \cos t & \sin t + t\cos t \end{bmatrix}\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ t\sin t \end{bmatrix}$, so $\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \frac{1}{\sin^2 t}\begin{bmatrix} \sin t + t\cos t & -t\sin t \\ -\cos t & \sin t \end{bmatrix}\begin{bmatrix} 0 \\ t\sin t \end{bmatrix} = \begin{bmatrix} -t^2 \\ t \end{bmatrix}$.

Antidifferentiation gives us $u_1 = -\frac{t^3}{3}$, $u_2 = \frac{t^2}{2}$, $y_P = -\frac{t^3}{3}\sin t + \frac{t^3}{2}\sin t = \frac{t^3}{6}\sin t$, and

$y = c_1\sin t + c_2t\sin t + \frac{t^3}{6}\sin t$.

9 (c). The method of undetermined coefficients is not applicable.

10 (a). $y_C = c_1t + c_2t\ln|t|$

10 (b). $\begin{bmatrix} t & t\ln|t| \\ 1 & \ln|t|+1 \end{bmatrix}\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{t}\ln|t| \end{bmatrix}$, so $\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \frac{1}{t}\begin{bmatrix} \ln|t|+1 & -t\ln|t| \\ -1 & t \end{bmatrix}\begin{bmatrix} 0 \\ \frac{1}{t}\ln|t| \end{bmatrix} = \begin{bmatrix} -\frac{1}{t}\left(\ln|t|\right)^2 \\ \frac{1}{t}\ln|t| \end{bmatrix}$.

Antidifferentiation gives us $u_1 = -\frac{1}{3}\left(\ln|t|\right)^3$ and $u_2 = \frac{1}{2}\left(\ln|t|\right)^2$. Thus

$y_P = -\frac{t}{3}\left(\ln|t|\right)^3 + \frac{t}{2}\left(\ln|t|\right)^3 = \frac{t}{6}\left(\ln|t|\right)^3$ and $y = c_1t + c_2t\ln|t| + \frac{t}{6}\left(\ln|t|\right)^3$.

10 (c). The method of undetermined coefficients is not applicable.

11 (a). $y_C = c_1t + c_2e^t$

11 (b). $\begin{bmatrix} t & e^t \\ 1 & e^t \end{bmatrix}\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ (t-1)e^t \end{bmatrix}$, so $\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \dfrac{1}{(t-1)e^t}\begin{bmatrix} e^t & -e^t \\ -1 & t \end{bmatrix}\begin{bmatrix} 0 \\ (t-1)e^t \end{bmatrix} = \begin{bmatrix} -e^t \\ t \end{bmatrix}$. Antidifferentiation gives us $u_1 = -e^t$, $u_2 = \dfrac{t^2}{2}$, $y_P = -te^t + \dfrac{t^2}{2}e^t$, and $y = c_1 t + c_2 e^t + \left(\dfrac{t^2}{2} - t\right)e^t$.

11 (c). The method of undetermined coefficients is not applicable.

12 (a). $y_C = c_1 e^{-t^2} + c_2 t e^{-t^2}$

12 (b). $\begin{bmatrix} e^{-t^2} & te^{-t^2} \\ -2te^{-t^2} & (1-2t^2)e^{-t^2} \end{bmatrix}\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ t^2 e^{-t^2} \end{bmatrix}$, so $\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = e^{2t^2}\begin{bmatrix} (1-2t^2)e^{-t^2} & -te^{-t^2} \\ 2te^{-t^2} & e^{-t^2} \end{bmatrix}\begin{bmatrix} 0 \\ t^2 e^{-t^2} \end{bmatrix} = \begin{bmatrix} -t^3 \\ t^2 \end{bmatrix}$.

Antidifferentiation gives us $u_1 = -\dfrac{t^4}{4}$ and $u_2 = \dfrac{t^3}{3}$. Thus $y_P = -\dfrac{t^4}{4}e^{-t^2} + \dfrac{t^4}{3}e^{-t^2} = \dfrac{t^4}{12}e^{-t^2}$ and $y = c_1 e^{-t^2} + c_2 t e^{-t^2} + \dfrac{t^4}{12}e^{-t^2}$.

12 (c). The method of undetermined coefficients is not applicable.

13 (a). $y_1 = (t-1)^2$, and using reduction of order we have $y_2 = (t-1)^2 v$. Differentiation yields $y_2' = 2(t-1)v + (t-1)^2 v'$ and $y_2'' = 2v + 4(t-1)v' + (t-1)^2 v''$. Then we have $(t-1)^4 v'' + 4(t-1)^3 v' + 2(t-1)^2 v - 4(t-1)\left[2(t-1)v + (t-1)^2 v'\right] + 6(t-1)^2 v = 0$. Thus $(t-1)^4 v'' = 0$, and antidifferentiation of $v'' = 0$ gives us $v = k_1(t-1) + k_2$. Then $y_2 = (t-1)^3$, and so $y_C = c_1(t-1)^2 + c_2(t-1)^3$

13 (b). $\begin{bmatrix} (t-1)^2 & (t-1)^3 \\ 2(t-1) & 3(t-1)^2 \end{bmatrix}\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ \dfrac{t}{(t-1)^2} \end{bmatrix}$, so

$\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \dfrac{1}{(t-1)^4}\begin{bmatrix} 3(t-1)^2 & -(t-1)^3 \\ -2(t-1) & (t-1)^2 \end{bmatrix}\begin{bmatrix} 0 \\ \dfrac{t}{(t-1)^2} \end{bmatrix} = \begin{bmatrix} -\dfrac{t}{(t-1)^3} \\ \dfrac{t}{(t-1)^4} \end{bmatrix}$. Antidifferentiation gives us

$u_1 = (t-1)^{-1} + \dfrac{1}{2}(t-1)^{-2}$, $u_2 = -\dfrac{1}{2}(t-1)^{-2} - \dfrac{1}{3}(t-1)^{-3}$, $y_P = (t-1) + \dfrac{1}{2} - \dfrac{1}{2}(t-1) - \dfrac{1}{3} = \dfrac{t}{2} - \dfrac{1}{3}$,

and $y = c_1(t-1)^2 + c_2(t-1)^3 + \dfrac{t}{2} - \dfrac{1}{3}$.

13 (c). The method of undetermined coefficients is not applicable.

14 (a). $y_1 = e^t$. Use Reduction of Order to obtain $y_2(t)$. $y_2 = e^t v,\ y_2' = e^t v + e^t v',\ y_2'' = e^t(v + 2v' + v'')$

Therefore, $v'' + 2v' + v - \left(2 + \frac{2}{t}\right)(v + v') + \left(1 + \frac{2}{t}\right)v = 0 \Rightarrow v'' - \frac{2}{t}v' = 0$

$$\Rightarrow (t^{-2}v')' = 0 \Rightarrow t^{-2}v' = k_1 \Rightarrow v = k_1\frac{t^3}{3} + k_2.$$

Using $y_2 = t^3 e^t$, $y_c = c_1 e^t + c_2 t^3 e^t$.

14 (b). $\begin{bmatrix} e^t & t^3 e^t \\ e^t & (t^3 + 3t^2)e^t \end{bmatrix}\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ e^t \end{bmatrix}$, so $\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \dfrac{e^{-2t}}{3t^2}\begin{bmatrix} (t^3 + 3t^2)e^t & -t^3 e^t \\ -e^t & e^t \end{bmatrix}\begin{bmatrix} 0 \\ e^t \end{bmatrix} = \begin{bmatrix} -\frac{t}{3} \\ \frac{1}{3t^2} \end{bmatrix}$.

Antidifferentiation gives us $u_1 = -\dfrac{t^2}{6}$ and $u_2 = -\dfrac{t^{-1}}{3}$. Thus $y_P = -\dfrac{t^2}{6}e^t - \dfrac{t^2}{3}e^t = -\dfrac{t^2}{2}e^t$ and

$$y = c_1 e^t + c_2 t^3 e^t - \frac{t^2}{2}e^t.$$

14 (c). The method of undetermined coefficients is not applicable.

15. $\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ g \end{bmatrix} \Rightarrow \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \dfrac{1}{W}\begin{bmatrix} y_2' & -y_2 \\ -y_1' & y_1 \end{bmatrix}\begin{bmatrix} 0 \\ g \end{bmatrix} = \begin{bmatrix} -\dfrac{y_2 g}{W} \\ \dfrac{y_1 g}{W} \end{bmatrix}$. Antidifferentiation yields

$u_1 = -\displaystyle\int_0^t \frac{y_2(\lambda)g(\lambda)}{W(\lambda)}d\lambda$ and $u_2 = \displaystyle\int_0^t \frac{y_1(\lambda)g(\lambda)}{W(\lambda)}d\lambda$, and so $y_P = \displaystyle\int_0^t \frac{\left[y_2(t)y_1(\lambda) - y_1(t)y_2(\lambda)\right]}{W(\lambda)}g(\lambda)d\lambda$

and $y = c_1 y_1 + c_2 y_2 + \displaystyle\int_0^t \frac{\left[y_2(t)y_1(\lambda) - y_1(t)y_2(\lambda)\right]}{W(\lambda)}g(\lambda)d\lambda$. $y(0) = c_1 y_1(0) + c_2 y_2(0) = y_0$ and

$y'(0) = c_1 y_1'(0) + c_2 y_2'(0) = y_0'$, since $y_P(0) = y_P'(0) = 0$.

16. For this problem, we have $y_1 = \cos 2t,\ y_2 = \sin 2t$, and $W = \begin{vmatrix} \cos 2t & \sin 2t \\ -2\sin 2t & 2\cos 2t \end{vmatrix} = 2$. Then we

have $y_P = \displaystyle\int_0^t \frac{\sin 2t\cos 2\lambda - \cos 2t\sin 2\lambda}{2}g(\lambda)d\lambda = \frac{1}{2}\int_0^t \sin(2(t-\lambda))g(\lambda)d\lambda$. Since $y = y_P$,

$\alpha = 0,\ \beta = 4,\ y_0 = 0,\ y_0' = 0$.

17. For this problem, we have $y_1 = e^{-t},\ y_2 = e^t$, and $W = \begin{vmatrix} e^{-t} & e^t \\ -e^{-t} & e^t \end{vmatrix} = 2$. Then we have

$y_P = \displaystyle\int_0^t \frac{\left[e^t e^{-\lambda} - e^{-t}e^{\lambda}\right]}{2}g(\lambda)d\lambda = \int_0^t \sinh(t-\lambda)g(\lambda)d\lambda$. Thus we can see that $y = e^{-t} + y_P$, and so

$\alpha = 0,\ \beta = -1,\ y_0 = 1,\ y_0' = -1$.

18. For this problem, we have $y_1 = 1,\ y_2 = t$, and $W = \begin{vmatrix} 1 & t \\ 0 & 1 \end{vmatrix} = 1$. Then we have

$y_P = \int_0^t [t-\lambda] g(\lambda) d\lambda$. Thus we can see that $y = t + y_P(t)$, and so $\alpha = 0,\ \beta = 0,\ y_0 = 0,\ y_0' = 1$.

Section 4.11

1 (a). $y_C = c_1 \cos\omega_0 t + c_2 \sin\omega_0 t$

1 (b). Case i: $\omega \neq \omega_0$. $y_P = A\cos\omega t + B\sin\omega t$, and differentiation yields

$y_P'' = -\omega^2 A\cos\omega t - \omega^2 B\sin\omega t$. Then we have $\left(\omega_0{}^2 - \omega^2\right) A\cos\omega t + \left(\omega_0{}^2 - \omega^2\right) B\sin\omega t = F\cos\omega t$,

and thus $B = 0$ and $A = \dfrac{F}{\omega_0{}^2 - \omega^2}$. The particular solution is then $y_P = \dfrac{F}{\omega_0{}^2 - \omega^2}\cos\omega t$.

Case ii: : $\omega = \omega_0$. $y_P = At\cos\omega_0 t + Bt\sin\omega_0 t$, and differentiation yields

$y_P'' = \omega_0 B\cos\omega_0 t - \omega_0\left(A + \omega_0 Bt\right)\sin\omega_0 t - \omega_0 A\sin\omega_0 t + \omega_0\left(B - \omega_0 At\right)\cos\omega_0 t$

$= \left(2\omega_0 B - \omega_0{}^2 At\right)\cos\omega_0 t + \left(-2\omega_0 A - \omega_0{}^2 Bt\right)\sin\omega_0 t$. Then we have

$2\omega_0 B\cos\omega_0 t - 2\omega_0 A\sin\omega_0 t = F\cos\omega_0 t$, and thus $A = 0$ and $B = \dfrac{F}{2\omega_0}$. The particular solution is

then $y_P = \dfrac{F}{2\omega_0} t\sin\omega_0 t$.

2 (a). $ky = mg,\ k = \dfrac{10\cdot 9.8}{0.098} = 1000\ \mathrm{N/m}$.

2 (b). $10y'' + 1000y = 20\cos(10t);\ \ y'' + 100y = 2\cos(10t),\ \ y(0) = 0,\ \ y'(0) = 0$.

$y_P = At\cos(10t) + Bt\sin(10t)$ and differentiation yields

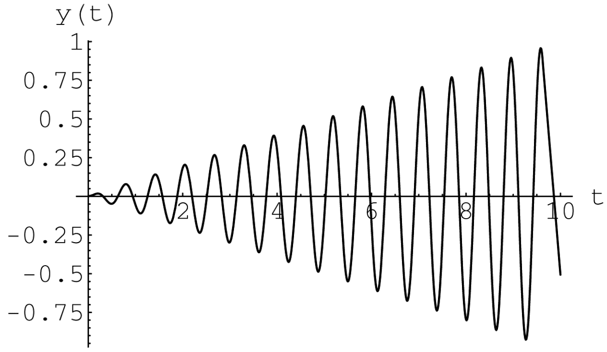
$y_P'' = (20B - 100At)\cos(10t) + (-20A - 100Bt)\sin(10t)$. Then we have

$y_p'' + 100y_p = 20B\cos(10t) - 20A\sin(10t) = 2\cos(10t) \Rightarrow B = \dfrac{1}{10},\ \ A = 0$, and

so $y_P = \dfrac{t}{10}\sin(10t)$, and $y = c_1\cos 10t + c_2\sin 10t + \dfrac{t}{10}\sin(10t)$. From the initial conditions, we

have $y(0) = c_1 = 0$ and $y'(0) = 10c_2 = 0$. Thus the unique solution is $y = \dfrac{t}{10}\sin(10t)$.

2 (c). There is no maximum excursion.



3 (a). $ky = mg,\ k = \dfrac{10 \cdot 9.8}{0.098} = 1000\,\mathrm{N/m}$.

3 (b). $10y'' + 1000y = 20e^{-t};\ y'' + 100y = 2e^{-t},\ y(0) = 0,\ y'(0) = 0$.

$y_P = Ae^{-t}$, and differentiation yields $y_P'' = Ae^{-t}$. Then we have $Ae^{-t} + 100Ae^{-t} = 2e^{-t}$, and so $A = \dfrac{2}{101}$, $y_P = \dfrac{2}{101}e^{-t}$, and $y = c_1\cos 10t + c_2\sin 10t + \dfrac{2}{101}e^{-t}$. From the initial conditions, we have $y(0) = c_1 + \dfrac{2}{101} = 0$ and $y'(0) = 10c_2 - \dfrac{2}{101} = 0$, and thus $c_1 = -\dfrac{2}{101}$ and $c_2 = \dfrac{1}{10}\left(\dfrac{2}{101}\right)$.

Thus the unique solution is $y = \dfrac{2}{101}\left(-\cos 10t + \dfrac{1}{10}\sin 10t + e^{-t}\right)$.

3 (c). $|y|_{\max} \approx 0.035\mathrm{m}$.

4 (a). $ky = mg,\ k = \dfrac{10 \cdot 9.8}{0.098} = 1000\,\mathrm{N/m}$.

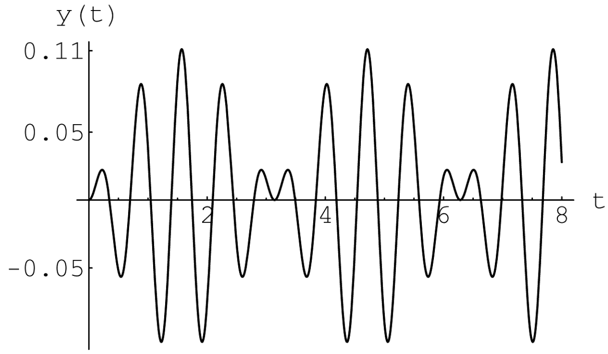
4 (b). $10y'' + 1000y = 20\cos(8t);\ y'' + 100y = 2\cos(8t),\ y(0) = 0,\ y'(0) = 0$.

$y_P = A\cos(8t) + B\sin(8t)$ and differentiation yields $y_P'' = -64A\cos(8t) - 64B\sin(8t)$. Then we have $y_p'' + 100y_p = 36A\cos(8t) + 36B\sin(8t) = 2\cos(8t) \Rightarrow A = \dfrac{1}{18},\ B = 0$, and

so $y_P = \dfrac{1}{18}\cos(8t)$, and $y = c_1\cos 10t + c_2\sin 10t + \dfrac{1}{18}\cos(8t)$. From the initial conditions, we have $y(0) = c_1 + \dfrac{1}{18} = 0$ and $y'(0) = 10c_2 = 0 \Rightarrow c_1 = -\dfrac{1}{18},\ c_2 = 0$. Thus the unique solution is

$y = -\dfrac{1}{18}\cos(10t) + \dfrac{1}{18}\cos(8t)$.

4 (c). $|y|_{\max} \approx 0.11\text{m}$.



5 (a). See 3 (a)

5 (b). On $0 \le t \le \pi$: $y'' + 100y = 2,\ y(0) = 0,\ y'(0) = 0$. $y_P = A$, and differentiation yields $y_P'' = 0$. Then we have $0 + 100A = 2$, and so $A = \frac{1}{50}$, $y_P = \frac{1}{50}$, and $y = c_1 \cos 10t + c_2 \sin 10t + \frac{1}{50}$. From the initial conditions, we have $y(0) = c_1 + \frac{1}{50} = 0$ and $y'(0) = 10c_2 = 0$, and thus $c_1 = -\frac{1}{50}$ and $c_2 = 0$. Thus the unique solution is $y = \frac{1}{50} - \frac{1}{50}\cos 10t$. At $t = \pi,\ y(\pi) = \frac{1}{50} - \frac{1}{50}\cos(10\pi) = 0$ and $y'(\pi) = \frac{10}{50}\sin(10\pi) = 0$. Then we have

$y'' + 100y = 0,\ y(\pi) = 0,\ y'(\pi) = 0$ for $t > \pi$.

$y = c_1 \cos 10t + c_2 \sin 10t \Rightarrow y(\pi) = c_1 = 0,\ y'(\pi) = 10c_2 = 0$. Thus $y = 0$ for this region.

5 (c). $|y|_{\max} = \frac{2}{50} = 0.04\text{m}$.

6. $$y = 0.1\sin(\pi t)\sin(7\pi t) = 0.1\left[\frac{1}{2}(\cos(6\pi t) - \cos(8\pi t))\right] = 0.05[\cos(6\pi t) - \cos(8\pi t]$$

Therefore, $y_c = A\cos(6\pi t) + B\sin(6\pi t)$ and $\frac{k}{m} = (6\pi)^2$. If $y_p = A\cos(8\pi t) + B\sin(8\pi t)$, then

$$-(8\pi)^2[A\cos(8\pi t) + B\sin(8\pi t)] + (6\pi)^2[A\cos(8\pi t) + B\sin(8\pi t)] = \frac{20}{m}\cos(8\pi t).$$

Therefore, $(-64 + 36)\pi^2 A = \frac{20}{m}$, $(-64 + 36)\pi^2 B = 0 \Rightarrow A = -\frac{1}{28\pi^2}\frac{20}{m} = -\frac{5}{7\pi^2 m}$

and $\frac{k}{m} = 36\pi^2$, $-\frac{5}{7\pi^2 m} = -0.05 \Rightarrow m = 1.447..kg,\ k = 36\pi^2 m = 514.2857..N/m$.

7 (a). $2y''+8y'+80y=20\cos 8t,\ y(0)=0,\ y'(0)=0$. For the complementary solution, we have $\lambda^2+4\lambda+40=0$, and so $\lambda=-2\pm i6$. Thus $y_C=c_1e^{-2t}\cos 6t+c_2e^{-2t}\sin 6t$. For the particular solution, we have $y_P=A\cos 8t+B\sin 8t$, and differentiation yields $y_P'=-8A\sin 8t+8B\cos 8t$ and $y_P''=-64A\cos 8t-64B\sin 8t$. Then we have $-64A\cos 8t-64B\sin 8t+4(-8A\sin 8t+8B\cos 8t)+40(A\cos 8t+B\sin 8t)=10\cos 8t$. Solving for A and B yields $A=-\frac{3}{20}$ and $B=\frac{1}{5}$. Thus $y=c_1e^{-2t}\cos 6t+c_2e^{-2t}\sin 6t-\frac{3}{20}\cos 8t+\frac{1}{5}\sin 8t$. From the initial conditions, we have $y(0)=c_1-\frac{3}{20}=0$ and $y'(0)=-2c_1+6c_2+\frac{8}{5}=0$. Solving these simultaneous equations yields $c_1=\frac{3}{20}$ and $c_2=-\frac{13}{60}$, so
$y=\frac{3}{20}e^{-2t}\cos 6t-\frac{13}{60}e^{-2t}\sin 6t-\frac{3}{20}\cos 8t+\frac{1}{5}\sin 8t$.

7 (b). For sufficiently large t, $y(t)\approx-\frac{3}{20}\cos 8t+\frac{1}{5}\sin 8t$, and so the limit does not exist. This equation is called the steady state solution.

8 (a). $y_C=c_1e^{-2t}\cos 6t+c_2e^{-2t}\sin 6t$. For the particular solution, we have $y_P=Ae^{-t}$,
$y_p''+4y_p'+40y_p=10e^{-t}\Rightarrow A-4A+40A=10\Rightarrow A=\frac{10}{37}$. Thus
$y=c_1e^{-2t}\cos 6t+c_2e^{-2t}\sin 6t+\frac{10}{37}e^{-t}$. From the initial conditions, we have
$y(0)=c_1+\frac{10}{37}=0$ and $y'(0)=-2c_1+6c_2-\frac{10}{37}=0\Rightarrow c_1=-\frac{10}{37},\ c_2=-\frac{5}{111}$.
$y=-\frac{10}{37}e^{-2t}\cos 6t-\frac{5}{111}e^{-2t}\sin 6t+\frac{10}{37}e^{-t}$.

8 (b). $\lim_{t\to\infty}y(t)=0$.

9 (a). $y_C=c_1e^{-2t}\cos 6t+c_2e^{-2t}\sin 6t$. For the particular solution, we have $y_P=A\cos 6t+B\sin 6t$, and differentiation yields $y_P'=-6A\sin 6t+6B\cos 6t$ and $y_P''=-36A\cos 6t-36B\sin 6t$. Then we have $-36A\cos 6t-36B\sin 6t+4(-6A\sin 6t+6B\cos 6t)+40(A\cos 6t+B\sin 6t)=10\sin 6t$. Solving for A and B yields $A=-\frac{30}{74}$ and $B=\frac{5}{74}$. Thus $y=c_1e^{-2t}\cos 6t+c_2e^{-2t}\sin 6t-\frac{30}{74}\cos 6t+\frac{5}{74}\sin 6t$.
From the initial conditions, we have $y(0)=c_1-\frac{30}{74}=0$ and $y'(0)=-2c_1+6c_2+\frac{30}{74}=0$.

Solving these simultaneous equations yields $c_1 = \dfrac{30}{74}$ and $c_2 = \dfrac{5}{74}$, so

$$y = \frac{30}{74}e^{-2t}\cos 6t + \frac{5}{74}e^{-2t}\sin 6t - \frac{30}{74}\cos 6t + \frac{5}{74}\sin 6t.$$

9 (b). For sufficiently large t, $y(t) \approx -\dfrac{30}{74}\cos 6t + \dfrac{5}{74}\sin 6t$, and so the limit does not exist. This equation is called the steady state solution.

10 (a). On $0 \le t \le \frac{\pi}{2}$:

$y_C = c_1 e^{-2t}\cos 6t + c_2 e^{-2t}\sin 6t$. For the particular solution, we have $y_P = \dfrac{1}{4}$. Thus

$y = c_1 e^{-2t}\cos 6t + c_2 e^{-2t}\sin 6t + \dfrac{1}{4}$. From the initial conditions, we have

$$y(0) = c_1 + \frac{1}{4} = 0 \text{ and } y'(0) = -2c_1 + 6c_2 = 0 \Rightarrow c_1 = -\frac{1}{4},\ c_2 = -\frac{1}{12}.$$

$$y = -\frac{1}{4}e^{-2t}\cos 6t - \frac{1}{12}e^{-2t}\sin 6t + \frac{1}{4},\ y\left(\frac{\pi}{2}\right) = \frac{1}{4}e^{-\pi} + \frac{1}{4},\ y'\left(\frac{\pi}{2}\right) = -\frac{1}{2}e^{-\pi} + \frac{1}{2}e^{-\pi} = 0.$$

On $\frac{\pi}{2} < t < \infty$:

$y = d_1 e^{-2t}\cos 6t + d_2 e^{-2t}\sin 6t$. From the initial conditions, we have

$$y\left(\frac{\pi}{2}\right) = -d_1 e^{-\pi} = \frac{1}{4}\left(1 + e^{-\pi}\right) \text{ and } y'\left(\frac{\pi}{2}\right) = 2d_1 e^{-\pi} - 6d_2 e^{-\pi} = 0$$

$$\Rightarrow d_1 = -\frac{1}{4}\left(e^{\pi} + 1\right),\ d_2 = -\frac{1}{12}\left(e^{\pi} + 1\right).\ y = -\frac{1}{4}\left(e^{\pi} + 1\right)\left[e^{-2t}\cos 6t + \frac{1}{3}e^{-2t}\sin 6t\right].$$

10 (b). $\lim_{t\to\infty} y(t) = 0$.

11 (a). $y'' + 2\delta y' + \omega_0^{\,2} y = F\cos\omega_0 t,\ y(0) = 0,\ y'(0) = 0$. $\lambda = \dfrac{-2\delta \pm \sqrt{4\delta^2 - 4\omega_0^{\,2}}}{2} = -\delta \pm i\sqrt{\omega_0^{\,2} - \delta^2}$. Thus

$y_C = c_1 e^{-\delta t}\cos\left(t\sqrt{\omega_0^{\,2} - \delta^2}\right) + c_2 e^{-\delta t}\sin\left(t\sqrt{\omega_0^{\,2} - \delta^2}\right)$. $y_P = A\cos\omega_0 t + B\sin\omega_0 t$, and differentiation yields $y_P' = -\omega_0 A\sin\omega_0 t + \omega_0 B\cos\omega_0 t$ and $y_P'' = -\omega_0^{\,2} A\cos\omega_0 t - \omega_0^{\,2} B\sin\omega_0 t$.

Then we have $y_P'' + 2\delta y_P' + \omega_0^{\,2} y_P = 2\delta\left[-\omega_0 A\sin\omega_0 t + \omega_0 B\cos\omega_0 t\right] = F\cos\omega_0 t$. Solving for A and B yields $A = 0$ and $B = \dfrac{F}{2\delta\omega_0}$. Thus

$$y = c_1 e^{-\delta t}\cos\left(t\sqrt{\omega_0^{\,2} - \delta^2}\right) + c_2 e^{-\delta t}\sin\left(t\sqrt{\omega_0^{\,2} - \delta^2}\right) + \frac{F}{2\delta\omega_0}\sin\omega_0 t.$$

From the initial conditions, we have $y(0) = c_1 = 0$ and $y'(0) = c_2\sqrt{\omega_0^{\,2} - \delta^2} + \dfrac{F}{2\delta} = 0$. Thus

$c_1 = 0$ and $c_2 = -\dfrac{F}{2\delta\sqrt{\omega_0^{\,2} - \delta^2}}$, and so $y = \dfrac{F}{2\delta}\left[\dfrac{\sin\omega_0 t}{\omega_0} - \dfrac{e^{-\delta t}\sin\left(t\sqrt{\omega_0^{\,2} - \delta^2}\right)}{\sqrt{\omega_0^{\,2} - \delta^2}}\right]$.

11 (b). First, let us rewrite $y = \dfrac{F}{2}\left[\dfrac{\sqrt{\omega_0^{\,2} - \delta^2}\sin\omega_0 t - \omega_0 e^{-\delta t}\sin\left(t\sqrt{\omega_0^{\,2} - \delta^2}\right)}{\delta\omega_0\sqrt{\omega_0^{\,2} - \delta^2}}\right] \equiv \dfrac{F}{2}\dfrac{N(\delta)}{D(\delta)}$. To use

L'Hopital's Rule to find the limit, we need

$$\frac{dN}{d\delta} = \frac{1}{2}\left(\omega_0^{\,2} - \delta^2\right)^{-\frac{1}{2}}(-2\delta)\sin\omega_0 t + \omega_0 t e^{-\delta t}\sin\left(t\sqrt{\omega_0^{\,2} - \delta^2}\right)$$

$$-\omega_0 e^{-\delta t}\cos\left(t\sqrt{\omega_0^{\,2} - \delta^2}\right)\cdot\frac{1}{2}\left(\omega_0^{\,2} - \delta^2\right)^{-\frac{1}{2}}(-2\delta)t.$$ Thus $\dfrac{dN}{d\delta} \to 0 + \omega_0 t\sin\omega_0 t + 0$ as $\delta \to 0$.

$\dfrac{dD}{d\delta} = \omega_0\sqrt{\omega_0^{\,2} - \delta^2} + \delta\omega_0\cdot\dfrac{1}{2}\left(\omega_0^{\,2} - \delta^2\right)^{-\frac{1}{2}}(-2\delta)$. Thus $\dfrac{dD}{d\delta} \to \omega_0^{\,2}$ as $\delta \to 0$. Therefore,

$\lim\limits_{\delta \to 0^+}\dfrac{F}{2}\dfrac{N(\delta)}{D(\delta)} = \dfrac{F}{2\omega_0}t\sin\omega_0 t$.

11 (c). For sufficiently large t, $y \approx \dfrac{F}{2\delta}\dfrac{\sin\omega_0 t}{\omega_0}$. Knowing m and k means that we know $\omega_0 = \sqrt{\dfrac{k}{m}}$.

Therefore, by measuring the amplitude $\dfrac{F}{2\delta\omega_0}$ of the steady state solution and knowing $F = \dfrac{\overline{F}}{m}$, we can determine δ.

12 (a). $y'' + 2\delta y' + \omega_0^{\,2} y = F\cos\omega_1 t,\ y(0) = 0,\ y'(0) = 0$.

$y_C = c_1 e^{-\delta t}\cos\left(t\sqrt{\omega_0^{\,2} - \delta^2}\right) + c_2 e^{-\delta t}\sin\left(t\sqrt{\omega_0^{\,2} - \delta^2}\right)$. $y_P = A\cos\omega_1 t + B\sin\omega_1 t$, and differentiation yields $y_P' = -\omega_1 A\sin\omega_1 t + \omega_1 B\cos\omega_1 t$ and $y_P'' = -\omega_1^{\,2} A\cos\omega_1 t - \omega_1^{\,2} B\sin\omega_1 t$.

Then we have

$$y_P'' + 2\delta y_P' + \omega_0^{\,2} y_P = \left(\omega_0^{\,2} - \omega_1^{\,2}\right)\left[A\cos\omega_1 t + B\sin\omega_1 t\right] + 2\delta\left[-\omega_1 A\sin\omega_1 t + \omega_1 B\cos\omega_1 t\right] = F\cos\omega_1 t.$$

Solving for A and B yields $A = \dfrac{\left(\omega_0^{\,2} - \omega_1^{\,2}\right)F}{\left(\omega_0^{\,2} - \omega_1^{\,2}\right)^2 + \left(2\delta\omega_1\right)^2}$ and $B = \dfrac{2\delta\omega_1 F}{\left(\omega_0^{\,2} - \omega_1^{\,2}\right)^2 + \left(2\delta\omega_1\right)^2}$. Thus

$y = c_1 e^{-\delta t}\cos\left(t\sqrt{\omega_0^{\,2} - \delta^2}\right) + c_2 e^{-\delta t}\sin\left(t\sqrt{\omega_0^{\,2} - \delta^2}\right) + A\cos\omega_1 t + B\sin\omega_1 t$. From the initial conditions, we have $y(0) = c_1 + A = 0$ and $y'(0) = -\delta c_1 + c_2\sqrt{\omega_0^{\,2} - \delta^2} + \omega_1 B = 0$.

Thus $c_1 = -A$ and $c_2 = -\dfrac{\delta A + \omega_1 B}{\sqrt{\omega_0^{\,2} - \delta^2}}$, and so

$$y = \frac{F}{\left(\omega_0^{\,2} - \omega_1^{\,2}\right)^2 + \left(2\delta\omega_1\right)^2}\left[\left(\omega_0^{\,2} - \omega_1^{\,2}\right)\cos\omega_1 t + 2\delta\omega_1 \sin\omega_1 t\right] -$$

$$\frac{Fe^{-\delta t}}{\left(\omega_0^{\,2} - \omega_1^{\,2}\right)^2 + \left(2\delta\omega_1\right)^2}\left\{\left(\omega_0^{\,2} - \omega_1^{\,2}\right)\cos\left(t\sqrt{\omega_0^{\,2} - \delta^2}\right) + \left[\delta\left(\omega_0^{\,2} - \omega_1^{\,2}\right) + \omega_1(2\delta\omega_1)\right]\frac{\sin\left(t\sqrt{\omega_0^{\,2} - \delta^2}\right)}{\sqrt{\omega_0^{\,2} - \delta^2}}\right\}$$

12 (b). Using $\delta\left(\omega_0^{\,2} - \omega_1^{\,2}\right) + \omega_1(2\delta\omega_1) = \delta\left(\omega_0^{\,2} + \omega_1^{\,2}\right)$,

$$\lim_{\omega_1 \to \omega_0} y(t) = \frac{F}{\left(2\delta\omega_0\right)^2} 2\delta\omega_0 \sin\omega_0 t - \frac{Fe^{-\delta t}}{\left(2\delta\omega_0\right)^2}\left\{\left[2\delta\omega_0^{\,2}\right]\frac{\sin\left(t\sqrt{\omega_0^{\,2} - \delta^2}\right)}{\sqrt{\omega_0^{\,2} - \delta^2}}\right\}$$

$$= \frac{F}{2\delta}\left\{\frac{\sin\omega_0 t}{\omega_0} - e^{-\delta t}\,\frac{\sin\left(t\sqrt{\omega_0^{\,2} - \delta^2}\right)}{\sqrt{\omega_0^{\,2} - \delta^2}}\right\}$$

12 (c).

$$\lim_{\delta \to 0^+} y(t) = \frac{F}{\left(\omega_0^{\,2} - \omega_1^{\,2}\right)^2}\left\{\left(\omega_0^{\,2} - \omega_1^{\,2}\right)\cos\omega_1 t\right\} - \frac{F}{\left(\omega_0^{\,2} - \omega_1^{\,2}\right)^2}\left\{\left(\omega_0^{\,2} - \omega_1^{\,2}\right)\cos\omega_0 t\right\}$$

$$= \frac{F}{\left(\omega_0^{\,2} - \omega_1^{\,2}\right)}\left\{\cos\omega_1 t - \cos\omega_0 t\right\} = \frac{F}{\left(\omega_1^{\,2} - \omega_0^{\,2}\right)}\left\{\cos\omega_0 t - \cos\omega_1 t\right\}.$$

13 (a). $mx'' = -kx - mg\cos\left(\dfrac{\pi}{4}\right),\ x(0) = -10,\ x'(0) = 0$.

13 (b). $x'' + \dfrac{k}{m}x = -\dfrac{g}{\sqrt{2}}$. $\dfrac{k}{m} = \dfrac{150}{\left(150/32\right)} = 32\text{s}^{-2}$. Thus the complementary solution is

$x_C = c_1\cos\left(t\sqrt{32}\right) + c_2\sin\left(t\sqrt{32}\right)$. $x_P = A$, so $\dfrac{k}{m}A = -\dfrac{g}{\sqrt{2}}$, and so $A = -\dfrac{1}{\sqrt{2}}$ and

$x = c_1\cos\left(t\sqrt{32}\right) + c_2\sin\left(t\sqrt{32}\right) - \dfrac{1}{\sqrt{2}}$.

From the initial conditions, we have $x(0) = c_1 - \dfrac{1}{\sqrt{2}} = -10$ and $x'(0) = \sqrt{32}c_2 = 0$. Thus

$c_1 = \dfrac{1}{\sqrt{2}} - 10,\ c_2 = 0$, and $x = \left(\dfrac{1}{\sqrt{2}} - 10\right)\cos\left(t\sqrt{32}\right) - \dfrac{1}{\sqrt{2}}$.

Differentiation gives us $x'(t)=\sqrt{32}\left(10-\frac{1}{\sqrt{2}}\right)\sin\left(t\sqrt{32}\right)$. We need x' when $x=0$, so

$\cos\left(t\sqrt{32}\right)=\dfrac{\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}-10}$. Solving for $\sin\left(t\sqrt{32}\right)=.9971$, and so $x'=52.416\,\text{ft}/\text{s}$.

13 (c). Letting x and y represent horizontal and vertical coordinates which have their origin at the mouth o the "cannon," we have $y''=-g,\ y(0)=0,\ y'(0)=\frac{v_0}{\sqrt{2}}$ and $x''=0,\ x(0)=0,\ x'(0)=\frac{v_0}{\sqrt{2}}$. For the y initial value problem, we have $y'=-gt+\frac{v_0}{\sqrt{2}}$ and antidifferentiation yields $y=-\frac{gt^2}{2}+\frac{v_0t}{\sqrt{2}}$. Settin $y=0$ gives us $t=0,\ \frac{\sqrt{2}v_0}{g}$. Substituting the second value for t into the solution of the initial value problem for x gives us $x(t)=\frac{v_0}{\sqrt{2}}t=\frac{v_0}{\sqrt{2}}\sqrt{2}\frac{v_0}{g}=\frac{v_0{}^2}{g}=\frac{(52.416)^2}{32}=85.857\text{ft}.$

14 (b). (i). $\frac{v^2}{2}=-gy+C_1,\ \frac{30^2}{2}=0+C_1\Rightarrow v^2=-2gy+900,\ v(2)=(900-128)^{\frac{1}{2}}=\sqrt{772}\,ft/\sec$.

(ii). $\frac{v^2}{2}=-\frac{k}{2m}(y-2)^2-g(y-2)+C_2,\ C_2=\frac{772}{2}\Rightarrow v^2=-\frac{k}{m}(y-2)^2-2g(y-2)+772$

$v(3)=0\Rightarrow 0=-\frac{k}{m}-2g+772\Rightarrow\frac{k}{m}=772-64=708,\ m=\frac{\frac{1}{16}}{32}\Rightarrow k=\frac{708}{16(32)}=1.3828\ \ lb/ft$.

15. $V_s=L\frac{dI}{dt}+\frac{1}{C}\int_0^t I(\lambda)d\lambda$, so $\frac{dV_s}{dt}=LI''+\frac{1}{C}I,\ I(0)=0,\ I'(0)=0$, and therefore $I''+\frac{1}{4}I=\frac{dV_s}{dt}$.

$V_s=5\sin 3t,$ so $I''+\frac{1}{4}I=15\cos 3t$. Thus $I_C=c_1\cos\left(\frac{t}{2}\right)+c_2\sin\left(\frac{t}{2}\right)$ and $I_P=A\cos 3t+B\sin 3t$.

Differentiation gives us $I_P''=-9A\cos 3t-9B\sin 3t$, and then we have

$\left(-9+\frac{1}{4}\right)A\cos 3t+\left(-9+\frac{1}{4}\right)B\sin 3t=15\cos 3t$. Thus $B=0$ and $A=-\frac{12}{7}$, and so

$I_P=-\frac{12}{7}\cos 3t$ and $I=c_1\cos\left(\frac{t}{2}\right)+c_2\sin\left(\frac{t}{2}\right)-\frac{12}{7}\cos 3t$. From the initial conditions, we have

$I(0)=c_1-\frac{12}{7}=0$ and $I'(0)=\frac{1}{2}c_2=0$. Thus $c_1=\frac{12}{7}$ and $c_2=0$ and $I(t)=\frac{12}{7}\left(\cos\frac{t}{2}-\cos 3t\right)$.

16. $I''+\frac{1}{4}I=\frac{dV_s}{dt}$. $V_s=10te^{-t}$, so $I''+\frac{1}{4}I=10te^{-t}$. Thus $I_C=c_1\cos\left(\frac{t}{2}\right)+c_2\sin\left(\frac{t}{2}\right)$ and

$I_P=(At+B)e^{-t}$. Differentiation gives us $I_P''=(At-2A+B)e^{-t}$, and then we have

$At-2A+B+\frac{1}{4}(At+B)=-10t+10 \Rightarrow A=-8,\ B=-\frac{24}{5}$.

Thus, $I_P=\left(-8t-\frac{24}{5}\right)e^{-t}$ and $I=c_1\cos\left(\frac{t}{2}\right)+c_2\sin\left(\frac{t}{2}\right)+\left(-8t-\frac{24}{5}\right)e^{-t}$. From the initial conditions,

we have $I(0)=c_1-\frac{24}{5}=0$ and $I'(0)=\frac{1}{2}c_2-8+\frac{24}{5}=0$. Thus $c_1=\frac{24}{7}$ and $c_2=\frac{32}{5}$ and

$I(t)=\frac{24}{5}\cos\left(\frac{t}{2}\right)+\frac{32}{5}\sin\left(\frac{t}{2}\right)-\left(8t+\frac{24}{5}\right)e^{-t}$.

17. $I_S=\frac{V}{R}+\frac{1}{L}\int_0^t V(\lambda)d\lambda+C\frac{dV}{dt}$, and then we have $CV''+\frac{1}{R}V'+\frac{1}{L}V=\frac{dI_s}{dt}$, $V(0)=0,\ V'(0)=0$.

$R=1\text{k}\Omega,\ L=1\text{H},\ \text{C}=\frac{1}{2}\mu\text{F}$, and so $V''+2V'+2V=2\frac{dI_s}{dt}$. For this problem,

$I_s=1-e^{-t}\Rightarrow 2\frac{dI_s}{dt}=2e^{-t}$. For the complementary solution, we have $\lambda=\frac{-2\pm\sqrt{4-8}}{2}=-1\pm i$, and

so $V_C=c_1e^{-t}\cos t+c_2e^{-t}\sin t$. $V_P=Ae^{-t}$, and substituting this into the original differential equation,

we have $A-2A+2A=2$. Thus $A=2$, and so $V=c_1e^{-t}\cos t+c_2e^{-t}\sin t+2e^{-t}$. From the initial

conditions, we have $V(0)=c_1+2=0$ and $V'(0)=-c_1+c_2-2=0$. Solving these simultaneous

equations yields $c_1=-2$ and $c_2=0$, and so $V=-2e^{-t}\cos t+2e^{-t}$.

18. $V''+2V'+2V=2\frac{dI_s}{dt}$, $V(0)=0,\ V'(0)=0$. For this problem, $I_s=5\sin t\Rightarrow 2\frac{dI_s}{dt}=10\cos t$. For the

complementary solution, $V_C=c_1e^{-t}\cos t+c_2e^{-t}\sin t$.

$V_P=A\cos t+B\sin t,\ V_P'=-A\sin t+B\cos t,\ V_P''=-A\cos t-B\sin t$, and substituting this into the

original differential equation, we have $(-2A+B)\sin t+(A+2B)\cos t=10\cos t\Rightarrow A=2,\ B=4$.

Thus, $V=c_1e^{-t}\cos t+c_2e^{-t}\sin t+2\cos t+4\sin t$. From the initial conditions, we have

$V(0)=c_1+2=0$ and $V'(0)=-c_1+c_2+4=0$. Solving these simultaneous equations yields

$c_1=-2$ and $c_2=-6$, and so $V=-2e^{-t}\cos t-6e^{-t}\sin t+2\cos t+4\sin t$.

Chapter 5
Higher Order Linear Differential Equations

Section 5.1

1-5 Verify by substituting into differential equation.

6. Discontinuities for the relevant functions exist at $t = -3, -1, 3$.

$y''' - \frac{1}{t^2 - 9}y'' + \ln(t+1)y' + \cos ty = 0$; Initial condition at $t = 0$. $-1 < t < 3$.

7. Discontinuities for the relevant functions exist at $t = -1$ and $n\pi + \frac{\pi}{2}$. Since $t_0 = 0$, the largest interval on which Theorem 5.1 guarantees a unique solution is $-1 < t < \frac{\pi}{2}$.

8. Discontinuities for the relevant functions exist at $t = \pm 4$ and $\pm\frac{\pi}{2}$. $\left(t^2 - 16\right)y^{(4)} + 2y'' + t^2 y = \sec t$; Initial condition at $t = 3$. $\frac{\pi}{2} < t < 4$.

9. There are no discontinuities for the relevant functions and $t_0 = 0$. Thus the largest interval on whic Theorem 5.1 guarantees a unique solution is $-\infty < t < \infty$.

10. $y^{(4)} - 5y'' + 4y = 0$; $\lambda^4 - 5\lambda^2 + 4 = \left(\lambda^2 - 1\right)\left(\lambda^2 - 4\right) = 0$ $\lambda = -1,\ 1,\ -2,\ 2$.

11. $\lambda^3 - \lambda = \lambda(\lambda - 1)(\lambda + 1) = 0$, so $\lambda = 0,\ \pm 1$.

12. $y''' - 2y'' - y' + 2y = 0$; $\lambda^3 - 2\lambda^2 - \lambda + 2 = \lambda^2(\lambda - 2) - (\lambda - 2) = (\lambda - 2)(\lambda^2 - 1) = 0$
$\lambda = -1,\ 1,\ 2$.

13. $\lambda^4 - 2\lambda^2 + 1 = (\lambda^2 - 1)^2 = (\lambda - 1)^2(\lambda + 1)^2 = 0$, so $\lambda = \pm 1$.

Section 5.2

1 (a). $y''' = 0$, and antidifferentiation yields $y'' = c_1$, $y' = c_1 t + c_2$, and $y = \frac{c_1}{2}t^2 + c_2 t + c_3$.

1 (b). $W = \begin{vmatrix} 1 & t & t^2 \\ 0 & 1 & 2t \\ 0 & 0 & 2 \end{vmatrix} = 2 \neq 0$, and thus the three functions form a fundamental set of solutions.

2. $W = \begin{vmatrix} 1 & e^t & e^{-t} \\ 0 & e^t & -e^{-t} \\ 0 & e^t & e^{-t} \end{vmatrix} = 1(2) = 2 \neq 0$.

3. $W = \begin{vmatrix} 1 & t & e^{-t} \\ 0 & 1 & -e^{-t} \\ 0 & 0 & e^{-t} \end{vmatrix} = e^{-t} \neq 0$, and thus the three functions form a fundamental set of solutions.

4. $W = \begin{vmatrix} 1 & t & \cos t & \sin t \\ 0 & 1 & -\sin t & \cos t \\ 0 & 0 & -\cos t & -\sin t \\ 0 & 0 & \sin t & -\cos t \end{vmatrix} = 1 \cdot 1 \cdot \left(\cos^2 t + \sin^2 t\right) = 1 \neq 0$.

5. $W = \begin{vmatrix} 1 & t & t^{-1} \\ 0 & 1 & -t^{-2} \\ 0 & 0 & 2t^{-3} \end{vmatrix} = 2t^{-3} \neq 0,\ t > 0$, and thus the three functions form a fundamental set of solutions.

6. $W = \begin{vmatrix} 1 & \ln t & t^2 \\ 0 & t^{-1} & 2t \\ 0 & -t^{-2} & 2 \end{vmatrix} = 1 \cdot (3t^{-1}) = 3t^{-1} \neq 0,\ t > 0$.

7. $y = c_1 + c_2 e^t + c_3 e^{-t}$, and differentiation yields $y' = c_2 e^t - c_3 e^{-t}$ and $y'' = c_2 e^t + c_3 e^{-t}$. From the initial conditions, we have $y(0) = c_1 + c_2 + c_3 = 3$, $y'(0) = c_2 - c_3 = -3$, and $y''(0) = c_2 + c_3 = 1$. Solving these simultaneous equations yields $c_1 = 2,\ c_2 = -1,$ and $c_3 = 2$, and so the unique solution is $y = 2 - e^t + 2e^{-t}$.

8. $y = c_1 + c_2 t + c_3 e^{-t}$, $y(1) = c_1 + c_2 + c_3 e^{-1} = 4$, $y'(1) = c_2 - c_3 e^{-1} = 3$, $y''(1) = c_3 e^{-1} = 0$
$\therefore\ c_3 = 0,\ c_2 = 3,\ c_1 = 1$ and $y(t) = 1 + 3t$.

9. $y = c_1 + c_2 t + c_3 \cos t + c_4 \sin t$, and differentiation yields $y' = c_2 - c_3 \sin t + c_4 \cos t$, $y'' = -c_3 \cos t - c_4 \sin t$, and $y''' = c_3 \sin t - c_4 \cos t$. From the initial conditions, we have
$y\left(\frac{\pi}{2}\right) = c_1 + c_2 \frac{\pi}{2} + c_4 = 2 + \pi$, $y'\left(\frac{\pi}{2}\right) = c_2 - c_3 = 3$, $y''\left(\frac{\pi}{2}\right) = -c_4 = -3$, and $y'''\left(\frac{\pi}{2}\right) = c_3 = 1$.
Solving these simultaneous equations yields $c_1 = -1 - \pi,\ c_2 = 4,\ c_3 = 1,$ and $c_4 = 3$, and so the unique solution is $y = -1 - \pi + 4t + \cos t + 3\sin t$.

10. $y = c_1 + c_2 t + c_3 t^{-1}$, $y' = c_2 - c_3 t^{-2}$, $y'' = 2c_3 t^{-3}$, $y(2) = c_1 + 2c_2 + \frac{1}{2} c_3 = -1$, $y'(2) = c_2 - \frac{1}{4} c_3 = \frac{3}{2}$,
$y''(2) = \frac{2}{8} c_3 = -\frac{1}{2} \Rightarrow c_3 = -2$, $c_2 = \frac{3}{2} - \frac{1}{2} = 1$, $c_1 + 2(1) + \frac{1}{2}(-2) = -1 \Rightarrow c_1 = -2$
$y = -2 + t - 2t^{-1}$.

11. $y = c_1 + c_2 \ln t + c_3 t^2$, and differentiation yields $y' = c_2 t^{-1} + 2c_3 t$ and $y'' = -c_2 t^{-2} + 2c_3$. From the initial conditions, we have $y(1) = c_1 + c_3 = 1$, $y'(1) = c_2 + 2c_3 = 2$, and $y''(1) = -c_2 + 2c_3 = -6$. Solving these simultaneous equations yields $c_1 = 2,\ c_2 = 4,$ and $c_3 = -1$, and so the unique solution is $y = 2 + 4\ln t - t$.

12. $y''' - y' = 0;\ \ p_{n-1}(t) = p_2(t) = 0$. Abel's theorem predicts $W(t) = W(t_0)$.
If $t_0 = -1$, then $W(t) = W(-1) = \text{constant}$. From exercise 2, $W(t) = 2$.

13. $p_{n-1}(t) = p_2(t) = 1$, and so Abel's Theorem predicts $W(t) = W(0)e^{-t}$ with $t_0 = 0$. From Exercise 3, $W(t) = e^{-t} = W(0)e^{-t}$ since $W(0) = 1$.

14. $y^{(4)} + y'' = 0;\ \ p_{n-1}(t) = p_3(t) = 0$. If $t_0 = 1$, Abel's theorem predicts $W(t) = W(1) = \text{constant}$. From exercise 4, $W(t) = 1$.

15. $p_{n-1}(t) = p_2(t) = \dfrac{3}{t}$, and so Abel's Theorem predicts $W(t) = W(1)e^{-\int_1^t \frac{3}{s} ds} = W(1)e^{\{-3[\ln t - \ln 1]\}} = \dfrac{W(1)}{t^3}$
with $t_0 = 1$. From Exercise 5, $W(t) = 2t^{-3} = W(1)t^{-3}$ since $W(1) = 2$.

16. $t^2 y''' + t y'' - y' = 0;\ \ p_{n-1}(t) = p_2(t) = \dfrac{1}{t}$. With $t_0 = 2$, Abel's theorem predicts

$$W(t) = W(2)\exp\left\{-\int_2^t \frac{1}{s} ds\right\} = W(2)\exp\{-\ln t + \ln 2\} = W(2)\exp\left\{\ln\left(\frac{2}{t}\right)\right\} = 2\frac{W(2)}{t}.$$

From exercise 6, $W(t) = 3t^{-1}\quad \therefore\ W(2) = \dfrac{3}{2}$ and $W(t) = 2\dfrac{W(2)}{t}$.

17. $p_{n-1}(t) = p_2(t) = -3$. $W(t) = W(1)e^{-\int_1^t (-3) ds} = e^{3(t-1)}$.

18. $p_{n-1}(t) = p_2(t) = \sin t$. $\quad W(t) = W(1)\exp\left\{-\int_1^t \sin s\, ds\right\} = 0$ since $W(1) = 0$.

19. $p_{n-1}(t) = p_2(t) = \dfrac{1}{t}$. $W(t) = W(1)e^{-\int_1^t s^{-1} ds} = 3t^{-1},\ t > 0$.

20. $p_{n-1}(t) = p_2(t) = 0$. $\ \therefore\ W(t) = W(1) = 3$.

21. $u'' - u = 0$, so $\lambda^2 - 1 = (\lambda + 1)(\lambda - 1) = 0$. Then we have $u = c_1 e^{-t} + c_2 e^{t} = y'$. Antidifferentiation yields $y = -c_1 e^{-t} + c_2 e^{t} + c_3 \equiv Ae^{-t} + Be^{t} + C(1)$.

22 (a). $u = y';\ \ u'' + u' = 0,\ \lambda^2 + \lambda = 0\ \Rightarrow\ \lambda = 0, -1 \qquad u = y' = c_1 + c_2 e^{-t}$

$\therefore\ y = c_1 t - c_2 e^{-t} + c_3\ \equiv\ c_1 t + c_2 e^{-t} + c_3 \cdot 1$

22 (b). $v = y'' \qquad v' + v = 0\ \Rightarrow\ v = y'' = k_1 e^{-t}\ \Rightarrow\ y' = -k_1 e^{-t} + k_2\ \Rightarrow\ y = k_1 e^{-t} + k_2 t + k_3$.

23. $v'' + v = 0$, so we have $v = c_1\cos t + c_2\sin t = y''$. Antidifferentiation twice yields $y = -c_1\cos t - c_2\sin t + c_3 t + c_4 \equiv A\cos t + B\sin t + Ct + D$.

24 (a). $W = \begin{vmatrix} 1 & t^2 & t^4 \\ 0 & 2t & 4t^3 \\ 0 & 2 & 12t^2 \end{vmatrix} = 1\cdot\left[24t^3 - 8t^3\right] = 16t^3$. Note that $W(0) = 0$ but $W(t)$ is not identically zero on $(-1,\ 1)$. If $y''' + p_2(t)y'' + p_1(t)y = 0$ were to have $1,\ t^2,\ t^4$ solutions with $p_2(t),\ p_1(t)$ continuous on $(-1,\ 1)$, we would contradict Abel's theorem. $\therefore$ No.

24 (b). If $y = 1,$ then $y''' + p_2 y'' + p_1 y' = 0$ is satisfied for any $p_1,\ p_2$.

If $y = t^2,\ y' = 2t,\ y'' = 2 \Rightarrow 0 + 2p_2 + 2tp_1 = 0 \Rightarrow p_2 + tp_1 = 0$.

If $y = t^4,\ y' = 4t^3,\ y'' = 12t^2,\ y''' = 24t \Rightarrow 24t + 12t^2 p_2 + 4t^3 p_1 = 0 \Rightarrow 6 + 3tp_2 + t^2 p_1 = 0$.

Therefore, $\begin{bmatrix} t & 1 \\ t^2 & 3t \end{bmatrix}\begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \begin{bmatrix} 0 \\ -6 \end{bmatrix} \Rightarrow \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = \dfrac{1}{2t^2}\begin{bmatrix} 3t & -1 \\ -t^2 & t \end{bmatrix}\begin{bmatrix} 0 \\ -6 \end{bmatrix} = \begin{bmatrix} 3/t^2 \\ -3/t \end{bmatrix}$.

Both functions fail to be continuous at $t = 0$.

25 (a). Differentiation yields $y' = c_2 e^t - c_3 e^{-t}$ and $y'' = c_2 e^t + c_3 e^{-t}$. From the initial conditions, we have $y(0) = c_1 + c_2 + c_3 = \alpha,\ y'(0) = c_2 - c_3 = \beta$, and $y''(0) = c_2 + c_3 = 4$. Solving these simultaneous equations in terms of α and β yields $c_1 = \alpha - 4,\ c_2 = 2 + \frac{1}{2}\beta,$ and $c_3 = 2 - \frac{1}{2}\beta$. Then we have $y(t) = (\alpha - 4) + \left(2 + \frac{1}{2}\beta\right)e^t + \left(2 - \frac{1}{2}\beta\right)e^{-t}$. Since the third term goes to zero as *t* gets large, we must set $\alpha = 4$ and $\beta = -4$ so that the first two terms also become zero (for all *t*).

25 (b). *y* will be bounded on $0 \le t < \infty$ if $\beta = -4$ (α can be arbitrary). No choice of α and β will produce a solution that is bounded on $-\infty < t < \infty$ since $2 + \frac{1}{2}\beta$ and $2 - \frac{1}{2}\beta$ cannot simultaneously be zero.

Section 5.3

1. Antidifferentiation yields $y = c_1 + c_2 t + c_3 t^2$. Since $t_0 = 0$, we have $y_1 = 1,\ y_2 = t,$ and $y_3 = \dfrac{t^2}{2}$ from the initial conditions provided.

2. $y = c_1 + c_2 t + c_3 t^2,\ \ y' = c_2 + 2c_3 t,\ \ y'' = 2c_3$

$t_0 = 1{:}\ \ y_1{:}\ c_1 + c_2 + c_3 = 1,\ c_2 + 2c_3 = 0,\ 2c_3 = 0 \quad \therefore\ c_1 = 1,\ c_2 = c_3 = 0,\ y_1(t) = 1.$

$y_2{:}\ c_1 + c_2 + c_3 = 0,\ c_2 + 2c_3 = 1,\ 2c_3 = 0 \ \Rightarrow\ c_3 = 0,\ c_2 = -1,\ c_1 = -1,\ y_2(t) = t - 1,$

y_3: $c_1 + c_2 + c_3 = 0,\ c_2 + 2c_3 = 0,\ 2c_3 = 1 \ \Rightarrow\ c_3 = \frac{1}{2},\ c_2 = -1,\ c_1 = \frac{1}{2} \qquad y_3(t) = \frac{1}{2}(t-1)^2$

$y_1(t) = 1,\ y_2(t) = t-1,\ y_3(t) = \frac{1}{2}(t-1)^2.$

3. Since $t_0 = 0$, we have $y_1(0) = c_1 + c_3 = 1,\ y_1'(0) = c_2 - c_3 = 0,$ and $y_1'' = c_3 = 0$ from the initial conditions provided. Thus $c_1 = 1$ and $c_2 = c_3 = 0$, and $y_1(t) = 1$. Similarly, we have $y_2(0) = c_1 + c_3 = 0,\ y_2'(0) = c_2 - c_3 = 1,$ and $y_2'' = c_3 = 0$ from the initial conditions provided. Thus $c_2 = 1$ and $c_1 = c_3 = 0$, and $y_2(t) = t$. Finally, we have $y_3(0) = c_1 + c_3 = 0,\ y_3'(0) = c_2 - c_3 = 0,$ and $y_3'' = c_3 = 1$ from the initial conditions provided. Thus $c_1 = -1$ and $c_2 = c_3 = 1$, and $y_3(t) = -1 + t + e^{-t}$.

4. $y = c_1 + c_2 t + c_3 e^{-t},\ y' = c_2 - c_3 e^{-t},\ y'' = c_3 e^{-t}$

$t_0 = 1$: $c_1 + c_2 + c_3 e^{-1} = 1,\ c_2 - c_3 e^{-1} = 0,\ c_3 e^{-1} = 0 \ \Rightarrow\ c_1 = 1,\ c_2 = c_3 = 0,\ y_1(t) = 1$

y_2: $c_1 + c_2 + c_3 e^{-1} = 0,\ c_2 - c_3 e^{-1} = 1,\ c_3 e^{-1} = 0,\ c_3 = 0,\ c_2 = 1,\ c_1 = -1,\ y_2(t) = t-1$

y_3: $c_1 + c_2 + c_3 e^{-1} = 0,\ c_2 - c_3 e^{-1} = 0,\ c_3 e^{-1} = 1 \ \Rightarrow\ c_3 = e,\ c_2 = 1,\ c_1 = -2,\ y_3(t) = -2 + t + e^{-(t-1)}$

$y_1(t) = 1,\ y_2(t) = t-1,\ y_3(t) = -2 + t + e^{-(t-1)}.$

5 (a). $\{\cosh t,\ 1 - \sinh t,\ 2 + \sinh t\}$ is a solution set.

5 (b). $\cosh t = 0 \cdot 1 + \frac{1}{2}e^{t} + \frac{1}{2}e^{-t}$, $1 - \sinh t = 1 - \frac{1}{2}e^{t} + \frac{1}{2}e^{-t}$, and $2 + \sinh t = 2 + \frac{1}{2}e^{t} - \frac{1}{2}e^{-t}$. Thus

$$A = \begin{bmatrix} 0 & 1 & 2 \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \end{bmatrix}.$$

5 (c). $\det A = 0\begin{vmatrix} -\frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} - 1\begin{vmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{vmatrix} + 2\begin{vmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{vmatrix} = \frac{3}{2} \neq 0$, so the three functions form a fundamental set.

6 (a). $\{1 - 2t,\ t + 2,\ e^{-(t+2)}\}$ is a solution set.

6 (b). $1 - 2t = 1 \cdot 1 - 2 \cdot t + 0 \cdot e^{-t},\ t + 2 = 2 \cdot 1 + 1 \cdot t + 0 \cdot e^{-t},\ e^{-(t+2)} = 0 \cdot 1 + 0 \cdot t + e^{-2} \cdot e^{-t}$

$$A = \begin{bmatrix} 1 & 2 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & e^{-2} \end{bmatrix}$$

6 (c). $\det A = e^{-2}(5) = 5e^{-2} \neq 0 \quad \therefore$ fundamental set.

7 (a). $\bar{y}_1 = 1+t$ and $\bar{y}_2 = \dfrac{t+1}{t}$ are solutions. However, $\bar{y}_3 = (t+1)^{-1}$ is not a solution and so

$\left\{1+t,\ \dfrac{t+1}{t},\ (t+1)^{-1}\right\}$ is not a solution set.

8 (a). $\{2t^2-1,\ 3,\ \ln(t^3)\}$ is a solution set; $\ln(t^3) = 3\ln t$.

8 (b). $2t^2-1 = -1\cdot 1 + 0\cdot \ln t + 2\cdot t^2,\ \ 3 = 3\cdot 1 + 0\cdot \ln t + 0\cdot t^2,\ \ 3\ln t = 0\cdot 1 + 3\cdot \ln t + 0\cdot t^2$

$$A = \begin{bmatrix} -1 & 3 & 0 \\ 0 & 0 & 3 \\ 2 & 0 & 0 \end{bmatrix}$$

8 (c). $\det A = -3\begin{bmatrix} -1 & 3 \\ 2 & 0 \end{bmatrix} = 18 \neq 0 \quad \therefore$ fundamental set.

9. Setting $c_1\cdot 1 + c_2 t + c_3 t^2 = 0$ and evaluating at $t = -1,\ 0,\ 1$ we have $c_1 - c_2 + c_3 = 0,\ c_1 = 0,$ and $c_1 + c_2 + c_3 = 0$. Thus $c_1 = c_2 = c_3 = 0$, and the three functions are linearly independent on the interval.

10. $c_1\cdot 1 + c_2\cdot(1+t) + c_3(1+t+t^2) = 0 \quad \therefore \ (c_1+c_2+c_3)\cdot 1 + (c_2+c_3)\cdot t + c_3\cdot t^2 = 0$

The argument of 9 leads to $c_1+c_2+c_3 = 0,\ c_2+c_3 = 0,\ c_3 = 0 \ \Rightarrow\ c_1 = c_2 = c_3 = 0$

$\therefore$ linearly independent on $-\infty < t < \infty$.

11. Setting $c_1\cos^2 t + c_2 2\cos 2t + c_3 2\sin^2 t = 0$ and using the identity $\cos^2 t - \sin^2 t = \cos 2t$, we have $c_1\cos^2 t + 2c_2(\cos^2 t - \sin^2 t) + 2c_3\sin^2 t = (c_1+2c_2)\cos^2 t + (-2c_2+2c_3)\sin^2 t = 0$. Taking $c_3 = 1,\ c_2 = 1,$ and $c_1 = -2$ to be one nontrivial solution, we can conclude that the three functions are linearly dependent on the interval.

12. $c_1(t^2+2t) + c_2(\alpha t+1) + c_3(t+\alpha) = c_1 t^2 + (2c_1 + \alpha c_2 + c_3)t + (c_2 + \alpha c_3) = 0$

$\therefore\ c_1 = 0,\ \alpha c_2 + c_3 = 0,\ c_2 + \alpha c_3 = 0$ or $\begin{bmatrix} \alpha & 1 \\ 1 & \alpha \end{bmatrix}\begin{bmatrix} c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. $\det = \alpha^2 - 1$

$\therefore$ linearly dependent on $-\infty < t < \infty$ if $\alpha = \pm 1$ and linearly independent on $-\infty < t < \infty$ otherwise. (nontrivial $c_2,\ c_3$ if $\alpha = \pm 1$).

13. On $0 \le t < \infty,\ t|t| + 1 = t^2 + 1$. Then we have $c_1(t^2+1) + c_2(t^2-1) + c_3 t = (c_1+c_2)t^2 + c_3 t + (c_1 - c_2) = 0$. Thus $c_1 = c_2 = c_3 = 0$, and so the three functions are linearly independent on the interval.

14. On $-\infty < t \le 0$, $t|t| + 1 = -t^2 + 1 \Rightarrow 1(t|t|+1) + 1(t^2-1) + 0(t) = 0$, and so the three functions are linearly dependent on the interval.

15. Since the three functions are linearly independent on half of the interval (see 13), the functions are linearly independent on the entire interval.

Section 5.4

1 (a). $\lambda^3 - 4\lambda = 0$

1 (b). $\lambda^3 - 4\lambda = \lambda(\lambda+2)(\lambda-2) = 0$. Thus $\lambda = 0, \pm 2$.

1 (c). $y = c_1 + c_2 e^{-2t} + c_3 e^{2t}$, since the roots are distinct.

2 (a). $\lambda^3 + \lambda^2 - \lambda - 1$

2 (b). $\lambda^2(\lambda+1) - (\lambda+1) = (\lambda^2-1)(\lambda+1) = 0 \qquad \lambda = -1, -1, 1$

2 (c). $y = c_1 e^{-t} + c_2 te^{-t} + c_3 e^t; \qquad W = \begin{vmatrix} e^{-t} & te^{-t} & e^t \\ -e^{-t} & (1-t)e^{-t} & e^t \\ e^{-t} & (-2+t)e^{-t} & e^t \end{vmatrix} =$

$$e^{-t}[3-2t] - te^{-t}[-2] + e^t[2-t-1+t]e^{-2t} = e^{-t}[3-2t+2t+1] = 4e^{-t} \neq 0$$

3 (a). $\lambda^3 + \lambda^2 + 4\lambda + 4 = 0$

3 (b). $\lambda^3 + \lambda^2 + 4\lambda + 4 = \lambda^2(\lambda+1) + 4(\lambda+1) = (\lambda^2+4)(\lambda+1) = 0$. Thus $\lambda = -1, \pm i2$.

3 (c). $y = c_1 e^{-t} + c_2 \cos 2t + c_3 \sin 2t$, since $W = \begin{vmatrix} e^{-t} & \cos 2t & \sin 2t \\ -e^{-t} & -2\sin 2t & 2\cos 2t \\ e^{-t} & -4\cos 2t & -4\sin 2t \end{vmatrix} = 10e^{-t} \neq 0$.

4 (a). $16\lambda^4 - 8\lambda^2 + 1$

4 (b). $(4\lambda^2 - 1)^2 = 0; \qquad \lambda = -\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{1}{2}$

4 (c). $y = c_1 e^{-t/2} + c_2 te^{-t/2} + c_3 e^{t/2} + c_4 te^{t/2} \qquad W = \begin{vmatrix} e^{-t/2} & te^{-t/2} & e^{t/2} & te^{t/2} \\ -\frac{1}{2}e^{-t/2} & \left(1-\frac{t}{2}\right)e^{-t/2} & \frac{1}{2}e^{t/2} & \left(1+\frac{t}{2}\right)e^{t/2} \\ \frac{1}{4}e^{-t/2} & \left(-1+\frac{t}{4}\right)e^{-t/2} & \frac{1}{4}e^{t/2} & \left(1+\frac{t}{4}\right)e^{t/2} \\ -\frac{1}{8}e^{-t/2} & \left(\frac{3}{4}-\frac{t}{8}\right)e^{-t/2} & \frac{1}{8}e^{t/2} & \left(\frac{3}{4}+\frac{t}{8}\right)e^{t/2} \end{vmatrix}$

$$W(0) = \begin{vmatrix} 1 & 0 & 1 & 0 \\ -\frac{1}{2} & 1 & \frac{1}{2} & 1 \\ \frac{1}{4} & -1 & \frac{1}{4} & 1 \\ -\frac{1}{8} & \frac{3}{4} & \frac{1}{8} & \frac{3}{4} \end{vmatrix} \to \begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & -1 & 0 & 1 \\ 0 & \frac{3}{4} & 0 & \frac{3}{4} \end{vmatrix} \to \begin{vmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & -\frac{3}{4} & 0 \end{vmatrix} = 1 \cdot 1\left(\frac{3}{2}\right) \neq 0.$$

5 (a). $16\lambda^4 + 8\lambda^2 + 1 = 0$

5 (b). $16\lambda^4 + 8\lambda^2 + 1 = (4\lambda^2+1)^2 = 0$. Thus $\lambda = \pm\frac{i}{2}, \pm\frac{i}{2}$.

5 (c). $y = c_1\cos\frac{t}{2} + c_2 t\cos\frac{t}{2} + c_3\sin\frac{t}{2} + c_4 t\sin\frac{t}{2}$. To verify this,

$$W = \begin{vmatrix} \cos\left(\frac{t}{2}\right) & t\cos\left(\frac{t}{2}\right) & \sin\left(\frac{t}{2}\right) & t\sin\left(\frac{t}{2}\right) \\ -\frac{1}{2}\sin\left(\frac{t}{2}\right) & \cos\left(\frac{t}{2}\right) - \frac{t}{2}\sin\left(\frac{t}{2}\right) & \frac{1}{2}\cos\left(\frac{t}{2}\right) & \sin\left(\frac{t}{2}\right) + \frac{t}{2}\cos\left(\frac{t}{2}\right) \\ -\frac{1}{4}\cos\left(\frac{t}{2}\right) & -\sin\left(\frac{t}{2}\right) - \frac{t}{4}\cos\left(\frac{t}{2}\right) & -\frac{1}{4}\sin\left(\frac{t}{2}\right) & \cos\left(\frac{t}{2}\right) - \frac{t}{4}\sin\left(\frac{t}{2}\right) \\ \frac{1}{8}\sin\left(\frac{t}{2}\right) & -\frac{3}{4}\cos\left(\frac{t}{2}\right) + \frac{t}{8}\sin\left(\frac{t}{2}\right) & -\frac{1}{8}\cos\left(\frac{t}{2}\right) & -\frac{3}{4}\sin\left(\frac{t}{2}\right) - \frac{t}{8}\cos\left(\frac{t}{2}\right) \end{vmatrix}.$$

$$W(0) = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & \frac{1}{2} & 0 \\ -\frac{1}{4} & 0 & 0 & 1 \\ 0 & -\frac{3}{4} & -\frac{1}{8} & 0 \end{vmatrix} = -\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & \frac{1}{2} \\ 0 & -\frac{3}{4} & -\frac{1}{8} \end{vmatrix} = -\frac{1}{4} \neq 0.$$

6 (a). $\lambda^3 - 1$

6 (b). $\lambda^3 = e^{i2k\pi} \Rightarrow \lambda_k = e^{i2k\pi/3},\ k = 0,\ 1,\ 2;\quad \lambda = 1,\ e^{i2\pi/3},\ e^{i4\pi/3} = 1, -\frac{1}{2} + i\frac{\sqrt{3}}{2}, -\frac{1}{2} - i\frac{\sqrt{3}}{2}$

6 (c). $y = c_1 e^t + c_2 e^{-t/2}\cos\left(\frac{\sqrt{3}}{2}t\right) + c_3 e^{-t/2}\sin\left(\frac{\sqrt{3}}{2}t\right)$

7 (a). $\lambda^3 - 2\lambda^2 - \lambda + 2 = 0$

7 (b). $\lambda^3 - 2\lambda^2 - \lambda + 2 = \lambda^2(\lambda - 2) - (\lambda - 2) = (\lambda + 1)(\lambda - 1)(\lambda - 2)$. Thus $\lambda = 2, \pm 1$.

7 (c). $y = c_1 e^{-t} + c_2 e^t + c_3 e^{2t}$, since the roots are distinct.

8 (a). $\lambda^4 + 16$

8 (b). $\lambda^4 = -16 = 16e^{i(\pi + 2k\pi)} \Rightarrow \lambda_k = 2e^{i\left(\frac{\pi}{4} + k\frac{\pi}{2}\right)},\ k = 0,\ 1,\ 2,\ 3;\quad \lambda_1 = 2e^{i\frac{\pi}{4}} = \sqrt{2} + i\sqrt{2},\ \lambda_2 =$ $2e^{i\frac{3\pi}{4}} = -\sqrt{2} + i\sqrt{2},\ \lambda_3 = 2e^{i\frac{5\pi}{4}} = -\sqrt{2} - i\sqrt{2},\ \lambda_4 = 2e^{i\frac{7\pi}{4}} = \sqrt{2} - i\sqrt{2}\quad \therefore\ \lambda = \pm\sqrt{2} \pm i\sqrt{2}$

$y = c_1 e^{-\sqrt{2}t}\cos\left(\sqrt{2}t\right) + c_2 e^{-\sqrt{2}t}\sin\left(\sqrt{2}t\right) + c_3 e^{\sqrt{2}t}\cos\left(\sqrt{2}t\right) + c_4 e^{\sqrt{2}t}\sin\left(\sqrt{2}t\right)$.

9 (a). $\lambda^3 + 4\lambda = 0$

9 (b). $\lambda^3 + 4\lambda = \lambda\left(\lambda^2 + 4\right) = 0$. Thus $\lambda = 0, \pm i2$.

9 (c). $y = c_1 + c_2\cos 2t + c_3\sin 2t$

9 (d). Differentiation gives us $y' = -2c_2\sin 2t + 2c_3\cos 2t$ and $y'' = -4c_2\cos 2t - 4c_3\sin 2t$. From the initial conditions, we have $y(0) = c_1 + c_2 = 1$, $y'(0) = 2c_3 = 6$, and $y''(0) = -4c_2 = 4$. Thus $c_1 = 2$, $c_2 = -1$, and $c_3 = 3$, and so $y = 2 - \cos 2t + 3\sin 2t$

10 (a). $\lambda^3 + 3\lambda^2 + 3\lambda + 1 = (\lambda + 1)^3$

10 (b). $\lambda = -1,\ -1,\ -1$

10 (c). $y = c_1 e^{-t} + c_2 t e^{-t} + c_3 t^2 e^{-t}$

10 (d). $y(0) = 0,\ y'(0) = 1,\ y''(0) = 0$

$y' = -c_1 e^{-t} + c_2 (1-t) e^{-t} + c_3 \left(2t - t^2\right) e^{-t},\ y'' = c_1 e^{-t} + c_2(-2+t) e^{-t} + c_3\left(t^2 - 4t + 2\right) e^{-t}$

$y(0) = c_1 = 0,\ y'(0) = c_2 = 1,\ y''(0) = c_2(-2) + c_3(2) = 0 \ \Rightarrow\ c_3 = c_2 = 1 \qquad y = \left(t + t^2\right) e^{-t}$

11. $\lambda^2\left(\lambda^2 + 9\right) = \lambda^4 + 9\lambda^2 = 0$. Thus the differential equation is $y^{(4)} + 9y'' = 0$, and so $a_3 = 0,\ a_2 = 9,\ a_1 = 0$, and $a_0 = 0$.

12. $y = c_1 \cos t + c_2 \sin t + c_3 \cos 2t + c_4 \sin 2t; \quad \left(\lambda^2 + 1\right)\left(\lambda^2 + 4\right) = \lambda^4 + 5\lambda^2 + 4 = 0$

$y^{(4)} + 5y'' + 4y = 0; \quad a_3 = 0,\ a_2 = 5,\ a_1 = 0,\ a_0 = 4$.

13. $(\lambda - 1)^2 (\lambda + 1)^2 = \left(\lambda^2 - 1\right)^2 = \lambda^4 - 2\lambda^2 + 1 = 0$. Thus the differential equation is $y^{(4)} - 2y'' + y = 0$, an so $a_3 = 0,\ a_2 = -2,\ a_1 = 0$, and $a_0 = 1$.

14.

$y = c_1 e^{-t} \sin t + c_2 e^{-t} \cos t + c_3 e^{t} \sin t + c_4 e^{t} \cos t; \quad \lambda = -1 \pm i1,\ 1 + i1$.

$(\lambda + 1 - i1)(\lambda + 1 + i1) = (\lambda + 1)^2 + 1 = \lambda^2 + 2\lambda + 2; \quad (\lambda - 1 - i1)(\lambda - 1 + i1) = (\lambda - 1)^2 + 1 = \lambda^2 - 2\lambda + 2$

$\left(\lambda^2 + 2\lambda + 2\right)\left(\lambda^2 - 2\lambda + 2\right) = \left(\lambda^2 + 2\right)^2 - 4\lambda^2 = \lambda^4 + 4\lambda^2 + 4 - 4\lambda^2 = \lambda^4 + 4 = 0$

$y^{(4)} + 4y = 0; \quad a_3 = 0,\ a_2 = 0,\ a_1 = 0,\ a_0 = 4$.

15. $(\lambda - 1)^4 = \lambda^4 - 4\lambda^3 + 6\lambda^2 - 4\lambda + 1 = 0$. Thus the differential equation is $y^{(4)} - 4y''' + 6y'' - 4y' + y = 0$, and so $a_3 = -4,\ a_2 = 6,\ a_1 = -4$, and $a_0 = 1$.

16 (a). $n = 5$

16 (b). $\{1,\ t,\ e^t,\ \cos t,\ \sin t\}$

17 (a). $n = 5$

17 (b). $\{e^t,\ e^t \cos 2t,\ e^t \sin 2t,\ e^{-t} \cos 2t,\ e^{-t} \sin 2t\}$

18 (a). $n = 8$

18 (b). $\{\sin t,\ \cos t,\ t\sin t,\ t\cos t,\ t^2 \sin t,\ t^2 \cos t,\ e^t \sin t,\ e^t \cos t\}$

19 (a). $n = 7$

19 (b). $\{\sin t,\ \cos t,\ t\sin t,\ t\cos t,\ e^t,\ te^t,\ t^2 e^t\}$

20 (a). $n = 4$

20 (b). $\{1,\ t,\ t^2, e^{2t}\}$

21. $n = 1,\ a = 1,\ y(t) = Ce^{-t}$

22. $n = 1,\ a = \pm 2;\quad y' \pm 2y = 0,\ y(t) = Ce^{\mp t}$

23. $n = 4,\ a = 0,\ y(t) = c_1 + c_2 t + c_3 t^2 + c_4 t^3$

24. $n = 2,\ a = 4;\quad y'' + 4y = 0,\ y = c_1 \cos 2t + c_2 \sin 2t.$

25. Three values for λ must be 3 and $-\frac{3}{2} \pm i\frac{3\sqrt{3}}{2}$. Using these values to reach the characteristic equation gives us $(\lambda - 3)(\lambda^2 + 3\lambda + 9) = \lambda^3 + 3\lambda^2 + 9\lambda - 3\lambda^2 - 9\lambda - 27 = \lambda^3 - 27$. Thus $n = 3$ and $a = -27$.

Section 5.5

1 (a). $\lambda^3 - \lambda = \lambda(\lambda + 1)(\lambda - 1) = 0$. Thus $y_C = c_1 e^{-t} + c_2 + c_3 e^t$.

1 (b). $y_P = Ae^{2t}$, and substituting this into the original differential equation yields $(8A - 2A)e^{2t} = e^{2t}$. Thus $A = \frac{1}{6}$ and so $y_P = \frac{1}{6}e^{2t}$.

1 (c). $y = c_1 e^{-t} + c_2 + c_3 e^t + \frac{1}{6}e^{2t}$

2 (a). $y_c = c_1 e^{-t} + c_2 + c_3 e^t$

2 (b). $y_p = At + B\cos 2t + C\sin 2t,\ y_p{}' = A - 2B\sin 2t + 2C\cos 2t,\ y'' =$

$-4B\cos 2t - 4C\sin 2t,\ y_p{}''' = 8B\sin 2t - 8C\cos 2t$

$\therefore\ 8B\sin 2t - 8C\cos 2t - A + 2B\sin 2t - 2C\cos 2t = 4 + 2\cos 2t$

$10B = 0,\ -10C = 2,\ -A = 4\quad \therefore\ A = -4,\ B = 0,\ C = -\frac{1}{5};\qquad y_p = -4t - \frac{1}{5}\sin 2t$

2 (c). $y = c_1 e^{-t} + c_2 + c_3 e^t - 4t - \frac{1}{5}\sin 2t$

3 (a). $\lambda^3 - \lambda = \lambda(\lambda + 1)(\lambda - 1) = 0$. Thus $y_C = c_1 e^{-t} + c_2 + c_3 e^t$.

3 (b). $y_P = t(At + B) = At^2 + Bt$. Differentiation yields $y_P' = 2At + B,\ y_P'' = 2A,$ and $y_P''' = 0$, and substituting into the original differential equation yields $0 - 2At - B = 4t$. Thus $A = -2$ and $B = 0$ and so $y_P = -2t^2$.

3 (c). $y = c_1 e^{-t} + c_2 + c_3 e^t - 2t^2$

4 (a). $y_c = c_1 e^{-t} + c_2 + c_3 e^t$

4 (b). $y_p = Ate^t,\ y_p' = A(t+1)e^t,\ y_p'' = A(t+2)e^t,\ y_p''' = A(t+3)e^t;$

$y_p''' - y_p' = A[t + 3 - t - 1]e^t = -4e^t\ \Rightarrow\ 2A = -4\ \text{or}\ A = -2\quad \therefore\ y_p = -2te^t$

4 (c). $y = c_1e^{-t} + c_2 + c_3e^t - 2te^t$

5 (a). $\lambda^3 + \lambda^2 = \lambda^2(\lambda+1) = 0$. Thus $y_C = c_1 + c_2t + c_3e^{-t}$.

5 (b). $y_P = Ate^{-t}$. Differentiation yields $y_P' = A(1-t)e^{-t}$, $y_P'' = A(t-2)e^{-t}$, and $y_P''' = A(-t+3)e^{-t}$, and substituting into the original differential equation yields $A(-t+3+t-2)e^{-t} = 6e^{-t}$. Thus $A = 6$ and so $y_P = 6te^{-t}$.

5 (c). $y = c_1 + c_2t + c_3e^{-t} + 6te^{-t}$

6 (a). $\lambda^3 - \lambda^2 = 0,\ \lambda = 0,\ 0,\ 1;\qquad y_c = c_1 + c_2t + c_3e^t$

6 (b). $y_p = Ae^{-2t},\ y_p' = -2Ae^{-2t},\ y_p'' = 4Ae^{-2t},\ y_p''' = -8Ae^{-2t}\quad \therefore\ (-8A - 4A)e^{-2t} = 4e^{-2t}$

$A = -\frac{1}{3},\ y_p = -\frac{1}{3}e^{-2t}$

6 (c). $y = c_1 + c_2t + c_3e^t - \frac{1}{3}e^{-2t}$

7 (a). $\lambda^3 - 2\lambda^2 + \lambda = \lambda(\lambda-1)^2 = 0$. Thus $y_C = c_1 + c_2e^t + c_3te^t$.

7 (b). $y_P = t(At+B) + Ct^2e^t = At^2 + Bt + Ct^2e^t$. Differentiation yields

$y_P' = 2At + B + C(t^2+2t)e^t$, $y_P'' = 2A + C(t^2+4t+2)e^t$, and $y_P''' = C(t^2+6t+6)e^t$, and substituting into the original differential equation yields

$C(t^2+6t+6)e^t - 2\left[2A + C(t^2+4t+2)e^t\right] + 2At + B + C(t^2+2t)e^t = t + 4e^t$. Thus

$A = \frac{1}{2}$, $B = 2$, and $C = 2$ and so $y_P = \frac{1}{2}t^2 + 2t + 2t^2e^t$.

7 (c). $y = c_1 + c_2e^t + c_3te^t + \frac{1}{2}t^2 + 2t + 2t^2e^t$

8 (a). $\lambda^3 - 3\lambda^2 + 3\lambda - 1 = (\lambda-1)^3 = 0;\qquad y_c = c_1e^t + c_2te^t + c_3t^2e^t$

8 (b). $y_p = At^3e^t,\ y_p' = A(t^3+3t^2)e^t,\ y_p'' = A(t^3+6t^2+6t)e^t,\ y_p''' = A(t^3+9t^2+18t+6)e^t$

$\therefore\ A\left[t^3+9t^2+18t+6-3(t^3+6t^2+6t)+3(t^3+3t^2)-t^3\right]e^t = 12e^t\ \Rightarrow\ A\cdot 6 = 12;\qquad y_p = 2t^3e^t$

8 (c). $y = c_1e^t + c_2te^t + c_3t^2e^t + 2t^3e^t$

9 (a). $\lambda^3 - 1 = 0$. Thus $\lambda = 1,\ -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$, and so $y_C = c_1e^t + c_2e^{-\frac{t}{2}}\cos\left(\frac{\sqrt{3}}{2}t\right) + c_3e^{-\frac{t}{2}}\sin\left(\frac{\sqrt{3}}{2}t\right)$.

9 (b). $y_P = Ate^t$. Differentiation yields $y_P' = A(t+1)e^t$, $y_P'' = A(t+2)e^t$, and $y_P''' = A(t+3)e^t$, and substituting into the original differential equation yields $A(t+3-t)e^t = e^t$. Thus $A = \frac{1}{3}$ and so $y_P = \frac{1}{3}te^t$.

9 (c). $y_C = c_1 e^t + c_2 e^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}}{2} t\right) + c_3 e^{-\frac{t}{2}} \sin\left(\frac{\sqrt{3}}{2} t\right) + \frac{1}{3} t e^t$

10 (a). $\lambda^3 + 1 = 0,\ \lambda^3 = -1 = e^{i(\pi + 2k\pi)},\ \lambda_k = e^{i(\pi+2k\pi)/3},\ \lambda_1 = e^{i\pi/3} = \frac{1}{2} + i\frac{\sqrt{3}}{2},$

$\lambda_2 = -1,\ \lambda_3 = \frac{1}{2} - i\frac{\sqrt{3}}{2};\qquad y_c = c_1 e^{-t} + c_2 e^{t/2} \cos\left(\frac{\sqrt{3}}{2} t\right) + c_3 e^{t/2} \sin\left(\frac{\sqrt{3}}{2} t\right)$

10 (b). $y_p = Ae^t + B\cos t + C\sin t,\ y_p' = Ae^t - B\sin t + C\cos t,\ y_p'' = Ae^t - B\cos t - C\sin t,$

$y_p''' = Ae^t + B\sin t - C\cos t \quad \therefore \quad Ae^t + B\sin t - C\cos t + Ae^t + B\cos t + C\sin t =$

$= e^t + \cos t \quad \therefore \quad 2A = 1,\ B + C = 0,\ -C + B = 1,\ A = \frac{1}{2},\ B = \frac{1}{2},\ C = -\frac{1}{2},$

$y_p = \frac{1}{2} e^t + \frac{1}{2}\cos t - \frac{1}{2}\sin t$

10 (c). $y = c_1 e^{-t} + c_2 e^{t/2} \cos\left(\frac{\sqrt{3}}{2} t\right) + c_3 e^{t/2} \sin\left(\frac{\sqrt{3}}{2} t\right) + \frac{1}{2} e^t + \frac{1}{2}\cos t - \frac{1}{2}\sin t$

11 (a). $\lambda^3 - 4\lambda^2 + 4\lambda = \lambda(\lambda - 2)^2 = 0$. Thus $y_C = c_1 + c_2 e^{2t} + c_3 t e^{2t}$.

11 (b). $y_P = t\left(A_3 t^3 + A_2 t^2 + A_1 t + A_0\right) + t^2\left(B_2 t^2 + B_1 t + B_0\right)e^{2t}$

$= A_3 t^4 + A_2 t^3 + A_1 t^2 + A_0 t + \left(B_2 t^4 + B_1 t^3 + B_0 t^2\right)e^{2t}$

12 (a). $\lambda^3 - 3\lambda^2 + 3\lambda - 1 = (\lambda - 1)^3 = 0 \qquad y_c = c_1 e^t + c_2 t e^t + c_3 t^2 e^t$

12 (b). $y_p = At^3 e^t + Be^t \cos 3t + Ce^t \sin 3t + D$

13(a). $\lambda^4 - 16 = (\lambda^2 + 4)(\lambda - 2)(\lambda + 2) = 0,\ \lambda = \pm 2,\ \pm i2$, and so

$y_C = c_1 e^{-2t} + c_2 e^{2t} + c_3 \cos 2t + c_4 \sin 2t$.

13 (b). $y_P = t\left(A_1 t + A_0\right)e^{2t} + t\left(B_1 t + B_0\right)e^{-2t} + t\left(C_1 t + C_0\right)\cos 2t + t\left(D_1 t + D_0\right)\sin 2t$

14 (a). $\lambda^4 + 8\lambda^2 + 16 = \left(\lambda^2 + 4\right)^2 = 0 \quad \therefore \quad \lambda = \pm i2,\ \pm i2$

$y_c = c_1 \cos 2t + c_2 \sin 2t + c_3 t\cos 2t + c_4 t\sin 2t$

14 (b). $y_p = t^2\left(A_1 t + A_0\right)\cos 2t + t^2\left(B_1 t + B_0\right)\sin 2t$

15 (a). $\lambda^4 - 1 = \left(\lambda^2 + 1\right)\left(\lambda + 1\right)\left(\lambda - 1\right) = 0$. Thus $y_C = c_1 e^{-t} + c_2 e^t + c_3 \cos t + c_4 \sin t$.

15 (b). $y_P = t\left(A_1 t + A_0\right)e^{-t} + t\left(C_1 t + C_0\right)\cos t + t\left(D_1 t + D_0\right)\sin t$.

16. $y = c_1 + c_2 t + c_3 e^{2t} + 4\sin 2t;\qquad \lambda^2(\lambda - 2) = \lambda^3 - 2\lambda^2 = 0;\qquad y''' - 2y'' = g(t)$

$\therefore\ a = -2,\ b = 0,\ c = 0.\qquad y_p = 4\sin 2t,\ y_p' = 8\cos 2t,\ y_p'' = -16\sin 2t,\ y_p''' =$

$-32\cos 2t$.

Substitute: $-32\cos 2t - 2(-16\sin 2t) = g(t)$

$\therefore\ a = -2,\ b = 0,\ c = 0,\ g(t) = -32(\cos 2t - \sin 2t)$.

17. $(\lambda^2+4)(\lambda-1) = \lambda^3 - \lambda^2 + 4\lambda - 4 = 0$. Therefore, $g(t) = y''' - y'' + 4y' - 4y$. $y_P = t^2$, and differentiation yields $y_P' = 2t,\ y_P'' = 2,$ and $y_P''' = 0$. Then we have $g(t) = 0 - 2 + 4(2t) - 4t^2 = -4t^2 + 8t - 2$ and $a = -1,\ b = 4,\ c = -4$.

18. $y = c_1 + c_2 t + c_3 t^2 - 2t^3;\quad \lambda^3 = 0,\ y''' = g(t);\quad y_p = -2t^3,\ y_p' = -6t^2,$
$y_p'' = -12t,\ y_p''' = -12;\quad a = b = c = 0,\ g(t) = -12$

19. $y = c_1 + c_2 t + c_3 t^3 + t^4$, and so $1,\ t,\ t^3$ are solutions of the homogeneous equation.
$0 + 0 + 0 + c\cdot 1 = 0$, so c $= 0$. $0 + 0 + bt\cdot 1 = 0$, so $b = 0$. $t^3\cdot 6 + at^2(6t) = 0$, so $a = -1$.
$t^3 y''' - t^2 y'' = g(t)$ and $y_P = t^4$, so $g(t) = t^3\cdot 24t - t^2\cdot 12t^2 = 12t^4$.

20. $y = c_1 t + c_2 t^2 + c_3 t^4 + 2\ln t;\quad t,\ t^2,\ t^4$ are solutions of homogeneous equation.

$0 + 0 + bt + ct = 0\quad \therefore\ b + c = 0,\ 0 + at^2(2) + bt(2t) + c(t^2) = 0\quad \therefore\ 2a + 2b + c = 0$

$(t^4)' = 4t^3,\ (t^4)'' = 12t^2,\ (t^4)''' = 24t.\quad t^3(24t) + at^2(12t^2) + bt(4t^3) + c(t^4) = 0$

$24 + 12a + 4b + c = 0\quad \therefore\ c = -b\ \Rightarrow\ 2a + b = 0$ and $24 + 12a + 3b = 0$

$\therefore\ b = -2a\ \Rightarrow\ 24 + 12a - 6a = 0\ \Rightarrow\ a = -4,\ b = 8,\ c = -8$

$t^3 y''' - 4t^2 y'' + 8ty' - 8y = g(t)$

$(2\ln t)' = \dfrac{2}{t},\ (2\ln t)'' = -\dfrac{2}{t^2},\ (2\ln t)''' = \dfrac{4}{t^3}$

$t^3\left(\dfrac{4}{t^3}\right) - 4t^2\left(-\dfrac{2}{t^2}\right) + 8t\left(\dfrac{2}{t}\right) - 16\ln t = g\ \Rightarrow\ g = 28 - 16\ln t$

$a = -4,\ b = 8,\ c = -8,\ g(t) = 28 - 16\ln(t),\ t > 0$.

21 (a). The three solutions can be verified by substitution.

21 (b). $\begin{bmatrix} t & t^2 & t^4 \\ 1 & 2t & 4t^3 \\ 0 & 2 & 12t^2 \end{bmatrix}\begin{bmatrix} u_1' \\ u_2' \\ u_3' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ t^{-3}g \end{bmatrix}$ where $y_P = tu_1 + t^2u_2 + t^4u_3$. $\det = t[24t^3 - 8t^3] - 1[12t^4 - 2t^4] = 6t^4$.

Thus $u_1' = \begin{vmatrix} 0 & t^2 & t^4 \\ 0 & 2t & 4t^3 \\ t^{-3}g & 2 & 12t^2 \end{vmatrix}\cdot\dfrac{1}{6t^4} = \dfrac{1}{6}t^{-7}g[2t^5] = \dfrac{1}{3}t^{-2}g$,

$u_2' = \begin{vmatrix} t & 0 & t^4 \\ 1 & 0 & 4t^3 \\ 0 & t^{-3}g & 12t^2 \end{vmatrix}\cdot\dfrac{1}{6t^4} = -\dfrac{1}{6}t^{-7}g[3t^4] = -\dfrac{1}{2}t^{-3}g$, and $u_3' = \begin{vmatrix} t & t^2 & 0 \\ 1 & 2t & 0 \\ 0 & 2 & t^{-3}g \end{vmatrix} \cdot \dfrac{1}{6t^4} = \dfrac{1}{6}t^{-7}g\left[t^2\right] = \dfrac{1}{6}t^{-5}g$. $g = 2t^{\frac{1}{2}}$. Making this substitution,

antidifferentiating the three u' equations yields $u_1 = -\dfrac{4}{3}t^{-\frac{1}{2}}$, $u_2 = \dfrac{2}{3}t^{-\frac{3}{2}}$, and $u_3 = -\dfrac{2}{21}t^{-\frac{7}{2}}$. Thus

$y_P = -\dfrac{4}{3}t^{-\frac{1}{2}}(t) + \dfrac{2}{3}t^{-\frac{3}{2}}\left(t^2\right) - \dfrac{2}{21}t^{-\frac{7}{2}}\left(t^4\right) = -\dfrac{16}{21}t^{\frac{1}{2}}$. Finally, we have the general solution:

$y = c_1 t + c_2 t^2 + c_3 t^4 - \dfrac{16}{21}t^{\frac{1}{2}}$.

22. $g = 2t;\quad u_1' = \dfrac{2}{3}t^{-1} \;\Rightarrow\; u_1 = \dfrac{2}{3}\ln t;\quad u_2' = -t^{-2} \;\Rightarrow\; u_2 = t^{-1};$

$u_3' = \dfrac{1}{3}t^{-4} \;\Rightarrow\; u_3 = -\dfrac{1}{9}t^{-3};\quad y_p = \dfrac{2}{3}t\ln t + t - \dfrac{1}{9}t$

$y = c_1' t + c_2' t^2 + c_3' t^4 + \dfrac{2}{3}t\ln t + t - \dfrac{1}{9}t \;=\; c_1 t + c_2 t^2 + c_3 t^4 + \dfrac{2}{3}t\ln t$

23. $g = t^3 + 2t^2 + 1$, and so $u_1' = \dfrac{1}{3}t^{-2}\left(t^3 + 2t^2 + 1\right) = \dfrac{1}{3}t + \dfrac{2}{3} + \dfrac{1}{3}t^{-2}$, and antidifferentiation yields

$u_1 = \dfrac{t^2}{6} + \dfrac{2}{3}t - \dfrac{1}{3}t^{-1}$. $u_2' = -\dfrac{1}{2}t^{-3}\left(t^3 + 2t^2 + 1\right) = -\dfrac{1}{2} - t^{-1} - \dfrac{1}{2}t^{-3}$, and antidifferentiation yields

$u_2 = -\dfrac{t}{2} - \ln t + \dfrac{1}{4}t^{-2}$. $u_3' = \dfrac{1}{6}t^{-5}\left(t^3 + 2t^2 + 1\right) = \dfrac{1}{6}t^{-2} + \dfrac{1}{3}t^{-3} + \dfrac{1}{6}t^{-5}$, and antidifferentiation yields

$u_3 = -\dfrac{1}{6}t^{-1} - \dfrac{1}{6}t^{-2} - \dfrac{1}{24}t^{-4}$. Thus

$y_P = t\left(\dfrac{t^2}{6} + \dfrac{2}{3}t - \dfrac{1}{3}t^{-1}\right) + t^2\left(-\dfrac{t}{2} - \ln t + \dfrac{1}{4}t^{-2}\right) + t^4\left(-\dfrac{1}{6}t^{-1} - \dfrac{1}{6}t^{-2} - \dfrac{1}{24}t^{-4}\right)$

$= -\dfrac{1}{2}t^3 + \dfrac{1}{2}t^2 - \dfrac{1}{8} - t^2\ln t$. Finally, we have the general solution:

$y = c_1 t + c_2 t^2 + c_3 t^4 - \dfrac{1}{2}t^3 - \dfrac{1}{8} - t^2\ln t$.

Chapter 6
First Order Linear Systems

Section 6.1

1. $2A(t) - 3tB(t) = 2\begin{bmatrix} t-1 & t^2 \\ 2 & 2t+1 \end{bmatrix} - 3t\begin{bmatrix} t & -1 \\ 0 & t+2 \end{bmatrix} = \begin{bmatrix} 2t-2 & 2t^2 \\ 4 & 4t+2 \end{bmatrix} - \begin{bmatrix} 3t^2 & -3t \\ 0 & 3t^2+6t \end{bmatrix}$
$= \begin{bmatrix} 2t-2-3t^2 & 2t^2+3t \\ 4 & 2-2t-3t^2 \end{bmatrix}$

2. $A(t)B(t) - B(t)A(t) = \begin{bmatrix} 2 & 2t^2+t+2 \\ -4 & -2 \end{bmatrix}$

3. $A(t)\mathbf{c}(t) = \begin{bmatrix} t-1 & t^2 \\ 2 & 2t+1 \end{bmatrix}\begin{bmatrix} t+1 \\ -1 \end{bmatrix} = \begin{bmatrix} (t-1)(t+1)+t^2(-1) \\ 2(t+1)+(2t+1)(-1) \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

4. $\det[tA(t)] = -t^3 - t^2$

5. There are two natural ways to do this problem. We can form the matrix $A(t)B(t)$ and then calculate $\det[A(t)B(t)]$. Alternatively, we can separately calculate $\det[A(t)]$ and $\det[B(t)]$ and use the fact that $\det[A(t)B(t)] = \det[A(t)]\det[B(t)]$.
Taking the latter course, $\det[A(t)] = (t-1)(2t+1) - 2t^2 = -(t+1)$, and $\det[B(t)] = t(t+2) = t^2+2t$. Thus, $\det[A(t)B(t)] = -(t+1)(t^2+2t) = -(t^3+3t^2+2t)$.

6. $\det[A(t)] = 2t+1$ and so the matrix $A(t)$ is invertible for every value t except $t = -\frac{1}{2}$. The inverse of $A(t)$ is given by $A^{-1}(t) = \frac{1}{2t+1}\begin{bmatrix} t+1 & -t \\ -t & t+1 \end{bmatrix}$, $t \neq -\frac{1}{2}$.

7. As noted in Example 2, a square matrix is invertible if and only if its determinant is nonzero. Now, $\det[A(t)] = t(t-3) - 4 = t^2 - 3t - 4 = (t-4)(t+1)$ and so the matrix $A(t)$ is invertible for every value t except $t = 4$ and $t = -1$. The inverse of $A(t)$ is given by $A^{-1}(t) = \frac{1}{(t-4)(t+1)}\begin{bmatrix} t-3 & -2 \\ -2 & t \end{bmatrix}$, $t \neq 4, t \neq -1$.

8. $\det[A(t)] = 2\sin t\cos t = \sin 2t$ and so the matrix $A(t)$ is invertible for every value t except $2t = n\pi \Rightarrow t = \dfrac{n\pi}{2}$, $n = 0, \pm 1, \pm 2, \pm 3, \ldots$. The inverse of $A(t)$ is given by

$$A^{-1}(t) = \frac{1}{2\sin t\cos t}\begin{bmatrix} \cos t & \cos t \\ -\sin t & \sin t \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\csc t & \frac{1}{2}\csc t \\ -\frac{1}{2}\sec t & \frac{1}{2}\sec t \end{bmatrix}, \quad t \neq \frac{n\pi}{2},\ n = 0, \pm 1, \pm 2, \pm 3, \ldots$$

9. In this case, $\det[A(t)] = e^t e^{4t} - e^{3t}e^{2t} = e^{5t} - e^{5t}$ and so $\det[A(t)]$ is zero for every value of t. Hence, the given matrix $A(t)$ is never invertible.

10. $$\lim_{t\to 0} A(t) = \lim_{t\to 0}\begin{bmatrix} \frac{\sin t}{t} & t\cos t & \frac{3}{t+1} \\ e^{3t} & \sec t & \frac{2t}{t^2-1} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 3 \\ 1 & 1 & 0 \end{bmatrix}.$$

11. $$\lim_{t\to 0} A(t) = \lim_{t\to 0}\begin{bmatrix} te^{-t} & \tan t \\ t^2-2 & e^{\sin t} \end{bmatrix} = \begin{bmatrix} \lim_{t\to 0} te^{-t} & \lim_{t\to 0}\tan t \\ \lim_{t\to 0}[t^2-2] & \lim_{t\to 0} e^{\sin t} \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ -2 & 1 \end{bmatrix}.$$

12. Differentiating $A(t)$ component wise, we have $A'(t) = \begin{bmatrix} \cos t & 3 \\ 2t & 0 \end{bmatrix}$ and

$A''(t) = \begin{bmatrix} -\sin t & 0 \\ 2 & 0 \end{bmatrix}$. $A(t), A'(t)$ and $A''(t)$ are defined for $-\infty < t < \infty$.

13. Differentiating $A(t)$ component wise, we have $A'(t) = \begin{bmatrix} 0 & t^{-1} \\ -0.5(1-t)^{-1/2} & 3e^{3t} \end{bmatrix}$ and

$A''(t) = \begin{bmatrix} 0 & -t^{-2} \\ -0.25(1-t)^{-3/2} & 9e^{3t} \end{bmatrix}$. $A(t)$ is defined for $-\infty < t < 0$ and $0 < t \le 1$.

$A'(t)$ and $A''(t)$ are defined for $-\infty < t < 0$ and $0 < t < 1$.

14. $P(t) = \begin{bmatrix} t^2 & 3 \\ \sin t & t \end{bmatrix}$ and $\mathbf{g}(t) = \begin{bmatrix} \sec t \\ -5 \end{bmatrix}$.

15. $$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} t^{-1}y_1 + (t^2+1)y_2 + t \\ 4y_1 + t^{-1}y_2 + 8t\ln t \end{bmatrix} = \begin{bmatrix} t^{-1}y_1 + (t^2+1)y_2 \\ 4y_1 + t^{-1}y_2 \end{bmatrix} + \begin{bmatrix} t \\ 8t\ln t \end{bmatrix} =$$

$\begin{bmatrix} t^{-1} & t^2+1 \\ 4 & t^{-1} \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix} + \begin{bmatrix} t \\ 8t\ln t \end{bmatrix}$. Therefore, $P(t) = \begin{bmatrix} t^{-1} & t^2+1 \\ 4 & t^{-1} \end{bmatrix}$ and $\mathbf{g}(t) = \begin{bmatrix} t \\ 8t\ln t \end{bmatrix}$.

16. Let $A'(t) = \begin{bmatrix} 2t & 1 \\ \cos t & 3t^2 \end{bmatrix}$. Integrating component wise, we find

$$A(t) = \begin{bmatrix} t^2 + C_{11} & t + C_{12} \\ \sin t + C_{21} & t^3 + C_{22} \end{bmatrix}.$$

Since $A(0)=\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}=\begin{bmatrix} 2 & 5 \\ 1 & -2 \end{bmatrix}$, we obtain $A(t)=\begin{bmatrix} t^2+2 & t+5 \\ \sin t+1 & t^3-2 \end{bmatrix}$.

17. Let $A'(t)=\begin{bmatrix} t^{-1} & 4t \\ 5 & 3t^2 \end{bmatrix}$. Integrating component wise, we find $A(t)=\begin{bmatrix} \ln|t|+C_{11} & 2t^2+C_{12} \\ 5t+C_{21} & t^3+C_{22} \end{bmatrix}$. Since $A(1)=\begin{bmatrix} C_{11} & 2+C_{12} \\ 5+C_{21} & 1+C_{22} \end{bmatrix}=\begin{bmatrix} 2 & 5 \\ 1 & -2 \end{bmatrix}$, we obtain $A(t)=\begin{bmatrix} \ln|t|+2 & 2t^2+3 \\ 5t-4 & t^3-3 \end{bmatrix}$.

18. Let $A''(t)=\begin{bmatrix} 1 & t \\ 0 & 0 \end{bmatrix}$. Integrating component wise, we find

$$A'(t)=\begin{bmatrix} t+C_{11} & \dfrac{t^2}{2}+C_{12} \\ C_{21} & C_{22} \end{bmatrix} \Rightarrow A(t)=\begin{bmatrix} \dfrac{t^2}{2}+C_{11}t+D_{11} & \dfrac{t^3}{6}+C_{12}t+D_{12} \\ C_{21}t+D_{21} & C_{22}t+D_{22} \end{bmatrix}.$$

Since $A(0)=\begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}$ and $A(1)=\begin{bmatrix} -1 & 2 \\ -2 & 3 \end{bmatrix}$, we obtain $A(t)=\begin{bmatrix} \dfrac{t^2}{2}-\dfrac{5}{2}t+1 & \dfrac{t^3}{6}+\dfrac{5}{6}t+1 \\ -2 & 2t+1 \end{bmatrix}$.

19. Integrating component wise, we obtain

$$\int_0^t B(s)\,ds=\begin{bmatrix} \int_0^t 2s\,ds & \int_0^t \cos s\,ds & \int_0^t 2\,ds \\ \int_0^t 5\,ds & \int_0^t (s+1)^{-1}\,ds & \int_0^t 3s^2\,ds \end{bmatrix}=\begin{bmatrix} t^2 & \sin t & 2t \\ 5t & \ln|t+1| & t^3 \end{bmatrix}.$$

20. Integrating component wise, we obtain $\int_0^t B(s)\,ds=\begin{bmatrix} e^t-1 & 3t^2 \\ \dfrac{\sin 2\pi t}{2\pi} & \dfrac{1-\cos 2\pi t}{2\pi} \end{bmatrix}$.

21 (a). One example is $A=\begin{bmatrix} 1 & t \\ t^2 & 0 \end{bmatrix}$.

22. One example is $A=\begin{bmatrix} 0 & t \\ 0 & 0 \end{bmatrix}$.

Section 6.2

1. The given problem can be written as $\mathbf{y}'(t)=P(t)\mathbf{y}(t)+\mathbf{g}(t), \quad \mathbf{y}(3)=\mathbf{y}_0$ where $P(t)=\begin{bmatrix} t^{-1} & \tan t \\ \ln|t| & e^t \end{bmatrix}$, $\mathbf{g}(t)=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\mathbf{y}_0=\begin{bmatrix} 0 \\ 1 \end{bmatrix}$. The coefficient functions $p_{11}(t)=t^{-1}$ and $p_{21}(t)=\ln|t|$ are discontinuous at $t=0$. The coefficient function $p_{12}(t)=\tan t$ has discontinuities at $\pm\pi/2, \pm 3\pi/2, \ldots$. The largest interval containing $t_0=3$ but containing no discontinuities of any coefficient function is the interval $\pi/2<t<3\pi/2$.

2. In standard form, the problem is $\mathbf{y}' = \begin{bmatrix} 1 & \tan t \\ t^2-2 & 4 \end{bmatrix}\mathbf{y} + \begin{bmatrix} (t+1)^{-2} \\ 0 \end{bmatrix}, \mathbf{y}(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

$\tan t$ is discontinuous at $t = \pm\pi/2$ and $(t+1)^{-2}$ is discontinuous at $t = -1$. The largest interval containing $t_0 = 0$ but containing no discontinuities of any coefficient function is the interval $-1 < t < \pi/2$.

3. In standard form, the problem is $\mathbf{y}' = \begin{bmatrix} (\cos t)/t^2 & 1/t^2 \\ 2 & 4t \end{bmatrix}\mathbf{y} + \begin{bmatrix} 1/t^2 \\ \sec t \end{bmatrix}, \mathbf{y}(1) = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$.

The only discontinuities of $p_{11}(t)$ and $p_{12}(t)$ are at $t = 0$, while $g_2(t)$ is discontinuous at $t = \pm\pi/2, \pm 3\pi/2, \ldots$. The largest interval containing $t_0 = 1$ but containing no discontinuities of any coefficient function is the interval $0 < t < \pi/2$.

4. In standard form, the problem is $\mathbf{y}' = \begin{bmatrix} \dfrac{3t}{t+2} & \dfrac{5}{t+2} \\ \dfrac{2}{t-2} & \dfrac{4t}{t-2} \end{bmatrix}\mathbf{y}, \mathbf{y}(1) = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$.

The largest interval containing $t_0 = 1$ but containing no discontinuities of any coefficient function is the interval $-2 < t < 2$.

5. Differentiating, $y_1' = 5c_1e^{5t} + 3c_2e^{3t}$ and $y_2' = 5c_1e^{5t} - 3c_2e^{3t}$. Calculating the right hand sides, $4y_1 + y_2 = 4(c_1e^{5t} + c_2e^{3t}) + (c_1e^{5t} - c_2e^{3t}) = 5c_1e^{5t} + 3c_2e^{3t} = y_1'$ and $y_1 + 4y_2 = (c_1e^{5t} + c_2e^{3t}) + 4(c_1e^{5t} - c_2e^{3t}) = 5c_1e^{5t} - 3c_2e^{3t} = y_2'$.

7 (a). $\mathbf{y}' = \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix}\mathbf{y}$

7 (b). $\mathbf{y} = c_1e^{5t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2e^{3t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

8 (a). $\mathbf{y}' = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\mathbf{y}$

8 (b). $\mathbf{y} = c_1\begin{bmatrix} e^t\cos t \\ -e^t\sin t \end{bmatrix} + c_2\begin{bmatrix} e^t\sin t \\ e^t\cos t \end{bmatrix}$

9. For $\mathbf{y} = c_1e^{2t}\begin{bmatrix} 2 \\ -1 \end{bmatrix} + c_2e^{3t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$, we have $\mathbf{y}' = 2c_1e^{2t}\begin{bmatrix} 2 \\ -1 \end{bmatrix} + 3c_2e^{3t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$. On the other hand, $A\mathbf{y} = A\left(c_1e^{2t}\begin{bmatrix} 2 \\ -1 \end{bmatrix} + c_2e^{3t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right) = c_1e^{2t}A\begin{bmatrix} 2 \\ -1 \end{bmatrix} + c_2e^{3t}A\begin{bmatrix} 1 \\ -1 \end{bmatrix} =$ $c_1e^{2t}\begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} 2 \\ -1 \end{bmatrix} + c_2e^{3t}\begin{bmatrix} 1 & -2 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = c_1e^{2t}\begin{bmatrix} 4 \\ -2 \end{bmatrix} + c_2e^{3t}\begin{bmatrix} 3 \\ -3 \end{bmatrix}$. Thus, $\mathbf{y}' = A\mathbf{y}$ for every choice of c_1 and c_2.

In order to solve the initial value problem, we first note that

$\mathbf{y}(0)=c_1\begin{bmatrix}2\\-1\end{bmatrix}+c_2\begin{bmatrix}1\\-1\end{bmatrix}=\begin{bmatrix}2&1\\-1&-1\end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix}$. Thus, solving $\begin{bmatrix}2&1\\-1&-1\end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix}=\begin{bmatrix}4\\-3\end{bmatrix}$, we obtain $c_1=1$ and $c_2=2$. Therefore, $\mathbf{y}(t)=e^{2t}\begin{bmatrix}2\\-1\end{bmatrix}+2e^{3t}\begin{bmatrix}1\\-1\end{bmatrix}=\begin{bmatrix}2e^{2t}+2e^{3t}\\-e^{2t}-2e^{3t}\end{bmatrix}$ is the solution of the given initial value problem.

10. For $\mathbf{y}'=c_1e^{5t}\begin{bmatrix}5\\5\end{bmatrix}+c_2e^{-t}\begin{bmatrix}1\\-2\end{bmatrix}$, $A\mathbf{y}=c_1e^{5t}\begin{bmatrix}3+2\\4+1\end{bmatrix}+c_2e^{-t}\begin{bmatrix}-3+4\\-4+2\end{bmatrix}$

Solving $\begin{bmatrix}1&-1\\1&2\end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix}=\begin{bmatrix}-1\\8\end{bmatrix}$, we obtain $c_1=2$ and $c_2=3$. Therefore, $\mathbf{y}(t)=\begin{bmatrix}2e^{5t}-3e^{-t}\\2e^{5t}+6e^{-t}\end{bmatrix}$ is the solution of the given initial value problem.

11. Let $\mathbf{Y}(t)=\begin{bmatrix}y(t)\\y'(t)\end{bmatrix}$. Calculating $\mathbf{Y}'(t)$, we find

$\mathbf{Y}'(t)=\begin{bmatrix}y'(t)\\y''(t)\end{bmatrix}=\begin{bmatrix}y'(t)\\-t^2y'(t)-4y(t)+\sin t\end{bmatrix}=\begin{bmatrix}0&1\\-4&-t^2\end{bmatrix}\begin{bmatrix}y(t)\\y'(t)\end{bmatrix}+\begin{bmatrix}0\\\sin t\end{bmatrix}$. Therefore, the scalar equation can be written as $\mathbf{Y}'=P(t)\mathbf{Y}+\mathbf{G}(t)$ where $P(t)=\begin{bmatrix}0&1\\-4&-t^2\end{bmatrix}$ and $\mathbf{G}(t)=\begin{bmatrix}0\\\sin t\end{bmatrix}$.

12. Let $\mathbf{Y}(t)=\begin{bmatrix}y(t)\\y'(t)\end{bmatrix}$. Calculating $\mathbf{Y}'(t)$, we find

$\mathbf{Y}'(t)=\begin{bmatrix}y'(t)\\y''(t)\end{bmatrix}=\begin{bmatrix}0&1\\-\sqrt{t}\sec t&3t\sec t\end{bmatrix}\begin{bmatrix}y(t)\\y'(t)\end{bmatrix}+\begin{bmatrix}0\\(t^2+1)\sec t\end{bmatrix}$. Therefore, the scalar equation can be written as $\mathbf{Y}'=P(t)\mathbf{Y}+\mathbf{G}(t)$ where $P(t)=\begin{bmatrix}0&1\\-\sqrt{t}\sec t&3t\sec t\end{bmatrix}$ and $\mathbf{G}(t)=\begin{bmatrix}0\\(t^2+1)\sec t\end{bmatrix}$.

13. Let $\mathbf{Y}(t)=\begin{bmatrix}y(t)\\y'(t)\\y''(t)\end{bmatrix}$. We solve for y''' by multiplying the equation by e^{-t} and find

$\mathbf{Y}'(t)=\begin{bmatrix}y'(t)\\y''(t)\\-5e^{-t}y''(t)-e^{-t}t^{-1}y'(t)-(e^{-t}\tan t)y(t)+e^{-t}\end{bmatrix}$. Expressing $\mathbf{Y}'(t)$ in matrix terms, we have $\mathbf{Y}'(t)=\begin{bmatrix}0&1&0\\0&0&1\\-e^{-t}\tan t&-e^{-t}t^{-1}&-5e^{-t}\end{bmatrix}\begin{bmatrix}y(t)\\y'(t)\\y''(t)\end{bmatrix}+\begin{bmatrix}0\\0\\e^{-t}\end{bmatrix}$. Therefore, the scalar equation can be written as $\mathbf{Y}'=P(t)\mathbf{Y}+\mathbf{G}(t)$ where

$$P(t)=\begin{bmatrix}0&1&0\\0&0&1\\-e^{-t}\tan t&-e^{-t}t^{-1}&-5e^{-t}\end{bmatrix}\text{ and }\mathbf{G}(t)=\begin{bmatrix}0\\0\\e^{-t}\end{bmatrix}.$$

14. Let $\mathbf{Y}(t)=\begin{bmatrix} y(t) \\ y'(t) \\ y''(t) \end{bmatrix}$. $\mathbf{Y}'(t)=\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ t & -\cos t & 2 \end{bmatrix}\begin{bmatrix} y(t) \\ y'(t) \\ y''(t) \end{bmatrix}+\begin{bmatrix} 0 \\ 0 \\ e^{3t} \end{bmatrix}$. Therefore, the scalar equation can be written as $\mathbf{Y}'=P(t)\mathbf{Y}+\mathbf{G}(t)$ where $P(t)=\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ t & -\cos t & 2 \end{bmatrix}$ and $\mathbf{G}(t)=\begin{bmatrix} 0 \\ 0 \\ e^{3t} \end{bmatrix}$.

15. Let $\mathbf{Y}(t)=\begin{bmatrix} y(t) \\ y'(t) \end{bmatrix}$ so that $\mathbf{Y}'(t)=\begin{bmatrix} y'(t) \\ y''(t) \end{bmatrix}$. We are given that

$$\mathbf{Y}'(t)=\begin{bmatrix} 0 & 1 \\ -3 & 2 \end{bmatrix}\begin{bmatrix} y(t) \\ y'(t) \end{bmatrix}+\begin{bmatrix} 0 \\ 2\cos 2t \end{bmatrix}=\begin{bmatrix} y'(t) \\ -3y(t)+2y'(t)+2\cos 2t \end{bmatrix}.$$

Therefore, equating components of the vector $\mathbf{Y}'(t)$, we see that the scalar equation is $y''=-3y+2y'+2\cos 2t, \quad y(-1)=1, y'(-1)=4$.

16. $y'''-4y''+2y=e^{3t}, \quad y(0)=1, y'(0)=-2, \ y''(0)=3$.

17. Let $\mathbf{Y}(t)=\begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \\ y_4(t) \end{bmatrix}=\begin{bmatrix} y(t) \\ y'(t) \\ y''(t) \\ y'''(t) \end{bmatrix}$ so that $\mathbf{Y}'(t)=\begin{bmatrix} y'(t) \\ y''(t) \\ y'''(t) \\ y^{(4)}(t) \end{bmatrix}$. We are given that

$$\mathbf{Y}'(t)=\begin{bmatrix} y_2 \\ y_3 \\ y_4 \\ y_2+y_3\sin(y_1)+y_3^2 \end{bmatrix}=\begin{bmatrix} y' \\ y'' \\ y''' \\ y'+y''\sin(y)+(y'')^2 \end{bmatrix}.$$

Equating components of the vector $\mathbf{Y}'(t)$, we see that the scalar equation is $y^{(4)}=y'+y''\sin(y)+(y'')^2, \quad y(1)=0, y'(1)=0, y''(1)=-1, y'''(1)=2$.

18. Making the indicated change of variables, the system of differential equations is

$$\begin{aligned} Y_2' &= Y_2+Y_3+tY_4 \\ Y_4' &= 2tY_1+Y_2+Y_4 \end{aligned}$$

. Therefore, the system can be expressed in the form $\mathbf{Y}'=P(t)\mathbf{Y}+\mathbf{G}(t)$ where

$$P(t)=\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & t \\ 0 & 0 & 0 & 1 \\ 2t & 1 & 0 & 1 \end{bmatrix} \text{ and } \mathbf{G}(t)=\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

19. Making the indicated change of variables, the system of differential equations is

$$\begin{aligned} Y_2' &= t^{-1}Y_2+4Y_1-tY_3+(\sin t)Y_4+e^{2t} \\ Y_4' &= Y_1-5Y_4 \end{aligned}.$$

Therefore, the system can be expressed in the form $\mathbf{Y}' = P(t)\mathbf{Y} + \mathbf{G}(t)$ where

$$P(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 4 & t^{-1} & -t & \sin t \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & -5 \end{bmatrix} \text{ and } \mathbf{G}(t) = \begin{bmatrix} 0 \\ e^{2t} \\ 0 \\ 0 \end{bmatrix}.$$

20. Making the indicated change of variables, the system of differential equations is

$$\begin{aligned} Y_2' &= 4Y_1 + 7Y_2 - 8Y_3 + 6Y_4 + t^2 \\ Y_4' &= 3Y_1 - 6Y_2 + 2Y_3 + 5Y_4 - \sin t \end{aligned}$$

. Therefore, the system can be expressed in the form $\mathbf{Y}' = P(t)\mathbf{Y} + \mathbf{G}(t)$ where

$$P(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 4 & 7 & -8 & 6 \\ 0 & 0 & 0 & 1 \\ 3 & -6 & 2 & 5 \end{bmatrix} \text{ and } \mathbf{G}(t) = \begin{bmatrix} 0 \\ t^2 \\ 0 \\ -\sin t \end{bmatrix}.$$

21. Making the indicated change of variables, the system of differential equations is

$$\begin{aligned} 15Y_3 + 9Y_2 + 3Y_2' &= 12Y_1 - 6Y_4 + 3t^2 \\ Y_4 + 5Y_1 - Y_4' &= 2Y_3 - 6Y_2 + t \end{aligned}.$$

Writing this system in standard form,

$$\begin{aligned} Y_2' &= 4Y_1 - 3Y_2 - 5Y_3 - 2Y_4 + t^2 \\ Y_4' &= 5Y_1 + 6Y_2 - 2Y_3 + Y_4 - t \end{aligned}.$$

Therefore, the system can be expressed in the form $\mathbf{Y}' = P(t)\mathbf{Y} + \mathbf{G}(t)$ where

$$P(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 4 & -3 & -5 & -2 \\ 0 & 0 & 0 & 1 \\ 5 & 6 & -2 & 1 \end{bmatrix} \text{ and } \mathbf{G}(t) = \begin{bmatrix} 0 \\ t^2 \\ 0 \\ -t \end{bmatrix}.$$

Section 6.3

1 (a). In matrix terms, the system has the form $\mathbf{y}' = A\mathbf{y}$ where

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 9 & -4 \\ 15 & -7 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \text{ or } \mathbf{y}' = \begin{bmatrix} 9 & -4 \\ 15 & -7 \end{bmatrix} \mathbf{y}.$$

1 (b). We have

$\mathbf{y}' = \begin{bmatrix} 6e^{3t} \\ 9e^{3t} \end{bmatrix}$. Calculating $A\mathbf{y}$, we obtain $\begin{bmatrix} 9 & -4 \\ 15 & -7 \end{bmatrix} \begin{bmatrix} 2e^{3t} \\ 3e^{3t} \end{bmatrix} = \begin{bmatrix} 18e^{3t} - 12e^{3t} \\ 30e^{3t} - 21e^{3t} \end{bmatrix}$ and

therefore, $A\mathbf{y} = \begin{bmatrix} 6e^{3t} \\ 9e^{3t} \end{bmatrix}$, showing that the function $\mathbf{y}(t)$ is a solution of $\mathbf{y}' = A\mathbf{y}$.

2 (a). $\mathbf{y}' = \begin{bmatrix} -3 & -2 \\ 4 & 3 \end{bmatrix} \mathbf{y}$.

3 (a). In matrix terms, the system has the form $\mathbf{y}' = A\mathbf{y}$ where

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \text{ or } \mathbf{y}' = \begin{bmatrix} 1 & 4 \\ -1 & 1 \end{bmatrix} \mathbf{y}.$$

3 (b). We have

$$\mathbf{y}' = \begin{bmatrix} 2e^t \cos 2t - 4e^t \sin 2t \\ -e^t \sin 2t - 2e^t \cos 2t \end{bmatrix}. \text{ Calculating } A\mathbf{y}, \text{ we obtain}$$

$$\begin{bmatrix} 1 & 4 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 2e^t \cos 2t \\ -e^t \sin 2t \end{bmatrix} = \begin{bmatrix} 2e^t \cos 2t - 4e^t \sin 2t \\ -2e^t \cos 2t - e^t \sin 2t \end{bmatrix}. \text{ Therefore,}$$

the function $\mathbf{y}(t)$ is a solution of $\mathbf{y}' = A\mathbf{y}$.

4 (a). $\mathbf{y}' = \begin{bmatrix} 0 & 1 \\ \frac{2}{t^2} & -\frac{2}{t} \end{bmatrix} \mathbf{y}.$

5 (a). In matrix terms, the system has the form $\mathbf{y}' = A\mathbf{y}$ where

$$\begin{bmatrix} y_1' \\ y_2' \\ y_3' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 \\ -6 & -3 & 1 \\ -8 & -2 & 4 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \text{ or } \mathbf{y}' = \begin{bmatrix} 0 & 1 & 1 \\ -6 & -3 & 1 \\ -8 & -2 & 4 \end{bmatrix} \mathbf{y}.$$

5 (b). We have

$$\mathbf{y}' = \begin{bmatrix} e^t \\ -e^t \\ 2e^t \end{bmatrix}. \text{ Calculating } A\mathbf{y}, \text{ we obtain}$$

$$\begin{bmatrix} 0 & 1 & 1 \\ -6 & -3 & 1 \\ -8 & -2 & 4 \end{bmatrix} \begin{bmatrix} e^t \\ -e^t \\ 2e^t \end{bmatrix} = \begin{bmatrix} -e^t + 2e^t \\ -6e^t + 3e^t + 2e^t \\ -8e^t + 2e^t + 8e^t \end{bmatrix} = \begin{bmatrix} e^t \\ -e^t \\ 2e^t \end{bmatrix} \text{ and therefore}$$

the function $\mathbf{y}(t)$ is a solution of $\mathbf{y}' = A\mathbf{y}$.

6 (a). $\mathbf{y}' = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix} \mathbf{y}.$

7 (a). $\mathbf{y}_1' = \begin{bmatrix} 6e^{3t} \\ 9e^{3t} \end{bmatrix}$ and also $A\mathbf{y}_1 = \begin{bmatrix} 9 & -4 \\ 15 & -7 \end{bmatrix} \begin{bmatrix} 2e^{3t} \\ 3e^{3t} \end{bmatrix} = \begin{bmatrix} 18e^{3t} - 12e^{3t} \\ 30e^{3t} - 21e^{3t} \end{bmatrix} = \mathbf{y}_1'.$

Similarly for $\mathbf{y}_2'$.

7 (b). The Wronskian $W(t)$ is given by

$$W(t) = \det[\Psi(t)] \text{ where } \Psi(t) = \begin{bmatrix} 2e^{3t} & 2e^{-t} \\ 3e^{3t} & 5e^{-t} \end{bmatrix}. \text{ Thus, } W(t) = 10e^{2t} - 6e^{2t} = 4e^{2t}.$$

7 (c). $\mathbf{y}(t) = \begin{bmatrix} 2e^{3t} & 2e^{-t} \\ 3e^{3t} & 5e^{-t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

7 (d). Given the general solution in part (c), $\mathbf{y}(0)=\begin{bmatrix}2 & 2\\ 3 & 5\end{bmatrix}\begin{bmatrix}c_1\\ c_2\end{bmatrix}=\begin{bmatrix}1\\ 1\end{bmatrix}$. Solving, we find

$\begin{bmatrix}c_1\\ c_2\end{bmatrix}=\begin{bmatrix}3/4\\ -1/4\end{bmatrix}$. Therefore, the solution of the initial value problem is

$$\mathbf{y}(t)=(3/4)\begin{bmatrix}2e^{3t}\\ 3e^{3t}\end{bmatrix}-(1/4)\begin{bmatrix}2e^{-t}\\ 5e^{-t}\end{bmatrix}=(1/4)\begin{bmatrix}6e^{3t}-2e^{-t}\\ 9e^{3t}-5e^{-t}\end{bmatrix}.$$

8 (b). The Wronskian $W(t)$ is given by

$W(t)=\det[\Psi(t)]$ where $\Psi(t)=\begin{bmatrix}2e^{3t}-4e^{-t} & 4e^{3t}+2e^{-t}\\ 3e^{3t}-10e^{-t} & 6e^{3t}+5e^{-t}\end{bmatrix}$. Thus, $W(t)=20e^{2t}\neq 0$.

8 (c). $\mathbf{y}(t)=\begin{bmatrix}2e^{3t}-4e^{-t} & 4e^{3t}+2e^{-t}\\ 3e^{3t}-10e^{-t} & 6e^{3t}+5e^{-t}\end{bmatrix}\begin{bmatrix}c_1\\ c_2\end{bmatrix}$

8 (d). Given the general solution in part (c), $\mathbf{y}(0)=\begin{bmatrix}-2 & 6\\ -7 & 11\end{bmatrix}\begin{bmatrix}c_1\\ c_2\end{bmatrix}=\begin{bmatrix}0\\ 1\end{bmatrix}$. Solving, we find

$\begin{bmatrix}c_1\\ c_2\end{bmatrix}=\begin{bmatrix}-3/10\\ -1/10\end{bmatrix}$. Therefore, the solution of the initial value problem is

$$\mathbf{y}(t)=(-3/10)\begin{bmatrix}2e^{3t}-4e^{-t}\\ 3e^{3t}-10e^{-t}\end{bmatrix}-(1/10)\begin{bmatrix}4e^{3t}+2e^{-t}\\ 6e^{3t}+5e^{-t}\end{bmatrix}=\begin{bmatrix}-e^{3t}+e^{-t}\\ -\frac{3}{2}e^{3t}+\frac{5}{2}e^{-t}\end{bmatrix}.$$

9 (a). $\mathbf{y}_1'=\begin{bmatrix}-e^{-t}\\ 2e^{-t}\end{bmatrix}$ and also $A\mathbf{y}_1=\begin{bmatrix}3 & 2\\ -4 & -3\end{bmatrix}\begin{bmatrix}e^{-t}\\ -2e^{-t}\end{bmatrix}=\begin{bmatrix}3e^{-t}-4e^{-t}\\ -4e^{-t}+6e^{-t}\end{bmatrix}=\mathbf{y}_1'$.

Similarly for $\mathbf{y}_2'$.

9 (b). The Wronskian $W(t)$ is given by

$W(t)=\det[\Psi(t)]$ where $\Psi(t)=\begin{bmatrix}e^{-t} & -3e^{-t}\\ -2e^{-t} & 6e^{-t}\end{bmatrix}$. Thus, $W(t)=6e^{-2t}-6e^{-2t}\equiv 0$

and therefore, the given set of solutions is not a fundamental set of solutions.

10 (b). The Wronskian $W(t)$ is given by

$W(t)=\det[\Psi(t)]$ where $\Psi(t)=\begin{bmatrix}-5e^{-2t}\cos 3t & -5e^{-2t}\sin 3t\\ e^{-2t}(\cos 3t-3\sin 3t) & e^{-2t}(3\cos 3t+\sin 3t)\end{bmatrix}$. Thus,

$W(t)=-15e^{-4t}\neq 0$.

10 (c). $\mathbf{y}(t)=\begin{bmatrix}-5e^{-2t}\cos 3t & -5e^{-2t}\sin 3t\\ e^{-2t}(\cos 3t-3\sin 3t) & e^{-2t}(3\cos 3t+\sin 3t)\end{bmatrix}\begin{bmatrix}c_1\\ c_2\end{bmatrix}$

10 (d). Given the general solution in part (c), $\mathbf{y}(0)=\begin{bmatrix}-5 & 0\\ 1 & 3\end{bmatrix}\begin{bmatrix}c_1\\ c_2\end{bmatrix}=\begin{bmatrix}5\\ 2\end{bmatrix}$. Solving, we find $\begin{bmatrix}c_1\\ c_2\end{bmatrix}=\begin{bmatrix}-1\\ 1\end{bmatrix}$.

Therefore, the solution of the initial value problem is $\mathbf{y}(t)=\begin{bmatrix}5e^{-2t}(\cos 3t-\sin 3t)\\ e^{-2t}(2\cos 3t+4\sin 3t)\end{bmatrix}$.

11 (a). $\mathbf{y}_1'=\begin{bmatrix}e^{t}\\ -2e^{t}\end{bmatrix}$ and also $A\mathbf{y}_1=\begin{bmatrix}-3 & -2\\ 4 & 3\end{bmatrix}\begin{bmatrix}e^{t}\\ -2e^{t}\end{bmatrix}=\begin{bmatrix}-3e^{t}+4e^{t}\\ 4e^{t}-6e^{t}\end{bmatrix}=\mathbf{y}_1'$.

Similarly for $\mathbf{y}_2'$.

11 (b). The Wronskian $W(t)$ is given by

$$W(t)=\det[\Psi(t)] \text{ where } \Psi(t)=\begin{bmatrix} e^{t} & e^{-t} \\ -2e^{t} & -e^{-t} \end{bmatrix}. \text{ Thus, } W(t)=-1+2=1.$$

11 (c). $\mathbf{y}(t)=\begin{bmatrix} e^{t} & e^{-t} \\ -2e^{t} & -e^{-t} \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

11 (d). Given the general solution in part (c), $\mathbf{y}(1)=\begin{bmatrix} e & e^{-1} \\ -2e & -e^{-1} \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}=\begin{bmatrix} 1 \\ -3 \end{bmatrix}$. Solving, we find

$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}=\begin{bmatrix} 2e^{-1} \\ -e \end{bmatrix}$. Therefore, the solution of the initial value problem is

$$\mathbf{y}(t)=2e^{-1}\begin{bmatrix} e^{t} \\ -2e^{t} \end{bmatrix}-e\begin{bmatrix} e^{-t} \\ -e^{-t} \end{bmatrix}=\begin{bmatrix} 2e^{t-1}-e^{1-t} \\ -4e^{t-1}+e^{1-t} \end{bmatrix}.$$

12 (b). The Wronskian $W(t)$ is given by

$$W(t)=\det[\Psi(t)] \text{ where } \Psi(t)=\begin{bmatrix} 1 & e^{3t} \\ 1 & -2e^{3t} \end{bmatrix}. \text{ Thus, } W(t)=-3e^{-3t}\neq 0.$$

12 (c). $\mathbf{y}(t)=\begin{bmatrix} 1 & e^{3t} \\ 1 & -2e^{3t} \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

12 (d). Given the general solution in part (c), $\mathbf{y}(-1)=\begin{bmatrix} 1 & e^{-3} \\ 1 & -2e^{-3} \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}=\begin{bmatrix} -2 \\ 4 \end{bmatrix}$. Solving, we find

$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}=-\frac{e^{3}}{3}\begin{bmatrix} 0 \\ 6 \end{bmatrix}$. Therefore, the solution of the initial value problem is $\mathbf{y}(t)=\begin{bmatrix} -2e^{3(t+1)} \\ 4e^{3(t+1)} \end{bmatrix}$.

13 (a).

$$\mathbf{y}_1'=\begin{bmatrix} 2t-2 \\ 2 \end{bmatrix} \text{ and also } A\mathbf{y}_1=\begin{bmatrix} 2t^{-2} & 1-2t^{-1}+2t^{-2} \\ -2t^{-2} & 2t^{-1}-2t^{-2} \end{bmatrix}\begin{bmatrix} t^2-2t \\ 2t \end{bmatrix}.$$

$$=\begin{bmatrix} (2-4t^{-1})+(2t-4+4t^{-1}) \\ (-2+4t^{-1})+(4-4t^{-1}) \end{bmatrix}=\mathbf{y}_1'$$

Similarly for $\mathbf{y}_2'$.

13 (b). The Wronskian $W(t)$ is given by

$$W(t)=\det[\Psi(t)] \text{ where } \Psi(t)=\begin{bmatrix} t^2-2t & t-1 \\ 2t & 1 \end{bmatrix}. \text{ Thus, } W(t)=-t^2.$$

13 (c). $\mathbf{y}(t)=\begin{bmatrix} t^2-2t & t-1 \\ 2t & 1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

13 (d). Given the general solution in part (c), $\mathbf{y}(2)=\begin{bmatrix} 0 & 1 \\ 4 & 1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}=\begin{bmatrix} -2 \\ 2 \end{bmatrix}$. Solving, we find $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}=\begin{bmatrix} 1 \\ -2 \end{bmatrix}$.

Therefore, the solution of the initial value problem is $\mathbf{y}(t)=\begin{bmatrix} t^2-2t \\ 2t \end{bmatrix}-2\begin{bmatrix} t-1 \\ 1 \end{bmatrix}=\begin{bmatrix} t^2-4t+2 \\ 2t-2 \end{bmatrix}$.

14 (b). The Wronskian $W(t)$ is given by

$$W(t)=\det[\Psi(t)] \text{ where } \Psi(t)=\begin{bmatrix} e^{-2t} & 0 & 0 \\ 0 & 2e^t\cos 2t & 2e^t\sin 2t \\ 0 & -e^t\sin 2t & e^t\cos 2t \end{bmatrix}. \text{ Thus, } W(t)=2.$$

14 (c). $\mathbf{y}(t)=\begin{bmatrix} e^{-2t} & 0 & 0 \\ 0 & 2e^t\cos 2t & 2e^t\sin 2t \\ 0 & -e^t\sin 2t & e^t\cos 2t \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}.$

14 (d). Given the general solution on part (c), $\mathbf{y}(0)=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}=\begin{bmatrix} 3 \\ 4 \\ -2 \end{bmatrix}.$

Solving, we find $\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}=\begin{bmatrix} 3 \\ 2 \\ -2 \end{bmatrix}$. Therefore, the solution of the initial value problem

is $\mathbf{y}(t)=\begin{bmatrix} 3e^{-2t} \\ 4e^t(\cos 2t-\sin 2t) \\ 2e^t(-\cos 2t+\sin 2t) \end{bmatrix}.$

15 (a). $\mathbf{y}_1'=\begin{bmatrix} 5e^t \\ -11e^t \\ 0 \end{bmatrix}$ and also $A\mathbf{y}_1=\begin{bmatrix} -21 & -10 & 2 \\ 22 & 11 & -2 \\ -110 & -50 & 11 \end{bmatrix}\begin{bmatrix} 5e^t \\ -11e^t \\ 0 \end{bmatrix}=\begin{bmatrix} 5e^t \\ -11e^t \\ 0 \end{bmatrix}=\mathbf{y}_1'.$

Similarly for $\mathbf{y}_2'$ and $\mathbf{y}_3'$.

15 (b). The Wronskian $W(t)$ is given by

$$W(t)=\det[\Psi(t)] \text{ where } \Psi(t)=\begin{bmatrix} 5e^t & e^t & e^{-t} \\ -11e^t & 0 & -e^{-t} \\ 0 & 11e^t & 5e^{-t} \end{bmatrix}. \text{ Thus, } W(t)=-11e^t.$$

15 (c). $\mathbf{y}(t)=\begin{bmatrix} 5e^t & e^t & e^{-t} \\ -11e^t & 0 & -e^{-t} \\ 0 & 11e^t & 5e^{-t} \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}.$

15 (d). Given the general solution on part (c), $\mathbf{y}(0)=\begin{bmatrix} 5 & 1 & 1 \\ -11 & 0 & -1 \\ 0 & 11 & 5 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}=\begin{bmatrix} 3 \\ -10 \\ -16 \end{bmatrix}.$

Solving, we find $\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}=\begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$. Therefore, the solution of the initial value problem

is $\mathbf{y}(t)=\begin{bmatrix} 5e^t \\ -11e^t \\ 0 \end{bmatrix}-\begin{bmatrix} e^t \\ 0 \\ 11e^t \end{bmatrix}-\begin{bmatrix} e^{-t} \\ -e^{-t} \\ 5e^{-t} \end{bmatrix}=\begin{bmatrix} 4e^t-e^{-t} \\ -11e^t+e^{-t} \\ -11e^t-5e^{-t} \end{bmatrix}.$

16 (a). $W(t)=\det\begin{bmatrix}5e^{-t} & e^{t}\\ -7e^{-t} & -e^{t}\end{bmatrix}=2$

16 (b). The trace of A is equal to $6-6=0$.

16 (c). For $t_0=-1, W(t_0)e^{\int_{t_0}^{t} tr[P(s)]ds}=2e^{\int_{-1}^{t}0ds}=2$.

17 (a). $W(t)=\det\begin{bmatrix}5e^{2t} & e^{4t}\\ -7e^{2t} & -e^{4t}\end{bmatrix}=2e^{6t}$

17 (b). The trace of $A=\begin{bmatrix}9 & 5\\ -7 & -3\end{bmatrix}$ is equal to $9+(-3)=6$.

17 (c). For $t_0=0, W(t_0)e^{\int_{t_0}^{t} tr[P(s)]ds}=2e^{\int_{0}^{t}6ds}=2e^{6t}$.

18 (a). $W(t)=\det\begin{bmatrix}-1 & e^{t}\\ t^{-1} & 0\end{bmatrix}=-t^{-1}e^{t}$

18 (b). The trace of A is equal to $1-t^{-1}$.

18 (c). For $t_0=1, W(t_0)e^{\int_{t_0}^{t} tr[P(s)]ds}=-e\left(e^{\int_{1}^{t}(1-s^{-1})ds}\right)=-ee^{s-\ln s|_1^t}=-ee^{t-\ln t-1}=-e^{t-\ln t}=-t^{-1}e^{t}$.

19 (a). $W(t)=\det\begin{bmatrix}2e^{t} & 0 & e^{4t}\\ -e^{t} & -e^{-t} & e^{4t}\\ -e^{t} & e^{-t} & e^{4t}\end{bmatrix}=e^{t}e^{-t}e^{4t}\det\begin{bmatrix}2 & 0 & 1\\ -1 & -1 & 1\\ -1 & 1 & 1\end{bmatrix}=-6e^{4t}$

19 (b). The trace of $A=\begin{bmatrix}2 & 1 & 1\\ 1 & 1 & 2\\ 1 & 2 & 1\end{bmatrix}$ is equal to $2+1+1=4$.

19 (c). For $t_0=0, W(t_0)e^{\int_{t_0}^{t} tr[P(s)]ds}=-6e^{\int_{0}^{t}4ds}=-6e^{4t}$.

20 (a). $W(t)=\det\begin{bmatrix}5 & 2e^{3t}\\ 1 & e^{3t}\end{bmatrix}=3e^{3t}\neq 0$.

20 (b). $3e^{3t}=3e^{\int_0^t tr[A]ds}\Rightarrow tr[A]=3$.

20 (c). $\psi=\begin{bmatrix}5 & 2e^{3t}\\ 1 & e^{3t}\end{bmatrix}\Rightarrow \psi'=\begin{bmatrix}0 & 6e^{3t}\\ 0 & 3e^{3t}\end{bmatrix}=A\begin{bmatrix}5 & 2e^{3t}\\ 1 & e^{3t}\end{bmatrix}$.

20 (d). $A=\begin{bmatrix}0 & 6e^{3t}\\ 0 & 3e^{3t}\end{bmatrix}\cdot\dfrac{1}{3e^{3t}}\begin{bmatrix}e^{3t} & -2e^{3t}\\ -1 & 5\end{bmatrix}=\begin{bmatrix}-2 & 10\\ -1 & 5\end{bmatrix}$.

The results are consistent since $tr[A]=-2+5=3$

21. If $W(t)$ is constant, then by Abel's Theorem, the function $e^{\int_{t_0}^{t} tr[P(s)]ds}$ must also be constant. Therefore, $g(t)=\int_{t_0}^{t} tr[P(s)]ds$ must be constant and hence the derivative of $g(t)$ is identically zero. However, by the fundamental theorem of calculus, $g'(t)=tr[P(t)]$ and hence the trace of $P(t)$ must be zero. Since the trace is equal to $3+\alpha$ we conclude that $\alpha=-3$.

Section 6.4

1 (a). Let $F(t)=[\mathbf{f}_1(t),\mathbf{f}_2(t)]=\begin{bmatrix} t & t^2 \\ 1 & 2 \end{bmatrix}$. Then, $\det[F(t)]=2t-t^2$.

1 (b). No, because we do not know whether the functions $\mathbf{f}_1(t)$ and $\mathbf{f}_2(t)$ form a fundamental set of solutions for a linear system.

1 (c). Yes. At $t=1$, the determinant is $2-1=1\neq 0$. Therefore, $[\mathbf{f}_1(t),\mathbf{f}_2(t)]\mathbf{k}=\mathbf{0}\Rightarrow \mathbf{k}=\mathbf{0}$.

2 (a). $\det[F(t)]=t^2e^t-t\sin t$.

2 (b). No

2 (c). Yes. At $t=1$, the determinant is e - $\sin 1\neq 0$.

3 (a). Let $F(t)=[\mathbf{f}_1(t),\mathbf{f}_2(t)]=\begin{bmatrix} te^t & \sin^2 t \\ t-1 & 2 \end{bmatrix}$. Then, $\det[F(t)]=2te^t-(t-1)\sin^2 t$.

3 (b). No, because we do not know whether the functions $\mathbf{f}_1(t)$ and $\mathbf{f}_2(t)$ form a fundamental set of solutions for a linear system.

3 (c). Yes. At $t=1$, the determinant is $2e\neq 0$.

4. $k_1\begin{bmatrix} t \\ 1 \end{bmatrix}+k_2\begin{bmatrix} t^2 \\ 1 \end{bmatrix}=\begin{bmatrix} t & t^2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} k_1 \\ k_2 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$; det $=\mathrm{t}-\mathrm{t}^2\neq 0$ at $t=2$ for example $\Rightarrow \mathbf{k}=\mathbf{0}$. Therefore, the given set of functions is linearly independent.

5. We need to solve the equation $k_1\begin{bmatrix} e^t \\ 1 \end{bmatrix}+k_2\begin{bmatrix} e^{-t} \\ 0 \end{bmatrix}\equiv\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ or $\begin{bmatrix} k_1e^t+k_2e^{-t} \\ k_1 \end{bmatrix}\equiv\begin{bmatrix} 0 \\ 0 \end{bmatrix}$. This vector equation requires $k_1e^t+k_2e^{-t}\equiv 0$ and $k_1\equiv 0$. By the second equation, $k_1=0$ and hence, using this fact in the first equation, $k_2e^{-t}\equiv 0$. Multiplying this identity by the nonzero function e^t, we see that $k_2=0$ as well. Hence, the only way to satisfy $k_1\begin{bmatrix} e^t \\ 1 \end{bmatrix}+k_2\begin{bmatrix} e^{-t} \\ 0 \end{bmatrix}\equiv\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is to choose $k_1=k_2=0$. This means the given set of functions is linearly independent.

6. $k_1\begin{bmatrix} e^t \\ 1 \end{bmatrix}+k_2\begin{bmatrix} e^{-t} \\ 1 \end{bmatrix}+k_3\begin{bmatrix} \sinh t \\ 0 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$; Let $k_1=1,\ k_2=-1,\ k_3=-2$. The given set of functions is linearly dependent.

7. We need to solve the equation $k_1\begin{bmatrix} 1 \\ t \\ 0 \end{bmatrix}+k_2\begin{bmatrix} 0 \\ 1 \\ t^2 \end{bmatrix}\equiv\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ or $\begin{bmatrix} k_1 \\ k_1t+k_2 \\ k_2t^2 \end{bmatrix}\equiv\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. The first component of this vector identity cannot be satisfied unless $k_1=0$ and the third component cannot be satisfied unless $k_2=0$. Hence, the only way to satisfy the identity $k_1\begin{bmatrix} 1 \\ t \\ 0 \end{bmatrix}+k_2\begin{bmatrix} 0 \\ 1 \\ t^2 \end{bmatrix}\equiv\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ is to choose $k_1=k_2=0$. This means the given set of functions is linearly independent.

8. $k_1\begin{bmatrix} 1 \\ t \\ 0 \end{bmatrix}+k_2\begin{bmatrix} 0 \\ 1 \\ t^2 \end{bmatrix}+k_3\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$; Let $k_1=0,\ k_2=0,\ k_3=1$. The given set of functions is linearly dependent.

9. We need to solve the equation $k_1\begin{bmatrix}1\\t\\0\end{bmatrix}+k_2\begin{bmatrix}0\\1\\t^2\end{bmatrix}+k_3\begin{bmatrix}0\\0\\1\end{bmatrix}\equiv\begin{bmatrix}0\\0\\0\end{bmatrix}$ or $\begin{bmatrix}k_1\\k_1t+k_2\\k_2t^2+k_3\end{bmatrix}\equiv\begin{bmatrix}0\\0\\0\end{bmatrix}$. The first component of this vector identity cannot be satisfied unless $k_1=0$ and therefore the second component requires $k_2=0$. Given that k_1 and k_2 must both be zero, the third component then requires that $k_3=0$. Hence, the only way to satisfy the identity $k_1\begin{bmatrix}1\\t\\0\end{bmatrix}+k_2\begin{bmatrix}0\\1\\t^2\end{bmatrix}+k_3\begin{bmatrix}0\\0\\1\end{bmatrix}\equiv\begin{bmatrix}0\\0\\0\end{bmatrix}$ is to choose $k_1=k_2=k_3=0$. This means the given set of functions is linearly independent.

10. $k_1\begin{bmatrix}1\\\sin^2 t\\0\end{bmatrix}+k_2\begin{bmatrix}0\\2-2\cos^2 t\\-2\end{bmatrix}+k_3\begin{bmatrix}1\\0\\1\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$; Let $k_1=1,\ k_2=-\frac{1}{2},\ k_3=-1$. The given set of functions is linearly dependent.

11 (a). Let $F(t)=\begin{bmatrix}e^t & t^2\\0 & t\end{bmatrix}$. Then, $\det[F(t)]=te^t$.

11 (b). Since the determinant is zero at $t=0$, $F(t)$ cannot be a fundamental matrix for a linear system defined on an interval containing $t=0$.

11 (c). A fundamental matrix $\Psi(t)$ satisfies the matrix differential equation $\Psi'=P(t)\Psi$. Given that $\Psi(t)=\begin{bmatrix}e^t & t^2\\0 & t\end{bmatrix}$, we know that $\Psi'(t)=\begin{bmatrix}e^t & 2t\\0 & 1\end{bmatrix}$. Therefore, the equation $\Psi'=P(t)\Psi$ implies that $\begin{bmatrix}e^t & 2t\\0 & 1\end{bmatrix}=P(t)\begin{bmatrix}e^t & t^2\\0 & t\end{bmatrix}$. Postmultiplying by Ψ^{-1}, we see that $\Psi'\Psi^{-1}=P(t)$. Therefore, $1/(te^t)\begin{bmatrix}e^t & 2t\\0 & 1\end{bmatrix}\begin{bmatrix}t & -t^2\\0 & e^t\end{bmatrix}=P(t)$ and so $P(t)=1/(te^t)\begin{bmatrix}te^t & (2t-t^2)e^t\\0 & e^t\end{bmatrix}$. Canceling the nonzero term e^t we have $P(t)=t^{-1}\begin{bmatrix}t & (2t-t^2)\\0 & 1\end{bmatrix}$.

12 (a). Let $F(t)=\begin{bmatrix}t^2 & 2t\\0 & 1\end{bmatrix}$. Then, $\det[F(t)]=t^2$.

12 (b). Since the determinant is zero at $t=0$, $F(t)$ cannot be a fundamental matrix for a linear system defined on an interval containing $t=0$.

12 (c). A fundamental matrix $\Psi(t)$ satisfies the matrix differential equation $\Psi'=P(t)\Psi$. Given that $\Psi(t)=\begin{bmatrix}t^2 & 2t\\0 & 1\end{bmatrix}$, we know that $\Psi'(t)=\begin{bmatrix}2t & 2\\0 & 0\end{bmatrix}$. Therefore, the equation $\Psi'=P(t)\Psi$ implies that $\begin{bmatrix}2t & 2\\0 & 0\end{bmatrix}=P(t)\begin{bmatrix}t^2 & 2t\\0 & 1\end{bmatrix}$. Postmultiplying by Ψ^{-1}, we see that $\Psi'\Psi^{-1}=P(t)$. Therefore, $1/(t^2)\begin{bmatrix}2t & 2\\0 & 0\end{bmatrix}\begin{bmatrix}1 & -2t\\0 & t^2\end{bmatrix}=P(t)$ and so $P(t)=\begin{bmatrix}2t^{-1} & -2\\0 & 0\end{bmatrix}$ which is continuous on $(-\infty,0)$ and $(0,\infty)$.

13 (a). We first show that $\Psi' = P(t)\Psi$. Now, $\Psi'(t) = \begin{bmatrix} e^t & -e^{-t} \\ e^t & e^{-t} \end{bmatrix}$ whereas

$P(t)\Psi(t) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} e^t & e^{-t} \\ e^t & -e^{-t} \end{bmatrix} = \begin{bmatrix} e^t & -e^{-t} \\ e^t & e^{-t} \end{bmatrix}$. Thus, since $\Psi' = P(t)\Psi$, we know that $\Psi(t)$ is a solution matrix. To show $\Psi(t)$ is a fundamental matrix, we need to verify that $\det[\Psi(t)] \neq 0$. Since $\det[\Psi(t)] \equiv -2$, we know $\Psi(t)$ is a fundamental matrix.

13 (b). $\hat{\Psi}(t) = \begin{bmatrix} \sinh t & \cosh t \\ \cosh t & \sinh t \end{bmatrix} = \frac{1}{2}\begin{bmatrix} e^t - e^{-t} & e^t + e^{-t} \\ e^t + e^{-t} & e^t - e^{-t} \end{bmatrix}$. Thus, we need a matrix $C = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ such that

$\frac{1}{2}\begin{bmatrix} e^t - e^{-t} & e^t + e^{-t} \\ e^t + e^{-t} & e^t - e^{-t} \end{bmatrix} = \begin{bmatrix} e^t & e^{-t} \\ e^t & -e^{-t} \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Expanding the right-hand side of this matrix equation, we arrive at the requirement

$\frac{1}{2}\begin{bmatrix} e^t - e^{-t} & e^t + e^{-t} \\ e^t + e^{-t} & e^t - e^{-t} \end{bmatrix} = \begin{bmatrix} ae^t + ce^{-t} & be^t + de^{-t} \\ ae^t - ce^{-t} & be^t - de^{-t} \end{bmatrix}$. Comparing entries, we see that

$a = 1/2, c = -1/2, b = 1/2,$ and $d = 1/2$. Thus, $C = \begin{bmatrix} 1/2 & 1/2 \\ -1/2 & 1/2 \end{bmatrix}$.

13 (c). $\det[C] = 1/2$ and thus, $\hat{\Psi}(t)$ is a fundamental matrix.

14 (a). Since $\det[\Psi(t)] \neq 0$, we know $\Psi(t)$ is a fundamental matrix.

14 (b). $\hat{\Psi}(t) = \begin{bmatrix} 2e^t - e^{-t} & e^t + 3e^{-t} \\ 2e^t + e^{-t} & e^t - 3e^{-t} \end{bmatrix} = \begin{bmatrix} e^t & -e^{-t} \\ e^t & e^{-t} \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & -3 \end{bmatrix}$. Thus, $C = \begin{bmatrix} 2 & 1 \\ 1 & -3 \end{bmatrix}$.

14 (c). $\det[C] = -7$ and thus, $\hat{\Psi}(t)$ is a fundamental matrix.

15 (a). We first show that $\hat{\Psi} = P(t)\Psi$. Now, $\Psi'(t) = \begin{bmatrix} e^t & -2e^{-2t} \\ 0 & 6e^{-2t} \end{bmatrix}$ whereas

$P(t)\Psi(t) = \begin{bmatrix} 1 & 1 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} e^t & e^{-2t} \\ 0 & -3e^{-2t} \end{bmatrix} = \begin{bmatrix} e^t & -2e^{-2t} \\ 0 & 6e^{-2t} \end{bmatrix}$. Thus, since $\Psi' = P(t)\Psi$, we know that $\Psi(t)$ is a solution matrix. To show $\Psi(t)$ is a fundamental matrix, we need to verify that $\det[\Psi(t)] \neq 0$. Now $\det[\Psi(t)] = -3e^{-t}$ and thus is never zero for any value t. Therefore, $\Psi(t)$ is a fundamental matrix.

15 (b). $\hat{\Psi}(t) = \begin{bmatrix} 2e^{-2t} & 0 \\ -6e^{-2t} & 0 \end{bmatrix}$ and so we need a matrix $C = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ such that

$\begin{bmatrix} 2e^{-2t} & 0 \\ -6e^{-2t} & 0 \end{bmatrix} = \begin{bmatrix} e^t & e^{-2t} \\ 0 & -3e^{-2t} \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Expanding the right-hand side of this matrix equation, we arrive at the requirement

$\begin{bmatrix} 2e^{-2t} & 0 \\ -6e^{-2t} & 0 \end{bmatrix} = \begin{bmatrix} ae^t + ce^{-2t} & be^t + de^{-2t} \\ -3ce^{-2t} & -3de^{-2t} \end{bmatrix}$. Comparing entries in the second column, we see that $d = 0$ and $b = 0$. Comparing entries in the first column, we see $c = 2$ and $a = 0$. Thus, $C = \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}$.

15 (c). $\det[C]=0$ and thus, $\widehat{\Psi}(t)$ is a solution matrix but not a fundamental matrix.

16 (a). Since $\det[\Psi(t)]=-6e^{2t}\neq 0$, we know $\Psi(t)$ is a fundamental matrix.

16 (b). $\widehat{\Psi}(t)=\begin{bmatrix} e^t+e^{-t} & 4e^{2t} & e^t+4e^{2t} \\ -2e^{-t} & e^{2t} & e^{2t} \\ 0 & 3e^{2t} & 3e^{2t} \end{bmatrix}=\begin{bmatrix} e^t & e^{-t} & 4e^{2t} \\ 0 & -2e^{-t} & e^{2t} \\ 0 & 0 & 3e^{2t} \end{bmatrix}\begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$. Thus, $C=\begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$.

16 (c). $\det[C]=1$ and thus, $\widehat{\Psi}(t)$ is a fundamental matrix.

17. For $\Psi(t)=\begin{bmatrix} e^t & e^{-t} \\ e^t & -e^{-t} \end{bmatrix}$, we need a matrix C such that $\widehat{\Psi}(t)=\Psi(t)C$ where $\widehat{\Psi}(0)=I$. This requirement means that $I=\widehat{\Psi}(0)=\Psi(0)C$.
Equivalently, C is the inverse of $\Psi(0)=\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$. Thus, $C=\Psi(0)^{-1}=\frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$.

18. For $\Psi(t)=\begin{bmatrix} e^t & e^{-2t} \\ 0 & -3e^{-2t} \end{bmatrix}$, we need a matrix C such that $\widehat{\Psi}(t)=\Psi(t)C$ where $\widehat{\Psi}(0)=I$. This requirement means that $I=\widehat{\Psi}(0)=\Psi(0)C$.
Equivalently, C is the inverse of $\Psi(0)=\begin{bmatrix} 1 & 1 \\ 0 & -3 \end{bmatrix}$. Thus, $C=\Psi(0)^{-1}=-\frac{1}{3}\begin{bmatrix} -3 & -1 \\ 0 & 1 \end{bmatrix}$.

Section 6.5

1 (a). $A\mathbf{x}_1=\begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix}=\begin{bmatrix} 2 \\ -2 \end{bmatrix}=2\mathbf{x}_1$. Thus, $\mathbf{x}_1$ is an eigenvector corresponding to the eigenvalue $\lambda_1=2$. Similarly, $A\mathbf{x}_2=\begin{bmatrix} 4 & 2 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} -2 \\ 1 \end{bmatrix}=\begin{bmatrix} -6 \\ 3 \end{bmatrix}=3\mathbf{x}_2$. Thus, $\mathbf{x}_2$ is an eigenvector corresponding to the eigenvalue $\lambda_2=3$.

1 (b). Solutions are $\mathbf{y}_1(t)=e^{2t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\mathbf{y}_2(t)=e^{3t}\begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

1 (c). The Wronskian is $W(t)=\det\begin{bmatrix} e^{2t} & -2e^{3t} \\ -e^{2t} & e^{3t} \end{bmatrix}=e^{5t}-2e^{5t}=-e^{5t}$. Since $W(t)$ is nonzero for any value t, the two solutions form a fundamental set of solutions.

2 (a). $A\mathbf{x}_1=\begin{bmatrix} 7 & -3 \\ 16 & -7 \end{bmatrix}\begin{bmatrix} 3 \\ 8 \end{bmatrix}=\begin{bmatrix} -3 \\ -8 \end{bmatrix}=-1\mathbf{x}_1$. Thus, $\mathbf{x}_1$ is an eigenvector corresponding to the eigenvalue $\lambda_1=-1$. Similarly, $A\mathbf{x}_2=\begin{bmatrix} 7 & -3 \\ 16 & -7 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix}=\begin{bmatrix} 1 \\ 2 \end{bmatrix}=1\mathbf{x}_2$. Thus, $\mathbf{x}_2$ is an eigenvector corresponding to the eigenvalue $\lambda_2=1$.

2 (b). Solutions are $\mathbf{y}_1(t)=e^{-t}\begin{bmatrix} 3 \\ 8 \end{bmatrix}$ and $\mathbf{y}_2(t)=e^{t}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

2 (c). The Wronskian is $W(t) = \det\begin{bmatrix} 3e^{-t} & e^{t} \\ 8e^{-t} & 2e^{t} \end{bmatrix} = -2 \neq 0$. Since $W(t)$ is nonzero for any value t, the two solutions form a fundamental set of solutions.

3 (a). The vector $\mathbf{x}_1 = \mathbf{0}$ cannot be an eigenvector since an eigenvector must be nonzero. Considering the other vector, $A\mathbf{x}_2 = \begin{bmatrix} 11 & 5 \\ -22 & -10 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} = \mathbf{x}_2$. Thus, $\mathbf{x}_2$ is an eigenvector corresponding to the eigenvalue $\lambda_2 = 1$.

3 (b). The solution is $\mathbf{y}_2(t) = e^{t}\begin{bmatrix} 1 \\ -2 \end{bmatrix}$.

4 (a). $A\mathbf{x}_1 = \begin{bmatrix} -5 & 2 \\ -18 & 7 \end{bmatrix}\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} = 1\mathbf{x}_1$, $A\mathbf{x}_2 = \begin{bmatrix} -5 & 2 \\ -18 & 7 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ -4 \end{bmatrix}$. Thus, $\mathbf{x}_1$ is an eigenvector corresponding to the eigenvalue $\lambda_1 = 1$, but $\mathbf{x}_2$ is not an eigenvector.

4 (b). The solution is $\mathbf{y}_1(t) = e^{t}\begin{bmatrix} 1 \\ 3 \end{bmatrix}$.

5 (a). $A\mathbf{x}_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix} = -\mathbf{x}_1$. Thus, $\mathbf{x}_1$ is an eigenvector corresponding to the eigenvalue $\lambda_1 = -1$. Similarly, $A\mathbf{x}_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \mathbf{x}_2$. Thus, $\mathbf{x}_2$ is an eigenvector corresponding to the eigenvalue $\lambda_2 = 1$.

5 (b). Solutions are $\mathbf{y}_1(t) = e^{-t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\mathbf{y}_2(t) = e^{t}\begin{bmatrix} 2 \\ 2 \end{bmatrix}$.

5 (c). The Wronskian is $W(t) = \det\begin{bmatrix} e^{-t} & 2e^{t} \\ -e^{-t} & 2e^{t} \end{bmatrix} = 2 + 2 = 4$. Since $W(t)$ is nonzero for any value t, the two solutions form a fundamental set of solutions.

6 (a). $A\mathbf{x}_1 = \begin{bmatrix} 2 & -1 \\ -4 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ -2 \end{bmatrix} = \begin{bmatrix} 4 \\ -8 \end{bmatrix} = 4\mathbf{x}_1$. Thus, $\mathbf{x}_1$ is an eigenvector corresponding to the eigenvalue $\lambda_1 = 4$. Similarly, $A\mathbf{x}_2 = \begin{bmatrix} 2 & -1 \\ -4 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0\mathbf{x}_2$. Thus, $\mathbf{x}_2$ is an eigenvector corresponding to the eigenvalue $\lambda_2 = 0$.

6 (b). Solutions are $\mathbf{y}_1(t) = e^{4t}\begin{bmatrix} 1 \\ -2 \end{bmatrix}$ and $\mathbf{y}_2(t) = e^{0}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

6 (c). The Wronskian is $W(t) = \det\begin{bmatrix} e^{4t} & 1 \\ -2e^{4t} & 2 \end{bmatrix} = 4e^{4t} \neq 0$. Since $W(t)$ is nonzero for any value t, the two solutions form a fundamental set of solutions.

7. For $A=\begin{bmatrix}-4 & 3\\ -4 & 4\end{bmatrix}$ the equation $(A-2I)\mathbf{x}=\mathbf{0}$ has the form $\begin{bmatrix}-6 & 3\\ -4 & 2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$. Elementary row operations $[-(1/3)R_1$ then $R_2+2R_1]$ can be used to row reduce the system to $\begin{bmatrix}2 & -1\\ 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$ or $2x_1=x_2$. Thus an eigenvector is $\mathbf{x}=\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}x_1\\ 2x_1\end{bmatrix}=x_1\begin{bmatrix}1\\ 2\end{bmatrix}, x_1\neq 0$.

8. For $A=\begin{bmatrix}5 & 3\\ -4 & -3\end{bmatrix}$ the equation $(A+I)\mathbf{x}=\mathbf{0}$ has the form $\begin{bmatrix}6 & 3\\ -4 & -2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$. Thus an eigenvector is $\mathbf{x}=\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}x_1\\ -2x_1\end{bmatrix}=x_1\begin{bmatrix}1\\ -2\end{bmatrix}, x_1\neq 0$.

9. For $A=\begin{bmatrix}1 & 1\\ -4 & 6\end{bmatrix}$ the equation $(A-5I)\mathbf{x}=\mathbf{0}$ has the form $\begin{bmatrix}-4 & 1\\ -4 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$. Elementary row operations $[-(1/4)R_1$ then $R_2+4R_1]$ can be used to row reduce the system to $\begin{bmatrix}1 & -1/4\\ 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$ or $4x_1=x_2$. Thus an eigenvector is $\mathbf{x}=\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}x_1\\ 4x_1\end{bmatrix}=x_1\begin{bmatrix}1\\ 4\end{bmatrix}, x_1\neq 0$.

10. For $A=\begin{bmatrix}1 & -7 & 3\\ -1 & -1 & 1\\ 4 & -4 & 0\end{bmatrix}$ the equation $(A+4I)\mathbf{x}=\mathbf{0}$ has the form $\begin{bmatrix}5 & -7 & 3\\ -1 & 3 & 1\\ 4 & -4 & 4\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$. Thus an eigenvector is $\mathbf{x}=\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}2x_2\\ x_2\\ -x_2\end{bmatrix}=x_2\begin{bmatrix}2\\ 1\\ -1\end{bmatrix}, x_2\neq 0$.

11. For $A=\begin{bmatrix}3 & 1 & 1\\ -1 & 1 & -1\\ 2 & 1 & 2\end{bmatrix}$ the equation $(A-2I)\mathbf{x}=\mathbf{0}$ has the form $\begin{bmatrix}1 & 1 & 1\\ -1 & -1 & -1\\ 2 & 1 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$. Elementary row operations $(R_2+R_1$, then $R_3-2R_1)$ followed by R_1+R_3 and $(-R_3)$ can be used to row reduce the system to $\begin{bmatrix}1 & 0 & -1\\ 0 & 0 & 0\\ 0 & 1 & 2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$, or $\begin{matrix}x_1=x_3\\ x_2=-2x_3\end{matrix}$. Thus an eigenvector is $\mathbf{x}=\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}x_3\\ -2x_3\\ x_3\end{bmatrix}=x_3\begin{bmatrix}1\\ -2\\ 1\end{bmatrix}, x_3\neq 0$.

12. For $A=\begin{bmatrix}1 & 3 & 1\\ 2 & 1 & 2\\ 4 & 3 & -2\end{bmatrix}$ the equation $(A-5I)\mathbf{x}=\mathbf{0}$ has the form $\begin{bmatrix}-4 & 3 & 1\\ 2 & -4 & 2\\ 4 & 3 & -7\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix}$.

Thus an eigenvector is $\mathbf{x}=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}x_1\\x_1\\x_1\end{bmatrix}=x_1\begin{bmatrix}1\\1\\1\end{bmatrix}, x_1\neq 0$.

13. For $A=\begin{bmatrix}-2&3&1\\-8&13&5\\11&-17&-6\end{bmatrix}$ the equation $A\mathbf{x}=\mathbf{0}$ has the form

$\begin{bmatrix}-2&3&1\\-8&13&5\\11&-17&-6\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$. Since the arithmetic looks forbidding, we turned to MATLAB and used the RREF command. MATLAB says the system is equivalent to $\begin{bmatrix}1&0&1\\0&1&1\\0&0&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$, or

$\begin{matrix}x_1=-x_3\\x_2=-x_3\end{matrix}$. Thus, an eigenvector is $\mathbf{x}=\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}-x_3\\-x_3\\x_3\end{bmatrix}=x_3\begin{bmatrix}-1\\-1\\1\end{bmatrix}, x_3\neq 0$.

14. The characteristic polynomial is $p(\lambda)=\begin{vmatrix}-5-\lambda&1\\0&4-\lambda\end{vmatrix}$, or $p(\lambda)=(\lambda+5)(\lambda-4)$. Thus, the eigenvalues are $\lambda_1=-5$ and $\lambda_2=4$.

15. For $A=\begin{bmatrix}8&0\\3&2\end{bmatrix}$, the characteristic polynomial is $p(\lambda)=\begin{vmatrix}8-\lambda&0\\3&2-\lambda\end{vmatrix}$, or $p(\lambda)=(8-\lambda)(2-\lambda)$. Thus, the eigenvalues are $\lambda_1=8$ and $\lambda_2=2$.

16. The characteristic polynomial is $p(\lambda)=\begin{vmatrix}3-\lambda&-3\\-6&6-\lambda\end{vmatrix}$, or $p(\lambda)=(\lambda)(\lambda-9)$. Thus, the eigenvalues are $\lambda_1=0$ and $\lambda_2=9$.

17. For $A=\begin{bmatrix}5&2\\4&3\end{bmatrix}$, the characteristic polynomial is $p(\lambda)=\begin{vmatrix}5-\lambda&2\\4&3-\lambda\end{vmatrix}$, or

$p(\lambda)=(5-\lambda)(3-\lambda)-8=\lambda^2-8\lambda+7=(\lambda-7)(\lambda-1)$. Thus, the eigenvalues are $\lambda_1=7$ and $\lambda_2=1$.

18. The characteristic polynomial is

$$p(\lambda)=\begin{vmatrix}5-\lambda&0&0\\0&1-\lambda&3\\0&2&2-\lambda\end{vmatrix}$$

or $p(\lambda)=(5-\lambda)(\lambda^2-3\lambda-4)=(5-\lambda)(\lambda-4)(\lambda+1)=-\lambda^3+8\lambda^2-11\lambda-20$ and the eigenvalues are $\lambda_1=-1, \lambda_2=4$, and $\lambda_3=5$.

19. For $A=\begin{bmatrix}-2 & 3 & 1\\ -8 & 13 & 5\\ 11 & -17 & -6\end{bmatrix}$, the characteristic polynomial is

$p(\lambda)=\begin{vmatrix}-2-\lambda & 3 & 1\\ -8 & 13-\lambda & 5\\ 11 & -17 & -6-\lambda\end{vmatrix}$. Given the arithmetic required to find the characteristic polynomial, it is advisable to use a computer routine such as poly(A) from MATLAB. However, it is possible to find $p(\lambda)$ by hand:

$$\begin{vmatrix}-2-\lambda & 3 & 1\\ -8 & 13-\lambda & 5\\ 11 & -17 & -6-\lambda\end{vmatrix}=(-2-\lambda)\begin{vmatrix}13-\lambda & 5\\ -17 & -6-\lambda\end{vmatrix}-3\begin{vmatrix}-8 & 5\\ 11 & -6-\lambda\end{vmatrix}+\begin{vmatrix}-8 & 13-\lambda\\ 11 & -17\end{vmatrix}$$

or $p(\lambda)=(-2-\lambda)(\lambda^2-7\lambda+7)-3(8\lambda-7)+(11\lambda-7)=-\lambda^3+5\lambda^2-6\lambda$.

Thus, $p(\lambda)=-\lambda(\lambda^2-5\lambda+6)=-\lambda(\lambda-3)(\lambda-2)$ and hence the eigenvalues are $\lambda_1=0, \lambda_2=3$, and $\lambda_3=2$.

20. The characteristic polynomial is

$p(\lambda)=\begin{vmatrix}1-\lambda & -7 & 3\\ -1 & -1-\lambda & 1\\ 4 & -4 & -\lambda\end{vmatrix}$

or $p(\lambda)=-\lambda(\lambda^2-16)=(-\lambda)(\lambda-4)(\lambda+4)=-\lambda^3+16\lambda$ and the eigenvalues are $\lambda_1=0, \lambda_2=4$, and $\lambda_3=-4$.

21. The eigenvalues are $\lambda_1=-2$ and $\lambda_2=2$ with corresponding eigenvectors

$\mathbf{x}_1=\begin{bmatrix}3\\2\end{bmatrix}$ and $\mathbf{x}_2=\begin{bmatrix}1\\2\end{bmatrix}$. A fundamental set of solutions consists of the functions

$\mathbf{y}_1(t)=e^{-2t}\begin{bmatrix}3\\2\end{bmatrix}$ and $\mathbf{y}_2(t)=e^{2t}\begin{bmatrix}1\\2\end{bmatrix}$. Therefore, the general solution is

$\mathbf{y}(t)=c_1e^{-2t}\begin{bmatrix}3\\2\end{bmatrix}+c_2e^{2t}\begin{bmatrix}1\\2\end{bmatrix}$. The solution of the initial value problem is $\mathbf{y}(t)=e^{-2t}\begin{bmatrix}3\\2\end{bmatrix}+e^{2t}\begin{bmatrix}1\\2\end{bmatrix}$.

22. The eigenvalues are $\lambda_1=-1$ and $\lambda_2=3$ with corresponding eigenvectors

$\mathbf{x}_1=\begin{bmatrix}1\\-2\end{bmatrix}$ and $\mathbf{x}_2=\begin{bmatrix}3\\-2\end{bmatrix}$. A fundamental set of solutions consists of the functions

$\mathbf{y}_1(t)=e^{-t}\begin{bmatrix}1\\-2\end{bmatrix}$ and $\mathbf{y}_2(t)=e^{3t}\begin{bmatrix}3\\-2\end{bmatrix}$. Therefore, the general solution is

$\mathbf{y}(t)=c_1e^{-t}\begin{bmatrix}1\\-2\end{bmatrix}+\mathrm{c}_2e^{3t}\begin{bmatrix}3\\-2\end{bmatrix}$. The solution of the initial value problem is

$\mathbf{y}(t)=\begin{bmatrix}3e^{3(t-1)}-e^{-(t-1)}\\ -2e^{3(t-1)}+2e^{-(t-1)}\end{bmatrix}$.

23. The eigenvalues are $\lambda_1 = 3$ and $\lambda_2 = 5$ with corresponding eigenvectors $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\mathbf{x}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. A fundamental set of solutions consists of the functions $\mathbf{y}_1(t) = e^{3t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\mathbf{y}_2(t) = e^{5t}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Therefore, the general solution is $\mathbf{y}(t) = c_1 e^{3t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{5t}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.
The solution of the initial value problem is $\mathbf{y}(t) = 3e^{3t}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + 4e^{5t}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

24. The eigenvalues are $\lambda_1 = -0.11$ and $\lambda_2 = -0.05$ with corresponding eigenvectors $\mathbf{x}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\mathbf{x}_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. A fundamental set of solutions consists of the functions $\mathbf{y}_1(t) = e^{-0.11t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\mathbf{y}_2(t) = e^{-0.05t}\begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

25. The eigenvalues are $\lambda_1 = 1, \lambda_2 = 2,$ and $\lambda_3 = 3$ with corresponding eigenvectors $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$, $\mathbf{x}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, and $\mathbf{x}_3 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$. A fundamental set of solutions consists of the functions $\mathbf{y}_1(t) = e^{t}\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$, $\mathbf{y}_2(t) = e^{2t}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, and $\mathbf{y}_3(t) = e^{3t}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$. The solution of the initial value problem is $\mathbf{y}(t) = e^{t}\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} + e^{2t}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 2e^{3t}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$.

26. The eigenvalues are $\lambda_1 = -2, \lambda_2 = 1,$ and $\lambda_3 = 4$ with corresponding eigenvectors $\mathbf{x}_1 = \begin{bmatrix} 1 \\ -3 \\ 3 \end{bmatrix}$, $\mathbf{x}_2 = \begin{bmatrix} 2 \\ -3 \\ 0 \end{bmatrix}$, and $\mathbf{x}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. A fundamental set of solutions consists of the functions $\mathbf{y}_1(t) = e^{-2t}\begin{bmatrix} 1 \\ -3 \\ 3 \end{bmatrix}$, $\mathbf{y}_2(t) = e^{t}\begin{bmatrix} 2 \\ -3 \\ 0 \end{bmatrix}$, and $\mathbf{y}_3(t) = e^{4t}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. The solution of the initial value problem is $\mathbf{y}(t) = e^{-2t}\begin{bmatrix} 1 \\ -3 \\ 3 \end{bmatrix} + e^{t}\begin{bmatrix} -2 \\ 3 \\ 0 \end{bmatrix} + e^{4t}\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$.

27. The eigenvalues are $\lambda_1 = -2, \lambda_2 = 2$, and $\lambda_3 = 4$ with corresponding eigenvectors $\mathbf{x}_1 = \begin{bmatrix} 3 \\ 4 \\ -8 \end{bmatrix}$, $\mathbf{x}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, and $\mathbf{x}_3 = \begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix}$. A fundamental set of solutions consists of the functions $\mathbf{y}_1(t) = e^{-2t}\begin{bmatrix} 3 \\ 4 \\ -8 \end{bmatrix}$, $\mathbf{y}_2(t) = e^{2t}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, and $\mathbf{y}_3(t) = e^{4t}\begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix}$.

28. The eigenvalues are $\lambda_1 = 1, \lambda_2 = 3$, and $\lambda_3 = 5$ with corresponding eigenvectors $\mathbf{x}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, $\mathbf{x}_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, and $\mathbf{x}_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$. A fundamental set of solutions consists of the functions $\mathbf{y}_1(t) = e^{t}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, $\mathbf{y}_2(t) = e^{3t}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, and $\mathbf{y}_3(t) = e^{5t}\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$.

29. The eigenvalues are $\lambda_1 = -1, \lambda_2 = 1$, and $\lambda_3 = 2$ with corresponding eigenvectors $\mathbf{x}_1 = \begin{bmatrix} 1 \\ -4 \\ -5 \end{bmatrix}$, $\mathbf{x}_2 = \begin{bmatrix} 1 \\ -2 \\ -3 \end{bmatrix}$, and $\mathbf{x}_3 = \begin{bmatrix} 2 \\ -2 \\ -1 \end{bmatrix}$. A fundamental set of solutions consists of the functions $\mathbf{y}_1(t) = e^{-t}\begin{bmatrix} 1 \\ -4 \\ -5 \end{bmatrix}$, $\mathbf{y}_2(t) = e^{t}\begin{bmatrix} 1 \\ -2 \\ -3 \end{bmatrix}$, and $\mathbf{y}_3(t) = e^{2t}\begin{bmatrix} 2 \\ -2 \\ -1 \end{bmatrix}$.

30. The eigenvalues are $\lambda_1 = -2, \lambda_2 = 1$, and $\lambda_3 = 2$ with corresponding eigenvectors $\mathbf{x}_1 = \begin{bmatrix} 1 \\ -5 \\ -6 \end{bmatrix}$, $\mathbf{x}_2 = \begin{bmatrix} 1 \\ -2 \\ -3 \end{bmatrix}$, and $\mathbf{x}_3 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$. A fundamental set of solutions consists of the functions $\mathbf{y}_1(t) = e^{-2t}\begin{bmatrix} 1 \\ -5 \\ -6 \end{bmatrix}$, $\mathbf{y}_2(t) = e^{t}\begin{bmatrix} 1 \\ -2 \\ -3 \end{bmatrix}$, and $\mathbf{y}_3(t) = e^{2t}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$.

31. We need to have $\begin{bmatrix} 2 & x \\ 1 & -5 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \end{bmatrix} = \lambda\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ for some value λ. Therefore, equating vectors, it follows that we $2 - x = \lambda$ and $1 + 5 = -\lambda$. This requires $\lambda = -6$ and $x = 8$.

32. We need to have $\begin{bmatrix} x & y \\ 2x & -y \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = 1\begin{bmatrix} -1 \\ 1 \end{bmatrix}$. Therefore, it follows that $-x + y = -1$ and $-2x$ - y $= 1$. This requires $x = 0,\ y = -1$.

39 (a). The eigenvalues are $\lambda_1 = -3$ and $\lambda_2 = -1$. Corresponding eigenvectors are $\mathbf{x}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\mathbf{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. The general solution is $\mathbf{y}(t) = c_1 e^{-3\tau}\begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 e^{-\tau}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

39 (b). The solution of the initial value problem is $\mathbf{y}(t) = -\frac{Q_0}{2}e^{-3\tau}\begin{bmatrix}1\\-1\end{bmatrix} + 3\frac{Q_0}{2}e^{-\tau}\begin{bmatrix}1\\1\end{bmatrix}$.

Therefore, $Q_1(t) = \frac{Q_0}{2}\left(-e^{-3\tau} + 3e^{-\tau}\right)$ and $Q_2(t) = \frac{Q_0}{2}\left(e^{-3\tau} + 3e^{-\tau}\right)$.

Note that $0 < Q_1(\tau) < Q_2(\tau)$. Therefore, we need τ such that $\frac{Q_0}{2}(e^{-3\tau} + 3e^{-\tau}) < .01Q_0 \Rightarrow (e^{-3\tau} + 3e^{-\tau}) < .02$. Graphically, we find that a value $\tau \approx 5.011$ will suffice. Since $t = (V/r)\tau = 50\tau$, we obtain a value of $t \approx 250.55$ sec or $t \approx 4.18$ min.

40 (a). The eigenvalues are $\lambda_1 = -1$ and $\lambda_2 = \lambda_3 = -4$. Corresponding eigenvectors are $\mathbf{x}_1 = \begin{bmatrix}1\\1\\1\end{bmatrix}$, $\mathbf{x}_2 = \begin{bmatrix}1\\-1\\0\end{bmatrix}$, and $\mathbf{x}_3 = \begin{bmatrix}1\\0\\-1\end{bmatrix}$. The general solution is

$$\mathbf{Q}(t) = c_1e^{-\tau}\begin{bmatrix}1\\1\\1\end{bmatrix} + c_2e^{-4\tau}\begin{bmatrix}1\\-1\\0\end{bmatrix} + c_3e^{-4\tau}\begin{bmatrix}1\\0\\-1\end{bmatrix}.$$

40 (b). The solution of the initial value problem is $\mathbf{y}(t) = 2Q_0e^{-\tau}\begin{bmatrix}1\\1\\1\end{bmatrix} - Q_0e^{-4\tau}\begin{bmatrix}1\\0\\-1\end{bmatrix}$.

Therefore, $\mathbf{Q}(t) = Q_0\begin{bmatrix}2e^{-\frac{r}{v}t} - e^{-4\frac{r}{v}t}\\ 2e^{-\frac{r}{v}t}\\ 2e^{-\frac{r}{v}t} + e^{-4\frac{r}{v}t}\end{bmatrix}$.

Section 6.6

1. For $A = \begin{bmatrix}2 & 1\\-1 & 2\end{bmatrix}$, the characteristic polynomial is $p(\lambda) = \begin{vmatrix}2-\lambda & 1\\-1 & 2-\lambda\end{vmatrix} = \lambda^2 - 4\lambda + 5$.

Therefore, the eigenvalues are $\lambda_1 = 2+i$ and $\lambda_2 = 2-i$. We find an eigenvector $\mathbf{x}_1$ by solving $(A - \lambda_1 I)\mathbf{x} = \mathbf{0}$ or

$\begin{bmatrix}2-(2+i) & 1\\-1 & 2-(2+i)\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$. This equation reduces to $\begin{bmatrix}-i & 1\\-1 & -i\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$. The elementary row operation $R_2 + iR_1$ reduces the system to $\begin{bmatrix}-i & 1\\0 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$ or $-ix_1 + x_2 = 0$.

Thus, an eigenvector is $\mathbf{x} = \begin{bmatrix}x_1\\x_2\end{bmatrix} = \begin{bmatrix}x_1\\ix_1\end{bmatrix} = x_1\begin{bmatrix}1\\i\end{bmatrix}$, $x_1 \neq 0$. Since the eigenvalues and eigenvectors occur in conjugate pairs, the eigenpairs are

$\lambda_1 = 2+i$, $\mathbf{x}_1 = \begin{bmatrix}1\\i\end{bmatrix}$ and $\lambda_2 = 2-i$, $\mathbf{x}_2 = \begin{bmatrix}1\\-i\end{bmatrix}$.

2. The characteristic polynomial is $p(\lambda)=\begin{vmatrix}-\lambda & -9\\ 1 & -\lambda\end{vmatrix}=\lambda^2+9$. Therefore, the eigenvalues are $\lambda_1=3i$ and $\lambda_2=-3i$. We find an eigenvector $\mathbf{x}_1$ by solving $(A-\lambda_1 I)\mathbf{x}=\mathbf{0}$ or $\begin{bmatrix}-3i & -9\\ 1 & -3i\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$. Thus, an eigenvector is $\mathbf{x}_1=\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}3ix_2\\ x_2\end{bmatrix}=x_2\begin{bmatrix}3i\\ 1\end{bmatrix}$, $x_2\neq 0$. Since the eigenvalues and eigenvectors occur in conjugate pairs, the eigenpairs are

$$\lambda_1=3i,\ \mathbf{x}_1=\begin{bmatrix}3i\\ 1\end{bmatrix} \text{ and } \lambda_2=-3i,\ \mathbf{x}_2=\begin{bmatrix}-3i\\ 1\end{bmatrix}.$$

3. For $A=\begin{bmatrix}6 & -13\\ 1 & 0\end{bmatrix}$, the characteristic polynomial is $p(\lambda)=\begin{vmatrix}6-\lambda & -13\\ 1 & -\lambda\end{vmatrix}=\lambda^2-6\lambda+13$. Therefore, the eigenvalues are $\lambda_1=3+2i$ and $\lambda_2=3-2i$. We find an eigenvector $\mathbf{x}_1$ by solving $(A-\lambda_1 I)\mathbf{x}=\mathbf{0}$ or $\begin{bmatrix}6-(3+2i) & -13\\ 1 & -(3+2i)\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$. This equation reduces to $\begin{bmatrix}3-2i & -13\\ 1 & -3-2i\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$. The elementary row operations $R_1\leftrightarrow R_2$, then $R_2-(3-2i)R_1$ reduces the system to $\begin{bmatrix}1 & -(3+2i)\\ 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$ or $x_1-(3+2i)x_2=0$. Thus, an eigenvector is $\mathbf{x}=\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}(3+2i)x_2\\ x_2\end{bmatrix}=x_2\begin{bmatrix}3+2i\\ 1\end{bmatrix}$, $x_2\neq 0$. Choosing $x_2=1$, we obtain the eigenvector $\mathbf{x}_1=\begin{bmatrix}3+2i\\ 1\end{bmatrix}$. Since the eigenvalues and eigenvectors occur in conjugate pairs, the eigenpairs are

$$\lambda_1=3+2i,\ \mathbf{x}_1=\begin{bmatrix}3+2i\\ 1\end{bmatrix} \text{ and } \lambda_2=3-2i,\ \mathbf{x}_2=\begin{bmatrix}3-2i\\ 1\end{bmatrix}.$$

4. The characteristic polynomial is $p(\lambda)=\begin{vmatrix}3-\lambda & 1\\ -2 & 1-\lambda\end{vmatrix}=(\lambda-2)^2+1$. Therefore, the eigenvalues are $\lambda_1=2+i$ and $\lambda_2=2-i$. We find an eigenvector $\mathbf{x}_1$ by solving $(A-\lambda_1 I)\mathbf{x}=\mathbf{0}$ or $\begin{bmatrix}1-i & 1\\ -2 & -1-i\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$. Thus, an eigenvector is $\mathbf{x}_1=\begin{bmatrix}1\\ -1+i\end{bmatrix}$. Since the eigenvalues and eigenvectors occur in conjugate pairs, the eigenpairs are

$$\lambda_1=2+i,\ \mathbf{x}_1=\begin{bmatrix}1\\ -1+i\end{bmatrix} \text{ and } \lambda_2=2-i,\ \mathbf{x}_2=\begin{bmatrix}1\\ -1-i\end{bmatrix}.$$

5. Using the EIG command in MATLAB, we find eigenvalues $\lambda_1=1, \lambda_2=1+i$, and $\lambda_3=1-i$. For each eigenvalue λ, we use the RREF command in MATLAB to solve $(A-\lambda I)\mathbf{x}=\mathbf{0}$, finding $\mathbf{x}_1=\begin{bmatrix}-3\\ 0\\ 1\end{bmatrix}$, $\mathbf{x}_2=\begin{bmatrix}3+2i\\ 1\\ -1-i\end{bmatrix}$, and $\mathbf{x}_3=\begin{bmatrix}3-2i\\ 1\\ -1+i\end{bmatrix}$.

Note that another possible eigenvector for λ_2 is $\mathbf{x}_2 = (-1+i)\begin{bmatrix} 3+2i \\ 1 \\ -1-i \end{bmatrix} = \begin{bmatrix} -5+i \\ -1+i \\ 2 \end{bmatrix}$.

6. The eigenvalues are $\lambda_1 = 2, \lambda_2 = 2+3i,$ and $\lambda_3 = 2-3i$. The corresponding eigenvectors are $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$, $\mathbf{x}_2 = \begin{bmatrix} -4-i \\ 3i \\ 1+i \end{bmatrix}$, and $\mathbf{x}_3 = \begin{bmatrix} -4+i \\ -3i \\ 1-i \end{bmatrix}$.

7. As in Example 1, $\mathbf{y}(t) = e^{(4+2i)t}\begin{bmatrix} 4 \\ -1+i \end{bmatrix} = e^{4t}(\cos 2t + i\sin 2t)\begin{bmatrix} 4 \\ -1+i \end{bmatrix}$ is one solution of $\mathbf{y}' = A\mathbf{y}$. Expanding and collecting real and imaginary parts, we obtain $\mathbf{y}(t) = e^{4t}\begin{bmatrix} 4\cos 2t \\ -\cos 2t - \sin 2t \end{bmatrix} + ie^{4t}\begin{bmatrix} 4\sin 2t \\ \cos 2t - \sin 2t \end{bmatrix}$. Thus, a fundamental set of solutions can be formed from $\mathbf{y}_1(t) = e^{4t}\begin{bmatrix} 4\cos 2t \\ -\cos 2t - \sin 2t \end{bmatrix}$ and $\mathbf{y}_2(t) = e^{4t}\begin{bmatrix} 4\sin 2t \\ \cos 2t - \sin 2t \end{bmatrix}$.

8. $\mathbf{y}(t) = e^{it}\begin{bmatrix} -2+i \\ 5 \end{bmatrix} = (\cos t + i\sin t)\begin{bmatrix} -2+i \\ 5 \end{bmatrix}$ is one solution of $\mathbf{y}' = A\mathbf{y}$. Expanding and collecting real and imaginary parts, we obtain $\mathbf{y}(t) = \begin{bmatrix} -2\cos t - \sin t \\ 5\cos t \end{bmatrix} + i\begin{bmatrix} \cos t - 2\sin t \\ 5\sin t \end{bmatrix}$. Thus, a fundamental set of solutions can be formed from $\mathbf{y}_1(t) = \begin{bmatrix} -2\cos t - \sin t \\ 5\cos t \end{bmatrix}$ and $\mathbf{y}_2(t) = \begin{bmatrix} \cos t - 2\sin t \\ 5\sin t \end{bmatrix}$.

9. As in Example 1, $\mathbf{y}(t) = e^{2it}\begin{bmatrix} -1-i \\ 1 \end{bmatrix} = (\cos 2t + i\sin 2t)\begin{bmatrix} -1-i \\ 1 \end{bmatrix}$ is one solution of $\mathbf{y}' = A\mathbf{y}$. Expanding and collecting real and imaginary parts, we obtain $\mathbf{y}(t) = \begin{bmatrix} -\cos 2t + \sin 2t \\ \cos 2t \end{bmatrix} + i\begin{bmatrix} -\cos 2t - \sin 2t \\ \sin 2t \end{bmatrix}$. Thus, a fundamental set of solutions can be formed from $\mathbf{y}_1(t) = \begin{bmatrix} -\cos 2t + \sin 2t \\ \cos 2t \end{bmatrix}$ and $\mathbf{y}_2(t) = \begin{bmatrix} -\cos 2t - \sin 2t \\ \sin 2t \end{bmatrix}$.

10. $\mathbf{y}(t) = e^{t}(\cos t + i\sin t)\begin{bmatrix} -1+i \\ i \end{bmatrix}$ is one solution of $\mathbf{y}' = A\mathbf{y}$. Expanding and collecting real and imaginary parts, we obtain $\mathbf{y}(t) = e^{t}\begin{bmatrix} -\cos t - \sin t \\ -\sin t \end{bmatrix} + ie^{t}\begin{bmatrix} \cos t - \sin t \\ \cos t \end{bmatrix}$. Thus, a fundamental set of solutions can be formed from $\mathbf{y}_1(t) = e^{t}\begin{bmatrix} -\cos t - \sin t \\ -\sin t \end{bmatrix}$ and $\mathbf{y}_2(t) = e^{t}\begin{bmatrix} \cos t - \sin t \\ \cos t \end{bmatrix}$.

11. As in Example 1, $\mathbf{y}(t)=e^{(2+3i)t}\begin{bmatrix}-5+3i\\3+3i\\2\end{bmatrix}=e^{2t}(\cos 3t+i\sin 3t)\begin{bmatrix}-5+3i\\3+3i\\2\end{bmatrix}$ is one solution of $\mathbf{y}'=A\mathbf{y}$. Expanding and collecting real and imaginary parts, we obtain $\mathbf{y}(t)=e^{2t}\begin{bmatrix}-5\cos 3t-3\sin 3t\\3\cos 3t-3\sin 3t\\2\cos 3t\end{bmatrix}+ie^{2t}\begin{bmatrix}3\cos 3t-5\sin 3t\\3\cos 3t+3\sin 3t\\2\sin 3t\end{bmatrix}$. Thus, two linearly independent solutions are $\mathbf{y}_1(t)=e^{2t}\begin{bmatrix}-5\cos 3t-3\sin 3t\\3\cos 3t-3\sin 3t\\2\cos 3t\end{bmatrix}$ and $\mathbf{y}_2(t)=e^{2t}\begin{bmatrix}3\cos 3t-5\sin 3t\\3\cos 3t+3\sin 3t\\2\sin 3t\end{bmatrix}$. The third solution needed to complete the fundamental set is obtained from the real eigenvalue $\lambda=2$, $\mathbf{y}_3(t)=e^{2t}\begin{bmatrix}1\\0\\-1\end{bmatrix}$.

12. $e^t(\cos 5t+i\sin 5t)\begin{bmatrix}i\\1\\0\\0\end{bmatrix}=e^t\begin{bmatrix}-\sin 5t\\\cos 5t\\0\\0\end{bmatrix}+ie^t\begin{bmatrix}\cos 5t\\\sin 5t\\0\\0\end{bmatrix}$. Also,

$e^t(\cos 2t+i\sin 2t)\begin{bmatrix}0\\0\\i\\1\end{bmatrix}=e^t\begin{bmatrix}0\\0\\-\sin 2t\\\cos 2t\end{bmatrix}+ie^t\begin{bmatrix}0\\0\\\cos 2t\\\sin 2t\end{bmatrix}$ Thus, a fundamental set of solutions can be formed from $e^t\begin{bmatrix}-\sin 5t\\\cos 5t\\0\\0\end{bmatrix}$, $e^t\begin{bmatrix}\cos 5t\\\sin 5t\\0\\0\end{bmatrix}$, $e^t\begin{bmatrix}0\\0\\-\sin 2t\\\cos 2t\end{bmatrix}$, $e^t\begin{bmatrix}0\\0\\\cos 2t\\\sin 2t\end{bmatrix}$.

13. Proceeding as in Exercises 7-12, we find the general solution of $\mathbf{y}'=A\mathbf{y}$ is $\mathbf{y}(t)=c_1e^{2t}\begin{bmatrix}\cos t\\-\sin t\end{bmatrix}+c_2e^{2t}\begin{bmatrix}\sin t\\\cos t\end{bmatrix}$. Imposing the initial condition, $\mathbf{y}(0)=c_1\begin{bmatrix}1\\0\end{bmatrix}+c_2\begin{bmatrix}0\\1\end{bmatrix}=\begin{bmatrix}4\\7\end{bmatrix}$, we obtain the solution $\mathbf{y}(t)=e^{2t}\begin{bmatrix}4\cos t+7\sin t\\-4\sin t+7\cos t\end{bmatrix}$.

14. Proceeding as in Exercises 7-12, we find the general solution of $\mathbf{y}'=A\mathbf{y}$ is $\mathbf{y}(t)=c_1\begin{bmatrix}3\sin 3t\\\cos 3t\end{bmatrix}+c_2\begin{bmatrix}3\cos 3t\\\sin 3t\end{bmatrix}$. Imposing the initial condition, $\mathbf{y}(0)=c_1\begin{bmatrix}0\\1\end{bmatrix}+c_2\begin{bmatrix}3\\0\end{bmatrix}=\begin{bmatrix}6\\2\end{bmatrix}$, we obtain the solution $\mathbf{y}(t)=\begin{bmatrix}-6\sin 3t+6\cos 3t\\2\cos 3t+2\sin 3t\end{bmatrix}$.

15. Proceeding as in Exercises 7-12, we find the general solution of $\mathbf{y}' = A\mathbf{y}$ is
$\mathbf{y}(t) = c_1 e^{3t}\begin{bmatrix} 3\cos 2t - 2\sin 2t \\ \cos 2t \end{bmatrix} + c_2 e^{3t}\begin{bmatrix} 2\cos 2t + 3\sin 2t \\ \sin 2t \end{bmatrix}$. Imposing the initial condition,
$\mathbf{y}(0) = c_1\begin{bmatrix} 3 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, we find $c_1 = 3$ and $c_2 = -4$ and the solution is
$\mathbf{y}(t) = e^{3t}\begin{bmatrix} \cos 2t - 18\sin 2t \\ 3\cos 2t - 4\sin 2t \end{bmatrix}$.

16. Proceeding as in Exercises 7-12, we find the general solution of $\mathbf{y}' = A\mathbf{y}$ is
$\mathbf{y}(t) = c_1 e^{2t}\begin{bmatrix} \cos t \\ -\cos t - \sin t \end{bmatrix} + c_2 e^{2t}\begin{bmatrix} \sin t \\ \cos t - \sin t \end{bmatrix}$. Imposing the initial condition,
$\mathbf{y}(0) = c_1\begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ 6 \end{bmatrix}$, we obtain the solution $\mathbf{y}(t) = e^{2t}\begin{bmatrix} 8\cos t + 14\sin t \\ 6\cos t - 22\sin t \end{bmatrix}$.

17. Proceeding as in Exercises 7-12, we find the general solution of $\mathbf{y}' = A\mathbf{y}$ is
$$\mathbf{y}(t) = c_1 e^{t}\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} + c_2 e^{t}\begin{bmatrix} 3\cos t - 2\sin t \\ \cos t \\ -\cos t + \sin t \end{bmatrix} + c_3 e^{t}\begin{bmatrix} 2\cos t + 3\sin t \\ \sin t \\ -\cos t - \sin t \end{bmatrix}.$$
Imposing the initial condition, $\mathbf{y}(0) = c_1\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} + c_2\begin{bmatrix} 3 \\ 1 \\ -1 \end{bmatrix} + c_3\begin{bmatrix} 2 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 6 \\ 1 \\ 2 \end{bmatrix}$, we find
$c_1 = -9$, $c_2 = 1$, and $c_3 = -12$, and the solution $\mathbf{y}(t) = e^{t}\begin{bmatrix} 27 - 21\cos t - 38\sin t \\ \cos t - 12\sin t \\ -9 + 11\cos t + 13\sin t \end{bmatrix}$.

18. Proceeding as in Exercises 7-12, we find the general solution of $\mathbf{y}' = A\mathbf{y}$ is
$$\mathbf{y}(t) = c_1 e^{2t}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + c_2 e^{2t}\begin{bmatrix} -4\cos 3t + \sin 3t \\ -3\sin 3t \\ \cos 3t - \sin 3t \end{bmatrix} + c_3 e^{2t}\begin{bmatrix} -\cos 3t - 4\sin 3t \\ 3\cos 3t \\ \cos 3t + \sin 3t \end{bmatrix}.$$
Imposing the initial condition, $\mathbf{y}(0) = c_1\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + c_2\begin{bmatrix} -4 \\ 0 \\ 1 \end{bmatrix} + c_3\begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 9 \\ 4 \end{bmatrix}$, we find
$c_1 = -2$, $c_2 = -1$, and $c_3 = 3$, and the solution $\mathbf{y}(t) = e^{2t}\begin{bmatrix} -2 + \cos 3t - 13\sin 3t \\ 9\cos 3t + 3\sin 3t \\ 2 + 2\cos 3t + 4\sin 3t \end{bmatrix}$.

22. The eigenvalues of A are $\lambda = \left(-1 \pm \sqrt{9 + 12\mu}\right)/2$. If $9 + 12\mu < 0$, $\lambda = -\frac{1}{2} \pm i\beta$ $(\beta \neq 0)$, therefore distinct and $y(t) \to 0$. If $0 < 9 + 12\mu < 1$, , the eigenvalues are distinct, real and negative. Therefore, $-\infty < 9 + 12\mu < 1$ $-\infty < \mu < -\frac{2}{3}$.

23. The eigenvalues of A are $\lambda = \left(-5 \pm \sqrt{1+4\mu}\right)/2$. In order that both components of $\mathbf{y}(t)$ go to zero as $t \to \infty$, we need each of these (real) eigenvalues to be negative. Therefore, we need $\left(-5 + \sqrt{1+4\mu}\right)/2 < 0$ or $\sqrt{1+4\mu} < 5$. This inequality holds if and only if $1+4\mu < 25$ or $-\infty < \mu < 6$.

24. The eigenvalues of A are $\lambda = \left(-2 \pm \sqrt{16-4\mu^2}\right)/2 = -1 \pm \sqrt{4-\mu^2}$. Require $-\infty < 4-\mu^2 < 1 \Rightarrow 3 < \mu^2 < \infty$. Therefore, $-\infty < \mu < -\sqrt{3}$ and $\sqrt{3} < \mu < \infty$.

25. The eigenvalues of A are $\lambda = -1 \pm \sqrt{4+\mu^2}$. In order that both components of $\mathbf{y}(t)$ go to zero as $t \to \infty$, we need each of these (real) eigenvalues to be negative. Therefore, we need $-1+\sqrt{4+\mu^2} < 0$ or $\sqrt{4+\mu^2} < 1$. This inequality cannot hold for any real value of μ.

26 (a). $\frac{d}{dt}\mathbf{v} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\mathbf{v} \Rightarrow \mathbf{v}(t) = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$.

26 (b). $\mathbf{v}(0) = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \Rightarrow \mathbf{v}(t) = \begin{bmatrix} \cos t + 2\sin t \\ -\sin t + 2\cos t \end{bmatrix}$. $\mathbf{r}(t) = \begin{bmatrix} \sin t - 2\cos t \\ \cos t + 2\sin t \end{bmatrix} + \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$.

$\mathbf{r}(0) = \begin{bmatrix} -2+d_1 \\ 1+d_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \Rightarrow \mathbf{r}(t) = \begin{bmatrix} \sin t - 2\cos t + 4 \\ \cos t + 2\sin t \end{bmatrix}$. $\mathbf{v}\left(\frac{3\pi}{2}\right) = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ and $\mathbf{r}\left(\frac{3\pi}{2}\right) = \begin{bmatrix} 3 \\ -2 \end{bmatrix}$.

27 (d). If the charged particle is launched with initial velocity parallel to the magnetic field, it will move with constant velocity.

28 (b). The eigenpairs are $-\frac{\gamma}{m}+\lambda_1, \mathbf{x}_1$ and $-\frac{\gamma}{m}+\lambda_2, \mathbf{x}_2$ and $-\frac{\gamma}{m}+\lambda_3, \mathbf{x}_3$.

The corresponding fundamental matrix is $e^{-\frac{\gamma}{m}t}\psi(t)$.

Section 6.7

1 (a). For $A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$, the characteristic polynomial is $p(\lambda) = (\lambda-2)^2$. The eigenvalue $\lambda_1 = 2$ has algebraic multiplicity 2. Corresponding eigenvectors are obtained by solving $(A-2I)\mathbf{x} = \mathbf{0}$ or $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Therefore, all the eigenvectors corresponding to $\lambda_1 = 2$ have the form $\mathbf{x} = \begin{bmatrix} x_1 \\ 0 \end{bmatrix} = x_1\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $x_1 \neq 0$. The geometric multiplicity of $\lambda_1 = 2$ is 1.

1 (b). We find a generalized eigenvector corresponding to $\lambda_1 = 2$ by solving the equation $(A-2I)\mathbf{x} = \mathbf{x}_1$ where $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. The solution is $\mathbf{x} = \begin{bmatrix} x_1 \\ 1 \end{bmatrix}$ where x_1 is arbitrary. Choosing $x_1 = 0$, we obtain the generalized eigenvector $\mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

Thus, we have solutions $\mathbf{y}_1(t) = e^{2t}\mathbf{x}_1$ and, as in equation (6), $\mathbf{y}_2(t) = te^{2t}\mathbf{x}_1 + e^{2t}\mathbf{x}_2$. A fundamental matrix is $\Psi(t) = [\mathbf{y}_1(t), \mathbf{y}_2(t)] = \begin{bmatrix} e^{2t} & te^{2t} \\ 0 & e^{2t} \end{bmatrix}$.

1 (c). The general solution is $\Psi(t)\mathbf{c}$. Imposing the initial condition, $\Psi(0)\mathbf{c} = \mathbf{y}_0$. We find $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ or $\mathbf{c} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Thus, the solution of the initial value problem is $\mathbf{y}(t) = e^{2t}\begin{bmatrix} 1-t \\ -1 \end{bmatrix}$.

2 (a). The characteristic polynomial is $p(\lambda) = (3-\lambda)^2$. The eigenvalue $\lambda_1 = 3$ has algebraic multiplicity 2. Corresponding eigenvectors are obtained by solving $(A-3I)\mathbf{x} = \mathbf{0}$. Therefore, all the eigenvectors corresponding to $\lambda_1 = 3$ have the form $\mathbf{x} = \begin{bmatrix} x_1 \\ 0 \end{bmatrix} = x_1\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $x_1 \neq 0$. The geometric multiplicity of $\lambda_1 = 3$ is 1.

2 (b). $\mathbf{y}_1(t) = e^{3t}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{y}_2(t) = te^{3t}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + e^{3t}\begin{bmatrix} 0 \\ \frac{1}{2} \end{bmatrix} = e^{3t}\begin{bmatrix} t \\ \frac{1}{2} \end{bmatrix}$. A fundamental matrix is $\Psi(t) = [\mathbf{y}_1(t), \mathbf{y}_2(t)] = \begin{bmatrix} e^{3t} & te^{3t} \\ 0 & \frac{1}{2}e^{3t} \end{bmatrix}$.

2 (c). The general solution is $\Psi(t)\mathbf{c}$. Imposing the initial condition, $\Psi(0)\mathbf{c} = \mathbf{y}_0$. We find $\begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}$ or $\mathbf{c} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$. Thus, the solution of the initial value problem is $\mathbf{y}(t) = e^{3t}\begin{bmatrix} 2t+4 \\ 1 \end{bmatrix}$.

3 (a). For $A = \begin{bmatrix} 6 & 0 \\ 2 & 6 \end{bmatrix}$, the characteristic polynomial is $p(\lambda) = (\lambda-6)^2$. The eigenvalue $\lambda_1 = 6$ has algebraic multiplicity 2. Corresponding eigenvectors are obtained by solving $(A-6I)\mathbf{x} = \mathbf{0}$ or $\begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Therefore, all the eigenvectors corresponding to $\lambda_1 = 6$ have the form $\mathbf{x} = \begin{bmatrix} 0 \\ x_2 \end{bmatrix} = x_2\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $x_2 \neq 0$. The geometric multiplicity of $\lambda_1 = 6$ is 1.

3 (b). We find a generalized eigenvector corresponding to $\lambda_1 = 6$ by solving the equation $(A-6I)\mathbf{x} = \mathbf{x}_1$ where $\mathbf{x}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. The solution is $\mathbf{x} = \begin{bmatrix} 0.5 \\ x_2 \end{bmatrix}$ where x_2 is arbitrary. Choosing $x_2 = 0$, we obtain the generalized eigenvector $\mathbf{x}_2 = \begin{bmatrix} 0.5 \\ 0 \end{bmatrix}$. Thus, we have solutions $\mathbf{y}_1(t) = e^{6t}\mathbf{x}_1$ and, as in equation (6), $\mathbf{y}_2(t) = te^{6t}\mathbf{x}_1 + e^{6t}\mathbf{x}_2$. A fundamental matrix is $\Psi(t) = [\mathbf{y}_1(t), \mathbf{y}_2(t)] = \begin{bmatrix} 0 & 0.5e^{6t} \\ e^{6t} & te^{6t} \end{bmatrix}$.

3 (c). The general solution is $\Psi(t)\mathbf{c}$. Imposing the initial condition requires $\Psi(0)\mathbf{c} = \mathbf{y}_0$. We find $\begin{bmatrix} 0 & 0.5 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$ or $\mathbf{c} = \begin{bmatrix} 0 \\ -4 \end{bmatrix}$. Thus, the solution of the initial value problem is $\mathbf{y}(t) = e^{6t}\begin{bmatrix} -2 \\ -4t \end{bmatrix}$.

4 (a). The characteristic polynomial is $p(\lambda) = (3-\lambda)^2$. The eigenvalue $\lambda_1 = 3$ has algebraic multiplicity 2. Corresponding eigenvectors are obtained by solving $(A - 3I)\mathbf{x} = \mathbf{0}$. Therefore, all the eigenvectors corresponding to $\lambda_1 = 3$ have the form $\mathbf{x} = \begin{bmatrix} 0 \\ x_2 \end{bmatrix} = x_2\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $x_2 \neq 0$. The geometric multiplicity of $\lambda_1 = 3$ is 1.

4 (b). $\mathbf{y}_1(t) = e^{3t}\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $\mathbf{y}_2(t) = te^{3t}\begin{bmatrix} 0 \\ 1 \end{bmatrix} + e^{3t}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = e^{3t}\begin{bmatrix} 1 \\ t \end{bmatrix}$. A fundamental matrix is $\Psi(t) = [\mathbf{y}_1(t), \mathbf{y}_2(t)] = \begin{bmatrix} 0 & e^{3t} \\ e^{3t} & te^{3t} \end{bmatrix}$.

4 (c). The general solution is $\Psi(t)\mathbf{c}$. Imposing the initial condition, $\Psi(0)\mathbf{c} = \mathbf{y}_0$. We find $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$ or $\mathbf{c} = \begin{bmatrix} -3 \\ 2 \end{bmatrix}$. Thus, the solution of the initial value problem is $\mathbf{y}(t) = e^{3t}\begin{bmatrix} 2 \\ 2t-3 \end{bmatrix}$.

5 (a). For $A = \begin{bmatrix} 5 & -1 \\ 4 & 1 \end{bmatrix}$, the characteristic polynomial is $p(\lambda) = (\lambda - 3)^2$. The eigenvalue $\lambda_1 = 3$ has algebraic multiplicity 2. Corresponding eigenvectors are obtained by solving $(A - 3I)\mathbf{x} = \mathbf{0}$ or $\begin{bmatrix} 2 & -1 \\ 4 & -2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Therefore, all the eigenvectors corresponding to $\lambda_1 = 3$ have the form $\mathbf{x} = \begin{bmatrix} x_1 \\ 2x_1 \end{bmatrix} = x_1\begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $x_1 \neq 0$. The geometric multiplicity of $\lambda_1 = 3$ is 1.

5 (b). We find a generalized eigenvector corresponding to $\lambda_1 = 3$ by solving the equation $(A - 3I)\mathbf{x} = \mathbf{x}_1$ where $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. The solution is $\mathbf{x} = \begin{bmatrix} .5x_2 + .5 \\ x_2 \end{bmatrix}$ where x_2 is arbitrary. Choosing $x_2 = 0$, we obtain the generalized eigenvector $\mathbf{x}_2 = \begin{bmatrix} .5 \\ 0 \end{bmatrix}$. Thus, we have solutions $\mathbf{y}_1(t) = e^{3t}\mathbf{x}_1$ and, as in equation (6), $\mathbf{y}_2(t) = te^{3t}\mathbf{x}_1 + e^{3t}\mathbf{x}_2$. A fundamental matrix is $\Psi(t) = [\mathbf{y}_1(t), \mathbf{y}_2(t)] = \begin{bmatrix} e^{3t} & (t+.5)e^{3t} \\ 2e^{3t} & 2te^{3t} \end{bmatrix}$.

5 (c). The general solution is $\Psi(t)\mathbf{c}$. Imposing the initial condition requires $\Psi(0)\mathbf{c} = \mathbf{y}_0$. We find $\begin{bmatrix} 1 & .5 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ or $\mathbf{c} = \begin{bmatrix} .5 \\ 1 \end{bmatrix}$. Thus, the solution of the initial value problem is $\mathbf{y}(t) = e^{3t}\begin{bmatrix} t+1 \\ 2t+1 \end{bmatrix}$.

6 (a). The characteristic polynomial is $p(\lambda) = (3-\lambda)^2$. The eigenvalue $\lambda_1 = 3$ has algebraic multiplicity 2. Corresponding eigenvectors are obtained by solving $(A-3I)\mathbf{x} = \mathbf{0}$. Use $\mathbf{x}_1 = \begin{bmatrix} 6 \\ -1 \end{bmatrix}$. The geometric multiplicity of $\lambda_1 = 3$ is 1.

6 (b). $\mathbf{y}_1(t) = e^{3t}\begin{bmatrix} 6 \\ -1 \end{bmatrix}$ and $\mathbf{y}_2(t) = te^{3t}\begin{bmatrix} 6 \\ -1 \end{bmatrix} + e^{3t}\begin{bmatrix} -1 \\ 0 \end{bmatrix} = e^{3t}\begin{bmatrix} 6t-1 \\ -t \end{bmatrix}$. A fundamental matrix is

$$\Psi(t) = [\mathbf{y}_1(t), \mathbf{y}_2(t)] = \begin{bmatrix} 6e^{3t} & (6t-1)e^{3t} \\ -e^{3t} & -te^{3t} \end{bmatrix}.$$

6 (c). The general solution is $\Psi(t)\mathbf{c}$. Imposing the initial condition, $\Psi(0)\mathbf{c} = \mathbf{y}_0$. We find $\begin{bmatrix} 6 & -1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$ or $\mathbf{c} = \begin{bmatrix} -2 \\ -12 \end{bmatrix}$. Thus, the solution of the initial value problem is $\mathbf{y}(t) = e^{3t}\begin{bmatrix} -72t \\ 12t+2 \end{bmatrix}$.

7 (a). For $A = \begin{bmatrix} 1 & -1 \\ 1 & 3 \end{bmatrix}$, the characteristic polynomial is $p(\lambda) = (\lambda-2)^2$. The eigenvalue $\lambda_1 = 2$ has algebraic multiplicity 2. Corresponding eigenvectors are obtained by solving $(A-2I)\mathbf{x} = \mathbf{0}$ or $\begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Therefore, all the eigenvectors corresponding to $\lambda_1 = 2$ have the form $\mathbf{x} = \begin{bmatrix} x_1 \\ -x_1 \end{bmatrix} = x_1\begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $x_1 \neq 0$. The geometric multiplicity of $\lambda_1 = 2$ is 1.

7 (b). We find a generalized eigenvector corresponding to $\lambda_1 = 2$ by solving the equation $(A-2I)\mathbf{x} = \mathbf{x}_1$ where $\mathbf{x}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. The solution is $\mathbf{x} = \begin{bmatrix} -1-x_2 \\ x_2 \end{bmatrix}$ where x_2 is arbitrary. Choosing $x_2 = 0$, we obtain the generalized eigenvector $\mathbf{x}_2 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$. Thus, we have solutions $\mathbf{y}_1(t) = e^{2t}\mathbf{x}_1$ and, as in equation (6), $\mathbf{y}_2(t) = te^{2t}\mathbf{x}_1 + e^{2t}\mathbf{x}_2$. A fundamental matrix is

$$\Psi(t) = [\mathbf{y}_1(t), \mathbf{y}_2(t)] = \begin{bmatrix} e^{2t} & (t-1)e^{2t} \\ -e^{2t} & -te^{2t} \end{bmatrix}.$$

7 (c). The general solution is $\Psi(t)\mathbf{c}$. Imposing the initial condition requires $\Psi(0)\mathbf{c} = \mathbf{y}_0$. We find $\begin{bmatrix} 1 & -1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$ or $\mathbf{c} = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$. Thus, the solution of the initial value problem is $\mathbf{y}(t) = e^{2t}\begin{bmatrix} 4-3t \\ 3t-1 \end{bmatrix}$.

8 (a). The characteristic polynomial is $p(\lambda)=(\lambda-5)^2$. The eigenvalue $\lambda_1=5$ has algebraic multiplicity 2. Corresponding eigenvectors are obtained by solving $(A-5I)\mathbf{x}=\mathbf{0}$. Use $\mathbf{x}_1=\begin{bmatrix}1\\-1\end{bmatrix}$. The geometric multiplicity of $\lambda_1=5$ is 1.

8 (b). $\mathbf{y}_1(t)=e^{5t}\begin{bmatrix}1\\-1\end{bmatrix}$ and $\mathbf{y}_2(t)=te^{5t}\begin{bmatrix}1\\-1\end{bmatrix}+e^{5t}\begin{bmatrix}1\\0\end{bmatrix}=e^{5t}\begin{bmatrix}t+1\\-t\end{bmatrix}$. A fundamental matrix is

$$\Psi(t)=[\mathbf{y}_1(t),\mathbf{y}_2(t)]=\begin{bmatrix}e^{5t} & (t+1)e^{5t}\\ -e^{5t} & -te^{5t}\end{bmatrix}.$$

8 (c). The general solution is $\Psi(t)\mathbf{c}$. Imposing the initial condition, $\Psi(0)\mathbf{c}=\mathbf{y}_0$. We find $\begin{bmatrix}1 & 1\\ -1 & 0\end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix}=\begin{bmatrix}4\\-4\end{bmatrix}$ or $\mathbf{c}=\begin{bmatrix}4\\0\end{bmatrix}$. Thus, the solution of the initial value problem is

$$\mathbf{y}(t)=e^{5t}\begin{bmatrix}4\\-4\end{bmatrix}.$$

9 (a). For $A=\begin{bmatrix}2 & 1 & 0\\ 0 & 2 & 1\\ 0 & 0 & 2\end{bmatrix}$, the characteristic polynomial is $p(\lambda)=(\lambda-2)^3$. The eigenvalue $\lambda_1=2$ has algebraic multiplicity 3. Corresponding eigenvectors are obtained by solving $(A-2I)\mathbf{x}=\mathbf{0}$ or $\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$. Therefore, all the eigenvectors corresponding to $\lambda_1=2$ have the form $\mathbf{x}=\begin{bmatrix}x_1\\0\\0\end{bmatrix}=x_1\begin{bmatrix}1\\0\\0\end{bmatrix}$, $x_1\neq 0$. The geometric multiplicity of $\lambda_1=2$ is 1.

9 (b). For $A=\begin{bmatrix}2 & 1 & 0\\ 0 & 2 & 0\\ 0 & 0 & 2\end{bmatrix}$, the characteristic polynomial is $p(\lambda)=(\lambda-2)^3$. The eigenvalue $\lambda_1=2$ has algebraic multiplicity 3. Corresponding eigenvectors are obtained by solving $(A-2I)\mathbf{x}=\mathbf{0}$ or $\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$. Therefore, all the eigenvectors corresponding to $\lambda_1=2$ have the form $\mathbf{x}=\begin{bmatrix}x_1\\0\\x_3\end{bmatrix}=x_1\begin{bmatrix}1\\0\\0\end{bmatrix}+x_3\begin{bmatrix}0\\0\\1\end{bmatrix}$, where $\mathbf{x}$ is nonzero. The geometric multiplicity of $\lambda_1=2$ is 2.

13. For $A=\begin{bmatrix}5 & 0 & 0\\ 1 & 5 & 0\\ 1 & 1 & 5\end{bmatrix}$, the characteristic polynomial is $p(\lambda)=(\lambda-5)^3$. The eigenvalue $\lambda_1=5$ has algebraic multiplicity 3.

Corresponding eigenvectors are obtained by solving $(A-5I)\mathbf{x}=\mathbf{0}$ or $\begin{bmatrix}0&0&0\\1&0&0\\1&1&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$.

Therefore, all the eigenvectors corresponding to $\lambda_1=5$ have the form
$\mathbf{x}=\begin{bmatrix}0\\0\\x_3\end{bmatrix}=x_3\begin{bmatrix}0\\0\\1\end{bmatrix}$, $x_3\neq 0$. The geometric multiplicity of $\lambda_1=5$ is 1, so A does not have a full set of eigenvectors.

14. The characteristic polynomial is $p(\lambda)=(\lambda-5)^3$. The eigenvalue $\lambda_1=5$ has algebraic multiplicity 3. Corresponding eigenvectors are obtained by solving $(A-5I)\mathbf{x}=\mathbf{0}$. Use $\mathbf{x}_1=\begin{bmatrix}0\\1\\0\end{bmatrix}$, $\mathbf{x}_2=\begin{bmatrix}0\\0\\1\end{bmatrix}$. The geometric multiplicity of $\lambda_1=5$ is 2, so A does not have a full set of eigenvectors.

15. For $A=\begin{bmatrix}5&0&1\\0&5&0\\0&0&5\end{bmatrix}$, the characteristic polynomial is $p(\lambda)=(\lambda-5)^3$. The eigenvalue $\lambda_1=5$ has algebraic multiplicity 3. Corresponding eigenvectors are obtained by solving $(A-5I)\mathbf{x}=\mathbf{0}$ or $\begin{bmatrix}0&0&1\\0&0&0\\0&0&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$. Therefore, all the eigenvectors corresponding to $\lambda_1=5$ have the form
$\mathbf{x}=\begin{bmatrix}x_1\\x_2\\0\end{bmatrix}=x_1\begin{bmatrix}1\\0\\0\end{bmatrix}+x_2\begin{bmatrix}0\\1\\0\end{bmatrix}$, where $\mathbf{x}$ is nonzero. The geometric multiplicity of $\lambda_1=5$ is 2, so A does not have a full set of eigenvectors.

16. The characteristic polynomial is $p(\lambda)=(\lambda-5)^3$. The eigenvalue $\lambda_1=5$ has algebraic multiplicity 3. Corresponding eigenvectors are obtained by solving $(A-5I)\mathbf{x}=0\mathbf{x}=\mathbf{0}$. Use $\mathbf{x}_1=\begin{bmatrix}1\\0\\0\end{bmatrix}$, $\mathbf{x}_2=\begin{bmatrix}0\\1\\0\end{bmatrix}$, $\mathbf{x}_3=\begin{bmatrix}0\\0\\1\end{bmatrix}$. The geometric multiplicity of $\lambda_1=5$ is 3, so A does have a full set of eigenvectors.

17. For $A=\begin{bmatrix}2&0&0&0\\1&2&0&0\\0&0&3&0\\0&0&1&3\end{bmatrix}$, the characteristic polynomial is $p(\lambda)=(\lambda-2)^2(\lambda-3)^2$. The eigenvalue $\lambda_1=2$ has algebraic multiplicity 2 as does $\lambda_2=3$.

Corresponding eigenvectors for $\lambda_1 = 2$ have the form $\mathbf{x} = \begin{bmatrix} 0 \\ x_2 \\ 0 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $x_2 \neq 0$. Therefore, the geometric multiplicity of $\lambda_1 = 2$ is 1. Similarly, eigenvectors corresponding to $\lambda_2 = 3$ have the form $\mathbf{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ x_4 \end{bmatrix} = x_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$, $x_4 \neq 0$ and so $\lambda_2 = 3$ has geometric multiplicity 1. *A* does not have a full set of eigenvectors.

18. The characteristic polynomial is $p(\lambda) = (2 - \lambda)^4$. The eigenvalue $\lambda_1 = 2$ has algebraic multiplicity 4. Corresponding eigenvectors are obtained by solving $(A - 2I)\mathbf{x} = \mathbf{0}$. Use $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{x}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{x}_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$. The geometric multiplicity of $\lambda = 2$ is 3, so *A* does not have a full set of eigenvectors.

19. For $A = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 3 \end{bmatrix}$, the characteristic polynomial is $p(\lambda) = (\lambda - 2)^3(\lambda - 3)$. The eigenvalue $\lambda_1 = 2$ has algebraic multiplicity 3 while $\lambda_2 = 3$ has algebraic multiplicity 1. Corresponding eigenvectors for $\lambda_1 = 2$ have the form $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ -x_3 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}$. Therefore, the geometric multiplicity of $\lambda_1 = 2$ is 3. Eigenvectors corresponding to $\lambda_2 = 3$ have the form $\mathbf{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ x_4 \end{bmatrix} = x_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$, $x_4 \neq 0$ and so $\lambda_2 = 3$ has geometric multiplicity 1. Since every eigenvalue of *A* has geometric multiplicity equal to its algebraic multiplicity, *A* has a full set of eigenvectors.

20. *A* must have $\lambda_1 = a + ib$ and $\lambda_2 = a - ib$ as two distinct eigenvalues. Therefore, *A* cannot have a repeated eigenvalue and cannot be defective.

21. In order for *A* to be symmetric, $a_{12} = x$ must be the same as $a_{21} = 9$. Thus, $x = 9$. Similarly, $a_{23} = y$ must equal $a_{32} = 4$.

22. $x = 6$, $y = 1$.

23. In order for A to be symmetric, $a_{13} = x^2 - 1$ must be the same as $a_{31} = 0$. Thus, we can have either $x = 1$ or $x = -1$. Similarly, $a_{21} = 2/y$ must equal $a_{12} = 1$. Hence, $y = 2$.

24. In order for A to be Hermitian, $\bar{a}_{12} = x - 3i = 9 - 3i \Rightarrow x = 9$ and $\bar{a}_{23} = 2 - yi = 2 + 5i \Rightarrow y = -5$.

25. In order for A to be Hermitian, $a_{11} = 2 + xi$ must be the same as $\bar{a}_{11} = 2 - xi$. Thus, we need $x = 0$. Similarly, $a_{21} = 1 + yi$ must equal $\bar{a}_{12} = 1 - 2i$. Hence, $y = 2$. These choices are consistent with the remaining undetermined entries, a_{22} and a_{23}.

26 (a). $A = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$, for example.

26 (b). $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, for example.

30. The equation $(A - 2I)\mathbf{x} = \mathbf{v}_1$ is $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$. Choose $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$.

The equation $(A - 2I)\mathbf{x} = \mathbf{v}_2$ is $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$. Choose $\mathbf{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. A fundamental set of solutions can be formed from $\mathbf{y}_1(t) = e^{2t}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{y}_2(t) = e^{2t}\begin{bmatrix} t \\ 1 \\ 0 \end{bmatrix}$, and $\mathbf{y}_3(t) = e^{2t}\begin{bmatrix} \frac{t^2}{2} \\ t \\ 1 \end{bmatrix}$.

31. For $A = \begin{bmatrix} 4 & 0 & 0 \\ 2 & 4 & 0 \\ 1 & 3 & 4 \end{bmatrix}$, the equation $(A - 4I)\mathbf{x} = \mathbf{v}_1$ is $\begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 1 & 3 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. The solution is $\mathbf{x} = \begin{bmatrix} 0 \\ 1/3 \\ x_3 \end{bmatrix}$ where x_3 is arbitrary. Choosing $x_3 = 0$ we have $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1/3 \\ 0 \end{bmatrix}$.

The equation $(A - 4I)\mathbf{x} = \mathbf{v}_2$ is $\begin{bmatrix} 0 & 0 & 0 \\ 2 & 0 & 0 \\ 1 & 3 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1/3 \\ 0 \end{bmatrix}$. The solution is $\mathbf{x} = \begin{bmatrix} 1/6 \\ -1/18 \\ x_3 \end{bmatrix}$ where x_3 is arbitrary. Choosing $x_3 = 0$ we have $\mathbf{v}_3 = \begin{bmatrix} 1/6 \\ -1/18 \\ 0 \end{bmatrix}$. By equation (12), a fundamental set of solutions can be formed from $\mathbf{y}_1(t) = e^{4t}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, $\mathbf{y}_2(t) = e^{4t}(\mathbf{v}_2 + t\mathbf{v}_1) = e^{4t}\begin{bmatrix} 0 \\ 1/3 \\ t \end{bmatrix}$, and

$$\mathbf{y}_3(t) = e^{4t}(\mathbf{v}_3 + t\mathbf{v}_2 + 0.5t^2\mathbf{v}_1) = \frac{e^{4t}}{18}\begin{bmatrix} 3 \\ -1 + 6t \\ 9t^2 \end{bmatrix}.$$

32. The equation $(A-I)\mathbf{x}=\mathbf{v}_1$ is $\begin{bmatrix} 2 & -8 & -10 \\ -2 & 6 & 8 \\ 2 & -6 & -8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$. Choose $\mathbf{v}_2=\begin{bmatrix} \frac{1}{2} \\ 0 \\ 0 \end{bmatrix}$.

The equation $(A-I)\mathbf{x}=\mathbf{v}_2$ is $\begin{bmatrix} 2 & -8 & -10 \\ -2 & 6 & 8 \\ 2 & -6 & -8 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} \frac{1}{2} \\ 0 \\ 0 \end{bmatrix}$. Choose $\mathbf{v}_3=\begin{bmatrix} -\frac{3}{4} \\ -\frac{1}{4} \\ 0 \end{bmatrix}$. A fundamental set of solutions can be formed from $\mathbf{y}_1(t)=e^t\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$, $\mathbf{y}_2(t)=e^t\begin{bmatrix} \frac{1}{2}+t \\ -t \\ t \end{bmatrix}$, and $\mathbf{y}_3(t)=e^t\begin{bmatrix} -\frac{3}{4}+\frac{t}{2}+\frac{t^2}{2} \\ -\frac{1}{4}-\frac{t^2}{2} \\ \frac{t^2}{2} \end{bmatrix}$.

33. For $A=\begin{bmatrix} -6 & -8 & 22 \\ 2 & 4 & -4 \\ -2 & -2 & 8 \end{bmatrix}$, the equation $(A-2I)\mathbf{x}=\mathbf{v}_1$ is given by $\begin{bmatrix} -8 & -8 & 22 \\ 2 & 2 & -4 \\ -2 & -2 & 6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$. A convenient solution is $\mathbf{v}_2=\begin{bmatrix} -1.5 \\ 0 \\ -0.5 \end{bmatrix}$. The equation $(A-2I)\mathbf{x}=\mathbf{v}_2$ is $\begin{bmatrix} -8 & -8 & 22 \\ 2 & 2 & -4 \\ -2 & -2 & 6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} -1.5 \\ 0 \\ -0.5 \end{bmatrix}$. One solution is $\mathbf{v}_3=\begin{bmatrix} -0.5 \\ 0 \\ -0.25 \end{bmatrix}$. A fundamental set consists of

$\mathbf{y}_1(t)=e^{2t}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, $\mathbf{y}_2(t)=e^{2t}(\mathbf{v}_2+t\mathbf{v}_1)=e^{2t}\begin{bmatrix} t-1.5 \\ -t \\ -0.5 \end{bmatrix}$, and

$\mathbf{y}_3(t)=e^{2t}(\mathbf{v}_3+t\mathbf{v}_2+0.5t^2\mathbf{v}_1)=\dfrac{e^{2t}}{4}\begin{bmatrix} -2-6t+2t^2 \\ -2t^2 \\ -1-2t \end{bmatrix}$.

36 (a). Two linearly independent solutions are $\mathbf{x}_1=\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, $\mathbf{x}_2=\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$.

36 (b). Choose $\mathbf{x}_3=\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

36 (c). $\mathbf{Q}(t)=c_1e^{-4t}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}+c_2e^{-4t}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}+c_3e^{-t}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

Section 6.8

1 (a). For $A=\begin{bmatrix}-2 & 1\\ 1 & -2\end{bmatrix}$, the characteristic polynomial is $p(\lambda)=\lambda^2+4\lambda+3=(\lambda+3)(\lambda+1)$. The eigenvalues are $\lambda_1=-3$ and $\lambda_2=-1$, with corresponding eigenvectors $\mathbf{x}_1=\begin{bmatrix}1\\ -1\end{bmatrix}$ and $\mathbf{x}_2=\begin{bmatrix}1\\ 1\end{bmatrix}$. Thus, the complementary solution of $\mathbf{y}'=A\mathbf{y}$ is $\mathbf{y}_C=\begin{bmatrix}e^{-3t} & e^{-t}\\ -e^{-3t} & e^{-t}\end{bmatrix}\begin{bmatrix}c_1\\ c_2\end{bmatrix}$.

1 (b). Inserting the suggested trial form $\mathbf{y}_P=\begin{bmatrix}a_1\\ a_2\end{bmatrix}$ into the nonhomogeneous equation leads to $\mathbf{y}_P'=A\mathbf{y}_P+\mathbf{g}(t)$ or $\begin{bmatrix}0\\ 0\end{bmatrix}=\begin{bmatrix}-2 & 1\\ 1 & -2\end{bmatrix}\begin{bmatrix}a_1\\ a_2\end{bmatrix}+\begin{bmatrix}1\\ 1\end{bmatrix}$. Solving this system, we obtain $\mathbf{y}_P=\begin{bmatrix}1\\ 1\end{bmatrix}$.

1 (c). The general solution of the nonhomogeneous problem is $\mathbf{y}_C+\mathbf{y}_P=\begin{bmatrix}e^{-3t} & e^{-t}\\ -e^{-3t} & e^{-t}\end{bmatrix}\begin{bmatrix}c_1\\ c_2\end{bmatrix}+\begin{bmatrix}1\\ 1\end{bmatrix}$.

1 (d). Imposing the initial condition, $\begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix}\begin{bmatrix}c_1\\ c_2\end{bmatrix}+\begin{bmatrix}1\\ 1\end{bmatrix}=\begin{bmatrix}3\\ 1\end{bmatrix}$. Solving, we find $c_1=1$ and $c_2=1$. Thus, $\mathbf{y}(t)=\begin{bmatrix}e^{-3t}+e^{-t}+1\\ -e^{-3t}+e^{-t}+1\end{bmatrix}$ is the unique solution of the given initial value problem.

2 (a). For $A=\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}$, the characteristic polynomial is $p(\lambda)=\lambda^2-4\lambda+3=(\lambda-1)(\lambda-3)$. The eigenvalues are $\lambda_1=1$ and $\lambda_2=3$, with corresponding eigenvectors $\mathbf{x}_1=\begin{bmatrix}1\\ -1\end{bmatrix}$ and $\mathbf{x}_2=\begin{bmatrix}1\\ 1\end{bmatrix}$. Thus, the complementary solution of $\mathbf{y}'=A\mathbf{y}$ is $\mathbf{y}_C=\begin{bmatrix}e^{t} & e^{3t}\\ -e^{t} & e^{3t}\end{bmatrix}\begin{bmatrix}c_1\\ c_2\end{bmatrix}$.

2 (b). Inserting the suggested trial form $\mathbf{y}_P=e^{-t}\begin{bmatrix}a_1\\ a_2\end{bmatrix}$ into the nonhomogeneous equation and solving the system, we obtain $\mathbf{y}_P=e^{-t}\begin{bmatrix}-\frac{3}{8}\\ \frac{1}{8}\end{bmatrix}$.

2 (c). The general solution of the nonhomogeneous problem is $\mathbf{y}_C+\mathbf{y}_P=\begin{bmatrix}e^{t} & e^{3t}\\ -e^{t} & e^{3t}\end{bmatrix}\begin{bmatrix}c_1\\ c_2\end{bmatrix}+e^{-t}\begin{bmatrix}-\frac{3}{8}\\ \frac{1}{8}\end{bmatrix}$.

2 (d). Imposing the initial condition, $\begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix}\begin{bmatrix}c_1\\ c_2\end{bmatrix}+\begin{bmatrix}-\frac{3}{8}\\ \frac{1}{8}\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$. Solving, we find $c_1=\dfrac{1}{4}$ and $c_2=\dfrac{1}{8}$. Thus, $\mathbf{y}(t)=\begin{bmatrix}\frac{1}{4}e^{t}+\frac{1}{8}e^{3t}-\frac{3}{8}e^{-t}\\ -\frac{1}{4}e^{t}+\frac{1}{8}e^{3t}+\frac{1}{8}e^{-t}\end{bmatrix}$ is the unique solution of the given initial value problem.

3 (a). For $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, the characteristic polynomial is $p(\lambda) = \lambda^2 - 1 = (\lambda + 1)(\lambda - 1)$. The eigenvalues are $\lambda_1 = -1$ and $\lambda_2 = 1$, with corresponding eigenvectors $\mathbf{x}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\mathbf{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Thus, the complementary solution of $\mathbf{y}' = A\mathbf{y}$ is $\mathbf{y}_C = \begin{bmatrix} e^{-t} & e^t \\ -e^{-t} & e^t \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$.

3 (b). Inserting the suggested trial form $\mathbf{y}_P = t\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ into the nonhomogeneous equation leads to $\mathbf{y}'_P = A\mathbf{y}_P + \mathbf{g}(t)$ or $\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} ta_1 + b_1 \\ ta_2 + b_2 \end{bmatrix} + \begin{bmatrix} t \\ -1 \end{bmatrix}$. Solving this system, we obtain $\mathbf{y}_P = \begin{bmatrix} 0 \\ -t \end{bmatrix}$.

3 (c). The general solution of the nonhomogeneous problem is $\mathbf{y}_C + \mathbf{y}_P = \begin{bmatrix} e^{-t} & e^t \\ -e^{-t} & e^t \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + \begin{bmatrix} 0 \\ -t \end{bmatrix}$.

3 (d). Imposing the initial condition, $\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$. Solving, we find $c_1 = 1.5$ and $c_2 = 0.5$. Thus, $\mathbf{y}(t) = 0.5\begin{bmatrix} 3e^{-t} + e^t \\ -3e^{-t} + e^t - t \end{bmatrix}$ is the unique solution of the given initial value problem.

4 (a). For $A = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$, the characteristic polynomial is $p(\lambda) = \lambda^2 - 1$. The eigenvalues are $\lambda_1 = -1$ and $\lambda_2 = 1$, with corresponding eigenvectors $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\mathbf{x}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Thus, the complementary solution of $\mathbf{y}' = A\mathbf{y}$ is $\mathbf{y}_C = \begin{bmatrix} e^{-t} & e^t \\ e^{-t} & -e^t \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$.

4 (b). Inserting the suggested trial form $\mathbf{y}_P = e^{2t}\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} + t\begin{bmatrix} b_1 \\ b_2 \end{bmatrix} + \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ into the nonhomogeneous equation and solving the system, we obtain $\mathbf{y}_P = \begin{bmatrix} -\frac{1}{3}e^{2t} - 1 \\ \frac{2}{3}e^{2t} + t \end{bmatrix}$.

4 (c). The general solution of the nonhomogeneous problem is
$$\mathbf{y}_C + \mathbf{y}_P = \begin{bmatrix} e^{-t} & e^t \\ e^{-t} & -e^t \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + \begin{bmatrix} -\frac{1}{3}e^{2t} - 1 \\ \frac{2}{3}e^{2t} + t \end{bmatrix}.$$

4 (d). Imposing the initial condition, $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + \begin{bmatrix} -\frac{4}{3} \\ \frac{2}{3} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Solving, we find $c_1 = \dfrac{5}{6}$ and $c_2 = \dfrac{1}{2}$. Thus, $\mathbf{y}(t) = \begin{bmatrix} \frac{5}{6}e^{-t} + \frac{1}{2}e^t - \frac{1}{3}e^{2t} - 1 \\ \frac{5}{6}e^{-t} - \frac{1}{2}e^t + \frac{2}{3}e^{2t} + t \end{bmatrix}$ is the unique solution of the given initial value problem.

5 (a). For $A=\begin{bmatrix}-3 & -2\\ 4 & 3\end{bmatrix}$, the characteristic polynomial is $p(\lambda)=\lambda^2-1=(\lambda+1)(\lambda-1)$. The eigenvalues are $\lambda_1=-1$ and $\lambda_2=1$, with corresponding eigenvectors $\mathbf{x}_1=\begin{bmatrix}1\\ -1\end{bmatrix}$ and $\mathbf{x}_2=\begin{bmatrix}1\\ -2\end{bmatrix}$. Thus, the complementary solution of $\mathbf{y}'=A\mathbf{y}$ is $\mathbf{y}_C=\begin{bmatrix}e^{-t} & e^{t}\\ -e^{-t} & -2e^{t}\end{bmatrix}\begin{bmatrix}c_1\\ c_2\end{bmatrix}$.

5 (b). Inserting the suggested trial form $\mathbf{y}_P=\sin t\begin{bmatrix}a_1\\ a_2\end{bmatrix}+\cos t\begin{bmatrix}b_1\\ b_2\end{bmatrix}$ into the nonhomogeneous equation leads to $\mathbf{y}_P'=A\mathbf{y}_P+\mathbf{g}(t)$ or $\begin{bmatrix}a_1\cos t-b_1\sin t\\ a_2\cos t-b_2\sin t\end{bmatrix}=\begin{bmatrix}-3 & -2\\ 4 & 3\end{bmatrix}\begin{bmatrix}a_1\sin t+b_1\cos t\\ a_2\sin t+b_2\cos t\end{bmatrix}+\begin{bmatrix}\sin t\\ 0\end{bmatrix}$. Solving this system, we obtain $\mathbf{y}_P=0.5\begin{bmatrix}3\sin t-\cos t\\ -4\sin t\end{bmatrix}$.

5 (c). The general solution of the nonhomogeneous problem is

$$\mathbf{y}_C+\mathbf{y}_P=\begin{bmatrix}e^{-t} & e^{t}\\ -e^{-t} & -2e^{t}\end{bmatrix}\begin{bmatrix}c_1\\ c_2\end{bmatrix}+0.5\begin{bmatrix}3\sin t-\cos t\\ -4\sin t\end{bmatrix}.$$

5 (d). Imposing the initial condition, $\begin{bmatrix}1 & 1\\ -1 & -2\end{bmatrix}\begin{bmatrix}c_1\\ c_2\end{bmatrix}+\begin{bmatrix}-.5\\ 0\end{bmatrix}=\begin{bmatrix}0\\ 0\end{bmatrix}$. Solving, we find $c_1=1$ and $c_2=-0.5$. Thus, $\mathbf{y}(t)=0.5\begin{bmatrix}2e^{-t}-e^{t}+3\sin t-\cos t\\ -2e^{-t}+2e^{t}-4\sin t\end{bmatrix}$ is the unique solution of the given initial value problem.

6 (a). For $A=\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}$, the characteristic polynomial is $p(\lambda)=\lambda^2-2\lambda$. The eigenvalues are $\lambda_1=0$ and $\lambda_2=2$, with corresponding eigenvectors $\mathbf{x}_1=\begin{bmatrix}1\\ -1\end{bmatrix}$ and $\mathbf{x}_2=\begin{bmatrix}1\\ 1\end{bmatrix}$. Thus, the complementary solution of $\mathbf{y}'=A\mathbf{y}$ is $\mathbf{y}_C=\begin{bmatrix}1 & e^{2t}\\ -1 & e^{2t}\end{bmatrix}\begin{bmatrix}c_1\\ c_2\end{bmatrix}$.

7. Given $\mathbf{y}(t)=\begin{bmatrix}1+\sin 2t\\ e^{t}+\cos 2t\end{bmatrix}$ it follows that $\mathbf{y}_0=\mathbf{y}(\pi/2)=\begin{bmatrix}1+\sin\pi\\ e^{\pi/2}+\cos\pi\end{bmatrix}=\begin{bmatrix}1\\ e^{\pi/2}-1\end{bmatrix}$. Inserting $\mathbf{y}(t)$ into the differential equation, we see that $\mathbf{y}'(t)=A\mathbf{y}(t)+\mathbf{g}(t)$ and thus $\begin{bmatrix}2\cos 2t\\ e^{t}-2\sin 2t\end{bmatrix}=\begin{bmatrix}0 & 2\\ -2 & 0\end{bmatrix}\begin{bmatrix}1+\sin 2t\\ e^{t}+\cos 2t\end{bmatrix}+\mathbf{g}(t)$. Solving for $\mathbf{g}(t)$, we obtain $\mathbf{g}(t)=\begin{bmatrix}-2e^{t}\\ e^{t}+2\end{bmatrix}$.

8. Given $\mathbf{y}(t)=\begin{bmatrix}t+\alpha\\ t^2+\beta\end{bmatrix}$ it follows that $\mathbf{y}(1)=\begin{bmatrix}1+\alpha\\ 1+\beta\end{bmatrix}=\begin{bmatrix}2\\ -1\end{bmatrix}\Rightarrow\alpha=1,\ \beta=-2$. Inserting $\mathbf{y}(t)$ into the differential equation, we see that $\mathbf{y}'(t)=A\mathbf{y}(t)+\mathbf{g}(t)$ and thus $\mathbf{y}'=\begin{bmatrix}1\\ 2t\end{bmatrix}=\begin{bmatrix}1 & t\\ t^2 & 1\end{bmatrix}\begin{bmatrix}t+1\\ t^2-2\end{bmatrix}+\mathbf{g}(t)$. Solving for $\mathbf{g}(t)$, we obtain $\mathbf{g}(t)=\begin{bmatrix}-t^3+t\\ -t^3-2t^2+2t+2\end{bmatrix}$.

9. Following the hint, we form $[\mathbf{y}_1', \mathbf{y}_2'] = P(t)[\mathbf{y}_1, \mathbf{y}_2] + [\mathbf{g}_1(t), \mathbf{g}_2(t)]$ which has the form $\begin{bmatrix} 0 & e^t \\ -e^{-t} & 0 \end{bmatrix} = P(t)\begin{bmatrix} 1 & e^t \\ e^{-t} & -1 \end{bmatrix} + \begin{bmatrix} -2 & e^t \\ 0 & -1 \end{bmatrix}$. Solving for $P(t)$, we have

$$P(t) = \begin{bmatrix} 2 & 0 \\ -e^{-t} & 1 \end{bmatrix}\begin{bmatrix} 1 & e^t \\ e^{-t} & -1 \end{bmatrix}^{-1} = \begin{bmatrix} 2 & 0 \\ -e^{-t} & 1 \end{bmatrix}(-1/2)\begin{bmatrix} -1 & -e^t \\ -e^{-t} & 1 \end{bmatrix} = \begin{bmatrix} 1 & e^t \\ 0 & -1 \end{bmatrix}.$$

10. If A^{-1} exists, $\mathbf{y}_2 = -A^{-1}\mathbf{b}$ is the unique solution.
If A^{-1} does not exist, the matrix equation $A\mathbf{y} = -\mathbf{b}$ will either have no solution or a non-unique solution. Therefore, either no equilibrium solution or a non-unique equilibrium solution.

11. An equilibrium solution of $\mathbf{y}' = A\mathbf{y} + \mathbf{b}$ is a constant solution. Therefore, since $\mathbf{y}' = \mathbf{0}$ we need $A\mathbf{y}_e = -\mathbf{b}$. For $A = \begin{bmatrix} 1 & 4 \\ -1 & -3 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ we see that $\mathbf{y}_e = -A^{-1}\mathbf{b} = -\begin{bmatrix} -3 & -4 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 10 \\ -3 \end{bmatrix}$.

12. A^{-1} exists and $\mathbf{y}_e = \begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$.

13. As noted in the solution of Exercise 11, we need $A\mathbf{y}_e = -\mathbf{b}$. For $A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$ we see that $\mathbf{y}_e = \begin{bmatrix} 0 \\ 2 \end{bmatrix} + a\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ where a is arbitrary.

14. A^{-1} exists and $\begin{bmatrix} 1 & 1 & 0 \\ 0 & -1 & 2 \\ 0 & 0 & 1 \end{bmatrix}\mathbf{y}_e = \begin{bmatrix} -2 \\ -3 \\ -2 \end{bmatrix} \Rightarrow \mathbf{y}_e = \begin{bmatrix} -1 \\ -1 \\ -2 \end{bmatrix}$.

15. As noted in the solution of Exercise 11, we need $A\mathbf{y}_e = -\mathbf{b}$. For $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} -2 \\ 0 \\ 0 \end{bmatrix}$ we see that $\mathbf{y}_e = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} + a\begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$ where a is arbitrary.

16. The characteristic polynomial is $p(\lambda) = \lambda^2 - 2\lambda$. The eigenvalues are $\lambda_1 = 0$ and $\lambda_2 = 2$, with corresponding eigenvectors $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\mathbf{x}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.
Thus, one fundamental matrix is $\hat{\Psi}(t) = \begin{bmatrix} 1 & e^{2t} \\ 1 & -e^{2t} \end{bmatrix}$. Set

$$\Psi = \hat{\Psi}C.\ \Psi(1) = I = \hat{\Psi}(1)C.\ \therefore C = \hat{\Psi}(1)^{-1} = \begin{bmatrix} 1 & e^2 \\ 1 & -e^2 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2}e^{-2} & -\frac{1}{2}e^{-2} \end{bmatrix}.$$

$$\Psi(t) = \begin{bmatrix} 1 & e^{2t} \\ 1 & -e^{2t} \end{bmatrix}\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2}e^{-2} & -\frac{1}{2}e^{-2} \end{bmatrix} = \begin{bmatrix} \frac{1}{2}\left(1+e^{2(t-1)}\right) & \frac{1}{2}\left(1-e^{2(t-1)}\right) \\ \frac{1}{2}\left(1-e^{2(t-1)}\right) & \frac{1}{2}\left(1+e^{2(t-1)}\right) \end{bmatrix}.$$

17. For $A=\begin{bmatrix}0 & 2\\ -2 & 0\end{bmatrix}$, the characteristic polynomial is $p(\lambda)=\lambda^2+4=(\lambda+2i)(\lambda-2i)$. The eigenvalues are $\lambda_1=-2i$ and $\lambda_2=2i$, with corresponding eigenvectors $\mathbf{x}_1=\begin{bmatrix}-1\\ i\end{bmatrix}$ and $\mathbf{x}_2=\begin{bmatrix}-1\\ -i\end{bmatrix}$. Converting to real solutions, we have $\mathbf{y}(t)=e^{-2it}\begin{bmatrix}-1\\ i\end{bmatrix}=(\cos 2t-i\sin 2t)\begin{bmatrix}-1\\ i\end{bmatrix}$. Therefore, a fundamental set of solutions is $\mathbf{y}_1(t)=\begin{bmatrix}-\cos 2t\\ \sin 2t\end{bmatrix}$ and $\mathbf{y}_2(t)=\begin{bmatrix}\sin 2t\\ \cos 2t\end{bmatrix}$.

Thus, one fundamental matrix is $\Psi(t)=\begin{bmatrix}-\cos 2t & \sin 2t\\ \sin 2t & \cos 2t\end{bmatrix}$. The solution of the given initial value problem has the form $\hat{\Psi}(t)=\Psi(t)C=\begin{bmatrix}-\cos 2t & \sin 2t\\ \sin 2t & \cos 2t\end{bmatrix}C$ where C is a (2×2) matrix chosen so that $\hat{\Psi}(\pi/4)=\begin{bmatrix}1 & -1\\ 0 & 1\end{bmatrix}$. Imposing this condition, we have $\begin{bmatrix}1 & -1\\ 0 & 1\end{bmatrix}=\Psi(\pi/4)C=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}C$. Solving for C, we obtain $C=\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}1 & -1\\ 0 & 1\end{bmatrix}=\begin{bmatrix}0 & 1\\ 1 & -1\end{bmatrix}$ and thus $\hat{\Psi}(t)=\begin{bmatrix}-\cos 2t & \sin 2t\\ \sin 2t & \cos 2t\end{bmatrix}\begin{bmatrix}0 & 1\\ 1 & -1\end{bmatrix}$ or $\hat{\Psi}(t)=\begin{bmatrix}\sin 2t & -\cos 2t-\sin 2t\\ \cos 2t & \sin 2t-\cos 2t\end{bmatrix}$.

18. The characteristic polynomial is $p(\lambda)=\lambda^2-2\lambda$. The eigenvalues are $\lambda_1=0$ and $\lambda_2=2$, with corresponding eigenvectors $\mathbf{x}_1=\begin{bmatrix}1\\ 1\end{bmatrix}$ and $\mathbf{x}_2=\begin{bmatrix}1\\ -1\end{bmatrix}$.

Thus, one fundamental matrix is $\hat{\Psi}(t)=\begin{bmatrix}1 & e^{2t}\\ 1 & -e^{2t}\end{bmatrix}$. Set

$\Psi=\hat{\Psi}C$. $\Psi(0)=\begin{bmatrix}1 & 0\\ 2 & 1\end{bmatrix}=\hat{\Psi}(0)C$. $\therefore C=\hat{\Psi}(0)^{-1}\Psi(0)=\begin{bmatrix}1 & 1\\ 1 & -1\end{bmatrix}^{-1}\begin{bmatrix}1 & 0\\ 2 & 1\end{bmatrix}=\begin{bmatrix}\frac{3}{2} & \frac{1}{2}\\ -\frac{1}{2} & -\frac{1}{2}\end{bmatrix}$.

$$\Psi(t)=\begin{bmatrix}1 & e^{2t}\\ 1 & -e^{2t}\end{bmatrix}\begin{bmatrix}\frac{3}{2} & \frac{1}{2}\\ -\frac{1}{2} & -\frac{1}{2}\end{bmatrix}=\begin{bmatrix}\frac{3}{2}-\frac{1}{2}e^{2t} & \frac{1}{2}-\frac{1}{2}e^{2t}\\ \frac{3}{2}+\frac{1}{2}e^{2t} & \frac{1}{2}+\frac{1}{2}e^{2t}\end{bmatrix}.$$

19. For $A=\begin{bmatrix}3 & -4\\ 2 & -3\end{bmatrix}$, the characteristic polynomial is $p(\lambda)=\lambda^2-1=(\lambda+1)(\lambda-1)$. The eigenvalues are $\lambda_1=-1$ and $\lambda_2=1$, with corresponding eigenvectors $\mathbf{x}_1=\begin{bmatrix}1\\ 1\end{bmatrix}$ and $\mathbf{x}_2=\begin{bmatrix}2\\ 1\end{bmatrix}$. Therefore, a fundamental set of solutions is $\mathbf{y}_1(t)=e^{-t}\begin{bmatrix}1\\ 1\end{bmatrix}$ and $\mathbf{y}_2(t)=e^{t}\begin{bmatrix}2\\ 1\end{bmatrix}$. Thus, one fundamental matrix is $\Psi(t)=\begin{bmatrix}e^{-t} & 2e^{t}\\ e^{-t} & e^{t}\end{bmatrix}$.

The solution of the given initial value problem has the form $\hat{\Psi}(t)=\Psi(t)C=\begin{bmatrix} e^{-t} & 2e^{t} \\ e^{-t} & e^{t}\end{bmatrix}C$

where C is a (2×2) matrix chosen so that $\hat{\Psi}(0)=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}$. Imposing this condition, we have

$\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=\Psi(0)C=\begin{bmatrix}1 & 2\\ 1 & 1\end{bmatrix}C$. Solving for C, we obtain $C=\begin{bmatrix}-1 & 2\\ 1 & -1\end{bmatrix}$ and thus

$\hat{\Psi}(t)=\begin{bmatrix} e^{-t} & 2e^{t} \\ e^{-t} & e^{t}\end{bmatrix}\begin{bmatrix}-1 & 2\\ 1 & -1\end{bmatrix}$ or $\hat{\Psi}(t)=\begin{bmatrix} -e^{-t}+2e^{t} & 2e^{-t}-2e^{t} \\ -e^{-t}+e^{t} & 2e^{-t}-e^{t}\end{bmatrix}$.

20. The characteristic polynomial is $p(\lambda)=\lambda^2-2\lambda+5$. The eigenvalues are $\lambda_1=1+2i$ and $\lambda_2=1-2i$, with corresponding eigenvectors $\mathbf{x}=\begin{bmatrix}-2i\\ 1\end{bmatrix}$. Then

$\mathbf{y}(t)=e^{t}(\cos 2t+i\sin 2t)\begin{bmatrix}-2i\\ 1\end{bmatrix}=\begin{bmatrix}2e^{t}\sin 2t\\ e^{t}\cos 2t\end{bmatrix}+i\begin{bmatrix}-2e^{t}\cos 2t\\ e^{t}\sin 2t\end{bmatrix}$.

Thus, one fundamental matrix is $\hat{\Psi}(t)=\begin{bmatrix}2e^{t}\sin 2t & -2e^{t}\cos 2t\\ e^{t}\cos 2t & e^{t}\sin 2t\end{bmatrix}$. Set

$\Psi=\hat{\Psi}C$. $\Psi\left(\frac{\pi}{4}\right)=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}=\begin{bmatrix}2e^{\frac{\pi}{4}} & 0\\ 0 & e^{\frac{\pi}{4}}\end{bmatrix}C$. $\therefore C=\begin{bmatrix}\frac{1}{2}e^{-\frac{\pi}{4}} & 0\\ 0 & e^{-\frac{\pi}{4}}\end{bmatrix}$.

$\Psi(t)=\begin{bmatrix}e^{\left(t-\frac{\pi}{4}\right)}\sin 2t & -2e^{\left(t-\frac{\pi}{4}\right)}\cos 2t\\ \frac{1}{2}e^{\left(t-\frac{\pi}{4}\right)}\cos 2t & e^{\left(t-\frac{\pi}{4}\right)}\sin 2t\end{bmatrix}$.

21. For $A=\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix}$, the characteristic polynomial is $p(\lambda)=\lambda^2-2\lambda=\lambda(\lambda-2)$. The eigenvalues are $\lambda_1=0$ and $\lambda_2=2$, with corresponding eigenvectors $\mathbf{x}_1=\begin{bmatrix}1\\ -1\end{bmatrix}$ and $\mathbf{x}_2=\begin{bmatrix}1\\ 1\end{bmatrix}$. Therefore, a fundamental set of solutions is $\mathbf{y}_1(t)=\begin{bmatrix}1\\ -1\end{bmatrix}$ and $\mathbf{y}_2(t)=e^{2t}\begin{bmatrix}1\\ 1\end{bmatrix}$. A fundamental matrix is

$\Psi(t)=\begin{bmatrix}1 & e^{2t}\\ -1 & e^{2t}\end{bmatrix}$ and therefore, $\Psi^{-1}(s)=0.5e^{-2s}\begin{bmatrix}e^{2s} & -e^{2s}\\ 1 & 1\end{bmatrix}=0.5\begin{bmatrix}1 & -1\\ e^{-2s} & e^{-2s}\end{bmatrix}$. From equation (11), the solution is $\mathbf{y}(t)=\Psi(t)\Psi^{-1}(t_0)\mathbf{y}_0+\Psi(t)\int_{t_0}^{t}\Psi^{-1}(s)\mathbf{g}(s)\,ds$. Since $\mathbf{y}_0=\mathbf{0}$ and $t_0=0$, we have

$$\mathbf{y}(t)=\Psi(t)\int_0^t\Psi^{-1}(s)\mathbf{g}(s)\,ds=\begin{bmatrix}1 & e^{2t}\\ -1 & e^{2t}\end{bmatrix}\int_0^t 0.5\begin{bmatrix}e^{2s}\\ 1\end{bmatrix}ds=\begin{bmatrix}1 & e^{2t}\\ -1 & e^{2t}\end{bmatrix}0.25\begin{bmatrix}e^{2t}-1\\ 2t\end{bmatrix}$$

$$=0.25\begin{bmatrix}e^{2t}-1+2te^{2t}\\ -(e^{2t}-1)+2te^{2t}\end{bmatrix}.$$

22. For $A = \begin{bmatrix} 9 & -4 \\ 15 & -7 \end{bmatrix}$, the characteristic polynomial is $p(\lambda) = \lambda^2 - 2\lambda - 3 = (\lambda - 3)(\lambda + 1)$. The eigenvalues are $\lambda_1 = -1$ and $\lambda_2 = 3$, with corresponding eigenvectors $\mathbf{x}_1 = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$ and $\mathbf{x}_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$.

A fundamental matrix is $\Psi(t) = \begin{bmatrix} 2e^{-t} & 2e^{3t} \\ 5e^{-t} & 3e^{3t} \end{bmatrix}$ and therefore,

$$\Psi^{-1}(s) = -\frac{1}{4}e^{-2s}\begin{bmatrix} 3e^{3s} & -2e^{3s} \\ -5e^{-s} & 2e^{-s} \end{bmatrix} = \begin{bmatrix} -\frac{3}{4}e^{s} & \frac{1}{2}e^{s} \\ \frac{5}{4}e^{-3s} & -\frac{1}{2}e^{-3s} \end{bmatrix}.$$

$$\int_0^t \Psi^{-1}(s)\mathbf{g}(s)\,ds = \int_0^t \begin{bmatrix} -\frac{3}{4}e^{2s} \\ \frac{5}{4}e^{-2s} \end{bmatrix} ds = \begin{bmatrix} -\frac{3}{8}\left(e^{2t} - 1\right) \\ -\frac{5}{8}\left(e^{-2t} - 1\right) \end{bmatrix}.$$

$$\Psi(t)\int_0^t \Psi^{-1}(s)\mathbf{g}(s)\,ds = \begin{bmatrix} -\frac{3}{4}(e^{t} - e^{-t}) - \frac{5}{4}(e^{t} - e^{3t}) \\ -\frac{15}{8}(e^{t} - e^{-t}) - \frac{15}{8}(e^{t} - e^{3t}) \end{bmatrix} = \begin{bmatrix} \frac{3}{4}e^{-t} - 2e^{t} + \frac{5}{4}e^{3t} \\ \frac{15}{8}e^{-t} - \frac{15}{4}e^{t} + \frac{15}{8}e^{3t} \end{bmatrix}.$$

Then, $\mathbf{y}(t) = \Psi(t)\mathbf{y}_0 + \Psi(t)\int_0^t \Psi^{-1}(s)\mathbf{g}(s)\,ds$, $\mathbf{y}(0) = \Psi(0)\mathbf{y}_0 + \mathbf{0} = \begin{bmatrix} 2 \\ 5 \end{bmatrix} \Rightarrow$

$\begin{bmatrix} 2 \\ 5 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 5 & 3 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \Rightarrow \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Therefore,

$$\mathbf{y}(t) = \begin{bmatrix} 2e^{-t} \\ 5e^{-t} \end{bmatrix} + \begin{bmatrix} \frac{3}{4}e^{-t} - 2e^{t} + \frac{5}{4}e^{3t} \\ \frac{15}{8}e^{-t} - \frac{15}{4}e^{t} + \frac{15}{8}e^{3t} \end{bmatrix} = \begin{bmatrix} \frac{11}{4}e^{-t} - 2e^{t} + \frac{5}{4}e^{3t} \\ \frac{55}{8}e^{-t} - \frac{15}{4}e^{t} + \frac{15}{8}e^{3t} \end{bmatrix}.$$

23. For $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, the characteristic polynomial is $p(\lambda) = \lambda^2 + 1 = (\lambda - i)(\lambda + i)$. The eigenvalues are $\lambda_1 = i$ and $\lambda_2 = -i$, with corresponding eigenvectors $\mathbf{x}_1 = \begin{bmatrix} 1 \\ i \end{bmatrix}$ and $\mathbf{x}_2 = \begin{bmatrix} 1 \\ -i \end{bmatrix}$.

$\mathbf{x}_1 = \begin{bmatrix} -1 \\ i \end{bmatrix}$ and $\mathbf{x}_2 = \begin{bmatrix} -1 \\ -i \end{bmatrix}$. Converting to real solutions, we have $\mathbf{y}(t) = e^{it}\begin{bmatrix} 1 \\ i \end{bmatrix} = (\cos t + i\sin t)\begin{bmatrix} 1 \\ i \end{bmatrix}$.

Therefore, a fundamental set of solutions is $\mathbf{y}_1(t) = \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix}$ and $\mathbf{y}_2(t) = \begin{bmatrix} \sin t \\ \cos t \end{bmatrix}$. A fundamental matrix is $\Psi(t) = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}$. We have $\Psi^{-1}(s) = \begin{bmatrix} \cos s & -\sin s \\ \sin s & \cos s \end{bmatrix}$.

From equation (11), the solution is $\mathbf{y}(t) = \Psi(t)\Psi^{-1}(t_0)\mathbf{y}_0 + \Psi(t)\int_{t_0}^t \Psi^{-1}(s)\mathbf{g}(s)\,ds$. Since $\mathbf{y}_0 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\mathbf{g}(s) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, and $t_0 = 0$, we have

$$\mathbf{y}(t) = \Psi(t)\Psi^{-1}(t_0)\mathbf{y}_0 + \Psi(t)\int_0^t \Psi^{-1}(s)\mathbf{g}(s)\,ds = \begin{bmatrix} \sin t \\ \cos t \end{bmatrix} + \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}\int_0^t \begin{bmatrix} 2\cos s - \sin s \\ 2\sin s + \cos s \end{bmatrix} ds$$

$$= \begin{bmatrix} \sin t \\ \cos t \end{bmatrix} + \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}\begin{bmatrix} 2\sin t + \cos t - 1 \\ -2\cos t + \sin t + 2 \end{bmatrix} = \begin{bmatrix} 1 - \cos t + 3\sin t \\ -2 + 3\cos t + \sin t \end{bmatrix}.$$

24. For $A=\begin{bmatrix}1 & 1\\ 0 & 1\end{bmatrix}$, the characteristic polynomial is $p(\lambda)=(1-\lambda)^2$. The eigenvalue is $\lambda=1$, with corresponding eigenvector $\mathbf{x}=\begin{bmatrix}1\\0\end{bmatrix}$. Then

$\mathbf{y}_1=\begin{bmatrix}e^t\\0\end{bmatrix}$. Let $\mathbf{y}_2=e^t\left(t\mathbf{v}_1+\mathbf{v}_2\right)$. $\mathbf{v}_1=\begin{bmatrix}1\\0\end{bmatrix}$, $\mathbf{v}_2=\begin{bmatrix}0\\1\end{bmatrix}\Rightarrow \mathbf{y}_2=e^t\begin{bmatrix}t\\1\end{bmatrix}$.

A fundamental matrix is $\Psi(t)=\begin{bmatrix}e^t & te^t\\ 0 & e^t\end{bmatrix}$ and therefore,

$\Psi^{-1}(s)=e^{-2s}\begin{bmatrix}e^s & -se^s\\ 0 & e^s\end{bmatrix}=\begin{bmatrix}e^{-s} & -se^{-s}\\ 0 & e^{-s}\end{bmatrix}$. $\int_0^t\Psi^{-1}(s)\mathbf{g}(s)\,ds=\int_0^t\begin{bmatrix}e^{-s}(1-s)\\ e^{-s}\end{bmatrix}ds=\begin{bmatrix}te^{-t}\\ 1-e^{-t}\end{bmatrix}$.

Since $\mathbf{y}(0)=\begin{bmatrix}0\\0\end{bmatrix}$, $\mathbf{y}(t)=\Psi(t)\int_0^t\Psi^{-1}(s)\mathbf{g}(s)\,ds=\begin{bmatrix}t+te^t-t\\ e^t-1\end{bmatrix}=\begin{bmatrix}te^t\\ e^t-1\end{bmatrix}$.

25 (a). $\mathbf{Q}(t)$ is an equilibrium solution if $\frac{r}{V}\begin{bmatrix}-2 & 1\\ 1 & -2\end{bmatrix}\mathbf{Q}+\begin{bmatrix}cr\\0\end{bmatrix}=\mathbf{0}$. Solving for $\mathbf{Q}$, we obtain

$\mathbf{Q}_e(t)=\frac{Cv}{3}\begin{bmatrix}2\\1\end{bmatrix}$.

25 (b). The characteristic polynomial is $p(\lambda)=\lambda^2+(4r/V)\lambda+3r^2/V^2=(\lambda+3r/V)(\lambda+r/V)$. The eigenvalues are $\lambda_1=-3r/V$ and $\lambda_2=-r/V$, with corresponding eigenvectors

$\mathbf{x}_1=\begin{bmatrix}1\\-1\end{bmatrix}$ and $\mathbf{x}_2=\begin{bmatrix}1\\1\end{bmatrix}$. Therefore, a fundamental set of solutions is

$\mathbf{y}_1(t)=e^{-3rt/V}\begin{bmatrix}1\\-1\end{bmatrix}$ and $\mathbf{y}_2(t)=e^{-rt/V}\begin{bmatrix}1\\1\end{bmatrix}$. The complementary solution is

$\mathbf{Q}_C(t)=\begin{bmatrix}e^{-3rt/V} & e^{-rt/V}\\ -e^{-3rt/V} & e^{-rt/V}\end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix}$.

25 (c). Finding a constant particular solution is equivalent to finding an equilibrium solution, as in part (a).

25 (d). The general solution is $\mathbf{Q}(t)=\mathbf{Q}_C(t)+\mathbf{Q}_e(t)=\begin{bmatrix}e^{-3rt/V} & e^{-rt/V}\\ -e^{-3rt/V} & e^{-rt/V}\end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix}+\frac{cV}{3}\begin{bmatrix}2\\1\end{bmatrix}$. Imposing the initial condition leads to $\mathbf{Q}(0)=\mathbf{0}$ or $\begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix}+\frac{cV}{3}\begin{bmatrix}2\\1\end{bmatrix}=\begin{bmatrix}0\\0\end{bmatrix}$. The solution of the initial value problem is $\mathbf{Q}(t)=\frac{cV}{6}\begin{bmatrix}4-e^{-3rt/V}-3e^{-rt/V}\\ 2+e^{-3rt/V}-3e^{-rt/V}\end{bmatrix}$.

25 (e). $\frac{1}{V}\lim_{t\to\infty}\mathbf{Q}(t)=\frac{c}{3}\begin{bmatrix}2\\1\end{bmatrix}$.

26 (a). $-V_s + \frac{1}{2}I_1' + I_1 + 2(I_1 - I_2) = 0,\ 2(I_2 - I_1) + I_2 + \frac{1}{2}I_2' = 0$. Therefore,

$$\frac{d}{dt}\begin{bmatrix} I_1' \\ I_2' \end{bmatrix} = \begin{bmatrix} -6I_1 + 4I_2 + 2V_s \\ 4I_1 - 6I_2 \end{bmatrix} = \begin{bmatrix} -6 & 4 \\ 4 & -6 \end{bmatrix}\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} + \begin{bmatrix} 2V_s \\ 0 \end{bmatrix},\ t > 0,\ \begin{bmatrix} I_1(0) \\ I_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

26 (b). The characteristic polynomial is $p(\lambda) = \lambda^2 + 12\lambda + 20 = (\lambda + 10)(\lambda + 2)$. The eigenvalues are $\lambda_1 = -10$ and $\lambda_2 = -2$, with corresponding eigenvectors $\mathbf{x}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\mathbf{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Therefore, a fundamental matrix is $\Psi(t) = \begin{bmatrix} e^{-10t} & e^{-2t} \\ -e^{-10t} & e^{-2t} \end{bmatrix}$.

26 (c). $\mathbf{I}(t) = \Psi(t)\int_0^t \Psi^{-1}(s)\begin{bmatrix} 2V_s \\ 0 \end{bmatrix} ds$ since $\mathbf{I}(0) = \mathbf{0}$. $\Psi^{-1}(s) = \frac{1}{2e^{-12s}}\begin{bmatrix} e^{-2s} & -e^{-2s} \\ e^{-10s} & e^{-10s} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} e^{10s} & -e^{10s} \\ e^{2s} & e^{2s} \end{bmatrix}$.

With $V_s(t) = 1,\ t > 0,\ \int_0^t \Psi^{-1}(s)\begin{bmatrix} 2V_s \\ 0 \end{bmatrix} ds = \int_0^t \begin{bmatrix} \mathrm{e}^{10s} \\ \mathrm{e}^{2s} \end{bmatrix} = \begin{bmatrix} \frac{1}{10}(e^{10t} - 1) \\ \frac{1}{2}(e^{2t} - 1) \end{bmatrix}$, Therefore,

$$\mathbf{I}(t) = \begin{bmatrix} \frac{1}{10}(1 - e^{-10t}) + \frac{1}{2}(1 - e^{-2t}) \\ -\frac{1}{10}(1 - e^{-10t}) + \frac{1}{2}(1 - e^{-2t}) \end{bmatrix} = \begin{bmatrix} -\frac{1}{10}e^{-10t} - \frac{1}{2}e^{-2t} + \frac{3}{5} \\ \frac{1}{10}e^{-10t} - \frac{1}{2}e^{-2t} + \frac{2}{5} \end{bmatrix}.$$

27 (a). In the vector system $\mathbf{v}' = -\mathbf{v} + (\mathbf{v} \times \mathbf{k}) + \mathbf{f}$, the term $\mathbf{v} \times \mathbf{k}$ is given by $(v_x\mathbf{i} + v_y\mathbf{j}) \times \mathbf{k} = -v_x\mathbf{j} + v_y\mathbf{i}$. Therefore, the system is $\begin{bmatrix} v_x' \\ v_y' \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} v_x \\ v_y \end{bmatrix} + \begin{bmatrix} f_x \\ f_y \end{bmatrix}$.

27 (b). For $\mathbf{f} = 0.5\begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix}$, we seek a constant solution $\mathbf{v}$; that is, an equilibrium solution. Thus, we need to solve, if possible, $\begin{bmatrix} -1 & 1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} v_x \\ v_y \end{bmatrix} + 0.5\begin{bmatrix} 1 \\ \sqrt{3} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. This system does indeed have a solution, namely $\mathbf{v}_e = 0.25\begin{bmatrix} 1 + \sqrt{3} \\ -1 + \sqrt{3} \end{bmatrix}$. If we choose the initial velocity equal to the "equilibrium velocity," $\mathbf{v}_e$, then the particle will move at that constant velocity.

28 (b). The characteristic polynomial is $p(\lambda) = -\lambda(\lambda^2 + \omega_c^2)$. The eigenvalues are $\lambda_1 = 0,\ \lambda_2 = i\omega_c,\ \lambda_3 = -i\omega_c$, with corresponding eigenvectors $\mathbf{x}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} 1 \\ i \\ 0 \end{bmatrix}$. Therefore,

A fundamental matrix is $\Psi(t) = \begin{bmatrix} 0 & \cos\omega_c t & \sin\omega_c t \\ 0 & -\sin\omega_c t & \cos\omega_c t \\ 1 & 0 & 0 \end{bmatrix}$.

28 (c). $\Phi(t) = \Psi(t)\Psi^{-1}(0) = \begin{bmatrix} 0 & \cos\omega_c t & \sin\omega_c t \\ 0 & -\sin\omega_c t & \cos\omega_c t \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}^{-1} =$

$$\begin{bmatrix} 0 & \cos\omega_c t & \sin\omega_c t \\ 0 & -\sin\omega_c t & \cos\omega_c t \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} \cos\omega_c t & \sin\omega_c t & 0 \\ -\sin\omega_c t & \cos\omega_c t & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

28 (d). From equation 11, $\mathbf{v}(t) = \Phi(t)\mathbf{v}_0 + \Phi(t)\int_0^t \Phi^{-1}(s)\mathbf{g}(s)\,ds$, using $\Phi(t)$ as a fundamental matrix and noting that $\Phi^{-1}(0) = \mathbf{I}$. Therefore, $\mathbf{v}(t) = \Phi(t)\mathbf{v}_0 + \mathbf{f}(t)$.

28 (e). $\mathbf{r}(t) = \int_0^\tau \mathbf{v}(t)dt = \left[\int_0^\tau \Phi(t)dt\right]\mathbf{v}_0 + \int_0^\tau \mathbf{f}(t)dt = \hat{\mathbf{r}}$. $\therefore \left[\int_0^\tau \Phi(t)dt\right]\mathbf{v}_0 = \hat{\mathbf{r}} - \int_0^\tau \mathbf{f}(t)dt$

$$\int_0^\tau \Phi(t)dt = \begin{bmatrix} \omega_c^{-1}\sin\omega_c t & \omega_c^{-1}(1-\cos\omega_c t) & 0 \\ -\omega_c^{-1}(1-\cos\omega_c t) & \omega_c^{-1}\sin\omega_c t & 0 \\ 0 & 0 & \tau \end{bmatrix}.$$

$$D = \det\left\{\int_0^\tau \Phi(t)dt\right\} = \frac{2\tau}{\omega_c^2}\left(1-\cos\omega_c\tau\right) = \frac{4\tau}{\omega_c^2}\sin^2\left(\frac{\omega_c\tau}{2}\right) \quad \therefore \frac{\omega_c\tau}{2} \neq n\pi \Rightarrow \tau \neq \frac{2n\pi}{\omega_c}.$$

Section 6.9

1 (a). For $\mathbf{y}' = P(t)\mathbf{y} + \mathbf{g}(t)$, $\mathbf{y}(t_0) = \mathbf{y}_0$, Euler's method has the form $\mathbf{y}_{n+1} = \mathbf{y}_n + h\left[P(t_n)\mathbf{y}_n + \mathbf{g}(t_n)\right]$.

For $P(t) = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$, $\mathbf{g}(t) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\mathbf{y}_0 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, and $t_0 = 0$ the iteration is

$$\mathbf{y}_{n+1} = \mathbf{y}_n + h\left[\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}\mathbf{y}_n + \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right], \; \mathbf{y}_0 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

1 (b). In general, $t_k = t_0 + kh, k = 0,1,\ldots$. Since $t_0 = 0$, we have $t_k = kh, k = 0,1,\ldots$. In general, $h = (b-a)/N$. So, for $a = 0, b = 1$, and $h = 0.01$, we obtain $N = 1/h = 100$.

2 (a). $\mathbf{y}_{n+1} = \mathbf{y}_n + h\left[\begin{bmatrix} 1 & t_n \\ 2+t_n & 2 \end{bmatrix}\mathbf{y}_n + \begin{bmatrix} 1 \\ t_n \end{bmatrix}\right], \; \mathbf{y}_0 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$

2 (b). $t_k = 1 + kh, k = 0,1,\ldots$. $N = .5/h = 50$.

3 (a). For $\mathbf{y}' = P(t)\mathbf{y} + \mathbf{g}(t)$, $\mathbf{y}(t_0) = \mathbf{y}_0$, Euler's method has the form $\mathbf{y}_{n+1} = \mathbf{y}_n + h\left[P(t_n)\mathbf{y}_n + \mathbf{g}(t_n)\right]$.

For $P(t) = \begin{bmatrix} -t^2 & t \\ 2-t & 0 \end{bmatrix}$, $\mathbf{g}(t) = \begin{bmatrix} 1 \\ t \end{bmatrix}$, $\mathbf{y}_0 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$, and $t_0 = 1$ the iteration is

$$\mathbf{y}_{n+1} = \mathbf{y}_n + h\left[\begin{bmatrix} -t_n^2 & t_n \\ 2-t_n & 0 \end{bmatrix}\mathbf{y}_n + \begin{bmatrix} 1 \\ t_n \end{bmatrix}\right], \; \mathbf{y}_0 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}.$$

3 (b). In general, $t_k = t_0 + kh, k = 0,1,\ldots$. Since $t_0 = 1$, we have $t_k = 1 + kh, k = 0,1,\ldots$. In general, $h = (b-a)/N$. So, for $a = 1, b = 4$, and $h = 0.01$, we obtain $N = 3/h = 300$.

4 (a). $\mathbf{y}_{n+1} = \mathbf{y}_n + h\left[\begin{bmatrix} 1 & 0 & 1 \\ 3 & 2 & 1 \\ 1 & 2 & 0 \end{bmatrix}\mathbf{y}_n + \begin{bmatrix} 0 \\ 2 \\ t_n \end{bmatrix}\right],\ \mathbf{y}_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$

4 (b). $t_k = -1 + kh, k = 0,1,\ldots.\quad N = 1/h = 100.$

5 (a). For $\mathbf{y}' = P(t)\mathbf{y} + \mathbf{g}(t),\ \mathbf{y}(t_0) = \mathbf{y}_0$, Euler's method has the form $\mathbf{y}_{n+1} = \mathbf{y}_n + h\left[P(t_n)\mathbf{y}_n + \mathbf{g}(t_n)\right]$.

For $P(t) = \begin{bmatrix} t^{-1} & \sin t \\ 1-t & 1 \end{bmatrix}, \mathbf{g}(t) = \begin{bmatrix} 0 \\ t^2 \end{bmatrix}, \mathbf{y}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, and $t_0 = 1$ the iteration is

$$\mathbf{y}_{n+1} = \mathbf{y}_n + h\left[\begin{bmatrix} t_n^{-1} & \sin t_n \\ 1-t_n & 1 \end{bmatrix}\mathbf{y}_n + \begin{bmatrix} 0 \\ t_n^2 \end{bmatrix}\right],\ \mathbf{y}_0 = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

5 (b). In general, $t_k = t_0 + kh, k = 0,1,\ldots$. Since $t_0 = 1$, we have $t_k = 1 + kh, k = 0,1,\ldots$. In general, $h = (b-a)/N$. So, for $a = 1, b = 6$, and $h = 0.01$, we obtain $N = 5/h = 500$.

6. $\mathbf{y}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} + 0.01\left[\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right] = \begin{bmatrix} -0.98 \\ 1.02 \end{bmatrix}$ and

$$\mathbf{y}_2 = \begin{bmatrix} -0.98 \\ 1.02 \end{bmatrix} + 0.01\left[\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}\begin{bmatrix} -0.98 \\ 1.02 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right] = \begin{bmatrix} -0.9594 \\ 1.041 \end{bmatrix}$$

7. The iteration has the form $\mathbf{y}_{n+1} = \mathbf{y}_n + h\left[\begin{bmatrix} 1 & t_n \\ 2+t_n & 2 \end{bmatrix}\mathbf{y}_n + \begin{bmatrix} 1 \\ t_n \end{bmatrix}\right],\ \mathbf{y}_0 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ where

$t_0 = 1$ and $t_1 = 1.01$. Therefore, $\mathbf{y}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix} + 0.01\left[\begin{bmatrix} 1 & 1 \\ 3 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right] = \begin{bmatrix} 2 \\ 1 \end{bmatrix} + 0.01\begin{bmatrix} 4 \\ 9 \end{bmatrix} = \begin{bmatrix} 2.04 \\ 1.09 \end{bmatrix}$ and

$$\mathbf{y}_2 = \begin{bmatrix} 2.04 \\ 1.09 \end{bmatrix} + 0.01\left[\begin{bmatrix} 1 & 1.01 \\ 3.01 & 2 \end{bmatrix}\begin{bmatrix} 2.04 \\ 1.09 \end{bmatrix} + \begin{bmatrix} 1 \\ 1.01 \end{bmatrix}\right] = \begin{bmatrix} 2.04 \\ 1.09 \end{bmatrix} + 0.01\begin{bmatrix} 4.1409 \\ 9.3304 \end{bmatrix}$$

$$= \begin{bmatrix} 2.081409 \\ 1.183304 \end{bmatrix}.$$

8. $\mathbf{y}_1 = \begin{bmatrix} 2 \\ 0 \end{bmatrix} + 0.01\left[\begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right] = \begin{bmatrix} 1.99 \\ 0.03 \end{bmatrix}$ and

$$\mathbf{y}_2 = \begin{bmatrix} 1.99 \\ 0.03 \end{bmatrix} + 0.01\left[\begin{bmatrix} -(1.01)^2 & 1.01 \\ 2-1.01 & 0 \end{bmatrix}\begin{bmatrix} 1.99 \\ 0.03 \end{bmatrix} + \begin{bmatrix} 1 \\ 1.01 \end{bmatrix}\right] = \begin{bmatrix} 1.9800030 \\ 0.059801 \end{bmatrix}$$

9. The iteration has the form $\mathbf{y}_{n+1} = \mathbf{y}_n + h\left[\begin{bmatrix} 1 & 0 & 1 \\ 3 & 2 & 1 \\ 1 & 2 & 0 \end{bmatrix}\mathbf{y}_n + \begin{bmatrix} 0 \\ 2 \\ t_n \end{bmatrix}\right],\ \mathbf{y}_0 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ where

$t_0 = -1$ and $t_1 = -0.99$. Therefore,

$$\mathbf{y}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + 0.01\left[\begin{bmatrix} 1 & 0 & 1 \\ 3 & 2 & 1 \\ 1 & 2 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}\right] = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + 0.01\begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 0.01 \\ 0.03 \\ 0.99 \end{bmatrix}$$

and

$$\mathbf{y}_2 = \begin{bmatrix} 0.01 \\ 0.03 \\ 0.99 \end{bmatrix} + 0.01\left[\begin{bmatrix} 1 & 0 & 1 \\ 3 & 2 & 1 \\ 1 & 2 & 0 \end{bmatrix}\begin{bmatrix} 0.01 \\ 0.03 \\ 0.99 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \\ -0.99 \end{bmatrix}\right] = \begin{bmatrix} 0.01 \\ 0.03 \\ 0.99 \end{bmatrix} + 0.01\begin{bmatrix} 1 \\ 3.08 \\ -0.92 \end{bmatrix}$$

$$= \begin{bmatrix} 0.02 \\ 0.0608 \\ 0.9808 \end{bmatrix}.$$

10. $\mathbf{y}_1 = \begin{bmatrix} 0 \\ 0 \end{bmatrix} + 0.01\left[\begin{bmatrix} 1 & \sin(1) \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right] = \begin{bmatrix} 0 \\ 0.01 \end{bmatrix}$ and

$$\mathbf{y}_2 = \begin{bmatrix} 0 \\ 0.01 \end{bmatrix} + 0.01\left[\begin{bmatrix} \frac{1}{1.01} & \sin(1.01) \\ 1-1.01 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 0.01 \end{bmatrix} + \begin{bmatrix} 0 \\ (1.01)^2 \end{bmatrix}\right] = \begin{bmatrix} 0.00008468318 \\ 0.020301 \end{bmatrix}$$

11 (a). Let $\mathbf{z}(t) = \begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} = \begin{bmatrix} y(t) \\ y'(t) \end{bmatrix}$. With this,

$$\mathbf{z}'(t) = \begin{bmatrix} z_1'(t) \\ z_2'(t) \end{bmatrix} = \begin{bmatrix} y'(t) \\ y''(t) \end{bmatrix} = \begin{bmatrix} z_2(t) \\ -z_1(t) + t^{3/2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ t^{3/2} \end{bmatrix}, \mathbf{z}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

11 (b). Thus, the iteration has the form $\mathbf{z}_{n+1} = \mathbf{z}_n + h\left[\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\mathbf{z}_n + \begin{bmatrix} 0 \\ t_n^{3/2} \end{bmatrix}\right]$, $\mathbf{z}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ where

$t_0 = 0$ and $t_1 = 0.01$.

11 (c). Therefore, $\mathbf{z}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 0.01\left[\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right] = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 0.01\begin{bmatrix} 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -0.01 \end{bmatrix}$ and

$$\mathbf{z}_2 = \begin{bmatrix} 1 \\ -0.01 \end{bmatrix} + 0.01\left[\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ -0.01 \end{bmatrix} + \begin{bmatrix} 0 \\ 0.001 \end{bmatrix}\right] = \begin{bmatrix} 1 \\ -0.01 \end{bmatrix} + 0.01\begin{bmatrix} -.01 \\ -.999 \end{bmatrix}$$

$$= \begin{bmatrix} .9999 \\ -.01999 \end{bmatrix}.$$

12 (a). $\mathbf{z}'(t) = \begin{bmatrix} z_1'(t) \\ z_2'(t) \end{bmatrix} = \begin{bmatrix} z_2(t) \\ -t^2 z_1(t) - z_2(t) + 2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -t^2 & -1 \end{bmatrix}\begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \mathbf{z}(1) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$

12 (b). $\mathbf{z}_{n+1} = \mathbf{z}_n + h\left[\begin{bmatrix} 0 & 1 \\ -t^2 & -1 \end{bmatrix}\mathbf{z}_n + \begin{bmatrix} 0 \\ 2 \end{bmatrix}\right]$, $\mathbf{z}_0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ where $t_0 = 1$.

12 (c). $\mathbf{z}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + 0.01\left[\begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix}\right] = \begin{bmatrix} 1.01 \\ 1 \end{bmatrix}$ and

$$\mathbf{z}_2 = \begin{bmatrix} 1.01 \\ 1 \end{bmatrix} + 0.01\left[\begin{bmatrix} 0 & 1 \\ -(1.01)^2 & -1 \end{bmatrix}\begin{bmatrix} 1.01 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix}\right] = \begin{bmatrix} 1.02 \\ 0.999696699 \end{bmatrix}$$

13 (a). Let $\mathbf{z} = \begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} = \begin{bmatrix} y \\ y' \\ y'' \end{bmatrix}$. With this,

$$\mathbf{z}' = \begin{bmatrix} z_1' \\ z_2' \\ z_3' \end{bmatrix} = \begin{bmatrix} y' \\ y'' \\ y''' \end{bmatrix} = \begin{bmatrix} z_2 \\ z_3 \\ -2z_2 - tz_1 + t + 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -t & -2 & 0 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \\ z_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ t+1 \end{bmatrix}, \mathbf{z}(0) = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}.$$

13 (b). Thus, the iteration has the form $\mathbf{z}_{n+1} = \mathbf{z}_n + h\left[\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -t_n & -2 & 0 \end{bmatrix}\mathbf{z}_n + \begin{bmatrix} 0 \\ 0 \\ t_n + 1 \end{bmatrix}\right]$, $\mathbf{z}_0 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ where $t_0 = 0$ and $t_1 = 0.01$.

13 (c). Therefore, $\mathbf{z}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + 0.01\left[\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -2 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right] = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + 0.01\begin{bmatrix} -1 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 0.99 \\ -1 \\ 0.03 \end{bmatrix}$ and

$$\mathbf{z}_2 = \begin{bmatrix} 0.99 \\ -1 \\ 0.03 \end{bmatrix} + 0.01\left[\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -0.01 & -2 & 0 \end{bmatrix}\begin{bmatrix} 0.99 \\ -1 \\ 0.03 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1.01 \end{bmatrix}\right] = \begin{bmatrix} 0.99 \\ -1 \\ 0.03 \end{bmatrix} + 0.01\begin{bmatrix} -1 \\ 0.03 \\ 3.0001 \end{bmatrix}$$

$$= \begin{bmatrix} 0.98 \\ -0.9997 \\ 0.060001 \end{bmatrix}.$$

14 (a). $\mathbf{z}'(t) = \begin{bmatrix} z_1'(t) \\ z_2'(t) \end{bmatrix} = \begin{bmatrix} z_2(t) \\ -e^{-t}z_1(t) - z_2(t) + 2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -e^{-t} & -1 \end{bmatrix}\begin{bmatrix} z_1(t) \\ z_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \mathbf{z}(0) = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$

14 (b). $\mathbf{z}_{n+1} = \mathbf{z}_n + h\left[\begin{bmatrix} 0 & 1 \\ -e^{-t} & -1 \end{bmatrix}\mathbf{z}_n + \begin{bmatrix} 0 \\ 2 \end{bmatrix}\right]$, $\mathbf{z}_0 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ where $t_0 = 0$.

14 (c). $\mathbf{z}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix} + 0.01\left[\begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix}\right] = \begin{bmatrix} -0.99 \\ 1.02 \end{bmatrix}$ and

$$\mathbf{z}_2 = \begin{bmatrix} -0.99 \\ 1.02 \end{bmatrix} + 0.01\left[\begin{bmatrix} 0 & 1 \\ -e^{-0.01} & -1 \end{bmatrix}\begin{bmatrix} -0.99 \\ 1.02 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \end{bmatrix}\right] = \begin{bmatrix} -0.9798 \\ 1.039601493 \end{bmatrix}$$

18. Actual error: $\mathbf{y}(1) - \overline{\mathbf{y}}_{200} = \begin{bmatrix} -0.00807508729 \\ 0.0759433736... \end{bmatrix}$

Estimated error: $\overline{\mathbf{y}}_{200} - \mathbf{y}_{100} = \begin{bmatrix} -0.0086591617... \\ 0.0764878206... \end{bmatrix}$

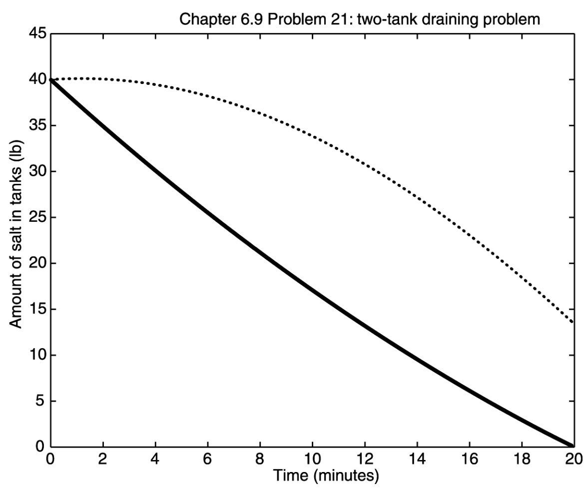
20. Actual error: $\mathbf{y}(1) - \overline{\mathbf{y}}_{200} = \begin{bmatrix} 0.0027112167... \\ -0.0027112167... \end{bmatrix}$

Estimated error: $\overline{\mathbf{y}}_{200} - \mathbf{y}_{100} = \begin{bmatrix} 0.0027202379... \\ -0.0027202379... \end{bmatrix}$

21 (a). $\dfrac{dQ_1}{dt} = -15\dfrac{Q_1}{V_1} + \dfrac{5}{V_2}Q_2 = \dfrac{-15}{200-10t}Q_1 + \dfrac{5}{500-20t}Q_2$

$\dfrac{dQ_2}{dt} = \dfrac{15}{200-10t}Q_1 - \dfrac{35}{500-20t}Q_2$

21 (b).
```
t=0:.01:19.9;
Q1(1)=40;Q2(1)=40;
h=0.01;
V1=200-10*t;
V2=500-20*t;
N=19.9/h;
for i=1:N
Q1(i+1)=Q1(i)+h*(-(15/V1(i))*Q1(i)+(5/V2(i))*Q2(i));
Q2(i+1)=Q2(i)+h*((15/V1(i))*Q1(i)-(35/V2(i))*Q2(i));
end
plot(t,Q1,t,Q2,':')
ylabel('Amount of salt in tanks (lb)')
xlabel('time (minutes)')
title('Chapter 6.9 Problem 21: two-tank draining problem')
```
21 (c).

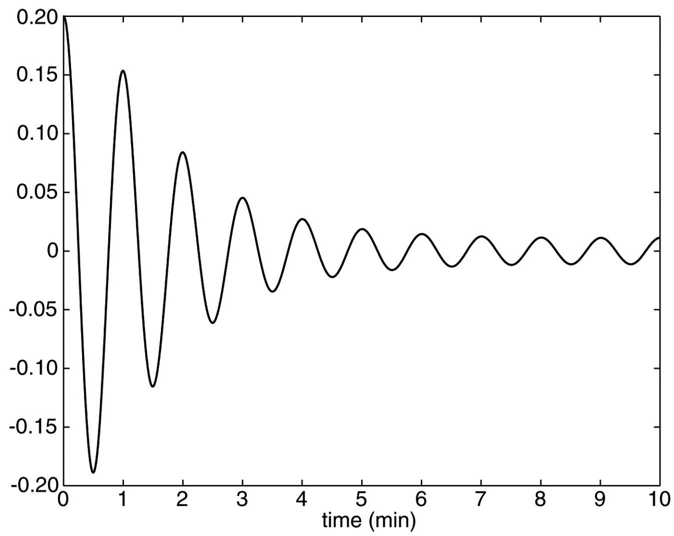


21 (d). The coefficients $\pm\dfrac{15}{200-10t}$ are not continuous at $t = 20$. Therefore, Existence-Uniqueness Theorem 6.1 does not apply to any interval containing $t = 20$.

22 (a). $my'' + \gamma y' + ky = 0,\ m = 1,\ \gamma = 2te^{-\frac{t}{2}},\ k = 4\pi^2,\ y(0) = \dfrac{1}{5}\,meters,\ y'(0) = 0$.

$$\mathbf{y}'(t) = \begin{bmatrix} y_1'(t) \\ y_2'(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -4\pi^2 & -2te^{-\frac{t}{2}} \end{bmatrix}\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}, \mathbf{y}(0) = \begin{bmatrix} 0.2 \\ 0 \end{bmatrix}.$$

22 (b).
```
t=0:.005:10;
h=.005;
N=2000;
y1(1)=0.2;y2(1)=0;
gamma=2*t.*exp(-0.5*t);
k=4*(pi^2);
for i=1:N
y1(i+1)=y1(i)+h*y2(i);
y2(i+1)=y2(i)+h*(-k*y1(i)-gamma(i)*y2(i));
end
plot(t,y1)
```

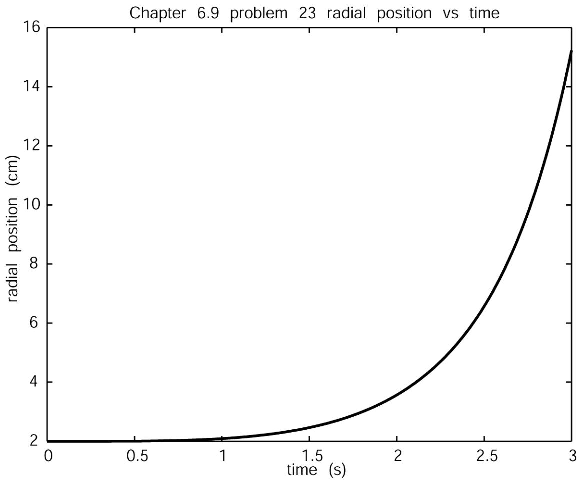


22 (c).

The amplitude of displacement decreases significantly during the time when damping is significant. As damping dimishes, the vibration amplitude seems to settle down to a constant value.

23 (b).
```
t=0:0.01:3;
h=0.01;
N=300;
y1(1)=2;y2(1)=0;
for i=1:N
y1(i+1)=y1(i)+h*y2(i);
y2(i+1)=y2(i)+h*(((pi/4)^2)*(t(i)^2)*y1(i)-0.5*y2(i));
end
plot(t,y1)
xlabel('time (s)');
ylabel('radial position (cm)');
title('Chapter 6.9 problem 23 radial position vs time')
y1(301)
```

23 (c).



$r(3) = 15.2268..cm$

Section 6.10

1. For $A = \begin{bmatrix} 5 & -6 \\ 3 & -4 \end{bmatrix}$ the characteristic polynomial is $p(\lambda) = \lambda^2 - \lambda - 2$. Eigenvalues are $\lambda_1 = -1$ and $\lambda_2 = 2$. Corresponding eigenvectors are $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\mathbf{x}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. As in Example 1, we can construct a diagonalizing matrix T from the eigenvectors of A, $T = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$. The corresponding matrix of eigenvalues, $D = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}$, is such that $T^{-1}AT = D$.

2. The characteristic polynomial is $p(\lambda) = \lambda^2 - 1$. Eigenvalues are $\lambda_1 = -1$ and $\lambda_2 = 1$. Corresponding eigenvectors are $\mathbf{x}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\mathbf{x}_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$. Therefore, $T = \begin{bmatrix} 1 & 2 \\ -1 & -1 \end{bmatrix}$ and $D = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$.

3. For $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ the characteristic polynomial is $p(\lambda) = \lambda^2 - 2\lambda$. Eigenvalues are $\lambda_1 = 0$ and $\lambda_2 = 2$. Corresponding eigenvectors are $\mathbf{x}_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and $\mathbf{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. As in Example 1, we can construct a diagonalizing matrix T from the eigenvectors of A, $T = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$. The corresponding matrix of eigenvalues, $D = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$, is such that $T^{-1}AT = D$.

4. The characteristic polynomial is $p(\lambda)=\lambda^2-5\lambda$. Eigenvalues are $\lambda_1=0$ and $\lambda_2=5$. Corresponding eigenvectors are $\mathbf{x}_1=\begin{bmatrix}3\\-2\end{bmatrix}$ and $\mathbf{x}_2=\begin{bmatrix}1\\1\end{bmatrix}$. Therefore, $T=\begin{bmatrix}3&1\\-2&1\end{bmatrix}$ and $D=\begin{bmatrix}0&0\\0&5\end{bmatrix}$.

5. For $A=\begin{bmatrix}2&3\\3&2\end{bmatrix}$ the characteristic polynomial is $p(\lambda)=\lambda^2-4\lambda-5$. Eigenvalues are $\lambda_1=-1$ and $\lambda_2=5$. Corresponding eigenvectors are $\mathbf{x}_1=\begin{bmatrix}-1\\1\end{bmatrix}$ and $\mathbf{x}_2=\begin{bmatrix}1\\1\end{bmatrix}$. As in Example 1, we can construct a diagonalizing matrix T from the eigenvectors of A, $T=\begin{bmatrix}-1&1\\1&1\end{bmatrix}$. The corresponding matrix of eigenvalues, $D=\begin{bmatrix}-1&0\\0&5\end{bmatrix}$, is such that $T^{-1}AT=D$.

6. The characteristic polynomial is $p(\lambda)=\lambda^2-2\lambda-3=(\lambda+1)(\lambda-3)$. Eigenvalues are $\lambda_1=-1$ and $\lambda_2=3$. Corresponding eigenvectors are $\mathbf{x}_1=\begin{bmatrix}1\\-1\end{bmatrix}$ and $\mathbf{x}_2=\begin{bmatrix}1\\1\end{bmatrix}$. Therefore, $T=\begin{bmatrix}1&1\\-1&1\end{bmatrix}$ and $D=\begin{bmatrix}-1&0\\0&3\end{bmatrix}$.

7. For $A=\begin{bmatrix}2&0\\1&1\end{bmatrix}$ the characteristic polynomial is $p(\lambda)=\lambda^2-3\lambda+2$. Eigenvalues are $\lambda_1=1$ and $\lambda_2=2$. Corresponding eigenvectors are $\mathbf{x}_1=\begin{bmatrix}0\\1\end{bmatrix}$ and $\mathbf{x}_2=\begin{bmatrix}1\\1\end{bmatrix}$. As in Example 1, we can construct a diagonalizing matrix T from the eigenvectors of A, $T=\begin{bmatrix}0&1\\1&1\end{bmatrix}$. The corresponding matrix of eigenvalues, $D=\begin{bmatrix}1&0\\0&2\end{bmatrix}$, is such that $T^{-1}AT=D$.

8. The characteristic polynomial is $p(\lambda)=\lambda^2-\lambda-6=(\lambda+2)(\lambda-3)$. Eigenvalues are $\lambda_1=-2$ and $\lambda_2=3$. Corresponding eigenvectors are $\mathbf{x}_1=\begin{bmatrix}1\\0\end{bmatrix}$ and $\mathbf{x}_2=\begin{bmatrix}2\\1\end{bmatrix}$. Therefore, $T=\begin{bmatrix}1&2\\0&1\end{bmatrix}$ and $D=\begin{bmatrix}-2&0\\0&3\end{bmatrix}$.

9. For $A=\begin{bmatrix}25&-8&30\\24&-7&30\\-12&4&-14\end{bmatrix}$, the eigenvalues are $\lambda_1=1$ and $\lambda_2=2$. From the characteristic polynomial given, it follows that λ_1 has algebraic multiplicity 2 and λ_2 has algebraic multiplicity 1.

In order to find the eigenvectors corresponding to λ_1, we solve $(A-\lambda_1 I)\mathbf{x}=\mathbf{0}$ or $\begin{bmatrix} 24 & -8 & 30 \\ 24 & -8 & 30 \\ -12 & 4 & -15 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. This system reduces to $\begin{bmatrix} 12 & -4 & 15 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ and hence eigenvectors corresponding to λ_1 all have the form

$\mathbf{x}=\begin{bmatrix} (4x_2-15x_3)/12 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} x_2/3 \\ x_2 \\ 0 \end{bmatrix}+\begin{bmatrix} -5x_3/4 \\ 0 \\ x_3 \end{bmatrix}$. Thus, we find two linearly independent eigenvectors, $\mathbf{x}_1=\begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$ and $\mathbf{x}_2=\begin{bmatrix} -5 \\ 0 \\ 4 \end{bmatrix}$ corresponding to λ_1 and therefore, λ_1 has geometric multiplicity 2. Since λ_2 has algebraic multiplicity 1, it also has geometric multiplicity 1. Thus, A is not defective (that is, A is diagonalizable). In order to find the eigenvectors corresponding to λ_2, we solve $(A-\lambda_2 I)\mathbf{x}=\mathbf{0}$ or $\begin{bmatrix} 23 & -8 & 30 \\ 24 & -9 & 30 \\ -12 & 4 & -16 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. Solving this system, we obtain an eigenvector corresponding to λ_2, $\mathbf{x}=\begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$. Therefore, if $T=\begin{bmatrix} 1 & -5 & 2 \\ 3 & 0 & 2 \\ 0 & 4 & -1 \end{bmatrix}$, and $D=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$, then $T^{-1}AT=D$.

10. $\lambda_1=-1$ has algebraic multiplicity 1 and $\lambda_2=3$ has algebraic multiplicity 2. The corresponding eigenvectors are $\mathbf{x}=\begin{bmatrix} 1 \\ 2 \\ -2 \end{bmatrix}$ for λ_1 and $\mathbf{x}_1=\begin{bmatrix} 1 \\ 0 \\ -2 \end{bmatrix}$ and $\mathbf{x}_2=\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ for λ_2. Therefore, λ_1 has geometric multiplicity 1 and λ_2 has geometric multiplicity 2. A is diagonalizable and $T=\begin{bmatrix} 1 & 1 & 1 \\ 2 & 0 & 2 \\ -2 & -2 & 0 \end{bmatrix}$, and $D=\begin{bmatrix} -1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}$.

11. For $A=\begin{bmatrix} 1 & 0 & 1 \\ 2 & 2 & -3 \\ 0 & 0 & 1 \end{bmatrix}$, the eigenvalues are $\lambda_1=1$ and $\lambda_2=2$. From the characteristic polynomial given, it follows that λ_1 has algebraic multiplicity 2 and λ_2 has algebraic multiplicity 1.

In order to find the eigenvectors corresponding to λ_1, we solve $(A-\lambda_1 I)\mathbf{x}=\mathbf{0}$ or $\begin{bmatrix}0&0&1\\2&1&-3\\0&0&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$. This system reduces to $\begin{bmatrix}0&0&1\\2&1&0\\0&0&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$ and hence eigenvectors corresponding to λ_1 all have the form $\mathbf{x}=\begin{bmatrix}x_1\\-2x_1\\0\end{bmatrix}=x_1\begin{bmatrix}1\\-2\\0\end{bmatrix}$.

Thus, there is only one linearly independent eigenvector corresponding to λ_1. Therefore, λ_1 has geometric multiplicity 1 and consequently A is defective (not diagonalizable).

12. $\lambda_1=2$ has algebraic multiplicity 2 and $\lambda_2=3$ has algebraic multiplicity 1. The corresponding eigenvectors are $\mathbf{x}=\begin{bmatrix}1\\4\\1\end{bmatrix}$ for λ_1 and $\mathbf{x}=\begin{bmatrix}0\\1\\1\end{bmatrix}$ for λ_2. Therefore, λ_1 has geometric multiplicity 1 and λ_2 has geometric multiplicity1 and A is not diagonalizable.

13. For $A=\begin{bmatrix}4&-1&1\\10&-2&3\\1&0&1\end{bmatrix}$, the only eigenvalue is $\lambda_1=1$. From the characteristic polynomial given, it follows that λ_1 has algebraic multiplicity 3. In order to find the eigenvectors corresponding to λ_1, we solve $(A-\lambda_1 I)\mathbf{x}=\mathbf{0}$ or $\begin{bmatrix}3&-1&1\\10&-3&3\\1&0&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$. This system reduces to $\begin{bmatrix}1&0&0\\0&1&-1\\0&0&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0\\0\\0\end{bmatrix}$ and hence eigenvectors corresponding to λ_1 all have the form $\mathbf{x}=\begin{bmatrix}0\\x_3\\x_3\end{bmatrix}=x_3\begin{bmatrix}0\\1\\1\end{bmatrix}$. Thus, there is only one linearly independent eigenvector corresponding to λ_1. Therefore, λ_1 has geometric multiplicity 1 and consequently A is defective (not diagonalizable).

14. All four matrices are diagonalizable.
Matrices (a) and (d) have distinct eigenvalues.
Matrix (b) is a real, symmetric matrix.
Matrix (c) is lower triangular and has distinct eigenvalues.

15. For $A=\begin{bmatrix}6&-6\\2&-1\end{bmatrix}$ the eigenvalues are $\lambda_1=2$ and $\lambda_2=3$ with corresponding eigenvectors $\mathbf{x}_1=\begin{bmatrix}3\\2\end{bmatrix}$ and $\mathbf{x}_2=\begin{bmatrix}2\\1\end{bmatrix}$. Make the substitution $\mathbf{y}=T\mathbf{z}=\begin{bmatrix}3&2\\2&1\end{bmatrix}\mathbf{z}$ to obtain $T\mathbf{z}'=AT\mathbf{z}+\mathbf{g}(t)$.

Multiplying by T^{-1} gives $\mathbf{z}' = T^{-1}AT\mathbf{z} + T^{-1}\mathbf{g}(t)$ or

$\mathbf{z}' = D\mathbf{z} + T^{-1}\mathbf{g}(t) = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}\mathbf{z} + \begin{bmatrix} -1 & 2 \\ 2 & -3 \end{bmatrix}\begin{bmatrix} 4+3e^t \\ 2+2e^t \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}\mathbf{z} + \begin{bmatrix} e^t \\ 2 \end{bmatrix}$. Thus, the system uncouples

into $\begin{bmatrix} z_1' \\ z_2' \end{bmatrix} = \begin{bmatrix} 2z_1 + e^t \\ 3z_2 + 2 \end{bmatrix}$. Solving these uncoupled first order equations, we obtain

$\mathbf{z} = \begin{bmatrix} -e^t + c_1 e^{2t} \\ -(2/3) + c_2 e^{3t} \end{bmatrix}$. Finally, forming $\mathbf{y} = T\mathbf{z}$, we obtain the general solution

$$\mathbf{y} = \begin{bmatrix} 3 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} -e^t + c_1 e^{2t} \\ -(2/3) + c_2 e^{3t} \end{bmatrix} = \begin{bmatrix} 3e^{2t} & 2e^{3t} \\ 2e^{2t} & e^{3t} \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} - \begin{bmatrix} 3e^t + 4/3 \\ 2e^t + 2/3 \end{bmatrix}.$$

16. The eigenvalues are $\lambda_1 = -1$ and $\lambda_2 = 2$ with corresponding eigenvectors $\mathbf{x}_1 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ and

$\mathbf{x}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Make the substitution $\mathbf{y} = T\mathbf{z} = \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}\mathbf{z}$.

$\mathbf{z}' = D\mathbf{z} + T^{-1}\mathbf{g}(t) = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}\mathbf{z} + \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} e^{2t} - 2e^t \\ -e^{2t} + e^t \end{bmatrix} = \begin{bmatrix} -e^t \\ e^{2t} \end{bmatrix}$.

Solving these first order equations, we obtain $\mathbf{z} = \begin{bmatrix} -(1/2)e^t + c_1 e^{-t} \\ te^{2t} + c_2 e^{2t} \end{bmatrix}$. Finally, forming $\mathbf{y} = T\mathbf{z}$,

we obtain the solution $\mathbf{y} = \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} -(1/2)e^t + c_1 e^{-t} \\ te^{2t} + c_2 e^{2t} \end{bmatrix} = \begin{bmatrix} 2e^{-t} & e^{2t} \\ -e^{-t} & -e^{2t} \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + \begin{bmatrix} -e^t + te^{2t} \\ \frac{1}{2}e^t - te^{2t} \end{bmatrix}$.

17. For $A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}$ the eigenvalues are $\lambda_1 = 0$ and $\lambda_2 = 3$ with corresponding eigenvectors

$\mathbf{x}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Make the substitution $\mathbf{y} = T\mathbf{z} = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}\mathbf{z}$ to obtain $T\mathbf{z}' = AT\mathbf{z} + \mathbf{g}(t)$.

Multiplying by T^{-1} gives $\mathbf{z}' = T^{-1}AT\mathbf{z} + T^{-1}\mathbf{g}(t)$ or

$\mathbf{z}' = D\mathbf{z} + T^{-1}\mathbf{g}(t) = \begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix}\mathbf{z} + \begin{bmatrix} 2/3 & -1/3 \\ 1/3 & 1/3 \end{bmatrix}\begin{bmatrix} t \\ 3-t \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix}\mathbf{z} + \begin{bmatrix} t-1 \\ 1 \end{bmatrix}$.

Thus, the system uncouples into $\begin{bmatrix} z_1' \\ z_2' \end{bmatrix} = \begin{bmatrix} t-1 \\ 3z_2 + 1 \end{bmatrix}$. Solving these uncoupled first order

equations, we obtain $\mathbf{z} = \begin{bmatrix} (1/2)t^2 - t + c_1 \\ -(1/3) + c_2 e^{3t} \end{bmatrix}$. Finally, forming $\mathbf{y} = T\mathbf{z}$, we obtain the solution

$$\mathbf{y} = \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} (1/2)t^2 - t + c_1 \\ -(1/3) + c_2 e^{3t} \end{bmatrix} = \begin{bmatrix} 1 & e^{3t} \\ -1 & 2e^{3t} \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} - \begin{bmatrix} (1/2)t^2 - t - (1/3) \\ -(1/2)t^2 + t - (2/3) \end{bmatrix}.$$

18. The eigenvalues are $\lambda_1 = 2$ and $\lambda_2 = 5$ with corresponding eigenvectors $\mathbf{x}_1 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$ and $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Make the substitution $\mathbf{y} = T\mathbf{z} = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}\mathbf{z}$.

$$\mathbf{z}' = D\mathbf{z} + T^{-1}\mathbf{g}(t) = \begin{bmatrix} 2 & 0 \\ 0 & 5 \end{bmatrix}\mathbf{z} + \begin{bmatrix} \frac{1}{3} & -\frac{1}{3} \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix}\begin{bmatrix} 4t+4 \\ -2t+1 \end{bmatrix} = \begin{bmatrix} 2t+1 \\ 2 \end{bmatrix}.$$

Solving these first order equations, we obtain $\mathbf{z} = \begin{bmatrix} -t-1+c_1e^{2t} \\ -\frac{2}{5}+c_2e^{5t} \end{bmatrix}$. Finally, forming $\mathbf{y} = T\mathbf{z}$, we obtain the solution $\mathbf{y} = \begin{bmatrix} 2 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} -t-1+c_1e^{2t} \\ -\frac{2}{5}+c_2e^{5t} \end{bmatrix} = \begin{bmatrix} 2e^{2t} & e^{5t} \\ -e^{2t} & e^{5t} \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + \begin{bmatrix} -2t-\frac{12}{5} \\ t+\frac{3}{5} \end{bmatrix}$.

19. For $A = \begin{bmatrix} -9 & -5 \\ 8 & 4 \end{bmatrix}$ the eigenvalues are $\lambda_1 = -1$ and $\lambda_2 = -4$ with corresponding eigenvectors $\mathbf{x}_1 = \begin{bmatrix} 5 \\ -8 \end{bmatrix}$ and $\mathbf{x}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Make the substitution $\mathbf{y} = T\mathbf{z} = \begin{bmatrix} 5 & 1 \\ -8 & -1 \end{bmatrix}\mathbf{z}$ to obtain $T\mathbf{z}'' = AT\mathbf{z}$.

Multiplying by T^{-1} gives $\mathbf{z}'' = T^{-1}AT\mathbf{z}$ or $\mathbf{z}'' = D\mathbf{z} = \begin{bmatrix} -1 & 0 \\ 0 & -4 \end{bmatrix}\mathbf{z}$. Thus, the system uncouples into $\begin{bmatrix} z_1'' \\ z_2'' \end{bmatrix} = \begin{bmatrix} -z_1 \\ -4z_2 \end{bmatrix}$.

Solving these uncoupled equations, we obtain $\mathbf{z} = \begin{bmatrix} c_1\cos t + d_1\sin t \\ c_2\cos 2t + d_2\sin 2t \end{bmatrix}$. Finally, forming $\mathbf{y} = T\mathbf{z}$, we obtain the solution

$$\mathbf{y} = \begin{bmatrix} 5 & 1 \\ -8 & -1 \end{bmatrix}\begin{bmatrix} c_1\cos t + d_1\sin t \\ c_2\cos 2t + d_2\sin 2t \end{bmatrix} = \begin{bmatrix} 5(c_1\cos t + d_1\sin t) + c_2\cos 2t + d_2\sin 2t \\ -8(c_1\cos t + d_1\sin t) - c_2\cos 2t - d_2\sin 2t \end{bmatrix}.$$

20. The eigenvalues are $\lambda_1 = -9$ and $\lambda_2 = -1$ with corresponding eigenvectors $\mathbf{x}_1 = \begin{bmatrix} 7 \\ -15 \end{bmatrix}$ and $\mathbf{x}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Make the substitution $\mathbf{x} = T\mathbf{z} = \begin{bmatrix} 7 & 1 \\ -15 & -1 \end{bmatrix}\mathbf{z}$ to obtain $\mathbf{z}'' + \begin{bmatrix} -9 & 0 \\ 0 & -1 \end{bmatrix}\mathbf{z} = \mathbf{0}$. Solving the equations, we obtain $\mathbf{z} = \begin{bmatrix} c_1e^{-3t} + c_2e^{3t} \\ k_1e^{-t} + k_2e^{t} \end{bmatrix}$. Finally, we obtain the solution $\mathbf{x} = \begin{bmatrix} 7 & 1 \\ -15 & -1 \end{bmatrix}\begin{bmatrix} c_1e^{-3t} + c_2e^{3t} \\ k_1e^{-t} + k_2e^{t} \end{bmatrix} = \begin{bmatrix} 7(c_1e^{-3t} + c_2e^{3t}) + k_1e^{-t} + k_2e^{t} \\ -15(c_1e^{-3t} + c_2e^{3t}) - (k_1e^{-t} + k_2e^{t}) \end{bmatrix}$.

21. For $A = \begin{bmatrix} -2 & -1 \\ 3 & 2 \end{bmatrix}$ the eigenvalues are $\lambda_1 = -1$ and $\lambda_2 = 1$ with corresponding eigenvectors $\mathbf{x}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\mathbf{x}_2 = \begin{bmatrix} 1 \\ -3 \end{bmatrix}$. Make the substitution $\mathbf{y} = T\mathbf{z} = \begin{bmatrix} 1 & 1 \\ -1 & -3 \end{bmatrix}\mathbf{z}$ to obtain $T\mathbf{z}'' = AT\mathbf{z}$.

Multiplying by T^{-1} gives $\mathbf{z}'' = T^{-1}AT\mathbf{z}$ or $\mathbf{z}'' = D\mathbf{z} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\mathbf{z}$. Thus, the system uncouples into $\begin{bmatrix} z_1'' \\ z_2'' \end{bmatrix} = \begin{bmatrix} -z_1 \\ z_2 \end{bmatrix}$.

Solving these uncoupled equations, we obtain $\mathbf{z} = \begin{bmatrix} c_1 \cos t + d_1 \sin t \\ c_2 e^{-t} + d_2 e^{t} \end{bmatrix}$. Finally, forming $\mathbf{y} = T\mathbf{z}$, we obtain the solution $\mathbf{y} = \begin{bmatrix} 1 & 1 \\ -1 & -3 \end{bmatrix}\begin{bmatrix} c_1 \cos t + d_1 \sin t \\ c_2 e^{-t} + d_2 e^{t} \end{bmatrix} = \begin{bmatrix} c_1 \cos t + d_1 \sin t + c_2 e^{-t} + d_2 e^{t} \\ -(c_1 \cos t + d_1 \sin t) - 3(c_2 e^{-t} + d_2 e^{t}) \end{bmatrix}$.

22. The eigenvalues are $\lambda_1 = 0$ and $\lambda_2 = 5$ with corresponding eigenvectors $\mathbf{x}_1 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$ and $\mathbf{x}_2 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. Make the substitution $\mathbf{x} = T\mathbf{z} = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}\mathbf{z}$ to obtain $\mathbf{z}'' + \begin{bmatrix} 0 & 0 \\ 0 & 5 \end{bmatrix}\mathbf{z} = \mathbf{0}$.

Solving the equations, we obtain $\mathbf{z} = \begin{bmatrix} c_1 t + c_2 \\ k_1 \cos(\sqrt{5}t) + k_2 \sin(\sqrt{5}t) \end{bmatrix}$. Finally, we obtain the solution

$$\mathbf{x} = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} c_1 t + c_2 \\ k_1 \cos(\sqrt{5}t) + k_2 \sin(\sqrt{5}t) \end{bmatrix} = \begin{bmatrix} (c_1 t + c_2) + 2\left[k_1 \cos(\sqrt{5}t) + k_2 \sin(\sqrt{5}t)\right] \\ -2(c_1 t + c_2) + \left[k_1 \cos(\sqrt{5}t) + k_2 \sin(\sqrt{5}t)\right] \end{bmatrix}.$$

27 (a). For $A = \begin{bmatrix} 500 & -200 \\ -200 & 200 \end{bmatrix}$ the eigenvalues are $\lambda_1 = 100$ and $\lambda_2 = 600$ with corresponding eigenvectors $\mathbf{x}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\mathbf{x}_2 = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$.

27 (b). Make the substitution $\mathbf{y} = T\mathbf{z} = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}\mathbf{z}$ to obtain $T\mathbf{z}'' + AT\mathbf{z} = \mathbf{0}$. Multiplying by T^{-1} gives $\mathbf{z}'' + T^{-1}AT\mathbf{z} = \mathbf{0}$ or $\mathbf{z}'' + \begin{bmatrix} 100 & 0 \\ 0 & 600 \end{bmatrix}\mathbf{z} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Thus, the system uncouples into $\begin{bmatrix} z_1'' + 100z_1 \\ z_2'' + 600z_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. The initial condition is $\mathbf{z}(0) = T^{-1}\mathbf{y}(0) = \begin{bmatrix} 0.2 & 0.4 \\ 0.4 & -0.2 \end{bmatrix}\begin{bmatrix} 0.1 \\ 0.15 \end{bmatrix} = \begin{bmatrix} 0.08 \\ 0.01 \end{bmatrix}$.

27 (c). Solving the uncoupled equations $\mathbf{z}'' + D\mathbf{z} = \mathbf{0}$, we obtain $\mathbf{z} = \begin{bmatrix} c_1 \cos 10t + d_1 \sin 10t \\ c_2 \cos 10\sqrt{6}t + d_2 \sin 10\sqrt{6}t \end{bmatrix}$.

Imposing the initial condition, we find $\mathbf{z} = \begin{bmatrix} 0.08 \cos 10t \\ 0.01 \cos 10\sqrt{6}t \end{bmatrix}$. Finally, forming $\mathbf{y} = T\mathbf{z}$, we obtain the solution of the initial value problem:

$$\mathbf{y} = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 0.08 \cos 10t \\ 0.01 \cos 10\sqrt{6}t \end{bmatrix} = \begin{bmatrix} 0.08 \cos 10t + 0.02 \cos\left(10\sqrt{6}t\right) \\ 0.16 \cos 10t - 0.01 \cos\left(10\sqrt{6}t\right) \end{bmatrix}.$$

Section 6.11

1 (a). We proceed as in Example 2. For $A=\begin{bmatrix}5 & -4\\ 5 & -4\end{bmatrix}$, the characteristic polynomial is $p(\lambda)=\lambda^2-\lambda$.

Eigenvalues are $\lambda_1=0$ and $\lambda_2=1$ with corresponding eigenvectors $\mathbf{x}_1=\begin{bmatrix}4\\ 5\end{bmatrix}$ and $\mathbf{x}_2=\begin{bmatrix}1\\ 1\end{bmatrix}$.

Since A is diagonalizable, we obtain from equation (7) $e^{tA}=T\Lambda(t)T^{-1}$ where

$T=[\mathbf{x}_1,\mathbf{x}_2]=\begin{bmatrix}4 & 1\\ 5 & 1\end{bmatrix}$ and $\Lambda(t)=\begin{bmatrix}e^{\lambda_1 t} & 0\\ 0 & e^{\lambda_2 t}\end{bmatrix}=\begin{bmatrix}1 & 0\\ 0 & e^t\end{bmatrix}$. Thus,

$$\mathbf{\Phi}(t)=e^{tA}=\begin{bmatrix}4 & 1\\ 5 & 1\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & e^t\end{bmatrix}\begin{bmatrix}-1 & 1\\ 5 & -4\end{bmatrix}=\begin{bmatrix}-4+5e^t & 4-4e^t\\ -5+5e^t & 5-4e^t\end{bmatrix}.$$

1 (b). The solution of $\mathbf{y}'=A\mathbf{y},\mathbf{y}(-1)=\mathbf{y}_0$ is given by $\mathbf{y}(t)=e^{(t+1)A}\mathbf{y}_0$. Therefore,

$$\mathbf{y}(2)=e^{(2+1)A}\mathbf{y}_0=e^{3A}\begin{bmatrix}1\\ 0\end{bmatrix}=\begin{bmatrix}-4+5e^3 & 4-4e^3\\ -5+5e^3 & 5-4e^3\end{bmatrix}\begin{bmatrix}1\\ 0\end{bmatrix}=\begin{bmatrix}-4+5e^3\\ -5+5e^3\end{bmatrix}.$$

2 (a). The characteristic polynomial is $p(\lambda)=(\lambda-2)^2$. Eigenvalues are $\lambda_1=\lambda_2=2$ with corresponding eigenvector $\mathbf{x}_1=\begin{bmatrix}1\\ 0\end{bmatrix}$. Therefore,

$\mathbf{y}_1(t)=\begin{bmatrix}e^{2t}\\ 0\end{bmatrix}$. Let $\mathbf{y}_2(t)=e^{2t}(t\xi+7)$, $\xi=\begin{bmatrix}1\\ 0\end{bmatrix}$, $(A-2\mathbf{I})\eta=\xi\Rightarrow\eta=\begin{bmatrix}0\\ 1\end{bmatrix}\Rightarrow\mathbf{y}_2(t)=e^{2t}\begin{bmatrix}t\\ 1\end{bmatrix}$.

$\Psi(t)=\begin{bmatrix}e^{2t} & te^{2t}\\ 0 & e^{2t}\end{bmatrix}=\mathbf{\Phi}(t)$ since $\Psi(0)=\mathbf{I}$.

2 (b). $\mathbf{y}(2)=\mathbf{\Phi}(1)\mathbf{y}(1)=\begin{bmatrix}e^2 & e^2\\ 0 & e^2\end{bmatrix}\begin{bmatrix}1\\ 2\end{bmatrix}=\begin{bmatrix}3e^2\\ 2e^2\end{bmatrix}$.

3 (a). We proceed as in Example 2. For $A=\begin{bmatrix}6 & 5\\ 1 & 2\end{bmatrix}$, the characteristic polynomial is

$p(\lambda)=\lambda^2-8\lambda+7$. Eigenvalues are $\lambda_1=1$ and $\lambda_2=7$ with corresponding eigenvectors

$\mathbf{x}_1=\begin{bmatrix}1\\ -1\end{bmatrix}$ and $\mathbf{x}_2=\begin{bmatrix}5\\ 1\end{bmatrix}$. Since A is diagonalizable, we obtain from equation (7) $e^{tA}=T\Lambda(t)T^{-1}$

where $T=[\mathbf{x}_1,\mathbf{x}_2]=\begin{bmatrix}1 & 5\\ -1 & 1\end{bmatrix}$ and $\Lambda(t)=\begin{bmatrix}e^{\lambda_1 t} & 0\\ 0 & e^{\lambda_2 t}\end{bmatrix}=\begin{bmatrix}e^t & 0\\ 0 & e^{7t}\end{bmatrix}$. Thus,

$$\mathbf{\Phi}(t)=e^{tA}=\begin{bmatrix}1 & 5\\ -1 & 1\end{bmatrix}\begin{bmatrix}e^t & 0\\ 0 & e^{7t}\end{bmatrix}\begin{bmatrix}1/6 & -5/6\\ 1/6 & 1/6\end{bmatrix}=(1/6)\begin{bmatrix}e^t+5e^{7t} & -5e^t+5e^{7t}\\ -e^t+e^{7t} & 5e^t+e^{7t}\end{bmatrix}.$$

3 (b). The solution of $\mathbf{y}'=A\mathbf{y},\mathbf{y}(0)=\mathbf{y}_0$ is given by $\mathbf{y}(t)=e^{tA}\mathbf{y}_0$. Therefore,

$$\mathbf{y}(-1)=e^{(-1)A}\begin{bmatrix}1\\ 1\end{bmatrix}=(1/6)\begin{bmatrix}e^{-1}+5e^{-7} & -5e^{-1}+5e^{-7}\\ -e^{-1}+e^{-7} & 5e^{-1}+e^{-7}\end{bmatrix}\begin{bmatrix}1\\ 1\end{bmatrix}=(1/6)\begin{bmatrix}-4e^{-1}+10e^{-7}\\ 4e^{-1}+2e^{-7}\end{bmatrix}.$$

4 (a). The characteristic polynomial is $p(\lambda)=(1-\lambda)(2-\lambda)(-1-\lambda)$. Eigenvalues are $\lambda_1=-1,\ \lambda_2=1,\ \lambda_3=2$ with corresponding eigenvectors $\mathbf{x}_1=\begin{bmatrix}1\\1\\-3\end{bmatrix},\ \mathbf{x}_2=\begin{bmatrix}1\\0\\0\end{bmatrix},\ \mathbf{x}_3=\begin{bmatrix}1\\1\\0\end{bmatrix}$.

Therefore, $\Psi(t)=\begin{bmatrix}e^{-t} & e^{t} & e^{2t}\\ e^{-t} & 0 & e^{2t}\\ -3e^{-t} & 0 & 0\end{bmatrix},\ \Psi(0)=\begin{bmatrix}1 & 1 & 1\\ 1 & 0 & 1\\ -3 & 0 & 0\end{bmatrix}\Rightarrow\Psi^{-1}(0)=\begin{bmatrix}0 & 0 & -\frac{1}{3}\\ 1 & -1 & 0\\ 0 & 1 & \frac{1}{3}\end{bmatrix}$

and $\Phi(t)=\begin{bmatrix}e^{-t} & e^{t} & e^{2t}\\ e^{-t} & 0 & e^{2t}\\ -3e^{-t} & 0 & 0\end{bmatrix}\begin{bmatrix}0 & 0 & -\frac{1}{3}\\ 1 & -1 & 0\\ 0 & 1 & \frac{1}{3}\end{bmatrix}=\begin{bmatrix}e^{t} & -e^{t}+e^{2t} & \frac{1}{3}(-e^{-t}+e^{2t})\\ 0 & e^{2t} & \frac{1}{3}(-e^{-t}+e^{2t})\\ 0 & 0 & e^{-t}\end{bmatrix}$

4 (b). $\mathbf{y}(1)=\Phi(1)\mathbf{y}(0)=\Phi(1)\begin{bmatrix}1\\1\\0\end{bmatrix}=\begin{bmatrix}e^{2}\\e^{2}\\0\end{bmatrix}$.

5 (a). From Theorem 6.15, $\Phi(t,s)=\Psi(t)\Psi^{-1}(s)=\begin{bmatrix}t & t^2\\ 1 & 2t\end{bmatrix}\begin{bmatrix}2s^{-1} & -1\\ -s^{-2} & s^{-1}\end{bmatrix}$, and thus

$\Phi(t,s)=\begin{bmatrix}2s^{-1}t-s^{-2}t^2 & -t+t^2s^{-1}\\ 2s^{-1}-2s^{-2}t & -1+2ts^{-1}\end{bmatrix}$; $\Phi(t,s)$ is not a function of $t-s$.

5 (b). From Theorem 6.15, $\mathbf{y}(3)=\Phi(3,1)\mathbf{y}(1)=\begin{bmatrix}6-9 & -3+9\\ 2-6 & -1+6\end{bmatrix}\begin{bmatrix}1\\-1\end{bmatrix}=\begin{bmatrix}-9\\-9\end{bmatrix}$.

6. $B=T^{-1}p(A)T=T^{-1}(2A^3-A+3I)T=2T^{-1}A^3T-T^{-1}AT+3T^{-1}T=2D^3-D+3I$.

Therefore, $B=\begin{bmatrix}2\lambda_1^3-\lambda_1+3 & 0\\ 0 & 2\lambda_2^3-\lambda_2+3\end{bmatrix}$.

9 (a). As we saw in equation (6), if $T^{-1}AT=D$ then $A^n=TD^nT^{-1}$. (For this present case, $T=\begin{bmatrix}3 & 2\\ 4 & 3\end{bmatrix}$ and $D=\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}$.) Since $A^n=T\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}^n T^{-1}$ and since $\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}^n=I$ when n is even, it follows that $A^n=I$.

9 (b). $A^n=TD^nT^{-1}=T\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}^n T^{-1}=TDT^{-1}=A$ when n is odd.

9 (c). As in parts (a) and (b), we see that $A^{-n}=I$ when n is even and $A^{-n}=A$ when n is odd.

10. $A=T\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}T^{-1}$. The four matrices are: $D=\begin{bmatrix}\pm1 & 0\\ 0 & \pm i\end{bmatrix}$

$D_1=\begin{bmatrix}1 & 0\\ 0 & i\end{bmatrix},\ D_2=\begin{bmatrix}-1 & 0\\ 0 & i\end{bmatrix},\ D_3=\begin{bmatrix}1 & 0\\ 0 & -i\end{bmatrix},\ D_4=\begin{bmatrix}-1 & 0\\ 0 & -i\end{bmatrix}$.

11. For the given matrix, $A^{-1}=A$. Thus, if $B=A^{1/2}$, then $B^2=A=A^{-1}$ as requested. Exercise 10 asks for four different square roots of A and any one of these will serve as B.

12. $A^2+A^{\frac{1}{2}}=TBT^{-1}\Rightarrow B=T^{-1}A^2T+T^{-1}A^{\frac{1}{2}}T$. Since $A^2=I,\ B=I+D=\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix}+\begin{bmatrix}\pm1 & 0\\ 0 & \pm i\end{bmatrix}$.

13. Since $A = TDT^{-1}$, it follows that $A^3 = TD^3T^{-1} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} -2 & 0 \\ 0 & 2 \end{bmatrix}^3\begin{bmatrix} -1 & 1 \\ 2 & -1 \end{bmatrix} = \begin{bmatrix} 24 & -16 \\ 32 & -24 \end{bmatrix}$.

14. $f_1(A) = \cos(\pi A) = T\begin{bmatrix} \cos(\pi\lambda_1) & 0 \\ 0 & \cos(\pi\lambda_2) \end{bmatrix}T^{-1}$. $T = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}$, $T^{-1} = \begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix}$.

$\cos(\pi\lambda_1) = \cos\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$, $\cos(\pi\lambda_2) = \cos\left(\frac{\pi}{2}\right) = 0$. Therefore,

$$f_1(A) = \cos(\pi A) = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix} = \begin{bmatrix} 3\sqrt{2} & -\sqrt{2} \\ \frac{15}{2}\sqrt{2} & -\frac{5}{2}\sqrt{2} \end{bmatrix}.$$

$\sin(\pi\lambda_1) = \sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}$, $\sin(\pi\lambda_2) = \sin\left(\frac{\pi}{2}\right) = 1$. Therefore,

$$f_2(A) = \sin(\pi A) = \begin{bmatrix} 2 & 1 \\ 5 & 3 \end{bmatrix}\begin{bmatrix} \frac{1}{\sqrt{2}} & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 3 & -1 \\ -5 & 2 \end{bmatrix} = \begin{bmatrix} 3\sqrt{2} - 5 & -\sqrt{2} + 2 \\ \frac{15}{2}\sqrt{2} - 15 & -\frac{5}{2}\sqrt{2} + 6 \end{bmatrix}.$$

15. As we saw following Theorem 6.16, $\cos(tA) = T\begin{bmatrix} \cos\lambda_1 t & 0 \\ 0 & \cos\lambda_2 t \end{bmatrix}T^{-1}$ when A is a (2×2) diagonalizable matrix with eigenvalues λ_1 and λ_2. Thus, with $t = \pi$ and the given eigenvalues, we have

$$\cos(\pi A) = T\begin{bmatrix} \cos(\pi/3) & 0 \\ 0 & \cos(7\pi/3) \end{bmatrix}T^{-1} = T\begin{bmatrix} 1/2 & 0 \\ 0 & 1/2 \end{bmatrix}T^{-1}$$
$$= (1/2)T\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}T^{-1} = (1/2)I.$$

Similarly, we find $\sin(\pi A) = (\sqrt{3}/2)I$.

16. Let $T = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}$. Make the substitution $\mathbf{y} = T\mathbf{z}$. Premultiplying by $T^{-1} = \begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix}$ gives $\mathbf{z}'' + D\mathbf{z} = T^{-1}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$, $D = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$. The solution is $\mathbf{z}(t) = \begin{bmatrix} c_1e^{-t} + c_2e^{t} + 1 \\ k_1\cos t + k_2\sin t + 2 \end{bmatrix}$. Converting to the original variables, we obtain

$$\mathbf{y}(t) = T\mathbf{z}(t) = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}\begin{bmatrix} c_1e^{-t} + c_2e^{t} + 1 \\ k_1\cos t + k_2\sin t + 2 \end{bmatrix} = \begin{bmatrix} c_1e^{-t} + c_2e^{t} + k_1\cos t + k_2\sin t + 3 \\ -2c_1e^{-t} - 2c_2e^{t} - k_1\cos t - k_2\sin t - 4 \end{bmatrix}.$$

17. Let $T = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}$. Making the substitution $\mathbf{y} = T\mathbf{z}$, the system becomes $AT\mathbf{z}' + T\mathbf{z} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Premultiplying by $T^{-1} = \begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix}$ gives $T^{-1}AT\mathbf{z}' + \mathbf{z} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$ or $\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\mathbf{z}' + \mathbf{z} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$. Thus, the system uncouples into $\begin{bmatrix} -z_1' + z_1 \\ z_2' + z_2 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$. The solution is $\mathbf{z}(t) = \begin{bmatrix} c_1e^{t} - 2 \\ c_2e^{-t} + 3 \end{bmatrix}$. Converting to the original variables, we obtain $\mathbf{y} = T\mathbf{z} = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}\begin{bmatrix} c_1e^{t} - 2 \\ c_2e^{-t} + 3 \end{bmatrix} = \begin{bmatrix} c_1e^{t} + c_2e^{-t} + 1 \\ -2c_1e^{t} - c_2e^{-t} + 1 \end{bmatrix}$.

18. Make the substitution $\mathbf{y} = T\mathbf{z}$. $\mathbf{z}'' + \mathbf{z}' + D\mathbf{z} = \mathbf{0}$.. The solution is
$\mathbf{z}(t) = \begin{bmatrix} c_1 e^{\left(-\frac{1}{2}-\frac{\sqrt{5}}{2}\right)t} + c_2 e^{\left(-\frac{1}{2}+\frac{\sqrt{5}}{2}\right)t} \\ k_1 e^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}}{2} t\right) + k_2 e^{-\frac{t}{2}} \sin\left(\frac{\sqrt{3}}{2} t\right) \end{bmatrix}$. Converting to the original variables, we obtain

$$\mathbf{y}(t) = T\mathbf{z}(t) = \begin{bmatrix} c_1 e^{\left(-\frac{1}{2}-\frac{\sqrt{5}}{2}\right)t} + c_2 e^{\left(-\frac{1}{2}+\frac{\sqrt{5}}{2}\right)t} + k_1 e^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}}{2} t\right) + k_2 e^{-\frac{t}{2}} \sin\left(\frac{\sqrt{3}}{2} t\right) \\ -2c_1 e^{\left(-\frac{1}{2}-\frac{\sqrt{5}}{2}\right)t} - 2c_2 e^{\left(-\frac{1}{2}+\frac{\sqrt{5}}{2}\right)t} - k_1 e^{-\frac{t}{2}} \cos\left(\frac{\sqrt{3}}{2} t\right) - k_2 e^{-\frac{t}{2}} \sin\left(\frac{\sqrt{3}}{2} t\right) \end{bmatrix}.$$

19. Let $T = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}$. Making the substitution $\mathbf{y} = T\mathbf{z}$, the system becomes $T\mathbf{z}'' + 2AT\mathbf{z}' = \mathbf{0}$.
Premultiplying by $T^{-1} = \begin{bmatrix} -1 & -1 \\ 2 & 1 \end{bmatrix}$ gives $\mathbf{z}'' + 2T^{-1}AT\mathbf{z}' = \mathbf{0}$ or $\mathbf{z}'' + 2\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}\mathbf{z}' = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Thus, the system uncouples into $\begin{bmatrix} z_1'' - 2z_1' \\ z_2'' + 2z_2' \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. The solution is $\mathbf{z}(t) = \begin{bmatrix} c_1 + c_2 e^{2t} \\ d_1 + d_2 e^{-2t} \end{bmatrix}$. Converting to the original variables, we obtain

$$\mathbf{y}(t) = T\mathbf{z}(t) = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}\begin{bmatrix} c_1 + c_2 e^{2t} \\ d_1 + d_2 e^{-2t} \end{bmatrix} = \begin{bmatrix} c_1 + c_2 e^{2t} + d_1 + d_2 e^{-2t} \\ -2(c_1 + c_2 e^{2t}) - (d_1 + d_2 e^{-2t}) \end{bmatrix}.$$

20 (a). $m_1 x_1'' = k_1(x_2 - x_1)$; $m_2 x_2'' = k_2(x_3 - x_2) - k_1(x_2 - x_1)$; $m_3 x_3'' = -k_2(x_3 - x_2)$. Therefore,
$m_1 x_1'' + k_1(x_1 - x_2) = 0$; $m_2 x_2'' - k_1 x_1 + (k_1 + k_2)x_2 - k_2 x_3 = 0$; $m_3 x_3'' - k_2 x_2 + k_2 x_3 = 0$.
The result follows.

20 (b). $K\mathbf{v}_0 = \mathbf{0}$, where $\mathbf{v}_0$ is any nonzero multiple of $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. Therefore, $0, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is an eigenpair.

20 (c). Let $\mathbf{x} = f(t)\mathbf{v}_0$. $M\mathbf{x}'' + K\mathbf{x} = M(f''(t)\mathbf{v}_0) + Kf(t)\mathbf{v}_0 = \mathbf{0}$. Therefore, since $K(f(t)\mathbf{v}_0) = f(t)K\mathbf{v}_0 = \mathbf{0}$, $Mf''(t)\mathbf{v}_0 = \mathbf{0}$ *or* $m_j f''(t) = 0$, $j = 1,2,3 \Rightarrow f''(t) = 0$. Therefore, $f(t) = c_1 t + c_2$ and $\mathbf{x}(t) = (c_1 t + c_2)\mathbf{v}_0$. $\mathbf{x}(0) = c_2\mathbf{v}_0 = \mathbf{0} \Rightarrow c_2 = 0$, $\mathbf{x}(t) = c_1\mathbf{v}_0 = \mathbf{v}_0 \Rightarrow c_1 = 1$.
Therefore, $\mathbf{x}(t) = t\mathbf{v}_0$. The system is executing motion at constant velocity $\mathbf{v}_0$. There is no relative motion; the three-mass system is translating like a rigid body.

21 (a). For this case, we have $A = M^{-1}K = \frac{k}{m}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$. Using MATLAB, we find the eigenvalues of $B = \begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$ are $\gamma_1 = 0, \gamma_2 = 1$, and $\gamma_3 = 3$ with corresponding eigenvectors $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \mathbf{u}_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$, and $\mathbf{u}_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$. Since $A = (k/m)B$, the eigenvalues of A are multiples of *k*/*m* times the eigenvalues of B while corresponding eigenvectors can be chosen to be the same as those of B.

21 (b). Making the substitution $\mathbf{x} = T\mathbf{z}$, the system becomes $T\mathbf{z}'' + AT\mathbf{z} = \mathbf{0}$. Premultiplying by gives $\mathbf{z}'' + T^{-1}AT\mathbf{z} = \mathbf{0}$ or $\mathbf{z}'' + \begin{bmatrix} 0 & 0 & 0 \\ 0 & km^{-1} & 0 \\ 0 & 0 & 3km^{-1} \end{bmatrix}\mathbf{z} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. Thus, the system uncouples into $\begin{bmatrix} z_1'' \\ z_2'' + km^{-1}z_2 \\ z_3'' + 3km^{-1}z_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$. The solution is $\mathbf{z}(t) = \begin{bmatrix} c_1 t + c_2 \\ d_1\cos\omega t + d_2 \sin\omega t \\ e_1\cos\sqrt{3}\omega t + e_2\sin\sqrt{3}\omega t \end{bmatrix}$, where $\omega = \sqrt{km^{-1}}$.

Converting to the original variables, we obtain

$$\mathbf{x}(t) = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & -2 \\ 1 & -1 & 1 \end{bmatrix}\begin{bmatrix} c_1 t + c_2 \\ d_1\cos\omega t + d_2 \sin\omega t \\ e_1\cos\sqrt{3}\omega t + e_2\sin\sqrt{3}\omega t \end{bmatrix}$$

$$= \begin{bmatrix} c_1 t + c_2 + d_1\cos\omega t + d_2\sin\omega t + e_1\cos\sqrt{3}\omega t + e_2\sin\sqrt{3}\omega t \\ c_1 t + c_2 - 2(e_1\cos\sqrt{3}\omega t + e_2\sin\sqrt{3}\omega t) \\ c_1 t + c_2 - (d_1\cos\omega t + d_2\sin\omega t) + e_1\cos\sqrt{3}\omega t + e_2\sin\sqrt{3}\omega t \end{bmatrix}.$$

Chapter 7
Laplace Transforms

Section 7.1

1. $\mathcal{L}\{1\} = \lim_{T\to\infty}\int_0^T 1\cdot e^{-st}dt = \lim_{T\to\infty}\left.\frac{-e^{-st}}{s}\right|_0^T = \lim_{T\to\infty}\frac{1}{s}\left(1-e^{-sT}\right) = \frac{1}{s},\ s>0$

2. $\mathcal{L}\{e^{3t}\} = \lim_{T\to\infty}\int_0^T e^{3t}\cdot e^{-st}dt = \lim_{T\to\infty}\int_0^T e^{-(s-3)t}dt = \lim_{T\to\infty}-\frac{1}{s-3}e^{-(s-3)t}\Big|_0^T = \lim_{T\to\infty}\frac{1}{s-3}\left(1-e^{-(s-3)T}\right)$

   $= \frac{1}{s-3},\ s>3.$

3. $\mathcal{L}\{te^{-t}\} = \lim_{T\to\infty}\int_0^T te^{-t}\cdot e^{-st}dt = \lim_{T\to\infty}\int_0^T te^{-(s+1)t}dt$. For integration by parts, we will use

   $u=t,\ du=dt,\ dv=e^{-(s+1)t}dt,$ and $v=-\frac{e^{-(s+1)t}}{s+1}$. Then we have

   $\lim_{T\to\infty}\int_0^T te^{-(s+1)t}dt = \lim_{T\to\infty}\left\{\left.\frac{-te^{-(s+1)t}}{s+1}\right|_0^T + \frac{1}{s+1}\int_0^T e^{-(s+1)t}dt\right\} = \lim_{T\to\infty}\left\{\frac{-Te^{-(s+1)T}}{s+1}+\frac{1}{(s+1)^2}\left(1-e^{-(s+1)T}\right)\right\}$

   $= \frac{1}{(s+1)^2},\ s>-1.$

4. $\mathcal{L}\{t-5\} = \lim_{T\to\infty}\int_0^T (t-5)\cdot e^{-st}dt$. For integration by parts, we will use

   $u=t-5,\ du=dt,\ dv=e^{-st}dt,$ and $v=-\frac{e^{-st}}{s}$. Then we have

   $\lim_{T\to\infty}\int_0^T (t-5)e^{-st}dt = \lim_{T\to\infty}\left\{\left.\frac{-(t-5)e^{-st}}{s}\right|_0^T + \frac{1}{s}\int_0^T e^{-st}dt\right\} = \lim_{T\to\infty}\left\{\frac{-(T-5)e^{-sT}+5}{s} - \left.\frac{e^{-st}}{s^2}\right|_0^T\right\}$

   $= \frac{1}{s^2}-\frac{5}{s},\ s>0.$

5. $\mathcal{L}\{f(t)\}$ does not exist because $\lim_{t\to\infty} te^{t\sqrt{t}}e^{-st} = \infty$ for all $s>0$.

6. $\mathcal{L}\{f(t)\}$ does not exist because $\lim_{t\to\infty} e^{(t-1)^2}e^{-st} = \infty$ for all $s>0$.

7. $\mathcal{L}\{|t-1|\} = \underbrace{\int_0^1 (1-t)e^{-st}dt}_{(1)} + \underbrace{\int_1^\infty (t-1)e^{-st}dt}_{(2)}$.

(1): $u = 1-t,\ du = -dt,\ dv = e^{-st}dt,\ v = -\frac{e^{-st}}{s}$. Then we have

$$(1) = -(1-t)\frac{e^{-st}}{s}\Big|_0^1 - \int_0^1 \frac{e^{-st}}{s}dt = \frac{1}{s} + \frac{1}{s^2}e^{-st}\Big|_0^1 = \frac{1}{s} + \frac{1}{s^2}\left(e^{-s}-1\right)$$

(2): $u = t-1,\ du = dt,\ dv = e^{-st}dt,\ v = -\frac{e^{-st}}{s}$. Then we have

$$(2) = -\frac{t-1}{s}e^{-st}\Big|_1^\infty + \frac{1}{s}\int_1^\infty e^{-st}dt = \frac{e^{-s}}{s^2}$$

$(1)+(2) = \frac{1}{s} - \frac{1}{s^2} + \frac{2}{s^2}e^{-s},\ s > 0$.

8. $\mathcal{L}\{(t-2)^2\} = \lim_{T\to\infty}\int_0^T (t-2)^2 \cdot e^{-st}dt$. Using

$u = (t-2)^2,\ du = 2(t-2)dt,\ dv = e^{-st}dt,$ and $v = -\frac{e^{-st}}{s},$ we have

$$\lim_{T\to\infty}\int_0^T (t-2)^2 \cdot e^{-st}dt = \lim_{T\to\infty}\left\{\frac{-(t-2)^2 e^{-st}}{s}\Big|_0^T + \frac{2}{s}\int_0^T (t-2)e^{-st}dt\right\} = \frac{4}{s} + \frac{2}{s}\lim_{T\to\infty}\int_0^T (t-2)e^{-st}dt.$$ Using

parts with $u = t-2,\ du = dt,\ dv = e^{-st}dt,$ and $v = -\frac{e^{-st}}{s}$, we have

$$\lim_{T\to\infty}\int_0^T (t-2)^2 \cdot e^{-st}dt = \frac{4}{s} + \frac{2}{s}\lim_{T\to\infty}\left\{\left(\frac{-(t-2)e^{-st}}{s} + \frac{1}{s^2}e^{-st}\right)\Big|_0^T\right\} = \frac{4}{s} - \frac{4}{s^2} + \frac{2}{s^3},\ s > 0.$$

9. $\mathcal{L}\{f(t)\} = \lim_{T\to\infty}\int_1^T 1\cdot e^{-st}dt = \lim_{T\to\infty}\frac{-e^{-st}}{s}\Big|_1^T = \lim_{T\to\infty}\frac{1}{s}\left(e^{-s}-e^{-sT}\right) = \frac{e^{-s}}{s},\ s > 0$.

10. $\mathcal{L}\{f(t)\} =$

$$\lim_{T\to\infty}\int_1^T (t-1)\cdot e^{-st}dt = \lim_{T\to\infty}\left\{\frac{-(t-1)e^{-st}}{s}\Big|_1^T + \frac{1}{s}\int_1^T e^{-st}dt\right\} = \lim_{T\to\infty}\left\{-\frac{(T-1)e^{-sT}}{s} + \frac{e^{-s}-e^{-sT}}{s^2}\right\}$$

$= \frac{e^{-s}}{s^2},\ s > 0.$

11. $\mathcal{L}\{f(t)\} = \int_1^2 1\cdot e^{-st}dt = \frac{-e^{-st}}{s}\Big|_1^2 = \frac{1}{s}\left(e^{-s}-e^{-2s}\right),\ s \neq 0;\ = 1,\ s = 0$.

12. $\mathcal{L}\{f(t)\} = \int_1^2 (t-1)\cdot e^{-st}dt = \left.\frac{-(t-1)e^{-st}}{s}\right|_1^2 + \frac{1}{s}\int_1^2 e^{-st}dt = -\frac{e^{-2s}}{s} + \frac{1}{s^2}\left(e^{-s}-e^{-2s}\right),\ s \neq 0.$

13. $\int t^n e^{-st}dt$. $u = t^n$, $du = nt^{n-1}dt$, $dv = e^{-st}dt$, and $v = -\frac{e^{-st}}{s}$. Then we have

$$-\frac{t^n e^{-st}}{s} + \frac{n}{s}\int t^{n-1}e^{-st}dt,\ s>0.$$

14 (a). Since $\lim_{t\to 0^+} t^n = 0$ and $\lim_{t\to 0^+} e^{-st} = 1$, $\lim_{t\to 0^+} t^n e^{-st} = 0$.

14 (b). Using L'Hopital's rule:

$$\lim_{t\to\infty} t^n e^{-st} = \lim_{t\to\infty}\frac{t^n}{e^{st}} = \lim_{t\to\infty}\frac{nt^{n-1}}{se^{st}} = \ldots = \lim_{t\to\infty}\frac{n!}{s^n e^{st}} = 0.$$

15 (a). Using 13 and 14,

$$\mathcal{L}\{t^n\} = \lim_{T\to\infty}\left\{\left.\frac{-t^n e^{-st}}{s}\right|_0^T + \frac{n}{s}\int_0^T t^{n-1}e^{-st}dt\right\} = 0 + \frac{n}{s}\lim_{T\to\infty}\int_0^T t^{n-1}e^{-st}dt = \frac{n}{s}\mathcal{L}\{t^{n-1}\},\ s>0.$$

15 (b).

$$\mathcal{L}\{t^2\} = \frac{2}{s}\mathrm{L}\{t\} = \frac{2}{s^3},\ \mathcal{L}\{t^3\} = \frac{3}{s}\mathcal{L}\{t^2\} = \frac{3!}{s^4},\ \mathcal{L}\{t^4\} = \frac{4}{s}\mathcal{L}\{t^3\} = \frac{4!}{s^5},$$
$$\mathcal{L}\{t^5\} = \frac{5}{s}\mathcal{L}\{t^4\} = \frac{5!}{s^6},\ s>0.$$

15 (c). $\mathcal{L}\{t^m\} = \frac{m!}{s^{m+1}},\ s>0.$

16. $\mathcal{L}\{f(t)\} = \lim_{T\to\infty}\left\{e^{-st}\left.\left(\frac{-s\cos\omega t - \omega\sin\omega t}{s^2+\omega^2}\right)\right|_0^T\right\} = \frac{s}{s^2+\omega^2},\ s>0.$

17. $\mathcal{L}\{f(t)\} = \lim_{T\to\infty}\left\{e^{-st}\left.\left(\frac{-s\sin\omega t - \omega\cos\omega t}{s^2+\omega^2}\right)\right|_0^T\right\} = \frac{\omega}{s^2+\omega^2},\ s>0.$

18. $f(t) = \cos(\omega(t-2)) = \cos(\omega t)\cos(2\omega) + \sin(\omega t)\sin(2\omega)$. Using 16 and 17,

$$\mathcal{L}\{f(t)\} = \frac{s\cos(2\omega) + \omega\sin(2\omega)}{s^2+\omega^2},\ s>0.$$

19. $f(t) = \sin(\omega(t-2)) = \sin\omega t\cos 2\omega - \cos\omega t\sin 2\omega$. Then we have

$$\mathcal{L}\{f(t)\} = \frac{\omega\cos 2\omega - s\sin 2\omega}{s^2+\omega^2},\ s>0.$$

20. $\mathcal{L}\{f(t)\} = \lim_{T\to\infty}\int_0^T e^{-(s-3)t}\sin t\,dt = \lim_{T\to\infty}\left\{e^{-(s-3)t}\left.\left(\frac{-(s-3)\sin t - \cos t}{(s-3)^2+1}\right)\right|_0^T\right\} = \frac{1}{(s-3)^2+1},\ s>3.$

21. $\mathcal{L}\{f(t)\} = \lim_{T\to\infty}\int_0^T e^{-(s+2)t}\cos 4t\,dt = \lim_{T\to\infty}\left\{ e^{-(s+2)t}\left(\frac{-(s+2)\cos 4t + 4\sin 4t}{(s+2)^2+16}\right)\Bigg|_0^T \right\}$

$= \frac{s+2}{(s+2)^2+16},\ s > -2.$

22. $\mathcal{L}\{2e^{-5t}\} = \frac{2}{s+5},\ s > -5$. $\mathcal{L}\{6t\} = \frac{6}{s^2},\ s > 0$. Then $\mathcal{L}\{r(t)\} = \frac{2}{s+5} + \frac{6}{s^2},\ s > 0$.

23. $\mathcal{L}\{5e^{-7t}\} = \frac{5}{s+7},\ s > -7$. $\mathcal{L}\{t\} = \frac{1}{s^2},\ s > 0$, and $\mathcal{L}\{2e^{2t}\} = \frac{2}{s-2},\ s > 2$. Then

$\mathcal{L}\{r(t)\} = \frac{5}{s+7} + \frac{1}{s^2} + \frac{2}{s-2},\ s > 2$.

24 (a). The function is discontinuous on $0 \le t < \infty$ because the one-sided limits do not exist at the vertical asymptotes.

24 (b). The function is not exponentially bounded on $0 \le t < \infty$.

25 (a). The function is continuous on $0 \le t < \infty$.

25 (b). The function is exponentially bounded on $0 \le t < \infty$. $|f(t)| \le e^t$, so we can take $M = 1,\ a = 1$.

26 (a). The function is continuous on $0 \le t < \infty$.

26 (b). The function is exponentially bounded on $0 \le t < \infty$.

If $f(t) = t^2 e^{-t}$, then $f'(t) = (2t - t^2)e^{-t} = 0 \Rightarrow t = 2$, is a maximum point, so we can take $M = f(2) = 4e^{-2},\ a = 0$.

27 (a). The function is continuous on $0 \le t < \infty$.

27 (b). The function is exponentially bounded on $0 \le t < \infty$, since $\cosh 2t \le \frac{e^{2t}+1}{2} \le e^{2t}$ on $0 \le t < \infty$, so we can take $M = 1,\ a = 2$.

28 (a). The function is piecewise continuous on $0 \le t < \infty$.

28 (b). Consider $g(t) = te^{-t},\ g'(t) = (1-t)e^{-t} = 0 \Rightarrow t = 1$. $\therefore t = 1$ is a maximum and $g(t) \le e^{-1} \Rightarrow t \le e^{-1}e^t,\ 0 \le t < \infty$. Since $[[t]] \le t,\ 0 \le t < \infty$, the function is exponentially bounded on $0 \le t < \infty$, taking $M = e^{-1},\ a = 1$.

29 (a). The function is piecewise continuous on $0 \le t < \infty$.

29 (b). $|f(t)| \le e^{2t}$, and so the function is exponentially bounded on $0 \le t < \infty$, taking $M = 1,\ a = 2$.

30 (a). The function is continuous on $0 \le t < \infty$.

30 (b). The function is not exponentially bounded on $0 \le t < \infty$ because

$$f(t) \ge \frac{e^{t^2}}{e^{2t} + e^{2t}} = \frac{1}{2}e^{t^2-2t} \text{ and } e^{t^2-2t} > e^{\frac{t^2}{2}},\ t > 4.$$

31 (a). The function is discontinuous and not piecewise continuous on $0 \le t < \infty$.

31 (b). The function is not exponentially bounded on $0 \le t < \infty$.

32. $\lim_{T\to\infty}\int_0^T \frac{1}{1+t^2}\,dt = \lim_{T\to\infty}\left(\tan^{-1} t\Big|_0^T\right) = \lim_{T\to\infty}\left(\tan^{-1} T\right) = \frac{\pi}{2}$, so the improper integral converges.

33. $\int_0^T \frac{t}{1+t^2}\,dt = \frac{1}{2}\ln\left(1+t^2\right)\Big|_0^T = \frac{1}{2}\ln\left(1+T^2\right)$. Since $\lim_{T\to\infty}\frac{1}{2}\ln\left(1+T^2\right) = \infty$, the improper integral diverges.

34. $\lim_{T\to\infty}\int_0^T e^{-t}\cos(-e^{-t})dt = \lim_{T\to\infty}\int_1^{e^{-T}} -\cos(u)du = \int_0^1 \cos u\,du = \sin u\Big|_0^1 = \sin(1)$, so the improper integral converges.

35. $\int_0^\infty te^{-t^2}dt = \lim_{T\to\infty}\int_0^T te^{-t^2}dt = \lim_{T\to\infty}\int_0^{T^2} e^{-u}\left(\frac{1}{2}du\right) = \lim_{T\to\infty}\frac{1}{2}\int_0^{T^2} e^{-u}du = \lim_{T\to\infty}\frac{1}{2}\left(1-e^{-T^2}\right) = \frac{1}{2}$, so the integral converges to $\frac{1}{2}$.

36. $f(t) = 3\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} = 3e^{2t},\ t \ge 0$.

37. $f(t) = -2\mathcal{L}^{-1}\left\{\frac{1}{s^2}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} = -2t + e^{-t},\ t \ge 0$.

38. $f(t) = 2\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} + 2\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} = 2e^{-2t} + 2e^{2t} = 4\cosh(2t),\ t \ge 0$.

39. $f(t) = \mathcal{L}^{-1}\left\{\frac{1}{s-1}\right\} - \mathcal{L}^{-1}\left\{\frac{1}{s+1}\right\} = e^{t} - e^{-t} = 2\sinh t,\ t \ge 0$.

Section 7.2

1. $\mathcal{L}\{f(t)\} = 3\left(\frac{2}{s^3}\right) + \frac{2}{s^2} + \frac{1}{s} = \frac{6}{s^3} + \frac{2}{s^2} + \frac{1}{s},\ s > 0$

2. $\mathcal{L}\{f(t)\} = \frac{2}{s-1} + \frac{5}{s},\ s > 1$

3. $\mathcal{L}\{f(t)\} = \frac{1}{s} + \frac{3}{s^2+9},\ s > 0$

4. $\mathcal{L}\{f(t)\} = e^{-s}\mathcal{L}\{e^{3t}\} = \frac{e^{-s}}{s-3},\ s > 3$

5. $\mathcal{L}\{f(t)\} = e^{-s}\mathcal{L}\{t^2\} = \dfrac{2e^{-s}}{s^3},\ s > 0$

6. $\mathcal{L}\{\sin^2 \omega t\} = \mathcal{L}\left\{\frac{1}{2} - \frac{1}{2}\cos(2\omega t)\right\} = \dfrac{1}{2} \cdot \dfrac{1}{s} - \dfrac{1}{2} \cdot \dfrac{s}{s^2 + 4\omega^2} = \dfrac{1}{2}\left(\dfrac{1}{s} - \dfrac{s}{s^2 + 4\omega^2}\right),\ s > 0$

7. $\mathcal{L}\{f(t)\} = \mathcal{L}\{2t\}\big|_{s \to s+2} = \dfrac{2}{(s+2)^2},\ s > -2$

8. $\mathcal{L}\{\sin 3t \cos 3t\} = \mathcal{L}\left\{\frac{1}{2}\sin 6t\right\} = \dfrac{3}{s^2 + 36},\ s > 0$

9. $\mathcal{L}\{f(t)\} = \mathcal{L}\{2(t-2)h(t-2) + 4h(t-2)\}$

$$= e^{-2s}\left[\mathcal{L}\{2t\} + \mathcal{L}\{4\}\right] = e^{-2s}\left[\frac{2}{s^2} + \frac{4}{s}\right],\ s > 0$$

10. $\mathcal{L}\{e^{2t}\cos 3t\} = \mathcal{L}\{\cos 3t\}\big|_{s \to s-2} = \dfrac{s-2}{(s-2)^2 + 9},\ s > 2$

11. $\mathcal{L}\{f(t)\} = e^3\mathcal{L}\{e^{3(t-1)}h(t-1)\} = e^3 e^{-s}\mathcal{L}\{e^{3t}\} = \dfrac{e^{3-s}}{s-3},\ s > 3$

12. $\mathcal{L}\{f(t)\} = \mathcal{L}\left\{t^2 + 3t + 5\right\}\Big|_{s \to s-4} = \dfrac{2}{(s-4)^3} + \dfrac{3}{(s-4)^2} + \dfrac{5}{(s-4)},\ s > 4$

13. $\mathcal{L}^{-1}\{F(s)\} = 3 + \dfrac{24}{3!}t^3 = 3 + 4t^3,\ t \ge 0$

14. $\mathcal{L}^{-1}\{F(s)\} = 2\sin 5t + 4e^{3t},\ t \ge 0$

15. $\mathcal{L}^{-1}\{F(s)\} = 2e^{2t}\cos 3t,\ t \ge 0$

16. $\mathcal{L}^{-1}\{F(s)\} = 5e^{3t}\dfrac{t^3}{3!} = \dfrac{5}{6}e^{3t}t^3,\ t \ge 0$

17. $\mathcal{L}^{-1}\{F(s)\} = \sin(3(t-2))h(t-2),\ t \ge 0$

18. $\mathcal{L}^{-1}\{F(s)\} = e^{9(t-2)}h(t-2),\ t \ge 0$

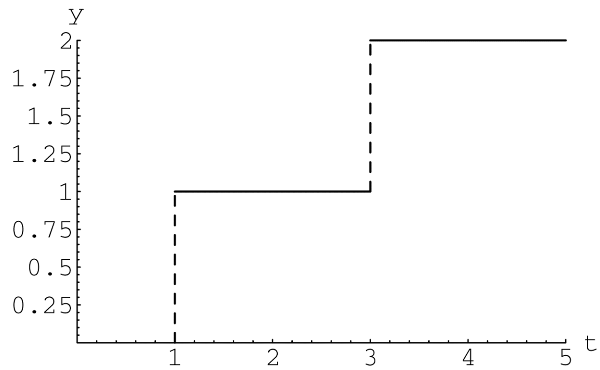
19. $\mathcal{L}^{-1}\{F(s)\} = \mathcal{L}^{-1}\left(\dfrac{4(s-1) - 2}{(s-1)^2 + 9}\right) = e^t\left(4\cos 3t - \dfrac{2}{3}\sin 3t\right),\ t \ge 0$

20. $\mathcal{L}^{-1}\{F(s)\} = \left[2\cos(4(t-3)) + \dfrac{7}{4}\sin(4(t-3))\right]h(t-3),\ t \ge 0$

21. $\mathcal{L}^{-1}\{F(s)\} = \dfrac{48}{4!}\left((t-3)^4 h(t-3) + 2(t-5)^4 h(t-5)\right)$

$$= 2(t-3)^4 h(t-3) + 4(t-5)^4 h(t-5),\ t \ge 0$$

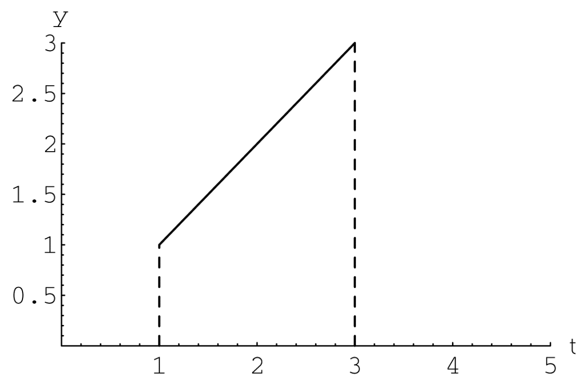
22. $\mathcal{L}\{f(t)\} = \dfrac{e^{-s} + e^{-3s}}{s}, \ s > 0$



23. $\mathcal{L}\{f(t)\} = e^{-2\pi s}\dfrac{1}{s^2+1}, \ s > 0$

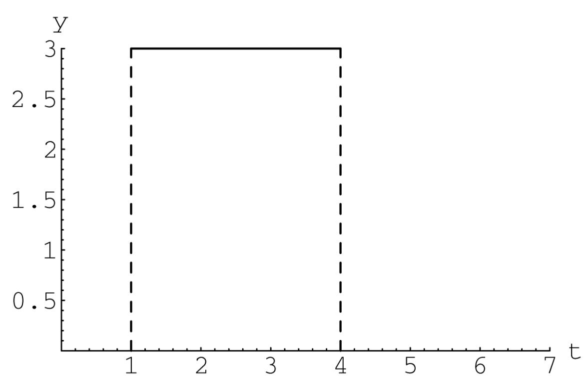
24. $f(t) = (t-1)h(t-1) + h(t-1) - (t-3)h(t-3) - 3h(t-3)$, and so

$\mathcal{L}\{f(t)\} = e^{-s}\left(\dfrac{1}{s^2} + \dfrac{1}{s}\right) - e^{-3s}\left(\dfrac{1}{s^2} + \dfrac{3}{s}\right), \ s \neq 0 \ (= 4, s = 0)$



25. $\mathcal{L}\{f(t)\} = \dfrac{\left(1 - e^{-3s}\right)}{s}, \ s > 0$

26. $\mathcal{L}\{f(t)\} = \dfrac{3\left(e^{-s} - e^{-4s}\right)}{s}, \ s \neq 0 \ (= 9, s = 0)$

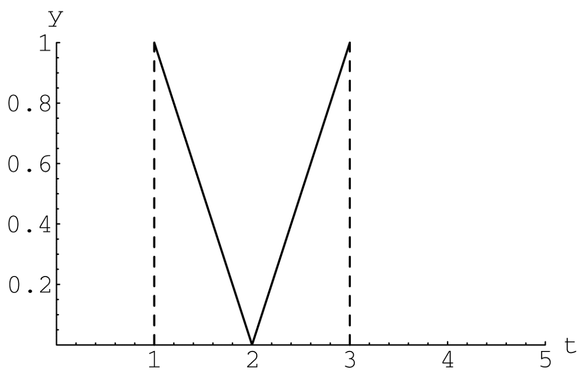


27. $f(t) = -(t-1)h(t-1) + h(t-1) + (t-3)h(t-3) + h(t-3)$, and so

$$\mathcal{L}\{f(t)\} = -\frac{e^{-s}}{s^2} + \frac{e^{-s}}{s} + \frac{e^{-3s}}{s^2} + \frac{e^{-3s}}{s},\ s \neq 0\ (=0, s=0)$$

28. $f(t) = -(t-1)h(t-1) + h(t-1) + (t-2)h(t-2) + (t-2)h(t-2) - (t-3)h(t-3) - h(t-3)$, and

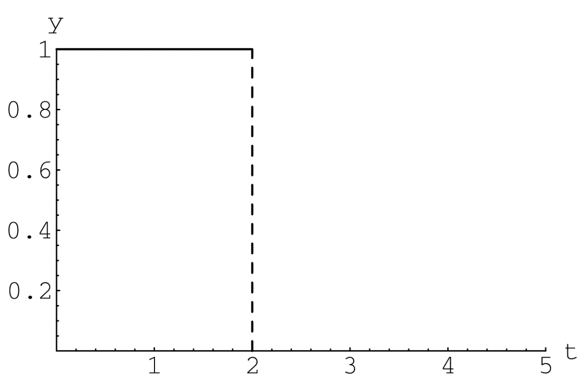
so $\mathcal{L}\{f(t)\} = -\frac{e^{-s}}{s^2} + \frac{e^{-s}}{s} + \frac{2e^{-2s}}{s^2} - \frac{e^{-3s}}{s^2} - \frac{e^{-3s}}{s},\ s \neq 0\ (=1, s=0)$



29. $$\mathcal{L}\{f(t)\} = \frac{e^{-s} - 2e^{-2s} + e^{-3s}}{s},\ s \neq 0\ (=0, s=0)$$

30. $$\mathcal{L}\{f(t)\} = \int_0^2 e^{-st}\,dt = -\frac{e^{-st}}{s}\bigg|_0^2 = \frac{1+e^{-2s}}{s},\ s \neq 0\ (=2, s=0)$$

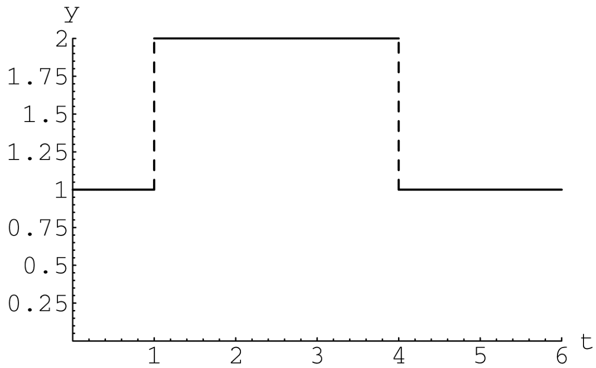
Note: $f(t) = 1 - h(t-2)$ for all t except $t = 2$.



31. $$\mathcal{L}\{f(t)\} = \int_0^1 e^{-(s-2)t}\,dt = -\frac{e^{-(s+2)t}}{s+2}\bigg|_0^1 = \frac{1-e^{-(s+2)}}{s+2},\ s \neq -2\ (=1, s=-2)$$

32. $$\mathcal{L}\{f(t)\} = \frac{1+e^{-s}-e^{-4s}}{s},\ s > 0$$

Note: $f(t) = 1 + [h(t-1) - h(t-4)]$ for all t except $t = 4$.



33. $\mathcal{L}\{f(t)\} = -\int_0^2 e^{-st}dt + \int_3^\infty e^{-st}dt = \left.\frac{e^{-st}}{s}\right|_0^2 - \left.\frac{e^{-st}}{s}\right|_3^\infty = \frac{-1+e^{-2s}+e^{-3s}}{s},\ s>0$

34. $f(t) = (t-2)[h(t-2)-h(t-3)] + [h(t-3)-h(t-4)]$

$= (t-2)h(t-2) - [(t-3)+1]h(t-3) + [h(t-3)-h(t-4)]$

$= (t-2)h(t-2) - (t-3)h(t-3) - h(t-4)$ and $\mathcal{L}\{f(t)\} = \frac{e^{-2s}-e^{-3s}}{s^2} - \frac{e^{-4s}}{s},\ s>0$

35. $f(t) = h(t-1) + h(t-2) - 2h(t-3)$ and $\mathcal{L}\{f(t)\} = \frac{e^{-s}+e^{-2s}-2e^{-3s}}{s},\ s\neq 0;\ =3,\ s=0$

36. $f(t) = (t-1)[h(t-1)-h(t-2)] + (3-t)[h(t-2)-h(t-3)]$

$= (t-1)h(t-1) - [(t-2)+1]h(t-2) + [-(t-2)+1]h(t-2) + (t-3)h(t-3)$

and $\mathcal{L}\{f(t)\} = \frac{e^{-s}}{s^2} - \frac{e^{-2s}}{s^2} - \frac{e^{-2s}}{s} - \frac{e^{-2s}}{s^2} + \frac{e^{-2s}}{s} + \frac{e^{-3s}}{s^2} = \frac{e^{-s}-2e^{-2s}+e^{-3s}}{s^2},\ s\neq 0\ (=1, s=0)$

37. $f(t) = (1-t)[h(t)-h(t-1)] + (2-t)[h(t-1)-h(t-2)] = 1-t+h(t-1)+(t-2)h(t-2),\ t\geq 0$

and $\mathcal{L}\{f(t)\} = \frac{1}{s} - \frac{1}{s^2} + \frac{e^{-s}}{s} + \frac{e^{-2s}}{s^2},\ s\neq 0;\ =1,\ s=0$

38. $F(s) = \frac{A_1}{s-3} + \frac{A_2}{s+1}$, and so $A_1(s+1) + A_2(s-3) = 12$ and $A_1 + A_2 = 0,\ A_1 - 3A_2 = 12$. Thus

$A_1 = 3,\ A_2 = -3$, and $F(s) = \frac{3}{s-3} - \frac{3}{s+1}$, and so $f(t) = -3e^{-t} + 3e^{3t},\ t\geq 0$.

39. $F(s) = \frac{A_1}{s} + \frac{A_2}{s+2}$, and so $A_1(s+2) + A_2 s = 4$ and $A_1 + A_2 = 0,\ 2A_1 = 4$. Thus

$A_1 = 2,\ A_2 = -2$, and $F(s) = \frac{2}{s} - \frac{2}{s+2}$, and so $f(t) = 2 - 2e^{-2t},\ t\geq 0$.

40. $F(s)=24e^{-5s}\left(\dfrac{A_1}{s-3}+\dfrac{A_2}{s+3}\right)$, and so $A_1(s+3)+A_2(s-3)=1$ and $A_1+A_2=0,\ 3A_1-3A_2=1$.

Thus $A_1=\dfrac{1}{6},\ A_2=-\dfrac{1}{6}$, and $F(s)=4e^{-5s}\left(\dfrac{1}{s-3}-\dfrac{1}{s+3}\right)$, and so

$f(t)=4\left[e^{3(t-5)}-e^{-3(t-5)}\right]h(t-5),\ t\ge 0$.

From (6), Table 7.1: $\mathcal{L}^{-1}\{F(s)\}=\mathcal{L}^{-1}\left(\dfrac{3}{s^2-9}\right)=\sinh 3t \Rightarrow f(t)=8\sinh(3(t-5))h(t-5),\ t\ge 0$.

Answers agree since $\sinh(3(t-5))=\dfrac{e^{3(t-5)}-e^{-3(t-5)}}{2}$.

41. $F(s)=\left[\dfrac{A_1}{(s-2)}+\dfrac{A_2}{(s-3)}\right]10e^{-s}$ and so $A_1(s-3)+A_2(s-2)=1$ and $A_1+A_2=0,\ -3A_1-2A_2=1$.

Thus $A_1=-1,\ A_2=1$, and $F(s)=10e^{-s}\left[\dfrac{-1}{s-2}+\dfrac{1}{s-3}\right]$, and so

$f(t)=10\left(e^{3(t-1)}-e^{2(t-1)}\right)h(t-1),\ t\ge 0$.

42. $g(t)=12[h(t-1)-h(t-3)]$, and so $sY(s)-y(0)+4Y=\dfrac{12}{s}\left(e^{-s}-e^{-3s}\right)$. Therefore,

$Y=\dfrac{2}{s+4}+\dfrac{12}{s(s+4)}\left(e^{-s}-e^{-3s}\right)$, and so $\dfrac{1}{s(s+4)}=\dfrac{A_1}{s}+\dfrac{A_2}{s+4}$. Thus

$A_1(s+4)+A_2s=1$ and $A_1+A_2=0,\ 4A_1=1$. Solving these simultaneous equations yields

$A_1=\dfrac{1}{4},\ A_2=-\dfrac{1}{4}$, and so $Y=\dfrac{2}{s+4}+3\left(e^{-s}-e^{-3s}\right)\left[\dfrac{1}{s}-\dfrac{1}{s+4}\right]$. Thus

$y(t)=2e^{-4t}+3[h(t-1)-h(t-3)]-3\left[e^{-4(t-1)}h(t-1)-e^{-4(t-3)}h(t-3)\right],\ t\ge 0$.

43. $g(t)=e^{3t}h(t-4)=e^{12}e^{3(t-4)}h(t-4)$, and so $sY-1-Y=e^{12}\dfrac{e^{-4s}}{s-3}$. Therefore,

$Y=\dfrac{1}{s-1}+\dfrac{e^{12}e^{-4s}}{(s-1)(s-3)}$, and so $\dfrac{1}{(s-1)(s-3)}=\dfrac{A_1}{s-1}+\dfrac{A_2}{s-3}$. Thus

$A_1(s-3)+A_2(s-1)=1$ and $A_1+A_2=0,\ -3A_1-A_2=1$. Solving these simultaneous equations

yields $A_1=-\dfrac{1}{2},\ A_2=\dfrac{1}{2}$, and so $Y=\dfrac{1}{s-1}+\dfrac{e^{12}}{2}e^{-4s}\left[\dfrac{-1}{s-1}+\dfrac{1}{s-3}\right]$. Thus

$y(t)=e^{t}+\dfrac{1}{2}e^{12}\left(-e^{t-4}+e^{3(t-4)}\right)h(t-4),\ t\ge 0$.

44. $s^2Y - sy(0) - y'(0) - 4Y = \dfrac{1}{s-3}$. Therefore, $Y = \dfrac{1}{(s-3)(s-2)(s+2)}$, and so

$\dfrac{1}{(s-3)(s+2)(s-2)} = \dfrac{A_1}{s-3} + \dfrac{A_2}{s+2} + \dfrac{A_3}{s-2}$. Thus,

$A_1 = \left.\dfrac{1}{(s-2)(s+2)}\right|_{s=3} = \dfrac{1}{5},\ A_2 = \left.\dfrac{1}{(s-3)(s-2)}\right|_{s=-2} = \dfrac{1}{20},\ A_3 = \left.\dfrac{1}{(s-3)(s+2)}\right|_{s=2} = -\dfrac{1}{4}$. Therefore,

$y(t) = \dfrac{1}{5}e^{3t} + \dfrac{1}{20}e^{-2t} - \dfrac{1}{4}e^{2t},\ t \ge 0$.

45. $s^2Y - s(0) - 1 - 2(sY - 0) - 8Y = \dfrac{1}{s-1}$. Therefore,

$Y\left(s^2 - 2s - 8\right) - 1 = \dfrac{1}{s-1} \Rightarrow Y\left(s^2 - 2s - 8\right) = \dfrac{1}{s-1} + 1 = \dfrac{s}{s-1}$, which means that

$Y = \dfrac{s}{(s-1)(s+2)(s-4)}$, and so $\dfrac{s}{(s-1)(s+2)(s-4)} = \dfrac{A_1}{s-1} + \dfrac{A_2}{s+2} + \dfrac{A_3}{s-4}$. Thus

$A_1\left(s^2 - 2s - 8\right) + A_2\left(s^2 - 5s + 4\right) + A_3\left(s^2 + s - 2\right) = s$, and so $A_1 = -\dfrac{1}{9},\ A_2 = -\dfrac{1}{9},\ A_3 = \dfrac{2}{9}$.

Finally, we have $Y = \dfrac{1}{9}\left[\dfrac{-1}{s-1} + \dfrac{-1}{s+2} + \dfrac{2}{s-4}\right]$, and so $y(t) = -\dfrac{1}{9}e^{t} - \dfrac{1}{9}e^{-2t} + \dfrac{2}{9}e^{4t},\ t \ge 0$.

46 (a). $\dfrac{d}{ds}F(s) = \int_0^\infty \dfrac{d}{ds}(e^{-st})f(t)dt = \int_0^\infty -te^{-st}f(t)dt = -\int_0^\infty e^{-st}(tf(t))dt = -\mathcal{L}\{tf(t)\}$.

46 (b). $\mathcal{L}\{f(t)\} = \mathcal{L}\left\{\dfrac{t\sin\omega t}{2\omega}\right\} = -\dfrac{1}{2\omega}\dfrac{d}{ds}\mathcal{L}\{\sin\omega t\} = -\dfrac{1}{2\omega}\dfrac{d}{ds}\left(\dfrac{\omega}{s^2+\omega^2}\right)$

$= -\dfrac{1}{2}(-1)\left(s^2+\omega^2\right)^{-2}(2s) = \dfrac{s}{\left(s^2+\omega^2\right)^2},\ s > 0$.

47. $\mathcal{L}\left(\int_0^t\left[\int_0^\lambda f(\sigma)d\sigma\right]d\lambda\right) = \dfrac{1}{s}\mathcal{L}\left(\int_0^t f(\sigma)d\sigma\right) = \dfrac{1}{s^2}F(s),\ s > \max\{a,0\}$

48. $\mathcal{L}\left\{\int_2^t f(\lambda)d\lambda\right\} = \mathcal{L}\left\{\int_0^t f(\lambda)d\lambda - \int_0^2 f(\lambda)d\lambda\right\} = \dfrac{1}{s}F(s) - \mathcal{L}\{3\} = \dfrac{1}{s}[F(s) - 3],\ s > \max\{a,0\}$

49 (a). $f(t) = h(t)h(3-t) = \begin{cases}1, & 0 \le t \le 3\\ 0, & 3 < t\end{cases}$, and $g(t) = h(t) - h(t-3) = \begin{cases}1, & 0 \le t < 3\\ 0, & 3 \le t\end{cases}$, and so the two functions are equal for all $t \ne 3$ and hence not identical.

49 (b). $\mathcal{L}\{f(t)\} = \mathcal{L}\{g(t)\} = \int_0^3 e^{-st}dt = \dfrac{1-e^{-3s}}{s},\ s \ne 0;\ = 3,\ s = 0$, and so the transformed functions are identical.

Section 7.3

1. $F(s)=\dfrac{A}{s-1}+\dfrac{B_2}{(s-2)^2}+\dfrac{B_1}{s-2}$

2. $F(s)=\dfrac{A_3}{(s-1)^3}+\dfrac{A_2}{(s-1)^2}+\dfrac{A_1}{s-1}+\dfrac{B_2}{(s-2)^2}+\dfrac{B_1}{s-2}$

3. $F(s)=\dfrac{A_2}{s^2}+\dfrac{A_1}{s}+\dfrac{Bs+C}{(s+1)^2+9}$

4. $F(s)=\dfrac{A}{s-2}+\dfrac{Bs+C}{s^2+16}$

5. $F(s)=\dfrac{A_2}{(s-3)^2}+\dfrac{A_1}{s-3}+\dfrac{B_2}{(s+3)^2}+\dfrac{B_1}{s+3}$

6. $F(s)=\dfrac{A_2}{(s+4)^2}+\dfrac{A_1}{s+4}+\dfrac{B_2s+C_2}{(s^2+1)^2}+\dfrac{B_1s+C_1}{s^2+1}$

7. $F(s)=\dfrac{Bs+C}{(s+4)^2+1}+\dfrac{Ds+E}{(s+3)^2+4}$

8. $F(s)=\dfrac{A_2}{(s-2)^2}+\dfrac{A_1}{s-2}+\dfrac{B_2s+C_2}{((s+4)^2+1)^2}+\dfrac{B_1s+C_1}{(s+4)^2+1}$

9. $f(t)=2e^{3t},\ t\ge 0$

10. $f(t)=\dfrac{1}{2}t^2e^{-t},\ t\ge 0$

11. $F(s)=4\left(\dfrac{s}{s^2+9}\right)+\dfrac{5}{3}\left(\dfrac{3}{s^2+9}\right)$, and so $f(t)=4\cos 3t+\dfrac{5}{3}\sin 3t,\ t\ge 0$.

12. $F(s)=\dfrac{A}{s-1}+\dfrac{B}{s-2}$. $A=\left.\dfrac{2s-3}{s-2}\right|_{s=1}=1$ and $B=\left.\dfrac{2s-3}{s-1}\right|_{s=2}=1$. Therefore, $f(t)=e^t+e^{2t},\ t\ge 0$.

13. $F(s)=\dfrac{A}{s+3}+\dfrac{B}{s+1}$. $A=\left.\dfrac{3s+7}{s+1}\right|_{s=-3}=\dfrac{-2}{-2}=1$ and $B=\left.\dfrac{3s+7}{s+3}\right|_{s=-1}=\dfrac{4}{2}=2$. Thus

$F(s)=\dfrac{1}{s+3}+\dfrac{2}{s+1}$ and so $f(t)=e^{-3t}+2e^{-t},\ t\ge 0$.

14. $F(s)=\dfrac{A}{s}+\dfrac{Bs+C}{s^2+1}=\dfrac{As^2+A+Bs^2+Cs}{s(s^2+1)}$. Then we have $A+B=4,\ C=1,\ A=1,\ B=3$.

Therefore, $F(s)=\dfrac{1}{s}+\dfrac{3s}{s^2+1}+\dfrac{1}{s^2+1}$ and so $f(t)=1+3\cos t+\sin t,\ t\ge 0$.

15. $F(s)=\dfrac{A}{s}+\dfrac{Bs+C}{s^2+4}$. Then we have $A+B=3,\ C=1,$ and $4A=8$, so $A=2,\ B=1,\ C=1$. Thus

$F(s)=\dfrac{2}{s}+\dfrac{s+1}{s^2+4}$ and so $f(t)=2+\cos 2t+\dfrac{1}{2}\sin 2t,\ t\ge 0$.

16. $F(s)=\dfrac{B_2s+C_2}{(s^2+4)^2}+\dfrac{B_1s+C_1}{(s^2+4)}$. Then we have $B_1=0,\ C_1=1,\ B_2=6,\ C_2=4$. Therefore,

$F(s)=\dfrac{6s+4}{(s^2+4)^2}+\dfrac{1}{s^2+4}$ and so

$$f(t)=6\left(\frac{t}{4}\sin 2t\right)+4\left(\frac{1}{16}[\sin 2t-2t\cos 2t]\right)+\frac{1}{2}\sin 2t=\frac{3}{2}t\sin 2t+\frac{3}{4}\sin 2t-\frac{1}{2}t\cos 2t,\ t\ge 0.$$

17. $F(s)=\dfrac{s}{(s-1)^3}=\dfrac{(s-1)+1}{(s-1)^3}=\dfrac{1}{(s-1)^2}+\dfrac{1}{(s-1)^3}$, so $f(t)=te^t+\dfrac{1}{2}t^2e^t,\ t\ge 0$.

18. $sY-3+2Y=26\left(\dfrac{3}{s^2+9}\right)$, and thus $Y=\dfrac{3}{s+2}+26\dfrac{3}{(s+2)(s^2+9)}$.

$\dfrac{1}{(s+2)(s^2+9)}=\dfrac{A}{s+2}+\dfrac{Bs+C}{s^2+9}$, and so $A=\left.\dfrac{s}{s^2+9}\right|_{s=-2}=\dfrac{1}{13}$.

$\dfrac{1}{(s+2)(s^2+9)}-\dfrac{1/13}{s+2}=\dfrac{1-1/13(s^2+9)}{(s+2)(s^2+9)}=\dfrac{-1/13(s^2-4)}{(s+2)(s^2+9)}=\dfrac{-1/13(s-2)}{(s^2+9)}=\dfrac{Bs+C}{s^2+9}$. Then,

$B=-\dfrac{1}{13},\ C=\dfrac{2}{13}$, and so

$Y=\dfrac{3}{s+2}+26\cdot 3\left(\dfrac{1/13}{s+2}-\dfrac{1}{13}\cdot\dfrac{s}{s^2+9}+\dfrac{2}{13}\cdot\dfrac{1}{s^2+9}\right)=\dfrac{9}{s+2}-6\left(\dfrac{s}{s^2+9}\right)+4\left(\dfrac{3}{s^2+9}\right)$. Finally, we have $y(t)=9e^{-2t}-6\cos 3t+4\sin 3t$.

19. $sY-1-3Y=13\left(\dfrac{s}{s^2+4}\right)$, and thus $Y=\dfrac{1}{s-3}+13\dfrac{s}{(s-3)(s^2+4)}$.

$\dfrac{s}{(s-3)(s^2+4)}=\dfrac{A}{s-3}+\dfrac{Bs+C}{s^2+4}$, and so $A=\left.\dfrac{s}{s^2+4}\right|_{s=3}=\dfrac{3}{13}$. Setting $s=0$ gives us

$0=-\dfrac{1}{13}+\dfrac{C}{4},\ C=\dfrac{4}{13}$. Setting $s=1$ gives us $\dfrac{1}{-2\cdot 5}=-\dfrac{1}{2}\left(\dfrac{3}{13}\right)+\dfrac{B+\frac{4}{13}}{5}$. Solving for B yields

$B=-\dfrac{3}{13}$, and so $Y=\dfrac{1}{s-3}+13\left(\dfrac{\frac{3}{13}}{s-3}+\dfrac{\frac{-3}{13}s+\frac{4}{13}}{s^2+4}\right)=\dfrac{4}{s-3}-3\left(\dfrac{s}{s^2+4}\right)+2\left(\dfrac{2}{s^2+4}\right)$. Finally, we have $y(t)=4e^{3t}-3\cos 2t+2\sin 2t$.

20. $sY - 3 + 2Y = \frac{4}{s^2}$, and thus $Y = \frac{3}{s+2} + \frac{4}{s^2(s+2)} = \frac{3s^2+4}{s^2(s+2)} = \frac{A}{s^2} + \frac{B}{s} + \frac{C}{s+2}$, and so

$C = \left.\frac{3s^2+4}{s^2}\right|_{s=-2} = 4$, $A = \left.\frac{3s^2+4}{s+2}\right|_{s=0} = 2$. Setting $s = 1$ gives us $\frac{7}{3} = 2 + B + \frac{4}{3} \Rightarrow B = -1$.

Therefore, $Y = \frac{2}{s^2} - \frac{1}{s} + \frac{4}{s+2}$. Finally, we have $y(t) = 4e^{-2t} + 2t - 1$.

21. $sY - 1 - 3Y = \frac{1}{s-3}$, so $Y = \frac{1}{s-3} + \frac{1}{(s-3)^2}$ and thus $y(t) = e^{3t} + te^{3t}$.

22. $s^2Y - s(1) - 2 + 3(sY-1) + 2Y = \frac{6}{s+1}$, and thus $Y = \frac{s+5}{(s+1)(s+2)} + \frac{6}{(s+1)^2(s+2)}$.

$\frac{s^2+6s+11}{(s+1)^2(s+2)} = \frac{A}{s+2} + \frac{B}{s+1} + \frac{C}{(s+1)^2}$, and so

$A = \left.\frac{s^2+6s+11}{(s+1)^2}\right|_{s=-2} = 3$, $C = \left.\frac{s^2+6s+11}{s+2}\right|_{s=-1} = 6$. Setting $s = 0$ gives us

$\frac{11}{2} = \frac{3}{2} + B + 6 \Rightarrow B = -2$. Therefore, $Y = \frac{3}{s+2} - \frac{2}{s+1} + \frac{6}{(s+1)^2}$. Finally, we have

$y(t) = 3e^{-2t} - 2e^{-t} + 6te^{-t}$.

23. $s^2Y - s(2) - 6 + 4Y = \frac{8}{s^2}$, so $Y = \frac{2s+6}{s^2+4} + \frac{8}{s^2(s^2+4)}$.

If $\frac{8}{s^2(s^2+4)} = \frac{A}{s^2} + \frac{B}{s} + \frac{Cs+D}{s^2+4}$, then $\frac{A(s^2+4) + Bs(s^2+4) + Cs^3 + Ds^2}{s^2(s^2+4)} = \frac{8}{s^2(s^2+4)}$, and so

we have $B + C = 0$, $A + D = 0$, $4B = 0$, $4A = 8$, which means that $B = C = 0$, $A = 2$, $D = -2$.

Then $Y = 2\left(\frac{s}{s^2+4}\right) + 3\left(\frac{2}{s^2+4}\right) + \frac{2}{s^2} - \left(\frac{2}{s^2+4}\right)$ and $y(t) = 2\cos 2t + 2\sin 2t + 2t$.

24. $s^2Y - s(1) - 1 + 4Y = \frac{s}{s^2+4}$, so $Y = \frac{s+1}{s^2+4} + \frac{s}{(s^2+4)^2}$. Therefore,

$y(t) = \cos 2t + \frac{1}{2}\sin 2t + \frac{t}{4}\sin 2t$.

25. $s^2Y - s(1) - 0 + 4Y = \frac{2}{s^2+4}$, so $Y = \frac{s}{s^2+4} + \frac{2}{(s^2+4)^2}$. Therefore,

$y(t) = \cos 2t + \frac{2}{2\cdot 8}(\sin 2t - 2t\cos 2t) = \cos 2t + \frac{1}{8}\sin 2t - \frac{t}{4}\cos 2t$.

26. $s^2Y - s(0) - 0 - 2(sY - 0) + Y = \dfrac{1}{s-2}$, and thus $Y = \dfrac{1}{(s-2)(s-1)^2} = \dfrac{A}{s-2} + \dfrac{B}{s-1} + \dfrac{C}{(s-1)^2}$, and

so $A = \left.\dfrac{1}{(s-1)^2}\right|_{s=2} = 1$, $C = \left.\dfrac{1}{s-2}\right|_{s=1} = -1$. Setting $s = 0$ gives us $-\dfrac{1}{2} = -\dfrac{1}{2} - B - 1 \Rightarrow B = -1$.

Therefore, $Y = \dfrac{1}{s-2} - \dfrac{1}{s-1} - \dfrac{1}{(s-1)^2}$. Finally, we have $y(t) = e^{2t} - e^t - te^t$.

27. $s^2Y - 1 + 2sY - 0 + Y = \dfrac{1}{s+1}$, so $Y = \dfrac{1}{(s+1)^2} + \dfrac{1}{(s+1)^3}$. Therefore, $y(t) = te^{-t} + \dfrac{t^2}{2}e^{-t}$.

28. $g(t) = 6[h(t) - h(t-\pi)]$, and then we have $s^2Y - s(1) - 3 + 9Y = \dfrac{6}{s}\left(1 - e^{-\pi s}\right)$, so

$Y = \dfrac{s+3}{s^2+9} + \dfrac{6}{s(s^2+9)}\left(1 - e^{-\pi s}\right)$. $\dfrac{6}{s(s^2+9)} = \dfrac{A}{s} + \dfrac{Bs + C}{s^2+9} = \dfrac{As^2 + 9A + Bs^2 + Cs}{s(s^2+9)}$. Thus

$A + B = 0,\ C = 0,\ A = \dfrac{2}{3},\ B = -\dfrac{2}{3}$, and so $Y = \dfrac{s}{s^2+9} + \dfrac{3}{s^2+9} + \left(\dfrac{2/3}{s} - \dfrac{2}{3}\cdot\dfrac{s}{s^2+9}\right)\left(1 - e^{-\pi s}\right)$. Then,

$$y(t) = \cos 3t + \sin 3t + \frac{2}{3}(1 - \cos 3t) - \frac{2}{3}(1 - \cos 3(t-\pi))h(t-\pi)$$
$$= \cos 3t + \sin 3t + \frac{2}{3}(1 - \cos 3t) - \frac{2}{3}(1 + \cos 3t)h(t-\pi).$$

29. $g(t) = t[1 - h(t-2)] = t - (t-2)h(t-2) - 2h(t-2)$, and then we have

$s^2Y - s(1) - 0 + Y = \dfrac{1}{s^2} - \dfrac{1}{s^2}e^{-2s} - \dfrac{2}{s}e^{-2s}$, so $Y = \dfrac{s}{s^2+1} + \dfrac{1}{s^2(s^2+1)}\left(1 - e^{-2s}\right) - \dfrac{2}{s(s^2+1)}e^{-2s}$.

$\dfrac{1}{s^2(s^2+1)} = \dfrac{1}{s^2} - \dfrac{1}{s^2+1}$ and $\dfrac{-2}{s(s^2+1)} = \dfrac{A}{s} + \dfrac{Bs+C}{s^2+1} = \dfrac{As^2 + A + Bs^2 + Cs}{s(s^2+1)}$. Thus

$A + B = 0,\ C = 0,\ A = -2 \ \Rightarrow\ B = 2$, and so

$Y = \dfrac{s}{s^2+1} + \left(\dfrac{1}{s^2} - \dfrac{1}{s^2+1}\right)\left(1 - e^{-2s}\right) + \left(-\dfrac{2}{s} + \dfrac{2s}{s^2+1}\right)e^{-2s}$. The inverse Laplace transform yields

$$y(t) = \cos t + t - (t-2)h(t-2) - \sin t + \sin(t-2)h(t-2) - 2h(t-2) + 2\cos(t-2)h(t-2)$$
$$= \cos t - \sin t + t + [-(t-2) + \sin(t-2) - 2 + 2\cos(t-2)]h(t-2)$$
$$= \cos t - \sin t + t + [-t + \sin(t-2) + 2\cos(t-2)]h(t-2).$$

30. $s^2Y - sy_0 - y_0' + \alpha(sY - y_0) + \beta Y = 0 \Rightarrow Y = \dfrac{sy_0 + (y_0' + \alpha y_0)}{s^2 + \alpha s + \beta} = \dfrac{2s - 1}{s^2 + s + 2}$, so

$\alpha = 1,\ \beta = 2,\ y_0 = 2,\ y_0' + \alpha y_0 = y_0' + 1(2) = -1 \Rightarrow y_0' = -3$.

31. $\dfrac{sy_0 + (y_0' + \alpha y_0)}{s^2 + \alpha s + \beta} = \dfrac{3}{s^2 - 4}$, so $\alpha = 0,\ \beta = -4,\ y_0 = 0,\ y_0' = 3$.

32. $\dfrac{sy_0+\left(y_0'+\alpha y_0\right)}{s^2+\alpha s+\beta}=\dfrac{s}{(s+1)^2}$, so $\alpha=2,\ \beta=1,\ y_0=1,\ y_0'=-2$.

Section 7.4

1. $T=4.\ 3\int_0^2 e^{-st}dt-3\int_2^4 e^{-st}dt=3\left[\dfrac{e^{-st}}{-s}\Big|_0^2-\dfrac{e^{-st}}{-s}\Big|_2^4\right]=\dfrac{3}{s}\left[1-e^{-2s}+e^{-4s}-e^{-2s}\right]=\dfrac{3}{s}\left(1-e^{-2s}\right)^2.$

Therefore, $\mathcal{L}\{f\}=\dfrac{3}{s}\dfrac{\left(1-e^{-2s}\right)^2}{1-e^{-4s}}=\dfrac{3}{s}\left(\dfrac{1-e^{-2s}}{1+e^{-2s}}\right).$

2. $T=2.\ 3\int_0^1 e^{-st}dt+\int_1^2 e^{-st}dt=3\dfrac{e^{-st}}{-s}\Big|_0^1+\dfrac{e^{-st}}{-s}\Big|_1^2=\dfrac{1}{s}\left(3-3e^{-s}+e^{-s}-e^{-2s}\right)=\dfrac{1}{s}\left(3-2e^{-s}-e^{-2s}\right).$

Therefore, $\mathcal{L}\{f\}=\dfrac{3-2e^{-s}-e^{-2s}}{s(1-e^{-2s})}=\dfrac{-3+e^{-s}}{s(1-e^{-s})}.$

3. $T=4.-3\int_0^2 e^{-st}dt+2\int_2^4 e^{-st}dt=\dfrac{3}{s}e^{-st}\Big|_0^2-\dfrac{2}{s}e^{-st}\Big|_2^4=\dfrac{3}{s}\left(e^{-2s}-1\right)-\dfrac{2}{s}\left(e^{-4s}-e^{-2s}\right)$

$=\dfrac{1}{s}\left(-2e^{-4s}+5e^{-2s}-3\right)$. Therefore, $\mathcal{L}\{f\}=\dfrac{-2e^{-4s}+5e^{-2s}-3}{s\left(1-e^{-4s}\right)}=\dfrac{-3+2e^{-2s}}{s\left(1+e^{-2s}\right)}.$

4. $T=4.\ 2\int_0^1 e^{-st}dt+\int_1^3 e^{-st}dt=2\dfrac{e^{-st}}{-s}\Big|_0^1+\dfrac{e^{-st}}{-s}\Big|_1^3=\dfrac{1}{s}\left(2-2e^{-s}+e^{-s}-e^{-3s}\right)=\dfrac{1}{s}\left(2-e^{-s}-e^{-3s}\right).$

Therefore, $\mathcal{L}\{f\}=\dfrac{2-e^{-s}-e^{-3s}}{s(1-e^{-4s})}=\dfrac{2+e^{-s}+e^{-2s}}{s(1+e^{-2s})(1+e^{-s})}.$

5. $T=2.\ \int_0^1 te^{-st}dt+\int_1^2(2-t)e^{-st}dt=-\dfrac{1}{s^2}(st+1)e^{-st}\Big|_0^1+\dfrac{1}{s^2}(st+1)e^{-st}\Big|_1^2+\dfrac{2}{-s}e^{-st}\Big|_1^2$

$=-\dfrac{1}{s^2}\left[(s+1)e^{-s}-1\right]+\dfrac{1}{s^2}\left[(2s+1)e^{-2s}-(s+1)e^{-s}\right]+\dfrac{2}{s}\left[e^{-s}-e^{-2s}\right]=\dfrac{1}{s^2}\left(1-e^{-s}\right)^2$. Therefore,

$\mathcal{L}\{f\}=\dfrac{1}{s^2}\dfrac{\left(1-e^{-s}\right)^2}{1-e^{-2s}}=\dfrac{1-e^{-s}}{s^2\left(1+e^{-s}\right)}.$

6. $T=2.\ \int_1^2(t-1)e^{-st}dt=\int_0^1 ue^{-s(u+1)}du=e^{-s}\int_0^1 ue^{-su}du=\dfrac{-e^{-s}}{s^2}(su+1)e^{-su}\Big|_0^1=\dfrac{-e^{-s}}{s^2}\left[(s+1)e^{-s}-1\right].$

Therefore, $\mathcal{L}\{f\}=\dfrac{e^{-s}\left[1-(s+1)e^{-s}\right]}{s^2\left(1-e^{-2s}\right)}.$

7. $T=2$. $\int_0^1 (1-t^2)e^{-st}dt = \frac{1}{s^3}\left[s^2t^2+2st+2-s^2\right]e^{-st}\Big|_0^1$

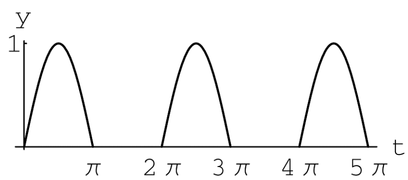
$= \frac{1}{s^3}\left[(s^2+2s+2-s^2)e^{-s}-2+s^2\right] = \frac{1}{s^3}\left[(2s+2)e^{-s}+s^2-2\right]$. Therefore,

$\mathcal{L}\{f\} = \frac{\left[(2s+2)e^{-s}+(s^2-2)\right]}{s^3(1-e^{-2s})}$.

8. $T=2$. $\int_0^2 te^{-st}dt = \frac{-1}{s^2}(st+1)e^{-st}\Big|_0^2 = \frac{-1}{s^2}\left[(2s+1)e^{-2s}-1\right]$. Therefore, $\mathcal{L}\{f\} = \frac{1-(2s+1)e^{-2s}}{s^2(1-e^{-2s})}$.

9. $T=\frac{\pi}{2}$, $\mathcal{L}\{f\} = \frac{2+2e^{-\frac{\pi}{2}s}}{(s^2+4)\left(1-e^{-\frac{\pi}{2}s}\right)}$

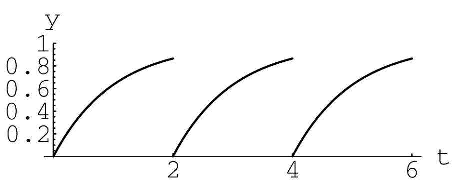
10. $T=2\pi$, $\mathcal{L}\{f\} = \frac{1+e^{-\pi s}}{(s^2+1)(1-e^{-2\pi s})} = \frac{1}{(s^2+1)(1-e^{-\pi s})}$



11. $T=1$, $\int_0^1 e^{-t}e^{-st}dt = -\frac{e^{-(s+1)t}}{s+1}\Big|_0^1 = \frac{1}{s+1}\left(1-e^{-(s+1)}\right)$, $\mathcal{L}\{f\} = \frac{1-e^{-(s+1)}}{(s+1)(1-e^{-s})}$.

12. $T=2$, $\int_0^2 (1-e^{-t})e^{-st}dt = \frac{e^{-st}}{-s}\Big|_0^2 + \frac{e^{-(s+1)t}}{s+1}\Big|_0^2 = \frac{1}{s}\left(1-e^{-2s}\right) - \frac{1}{s+1}\left(1-e^{-2(s+1)}\right)$

$\mathcal{L}\{f\} = \frac{1}{s} - \frac{1-e^{-2(s+1)}}{(s+1)(1-e^{-2s})}$.

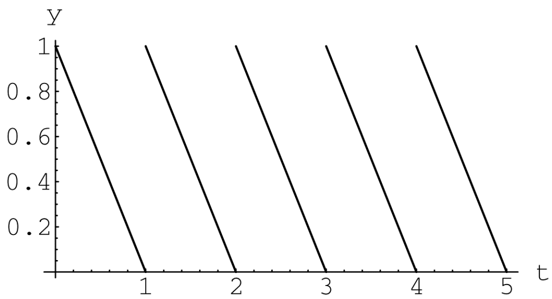


13. $\mathcal{L}^{-1}\left(\frac{e^{-\alpha s}}{s(1-e^{-\alpha s})}\right) = \mathcal{L}^{-1}\left(\frac{e^{-\alpha s}}{s}\left(1+e^{-\alpha s}+e^{-2\alpha s}+e^{-3\alpha s}+\ldots\right)\right)$

$= \mathcal{L}^{-1}\left(\frac{e^{-\alpha s}}{s}+\frac{e^{-2\alpha s}}{s}+\frac{e^{-3\alpha s}}{s}+\ldots\right) = h(t-\alpha)+h(t-2\alpha)+\ldots,$

14. $F(s)=\frac{1}{s}-\frac{1}{s^2}-\frac{e^{-s}}{s(1-e^{-s})}$

and so $f(t)=1-t+h(t-1)+h(t-2)+h(t-3)+...=1-t+\sum_{n=1}^{\infty}h(t-n)$.

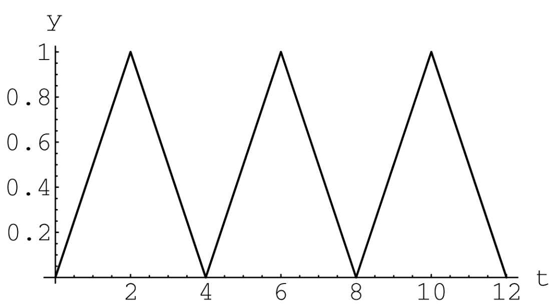


15. $F(s)=\frac{3}{s^2}-\frac{3e^{-2s}}{s(1-e^{-2s})}$ and so $f(t)=3[t-h(t-2)-h(t-4)-h(t-6)...]=3\left[t-\sum_{n=1}^{\infty}h(t-2n)\right]$.

16. $F(s)=\frac{1}{2s^2}-\frac{1}{s^2}\frac{e^{-2s}}{1+e^{-2s}}=\frac{1}{2s^2}-\frac{e^{-2s}}{s^2}\left(1-e^{-2s}+e^{-4s}-e^{-6s}+e^{-8s}+...\right)=\frac{1}{2s^2}+\sum_{n=1}^{\infty}(-1)^n\frac{e^{-2ns}}{s^2}$

and so $f(t)=\frac{t}{2}-(t-2)h(t-2)+(t-4)h(t-4)-(t-6)h(t-6)+...$

$=\frac{t}{2}+\sum_{n=1}^{\infty}(-1)^n(t-2n)h(t-2n)$.



17 (a). $\frac{dq}{dt}=rC_i(t)-\frac{r}{v}q$. With q in kg and t in days, we have $\frac{dq}{dt}+\frac{5(10^6)}{50(10^6)}q=\frac{5(10^6)}{10^6}C_i(t)$, where

$C_i(t)=\begin{cases}1, & 0\le t<1/2\\ 0, & 1/2\le t<1\end{cases}$, $C_i(t+1)=C_i(t)$. Then $\frac{dq}{dt}+\frac{1}{10}q=5C_i(t),\ q(0)=0$.

17 (b). $sQ+0.1Q=5\left\{\frac{\int_0^{0.5}e^{-st}dt}{1-e^{-s}}\right\}=\frac{5}{s}\left(\frac{1-e^{-\frac{s}{2}}}{1-e^{-s}}\right)=\frac{5}{s\left(1+e^{-\frac{s}{2}}\right)}$. Then

$Q=\frac{1}{s(s+0.1)}\cdot\frac{5}{1+e^{-\frac{s}{2}}}=50\left(\frac{1}{s}-\frac{1}{s+0.1}\right)\frac{1}{1+e^{-\frac{s}{2}}}$.

17 (c). $Q=50\left(\frac{1}{s}-\frac{1}{s+0.1}\right)\left(1-e^{-\frac{s}{2}}+e^{-s}-e^{-\frac{3s}{2}}+e^{-2s}+...\right)$. Noting that $\mathcal{L}^{-1}\left(\frac{1}{s}-\frac{1}{s+0.1}\right)=\left(1-e^{-0.1t}\right)$,

we have $q(t)=50\left[1-e^{-0.1t}-\left(1-e^{-0.1\left(t-\frac{1}{2}\right)}\right)h\left(t-\frac{1}{2}\right)+\left(1-e^{-0.1(t-1)}\right)h(t-1)-...\right]$. Thus for

$$1\le t<2,\ q(t)=\begin{cases}50\left[1-e^{-0.1t}+e^{-0.1\left(t-\frac{1}{2}\right)}-e^{-0.1(t-1)}\right],\ 1\le t<\frac{3}{2}\\ 50\left[-e^{-0.1t}+e^{-0.1\left(t-\frac{1}{2}\right)}-e^{-0.1(t-1)}\right],\ \frac{3}{2}\le t<2\end{cases}.$$

18 (a). $ms^2X(s)=\frac{f_0\int_0^{\frac{T}{2}}e^{-st}dt}{1-e^{-st}}=\frac{f_0}{s}\left(\frac{1-e^{-s\frac{T}{2}}}{1-e^{-sT}}\right)\Rightarrow X(s)=\frac{f_0}{m}\cdot\frac{1}{s^3}\cdot\frac{1}{1+e^{-s\frac{T}{2}}}$ and

$$V(s)=sX(s)=\frac{f_0}{m}\cdot\frac{1}{s^2}\cdot\frac{1}{1+e^{-s\frac{T}{2}}}=\frac{f_0}{m}\cdot\frac{1}{s^2}\left(1-e^{-s\frac{T}{2}}+e^{-sT}-e^{-s\frac{3T}{2}}+...\right)$$

$$v(t)=\frac{f_0}{m}\left[t-\left(t-\frac{T}{2}\right)h\left(t-\frac{T}{2}\right)+(t-T)h(t-T)-\left(t-\frac{3T}{2}\right)h\left(t-\frac{3T}{2}\right)+...\right]$$

$$=\frac{f_0}{m}\sum_{n=0}^{\infty}(-1)^n\left(t-\frac{nT}{2}\right)h\left(t-\frac{nT}{2}\right).$$

Similarly, $x(t)=\frac{f_0}{2m}\left[t^2-\left(t-\frac{T}{2}\right)^2h\left(t-\frac{T}{2}\right)+(t-T)^2h(t-T)-\left(t-\frac{3T}{2}\right)^2h\left(t-\frac{3T}{2}\right)+...\right]$

$$=\frac{f_0}{2m}\sum_{n=0}^{\infty}(-1)^n\left(t-\frac{nT}{2}\right)^2h\left(t-\frac{nT}{2}\right).$$

18 (b). $m=1,\ f_0=1,\ T=1,\ t=\frac{5}{4}\Rightarrow v\left(\frac{5}{4}\right)=\left[t-(t-\tfrac{1}{2})+(t-1)\right]\Big|_{t=\frac{5}{4}}=(t-\tfrac{1}{2})\Big|_{t=\frac{5}{4}}=\frac{3}{4}\ m/s$ and

$$x\left(\frac{5}{4}\right)=\frac{1}{2}\left[t^2-(t-\tfrac{1}{2})^2+(t-1)^2\right]\Big|_{t=\frac{5}{4}}=\frac{1}{2}\left(\frac{25}{16}-\frac{9}{16}+\frac{1}{16}\right)=\frac{17}{32}\ m.$$

19. We know that $ay_i''+by_i'+cy_i=f_i(t),\ i=1,\ 2$. Therefore,

$$a\left(c_1y_1+c_2y_2\right)''+b\left(c_1y_1+c_2y_2\right)'+c\left(c_1y_1+c_2y_2\right)=c_1\left[ay_1''+by_1'+cy_1\right]+c_2\left[ay_2''+by_2'+cy_2\right]$$

$$=c_1f_1+c_2f_2.$$

20 (a). $\mathcal{L}\{ay''+by'+cy\}=\mathcal{L}\{f\}$. Since

$y(0)=0,\ y'(0)=0,\ \left(as^2+bs+c\right)Y=F\Rightarrow\Phi(s)=\frac{1}{as^2+bs+c}=\frac{1}{2s^2+5s+2}$. Comparing:

$a=2,\ b=5,\ c=2$.

20 (b). If $f(t)=e^{-t}$, $F(s)=\dfrac{1}{s+1}$, $Y(s)=\dfrac{1}{\left(2s^2+5s+2\right)(s+1)}$. Since

$2s^2+5s+2=2(s+2)(s+½)$, $Y(s)=\dfrac{1}{2}\left[\dfrac{1}{(s+2)(s+1)(s+½)}\right]=\dfrac{1}{2}\left[\dfrac{A}{s+2}+\dfrac{B}{s+1}+\dfrac{C}{s+½}\right]$. Then

$A=\left.\dfrac{1}{(s+1)(s+½)}\right|_{s=-2}=\dfrac{2}{3}$, $B=\left.\dfrac{1}{(s+2)(s+½)}\right|_{s=-1}=-2$, $C=\left.\dfrac{1}{(s+2)(s+1)}\right|_{s=-½}=\dfrac{4}{3}$ and

$y(t)=\dfrac{1}{3}e^{-2t}-e^{-t}+\dfrac{2}{3}e^{-\frac{t}{2}}$.

21. $f(t)=t$, $F(s)=\dfrac{1}{s^2}$, $y(t)=2\left(e^{-t}-1\right)+t\left(e^{-t}+1\right)$,

$Y(s)=\dfrac{2}{s+1}-\dfrac{2}{s}+\dfrac{1}{(s+1)^2}+\dfrac{1}{s^2}=\dfrac{-2\left(s^2+s\right)+2s^2+2s+1}{s^2(s+1)^2}=\dfrac{1}{s^2(s+1)^2}$. Since $Y(s)=\Phi(s)F(s)$,

$\Phi(s)=\dfrac{1}{(s+1)^2}$.

22. From 21, $\Phi(s)=\dfrac{1}{(s+1)^2}$. If $f(t)=h(t)$, $F(s)=\dfrac{1}{s}$ and $Y(s)=\dfrac{1}{s(s+1)^2}=\dfrac{A}{s}+\dfrac{B}{s+1}+\dfrac{C}{(s+1)^2}$

$A=\left.\dfrac{1}{(s+1)^2}\right|_{s=0}=1$, $C=\left.\dfrac{1}{s}\right|_{s=-1}=-1$. Setting $s=1$, $\dfrac{1}{4}=1+\dfrac{B}{2}-\dfrac{1}{4}\Rightarrow B=-1$. Therefore,

$Y(s)=\dfrac{1}{s}-\dfrac{1}{s+1}-\dfrac{1}{(s+1)^2}\Rightarrow y(t)=1-e^{-t}-te^{-t}$.

23 (a). $s^2Y+4Y=F$, so $\Phi=\dfrac{1}{\left(s^2+4\right)}$.

23 (b). $F=\dfrac{2}{s^3}$, so $Y=\dfrac{2}{s^3\left(s^2+4\right)}$.

24 (a). $s^2Y+sY+Y=F$, so $\Phi=\dfrac{1}{\left(s^2+s+1\right)}$.

24 (b). $\int_0^2 e^{-st}f(t)dt=\int_0^1 e^{-st}dt-\int_1^2 e^{-st}dt=\left.\dfrac{e^{-st}}{-s}\right|_0^1-\left.\dfrac{e^{-st}}{-s}\right|_1^2=\dfrac{1}{s}(1-e^{-s})+\dfrac{1}{s}(e^{-2s}-e^{-s})=\dfrac{1}{s}(1-e^{-s})^2$.

Therefore, $F=\dfrac{(1-e^{-s})^2}{s(1-e^{-2s})}=\dfrac{(1-e^{-s})}{s(1+e^{-s})}$, so $Y=\dfrac{(1-e^{-s})}{s(1+e^{-s})(s^2+s+1)}$.

25 (a). $s^2Y+4sY+4Y=F$, so $\Phi=\dfrac{1}{\left(s^2+4s+4\right)}=\dfrac{1}{(s+2)^2}$.

25 (b). $\int_0^1 te^{-st}dt = -\frac{1}{s^2}(st+1)e^{-st}\Big|_0^1 = \frac{1}{s^2}\left[1-(s+1)e^{-s}\right]$, so $F = \frac{1-(s+1)e^{-s}}{s^2\left(1-e^{-s}\right)}$ and $Y = \frac{1-(s+1)e^{-s}}{s^2\left(1-e^{-s}\right)(s+2)^2}$.

26 (a). $s^3Y - 4Y = F$, so $\Phi = \frac{1}{\left(s^3-4\right)}$.

26 (b). $F = \frac{1}{s-1} + \frac{1}{s^2} = \frac{s^2+s-1}{s^2(s-1)}$, so $Y = \frac{s^2+s-1}{s^2(s-1)(s^3-4)}$.

27 (a). $s^3Y + 4sY = F$, so $\Phi = \frac{1}{\left(s^3+4s\right)}$.

27 (b). $F = \frac{s}{s^2+4}$, so $Y = \frac{s}{\left(s^2+4\right)\left(s^3+4s\right)} = \frac{1}{\left(s^2+4\right)^2}$.

28. $y'' + by' + cy = f \Rightarrow s^2Y - sy(0) - y'(0) + b(sY - y(0)) + cY = F$. Therefore,

$(s^2+bs+c)Y - sy_0 - y_0' - by_0 = F \Rightarrow Y = \frac{sy_0 + y_0' + by_0}{s^2+bs+c} + \frac{F(s)}{s^2+bs+c}$. If

$f(t) = h(t),\ F(s) = \frac{1}{s} \Rightarrow Y(s) = \frac{s^2y_0 + s(y_0' + by_0) + 1}{s^3+bs^2+cs} = \frac{s^2+2s+1}{s^3+3s^2+2s}$. Therefore,

$b = 3,\ c = 2,\ y_0 = 1,\ y_0' + by_0 = y_0' + 3 = 2 \Rightarrow y_0' = -1$.

29. $Y = \frac{sy_0 + y_0' + by_0}{s^2+bs+c} + \frac{F(s)}{s^2+bs+c}$. If $f(t) = e^{-t},\ F(s) = \frac{1}{s+1}$ and

$\frac{(s+1)\left(sy_0 + y_0' + by_0\right)+1}{(s+1)\left(s^2+bs+c\right)} = \frac{s^2+s+1}{(s+1)\left(s^2+4\right)}$. Therefore,

$b = 0,\ c = 4,$ and $(s+1)\left(sy_0 + y_0'\right) + 1 = s^2y_0 + sy_0' + sy_0 + y_0' = s^2 + s + 1$. Finally, $y_0 = 1,\ y_0' = 0$.

Section 7.5

1. $$\mathcal{L}\left\{\begin{bmatrix}\cos t\\ t\\ te^t\end{bmatrix}\right\} = \begin{bmatrix}\frac{s}{s^2+1}\\ \frac{1}{s^2}\\ \frac{1}{(s-1)^2}\end{bmatrix}$$

2. $$\mathcal{L}\left\{\frac{d}{dt}\begin{bmatrix}e^{-t}\cos 2t\\ 0\\ t+e^t\end{bmatrix}\right\} = s\mathcal{L}\left\{\frac{d}{dt}\begin{bmatrix}e^{-t}\cos 2t\\ 0\\ t+e^t\end{bmatrix}\right\} - \begin{bmatrix}1\\0\\1\end{bmatrix} = s\begin{bmatrix}\frac{s+1}{(s+1)^2+4}\\ 0\\ \frac{1}{s^2}+\frac{1}{s-1}\end{bmatrix} - \begin{bmatrix}1\\0\\1\end{bmatrix} = \begin{bmatrix}\frac{s(s+1)}{(s+1)^2+4}-1\\ 0\\ \frac{1}{s}+\frac{s}{s-1}-1\end{bmatrix}$$

3. $$\mathcal{L}\left\{\begin{bmatrix} 2t-h(t-2) \\ 2h(t-2) \end{bmatrix}\right\} = \begin{bmatrix} \dfrac{2}{s^2}-\dfrac{e^{-2s}}{s} \\ \dfrac{2e^{-2s}}{s} \end{bmatrix}$$

4. $$\mathcal{L}\left\{\int_0^t \begin{bmatrix} 1 \\ \lambda \\ e^{-\lambda} \end{bmatrix} d\lambda\right\} = \frac{1}{s}\mathcal{L}\left\{\begin{bmatrix} 1 \\ t \\ e^{-t} \end{bmatrix}\right\} = \begin{bmatrix} \dfrac{1}{s^2} \\ \dfrac{1}{s^3} \\ \dfrac{1}{s(s+1)} \end{bmatrix}$$

5. $$\mathcal{L}\left\{\begin{bmatrix} \sin(t-1)h(t-1) \\ e^{t-1}-2t \end{bmatrix}\right\} = \begin{bmatrix} \dfrac{e^{-s}}{s^2+1} \\ \dfrac{e^{-1}}{s-1}-\dfrac{2}{s^2} \end{bmatrix}$$

6. $$\mathcal{L}^{-1}\left\{\begin{bmatrix} 1/s \\ \dfrac{2}{s^2+2s+2} \\ \dfrac{1}{s^2+s} \end{bmatrix}\right\};\ \frac{2}{(s+1)^2+1} = \mathcal{L}\{[2e^{-t}\sin t]\};\ \frac{1}{s(s+1)} = \frac{1}{s}-\frac{1}{s+1}.\ \text{Therefore, } \begin{bmatrix} 1 \\ 2e^{-t}\sin t \\ 1-e^{-t} \end{bmatrix}.$$

7. $$\mathcal{L}^{-1}\left\{\begin{bmatrix} e^{-s}\left(\dfrac{1}{s}-\dfrac{1}{s^2+1}\right) \\ \dfrac{2e^{-s}}{s^2+1} \end{bmatrix}\right\} = \begin{bmatrix} (1-\sin(t-1))h(t-1) \\ 2\sin(t-1)h(t-1) \end{bmatrix}$$

8. $$\mathcal{L}^{-1}\{\mathbf{Y}(s)\} = \begin{bmatrix} t^3-e^{2t}+2\sin t \\ 2t^3+3\sin t \\ t^3-2e^{2t}+\sin t \end{bmatrix}$$

9. $$s\mathbf{Y}-\begin{bmatrix} 5 \\ 6 \end{bmatrix} = A\mathbf{Y},\ \text{so } \begin{bmatrix} s-5 & 4 \\ -5 & s+4 \end{bmatrix}\mathbf{Y} = \begin{bmatrix} 5 \\ 6 \end{bmatrix}.\ \text{Thus } \mathbf{Y} = \frac{1}{s^2-s}\begin{bmatrix} s+4 & -4 \\ 5 & s-5 \end{bmatrix}\begin{bmatrix} 5 \\ 6 \end{bmatrix} = \begin{bmatrix} \dfrac{5s-4}{s^2-s} \\ \dfrac{6s-5}{s^2-s} \end{bmatrix},\ \text{and}$$

since $\dfrac{5s-4}{s^2-s} = \dfrac{A}{s}+\dfrac{B}{s-1} \Rightarrow A=4,\ B=1$ and $\dfrac{6s-5}{s^2-s} = \dfrac{5}{s}+\dfrac{1}{s-1}$, we have $\mathbf{y}(t) = \begin{bmatrix} 4+e^t \\ 5+e^t \end{bmatrix}$.

10. $$s\mathbf{Y}-\mathbf{0} = A\mathbf{Y}+\begin{bmatrix} 0 \\ 1/s \end{bmatrix} = \begin{bmatrix} s-5 & 4 \\ -5 & s+4 \end{bmatrix}\mathbf{Y}+\begin{bmatrix} 0 \\ 1/s \end{bmatrix} \Rightarrow \mathbf{Y} = \frac{1}{s^2-s}\begin{bmatrix} s+4 & -4 \\ 5 & s-5 \end{bmatrix}\begin{bmatrix} 0 \\ 1/s \end{bmatrix} = \begin{bmatrix} \dfrac{-4}{s^2(s-1)} \\ \dfrac{s+4}{s^2(s-1)} \end{bmatrix},\ \text{and}$$

since $\dfrac{-4}{s^2(s-1)} = \dfrac{A}{s^2}+\dfrac{B}{s}+\dfrac{C}{s-1} \Rightarrow A=4,\ B=4,\ C=-4.$ then $\dfrac{-4}{s^2(s-1)} = \dfrac{4}{s^2}+\dfrac{4}{s}-\dfrac{4}{s-1}$

and $\frac{s-5}{s^2(s-1)}=\frac{-5}{s^2(s-1)}+\frac{1}{s(s-1)}=\frac{5}{s^2}+\frac{5}{s}-\frac{5}{s-1}+\frac{1}{s-1}-\frac{1}{s}=\frac{5}{s^2}+\frac{4}{s}-\frac{4}{s-1}$. Therefore,

$$\mathbf{Y}(s)=\begin{bmatrix}\frac{4}{s^2}+\frac{4}{s}-\frac{4}{s-1}\\ \frac{5}{s^2}+\frac{4}{s}-\frac{4}{s-1}\end{bmatrix}\Rightarrow \mathbf{y}(t)=\begin{bmatrix}4t+4-4e^t\\ 5t+4-4e^t\end{bmatrix}.$$

11. $s\mathbf{Y}=A\mathbf{Y}+\begin{bmatrix}\frac{1}{s^2}\\ \frac{1}{s}\end{bmatrix}$, so $\begin{bmatrix}s-5 & 4\\ -3 & s+2\end{bmatrix}\mathbf{Y}=\begin{bmatrix}\frac{1}{s^2}\\ \frac{1}{s}\end{bmatrix}$. Thus

$$\mathbf{Y}=\frac{1}{s^2-3s+2}\begin{bmatrix}s+2 & -4\\ 3 & s-5\end{bmatrix}\begin{bmatrix}\frac{1}{s^2}\\ \frac{1}{s}\end{bmatrix}=\frac{1}{s^2(s-1)(s-2)}\begin{bmatrix}-3s+2\\ s^2-5s+3\end{bmatrix}.$$

$Y_1=\frac{-3s+2}{s^2(s-1)(s-2)}=\frac{A}{s^2}+\frac{B}{s}+\frac{C}{s-1}+\frac{D}{s-2}\Rightarrow A=1,\ B=0,\ C=1,\ D=-1$, so $y_1=t+e^t-e^{2t}$.

$Y_2=\frac{s^2-5s+3}{s^2(s-1)(s-2)}=\frac{A}{s^2}+\frac{B}{s}+\frac{C}{s-1}+\frac{D}{s-2}\Rightarrow A=\frac{3}{2},\ B=-\frac{1}{4},\ C=1,\ D=-\frac{3}{4}$, so

$y_2=\frac{3}{2}t-\frac{1}{4}+e^t-\frac{3}{4}e^{2t}$. Finally, we have $\mathbf{y}(t)=\begin{bmatrix}t+e^t-e^{2t}\\ \frac{3}{2}t-\frac{1}{4}+e^t-\frac{3}{4}e^{2t}\end{bmatrix}$.

12. From 11, $\mathbf{Y}=\frac{1}{(s-1)(s-2)}\begin{bmatrix}s+2 & -4\\ 3 & s-5\end{bmatrix}\begin{bmatrix}3\\ 2\end{bmatrix}=\frac{1}{(s-1)(s-2)}\begin{bmatrix}3s-2\\ 2s-1\end{bmatrix}$.

$Y_1=\frac{3s-2}{(s-1)(s-2)}=\frac{A}{s-1}+\frac{B}{s-2}\Rightarrow A=-1,\ B=4$, so $y_1=-e^t+4e^{2t}$.

$Y_2=\frac{2s-1}{(s-1)(s-2)}=\frac{A}{s-1}+\frac{B}{s-2}\Rightarrow A=-1,\ B=3$, so $y_2=-e^t+3e^{2t}$. Finally, we have

$$\mathbf{y}(t)=\begin{bmatrix}-e^t+4e^{2t}\\ -e^t+3e^{2t}\end{bmatrix}.$$

13. $s\mathbf{Y}=A\mathbf{Y}+\begin{bmatrix}2\\ 0\end{bmatrix}$, so $\begin{bmatrix}s-1 & -4\\ 1 & s-1\end{bmatrix}\mathbf{Y}=\begin{bmatrix}2\\ 0\end{bmatrix}$. Thus $\mathbf{Y}=\frac{1}{(s-1)^2+4}\begin{bmatrix}s-1 & 4\\ -1 & s-1\end{bmatrix}\begin{bmatrix}2\\ 0\end{bmatrix}$.

$Y_1=\frac{2(s-1)}{(s-1)^2+4}$, so $y_1=2e^t\cos 2t$. $Y_2=\frac{-2}{(s-1)^2+4}$, so $y_2=-e^t\sin 2t$. Finally, we have

$$\mathbf{y}(t)=\begin{bmatrix}2e^t\cos 2t\\ -e^t\sin 2t\end{bmatrix}.$$

14. $s\mathbf{Y}-\begin{bmatrix}3\\0\end{bmatrix}=A\mathbf{Y}+\begin{bmatrix}0\\ \frac{3}{s-1}\end{bmatrix}$, so $\begin{bmatrix}s-1 & -4\\ 1 & s-1\end{bmatrix}\mathbf{Y}=\begin{bmatrix}0\\ \frac{3}{s-1}\end{bmatrix}$. Thus $\mathbf{Y}=\frac{1}{(s-1)^2+4}\begin{bmatrix}s-1 & 4\\ -1 & s-1\end{bmatrix}\begin{bmatrix}0\\ \frac{3}{s-1}\end{bmatrix}$.

$Y_1=\frac{3(s-1)+\frac{12}{s-1}}{(s-1)^2+4}=\frac{3(s-1)}{(s-1)^2+4}+\frac{12}{(s-1)\left[(s-1)^2+4\right]}$. For

$\frac{12}{(s-1)\left[(s-1)^2+4\right]}=\frac{A}{s-1}+\frac{B(s-1)+C}{(s-1)^2+4}$, $A=3,\ B=-3,\ C=0$. Therefore,

$Y_1=\frac{3(s-1)}{(s-1)^2+4}+\frac{3}{s-1}-\frac{3(s-1)}{(s-1)^2+4}=\frac{3}{s-1}$, so $y_1=3e^t$. $Y_2=0$, so $y_2=0$. Finally, we have

$\mathbf{y}(t)=\begin{bmatrix}3e^t\\0\end{bmatrix}$.

15. Letting $t=\tau+1;\ \tau=t-1$, we have $\frac{d\mathbf{y}}{d\tau}=\begin{bmatrix}6 & -3\\ 8 & -5\end{bmatrix}\mathbf{y}$, $\mathbf{y}|_{\tau=0}=\begin{bmatrix}5\\10\end{bmatrix}$. Then $s\mathbf{Y}=A\mathbf{Y}+\begin{bmatrix}5\\10\end{bmatrix}$, so

$\begin{bmatrix}s-6 & 3\\ -8 & s+5\end{bmatrix}\mathbf{Y}=\begin{bmatrix}5\\10\end{bmatrix}$. Thus $\mathbf{Y}=\frac{1}{s^2-s-6}\begin{bmatrix}s+5 & -3\\ 8 & s-6\end{bmatrix}\begin{bmatrix}5\\10\end{bmatrix}$.

$Y_1=\frac{5s-5}{(s-3)(s+2)}=\frac{A}{s+2}+\frac{B}{s-3}\Rightarrow A=3,\ B=2$, so $y_1=3e^{-2\tau}+2e^{3\tau}$.

$Y_2=\frac{10s-20}{(s-3)(s+2)}=\frac{A}{s+2}+\frac{B}{s-3}\Rightarrow A=8,\ B=2$, so $y_2=8e^{-2\tau}+2e^{3\tau}$.Finally, we have

$\mathbf{y}(t)=\begin{bmatrix}3e^{-2(t-1)}+2e^{3(t-1)}\\ 8e^{-2(t-1)}+2e^{3(t-1)}\end{bmatrix}$.

16. $s^2\mathbf{Y}-s\begin{bmatrix}1\\0\end{bmatrix}-\begin{bmatrix}0\\1\end{bmatrix}=A\mathbf{Y}$, so $\begin{bmatrix}s^2+3 & 2\\ -4 & s^2-3\end{bmatrix}\mathbf{Y}=\begin{bmatrix}s\\1\end{bmatrix}$. Thus $\mathbf{Y}=\frac{1}{s^4-1}\begin{bmatrix}s^2-3 & -2\\ 4 & s^2+3\end{bmatrix}\begin{bmatrix}s\\1\end{bmatrix}$.

$Y_1=\frac{s^3-3s-2}{(s-1)(s+1)(s^2+1)}=\frac{A}{s+1}+\frac{B}{s-1}+\frac{Cs+D}{s^2+1}\Rightarrow A=0,\ B=-1,\ C=2,\ D=1$, so

$y_1=-e^t+2\cos t+\sin t$.

$Y_2=\frac{s^2+4s+3}{(s-1)(s+1)(s^2+1)}=\frac{s+3}{(s-1)(s^2+1)}=\frac{A}{s-1}+\frac{Bs+C}{s^2+1}\Rightarrow A=2,\ B=-2,\ C=-1$, so

$y_2=2e^t-2\cos t-\sin t$. Finally, we have $\mathbf{y}(t)=\begin{bmatrix}-e^t+2\cos t+\sin t\\ 2e^t-2\cos t-\sin t\end{bmatrix}$.

17. $s^2\mathbf{Y} = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}\mathbf{Y} + \begin{bmatrix} \frac{1}{s^2} \\ \frac{1}{s} \end{bmatrix}$, so $\begin{bmatrix} s^2-1 & 1 \\ -1 & s^2+1 \end{bmatrix}\mathbf{Y} = \begin{bmatrix} \frac{1}{s^2} \\ \frac{1}{s} \end{bmatrix}$. Thus $\mathbf{Y} = \frac{1}{s^4}\begin{bmatrix} s^2+1 & -1 \\ 1 & s^2-1 \end{bmatrix}\begin{bmatrix} \frac{1}{s^2} \\ \frac{1}{s} \end{bmatrix}$.

$Y_1 = \frac{1}{s^4}\left(1 + \frac{1}{s^2} - \frac{1}{s}\right)$, so $y_1 = \frac{1}{3!}t^3 + \frac{1}{5!}t^5 - \frac{1}{4!}t^4 = \frac{t^5}{120} - \frac{t^4}{24} + \frac{t^3}{6}$. $Y_2 = \frac{1}{s^4}\left(\frac{1}{s^2} + s - \frac{1}{s}\right)$, so

$y_2 = \frac{1}{5!}t^5 + \frac{1}{2!}t^2 - \frac{1}{4!}t^4 = \frac{t^5}{120} - \frac{t^4}{24} + \frac{t^2}{2}$. Finally, we have

$$\mathbf{y}(t) = \begin{bmatrix} \frac{t^5}{120} - \frac{t^4}{24} + \frac{t^3}{6} \\ \frac{t^5}{120} - \frac{t^4}{24} + \frac{t^2}{2} \end{bmatrix} = \frac{1}{120}\begin{bmatrix} t^5 - 5t^4 + 20t^3 \\ t^5 - 5t^4 + 60t^2 \end{bmatrix}.$$

18. $s^2\mathbf{Y} - s\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix}\mathbf{Y} + \begin{bmatrix} \frac{2}{s} \\ \frac{1}{s} \end{bmatrix}$, so $\begin{bmatrix} s^2-1 & 1 \\ -1 & s^2+1 \end{bmatrix}\mathbf{Y} = \begin{bmatrix} \frac{2}{s} \\ \frac{1}{s} + s \end{bmatrix}$. Thus

$\mathbf{Y} = \frac{1}{s^4}\begin{bmatrix} s^2+1 & -1 \\ 1 & s^2-1 \end{bmatrix}\begin{bmatrix} \frac{2}{s} \\ \frac{1}{s} + s \end{bmatrix}$. $Y_1 = \frac{1}{s^4}\left(2s + \frac{2}{s} - \frac{1}{s} - s\right) = \frac{1}{s^3} + \frac{1}{s^5}$, so $y_1 = \frac{t^4}{4!} + \frac{t^2}{2!} = \frac{t^4}{24} + \frac{t^2}{2}$.

$Y_2 = \frac{1}{s^4}\left(\frac{2}{s} + s - \frac{1}{s} + s^3 - s\right) = \frac{1}{s} + \frac{1}{s^5}$, so $y_2 = 1 + \frac{t^4}{24}$. Finally, we have $\mathbf{y}(t) = \begin{bmatrix} \frac{t^2}{2} + \frac{t^4}{24} \\ 1 + \frac{t^4}{24} \end{bmatrix}$.

19. $s\mathbf{Y} - \mathbf{y}(0) = A\mathbf{Y}$, so $\begin{bmatrix} s-6 & -5 & 0 \\ 7 & s+6 & 0 \\ 0 & 0 & s+2 \end{bmatrix}\mathbf{Y} = \begin{bmatrix} 2 \\ -4 \\ -1 \end{bmatrix}$. Thus

$$\mathbf{Y} = \begin{bmatrix} \frac{s+6}{s^2-1} & \frac{5}{s^2-1} & 0 \\ \frac{-7}{s^2-1} & \frac{s-6}{s^2-1} & 0 \\ 0 & 0 & \frac{1}{s+2} \end{bmatrix}\begin{bmatrix} 2 \\ -4 \\ -1 \end{bmatrix} = \begin{bmatrix} \frac{2s-8}{s^2-1} \\ \frac{-4s+10}{s^2-1} \\ \frac{-1}{s+2} \end{bmatrix}.$$

$Y_1 = \frac{2s-8}{(s+1)(s-1)} = \frac{A}{s+1} + \frac{B}{s-1} \Rightarrow A = 5,\ B = -3$, so $y_1 = 5e^{-t} - 3e^t$.

$Y_2 = \frac{-4s+10}{(s+1)(s-1)} = \frac{A}{s+1} + \frac{B}{s-1} \Rightarrow A = -7,\ B = 3$, so $y_2 = -7e^{-t} + 3e^t$.

$Y_3 = \frac{-1}{s+2}$, so $y_3 = -e^{-2t}$. Finally, we have $\mathbf{y}(t) = \begin{bmatrix} 5e^{-t} - 3e^{t} \\ -7e^{-t} + 3e^{t} \\ -e^{-2t} \end{bmatrix}$.

20. $s\mathbf{Y} = A\mathbf{Y} + \begin{bmatrix} \frac{1}{s-1} \\ \frac{1}{s} \\ -\frac{2}{s^2} \end{bmatrix}$, so $\begin{bmatrix} s-1 & 0 & 0 \\ 0 & s+1 & -1 \\ 0 & 0 & s-2 \end{bmatrix}\mathbf{Y} = \begin{bmatrix} \frac{1}{s-1} \\ \frac{1}{s} \\ -\frac{2}{s^2} \end{bmatrix}$. Thus

$\mathbf{Y} = \begin{bmatrix} \frac{1}{s-1} & 0 & 0 \\ 0 & \frac{1}{s+1} & \frac{1}{(s+1)(s-2)} \\ 0 & 0 & \frac{1}{s-2} \end{bmatrix}\begin{bmatrix} \frac{1}{s-1} \\ \frac{1}{s} \\ -\frac{2}{s^2} \end{bmatrix}$. $Y_1 = \frac{1}{(s-1)^2}$, so $y_1 = te^{t}$.

$Y_2 = \frac{s(s-2)-2}{s^2(s+1)(s-2)} = \frac{A}{s^2} + \frac{B}{s} + \frac{C}{s+1} + \frac{D}{s-2} \Rightarrow A = 1,\ B = \frac{1}{2},\ C = -\frac{1}{3},\ D = -\frac{1}{6}$, so

$y_2 = t + \frac{1}{2} - \frac{1}{3}e^{-t} - \frac{1}{6}e^{2t}$.

$Y_3 = \frac{-2}{s^2(s-2)} = \frac{A}{s^2} + \frac{B}{s} + \frac{C}{s-2} \Rightarrow A = 1,\ B = \frac{1}{2},\ C = -\frac{1}{2}$, so $y_3 = t + \frac{1}{2} - \frac{1}{2}e^{2t}$. Finally, we have

$$\mathbf{y}(t) = \begin{bmatrix} te^{t} \\ t + \frac{1}{2} - \frac{1}{3}e^{-t} - \frac{1}{6}e^{2t} \\ t + \frac{1}{2} - \frac{1}{2}e^{2t} \end{bmatrix}.$$

21 (a). $s^2 - 9s + 18 = (s-3)(s-6) = 0 \Rightarrow \lambda = 3,\ 6$.

21 (b). $s\mathbf{Y} - \mathbf{y}(0) = A\mathbf{Y} \Rightarrow \mathbf{Y} = (sI - A)^{-1}\mathbf{y}_0$. Then $-A^{-1} = \frac{1}{18}\begin{bmatrix} -2 & -1 \\ 4 & -7 \end{bmatrix} \Rightarrow A^{-1} = \frac{1}{18}\begin{bmatrix} 2 & 1 \\ -4 & 7 \end{bmatrix}$, and

$\det A^{-1} = \left(\frac{1}{18}\right)^2 \cdot 18 = \frac{1}{18}$. Thus $A = \left(A^{-1}\right)^{-1} = \frac{18}{18}\begin{bmatrix} 7 & -1 \\ 4 & 2 \end{bmatrix} = \begin{bmatrix} 7 & -1 \\ 4 & 2 \end{bmatrix}$.

22 (a). $s\mathbf{Y}_1 = A\mathbf{Y}_1 + \mathbf{G}(s),\ s\mathbf{Y}_2 = A\mathbf{Y}_2 + \mathbf{Y}_1 \Rightarrow \mathbf{Y}_1 = (sI - A)^{-1}\mathbf{G},\ \mathbf{Y}_2 = (sI - A)^{-1}\mathbf{Y}_1$ and

$\mathbf{Y}_2 = (sI - A)^{-2}\mathbf{G}(s)\ \ \therefore \Omega(s) = (sI - A)^{-2}$.

22 (b). $(sI-A)=\begin{bmatrix} s-1 & 1 \\ -1 & s+1 \end{bmatrix} \Rightarrow (sI-A)^{-1}=\dfrac{1}{s^2}\begin{bmatrix} s+1 & -1 \\ 1 & s-1 \end{bmatrix}$ and

$$\Omega(s)=(sI-A)^{-2}=\frac{1}{s^4}\begin{bmatrix} s+1 & -1 \\ 1 & s-1 \end{bmatrix}\begin{bmatrix} s+1 & -1 \\ 1 & s-1 \end{bmatrix}=\frac{1}{s^4}\begin{bmatrix} s^2+2s & -2s \\ 2s & s^2-2s \end{bmatrix}.$$

$$\mathbf{G}(s)=\begin{bmatrix} \frac{1}{s} \\ \frac{1}{s^2} \end{bmatrix} \Rightarrow \mathbf{Y}_2(s)=\frac{1}{s^4}\begin{bmatrix} s+2-\frac{2}{s} \\ 2+1-\frac{2}{s} \end{bmatrix}=\begin{bmatrix} -\frac{2}{s^5}+\frac{2}{s^4}+\frac{1}{s^3} \\ -\frac{2}{s^5}+\frac{3}{s^4} \end{bmatrix} \text{ Therefore,}$$

$$\mathbf{Y}_2(t)=\begin{bmatrix} -\frac{2t^4}{4!}+\frac{2t^3}{3!}+\frac{t^2}{2!} \\ -\frac{2t^4}{4!}+\frac{3t^3}{3!} \end{bmatrix}=\begin{bmatrix} -\frac{t^4}{12}+\frac{t^3}{3}+\frac{t^2}{2} \\ -\frac{t^4}{12}+\frac{t^3}{2} \end{bmatrix}.$$

23. $\mathbf{Y}_1=\dfrac{1}{s-3}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\mathbf{Y}_2=\begin{bmatrix} \frac{3}{s+2}-\frac{3}{s-3} \\ \frac{8}{s+2}-\frac{3}{s-3} \end{bmatrix}=\begin{bmatrix} \frac{-15}{(s+2)(s-3)} \\ \frac{5s-30}{(s+2)(s-3)} \end{bmatrix}$. Therefore,

$\begin{bmatrix} \frac{1}{s-3} & \frac{-15}{(s+2)(s-3)} \\ \frac{1}{s-3} & \frac{5s-30}{(s+2)(s-3)} \end{bmatrix}=(sI-A)^{-1}\begin{bmatrix} 1 & 0 \\ 1 & 5 \end{bmatrix}$, and so $\begin{bmatrix} \frac{-1}{3} & \frac{5}{2} \\ \frac{-1}{3} & 5 \end{bmatrix}=-A^{-1}\begin{bmatrix} 1 & 0 \\ 1 & 5 \end{bmatrix}$. Thus

$$A=-\begin{bmatrix} 1 & 0 \\ 1 & 5 \end{bmatrix}\begin{bmatrix} \frac{-1}{3} & \frac{5}{2} \\ \frac{-1}{3} & 5 \end{bmatrix}^{-1}=\frac{6}{5}\begin{bmatrix} 5 & \frac{-5}{2} \\ \frac{20}{3} & \frac{-25}{6} \end{bmatrix}=\begin{bmatrix} 6 & -3 \\ 8 & -5 \end{bmatrix}.$$

24. $\mathbf{Y}_1=\begin{bmatrix} \frac{1}{(s+2)^2} \\ \frac{1}{s+2} \end{bmatrix}$, $\mathbf{Y}_2=\begin{bmatrix} \frac{1}{s} \\ 0 \end{bmatrix}$, $\mathbf{G}(s)=\begin{bmatrix} \frac{2}{s} \\ 0 \end{bmatrix}$. Therefore, $\begin{bmatrix} \frac{1}{(s+2)^2} & \frac{1}{s} \\ \frac{1}{s+2} & 0 \end{bmatrix}=(sI-A)^{-1}\begin{bmatrix} 0 & 1+\frac{2}{s} \\ 1 & 0 \end{bmatrix}$, and so

$$(sI-A)=\begin{bmatrix} 0 & \frac{s+2}{s} \\ 1 & 0 \end{bmatrix}\begin{bmatrix} \frac{1}{(s+2)^2} & \frac{1}{s} \\ \frac{1}{s+2} & 0 \end{bmatrix}^{-1}=-s(s+2)\begin{bmatrix} 0 & \frac{s+2}{s} \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & -\frac{1}{s} \\ -\frac{1}{s+2} & \frac{1}{(s+2)^2} \end{bmatrix}$$

$$=-s(s+2)\begin{bmatrix} -\frac{1}{s} & \frac{1}{s(s+2)} \\ 0 & -\frac{1}{s} \end{bmatrix}=\begin{bmatrix} s+2 & -1 \\ 0 & s+2 \end{bmatrix}=sI-\begin{bmatrix} -2 & 1 \\ 0 & -2 \end{bmatrix};\ A=\begin{bmatrix} -2 & 1 \\ 0 & -2 \end{bmatrix}.$$

25 (a). $-V_1(s)+R_1I_1+LsI_1+R_2(I_1-I_2)=0$, $R_2(I_2-I_1)+\frac{1}{Cs}I_2+R_3I_2+V_2(s)=0$.

$$\begin{bmatrix} R_1+R_2+sL & -R_2 \\ -R_2 & R_2+R_3+\frac{1}{Cs}\end{bmatrix}\begin{bmatrix} I_1 \\ I_2\end{bmatrix}=\begin{bmatrix} V_1 \\ -V_2\end{bmatrix}, \text{ so}$$

$$\begin{bmatrix} I_1 \\ I_2\end{bmatrix}=\frac{1}{(R_1+R_2+sL)\left(R_2+R_3+\frac{1}{Cs}\right)-R_2^{\,2}}\begin{bmatrix} R_2+R_3+\frac{1}{Cs} & R_2 \\ R_2 & R_1+R_2+sL\end{bmatrix}\begin{bmatrix} V_1 \\ -V_2\end{bmatrix}.$$

25 (b). $$\begin{bmatrix} I_1 \\ I_2\end{bmatrix}=\frac{1}{(2+s)\left(2+\frac{1}{s}\right)-1}\begin{bmatrix} 2+\frac{1}{s} & 1 \\ 1 & 2+s\end{bmatrix}\begin{bmatrix} \frac{1}{(s+1)^2} \\ \frac{-1}{(s+1)^2}\end{bmatrix}=\frac{(s+1)^{-2}}{4+2s+\frac{2}{s}}\begin{bmatrix} \frac{s+1}{s} \\ -(s+1)\end{bmatrix}=\frac{s}{2(s+1)^4}\begin{bmatrix} \frac{s+1}{s} \\ -(s+1)\end{bmatrix}.$$

$I_1(s)=\frac{1}{2(s+1)^3}\Rightarrow i_1(t)=\frac{t^2}{4}e^{-t}$ and

$$I_2(s)=\frac{-s}{2(s+1)^3}=\frac{-(s+1)+1}{2(s+1)^3}=\frac{-1}{2(s+1)^2}+\frac{1}{2(s+1)^3}\Rightarrow i_2(t)=-\frac{t}{2}e^{-t}+\frac{t^2}{4}e^{-t}.$$

Section 7.6

1. First, let $\sigma=t-\lambda$, and differentiation yields $d\sigma=-d\lambda$. Then,

$$f*g=\int_0^t f(t-\lambda)g(\lambda)d\lambda=\int_t^0 f(\sigma)g(t-\sigma)(-d\sigma)=\int_0^t g(t-\sigma)f(\sigma)d\sigma=g*f.$$

2 (a). $f*g=\int_0^t h(t-\lambda)h(\lambda)d\lambda=\int_0^t 1d\lambda=t$

2 (b). $F=G=\frac{1}{s}$, so $f*g=\mathcal{L}^{-1}\left(\frac{1}{s^2}\right)=t$

3 (a). $f*g=\int_0^t (t-\lambda)\lambda^2d\lambda=\left(t\frac{\lambda^3}{3}-\frac{\lambda^4}{4}\right)\Big|_0^t=t^4\left(\frac{1}{3}-\frac{1}{4}\right)=\frac{t^4}{12}$

3 (b). $F=\frac{1}{s^2}$, $G=\frac{2}{s^3}$, so $f*g=\mathcal{L}^{-1}\left\{\frac{2}{s^5}\right\}=\frac{2t^4}{4!}=\frac{t^4}{12}$

4 (a). $f*g=\int_0^t e^{(t-\lambda)}e^{-2\lambda}d\lambda=\left(e^t\frac{e^{-3\lambda}}{-3}\right)\Big|_0^t=\frac{e^t}{3}\left(1-e^{-3t}\right)=\frac{e^t-e^{-2t}}{3}$

4 (b). $F=\frac{1}{s-1}$, $G=\frac{1}{s+2}$, so $f*g=\mathcal{L}^{-1}\left\{\frac{1}{(s-1)(s+2)}\right\}=\mathcal{L}^{-1}\left\{\frac{1}{3}\left(\frac{1}{s-1}-\frac{1}{s+2}\right)\right\}=\frac{1}{3}\left(e^t-e^{-2t}\right)$

5 (a). $f * g = \int_0^t (t-\lambda)\sin\lambda d\lambda = -t\cos\lambda\Big|_0^t - \int_0^t \lambda\sin\lambda d\lambda$

$= -t(\cos t - 1) - (-\lambda\cos\lambda + \sin\lambda)\Big|_0^t = -t\cos t + t + t\cos t - \sin t = t - \sin t$

5 (b). $F = \frac{1}{s^2},\ G = \frac{1}{s^2+1}$, so $f * g = \mathcal{L}^{-1}\left\{\frac{1}{s^2(s^2+1)}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{s^2} - \frac{1}{s^2+1}\right\} = t - \sin t$

6 (a). $f * g = \int_0^t \sin(t-\lambda)\cos\lambda d\lambda = \frac{1}{2}\int_0^t [\sin t + \sin(t-2\lambda)]d\lambda = \frac{1}{2}\left[t\sin t + \int_t^{-t}(\sin\sigma)\left(-\frac{1}{2}d\sigma\right)\right]$

$= \frac{1}{2}\left[t\sin t + \frac{1}{2}\cos\sigma\Big|_{-t}^{t}\right] = \frac{t}{2}\sin t$, where $\sigma = t - 2\lambda$.

6 (b). $F = \frac{1}{s^2+1},\ G = \frac{s}{s^2+1}$, so $f * g = \mathcal{L}^{-1}\left\{\frac{s}{(s^2+1)^2}\right\} = \frac{t}{2}\sin t$

7 (a). $$f * g = \int_0^t (t-\lambda)[h(\lambda) - h(\lambda-1)]d\lambda = \begin{cases} \int_0^t (t-\lambda)d\lambda = -\frac{(t-\lambda)^2}{2}\Big|_0^t = \frac{t^2}{2}, & 0 \le t \le 1 \\ \int_0^1 (t-\lambda)d\lambda = -\frac{(t-\lambda)^2}{2}\Big|_0^1 = \frac{t^2}{2} - \frac{(t-1)^2}{2}, & 1 \le t < \infty \end{cases}$$

7 (b). $F = \frac{1}{s^2},\ G = \frac{(1-e^{-s})}{s}$, so $f * g = \mathcal{L}^{-1}\left\{\frac{1}{s^3} - \frac{e^{-s}}{s^3}\right\} = \frac{t^2}{2} - \frac{(t-1)^2}{2}h(t-1)$

8. $$P * \mathbf{y} = \mathcal{L}^{-1}\left\{\begin{bmatrix} \frac{1}{s} & \frac{1}{s-1} \\ 0 & \frac{1}{s^2} \end{bmatrix}\begin{bmatrix} \frac{1}{s} \\ \frac{1}{s+1} \end{bmatrix}\right\} = \mathcal{L}^{-1}\left\{\begin{bmatrix} \frac{1}{s^2} + \frac{1}{(s-1)(s+1)} \\ \frac{1}{s^2(s+1)} \end{bmatrix}\right\}.$$

$\frac{1}{(s-1)(s+1)} = \frac{1}{2}\left(\frac{1}{s-1} - \frac{1}{s+1}\right)$; $\frac{1}{s^2(s+1)} = \frac{A}{s^2} + \frac{B}{s} + \frac{C}{s+1} \Rightarrow A = 1,\ B = -1,\ C = 1$. Therefore,

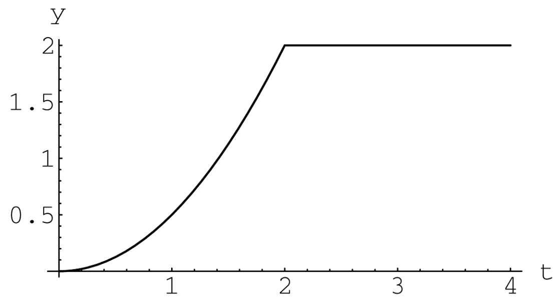
$$P * \mathbf{y} = \begin{bmatrix} t + \frac{1}{2}e^t - \frac{1}{2}e^{-t} \\ t - 1 + e^{-t} \end{bmatrix}$$

9. $t * \begin{bmatrix} t \\ \cos t \end{bmatrix} = \mathcal{L}^{-1}\left\{\begin{bmatrix} \frac{1}{s^4} \\ \frac{s}{s^2(s^2+1)} \end{bmatrix}\right\}$. $\mathcal{L}^{-1}\left\{\frac{1}{s^4}\right\} = \frac{t^3}{6}$. Then we have

$\frac{1}{s(s^2+1)} = \frac{A}{s} + \frac{Bs+C}{s^2+1} \Rightarrow A = 1,\ B = -1,\ C = 0$. $\mathcal{L}^{-1}\left\{\frac{1}{s} - \frac{s}{s^2+1}\right\} = 1 - \cos t$, so $t*\begin{bmatrix} t \\ \cos t \end{bmatrix} = \begin{bmatrix} \frac{t^3}{6} \\ 1-\cos t \end{bmatrix}$.

10. $g(t) = th(t) - (t-2)h(t-2) - 2h(t-2)$, $F(s) = \frac{1}{s}$, $G(s) = \frac{1}{s^2} - \frac{e^{-2s}}{s^2} - \frac{2e^{-2s}}{s}$,

so $FG = \frac{1}{s^3} - \frac{e^{-2s}}{s^3} - \frac{2e^{-2s}}{s^2}$. Then $f*g = \frac{t^2}{2} - \frac{(t-2)^2}{2}h(t-2) - 2(t-2)h(t-2)$.

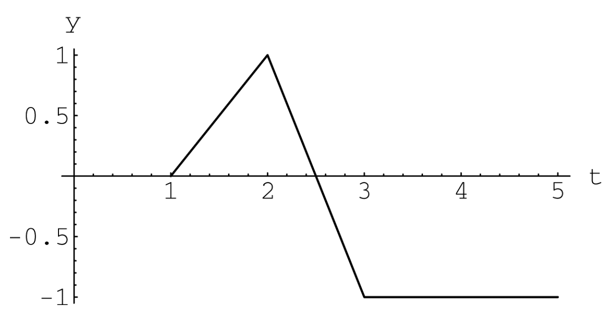


11. $F = G = \frac{e^{-s} - e^{-2s}}{s}$, so $FG = \frac{e^{-2s} - 2e^{-3s} + e^{-4s}}{s^2}$. Then

$f*g = (t-2)h(t-2) - 2(t-3)h(t-3) + (t-4)h(t-4)$.

12. $F(s) = \frac{1-e^{-s}}{s}$, $G(s) = \frac{e^{-s} - 2e^{-2s}}{s}$, so $FG = \frac{e^{-s} - 3e^{-2s} + 2e^{-3s}}{s^2}$. Then

$f*g = (t-1)h(t-1) - 3(t-2)h(t-2) + 2(t-3)h(t-3)$.



13. $\mathcal{L}\{t*t*t\} = \left(\frac{1}{s^2}\right)^3 = \frac{1}{s^6}$, so $t*t*t = \frac{t^5}{5!} = \frac{t^5}{120}$.

14. $\mathcal{L}\{h(t)*e^{-t}*e^{-2t}\} = \frac{1}{s(s+1)(s+2)} = \frac{A}{s} + \frac{B}{s+1} + \frac{C}{s+2}$. Thus $A = \frac{1}{2}$, $B = -1$, $C = \frac{1}{2}$ and

$h(t)*e^{-t}*e^{-2t} = \frac{1}{2} - e^{-t} + \frac{1}{2}e^{-2t}$.

15. $\mathcal{L}\{t*e^{-t}*e^{t}\}=\dfrac{1}{s^2(s+1)(s-1)}=\dfrac{A}{s^2}+\dfrac{B}{s}+\dfrac{C}{s-1}+\dfrac{D}{s+1}$. Thus $A=-1,\ B=0,\ C=\dfrac{1}{2},\ D=-\dfrac{1}{2}$

and $t*e^{-t}*e^{t}=\mathcal{L}^{-1}\left\{\dfrac{-1}{s^2}+\dfrac{\frac{1}{2}}{s-1}-\dfrac{\frac{1}{2}}{s+1}\right\}=-t+\dfrac{1}{2}\left(e^{t}-e^{-t}\right)$.

16. $\mathcal{L}\{h(t)*h(t)*...*h(t)\}=\dfrac{1}{s^n}$ and $\mathcal{L}^{-1}\left\{\dfrac{1}{s^n}\right\}=\dfrac{t^{n-1}}{(n-1)!}$.

Therefore, $\dfrac{t^{n-1}}{(n-1)!}=Ct^8\Rightarrow n=9,\ C=\dfrac{1}{8!}$.

17. $\mathcal{L}\left\{\underbrace{e^{-t}*e^{-t}*...*e^{-t}}_{\text{n times}}\right\}=\dfrac{1}{(s+1)^n},\ \mathcal{L}^{-1}\left\{\dfrac{1}{(s+1)^n}\right\}=\dfrac{t^{n-1}}{(n-1)!}e^{-t}=Ct^4e^{\alpha t}$. Thus

$n=5,\ C=\dfrac{1}{4!},\ \alpha=-1.$

18. $\int_0^t \sin(t-\lambda)y(\lambda)d\lambda=t^2=\sin t*y$. Therefore,

$\dfrac{2}{s^3}=\dfrac{1}{s^2+1}Y\Rightarrow Y=\dfrac{2\left(s^2+1\right)}{s^3}=\dfrac{2}{s}+\dfrac{2}{s^3}\Rightarrow y(t)=2+t^2.$

19. $t^2e^{-t}=\int_0^t \cos(t-\lambda)y(\lambda)d\lambda=\cos t*y$. Therefore,

$\dfrac{2}{(s+1)^3}=\dfrac{s}{s^2+1}Y\Rightarrow Y=\dfrac{2\left(s^2+1\right)}{s(s+1)^3}=\dfrac{A}{s}+\dfrac{B}{s+1}+\dfrac{C}{(s+1)^2}+\dfrac{D}{(s+1)^3}$. Thus

$A=2,\ B=-2,\ C=0,\ D=-4$, and so we have $Y=\dfrac{2}{s}-\dfrac{2}{s+1}-\dfrac{4}{(s+1)^3}$. Finally,

$y(t)=2-2e^{-t}-2t^2e^{-t}$.

20. $y(t)-\int_0^t e^{t-\lambda}y(\lambda)d\lambda=t\Rightarrow y-e^{t}*y=t,\ Y-\dfrac{1}{s-1}Y=\dfrac{1}{s^2}$. Therefore,

$\dfrac{1}{s^2}=\dfrac{s-2}{s-1}Y\Rightarrow Y=\dfrac{s-1}{s^2(s-2)}=\dfrac{A}{s}+\dfrac{B}{s^2}+\dfrac{C}{s-2}$. Thus $A=-\dfrac{1}{4},\ B=\dfrac{1}{2},\ C=\dfrac{1}{4}$, and so we have

$Y=\dfrac{-1/4}{s}+\dfrac{1/2}{s^2}+\dfrac{1/4}{s-2}$. Finally, $y(t)=-\dfrac{1}{4}+\dfrac{t}{2}+\dfrac{1}{4}e^{2t}$.

21. $\int_0^t y(t-\lambda)y(\lambda)d\lambda=6t^3=y*y\Rightarrow Y^2=(6)\dfrac{3!}{s^4}=\dfrac{36}{s^4}\Rightarrow Y=\pm\dfrac{6}{s^2}$ and $y(t)=\pm 6t$.

22. $\frac{1}{s^2}Y=\frac{2}{s^3}-\frac{2}{(s+1)^3}\Rightarrow Y=\frac{2}{s}-\frac{2s^2}{(s+1)^3}$.

$s^2=(s^2+2s+1)-(2s+1)=(s+1)^2-(2s+2-1)=(s+1)^2-2(s+1)+1$. Therefore,

$Y=\frac{2}{s}-2\left(\frac{1}{s+1}-\frac{2}{(s+1)^2}+\frac{1}{(s+1)^3}\right)$, so $y(t)=2-2\left(e^{-t}-2te^{-t}+\frac{t^2}{2}e^{-t}\right)=2-2\left(1-2t+\frac{t^2}{2}\right)e^{-t}$.

23. $sY-0+\frac{1}{s+2}Y=\frac{1}{s}\Rightarrow Y=\frac{1}{s}\left(\frac{s+2}{(s+1)^2}\right)=\frac{A}{s}+\frac{B}{s+1}+\frac{C}{(s+1)^2}$. Thus $A=2,\ B=-2, C=-1$, and

$Y=\frac{2}{s}-\frac{2}{s+1}-\frac{1}{(s+1)^2}$, so $y(t)=2-2e^{-t}-te^{-t}$.

24. $s\mathbf{Y}-\begin{bmatrix}1\\2\end{bmatrix}=\frac{1}{s}\mathbf{Y}\Rightarrow\left(s-\frac{1}{s}\right)\mathbf{Y}=\begin{bmatrix}1\\2\end{bmatrix}\Rightarrow\mathbf{Y}=\frac{s}{s^2-1}\begin{bmatrix}1\\2\end{bmatrix}=\frac{1}{2}\left(\frac{1}{s-1}+\frac{1}{s+1}\right)\begin{bmatrix}1\\2\end{bmatrix}$, so

$\mathbf{y}(t)=\frac{1}{2}\left(e^{t}+e^{-t}\right)\begin{bmatrix}1\\2\end{bmatrix}=\cosh t\begin{bmatrix}1\\2\end{bmatrix}$.

25. $sY-1=\frac{1}{s^2}\cdot\frac{1}{s^2}\Rightarrow Y=\frac{1}{s}+\frac{1}{s^5}\Rightarrow y(t)=1+\frac{t^4}{4!}=1+\frac{t^4}{24}$.

26. $sY-(-1)-Y=\frac{1}{s^2}\cdot\frac{1}{s-1}\Rightarrow(s-1)Y=-1+\frac{1}{s^2(s-1)}\Rightarrow Y=\frac{-1}{s-1}+\frac{1}{s^2(s-1)^2}$.

$\frac{1}{s^2(s-1)^2}=\frac{A}{s}+\frac{B}{s^2}+\frac{C}{s-1}+\frac{D}{(s-1)^2}\Rightarrow A=2,\ B=1,\ C=-2,\ D=1$. Thus,

$Y=-\frac{1}{s-1}+\frac{2}{s}+\frac{1}{s^2}-\frac{2}{s-1}+\frac{1}{(s-1)^2}=\frac{2}{s}+\frac{1}{s^2}-\frac{3}{s-1}+\frac{1}{(s-1)^2}$, so $y(t)=2+t-3e^{t}+te^{t}$.

Section 7.7

1 (a). $\int_0^3\left(1+e^{-t}\right)\delta(t-2)dt=1+e^{-2}$

1 (b). $\int_{-2}^1\left(1+e^{-t}\right)\delta(t-2)dt=0$ since t=2 lies outside the integration interval.

1 (c). $\int_{-1}^2\begin{bmatrix}\cos 2t\\te^{-t}\end{bmatrix}\delta(t)dt=\begin{bmatrix}1\\0\end{bmatrix}$

1 (d). $\int_{-3}^2\left(e^{2t}+t\right)\begin{bmatrix}\delta(t+2)\\\delta(t-1)\\\delta(t-3)\end{bmatrix}dt=\begin{bmatrix}e^{-4}-2\\e^2+1\\0\end{bmatrix}$

2. From Equation 7b, $f*\delta=f(t)$.

3. $\int_0^1 \sin^2[\pi(t-t_0)]\delta\left(t-\frac{1}{2}\right)dt = \sin^2\left[\pi\left(\frac{1}{2}-t_0\right)\right] = \frac{3}{4} \Rightarrow \left|\sin\left[\pi\left(\frac{1}{2}-t_0\right)\right]\right| = \frac{\sqrt{3}}{2}$

One possible t_0: $\pi\left(\frac{1}{2}-t_0\right) = \frac{\pi}{3} \Rightarrow \frac{1}{2}-t_0 = \frac{1}{3} \Rightarrow t_0 = \frac{1}{6}$.

4. $\int_1^5 t^n \delta(t-2)dt = 2^n = 8 \Rightarrow n = 3.$

5. $f(t) = 1 - h(1-t) = h(t-1)$ for all t except t=1.

6. $g(t) = \int_0^t h(\lambda-1)d\lambda = \begin{cases} 0, & t<1 \\ t-1, & t \ge 1 \end{cases}$. Therefore, $g(t) = (t-1)h(t-1)$.

7. $k = h(2-t) - h(1-t) = h(t-1) - h(t-2)$ for all t except t=1, 2.

8. $g(t) = \int_0^t e^{\alpha t}\delta(t-t_0)dt = \begin{cases} 0, & 0 \le t \le t_0 \\ e^{\alpha t_0}, & t_0 < t < \infty \end{cases}$. Therefore, $t_0 = 2,\ e^{2\alpha} = e^{-2} \Rightarrow \alpha = -1$.

9 (a). $\left(e^{-t}y\right)' = e^{-t} \Rightarrow e^{-t}y = -e^{-t} + C \Rightarrow y = -1 + Ce^t$. From the initial condition, we have $y(0) = 0 = -1 + C$. Thus C=1 and $y = -1 + e^t$.

9 (b). $s\Phi - \Phi = 1 \Rightarrow \Phi = \frac{1}{s-1}$. Therefore, $\phi(t) = e^t$.

9 (c). $\phi * g = \int_0^t e^{(t-\lambda)}h(\lambda)d\lambda = \int_0^t e^{t-\lambda}d\lambda = e^t\left(-e^{-\lambda}\right)\Big|_0^t = -1 + e^t$

10 (a). $\left(e^{-t}y\right)' = 1 \Rightarrow e^{-t}y = t + C \Rightarrow y = te^t + Ce^t$. From the initial condition, we have $y(0) = 0 = C$. Thus $y = te^t$.

10 (b). From 9b, $\phi(t) = e^t$.

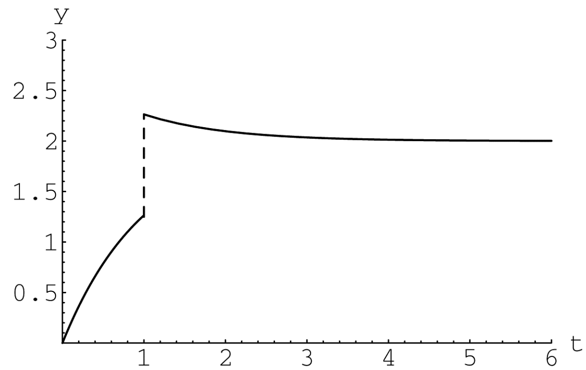
10 (c). $\phi * g = \int_0^t e^{(t-\lambda)}e^{\lambda}d\lambda = e^t\int_0^t d\lambda = te^t$.

11 (a). $\left(e^{-t}y\right)' = te^{-t} \Rightarrow e^{-t}y = -te^{-t} - e^{-t} + C \Rightarrow y = -(t+1) + Ce^t$. From the initial condition, we have $y(0) = 0 = -1 + C$. Thus C=1 and $y = e^t - (t+1)$.

11 (b). $s\Phi - \Phi = 1 \Rightarrow \Phi = \frac{1}{s-1}$. Therefore, $\phi(t) = e^t$.

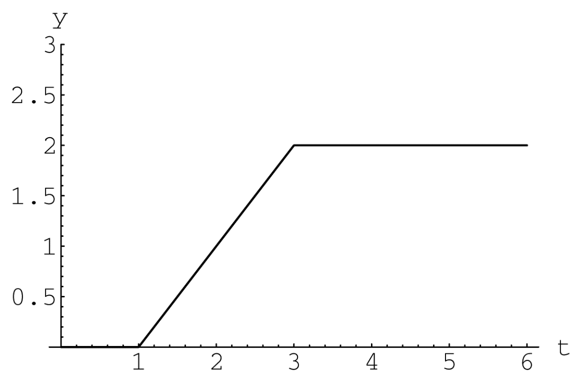
11 (c). $\phi * g = \int_0^t e^{(t-\lambda)}\lambda d\lambda = e^t\left(-\lambda e^{-\lambda} - e^{-\lambda}\right)\Big|_0^t = e^t - t - 1$

12. $sY + Y = \frac{2}{s} + e^{-s} \Rightarrow Y = \frac{2}{s(s+1)} + \frac{e^{-s}}{s+1},\ \frac{2}{s(s+1)} = \frac{A}{s} + \frac{B}{s+1} \Rightarrow A = 2,\ B = -2$. Therefore, $y(t) = 2 - 2e^{-t} + e^{-(t-1)}h(t-1)$.



13. $sY + Y = e^{-s} - e^{-2s} \Rightarrow Y = \dfrac{e^{-s}}{s+1} - \dfrac{e^{-2s}}{s+1}$. Therefore, $y(t) = e^{-(t-1)}h(t-1) - e^{-(t-2)}h(t-2)$.

14. $s^2Y = e^{-s} - e^{-3s} \Rightarrow Y = \dfrac{e^{-s}}{s^2} - \dfrac{e^{-3s}}{s^2}$. Therefore, $y(t) = (t-1)h(t-1) - (t-3)h(t-3)$.

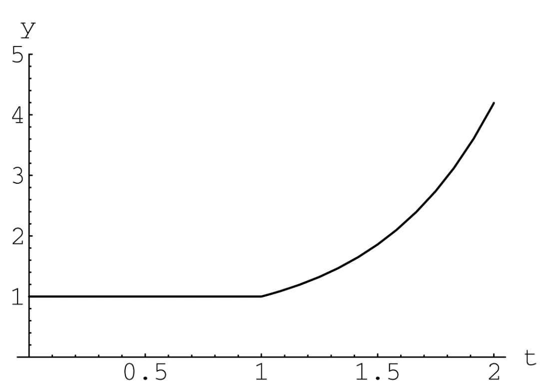


15. $\left(s^2 + 4\pi^2\right)Y = 2\pi e^{-2s} \Rightarrow Y = \dfrac{2\pi}{s^2 + 4\pi^2}e^{-2s}$. Therefore, $y(t) = \sin\left(2\pi(t-2)\right)h(t-2)$.

16. $s^2Y - s - 2(sY - 1) = e^{-s} \Rightarrow Y = \dfrac{s-2}{s(s-2)} + \dfrac{e^{-s}}{s(s-2)} = \dfrac{1}{s} + \dfrac{e^{-s}}{s(s-2)}$.

$\dfrac{1}{s(s-2)} = \dfrac{A}{s} + \dfrac{B}{s-2} \Rightarrow A = -\dfrac{1}{2},\ B = \dfrac{1}{2}$. Therefore, $Y = \dfrac{1}{s} + e^{-s}\left(-\dfrac{1}{2s} + \dfrac{1}{s(s-2)}\right)$ and

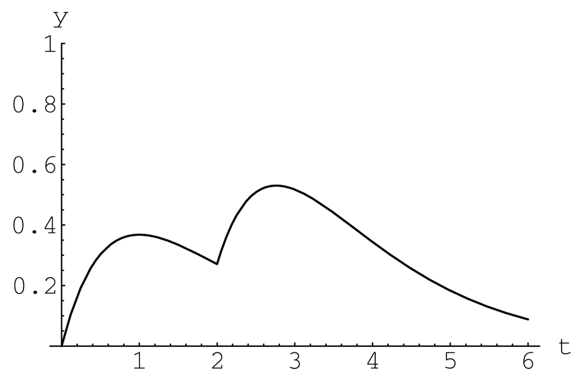
$y(t) = 1 - \dfrac{1}{2}h(t-1) + \dfrac{1}{2}e^{2(t-1)}h(t-1)$.



17. $\left(s^2+2s+2\right)Y=e^{-s}\Rightarrow Y=e^{-s}\dfrac{1}{(s+1)^2+1}$. Therefore, $y(t)=e^{-(t-1)}\sin(t-1)h(t-1)$.

18. $s^2Y-1+2sY+Y=e^{-2s}\Rightarrow\left(s^2+2s+1\right)Y=1+e^{-2s}\Rightarrow Y=\dfrac{1}{(s+1)^2}+\dfrac{e^{-2s}}{(s+1)^2}$. Therefore,

$y(t)=te^{-t}+(t-2)e^{-(t-2)}h(t-2)$.



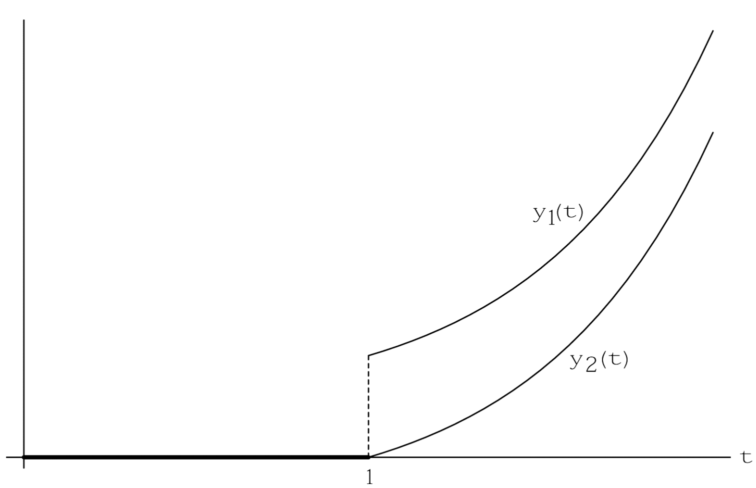
19. $s\mathbf{Y}=A\mathbf{Y}+e^{-s}\begin{bmatrix}1\\0\end{bmatrix}\Rightarrow\mathbf{Y}=e^{-s}(sI-A)^{-1}\begin{bmatrix}1\\0\end{bmatrix}$. $sI-A=\begin{bmatrix}s-1 & -1\\-1 & s-1\end{bmatrix}$, so

$(sI-A)^{-1}=\dfrac{1}{s^2-2s}\begin{bmatrix}s-1 & 1\\1 & s-1\end{bmatrix}$. Then $\mathbf{Y}=\dfrac{e^{-2s}}{s(s-2)}\begin{bmatrix}s-1 & 1\\1 & s-1\end{bmatrix}\begin{bmatrix}1\\0\end{bmatrix}=\dfrac{e^{-2s}}{s(s-2)}\begin{bmatrix}s-1\\1\end{bmatrix}$.

$\dfrac{s-1}{s(s-2)}=\dfrac{1}{s}+\dfrac{1}{s(s-2)}$, $\dfrac{1}{s(s-2)}=\dfrac{-\frac{1}{2}}{s}+\dfrac{\frac{1}{2}}{s-2}\Rightarrow\dfrac{s-1}{s(s-2)}=\dfrac{\frac{1}{2}}{s}+\dfrac{\frac{1}{2}}{s-2}$, so

$y_1(t)=\dfrac{1}{2}\left(1+e^{2(t-1)}\right)h(t-1)$ and $y_2(t)=\dfrac{1}{2}\left(-1+e^{2(t-1)}\right)h(t-1)$. Finally, we have

$$\mathbf{y}(t)=\begin{bmatrix}\frac{1}{2}\left(1+e^{2(t-1)}\right)h(t-1)\\ \frac{1}{2}\left(-1+e^{2(t-1)}\right)h(t-1)\end{bmatrix}.$$



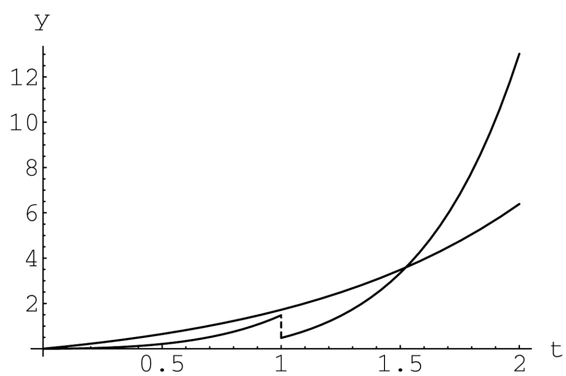
20. $s\mathbf{Y} = A\mathbf{Y} + \begin{bmatrix} 0 \\ \frac{1}{s} \end{bmatrix} - e^{-s}\begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow \mathbf{Y} = (sI - A)^{-1}\left(\begin{bmatrix} 0 \\ \frac{1}{s} \end{bmatrix} - e^{-s}\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$. $sI - A = \begin{bmatrix} s-2 & -1 \\ 0 & s-1 \end{bmatrix}$, so

$(sI - A)^{-1} = \frac{1}{(s-1)(s-2)}\begin{bmatrix} s-1 & 1 \\ 0 & s-2 \end{bmatrix}$. Then

$$\mathbf{Y} = \frac{1}{(s-1)(s-2)}\begin{bmatrix} \frac{1}{s} - e^{-s}(s-1) \\ \frac{s-2}{s} \end{bmatrix} = \begin{bmatrix} \frac{1}{s(s-1)(s-2)} - \frac{e^{-s}}{(s-2)} \\ \frac{1}{s(s-1)} \end{bmatrix}.$$

$\frac{1}{s(s-1)(s-2)} = \frac{A}{s} + \frac{B}{s-1} + \frac{C}{s-2} \Rightarrow A = \frac{1}{2},\ B = -1,\ C = \frac{1}{2};\ \frac{1}{s(s-1)} = -\frac{1}{s} + \frac{1}{s-1}$, so

$y_1(t) = \frac{1}{2} - e^t + \frac{1}{2}e^{2t} - e^{2(t-1)}h(t-1)$ and $y_2(t) = -1 + e^t$.



Chapter 8
Nonlinear Systems

Section 8.1

1 (a). For $y'' + ty = \sin y', y(0) = 0, y'(0) = 1,$ let $\mathbf{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} y(t) \\ y'(t) \end{bmatrix}$. Thus,

$$\mathbf{y}' = \begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} y' \\ y'' \end{bmatrix} = \begin{bmatrix} y' \\ -ty + \sin y' \end{bmatrix} = \begin{bmatrix} y_2 \\ -ty_1 + \sin y_2 \end{bmatrix}, \ \mathbf{y}(0) = \begin{bmatrix} y_1(0) \\ y_2(0) \end{bmatrix} = \begin{bmatrix} y(0) \\ y'(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

1 (b). From part (a), $\mathbf{f}(t,\mathbf{y}) = \begin{bmatrix} f_1(t,y_1,y_2) \\ f_2(t,y_1,y_2) \end{bmatrix} = \begin{bmatrix} y_2 \\ -ty_1 + \sin y_2 \end{bmatrix}$. Therefore, the requested partial derivatives are $\dfrac{\partial f_1}{\partial y_1} = 0,\ \dfrac{\partial f_1}{\partial y_2} = 1,\ \dfrac{\partial f_2}{\partial y_1} = -t,\ \dfrac{\partial f_2}{\partial y_2} = \cos y_2$.

1 (c). There are no points in 3-dimensional space where the hypotheses of Theorem 8.1 fail to be satisfied.

2 (a). For $y'' + (y')^3 + y^{1/3} = \tan(t/2), y(1) = 1, y'(1) = -2,$ let $\mathbf{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} y(t) \\ y'(t) \end{bmatrix}$. Thus,

$$\mathbf{y}' = \begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} y_2 \\ \tan(t/2) - y_1^{1/3} - y_2^3 \end{bmatrix}, \ \mathbf{y}(1) = \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

2 (b). For $\mathbf{f}(t,\mathbf{y}) = \begin{bmatrix} f_1(t,y_1,y_2) \\ f_2(t,y_1,y_2) \end{bmatrix}$, the requested partial derivatives are $\dfrac{\partial f_1}{\partial y_1} = 0,\ \dfrac{\partial f_1}{\partial y_2} = 1,\ \dfrac{\partial f_2}{\partial y_1} = -\dfrac{1}{3}y_1^{-2/3},\ \dfrac{\partial f_2}{\partial y_2} = -3y_2^2$.

2 (c). The hypotheses of Theorem 8.1 are not satisfied at $t = \pm(2n+1)\pi/2$ and $y_1 = 0$.

3 (a). For $y'' + t^{-1}(1 + y + 2y')^{-1} = t^{-1}e^{-t}, y(2) = 2, y'(2) = 1,$ let

$\mathbf{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} y(t) \\ y'(t) \end{bmatrix}$. Thus,

$$\mathbf{y}' = \begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} y' \\ y'' \end{bmatrix} = \begin{bmatrix} y' \\ -t^{-1}(1 + y + 2y')^{-1} + t^{-1}e^{-t} \end{bmatrix} = \begin{bmatrix} y_2 \\ -t^{-1}(1 + y_1 + 2y_2)^{-1} + t^{-1}e^{-t} \end{bmatrix},$$

$$\mathbf{y}(2) = \begin{bmatrix} y_1(2) \\ y_2(2) \end{bmatrix} = \begin{bmatrix} y(2) \\ y'(2) \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

3 (b). From part (a), $\mathbf{f}(t,\mathbf{y}) = \begin{bmatrix} f_1(t,y_1,y_2) \\ f_2(t,y_1,y_2) \end{bmatrix} = \begin{bmatrix} y_2 \\ -t^{-1}(1 + y_1 + 2y_2)^{-1} + t^{-1}e^{-t} \end{bmatrix}$.

Therefore, the requested partial derivatives are

$$\frac{\partial f_1}{\partial y_1} = 0,\ \frac{\partial f_1}{\partial y_2} = 1,\ \frac{\partial f_2}{\partial y_1} = t^{-1}(1 + y_1 + 2y_2)^{-2},\ \frac{\partial f_2}{\partial y_2} = 2t^{-1}(1 + y_1 + 2y_2)^{-2}.$$

3 (c). The hypotheses of Theorem 8.1 are satisfied everywhere except on the planes $t=0$ and $1+y_1+2y_2=0$.

4 (a). For $y'''+\cos(ty')=t(y'')^2, y(0)=1, y'(0)=1, y''(0)=-2$, let

$$\mathbf{y}(t)=\begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix}=\begin{bmatrix} y(t) \\ y'(t) \\ y''(t) \end{bmatrix}. \text{ Thus, } \mathbf{y}'=\begin{bmatrix} y_1' \\ y_2' \\ y_3' \end{bmatrix}=\begin{bmatrix} y_2 \\ y_3 \\ -\cos(ty_2)+y_3^2 \end{bmatrix}, \ \mathbf{y}(0)=\begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}.$$

4 (b). For $\mathbf{f}(t,\mathbf{y})=\begin{bmatrix} f_1(t,y_1,y_2,y_3) \\ f_2(t,y_1,y_2,y_3) \\ f_3(t,y_1,y_2,y_3) \end{bmatrix}$, the requested partial derivatives are

$$\frac{\partial f_1}{\partial y_1}=0,\ \frac{\partial f_1}{\partial y_2}=1,\ \frac{\partial f_1}{\partial y_3}=0,\ \frac{\partial f_2}{\partial y_1}=0,\ \frac{\partial f_2}{\partial y_2}=0, \frac{\partial f_2}{\partial y_3}=1,$$
$$\frac{\partial f_3}{\partial y_1}=0,\ \frac{\partial f_3}{\partial y_2}=t\sin(ty_2), \frac{\partial f_3}{\partial y_3}=2ty_3 .$$

4 (c). The hypotheses of Theorem 8.1 are satisfied in all of $ty_1y_2y_3$-space.

5 (a). For $y'''+2t^{1/3}(y-2)^{-1}(y''+2)^{-1}=0, y(0)=0, y'(0)=2, y''(0)=2$, let

$$\mathbf{y}(t)=\begin{bmatrix} y_1(t) \\ y_2(t) \\ y_3(t) \end{bmatrix}=\begin{bmatrix} y(t) \\ y'(t) \\ y''(t) \end{bmatrix}. \text{ Thus,}$$

$$\mathbf{y}'=\begin{bmatrix} y_1' \\ y_2' \\ y_3' \end{bmatrix}=\begin{bmatrix} y' \\ y'' \\ y''' \end{bmatrix}=\begin{bmatrix} y' \\ y'' \\ -2t^{1/3}(y-2)^{-1}(y''+2)^{-1} \end{bmatrix}=\begin{bmatrix} y_2 \\ y_3 \\ -2t^{1/3}(y_1-2)^{-1}(y_3+2)^{-1} \end{bmatrix},$$

$$\mathbf{y}(0)=\begin{bmatrix} y_1(0) \\ y_2(0) \\ y_3(0) \end{bmatrix}=\begin{bmatrix} y(0) \\ y'(0) \\ y''(0) \end{bmatrix}=\begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}.$$

5 (b). From part (a), $\mathbf{f}(t,\mathbf{y})=\begin{bmatrix} f_1(t,y_1,y_2,y_3) \\ f_2(t,y_1,y_2,y_3) \\ f_3(t,y_1,y_2,y_3) \end{bmatrix}=\begin{bmatrix} y_2 \\ y_3 \\ -2t^{1/3}(y_1-2)^{-1}(y_2+2)^{-1} \end{bmatrix}.$

Therefore, the requested partial derivatives are

$$\frac{\partial f_1}{\partial y_1}=0,\ \frac{\partial f_1}{\partial y_2}=1,\ \frac{\partial f_1}{\partial y_3}=0$$
$$\frac{\partial f_2}{\partial y_1}=0,\ \frac{\partial f_2}{\partial y_2}=0,\ \frac{\partial f_2}{\partial y_3}=1$$
$$\frac{\partial f_3}{\partial y_1}=2t^{1/3}(y_1-2)^{-2}(y_3+2)^{-1},\ \frac{\partial f_3}{\partial y_2}=0,\ \frac{\partial f_3}{\partial y_3}=2t^{1/3}(y_1-2)^{-1}(y_3+2)^{-2} .$$

5 (c). The hypotheses of Theorem 8.1 are satisfied everywhere except on the "hyperplanes" $y_1=2$ and $y_3=-2$.

6. Since $y_2'=t\cos^2(y_2)-3y_1+t^4$, it follows that the scalar problem is $y''=t\cos^2(y')-3y+t^4$, $y(2)=1, y'(2)=-1$.

7. Since $y_2'=y_2\tan y_1+e^{y_2}$, it follows that the scalar problem is $y''=y'\tan y+e^{y'}$, $y(0)=0, y'(0)=1$.

8. Since $y_3' = y_1y_2 + y_3^2$, it follows that the scalar problem is $y''' = yy' + (y'')^2$, $y(-1) = -1, y'(-1) = 2, y''(-1) = -4$.

9. Since $y_3' = (y_2y_3 + t^2)^{1/2}$, it follows that the scalar problem is $y''' = (y'y'' + t^2)^{1/2}$, $y(1) = 1, y'(1) = 1/2, y''(1) = 3$.

11. Laplace transforms cannot be productively used because the equation is nonlinear.

14 (a). Let $a = \pi/(2\delta)$. Then $\tan ax = ax + (1/3)a^3x^3 + (2/15)a^5x^5 + \cdots$. Retaining the first term of the Maclaurin series in equation (7), we have
$mx'' + (2k\delta/\pi)\tan(\pi x/2\delta) \approx mx'' + (2k\delta/\pi)(\pi x/2\delta) = mx'' + kx$.

14 (b). As in part (a), retaining the first two terms of the Maclaurin series in equation (7) results in equation (8).

14 (c). Equation (7) becomes $\mathbf{y}' = \begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} y_2 \\ -(2k\delta/m\pi)\tan(\pi y_1/2\delta) \end{bmatrix}$.

Equation (8) becomes $\mathbf{y}' = \begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} y_2 \\ -(k/m)(y_1 + (\pi^2/12\delta^2)y_1^3) \end{bmatrix}$.

14 (d). The system version of equation (7) satisfies the hypotheses of Theorem 8.1 everywhere except along $y_1 = \pm(2n+1)\pi/2$. The system version of equation (8) satisfies the hypotheses of Theorem 8.1 everywhere in ty_1y_2 - space

15 (a). Adding equations 3 and 4, we obtain $\frac{dc}{dt} + \frac{de}{dt} = 0$. Thus, using the linearity of differentiation, $\frac{d(c+e)}{dt} = 0$ and hence, $c(t) + e(t) \equiv e_0$ is a constant function.

15 (b). Substituting $e(t) = e_0 - c(t)$ in equations 1 and 3, we find
$\frac{da}{dt} = -k_1e_0a(t) + k_1c(t)a(t) + k_1'c(t)$ and $\frac{dc}{dt} = k_1e_0a(t) - k_1c(t)a(t) - (k_1' + k_2)c(t)$.

15 (c). The hypotheses of Theorem 8.1 are satisfied for all points in (t,a,c) - space.

16 (a). At the instant shown in the figure,
$$V_{\text{sub}} = (2/3)\pi R^3 + \int_0^{y(t)} \pi r^2 dy = (2/3)\pi R^3 + \int_0^{y(t)} \pi(R^2 - y^2)\,dy$$
$$= (2/3)\pi R^3 + \pi[R^2y(t) - (1/3)(y(t))^3].$$

16 (b). Equation (10) is physically relevant as long as $-R \le y(t) \le R$.

Section 8.2

1. For
$$x' = x(-1+y)$$
$$y' = y(1-x),$$
we see that $x' = 0$ if (a) $x = 0$ or (b) $y = 1$. In Case (a), we have $y' = 0$ only if $y = 0$, yielding the equilibrium point $(x,y) = (0,0)$. In Case (b), we have $y' = 0$ only if $x = 1$, yielding the equilibrium point $(x,y) = (1,1)$.

2. For

$$x' = y(x+3)$$
$$y' = (x-1)(y-2),$$

we see that $x' = 0$ if (a) $x = -3$ or (b) $y = 0$. In Case (a), we have $y' = 0$ only if $y = 2$, yielding the equilibrium point $(x,y) = (-3,2)$. In Case (b), we have $y' = 0$ only if $x = 1$, yielding the equilibrium point $(x,y) = (1,0)$.

3. For

$$x' = (x-2)(y+1)$$
$$y' = x^2 - 4x + 3,$$

we see that $x' = 0$ if (a) $x = 2$ or (b) $y = -1$. In Case (a), we cannot have $y' = 0$. In Case (b), we have $y' = 0$ only if $x = 1$ or $x = 3$, yielding the equilibrium points $(x,y) = (1,-1)$ and $(x,y) = (3,-1)$.

4. For

$$x' = (x-1)(y+1)$$
$$y' = (x-2)y,$$

we see that $x' = 0$ if (a) $x = 1$ or (b) $y = -1$. In Case (a), we have $y' = 0$ only if $y = 0$, yielding the equilibrium point $(x,y) = (1,0)$. In Case (b), we have $y' = 0$ only if $x = 2$, yielding the equilibrium point $(x,y) = (2,-1)$.

5. For

$$x' = x(x-2y)$$
$$y' = y(3x-y),$$

we see that $x' = 0$ if (a) $x = 0$ or (b) $x = 2y$. In Case (a), we have $y' = 0$ only if $y = 0$, yielding the equilibrium point $(x,y) = (0,0)$. In Case (b), we have $y' = 0$ only if $y = 0$, yielding the same equilibrium point as in Case (a), $(x,y) = (0,0)$.

6. For

$$x' = y(y-x)$$
$$y' = x(x+2y),$$

we see that $x' = 0$ if (a) $y = 0$ or (b) $y = x$. In Case (a), we have $y' = 0$ only if $x = 0$, yielding the equilibrium point $(x,y) = (0,0)$. In Case (b), we have $y' = 0$ only if $x = 0$, yielding the same equilibrium point $(x,y) = (0,0)$.

7. For

$$x' = x^2 + y^2 - 8$$
$$y' = x^2 - y^2,$$

we see that $y' = 0$ if $x^2 = y^2$. Using this requirement in the first equation, we see that $x' = 0$ requires $2x^2 - 8 = 0$ or $x = \pm 2$. Since $y = \pm x$, we find 4 equilibrium points, $(2,2), (2,-2), (-2,-2),$ and $(-2,2)$.

8. For

$$x' = x^2 + 2y^2 - 3$$
$$y' = 2x^2 + y^2 - 3,$$

we see that $x' = 0$ if $x^2 = 3 - 2y^2$. In this event, we have $y' = 0$ only if $2(3-2y^2) + y^2 - 3 = 0$. Solving for y we obtain $y = \pm 1$. Then, since $x^2 = 3 - 2y^2$, we see that $x = \pm 1$ for each choice of y. The equilibrium points are
$(x,y) = (1,1), (-1,1), (1,-1), (-1,-1)$.

9. For

$$x' = y - 1$$
$$y' = x(y + x)$$
$$z' = y(2 - z),$$

we see that $x' = 0$ requires $y = 1$. Using this requirement in the second equation, we see that $y' = 0$ requires $x(1 + x) = 0$. Thus, we need in Case (a) $x = 0$ or in Case (b), $x = -1$. Finally, $z' = 0$ requires $z = 2$ since y is nonzero. We obtain 2 equilibrium points, $(x,y,z) = (0,1,2)$ and $(x,y,z) = (-1,1,2)$.

10. For

$$x' = z^2 - 1$$
$$y' = z(1 - 2x + y)$$
$$z' = -(1 - x - y)^2,$$

we see that $x' = 0$ requires $z = \pm 1$. Using this requirement in the second equation, we see that $y' = 0$ requires $1 - 2x + y = 0$ while $z' = 0$ requires $1 - x - y = 0$. Satisfying $y' = 0$ and $z' = 0$ therefore requires $x = 2/3$ and $y = 1/3$. Combining this requirement with $z = \pm 1$, we obtain 2 equilibrium points,
$(x,y,z) = (2/3,1/3,1)$ and $(x,y,z) = (2/3,1/3,-1)$.

11. Making the substitution $y_1 = y$ and $y_2 = y'$ the scalar equation can be expressed as the system

$$y_1' = y_2$$
$$y_2' = -y_1 - y_1^3$$

Since $y_2' = -y_1(1 + y_1^2)$, we cannot have $y_2' = 0$ unless $y_1 = 0$. Similarly, from the first equation, $y_1' = 0$ requires $y_2 = 0$. Thus, the only equilibrium point is $(y_1, y_2) = (y, y') = (0,0)$.

12. Making the substitution $y_1 = y$ and $y_2 = y'$ the scalar equation can be expressed as the system

$$y_1' = y_2$$
$$y_2' = 1 - e^{y_1} y_2 - \sin^2(\pi y_1)$$

Thus, the equilibrium points are $(y_1, y_2) = (y, y') = (n + 0.5, 0), n = 0, \pm 1, \pm 2, \ldots$.

13. Making the substitution $y_1 = y$ and $y_2 = y'$ the scalar equation can be expressed as the system

$$y_1' = y_2$$
$$y_2' = 1 - y_1^2 - 2(1 + y_1^4)^{-1} y_2$$

From the first equation, $y_1' = 0$ requires $y_2 = 0$. Thus, in the second equation, $y_2' = 0$ requires $1 - y_1^2 = 0$ or $y_1 = \pm 1$. There are two equilibrium points
$(y_1, y_2) = (y, y') = (1,0)$ and $(y_1, y_2) = (y, y') = (-1,0)$.

14. Making the substitution $y_1 = y$, $y_2 = y'$, and $y_3 = y''$ the scalar equation can be expressed as the system

$$y_1' = y_2$$
$$y_2' = y_3$$
$$y_3' = 1 + y_3 - 2\sin y_1$$

Thus, the equilibrium points are

$$(y_1, y_2, y_3) = ((\pi/6) + 2n\pi, 0, 0) \text{ and}$$
$$(y_1, y_2, y_3) = ((5\pi/6) + 2n\pi, 0, 0), n = 0, \pm 1, \pm 2, \ldots.$$

15. Making the substitution $y_1 = y, y_2 = y'$ and $y_3 = y''$, the scalar equation can be expressed as the system

$$y_1' = y_2$$
$$y_2' = y_3$$
$$y_3' = y_2^2 + (y_1^2 - 4)(2 + y_2^2)^{-1} .$$

From the first equation, $y_1' = 0$ requires $y_2 = 0$ while (by the second equation) $y_2' = 0$ requires $y_3 = 0$. Having these requirements, the third equation tells us that $y_3' = 0$ only if $y_1 = \pm 2$. Hence, There are two equilibrium points
$(y_1, y_2, y_3) = (y, y', y'') = (2,0,0)$ and $(y_1, y_2, y_3) = (y, y', y'') = (-2,0,0)$.

16. Since (0,0) is an equilibrium point, we know $\beta = 0$ and $\delta = 0$. Similarly, since (2,1) is an equilibrium point, we know $2\alpha + 2 = 0$ and $\gamma - 6 = 0$. Thus, $\alpha = -1$ and $\gamma = 6$.

17. Since (1,1) is an equilibrium point, we know $\alpha + \beta + 2 = 0$ and $\gamma + \delta - 1 = 0$. Similarly, since (2,0) is an equilibrium point, we know $2\alpha + 2 = 0$ and $2\gamma - 1 = 0$. Thus, $\alpha = -1$ and $\gamma = 1/2$. Using the equations derived from the equilibrium point (1,1), we have $-1 + \beta + 2 = 0$ and $(1/2) + \delta - 1 = 0$. Therefore, $\beta = -1$ and $\delta = 1/2$.

18. The slope of a phase plane trajectory is given by $y'/x' = g(x,y)/f(x,y)$, see equation (9). As given, $g(2,1)/f(2,1) = 1$ and $g(1,-1)/f(1,-1) = 0$. Therefore, $g(1,-1) = 0$ and so $\beta = 2$. Knowing $\beta = 2$ and $g(2,1)/f(2,1) = 1$, we obtain $(3+\beta)/(2+\alpha) = 1$ or $5/(2+\alpha) = 1$. Thus, we obtain $\alpha = 3$.

19. The slope of a phase plane trajectory is given by $y'/x' = g(x,y)/f(x,y)$, see equation (9). As given, $g(1,1)/f(1,1) = 0$ and $g(1,-1)/f(1,-1) = 4$. Therefore, $g(1,1) = 0$ and so $2 + \gamma = 0$ or $\gamma = -2$. Knowing $\gamma = -2$ and $g(1,-1)/f(1,-1) = 4$, we obtain $(2-\gamma)/(\alpha - \beta + 1) = 4$ or $1/(\alpha - \beta + 1) = 1$. Finally, since there is a vertical tangent at (0,–1) we know $f(0,-1) = 0$, and thus $-\beta + 1 = 0$. Using $\beta = 1$ along with the prior equation $1/(\alpha - \beta + 1) = 1$, we obtain $\alpha = 1$.

20. The slope of a phase plane trajectory is given by $y'/x' = g(x,y)/f(x,y)$, see equation (9). As given, $g(1,2)/f(1,2) = 1/6$ and thus
$1/6 = g(1,2)/f(1,2) = (-1+0.5)/(5-2^n)$. Solving for n, we obtain $n = 3$.

21. Making the substitution $y_1 = y$ and $y_2 = y'$ the scalar equation can be expressed as the system

$$y_1' = y_2$$
$$y_2' = y_2 - 2y_1^2 + \alpha .$$

Since $(y_1, y_2) = (2,0)$ is an equilibrium point, it follows that $2y_1^2 = 8 = \alpha$.

22 (a). $\mathbf{v} = 4\mathbf{i} - 3\mathbf{j}$

22 (b). $\mathbf{v} = 15\mathbf{i} + \mathbf{j}$

22 (a). $\mathbf{v} = -\mathbf{j}$

24. For $A = \begin{bmatrix} -9 & 1 \\ 1 & -9 \end{bmatrix}$, the eigenvalues are $\lambda_1 = -10$ and $\lambda_2 = -8$ with corresponding eigenvectors $\mathbf{u}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\mathbf{u}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. The general solution is
$\mathbf{y}(t) = c_1 e^{-10t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + c_2 e^{-8t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and hence all solution points are attracted to the origin. Thus, the direction field corresponding to the given matrix is C.

25. For $A=\begin{bmatrix}-1 & -3\\ -3 & -1\end{bmatrix}$, the eigenvalues are $\lambda_1=-4$ and $\lambda_2=2$ with corresponding eigenvectors $\mathbf{u}_1=\begin{bmatrix}1\\1\end{bmatrix}$ and $\mathbf{u}_2=\begin{bmatrix}1\\-1\end{bmatrix}$. The general solution is $\mathbf{y}(t)=c_1e^{-4t}\begin{bmatrix}1\\1\end{bmatrix}+c_2e^{2t}\begin{bmatrix}1\\-1\end{bmatrix}$ and hence solution points that begin on the line $y=x$ are attracted to the origin whereas those that begin on the line $y=-x$ are repelled away from the origin. Thus, the direction field corresponding to the given matrix is B.

26. For $A=\begin{bmatrix}-4 & 6\\ 6 & -4\end{bmatrix}$, the eigenvalues are $\lambda_1=-10$ and $\lambda_2=2$ with corresponding eigenvectors $\mathbf{u}_1=\begin{bmatrix}1\\-1\end{bmatrix}$ and $\mathbf{u}_2=\begin{bmatrix}1\\1\end{bmatrix}$. The general solution is $\mathbf{y}(t)=c_1e^{-10t}\begin{bmatrix}1\\-1\end{bmatrix}+c_2e^{2t}\begin{bmatrix}1\\1\end{bmatrix}$ and hence solution points that begin on the line $y=x$ are repelled away from the origin whereas those that begin on the line $y=-x$ are attracted to the origin. Thus, the direction field corresponding to the given matrix is D.

27. For $A=\begin{bmatrix}4 & 2\\ 2 & 4\end{bmatrix}$, the eigenvalues are $\lambda_1=6$ and $\lambda_2=2$ with corresponding eigenvectors $\mathbf{u}_1=\begin{bmatrix}1\\1\end{bmatrix}$ and $\mathbf{u}_2=\begin{bmatrix}1\\-1\end{bmatrix}$. The general solution is $\mathbf{y}(t)=c_1e^{6t}\begin{bmatrix}1\\1\end{bmatrix}+c_2e^{2t}\begin{bmatrix}1\\-1\end{bmatrix}$ and hence solution points that begin on the line $y=x$ are repelled away from the origin as are those that begin on the line $y=-x$. Thus, the direction field corresponding to the given matrix is A.

28. The phase plane point $(\alpha,0)$ is an equilibrium point when α is a root of $f(y)=0$.

29 (a). Making the substitution $y_1=y$ and $y_2=y'$ the scalar equation can be expressed as the system

$$y_1'=y_2$$
$$y_2'=-y_1-y_1^3\ .$$

The nullclines are the lines $y_1=0$ and $y_2=0$. The only equilibrium point is the point $(0,0)$.

30 (a). Making the substitution $y_1=y$ and $y_2=y'$ the scalar equation can be expressed as the system

$$y_1'=y_2$$
$$y_2'=-y_1(1-y_1^2)\ .$$

The nullclines are the lines $y_1=0, y_1=\pm1$, and $y_2=0$. The equilibrium points are $(0,0),(-1,0),(1,0)$.

31 (a). Making the substitution $y_1 = y$ and $y_2 = y'$ the scalar equation can be expressed as the system

$$y_1' = y_2$$
$$y_2' = 1 - 2\sin^2 y_1 .$$

The nullclines are the lines $y_1 = \pm(\pi/4) + n\pi, n = 0, \pm 1, \pm 2, \ldots$ and the line $y_2 = 0$ The equilibrium points are $(\pm(\pi/4) + n\pi, 0), n = 0, \pm 1, \pm 2, \ldots$.

32 (a). The nullclines are the lines $y = 3x - 2$ and $y = x$. These lines intersect at the point $(1,1)$ yielding the only equilibrium point.

33 (a). The nullclines are the lines $y = 2 - x$ and $y = x$. These lines intersect at the point $(1,1)$ yielding the only equilibrium point.

34 (a). The nullclines are the lines $y = 2x - 2$ and $y = 4 - x$ where $f = 0$ and the line $y = (1/2)x$ where $g = 0$. The lines $f = 0$ and $g = 0$ intersect at the points $(4/3, 2/3)$ and $(8/3, 4/3)$ yielding the only equilibrium points.

35 (a). The nullclines are the lines $y = 2x - 6$ and $y = x$, where $f = 0$ and the line $y = -x$, where $g = 0$. The lines $f = 0$ and $g = 0$ intersect at the points $(0,0)$ and $(2,-2)$ yielding the only equilibrium points.

36 (a). The nullclines are the curves $y = 1 - x^2$ and $y = -1 + x^2$. These curves intersect at the equilibrium points $(-1,0)$ and $(1,0)$.

Section 8.3

1 (a). Given $x'' + 4x = 0$, multiply by x' to obtain $x'x'' + 4x'x = 0$. Integrating, we obtain $0.5(x')^2 + 2x^2 = C$.

1 (b). The equation $x'' + 4x = 0$ can be expressed as $\begin{matrix} x' = y \\ y' = -4x \end{matrix}$. With this notation, the conserved quantity found in part (a) is $0.5y^2 + 2x^2 = C$. The graph passes through the point $(x,y) = (1,1)$ when $C = 2.5$.

1 (c). At $(1,1)$, the velocity vector is $\mathbf{v} = x'\mathbf{i} + y'\mathbf{j} = \mathbf{i} - 4\mathbf{j}$. The velocity vector is tangent to the graph and indicates that the graph is traversed in the clockwise direction as t increases.

2 (a). Given $x'' - (x+1) = 0$, multiply by x' to obtain $x'x'' - x'(x+1) = 0$. Integrating, we obtain $(x')^2 - (x+1)^2 = C$.

2 (b). The equation $x'' - (x+1) = 0$ can be expressed as $\begin{matrix} x' = y \\ y' = x + 1 \end{matrix}$. With this notation, the conserved quantity found in part (a) is $y^2 - (x+1)^2 = C$. The graph passes through the point $(x,y) = (1,1)$ when $C = -3$.

2 (c). At $(1,1)$, the velocity vector is $\mathbf{v} = x'\mathbf{i} + y'\mathbf{j} = \mathbf{i} + 2\mathbf{j}$. The velocity vector indicates that the solution point moves upward and to the right along the right branch of the hyperbola as t increases.

3 (a). Given $x'' + x^3 = 0$, multiply by x' to obtain $x'x'' + x'x^3 = 0$. Integrating, we obtain $0.5(x')^2 + 0.25x^4 = C$.

3 (b). The equation $x'' + x^3 = 0$ can be expressed as $\begin{aligned} x' &= y \\ y' &= -x^3. \end{aligned}$ With this notation, the conserved quantity found in part (a) is $0.5y^2 + 0.25x^4 = C$. The graph passes through the point $(x,y) = (1,1)$ when $C = 0.75$.

3 (c). At $(1,1)$, the velocity vector is $\mathbf{v} = x'\mathbf{i} + y'\mathbf{j} = \mathbf{i} - \mathbf{j}$. The velocity vector is tangent to the graph and indicates that the graph is traversed in the clockwise direction as *t* increases.

4 (a). Given $x'' - (x^3 + \pi \sin \pi x) = 0$, multiply by x' to obtain $x'x'' - x'(x^3 + \pi \sin \pi x) = 0$. Integrating, we obtain $2(x')^2 - (x^4 - 4\cos \pi x) = C$.

4 (b). The equation $x'' - (x^3 + \pi \sin \pi x) = 0$ can be expressed as $\begin{aligned} x' &= y \\ y' &= x^3 + \pi \sin \pi x. \end{aligned}$ With this notation, the conserved quantity found in part (a) is $2y^2 - (x^4 - 4\cos \pi x) = C$.
The graph passes through the point $(x,y) = (1,1)$ when $C = -3$.

4 (c). At $(1,1)$, the velocity vector is $\mathbf{v} = x'\mathbf{i} + y'\mathbf{j} = \mathbf{i} + \mathbf{j}$. The velocity vector indicates that the solution point moves upward and to the right along the right branch of the graph as *t* increases.

5 (a). Given $x'' + x^2 = 0$, multiply by x' to obtain $x'x'' + x'x^2 = 0$. Integrating, we obtain $0.5(x')^2 + (1/3)x^3 = C$.

5 (b). The equation $x'' + x^2 = 0$ can be expressed as $\begin{aligned} x' &= y \\ y' &= -x^2. \end{aligned}$ With this notation, the conserved quantity found in part (a) is $0.5y^2 + (1/3)x^3 = C$. The graph passes through the point $(x,y) = (1,1)$ when $C = 5/6$.

5 (c). At $(1,1)$, the velocity vector is $\mathbf{v} = x'\mathbf{i} + y'\mathbf{j} = \mathbf{i} - \mathbf{j}$. The velocity vector is tangent to the graph and indicates that the solution point moves "down the graph" as *t* increases.

6 (a). Given $x'' + x/(1 + x^2) = 0$, multiply by x' to obtain $x'x'' + x'x/(1 + x^2) = 0$. Integrating, we obtain $(x')^2 + \ln(1 + x^2) = C$.

6 (b). The equation $x'' + x/(1 + x^2) = 0$ can be expressed as $\begin{aligned} x' &= y \\ y' &= -x/(1 + x^2). \end{aligned}$ With this notation, the conserved quantity found in part (a) is $y^2 + \ln(1 + x^2) = C$.
The graph passes through the point $(x,y) = (1,1)$ when $C = 1 + \ln 2$.

6 (c). At $(1,1)$, the velocity vector is $\mathbf{v} = x'\mathbf{i} + y'\mathbf{j} = \mathbf{i} - 0.5\mathbf{j}$. The velocity vector indicates that the solution point moves clockwise along the curve as *t* increases.

7. Rewriting the conservation law in terms of x and x', we have $(x')^2 + x^2 \cos x = C$. Differentiating with respect to *t*, we obtain $2x'x'' + 2x'x\cos x - x^2 x' \sin x = 0$ or $x'(2x'' + 2x\cos x - x^2 \sin x) = 0$. Therefore, the differential equation is $x'' + x\cos x - 0.5x^2 \sin x = 0$.

8. Rewriting the conservation law in terms of x and x', we have $(x')^2 - e^{-x^2} = C$. Differentiating with respect to *t*, we obtain $2x'x'' - (e^{-x^2})(-2xx') = 0$. Therefore, the differential equation is $x'' + xe^{-x^2} = 0$.

9 (a). The equation $x'' + x + x^3 = 0$ can be expressed as $\begin{aligned} x' &= y \\ y' &= -x - x^3. \end{aligned}$ The nullclines are the lines defined by $y = 0$ and $-x(1 + x^2) = 0$; the lines $y = 0$ and $x = 0$. Thus, the only equilibrium point is the point $(x,y) = (0,0)$.

9 (b). The velocity vector has the form $\mathbf{v}(x,y) = y\mathbf{i} - (x + x^3)\mathbf{j}$. Thus, we obtain $\mathbf{v}(1,1) = \mathbf{i} - 2\mathbf{j}$, $\mathbf{v}(1,-1) = -\mathbf{i} - 2\mathbf{j}$, $\mathbf{v}(-1,1) = \mathbf{i} + 2\mathbf{j}$, and $\mathbf{v}(-1,-1) = -\mathbf{i} + 2\mathbf{j}$.

9 (c). Multiplying by x', the equation becomes $x'x'' + x'(x + x^3) = 0$. Integrating, we obtain $0.5(x')^2 + 0.5x^2 + 0.25x^4 = C$ or $2y^2 + 2x^2 + x^4 = C_1$. The graph of the conserved quantity passes through the point $(1,1)$ when $C_1 = 5$. The graph passes through the other three points and is consistent with the sketch in part (b).

10. Since $x'' + \alpha x = 0$ it follows that $0.5(x')^2 + 0.5\alpha x^2 = C_1$ and hence $\alpha x^2 + y^2 = C$.

10 (a). Figure A is a circle of radius 2 and thus $\alpha = 1$ and $x^2 + y^2 = 4$.
Figure B is a hyperbola with asymptotes $y = \pm x$. Since (0, 2) is on the graph, we see that $\alpha = -1$ and $y^2 - x^2 = 4$.
Figure C shows horizontal lines, $y = \pm 2$. Thus, $\alpha = 0$.

10 (b). The solution point in Figure A travels clockwise around the circle. Solution points in Figure B move to the right on the upper branch and to the left on the lower branch. Solutions points in Figure C move to the right on the upper line and to the left on the lower line.

11. In analogy with Exercise 9, multiply the equation $y''' + f(y') = 0$ by y'', obtaining $y''y''' + y''f(y') = 0$. Integrating, we find $0.5y'' + F(y') = C$ where $F(u)$ is an antiderivative of $f(u)$. Thus, the differential equation has a conservation law given by $0.5(y'')^2 + F(y') = C$.

12. (a) From the definition of $E(t)$, it follows that $\dfrac{dE}{dt} = mx'x'' + kxx' = (mx'' + kx)x'$. From the differential equation, $mx'' + \gamma x' + kx = 0$ and hence $mx'' + kx = -\gamma x'$. Therefore, $\dfrac{dE}{dt} = (-\gamma x')x' \le 0$.
(b) Energy is not conserved. On t-intervals where $x'(t) \ne 0$, $E(t)$ is a decreasing function of t and energy is being lost.

13 (a). For the system

$$x' = 2x$$
$$y' = -2y$$

we have $f(x,y) = 2x$ and $g(x,y) = -2y$. Thus, $f_x = 2$ and $g_y = -2$. Since $f_x = -g_y$, the system is Hamiltonian.

13 (b). Let $H(x,y)$ denote the Hamiltonian function. Thus, $H_x(x,y) = -g(x,y) = 2y$. Integrating with respect to x, we obtain $H(x,y) = 2xy + p(y)$. Differentiating with respect to y in order to determine $p(y)$, we find $H_y(x,y) = 2x + p'(y) = f(x,y) = 2x$. Therefore, $p'(y) = 0$ and hence $p(y) = C$ is a constant function. Dropping the constant, we obtain a Hamiltonian function, $H(x,y) = 2xy$.

13 (c). From part (b), the phase-plane trajectories are defined by $2xy = C$. If a phase-plane trajectory passes through the point $(1,1)$, then $C = 2$ and the trajectory is given by $xy = 1$.

14 (a). For the system

$$x' = 2xy$$
$$y' = -y^2$$

we have $f(x,y) = 2xy$ and $g(x,y) = -y^2$. Thus, $f_x = 2y$ and $g_y = -2y$. Since $f_x = -g_y$, the system is Hamiltonian.

14 (b). Let $H(x,y)$ denote the Hamiltonian function. Thus, $H_x(x,y) = -g(x,y) = y^2$. Integrating with respect to x, we obtain $H(x,y) = xy^2 + p(y)$. Differentiating with respect to y in order to determine $p(y)$, we find $H_y(x,y) = 2xy + p'(y) = f(x,y) = 2xy$.

Therefore, $p'(y)=0$ and hence $p(y)=C$ is a constant function. Dropping the constant, we obtain a Hamiltonian function, $H(x,y)=xy^2$.

14 (c). From part (b), the phase-plane trajectories are defined by $xy^2=C$. If a phase-plane trajectory passes through the point $(1,1)$, then $C=1$ and the trajectory is given by $xy^2=1$.

15 (a). For the system

$$x' = x - x^2 + 1$$
$$y' = -y + 2xy + 4x$$

we have $f(x,y)=x-x^2+1$ and $g(x,y)=-y+2xy+4x$. Thus, $f_x=1-2x$ and $g_y=-1+2x$. Since $f_x=-g_y$, the system is Hamiltonian.

15 (b). Let $H(x,y)$ denote the Hamiltonian function. Thus, $H_x(x,y)=-g(x,y)=y-2xy-4x$. Integrating with respect to x, we obtain $H(x,y)=xy-x^2y-2x^2+p(y)$. Differentiating with respect to y in order to determine $p(y)$, we find $H_y(x,y)=x-x^2+p'(y)=f(x,y)=x-x^2+1$. Therefore, $p'(y)=1$ and hence $p(y)=y+C$. Dropping the additive constant, we obtain a Hamiltonian function, $H(x,y)=xy-x^2y-2x^2+y$.

15 (c). From part (b), the phase-plane trajectories are defined by $xy-x^2y-2x^2+y=C$. If a phase-plane trajectory passes through the point $(1,1)$, then $C=-1$ and the trajectory is given by $xy-x^2y-2x^2+y+1=0$.

16 (a). For the system

$$x' = -8y$$
$$y' = 2x$$

we have $f(x,y)=-8$ and $g(x,y)=2x$. Thus, $f_x=0$ and $g_y=0$. Since $f_x=-g_y$, the system is Hamiltonian.

16 (b). Let $H(x,y)$ denote the Hamiltonian function. Thus, $H_y(x,y)=f(x,y)=-8y$. Integrating with respect to y, we obtain $H(x,y)=-4y^2+q(x)$. Differentiating with respect to x in order to determine $q(x)$, we find $H_x(x,y)=q'(x)=-2x$. Therefore, $q(x)=-x^2+C$. Dropping the additive constant, we obtain a Hamiltonian function, $H(x,y)=-x^2-4y^2$.

16 (c). From part (b), the phase-plane trajectories are defined by $-x^2-4y^2=C$. If a phase-plane trajectory passes through the point $(1,1)$, then $C=-5$ and the trajectory is given by $x^2+4y^2=5$.

17 (a). For the system

$$x' = 2y\cos x$$
$$y' = y^2\sin x$$

we have $f(x,y)=2y\cos x$ and $g(x,y)=y^2\sin x$. Thus, $f_x=-2y\sin x$ and $g_y=2y\sin x$. Since $f_x=-g_y$, the system is Hamiltonian.

17 (b). Let $H(x,y)$ denote the Hamiltonian function. Thus, $H_x(x,y)=-g(x,y)=-y^2\sin x$. Integrating with respect to x, we obtain $H(x,y)=y^2\cos x+p(y)$. Differentiating with respect to y in order to determine $p(y)$, we find $H_y(x,y)=2y\cos x+p'(y)=f(x,y)=2y\cos x$. Therefore, $p'(y)=0$ and hence $p(y)=C$ is a constant function. Dropping the constant, we obtain a Hamiltonian function, $H(x,y)=y^2\cos x$.

17 (c). From part (b), the phase-plane trajectories are defined by $y^2\cos x = C$. If a phase-plane trajectory passes through the point $(1,1)$, then $C = \cos 1$ and the trajectory is given by $y^2\cos x = \cos 1$.

18 (a). For the system

$$x' = 2y - x + 3$$
$$y' = y + 4x^3 - 2x$$

we have $f_x = -1$ and $g_y = 1$. Since $f_x = -g_y$, the system is Hamiltonian.

18 (b). Let $H(x,y)$ denote the Hamiltonian function. Thus, $H_y(x,y) = f(x,y) = 2y - x + 3$. Integrating with respect to y, we obtain $H(x,y) = y^2 - xy - 3y + q(x)$. Differentiating with respect to x in order to determine $q(x)$, we find $H_x(x,y) = -y + q'(x) = -y - 4x^3 + 2x$. Therefore, $q(x) = -x^4 + x^2 + C$. Dropping the additive constant, we obtain a Hamiltonian function, $H(x,y) = y^2 - xy + 3y - x^4 + x^2$.

18 (c). If a phase-plane trajectory $H(x,y) = C$ passes through the point $(1,1)$, then the trajectory is given by $y^2 - xy + 3y - x^4 + x^2 = 8$.

19 (a). For the system

$$x' = -2y$$
$$y' = 3x^2$$

we have $f(x,y) = -2y$ and $g(x,y) = 3x^2$. Thus, $f_x = 0$ and $g_y = 0$. Since $f_x = -g_y$, the system is Hamiltonian.

19 (b). Let $H(x,y)$ denote the Hamiltonian function. Thus, $H_x(x,y) = -g(x,y) = -3x^2$. Integrating with respect to x, we obtain $H(x,y) = -x^3 + p(y)$. Differentiating with respect to y in order to determine $p(y)$, we find $H_y(x,y) = p'(y) = f(x,y) = -2y$. Therefore, $p'(y) = -2y$ and hence $p(y) = -y^2 + C$ is a constant function. Dropping the additive constant, we obtain a Hamiltonian function, $H(x,y) = -x^3 - y^2$.

19 (c). From part (b), the phase-plane trajectories are defined by $-x^3 - y^2 = C$. If a phase-plane trajectory passes through the point $(1,1)$, then $C = -2$ and the trajectory is given by $x^3 + y^2 = 2$.

20 (a). For the system

$$x' = xe^{xy}$$
$$y' = -2x - ye^{xy}$$

we have $f_x = e^{xy} + xye^{xy}$ and $g_y = -e^{xy} - xye^{xy}$. Since $f_x = -g_y$, the system is Hamiltonian.

20 (b). Let $H(x,y)$ denote the Hamiltonian function. Thus, $H_y(x,y) = f(x,y) = xe^{xy}$. Integrating with respect to y, we obtain $H(x,y) = e^{xy} + q(x)$. Differentiating with respect to x in order to determine $q(x)$, we find $H_x(x,y) = ye^{xy} + q'(x) = 2x + ye^{xy}$. Therefore, $q(x) = x^2 + C$. Dropping the additive constant, we obtain a Hamiltonian function, $H(x,y) = e^{xy} + x^2$.

20 (c). If a phase-plane trajectory $H(x,y) = C$ passes through the point $(1,1)$, then the trajectory is given by $e^{xy} + x^2 = 1 + e$.

21. Consider the system

$$x' = x^3 + 3\sin(2x+3y)$$
$$y' = -3x^2y - 2\sin(2x+3y)\ .$$

Calculating the partial derivatives, we have $f_x = 3x^2 + 6\cos(2x+3y)$ and $g_y = -3x^2 - 6\cos(2x+3y)$. Since $f_x = -g_y$, the system is Hamiltonian.
Let $H(x,y)$ denote the Hamiltonian function. Thus, $H_x(x,y) = -g(x,y) = 3x^2y + 2\sin(2x+3y)$. Integrating with respect to *x*, we obtain $H(x,y) = x^3y - \cos(2x+3y) + p(y)$. Differentiating with respect to *y* in order to determine $p(y)$, we find $H_y(x,y) = x^3 + 3\sin(2x+3y) + p'(y) = f(x,y) = x^3 + 3\sin(2x+3y)$. Therefore, $p'(y) = 0$ and hence $p(y) = C$ is a constant function. We obtain a Hamiltonian function, $H(x,y) = x^3y - \cos(2x+3y)$.

22. Consider the system

$$x' = e^{xy} + y^3$$
$$y' = -e^{xy} - x^3\ .$$

Calculating the partial derivatives, we have $f_x = ye^{xy}$ and $g_y = -xe^{xy}$. Since $f_x \neq -g_y$, the system is not Hamiltonian.

23. Consider the system

$$x' = -\sin(2xy) - x$$
$$y' = \sin(2xy) + y\ .$$

Calculating the partial derivatives, we have $f_x = -2y\cos(2xy) - 1$ and $g_y = 2x\cos(2xy) + 1$. Since $f_x \neq -g_y$, the system is not Hamiltonian.

24. Consider the system

$$x' = -3x^2 + xe^y$$
$$y' = 6xy + 3x - e^y\ .$$

Calculating the partial derivatives, we have $f_x = -6x + e^y$ and $g_y = 6x - e^y$. Since $f_x = -g_y$, the system is Hamiltonian. Let $H(x,y)$ denote the Hamiltonian function. Thus, $H_x(x,y) = -g(x,y) = -6xy - 3x + e^y$. Integrating with respect to *x*, we obtain $H(x,y) = -3x^2y - (3/2)x^2 + p(y)$. Differentiating with respect to *y* in order to determine $p(y)$, we find $H_y(x,y) = -3x^2 + p'(y) = f(x,y) = -3x^2 + xe^y$. Therefore, $p'(y) = xe^y$ and hence $p(y) = xe^y + C$. Dropping the additive constant, we obtain a Hamiltonian function, $H(x,y) = -3x^2y - (3/2)x^2 + xe^y$.

25. Consider the system

$$x' = y$$
$$y' = x - x^2\ .$$

Calculating the partial derivatives, we have $f_x = 0$ and $g_y = 0$. Since $f_x = -g_y$, the system is Hamiltonian.
Let $H(x,y)$ denote the Hamiltonian function. Thus, $H_x(x,y) = -g(x,y) = x^2 - x$. Integrating with respect to *x*, we obtain $H(x,y) = (1/6)(2x^3 - 3x^2) + p(y)$. Differentiating with respect to *y* in order to determine $p(y)$, we find $H_y(x,y) = p'(y) = f(x,y) = y$. Therefore, $p'(y) = y$ and hence $p(y) = 0.5y^2 + C$. Dropping the additive constant, we obtain a Hamiltonian function, $H(x,y) = (1/6)(2x^3 - 3x^2 + 3y^2)$.

26. Consider the system

$$x' = x + 2y$$
$$y' = x^3 - 2x + y \ .$$

Calculating the partial derivatives, we have $f_x = 1$ and $g_y = 1$. Since $f_x \neq -g_y$, the system is not Hamiltonian.

27. Consider the system

$$x' = f(y)$$
$$y' = g(x) \ .$$

Calculating the partial derivatives, we have $\partial_x[f(y)] = 0$ and $\partial_y[g(x)] = 0$. Since $\partial_x[f(y)] = -\partial_y[g(x)]$, the system is Hamiltonian.

Let $H(x,y)$ denote the Hamiltonian function. Thus, $H_x(x,y) = -g(x)$. Integrating with respect to x, we obtain $H(x,y) = -G(x) + p(y)$. Differentiating with respect to y in order to determine $p(y)$, we find $H_y(x,y) = p'(y) = f(y)$. Therefore, $p(y) = F(y) + C$. Dropping the additive constant, we obtain a Hamiltonian function, $H(x,y) = F(y) - G(x)$.

28. Consider the system

$$x' = f(y) + 2y$$
$$y' = g(x) + 6x \ .$$

Calculating the partial derivatives, we have $\partial_x[f(y) + 2y] = 0$ and $\partial_y[g(x) + 6x] = 0$. Since $\partial_x[f(y) + 2y] = -\partial_y[g(x) + 6x]$, the system is Hamiltonian. Let $H(x,y)$ denote the Hamiltonian function. Thus, $H_x(x,y) = -g(x) - 6x$. Integrating with respect to x, we obtain $H(x,y) = -G(x) - 3x^2 + p(y)$. Differentiating with respect to y in order to determine $p(y)$, we find $H_y(x,y) = p'(y) = f(y) + 2y$. Therefore, $p(y) = F(y) + y^2 + C$. Dropping the additive constant, we obtain a Hamiltonian function, $H(x,y) = -G(x) - 3x^2 + F(y) + y^2$.

29. Consider the system

$$x' = 3f(y) - 2xy$$
$$y' = g(x) + y^2 + 1 \ .$$

Calculating the partial derivatives, we have $\partial_x[3f(y) - 2xy] = -2y$ and $\partial_y[g(x) + y^2 + 1] = 2y$. Since $\partial_x[3f(y) - 2xy] = -\partial_y[g(x) + y^2 + 1]$, the system is Hamiltonian.

Let $H(x,y)$ denote the Hamiltonian function. Thus, $H_x(x,y) = -g(x) - y^2 - 1$. Integrating with respect to x, we obtain $H(x,y) = -G(x) - y^2x - x + p(y)$. Differentiating with respect to y in order to determine $p(y)$, we find $H_y(x,y) = -2yx + p'(y) = 3f(y) - 2xy$. Therefore, $p(y) = 3F(y) + C$. Dropping the additive constant, we obtain a Hamiltonian function, $H(x,y) = 3F(y) - G(x) - y^2x - x$.

30. Consider the system

$$x' = f(x - y) + 2y$$
$$y' = f(x - y) \ .$$

Calculating the partial derivatives, we have $\partial_x[f(x-y) + 2y] = f'(x-y)$ and $\partial_y[f(x-y)] = -f'(x-y)$. Since $\partial_x[f(x-y) + 2y] = -\partial_y[f(x-y)]$, the system is Hamiltonian. Let $H(x,y)$ denote the Hamiltonian function. Thus, $H_x(x,y) = -f(x-y)$. Integrating with respect to x, we obtain $H(x,y) = -F(x-y) + p(y)$. Differentiating with respect to y in order to determine $p(y)$, we find $H_y(x,y) = f(x-y) + p'(y) = f(x-y) + 2y$.

Therefore, $p(y) = y^2 + C$. Dropping the additive constant, we obtain a Hamiltonian function, $H(x,y) = -F(x-y) + y^2$.

31. Consider the composition $K(x(t),y(t))$. Differentiating with respect to t, we obtain $\frac{d}{dt}K(x(t),y(t)) = \frac{\partial K}{\partial x}\frac{dx}{dt} + \frac{\partial K}{\partial y}\frac{dy}{dt} = -(\mu g)f + (\mu f)g = 0$. Therefore, $K(x(t),y(t))$ is a conserved quantity.

Section 8.4

1 (a). All points lying within the ellipse E having semi-major axis ε and semi-minor axis $\varepsilon/2$ lie within the circle of radius ε. Likewise, all points lying within the circle of radius $\varepsilon/2$ lie within the ellipse E. Therefore, given $\varepsilon > 0$, choose $\delta = \varepsilon/2$.

1 (b). The origin is not an asymptotically stable equilibrium point since the solution points remain on an ellipse and do not approach the origin as $t \to \infty$.

2. The origin is an unstable equilibrium point. Any solution point starting near the origin will follow a branch of the hyperbola and will eventually exit any circle centered at the origin.

3 (a). Making the substitution $y = x'$, the scalar equation $x'' + \gamma x' + x = 0$ can be expressed as the system

$$x' = y$$
$$y' = -x - \gamma y.$$

The origin is the only equilibrium point for this system.

3 (b). We analyze stability by appealing to Theorem 8.3. The system in part (a) has the form $\mathbf{y}' = A\mathbf{y}$ where $A = \begin{bmatrix} 0 & 1 \\ -1 & -\gamma \end{bmatrix}$. The characteristic polynomial for A is $p(\lambda) = \lambda^2 + \gamma\lambda + 1$ and thus the eigenvalues of A are $\lambda_1 = 0.5\left(-\gamma - \sqrt{\gamma^2 - 4}\right)$ and $\lambda_2 = 0.5\left(-\gamma + \sqrt{\gamma^2 - 4}\right)$. When $\gamma^2 - 4 \geq 0$, we see that $\lambda_1 \leq \lambda_2$. Thus, if $2 \leq \gamma$, then $\lambda_1 \leq \lambda_2 < 0$ which shows the origin is asymptotically stable. On the other hand, if $\gamma \leq -2$, then $0 < \lambda_1 \leq \lambda_2$ which shows the origin is an unstable equilibrium point. For $-2 < \gamma < 2$, the eigenvalues are complex with nonzero imaginary parts. For $-2 < \gamma < 0$, the real parts of λ_1 and λ_2 are positive, which shows the origin is an unstable equilibrium point. Likewise, for $0 < \gamma < 2$, the origin is an asymptotically stable equilibrium point. When $\gamma = 0$, the origin is a stable (but not asymptotically stable) equilibrium point.

4. For the system $\mathbf{y}' = \begin{bmatrix} -3 & -2 \\ 4 & 3 \end{bmatrix}\mathbf{y}$, the coefficient matrix has eigenvalues $\lambda_1 = -1$ and $\lambda_2 = 1$. Thus, by Theorem 8.3, the origin is an unstable equilibrium point.

5. For the system $\mathbf{y}' = \begin{bmatrix} 5 & -14 \\ 3 & -8 \end{bmatrix}\mathbf{y}$, the coefficient matrix has eigenvalues $\lambda_1 = -1$ and $\lambda_2 = -2$. Thus, by Theorem 8.3, the origin is an asymptotically stable equilibrium point.

6. For the system $\mathbf{y}' = \begin{bmatrix} 0 & -2 \\ 2 & 0 \end{bmatrix}\mathbf{y}$, the coefficient matrix has eigenvalues $\lambda_1 = 2i$ and $\lambda_2 = -2i$. Thus, by Theorem 8.3, the origin is a stable equilibrium point but not an asymptotically stable equilibrium point.

7. For the system $\mathbf{y}' = \begin{bmatrix} 1 & 4 \\ -1 & 1 \end{bmatrix}\mathbf{y}$, the coefficient matrix has eigenvalues $\lambda_1 = 1 + 2i$ and $\lambda_2 = 1 - 2i$. Thus, by Theorem 8.3, the origin is an unstable equilibrium point.

8. For the system $\mathbf{y}' = \begin{bmatrix} -7 & -3 \\ 5 & 1 \end{bmatrix}\mathbf{y}$, the coefficient matrix has eigenvalues $\lambda_1 = -4$ and $\lambda_2 = -2$. Thus, by Theorem 8.3, the origin is an asymptotically stable equilibrium point.

9. For the system $\mathbf{y}' = \begin{bmatrix} 9 & 5 \\ -7 & -3 \end{bmatrix}\mathbf{y}$, the coefficient matrix has eigenvalues $\lambda_1 = 2$ and $\lambda_2 = 4$. Thus, by Theorem 8.3, the origin is an unstable equilibrium point.

10. For the system $\mathbf{y}' = \begin{bmatrix} -3 & -5 \\ 2 & -1 \end{bmatrix}\mathbf{y}$, the coefficient matrix has eigenvalues $\lambda_1 = -2 + 3i$ and $\lambda_2 = -2 - 3i$. Thus, by Theorem 8.3, the origin is an asymptotically stable equilibrium point.

11. For the system $\mathbf{y}' = \begin{bmatrix} 9 & -4 \\ 15 & -7 \end{bmatrix}\mathbf{y}$, the coefficient matrix has eigenvalues $\lambda_1 = 3$ and $\lambda_2 = -1$. Thus, by Theorem 8.3, the origin is an unstable equilibrium point.

12. For the system $\mathbf{y}' = \begin{bmatrix} -13 & -8 \\ 15 & 9 \end{bmatrix}\mathbf{y}$, the coefficient matrix has eigenvalues $\lambda_1 = -3$ and $\lambda_2 = -1$. Thus, by Theorem 8.3, the origin is an asymptotically stable equilibrium point.

13. For the system $\mathbf{y}' = \begin{bmatrix} 3 & -2 \\ 5 & -3 \end{bmatrix}\mathbf{y}$, the coefficient matrix has eigenvalues $\lambda_1 = i$ and $\lambda_2 = -i$. Thus, by Theorem 8.3, the origin is a stable (but not asymptotically stable) equilibrium point.

14. For the system $\mathbf{y}' = \begin{bmatrix} 1 & -5 \\ 1 & -3 \end{bmatrix}\mathbf{y}$, the coefficient matrix has eigenvalues $\lambda_1 = -1 + i$ and $\lambda_2 = -1 - i$. Thus, by Theorem 8.3, the origin is an asymptotically stable equilibrium point.

15. For the system $\mathbf{y}' = \begin{bmatrix} -3 & 3 \\ 1 & -5 \end{bmatrix}\mathbf{y}$, the coefficient matrix has eigenvalues $\lambda_1 = -6$ and $\lambda_2 = -2$. Thus, by Theorem 8.3, the origin is an asymptotically stable equilibrium point.

16. Eigenvalues are $\lambda_1 = -2$ and $\lambda_2 = 3$. Since one of the eigenvalues is real and positive, the origin is an unstable equilibrium point.

17. Eigenvalues are $\lambda_1 = 2$ and $\lambda_2 = 3$. Since the eigenvalues are real and positive, the origin is an unstable equilibrium point.

18. Eigenvalues are $\lambda_1 = -4$ and $\lambda_2 = -2$. Since the eigenvalues are real and negative, the origin is an asymptotically stable equilibrium point.

19. Eigenvalues are $\lambda_1 = 1 - 2i$ and $\lambda_2 = 1 + 2i$. Since the eigenvalues are complex with positive real parts, the origin is an unstable equilibrium point.

20. Eigenvalues are $\lambda_1 = -2i$ and $\lambda_2 = 2i$. Since the eigenvalues are purely imaginary, the origin is a stable equilibrium point but it is not an asymptotically stable equilibrium point.

21. Eigenvalues are $\lambda_1 = -2 - 2i$ and $\lambda_2 = -2 + 2i$. Since the eigenvalues are complex with negative real parts, the origin is an asymptotically stable equilibrium point.

22. Eigenvalues are $\lambda_1 = -2$ and $\lambda_2 = 3$. Since one of the eigenvalues is real and positive, the origin is an unstable equilibrium point.

23. Eigenvalues are $\lambda_1 = -2$ and $\lambda_2 = -3$. Since the eigenvalues are real and negative, the origin is an asymptotically stable equilibrium point.

24 (a). Solving $\mathbf{0} = A\mathbf{y}_e + \mathbf{g}_0$, it follows that $\mathbf{y}_e = -A^{-1}\mathbf{g}_0$ is the unique equilibrium point.

24 (b). Let $\mathbf{z}(t) = \mathbf{y}(t) - \mathbf{y}_e$. Then, $\mathbf{z}' = \mathbf{y}' = A\mathbf{y} + \mathbf{g}_0 = A\mathbf{y} - A\mathbf{y}_e = A\mathbf{z}$. Theorem 8.3 can be applied to the new system $\mathbf{z}' = A\mathbf{z}$.

25. For the system $\mathbf{y}' = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}\mathbf{y} + \begin{bmatrix} -4 \\ 2 \end{bmatrix}$, the unique equilibrium point is $\mathbf{y}_e = -A^{-1}\begin{bmatrix} -4 \\ 2 \end{bmatrix} = -(1/3)\begin{bmatrix} -2 & -1 \\ -1 & -2 \end{bmatrix}\begin{bmatrix} -4 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \end{bmatrix}$. With the change of variable $\mathbf{z}(t) = \mathbf{y}(t) - \mathbf{y}_e$ the system becomes $(\mathbf{z} + \mathbf{y}_e)' = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}(\mathbf{z} + \mathbf{y}_e) + \begin{bmatrix} -4 \\ 2 \end{bmatrix}$ or $\mathbf{z}' = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}\mathbf{z} + \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}\mathbf{y}_e + \begin{bmatrix} -4 \\ 2 \end{bmatrix}$. This last system reduces to the homogeneous system $\mathbf{z}' = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}\mathbf{z}$. The coefficient matrix has eigenvalues $\lambda_1 = -3$ and $\lambda_2 = -1$. By Theorem 8.3, the origin is an asymptotically stable equilibrium point of $\mathbf{z}' = A\mathbf{z}$ and therefore, $\mathbf{y}_e$ is an asymptotically stable equilibrium point of the nonhomogeneous system $\mathbf{y}' = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}\mathbf{y} + \begin{bmatrix} -4 \\ 2 \end{bmatrix}$.

26. For the system $\mathbf{y}' = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\mathbf{y} + \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, the unique equilibrium point is $\mathbf{y}_e = -A^{-1}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$. With the change of variable $\mathbf{z}(t) = \mathbf{y}(t) - \mathbf{y}_e$ the system reduces to the homogeneous system $\mathbf{z}' = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\mathbf{z}$. The coefficient matrix has eigenvalues $\lambda_1 = i$ and $\lambda_2 = -i$. By Theorem 8.3, the origin is a stable but not an asymptotically stable equilibrium point of $\mathbf{z}' = A\mathbf{z}$. Therefore, $\mathbf{y}_e$ is a stable but not an asymptotically stable equilibrium point of the nonhomogeneous system.

27. For the system $\mathbf{y}' = \begin{bmatrix} 3 & 2 \\ -4 & -3 \end{bmatrix}\mathbf{y} + \begin{bmatrix} -2 \\ 2 \end{bmatrix}$, the unique equilibrium point is $\mathbf{y}_e = -A^{-1}\begin{bmatrix} -2 \\ 2 \end{bmatrix} = \begin{bmatrix} -3 & -2 \\ 4 & 3 \end{bmatrix}\begin{bmatrix} -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$. With the change of variable $\mathbf{z}(t) = \mathbf{y}(t) - \mathbf{y}_e$ the system becomes $(\mathbf{z} + \mathbf{y}_e)' = \begin{bmatrix} 3 & 2 \\ -4 & -3 \end{bmatrix}(\mathbf{z} + \mathbf{y}_e) + \begin{bmatrix} -2 \\ 2 \end{bmatrix}$ or $\mathbf{z}' = \begin{bmatrix} 3 & 2 \\ -4 & -3 \end{bmatrix}\mathbf{z} + \begin{bmatrix} 3 & 2 \\ -4 & -3 \end{bmatrix}\mathbf{y}_e + \begin{bmatrix} -2 \\ 2 \end{bmatrix}$. This last system reduces to the homogeneous system $\mathbf{z}' = \begin{bmatrix} 3 & 2 \\ -4 & -3 \end{bmatrix}\mathbf{z}$. The coefficient matrix has eigenvalues $\lambda_1 = -1$ and $\lambda_2 = 1$. By Theorem 8.3, the origin is an unstable equilibrium point of $\mathbf{z}' = A\mathbf{z}$ and therefore, $\mathbf{y}_e$ is an unstable equilibrium point of the nonhomogeneous system $\mathbf{y}' = \begin{bmatrix} 3 & 2 \\ -4 & -3 \end{bmatrix}\mathbf{y} + \begin{bmatrix} -2 \\ 2 \end{bmatrix}$.

28. For the system $\mathbf{y}' = \begin{bmatrix} -1 & 1 \\ -10 & 5 \end{bmatrix}\mathbf{y} + \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, the unique equilibrium point is $\mathbf{y}_e = -A^{-1}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -3/5 \\ -8/5 \end{bmatrix}$. With the change of variable $\mathbf{z}(t) = \mathbf{y}(t) - \mathbf{y}_e$ the system reduces to the homogeneous system

$\mathbf{z}' = \begin{bmatrix} -1 & 1 \\ -10 & 5 \end{bmatrix}\mathbf{z}$. The coefficient matrix has eigenvalues $\lambda_1 = 2 + i$ and $\lambda_2 = 2 - i$. By Theorem 8.3, the origin is an unstable equilibrium point of $\mathbf{z}' = A\mathbf{z}$. Therefore, $\mathbf{y}_e$ is an unstable equilibrium point of the nonhomogeneous system.

29. For the system $\mathbf{y}' = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}\mathbf{y}$, the coefficient matrix has eigenvalues $\lambda_1 = -1, \lambda_2 = 2,$ and $\lambda_3 = 3$. Thus, by the discussion following Theorem 8.3, the origin is an unstable equilibrium point.

30. For the system $\mathbf{y}' = \begin{bmatrix} 1 & -1 & 0 \\ 0 & -1 & 2 \\ 0 & 0 & -1 \end{bmatrix}\mathbf{y} + \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}$, the unique equilibrium point is $\mathbf{y}_e = -A^{-1}\begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ 3 \end{bmatrix}$. With the change of variable $\mathbf{z}(t) = \mathbf{y}(t) - \mathbf{y}_e$ the system reduces to the homogeneous system $\mathbf{z}' = \begin{bmatrix} 1 & -1 & -2 \\ 0 & -1 & -2 \\ 0 & 0 & -1 \end{bmatrix}\mathbf{z}$. The coefficient matrix has eigenvalues $\lambda_1 = 1, \lambda_1 = -1,$ and $\lambda_3 = -1$. By Theorem 8.3, the origin is an unstable equilibrium point of $\mathbf{z}' = A\mathbf{z}$. Therefore, $\mathbf{y}_e$ is an unstable equilibrium point of the nonhomogeneous system.

31. For the system $\mathbf{y}' = \begin{bmatrix} -3 & -5 & 0 & 0 \\ 2 & -1 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & -2 & 0 \end{bmatrix}\mathbf{y}$, the coefficient matrix has eigenvalues $\lambda_1 = -2 + 3i, \lambda_2 = -2 - 3i, \lambda_3 = 2i,$ and $\lambda_4 = -2i$. Thus, by the discussion following Theorem 8.3, the origin is a stable (but not asymptotically stable) equilibrium point.

32. For the system $\mathbf{y}' = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}\mathbf{y} + \begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \end{bmatrix}$, unique equilibrium point is given by $\mathbf{y}_e = -A^{-1}\begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix}$. With the change of variables $\mathbf{z}(t) = \mathbf{y}(t) - \mathbf{y}_e$, the system reduces to the homogeneous system $\mathbf{z}' = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}\mathbf{z}$. The coefficient matrix has eigenvalues $\lambda_1 = -1, \lambda_2 = -1, \lambda_3 = -1,$ and $\lambda_4 = 1$. Thus, by the discussion following Theorem 8.3, the origin is an unstable equilibrium point.

34 (a). Since the coefficient matrix A is real and symmetric, it has real eigenvalues and a full set of eigenvectors.

34 (b). From the discussion following Theorem 8.3, the equilibrium point $\mathbf{y}_e = \mathbf{0}$ is isolated if and only if $\det[A] \neq 0$. Now, $\det[A] = 1 - \alpha^2$ and therefore, $\mathbf{y}_e = \mathbf{0}$ is an isolated equilibrium point if and only if $\alpha \neq \pm 1$.

34 (c). When $\alpha = 1$ the equilibrium points lie on the line $y = x$. When $\alpha = -1$ the equilibrium points lie on the line $y = -x$.

34 (d). No, since the eigenvalues of A are real and not purely imaginary; see Theorem 8.3.

34 (e). The eigenvalues of A are $\lambda_1 = -1 + \alpha$, and $\lambda_2 = -1 - \alpha$. By part (b), if $\mathbf{y}_e = \mathbf{0}$ is an isolated equilibrium point, then $\alpha \neq \pm 1$. Clearly, both eigenvalues are negative when $-1 < \alpha < 1$ whereas one of the eigenvalues is positive when $|\alpha| > 1$.

35. Since $\begin{bmatrix} 1 & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 2 \end{bmatrix}$, it follows that $1 + 2a_{12} = 2$ and $a_{21} + 2a_{22} = 4$. From the first equation, we have $a_{12} = 1/2$. Since $\mathbf{y} = \mathbf{0}$ is not an isolated equilibrium point, it follows that $\det[A] = 0$. Thus, $a_{22} - a_{12}a_{21} = 0$ or $a_{22} - (1/2)a_{21} = 0$. This last equation, together with the prior equation $a_{21} + 2a_{22} = 4$ tells us that $a_{21} = 2$ and $a_{22} = 1$. Thus, $A = \begin{bmatrix} 1 & 1/2 \\ 2 & 1 \end{bmatrix}$.

Section 8.5

1 (a). For the system

$$x' = x^2 + y^2 - 32$$
$$y' = y - x,$$

the equilibrium points are $\mathbf{y}_e = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$ and $\mathbf{y}_e = \begin{bmatrix} -4 \\ -4 \end{bmatrix}$.

1 (b). At an equilibrium point, the linearized system $\mathbf{z}' = A\mathbf{z}$ has coefficient matrix $A = \begin{bmatrix} 2x & 2y \\ -1 & 1 \end{bmatrix}$.

Thus, the linearized systems are (i) $\mathbf{z}' = \begin{bmatrix} 8 & 8 \\ -1 & 1 \end{bmatrix}\mathbf{z}$

and (ii) $\mathbf{z}' = \begin{bmatrix} -8 & -8 \\ -1 & 1 \end{bmatrix}\mathbf{z}$.

1 (c). In case (i), the eigenvalues are $\lambda_1 = 2.438\ldots$ and $\lambda_2 = 6.561\ldots$ and thus the nonlinear system is unstable at the corresponding equilibrium point $\mathbf{y}_e$. For case (ii), the eigenvalues are $\lambda_1 = -8.815\ldots$ and $\lambda_2 = 1.815\ldots$ and thus the nonlinear system is unstable at the corresponding equilibrium point $\mathbf{y}_e$.

2 (a). For the system

$$x' = x^2 + 9y^2 - 9$$
$$y' = x,$$

the equilibrium points are $\mathbf{y}_e = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $\mathbf{y}_e = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$.

2 (b). At an equilibrium point, the linearized system $\mathbf{z}' = A\mathbf{z}$ has coefficient matrix $A = \begin{bmatrix} 2x & 18y \\ 1 & 0 \end{bmatrix}$.

Thus, the linearized systems are (i) $\mathbf{z}' = \begin{bmatrix} 0 & 18 \\ 1 & 0 \end{bmatrix}\mathbf{z}$ and (ii) $\mathbf{z}' = \begin{bmatrix} 0 & -18 \\ 1 & 0 \end{bmatrix}\mathbf{z}$

2 (c). In case (i), the eigenvalues are $\lambda_1 = 4.242\ldots$ and $\lambda_2 = -4.242\ldots$ and thus the nonlinear system is unstable at the corresponding equilibrium point $\mathbf{y}_e$. For case (ii), the eigenvalues are $\pm 3\sqrt{2}\,i$ and thus nothing can be inferred about the stability of the nonlinear system.

3 (a). For the system

$$x' = 1 - x^2$$
$$y' = x^2 + y^2 - 2,$$

the equilibrium points are $\mathbf{y}_e = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\mathbf{y}_e = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, $\mathbf{y}_e = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$, and $\mathbf{y}_e = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

3 (b). At an equilibrium point, the linearized system $\mathbf{z}' = A\mathbf{z}$ has coefficient matrix $A = \begin{bmatrix} -2x & 0 \\ 2x & 2y \end{bmatrix}$.

Thus, the linearized systems are (i) $\mathbf{z}' = \begin{bmatrix} -2 & 0 \\ 2 & 2 \end{bmatrix}\mathbf{z}$,

(ii) $\mathbf{z}' = \begin{bmatrix} 2 & 0 \\ -2 & 2 \end{bmatrix}\mathbf{z}$, (iii) $\mathbf{z}' = \begin{bmatrix} 2 & 0 \\ -2 & -2 \end{bmatrix}\mathbf{z}$, and (iv) $\mathbf{z}' = \begin{bmatrix} -2 & 0 \\ 2 & -2 \end{bmatrix}\mathbf{z}$.

3 (c). In cases (i) – (iii), $\lambda = 2$ is an eigenvalue and thus the nonlinear system is unstable at each of the corresponding equilibrium points $\mathbf{y}_e$. For case (iv), the eigenvalues are $\lambda_1 = -2$ and $\lambda_2 = -2$ and thus the nonlinear system is asymptotically stable at the corresponding equilibrium point $\mathbf{y}_e$.

4 (a). For the system

$$x' = x - y - 1$$
$$y' = x^2 - y^2 + 1,$$

the equilibrium point is $\mathbf{y}_e = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$.

4 (b). At the equilibrium point, the linearized system $\mathbf{z}' = A\mathbf{z}$ has coefficient matrix $A = \begin{bmatrix} 1 & -1 \\ 2x & -2y \end{bmatrix}$.

Thus, the linearized system is $\mathbf{z}' = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}\mathbf{z}$.

4 (c). The eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = 2$ and thus the nonlinear system is unstable at the equilibrium point $\mathbf{y}_e$.

5 (a). For the system

$$x' = (x-2)(y-3)$$
$$y' = (x+2y)(y-1),$$

the equilibrium points are $\mathbf{y}_e = \begin{bmatrix} 2 \\ -1 \end{bmatrix}, \mathbf{y}_e = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, and $\mathbf{y}_e = \begin{bmatrix} -6 \\ 3 \end{bmatrix}$.

5 (b). At an equilibrium point, the linearized system $\mathbf{z}' = A\mathbf{z}$ has coefficient matrix $A = \begin{bmatrix} y-3 & x-2 \\ y-1 & x+4y-2 \end{bmatrix}$. Thus, the linearized systems are (i) $\mathbf{z}' = \begin{bmatrix} -4 & 0 \\ -2 & -4 \end{bmatrix}\mathbf{z}$, (ii) $\mathbf{z}' = \begin{bmatrix} -2 & 0 \\ 0 & 4 \end{bmatrix}\mathbf{z}$, and (iii) $\mathbf{z}' = \begin{bmatrix} 0 & -8 \\ 2 & 4 \end{bmatrix}\mathbf{z}$.

5 (c). In case (i), the eigenvalues are $\lambda_1 = -4$ and $\lambda_2 = -4$ and thus the nonlinear system is asymptotically stable at the corresponding equilibrium point $\mathbf{y}_e$. For case (ii), the eigenvalues are $\lambda_1 = -2$ and $\lambda_2 = 4$ and thus the nonlinear system is unstable at the corresponding equilibrium point $\mathbf{y}_e$. In case (iii), the eigenvalues are $\lambda_1 = 2 + 2\sqrt{3}\,i$ and $\lambda_2 = 2 - 2\sqrt{3}\,i$. Thus the nonlinear system is unstable at the corresponding equilibrium point $\mathbf{y}_e$.

6 (a). For the system

$$x' = (x-y)(y+1)$$
$$y' = (x+2)(y-4),$$

the equilibrium points are $\mathbf{y}_e = \begin{bmatrix} -2 \\ -2 \end{bmatrix}, \mathbf{y}_e = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$, and $\mathbf{y}_e = \begin{bmatrix} -2 \\ -1 \end{bmatrix}$.

6 (b). At an equilibrium point, the linearized system $\mathbf{z}' = A\mathbf{z}$ has coefficient matrix $A = \begin{bmatrix} y+1 & x-2y-1 \\ y-4 & x+2 \end{bmatrix}$. Thus, the linearized systems are (i) $\mathbf{z}' = \begin{bmatrix} -1 & 1 \\ -6 & 0 \end{bmatrix}\mathbf{z}$, (ii) $\mathbf{z}' = \begin{bmatrix} 5 & -5 \\ 0 & 6 \end{bmatrix}\mathbf{z}$, and (iii) $\mathbf{z}' = \begin{bmatrix} 0 & -1 \\ -5 & 0 \end{bmatrix}\mathbf{z}$.

6 (c). In case (i), the eigenvalues are $-0.5 \pm 0.5i\sqrt{23}$ and thus the nonlinear system is asymptotically stable at the corresponding equilibrium point $\mathbf{y}_e$. For case (ii), the eigenvalues are $\lambda_1 = 5$ and $\lambda_2 = 6$ and thus the nonlinear system is unstable at the corresponding equilibrium point $\mathbf{y}_e$. In case (iii), the eigenvalues are $\pm\sqrt{5}$. Thus the nonlinear system is unstable at the corresponding equilibrium point $\mathbf{y}_e$.

7 (a). For the system

$$x' = (x-2y)(y+4)$$
$$y' = 2x - y,$$

the equilibrium points are $\mathbf{y}_e = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $\mathbf{y}_e = \begin{bmatrix} -2 \\ -4 \end{bmatrix}$.

7 (b). At an equilibrium point, the linearized system $\mathbf{z}' = A\mathbf{z}$ has coefficient matrix $A = \begin{bmatrix} y+4 & x-4y-8 \\ 2 & -1 \end{bmatrix}$. Thus, the linearized systems are (i) $\mathbf{z}' = \begin{bmatrix} 4 & -8 \\ 2 & -1 \end{bmatrix}\mathbf{z}$, and (ii) $\mathbf{z}' = \begin{bmatrix} 0 & 6 \\ 2 & -1 \end{bmatrix}\mathbf{z}$.

7 (c). In case (i), the eigenvalues are $\lambda_1 = 0.5(3+\sqrt{39}\,i)$ and $\lambda_2 = 0.5(3-\sqrt{39}\,i)$ and thus the nonlinear system is unstable at the corresponding equilibrium point $\mathbf{y}_e$. For case (ii), the eigenvalues are $\lambda_1 = -4$ and $\lambda_2 = 3$ and thus the nonlinear system is unstable at the corresponding equilibrium point $\mathbf{y}_e$.

8 (a). For the system

$$x' = xy - 1$$
$$y' = (x+4y)(x-1),$$

the equilibrium point is $\mathbf{y}_e = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

8 (b). At the equilibrium point, the linearized system $\mathbf{z}' = A\mathbf{z}$ has coefficient matrix $A = \begin{bmatrix} y & x \\ 2x+4y-1 & 4(x-1) \end{bmatrix}$. Thus, the linearized system is $\mathbf{z}' = \begin{bmatrix} 1 & 1 \\ 5 & 0 \end{bmatrix}\mathbf{z}$.

8 (c). The eigenvalues are $0.5\left(1 \pm \sqrt{21}\right)$ and thus the nonlinear system is unstable at the equilibrium point $\mathbf{y}_e$.

9 (a). For the system

$$x' = y^2 - x$$
$$y' = x^2 - y,$$

the equilibrium points are $\mathbf{y}_e = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ and $\mathbf{y}_e = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

9 (b). At an equilibrium point, the linearized system $\mathbf{z}' = A\mathbf{z}$ has coefficient matrix $A = \begin{bmatrix} -1 & 2y \\ 2x & -1 \end{bmatrix}$.

Thus, the linearized systems are (i) $\mathbf{z}' = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\mathbf{z}$,

and (ii) $\mathbf{z}' = \begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix}\mathbf{z}$.

9 (c). In case (i), the eigenvalues are $\lambda_1 = -1$ and $\lambda_2 = -1$ and thus the nonlinear system is asymptotically stable at the corresponding equilibrium point $\mathbf{y}_e$. For case (ii), the eigenvalues are $\lambda_1 = -3$ and $\lambda_2 = 1$ and thus the nonlinear system is unstable at the corresponding equilibrium point $\mathbf{y}_e$.

10. At an equilibrium point, the linearized system $\mathbf{z}' = A\mathbf{z}$ has coefficient matrix $A = \begin{bmatrix} (1/2)[1-x-(1/2)y] & -(1/4)x \\ -(1/12)y & (1/4)[1-(1/3)x-(4/3)y] \end{bmatrix}$. Thus, the linearized systems are: (i) at $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, $\mathbf{z}' = \begin{bmatrix} 1/2 & 0 \\ 0 & 1/4 \end{bmatrix}\mathbf{z}$, (ii) at $\begin{bmatrix} 0 \\ 3/2 \end{bmatrix}$, $\mathbf{z}' = \begin{bmatrix} 1/8 & 0 \\ -1/8 & -1/4 \end{bmatrix}\mathbf{z}$,
(iii) at $\begin{bmatrix} 2 \\ 0 \end{bmatrix}$, $\mathbf{z}' = \begin{bmatrix} -1/2 & -1/2 \\ 0 & 1/12 \end{bmatrix}\mathbf{z}$. Thus, in all three of these cases, the system is unstable at the corresponding equilibrium point.

11 (c). By Taylor's theorem, $f(z) = f(0) + f'(0)z + f''(\gamma)z^2/2$ where γ is between z and 0. For $f(z) = \sin z$, we have $\sin z_1 - z_1 = (-\sin\gamma)z_1^2/2$ where γ is between z_1 and 0. Now, $\|\mathbf{g}(\mathbf{z})\|/\|\mathbf{z}\| = |z_1 - \sin z_1|/\sqrt{z_1^2 + z_2^2} \le |z_1 - \sin z_1|/|z_1|$. So, by the remarks above, $\|\mathbf{g}(\mathbf{z})\|/\|\mathbf{z}\| \le |z_1^2/2|/|z_1| = |z_1|/2$. Hence, since $|z_1|/2$ goes to 0 as $\mathbf{z}$ goes to $\mathbf{0}$, the system is almost linear at both equilibrium points.

12 (a). For the given system $\mathbf{z}' = A\mathbf{z} + \mathbf{g}(\mathbf{z})$, the coefficient matrix A is $A = \begin{bmatrix} 9 & -4 \\ 15 & -7 \end{bmatrix}$, while $\mathbf{g}(\mathbf{z}) = \begin{bmatrix} z_2^2 \\ 0 \end{bmatrix}$.

12 (b). $\|\mathbf{g}(\mathbf{z})\| = z_2^2$, or using polar coordinates with $z_1 = r\cos\theta$ and $z_2 = r\sin\theta$, we obtain $\|\mathbf{g}(\mathbf{z})\| = r^2\sin^2\theta$.

12 (c). From part (b), $\|\mathbf{g}(\mathbf{z})\|/\|\mathbf{z}\| = r^2\sin^2\theta/r = r\sin^2\theta$. Thus, $\|\mathbf{g}(\mathbf{z})\|/\|\mathbf{z}\| \to 0$ as $\|\mathbf{z}\| \to 0$. In addition to the limit requirement, the system satisfies the other necessary conditions to be an almost linear system.

12 (d). The eigenvalues of A are $\lambda_1 = -1$ and $\lambda_2 = 3$. Thus, by Theorem 8.4, $\mathbf{z} = \mathbf{0}$ is an unstable equilibrium point.

13 (a). For the system $\mathbf{z}' = A\mathbf{z} + \mathbf{g}(\mathbf{z})$,

$$z_1' = 5z_1 - 14z_2 + z_1z_2$$
$$z_2' = 3z_1 - 8z_2 + z_1^2 + z_2^2,$$

the coefficient matrix A is given by $A = \begin{bmatrix} 5 & -14 \\ 3 & -8 \end{bmatrix}$, while $\mathbf{g}(\mathbf{z}) = \begin{bmatrix} z_1z_2 \\ z_1^2 + z_2^2 \end{bmatrix}$.

13 (b). Using polar coordinates with $z_1 = r\cos\theta$ and $z_2 = r\sin\theta$, we obtain $\|\mathbf{g}(\mathbf{z})\| = \sqrt{(z_1z_2)^2 + (z_1^2 + z_2^2)^2} = \sqrt{(r^2\cos\theta\sin\theta)^2 + (r^2)^2}$ or $\|\mathbf{g}(\mathbf{z})\| = \sqrt{r^4(\cos^2\theta\sin^2\theta + 1)}$. (Also note that $\|\mathbf{z}\| = r$.)

13 (c). From part (b), $\|\mathbf{g}(\mathbf{z})\|/\|\mathbf{z}\| = \sqrt{r^4(\cos^2\theta\sin^2\theta + 1)}/r \le r^2\sqrt{2}/r = r\sqrt{2}$. Thus, $\|\mathbf{g}(\mathbf{z})\|/\|\mathbf{z}\| \to 0$ as $\|\mathbf{z}\| \to 0$. In addition to the limit requirement, the system satisfies the other necessary conditions to be an almost linear system.

13 (d). The eigenvalues of A are $\lambda_1 = -2$ and $\lambda_2 = -1$. Thus, by Theorem 8.4, $\mathbf{z} = \mathbf{0}$ is an asymptotically stable equilibrium point.

14 (a). For the given system $\mathbf{z}' = A\mathbf{z} + \mathbf{g}(\mathbf{z})$, the coefficient matrix A is $A = \begin{bmatrix} -3 & 1 \\ 2 & -2 \end{bmatrix}$, while $\mathbf{g}(\mathbf{z}) = \begin{bmatrix} z_1^2 + z_2^2 \\ (z_1^2 + z_2^2)^{1/3} \end{bmatrix}$.

14 (b). Using polar coordinates with $z_1 = r\cos\theta$ and $z_2 = r\sin\theta$, we obtain $\|\mathbf{g}(\mathbf{z})\| = r^{2/3}\sqrt{1 + r^{8/3}}$.

14 (c). From part (b), $\|\mathbf{g}(\mathbf{z})\|/\|\mathbf{z}\| = r^{2/3}\sqrt{1 + r^{8/3}}/r = \sqrt{1 + r^{8/3}}/r^{1/3}$. Thus, $\|\mathbf{g}(\mathbf{z})\|/\|\mathbf{z}\|$ does not exist as $\|\mathbf{z}\| \to 0$. The system is not almost linear at $\mathbf{z} = \mathbf{0}$.

15 (a). For the system $\mathbf{z}' = A\mathbf{z} + \mathbf{g}(\mathbf{z})$,

$$z_1' = -z_1 + 3z_2 + z_2\cos\sqrt{z_1^2 + z_2^2}$$
$$z_2' = -z_1 - 5z_2 + z_1\cos\sqrt{z_1^2 + z_2^2},$$

the coefficient matrix A is given by $A = \begin{bmatrix} -1 & 3 \\ -1 & -5 \end{bmatrix}$, while $\mathbf{g}(\mathbf{z}) = \begin{bmatrix} z_2\cos\sqrt{z_1^2 + z_2^2} \\ z_1\cos\sqrt{z_1^2 + z_2^2} \end{bmatrix}$.

15 (b). Using polar coordinates with $z_1 = r\cos\theta$ and $z_2 = r\sin\theta$, we obtain $\|\mathbf{g}(\mathbf{z})\| = \sqrt{(z_1^2 + z_2^2)\cos^2\sqrt{z_1^2 + z_2^2}} = \sqrt{r^2\cos^2 r}$ or $\|\mathbf{g}(\mathbf{z})\| = r|\cos r|$. (Also note that $\|\mathbf{z}\| = r$.)

15 (c). From part (b), $\|\mathbf{g}(\mathbf{z})\| / \|\mathbf{z}\| = r|\cos r| / r = |\cos r|$. Thus, $\|\mathbf{g}(\mathbf{z})\| / \|\mathbf{z}\| \to 1$ as $\|\mathbf{z}\| \to 0$. Therefore, the system is not an almost linear system.

16 (a). For the given system $\mathbf{z}' = A\mathbf{z} + \mathbf{g}(\mathbf{z})$, the coefficient matrix A is $A = \begin{bmatrix} -2 & 2 \\ 1 & -3 \end{bmatrix}$, while $\mathbf{g}(\mathbf{z}) = \begin{bmatrix} z_1 z_2\cos z_2 \\ z_1 z_2\sin z_2 \end{bmatrix}$.

16 (b). Using polar coordinates with $z_1 = r\cos\theta$ and $z_2 = r\sin\theta$, we obtain $\|\mathbf{g}(\mathbf{z})\| = r^2|\cos\theta\sin\theta|$.

16 (c). From part (b), $\|\mathbf{g}(\mathbf{z})\| / \|\mathbf{z}\| = r^2|\sin\theta\cos\theta| / r \le r$. Thus, $\|\mathbf{g}(\mathbf{z})\| / \|\mathbf{z}\| \to 0$ as $\|\mathbf{z}\| \to 0$. In addition to the limit requirement, the system satisfies the other necessary conditions to be an almost linear system.

16 (d). The eigenvalues of A are $\lambda_1 = -4$ and $\lambda_2 = -1$. Thus, by Theorem 8.4, $\mathbf{z} = \mathbf{0}$ is an asymptotically stable equilibrium point.

17 (a). For the system $\mathbf{z}' = A\mathbf{z} + \mathbf{g}(\mathbf{z})$,

$$z_1' = 2z_2 + z_2^2$$
$$z_2' = -2z_1 + z_1 z_2,$$

the coefficient matrix A is given by $A = \begin{bmatrix} 0 & 2 \\ -2 & 0 \end{bmatrix}$, while $\mathbf{g}(\mathbf{z}) = \begin{bmatrix} z_2^2 \\ z_1 z_2 \end{bmatrix}$.

17 (b). Using polar coordinates with $z_1 = r\cos\theta$ and $z_2 = r\sin\theta$, we obtain

$\|\mathbf{g}(\mathbf{z})\| = \sqrt{(z_1 z_2)^2 + z_2^4} = \sqrt{(r^2\cos\theta\sin\theta)^2 + r^4\sin^4\theta}$ or

$\|\mathbf{g}(\mathbf{z})\| = \sqrt{r^4\sin^2\theta(\cos^2\theta + \sin^2\theta)} = r^2|\sin\theta|$. (Also note that $\|\mathbf{z}\| = r$.)

17 (c). From part (b), $\|\mathbf{g}(\mathbf{z})\| / \|\mathbf{z}\| = r^2|\sin\theta| / r = r|\sin\theta|$. Thus, $\|\mathbf{g}(\mathbf{z})\| / \|\mathbf{z}\| \to 0$ as $\|\mathbf{z}\| \to 0$. In addition to the limit requirement, the system satisfies the other necessary conditions to be an almost linear system.

(d) The eigenvalues of A are $\lambda_1 = -2i$ and $\lambda_2 = 2i$. No conclusion can be drawn from Theorem 8.4 relative to the stability of $\mathbf{z}' = A\mathbf{z} + \mathbf{g}(\mathbf{z})$.

18 (a). For the given system $\mathbf{z}' = A\mathbf{z} + \mathbf{g}(\mathbf{z})$, the coefficient matrix A is $A = \begin{bmatrix} -3 & -5 \\ 2 & -1 \end{bmatrix}$, while $\mathbf{g}(\mathbf{z}) = \begin{bmatrix} z_1 e^{-\sqrt{z_1^2 + z_2^2}} \\ z_2 e^{-\sqrt{z_1^2 + z_2^2}} \end{bmatrix}$.

18 (b). Using polar coordinates with $z_1 = r\cos\theta$ and $z_2 = r\sin\theta$, we obtain $\|\mathbf{g}(\mathbf{z})\| = re^{-r}$.

18 (c). From part (b), $\|\mathbf{g}(\mathbf{z})\| / \|\mathbf{z}\| = e^{-r}$. Thus, $\|\mathbf{g}(\mathbf{z})\| / \|\mathbf{z}\| \to 1$ as $\|\mathbf{z}\| \to 0$; the system is not almost linear at $\mathbf{z} = \mathbf{0}$.

19 (a). For the system $\mathbf{z}' = A\mathbf{z} + \mathbf{g}(\mathbf{z})$,

$$z_1' = 9z_1 + 5z_2 + z_1 z_2$$
$$z_2' = -7z_1 - 3z_2 + z_1^2 ,$$

the coefficient matrix A is given by $A = \begin{bmatrix} 9 & 5 \\ -7 & -3 \end{bmatrix}$, while $\mathbf{g}(\mathbf{z}) = \begin{bmatrix} z_1 z_2 \\ z_1^2 \end{bmatrix}$.

19 (b). Using polar coordinates with $z_1 = r\cos\theta$ and $z_2 = r\sin\theta$, we obtain

$\|\mathbf{g}(\mathbf{z})\| = \sqrt{(z_1 z_2)^2 + z_1^4} = \sqrt{(r^2\cos\theta\sin\theta)^2 + r^4\cos^4\theta}$ or

$\|\mathbf{g}(\mathbf{z})\| = \sqrt{r^4\cos^2\theta(\cos^2\theta + \sin^2\theta)} = r^2|\cos\theta|$. (Also note that $\|\mathbf{z}\| = r$.)

19 (c). From part (b), $\|\mathbf{g}(\mathbf{z})\| / \|\mathbf{z}\| = r^2|\cos\theta| / r = r|\cos\theta|$. Thus, $\|\mathbf{g}(\mathbf{z})\| / \|\mathbf{z}\| \to 0$ as $\|\mathbf{z}\| \to 0$. In addition to the limit requirement, the system satisfies the other necessary conditions to be an almost linear system.

(d) The eigenvalues of A are $\lambda_1 = 2$ and $\lambda_2 = 4$. Thus, by Theorem 8.4, $\mathbf{z} = \mathbf{0}$ is an unstable equilibrium point of the system.

20 (a). For the given system $\mathbf{z}' = A\mathbf{z} + \mathbf{g}(\mathbf{z})$, the coefficient matrix A is $A = \begin{bmatrix} 2 & 2 \\ -5 & -2 \end{bmatrix}$, while $\mathbf{g}(\mathbf{z}) = \begin{bmatrix} 0 \\ z_1^2 \end{bmatrix}$.

20 (b). Using polar coordinates with $z_1 = r\cos\theta$ and $z_2 = r\sin\theta$, we obtain $\|\mathbf{g}(\mathbf{z})\| = r^2\cos^2\theta$.

20 (c). From part (b), $\|\mathbf{g}(\mathbf{z})\| / \|\mathbf{z}\| = r\cos^2\theta$. Thus, $\|\mathbf{g}(\mathbf{z})\| / \|\mathbf{z}\| \to 0$ as $\|\mathbf{z}\| \to 0$. In addition to the limit requirement, the system satisfies the other necessary conditions to be an almost linear system.

20 (d). The eigenvalues of A are $\lambda_1 = i\sqrt{6}$ and $\lambda_2 = -i\sqrt{6}$. Thus, no conclusions can be drawn by using Theorem 8.4.

21 (a). The system

$$x' = -x + xy + y$$
$$y' = x - xy - 2y$$

can be expressed as $\mathbf{z}' = A\mathbf{z} + \mathbf{g}(\mathbf{z})$ where the coefficient matrix A is given by $A = \begin{bmatrix} -1 & 1 \\ 1 & -2 \end{bmatrix}$, $\mathbf{z} = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$, and $\mathbf{g}(\mathbf{z}) = \begin{bmatrix} z_1 z_2 \\ -z_1 z_2 \end{bmatrix}$. Since A is invertible, the solutions of $A\mathbf{z} + \mathbf{g}(\mathbf{z}) = \mathbf{0}$ are vectors $\mathbf{z}_e$ such that $\mathbf{0} = -A^{-1}\mathbf{g}(\mathbf{z}_e)$ and therefore, we need $\mathbf{g}(\mathbf{z}_e) = \mathbf{0}$. Clearly, the only solution of $\mathbf{g}(\mathbf{z}) = \mathbf{0}$ is $\mathbf{z}_e = \mathbf{0}$.

21 (b). The linearized system is $\mathbf{z}' = A\mathbf{z}$ and we find that A has eigenvalues $\lambda_1 = -2.618\ldots$ and $\lambda_2 = -0.382\ldots$ we see that $\mathbf{z} = \mathbf{0}$ is an asymptotically stable equilibrium point of $\mathbf{z}' = A\mathbf{z}$.

21 (c). Using polar coordinates with $z_1 = r\cos\theta$ and $z_2 = r\sin\theta$, we obtain $\|\mathbf{g}(\mathbf{z})\| = \sqrt{2(z_1z_2)^2} = \sqrt{2r^4\cos^2\theta\sin^2\theta} = \sqrt{2}\,r^2|\cos\theta\sin\theta|$. (Also note that $\|\mathbf{z}\| = r$.) Therefore, $\|\mathbf{g}(\mathbf{z})\|/\|\mathbf{z}\| = \sqrt{2}\,r^2|\cos\theta\sin\theta|/r = \sqrt{2}\,r|\cos\theta|$. Thus, $\|\mathbf{g}(\mathbf{z})\|/\|\mathbf{z}\| \to 0$ as $\|\mathbf{z}\| \to 0$. In addition to the limit requirement, the system satisfies the other necessary conditions to be an almost linear system.

21 (d). By Theorem 8.4, $\mathbf{z} = \mathbf{0}$ is an asymptotically stable equilibrium point of the original system.

22 (a). The system has the form

$$x' = y$$
$$y' = 1 - (1+x)^{3/2}\ .$$

22 (c). At an equilibrium point, the linearized system $\mathbf{z}' = A\mathbf{z}$ has coefficient matrix $A = \begin{bmatrix} 0 & 1 \\ -(3/2)(1+x)^{1/2} & 0 \end{bmatrix}$. Thus, at $\mathbf{z} = \mathbf{0}$, $A = \begin{bmatrix} 0 & 1 \\ -3/2 & 0 \end{bmatrix}$. The eigenvalues of A are $\lambda_1 = i\sqrt{3/2}$ and $\lambda_2 = -i\sqrt{3/2}$ and hence the linearized system is stable but not asymptotically stable at $\mathbf{z} = \mathbf{0}$.

22 (d). Theorem 8.4 does not provide any information about the stability of the nonlinear system since the eigenvalues of the linearized system $\mathbf{z}' = A\mathbf{z}$ are purely imaginary.

23 (a). Multiplying by x' we obtain $x'x'' = x'[1-(1+x)^{3/2}]$. Integrating, we obtain $0.5(x')^2 = x - 0.4(1+x)^{5/2}$. Therefore, with $y = x'$ we have $y^2 = 2x - 0.8(1+x)^{5/2} + C$.

24 (a). At the equilibrium point $(0, 0)$, the linearized system $\mathbf{z}' = A\mathbf{z}$ has coefficient matrix $A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$. Since A is not invertible, Theorem 8.4 does not apply.

24 (b). Let $\mathbf{z} = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}$. For the given system $\mathbf{z}' = A\mathbf{z} + \mathbf{g}(\mathbf{z})$, $\mathbf{g}(\mathbf{z}) = \begin{bmatrix} -z_1^{2/3} \\ 2z_2^{1/3} \end{bmatrix}$. Using polar coordinates, $\|\mathbf{g}(\mathbf{z})\|/\|\mathbf{z}\| = \sqrt{r^{-2/3}\cos^{4/3}\theta + 4r^{-4/3}\sin^{2/3}\theta}$. Thus, the limit of $\|\mathbf{g}(\mathbf{z})\|/\|\mathbf{z}\|$ does not exist as $\|\mathbf{z}\| \to 0$; The system is not almost linear at $(0, 0)$.

27. In this case, $a_{11} = 0, a_{12} = 1, a_{21} = -1, a_{22} = 0, g_1 = \alpha r^3\cos\theta,$ and $g_2 = \alpha r^3\sin\theta$. Thus, $h(r) = \alpha r^2$ and we obtain the system

$$r' = \alpha r^3$$
$$\theta' = -1\ .$$

Solving, $r(t) = (C_1 - 2\alpha t)^{-1/2}$ and $\theta(t) = -t + C_2$. Hence, $x = (C_1 - 2\alpha t)^{-1/2}\cos(-t + C_2)$ and $y = (C_1 - 2\alpha t)^{-1/2}\sin(-t + C_2)$.

28. So, $a_{11} = 1, a_{12} = 0, a_{21} = 0, a_{22} = 1, g_1 = r^2\cos\theta,$ and $g_2 = r^2\sin\theta$. Thus, $h(r) = r$ and we obtain the initial value problem

$$r' = r + r^2\ ,\ r(0) = 1$$
$$\theta' = 0\ ,\ \theta(0) = \sqrt{3}\ .$$

The solution is $r = (2/3)e^t/[1-(2/3)e^t]$, $\theta = \pi/3$. However, the denominator in the expression for *r*, $1-(2/3)e^t$, vanishes at $3/2 = e^t$. Solving for *t*, we have $t = \ln 1.5 = 0.405\ldots$. Thus, the solution does not exist at $t = 1$.

29. So, $a_{11}=0, a_{12}=1, a_{21}=-1, a_{22}=0, g_1=-r\cos\theta\ln r^2$, and $g_2=-r\sin\theta\ln r^2$. Thus, $h(r)=-\ln r^2$ and we obtain the initial value problem

$$r'=-2r\ln r\,,\ r(0)=1$$
$$\theta'=1\,,\ \theta(0)=\pi/4\,.$$

The general solution is $r=C_1\exp(e^{-2t})$, $\theta=t+C_2$. Imposing the initial conditions we arrive at $r=\exp(e^{-2t}-1)$, $\theta=t+\pi/4$. Hence, at $t=1$, we find $x=\exp(e^{-2}-1)\cos(1+\pi/4)\approx -0.0896\ldots$ and $y=\exp(e^{-2}-1)\sin(1+\pi/4)\approx 0.411\ldots$

Section 8.6

1 (a). Since the eigenvalues are real and have opposite signs, $\mathbf{y}=\mathbf{0}$ is an unstable saddle point.

1 (d). We have $\Psi(t)=[e^{\lambda_1 t}\mathbf{x}_1, e^{\lambda_2 t}\mathbf{x}_2]=\begin{bmatrix} e^{2t} & e^{-t} \\ e^{2t} & -e^{-t}\end{bmatrix}$ and $\Psi'(t)=\begin{bmatrix} 2e^{2t} & -e^{-t} \\ 2e^{2t} & e^{-t}\end{bmatrix}$.

Therefore, $A=\Psi'(t)\Psi^{-1}(t)=\begin{bmatrix} 2e^{2t} & -e^{-t} \\ 2e^{2t} & e^{-t}\end{bmatrix}\begin{bmatrix} 0.5e^{-2t} & 0.5e^{-2t} \\ 0.5e^{t} & -0.5e^{t}\end{bmatrix}=\begin{bmatrix} 0.5 & 1.5 \\ 1.5 & 0.5\end{bmatrix}$.

2 (a). Since the eigenvalues are real and positive, $\mathbf{y}=\mathbf{0}$ is an unstable node.

2 (d). We have $\Psi(t)=[e^{\lambda_1 t}\mathbf{x}_1, e^{\lambda_2 t}\mathbf{x}_2]=\begin{bmatrix} e^{t} & 2e^{2t} \\ 2e^{t} & -e^{2t}\end{bmatrix}$ and $\Psi'(t)=\begin{bmatrix} e^{t} & 4e^{2t} \\ 2e^{t} & -2e^{2t}\end{bmatrix}$.

Therefore, $A=\Psi'(t)\Psi^{-1}(t)=\begin{bmatrix} 9/5 & -2/5 \\ -2/5 & 6/5\end{bmatrix}$.

3 (a). Since both eigenvalues are real and positive, $\mathbf{y}=\mathbf{0}$ is an unstable improper node.

3 (d). We have $\Psi(t)=[e^{\lambda_1 t}\mathbf{x}_1, e^{\lambda_2 t}\mathbf{x}_2]=\begin{bmatrix} 2e^{2t} & 0 \\ 0 & 2e^{t}\end{bmatrix}$ and $\Psi'(t)=\begin{bmatrix} 4e^{2t} & 0 \\ 0 & 2e^{t}\end{bmatrix}$.

Therefore, $A=\Psi'(t)\Psi^{-1}(t)=\begin{bmatrix} 4e^{2t} & 0 \\ 0 & 2e^{t}\end{bmatrix}\begin{bmatrix} 0.5e^{-2t} & 0 \\ 0 & 0.5e^{-t}\end{bmatrix}=\begin{bmatrix} 2 & 0 \\ 0 & 1\end{bmatrix}$.

4 (a). Since the eigenvalues are real and negative, $\mathbf{y}=\mathbf{0}$ is an asymptotically stable node.

4 (d). We have $\Psi(t)=[e^{\lambda_1 t}\mathbf{x}_1, e^{\lambda_2 t}\mathbf{x}_2]=\begin{bmatrix} e^{-2t} & e^{-t} \\ 0 & e^{-t}\end{bmatrix}$ and $\Psi'(t)=\begin{bmatrix} -2e^{-2t} & -e^{-t} \\ 0 & -e^{-t}\end{bmatrix}$.

Therefore, $A=\Psi'(t)\Psi^{-1}(t)=\begin{bmatrix} -2 & 1 \\ 0 & -1\end{bmatrix}$.

5 (a). Since the eigenvalues are real and have opposite signs, $\mathbf{y}=\mathbf{0}$ is an unstable saddle point.

5 (d). We have $\Psi(t)=[e^{\lambda_1 t}\mathbf{x}_1, e^{\lambda_2 t}\mathbf{x}_2]=\begin{bmatrix} e^{t} & 2e^{-t} \\ 0 & e^{-t}\end{bmatrix}$ and $\Psi'(t)=\begin{bmatrix} e^{t} & -2e^{-t} \\ 0 & -e^{-t}\end{bmatrix}$.

Therefore, $A=\Psi'(t)\Psi^{-1}(t)=\begin{bmatrix} e^{t} & -2e^{-t} \\ 0 & -e^{-t}\end{bmatrix}\begin{bmatrix} e^{-t} & -2e^{-t} \\ 0 & e^{t}\end{bmatrix}=\begin{bmatrix} 1 & -4 \\ 0 & -1\end{bmatrix}$.

6 (a). For $A=\begin{bmatrix} 1 & -6 \\ 1 & -4\end{bmatrix}$, the eigenvalues are $\lambda_1=-1$ and $\lambda_2=-2$.

6 (b). Since the eigenvalues are real and negative, $\mathbf{y}=\mathbf{0}$ is an asymptotically stable improper node.

7 (a). For $A=\begin{bmatrix}6 & -10\\ 2 & -3\end{bmatrix}$, the eigenvalues are $\lambda_1=1$ and $\lambda_2=2$.

7 (b). Since the eigenvalues are real and positive, $\mathbf{y}=\mathbf{0}$ is an unstable improper node.

8 (a). For $A=\begin{bmatrix}-6 & 14\\ -2 & 5\end{bmatrix}$, the eigenvalues are $\lambda_1=1$ and $\lambda_2=-2$.

8 (b). Since the eigenvalues have opposite sign, $\mathbf{y}=\mathbf{0}$ is an unstable saddle point.

9 (a). For $A=\begin{bmatrix}1 & 2\\ -5 & -1\end{bmatrix}$, the eigenvalues are $\lambda_1=3i$ and $\lambda_2=-3i$.

9 (b). Since the eigenvalues are complex with zero real part, $\mathbf{y}=\mathbf{0}$ is a stable, but not asymptotically stable, center.

10 (a). For $A=\begin{bmatrix}-1 & 1\\ -1 & -1\end{bmatrix}$, the eigenvalues are $\lambda_1=-1+i$ and $\lambda_2=-1-i$.

10 (b). Since the eigenvalues are complex with negative real part, $\mathbf{y}=\mathbf{0}$ is an asymptotically stable spiral point.

11 (a). For $A=\begin{bmatrix}1 & -6\\ 2 & -6\end{bmatrix}$, the eigenvalues are $\lambda_1=-3$ and $\lambda_2=-2$.

11 (b). Since the eigenvalues are real and negative, $\mathbf{y}=\mathbf{0}$ is an asymptotically stable improper node.

12 (a). For $A=\begin{bmatrix}2 & -3\\ 3 & 2\end{bmatrix}$, the eigenvalues are $\lambda_1=2+3i$ and $\lambda_2=2-3i$.

12 (b). Since the eigenvalues are complex with positive real part, $\mathbf{y}=\mathbf{0}$ is an unstable spiral point.

13 (a). For $A=\begin{bmatrix}-2 & -4\\ 5 & 2\end{bmatrix}$, the eigenvalues are $\lambda_1=4i$ and $\lambda_2=-4i$.

13 (b). Since the eigenvalues are complex with zero real part, $\mathbf{y}=\mathbf{0}$ is a stable, but not asymptotically stable, center.

14 (a). For $A=\begin{bmatrix}7 & -24\\ 2 & -7\end{bmatrix}$, the eigenvalues are $\lambda_1=1$ and $\lambda_2=-1$.

14 (b). Since the eigenvalues are real with opposite sigen, $\mathbf{y}=\mathbf{0}$ is an unstable saddle point.

15 (a). For $A=\begin{bmatrix}-1 & 8\\ -1 & 5\end{bmatrix}$, the eigenvalues are $\lambda_1=1$ and $\lambda_2=3$.

15 (b). Since the eigenvalues are real and positive, $\mathbf{y}=\mathbf{0}$ is an unstable improper node.

16 (a). For $A=\begin{bmatrix}-2 & 1\\ -1 & -2\end{bmatrix}$, the eigenvalues are $\lambda_1=-2+i$ and $\lambda_2=-2-i$.

16 (b). Since the eigenvalues are complex with negative real part, $\mathbf{y}=\mathbf{0}$ is an asymptotically stable spiral point.

17 (a). For $A=\begin{bmatrix}2 & 4\\ -4 & -6\end{bmatrix}$, the eigenvalues are $\lambda_1=-2$ and $\lambda_2=-2$.

17 (b). Since the eigenvalues are real and negative and A is not a multiple of the identity, $\mathbf{y}=\mathbf{0}$ is an asymptotically stable improper node.

18 (a). For $A=\begin{bmatrix}3 & 0\\ 0 & 3\end{bmatrix}$, the eigenvalues are $\lambda_1=3$ and $\lambda_2=3$.

18 (b). Since the eigenvalues are real and positive and A is a multiple of the identity, $\mathbf{y} = \mathbf{0}$ is an unstable proper node.

19 (a). For $A = \begin{bmatrix} 1 & 2 \\ -8 & 1 \end{bmatrix}$, the eigenvalues are $\lambda_1 = 1 + 4i$ and $\lambda_2 = 1 - 4i$.

19 (b). Since the eigenvalues are complex with positive real part, $\mathbf{y} = \mathbf{0}$ is an unstable spiral point.

20 (a). For $A = \begin{bmatrix} -1 & -2 \\ 2 & 3 \end{bmatrix}$, the eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = 1$.

20 (b). Since the eigenvalues are real and positive and A is not a multiple of the identity, $\mathbf{y} = \mathbf{0}$ is an unstable improper node.

21 (a). For $A_1 = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$, the eigenvalues are $\lambda_1 = -3$ and $\lambda_2 = -1$. Since the eigenvalues are real and negative, $\mathbf{y} = \mathbf{0}$ is an asymptotically stable equilibrium point. Therefore, A_1 corresponds to Direction Field 2.

21 (b). For $A_2 = \begin{bmatrix} 1 & 2 \\ -2 & -1 \end{bmatrix}$, the eigenvalues are $\lambda_1 = -\sqrt{3}i$ and $\lambda_2 = \sqrt{3}i$. Since the eigenvalues are complex with zero real part, $\mathbf{y} = \mathbf{0}$ is a stable, but not asymptotically stable, center. Therefore, A_2 corresponds to Direction Field 4.

21 (c). For $A_3 = \begin{bmatrix} 2 & 1 \\ -1 & -2 \end{bmatrix}$, the eigenvalues are $\lambda_1 = -\sqrt{3}$ and $\lambda_2 = \sqrt{3}$. Since the eigenvalues are real and have opposite sign, $\mathbf{y} = \mathbf{0}$ is an unstable saddle point. Therefore, A_3 corresponds to Direction Field 1.

21 (d). For $A_4 = \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix}$, the eigenvalues are $\lambda_1 = 1 - 2i$ and $\lambda_2 = 1 + 2i$. Since the eigenvalues are complex with positive real part, $\mathbf{y} = \mathbf{0}$ is an unstable spiral point. Therefore, A_4 corresponds to Direction Field 3.

22. For a center, eigenvalues are purely imaginary. Therefore, $\alpha = -2$.

23. Consider $A = \begin{bmatrix} -4 & \alpha \\ -2 & 2 \end{bmatrix}$. The characteristic polynomial is $p(\lambda) = \lambda^2 + 2\lambda + (2\alpha - 8)$. Thus, the eigenvalues are $\lambda = -1 \pm \sqrt{9 - 2\alpha}$. In order to have an asymptotically stable spiral point at $\mathbf{y} = \mathbf{0}$, we need complex eigenvalues with negative real parts. Thus, we need $9 - 2\alpha < 0$ or $9/2 < \alpha$.

24. Note that $\lambda_1 = -2$ and $\lambda_2 = -2$ no matter the value of α. Thus, $\mathbf{y} = \mathbf{0}$ is always an asymptotically stable equilibrium point; it will be a proper node if $\alpha = 0$.

25. Consider $A = \begin{bmatrix} 4 & -2 \\ \alpha & -4 \end{bmatrix}$. The characteristic polynomial is $p(\lambda) = \lambda^2 + (2\alpha - 16)$. Thus, the eigenvalues are $\lambda = \pm\sqrt{16 - 2\alpha}$. In order to have a saddle point at $\mathbf{y} = \mathbf{0}$, we need real eigenvalues with opposite signs. Thus, we need $16 - 2\alpha > 0$ or $\alpha < 8$.

26. Consider the nonhomogeneous system $\mathbf{y}' = \begin{bmatrix} 1 & 4 \\ -1 & 1 \end{bmatrix}\mathbf{y} + \begin{bmatrix} 3 \\ 2 \end{bmatrix}$. The system has a unique equilibrium point given by $\mathbf{y}_e = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. Making the substitution $\mathbf{z} = \mathbf{y} - \mathbf{y}_e$, we obtain $\mathbf{z}' = \begin{bmatrix} 1 & -4 \\ -1 & 1 \end{bmatrix}\mathbf{z}$. The eigenvalues of the coefficient matrix are $\lambda_1 = 1 + 2i$ and $\lambda_2 = 1 - 2i$. Therefore, $\mathbf{z} = \mathbf{0}$ is an unstable spiral point and consequently, $\mathbf{y} = \mathbf{y}_e$ is an unstable spiral point of the original system.

27. Consider the nonhomogeneous system $\mathbf{y}' = \begin{bmatrix} 6 & 5 \\ -7 & -6 \end{bmatrix}\mathbf{y} + \begin{bmatrix} 4 \\ -6 \end{bmatrix}$. The system has a unique equilibrium point given by $\mathbf{y}_e = -\begin{bmatrix} 6 & 5 \\ -7 & -6 \end{bmatrix}^{-1}\begin{bmatrix} 4 \\ -6 \end{bmatrix} = \begin{bmatrix} -6 & -5 \\ 7 & 6 \end{bmatrix}\begin{bmatrix} 4 \\ -6 \end{bmatrix} = \begin{bmatrix} 6 \\ -8 \end{bmatrix}$. Making the substitution $\mathbf{z} = \mathbf{y} - \mathbf{y}_e$, we obtain $\mathbf{z}' = \begin{bmatrix} 6 & 5 \\ -7 & -6 \end{bmatrix}\mathbf{z}$. The eigenvalues of the coefficient matrix are $\lambda_1 = -1$ and $\lambda_2 = 1$. Therefore, $\mathbf{z} = \mathbf{0}$ is an unstable saddle point and consequently, $\mathbf{y} = \mathbf{y}_e$ is an unstable saddle point of the original system.

28. Consider the nonhomogeneous system $\mathbf{y}' = \begin{bmatrix} 5 & -14 \\ 3 & -8 \end{bmatrix}\mathbf{y} + \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. The system has a unique equilibrium point given by $\mathbf{y}_e = \begin{bmatrix} 1 \\ 0.5 \end{bmatrix}$. Making the substitution $\mathbf{z} = \mathbf{y} - \mathbf{y}_e$, we obtain $\mathbf{z}' = \begin{bmatrix} 5 & -14 \\ 3 & -8 \end{bmatrix}\mathbf{z}$. The eigenvalues of the coefficient matrix are $\lambda_1 = -2$ and $\lambda_2 = -1$. Therefore, $\mathbf{z} = \mathbf{0}$ is an asymptotically stable improper node and consequently, $\mathbf{y} = \mathbf{y}_e$ is an asymptotically stable improper node of the original system.

29. Consider the nonhomogeneous system $\mathbf{y}' = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}\mathbf{y} + \begin{bmatrix} 2 \\ -4 \end{bmatrix}$. The system has a unique equilibrium point given by $\mathbf{y}_e = -\begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}^{-1}\begin{bmatrix} 2 \\ -4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -0.5 \end{bmatrix}\begin{bmatrix} 2 \\ -4 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$. Making the substitution $\mathbf{z} = \mathbf{y} - \mathbf{y}_e$, we obtain $\mathbf{z}' = \begin{bmatrix} -1 & 0 \\ 0 & 2 \end{bmatrix}\mathbf{z}$. The eigenvalues of the coefficient matrix are $\lambda_1 = -1$ and $\lambda_2 = 2$. Therefore, $\mathbf{z} = \mathbf{0}$ is an unstable saddle point and consequently, $\mathbf{y} = \mathbf{y}_e$ is an unstable saddle point of the original system.

30 (a). The characteristic equation is $\lambda^2 - (a_{11} + a_{22})\lambda + a_{11}a_{22} - a_{12}a_{21} = 0$. The origin is a center if the roots are purely imaginary. That is, if $a_{11} + a_{22} = 0$ and $a_{11}a_{22} - a_{12}a_{21} < 0$.

30 (b). Note that $f(x,y) = a_{11}x + a_{12}y$ and $g(x,y) = a_{21}x + a_{22}y$. Thus, $f_x = a_{11}$ and $g_y = a_{22}$. By part (a), $f_x = -g_y$ and hence the system is Hamiltonian.

30 (c). The converse is not true since the system can be Hamiltonian even though $a_{11}a_{22} - a_{12}a_{21} = 0$.

32 (a). The eigenvalues of the coefficient matrix $A = \begin{bmatrix} -2 & 1 \\ 5 & 2 \end{bmatrix}$ are $\lambda_1 = 3$ and $\lambda_2 = -3$.

32 (b). Since the eigenvalues are real with opposite sign, $\mathbf{y} = \mathbf{0}$ is an (unstable) saddle point.

32 (c). Since the system is Hamiltonian, we know that $H_y(x,y) = -2x + y$. Therefore, $H(x,y) = -2xy + 0.5y^2 + q(x)$. We determine $q(x)$ by differentiating $H(x,y)$ with respect to x, finding $H_x(x,y) = -2y + q'(x) = -5x - 2y$. Thus, $q'(x) = -5x$ and so $q(x) = -2.5x^2 + C$. Dropping the additive constant, we obtain a Hamiltonian function, $H(x,y) = -2.5x^2 - 2xy + 0.5y^2$. The conservation law for the system is $H(x,y) = C$.

33 (a). The eigenvalues of the coefficient matrix $A = \begin{bmatrix} 1 & 3 \\ -3 & -1 \end{bmatrix}$ are $\lambda_1 = -2\sqrt{2}i$ and $\lambda_2 = -2\sqrt{2}i$.

33 (b). Since the eigenvalues are complex with zero real part, $\mathbf{y} = \mathbf{0}$ is a stable, but not asymptotically stable, center.

33 (c). Since the system is Hamiltonian, we know that $H_y(x,y) = x + 3y$. Therefore, $H(x,y) = xy + 1.5y^2 + q(x)$. We determine $q(x)$ by differentiating $H(x,y)$ with respect to x, finding $-3x - y = -H_x(x,y) = -y - q'(x)$. Thus, $q'(x) = 3x$ and so $q(x) = 1.5x^2 + C$. Dropping the additive constant, we obtain a Hamiltonian function, $H(x,y) = xy + 1.5(x^2 + y^2)$. The conservation law for the system is $H(x,y) = C$.

34 (a). The eigenvalues of the coefficient matrix $A = \begin{bmatrix} 2 & 1 \\ 0 & -2 \end{bmatrix}$ are $\lambda_1 = 2$ and $\lambda_2 = -2$.

34 (b). Since the eigenvalues are real with opposite sign, $\mathbf{y} = \mathbf{0}$ is an (unstable) saddle point.

34 (c). Since the system is Hamiltonian, we know that $H_y(x,y) = 2x + y$. Therefore, $H(x,y) = 2xy + 0.5y^2 + q(x)$. We determine $q(x)$ by differentiating $H(x,y)$ with respect to x, finding $H_x(x,y) = 2y + q'(x) = 2y$. Thus, $q'(x) = 0$ and so $q(x) = C$. Dropping the additive constant, we obtain a Hamiltonian function, $H(x,y) = 2xy + 0.5y^2$. The conservation law for the system is $H(x,y) = C$.

Section 8.7

1 (a). Consider the system

$$x' = x - x^2 - xy$$

$$y' = y - 3y^2 - 0.5xy \,.$$

If $y = 0$, then all direction field filaments on the positive x-axis point towards $x = 1$. Thus, x approaches an equilibrium value of $x_e = 1$ as t increases. Similarly, if $x = 0$, then y approaches an equilibrium value of $y_e = 1/3$ as t increases.

In each case, the presence of the xy term causes the derivative to decrease. Therefore, the presence of the other species is harmful in each case.

1 (b). Rewriting the system as

$$x' = x(1 - x - y)$$

$$y' = y(1 - 3y - 0.5x)\,,$$

we see that $x' = 0$ if (i) $x = 0$ or (ii) $1 - x - y = 0$. In case (i), $y' = 0$ if $y = 0$ or $y = 1/3$. Thus, two equilibrium points are $(x,y) = (0,0)$ and $(x,y) = (0,1/3)$. In case (ii), $y' = 0$ if $y = 0$ (and hence, $x = 1$) or if $1 - 3y - 0.5x = 0$ (and hence $x + y = 1$ and $0.5x + 3y = 1$). Thus, case (ii) leads us to two more equilibrium points $(x,y) = (1,0)$ and $(x,y) = (0.8,0.2)$.

1 (c). At the equilibrium point $\mathbf{z} = \mathbf{0}$, the linearized system takes the form $\mathbf{z}' = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\mathbf{z}$. The eigenvalues of the coefficient matrix are $\lambda_1 = 1$ and $\lambda_2 = 1$. Since, $\mathbf{z} = \mathbf{0}$ is an unstable proper node of the linearized system, the original system is also unstable at $\mathbf{y} = \mathbf{0}$.

2 (a). Consider the system

$$x' = -x - x^2$$

$$y' = -y + xy \ .$$

If $y = 0$, then x approaches an equilibrium value of $x_e = 0$ as t increases. If $x = 0$, then y approaches an equilibrium value of $y_e = 0$ as t increases.
The presence of y is a matter of indifference to x. The presence of x is beneficial to y.

2 (b). The only equilibrium point in the first quadrant is $(x,y) = (0,0)$.

2 (c). At the equilibrium point $\mathbf{z} = \mathbf{0}$, the linearized system takes the form $\mathbf{z}' = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\mathbf{z}$. The eigenvalues of the coefficient matrix are $\lambda_1 = -1$ and $\lambda_2 = -1$. Since, $\mathbf{z} = \mathbf{0}$ is an asymptotically stable proper node of the linearized system, the original system is also asymptotically stable at $\mathbf{y} = \mathbf{0}$.

3 (a). Consider the system

$$x' = x - x^2 - xy$$

$$y' = -y - y^2 + xy \ .$$

If $y = 0$, then all direction field filaments on the positive x-axis point towards $x = 1$. Thus, x approaches an equilibrium value of $x_e = 1$ as t increases. Similarly, if $x = 0$, then y approaches an equilibrium value of $y_e = 0$ as t increases. The presence of the xy term in the first equation causes the derivative to decrease. Therefore, the presence of y is harmful to x. On the other hand, the presence of the xy term in the second equation causes the derivative to increase. Therefore, the presence of x is beneficial to y.

3 (b). Rewriting the system as

$$x' = x(1 - x - y)$$

$$y' = -y(1 + y - x) \ ,$$

we see that $x' = 0$ if (i) $x = 0$ or (ii) $1 - x - y = 0$. In case (i), $y' = 0$ if $y = 0$ or $y = -1$. The latter possibility has been excluded and thus case (i) leads to a single equilibrium point, $(x,y) = (0,0)$. In case (ii), $y' = 0$ if $y = 0$ (and hence, $x = 1$) or if $1 + y - x = 0$ (and hence $x + y = 1$ and $x - y = 1$). This second set of equations also has solution $x = 1$ and $y = 0$. Thus, case (ii) leads us to one more equilibrium point $(x,y) = (1,0)$.

3 (c). At the equilibrium point $\mathbf{z} = \mathbf{0}$, the linearized system takes the form $\mathbf{z}' = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\mathbf{z}$. The eigenvalues of the coefficient matrix are $\lambda_1 = -1$ and $\lambda_2 = 1$. Since, $\mathbf{z} = \mathbf{0}$ is an unstable saddle point of the linearized system, the original system is also unstable at $\mathbf{y} = \mathbf{0}$.

4 (a). Consider the system

$$x' = x - x^2 + xy$$
$$y' = y - y^2 + xy\ .$$

If $y = 0$, then x approaches an equilibrium value of $x_e = 1$ as t increases. If $x = 0$, then y approaches an equilibrium value of $y_e = 1$ as t increases.
In both cases, the presence of one species is beneficial to the other species.

4 (b). The only equilibrium points in the first quadrant are $(x,y) = (0,0)$, $(x,y) = (0,1)$, and $(x,y) = (1,0)$.

4 (c). At the equilibrium point $\mathbf{z} = \mathbf{0}$, the linearized system takes the form $\mathbf{z}' = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\mathbf{z}$. The eigenvalues of the coefficient matrix are $\lambda_1 = 1$ and $\lambda_2 = 1$. Since, $\mathbf{z} = \mathbf{0}$ is an unstable proper node of the linearized system, the original system is also unstable at $\mathbf{y} = \mathbf{0}$.

5 (a). When $y = 0$, the assumed model reduces to $x' = r_1(1 + \alpha_1 x)x$. In this case, we see from the figure, that $\ln x(t) = 0.5t + \ln x(0)$. Differentiating, we obtain $\dfrac{x'(t)}{x(t)} = 0.5$ or $x' = 0.5x$. Thus, $\alpha_1 = 0$ and $r_1 = 0.5$. Similarly, when $x = 0$, the model reduces to $y' = r_2(1 + \alpha_2 y)y$. In this case, we see from the figure, that $\ln y(t) = -t + \ln y(0)$. Differentiating, we obtain $\dfrac{y'(t)}{y(t)} = -1$ or $y' = -y$. Thus, $\alpha_2 = 0$ and $r_2 = -1$. So far, we have deduced that the assumptions of the population model imply it has the form

$$x' = 0.5(1 + \beta_1 y)x$$
$$y' = -(1 + \beta_2 x)y\ .$$

Knowing the equilibrium point $(x_e, y_e) = (2,3)$, allows us to determine the last remaining model parameters, β_1 and β_2. In particular, we know from the first equation that $0.5(1 + 3\beta_1)2 = 0$ while the second equation gives $-(1 + 2\beta_2)3 = 0$. Consequently, $\beta_1 = -1/3$ and $\beta_2 = -1/2$.

5 (b). From part (a), the model is given by

$$x' = (1/2)x - (1/6)xy$$
$$y' = -y + (1/2)xy\ .$$

The presence of y causes x' to decrease and hence y is harmful to x. The presence of x causes y' to increase and hence x is beneficial to y.

6 (a). Consider the system

$$x' = r(1 - \alpha x - \beta y)x + \mu x$$
$$y' = r(1 - \alpha y - \beta x)y\ .$$

The equilibrium points are $(x,y) = (0,0)$, $(x,y) = (0,\alpha^{-1})$, $(x,y) = (\alpha^{-1}(1 + \mu r^{-1}),0)$, and $(x,y) = \delta^{-1}(\alpha(1 + \mu r^{-1}) - \beta, \alpha - \beta(1 + \mu r^{-1}))$ where $\delta = \alpha^2 - \beta^2$.

6 (b). If μ is chosen large enough so that $\beta(1 + \mu r^{-1}) > \alpha$ then we see from part (a) that the "coexisting species" equilibrium point is moved into the fourth quadrant and is therefore physically irrelevant.

6 (c). At $\mathbf{z}=\mathbf{0}$, the linearized system has the form $\mathbf{z}'=\begin{bmatrix} r+\mu & 0 \\ 0 & r \end{bmatrix}\mathbf{z}$. The point $\mathbf{z}=\mathbf{0}$ is an unstable improper node. At the equilibrium point $\mathbf{z}=\begin{bmatrix} 0 \\ 1/\alpha \end{bmatrix}$, the linearized system is $\mathbf{z}'=\begin{bmatrix} r(1+\mu r^{-1}-\beta\alpha^{-1}) & 0 \\ -r\beta\alpha^{-1} & -r \end{bmatrix}\mathbf{z}$. The eigenvalues are $\lambda_1=-r$ and $\lambda_2=r(1+\mu r^{-1}-\beta\alpha^{-1})$. Since the eigenvalues have opposite sign, the equilibrium point is an unstable saddle point. The equilibrium point $(x,y)=(\alpha^{-1}(1+\mu r^{-1}),0)$ is an asymptotically stable improper node since the eigenvalues of the linearized system are negative and different:
$\lambda_1=-r(1+\mu r^{-1})$ and $\lambda_2=r[1-\beta\mu(\alpha r)^{-1}-\beta\alpha^{-1}]$.

6 (d). For the nonlinear system, (0,0) and $(0,\alpha^{-1})$ are unstable equilibrium points. The equilirium point $(x,y)=(\alpha^{-1}(1+\mu r^{-1}),0)$ is stable.

6 (e). It appears that the *y* species will be driven to extinction with the *x* species approaching the limiting value $\alpha^{-1}(1+\mu r^{-1})$.

7 (a). Consider the system

$$x'=r(1-\alpha x-\beta y)x$$

$$y'=r(1-\alpha y-\beta x)y-\mu y\ .$$

We see that $x'=0$ if (i) $x=0$ or (ii) $1-\alpha x-\beta y=0$. In case (i), $y'=0$ if $y=0$ or $y=(r-\mu)/(\alpha r)$. Thus case (i) leads to two equilibrium points, $(x,y)=(0,0)$ and $(x,y)=(0,(r-\mu)/(\alpha r))$. In case (ii), $y'=0$ if $y=0$ or if $1-(\mu/r)-\alpha y-\beta x=0$. Thus case (ii) leads to two equilibrium points, $(x,y)=(1/\alpha,0)$ and
$(x,y)=(\delta^{-1}[\alpha-\beta(1-\mu r^{-1})],\delta^{-1}[-\beta+\alpha(1-\mu r^{-1})])$ where $\delta=\alpha^2-\beta^2$.

7 (b). If $\mu>r$, then $1-\mu r^{-1}<0$. In this case, we see from part (a) that the only physically relevant equilibrium points are $(x,y)=(0,0)$ and $(x,y)=(1/\alpha,0)$.

7 (c). At $\mathbf{z}=\mathbf{0}$, the linearized system has the form $\mathbf{z}'=\begin{bmatrix} r & 0 \\ 0 & r-\mu \end{bmatrix}\mathbf{z}$. Since we are assuming $\mu>r$, the point $\mathbf{z}=\mathbf{0}$ is an unstable saddle point. At the equilibrium point $\mathbf{z}=\begin{bmatrix} 1/\alpha \\ 0 \end{bmatrix}$, the linearized system is $\mathbf{z}'=\begin{bmatrix} -r & -r\beta\alpha^{-1} \\ 0 & r-\mu-r\beta\alpha^{-1} \end{bmatrix}\mathbf{z}$. The eigenvalues are $\lambda_1=-r$ and $\lambda_2=r-\mu-r\beta\alpha^{-1}$.
Since both eigenvalues are negative, the equilibrium point is an asymptotically stable improper node.

7 (d). For the nonlinear system, (0,0) is unstable and $(\alpha^{-1},0)$ is stable.

7 (e). If $\mu>r$, it appears that the *y* species will be driven to extinction with the *x* species approaching the limiting value α^{-1}.

8. The strategy of nurturing the desirable species leads to an equilibrium *x*-population of $\alpha^{-1}(1+\mu r^{-1})$. This is greater than the equilibrium *x*-population of α^{-1} that results from harvesting the undesirable species.

9. Consider the population model

$$x' = \pm a_1 x \pm b_1 x^2 \pm c_1 xy \pm d_1 xz$$
$$y' = \pm a_2 y \pm b_2 y^2 \pm c_2 xy \pm d_2 yz$$
$$z' = \pm a_3 z \pm c_3 xz \pm d_3 yz\ .$$

Since x and y are mutually competitive, we need to choose a negative sign for c_1 and c_2 (the presence of x reduces the growth rate y' and similarly the presence of y reduces the growth rate x'). The same argument applies to the signs of d_1 and d_2 since the predator is harmful to x and to y. The presence of the prey is beneficial to the predator z and thus we need to choose a positive sign for c_3 and d_3.

So far, we have deduced

$$x' = \pm a_1 x \pm b_1 x^2 - c_1 xy - d_1 xz$$
$$y' = \pm a_2 y \pm b_2 y^2 - c_2 xy - d_2 yz$$
$$z' = \pm a_3 z + c_3 xz + d_3 yz\ .$$

We also know that, in the absence of the other two species, x and y each evolve towards a nonzero equilibrium value. Thus, from the first equation, we know the term $\pm a_1 x \pm b_1 x^2 = x(\pm a_1 \pm b_1 x)$ has a positive zero, as does the corresponding term in the second equation, $\pm a_2 y \pm b_2 y^2 = y(\pm a_2 \pm b_2 y)$. From this fact, we infer that a_1 and b_1 have opposite signs, as do a_2 and b_2. The general solution of an equation of the form $u' = au + bu^2$ is $u = Ae^{-at} + Bt^2 + Ct + D$. If a is negative, then $u(t) \to \infty$ as $t \to \infty$. Hence, there cannot be a nonzero equilibrium solution when a is negative. Applying this observation to the equations $x' = \pm a_1 x \pm b_1 x^2$ and $y' = \pm a_2 y \pm b_2 y^2$, we deduce that a_1 and a_2 are positive and b_1 and b_2 are negative. Likewise, in order that z decrease to zero in the absence of x and y, we need to have a_3 negative. In summary, we arrive at the following model which will support the observations:

$$x' = a_1 x - b_1 x^2 - c_1 xy - d_1 xz$$
$$y' = a_2 y - b_2 y^2 - c_2 xy - d_2 yz$$
$$z' = -a_3 z + c_3 xz + d_3 yz\ .$$

10 (a). Consider the system

$$s' = -\alpha si + \gamma r$$
$$i' = \alpha si - \beta i$$
$$r' = \beta i - \gamma r\ .$$

Summing these three equations, we obtain $s'(t) + i'(t) + r'(t) = 0$. Hence, $s(t) + i(t) + r(t)$ is constant, say $s(t) + i(t) + r(t) = N$ where N denotes the size of the population.

10 (b). If those who recover are permanently immunized, then

$$s' = -\alpha si$$
$$i' = \alpha si - \beta i$$
$$r' = \beta i\ .$$

As in part (a), we can sum these equations and again conclude that $s(t) + i(t) + r(t) = N$.

10 (c). If some infected members perish, then

$$s' = -\alpha si$$
$$i' = \alpha si - \beta i$$
$$r' = \beta i - \gamma r \ .$$

In this case, $s'(t) + i'(t) + r'(t) = -\gamma r(t)$. Thus, the population is not constant but rather is decreasing.

11 (a). Consider the system

$$s' = -\alpha si + \gamma r$$
$$i' = \alpha si - \beta i$$
$$r' = \beta i - \gamma r \ .$$

Using the fact, from Exercise 10, that $s + i + r = N$, we obtain a reduced system,

$$s' = -\alpha si + \gamma(N - i - s)$$
$$i' = \alpha si - \beta i \ .$$

11 (b). For the given values, $\alpha = \beta = \gamma = 1$ and $N = 9$, the reduced system has the form

$$s' = -si + (9 - i - s)$$
$$i' = si - i \ .$$

Rewriting this system slightly,

$$s' = -si + 9 - i - s$$
$$i' = i(s - 1) \ .$$

We see that $i' = 0$ if (i) $i = 0$ or (ii) $s = 1$. In case (i), $s' = 0$ if $s = 9$. Thus case (i) leads to the equilibrium point $(s,i) = (9,0)$. In case (ii), $s' = 0$ if $i = 4$. Thus case (ii) leads to the equilibrium point $(s,i) = (1,4)$.

11 (c). At $\mathbf{z} = \begin{bmatrix} 9 \\ 0 \end{bmatrix}$, the linearized system has the form $\mathbf{z}' = \begin{bmatrix} -1 & -10 \\ 0 & 8 \end{bmatrix}\mathbf{z}$. The eigenvalues are $\lambda_1 = -1$ and $\lambda_2 = 8$. This equilibrium point is an unstable saddle point. At $\mathbf{z} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$, the linearized system has the form $\mathbf{z}' = \begin{bmatrix} -5 & -2 \\ 4 & 0 \end{bmatrix}\mathbf{z}$. The eigenvalues are $\lambda_1 = (-5 - i\sqrt{7})/2$ and $(-5 + i\sqrt{7})/2$. This equilibrium point is an asymptotically stable spiral point.

11 (d). $(9,0)$ is an unstable equilibrium point while $(1,4)$ is stable.

Chapter 9
Numerical Methods

Section 9.1

Unless indicated by $\ldots$, all results are rounded to the places shown.

1 (a). Integrating $y' = 2t - 1$, we find $y = t^2 - t + C$. Imposing the initial condition $y(1) = 0$, we obtain $y = t^2 - t$.

1 (b). Since $f(t, y) = 2t - 1$, it follows that $f(t + h, y + hf(t, y)) = 2(t + h) - 1$. Therefore, Heun's method takes the form

$$y_{n+1} = y_n + (h/2)[(2t_n - 1) + (2t_{n+1} - 1)].$$

1 (c). As in part (b), we find the modified Euler's method takes the form

$$y_{n+1} = y_n + h(2(t_n + h/2) - 1).$$

1 (d).

1.0000	0
1.1000	0.1100
1.2000	0.2400
1.3000	0.3900

1 (e).

1.0000	0
1.1000	0.1100
1.2000	0.2400
1.3000	0.3900

1 (f).

1.0000	0
1.1000	0.1100
1.2000	0.2400
1.3000	0.3900

2 (a). Integrating $y' = -y$ and imposing the initial condition, we obtain $y = e^{-t}$.

2 (b). Heun's method takes the form $y_{n+1} = (1 - h + 0.5h^2)y_n$.

2 (c). The modified Euler's method takes the form $y_{n+1} = (1 - h + 0.5h^2)y_n$.

2 (d).

0.0000	1.0000
0.1000	0.9050
0.2000	0.8190
0.3000	0.7412

2 (e).

0.0000	1.0000
0.1000	0.9050
0.2000	0.8190
0.3000	0.7412

2 (f).

0.0000	1.0000
0.1000	0.9048
0.2000	0.8187
0.3000	0.7408

3 (a). Solving the separable equation $y' = -ty$, we find $y = Ce^{-t^2/2}$. Imposing the initial condition $y(0) = 1$, we obtain $y = e^{-t^2/2}$.

3 (b). Since $f(t,y) = -ty$, it follows that $f(t+h, y+hf(t,y)) = -(t+h)[y + h(-ty)]$. Therefore, Heun's method takes the form

$$y_{n+1} = y_n + (h/2)[-t_n y_n - t_{n+1}(y_n - ht_n y_n)].$$

3 (c). As in part (b), we find the modified Euler's method takes the form

$$y_{n+1} = y_n - h(t_n + 0.5h)(y_n - 0.5ht_n y_n).$$

3 (d).

0	1.0000
0.1000	0.9950
0.2000	0.9802
0.3000	0.9560

3 (e).

0	1.0000
0.1000	0.9950
0.2000	0.9801
0.3000	0.9559

3 (f).

0	1.0000
0.1000	0.9950
0.2000	0.9802
0.3000	0.9560

4 (a). Integrating $y' = -y + t$ and imposing the initial condition, we obtain $y = t - 1 + e^{-t}$.

4 (b). Heun's method takes the form $y_{n+1} = y_n + 0.5h[-y_n + t_n - (y_n + h(-y_n + t_n)) + t_{n+1}]$.

4 (c). The modified Euler's method takes the form $y_{n+1} = y_n + h[-(y_n + 0.5h(-y_n + t_n)) + t_n + 0.5h]$.

4 (d).

0.0000	0.0000
0.1000	0.0050
0.2000	0.0190
0.3000	0.0412

4 (e).

0.0000	0.0000
0.1000	0.0050
0.2000	0.0190
0.3000	0.0412

4 (f).

0.0000	0.0000
0.1000	0.0048
0.2000	0.0187
0.3000	0.0408

5 (a). Solving the separable equation $y^2 y' + t = 0$, we find $y^3 = -1.5t^2 + C$. Imposing the initial condition $y(0) = 1$, we obtain $y^3 = -1.5t^2 + 1$ or $y = (1 - 1.5t^2)^{1/3}$.

5 (b). Since $f(t,y) = -ty^{-2}$, it follows that $f(t+h, y+hf(t,y)) = -(t+h)[y + h(-ty)]^{-2}$. Therefore, Heun's method takes the form

$$y_{n+1} = y_n + (h/2)[-t_n y_n^{-2} - t_{n+1}(y_n - ht_n y_n^{-2})^{-2}].$$

5 (c). As in part (b), we find the modified Euler's method takes the form

$$y_{n+1} = y_n - h(t_n + 0.5h)(y_n - 0.5ht_n y_n^{-2})^{-2}.$$

5 (d).

0	1.0000
0.1000	0.9950
0.2000	0.9796
0.3000	0.9529

5 (e).

0	1.0000
0.1000	0.9950
0.2000	0.9797
0.3000	0.9531

5 (f).

0	1.0000
0.1000	0.9950
0.2000	0.9796
0.3000	0.9528

6 (a). Solving the separable equation $y' = 1 + y^2$ and imposing the initial condition we obtain $y = \tan(t - \pi / 4)$.

6 (b).

T	*E*	*H*	*IE*
0.0000	-1.0000	-1.0000	-1.0000
0.0500	-0.9000	-0.9047	-0.9049
0.1000	-0.8095	-0.8177	-0.8179
0.1500	-0.7267	-0.7375	-0.7378
0.2000	-0.6503	-0.6630	-0.6634
0.2500	-0.5792	-0.5933	-0.5937
0.3000	-0.5124	-0.5276	-0.5280
0.3500	-0.4493	-0.4653	-0.4657
0.4000	-0.3892	-0.4058	-0.4062
0.4500	-0.3316	-0.3486	-0.3491
0.5000	-0.2761	-0.2934	-0.2940
0.5500	-0.2223	-0.2399	-0.2404
0.6000	-0.1698	-0.1876	-0.1881
0.6500	-0.1184	-0.1362	-0.1368
0.7000	-0.0677	-0.0856	-0.0862
0.7500	-0.0175	-0.0354	-0.0360
0.8000	0.0326	0.0147	0.0140
0.8500	0.0826	0.0648	0.0641
0.9000	0.1329	0.1152	0.1145
0.9500	0.1838	0.1662	0.1654
1.0000	0.2355	0.2181	0.2173

The errors at $t = 1$ are, respectively, 0.0176, 1.6495e-004, and 6.8239e-004.

7 (a). Solving the separable equation $yy' + t = 0$, we find $y^2 = -t^2 + C$. Imposing the initial condition $y(0) = 3$, we obtain $y^2 = -t^2 + 9$ or $y = (9 - t^2)^{1/2}$.

7 (b).

T	E	H	IE
0	3.0000	3.0000	3.0000
0.0500	3.0000	2.9996	2.9996
0.1000	2.9992	2.9983	2.9983
0.1500	2.9975	2.9962	2.9962
0.2000	2.9950	2.9933	2.9933
0.2500	2.9917	2.9896	2.9896
0.3000	2.9875	2.9850	2.9850
0.3500	2.9825	2.9795	2.9795
0.4000	2.9766	2.9732	2.9732
0.4500	2.9699	2.9661	2.9661
0.5000	2.9623	2.9580	2.9580
0.5500	2.9539	2.9492	2.9492
0.6000	2.9445	2.9394	2.9394
0.6500	2.9344	2.9287	2.9287
0.7000	2.9233	2.9172	2.9172
0.7500	2.9113	2.9047	2.9047
0.8000	2.8984	2.8914	2.8914
0.8500	2.8846	2.8771	2.8771
0.9000	2.8699	2.8618	2.8618
0.9500	2.8542	2.8456	2.8456
1.0000	2.8376	2.8284	2.8284

The errors at $t = 1$ are, respectively, -9.1466e-003, -6.9021e-007, and -1.3752e-005.

8 (a). Solving the equation $y' + 2y = 4$ and imposing the initial condition we obtain $y = 2 + e^{-2t}$.

8 (b).

T	E	H	IE
0	3.0000	3.0000	3.0000
0.0500	2.9000	2.9050	2.9050
0.1000	2.8100	2.8190	2.8190
0.1500	2.7290	2.7412	2.7412
0.2000	2.6561	2.6708	2.6708
0.2500	2.5905	2.6071	2.6071
0.3000	2.5314	2.5494	2.5494
0.3500	2.4783	2.4972	2.4972
0.4000	2.4305	2.4500	2.4500
0.4500	2.3874	2.4072	2.4072
0.5000	2.3487	2.3685	2.3685
0.5500	2.3138	2.3335	2.3335
0.6000	2.2824	2.3018	2.3018
0.6500	2.2542	2.2732	2.2732
0.7000	2.2288	2.2472	2.2472
0.7500	2.2059	2.2237	2.2237
0.8000	2.1853	2.2025	2.2025
0.8500	2.1668	2.1832	2.1832
0.9000	2.1501	2.1658	2.1658
0.9500	2.1351	2.1501	2.1501
1.0000	2.1216	2.1358	2.1358

The errors at $t = 1$ are, respectively, 1.3758e-002, 4.8717e-004, and 4.8717e-004.

9 (a). Solving the separable equation $y' + 2ty = 0$, we find $y = Ce^{-t^2}$. Imposing the initial condition $y(0) = 2$, we obtain $y = 2e^{-t^2}$.

9 (b).

T	E	H	IE
0	2.0000	2.0000	2.0000
0.0500	2.0000	1.9950	1.9950
0.1000	1.9900	1.9801	1.9801
0.1500	1.9701	1.9555	1.9554
0.2000	1.9405	1.9216	1.9215
0.2500	1.9017	1.8788	1.8787
0.3000	1.8542	1.8278	1.8277
0.3500	1.7986	1.7694	1.7692
0.4000	1.7356	1.7043	1.7040
0.4500	1.6662	1.6334	1.6330
0.5000	1.5912	1.5576	1.5572
0.5500	1.5117	1.4780	1.4775
0.6000	1.4285	1.3955	1.3949
0.6500	1.3428	1.3110	1.3103
0.7000	1.2555	1.2255	1.2247
0.7500	1.1676	1.1398	1.1390
0.8000	1.0801	1.0549	1.0541
0.8500	0.9937	0.9715	0.9706
0.9000	0.9092	0.8902	0.8893
0.9500	0.8274	0.8116	0.8107
1.0000	0.7488	0.7364	0.7354

The errors at $t = 1$ are, respectively, -1.3009e-002, -6.0218e-004, and 3.3293e-004.

10. The iteration is Euler's method, with $t_0 = 2, T = 1$, and $f(t, y) = y + t^2 y^3$.

11. Since $t_n = 1 + nh,\ h = 0.05,\ n = 0, 1, \dots, 99$, it follows that $t_0 = 1$ and $N - 1 = 99$. Thus, $N = 100$, and $T = t_N = 1 + Nh = 1 + 100h = 1 + (100)(0.05) = 5$. From the form of the iteration, it must be Heun's method. Therefore, $f(t, y) = ty^2 + 1$.

12. The iteration is the modified Euler's method, with $t_0 = 0, T = 2$, and $f(t, y) = t\sin^2 y$.

13. Since $t_n = 2 + nh,\ h = 0.01,\ n = 0, 1, \dots, 99$, it follows that $t_0 = 2$ and $N - 1 = 99$. Thus, $N = 100$, and $T = t_N = 2 + Nh = 2 + 100h = 2 + (100)(0.01) = 1$. From the form of the iteration, it must be Euler's method. Therefore, $f(t, y) = y/(t^2 + y^2)$.

14. The iteration is the modified Euler's method, with $t_0 = -1, T = 10$, and $f(t, y) = \sin(t + y)$.

16. (a) The initial value problem is $Q'(t) = 6(2 - \cos \pi t) - \dfrac{Q(t)}{V(t)}$, $Q(0) = 0$ where $V(t) = 90 + 5t$.

(c) The tank contains 100 gallons when $t = 2$ minutes. As estimated by Heun's method, $Q(2) = 23.7538\dots$ pounds.

17 (a). From Exercise 16, part (a), the problem to be solved is
$Q' = 12 - 6\cos \pi t - Q/(90+5t),\ Q(0)=0,\ 0 \le t \le 2$.

17 (b). Using the modified Euler's method with $h = 0.05$, we obtain

t	$Q(t)$
0	0
0.0500	0.3008
0.1000	0.6089
0.1500	0.9313
0.2000	1.2749
0.2500	1.6460
0.3000	2.0501
0.3500	2.4922
0.4000	2.9759
0.4500	3.5041
0.5000	4.0785
0.5500	4.6997
0.6000	5.3670
0.6500	6.0787
0.7000	6.8320
0.7500	7.6230
0.8000	8.4469
0.8500	9.2979
0.9000	10.1700
0.9500	11.0561
1.0000	11.9491
1.0500	12.8416
1.1000	13.7264
1.1500	14.5961
1.2000	15.4441
1.2500	16.2640
1.3000	17.0502
1.3500	17.7979
1.4000	18.5033
1.4500	19.1636
1.5000	19.7772
1.5500	20.3434
1.6000	20.8628
1.6500	21.3372
1.7000	21.7695
1.7500	22.1635
1.8000	22.5241
1.8500	22.8569
1.9000	23.1681
1.9500	23.4647
2.0000	23.7538

18. The Heun's method estimate is $P(2) = 1.5005$ million individuals.

19. Using the modified Euler's method, we estimate $P(2) = 1.5003$ million individuals.

20 (a). The results are listed below. The columns headed $H1$, $H2$, and $H3$ are the results obtained using step sizes $h = 0.05, h = 0.025,$ and $h = 0.0125$ respectively.

t	*H*1	*H*2	*H*3	*True*
0.0000	1.0000	1.0000	1.0000	1.0000
0.0500	1.0526	1.0526	1.0526	1.0526
0.1000	1.1109	1.1111	1.1111	1.1111
0.1500	1.1762	1.1764	1.1765	1.1765
0.2000	1.2495	1.2499	1.2500	1.2500
0.2500	1.3326	1.3332	1.3333	1.3333
0.3000	1.4275	1.4283	1.4285	1.4286
0.3500	1.5370	1.5381	1.5384	1.5385
0.4000	1.6645	1.6661	1.6665	1.6667
0.4500	1.8151	1.8174	1.8180	1.8182
0.5000	1.9954	1.9988	1.9997	2.0000
0.5500	2.2153	2.2204	2.2218	2.2222
0.6000	2.4894	2.4972	2.4993	2.5000
0.6500	2.8402	2.8527	2.8560	2.8571
0.7000	3.3049	3.3257	3.3314	3.3333
0.7500	3.9488	3.9860	3.9964	4.0000
0.8000	4.8975	4.9714	4.9925	5.0000
0.8500	6.4264	6.5969	6.6480	6.6667
0.9000	9.2615	9.7669	9.9353	10.0000
0.9500	15.9962	18.4267	19.5053	20.0000

The error ratios are denoted, respectively, by $R1$ and $R2$ where
$R1 = (H1 - True)/(H2 - True)$ and $R2 = (H2 - True)/(H3 - True)$

t	*R*1	*R*2
0.0500	3.8916	3.9471
0.1000	3.8879	3.9454
0.1500	3.8838	3.9436
0.2000	3.8791	3.9414
0.2500	3.8737	3.9390
0.3000	3.8674	3.9362
0.3500	3.8601	3.9329
0.4000	3.8513	3.9291
0.4500	3.8408	3.9244
0.5000	3.8279	3.9187
0.5500	3.8117	3.9117
0.6000	3.7909	3.9026
0.6500	3.7634	3.8907
0.7000	3.7255	3.8743
0.7500	3.6707	3.8503
0.8000	3.5860	3.8126
0.8500	3.4428	3.7459
0.9000	3.1683	3.6039
0.9500	2.5450	3.1800

20 (b). An error monitor is $y(t^*) - \hat{y}_{2n} = (\hat{y}_{2n} - y_n)/3$.

20 (c). The column headed *est* gives the estimated error using the error monitor from part (b). The column headed *true* gives the actual error. The error monitor used step sizes of $h = 0.025$ and $h = 0.0125$.

t	*est*	*true*
0.0500	4.4179e-006	4.4972e-006
0.1000	1.0382e-005	1.0574e-005
0.1500	1.8466e-005	1.8820e-005
0.2000	2.9497e-005	3.0084e-005
0.2500	4.4686e-005	4.5614e-005
0.3000	6.5850e-005	6.7281e-005
0.3500	9.5774e-005	9.7965e-005
0.4000	1.3886e-004	1.4222e-004
0.4500	2.0228e-004	2.0751e-004
0.5000	2.9820e-004	3.0650e-004
0.5500	4.4818e-004	4.6178e-004
0.6000	6.9264e-004	7.1587e-004
0.6500	1.1126e-003	1.1547e-003
0.7000	1.8854e-003	1.9679e-003
0.7500	3.4446e-003	3.6255e-003
0.8000	7.0273e-003	7.4956e-003
0.8500	1.7050e-002	1.8628e-002
0.9000	5.6137e-002	6.4677e-002
0.9500	3.5951e-001	4.9473e-001

Section 9.2

Unless indicated by $\ldots$, all results are rounded to the places shown.

1 (a). From the given equation $y' = -y + 2$, we know $y'(t) = -y(t) + 2$, $y''(t) = -y'(t)$, $y'''(t) = -y''(t)$, and $y^{(4)}(t) = -y'''(t)$. We also know that $y(0) = 1$. Therefore, $y'(0) = -y(0) + 2 = -1 + 2 = 1$, $y''(0) = -y'(0) = -1$, $y'''(0) = -y''(0) = 1$, and $y^{(4)}(0) = -y'''(0) = -1$. Thus,

$$P_4(t) = 1 + t - (1/2)t^2 + (1/6)t^3 - (1/24)t^4 .$$

2 (a). From the given equation $y' = 2ty$, we know, $y'(0) = 0$, $y''(0) = 2$, $y'''(0) = 0$, and $y^{(4)}(0) = 12$. Thus,

$$P_4(t) = 1 + t^2 + (1/2)t^4 .$$

3 (a). From the given equation $y' = ty^2$, we know $y'(t) = ty^2(t)$, $y''(t) = y^2(t) + 2ty(t)y'(t)$, $y'''(t) = 4y(t)y'(t) + 2ty'(t)y'(t) + 2ty(t)y''(t)$, and $y^{(4)}(t) = 6y'(t)y'(t) + 6y(t)y''(t) + 6ty'(t)y''(t) + 2ty(t)y'''(t)$. We also know that $y(0) = 1$. Therefore, $y'(0) = 0$, $y''(0) = 1$, $y'''(0) = 0$, and $y^{(4)}(0) = 6$. Thus,

$$P_4(t) = 1 + (1/2)t^2 + (1/4)t^4 .$$

4 (a). From the given equation $y' = t^2 + y$, we know, $y'(0) = 1$, $y''(0) = 1$, $y'''(0) = 3$, and $y^{(4)}(0) = 3$. Thus,

$$P_4(t) = 1 + t + (1/2)t^2 + (1/2)t^3 + (1/8)t^4.$$

5 (a). From the given equation $y' = y^{1/2}$, we know $y'(t) = y^{1/2}(t)$, $y''(t) = (1/2)y^{-1/2}(t)y'(t)$, $y'''(t) = -(1/4)y^{-3/2}(t)y'(t) + (1/2)y^{-1/2}(t)y''(t)$, and $y^{(4)}(t) = (3/8)y^{-5/2}(t)y'(t)y'(t) - (3/4)y^{-3/2}(t)y''(t)y'(t) + (1/2)y^{-1/2}(t)y'''(t)$. We also know that $y(0) = 1$. Therefore, $y'(0) = 1$, $y''(0) = 1/2$, $y'''(0) = -(1/4) + (1/2)(1/2) = 0$, and $y^{(4)}(0) = (3/8) - (3/4)(1/2) = 0$. Thus,

$$P_4(t) = 1 + t + (1/4)t^2.$$

6 (a). From the given equation $y' = ty^{-1}$, we know, $y'(0) = 0$, $y''(0) = 1$, $y'''(0) = 0$, and $y^{(4)}(0) = -3$. Thus,

$$P_4(t) = 1 + (1/2)t^2 - (1/8)t^4.$$

7 (a). From the given equation $y' = y + \sin t$, we know $y'(t) = y(t) + \sin t$, $y''(t) = y'(t) + \cos t$, $y'''(t) = y''(t) - \sin t$, and $y^{(4)}(t) = y'''(t) - \cos t$. We also know that $y(0) = 1$. Therefore, $y'(0) = 1$, $y''(0) = 1 + 1 = 2$, $y'''(0) = 2 - 0 = 2$, and $y^{(4)}(0) = 2 - 1 = 1$. Thus,

$$P_4(t) = 1 + t + t^2 + (1/3)t^3 + (1/24)t^4.$$

8 (a). From the given equation $y' = y^{3/4}$, we know, $y'(0) = 1$, $y''(0) = 3/4$, $y'''(0) = 3/8$, and $y^{(4)}(0) = 3/32$. Thus,

$$P_4(t) = 1 + t + (3/8)t^2 + (1/16)t^3 + (1/256)t^4.$$

9 (a). From the given equation $y' = 1 + y^2$, we know $y'(t) = 1 + y^2(t)$, $y''(t) = 2y(t)y'(t)$, $y'''(t) = 2y'(t)y'(t) + 2y(t)y''(t)$, and $y^{(4)}(t) = 6y''(t)y'(t) + 2y(t)y''(t)$. We also know that $y(0) = 1$. Therefore, $y'(0) = 2$, $y''(0) = (2)(1)(2) = 4$, $y'''(0) = (2)(2)(2) + (2)(1)(4) = 16$, and $y^{(4)}(0) = (6)(4)(2) + (2)(1)(16) = 80$. Thus,

$$P_4(t) = 1 + 2t + 2t^2 + (8/3)t^3 + (10/3)t^4.$$

10 (a). From the given equation $y' = -4t^3y$, we know, $y'(0) = 0$, $y''(0) = 0$, $y'''(0) = 0$, and $y^{(4)}(0) = -24$. Thus,

$$P_4(t) = 1 - t^4.$$

11 (a). From the given equation $y'' = 3y' - 2y$, we know $y''(t) = 3y'(t) - 2y(t)$, $y'''(t) = 3y''(t) - 2y'(t)$, $y^{(4)}(t) = 3y'''(t) - 2y''(t)$, and $y^{(5)}(t) = 3y^{(4)}(t) - 2y'''(t)$. We also know that $y(0) = 1$ and $y'(0) = 0$. Therefore, $y''(0) = (3)(0) - (2)(1) = -2$, $y'''(0) = (3)(-2) - (2)(0) = -6$, $y^{(4)}(0) = (3)(-6) - (2)(-2) = -14$, and $y^{(5)}(0) = (3)(-14) - (2)(-6) = -30$. Thus,

$$P_5(t) = 1 - t^2 - t^3 - (7/12)t^4 - (1/4)t^5.$$

12 (a). From the given equation $y'' - y' = 0$, we know, $y''(1) = 2$, $y'''(1) = 2$, $y^{(4)}(1) = 2$, and $y^{(5)}(1) = 2$. Thus,

$$P_5(t) = 1 + 2(t-1) + (t-1)^2 + (1/3)(t-1)^3 + (1/12)(t-1)^4 + (1/60)(t-1)^5.$$

13 (a). From the given equation $y''' = y'$, we know $y'''(t) = y'(t)$, $y^{(4)}(t) = y''(t)$, and $y^{(5)}(t) = y'''(t)$. We also know that $y(0) = 1, y'(0) = 2$ and $y''(0) = 0$. Therefore, $y'''(0) = 2$, $y^{(4)}(0) = 0$, $y^{(5)}(0) = 2$. Thus,

$$P_5(t) = 1 + 2t + (1/3)t^3 + (1/60)t^5.$$

14 (a). From the given equation $y'' + y + y^3 = 0$, we know, $y''(0) = -2$, $y'''(0) = 0$, $y^{(4)}(0) = 8$, and $y^{(5)}(0) = 0$. Thus,

$$P_5(t) = 1 - t^2 + (1/3)t^4.$$

15. The function $q(h) = \sin 2h$ has a Maclaurin expansion given by $\sin 2h = 2h - (1/6)(2h)^3 + \cdots$. Therefore, $q(h) = O(h)$.

16. $q(h) = O(h)$.

17. The function $q(h) = 1 - \cos h$ has a Maclaurin expansion given by $1 - \cos h = 1 - (1 - (1/2)h^2 + (1/24)h^4 + \cdots = (1/2)h^2 + \cdots$. Therefore, $q(h) = O(h^2)$.

18. The function $q(h) = e^h - (1 + h)$ has a Maclaurin expansion given by $e^h - (1 + h) = [1 + h + (1/2)h^2 + \cdots] - (1 + h) = (1/2)h^2 + \cdots$. Therefore, $q(h) = O(h^2)$.

20 (a).

t	ts-1	ts-2	ts-3
0.0000	1.0000	1.0000	1.0000
0.0500	1.0000	1.0006	1.0006
0.1000	1.0013	1.0025	1.0025
0.1500	1.0037	1.0056	1.0056
0.2000	1.0075	1.0100	1.0100
0.2500	1.0125	1.0156	1.0156
0.3000	1.0187	1.0224	1.0224
0.3500	1.0261	1.0304	1.0304
0.4000	1.0348	1.0396	1.0396
0.4500	1.0446	1.0500	1.0500
0.5000	1.0556	1.0616	1.0616
0.5500	1.0677	1.0743	1.0742
0.6000	1.0810	1.0881	1.0881
0.6500	1.0955	1.1030	1.1030
0.7000	1.1110	1.1190	1.1190
0.7500	1.1276	1.1360	1.1360
0.8000	1.1452	1.1541	1.1541
0.8500	1.1638	1.1732	1.1731
0.9000	1.1835	1.1932	1.1932
0.9500	1.2041	1.2142	1.2142
1.0000	1.2256	1.2361	1.2361

20 (b). At $t = 1$, the errors are (respectively): -1.0441e-002, 5.5071e-005, and -1.0500e-006.

21 (a).

t	*ts-1*	*ts-2*	*ts-3*
0.0000	-1.0000	-1.0000	-1.0000
0.0500	-1.0000	-0.9975	-0.9975
0.1000	-0.9950	-0.9901	-0.9901
0.1500	-0.9851	-0.9779	-0.9780
0.2000	-0.9705	-0.9614	-0.9615
0.2500	-0.9517	-0.9409	-0.9412
0.3000	-0.9291	-0.9171	-0.9174
0.3500	-0.9032	-0.8905	-0.8908
0.4000	-0.8746	-0.8616	-0.8620
0.4500	-0.8440	-0.8311	-0.8316
0.5000	-0.8120	-0.7994	-0.8000
0.5500	-0.7790	-0.7672	-0.7677
0.6000	-0.7456	-0.7347	-0.7353
0.6500	-0.7123	-0.7024	-0.7030
0.7000	-0.6793	-0.6705	-0.6711
0.7500	-0.6470	-0.6394	-0.6400
0.8000	-0.6156	-0.6092	-0.6098
0.8500	-0.5853	-0.5800	-0.5806
0.9000	-0.5562	-0.5520	-0.5525
0.9500	-0.5283	-0.5252	-0.5256
1.0000	-0.5018	-0.4996	-0.5000

21 (b). At $t = 1$, the errors are (respectively): -1.8055e-003, 4.0475e-004, and -6.8372e-006

22 (a).

t	*ts*-1	*ts*-2	*ts*-3
0.0000	1.0000	1.0000	1.0000
0.0500	1.0250	1.0247	1.0247
0.1000	1.0494	1.0488	1.0488
0.1500	1.0732	1.0724	1.0724
0.2000	1.0965	1.0954	1.0954
0.2500	1.1193	1.1180	1.1180
0.3000	1.1416	1.1401	1.1402
0.3500	1.1635	1.1619	1.1619
0.4000	1.1850	1.1832	1.1832
0.4500	1.2061	1.2041	1.2042
0.5000	1.2269	1.2247	1.2247
0.5500	1.2472	1.2449	1.2450
0.6000	1.2673	1.2649	1.2649
0.6500	1.2870	1.2845	1.2845
0.7000	1.3064	1.3038	1.3038
0.7500	1.3256	1.3228	1.3229
0.8000	1.3444	1.3416	1.3416
0.8500	1.3630	1.3601	1.3601
0.9000	1.3814	1.3783	1.3784
0.9500	1.3995	1.3964	1.3964
1.0000	1.4173	1.4142	1.4142

22 (b). At $t = 1$, the errors are (respectively): 3.1075e-003, -5.7087e-005, and 1.3615e-006.

23 (a).

t	*ts-1*	*ts-2*	*ts-3*
0.0000	0.0000	0.0000	0.0000
0.0500	0.0500	0.0488	0.0488
0.1000	0.0977	0.0955	0.0956
0.1500	0.1436	0.1405	0.1407
0.2000	0.1880	0.1841	0.1844
0.2500	0.2311	0.2266	0.2269
0.3000	0.2733	0.2682	0.2686
0.3500	0.3146	0.3091	0.3095
0.4000	0.3553	0.3494	0.3498
0.4500	0.3955	0.3892	0.3897
0.5000	0.4354	0.4288	0.4293
0.5500	0.4751	0.4682	0.4687
0.6000	0.5146	0.5075	0.5080
0.6500	0.5541	0.5468	0.5473
0.7000	0.5937	0.5862	0.5868
0.7500	0.6335	0.6258	0.6264
0.8000	0.6736	0.6657	0.6664
0.8500	0.7139	0.7060	0.7067
0.9000	0.7547	0.7467	0.7475
0.9500	0.7960	0.7879	0.7888
1.0000	0.8379	0.8298	0.8307

23 (b). At $t = 1$, the errors are (respectively): 7.2979e-003, -8.2708e-004, and 3.0263e-005.

24. We find $E_1 = -1.0499 \times 10^{-6}$ and $E_2 = -1.2939 \times 10^{-7}$. The error ratio is 0.12323 while 1/8 is equal to 0.125. Thus, the error ratio is close to 1/8.

25. We find $E_1 = -6.8372 \times 10^{-6}$ and $E_2 = -8.4649 \times 10^{-7}$. The error ratio is 0.12381 while 1/8 is equal to 0.125. Thus, the error ratio is close to 1/8.

26. We find $E_1 = -1.3615 \times 10^{-6}$ and $E_2 = -1.6598 \times 10^{-7}$. The error ratio is 0.12191 while 1/8 is equal to 0.125. Thus, the error ratio is close to 1/8.

27. We find $E_1 = 3.0263 \times 10^{-5}$ and $E_2 = 3.6501 \times 10^{-6}$. The error ratio is 0.12061 while 1/8 is equal to 0.125. Thus, the error ratio is fairly close to 1/8.

Section 9.3

Unless indicated by …, all results are rounded to the places shown.

1 (a). For the given initial value problem $y' = -y + 2$, $y(0) = 1$, we have

$$K_1 = f(t_0, y_0) = f(0,1) = 1$$
$$K_2 = f(t_0 + h/2, y_0 + (h/2)K_1) = f(0.05, 1 + 0.05(1)) = 0.95$$
$$K_3 = f(t_0 + h, y_0 - hK_1 + 2hK_2) = 0.91$$
$$y_1 = y_0 + h(K_1 + 4K_2 + K_3)/6 = 1 + (0.1)(1 + 3.8 + 0.91)/6 = 1.095166\ldots$$

1 (b). As in (a), we find $K_1 = 1, K_2 = 0.95, K_3 = 0.9525, K_4 = 0.90475$ and thus $y_1 = 1.0951625$.

1 (c). A *k*th order Runge-Kutta method will give the exact solution if the solution is a polynomial of degree *k*. In this case, the Runge-Kutta method will not give the exact solution.

2 (a). For the given initial value problem $y' = 2ty$, $y(0) = 1$, we have $y_1 = 1.01006667$

2 (b). $y_1 = 1.01005017$.

2 (c). Neither Runge-Kutta method will give the exact solution.

3 (a). For the given initial value problem $y' = ty^2$, $y(0) = 1$, we have

$K_1 = f(t_0, y_0) = f(0,1) = 0$

$K_2 = f(t_0 + h/2, y_0 + (h/2)K_1) = f(0.05, 1 + 0.05(0)) = 0.05$

$K_3 = f(t_0 + h, y_0 - hK_1 + 2hK_2) = 0.1020\ldots$

$y_1 = y_0 + h(K_1 + 4K_2 + K_3)/6 = 1 + (0.1)(0 + 0.05 + 0.1020)/6 = 1.0050335\ldots$

3 (b). As in (a), we find $K_1 = 0, K_2 = 0.05, K_3 = 0.0503\ldots, K_4 = 0.1010\ldots$ and thus $y_1 = 1.0050251\ldots$.

3 (c). A *k*th order Runge-Kutta method will give the exact solution if the solution is a polynomial of degree *k*. In this case, the Runge-Kutta method will not give the exact solution.

4 (a). For the given initial value problem $y' = t^2 + y$, $y(0) = 1$, we have $y_1 = 1.10550833$

4 (b). $y_1 = 1.10551271$.

4 (c). Neither Runge-Kutta method will give the exact solution.

5 (a). For the given initial value problem $y' = \sqrt{y}$, $y(0) = 1$, we have

$K_1 = f(t_0, y_0) = f(0,1) = 1$

$K_2 = f(t_0 + h/2, y_0 + (h/2)K_1) = f(0.05, 1 + 0.05(1)) = 1.0246\ldots$

$K_3 = f(t_0 + h, y_0 - hK_1 + 2hK_2) = 1.0511\ldots$

$y_1 = y_0 + h(K_1 + 4K_2 + K_3)/6 = 1.1024990\ldots$

5 (b). As in (a), we find $K_1 = 1, K_2 = 1.0246\ldots, K_3 = 1.0252\ldots, K_4 = 1.0500\ldots$ and thus $y_1 = 1.1024999\ldots$.

5 (c). A *k*th order Runge-Kutta method will give the exact solution if the solution is a polynomial of degree *k*. In this case, since the solution is a quadratic polynomial, both Runge-Kutta methods will give the exact solution.

6 (a). For the given initial value problem $y' = t/y$, $y(0) = 1$, we have $y_1 = 1.00498350$

6 (b). $y_1 = 1.00498757$.

6 (c). Neither Runge-Kutta method will give the exact solution.

7 (a). For the given initial value problem $y' = y + \sin t$, $y(0) = 1$, we have

$K_1 = f(t_0, y_0) = f(0,1) = 1$

$K_2 = f(t_0 + h/2, y_0 + (h/2)K_1) = f(0.05, 1 + 0.05(1)) = 1.0999\ldots$

$K_3 = f(t_0 + h, y_0 - hK_1 + 2hK_2) = 1.2198\ldots$

$y_1 = y_0 + h(K_1 + 4K_2 + K_3)/6 = 1.110329\ldots$

7 (b). As in (a), we find $K_1 = 1, K_2 = 1.0999\ldots, K_3 = 1.1049\ldots, K_4 = 1.2103\ldots$ and thus $y_1 = 1.110337\ldots$.

7 (c). A *k*th order Runge-Kutta method will give the exact solution if the solution is a polynomial of degree *k*. In this case, a Runge-Kutta method will not give the exact solution.

8 (a). For the given initial value problem $y' = y^{3/4}$, $y(0) = 1$, we have $y_1 = 1.10381059$

8 (b). $y_1 = 1.10381285$.

8 (c). The 4*th* order Runge-Kutta method will give the exact solution, but not the 3*rd* order.

9 (a). For the given initial value problem $y' = 1 + y^2$, $y(0) = 1$, we have
$K_1 = f(t_0, y_0) = f(0,1) = 2$
$K_2 = f(t_0 + h/2, y_0 + (h/2)K_1) = f(0.05, 1 + 0.05(2)) = 2.21$
$K_3 = f(t_0 + h, y_0 - hK_1 + 2hK_2) = 2.5425\ldots$
$y_1 = y_0 + h(K_1 + 4K_2 + K_3)/6 = 1.2230427\ldots$

9 (b). As in (a), we find $K_1 = 2, K_2 = 2.21, K_3 = 2.2332\ldots, K_4 = 2.4965\ldots$ and thus $y_1 = 1.2230489\ldots$.

9 (c). A *k*th order Runge-Kutta method will give the exact solution if the solution is a polynomial of degree *k*. In this case, a Runge-Kutta method will not give the exact solution.

10 (a). For the given initial value problem $y' = -4t^3 y$, $y(0) = 1$, we have $y_1 = 0.99990001$

10 (b). $y_1 = 0.99990000$.

10 (c). Neither Runge-Kutta method will give the exact solution.

11. Rewriting the given initial value problem, $y'' + ty' + y$, $y(0) = 1, y'(0) = -1$, as a first order system, we have

$$\begin{aligned} y_1' &= y_2, \quad y_1(0) = 1 \\ y_2' &= -ty_2 - y_1, \quad y_2(0) = -1, \end{aligned} \quad \text{or} \quad \mathbf{y}' = \mathbf{f}(t,\mathbf{y}), \mathbf{y}(0) = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \mathbf{f}(t,\mathbf{y}) = \begin{bmatrix} y_2 \\ -ty_2 - y_1 \end{bmatrix}.$$

Therefore,

$$\mathbf{K}_1 = \mathbf{f}(t_0, \mathbf{y}_0) = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

$$\mathbf{K}_2 = \mathbf{f}(t_0 + h/2, \mathbf{y}_0 + (h/2)\mathbf{K}_1) = \begin{bmatrix} -1.05 \\ -0.8975 \end{bmatrix}$$

$$\mathbf{K}_3 = \mathbf{f}(t_0 + h/2, \mathbf{y}_0 + (h/2)\mathbf{K}_2) = \begin{bmatrix} -1.0448\ldots \\ -0.8952\ldots \end{bmatrix}$$

$$\mathbf{K}_4 = \mathbf{f}(t_0 + h, \mathbf{y}_0 + h\mathbf{K}_3) = \begin{bmatrix} -1.0895\ldots \\ -0.7865\ldots \end{bmatrix}$$

$$\mathbf{y}_1 = \mathbf{y}_0 + h(\mathbf{K}_1 + 2\mathbf{K}_2 + 2\mathbf{K}_3 + \mathbf{K}_4)/6 = \begin{bmatrix} 0.895345\ldots \\ -1.089534\ldots \end{bmatrix}.$$

12. $\mathbf{y}_1 = \begin{bmatrix} 1.194834\ldots \\ 1.895042\ldots \end{bmatrix}$

13. For the given initial value problem, $\mathbf{y}' = \begin{bmatrix} 0 & t \\ e^t & 0 \end{bmatrix}\mathbf{y} + \begin{bmatrix} 1 \\ t \end{bmatrix}, \mathbf{y}(0) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, we have

$\mathbf{f}(t,\mathbf{y}) = \begin{bmatrix} 1 + ty_2 \\ t + e^t y_1 \end{bmatrix}$. Therefore,

$$\mathbf{K}_1 = \mathbf{f}(t_0, \mathbf{y}_0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$\mathbf{K}_2 = \mathbf{f}(t_0 + h/2, \mathbf{y}_0 + (h/2)\mathbf{K}_1) = \begin{bmatrix} 1.055 \\ 2.2051\ldots \end{bmatrix}$$

$$\mathbf{K}_3 = \mathbf{f}(t_0 + h/2, \mathbf{y}_0 + (h/2)\mathbf{K}_2) = \begin{bmatrix} 1.0555\ldots \\ 2.2079\ldots \end{bmatrix}$$

$$\mathbf{K}_4 = \mathbf{f}(t_0 + h, \mathbf{y}_0 + h\mathbf{K}_3) = \begin{bmatrix} 1.1220\ldots \\ 2.4269\ldots \end{bmatrix}$$

$$\mathbf{y}_1 = \mathbf{y}_0 + h(\mathbf{K}_1 + 2\mathbf{K}_2 + 2\mathbf{K}_3 + \mathbf{K}_4)/6 = \begin{bmatrix} 2.105718\ldots \\ 1.220886\ldots \end{bmatrix}.$$

14. $\mathbf{y}_1 = \begin{bmatrix} -0.900625\ldots \\ 0.809968\ldots \end{bmatrix}$

15. Rewriting the given initial value problem, $y''' = ty$, $y(0) = 1, y'(0) = 0, y''(0) = -1$, as a first order system, we have

$$\begin{aligned} y_1' &= y_2, \quad y_1(0) = 1 \\ y_2' &= y_3, \quad y_2(0) = 0 \\ y_3' &= ty_1, \quad y_3(0) = -1, \end{aligned} \quad \text{or} \quad \mathbf{y}' = \mathbf{f}(t,\mathbf{y}), \mathbf{y}(0) = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \mathbf{f}(t,\mathbf{y}) = \begin{bmatrix} y_2 \\ y_3 \\ ty_1 \end{bmatrix}.$$

Therefore,

$$\mathbf{K}_1 = \mathbf{f}(t_0, \mathbf{y}_0) = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$$

$$\mathbf{K}_2 = \mathbf{f}(t_0 + h/2, \mathbf{y}_0 + (h/2)\mathbf{K}_1) = \begin{bmatrix} -0.05 \\ -1.0 \\ 0.05 \end{bmatrix}$$

$$\mathbf{K}_3 = \mathbf{f}(t_0 + h/2, \mathbf{y}_0 + (h/2)\mathbf{K}_2) = \begin{bmatrix} -0.05 \\ -0.9975 \\ 0.0498\ldots \end{bmatrix}$$

$$\mathbf{K}_4 = \mathbf{f}(t_0 + h, \mathbf{y}_0 + h\mathbf{K}_3) = \begin{bmatrix} -0.0997\ldots \\ -0.9950\ldots \\ 0.0995 \end{bmatrix}$$

$$\mathbf{y}_1 = \mathbf{y}_0 + h(\mathbf{K}_1 + 2\mathbf{K}_2 + 2\mathbf{K}_3 + \mathbf{K}_4)/6 = \begin{bmatrix} 0.995004\ldots \\ -0.099833\ldots \\ -0.995012\ldots \end{bmatrix}.$$

16. $\mathbf{y}_1 = \begin{bmatrix} 1.199637\ldots \\ 1.988834\ldots \\ 0.114991\ldots \end{bmatrix}$

19 (a). For the given initial value problem $y' = t/(1+y)$, $y(0) = 1$ and for the step size $h = 0.05$, we obtain $y_{20} = 1.2360679786\ldots$ as our estimate of $y(1)$.

19 (b). The actual value of the solution is $y(1) = 1.2360679749\ldots$.

20 (a). For the given initial value problem $y' = 2ty^2$, $y(0) = -1$ and for the step size $h = 0.05$, we obtain $y_{20} = -0.5000000409\ldots$ as our estimate of $y(1)$.

20 (b). The actual value of the solution is $y(1) = -0.5$.

21 (a). For the given initial value problem $y' = 1/(2y)$, $y(0) = 1$ and for the step size $h = 0.05$, we obtain $y_{20} = 1.4142135632\ldots$ as our estimate of $y(1)$.

21 (b). The actual value of the solution is $y_{20} = 1.4142135623\ldots$.

22 (a). For the given initial value problem $y' = (1+y^2)/(1+t)$, $y(0) = 0$ and for the step size $h = 0.05$, we obtain $y_{20} = 0.83064092\ldots$ as our estimate of $y(1)$.

22 (b). The actual value of the solution is $y(1) = 0.83064087\ldots$.

23 (a). Rewriting the given initial value problem, $y'' + 2y' + 2y = -2$, $y(0) = 0, y'(0) = 1$, as a first order system, we have

$$\begin{aligned} y_1' &= y_2, \quad y_1(0) = 0 \\ y_2' &= -2y_2 - 2y_1 - 2, \quad y_2(0) = 1, \end{aligned} \quad \text{or} \quad \mathbf{y}' = \mathbf{f}(t,\mathbf{y}), \mathbf{y}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \mathbf{f}(t,\mathbf{y}) = \begin{bmatrix} y_2 \\ -2y_2 - 2y_1 - 2 \end{bmatrix}.$$

23 (b). Using the step size $h = 0.1$, we obtain $\mathbf{y}_{20} = \begin{bmatrix} -0.810202\ldots \\ -0.425496\ldots \end{bmatrix}$ as our estimate to the solution value $\mathbf{y}(2) = \begin{bmatrix} -0.810199\ldots \\ -0.425499\ldots \end{bmatrix}$.

24 (b). Using the step size $h = 0.1$, we obtain $\mathbf{y}_{10} = \begin{bmatrix} 0.829662\ldots \\ 0.383398\ldots \end{bmatrix}$ as our estimate to the solution value $\mathbf{y}(1) = \begin{bmatrix} 0.829660\ldots \\ 0.383400\ldots \end{bmatrix}$.

25 (a). Rewriting the given initial value problem, $t^2y'' - ty' + y = t^2$, $y(1) = 2, y'(1) = 2$, as a first order system, we have

$$\begin{aligned} y_1' &= y_2, \quad y_1(1) = 2 \\ y_2' &= (ty_2 - y_1 + t^2)/t^2, \quad y_2(1) = 2, \end{aligned} \quad \text{or}$$

$$\mathbf{y}' = \mathbf{f}(t,\mathbf{y}), \mathbf{y}(0) = \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \mathbf{f}(t,\mathbf{y}) = \begin{bmatrix} y_2 \\ (ty_2 - y_1 + t^2)/t^2 \end{bmatrix}.$$

25 (b). Using the step size $h = 0.1$, we obtain $\mathbf{y}_{10} = \begin{bmatrix} 4.6137054\ldots \\ 3.3068527\ldots \end{bmatrix}$ as our estimate to the solution value $\mathbf{y}_{10} = \begin{bmatrix} 4.6137056\ldots \\ 3.3068528\ldots \end{bmatrix}$.

Chapter 10
Series Solutions of Linear Differential Equations

Section 10.1

1. Consider the power series $\sum_{n=0}^{\infty} \frac{t^n}{2^n}$. Applying the ratio test at an arbitrary value of t, $t \neq 0$, we obtain $\lim_{n\to\infty}\left|\frac{2^n t^{n+1}}{2^{n+1} t^n}\right| = \lim_{n\to\infty}\left|\frac{t}{2}\right| = \left|\frac{t}{2}\right|$. The limiting ratio is less than 1 if $|t| < 2$. Therefore, the radius of convergence is $R = 2$.

2. $\lim_{n\to\infty}\left|\frac{t^{n+1}n^2}{t^n(n+1)^2}\right| = \lim_{n\to\infty}\left|\frac{t}{\left(1+\frac{1}{n}\right)^2}\right| = |t|$. Therefore, the radius of convergence is $R = 1$.

3. Consider the power series $\sum_{n=0}^{\infty}(t-2)^n$. Applying the ratio test at an arbitrary value of t, $t \neq 2$, we obtain $\lim_{n\to\infty}\left|\frac{(t-2)^{n+1}}{(t-2)^n}\right| = \lim_{n\to\infty}|t-2| = |t-2|$. The limiting ratio is less than 1 if $|t-2| < 1$. Therefore, the radius of convergence is $R = 1$.

4. $\lim_{n\to\infty}\left|\frac{(3t-1)^{n+1}}{(3t-1)^n}\right| = |3t-1| < 1 \Rightarrow -1 < 3t-1 < 1 \Rightarrow 0 < t < \frac{2}{3}$. Therefore, the radius of convergence is $R = \frac{1}{3}$.

5. Consider the power series $\sum_{n=0}^{\infty}\frac{(t-1)^n}{n!}$. Applying the ratio test at an arbitrary value of t, $t \neq 1$, we obtain $\lim_{n\to\infty}\left|\frac{n!(t-1)^{n+1}}{(n+1)!(t-1)^n}\right| = \lim_{n\to\infty}\left|\frac{t-1}{n+1}\right| = 0$. The limiting ratio is less than 1 for all t, $t \neq 1$. Therefore, the radius of convergence is $R = \infty$.

6. $\lim_{n\to\infty}\left|\frac{(n+1)!(t-1)^{n+1}}{n!(t-1)^n}\right| = \lim_{n\to\infty}|(n+1)(t-1)| = \infty$, $t \neq 1$. Therefore, the radius of convergence is $R = 0$.

7. Consider the power series $\sum_{n=1}^{\infty}\frac{(-1)^n t^n}{n}$. Applying the ratio test at an arbitrary value of t, $t \neq 0$, we obtain $\lim_{n\to\infty}\left|\frac{nt^{n+1}}{(n+1)t^n}\right| = \lim_{n\to\infty}\left|\frac{nt}{n+1}\right| = |t|$. The limiting ratio is less than 1 if $|t| < 1$. Therefore, the radius of convergence is $R = 1$.

8. $\lim_{n\to\infty}\left|\frac{(-1)^{n+1}(t-3)^{n+1}4^n}{(-1)^n(t-3)^n4^{n+1}}\right| = \left|\frac{t-3}{4}\right| < 1 \Rightarrow -4 < t-3 < 4 \Rightarrow -1 < t < 7$. Therefore, the radius of convergence is $R = 4$.

9. Consider the power series $\sum_{n=1}^{\infty}(\ln n)(t+2)^n$. Applying the ratio test at an arbitrary value of t, $t \neq -2$, we obtain $\lim_{n\to\infty}\left|\frac{(\ln(n+1))(t+2)^{n+1}}{(\ln n)(t+2)^n}\right| = \lim_{n\to\infty}\left|\frac{(\ln(n+1))(t+2)}{\ln n}\right| = \left|t+2\right|\lim_{n\to\infty}\frac{\ln(n+1)}{\ln n} = \left|t+2\right|$. (The last limit can be found using L'Hôpital's Rule.) The limiting ratio is less than 1 if $\left|t+2\right| < 1$. Therefore, the radius of convergence is $R = 1$.

10. $\lim_{n\to\infty}\left|\frac{(n+1)^3(t-1)^{n+1}}{n^3(t-1)^n}\right| = \left|t-1\right| < 1 \Rightarrow -1 < t-1 < 1 \Rightarrow 0 < t < 2$. Therefore, the radius of convergence is $R = 1$.

11. Consider the power series $\sum_{n=1}^{\infty}\frac{\sqrt{n}\,(t-4)^n}{2^n}$. Applying the ratio test at an arbitrary value of t, $t \neq 4$, we obtain $\lim_{n\to\infty}\left|\frac{2^n\sqrt{n+1}\,(t-4)^{n+1}}{2^{n+1}\sqrt{n}\,(t-4)^n}\right| = \lim_{n\to\infty}\left|\frac{\sqrt{n+1}\,(t-4)}{2\sqrt{n}}\right| = \left|\frac{t-4}{2}\right|$. The limiting ratio is less than 1 if $\left|t-4\right| < 2$. Therefore, the radius of convergence is $R = 2$.

12. $\lim_{n\to\infty}\left|\frac{(t-2)^{n+1}\arctan(n)}{(t-2)^n\arctan(n+1)}\right| = \left|t-2\right| < 1 \Rightarrow -1 < t-2 < 1 \Rightarrow 1 < t < 3$ $\left(\text{recall } \lim_{n\to\infty}\arctan(n) = \frac{\pi}{2}\right)$. Therefore, the radius of convergence is $R = 1$.

13. Applying the ratio test, we see the power series for $f(t)$ and $g(t)$ both have radius of convergence $R = 1$. Therefore, each series converges in the interval $-1 < t < 1$.
(a) $f(t) = 1 + t + t^2 + t^3 + t^4 + t^5 + \cdots$
$g(t) = 0 + t + 4t^2 + 9t^3 + 16t^4 + 25t^5 + \cdots$
(b) $f(t) + g(t) = 1 + 2t + 5t^2 + 10t^3 + 17t^4 + 26t^5 + \cdots$
(c) $f(t) - g(t) = 1 - 3t^2 - 8t^3 - 15t^4 - 24t^5 - \cdots$
(d) $f'(t) = 1 + 2t + 3t^2 + 4t^3 + 5t^4 + 6t^5 + \cdots$
(e) $f''(t) = 2 + 6t + 12t^2 + 20t^3 + 30t^4 + 42t^5 + \cdots$

14. Applying the ratio test, we see the power series for $f(t)$ and $g(t)$ both have radius of convergence $R = 1$. Therefore, each series converges in the interval $-1 < t < 1$.
(a) $f(t) = t + 2t^2 + 3t^3 + 4t^4 + 5t^5 + 6t^6 + \cdots$
$g(t) = -t + 2t^2 - 3t^3 + 4t^4 - 5t^5 + 6t^6 - \cdots$
(b) $f(t) + g(t) = 4t^2 + 8t^4 + 12t^6 + 16t^8 + 20t^{10} + \cdots$
(c) $f(t) - g(t) = 2t + 6t^3 + 10t^5 + 14t^7 + 18t^9 + 22t^{11} + \cdots$
(d) $f'(t) = 1 + 4t + 9t^2 + 16t^3 + 25t^4 + 36t^5 + \cdots$
(e) $f''(t) = 4 + 18t + 48t^2 + 100t^3 + 180t^4 + 294t^5 + \cdots$

15. Applying the ratio test, we see the power series for $f(t)$ has radius of convergence $R = 1/2$ while the series for $g(t)$ has radius of convergence $R = 1$. Therefore, each series converges in the interval $|t-1| < 1/2$, or $1/2 < t < 3/2$.

(a) $f(t) = 1 - 2(t-1) + 4(t-1)^2 - 8(t-1)^3 + 16(t-1)^4 - 32(t-1)^5 + \cdots$
$g(t) = 1 + (t-1) + (t-1)^2 + (t-1)^3 + (t-1)^4 + (t-1)^5 + \cdots$

(b) $f(t) + g(t) = 2 - (t-1) + 5(t-1)^2 - 7(t-1)^3 + 17(t-1)^4 - 31(t-1)^5 + \cdots$

(c) $f(t) - g(t) = -3(t-1) + 3(t-1)^2 - 9(t-1)^3 + 15(t-1)^4 - 33(t-1)^5 + \cdots$

(d) $f'(t) = -2 + 8(t-1) - 24(t-1)^2 + 64(t-1)^3 - 160(t-1)^4 + 384(t-1)^5 \cdots$

(e) $f''(t) = 8 - 48(t-1) + 192(t-1)^2 - 640(t-1)^3 + 1920(t-1)^4 - 5376(t-1)^5 \cdots$

16. Applying the ratio test, we see the power series for $f(t)$ is 1/2 and $g(t)$ is 1. Therefore, $R = \frac{1}{2}$.

(a) $f(t) = 1 + 2(t+1) + 4(t+1)^2 + 8(t+1)^3 + 16(t+1)^4 + 32(t+1)^5 + \cdots$
$g(t) = (t+1) + 2(t+1)^2 + 3(t+1)^3 + 4(t+1)^4 + 5(t+1)^5 + 6(t+1)^6 + \cdots$

(b) $f(t) + g(t) = 1 + 3(t+1) + 6(t+1)^2 + 11(t+1)^3 + 20(t+1)^4 + 37(t+1)^5 + \cdots$

(c) $f(t) - g(t) = 1 + (t+1) + 2(t+1)^2 + 5(t+1)^3 + 12(t+1)^4 + 27(t+1)^5 + \cdots$

(d) $f'(t) = 2 + 8(t+1) + 24(t+1)^2 + 64(t+1)^3 + 160(t+1)^4 + 384(t+1)^5 + \cdots$

(e) $f''(t) = 8 + 48(t+1) + 192(t+1)^2 + 640(t+1)^3 + 1920(t+1)^4 + 5376(t+1)^5 + \cdots$

17. Consider the power series $\sum_{n=0}^{\infty} 2^n t^{n+2}$. Make the change of index $k = n+2$. With this change, the lower limit of $n = 0$ transforms to $k = 2$ while the upper limit remains at ∞. Thus, the power series can be rewritten as $\sum_{k=2}^{\infty} 2^{k-2} t^k$. Finally, changing to the original summation index, n, we obtain $\sum_{n=2}^{\infty} 2^{n-2} t^n$.

18. Make the change of index $k = n+3$. The power series can be rewritten as $\sum_{k=3}^{\infty} (k-2)(k-1)t^k$.
Finally, changing to the original summation index, n, we obtain $\sum_{n=3}^{\infty} (n-2)(n-1)t^n$.

19. Consider the power series $\sum_{n=0}^{\infty} a_n t^{n+2}$. Make the change of index $k = n+2$. With this change, the lower limit of $n = 0$ transforms to $k = 2$ while the upper limit remains at ∞. Thus, the power series can be rewritten as $\sum_{k=2}^{\infty} a_{k-2} t^k$. Finally, changing to the original summation index, n, we obtain $\sum_{n=2}^{\infty} a_{n-2} t^n$.

20. Make the change of index $k = n-1$. The power series can be rewritten as $\sum_{k=0}^{\infty} (k+1)a_{k+1} t^k$.
Finally, changing to the original summation index, n, we obtain $\sum_{n=0}^{\infty} (n+1)a_{n+1} t^n$.

21. Consider the power series $\sum_{n=2}^{\infty} n(n-1)a_n t^{n-2}$. Make the change of index $k = n-2$. With this change, the lower limit of $n = 2$ transforms to $k = 0$ while the upper limit remains at ∞. Thus, the power series can be rewritten as $\sum_{k=0}^{\infty}(k+2)(k+1)a_{k+2}t^k$. Finally, changing to the original summation index, n, we obtain $\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}t^n$.

22. Make the change of index $k = n+3$. The power series can be rewritten as $\sum_{k=3}^{\infty}(-1)^{k-3}a_{k-3}t^k$. Finally, changing to the original summation index, n, we obtain $\sum_{n=3}^{\infty}(-1)^{n-3}a_{n-3}t^n$.

23. Consider the power series $\sum_{n=0}^{\infty}(-1)^{n+1}(n+1)a_n t^{n+2}$. Make the change of index $k = n+2$. With this change, the lower limit of $n = 0$ transforms to $k = 2$ while the upper limit remains at ∞. Thus, the power series can be rewritten as $\sum_{k=2}^{\infty}(-1)^{k-1}(k-1)a_{k-2}t^k$. Finally, changing to the original summation index, n, we obtain $\sum_{n=2}^{\infty}(-1)^{n-1}(n-1)a_{n-2}t^n$.

24. Let $f(t) = t^2(t-\sin t)$. $t-\sin t = -\sum_{n=1}^{\infty}\frac{(-1)^n t^{2n+1}}{(2n+1)!}$. Therefore, $f(t) = \sum_{n=1}^{\infty}\frac{(-1)^{n+1}t^{2n+3}}{(2n+1)!}$.
$\lim_{n\to\infty}\left|\frac{(-1)^{n+2}(2n+1)!(t)^{2n+5}}{(-1)^{n+1}(2n+3)!(t)^{2n+3}}\right| = 0$. Thus, the radius of convergence is $R = \infty$.

25. Let $f(t) = 1-\cos 3t$. From the Maclaurin series for $\cos u$ we have $\cos u = \sum_{n=0}^{\infty}(-1)^n\frac{u^{2n}}{(2n)!}$.
Therefore, $\cos 3t = 1-\frac{9t^2}{2!}+\frac{81t^4}{4!}-\frac{729t^6}{6!}+\cdots$. Hence,
$f(t) = \frac{9t^2}{2!}-\frac{81t^4}{4!}+\frac{729t^6}{6!}-\cdots = \sum_{n=1}^{\infty}(-1)^{n+1}\frac{(3t)^{2n}}{(2n)!}$. We calculate the radius of convergence by using the ratio test. For an arbitrary value of t, $t \neq 0$, we have
$\lim_{n\to\infty}\left|\frac{(2n)!(3t)^{2n+2}}{(2n+2)!(3t)^{2n}}\right| = \lim_{n\to\infty}\left|\frac{9t^2}{(2n+2)(2n+1)}\right| = 0$. Thus, the radius of convergence is $R = \infty$.

26. Let $f(t) = \frac{1}{1+2t} = \frac{1}{1-(-2t)}$. $\frac{1}{1-(-2t)} = \sum_{n=0}^{\infty}(-2t)^n = \sum_{n=0}^{\infty}(-2)^n t^n$. $\lim_{n\to\infty}\left|\frac{(-2t)^{n+1}}{(-2t)^n}\right| = 2|t| < 1$.
Thus, the radius of convergence is $R = \frac{1}{2}$.

27. Let $f(t) = 1/(1-t^2)$. From the Maclaurin series for $1/(1-u)$ we have $\frac{1}{1-u} = \sum_{n=0}^{\infty}u^n$. Therefore,
$\frac{1}{1-t^2} = 1+t^2+t^4+t^6+\cdots$. Hence, $f(t) = \sum_{n=0}^{\infty}t^{2n}$.

We calculate the radius of convergence by using the ratio test. For an arbitrary value of $t,\ t \neq 0$, we have $\lim_{n\to\infty}\left|\frac{t^{2n+2}}{t^{2n}}\right| = \lim_{n\to\infty}\left|t^2\right| = t^2$. Thus, the radius of convergence is $R = 1$.

28 (a). $e^t = \sum_{n=0}^{\infty}\frac{t^n}{n!} = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \frac{t^5}{5!} + \ldots.$

$e^{-t} = \sum_{n=0}^{\infty}\frac{(-t)^n}{n!} = 1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \frac{t^4}{4!} - \frac{t^5}{5!} + \ldots$

28 (b). $\sinh(t) = \frac{1}{2}\left\{\left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \frac{t^5}{5!} + \ldots\right) - \left(1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \frac{t^4}{4!} - \frac{t^5}{5!} + \ldots\right)\right\} = t + \frac{t^3}{3!} + \frac{t^5}{5!} + \ldots.$

$\cosh(t) = \frac{1}{2}\left\{\left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \frac{t^5}{5!} + \ldots\right) + \left(1 - t + \frac{t^2}{2!} - \frac{t^3}{3!} + \frac{t^4}{4!} - \frac{t^5}{5!} + \ldots\right)\right\} = 1 + \frac{t^2}{2!} + \frac{t^4}{4!} + \ldots$

29 (a). Consider the differential equation $y'' - \omega^2 y = 0$ and assume there is solution of the form $y(t) = \sum_{n=0}^{\infty} a_n t^n$. Differentiating, we obtain $y'(t) = \sum_{n=1}^{\infty} n a_n t^{n-1}$ and $y''(t) = \sum_{n=2}^{\infty} n(n-1)a_n t^{n-2}$.

Inserting these series into the differential equation, we have $\sum_{n=2}^{\infty} n(n-1)a_n t^{n-2} - \omega^2 \sum_{n=0}^{\infty} a_n t^n = 0$.

Making the change of index $k = n - 2$ in the series for $y''(t)$, we obtain

$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}t^n - \omega^2\sum_{n=0}^{\infty}a_n t^n = 0$, or $\sum_{n=0}^{\infty}[(n+2)(n+1)a_{n+2} - \omega^2 a_n]t^n = 0$. Equating the coefficients to zero, we find the recurrence relation $a_{n+2} = \frac{\omega^2 a_n}{(n+2)(n+1)}, n = 0, 1, \ldots$

29 (b). The recurrence relation in part (a) leads us to

$a_2 = \omega^2 a_0 / 2,\ a_4 = \omega^2 a_2 / 12 = \omega^4 a_0 / 24,\ a_6 = \omega^2 a_4 / 30 = \omega^6 a_0 / 720,\ \ldots$

$a_3 = \omega^2 a_1 / 6,\ a_5 = \omega^2 a_3 / 20 = \omega^4 a_1 / 120,\ a_7 = \omega^2 a_5 / 42 = \omega^6 a_1 / 5040,\ \ldots$

Thus, $y(t) = a_0[1 + \frac{(\omega t)^2}{2} + \frac{(\omega t)^4}{24} + \frac{(\omega t)^6}{720} + \cdots] + \frac{a_1}{\omega}[\omega t + \frac{(\omega t)^3}{6} + \frac{(\omega t)^5}{120} + \frac{(\omega t)^7}{5040} + \cdots]$.

By Exercise 28, $y_1(t) = \cosh \omega t$ and $y_2(t) = \sinh \omega t$.

30 (a). $y(t) = \int_0^t \sum_{n=1}^{\infty} n\lambda^{n-1} d\lambda + C = \sum_{n=1}^{\infty} t^n + C,\ y(0) = C = 1 \Rightarrow y(t) = 1 + \sum_{n=1}^{\infty} t^n = \sum_{n=0}^{\infty} t^n$.

30 (b). $R = 1$.

30 (c). $y(t) = \frac{1}{1-t}$.

31 (a). Consider the function given by $y'(t) = \sum_{n=0}^{\infty}\frac{(t-1)^n}{n!},\ y(1) = 1$. Integrating the series termwise, we obtain $y(t) = C + \sum_{n=0}^{\infty}\frac{(t-1)^{n+1}}{(n+1)!}$. Imposing the condition $y(1) = 1$, it follows that $C = 1$.

Adjusting the index of summation, we can write $y(t) = 1 + \sum_{n=1}^{\infty}\frac{(t-1)^n}{n!} = \sum_{n=0}^{\infty}\frac{(t-1)^n}{n!}$.

31 (b). Applying the ratio test, $\lim_{n\to\infty}\left|\dfrac{n!(t-1)^{n+1}}{(n+1)!(t-1)^n}\right| = \lim_{n\to\infty}\left|\dfrac{t-1}{n+1}\right| = 0$. Therefore, the radius of convergence is $R=\infty$.

31 (c). From the power series (7a), we see that $y(t)=e^{t-1}$.

32 (a). $y'(t) = -1+\int_0^t \sum_{n=0}^{\infty}(-1)^n \dfrac{\lambda^n}{n!}\,d\lambda = -1+\sum_{n=0}^{\infty}(-1)^n\dfrac{t^{n+1}}{(n+1)!} = -1+\sum_{n=1}^{\infty}(-1)^{n-1}\dfrac{t^n}{n!} = -\left\{1+\sum_{n=1}^{\infty}(-1)^n\dfrac{t^n}{n!}\right\}$

$y' = -\sum_{n=0}^{\infty}(-1)^n\dfrac{t^n}{n!}$. Then, $y(t) = -\sum_{n=0}^{\infty}(-1)^n\dfrac{t^{n+1}}{(n+1)!}+1 = 1+\sum_{n=0}^{\infty}(-1)^{n+1}\dfrac{t^{n+1}}{(n+1)!} = \sum_{n=0}^{\infty}(-1)^n\dfrac{t^n}{n!}$.

32 (b). $R=\infty$.

32 (c). $y(t)=e^{-t}$.

33 (a). Consider the function given by $y'(t)=\sum_{n=2}^{\infty}(-1)^n\dfrac{(t-1)^n}{n!}$, $y(1)=0$. Integrating the series termwise, we obtain $y(t)=C+\sum_{n=2}^{\infty}(-1)^n\dfrac{(t-1)^{n+1}}{(n+1)!}$. Imposing the condition $y(1)=0$, it follows that $C=0$. Adjusting the index of summation, we can write

$y(t)=\sum_{n=3}^{\infty}(-1)^{n+1}\dfrac{(t-1)^n}{n!} = -\sum_{n=3}^{\infty}(-1)^n\dfrac{(t-1)^n}{n!}$.

33 (b). Applying the ratio test, $\lim_{n\to\infty}\left|\dfrac{(-1)^{n+1}n!(t-1)^{n+1}}{(-1)^n(n+1)!(t-1)^n}\right| = \lim_{n\to\infty}\left|\dfrac{t-1}{n+1}\right| = 0$. Therefore, the radius of convergence is $R=\infty$.

33 (c). From the power series (7a), we see that $\sum_{n=0}^{\infty}(-1)^n\dfrac{(t-1)^n}{n!} = e^{-(t-1)}$. Thus,

$1-\dfrac{(t-1)}{1!}+\dfrac{(t-1)^2}{2!}+\sum_{n=3}^{\infty}(-1)^n\dfrac{(t-1)^n}{n!} = e^{-(t-1)}$. Or, using the results of part (a),

$1-\dfrac{(t-1)}{1!}+\dfrac{(t-1)^2}{2!}-e^{-(t-1)} = y(t)$.

34 (a). $y(t)=\int_0^t\sum_{n=0}^{\infty}(-1)^n s^{2n}\,ds = \sum_{n=0}^{\infty}(-1)^n\dfrac{t^{2n+1}}{2n+1}$.

34 (b). $\lim_{n\to\infty}\left|\dfrac{(-1)^{n+1}t^{2n+3}(2n+1)}{(-1)^n t^{2n+1}(2n+3)}\right| = \left|t^2\right|<1 \Rightarrow R=1$.

34 (c). $y(t)=\tan^{-1}(t)$.

35 (a). Consider the function $y(t)$ where $\int_0^t y(s)\,ds=\sum_{n=1}^{\infty}\dfrac{t^n}{n}$. Differentiating both sides, we obtain $y(t)=\sum_{n=1}^{\infty}t^{n-1}$. Adjusting the index of summation, we can write $y(t)=\sum_{n=0}^{\infty}t^n$.

35 (b). Applying the ratio test, $\lim_{n\to\infty}\left|\dfrac{t^{n+1}}{t^n}\right| = |t|$. Therefore, the radius of convergence is $R=1$.

35 (c). From the power series (7d), we see that $y(t)=\sum_{n=0}^{\infty}t^n = \dfrac{1}{1-t}$.

36. Assume there is solution of the form $y(t)=\sum_{n=0}^{\infty}a_nt^n$. Differentiating, we obtain

$y'(t)=\sum_{n=1}^{\infty}na_nt^{n-1}$ and $y''(t)=\sum_{n=2}^{\infty}n(n-1)a_nt^{n-2}=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}t^n$, $ty'=\sum_{n=0}^{\infty}na_nt^n$.

Therefore, $\sum_{n=0}^{\infty}[(n+2)(n+1)a_{n+2}-(n+1)a_n]t^n=0$. Equating the coefficients to zero, we find the recurrence relation $a_{n+2}=\dfrac{(n+1)a_n}{(n+2)(n+1)}=\dfrac{a_n}{n+2}$ The recurrence leads us to

$a_2=\dfrac{a_0}{2},\ a_3=\dfrac{a_1}{3},\ a_4=\dfrac{a_2}{4}=\dfrac{a_0}{8},\ a_5=\dfrac{a_3}{5}=\dfrac{a_1}{15}$

Therefore, $y(t)=a_0\left\{1+\dfrac{t^2}{2}+\dfrac{t^4}{8}+...\right\}+a_1\left\{t+\dfrac{t^3}{3}+\dfrac{t^5}{15}+...\right\}$, $y(0)=a_0=1,\ y'(0)=a_1=-1$.

Finally, $y(t)=\left\{1+\dfrac{t^2}{2}+\dfrac{t^4}{8}+...\right\}-\left\{t+\dfrac{t^3}{3}+\dfrac{t^5}{15}+...\right\}$.

37. Consider the initial value problem $y''+ty'-2y=0$, $y(0)=0, y'(0)=1$ and assume there is solution of the form $y(t)=\sum_{n=0}^{\infty}a_nt^n$. Differentiating, we obtain

$y'(t)=\sum_{n=1}^{\infty}na_nt^{n-1}$ and $y''(t)=\sum_{n=2}^{\infty}n(n-1)a_nt^{n-2}$. Inserting these series into the differential equation, we have $\sum_{n=2}^{\infty}n(n-1)a_nt^{n-2}+t\sum_{n=1}^{\infty}na_nt^{n-1}-2\sum_{n=0}^{\infty}a_nt^n=0$. Making the change of index $k=n-2$ in the series for $y''(t)$, we obtain $\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}t^n+\sum_{n=1}^{\infty}na_nt^n-2\sum_{n=0}^{\infty}a_nt^n=0$, or

$\sum_{n=0}^{\infty}[(n+2)(n+1)a_{n+2}+(n-2)a_n]t^n=0$. Equating the coefficients to zero, we find the recurrence relation $a_{n+2}=\dfrac{-(n-2)a_n}{(n+2)(n+1)}$, $n=0,1,\ldots$ The recurrence leads us to

$a_2=2a_0/2=a_0,\ a_4=0a_2/12=0,\ a_6=-2a_4/30=0,\ \ldots$

$a_3=a_1/6,\ a_5=-a_3/20=-a_1/120,\ a_7=-3a_5/42=a_1/1680,\ \ldots$

Imposing the initial conditions, we have $a_0=0$ and $a_1=1$. Thus,

$y(t)=t+\dfrac{t^3}{6}-\dfrac{t^5}{120}+\dfrac{t^7}{1680}+\cdots$.

38. Assume there is solution of the form $y(t)=\sum_{n=0}^{\infty}a_nt^n$. Differentiating, we obtain

$y'(t)=\sum_{n=1}^{\infty}na_nt^{n-1}$ and $y''(t)=\sum_{n=2}^{\infty}n(n-1)a_nt^{n-2}=\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}t^n$, $ty=\sum_{n=0}^{\infty}a_nt^{n+1}=\sum_{n=1}^{\infty}a_{n-1}t^n$

Therefore, $2a_2+\sum_{n=1}^{\infty}[(n+2)(n+1)a_{n+2}+a_{n-1}]t^n=0$. Equating the coefficients to zero, we find the recurrence relation $a_{n+2}=\dfrac{-a_{n-1}}{(n+2)(n+1)}$, $n=1,2,...$

The recurrence leads us to

$$a_3 = \frac{-a_0}{3\cdot 2},\ a_4 = \frac{-a_1}{4\cdot 3},\ a_5 = \frac{-a_2}{5\cdot 4} = 0$$

Therefore, $y(t) = a_0\left\{1 - \frac{t^3}{6} + ...\right\} + a_1\left\{t - \frac{t^4}{12} + ...\right\},\ a_0 = 1,\ a_1 = 2$.

Finally, $y(t) = \left\{1 - \frac{t^3}{6} + ...\right\} + 2\left\{t - \frac{t^4}{12} + ...\right\}$.

39. Consider the initial value problem $y'' + (1+t)y' + y = 0$, $y(0) = -1, y'(0) = 1$ and assume there is solution of the form $y(t) = \sum_{n=0}^{\infty} a_n t^n$. Differentiating, we obtain $y'(t) = \sum_{n=1}^{\infty} n a_n t^{n-1}$ and $y''(t) = \sum_{n=2}^{\infty} n(n-1)a_n t^{n-2}$. Inserting these series into the differential equation, we have $\sum_{n=2}^{\infty} n(n-1)a_n t^{n-2} + (1+t)\sum_{n=1}^{\infty} n a_n t^{n-1} + \sum_{n=0}^{\infty} a_n t^n = 0$ or $\sum_{n=2}^{\infty} n(n-1)a_n t^{n-2} + \sum_{n=1}^{\infty} n a_n t^{n-1} + \sum_{n=0}^{\infty} (1+n)a_n t^n = 0$. Making the change of index $k = n-2$ in the series for $y''(t)$ and $k = n-1$ in the series for $y'(t)$, we obtain $\sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} t^n + \sum_{n=0}^{\infty} (n+1)a_{n+1} t^n + \sum_{n=0}^{\infty} (1+n)a_n t^n = 0$, or $\sum_{n=0}^{\infty} [(n+2)(n+1)a_{n+2} + (n+1)a_{n+1} + (n+1)a_n]t^n = 0$. Equating the coefficients to zero, we find the recurrence relation $a_{n+2} = \frac{-(n+1)a_{n+1} - (n+1)a_n}{(n+2)(n+1)} = \frac{-a_{n+1} - a_n}{n+2}$. The recurrence leads us to

$a_2 = -(a_0 + a_1)/2,\ a_3 = -(a_2 + a_1)/3,\ a_4 = -(a_3 + a_2)/4 = 0,\ a_5 = -(a_4 + a_3)/5$.

Imposing the initial conditions, we have $a_0 = -1$ and $a_1 = 1$. Thus,

$a_2 = 0,\ a_3 = -1/3,\ a_4 = 1/12,\ a_5 = 1/20$ and so we find

$$y(t) = -1 + t - \frac{1}{3}t^3 + \frac{1}{12}t^4 + \frac{1}{20}t^5 + \cdots.$$

40. Assume there is solution of the form $y(t) = \sum_{n=0}^{\infty} a_n t^n$. Differentiating, we obtain $y'(t) = \sum_{n=0}^{\infty} (n+1)a_{n+1} t^n$ and $y''(t) = \sum_{n=0}^{\infty} (n+2)(n+1)a_{n+2} t^n$. Inserting these series into the differential equation, we have $\sum_{n=0}^{\infty}\left\{(n+2)(n+1)a_{n+2} - 5(n+1)a_{n+1} + 6a_n\right\}t^n = 0$. Equating the coefficients to zero, we find the recurrence relation $a_{n+2} = \frac{5(n+1)a_{n+1} - 6a_n}{(n+2)(n+1)},\ n = 0,1,2,....$ The recurrence leads us to

$$a_2 = \frac{5a_1 - 6a_0}{2} = \frac{5(2) - 6(1)}{2} = 2,\ a_3 = \frac{5(2)a_2 - 6a_1}{3\cdot 2} = \frac{10(2) - 6(2)}{6} = \frac{4}{3}.$$

$$a_4 = \frac{5(3)a_3 - 6a_2}{4\cdot 3} = \frac{15(4/3) - 6(2)}{12} = \frac{2}{3},\ a_5 = \frac{5(4)a_4 - 6a_3}{5\cdot 4} = \frac{20(2/3) - 6(4/3)}{20} = \frac{4}{15}$$

Therrefore, $y(t) = 1 + 2t + 2t^2 + \frac{4}{3}t^3 + \frac{2}{3}t^4 + \frac{4}{15}t^5 + \cdots$.

41. Consider the initial value problem $y'' - 2y' + y = 0$, $y(0) = 0, y'(0) = 2$ and assume there is solution of the form $y(t) = \sum_{n=0}^{\infty} a_n t^n$. Differentiating, we obtain $y'(t) = \sum_{n=1}^{\infty} na_n t^{n-1}$ and $y''(t) = \sum_{n=2}^{\infty} n(n-1)a_n t^{n-2}$. Inserting these series into the differential equation, we have $\sum_{n=2}^{\infty} n(n-1)a_n t^{n-2} - 2\sum_{n=1}^{\infty} na_n t^{n-1} + \sum_{n=0}^{\infty} a_n t^n = 0$. Making the change of index $k = n-2$ in the series for $y''(t)$ and $k = n-1$ in the series for $y'(t)$, we obtain
$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}t^n - 2\sum_{n=0}^{\infty}(n+1)a_{n+1}t^n + \sum_{n=0}^{\infty} a_n t^n = 0$, or
$\sum_{n=0}^{\infty}[(n+2)(n+1)a_{n+2} - 2(n+1)a_{n+1} + a_n]t^n = 0$. Equating the coefficients to zero, we find the recurrence relation $a_{n+2} = \frac{2(n+1)a_{n+1} - a_n}{(n+2)(n+1)}$. The recurrence leads us to
$a_2 = (2a_1 - a_0)/2$, $a_3 = (4a_2 - a_1)/6$, $a_4 = (6a_3 - a_2)/12$, $a_5 = (8a_4 - a_3)/20$.
Imposing the initial conditions, we have $a_0 = 0$ and $a_1 = 2$. Thus,
$a_2 = 2$, $a_3 = 1$, $a_4 = 1/3$, $a_5 = 1/12$ and so we find $y(t) = 2t + 2t^2 + t^3 + \frac{1}{3}t^4 + \frac{1}{12}t^5 + \cdots$.

Section 10.2

1. Consider the differential equation $y'' + (\sec t)y' + t(t^2-4)^{-1}y = 0$. The coefficient function $p(t) = \sec t$ is not analytic at odd integer multiples of $\pi/2$. Thus, in the interval $-10 < t < 10$, $p(t)$ is not analytic at $\pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \pm\frac{5\pi}{2}$. Similarly, the coefficient function $q(t) = t(t^2-4)^{-1}$ is not analytic at $t = \pm 2$. These 8 points are the only singular points in $-10 < t < 10$.
2. The function $p(t) = t^{\frac{2}{3}}$ is not analytic at $t = 0$. The function $q(t) = \sin t$ is analytic everywhere. Therefore, $t = 0$ is the only singular point in $-10 < t < 10$.
3. Consider the differential equation $(1-t^2)y'' + ty' + (\csc t)y = 0$. Putting the differential equation into the form of equation (1), we see that the coefficient function $p(t) = t(1-t^2)^{-1}$ is not analytic at $t = \pm 1$. Similarly, the coefficient function $q(t) = (\csc t)(1-t^2)^{-1}$ is not analytic at integer multiples of π or at $t = \pm 1$. Thus, in the interval $-10 < t < 10$, the singular points are given by $t = 0, \pm 1, \pm\pi, \pm 2\pi, \pm 3\pi$.
4. The function $p(t) = \frac{e^t}{\sin 2t}$ is not analytic at $t = 0, \pm\frac{\pi}{2}, \pm\pi, \pm\frac{3\pi}{2}, \pm 2\pi, \pm\frac{5\pi}{2}, \pm 3\pi$. The function $q(t) = \frac{t}{(25-t^2)\sin 2t}$ is also not analytic at $t = \pm 5$. Therefore, $t = 0, \pm\frac{\pi}{2}, \pm\pi, \pm\frac{3\pi}{2}, \pm 2\pi, \pm\frac{5\pi}{2}, \pm 3\pi, \pm 5$ are the singular points in $-10 < t < 10$.

5. Consider the differential equation $(1+\ln|t|)y'' + y' + (1+t^2)y = 0$. Putting the differential equation into the form of equation (1), we see that the coefficient function $p(t) = (1+\ln|t|)^{-1}$ is not analytic at $t = 0$ or at $t = \pm e^{-1}$. Similarly, the coefficient function $q(t) = (1+t^2)(1+\ln|t|)^{-1}$ is not analytic $t = 0$ or at $t = \pm e^{-1}$. These three points are the only singular points in the interval $-10 < t < 10$.

6. The function $p(t) = \dfrac{t}{1+|t|}$ is not analytic at $t = 0$. The function $q(t) = \tan t$ is not analytic at $t = \pm\dfrac{\pi}{2}, \pm\dfrac{3\pi}{2}, \pm\dfrac{5\pi}{2},\ldots$. Therefore, $t = 0, \pm\dfrac{\pi}{2}, \pm\dfrac{3\pi}{2}, \pm\dfrac{5\pi}{2}$ are the singular points in $-10 < t < 10$.

7. Consider the differential equation $y'' + (1+2t)^{-1}y' + t(1-t^2)^{-1}y = 0$. Since the coefficient functions are rational functions, each is analytic with a radius of convergence R equal to the distance from $t_0 = 0$ to its nearest singularity; see Figure 10.2. The only singularity of $p(t) = (1+2t)^{-1}$ is $t = -1/2$ while the only singularities of $q(t) = t(1-t^2)^{-1}$ are $t = \pm 1$. Thus, the radius of convergence of the series for $p(t)$ is $R = 1/2$ while the series for $q(t)$ has radius of convergence $R = 1$. The given initial value problem is guaranteed to have a unique solution that is analytic in the interval $-1/2 < t < 1/2$.

8. $p(t) = 4(1-9t^2)^{-1}$ and $q(t) = t(1-9t^2)^{-1}$ are not analytic at $t = \pm 1/3$. Thus, for $t_0 = 1$, $R = \dfrac{2}{3}$.

9. Consider the differential equation $y'' + (4-3t)^{-1}y' + 3t(5+30t)^{-1}y = 0$. Since the coefficient functions are rational functions, each is analytic with a radius of convergence R equal to the distance from $t_0 = -1$ to its nearest singularity; see Figure 10.2. The only singularity of $p(t) = (4-3t)^{-1}$ is $t = 4/3$ while the only singularity of $q(t) = 3t(5+30t)^{-1}$ is $t = -1/6$. Thus, the radius of convergence of the series for $p(t)$ is $R = |-1-(4/3)| = 7/3$ while the series for $q(t)$ has radius of convergence $R = |-1-(-1/6)| = 5/6$. The given initial value problem is guaranteed to have a unique solution that is analytic in the interval $-5/6 < t+1 < 5/6$.

10. $p(t) = (1+4t^2)^{-1}$ is not analytic at $t = \pm\dfrac{i}{2}$ and $q(t) = t(4+t)^{-1}$ is not analytic at $t = -4$. Thus, for $t_0 = 0$, $R = \dfrac{1}{2}$.

11. Consider the differential equation $y'' + (1+3(t-2))^{-1}y' + (\sin t)y = 0$. The coefficient function $p(t) = (3t-5)^{-1}$ is a rational function and is analytic with a radius of convergence R equal to the distance from $t_0 = 2$ to its nearest singularity; see Figure 10.2. The only singularity of $p(t) = (3t-5)^{-1}$ is $t = 5/3$. The other coefficient function, $q(t) = \sin t$, is analytic everywhere with an infinite radius of convergence. The radius of convergence of the series for $p(t)$ is $R = |2-(5/3)| = 1/3$. Therefore, the given initial value problem is guaranteed to have a unique solution that is analytic in the interval $-1/3 < t-2 < 1/3$.

12. $p(t) = (t+3)(1+t^2)^{-1}$ is not analytic at $t = \pm i$ and $q(t) = t^2$ is analytic everywhere. Thus, for $t_0 = 1$, $R = \sqrt{2}$.

13 (a). Consider the differential equation $y'' + ty' + y = 0$. Let the solution be given by $y(t) = \sum_{n=0}^{\infty} a_n t^n$.

Differentiating, we obtain $y'(t) = \sum_{n=1}^{\infty} n a_n t^{n-1}$ and $y''(t) = \sum_{n=2}^{\infty} n(n-1)a_n t^{n-2}$.

Inserting these series into the differential equation, we have

$$\sum_{n=2}^{\infty} n(n-1)a_n t^{n-2} + t\sum_{n=1}^{\infty} na_n t^{n-1} + \sum_{n=0}^{\infty} a_n t^n = 0 \text{ or } \sum_{n=2}^{\infty} n(n-1)a_n t^{n-2} + \sum_{n=1}^{\infty} na_n t^n + \sum_{n=0}^{\infty} a_n t^n = 0.$$

Adjusting the indices, we obtain $\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}t^n + \sum_{n=1}^{\infty} na_n t^n + \sum_{n=0}^{\infty} a_n t^n = 0$ or

$2a_2 + a_0 + \sum_{n=1}^{\infty}[(n+2)(n+1)a_{n+2} + (n+1)a_n]t^n = 0$. Consequently, the recurrence relation is given by $a_2 = -a_0/2$ and $a_{n+2} = -a_n/(n+2)$, $n = 1, 2, \ldots$.

13 (b). The recurrence leads us to

$a_2 = -a_0/2$, $a_4 = -a_2/4 = a_0/8$, …

$a_3 = -a_1/3$, $a_5 = -a_3/5 = a_1/15$, …

Thus, the general solution is

$$y(t) = a_0[1 - \frac{t^2}{2} + \frac{t^4}{8} - \cdots] + a_1[t - \frac{t^3}{3} + \frac{t^5}{15} - \cdots] = y_1(t) + y_2(t).$$

13 (c). Since the coefficient functions are analytic for $-\infty < t < \infty$, the series converges for $-\infty < t < \infty$.

13 (d). The coefficient function $p(t) = t$ is odd and the coefficient function $q(t) = 1$ is even. Therefore, Theorem 10.2 guarantees that the given equation has even solutions and odd solutions.

14 (a). $\sum_{n=0}^{\infty}[(n+2)(n+1)a_{n+2} + 2na_n + 3a_n]t^n = 0$. Consequently, the recurrence relation is given by

$$a_{n+2} = \frac{-(2n+3)a_n}{(n+2)(n+1)},\ n = 0, 1, 2, \ldots.$$

14 (b). The recurrence leads us to

$a_2 = -3a_0/2$, $a_3 = -5a_1/6$, $a_4 = -7a_2/12 = 7a_0/8$, $a_5 = -9a_3/20 = 3a_1/8$ …

$a_3 = -a_1/3$, $a_5 = -a_3/5 = a_1/15$, …

Thus, the general solution is

$$y(t) = a_0[1 - \frac{3t^2}{2} + \frac{7t^4}{8} - \cdots] + a_1[t - \frac{5t^3}{6} + \frac{3t^5}{8} - \cdots].$$

14 (c). Since the coefficient functions are analytic for $-\infty < t < \infty$, $R = \infty$.

14 (d). $p(t) = 2t$ is odd and $q(t) = 3$ is even. Therefore, Theorem 10.2 guarantees that the given equation has even solutions and odd solutions.

15 (a). Consider the differential equation $(1+t^2)y'' + ty' + 2y = 0$. Let the solution be given by $y(t) = \sum_{n=0}^{\infty} a_n t^n$. Differentiating, we obtain $y'(t) = \sum_{n=1}^{\infty} na_n t^{n-1}$ and $y''(t) = \sum_{n=2}^{\infty} n(n-1)a_n t^{n-2}$.

Inserting these series into the differential equation, we have

$$(1+t^2)\sum_{n=2}^{\infty} n(n-1)a_n t^{n-2} + t\sum_{n=1}^{\infty} na_n t^{n-1} + 2\sum_{n=0}^{\infty} a_n t^n = 0 \text{ or}$$

$\sum_{n=2}^{\infty} n(n-1)a_n t^{n-2} + \sum_{n=2}^{\infty} n(n-1)a_n t^n + \sum_{n=1}^{\infty} na_n t^n + 2\sum_{n=0}^{\infty} a_n t^n = 0$. Adjusting the indices, we obtain

$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}t^n + \sum_{n=2}^{\infty} n(n-1)a_n t^n + \sum_{n=1}^{\infty} na_n t^n + 2\sum_{n=0}^{\infty} a_n t^n = 0$. Consequently, the recurrence relation is given by $a_2 = -a_0$, $a_3 = -a_1/2$, and $a_{n+2} = -(n^2+2)a_n/[(n+2)(n+1)]$, $n = 2, 3, \ldots$.

15 (b). The recurrence leads us to

$a_2 = -a_0$, $a_4 = -a_2/2 = a_0/2$,…

$a_3 = -a_1/2$, $a_5 = -11a_3/20 = 11a_1/40$,…

Thus, the general solution is

$$y(t) = a_0[1 - t^2 + \frac{t^4}{2} - \cdots] + a_1[t - \frac{t^3}{2} + \frac{11t^5}{40} - \cdots] = y_1(t) + y_2(t).$$

15 (c). The coefficient functions $p(t) = t(1+t^2)^{-1}$ and $q(t) = 2(1+t^2)^{-1}$ fail to be analytic at $t = \pm i$. Therefore, the radius of convergence for each coefficient function is $R = 1$. Consequently, Theorem 10.1 guarantees that the power series solution converges in the interval $-1 < t < 1$.

15 (d). The coefficient function $p(t) = t(1+t^2)^{-1}$ is odd and the coefficient function $q(t) = 2(1+t^2)^{-1}$ is even. Therefore, Theorem 10.2 guarantees that the given equation has even solutions and odd solutions.

16 (a). $\sum_{n=0}^{\infty}[(n+2)(n+1)a_{n+2} - 5(n+1)a_{n+1} + 6a_n]t^n = 0$. Consequently, the recurrence relation is given by $a_{n+2} = \dfrac{5(n+1)a_{n+1} - 6a_n}{(n+2)(n+1)}$, $n = 0,1,2,\ldots$.

16 (b). The recurrence leads us to

$a_2 = (5a_1 - 6a_0)/2 = 5a_1/2 - 3a_0$, $a_3 = (5(2)a_2 - 6a_1)/(3\cdot 2) = 19a_1/6 - 5a_0$

Thus, the general solution is

$$y(t) = a_0[1 - 3t^2 - 5t^3 - \cdots] + a_1[t + \frac{5t^2}{2} + \frac{19t^3}{6} + \cdots].$$

16 (c). Since the coefficient functions are analytic for $-\infty < t < \infty$, $R = \infty$.

16 (d). $p(t) = -5$ and $q(t) = 6$ are both even. Therefore, Theorem 10.2 does not apply.

17 (a). Consider the differential equation $y'' - 4y' + 4y = 0$. Let the solution be given by $y(t) = \sum_{n=0}^{\infty} a_n t^n$. Differentiating, we obtain $y'(t) = \sum_{n=1}^{\infty} n a_n t^{n-1}$ and $y''(t) = \sum_{n=2}^{\infty} n(n-1)a_n t^{n-2}$.

Inserting these series into the differential equation, we have

$\sum_{n=2}^{\infty} n(n-1)a_n t^{n-2} - 4\sum_{n=1}^{\infty} n a_n t^{n-1} + 4\sum_{n=0}^{\infty} a_n t^n = 0$. Adjusting the indices, we obtain

$\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}t^n - 4\sum_{n=0}^{\infty}(n+1)a_{n+1}t^n + 4\sum_{n=0}^{\infty} a_n t^n = 0$. Consequently, the recurrence relation is given by $a_{n+2} = [4(n+1)a_{n+1} - 4a_n]/[(n+2)(n+1)]$, $n = 0,1,\ldots$.

17 (b). The recurrence leads us to

$a_2 = 2a_1 - 2a_0$, $a_3 = (8a_2 - 4a_1)/6 = (16a_1 - 16a_0 - 4a_1)/6 = 2a_1 - (8/3)a_0$,…

Thus, the general solution is

$$y(t) = a_0[1 - 2t^2 - \frac{8t^3}{3} + \cdots] + a_1[t + 2t^2 + 2t^3 \cdots] = y_1(t) + y_2(t).$$

17 (c). The coefficient functions are constant and hence analytic everywhere. Consequently, Theorem 10.1 guarantees that the power series solution converges in the interval $-\infty < t < \infty$.

17 (d). The coefficient function $p(t) = -4$ is even and hence Theorem 10.2 does not apply.

18 (a). $\sum_{n=0}^{\infty}[(n+2)(n+1)a_{n+2}+(n+1)na_{n+1}+a_n]t^n=0$. Consequently, the recurrence relation is given by $a_{n+2}=\dfrac{-[(n+1)na_{n+1}+a_n]}{(n+2)(n+1)}$.

18 (b). The recurrence leads us to

$$a_2=\frac{-a_0}{2},\ a_3=\frac{-[(2)(1)a_2+a_1]}{3\cdot 2}=\frac{a_0}{6}-\frac{a_1}{6},\ a_4=\frac{-[(3)(2)a_3+a_2]}{4\cdot 3}=-\frac{a_0}{8}+\frac{a_1}{12}$$

Thus, the general solution is

$$y(t)=a_0[1-\frac{t^2}{2}-\frac{t^3}{6}-\cdots]+a_1[t-\frac{t^3}{6}+\frac{t^4}{12}+\cdots].$$

18 (c). $q(t)=\dfrac{1}{1+t}$ is not analytic at $t=-1$, $R=1$.

18 (d). $q(t)=\dfrac{1}{1+t}$ is neither even nor odd. Therefore, Theorem 10.2 does not apply.

19 (a). Consider the differential equation $(3+t)y''+3ty'+y=0$. Let the solution be given by $y(t)=\sum_{n=0}^{\infty}a_nt^n$. Differentiating, we obtain $y'(t)=\sum_{n=1}^{\infty}na_nt^{n-1}$ and $y''(t)=\sum_{n=2}^{\infty}n(n-1)a_nt^{n-2}$.

Inserting these series into the differential equation, we have

$(3+t)\sum_{n=2}^{\infty}n(n-1)a_nt^{n-2}+3t\sum_{n=1}^{\infty}na_nt^{n-1}+\sum_{n=0}^{\infty}a_nt^n=0$ or

$3\sum_{n=2}^{\infty}n(n-1)a_nt^{n-2}+\sum_{n=2}^{\infty}n(n-1)a_nt^{n-1}+3\sum_{n=1}^{\infty}na_nt^n+\sum_{n=0}^{\infty}a_nt^n=0$. Adjusting the indices, we obtain

$3\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}t^n+\sum_{n=1}^{\infty}(n+1)na_{n+1}t^n+3\sum_{n=1}^{\infty}na_nt^n+\sum_{n=0}^{\infty}a_nt^n=0$. Consequently, the recurrence relation is given by

$a_2=-a_0/6$ and $a_{n+2}=-[n(n+1)a_{n+1}+(3n+1)a_n]/[3(n+2)(n+1)]$, $n=1,2,\ldots$.

19 (b). The recurrence leads us to

$$a_2=-a_0/6,\ a_3=-(2a_2+4a_1)/18=-(-2a_0/6+4a_1)/18=(a_0-12a_1)/54,\ldots$$

Thus, the general solution is

$$y(t)=a_0[1-\frac{t^2}{6}+\frac{t^3}{54}+\cdots]+a_1[t-\frac{2t^3}{9}+\cdots]=y_1(t)+y_2(t).$$

19 (c). The coefficient functions $p(t)=3t(3+t)^{-1}$ and $q(t)=(3+t)^{-1}$ fail to be analytic at $t=-3$. Therefore, the radius of convergence for each coefficient function is $R=3$. Consequently, Theorem 10.1 guarantees that the power series solution converges in the interval $-3<t<3$.

19 (d). The coefficient function $p(t)=3t(3+t)^{-1}$ is neither even nor odd. Therefore, Theorem 10.2 does not apply.

20 (a). $\sum_{n=0}^{\infty}[2(n+2)(n+1)a_{n+2}+n(n-1)a_n+4a_n]t^n=0$. Consequently, the recurrence relation is given by $a_{n+2}=\dfrac{-[n(n-1)+4]a_n}{2(n+2)(n+1)}$.

20 (b). The recurrence leads us to

$$a_2 = -a_0, a_3 = -\frac{a_1}{3},\ a_4 = \frac{a_0}{4},\ a_5 = \frac{a_1}{12}$$

Thus, the general solution is

$$y(t) = a_0[1 - t^2 + \frac{t^4}{4} - \cdots] + a_1[t - \frac{t^3}{3} + \frac{t^5}{12} + \cdots].$$

20 (c). $R = \sqrt{2}$.

20 (d). $p(t) = 0$ can be considered odd and $q(t) = \dfrac{4}{t^2 + 2}$ is even. Therefore, Theorem 10.2 guarantees that the given equation has even solutions and odd solutions.

21 (a). Consider the differential equation $y'' + t^2 y = 0$. Let the solution be given by $y(t) = \sum_{n=0}^{\infty} a_n t^n$. Differentiating, we obtain $y'(t) = \sum_{n=1}^{\infty} n a_n t^{n-1}$ and $y''(t) = \sum_{n=2}^{\infty} n(n-1) a_n t^{n-2}$. Inserting these series into the differential equation, we have $\sum_{n=2}^{\infty} n(n-1) a_n t^{n-2} + t^2 \sum_{n=0}^{\infty} a_n t^n = 0$ or $\sum_{n=2}^{\infty} n(n-1) a_n t^{n-2} + \sum_{n=0}^{\infty} a_n t^{n+2} = 0$. Adjusting the indices, we obtain $\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} t^n + \sum_{n=2}^{\infty} a_{n-2} t^n = 0$. Consequently, the recurrence relation is given by $a_2 = 0, a_3 = 0,$ and $a_{n+2} = -a_{n-2}/[(n+2)(n+1)],\ n = 2, 3, \ldots$.

21 (b). The recurrence leads us to

$$a_2 = 0,\ a_3 = 0,\ a_4 = -a_0/12, a_5 = -a_1/20, \ldots$$

Thus, the general solution is

$$y(t) = a_0[1 - \frac{t^4}{12} + \cdots] + a_1[t - \frac{t^5}{20} + \cdots] = y_1(t) + y_2(t).$$

21 (c). The coefficient functions are polynomials and hence analytic everywhere. Consequently, Theorem 10.1 guarantees that the power series solution converges in the interval $-\infty < t < \infty$.

21 (d). The coefficient function $p(t) = 0$ can be considered an odd function while $q(t) = t^2$ is clearly an even function. Therefore, Theorem 10.2 guarantees that the given equation has even solutions and odd solutions.

22 (a). $\sum_{n=0}^{\infty} [(n+2)(n+1) a_{n+2} + n a_n + a_n](t-1)^n = 0$. Consequently, the recurrence relation is given by

$$a_{n+2} = \frac{-(n+1)a_n}{(n+2)(n+1)} = \frac{-a_n}{n+2}, n = 0,1,2,....$$

22 (b). The recurrence leads us to

$$a_2 = -\frac{a_0}{2},\ a_3 = -\frac{a_1}{3},\ a_4 = -\frac{a_2}{4} = \frac{a_0}{8},\ a_5 = -\frac{a_3}{5} = \frac{a_1}{15}$$

Thus, the general solution is

$$y(t) = a_0[1 - \frac{(t-1)^2}{2} + \frac{(t-1)^4}{8} + \cdots] + a_1[(t-1) - \frac{(t-1)^3}{3} + \frac{(t-1)^5}{15} + \cdots].$$

22 (c). The coefficient functions are analytic everywhere. Consequently, $R = \infty$.

23 (a). Consider the differential equation $y'' + y = 0$. Let the solution be given by $y(z) = \sum_{n=0}^{\infty} a_n z^n$ where $z = t-1$. Differentiating, we obtain $y'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}$ and $y''(z) = \sum_{n=2}^{\infty} n(n-1) a_n z^{n-2}$. Inserting these series into the differential equation, we have $\sum_{n=2}^{\infty} n(n-1) a_n z^{n-2} + \sum_{n=0}^{\infty} a_n z^n = 0$. Adjusting the indices, we obtain $\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} z^n + \sum_{n=0}^{\infty} a_n z^n = 0$. Consequently, the recurrence relation is given by $a_{n+2} = -a_n / [(n+2)(n+1)],\ n = 0, 1, \ldots.$

23 (b). The recurrence leads us to

$a_2 = -a_0/2, a_4 = -a_2/12 = a_0/24, \ldots$

$a_3 = -a_1/6, a_5 = -a_3/20 = a_1/120, \ldots$

Thus, the general solution is

$$y(t) = a_0[1 - \frac{(t-1)^2}{2} + \frac{(t-1)^4}{24} + \cdots] + a_1[(t-1) - \frac{(t-1)^3}{6} + \frac{(t-1)^5}{120} + \cdots].$$

23 (c). The coefficient functions are constants and hence analytic everywhere. Consequently, Theorem 10.1 guarantees that the power series solution converges in the interval $-\infty < t-1 < \infty$.

24 (a). $\sum_{n=0}^{\infty} [(n+1) n a_{n+1} - (n+2)(n+1) a_{n+2} + (n+1) a_{n+1} + a_n](t-1)^n = 0$. Consequently, the recurrence relation is given by $a_{n+2} = \frac{(n+1)^2 a_{n+1} + a_n}{(n+2)(n+1)}, n = 0,1,2,\ldots.$

24 (b). The recurrence leads us to

$$a_2 = \frac{a_1 + a_0}{2} = \frac{a_1}{2} + \frac{a_0}{2},\quad a_3 = \frac{4a_2 + a_1}{3\cdot 2} = \frac{a_1}{2} + \frac{a_0}{3}$$

Thus, the general solution is

$$y(t) = a_0[1 + \frac{(t-1)^2}{2} + \frac{(t-1)^3}{3} + \cdots] + a_1[(t-1) - \frac{(t-1)^2}{2} + \frac{(t-1)^3}{2} + \cdots].$$

24 (c). $p(t) = q(t) = \frac{1}{t-2}$ are not analytic at $t = 2$. Consequently, $R = 1$.

25 (a). Consider the differential equation $y'' + y' + (t-2)y = 0$ or $y'' + y' + [(t-1)-1]y = 0$. Let the solution be given by $y(z) = \sum_{n=0}^{\infty} a_n z^n$ where $z = t-1$. Differentiating, we obtain $y'(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}$ and $y''(z) = \sum_{n=2}^{\infty} n(n-1) a_n z^{n-2}$. Inserting these series into the differential equation, we have $\sum_{n=2}^{\infty} n(n-1) a_n z^{n-2} + \sum_{n=1}^{\infty} n a_n z^{n-1} + \sum_{n=0}^{\infty} a_n z^{n+1} - \sum_{n=0}^{\infty} a_n z^n = 0$. Adjusting the indices, we obtain $\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} z^n + \sum_{n=0}^{\infty} (n+1) a_{n+1} z^n + \sum_{n=1}^{\infty} a_{n-1} z^n - \sum_{n=0}^{\infty} a_n z^n = 0$.

Consequently, the recurrence relation is given by

$a_2 = (a_0 - a_1)/2$ and $a_{n+2} = -[(n+1)a_{n+1} - a_n + a_{n-1}] / [(n+2)(n+1)],\ n = 1, 2, \ldots.$

25 (b). The recurrence leads us to

$$a_3 = -(2a_2 - a_1 + a_0)/6 = -(a_0 - a_1)/3, \ldots$$

Thus, the general solution is

$$y(t) = a_0\left[1 + \frac{(t-1)^2}{2} - \frac{(t-1)^3}{3} + \cdots\right] + a_1\left[(t-1) - \frac{(t-1)^2}{2} + \frac{(t-1)^3}{3} + \cdots\right].$$

25 (c). The coefficient functions are polynomials and hence analytic everywhere. Consequently, Theorem 10.1 guarantees that the power series solution converges in the interval $-\infty < t-1 < \infty$.

26. $a_{n+2} = \dfrac{(n^2 - \mu^2)a_n}{(n+2)(n+1)}, n = 0,1,2,\ldots$

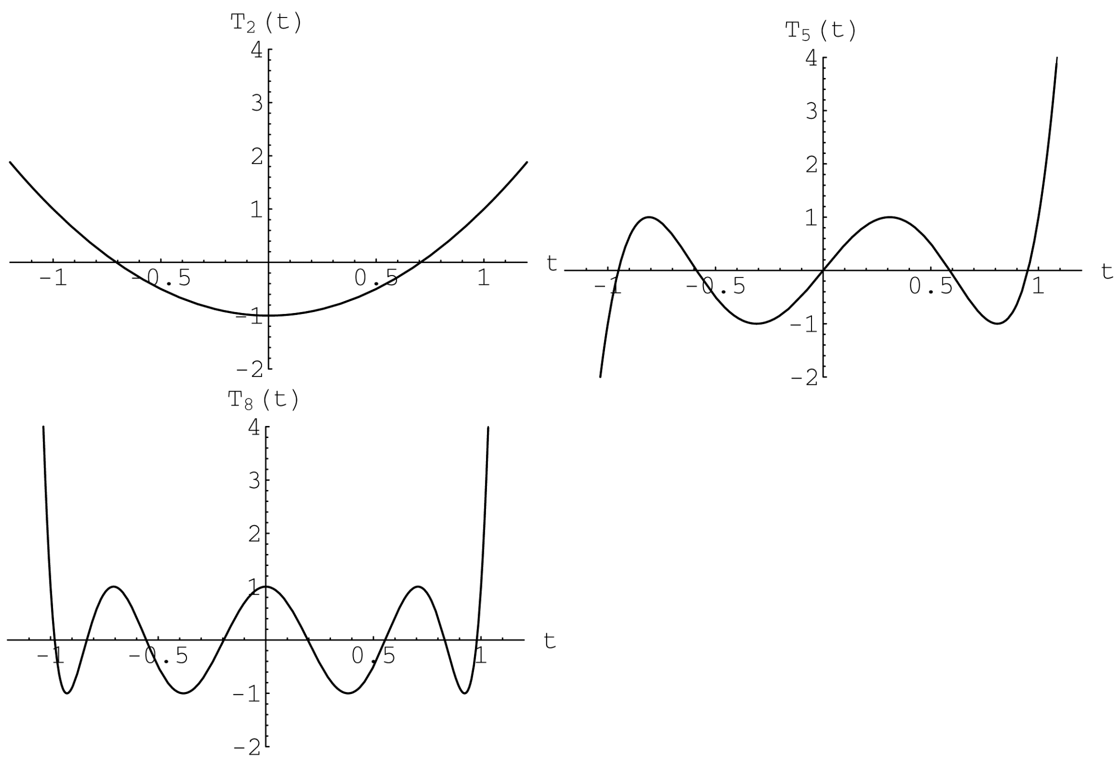
For $\mu = 5$, $a_3 = -4a_1$, $a_5 = \dfrac{16}{5}a_1$, $a_7 = a_9 = \ldots = 0$, $T_5(t) = a_1[t - 4t^3 + \frac{16}{5}t^5]$.

Set $T_5(1) = a_1[1 - 4 + \frac{16}{5}] = 1 \Rightarrow a_1 = 5$. Therefore, $T_5(t) = 16t^5 - 20t^3 + 5t$

For $\mu = 6$, $a_2 = -18a_0$, $a_4 = 48a_0$, $a_6 = -32a_0$, $T_6(t) = a_0[1 - 18t^2 + 48t^4 - 32t^6]$; $a_0 = -1$.

Therefore, $T_6(t) = 32t^6 - 48t^4 + 18t^2 - 1$

27 (c).



27 (d). $|T_N(t)| \le 1$ for $-1 < t < 1$. For $|t| \ge 1$, $\lim_{t \to \pm\infty} |T_N(t)| = \infty$.

28 (a). $\sum_{n=0}^{\infty}[(n+2)(n+1)a_{n+2} - n(n-1)a_n - 2na_n + \mu(\mu+1)a_n]t^n = 0$. Therefore the recurrence relation is $a_{n+2} = \dfrac{[n(n+1) - \mu(\mu+1)]a_n}{(n+2)(n+1)}, n = 0,1,2,\ldots$

28 (b). When $\mu = N$, $a_{N+2} = a_{N+4} = a_{N+6} = \ldots = 0$. Therefore, if $\mu = 2M$, a polynomial solution of the form $a_0 + a_2t^2 + \ldots + a_{2M}t^{2M}$ exists, while if $\mu = 2M+1$, a polynomial solution of the form $a_1t + a_3t^3 + \ldots + a_{2M+1}t^{2M+1}$ exists.

28 (c). If $\mu = 0$ and $y = 1$, $(1-t^2)(0) - 2t(0) + 0(1) = 0$.
If $\mu = 1$ and $y = t$, $(1-t^2)(0) - 2t(1) + 1(2)(t) = 0$.

28 (d). If $\mu = 2$, $a_{n+2} = \dfrac{[n(n+1)-6]a_n}{(n+2)(n+1)} \Rightarrow P_2(t) = \dfrac{3}{2}t^2 - \dfrac{1}{2}$.
If $\mu = 3$, $a_{n+2} = \dfrac{[n(n+1)-12]a_n}{(n+2)(n+1)} \Rightarrow P_3(t) = \dfrac{5}{2}t^3 - \dfrac{3}{2}t$.
If $\mu = 4$, $a_{n+2} = \dfrac{[n(n+1)-20]a_n}{(n+2)(n+1)} \Rightarrow P_4(t) = \dfrac{35}{8}t^4 - \dfrac{15}{4}t^2 + \dfrac{3}{8}$.
If $\mu = 5$, $a_{n+2} = \dfrac{[n(n+1)-30]a_n}{(n+2)(n+1)} \Rightarrow P_5(t) = \dfrac{63}{8}t^5 - \dfrac{35}{4}t^3 + \dfrac{15}{8}t$.

29 (a). Consider the differential equation $y'' - 2ty' + 2\mu y = 0$. Let the solution be given by $y(t) = \sum_{n=0}^{\infty} a_n t^n$. Differentiating, we obtain $y'(t) = \sum_{n=1}^{\infty} na_n t^{n-1}$ and $y''(t) = \sum_{n=2}^{\infty} n(n-1)a_n t^{n-2}$. Inserting these series into the differential equation, we have $\sum_{n=2}^{\infty} n(n-1)a_n t^{n-2} - 2\sum_{n=1}^{\infty} na_n t^n + 2\mu\sum_{n=0}^{\infty} a_n t^n = 0$. Adjusting the indices, we obtain $\sum_{n=0}^{\infty}(n+2)(n+1)a_{n+2}t^n - 2\sum_{n=1}^{\infty} na_n t^n + 2\mu\sum_{n=0}^{\infty} a_n t^n = 0$. Consequently, the recurrence relation is given by $a_2 = -\mu a_0$ and $a_{n+2} = (2n-2\mu)a_n / [(n+2)(n+1)]$, $n = 1, 2, \ldots$.

29 (d). For $\mu = 2$, the even indexed coefficients a_n vanish when $n > 2$. From the recurrence relation, $H_2(t) = a_0 - 2a_0 t^2 = -a_0(2t^2 - 1)$. Choosing $a_0 = -2$ leads us to $H_2(t) = 4t^2 - 2$. For $\mu = 3$, the odd indexed coefficients a_n vanish when $n > 3$. From the recurrence relation, $H_3(t) = a_1 t - (2/3)a_1 t^3 = -a_1[(2/3)t^3 - t)$. Choosing $a_1 = -12$ leads us to $H_3(t) = 8t^3 - 12t$. Similarly, $H_4(t) = 16t^4 - 48t^2 + 12$ and $H_5(t) = 32t^5 - 160t^3 + 120t$.

30 (a). Try $y(t) = \sum_{n=0}^{\infty} a_n t^n \Rightarrow \sum_{n=0}^{\infty}[(n+1)na_{n+1} + (n+1)a_{n+1} - a_n]t^n = 0$.
$\Rightarrow a_{n+1} = \dfrac{a_n}{(n+1)^2} \Rightarrow y(t) = a_0\sum_{n=0}^{\infty}\dfrac{t^n}{(n+1)^2}$. By the ratio test, $\lim_{n\to\infty}\left|\dfrac{t^{n+1}(n+1)^2}{t^n(n+2)^2}\right| = |t|$ and the series converges in $-1 < t < 1$.

30 (b). Try $y(t) = \sum_{n=0}^{\infty} a_n t^n \Rightarrow \sum_{n=0}^{\infty}[n(n-1)+1]a_n t^n = 0 \Rightarrow [n(n-1)+1]a_n = 0$.
The polynomial $x^2 - x + 1$ has roots $\dfrac{1 \pm \sqrt{1-4}}{2}$. Since there are no positive integer roots, the factor $[n(n-1)+1]$ is nonzero for all $n = 0,1,2,\ldots$ Therefore, $a_n = 0$, $n = 0,1,2,\ldots$ and $y(t) = 0$, The trivial solution results.

33. The coefficient function $p(t) = \sin t$ is odd and analytic everywhere. The coefficient function $q(t) = t^2$ is even and analytic everywhere. Thus, Theorem 10.2(b) applies. The differential equation has a general solution of the form (15).

34. No. $p(t) = \cos t$ is even; $q(t) = t$ is odd.

35. The coefficient function $p(t) = 0$ can be regarded as a function that is odd and analytic everywhere. The coefficient function $q(t) = t^2$ is even and analytic everywhere. Thus, Theorem 10.2(b) applies. The differential equation has a general solution of the form (15).

36. No. $p(t)=1$ and $q(t)=t^2$ are both even.
37. The coefficient function $q(t)=t$ is odd. Thus, Theorem 10.2(b) does not apply.
38. No. $p(t)=e^t$ is neither even nor odd and $q(t)=1$ is even.
39. Consider the differential equation $y''+ay'+by=0$. The coefficient function $p(t)=a$ can be regarded as an odd function if $a=0$, but is even if a is nonzero. The coefficient function $q(t)=b$ is even. Both coefficient functions are analytic everywhere. Thus, Theorem 10.2(b) applies if $a=0$ and b is arbitrary.

40 (a). $p(t)=0,\ q(t)=\dfrac{1}{1+t^2}$. The denominator of $q(t)$ vanishes at $t=\pm i \Rightarrow R=1$.

40 (b). $y(t)=\sum_{n=0}^{\infty} a_n t^n \Rightarrow \sum_{n=0}^{\infty}[(n+2)(n+1)a_{n+2}+n(n-1)a_n+a_n]t^n=0$

$\Rightarrow r(n)=(n+2)(n+1),\ s(n)=n(n-1)+1$. Then $\lim_{n\to\infty}\left|\dfrac{a_{n+2}}{a_n}\right|=\lim_{n\to\infty}\left|\dfrac{n(n-1)+1}{(n+2)(n+1)}\right|=1$. Therefore, the series diverges for $\left|t^2\right|>1 \Rightarrow |t|>1$ by the Ratio Test.

40 (c). No contradiction. The unique solution of the initial value problem exists for $-\infty<t<\infty$, but its Maclaurin series has a radius of convergence $R=1$.

Section 10.3

1 (a). $\lambda^2+(-2\alpha+1-1)\lambda+\alpha^2=\lambda^2-2\alpha\lambda+\alpha^2=0$

1 (b). Using the technique in Section 4.5, the general solution is $y=c_1t^\alpha+c_2t^\alpha\ln t, t>0$.

2. $$W=\begin{vmatrix} t^\gamma\cos(\delta\ln t) & t^\gamma\sin(\delta\ln t) \\ t^{\gamma-1}[\gamma\cos(\delta\ln t)-\delta\sin(\delta\ln t)] & t^{\gamma-1}[\gamma\sin(\delta\ln t)+\delta\cos(\delta\ln t)]\end{vmatrix}=\delta t^{2\gamma-1}\neq 0$$
in $0<t<\infty$ since $\delta\neq 0$.

3. When put in standard form, the differential equation is $y''-4t^{-1}y'+6t^{-2}y=0$. Thus, $t_0=0$ is the only singular point. The characteristic equation is $\lambda^2-5\lambda+6=0$ which has roots $\lambda_1=2$ and $\lambda_2=3$. Hence, the general solution is $y=c_1t^2+c_2t^3, t\neq 0$.
4. $t_0=0$. The characteristic equation is $\lambda^2-\lambda-6=0$ which has roots $\lambda_1=-2$ and $\lambda_2=3$. Hence, the general solution is $y=c_1t^{-2}+c_2t^3, t\neq 0$.
5. When put in standard form, the differential equation is $y''-3t^{-1}y'+4t^{-2}y=0$. Thus, $t_0=0$ is the only singular point. The characteristic equation is $\lambda^2-4\lambda+4=0$ which has roots $\lambda_1=2$ and $\lambda_2=2$. Hence, the general solution is $y=c_1t^2+c_2t^2\ln|t|, t\neq 0$.
6. $t_0=0$. The characteristic equation is $\lambda^2-2\lambda+5=0$ which has roots $\lambda_1=1+2i$ and $\lambda_2=1-2i$. Hence, the general solution is $y=c_1t\cos(2\ln|t|)+c_2t\sin(2\ln|t|), t\neq 0$.
7. When put in standard form, the differential equation is $y''-3t^{-1}y'+29t^{-2}y=0$. Thus, $t_0=0$ is the only singular point. The characteristic equation is $\lambda^2-4\lambda+29=0$ which has roots $\lambda_1=2+5i$ and $\lambda_2=2-5i$. Hence, the general solution is $y=c_1t^2\cos(5\ln|t|)+c_2t^2\sin(5\ln|t|), t\neq 0$.
8. $t_0=0$. The characteristic equation is $\lambda^2-6\lambda+9=0$ which has roots $\lambda_1=\lambda_2=3$. Hence, the general solution is $y=c_1t^3+c_2t^3\ln|t|, t\neq 0$.

9. When put in standard form, the differential equation is $y'' + t^{-1}y' + 9t^{-2}y = 0$. Thus, $t_0 = 0$ is the only singular point. The characteristic equation is $\lambda^2 + 9 = 0$ which has roots $\lambda_1 = 3i$ and $\lambda_2 = -3i$. Hence, the general solution is $y = c_1\cos(3\ln|t|) + c_2\sin(3\ln|t|), t \neq 0$.

10. $t_0 = 0$. The characteristic equation is $\lambda^2 + 2\lambda + 1 = 0$ which has roots $\lambda_1 = \lambda_2 = -1$. Hence, the general solution is $y = c_1t^{-1} + c_2t^{-1}\ln|t|, t \neq 0$.

11. When put in standard form, the differential equation is $y'' + 3t^{-1}y' + 17t^{-2}y = 0$. Thus, $t_0 = 0$ is the only singular point. The characteristic equation is $\lambda^2 + 2\lambda + 17 = 0$ which has roots $\lambda_1 = -1 + 4i$ and $\lambda_2 = -1 - 4i$. Hence, the general solution is $y = c_1t^{-1}\cos(4\ln|t|) + c_2t^{-1}\sin(4\ln|t|), t \neq 0$.

12. $t_0 = 0$. The characteristic equation is $\lambda^2 + 10\lambda + 25 = 0$ which has roots $\lambda_1 = \lambda_2 = -5$. Hence, the general solution is $y = c_1t^{-5} + c_2t^{-5}\ln|t|, t \neq 0$.

13. Consider the differential equation $y'' + 5t^{-1}y' + 40t^{-2}y = 0$. We see that, $t_0 = 0$ is the only singular point. The characteristic equation is $\lambda^2 + 4\lambda + 40 = 0$ which has roots $\lambda_1 = -2 + 6i$ and $\lambda_2 = -2 - 6i$. Hence, the general solution is $y = c_1t^{-2}\cos(6\ln|t|) + c_2t^{-2}\sin(6\ln|t|), t \neq 0$.

14. $t_0 = 0$. The characteristic equation is $\lambda^2 - 3\lambda = 0$ which has roots $\lambda_1 = 0,\ \lambda_2 = 3$. Hence, the general solution is $y = c_1 + c_2t^3, t \neq 0$.

15. When put in standard form, the differential equation is $y'' - (t-1)^{-1}y' - 3(t-1)^{-2}y = 0$. Thus, $t_0 = 1$ is the only singular point. The characteristic equation is $\lambda^2 - 2\lambda - 3 = 0$ which has roots $\lambda_1 = -3$ and $\lambda_2 = 1$. Hence, the general solution is $y = c_1(t-1)^3 + c_2(t-1)^{-1}, t \neq 1$.

16. $t_0 = 1$. The characteristic equation is $\lambda^2 + 2\lambda + 17 = 0$ which has roots $\lambda_1 = -1 + 4i,\ \lambda_2 = -1 - 4i$. Hence, the general solution is $y = c_1(t-1)^{-1}\cos(4\ln|t-1|) + c_2(t-1)^{-1}\sin(4\ln|t-1|), t \neq 1$.

17. When put in standard form, the differential equation is $y'' + 6(t+2)^{-1}y' + 6(t+2)^{-2}y = 0$. Thus, $t_0 = -2$ is the only singular point. The characteristic equation is $\lambda^2 + 5\lambda + 6 = 0$ which has roots $\lambda_1 = -3$ and $\lambda_2 = -2$. Hence, the general solution is $y = c_1(t+2)^{-3} + c_2(t+2)^{-2}, t \neq -2$.

18. $t_0 = 2$. The characteristic equation is $\lambda^2 + 4 = 0$ which has roots $\lambda_1 = 2i,\ \lambda_2 = -2i$. Hence, the general solution is $y = c_1\cos(2\ln|t-2|) + c_2\sin(2\ln|t-2|), t \neq 2$.

19. From the form of the general solution, $t_0 = -2$ and the characteristic equation has roots $\lambda_1 = 1$ and $\lambda_2 = -2$. Therefore, the characteristic equation is $\lambda^2 + \lambda - 2 = 0$. Matching the characteristic equation with the general form given in equation (3), we see that $\alpha - 1 = 1$ and $\beta = -2$. Thus, the differential equation is $(t+2)^2y'' + 2(t+2)y' - 2y = 0$.

20. $t_0 = 1,\ \lambda = 0,0.\ \therefore \lambda^2 = 0 \Rightarrow \alpha = 1,\ \beta = 0$.

21. From the form of the general solution, $t_0 = 0$ and the characteristic equation has roots $\lambda_1 = 2 + i$ and $\lambda_2 = 2 - i$. Therefore, the characteristic equation is $\lambda^2 - 4\lambda + 5 = 0$. Matching the characteristic equation with the general form given in equation (3), we see that $\alpha - 1 = -4$ and $\beta = 5$. Thus, the differential equation is $t^2y'' - 3ty' + 5y = 0$.

22. The characteristic equation has roots $\lambda_1 = 2$ and $\lambda_2 = -1$. Therefore, the characteristic equation is $\lambda^2 - \lambda - 2 = 0 \Rightarrow \alpha = 0,\ \beta = -2$. Thus, the differential equation is $t^2y'' + ty' - y = g(t)$. We can determine the nonhomogenous term $g(t)$ by inserting the given particular solution $y_P(t) = 2t + 1$. Doing so, we obtain $t^2(0) + t(2) - 2(2t+1) = -2t - 2 = g(t)$.

23. From the form of the general solution, the characteristic equation has roots $\lambda_1 = 2$ and $\lambda_2 = 3$. Therefore, the characteristic equation is $\lambda^2 - 5\lambda + 6 = 0$. Matching the characteristic equation with the general form given in equation (3), we see that $\alpha - 1 = -5$ and $\beta = 6$.
Thus, the differential equation is $t^2 y'' - 4ty' + 6y = g(t)$. We can determine the nonhomogenous term $g(t)$ by inserting the given particular solution $y_P(t) = \ln t$. Doing so, we obtain $t^2 y_P'' - 4ty_P' + 6y_P = g(t)$ or $t^2(-t^{-2}) - 4t(t^{-1}) + 6\ln t = g(t)$. Thus, $g(t) = -5 + 6\ln t$.

24. Under the change of variable $t = e^z$, the differential equation transforms into $Y''(z) - Y'(z) - 2Y(z) = 2$. The general solution is $Y(z) = c_1 e^{-z} + c_2 e^{2z} - 1 \Rightarrow y = c_1 t^{-1} + c_2 t^2 - 1$.

25. Under the change of variable $t = e^z$, the differential equation $t^2 y'' - ty' + y = t^{-1}$ transforms into $Y''(z) - 2Y'(z) + Y(z) = (e^z)^{-1}$ or $Y''(z) - 2Y'(z) + Y(z) = e^{-z}$. Solving this constant coefficient equation using the techniques of Chapter 4, we find the general solution $Y(z) = c_1 e^z + c_2 z e^z + 0.25e^{-z}$. Since $z = \ln t$, the solution can be converted to $y(t) = c_1 t + c_2 t \ln t + 0.25t^{-1}$.

26. Under the change of variable $t = e^z$, the differential equation transforms into $Y''(z) + 9Y(z) = 10e^z$.
The general solution is $Y(z) = c_1 \cos(3z) + c_2 \sin(3z) + e^z \Rightarrow y = c_1 \cos(3\ln t) + c_2 \sin(3\ln t) + t$.

27. Under the change of variable $t = e^z$, the differential equation $t^2 y'' - 6y = 10t^{-2} - 6$ transforms into $Y''(z) - Y'(z) - 6Y(z) = 10(e^z)^{-2} - 6$ or $Y''(z) - Y'(z) - 6Y(z) = 10e^{-2z} - 6$. Solving this constant coefficient equation using the techniques of Chapter 4, we find the general solution $Y(z) = c_1 e^{3z} + c_2 e^{-2z} - 2ze^{-2z} + 1$. Since $z = \ln t$, the solution can be converted to $y(t) = c_1 t^3 + c_2 t^{-2} - 2t^{-2} \ln t + 1$.

28. Under the change of variable $t = e^z$, the differential equation transforms into $Y''(z) - 5Y'(z) + 6Y(z) = 3z$. Therefore, $Y_c = c_1 e^{2z} + c_2 e^{3z}$, $Y_p = Az + B = \frac{1}{2}z + \frac{5}{12}$.
The general solution is $Y(z) = c_1 e^{2z} + c_2 e^{3z} + \frac{1}{2}z + \frac{5}{12} \Rightarrow y = c_1 t^2 + c_2 t^3 + \frac{1}{2}\ln t + \frac{5}{12}$.

29. Under the change of variable $t = e^z$, the differential equation $t^2 y'' + 8ty' + 10y = 36(t + t^{-1})$ transforms into $Y''(z) + 7Y'(z) + 10Y(z) = 36(e^z + e^{-z})$. Solving this constant coefficient equation using the techniques of Chapter 4, we find the general solution $Y(z) = c_1 e^{-5z} + c_2 e^{-2z} + 2e^z + 9e^{-z}$. Since $z = \ln t$, the solution can be converted to $y(t) = c_1 t^{-5} + c_2 t^{-2} + 2t + 9t^{-1}$.

30. The complementary solution is $y_C(t) = c_1 t^{-1} + c_2 t^3$. For a particular solution, use $y_P(t) = At + B$. Then, the general solution is $y(t) = c_1 t^{-1} + c_2 t^3 - 2t - 2$. Imposing the initial conditions, we obtain $y(1) = c_1 + c_2 - 2 - 2 = 1$ and $y'(1) = -c_1 + 3c_2 - 2 = 3$. Solving, we find the solution of the initial value problem is $y(t) = \frac{5}{2}t^{-1} + \frac{5}{2}t^3 - 2t - 2$. The interval of existence is $0 < t < \infty$.

31. Consider the initial value problem $t^2 y'' - 5ty' + 5y = 10, y(1) = 4, y'(1) = 6$. The complementary solution is $y_C(t) = c_1 t^5 + c_2 t$. By inspection, a particular solution is $y_P(t) = 2$. Thus, the general solution is $y(t) = c_1 t^5 + c_2 t + 2$. Imposing the initial conditions, we obtain $y(1) = c_1 + c_2 + 2 = 4$ and $y'(1) = 5c_1 + c_2 = 6$. Solving, we find the solution of the initial value problem is $y(t) = t^5 + t + 2$. The interval of existence is the entire t-axis.

32. The complementary solution is $y_C(t)=c_1t^{-1}+c_2t^{-1}\ln(-t)$. For a particular solution, use $y_P(t)=At+B$. Then, the general solution is $y_C(t)=c_1t^{-1}+c_2t^{-1}\ln(-t)+2t+9$. Imposing the initial conditions, we obtain $y(-1)=-c_1-2+9=1$ and $y'(-1)=-c_1+c_2+2=0$. Solving, we find the solution of the initial value problem is $y(t)=6t^{-1}+4t^{-1}\ln(-t)+2t+9$. The interval of existence is $-\infty<t<0$.

33. Consider the initial value problem $t^2y''+3ty'+y=2t^{-1}, y(1)=-2, y'(1)=1$. The complementary solution is $y_C(t)=c_1t^{-1}+c_2t^{-1}\ln t$. Using the change of variable $t=e^z$ as in Example 2, we find a particular solution $y_P(t)=t^{-1}(\ln t)^2$. Thus, the general solution is $y(t)=c_1t^{-1}+c_2t^{-1}\ln t+t^{-1}(\ln t)^2$. Imposing the initial conditions, we obtain $y(1)=c_1=-2$ and $y'(1)=-c_1+c_2=1$. Solving, we find the solution of the initial value problem is $y(t)=-2t^{-1}-t^{-1}\ln t+t^{-1}(\ln t)^2$. The interval of existence is the positive t-axis.

34. $$\frac{dy}{dt}=\frac{dy}{dz}\frac{dz}{dt}=\frac{1}{t}\frac{dy}{dz};\ \frac{d^2y}{dt^2}=-\frac{1}{t^2}\frac{dy}{dz}+\frac{1}{t}\frac{d^2y}{dz^2}\frac{1}{t}=\frac{1}{t^2}\left(\frac{d^2y}{dz^2}-\frac{dy}{dz}\right).$$

$$\frac{d^3y}{dt^3}=-\frac{2}{t^3}\left(\frac{d^2y}{dz^2}-\frac{dy}{dz}\right)+\frac{1}{t^3}\left(\frac{d^3y}{dz^3}-\frac{d^2y}{dz^2}\right)=\frac{1}{t^3}\left(\frac{d^3y}{dz^3}-3\frac{d^2y}{dz^2}+2\frac{dy}{dz}\right).$$ Therefore,

$$t^3y'''+\alpha t^2y''+\beta ty'+\gamma y=\frac{d^3Y}{dz^3}-3\frac{d^2Y}{dz^2}+2\frac{dY}{dz}+\alpha\left(\frac{d^2Y}{dz^2}-\frac{dY}{dz}\right)+\beta\left(\frac{dY}{dz}\right)+\gamma Y=0$$

$$\Rightarrow\frac{d^3Y}{dz^3}+(\alpha-3)\frac{d^2Y}{dz^2}+(\beta-\alpha+2)\frac{dY}{dz}+\gamma Y=0.$$

35. Consider the differential equation $t^3y'''+3t^2y''-3ty'=0$. Assuming a solution of the form $y(t)=t^\lambda$, we obtain the characteristic equation $\lambda^3-4\lambda=0$. The roots are $\lambda_1=0, \lambda_2=2$ and $\lambda_3=-2$. The general solution is $y(t)=c_1+c_2t^2+c_3t^{-2},\ t\neq 0$.

36. $\alpha=0,\ \beta=1,\ \gamma=-1\Rightarrow Y'''-3Y''+3Y'-Y=0$. The characteristic equation is $\lambda^3-3\lambda^2+3\lambda-1=(\lambda-1)^3=0$. The roots are $\lambda_1=\lambda_2=\lambda_3=1$. Therefore, $Y=c_1e^z+c_2ze^z+c_3z^2e^z\Rightarrow y=c_1t+c_2t\ln t+c_3t(\ln t)^2$.

37. Consider the differential equation $t^3y'''+3t^2y''+ty'=8t^2+12$. Using the change of variable $t=e^z$ as suggested in Exercise 34, the differential equation transforms to $Y'''(z)=8e^{2z}+12$. The general solution is $Y(z)=c_1+c_2z+c_3z^2+e^{2z}+2z^3$. Using the fact that $z=\ln t$, the general solution becomes $y(t)=c_1+c_2\ln t+c_3(\ln t)^2+t^2+2(\ln t)^3,\ t>0$.

38. $\alpha=6,\ \beta=7,\ \gamma=1\Rightarrow Y'''+3Y''+3Y'+Y=0$. The characteristic equation is $(\lambda+1)^3=0$. The roots are $\lambda_1=\lambda_2=\lambda_3=-1$. Therefore,
$Y_c=c_1e^{-z}+c_2ze^{-z}+c_3z^2e^{-z},\ Y_p=Az+B\Rightarrow Y=c_1e^{-z}+c_2ze^{-z}+c_3z^2e^{-z}+z-1$
$\Rightarrow y=c_1t^{-1}+c_2t^{-1}\ln t+c_3t^{-1}(\ln t)^2+\ln t-1$.

Section 10.4

1. When put in standard form, the differential equation is $y''+t^{-1}(\cos t)y'+t^{-1}y=0$. Thus, $t=0$ is the only singular point. The coefficient functions are $p(t)=t^{-1}(\cos t)$ and $q(t)=t^{-1}$. Clearly $tp(t)=\cos t$ and $t^2q(t)=t$ are analytic. Therefore, $t=0$ is a regular singular point.

2. $p(t)=\dfrac{\sin t}{t^2}$ and $q(t)=\dfrac{1}{t^2}$. Since $tp(t)=\dfrac{\sin t}{t}=1-\dfrac{t^2}{3!}+\dfrac{t^4}{5!}-\dfrac{t^6}{7!}+\ldots$ and $t^2q(t)=1$ are both analytic at $t=0$, then $t=0$ is a regular singular point.

3. When put in standard form, the differential equation is $y''+(t+1)^{-1}y'+(t^2-1)^{-1}y=0$. Thus, $t=1$ and $t=-1$ are singular points. The coefficient functions are $p(t)=(t+1)^{-1}$ and $q(t)=(t^2-1)^{-1}$. Clearly $(t-1)p(t)=(t-1)(t+1)^{-1}$ and $(t-1)^2q(t)=(t-1)(t+1)^{-1}$ are analytic at $t=1$. Therefore, $t=1$ is a regular singular point. Similarly, $t=-1$ is also a regular singular point.

4. $p(t)=\dfrac{t+1}{(t^2-1)^2}=\dfrac{1}{(t-1)^2(t+1)}$ and $q(t)=\dfrac{1}{(t-1)^2(t+1)^2}$.

At $t=-1$, $(t+1)p(t)=\dfrac{1}{(t-1)^2}\to\dfrac{1}{4}$ and $(t+1)^2q(t)=\dfrac{1}{(t-1)^2}\to\dfrac{1}{4}$ as $t\to-1$. Therefore, $t=-1$ is a regular singular point.

At $t=1$, $\lim\limits_{t\to1}(t-1)p(t)=\lim\limits_{t\to1}\dfrac{1}{(t-1)(t+1)}$ does not exist.. Therefore, $t=1$ is an irregular singular point.

5. When put in standard form, the differential equation is $y''+t^{-2}(1-\cos t)y'+t^{-2}y=0$. Thus, $t=0$ is the only singular point. The coefficient functions are $p(t)=t^{-2}(1-\cos t)$ and $q(t)=t^{-2}$. Using a Maclaurin series, $tp(t)=t^{-1}(1-\cos t)=\dfrac{t}{2!}-\dfrac{t^3}{4!}+\dfrac{t^5}{6!}-\cdots$ is analytic at $t=0$ as is $t^2q(t)=1$. Therefore, $t=0$ is a regular singular point.

6. $p(t)=q(t)=\dfrac{1}{|t|}$. Since neither $tp(t)=\dfrac{t}{|t|}$ nor $t^2q(t)=\dfrac{t^2}{|t|}$ are analytic at $t=0$, there is an irregular singular point at $t=0$.

7. When put in standard form, the differential equation is $y''+(1-e^t)^{-1}y'+(1-e^t)^{-1}y=0$. Thus, $t=0$ is the only singular point. The coefficient functions are $p(t)=(1-e^t)^{-1}$ and $q(t)=(1-e^t)^{-1}$. Using a Maclaurin series,

$tp(t)=t(1-e^t)^{-1}=t\left(-t-\dfrac{t^2}{2!}-\dfrac{t^3}{3!}-\cdots\right)^{-1}=\left(-1-\dfrac{t}{2!}-\dfrac{t^2}{3!}-\cdots\right)^{-1}$ is analytic at $t=0$ as is $t^2q(t)$..

Therefore, $t=0$ is a regular singular point.

8. $p(t)=\dfrac{t+2}{(2-t)(2+t)}=\dfrac{-1}{(t-2)}$ and $q(t)=\dfrac{1}{(4-t^2)^2}=\dfrac{1}{(t-2)^2(t+2)^2}$.

At $t=-2$, $(t+2)p(t)=\dfrac{-(t+2)}{(t-2)}\to0$ and $(t+2)^2q(t)=\dfrac{1}{(t-2)^2}\to\dfrac{1}{16}$ as $t\to-2$. Therefore, $t=-2$ is a regular singular point.

At $t=2$, $(t-2)p(t)=-1$ and $(t-2)^2q(t)=\dfrac{1}{(t+2)^2}\to\dfrac{1}{16}$ as $t\to2$. Therefore, $t=2$ is a regular singular point.

9. When put in standard form, the differential equation is $y''+(1-t^2)^{-1/3}y'+(1-t^2)^{-1/3}ty=0$. Thus, $t=1$ and $t=-1$ are singular points. The coefficient functions are $p(t)=(1-t^2)^{-1/3}$ and $q(t)=t(1-t^2)^{-1/3}$. Neither of the functions $(t\pm1)p(t)$ or $(t\pm1)^2q(t)$ is analytic at $t=\pm1$. Therefore, $t=1$ is an irregular singular point as is $t=-1$.

10. $p(t)=1$, $q(t)=t^{\frac{1}{3}}$. Since $tp(t)=t$ is analytic at $t=0$, but $t^2q(t)=t^{\frac{7}{3}}$ is not, there is an irregular singular point at $t=0$.

11. For this problem, $p(t)=(\sin 2t)/P(t)$. Since we know there are singular points at $t=0$ and $t=\pm 1$, we know that $P(t)$ must be zero at those points. Since $tp(t)$ is analytic at $t=0$ and since $(\sin 2t)/t$ tends to 2 as $t\to 0$, it follows that t^2 is a factor of $P(t)$. Similarly, $(t-1)p(t)$ is *not* analytic at $t=1$ and thus $(t-1)^2$ must be a factor of $P(t)$. The same argument applies at $t=-1$ and thus $(t+1)^2$ must be a factor of $P(t)$. In summary, $P(t)=t^2(t-1)^2(t+1)^2=t^2(t^2-1)^2$.

12. $P(t)=1$.

13. For this problem, $p(t)=[tP(t)]^{-1}$. Since we know there are singular points at $t=\pm 1$, we know that $P(t)$ must be zero at $t=\pm 1$. Since $t^2q(t)=1/t$, it follows [without any assumptions on $P(t)$] that $t=0$ is an irregular singular point. Since, $(t-1)p(t)$ is *not* analytic at $t=1$ it follows that $(t-1)^2$ must be a factor of $P(t)$. The same argument applies at $t=-1$ and thus $(t+1)^2$ must be a factor of $P(t)$. In summary, $P(t)=(t-1)^2(t+1)^2=(t^2-1)^2$.

14(a). $t=0$ is a regular singular point if $n=1$.

14(b). $t=0$ is an irregular singular point if $n\ge 2$.

15. For this problem, $tp(t)=t/(\sin t)$ and $t^2q(t)=1/t^{n-2}$. Since $t/(\sin t)$ is analytic at $t=0$, it follows that $t=0$ is a regular singular point if $n=0,1,2$ and an irregular singular point if $n>2$.

16 (a). $tp(t)=-\dfrac{1}{2}$ and $t^2q(t)=\dfrac{t+1}{2}\to\dfrac{1}{2}$ as $t\to 0$. Thus, $t=0$ is a regular singular point.

16 (b). Substituting the series $y=\sum_{n=0}^{\infty}a_nt^{\lambda+n}$ into the differential equation, we obtain

$[2\lambda(\lambda-1)-\lambda+1]a_0t^{\lambda}+\sum_{n=1}^{\infty}[(2(\lambda+n)(\lambda+n-1)-(\lambda+n)+1)a_n+a_{n-1}]t^{\lambda+n}=0$. Therefore, the indicial equation is $F(\lambda)=0$ where $F(\lambda)=2\lambda^2-3\lambda+1$. The roots of the indicial equation are $\lambda_1=\dfrac{1}{2}$ and $\lambda_2=1$.

16 (c). $a_n=\dfrac{-a_{n-1}}{F(\lambda+n)}=\dfrac{-a_{n-1}}{2(\lambda+n)^2-3(\lambda+n)+1}, n=1,2,\ldots$

For $\lambda_2=1$, the recurrence relation is $a_n=\dfrac{-a_{n-1}}{2(1+n)^2-3(1+n)+1}, n=1,2,\ldots.$.

16 (d). $y(t)=a_0\left[t-\dfrac{t^2}{3}+\dfrac{t^3}{30}+\cdots\right]$.

17 (a). For this problem, $tp(t)=1$ and $t^2q(t)=(t-1)/4$. Thus, $t=0$ is a regular singular point.

17 (b). Substituting the series $y=\sum_{n=0}^{\infty}a_nt^{\lambda+n}$ into the differential equation $4t^2y''+4ty'+(t-1)y=0$, we obtain $(4\lambda^2-1)a_0t^{\lambda}+\sum_{n=1}^{\infty}[(4(\lambda+n)^2-1)a_n+a_{n-1}]t^{\lambda+n}=0$. Therefore, the indicial equation is $F(\lambda)=0$ where $F(\lambda)=4\lambda^2-1$. The roots of the indicial equation are $\lambda_1=-1/2$ and $\lambda_2=1/2$.

17 (c). $a_n = \dfrac{-a_{n-1}}{F(\lambda+n)} = \dfrac{-a_{n-1}}{4(\lambda+n)^2-1}, n=1,2,\ldots$

For $\lambda = 1/2$, the recurrence relation is $a_n = -a_{n-1}/[4(n+0.5)^2-1], n=1,2,\ldots$.

17 (d). $y(t) = a_0[t^{1/2} - (1/8)t^{3/2} + (1/192)t^{5/2} - \cdots]$.

18 (a). $tp(t) = \dfrac{t}{16}$ and $t^2q(t) = \dfrac{3}{16}$. Both limits exist as $t \to 0$. Thus, $t = 0$ is a regular singular point.

18 (b). Substituting the series $y = \sum_{n=0}^{\infty} a_n t^{\lambda+n}$ into the differential equation, we obtain

$[16\lambda(\lambda-1)+3]a_0t^{\lambda} + \sum_{n=1}^{\infty}[(16(\lambda+n)(\lambda+n-1)+3)a_n + (\lambda+n-1)a_{n-1}]t^{\lambda+n} = 0$. Therefore, the indicial equation is $F(\lambda)=0$ where $F(\lambda) = 16\lambda^2 - 16\lambda + 3$. The roots of the indicial equation are $\lambda_1 = \dfrac{1}{4}$ and $\lambda_2 = \dfrac{3}{4}$.

18 (c). $a_n = \dfrac{-(\lambda+n-1)a_{n-1}}{F(\lambda+n)} = \dfrac{-(\lambda+n-1)a_{n-1}}{16(\lambda+n)(\lambda+n-1)+3}, n=1,2,\ldots$

For $\lambda_2 = \dfrac{3}{4}$, the recurrence relation is $a_n = \dfrac{-(3/4+n-1)a_{n-1}}{16(3/4+n)(3/4+n-1)+3}, n=1,2,\ldots$,..

18 (d). $y(t) = a_0\left[t^{\frac{3}{4}} - \dfrac{t^{\frac{7}{4}}}{32} + \dfrac{7t^{\frac{11}{4}}}{10240} + \cdots\right],\ t>0$.

19 (a). For this problem, $tp(t) = 1$ and $t^2q(t) = t-9$. Thus, $t=0$ is a regular singular point.

19 (b). Substituting the series $y = \sum_{n=0}^{\infty} a_n t^{\lambda+n}$ into the differential equation $t^2y'' + ty' + (t-9)y = 0$, we obtain $(\lambda^2-9)a_0t^{\lambda} + \sum_{n=1}^{\infty}[((\lambda+n)^2-9)a_n + a_{n-1}]t^{\lambda+n} = 0$. Therefore, the indicial equation is $F(\lambda)=0$ where $F(\lambda) = \lambda^2 - 9$. The roots of the indicial equation are $\lambda_1 = -3$ and $\lambda_2 = 3$.

19 (c). $a_n = \dfrac{-a_{n-1}}{F(\lambda+n)} = \dfrac{-a_{n-1}}{(\lambda+n)^2-9}, n=1,2,\ldots$

For $\lambda = 3$, the recurrence relation is $a_n = -a_{n-1}/[(n+3)^2-9], n=1,2,\ldots$.

19 (d). $y(t) = a_0[t^3 - (1/7)t^4 + (1/112)t^5 - \cdots]$.

20 (a). $tp(t) = t+2$ and $t^2q(t) = -t$. Both limits exist as $t \to 0$. Thus, $t=0$ is a regular singular point.

20 (b). Substituting the series $y = \sum_{n=0}^{\infty} a_n t^{\lambda+n}$ into the differential equation, we obtain

$[\lambda(\lambda-1)+2\lambda]a_0t^{\lambda-1} + \sum_{n=0}^{\infty}\{[(\lambda+n+1)(\lambda+n)+2(\lambda+n+1)]a_{n+1} + (\lambda+n-1)a_n\}t^{\lambda+n} = 0$.

Therefore, the indicial equation is $F(\lambda)=0$ where $F(\lambda) = \lambda^2 + \lambda$. The roots of the indicial equation are $\lambda_1 = -1$ and $\lambda_2 = 0$.

20 (c). $a_{n+1} = \dfrac{-(\lambda+n-1)a_n}{(\lambda+n+2)(\lambda+n+1)}, n=0,1,2,\ldots$

For $\lambda_2 = 0$, the recurrence relation is $a_n = \dfrac{-(n-1)a_n}{(n+2)(n+1)}, n=0,1,2,\ldots$,..

20 (d). $y(t) = a_0\left[1 + \frac{t}{2}\right]$.

21 (a). For this problem, $tp(t) = 3$ and $t^2q(t) = 2t + 1$. Thus, $t = 0$ is a regular singular point.

21 (b). Substituting the series $y = \sum_{n=0}^{\infty} a_n t^{\lambda+n}$ into the differential equation $t^2y'' + 3ty' + (2t+1)y = 0$, we obtain $(\lambda^2 + 2\lambda + 1)a_0t^{\lambda} + \sum_{n=1}^{\infty}[((\lambda+n)^2 + 2(\lambda+n) + 1)a_n + 2a_{n-1}]t^{\lambda+n} = 0$. Therefore, the indicial equation is $F(\lambda) = 0$ where $F(\lambda) = \lambda^2 + 2\lambda + 1$. The roots of the indicial equation are $\lambda_1 = \lambda_2 = -1$.

21 (c). $a_n = \frac{-2a_{n-1}}{F(\lambda+n)} = \frac{-2a_{n-1}}{((\lambda+n)+1)^2}, n = 1, 2, \ldots$

For $\lambda = -1$, the recurrence relation is $a_n = -2a_{n-1}/n^2, n = 1, 2, \ldots$.

21 (d). $y(t) = a_0[t^{-1} - 2 + t - \cdots]$.

22 (a). Both limits exist as $t \to 0$. Thus, $t = 0$ is a regular singular point.

22 (b). Substituting the series $y = \sum_{n=0}^{\infty} a_n t^{\lambda+n}$ into the differential equation, we obtain $[\lambda(\lambda-1) - \lambda - 3]a_0t^{\lambda} + \sum_{n=1}^{\infty}\{[(\lambda+n)^2 - 2(\lambda+n) - 3)]a_n + (\lambda+n-1)a_{n-1}\}t^{\lambda+n} = 0$. Therefore, the indicial equation is $F(\lambda) = 0$ where $F(\lambda) = \lambda^2 - 2\lambda - 3$. The roots of the indicial equation are $\lambda_1 = -1$ and $\lambda_2 = 3$.

22 (c). $a_n = \frac{-(\lambda+n-1)a_{n-1}}{F(\lambda+n)} = \frac{-(\lambda+n-1)a_{n-1}}{(\lambda+n)^2 - 2(\lambda+n) - 3}, n = 1, 2, \ldots$

For $\lambda_2 = 3$, the recurrence relation is $a_n = \frac{-(n+2)a_{n-1}}{n(n+4)}, n = 1, 2, \ldots,$.

22 (d). $y(t) = a_0\left[t^3 - \frac{3t^4}{5} + \frac{t^5}{5} + \cdots\right]$.

23 (a). For this problem, $tp(t) = t - 2$ and $t^2q(t) = t$. Thus, $t = 0$ is a regular singular point.

23 (b). Substituting the series $y = \sum_{n=0}^{\infty} a_n t^{\lambda+n}$ into the differential equation $ty'' + (t-2)y' + y = 0$, we obtain $(\lambda^2 - 3\lambda)a_0t^{\lambda-1} + \sum_{n=0}^{\infty}(\lambda+n+1)[(\lambda+n-2)a_n + a_{n-1}]t^{\lambda+n} = 0$. Therefore, the indicial equation is $F(\lambda) = 0$ where $F(\lambda) = \lambda^2 - 3\lambda$. The roots of the indicial equation are $\lambda_1 = 0$ and $\lambda_2 = 3$.

23 (c). $a_{n+1} = \frac{-(\lambda+n+1)a_n}{F(\lambda+n)} = \frac{-(\lambda+n+1)a_n}{(\lambda+n+1)(\lambda+n-2)} = \frac{-a_n}{(\lambda+n-2)}, n = 0, 1, 2, \ldots$

For $\lambda = 3$, the recurrence relation is $a_n = -a_{n-1}/(n+1), n = 0, 1, \ldots$.

23 (d). $y(t) = a_0[t^3 - t^4 + (1/2)t^5 - \cdots]$.

24 (a). $tp(t) = -\frac{2\sin t}{t} \to -2$ as $t \to 0$ and $t^2q(t) = 2 + t \to 2$ as $t \to 0$. Thus, $t = 0$ is a regular singular point.

24 (b). $t^2y'' - 2\sin ty' + (2+t)y = \left[\lambda(\lambda-1)a_0t^\lambda + (\lambda+1)\lambda a_1t^{\lambda+1} + (\lambda+2)(\lambda+1)a_2t^{\lambda+2} + \cdots\right]$

$$-2\left[t - \frac{t^3}{3!} + \ldots\right]\left[\lambda a_0t^{\lambda-1} + (\lambda+1)a_1t^\lambda + (\lambda+2)a_2t^{\lambda+1} + \cdots\right] + (2+t)\left[a_0t^\lambda + a_1t^{\lambda+1} + a_2t^{\lambda+2} + \ldots\right] = 0.$$

For $t^\lambda: \lambda(\lambda-1)a_0 - 2\lambda a_0 + 2a_0 = (\lambda^2 - 3\lambda + 2)a_0 = (\lambda-1)(\lambda-2)a_0 = 0$.

For $t^{\lambda+1}: \lambda(\lambda+1)a_1 - 2(\lambda+1)a_1 + 2a_1 + a_0 = [(\lambda+1)(\lambda-2)+2]a_1 + a_0 = 0$.

For $t^{\lambda+2}: (\lambda+2)(\lambda+1)a_2 - 2(\lambda+2)a_2 + \frac{2}{3!}\lambda a_0 + 2a_2 + a_1 = 0$.

Therefore, the indicial equation is $F(\lambda) = (\lambda-1)(\lambda-2) = 0$. The roots of the indicial equation are $\lambda_1 = 1$ and $\lambda_2 = 2$.

24 (c). $y(t) = a_0[t^2 - \frac{t^3}{2} - \frac{t^4}{6} - \cdots]$

25 (a). For this problem, $tp(t) = 4$ and $t^2q(t) = te^t$. Thus, $t = 0$ is a regular singular point.

25 (b). Given the series $y = \sum_{n=0}^{\infty} a_nt^{\lambda+n}$, we have $ty'' = \lambda(\lambda-1)a_0t^{\lambda-1} + (\lambda+1)\lambda a_1t^\lambda + \cdots$,

$-4y' = \lambda a_0t^{\lambda-1} + (\lambda+1)a_1t^\lambda + \cdots$, and

$e^ty = [1 + t + (1/2!)t^2 + \cdots][a_0t^\lambda + a_1t^{\lambda+1} + \cdots] = a_0t^\lambda + (a_1+1)t^{\lambda+1} + \cdots$.

Therefore, substituting the series into the differential equation $ty'' - 4y' + e^ty = 0$, we obtain $\lambda(\lambda-5)a_0t^{\lambda-1} + [(\lambda+1)(\lambda-4) + a_0]t^\lambda + \cdots = 0$. Therefore, the indicial equation is $\lambda^2 - 5\lambda = 0$. The roots of the indicial equation are $\lambda_1 = 0$ and $\lambda_2 = 5$.

25 (c). $y(t) = a_0[t^5 - (1/6)t^6 - (5/84)t^7 - \cdots]$

26 (a). $tp(t) = -\frac{t}{\sin t} \to -1$ as $t \to 0$ and $t^2q(t) = \frac{t^2}{\sin t} \to 0$ as $t \to 0$. Thus, $t = 0$ is a regular singular point.

26 (b). $(\sin t)y'' - y' + y =$

$$\left[t - \frac{t^3}{3!} + \frac{t^5}{5!}\ldots\right]\left[\lambda(\lambda-1)a_0t^{\lambda-2} + (\lambda+1)\lambda a_1t^{\lambda-1} + (\lambda+2)(\lambda+1)a_2t^\lambda + (\lambda+3)(\lambda+2)a_3t^{\lambda+1} + \cdots\right]$$

$$-\left[\lambda a_0t^{\lambda-1} + (\lambda+1)a_1t^\lambda + (\lambda+2)a_2t^{\lambda+1} + \cdots\right] + \left[a_0t^\lambda + a_1t^{\lambda+1} + a_2t^{\lambda+2} + \cdots\right] = 0..$$

For $t^{\lambda-1}: \lambda(\lambda-1)a_0 - \lambda a_0 = (\lambda^2 - 2\lambda)a_0 = \lambda(\lambda-2)a_0 = 0$.

For $t^\lambda: \lambda(\lambda+1)a_1 - (\lambda+1)a_1 + a_0 = (\lambda+1)(\lambda-1)a_1 + a_0 = 0$.

For $t^{\lambda+1}: (\lambda+2)(\lambda+1)a_2 + (\lambda+2)a_2 - \frac{1}{3!}\lambda(\lambda-1)a_0 + a_1 = (\lambda+2)^2a_2 + a_1 - \frac{1}{6}\lambda(\lambda-1)a_0 = 0$.

Therefore, the indicial equation is $F(\lambda) = \lambda(\lambda-2) = 0$. The roots of the indicial equation are $\lambda_1 = 0$ and $\lambda_2 = 2$.

26 (c). $y(t) = a_0[t^2 - \frac{t^3}{3} + \frac{t^4}{24} + \cdots]$

27 (a). For this problem, $tp(t) = t/(2-2e^t)$ and $t^2q(t) = t^2/(1-e^t)$. Thus, $t = 0$ is a regular singular point.

27 (b). Given the series $y = \sum_{n=0}^{\infty} a_nt^{\lambda+n}$, we have

$(1-e^t)y'' = -\lambda(\lambda-1)a_0t^{\lambda-1}[-0.5\lambda(\lambda-1)a_0 - (\lambda+1)\lambda a_1]t^\lambda + \cdots$,

$0.5y' = 0.5[\lambda a_0t^{\lambda-1} + (\lambda+1)a_1t^\lambda + \cdots]$.

Therefore, substituting the series into the differential equation $(1-e^t)y''+(1/2)y'+y=0$, we obtain $-\lambda(\lambda-1.5)a_0t^{\lambda-1}+[-(\lambda+1)(\lambda-0.5)a_1+0.5(-\lambda^2+\lambda+2)a_0]t^{\lambda}+\cdots=0$. Therefore, the indicial equation is $\lambda^2-1.5\lambda=0$. The roots of the indicial equation are $\lambda_1=0$ and $\lambda_2=1.5$.

27 (c). $y(t)=a_0[t^{3/2}+(1/2)t^{5/2}-(17/96)t^{7/2}+\cdots]$

Section 10.5

1 (a). When put in standard form, the differential equation is $y''-(2t)^{-1}(1+t)y'+t^{-1}y=0$. Therefore, $t=0$ is a regular singular point.

1 (b). Substituting the series $y=\sum_{n=0}^{\infty}a_nt^{n+\lambda}$ into the differential equation, we obtain

$$(2\lambda^2-3\lambda)a_0t^{\lambda-1}+\sum_{n=0}^{\infty}[(\lambda+n+1)(2(\lambda+n)-1)a_{n+1}-(\lambda+n-2)a_n]t^{n+\lambda}=0.$$

Therefore, the exponents at the singularity are $\lambda_1=0$ and $\lambda_2=1.5$.

1 (c). The recurrence relation is $a_{n+1}=[(\lambda+n-2)a_n]/[(\lambda+n+1)(2\lambda+2n-1)]$, $n=0,1,\ldots$.

1 (d). For $\lambda_1=0$, $y=a_0[1+2t-t^2]$ is a polynomial solution.
For $\lambda_2=3/2$, $y=a_0[t^{3/2}-(1/10)t^{5/2}-(1/280)t^{7/2}-\cdots]$.

1 (e). Note that $tp(t)$ and $t^2q(t)$ are analytic everywhere. Thus, see equations (18)-(21), the second series found in part (d) converges for $0<t$.

2 (b). Substituting the series into the differential equation, we obtain

$$\left[2\lambda(\lambda-1)+5\lambda\right]a_0t^{\lambda-1}+\left[2\lambda(\lambda+1)+5(\lambda+1)\right]a_1t^{\lambda}+\sum_{n=1}^{\infty}[2(\lambda+n+1)(\lambda+n+5/2)a_{n+1}+3a_{n-1}]t^{n+\lambda}$$

$=0$. Therefore, $F(\lambda)=2\lambda(\lambda+3/2)\Rightarrow\lambda_1=-\frac{3}{2},\ \lambda_2=0$.

2 (c). The recurrence relation is $a_{n+1}=\dfrac{-3a_{n-1}}{2(\lambda+n+1)(\lambda+n+5/2)}$, $n=1,2,\ldots$ and $(\lambda+1)(2\lambda+5)a_1=0$

2 (d). For $\lambda_1=-\frac{3}{2}$, $y=a_0[t^{-3/2}-(3/2)t^{1/2}+(9/40)t^{5/2}+\cdots]$.
For $\lambda_2=0$, $y=a_0[1-(3/14)t^2+(9/616)t^4-\cdots]$.

2 (e). The series converges for $0<t$.

3 (a). When put in standard form, the differential equation is $y''-(3t)^{-1}y'+(3t^2)^{-1}(1+t)y=0$. Therefore, $t=0$ is a regular singular point.

3 (b). Substituting the series $y=\sum_{n=0}^{\infty}a_nt^{n+\lambda}$ into the differential equation, we obtain

$$(3\lambda^2-4\lambda+1)a_0t^{\lambda}+\sum_{n=1}^{\infty}\{[3(\lambda+n)(\lambda+n-1)-\lambda-n+1]a_n+a_{n-1}\}t^{n+\lambda}=0.$$

Therefore, the exponents at the singularity are $\lambda_1=1/3$ and $\lambda_2=1$.

3 (c). The recurrence relation is $a_n=-a_{n-1}/[3(\lambda+n)(\lambda+n-1)-\lambda-n+1]$, $n=1,2,\ldots$.

3 (d). For $\lambda_1=1/3$, $y=a_0[t^{1/3}-t^{4/3}+(1/8)t^{7/3}+\cdots]$.
For $\lambda_2=1$, $y=a_0[t-(1/5)t^2+(1/80)t^3+\cdots]$.

3 (e). Note that $tp(t)$ and $t^2q(t)$ are analytic everywhere. Thus, see equations (18)-(21), the series found in part (d) converge for $0<t$.

4 (b). Substituting the series into the differential equation, we obtain

$[6\lambda(\lambda-1)+\lambda+1]a_0t^{\lambda}+\sum_{n=1}^{\infty}\{[6(\lambda+n)(\lambda+n-1)+(\lambda+n)+1]a_n-a_{n-1}\}t^{n+\lambda}=0$. Therefore,

$F(\lambda)=6\lambda^2-5\lambda+1 \Rightarrow \lambda_1=\frac{1}{3},\ \lambda_2=\frac{1}{2}$.

4 (c). The recurrence relation is $a_n=\dfrac{a_{n-1}}{6(\lambda+n)(\lambda+n-1)+(\lambda+n)+1},\ n=1,2,...$

4 (d). For $\lambda_1=\frac{1}{3}$, $y=a_0[t^{1/3}+(1/5)t^{4/3}+(1/110)t^{7/3}+\cdots]$.

For $\lambda_2=\frac{1}{2}$, $y=a_0[t^{1/2}+(1/7)t^{3/2}+(1/182)t^{5/2}+\cdots]$.

4 (e). The series converges for $0<t$.

5 (a). When put in standard form, the differential equation is $y''-5t^{-1}y'+t^{-2}(9+t^2)y=0$. Therefore, $t=0$ is a regular singular point.

5 (b). Substituting the series $y=\sum_{n=0}^{\infty}a_nt^{n+\lambda}$ into the differential equation, we obtain

$$(\lambda^2-6\lambda+9)a_0t^{\lambda}+[(\lambda+1)\lambda-5(\lambda+1)+9]a_1t^{\lambda+1}+$$
$$\sum_{n=2}^{\infty}\{[(\lambda+n)(\lambda+n-1)-5(\lambda+n)+9]a_n+a_{n-1}\}t^{n+\lambda}=0.$$

Therefore, the exponents at the singularity are $\lambda_1=\lambda_2=3$.

5 (c). The recurrence relation is $a_n=-a_{n-2}/(\lambda+n-3)^2,\ n=2,3,\ldots$.

5 (d). For $\lambda_1=3$, $y=a_0[t^3-(1/4)t^5+(1/64)t^7+\cdots]$.

5 (e). Note that $tp(t)$ and $t^2q(t)$ are analytic everywhere. Thus, see equations (18)-(21), the series found in part (d) converges for $0<t$.

6 (b). Substituting the series into the differential equation, we obtain

$[4\lambda(\lambda-1)+8\lambda+1]a_nt^{\lambda}+\sum_{n=1}^{\infty}\left\{[4(\lambda+n)^2+4(\lambda+n)+1]a_n-2a_{n-1}\right\}t^{n+\lambda}=0$. Therefore,

$F(\lambda)=4\lambda^2+4\lambda+1 \Rightarrow \lambda_1=\lambda_2=-\frac{1}{2}$.

6 (c). The recurrence relation is $a_n=\dfrac{2a_{n-1}}{(2(\lambda+n)+1)^2},\ n=1,2,...$

6 (d). For $\lambda_1=-\frac{1}{2}$, $y=a_0[t^{-1/2}+(1/2)t^{1/2}+(1/8)t^{3/2}+\cdots]$.

6 (e). The series converges for $0<t$.

7 (a). When put in standard form, the differential equation is $y''-2t^{-1}y'+t^{-2}(2+t)y=0$. Therefore, $t=0$ is a regular singular point.

7 (b). Substituting the series $y=\sum_{n=0}^{\infty}a_nt^{n+\lambda}$ into the differential equation, we obtain

$$(\lambda^2-3\lambda+2)a_0t^{\lambda}+\sum_{n=1}^{\infty}\{[(\lambda+n)^2-3(\lambda+n)+2]a_n+a_{n-1}\}t^{n+\lambda}=0.$$

Therefore, the exponents at the singularity are $\lambda_1=1$ and $\lambda_2=2$.

7 (c). The recurrence relation is $a_n=-a_{n-1}/[(\lambda+n-1)(\lambda+n-2)],\ n=1,2,\ldots$.

7 (d). For $\lambda_2=2$, $y=a_0[t^2-(1/2)t^3+(1/12)t^4+\cdots]$.

7 (e). Note that $tp(t)$ and $t^2q(t)$ are analytic everywhere. Thus, see equations (18)-(21), the series found in part (d) converges for $0 < t$.

8 (b). Substituting the series into the differential equation, we obtain

$$[\lambda(\lambda-1)+4\lambda]a_0t^\lambda + [\lambda(\lambda+1)+4(\lambda+1)]a_1t^{\lambda+1} + \sum_{n=1}^{\infty}\{[(\lambda+n+1)(\lambda+n+4)]a_{n+1} - 2a_{n-1}\}t^{n+\lambda} = 0$$

Therefore, $F(\lambda) = \lambda^2 + 3\lambda \Rightarrow \lambda_1 = -3,\ \lambda_2 = 0$.

8 (c). The recurrence relation is $a_{n+1} = \dfrac{2a_{n-1}}{(\lambda+n+1)(\lambda+n+4)}$, $n = 1,2,...$ and $(\lambda+1)(\lambda+4)a_1 = 0$

8 (d). For $\lambda_2 = 0$, $y = a_0[1 + (1/5)t^2 + (1/70)t^4 + \cdots]$.

8 (e). The series converges for $0 < t$.

9 (a). When put in standard form, the differential equation is $y'' + t^{-1}y' - t^{-2}(1+t^2)y = 0$. Therefore, $t = 0$ is a regular singular point.

9 (b). Substituting the series $y = \sum_{n=0}^{\infty} a_n t^{n+\lambda}$ into the differential equation, we obtain

$$(\lambda^2-1)a_0t^\lambda + [(\lambda+1)^2-1]a_1t^{\lambda+1} + \sum_{n=2}^{\infty}\{[(\lambda+n)^2-1]a_n - a_{n-2}\}t^{n+\lambda} = 0.$$

Therefore, the exponents at the singularity are $\lambda_1 = -1$ and $\lambda_2 = 1$.

9 (c). The recurrence relation is $a_n = a_{n-2}/[(\lambda+n)^2-1]$, $n = 2, 3, \ldots$.

9 (d). For $\lambda_2 = 1$, $y = a_0[t + (1/8)t^3 + (1/192)t^4 + \cdots]$.

9 (e). Note that $tp(t)$ and $t^2q(t)$ are analytic everywhere. Thus, see equations (18)-(21), the series found in part (d) converges for $0 < t$.

10 (b). Substituting the series into the differential equation, we obtain

$$[\lambda(\lambda-1)+5\lambda+4]a_0t^\lambda + [\lambda(\lambda+1)+5(\lambda+1)+4]a_1t^{\lambda+1}$$

$$+\sum_{n=2}^{\infty}\{[(\lambda+n)(\lambda+n+4)+4]a_n - a_{n-2}\}t^{n+\lambda} = 0.$$ Therefore, $F(\lambda) = \lambda^2 + 4\lambda + 4 \Rightarrow \lambda_1 = \lambda_2 = -2$.

10 (c). The recurrence relation is $a_n = \dfrac{a_{n-2}}{(\lambda+n+2)^2}$, $n = 2,3,...$ and $(\lambda+1)(\lambda+5)a_1 = 0$

10 (d). For $\lambda = -2$, $y = a_0[t^{-2} + (1/4) + (1/64)t^2 + \cdots]$.

10 (e). The series converges for $0 < t$.

11 (a). When put in standard form, the differential equation is $y'' + t^{-1}y' - t^{-2}(16+t)y = 0$. Therefore, $t = 0$ is a regular singular point.

11 (b). Substituting the series $y = \sum_{n=0}^{\infty} a_n t^{n+\lambda}$ into the differential equation, we obtain

$$(\lambda^2-16)a_0t^\lambda + \sum_{n=1}^{\infty}\{[(\lambda+n)^2-16]a_n - a_{n-1}\}t^{n+\lambda} = 0.$$

Therefore, the exponents at the singularity are $\lambda_1 = -4$ and $\lambda_2 = 4$.

11 (c). The recurrence relation is $a_n = a_{n-1}/[(\lambda+n)^2-16]$, $n = 1, 2, \ldots$.

11 (d). For $\lambda_2 = 4$, $y = a_0[t^4 + (1/9)t^5 + (1/180)t^6 + \cdots]$.

11 (e). Note that $tp(t)$ and $t^2q(t)$ are analytic everywhere. Thus, see equations (18)-(21), the series found in part (d) converges for $0 < t$.

12 (b). Substituting the series into the differential equation, we obtain

$$\left[8\lambda^2-2\lambda-1\right]a_0t^\lambda+\sum_{n=1}^{\infty}\left\{[8(\lambda+n)^2-2(\lambda+n)-1]a_n+a_{n-1}\right\}t^{n+\lambda}=0.$$ Therefore,

$$F(\lambda)=8\lambda^2-2\lambda-1\Rightarrow\lambda_1=-\frac{1}{4},\ \lambda_2=\frac{1}{2}.$$

12 (c). The recurrence relation is $a_n=\dfrac{-a_{n-1}}{(4(\lambda+n)+1)(2(\lambda+n)-1)},\ n=1,2,...$

12 (d). For $\lambda_1=-\dfrac{1}{4}$, $y=a_0[t^{-1/4}-(1/2)t^{3/4}+(1/40)t^{7/4}+\cdots]$.

For $\lambda_2=\dfrac{1}{2}$, $y=a_0[t^{1/2}-(1/14)t^{3/2}+(1/616)t^{5/2}+\cdots]$..

12 (e). The series converges for $0<t$.

13 (a). When put in standard form, the differential equation is

$y''-t^{-1}(t^2+1)^{-1}(1+t)y'+t^{-1}(t^2+1)^{-1}y=0$. Therefore, $t=0$ is a regular singular point and all other points are ordinary points.

13 (b). Substituting the series $y=\sum_{n=0}^{\infty}a_nt^{n+\lambda}$ into the differential equation, we obtain

$$\sum_{n=1}^{\infty}(\lambda+n-1)(\lambda+n-2)a_{n-1}t^{n+\lambda}+\sum_{n=-1}^{\infty}(\lambda+n+1)(\lambda+n-1)a_{n+1}t^{n+\lambda}$$
$$-\sum_{n=0}^{\infty}(\lambda+n-1)a_nt^{n+\lambda}=0$$

Therefore, indicial equation is $\lambda^2-2\lambda=0$. The exponents at the singularity are $\lambda_1=0$ and $\lambda_2=2$.

14 (a). $tp(t)=\dfrac{\sin 3t}{t}\to 3$ as $t\to 0$ and $t^2q(t)=\cos t\to 1$ as $t\to 0$. Thus, $t=0$ is a regular singular point.

14 (b). $t^2y''+\left(3t-\dfrac{(3t)^3}{3!}+...\right)y'+\left(1-\dfrac{t^2}{2!}+...\right)y=0$.

Therefore, indicial equation $(\lambda+1)^2=0\Rightarrow\lambda_1=\lambda_2=-1$.

15 (a). When put in standard form, the differential equation is $y''-(t^2-4)^{-2}y'+(t^2-4)^{-2}y=0$.
Therefore, $t=2$ and $t=-2$ are irregular singular points. All other points are ordinary points.

16 (a). $tp(t)=\dfrac{1}{(1-t)^{\frac{1}{3}}}\to 1$ as $t\to 0$ and $t^2q(t)=-\dfrac{1}{(1-t)^{\frac{1}{3}}}\to -1$ as $t\to 0$. Thus, $t=0$ is a regular singular point.

Neither $(t-1)p(t)$ nor $(t-1)^2q(t)$ are analytical at $t=1$, so $t=1$ is an irregular singular point.

16 (b). $(1-t)^{\frac{1}{3}}=1-\dfrac{1}{3}t-\dfrac{1}{9}t^2+...\Rightarrow t^2\left(1-\dfrac{1}{3}t-\dfrac{1}{9}t^2+...\right)y''+ty'-y=0$.

Therefore, indicial equation $\lambda^2-1=0\Rightarrow\lambda_1=-1,\ \lambda_2=1$.

17 (a). We need to substitute the series $y=\sum_{n=0}^{\infty} a_n (t-1)^{n+\lambda}$ into the differential equation. Before doing so, let us make the change of variable $\tau = t-1$. We now substitute the series $y=\sum_{n=0}^{\infty} a_n \tau^{n+\lambda}$ into the transformed equation, $-\tau(\tau+2)y'' - 2(\tau+1)y' + \alpha(\alpha+1)y = 0$, obtaining

$$-2\lambda^2 a_0 \tau^{\lambda-1} + \sum_{n=0}^{\infty}\{[-(\lambda+n)^2 - (\lambda+n) + \alpha(\alpha+1)]a_n - 2(\lambda+n+1)^2 a_{n+1}\tau^{\lambda+n} = 0.$$

Thus, the exponents at the singularity are $\lambda_1 = \lambda_2 = 0$.

17 (b). For $\lambda = 0$, the recurrence relation is $a_{n+1} = [-n^2 - n + \alpha(\alpha+1)]a_n / [2(n+1)^2]$.

Thus, $y(t) = a_0\left[1 + \dfrac{\alpha(\alpha+1)}{2}(t-1) + \dfrac{\alpha(\alpha+1)[-2+\alpha(\alpha+1)]}{16}(t-1)^2 + \cdots\right]$.

17 (c). When $\alpha = 1$, $y(t) = a_0 t$.

18 (a). $(1-t)^2 = -(t-1)(t+1) = -(t-1)((t-1)+2)$, $t = (t-1)+1$. Let $\tau = t-1$. We now substitute the series into the transformed equation, $-\tau(\tau+2)y'' - (\tau+1)y' + \alpha^2 y = 0$, obtaining

$$-[2\lambda(\lambda-1)+\lambda]a_0\tau^{\lambda-1} + \sum_{n=0}^{\infty}\{-[2(\lambda+n+1)(\lambda+n) + (\lambda+n+1)]a_{n+1} + [-(\lambda+n)^2 + \alpha^2]a_n\}\tau^{\lambda+n}.$$

Thus, $F(\lambda) = 2\lambda^2 - \lambda = 0$ and the exponents at the singularity are $\lambda_1 = 0$ and $\lambda_2 = \dfrac{1}{2}$.

18 (b). For $\lambda_1 = 0$, the recurrence relation is $a_{n+1} = \dfrac{[-n^2+\alpha^2]a_n}{(n+1)(2n+1)}$.

and $y(t) = a_0\left[1 + \alpha^2(t-1) + \dfrac{\alpha^2(\alpha^2-1)}{6}(t-1)^2 + \cdots\right]$.

For $\lambda_2 = \dfrac{1}{2}$, the recurrence relation is $a_{n+1} = \dfrac{[-(n+1/2)^2+\alpha^2]a_n}{(n+3/2)(2n+2)}$.

and $y(t) = a_0\left[(t-1)^{\frac{1}{2}} + \dfrac{(\alpha^2-\frac{1}{4})}{3}(t-1)^{\frac{3}{2}} + \dfrac{(\alpha^2-\frac{1}{4})(\alpha^2-\frac{9}{4})}{30}(t-1)^{\frac{5}{2}} + \cdots\right]$, $t-1>0$.

18 (c). By the Ratio Test, $\lim_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\dfrac{-(n+\lambda)^2+\alpha^2}{(n+\lambda+1)(2n+2\lambda+1)}\right| = \dfrac{1}{2}$

$\Rightarrow$ convergence for $\dfrac{1}{2}|\tau| < 1$ or $|t-1| < 2$ $\therefore R = 2$.

18 (d). When $\alpha = \dfrac{1}{2}$, one solution (with $\lambda = \dfrac{1}{2}$) reduces to $y(t) = a_0(t-1)^{\frac{1}{2}}$.

19 (a). Substituting the series $y=\sum_{n=0}^{\infty} a_n t^{n+\lambda}$ into the differential equation, we obtain

$$\lambda^2 a_0 t^{\lambda-1} + \sum_{n=0}^{\infty}\{(\lambda+n+1)^2 a_{n+1} - (\lambda+n-\alpha)a_n\}t^{n+\lambda} = 0.$$

19 (b). The recurrence relation is $a_{n+1} = (n-\alpha)a_n/(n+1)^2$. For $\alpha = 5$, the solution is $y(t) = a_0[1 - 5t + 5t^2 - (5/3)t^3 + (5/24)t^4 - (1/120)t^5]$.

19 (c). $y(t)$ is neither an even nor an odd function. Theorem 10.2 does not apply.

20. The indicial equation is $\lambda(\lambda-1) + \alpha\lambda + \beta = \lambda^2 + (\alpha-1)\lambda + \beta = 0$. Since $\lambda_1 = 1$, $\lambda_2 = 2$, then $\lambda^2 + (\alpha-1)\lambda + \beta = (\lambda-1)(\lambda-2) = \lambda^2 - 3\lambda + 2 \Rightarrow \alpha = -2$, $\beta = 2$.

21. The indicial equation is $\lambda^2 + (\alpha - 1)\lambda + \beta = 0$. In order to have $\lambda_1 = 1 + 2i$ and $\lambda_2 = 1 - 2i$, we need $(\lambda - \lambda_1)(\lambda - \lambda_2) = \lambda^2 - (\lambda_1 + \lambda_2)\lambda + \lambda_1\lambda_2 = \lambda^2 - 2\lambda + 5$. Therefore, $\alpha = -1$ and $\beta = 5$.

22. The indicial equation is $\lambda(\lambda - 1) + \alpha\lambda + 2 = 0$ has $\lambda = 2$ as a root. Therefore, $2(1) + 2\alpha + 2 = 0 \Rightarrow \alpha = -2$. Therefore,

$$t^2 y'' - 2ty' + (2 + \beta t)y = \sum_{n=0}^{\infty}\{(\lambda + n)(\lambda + n - 1) - 2(\lambda + n) + 2\}a_n t^{n+\lambda} + \beta\sum_{n=1}^{\infty} a_{n-1} t^{n+\lambda} = 0$$

$$\Rightarrow [\lambda(\lambda - 1) - 2\lambda + 2]a_0 t^{\lambda} + \sum_{n=1}^{\infty}\{[(\lambda + n)^2 - 3(\lambda + n) + 2]a_n + \beta a_{n-1}\}t^{n+\lambda} = 0.$$

For $\lambda = 2$, the recurrence relation becomes $\left[(n+2)^2 - 3(n+2) + 2\right]a_n + \beta a_{n-1} = 0,\ n = 1,2,...$

Therefore, $\left[n^2 + 4n + 4 - 3n - 6 + 2\right]a_n + \beta a_{n-1} = (n^2 + n)a_n + \beta a_{n-1} = 0 \Rightarrow \beta = -4$.

23. The indicial equation is $\lambda^2 = 0$ and the corresponding recurrence relation is $(n+1)^2 a_{n+1} + \alpha n a_n + \beta a_{n-1} = 0$. Therefore, $\alpha = -1$ and $\beta = 3$.

24 (a). $p(t)$ is odd and $q(t)$ is even, so we expect even and odd solutions.

24 (b). The indicial equation is $\lambda(\lambda - 1) + \lambda - \upsilon^2 = 0$ or $F(\lambda) = \lambda^2 - \upsilon^2 \Rightarrow \lambda_1 = -\upsilon,\ \lambda_2 = \upsilon$.
For the Bessel equation, $\lambda(\lambda - 1) + \lambda - \upsilon^2 = 0$ or $F(\lambda) = \lambda^2 - \upsilon^2$.
The indicial equation and exponents at the singularity are the same for both equations.

24 (c). $$[\lambda^2 - \upsilon^2]a_0 t^{\lambda} + [(\lambda + 1)^2 - \upsilon^2]a_1 t^{\lambda - 1} + \sum_{n=2}^{\infty}\{[(\lambda + n)^2 - \upsilon^2]a_n - a_{n-2}\}t^{n+\lambda} = 0$$

$$\Rightarrow a_n = \frac{a_{n-2}}{(\lambda + n)^2 - \upsilon^2},\ n = 2,3,...$$

For Bessel's equation, $a_n = \dfrac{-a_{n-2}}{(\lambda + n)^2 - \upsilon^2}$, $n = 2,3,...$ The minus sign creates a "term-to-term" change of sign in the series solution. This sign alteration is not present in the series solutions of the modified Bessel equation.

Chapter 11
Second Order Partial Differential Equations and Fourier Series

Chapter 11 Introduction

1. Differentiation gives us $u_t = 4,\ u_x = -2\alpha x,\ u_{xx} = -2\alpha$. Substitution yields $u_t - u_{xx} = 4 + 2\alpha = 0$, and thus $\alpha = -2$.
2. $u_t = -\alpha e^{-\alpha t}\sin 2x,\ u_x = 2e^{-\alpha t}\cos 2x,\ u_{xx} = -4e^{-\alpha t}\sin 2x$. $u_t - u_{xx} = -\alpha e^{-\alpha t}\sin 2x + 4e^{-\alpha t}\sin 2x = 0$. $\alpha = 4$.
3. Differentiation gives us $u_t = -2\alpha e^{-2\alpha t}\cos\alpha x,\ u_x = -\alpha e^{-2\alpha t}\sin\alpha x,\ u_{xx} = -\alpha^2 e^{-2\alpha t}\cos\alpha x$. Substitution yields $u_t - u_{xx} = \left(-2\alpha + \alpha^2\right)e^{-2\alpha t}\cos\alpha x = 0$, and thus $\alpha = 0,2$.
4. $u_t = \alpha\cos(x + \alpha t),\ u_{tt} = -\alpha^2\sin(x + \alpha t)\ \ u_x = \cos(x + \alpha t),\ u_{xx} = -\sin(x + \alpha t)$. $u_t - 4u_{xx} = -\left(\alpha^2 + 4\right)\sin(x + \alpha t) = 0$. $\alpha = \pm 2$.
5. Differentiation gives us $u_t = -2\alpha\sin(x + \alpha t),\ u_{tt} = -2\alpha^2\cos(x + \alpha t),\ u_x = -2\sin(x + \alpha t)$, $u_{xx} = -2\cos(x + \alpha t)$. Substitution yields $u_{tt} - u_{xx} + 2u = \left(-2\alpha^2 + 2 + 4\right)\cos(x + \alpha t) = 0$, and thus $\alpha = \pm\sqrt{3}$.
6. $u_{xx} = e^x\sin\alpha y,\ u_{yy} = -\alpha^2 e^x\sin\alpha y$. $u_{xx} + u_{yy} = \left(1 - \alpha^2\right)e^x\sin\alpha y = 0$. $\alpha = \pm 1$.
7. Differentiation gives us $u_{xx} = e^{x+\alpha y}\sin z,\ u_{yy} = \alpha^2 e^{x+\alpha y}\sin z,\ u_{zz} = -e^{x+\alpha y}\sin z$. Substitution yields $u_{xx} + u_{yy} + u_{zz}\left(1 + \alpha^2 - 1\right)e^{x+\alpha y}\sin z = 0$, and thus $\alpha = 0$.
8. $u_{xx} + u_{yy} - u_t = (-1 - 4 - \alpha)e^{\alpha t}\sin x\cos 2y = 0$. $\alpha = -5$.
9. Differentiation gives us $u_{xx} = -\alpha^2\sin\alpha x\cos 2t,\ u_{tt} = -4\sin\alpha x\cos 2t$. Substitution yields $u_{xx} - u_{tt} - 4\alpha u = \left(-\alpha^2 - 4 - 4\alpha\right)\sin\alpha x\cos 2t$, and thus $\alpha = -2$ or $\alpha = 0$.
10. $u_{xx} + u_{yy} - 2u - 4 = e^{-x}\cos y - e^{-x}\cos y - 2\alpha - 2e^{-x}\cos y - 4 \neq 0$. There is no possible choice for α.

11 (b). $u(x,0) = c_1\sin x + c_2\sin 2x = 3\sin 2x - \sin x$. Therefore, $c_1 = -1,\ c_2 = 3$.
12 (b). $u(x,0) = c_1 + c_2\cos x + c_3\cos 2x = 2 - \cos 2x$. $c_1 = 2,\ c_2 = 0,\ c_3 = -1$.
13 (b). $u(x,0) = c_2\sin x = -2\sin x,\ u_t(x,0) = 2c_1\sin x = 6\sin x$. Therefore, $c_1 = 3,\ c_2 = -2$.
14 (b). $u(x,0) = c_1\sin x + c_2\sin 2x = \sin x - 4\sin 2x$. $c_1 = 1,\ c_2 = -4$.
15 (b). $u(x,0) = c_1 + \left(c_2 + c_3\right)x = 1 + 2x,\ u_t(x,0) = -c_2 + c_3 = 0$. Solving these simultaneous equations yields $c_1 = c_2 = c_3 = 1$.

18. $4u_{xx} - u_{tt} = 4\left(-2\pi^2\sin\pi x\cos 2\pi t - 2t\right) = -8t = f(x,t)$.
19. $u_{xx} + u_{yy} = 4y^3 + 2\sinh x\sin y + 12x^2 y - 2\sinh x\sin y = 12x^2 y + 4y^3 = f(x,y)$.
20. $u_{xx} + u_{yy} - u_t = e^{x+2y} - e^{-5t}\sin x\sin 2y + 4e^{x+2y} - 4e^{-5t}\sin x\sin 2y - \left(-5e^{-5t}\sin x\sin 2y\right)$ $= 5e^{x+2y} = f(x,y,t)$.
21. $u_{xx} + u_{yy} = -2y^3 - 6x^2 y = f(x,y)$.

Section 11.1

2. $u(x,t_0)=x^2(2-x),\ u_x(x,t_0)=2x(2-x)-x^2=-3x^2+4x,\ u_{xx}(x,t_0)=-6x+4$. On $0<x<2,\ \kappa u_{xx}>0$ on $\left(0,\frac{2}{3}\right),\ u_{xx}=0$ at $x=\frac{2}{3},\ u_{xx}<0$ on $\frac{2}{3}<x<2$.

2 (a). $u_t(x,t_0)<0$ on $\frac{2}{3}<x<2$

2 (b). $u_t(x,t_0)=0$ at $\frac{2}{3}=x$

2 (c). $u_t(x,t_0)>0$ on $0<x<\frac{2}{3}$

3 (c). Noting that $\kappa=3$ and $u\to 5u$, we obtain $u(x,t)=5e^{-3\pi^2 t}\sin\pi x$.

4 (b). $\kappa=2$. $c_1+c_2\cos\pi x=3-\cos\pi x\Rightarrow u=3-e^{-2\pi^2 t}\cos\pi x$.

6 (a). (i) $u(x)=c_1x+c_2,\ u(0)=c_2=0,\ u(\ell)=c_1\ell+c_2=c_1\ell=0\Rightarrow c_1=0\Rightarrow u(x)=0$.
(ii) $u(x)=c_1x+c_2,\ u_x=c_1,\ u'(0)=u'(\ell)=0\Rightarrow c_1=0\Rightarrow u(x)=c_2$.

6 (b). The conjectured forms of the limiting distributions agree with the forms of the equilibrium solutions.

7 (a). If $u=e^{\delta t}w$, then we have $u_t=\delta e^{\delta t}w+e^{\delta t}w_t$. Since $u_{xx}=e^{\delta t}w_{xx}$, we have $\delta e^{\delta t}w+e^{\delta t}w_t=\kappa w_{xx}+\alpha e^{\delta t}w$ which leads us to $w_t=\kappa w_{xx}+(\alpha-\delta)w=\kappa w_{xx}$ if $\delta=\alpha$.

7 (c). $\alpha=4$, so $u(x,t)=e^{4t}e^{-4\pi^2 t}\cos 2\pi x=e^{-4(\pi^2-1)t}\cos 2\pi x$.

8. $XT'=\kappa X''T\Rightarrow\dfrac{T'}{\kappa T}=\dfrac{X''}{X}=\sigma\Rightarrow X''-\sigma X=0,\ T'-\sigma\kappa T=0$.
$u_x(0,t)=X'(0)T(t)=0$ and $u_x(\ell,t)=X'(\ell)T(t)=0\Rightarrow X'(0)=X'(\ell)=0$.

9 (a). $u_t(x,t)=\kappa u_{xx}(x,t),\ 0<x<\ell,\ 0<t<\infty;\ u(0,t)=0,\ u_x(l,t)=0,\ t\ge 0;\ u(x,0)=f(x),\ 0\le x\le\ell$.

9 (b). $X''-\sigma X=0,\ 0<x<\ell,\ X(0)=0,\ X'(\ell)=0;\ T'-\sigma\kappa T=0,\ t>0$.

10. $u_t=u_{xx}+u_x;\ u=XT$. $XT'=X''T+X'T$. $\dfrac{T'}{T}=\dfrac{X''+X'}{X}\equiv\sigma$.
$X''+X'-\sigma X=0$ and $T'-\sigma T=0$.

11. $u_t=u_{xx}+x^2u;\ u=XT$. Substitution and differentiation yields $XT'=X''T+x^2XT$. Division by XT gives us $\dfrac{T'}{T}=\dfrac{X''+x^2X}{X}\equiv\sigma$. Thus the appropriate separation equations are $X''+(x^2-\sigma)X=0$ and $T'-\sigma T=0$.

12. $u_t=(1+t^2)(1+x^2)u_{xx};\ u=XT$. $XT'=(1+t^2)(1+x^2)X''T$. $\dfrac{T'}{(1+t^2)T}=\dfrac{X''(1+x^2)}{X}\equiv\sigma$.
$(1+x^2)X''-\sigma X=0$ and $T'-\sigma(1+t^2)T=0$.

13. $u_{tt}=c^2u_{xx};\ u=XT$. Substitution and differentiation yields $XT''=c^2X''T$. Division by c^2XT gives us $\dfrac{T''}{c^2T}=\dfrac{X''}{X}\equiv\sigma$. Thus the appropriate separation equations are $X''-\sigma X=0$ and $T''-\sigma c^2T=0$.

14. $u_{tt}-u_t=u_{xx};\ u=XT$. $XT''-XT'=X''T$. $\dfrac{T''-T'}{T}=\dfrac{X''}{X}\equiv\sigma$.
$X''-\sigma X=0$ and $T''-T'-\sigma T=0$.

15. $u_{tt} = u_{xx} + xu_x$; $u = XT$. Substitution and differentiation yields $XT'' = X''T + xX'T$. Division by XT gives us $\frac{T''}{T} = \frac{X'' + xX'}{X} \equiv \sigma$. Thus the appropriate separation equations are $X'' + xX' - \sigma X = 0$ and $T'' - \sigma T = 0$.

16. $u_{xx} + u_{yy} = 0$; $u = XY$. $X''Y + XY'' = 0$. $\frac{-Y''}{Y} = \frac{X''}{X} \equiv \sigma$. $X'' - \sigma X = 0$ and $Y'' + \sigma Y = 0$.

17. $u_{xx} + e^{x+y}u_{yy} = 0$; $u = XY$. Substitution and differentiation yields $X''Y + e^x e^y XY'' = 0$. Division by $e^x XY$ and subtraction gives us $\frac{X''}{e^x X} = -\frac{e^y Y''}{Y} \equiv \sigma$. Thus the appropriate separation equations are $X'' - \sigma e^x X = 0$ and $e^y Y'' + \sigma Y = 0$.

18. $u_{xx} + e^{xy}u_{yy} = 0$; $u = XY$. $X''Y + e^{xy}XY'' = 0$. The variables cannot be separated.

19. $u_t = r^{-1}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)$; $u = RT$. Substitution and differentiation yields $RT' = r^{-1}(rR')'T$. Division by RT gives us $\frac{T'}{T} = \frac{1}{r}\left(\frac{(rR')'}{R}\right) \equiv \sigma$. Thus the appropriate separation equations are

$r^{-1}(rR')' - \sigma R = 0$ and $T' - \sigma T = 0$.

20. $u = R\theta$. $(rR')'\theta + \frac{1}{r}R\theta'' = 0$. $\frac{r(rR')'}{R} = \frac{-\theta''}{\theta} \equiv \sigma$. $(rR')' - \frac{\sigma}{r}R = 0$ and $\theta'' + \sigma\theta = 0$.

21 (a). $T' - \sigma T = 0$.

21 (b). We manipulate the given equation to read $\frac{X''}{X} = \sigma - \frac{Y''}{Y} \equiv \eta$. The resulting separation equations are then $X'' - \eta X = 0$ and $Y'' + (\eta - \sigma)Y = 0$.

22. $u_{xx} + u_{yy} + u_{zz} = 0$; $u = XYZ$. $X''YZ + XY''Z + XYZ'' = 0$. $-\frac{X''}{X} - \frac{Y''}{Y} = \frac{Z''}{Z} \equiv \sigma$. $Z'' - \sigma Z = 0$. $\frac{X''}{X} = -\frac{Y''}{Y} - \sigma = \eta$, $X'' - \eta X = 0$ and $Y'' + (\sigma + \eta)Y = 0$.

Section 11.2

1. From (13), $u(x,t) = \sum_{n=1}^{\infty} a_n e^{-(n\pi/\ell)^2 t}\sin\left(\frac{n\pi x}{\ell}\right)$. For $t = 0$, we have $f(x) = \sin\left(\frac{2\pi x}{\ell}\right) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{\ell}\right)$. Thus $a_n = 1$, $n = 2$; $a_n = 0$, $n \neq 2$, and so $u = e^{-(2\pi/\ell)^2 t}\sin\left(\frac{2\pi x}{\ell}\right)$.

2. $f(x) = 3\sin\left(\frac{3\pi x}{\ell}\right) = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{\ell}\right)$. $a_3 = 3$; $a_n = 0$, $n \neq 3$, and so $u = 3e^{-(3\pi/\ell)^2 t}\sin\left(\frac{3\pi x}{\ell}\right)$.

3. From (13), $u(x,t)=\sum_{n=1}^{\infty} a_n e^{-(n\pi/\ell)^2 t}\sin\left(\frac{n\pi x}{\ell}\right)$. For $t=0$, we have $f(x)=\sin\left(\frac{\pi x}{\ell}\right)-2\sin\left(\frac{2\pi x}{\ell}\right)=\sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{\ell}\right)$. Thus $a_1=1,\ a_2=-2;\ a_n=0,\ n>2$, and so $u=e^{-(\pi/\ell)^2 t}\sin\left(\frac{\pi x}{\ell}\right)-2e^{-(2\pi/\ell)^2 t}\sin\left(\frac{2\pi x}{\ell}\right)$.

4. $f(x)=\sin x+4\sin 2x=\sum_{n=1}^{\infty} a_n \sin(nx)$. $a_1=1,\ a_2=4;\ a_n=0,\ n>2$, and so $u=e^{-t}\sin x+4e^{-4t}\sin 2x$.

5. First, we note that $4\sin\left(\frac{\pi x}{\ell}\right)\cos\left(\frac{\pi x}{\ell}\right)=2\sin\left(\frac{2\pi x}{\ell}\right)$. From (13), $u(x,t)=\sum_{n=1}^{\infty} a_n e^{-(n\pi/\ell)^2 t}\sin\left(\frac{n\pi x}{\ell}\right)$. For $t=0$, we have $f(x)=2\sin\left(\frac{2\pi x}{\ell}\right)=\sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{\ell}\right)$. Thus $a_2=2;\ a_n=0,\ n\neq 2$, and so $u=2e^{-(2\pi/\ell)^2 t}\sin\left(\frac{2\pi x}{\ell}\right)$.

6. $f(x)=2\sin 3\pi x\cos \pi x=\sin 4\pi x+\sin 2\pi x=\sum_{n=1}^{\infty} a_n \sin(n\pi x)$. $a_2=a_4=1;\ a_n=0,\ n\neq 2,4$, and so $u=e^{-(2\pi)^2 t}\sin 2\pi x+e^{-(4\pi)^2 t}\sin 4\pi x$.

7. From (13), $u(x,t)=\sum_{n=1}^{\infty} a_n e^{-(n\pi/\ell)^2 t}\sin\left(\frac{n\pi x}{\ell}\right)$. For $t=0$, we have $f(x)=\sum_{n=1}^{3} n^{-1}\sin(n\pi x)=\sum_{n=1}^{\infty} a_n \sin(n\pi x)$. Thus $a_1=1,\ a_2=\frac{1}{2},\ a_3=\frac{1}{3};\ a_n=0,\ n>3$, and so $u=e^{-(\pi)^2 t}\sin(\pi x)+\frac{1}{2}e^{-(2\pi)^2 t}\sin(2\pi x)+\frac{1}{3}e^{-(3\pi)^2 t}\sin 3\pi x$.

8. $f(x)=\sin^3 \pi x=\frac{3}{4}\sin \pi x-\frac{1}{4}\sin 3\pi x=\sum_{n=1}^{\infty} a_n \sin(n\pi x)$. $a_1=\frac{3}{4},\ a_3=-\frac{1}{4};\ a_n=0,\ n\neq 1,3$, and so $u=\frac{3}{4}e^{-(\pi)^2 t}\sin \pi x-\frac{1}{4}e^{-(3\pi)^2 t}\sin 3\pi x$.

9. From (14), $u(x,t)=\sum_{n=0}^{\infty} a_n e^{-(n\pi/\ell)^2 t}\cos\left(\frac{n\pi x}{\ell}\right)$. For $t=0$, we have $f(x)=3+2\cos\left(\frac{\pi x}{\ell}\right)=\sum_{n=0}^{\infty} a_n \cos\left(\frac{n\pi x}{\ell}\right)$. Thus $a_0=3,\ a_1=2;\ a_n=0,\ n>1$, and so $u=3+2e^{-(\pi/\ell)^2 t}\cos\left(\frac{\pi x}{\ell}\right)$.

10. $f(x)=4-\cos\left(\frac{\pi x}{\ell}\right)+2\cos\left(\frac{2\pi x}{\ell}\right)=\sum_{n=0}^{\infty} a_n \cos\left(\frac{n\pi x}{\ell}\right)$. $a_0=4,\ a_1=-1,\ a_2=2;\ a_n=0,\ n>2$. $u=4-e^{-(\pi/\ell)^2 t}\cos\left(\frac{\pi x}{\ell}\right)+2e^{-(2\pi/\ell)^2 t}\cos\left(\frac{2\pi x}{\ell}\right)$.

11. From (14), $u(x,t)=\sum_{n=0}^{\infty} a_n e^{-(n\pi/\ell)^2 t}\cos\left(\frac{n\pi x}{\ell}\right)$. For $t=0$, we have $f(x)=\cos\left(\frac{\pi x}{2}\right)+2\cos(\pi x)=\sum_{n=0}^{\infty} a_n \cos\left(\frac{n\pi x}{2}\right)$. Thus $a_0=0,\ a_1=1,\ a_2=2;\ a_n=0,\ n>2$, and so $u=e^{-(\pi/2)^2 t}\cos\left(\frac{\pi x}{2}\right)+2e^{-\pi^2 t}\cos(\pi x)$.

12. $f(x)=3\cos x=\sum_{n=0}^{\infty} a_n \cos nx$. $a_1=3;\ a_n=0,\ n\neq 1$. $u=3e^{-t}\cos x$.

13. First, we note that $\cos\pi x+\cos^2\pi x=\frac{1}{2}+\cos\pi x+\frac{1}{2}\cos 2\pi x$. From (14), $u(x,t)=\sum_{n=0}^{\infty} a_n e^{-(n\pi/\ell)^2 t}\cos(n\pi x)$. For $t=0$, we have $f(x)=\frac{1}{2}+\cos(\pi x)+\frac{1}{2}\cos(2\pi x)=\sum_{n=0}^{\infty} a_n \cos(n\pi x)$. Thus $a_0=\frac{1}{2},\ a_1=1,\ a_2=\frac{1}{2};\ a_n=0,\ n>2$, and so $u=\frac{1}{2}+e^{-\pi^2 t}\cos(\pi x)+\frac{1}{2}e^{-(2\pi)^2 t}\cos(2\pi x)$.

14. $f(x)=2-\sin^2\pi x=\frac{3}{2}+\frac{1}{2}\cos 2\pi x=\sum_{n=0}^{\infty} a_n \cos\left(\frac{n\pi x}{2}\right)$. $a_0=\frac{3}{2},\ a_4=\frac{1}{2};\ a_n=0,\ n\neq 0,4$. $u=\frac{3}{2}+\frac{1}{2}e^{-(2\pi)^2 t}\cos 2\pi x$.

15. From (14), $u(x,t)=\sum_{n=0}^{\infty} a_n e^{-(n\pi/\ell)^2 t}\cos\left(\frac{n\pi x}{\ell}\right)$. For $t=0$, we have $f(x)=\frac{1}{2}+\sum_{n=1}^{3}\cos\left(\frac{n\pi x}{\ell}\right)=\sum_{n=0}^{\infty} a_n \cos\left(\frac{n\pi x}{\ell}\right)$. Thus $a_0=\frac{1}{2},\ a_1=a_2=a_3=1;\ a_n=0,\ n>3$, and so $u=\frac{1}{2}+\sum_{n=1}^{3} e^{-\left(\frac{n\pi}{\ell}\right)^2 t}\cos\left(\frac{n\pi x}{\ell}\right)$.

16. $f(x)=2\cos^3\left(\frac{\pi x}{\ell}\right)=\frac{3}{2}\cos\left(\frac{\pi x}{\ell}\right)+\frac{1}{2}\cos\left(\frac{3\pi x}{\ell}\right)=\sum_{n=0}^{\infty} a_n \cos\left(\frac{n\pi x}{\ell}\right)$.
$a_1=\frac{3}{2},\ a_3=\frac{1}{2};\ a_n=0,\ n\neq 1,3$. $u=\frac{3}{2}e^{-(\pi/\ell)^2 t}\cos\left(\frac{\pi x}{\ell}\right)+\frac{1}{2}e^{-(3\pi/\ell)^2 t}\cos\left(\frac{3\pi x}{\ell}\right)$.

17 (a). From the given conditions, we have $u(x,t)=100e^{-(\pi/2)^2 \kappa t}\sin\left(\frac{\pi x}{2}\right)$. Imposing the condition $u(1,1)=70=100e^{-(\pi/2)^2 \kappa}$ and solving for κ gives us $\kappa=\frac{4}{\pi^2}\ln\frac{10}{7}\approx 0.14455$.

17 (b). Differentiation gives us $u_x=100e^{-(\pi/2)^2 \kappa t}\cdot\frac{\pi}{2}\cos\left(\frac{\pi x}{2}\right)$, and substitution yields $u_x(2,1)=-70\left(\frac{\pi}{2}\right)\approx -109.95$.

18 (a). $u(x,t)=10+100e^{-(\pi/4)^2 \kappa t}\cos\left(\frac{\pi x}{4}\right)$. $u(1,1)=60=10+100e^{-(\pi/4)^2 \kappa}\cos\left(\frac{\pi}{4}\right)$. $\kappa=\frac{8}{\pi^2}\ln 2\approx 0.5618$.

18 (b). $u(0,1)=80.71$.

18 (c). $u(0,t)=10+100e^{-(\pi/4)^2 \kappa t}=40$. $t=3.4739$

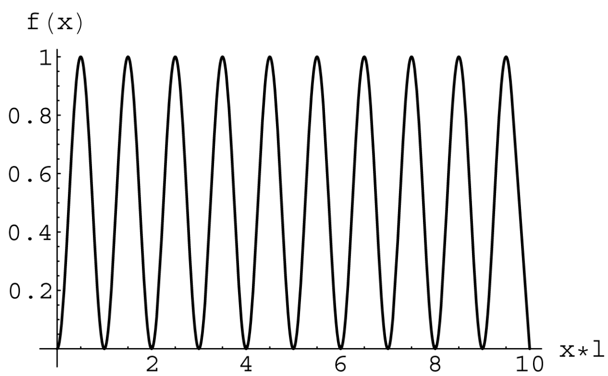
19 (d). Solving the equation given in part (c) for σ_n, we have $\sigma_n = -\left(\left(n-\frac{1}{2}\right)\frac{\pi}{\ell}\right)^2$. Thus the general solution of $X'' - \sigma_n X = 0$ is $X = A_n \cos\left(\left(n-\frac{1}{2}\right)\frac{\pi}{\ell}x\right) + B_n \sin\left(\left(n-\frac{1}{2}\right)\frac{\pi}{\ell}x\right)$. Since $X(0) = A_n = 0$ and $X'(\ell) = \left(n-\frac{1}{2}\right)\frac{\pi}{\ell}B_n \cos\left(\left(n-\frac{1}{2}\right)\frac{\pi}{\ell}\right) = 0$ for all B_n, we have $X_n(x) = \sin\left(\left(n-\frac{1}{2}\right)\frac{\pi}{\ell}x\right)$.

19 (e). $T_n' + \kappa\left(\left(n-\frac{1}{2}\right)\frac{\pi}{\ell}\right)^2 T_n = 0$ leads us to $T_n = e^{-\left(\left(n-\frac{1}{2}\right)\frac{\pi}{\ell}\right)^2 \kappa t}$, $n = 1,2,3....$ Therefore,

$U_n = X_n T_n = e^{-\left(\left(n-\frac{1}{2}\right)\frac{\pi}{\ell}\right)^2 \kappa t} \sin\left(\left(n-\frac{1}{2}\right)\frac{\pi x}{\ell}\right)$, $n = 1,2,3....$

Section 11.3

1 (a).



First, we note that $\sin^2\left(\frac{\pi x}{\ell}\right) = \frac{1}{2} - \frac{1}{2}\cos\left(\frac{2\pi x}{\ell}\right)$. From Equation (7), we have

$$a_n = \frac{2}{\ell}\int_0^\ell f(x)\sin\left(\frac{n\pi x}{\ell}\right)dx = \frac{1}{\ell}\int_0^\ell \sin\left(\frac{n\pi x}{\ell}\right)dx - \frac{1}{\ell}\int_0^\ell \sin\left(\frac{n\pi x}{\ell}\right)\cos\left(\frac{2\pi x}{\ell}\right)dx$$

$$= -\left.\frac{\cos\left(\frac{n\pi x}{\ell}\right)}{n\pi}\right|_0^\ell - \frac{1}{2\ell}\int_0^\ell \left[\sin\left(\frac{(n+2)\pi x}{\ell}\right) + \sin\left(\frac{(n-2)\pi x}{\ell}\right)\right]dx$$

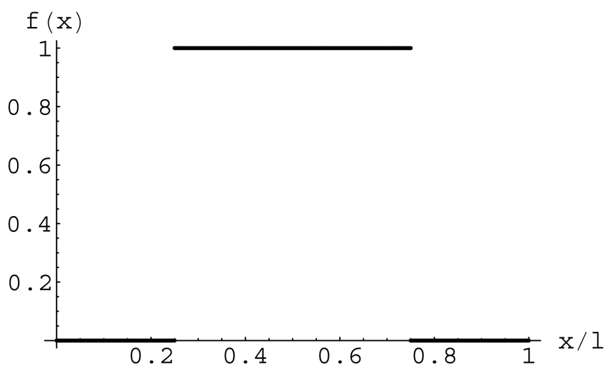
$$= \frac{1}{n\pi}\left(1-(-1)^n\right) + \frac{1}{2\ell}\begin{cases} \left.\left(\dfrac{\cos\left(\frac{(n+2)\pi x}{\ell}\right)}{(n+2)\pi/\ell} + \dfrac{\cos\left(\frac{(n-2)\pi x}{\ell}\right)}{(n-2)\pi/\ell}\right)\right|_0^\ell, & n \neq 2 \\ \left.\dfrac{\cos\left(\frac{4\pi x}{\ell}\right)}{4\pi/\ell}\right|_0^\ell, & n = 2 \end{cases}$$. Noting that

$(-1)^{n+2} - 1 = (-1)^{n-2} - 1 = 0$ for all even n, we find that $a_n = 0$, n even and

$a_n = \dfrac{2}{n\pi} - \left(\dfrac{1}{(n+2)\pi} + \dfrac{1}{(n-2)\pi}\right) = -\dfrac{8}{n(n^2-4)\pi}$, n odd .

1 (b). $$u(x,t) = -\frac{8}{\pi}\sum_{m=1}^{\infty}\frac{e^{-((2m-1)\pi/\ell)^2\kappa t}\sin((2m-1)\pi x/\ell)}{(2m-1)((2m-1)^2-4)}$$

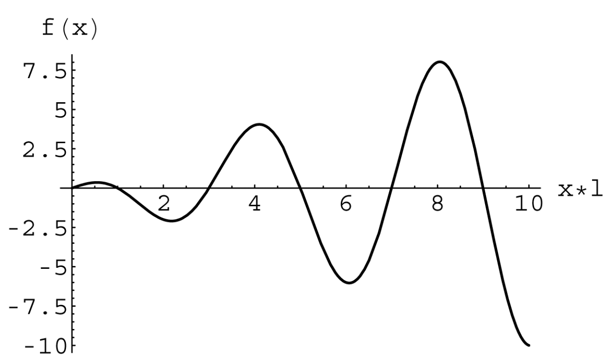
2 (a).



$$a_n = \frac{2}{\ell}\int_{\ell/4}^{3\ell/4}\sin\left(\frac{n\pi x}{\ell}\right)dx = \begin{cases} \dfrac{4}{n\pi}\cos\left(\dfrac{n\pi}{4}\right), & n \text{ odd} \\ 0, & n \text{ even} \end{cases}.$$

2 (b). $$u(x,t) = \frac{4}{\pi}\sum_{m=1}^{\infty}\frac{e^{-((2m-1)\pi/\ell)^2\kappa t}\cos((2m-1)\pi/4)\sin((2m-1)\pi x/\ell)}{(2m-1)}$$

3 (a).



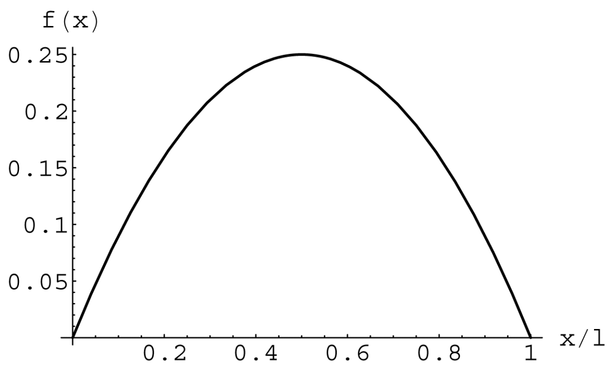
First, we note that $x\cos\left(\frac{\pi x}{2\ell}\right)\sin\left(\frac{n\pi x}{\ell}\right)=\frac{x}{2}\left(\sin\left(\frac{(n+1/2)\pi x}{\ell}\right)+\sin\left(\frac{(n-1/2)\pi x}{\ell}\right)\right)$. Next, let us note that $\int_0^\ell x\sin\alpha x\,dx=-\frac{\ell}{\alpha}\cos\alpha\ell+\frac{\sin\alpha\ell}{\alpha^2}$ (use integration by parts for this integral). Then, from Equation (7), we have

$$a_n=\frac{2}{\ell}\cdot\frac{1}{2}\left\{-\frac{1}{(n+1/2)\pi}\cos((n+1/2)\pi)+\frac{\sin((n+1/2)\pi)\ell^2}{(n+1/2)^2\pi^2}-\frac{1}{(n-1/2)\pi}\cos\left(\left(n-\frac{1}{2}\right)\pi\right)+\frac{\sin((n-1/2)\pi)\ell^2}{(n-1/2)^2\pi^2}\right\}.$$

Simplifying this expression for a_n, we have $a_n=\frac{2\ell}{\pi^2}\cdot\frac{n\sin((n-1/2)\pi)}{(n^2-1/4)^2}=\frac{2\ell}{\pi^2}\cdot\frac{n(-1)^{n+1}}{(n^2-1/4)^2}$.

3 (b). $u(x,t)=\frac{2\ell}{\pi^2}\sum_{n=1}^{\infty}\frac{n(-1)^{n+1}}{(n^2-1/4)^2}e^{-(n\pi/\ell)^2\kappa t}\sin\left(\frac{n\pi x}{\ell}\right)$

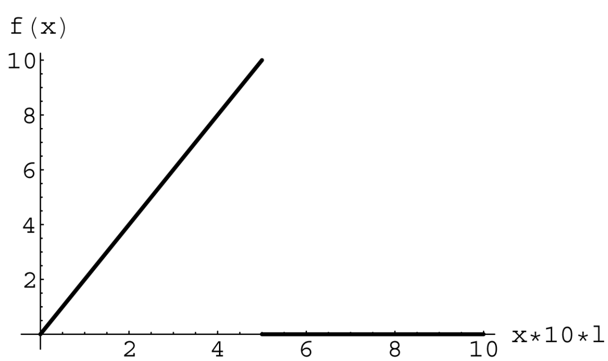
4 (a).



$$a_n=\frac{2}{\ell}\int_0^\ell x(\ell-x)\sin\left(\frac{n\pi x}{\ell}\right)dx=\begin{cases}\frac{8\ell^2}{(n\pi)^3}, & n \text{ odd}\\ 0, & n \text{ even}\end{cases}.$$

4 (b). $u(x,t)=\frac{8\ell^2}{\pi^3}\sum_{m=1}^{\infty}\frac{e^{-((2m-1)\pi/\ell)^2\kappa t}\sin((2m-1)\pi x/\ell)}{(2m-1)^3}$

5 (a).

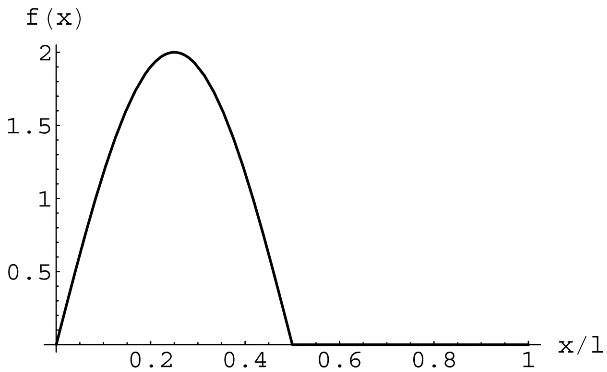


$a_n=\frac{4}{\ell}\int_0^{\ell/2}x\sin\left(\frac{n\pi x}{\ell}\right)dx$. Integration by parts and some simplification gives us

$$a_n=-\frac{2\ell}{n\pi}\cos\left(\frac{n\pi}{2}\right)+\frac{4\ell}{(n\pi)^2}\sin\left(\frac{n\pi}{2}\right).$$

5 (b). $u(x,t)=\sum_{n=1}^{\infty}\left[\frac{-2\ell}{n\pi}\cos\left(\frac{n\pi}{2}\right)+\frac{4\ell}{(n\pi)^2}\sin\left(\frac{n\pi}{2}\right)\right]e^{-\left(n\pi/\ell\right)^2\kappa t}\sin\left(\frac{n\pi x}{\ell}\right)$

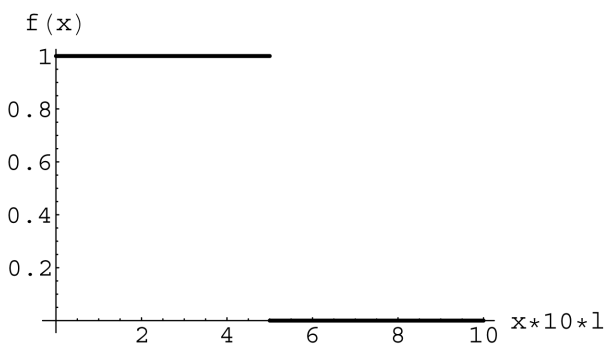
6 (a).



$$a_n=\frac{4}{\ell}\int_0^{\ell/2}\sin\left(\frac{2\pi x}{\ell}\right)\sin\left(\frac{n\pi x}{\ell}\right)dx=\begin{cases}1, & n=2\\ 0, & n \text{ even and } n>2\\ \frac{8}{\pi}(-1)^{m+1}\frac{1}{(2m-1)^2-4}, & n=2m-1\end{cases}.$$

6 (b). $u(x,t)=e^{-(2\pi/\ell)^2\kappa t}\sin\left(\frac{2\pi x}{\ell}\right)-\frac{8}{\pi}\sum_{m=1}^{\infty}\frac{(-1)^{m+1}e^{-((2m-1)\pi/\ell)^2\kappa t}\sin((2m-1)\pi x/\ell)}{(2m-1)^2-4}$

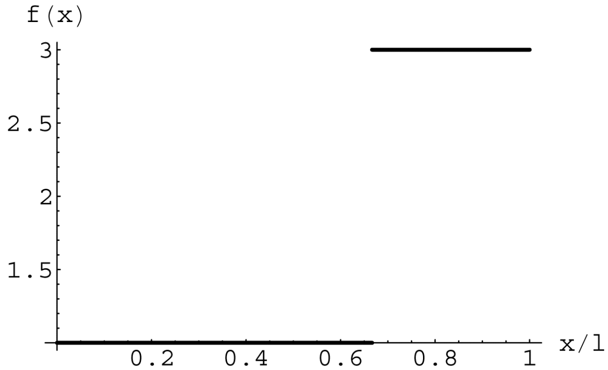
7 (a).



$$a_0=\frac{2}{\ell}\int_0^{\ell/2}dx=1,\ a_n=\frac{2}{\ell}\int_0^{\ell/2}\cos\left(\frac{n\pi x}{\ell}\right)dx=\frac{2}{n\pi}\sin\left(\frac{n\pi}{2}\right)=\begin{cases}0, & n=2m\\ \frac{2(-1)^{m+1}}{(2m-1)\pi}, & n=2m-1\end{cases}.$$

7 (b). $u(x,t)=\frac{1}{2}+\frac{2}{\pi}\sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{2m-1}e^{-\left((2m-1)\pi/\ell\right)^2\kappa t}\cos\left(\frac{(2m-1)\pi x}{\ell}\right)$

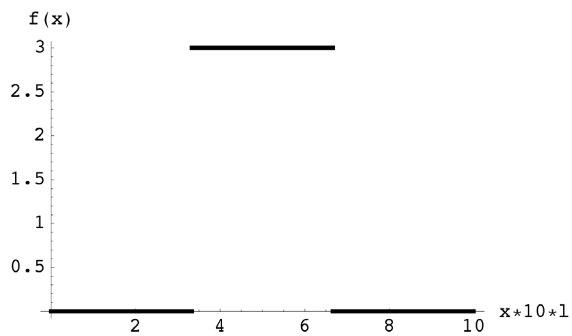
8 (a).



$$a_0 = \frac{2}{\ell}\left(\frac{2\ell}{3} + 3\frac{\ell}{3}\right) = \frac{10}{3},\ a_n = \frac{2}{\ell}\left(\int_0^{\ell}\cos\left(\frac{n\pi x}{\ell}\right)dx + 2\int_{2\ell/3}^{\ell}\cos\left(\frac{n\pi x}{\ell}\right)dx\right) = -\frac{4}{n\pi}\sin\left(\frac{2n\pi}{3}\right).$$

8 (b). $$u(x,t) = \frac{5}{3} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\sin\left(2n\pi/3\right)}{n}e^{-\left(n\pi/\ell\right)^2\kappa t}\cos\left(\frac{n\pi x}{\ell}\right)$$

9 (a).



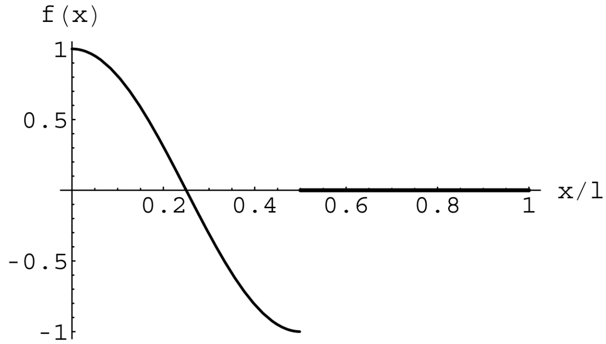
$$a_0 = \frac{2}{\ell}\int_0^{\ell/3}3dx = 2,\ a_n = \frac{2}{\ell}\cdot 3\int_{\ell/3}^{2\ell/3}\cos\left(\frac{n\pi x}{\ell}\right)dx = \frac{6}{n\pi}\left[\sin\left(\frac{2n\pi}{3}\right) - \sin\left(\frac{n\pi}{3}\right)\right].$$ Noting that

$$\sin\left(\frac{2n\pi}{3}\right) = \sin\left(n\pi - \frac{n\pi}{3}\right) = \sin n\pi\cos\left(\frac{n\pi}{3}\right) - \cos n\pi\sin\left(\frac{n\pi}{3}\right) = (-1)^{n+1}\sin\left(\frac{n\pi}{3}\right),$$ we have

$$a_n = \begin{cases} 0,\ n = 2m-1 \\ -\dfrac{6}{m\pi}\sin\left(\dfrac{2m\pi}{3}\right),\ n = 2m \end{cases}.$$

9 (b). $$u(x,t) = 1 - \frac{6}{\pi}\sum_{m=1}^{\infty}\frac{\sin\left(\dfrac{2m\pi}{3}\right)}{m}e^{-\left(2m\pi/\ell\right)^2\kappa t}\cos\left(\frac{2m\pi x}{\ell}\right)$$

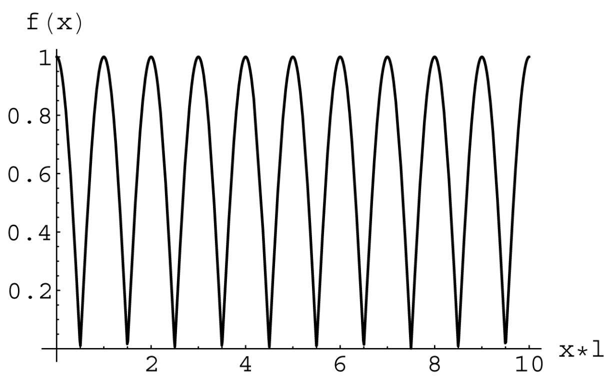
10 (a).



$$a_n = \frac{2}{\ell}\int_0^{\ell/2}\cos\left(\frac{2\pi x}{\ell}\right)\cos\left(\frac{n\pi x}{\ell}\right)dx = \begin{cases} \frac{1}{2}, & n=2 \\ 0, & n \text{ even and } n>2 \\ \dfrac{(-1)^m 2(2m-1)}{\pi\left((2m-1)^2-4\right)}, & n=2m-1 \end{cases}.$$

10 (b). $u(x,t) = \frac{1}{2}e^{-\left(2\pi/\ell\right)^2 \kappa t}\cos\left(\frac{2\pi x}{\ell}\right) + \frac{2}{\pi}\sum_{m=1}^{\infty}\frac{(-1)^m(2m-1)}{(2m-1)^2-4}e^{-\left((2m-1)\pi/\ell\right)^2\kappa t}\cos\left(\frac{(2m-1)\pi x}{\ell}\right)$

11 (a).



$$a_n = \frac{2}{\ell}\left[\int_0^{\ell/2}\cos\left(\frac{\pi x}{\ell}\right)\cos\left(\frac{n\pi x}{\ell}\right)dx - \int_{\ell/2}^{\ell}\cos\left(\frac{\pi x}{\ell}\right)\cos\left(\frac{n\pi x}{\ell}\right)dx\right]$$

$$= \frac{2}{\ell}\left[\int_0^{\ell/2}\frac{1}{2}\left(\cos\left(\frac{(n-1)\pi x}{\ell}\right)+\cos\left(\frac{(n+1)\pi x}{\ell}\right)\right)dx - \int_{\ell/2}^{\ell}\frac{1}{2}\left(\cos\left(\frac{(n-1)\pi x}{\ell}\right)+\cos\left(\frac{(n+1)\pi x}{\ell}\right)\right)dx\right].$$

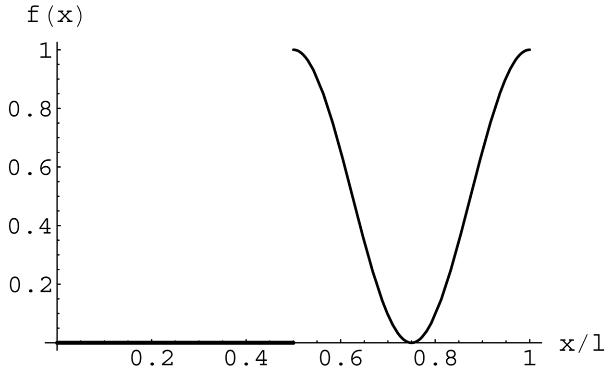
For $n \neq 1$, this expression simplifies to

$$a_n = 2\left[\frac{\sin\left(\frac{(n-1)\pi}{2}\right)}{(n-1)\pi} + \frac{\sin\left(\frac{(n+1)\pi}{2}\right)}{(n+1)\pi}\right] = \frac{4\sin\left(\frac{(n-1)\pi}{2}\right)}{\left(n^2-1\right)\pi} = -4\frac{\cos\left(\frac{n\pi}{2}\right)}{\left(n^2-1\right)\pi}.$$ For $n=1$, the

expression simplifies to $a_1 = 0$ (a quick examination of the integrals should reveal this).

11 (b). $u(x,t) = \frac{2}{\pi} + \frac{4}{\pi}\sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{4m^2-1}e^{-\left(2m\pi/\ell\right)^2\kappa t}\cos\left(\frac{2m\pi x}{\ell}\right).$

12 (a).



$$\cos^2\left(\frac{2\pi x}{\ell}\right)=\frac{1}{2}+\frac{1}{2}\cos\left(\frac{4\pi x}{\ell}\right).$$

$$a_n=\frac{2}{\ell}\int_{\ell/2}^{\ell}\left(\frac{1}{2}+\frac{1}{2}\cos\left(\frac{4\pi x}{\ell}\right)\right)\cos\left(\frac{n\pi x}{\ell}\right)dx=\begin{cases}0,\ n=0\\ 1/4,\ n=4\\ 0,\ n \text{ even and } n\neq 0,4\\ \dfrac{(-1)^m\left((2m-1)^2-8\right)}{(2m-1)\left((2m-1)^2-16\right)},\ n=2m-1\end{cases}.$$

12 (b). $u(x,t)=\frac{1}{4}+\frac{1}{4}e^{-(4\pi/\ell)^2\kappa t}\cos\left(\frac{4\pi x}{\ell}\right)+\frac{2}{\pi}\sum_{m=1}^{\infty}\frac{(-1)^m\left((2m-1)^2-8\right)}{(2m-1)\left((2m-1)^2-16\right)}e^{-\left((2m-1)\pi/\ell\right)^2\kappa t}\cos\left(\frac{(2m-1)\pi x}{\ell}\right)$

13. $\phi(x,t)=a_1u_1+a_2u_2$, since it is a combination of two solutions of the heat equation, is itself a solution for all values of a_1 and a_2. However, $\phi(0,t)=a_1T_0+a_2T_0=(a_1+a_2)T_0$ and $\phi(\ell,t)=a_1T_1+a_2T_1=(a_1+a_2)T_1$. Therefore, ϕ satisfies the two boundary conditions only if $a_1+a_2=1$.

14. $v(x)=25x,\ u=v+w$ where $w_t=\kappa w_{xx},\ w(0,t)=w(4,t)=0,\ t\geq 0$,
$w(x,0)=u(x,0)-v(x)=80\sin\pi x\cos\pi x=40\sin 2\pi x$.
$w(x,0)=\sum_{n=1}^{\infty}a_n\sin\left(\frac{n\pi x}{4}\right)=40\sin 2\pi x\Rightarrow a_8=40$. $w(x,t)=40e^{-2\pi^2 t}\sin 2\pi x$,
$u(x,t)=25x+40e^{-2\pi^2 t}\sin 2\pi x$.

15. Since $T_0=T_1=50,\ v(x)=50$. $w(x,t)=\sum_{n=1}^{\infty}a_ne^{-(n\pi)^2 t}\sin(n\pi x)$, and
$w(x,0)=-25\sin^2\pi x=\frac{-25}{2}+\frac{25}{2}\cos(2\pi x)=\sum_{n=1}^{\infty}a_n\sin n\pi x$.
$a_n=2\int_0^1\left(-\frac{25}{2}+\frac{25}{2}\cos 2\pi x\right)\sin n\pi x dx$, which simplifies to $a_n=\frac{100}{\pi}\cdot\frac{\left[1-(-1)^n\right]}{n\left(n^2-4\right)},\ n\neq 2,\ a_2=0$.
Therefore, $u(x,2)=50+w(x,t)=50+\frac{200}{\pi}\sum_{m=1}^{\infty}\frac{e^{-((2m-1)\pi)^2 t}}{(2m-1)\left((2m-1)^2-4\right)}\sin((2m-1)\pi x)$.

16. $v(x)=200-50x$. $w(x,0)=40\sin\pi x=\sum_{n=1}^{\infty}a_n\sin\left(\frac{n\pi x}{2}\right)\Rightarrow a_2=40,\ a_n=0,\ n\neq 2$.

$u(x,t)=200-50x+40e^{-\pi^2(0.1)t}\sin\pi x$.

17. $v(x)=50x$. $w(x,t)=\sum_{n=1}^{\infty}a_n e^{-\left(n\pi/2\right)^2 t}\sin\left(\frac{n\pi x}{2}\right)=50x-50x=0$. Thus all $a_n=0$, and so $u(x,t)=50x+0=50x$.

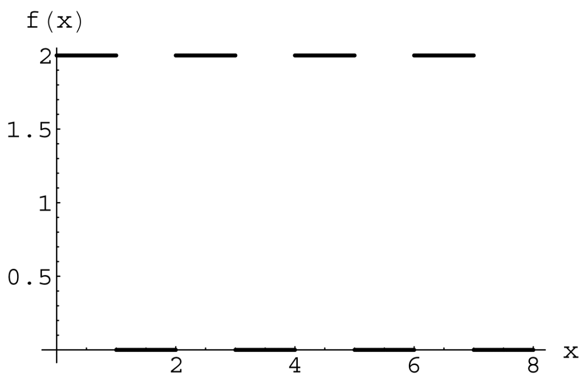
18. $v(x)=20+(T-20)x$. $w(x,0)=50\sin\pi x=\sum_{n=1}^{\infty}a_n\sin(n\pi x)\Rightarrow a_1=50,\ a_n=0,\ n\neq 1$.

$u(x,t)=20+(T_1-20)x+50e^{-\pi^2(\kappa)t}\sin\pi x$. $u\left(\frac{1}{2},t\right)=10+\frac{1}{2}T_1+50e^{-\pi^2\kappa t}$,

$u\left(\frac{1}{2},0\right)=135=60+\frac{1}{2}T_1\Rightarrow T_1=150$, $u\left(\frac{1}{2},2\right)=95=20+65+50e^{-4\pi^2\kappa}\Rightarrow\kappa=\frac{\ln 5}{4\pi^2}\approx .040767$.

Section 11.4

1 (a).

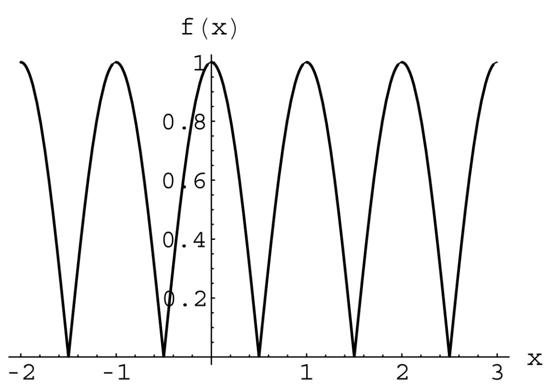


The graph is essentially odd, so $f(x)=1+\sum_{n=1}^{\infty}b_n\sin n\pi x$. $b_n=\int_0^1 2\sin n\pi x\,dx=\begin{cases}0, & n\text{ even}\\ \frac{4}{n\pi}, & n\text{ odd}\end{cases}$.

Therefore, $f(x)=1+\frac{4}{\pi}\sum_{m=1}^{\infty}\frac{\sin((2m-1)\pi x)}{2m-1}$.

1 (b). At $x=0$ and $x=1$, $f(x)=1$ and the series converges to $1/2$.

2 (a).

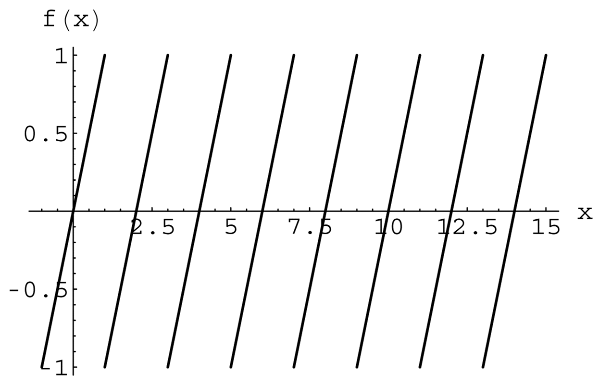


$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos 2n\pi x.\ a_n=4\int_0^{1/2}\cos\pi x\cos 2n\pi x dx=\frac{4}{\pi}\cdot\frac{(-1)^{n+1}}{4n^2-1}.$$

$$f(x)=\frac{2}{\pi}+\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{4n^2-1}\cos 2n\pi x.$$

2 (b). The Fourier series converges to $f(x)$ for all x.

3 (a).

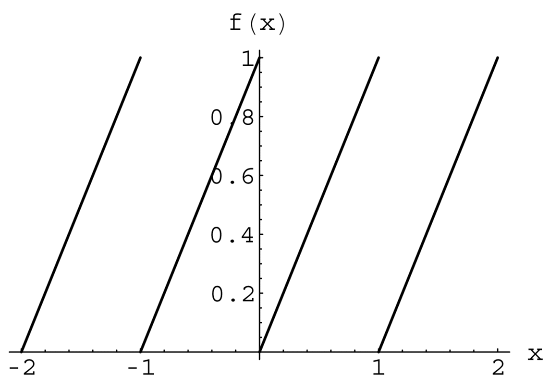


The graph is essentially odd, so $f(x)=\sum_{n=1}^{\infty}b_n\sin n\pi x$. $b_n=2\int_0^1 x\sin n\pi x dx=\frac{2(-1)^{n+1}}{n\pi}$.

Therefore, $f(x)=\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n}\sin(n\pi x)$.

3 (b). At $x=-1$ and $x=1$, $f(x)=-1$ and the series converges to 0.

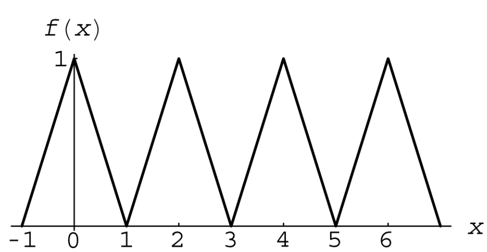
4 (a).



$$f(x)=\frac{1}{2}+\sum_{n=1}^{\infty}b_n\sin 2n\pi x.\ b_n=2\int_0^1 x\sin 2n\pi x dx=-\frac{1}{n\pi}.\ f(x)=\frac{1}{2}-\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{\sin 2n\pi x}{n}.$$

4 (b). At $x=0$ and $x=1$, $f(x)=0$ and the series converges to $1/2$.

5 (a).

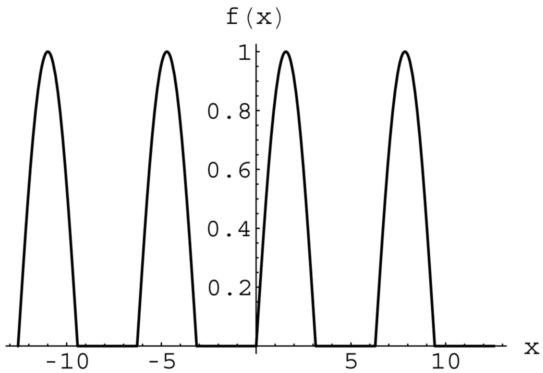


The graph is even, so $f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos n\pi x$. $a_n=2\int_0^1(1-x)\cos n\pi x dx=\begin{cases}0, & n \text{ even}\\ \frac{4}{(n\pi)^2}, & n \text{ odd}\end{cases}$.

$a_0=2\int_0^1(1-x)dx=1$. Therefore, $f(x)=\frac{1}{2}+\frac{4}{\pi^2}\sum_{m=1}^{\infty}\frac{\cos((2m-1)\pi x)}{(2m-1)^2}$.

5 (b). The Fourier series converges to $f(x)$ for all x.

6 (a).

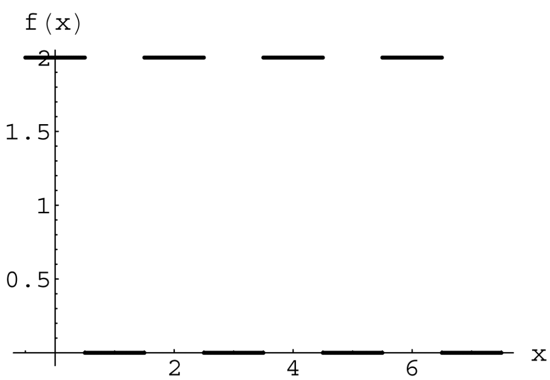


$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)$. $a_n=\frac{1}{\pi}\int_0^{\pi}\sin x\cos(nx)dx=\begin{cases}0, & n=2m-1\\ -\frac{2}{\pi(4m^2-1)}, & n=2m\end{cases}$.

$b_n=\frac{1}{\pi}\int_0^{\pi}\sin x\sin(nx)dx=\begin{cases}\frac{1}{2}, & n=1\\ 0, & n\neq 1\end{cases}$. $f(x)=\frac{1}{\pi}+\frac{1}{2}\sin x-\frac{2}{\pi}\sum_{m=1}^{\infty}\frac{\cos 2mx}{4m^2-1}$.

6 (b). The Fourier series converges to $f(x)$ for all x.

7 (a).



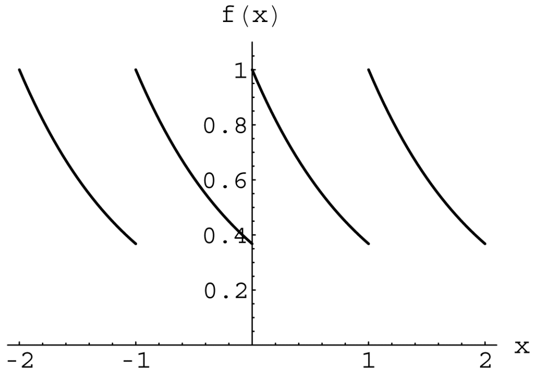
The graph is even, so $f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos n\pi x$.

$a_n=2\int_0^{1/2}2\cos n\pi x dx=\begin{cases}0, & n=2m\\ \frac{4(-1)^{m+1}}{(2m-1)\pi}, & n=2m-1\end{cases}$. $a_0=4\cdot\frac{1}{2}=2$. Therefore,

$f(x)=1+\frac{4}{\pi}\sum_{m=1}^{\infty}\frac{(-1)^{m+1}}{(2m-1)}\cos((2m-1)\pi x)$.

7 (b). At $x=-\frac{1}{2}$ and $x=\frac{1}{2}$, $f(x)=2$ and the series converges to 1.

8 (a).

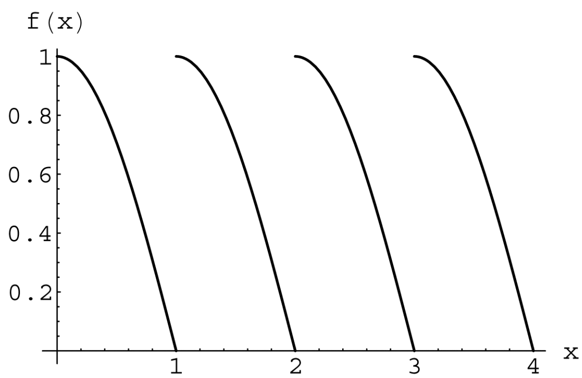


$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos 2n\pi x+b_n\sin 2n\pi x)$. $a_n=2\int_0^1 e^{-x}\cos(2n\pi x)dx=\frac{2(1-e^{-1})}{4n^2\pi^2+1}$.

$b_n=2\int_0^1 e^{-x}\sin(2n\pi x)dx=\frac{4n\pi(1-e^{-1})}{4n^2\pi^2+1}$. $f(x)=1-e^{-1}+2(1-e^{-1})\sum_{n=1}^{\infty}\frac{(\cos 2n\pi x+2n\pi\sin 2n\pi x)}{4n^2\pi^2+1}$.

8 (b). At $x=0$ and $x=1$, $f(x)=1$ and the series converges to $\frac{1}{2}(e^{-1}+1)$.

9 (a).



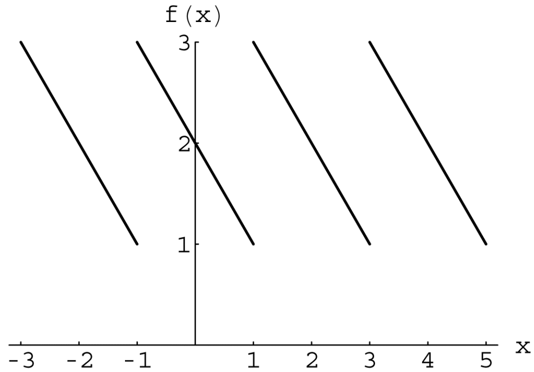
The graph is neither even nor odd, so $f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}(a_n\cos 2n\pi x+b_n\sin 2n\pi x)$.

$a_n=2\int_0^1\cos\left(\frac{\pi x}{2}\right)\cos(2n\pi x)dx=\frac{-1}{\pi(4n^2-1/4)}$. $b_n=2\int_0^1\cos\left(\frac{\pi x}{2}\right)\sin(2n\pi x)dx=\frac{4n}{\pi(4n^2-1/4)}$.

Therefore, $f(x)=\frac{2}{\pi}+\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{-\cos(2n\pi x)+4n\sin(2n\pi x)}{4n^2-1/4}$.

9 (b). At $x=0$ and $x=1$, $f(x)=1$ and the series converges to $\frac{1}{2}$.

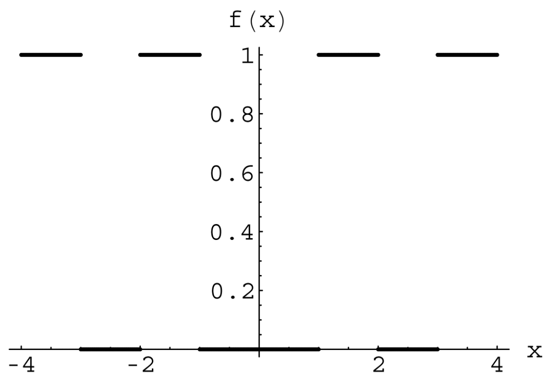
10 (a).



$f(x)=2+\sum_{n=1}^{\infty}b_n\sin n\pi x$. $b_n=\int_{-1}^{1}(2-x)\sin n\pi x dx=\frac{2(-1)^n}{n\pi}$. $f(x)=2+\frac{2}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin(n\pi x)$.

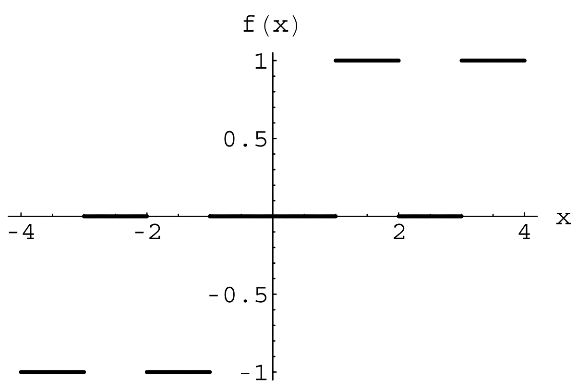
10 (b). At $x=-1$ and $x=1$, $f(x)=3$ and the series converges to 2.

11 (a).



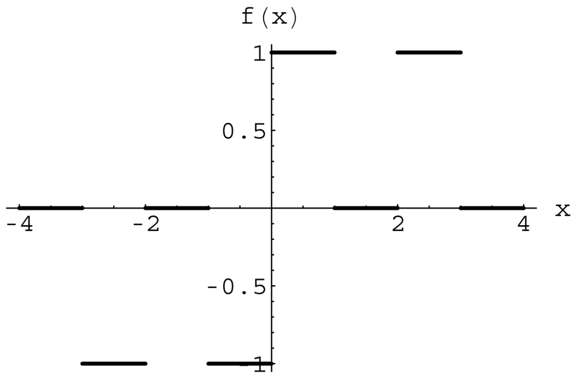
11 (b). $x=1,2$

12 (a).



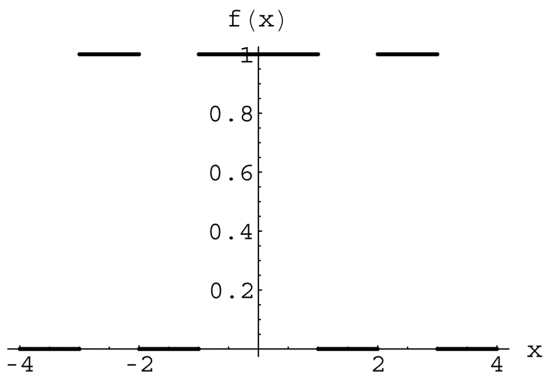
12 (b). $x=1,2$

13 (a).



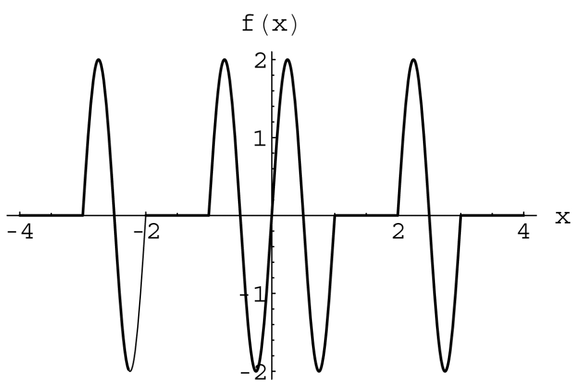
13 (b). $x = 1,2$

14 (a).



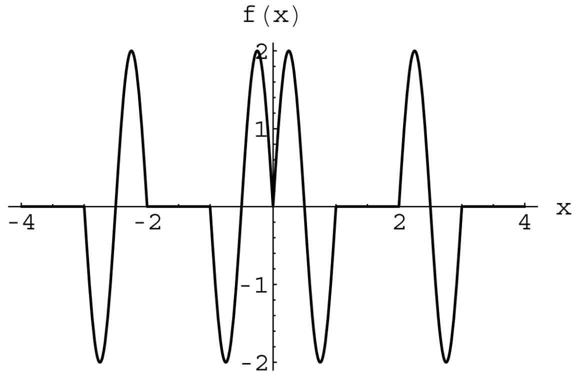
14 (b). $x = 0,1,2$

15 (a).



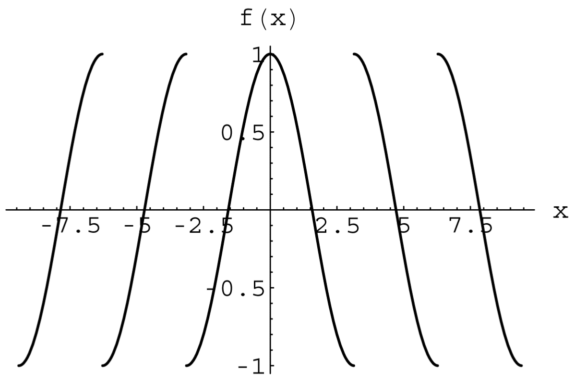
15 (b). The graph converges everywhere on the interval.

16 (a).



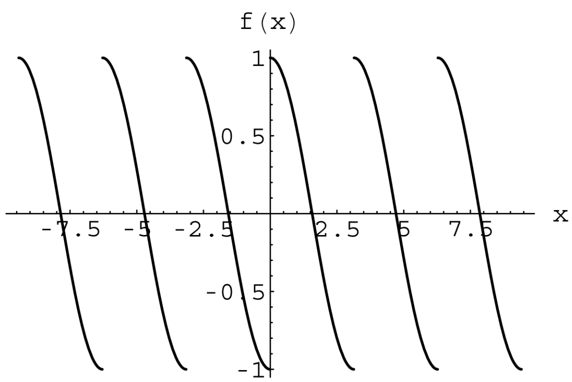
16 (b). The graph converges everywhere on the interval.

17 (a).



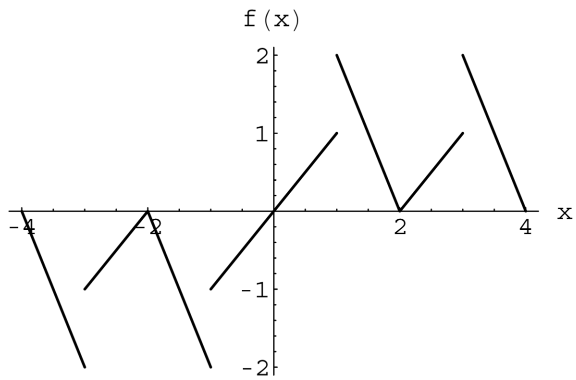
17 (b). $x = 0$ and $x = \pi$

18 (a).



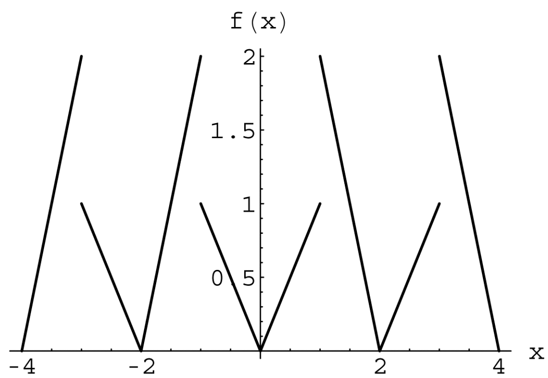
18(b). The graph converges everywhere on the interval.

19 (a).



19 (b). $x=1$

20 (a).



20 (b). $x=1$

21. $f(x)$ is even, $f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\left(\frac{n\pi x}{\ell}\right)$, $b_n=0$ for all n.

22. $f(x)=-2+\sum_{n=1}^{\infty}b_n\sin\left(\frac{n\pi x}{\ell}\right)$

23. Since $f(x)-3\sin\left(\frac{2\pi x}{\ell}\right)$ is even, $b_2=3$, all other $b_n=0$. Since $\int_{-\ell}^{\ell}f(x)dx=0$, $a_0=0$.

$$f(x)=3\sin\left(\frac{2\pi x}{\ell}\right)+\sum_{n=1}^{\infty}a_n\cos\left(\frac{n\pi x}{\ell}\right).$$

24. $f(x)=-2+\cos\left(\frac{\pi x}{\ell}\right)$

25. $\sum_{m=1}^{\infty}\frac{-2\cos((2m-1)\pi x)}{(2m-1)^2\pi^2}$

26. $\sum_{n=1}^{\infty}\frac{\sin 4\pi x}{2(n^2+1)}$

27. $\sum_{m=1}^{\infty}\frac{2\sin\left(\frac{(2m-1)\pi x}{3}\right)}{(2m-1)^2\pi^2}$

28. $\sum_{m=1}^{\infty}\frac{4(-1)^m\cos 2m\pi x}{4m^2\pi^2+1}$

29. Letting $n=\frac{1}{2}(m+1)$, we construct $\sum_{n=1}^{\infty}\frac{-2\sin((4n-1)\pi x)}{(2n-1)^2+1}$.

Section 11.5

For exercises 1-10:

$$u(x,t)=\sum_{n=1}^{\infty}\left[a_n\cos\left(\frac{n\pi ct}{\ell}\right)+b_n\sin\left(\frac{n\pi ct}{\ell}\right)\right]\sin\left(\frac{n\pi x}{\ell}\right),$$

$$u_t(x,t)=\sum_{n=1}^{\infty}\left[-a_n\frac{n\pi c}{\ell}\sin\left(\frac{n\pi ct}{\ell}\right)+b_n\frac{n\pi c}{\ell}\cos\left(\frac{n\pi ct}{\ell}\right)\right]\sin\left(\frac{n\pi x}{\ell}\right)$$

1 (a). From examination, $a_n=0$ for all n, $\frac{1}{2}b_1\pi=2\Rightarrow b_1=\frac{4}{\pi}$, all other $b_n=0$. Therefore, $u(x,t)=\frac{4}{\pi}\sin\left(\frac{\pi t}{2}\right)\sin\left(\frac{\pi x}{4}\right)$.

1 (b). Partial differentiation gives us $u_t=2\cos\left(\frac{\pi t}{2}\right)\sin\left(\frac{\pi x}{4}\right)$.

2 (a). $a_2=1,\ b_n=0$. $u(x,t)=\cos\pi t\sin\left(\frac{\pi x}{2}\right)$.

2 (b). $u_t=-\pi\sin\pi t\sin\left(\frac{\pi x}{2}\right)$.

3 (a). From examination, $a_1=1$, all other $a_n=0$, $3\pi b_1=-2\Rightarrow b_1=\frac{-2}{3\pi}$, all other $b_n=0$. Therefore, $u(x,t)=\left[\cos(3\pi t)-\frac{2}{3\pi}\sin(3\pi t)\right]\sin\pi x$.

3 (b). Partial differentiation gives us $u_t=[-3\pi\sin(3\pi t)-2\cos(3\pi t)]\sin\pi x$.

4 (a). $a_3=2,\ b_1=\frac{1}{\pi}$. $u(x,t)=2\cos 3\pi t\sin 3\pi x+\frac{1}{\pi}\sin\pi t\sin\pi x$.

4 (b). $u_t=-6\pi\sin 3\pi t\sin 3\pi x+\cos\pi t\sin\pi x$.

5 (a). From examination, $a_1=1$, $a_2=-1$, all other $a_n=0$, all $b_n=0$. Therefore, $u(x,t)=\cos(2\pi t)\sin\pi x-\cos(4\pi t)\sin(2\pi x)$.

5 (b). Partial differentiation gives us $u_t=-2\pi\sin(2\pi t)\sin\pi x+4\pi\sin(4\pi t)\sin(2\pi x)$.

6 (a). $a_1=-1,\ b_2=\frac{1}{4},\ b_3=\frac{1}{6}$. $u(x,t)=-\cos 2t\sin x+\frac{1}{4}\sin 4t\sin 2x+\frac{1}{6}\sin 6t\sin 3x$.

6 (b). $u_t=2\sin 2t\sin x+\cos t\sin 2x+\cos 6t\sin 3x$.

7 (a). First, we note that $u(x,0)=2\sin^2 x=1-\cos 2x=\sum_{n=1}^{\infty}a_n\sin nx$.

$$a_n=\frac{2}{\pi}\int_0^{\pi}(1-\cos 2x)\sin nx\,dx=\frac{\left[1-(-1)^n\right]}{\pi}\cdot\frac{-8}{n(n^2-4)},\ n\neq 2,$$

$a_2 = \frac{2}{\pi}\int_0^{\pi}\left[\sin 2x - \frac{1}{2}\sin 4x\right]dx = 0$. From examination, all $b_n = 0$. Therefore,

$$u(x,t) = \frac{-16}{\pi}\sum_{m=1}^{\infty}\frac{\cos((2m-1)t)\sin((2m-1)x)}{(2m-1)((2m-1^2)-4)}.$$

7 (b). Partial differentiation gives us $u_t(x,t) = \frac{16}{\pi}\sum_{m=1}^{\infty}\frac{\sin((2m-1)t)\sin((2m-1)x)}{(2m-1)^2-4}$.

8 (a). $a_n = 0,\ b_n = -\frac{4}{n}\cdot\frac{2}{\pi}\int_0^{\pi}\left(\frac{1}{2}-\frac{1}{2}\cos 2x\right)\sin nx dx = \begin{cases} 0, & n \text{ even} \\ \frac{32}{\pi n^2(n^2-4)}, & n \text{ odd}\end{cases}$.

$$u(x,t) = \frac{32}{\pi}\sum_{m=1}^{\infty}\frac{\sin((2m-1)t)\sin((2m-1)x)}{(2m-1)^2((2m-1)^2-4)}.$$

8 (b). $u_t = \frac{32}{\pi}\sum_{m=1}^{\infty}\frac{\cos((2m-1)t)\sin((2m-1)x)}{(2m-1)((2m-1)^2-4)}$.

9 (a). From examination, $a_1 = 1$, all other $a_n = 0$.

$n\pi b_n = 2\int_0^1\left(\frac{1}{2}-\frac{1}{2}\cos 2\pi x\right)\sin n\pi x dx = \begin{cases} -\frac{8}{\pi}\cdot\frac{1}{n(n^2-4)}, & n \text{ odd} \\ 0, & n \text{ even}\end{cases}$. Therefore,

$$u(x,t) = \cos\pi t\sin\pi x - \frac{8}{\pi^2}\sum_{m=1}^{\infty}\frac{\sin((2m-1)\pi t)\sin((2m-1)\pi x)}{(2m-1)^2\left[(2m-1)^2-4\right]}.$$

9 (b). Partial differentiation gives us $u_t(x,t) = -\pi\sin\pi t\sin\pi x - \frac{8}{\pi}\sum_{m=1}^{\infty}\frac{\cos((2m-1)\pi t)\sin((2m-1)\pi x)}{(2m-1)\left[(2m-1)^2-4\right]}$.

10 (a). $a_n = 2\int_0^1\left(\frac{1}{2}-\frac{1}{2}\cos 2\pi x\right)\sin n\pi x dx = \begin{cases} 0, & n \text{ even} \\ -\frac{8}{\pi n(n^2-4)}, & n \text{ odd}\end{cases}$. $b_2 = -\frac{1}{2\pi}$.

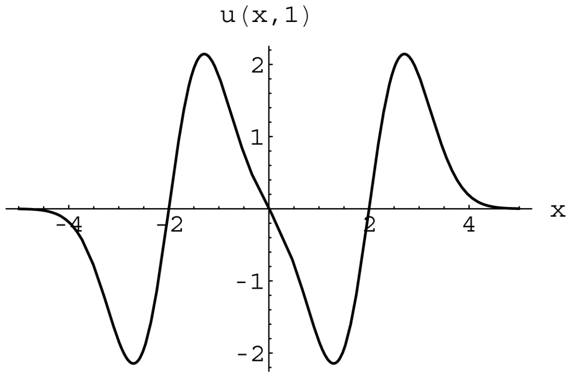
$$u(x,t) = -\frac{1}{2\pi}\sin 2\pi t\sin 2\pi x - \frac{8}{\pi}\sum_{m=1}^{\infty}\frac{\cos((2m-1)\pi t)\sin((2m-1)\pi x)}{(2m-1)((2m-1)^2-4)}.$$

10 (b). $u_t = -\cos 2\pi t\sin 2\pi x + \frac{8}{\pi}\sum_{m=1}^{\infty}\frac{\sin((2m-1)\pi t)\sin((2m-1)\pi x)}{(2m-1)^2-4}$

11 (a). $x = \frac{1}{2}(\xi+\eta),\ t = \frac{1}{2c}(-\xi+\eta) \Rightarrow a_1 = \frac{1}{2},\ a_2 = \frac{1}{2},\ b_1 = \frac{1}{2c},\ b_2 = \frac{-1}{2c}$.

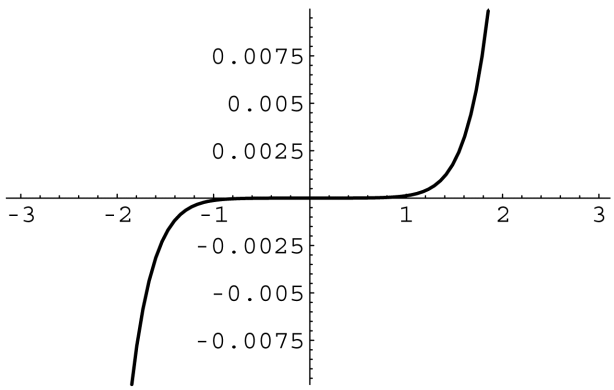
12 (a). $u(x,t) = 5\left((x-2t)^2 e^{-(x-2t)^2} + (x+2t)^2 e^{-(x+2t)^2}\right)$.

12 (b).



13 (a). $u(x,t)=\frac{1}{2c}\int_{x-ct}^{x+ct}8\lambda e^{-\lambda^2}\,d\lambda=\left[e^{-(x-2t)^2}-e^{-(x+2t)^2}\right]$

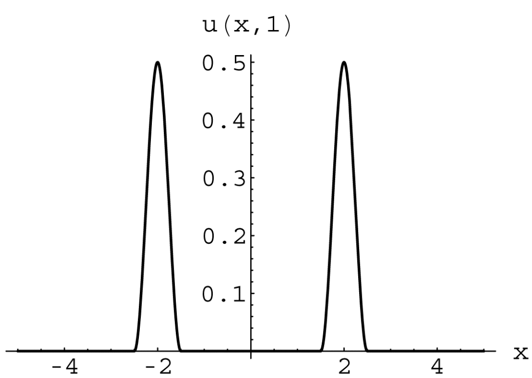
13 (b).



14 (a). Let $p_{1/2}(x)=\begin{cases}1, -1/2\le x\le 1/2\\ 0, \text{ otherwise}\end{cases}$.

$$u(x,t)=\frac{1}{2}\cos^2(\pi(x-ct))p_{1/2}(x-ct)+\frac{1}{2}\cos^2(\pi(x+ct))p_{1/2}(x+ct).$$

14 (b).



Section 11.6

5 (a). $v(x,y)=\alpha_1+\alpha_2x+\alpha_3y+\alpha_4xy$. Examination gives us $\alpha_1=0,\ \alpha_1+\alpha_2=1,\ \alpha_3=1,\ \alpha_2+\alpha_3+\alpha_4=1$. Solving these simultaneous equations gives us $\alpha_1=0,\ \alpha_2=1,\ \alpha_3=1,\ \alpha_4=-1$. Thus $v(x,y)=x+y-xy$.

5 (b). Since $U_{xx}+U_{yy}=0$ and $U=0$ on boundary, $U(x,y)=0$.

5 (c). $u(x,y)=v(x,y)=x+y-xy$.

6 (a). $v(x,y)=\alpha_1+\alpha_2 x+\alpha_3 y+\alpha_4 xy$. $\alpha_1=2,\ 2+2\alpha_2=0,\ 2+2\alpha_3=4,\ 2-2+2+4\alpha_4=2$.
$\alpha_1=2,\ \alpha_2=-1,\ \alpha_3=1,\ \alpha_4=0$. $v(x,y)=2-x+y$.

6 (b). $U_{xx}+U_{yy}=0$ and $U=0$ on boundary $\Rightarrow$ $U(x,y)=0$.

6 (c). $u(x,y)=v(x,y)=2-x+y$.

7 (a). $v(x,y)=\alpha_1+\alpha_2 x+\alpha_3 y+\alpha_4 xy$. Examination gives us
$\alpha_1=0,\ \alpha_2=0,\ \alpha_3=0,\ 0+2\alpha_4=8\Rightarrow\alpha_4=4$. Thus $v(x,y)=4xy$.

7 (b). Since $U_{xx}+U_{yy}=0$ and $U=0$ on boundary, $U(x,y)=0$.

7 (c). $u(x,y)=v(x,y)=4xy$.

8 (a). Since all of the corner values are 0, $v(x,y)=0$.

8 (b). $u(x,y)=U_2(x,y)=\sum_{n=1}^{\infty} b_n\sin(n\pi x)\sinh n\pi y$.

$$u(x,1)=\sum_{n=1}^{\infty} b_n\sinh n\pi\sin n\pi x=4\sin 2\pi x\Rightarrow b_2=\frac{4}{\sinh 2\pi}.\ U(x,y)=\frac{4}{\sinh 2\pi}\sin(2\pi x)\sinh 2\pi y.$$

8 (c). $u(x,y)=U+0=\dfrac{4}{\sinh 2\pi}\sin(2\pi x)\sinh 2\pi y$.

9 (a). Since all of the corner values are 0, $v(x,y)=0$.

9 (b). $u(x,y)=U_3(x,y)=\sum_{n=1}^{\infty} c_n\sinh(n\pi(1-x))\sin n\pi y$. $u(0,y)=\sum_{n=1}^{\infty} c_n\sinh n\pi\sin n\pi y=2\sin\pi y$.

Therefore, $c_1\sinh\pi=2$, all other $c_n=0$, and so $U(x,y)=\dfrac{2}{\sinh\pi}\sinh(\pi(1-x))\sin\pi y$.

9 (c). $u(x,y)=U+0=\dfrac{2}{\sinh\pi}\sinh(\pi(1-x))\sin\pi y$.

10 (a). $v(x,y)=\alpha_1+\alpha_2 x+\alpha_3 y+\alpha_4 xy$. $\alpha_1=1,\ 1+2\alpha_2=1,\ 1+\alpha_3=1,\ 1+0+0+2\alpha_4=1$.
$\alpha_1=1,\ \alpha_2=0,\ \alpha_3=0,\ \alpha_4=0$. $v(x,y)=1$.

10 (b). $U=U_4(x,y)=\sum_{n=1}^{\infty} d_n\sin\left(\dfrac{n\pi x}{2}\right)\sinh\left(\dfrac{n\pi}{2}(1-y)\right)=\cos 2\pi x-1$.

$$d_n\sinh\left(\frac{n\pi}{2}\right)=\int_0^2(\cos 2\pi x-1)\sin\left(\frac{n\pi x}{2}\right)dx=\begin{cases}0, & n=2m\\ \dfrac{64}{\pi(2m-1)\left((2m-1)^2-16\right)}, & n=2m-1\end{cases}\cdot$$

$$U(x,y)=\frac{64}{\pi}\sum_{m=1}^{\infty}\frac{\sin\left((2m-1)\frac{\pi}{2}x\right)\sinh\left((2m-1)\frac{\pi}{2}(1-y)\right)}{\sinh\left((2m-1)\frac{\pi}{2}\right)(2m-1)\left((2m-1)^2-16\right)}.$$

10 (c). $u(x,y)=U+1=1+\dfrac{64}{\pi}\sum_{m=1}^{\infty}\dfrac{\sin\left((2m-1)\frac{\pi}{2}x\right)\sinh\left((2m-1)\frac{\pi}{2}(1-y)\right)}{\sinh\left((2m-1)\frac{\pi}{2}\right)(2m-1)\left((2m-1)^2-16\right)}$.

11 (a). Since all of the corner values are 1, $v(x,y)=1$.

11 (b). $U_{xx}+U_{yy}=0$, so $U=U_1+U_2$. $U_1(x,y)=\sum_{n=1}^{\infty}a_n\sinh\left(\frac{n\pi x}{2}\right)\sin\left(\frac{n\pi y}{2}\right)$.

$\sum_{n=1}^{\infty}a_n\sinh\left(\frac{3n\pi}{2}\right)\sin\left(\frac{n\pi y}{2}\right)=\sin\pi y$. Therefore, $a_2\sinh 3\pi=1$, all other $a_n=0$.

$U_2(x,y)=\sum_{n=1}^{\infty}b_n\sinh\left(\frac{n\pi x}{3}\right)\sin\left(\frac{n\pi y}{3}\right)$. $\sum_{n=1}^{\infty}b_n\sinh\left(\frac{2n\pi}{3}\right)\sin\left(\frac{2n\pi y}{3}\right)=-2\sin\left(\frac{\pi x}{3}\right)$. Therefore,

$b_1\sinh\left(\frac{2\pi}{3}\right)=-2$, all other $b_n=0$, and so $U(x,y)=\frac{\sinh\pi x\sin\pi y}{\sinh 3\pi}-\frac{2\sin\left(\frac{\pi x}{3}\right)\sinh\left(\frac{\pi y}{3}\right)}{\sinh\left(\frac{2\pi}{3}\right)}$.

11 (c). $u(x,y)=U+v=1+\frac{\sinh\pi x\sin\pi y}{\sinh 3\pi}-\frac{2\sin\left(\frac{\pi x}{3}\right)\sinh\left(\frac{\pi y}{3}\right)}{\sinh\left(\frac{2\pi}{3}\right)}$.

12 (a). Since all of the corner values are 0, $v(x,y)=0$.

12 (b). $U=U_1+U_2+U_3+U_4$

$$=\frac{\sinh 2\pi x\sin 2\pi y}{\sinh 2\pi}+\frac{\sin 3\pi x\sinh 3\pi y}{\sinh 3\pi}+\frac{\sinh(2\pi(1-x))\sin 2\pi y}{\sinh 2\pi}+\frac{\sin 3\pi x\sinh 3\pi(1-y)}{\sinh 3\pi}.$$

12 (c). $u(x,y)=\frac{\sinh 2\pi x\sin 2\pi y}{\sinh 2\pi}+\frac{\sin 3\pi x\sinh 3\pi y}{\sinh 3\pi}+\frac{\sinh(2\pi(1-x))\sin 2\pi y}{\sinh 2\pi}+\frac{\sin 3\pi x\sinh 3\pi(1-y)}{\sinh 3\pi}$.

14. $u(r,\theta)=\frac{a_0}{2}+\sum_{n=1}^{\infty}r^n\left[a_n\cos n\theta+b_n\sin n\theta\right]$. $a_2=\frac{4}{9}$, $b_n=0$. $u(r,\theta)=4\left(\frac{r}{3}\right)^2\cos 2\theta$.

15. $u(r,\theta)=\frac{a_0}{2}+\sum_{n=1}^{\infty}r^n\left[a_n\cos n\theta+b_n\sin n\theta\right]$. Examination gives us $\frac{a_0}{2}=2$, $3^1b_1=-1$, all other $a_n,b_n=0$, and so $u(r,\theta)=2-\left(\frac{r}{3}\right)\sin\theta$.

16. $u(r,\theta)=\frac{a_0}{2}+\sum_{n=1}^{\infty}r^n\left[a_n\cos n\theta+b_n\sin n\theta\right]$. $a_0=4$, $a_2=-\frac{1}{2}$, $a_4=-\frac{1}{2}$, $b_n=0$.

$u(r,\theta)=2-\frac{r^2}{2}\cos 2\theta-\frac{r^4}{2}\cos 4\theta$.

17. Let $f(\theta)=\frac{a_0}{2}+\sum_{n=1}^{\infty}2^n\left[a_n\cos n\theta+b_n\sin n\theta\right]$. $a_n=\frac{1}{2^n\pi}\int_0^{\pi}\cos n\theta d\theta=\begin{cases}1, & n=0\\ 0, & n\ge 1\end{cases}$.

$b_n=\frac{1}{2^n\pi}\int_0^{\pi}\sin n\theta d\theta=\begin{cases}0, & n=2m\\ \frac{1}{(2m-1)\pi 2^{2m-2}}, & n=2m-1\end{cases}$. Therefore,

$u(r,\theta)=\frac{1}{2}+\frac{2}{\pi}\sum_{m=1}^{\infty}\left(\frac{r}{2}\right)^{2m-1}\frac{\sin((2m-1)\theta)}{(2m-1)}$.

18. $u(r,\theta)=\frac{a_0}{2}+\sum_{n=1}^{\infty}r^n\left[a_n\cos n\theta+b_n\sin n\theta\right]$. $a_n=\frac{1}{\pi}\int_0^{\pi}\sin\theta\cos n\theta d\theta=\begin{cases}0, & n=2m-1\\ \frac{-4}{2\pi(4m^2-1)}, & n=2m\end{cases}$.

$b_n=\frac{1}{\pi}\int_0^{\pi}\sin\theta\sin n\theta d\theta=\begin{cases}1/2, & n=1\\ 0, & n\neq 1\end{cases}$. $u(r,\theta)=\frac{1}{\pi}+\frac{r}{2}\sin\theta-\frac{2}{\pi}\sum_{m=1}^{\infty}\frac{r^{2m}\cos 2m\theta}{4m^2-1}$.

19. Since $f(\theta)$ is even, $b_n=0$ for all n. $a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}|\sin\theta|\cos n\theta d\theta=\begin{cases}0, & n=2m-1\\ \frac{4}{\pi(4m^2-1)}, & n=2m\end{cases}$ (note: $\int_{-\pi}^{\pi}|\sin\theta|\cos n\theta d\theta=2\int_0^{\pi}\sin\theta\cos n\theta d\theta$). Therefore, $u(r,\theta)=-\frac{2}{\pi}+\frac{4}{\pi}\sum_{m=1}^{\infty}r^{2m}\frac{\cos 2m\theta}{4m^2-1}$.

20. $a_n=0$. $b_n=\frac{1}{\pi\cdot 2^n}\int_{-\pi/2}^{\pi/2}\theta\sin n\theta d\theta=\begin{cases}0, & n=2m\\ \frac{2(-1)^{m+1}}{\pi(2m-1)^2 2^{2m-1}}, & n=2m-1\end{cases}$.

$u(r,\theta)=\frac{2}{\pi}\sum_{m=1}^{\infty}\left(\frac{r}{2}\right)^{2m-1}(-1)^{m+1}\frac{\sin((2m-1)\theta)}{(2m-1)^2}$.

23. $a_0+A_0\ln 1=\frac{1}{\pi}\cdot 2\pi,\ a_0+A_0\ln 3=\frac{1}{\pi}\cdot 3\cdot 2\pi$. Solving these simultaneous equations yields $a_0=2,\ A_0=\frac{4}{\ln 3}$ (all other $a_n,b_n=0$). Therefore, $u(r,\theta)=1+\frac{2\ln r}{\ln 3},\ 1\le r\le 3$.

24. $\frac{a_0}{2}+\frac{A_0}{2}\ln 1+\sum_{n=1}^{\infty}\left(\left[a_n+A_n\right]\cos n\theta+\left[b_n+B_n\right]\sin n\theta\right)=0$.

$\frac{a_0}{2}+\frac{A_0}{2}\ln 2+\sum_{n=1}^{\infty}\left(\left[a_n 2^n+A_n 2^{-n}\right]\cos n\theta+\left[b_n 2^n+B_n 2^{-n}\right]\sin n\theta\right)=1+\cos\theta$.

$\frac{a_0}{2}=0,\ \frac{A_0}{2}\ln 2=1,\ a_1+A_1=0,\ 2a_1+\frac{1}{2}A_1=1,\ b_n=B_n=0,\ a_0=0,\ A_0=2\ln 2,\ a_1=\frac{2}{3},\ A_1=-\frac{2}{3}$

$u(r,\theta)=\frac{\ln r}{\ln 2}+\frac{2}{3}\left(r-\frac{1}{r}\right)\cos\theta$.

25. $a_0+A_0\ln 1=\frac{1}{\pi}\cdot 2\cdot 2\pi,\ a_0+A_0\ln 2=\frac{1}{\pi}\cdot 1\cdot 2\pi$. Solving these simultaneous equations yields $a_0=4,\ A_0=\frac{-2}{\ln 2}$. $2a_1+\frac{1}{2}A_1=1,\ a_1+A_1=0$. Solving these simultaneous equations yields $a_1=\frac{2}{3},\ A_1=-\frac{2}{3}$. $4b_2+\frac{1}{4}B_2=0,\ b_2+B_2=1$. Solving these simultaneous equations yields $b_2=-\frac{1}{15},\ B_2=\frac{16}{15}$. (all other $a_n,b_n=0$). Therefore,

$u(r,\theta)=2-\frac{\ln r}{\ln 2}+\frac{2}{3}\left(r-r^{-1}\right)\cos\theta+\left(\frac{-r^2}{15}+\frac{16}{15r^2}\right)\sin 2\theta$.

26. $a_0 + A_0 \ln 2 = \frac{1}{\pi} 2\pi,\ a_0 + A_0 \ln 6 = 0 \Rightarrow A_0 = -\frac{2}{\ln 3},\ a_0 = 2 + \frac{2\ln 2}{\ln 3}$.

$4a_2 + \frac{1}{4} A_2 = -1,\ 36a_2 + \frac{1}{36} A_2 = 0 \Rightarrow A_2 = -\frac{1296}{320},\ a_0 = \frac{1}{320}$.

$u(r,\theta) = 1 + \frac{\ln 2}{\ln 3} - \frac{\ln r}{\ln 3} + \frac{1}{320}\left(r^2 - \frac{1296}{r^2}\right)\cos 2\theta = \frac{\ln 6 - \ln r}{\ln 3} + \frac{1}{320}\left(r^2 - \frac{1296}{r^2}\right)\cos 2\theta$.

Section 11.7

1. $f(x,y) = 4\sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{\pi y}{b}\right) = \sum\sum c_{mn} \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)$. Examination shows us that $c_{11} = 4,\ 0$ otherwise. Thus $u(x,y,t) = 4e^{-\left[(\pi/a)^2 + (\pi/b)^2\right]\kappa t} \sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{\pi y}{b}\right)$.

2. $f(x,y) = 8\sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{\pi y}{b}\right) - \sin\left(\frac{2\pi x}{a}\right)\sin\left(\frac{3\pi y}{b}\right) = \sum\sum c_{mn} \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)$.

$c_{11} = 8,\ c_{23} = -1$.

$u(x,y,t) = 8e^{-\left[(\pi/a)^2 + (\pi/b)^2\right]\kappa t} \sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{\pi y}{b}\right) - e^{-\left[(2\pi/a)^2 + (3\pi/b)^2\right]\kappa t} \sin\left(\frac{2\pi x}{a}\right)\sin\left(\frac{3\pi y}{b}\right)$.

3. First, we note that

$f(x,y) = 2\sin^2\left(\frac{\pi x}{a}\right)\sin\left(\frac{\pi y}{b}\right) = \left(1 - \cos\left(\frac{2\pi x}{a}\right)\right)\sin\left(\frac{\pi y}{b}\right) = \sum\sum c_{mn} \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)$.

$c_{m1} = \frac{2}{a}\int_0^a \left(1 - \cos\left(\frac{2\pi x}{a}\right)\right)\sin\left(\frac{m\pi x}{a}\right)dx,\ c_{mn} = 0$ otherwise. Working through this integral yields

$c_{m1} = \begin{cases} 0, & m = 2k \\ \dfrac{-16}{\pi(2k-1)\left[(2k-1)^2 - 4\right]}, & m = 2k-1 \end{cases}$. Therefore,

$$u(x,y,t) = -\frac{16}{\pi}\sum_{k=1}^{\infty} \frac{e^{-\left[\left((2k-1)\pi/a\right)^2 + (\pi/b)^2\right]\kappa t} \sin\left(\frac{(2k-1)\pi x}{a}\right)\sin\left(\frac{\pi y}{b}\right)}{(2k-1)\left[(2k-1)^2 - 4\right]}.$$

4. $f(x,y) = 8\sin^2\left(\frac{\pi x}{a}\right)\sin^2\left(\frac{\pi y}{b}\right) = \sum\sum c_{mn} \sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)$.

$$c_{mn} = 8\frac{4}{ab}\int_0^a \left(\frac{1 - \cos\left(\frac{2\pi x}{a}\right)}{2}\right)\sin\left(\frac{m\pi x}{a}\right)dx \int_0^b \left(\frac{1 - \cos\left(\frac{2\pi y}{b}\right)}{2}\right)\sin\left(\frac{n\pi y}{b}\right)dy$$

$$= \begin{cases} 2\left(\frac{16}{\pi}\right)^2 \frac{1}{(2k-1)\left[(2k-1)^2-4\right](2\ell-1)\left[(2\ell-1)^2-4\right]}, & m=2k-1,\ n=2\ell-1 \\ 0, \text{ otherwise} \end{cases}.$$

$$u(x,y,t) = \frac{512}{\pi^2}\sum_{k=1}^{\infty}\sum_{\ell=1}^{\infty} \frac{e^{-\left[\left(\frac{(2k-1)\pi}{a}\right)^2+\left(\frac{(2\ell-1)\pi}{b}\right)^2\right]\kappa t} \sin\left((2k-1)\pi x/a\right)\sin\left((2\ell-1)\pi y/b\right)}{(2k-1)\left[(2k-1)^2-4\right](2\ell-1)\left[(2\ell-1)^2-4\right]}.$$

5 (b). $u(x,y,0) = f(x,y) = c_{00} + \sum_{m=0}^{\infty} c_{m0}\cos\left(\frac{m\pi x}{a}\right) + \sum_{n=0}^{\infty} c_{0n}\cos\left(\frac{n\pi y}{b}\right) + \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} c_{mn}\cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)$

$c_{00} = \frac{1}{ab}\int_0^a\int_0^b f(x,y)\,dydx,\ c_{m0} = \frac{2}{ab}\int_0^a\int_0^b f(x,y)\cos\left(\frac{m\pi x}{a}\right)dydx,$

$c_{0n} = \frac{2}{ab}\int_0^a\int_0^b f(x,y)\cos\left(\frac{n\pi y}{b}\right)dydx,$

$c_{mn} = \frac{4}{ab}\int_0^a\int_0^b f(x,y)\cos\left(\frac{m\pi x}{a}\right)\cos\left(\frac{n\pi y}{b}\right)dydx,\ m,n = 1,2,3...$

6. $f(x,y) = 2 + \cos\left(\frac{\pi x}{a}\right) + 3\cos\left(\frac{\pi x}{a}\right)\cos\left(\frac{\pi y}{b}\right).\ c_{00} = 2,\ c_{10} = 1,\ c_{11} = 3.$

$u(x,y,t) = 2 + e^{-(\pi/a)^2\kappa t}\cos\left(\frac{\pi x}{a}\right) + 3e^{-\left[(\pi/a)^2+(\pi/b)^2\right]\kappa t}\cos\left(\frac{\pi x}{a}\right)\cos\left(\frac{\pi y}{b}\right).$

7. First, note that $f(x,y) = \cos\left(\frac{\pi x}{a}\right)\cos^2\left(\frac{\pi y}{b}\right) = \frac{1}{2}\cos\left(\frac{\pi x}{a}\right)\left(1+\cos\left(\frac{2\pi y}{b}\right)\right)$. Examination shows us that $c_{10} = \frac{1}{2},\ c_{12} = \frac{1}{2},\ c_{mn} = 0$ otherwise. Therefore,

$u(x,y,t) = \frac{1}{2}e^{-(\pi/a)^2\kappa t}\cos\left(\frac{\pi x}{a}\right) + \frac{1}{2}e^{-\left[(\pi/a)^2+(2\pi/b)^2\right]\kappa t}\cos\left(\frac{\pi x}{a}\right)\cos\left(\frac{2\pi y}{b}\right).$

8. $f(x,y) = 2 + \sin^2\left(\frac{\pi x}{a}\right) = \frac{3}{2} - \frac{1}{2}\cos\left(\frac{2\pi x}{a}\right).\ c_{00} = \frac{3}{2},\ c_{20} = -\frac{1}{2}.$

$u(x,y,t) = \frac{3}{2} - \frac{1}{2}e^{-(2\pi/a)^2\kappa t}\cos\left(\frac{2\pi x}{a}\right).$

9. By substituting into the equations from (5b) and evaluating the integrals, we have

$c_{00} = \frac{1}{4},\ c_{m0} = \begin{cases} 0, & m=2k \\ \frac{(-1)^{k+1}}{(2k-1)\pi}, & m=2k-1, \end{cases}\ c_{0n} = \begin{cases} 0, & n=2l \\ \frac{(-1)^{l+1}}{(2l-1)\pi}, & n=2l-1, \end{cases}$

$c_{mn} = \begin{cases} \frac{4(-1)^{k+l}}{(2k-1)(2l-1)\pi^2}, & m=2k-1 \text{ and } n=2l-1 \\ 0, \text{ otherwise} \end{cases}$. Therefore,

$$u(x,y,t)=\frac{1}{4}-\frac{1}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^k e^{-\left(\frac{(2k-1)\pi}{a}\right)^2\kappa t}\cos\left(\frac{(2k-1)\pi x}{a}\right)}{2k-1}-\frac{1}{\pi}\sum_{l=1}^{\infty}\frac{(-1)^l e^{-\left(\frac{(2l-1)\pi}{b}\right)^2\kappa t}\cos\left(\frac{(2l-1)\pi x}{b}\right)}{2l-1}$$
$$+\frac{4}{\pi^2}\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\frac{(-1)^{k+l}e^{-\left[\left(\frac{(2k-1)\pi}{a}\right)^2+\left(\frac{(2l-1)\pi}{b}\right)^2\right]\kappa t}\cos\left(\frac{(2k-1)\pi x}{a}\right)\cos\left(\frac{(2l-1)\pi y}{b}\right)}{(2k-1)(2l-1)}.$$

10. Let u_9 be the final answer to exercise 9.

$$u(x,y,t)=1-u_9=\frac{3}{4}+\frac{1}{\pi}\sum_{k=1}^{\infty}\frac{(-1)^k e^{-\left(\frac{(2k-1)\pi}{a}\right)^2\kappa t}\cos\left(\frac{(2k-1)\pi x}{a}\right)}{2k-1}$$
$$+\frac{1}{\pi}\sum_{l=1}^{\infty}\frac{(-1)^l e^{-\left(\frac{(2l-1)\pi}{b}\right)^2\kappa t}\cos\left(\frac{(2l-1)\pi x}{b}\right)}{2l-1}$$
$$-\frac{4}{\pi^2}\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\frac{(-1)^{k+l}e^{-\left[\left(\frac{(2k-1)\pi}{a}\right)^2+\left(\frac{(2l-1)\pi}{b}\right)^2\right]\kappa t}\cos\left(\frac{(2k-1)\pi x}{a}\right)\cos\left(\frac{(2l-1)\pi y}{b}\right)}{(2k-1)(2l-1)}.$$

11 (b). First, we rewrite u as follows: $u(x,y,t)=\sum_{n=1}^{\infty}c_{0n}u_{0n}(x,y,t)+\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}c_{mn}u_{mn}(x,y,t)$. Therefore,

$c_{0n}\cdot\frac{ab}{2}=\int_0^a\int_0^b f(x,y)\,dy\,dx\Rightarrow c_{0n}=\frac{2}{ab}\int_0^a\int_0^b f(x,y)\sin\left(\frac{n\pi y}{b}\right)dy\,dx$ and

$c_{mn}=\frac{4}{ab}\int_0^a\int_0^b f(x,y)\cos\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)dy\,dx,\ m,n=1,2,3\ldots$

11 (c). Because of the negative exponential t-dependence, we expect the limit to go to zero for all areas of the rectangle.

12. $c_{13}=1$. $u(x,y,t)=e^{-\left[\left(\frac{\pi}{a}\right)^2+\left(\frac{3\pi}{b}\right)^2\right]\kappa t}\cos\left(\frac{\pi x}{a}\right)\sin\left(\frac{3\pi y}{b}\right)$.

13. First, we note that $f(x)$ can be rewritten as $f(x)=\left(1+\cos\left(\frac{2\pi x}{a}\right)\right)\sin\left(\frac{2\pi y}{b}\right)$. Examination gives us $c_{02}=c_{22}=1,\ c_{mn}=0$ otherwise. Therefore,

$$u(x,y,t)=e^{-\left(\frac{2\pi}{b}\right)^2\kappa t}\sin\left(\frac{2\pi y}{b}\right)+e^{-\left[\left(\frac{2\pi}{a}\right)^2+\left(\frac{2\pi}{b}\right)^2\right]\kappa t}\cos\left(\frac{2\pi x}{a}\right)\sin\left(\frac{2\pi y}{b}\right).$$

14. $c_{0n}=\begin{cases}0,\ n=2k\\ \dfrac{4\cos\left(\frac{(2k-1)\pi}{3}\right)}{(2k-1)\pi},\ n=2k-1\end{cases},\quad c_{mn}=0.$

$$u(x,y,t)=\frac{4}{\pi}\sum_{k=1}^{\infty}\frac{\cos\left(\frac{(2k-1)\pi}{3}\right)e^{-\left(\frac{(2k-1)\pi}{b}\right)^2\kappa t}\sin\left(\frac{(2k-1)\pi y}{b}\right)}{2k-1}.$$

15. Substituting into the equations provided by (11b) and working through the integrals, we find

$$c_{0n}=\begin{cases}0, & n=2l\\ \dfrac{4}{3\pi}\cdot\dfrac{\cos\left(\frac{(2l-1)\pi}{3}\right)}{(2l-1)}, & n=2l-1\end{cases},$$

$$c_{mn}=\begin{cases}\dfrac{-16}{2k(2l-1)\pi^2}\sin\left(\dfrac{2k\pi}{3}\right)\cos\left(\dfrac{(2l-1)\pi}{3}\right), & m=2k,\ n=2l-1\\ 0 & \text{otherwise}\end{cases}.$$ Therefore,

$$u(x,y,t)=\frac{4}{3\pi}\sum_{k=1}^{\infty}\frac{\cos\left(\frac{(2k-1)\pi}{3}\right)e^{-\left((2k-1)\pi/b\right)^2\kappa t}\sin\left(\frac{(2k-1)\pi y}{b}\right)}{2k-1}-\frac{8}{\pi^2}\sum_{k=1}^{\infty}\sum_{l=1}^{\infty}\frac{e^{-\left[\left(2k\pi/a\right)^2+\left((2l-1)\pi/b\right)^2\right]\kappa t}}{k(2l-1)}$$
$$\cdot\sin\left(\frac{2k\pi}{3}\right)\cos\left(\frac{(2l-1)\pi}{3}\right)\cos\left(\frac{2k\pi x}{a}\right)\sin\left(\frac{(2l-1)\pi y}{b}\right).$$

17. $\displaystyle\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}A_{mn}\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)=\sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{\pi y}{b}\right)$,

$\displaystyle\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}c\sqrt{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2}B_{mn}\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)=0$. From examination,

$A_{11}=1,$ all other $A_{mn}=0,$ all $B_{mn}=0$. Therefore,

$$u(x,y,t)=\cos\left(\sqrt{\left(\frac{\pi}{a}\right)^2+\left(\frac{\pi}{b}\right)^2}\,ct\right)\sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{\pi y}{b}\right).$$

18. $\displaystyle\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}A_{mn}\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)=0$,

$\displaystyle\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}c\sqrt{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2}B_{mn}\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)=-\sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{2\pi y}{b}\right)$.

$$A_{mn}=0,\ B_{12}=-\frac{1}{c\sqrt{\left(\frac{\pi}{a}\right)^2+\left(\frac{2\pi}{b}\right)^2}}.$$

$$u(x,y,t)=-\frac{1}{c\sqrt{\left(\frac{\pi}{a}\right)^2+\left(\frac{2\pi}{b}\right)^2}}\sin\left(\sqrt{\left(\frac{\pi}{a}\right)^2+\left(\frac{2\pi}{b}\right)^2}\,ct\right)\sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{2\pi y}{b}\right).$$

19. $\displaystyle u(x,y,t)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[A_{mn}\cos\left(\sqrt{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2}\,ct\right)+B_{mn}\sin\left(\sqrt{\left(\frac{m\pi}{a}\right)^2+\left(\frac{n\pi}{b}\right)^2}\,ct\right)\right]\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)$

$\displaystyle u(x,y,0)=\sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{\pi y}{b}\right)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}A_{mn}\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)$, and examination gives us

$A_{11}=1,\ A_{mn}=0$ otherwise.

$u_t(x,y,0) = -2\sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{\pi y}{b}\right) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} c\sqrt{\left(\frac{m\pi}{a}\right)^2 + \left(\frac{n\pi}{b}\right)^2}\, B_{mn}\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)$, and examination gives us $\sqrt{\left(\frac{\pi}{a}\right)^2 + \left(\frac{\pi}{b}\right)^2}\, cB_{11} = -2,\ B_{mn} = 0$ otherwise. Therefore,

$$u(x,y,t) = \cos\left(\sqrt{\left(\frac{\pi}{a}\right)^2 + \left(\frac{\pi}{b}\right)^2}\, ct\right)\left(\frac{-2}{\sqrt{\left(\frac{\pi}{a}\right)^2 + \left(\frac{\pi}{b}\right)^2}\, c}\right)\left(\sin\left(\sqrt{\left(\frac{\pi}{a}\right)^2 + \left(\frac{\pi}{b}\right)^2}\, ct\right)\right)\sin\left(\frac{\pi x}{a}\right)\sin\left(\frac{\pi y}{b}\right).$$

20. $f(x,y) = 2\sin^2\left(\frac{\pi x}{a}\right)\sin\left(\frac{\pi y}{b}\right) = \left[1 - \cos\left(\frac{2\pi x}{a}\right)\right]\sin\left(\frac{\pi y}{b}\right)$.

$$B_{mn} = 0,\ A_{mn} = \frac{4}{ab}\int_0^a\int_0^b \left[1 - \cos\left(\frac{2\pi x}{a}\right)\right]\sin\left(\frac{\pi y}{b}\right)\sin\left(\frac{m\pi x}{a}\right)\sin\left(\frac{n\pi y}{b}\right)dydx$$

$$= \begin{cases} -\dfrac{16}{\pi(2k-1)\left[(2k-1)^2 - 4\right]}, & m = 2k-1,\ n = 1 \\ 0, & \text{otherwise} \end{cases}.$$

$$u(x,y,t) = -\frac{16}{\pi}\sum_{k=1}^{\infty}\frac{\cos\left(\sqrt{\left(\frac{(2k-1)\pi}{a}\right)^2 + \left(\frac{\pi}{b}\right)^2}\, ct\right)\sin\left(\frac{(2k-1)\pi x}{a}\right)\sin\left(\frac{\pi y}{b}\right)}{\pi(2k-1)\left[(2k-1)^2 - 4\right]}.$$

21 (a). Making the substitution

$\sum_{n=0}^{\infty} T_n'\cos\left(\frac{n\pi x}{\ell}\right) - \kappa\sum_{n=1}^{\infty} -\left(\frac{n\pi}{\ell}\right)^2 T_n\cos\left(\frac{n\pi x}{\ell}\right) = U_s h(\tau - t)\cdot\frac{1}{2}\left(1 - \cos\left(\frac{2\pi x}{\ell}\right)\right)$ gives us

$T_0' = \frac{U_s}{2}h(\tau - t),\ T_2' + \kappa\left(\frac{2\pi}{\ell}\right)^2 T_2 = -\frac{1}{2}U_s h(\tau - t),\ T_n' + \kappa\left(\frac{n\pi}{\ell}\right)^2 T_n = 0,\ n \ne 0,2$. Imposing the initial condition $u(x,0) = 0$ gives us $T_n = 0$ for all $n \ne 0,2$.

For T_0: $T_0' = \begin{cases} \frac{U_s}{2}, & 0 \le t \le \tau \\ 0, & t > \tau \end{cases}$, and antidifferentiation gives us $T_0 = \begin{cases} \frac{U_s}{2}t, & 0 \le t \le \tau \\ \frac{U_s}{2}\tau, & t > \tau \end{cases}$.

For T_2: $T_2' + \kappa\left(\frac{2\pi}{\ell}\right)^2 T_2 = \begin{cases} -\frac{1}{2}U_s, & 0 \le t \le \tau \\ 0, & t > \tau \end{cases}$. Therefore,

$$T_2 = \begin{cases} -\dfrac{U_s}{2\kappa}\left(\dfrac{\ell}{2\pi}\right)^2\left[1 - e^{-\kappa(2\pi/\ell)^2 t}\right], & 0 \le t \le \tau \\ -\dfrac{U_s}{2\kappa}\left(\dfrac{\ell}{2\pi}\right)^2\left[1 - e^{-\kappa(2\pi/\ell)^2 \tau}\right], & t > \tau \end{cases}.$$

21 (b). $u(x,t) = T_0 + T_2\cos\left(\frac{2\pi x}{\ell}\right)$. Therefore, $u\left(\frac{\ell}{2},t\right) = T_0 - T_2$.

For $0 \le t \le \tau$: $u\left(\frac{\ell}{2},t\right) = \frac{U_s}{2}t + \frac{U_s}{2\kappa}\left(\frac{\ell}{2\pi}\right)^2\left[1 - e^{-\kappa\left(2\pi/\ell\right)^2 t}\right] \approx 75t + \frac{75}{15.79}\left[1 - e^{-15.79t}\right]$. Setting this equal to zero and solving for τ yields $\tau = 0.936674$ hours.

22 (a). $\sum_{n=1}^{\infty} T_n{}'\sin\left(\frac{n\pi x}{\ell}\right) + \kappa\sum_{n=1}^{\infty} T_n\left(\frac{n\pi}{\ell}\right)^2\sin\left(\frac{n\pi x}{\ell}\right) = U_s\sin\left(\frac{\pi x}{\ell}\right)$.

$T_1' + \kappa\left(\frac{\pi}{\ell}\right)^2 T_1 = U_s$, $T_n{}' + \kappa\left(\frac{n\pi}{\ell}\right)^2 T_n = 0$. $T_1 = \left(\frac{\ell}{\pi}\right)^2\frac{U_s}{\kappa} + Ce^{-\kappa\left(\pi/\ell\right)^2}$, $U_0 = \left(\frac{\ell}{\pi}\right)^2\frac{U_s}{\kappa} + C$,

$T_1 = \left(\frac{\ell}{\pi}\right)^2\frac{U_s}{\kappa} + \left(U_0 - \left(\frac{\ell}{\pi}\right)^2\frac{U_s}{\kappa}\right)e^{-\kappa\left(\pi/\ell\right)^2 t}$, $T_n = 0$. $u(x,t) = T_1\sin\left(\frac{\pi x}{\ell}\right) = \left(\frac{\ell}{\pi}\right)^2\frac{U_s}{\kappa}\sin\left(\frac{\pi x}{\ell}\right)$.

22 (b). Assume $u = U_0\sin\left(\frac{\pi x}{\ell}\right)$. $0 - \kappa\left(-\left(\frac{\pi}{\ell}\right)^2\right)U_0\sin\left(\frac{\pi x}{\ell}\right) = U_s\sin\left(\frac{\pi x}{\ell}\right)$. $\kappa\left(\frac{\pi}{\ell}\right)^2 U_0 = U_s$.

24. $T_{1_C} = c_1\cos\left(\frac{\pi ct}{\ell}\right) + c_2\sin\left(\frac{\pi ct}{\ell}\right)$. $T_{1_P} = A\cos\omega t + B\sin\omega t$.

$T_{1_P}{}'' + \left(\frac{\pi c}{\ell}\right)^2 T_{1_P} = A\left(-\omega^2 + \left(\frac{\pi c}{\ell}\right)^2\right)\cos\omega t + B\left(-\omega^2 + \left(\frac{\pi c}{\ell}\right)^2\right)\sin\omega t = \cos\omega t$

$\Rightarrow A = \frac{1}{\left(-\omega^2 + \left(\frac{\pi c}{\ell}\right)^2\right)}$, $B = 0$. $T_1 = c_1\cos\left(\frac{\pi ct}{\ell}\right) + c_2\sin\left(\frac{\pi ct}{\ell}\right) + A\cos\omega t$. $T_1'(0) = \frac{\pi c}{\ell}c_2 = 0$,

$T_1(0) = c_1 + A = 0$. $T_1 = \frac{\cos\omega t - \cos\left(\frac{\pi ct}{\ell}\right)}{\left(\frac{\pi c}{\ell}\right)^2 - \omega^2}$, $u(x,t) = T_1\sin\left(\frac{\pi x}{\ell}\right)$, $u\left(\frac{\ell}{2},t\right) = T_1 = \frac{\cos 2t - \cos(2\pi t)}{4\pi^2 - 4}$.

25. The complementary solution is $T_{1_C} = c_1\cos(\pi t) + c_2\sin(\pi t)$. The particular solution is $T_{1_P} = t[A\cos\pi t + B\sin\pi t]$. Differentiation gives us

$T_{1_P}{}' = [A\cos\pi t + B\sin\pi t] + t[-\pi A\sin\pi t + \pi B\cos\pi t]$ and

$T_{1_P}{}'' = [-\pi A\sin\pi t + \pi B\cos\pi t] + [-\pi A\sin\pi t + \pi B\cos\pi t] + t\left[-\pi^2 A\cos\pi t - \pi^2 B\sin\pi t\right]$.

$T_{1_P}{}'' + \pi^2 T_{1_P} = -2\pi A\sin\pi t + 2\pi B\cos\pi t = \cos\pi t \Rightarrow A = 0,\ B = \frac{1}{2\pi}$. Thus

$T_{1_P} = \frac{t}{2\pi}\sin\pi t$, and so $T_1 = c_1\cos\pi t + c_2\sin\pi t + \frac{t}{2\pi}\sin\pi t$. Imposing the conditions

$T_1(0) = 0 = c_1$ and $T_1'(0) = 0 = \pi c_2$ gives us $T_1 = \frac{t}{2\pi}\sin\pi t$, and so

$u(x,t) = T_1\sin\left(\frac{\pi x}{\ell}\right) = \frac{t}{2\pi}\sin\pi t\sin\left(\frac{\pi x}{\ell}\right)$.

26 (b). $u(0,y)=\sum_{n=1}^{\infty} X_n(0)\sin\left(\frac{n\pi y}{b}\right),\ u(a,y)=\sum_{n=1}^{\infty} X_n(a)\sin\left(\frac{n\pi y}{b}\right),\ X_n(0)=X_n(a)=0$.

$X_2(x)=c_1e^{2\pi x/b}+c_2e^{-2\pi x/b}+X_{2_P}$.

$X_{2_P}=Ax+B \Rightarrow 0-\left(\frac{2\pi}{b}\right)^2(Ax+B)=x \Rightarrow X_2(x)=c_1e^{2\pi x/b}+c_2e^{-2\pi x/b}-\left(\frac{b}{2\pi}\right)^2 x$.

$c_1+c_2=0,\ c_1e^{2\pi a/b}+c_2e^{-2\pi a/b}=\left(\frac{b}{2\pi}\right)^2 a \Rightarrow c_1=-c_2=\frac{a\left(b/2\pi\right)^2}{e^{2\pi a/b}-e^{-2\pi a/b}}$. $X_n=0,\ n\neq 2$.

$$u(x,y)=\left[\frac{a\left(b/2\pi\right)^2\left(e^{2\pi x/b}-e^{-2\pi x/b}\right)}{e^{2\pi a/b}-e^{-2\pi a/b}}-\left(\frac{b}{2\pi}\right)^2 x\right]\sin\left(\frac{2\pi y}{b}\right).$$

27. $u(x,y)=\sum_{n=1}^{\infty}\sin(n\pi x)Y_n$. Making this substitution, we have $\sum_{n=1}^{\infty}-(n\pi)^2\sin(n\pi x)Y_n+\sum_{n=1}^{\infty}\sin(n\pi x)Y_n''=e^y\sin\pi x$. From this equation, we can obtain two initial value problems: $Y_1''-\pi^2Y_1=e^y,\ Y_1(0)=Y_1(1)=0$ and $Y_n''-\pi^2Y_n=0,\ Y_n(0)=Y_n(1)=0,\ n\geq 2$. For the first, IVP, the complementary solution is $Y_{1_C}=c_1e^{\pi y}+c_2e^{-\pi y}$, and the particular solution is $Y_{1_P}=Ae^y \Rightarrow A\left(1-\pi^2\right)=1 \Rightarrow A=\frac{-1}{\pi^2-1}$. Therefore, we have $Y_1=c_1e^{\pi y}+c_2e^{-\pi y}-\frac{e^y}{\pi^2-1}$. Imposing the initial conditions gives us $c_1+c_2=\frac{1}{\pi^2-1},\ e^{\pi}c_1+e^{-\pi}c_2=\frac{e}{\pi^2-1}$, and solving these simultaneous equations yields $c_1=\frac{1}{\pi^2-1}\left(\frac{e^{-\pi}-e}{e^{-\pi}-e^{\pi}}\right),\ c_2=\frac{1}{\pi^2-1}\left(\frac{-e^{\pi}+e}{e^{-\pi}-e^{\pi}}\right)$. Thus $Y_1=\frac{1}{\pi^2-1}\left(\frac{\left(e-e^{-\pi}\right)e^{\pi y}-\left(e-e^{\pi}\right)e^{-\pi y}}{e^{\pi}-e^{-\pi}}-e^y\right)$. For the second IVP, we can quickly see that the only unique solution is trivial, and so $Y_n=0,\ n\geq 2$. Finally, we have

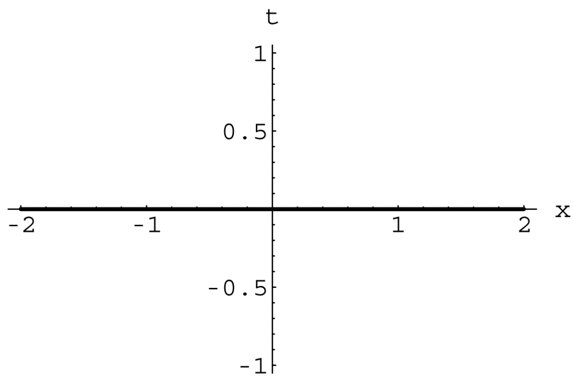
$$u(x,y)=Y_1\sin\pi x=\frac{\sin\pi x}{\pi^2-1}\left(\frac{\left(e-e^{-\pi}\right)e^{\pi y}-\left(e-e^{\pi}\right)e^{-\pi y}}{e^{\pi}-e^{-\pi}}-e^y\right).$$

Chapter 12
First Order Partial Differential Equations and the Method of Characteristics

Section 12.1

1 (a). Differentiation gives us $u_x = 3(x+\alpha t)^2$ and $u_t = 3\alpha(x+\alpha t)^2$. Therefore, $3(x+\alpha t)^2(1+\alpha) = 0 \Rightarrow \alpha = -1$.

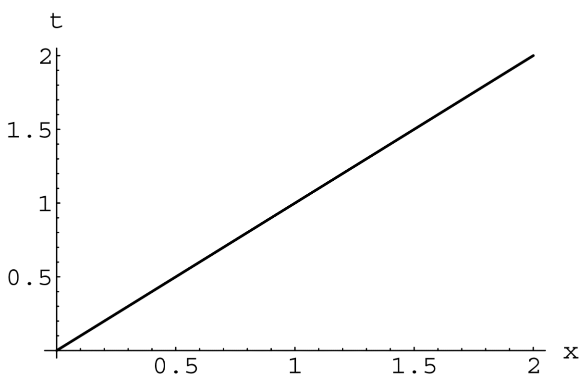
1 (b).



1 (c). $\omega(t) = (2\tau + \alpha \cdot 0)^3 = 8\tau^3$

2 (a). $u_x = \cos(x+\alpha t)$ and $u_t = \alpha\cos(x+\alpha t)$. $\cos(x+\alpha t)(-2+\alpha) = 0 \Rightarrow \alpha = 2$.

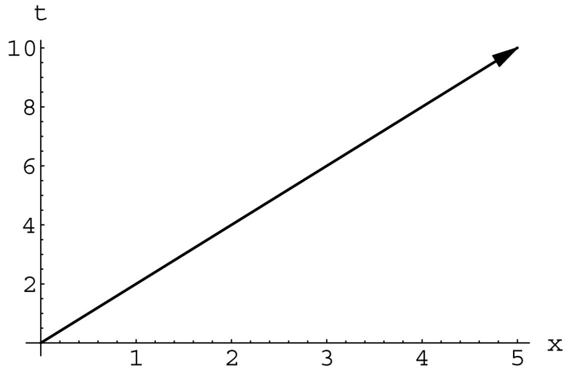
2 (b).



2 (c). $\omega(t) = \sin(\tau + 2\tau) = \sin 3\tau$.

3 (a). Differentiation gives us $u_x = 3(2x+t)^2 \cdot 2$ and $u_t = 3(2x+t)^2$. Therefore, $3(2x+t)^2(2+\alpha) = 0 \Rightarrow \alpha = -2$.

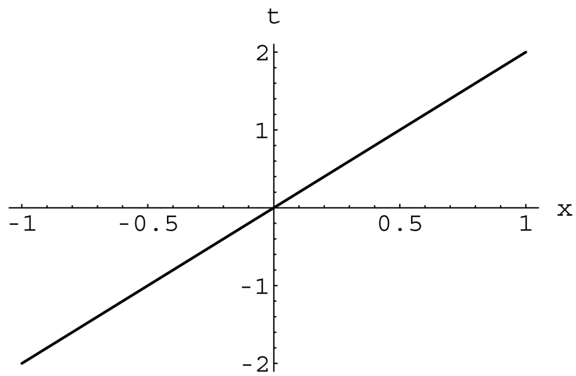
3 (b).



3 (c). $\omega(t) = (2\tau + 2\tau)^3 = 64\tau^3$

4 (a). $u_x = 0$ and $u_t = 2e^{2t}$. $0 + 2\alpha e^{2t} = 0 \Rightarrow \alpha = 0$.

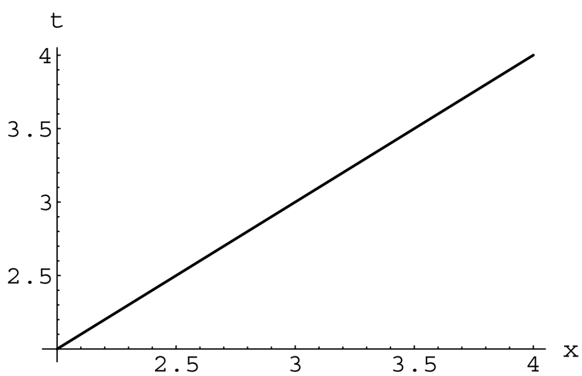
4 (b).



4 (c). $\omega(t) = e^{2(2\tau)} = e^{4\tau}$.

5 (a). Differentiation gives us $u_x = 3\left(xe^{\alpha t}\right)^2 e^{\alpha t}$ and $u_t = 3\left(xe^{\alpha t}\right)^2 \cdot x\alpha e^{\alpha t}$. Therefore, $3x\left(xe^{\alpha t}\right)^2 e^{\alpha t}(1+\alpha) = 0 \Rightarrow \alpha = -1$.

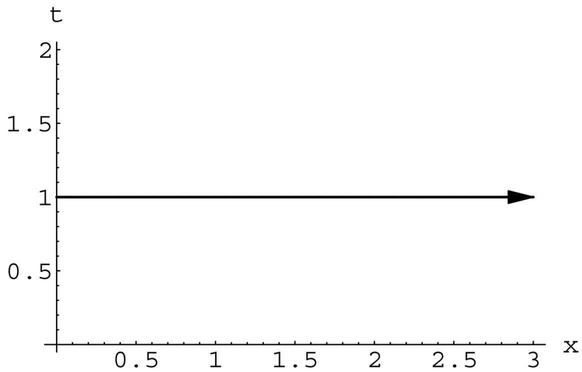
5 (b).



5 (c). $\omega(t) = \left(\tau e^{-\tau}\right)^3$

6 (a). $u_x = 1$ and $u_t = -\dfrac{2}{t}$. $1 - \alpha t\dfrac{2}{t} = 0 \Rightarrow \alpha = -\dfrac{1}{2}$.

6 (b).



6 (c). $\omega(t) = \tau - 2\ln 1 = \tau$.

7. Differentiation gives us $u_x = 2xt^3$ and $u_t = 3x^2t^2$. Therefore,

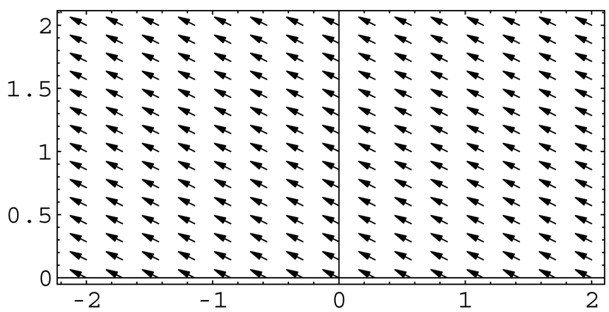
$$a(2xt^3) + xt^2(3x^2t^2) = xt^3(2a + 3x^2t) = 0 \Rightarrow a(x,t) = -\frac{3}{2}x^2t.$$

8. $u_x = e^{-t}$ and $u_t = -xe^{-t}$. $xe^{-t} + b(-xe^{-t}) = 0 \Rightarrow b(x,t) = -1$.

9. Differentiation gives us $u_x = f'(x^3 - t)\cdot 3x^2$ and $u_t = f'(x^3 - t)(-1)$. Therefore,

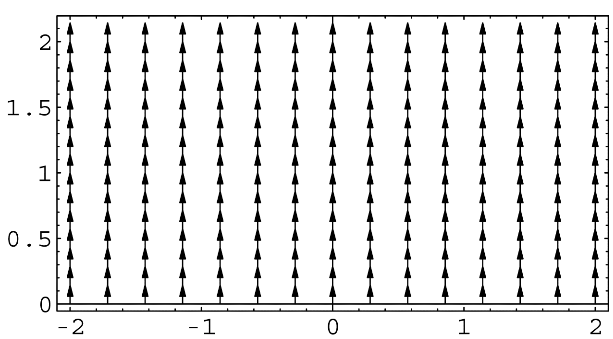
$$f'(x^3 - t)(3x^2 - b) = 0 \Rightarrow b(x,t) = 3x^2.$$

10 (a).



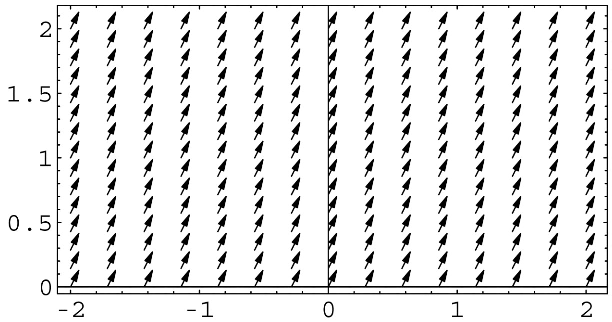
10 (b). We predict that $u(0,1) > 0$.

11 (a).



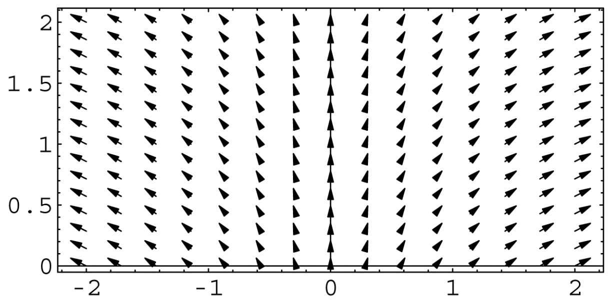
11 (b). We predict that $u(0,1) = 0$.

12 (a).



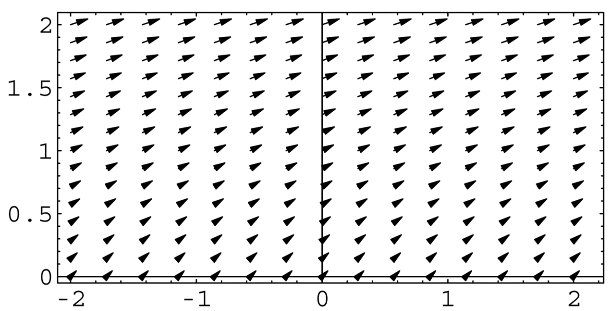
12 (b). We predict that $u(0,1) < 0$.

13 (a).



13 (b). We predict that $u(0,1) = 0$.

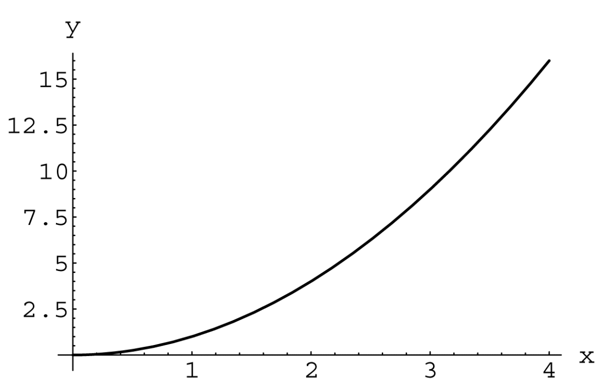
14 (a).



14 (b). We predict that $u(0,1) < 0$.

Section 12.2

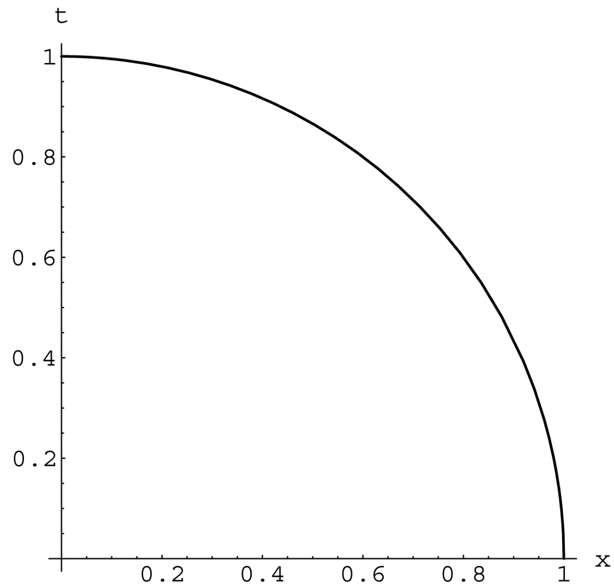
1 (a).



1 (b). $\phi' b - \psi' a = 1 \cdot 1 - 2\tau \cdot 1 = 0 \Rightarrow \tau = \frac{1}{2}$.

1 (c). Hypothesis *v* fails.

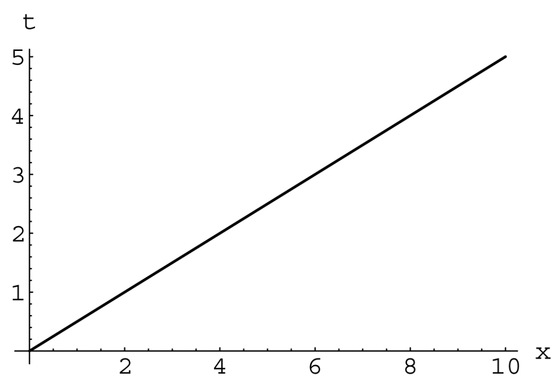
2 (a).



2 (b). $\phi' b - \psi' a = -\sin\tau(1) - \cos\tau(1) = 0 \Rightarrow \tau = \tan^{-1}(-1)$.

2 (c). Hypothesis *v* fails.

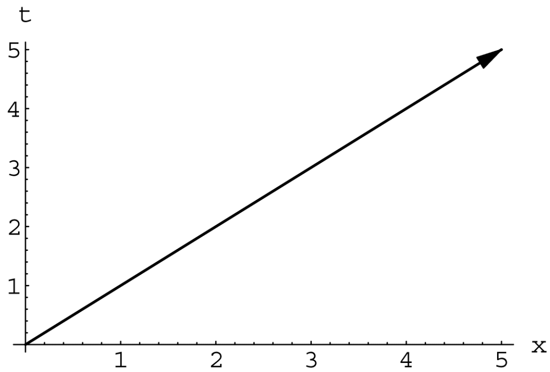
3 (a).



3 (b). $\phi' b - \psi' a = 1 \cdot 1 - \frac{1}{2} \cdot 2 = 0 \Rightarrow$ The transversality condition does not hold for any value of τ on the interval.

3 (c). Hypothesis *v* fails.

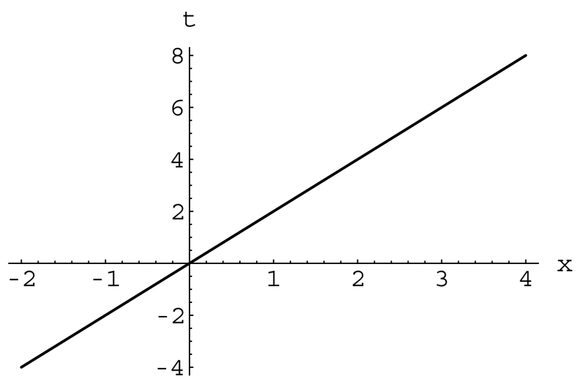
4 (a).



4 (b). $\phi' b - \psi' a = 2 - 1 = 1 \neq 0 \Rightarrow$ the transversality condition holds for all τ.

4 (c). The hypotheses are satisfied.

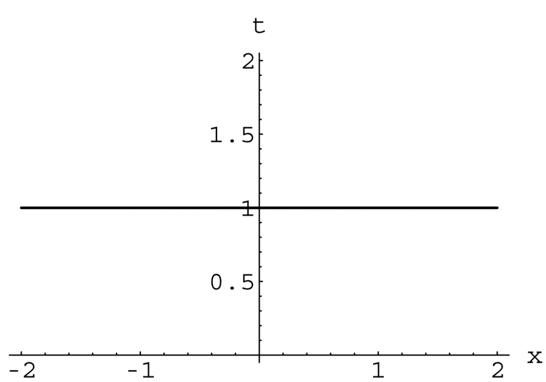
5 (a).



5 (b). $\phi' b - \psi' a = 1 \cdot 1 - 2 \cdot 2\tau = 0 \Rightarrow \tau = \dfrac{1}{4}$.

5 (c). Hypothesis *v* fails.

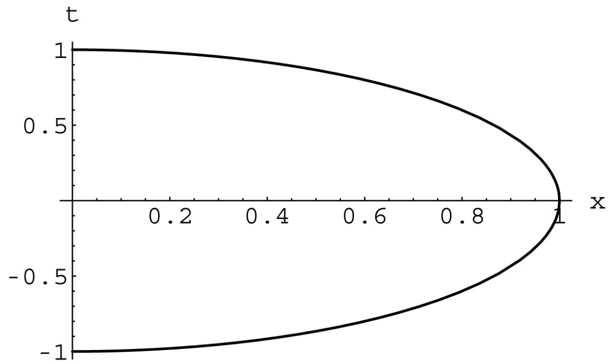
6 (a).



6 (b). $\phi' b - \psi' a = 1 - 0 = 1 \neq 0 \Rightarrow$ the transversality condition holds for all τ.

6 (c). The hypotheses are satisfied.

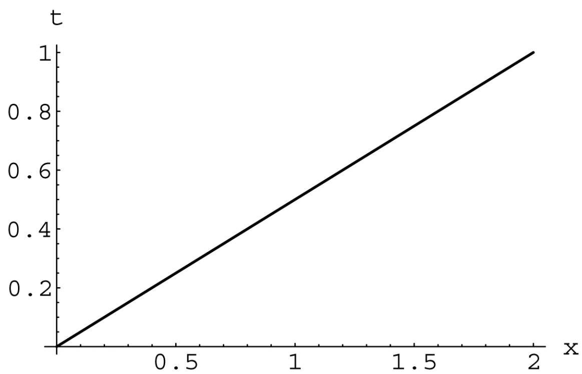
7 (a).



7 (b). $\phi' b - \psi' a = -\sin\tau(-\cos\tau) - \cos\tau(\sin\tau) = 0 \Rightarrow$ The transversality condition does not hold for any value of τ on the interval.

7 (c). Hypothesis *v* fails.

8 (a).



8 (b). $\phi' b - \psi' a = e^{-\tau} - \dfrac{1}{2} = 0 \Rightarrow \tau = \ln 2$.

8 (c). Hypothesis *v* fails.

9 (c). The solution is not unique. However, the hypotheses of Th. 12.1 are not satisfied, so the fact that the conclusions do not hold does not present any inconsistency.

Section 12.3

1. $x = \tau,\ t = 0,\ u(\tau,0) = \sin\tau$. Therefore,
$\dfrac{\partial x}{\partial s} = 1,\ x(0,\tau) = \tau;\ \dfrac{\partial t}{\partial s} = -2,\ t(0,\tau) = 0 \Rightarrow t = -2s \Rightarrow s = -\dfrac{t}{2} \Rightarrow \tau = x + \dfrac{t}{2}$. Finally,
$u(x,t) = \sin\left(x + \dfrac{t}{2}\right)$.

2. $x = 0,\ t = \tau,\ u(0,\tau) = e^{-\tau}$.
$\dfrac{\partial x}{\partial s} = 2,\ x(0,\tau) = 0;\ \dfrac{\partial t}{\partial s} = 3,\ t(0,\tau) = \tau \Rightarrow t = \tau + 3s \Rightarrow s = \dfrac{x}{2} \Rightarrow \tau = t - \dfrac{3x}{2}.\ u(x,t) = e^{-\left(t - \frac{3}{2}x\right)}$.

3. $x = \tau,\ t = 1,\ u(\tau,1) = \tau^2$. Therefore,
$\dfrac{\partial x}{\partial s} = x,\ x(0,\tau) = \tau;\ \dfrac{\partial t}{\partial s} = 1,\ t(0,\tau) = 1 \Rightarrow t = s + 1,\ x = \tau e^{s} \Rightarrow s = t - 1 \Rightarrow \tau = xe^{-(t-1)}$. Finally,
$u(x,t) = x^2 e^{-2(t-1)}$.

4. $x=\tau,\ t=\tau,\ u(\tau,\tau)=\tau^3$.
$\frac{\partial x}{\partial s}=1,\ x(0,\tau)=\tau;\ \frac{\partial t}{\partial s}=-2,\ t(0,\tau)=\tau\Rightarrow t=-2s+\tau\Rightarrow s=\frac{x-t}{3}\Rightarrow\tau=x-\frac{x-t}{3}=\frac{2x+t}{3}$.
$u(x,t)=\frac{(2x+t)^3}{27}$.

5. $x=\tau,\ t=2\tau,\ u(\tau,2\tau)=\tau^2$. Therefore,
$\frac{\partial x}{\partial s}=1;\ \frac{\partial t}{\partial s}=-1,\ \Rightarrow t=-s+2\tau,\ x=s+\tau\Rightarrow s=\frac{(2x-t)}{3}\Rightarrow\tau=\frac{x+t}{3}$. Finally,
$u(x,t)=\frac{(x+t)^2}{9}$.

6. $x=\tau,\ t=-1,\ u(\tau,-1)=\cos\pi\tau$. $\frac{\partial x}{\partial s}=x,\ x(0,\tau)=\tau;\ \frac{\partial t}{\partial s}=t,\ t(0,\tau)=-1\Rightarrow t=-e^s\Rightarrow\tau=-\frac{x}{t}$.
$u(x,t)=\cos\left(\frac{-\pi x}{t}\right)$. As t gets larger for fixed x, u undergoes increasingly rapid oscillations. The solution exists for $t<0$.

7. $x=\tau,\ t=0,\ u(\tau,0)=\tau$. Therefore,
$\frac{\partial x}{\partial s}=1-x,\ x(0,\tau)=\tau;\ \frac{\partial t}{\partial s}=1,\ t(0,\tau)=0\Rightarrow t=s,\ x=1+(\tau-1)e^{-s}\Rightarrow\tau=1+e^t(x-1)$. Finally,
$u(x,t)=1+e^t(x-1)$.

8. $x=\tau,\ t=1,\ u(\tau,1)=\tau$.
$\frac{\partial x}{\partial s}=1,\ x(0,\tau)=\tau;\ \frac{\partial t}{\partial s}=t^2,\ t(0,\tau)=1\Rightarrow s=1-\frac{1}{t}\Rightarrow\tau=x-s=x-1+\frac{1}{t}$. $u(x,t)=x-1+\frac{1}{t}$. The solution exists for $t>0$, all x.

9 (a). $x=\tau,\ t=0,\ u(\tau,0)=e^{-\tau^2}$. Therefore,
$\frac{\partial x}{\partial s}=1,\ x(0,\tau)=\tau;\ \frac{\partial t}{\partial s}=1,\ t(0,\tau)=0\Rightarrow t=s,\ x=\tau+s\Rightarrow\tau=x-t$. Then we have
$\frac{\partial u}{\partial s}=1,\ u(\tau,0)=e^{-\tau^2}\Rightarrow u=s+e^{-\tau^2}$. Finally, $u(x,t)=t+e^{-(x-t)^2}$.

9 (b). The solution exists on the entire half plane.

9 (c). The solution has a maximum at $x=1;\ u(1,1)=2$.

10 (a). $x=\tau,\ t=0,\ u(\tau,0)=e^{-\tau^2}$. $\frac{\partial x}{\partial s}=1,\ x(0,\tau)=\tau;\ \frac{\partial t}{\partial s}=1,\ t(0,\tau)=0\Rightarrow t=s,\ x=\tau+s\Rightarrow\tau=x-t$.
$\frac{\partial u}{\partial s}=s(\tau+s),\ u(0,\tau)=e^{-\tau^2}\Rightarrow u=\frac{s^2}{2}\tau+\frac{s^3}{3}+e^{-\tau^2}$. $u(x,t)=\frac{t^2}{2}(x-t)+\frac{t^3}{3}+e^{-(x-t)^2}$.

10 (b). The solution exists on the entire half plane.

10 (c). No finite maximum or minimum exists.

11 (a). $x=\tau,\ t=0,\ u(\tau,0)=e^{-\tau^2}$. Therefore,
$\frac{\partial x}{\partial s}=1,\ x(0,\tau)=\tau;\ \frac{\partial t}{\partial s}=2,\ t(0,\tau)=0\Rightarrow t=2s,\ x=\tau+s\Rightarrow\tau=x-\frac{t}{2}$. Then we have
$\frac{\partial u}{\partial s}=-u,\ u|_{s=0}=e^{-\tau^2}\Rightarrow u=e^{-\tau^2-s}$. Finally, $u(x,t)=e^{-(x-t/2)^2-t/2}$.

11 (b). The solution exists on the entire half plane.

11 (c). The solution has a maximum at $x=\frac{1}{2};\ u\left(\frac{1}{2},1\right)=e^{-1/2}$.

12 (a). $x=\tau,\ t=0,\ u(\tau,0)=e^{-\tau^2}$.

$\frac{\partial x}{\partial s}=1,\ x(0,\tau)=\tau;\ \frac{\partial t}{\partial s}=2,\ t(0,\tau)=0\Rightarrow t=2s,\ x=\tau+s\Rightarrow \tau=x-\frac{t}{2}$.

$\frac{\partial u}{\partial s}=2su,\ u(\tau,0)=e^{-\tau^2}\Rightarrow u=e^{-\tau^2+s^2}.\ u(x,t)=e^{-(x-t/2)^2+t^2/4}$.

12 (b). The solution exists on the entire half plane.

12 (c). The solution has a maximum at $x=\frac{1}{2};\ u\left(\frac{1}{2},1\right)=e^{1/4}$.

13 (a). $x=\tau,\ t=0,\ u(\tau,0)=e^{-\tau^2}$. Therefore,

$\frac{\partial x}{\partial s}=1,\ x(0,\tau)=\tau;\ \frac{\partial t}{\partial s}=2,\ t(0,\tau)=0\Rightarrow t=2s,\ x=\tau+s\Rightarrow \tau=x-\frac{t}{2}$. Then we have

$\frac{\partial u}{\partial s}=u+4s\Rightarrow u=-4(s+1)+Ce^s;\ -4+C=e^{-\tau^2}\Rightarrow C=e^{-\tau^2}+4$. Finally,

$u(x,t)=-4\left(\frac{t}{2}+1\right)+\left(e^{-(x-t/2)^2}+4\right)e^{t/2}$.

13 (b). The solution exists on the entire half plane.

13 (c). The solution has a maximum at $x=\frac{1}{2};\ u\left(\frac{1}{2},1\right)\approx 2.24$.

14 (a). $x=\tau,\ t=0,\ u(\tau,0)=e^{-\tau^2}$.

$\frac{\partial x}{\partial s}=t,\ x(0,\tau)=\tau;\ \frac{\partial t}{\partial s}=-1,\ t(0,\tau)=0\Rightarrow t=-s,\ x=-\frac{s^2}{2}+\tau\Rightarrow \tau=x+\frac{t^2}{2}$.

$\frac{\partial u}{\partial s}=-\frac{s^2}{2}+\tau,\ u|_{s=0}=e^{-\tau^2}\Rightarrow u=-\frac{s^3}{6}+\tau s+e^{-\tau^2}.\ u(x,t)=\frac{t^3}{6}-t\left(x+\frac{t^2}{2}\right)+e^{-\left(x+t^2/2\right)^2}$.

14 (b). The solution exists on the entire half plane.

14 (c). No finite maximum or minimum exists.

15 (a). $x=\tau,\ t=0,\ u(\tau,0)=e^{-\tau^2}$. Therefore,

$\frac{\partial x}{\partial s}=1,\ x(0,\tau)=\tau;\ \frac{\partial t}{\partial s}=(2t-1)^2,\ t(0,\tau)=0\Rightarrow x=\tau+s,\ \frac{dt}{(2t-1)^2}=ds\Rightarrow \frac{-1/2}{2t-1}=s+C$.

Substitution gives us $\frac{-1/2}{-1}=C=\frac{1}{2}$. Then we have

$s=\frac{-1/2}{2t-1}-\frac{1}{2}=-\frac{1}{2}\left(1+\frac{1}{2t-1}\right)=\frac{-t}{2t-1}=\frac{t}{1-2t}.\ \frac{\partial u}{\partial s}=1,\ u|_{s=0}=e^{-\tau^2}\Rightarrow u=s+e^{-\tau^2},\ \tau=x-s$.

Finally, $u(x,t)=\frac{t}{1-2t}+e^{-(x-t/(1-2t))^2}$.

15 (b). The solution exists on $-\infty<x<\infty,\ 0\le t<\frac{1}{2}$.

15 (c). The solution does not exist at $t=1$.

16 (a). $x=\tau,\ t=0,\ u(\tau,0)=e^{-\tau^2}$.

$\frac{\partial x}{\partial s}=v,\ x(0,\tau)=\tau;\ \frac{\partial t}{\partial s}=1,\ t(0,\tau)=0\Rightarrow t=s,\ x=\tau+vs\Rightarrow \tau=x-vt$.

$\frac{\partial u}{\partial s}=-cu,\ u|_{s=0}=e^{-\tau^2}\Rightarrow u=e^{-\tau^2-cs}.\ u(x,t)=e^{-(x-vt)^2-ct}$.

16 (b). If v=5 mi/hr, the peak value reaches 20 miles downstream at t=4 hours. At t=4 and x=20 mi, $u(20,4) = e^{-5c} = 0.05 \Rightarrow c = 0.599\text{hr}^{-1}$.

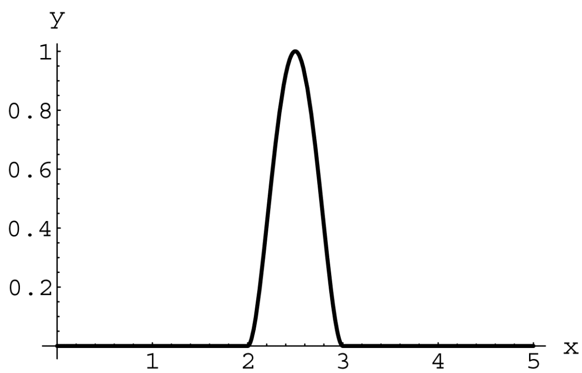
17 (b). $\frac{dx}{ds} = v,\ x(0,\tau) = 0,\ \frac{dt}{ds} = 1,\ t(0,\tau) = \tau \Rightarrow x = vs,\ t = s + \tau$.

$$\frac{du}{ds} = 0 \Rightarrow u = \omega(\tau) = \begin{cases} 16\tau^2(1-\tau)^2, & 0 \le \tau \le 1 \\ 0, & \tau > 1 \end{cases}.$$ Therefore,

$$u(x,t) = \begin{cases} 16\left(t - \frac{x}{v}\right)^2 \left(1 - \left(t - \frac{x}{v}\right)\right)^2, & 0 \le t - \frac{x}{v} \le 1 \\ 0, & t - \frac{x}{v} > 1 \end{cases}.$$

17 (c). $\frac{du}{ds} = 0,\ u = \omega(\tau) = 0 \Rightarrow u(x,t) = 0,\ t - \frac{x}{v} < 0$.

17 (d).



18 (a). $s = t,\ \tau = x - vt,\ \frac{\partial u}{\partial s} = -cu^2,\ u|_{s=0} = e^{-\tau^2}$. $\frac{du}{u^2} = -cds \Rightarrow -u^{-1} = -cs + C,\ C = -e^{\tau^2},\ u = \frac{1}{cs + e^{\tau^2}}$.

$$u(x,t) = \frac{1}{ct + e^{(x-vt)^2}}.$$

18 (b). As in (16 b), the peak value reaches 20 miles downstream at t=4 hours. At t=4 and x=20 mi, $u(20,4) = \frac{1}{4c+1} = 0.05 \Rightarrow c = \frac{19}{4} = 4.75\text{hr}^{-1}$.

Chapter 13
Linear Two-Point Boundary Value Problems

Section 13.1

Note: Part (a) of Exercises 1-7 are identical.

1 (a). $y(t) = c_1 \cos\left(\frac{t}{2}\right) + c_2 \sin\left(\frac{t}{2}\right) + 4$

1 (b). Applying the boundary conditions, we have $y(0) = c_1 + 4 = 0,\ y(\pi) = c_2 + 4 = 2$. Solving these simultaneous equations yields $c_1 = -4$ and $c_2 = -2$. Thus there is a unique solution: $y(t) = -4\cos\left(\frac{t}{2}\right) - 2\sin\left(\frac{t}{2}\right) + 4$.

2 (b). $y'(0) = \frac{c_2}{2} = 0,\ y'(\pi) = -\frac{c_1}{2} = 0$. $c_1 = c_2 = 0$. Thus there is a unique solution: $y(t) = 4$.

3 (b). Differentiation gives us $y' = -\frac{c_1}{2}\sin\left(\frac{t}{2}\right) + \frac{c_2}{2}\cos\left(\frac{t}{2}\right)$. Applying the boundary conditions, we have $y'(0) = \frac{c_2}{2} = -2,\ y(\pi) = c_2 + 4 = 0$. Solving these simultaneous equations yields c_1 arbitrary and $c_2 = -4$. Thus there are infinitely many solutions: $y(t) = c_1\cos\left(\frac{t}{2}\right) - 4\sin\left(\frac{t}{2}\right) + 4$.

4 (b). $y(0) = c_1 + 4 = 0,\ y'(\pi) = -\frac{c_1}{2} = 1$. There is no solution.

5 (b). Differentiation gives us $y' = -\frac{c_1}{2}\sin\left(\frac{t}{2}\right) + \frac{c_2}{2}\cos\left(\frac{t}{2}\right)$. Applying the boundary conditions, we have $y(0) + 2y'(0) = c_1 + 4 + c_2 = 0,\ y(\pi) + 2y'(\pi) = c_2 + 4 - c_1 = 0$. Solving these simultaneous equations yields $c_1 = 0$ and $c_2 = -4$. Thus there is a unique solution: $y(t) = -4\sin\left(\frac{t}{2}\right) + 4$.

6 (b). $y(0) + 2y'(0) = c_1 + 4 + c_2 = 0,\ y(\pi) - 2y'(\pi) = c_2 + 4 - (2)\frac{-c_1}{2} = 0$. There are infinitely many solutions: $y = c_1\cos\frac{t}{2} - (c_1 + 4)\sin\frac{t}{2} + 4$.

7 (b). Differentiation gives us $y' = -\frac{c_1}{2}\sin\left(\frac{t}{2}\right) + \frac{c_2}{2}\cos\left(\frac{t}{2}\right)$. Applying the boundary conditions, we have $y(0) + 2y'(0) = c_1 + 4 + c_2 = 4,\ y(\pi) - 2y'(\pi) = c_2 + 4 + c_1 = 0$. These simultaneous equations cannot be solved, and so there are no solutions.

8. $y'' - 4y = 0,\ y(0) = 0,\ y(2) = 1$. $\gamma = -4,\ \alpha = 0,\ \beta = 1$.

9. Differentiation gives us $y' = 2,\ y'' = 0$. Thus $\gamma = 0,\ \alpha = 2,\ \beta = 1$.

10. $y'' + \gamma y = e^t - \sin t + \gamma\left(e^t + \sin t\right) = 2e^t$, $y(0) = 1$, $y\left(\frac{\pi}{2}\right) = e^{\pi/2} + 1$. $\gamma = 1$, $\alpha = 1$, $\beta = 1 + e^{\pi/2}$.

11. The general solution is $y = c_1 \cos t + c_2 \sin t + 1$. Differentiation gives us $y' = -c_1 \sin t + c_2 \cos t$. From the boundary conditions,
$y(0) + a_1 y'(0) = 5 = c_1 + 1 + a_1 c_2$, $y\left(\frac{\pi}{2}\right) + y'\left(\frac{\pi}{2}\right) = \beta = c_2 + 1 - c_1$. From the graph, we can see that $y(0) = 1$, $y\left(\frac{\pi}{2}\right) = 3$, and so $c_1 + 1 = 1 \Rightarrow c_1 = 0$, $c_2 + 1 = 3 \Rightarrow c_2 = 2$. Finally,
$0 + a_1 \cdot 2 = 4 \Rightarrow a_1 = 2$ and $0 + 2 = \beta - 1 \Rightarrow \beta = 3$.

14 (a). $t^2 z'' - 2tz' + 2z = 0$, $z(1) + z'(1) = 0$, $z(2) - z'(2) = 0$

14 (b). $z = c_1 t + c_2 t^2$, $z' = c_1 + 2c_2 t$. $[c_1 + c_2] + [c_1 + 2c_2] = 0$, $[2c_1 + 4c_2] - [c_1 + 4c_2] = 0$. $c_1 = c_2 = 0$, $z = 0$.

14 (c). The given problem has a unique solution.

14 (d). $y = c_1 t + c_2 t^2$, $y' = c_1 + 2c_2 t$. $\begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 9 \\ 3 \end{bmatrix}$. $y = 3t + t^2$.

15 (a). $t^2 z'' - 2tz' + 2z = 0$, $2z(1) - z'(1) = 0$, $z(2) - z'(2) = 0$

15 (b). $z = c_1 t + c_2 t^2$, $z' = c_1 + 2c_2 t$. Applying the boundary conditions gives us $2[c_1 + c_2] - [c_1 + 2c_2] = 0$, $[2c_1 + 4c_2] - [c_1 + 4c_2] = 0$. Solving these simultaneous equations gives us $z = c_2 t^2$, c_2 arbitrary.

15 (c). By the Fredholm Alternative Theorem, the given problem does not have a unique solution.

15 (e). $y = c_1 t + c_2 t^2$, $y' = c_1 + 2c_2 t$. Applying the boundary conditions gives us $2y(1) - y'(1) = c_1 = 1$, $y(2) - y'(2) = c_1 = 1$. Solving these simultaneous equations gives us $y = t + c_2 t^2$, c_2 arbitrary.

16 (a). $t^2 z'' - 2tz' + 2z = 0$, $3z(1) - 2z'(1) = 0$, $5z(2) - 6z'(2) = 0$

16 (b). $z = c_1 t + c_2 t^2$, $z' = c_1 + 2c_2 t$. $3[c_1 + c_2] - 2[c_1 + 2c_2] = 0$, $5[2c_1 + 4c_2] - 6[c_1 + 4c_2] = 0$. $c_1 = c_2$.

16 (c). By the Fredholm Alternative Theorem, the given problem does not have a unique solution.

16 (e). $y = c_1 t + c_2 t^2$, $y' = c_1 + 2c_2 t$. $3y(1) - 2y'(1) = c_1 - c_2 = 2$, $5y(2) - 6y'(2) = 4c_1 - 4c_2 = 3$. There is no solution.

17 (a). $t^2 z'' - 2tz' + 2z = 0$, $z(1) - 2z'(1) = 0$, $2z(2) - z'(2) = 0$

17 (b). $z = c_1 t + c_2 t^2$, $z' = c_1 + 2c_2 t$. Applying the boundary conditions gives us $[c_1 + c_2] - 2[c_1 + 2c_2] = 0$, $2[2c_1 + 4c_2] - [c_1 + 4c_2] = 0$. Solving these simultaneous equations gives us $z = 0$, $c_1 = c_2 = 0$.

17 (c). By the Fredholm Alternative Theorem, the given problem has a unique solution.

17 (d). $y = c_1 t + c_2 t^2$, $y' = c_1 + 2c_2 t$. Applying the boundary conditions gives us $\begin{bmatrix} -1 & -3 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} -5 \\ 7 \end{bmatrix}$.
Solving this equation gives us $y = \frac{1}{5}t + \frac{8}{5}t^2$.

18 (a). $t^2 z'' - 2tz' + 2z = 0$, $z(1) - z'(1) = 0$, $z(2) - 2z'(2) = 0$

18 (b). $z = c_1 t + c_2 t^2$, $z' = c_1 + 2c_2 t$. $[c_1 + c_2] - [c_1 + 2c_2] = 0$, $[2c_1 + 4c_2] - 2[c_1 + 4c_2] = 0$. $c_2 = 0$, c_1 arbitrary.

18 (c). By the Fredholm Alternative Theorem, the given problem does not have a unique solution.

18 (e). $y = c_1 t + c_2 t^2$, $y' = c_1 + 2c_2 t$. $y(1) - y'(1) = -c_2 = 1$, $y(2) - 2y'(2) = -4c_2 = 4$. $y = c_1 t - t^2$, c_1 arbitrary.

19 (a). $t^2z'' - 2tz' + 2z = 0,\ 4z(1) - 3z'(1) = 0,\ 3z(2) - 4z'(2) = 0$

19 (b). $z = c_1t + c_2t^2,\ z' = c_1 + 2c_2t$. Applying the boundary conditions gives us $4[c_1 + c_2] - 3[c_1 + 2c_2] = 0,\ 3[2c_1 + 4c_2] - 4[c_1 + 4c_2] = 0$. Solving these simultaneous equations gives us $z = c_2(2t + t^2),\ c_2$ arbitrary.

19 (c). By the Fredholm Alternative Theorem, the given problem does not have a unique solution.

19 (e). $y = c_1t + c_2t^2,\ y' = c_1 + 2c_2t$. Applying the boundary conditions gives us $4y(1) - 3y'(1) = c_1 - 2c_2 = 1,\ 3y(2) - 4y'(2) = 2c_1 - 4c_2 = 3$. These simultaneous equations cannot be solved, and thus there is no solution.

20 (a). Both Theorems guarantee a unique solution.

20 (b). $c_1 + c_2 + 4 = 7,\ \frac{1}{2}c_1 + 2c_2 + 4 = 7 \Rightarrow y = 2e^{-t} + e^t + 4$.

21 (a). Since $a_0a_1 > 0$, Theorem 13.2 is not applicable. Likewise, the form of the boundary conditions makes Theorem 13.3 not applicable.

21 (c). $z'' - z = 0,\ z(0) + z'(0) = 0,\ z(\ln 2) + z'(\ln 2) = 0$. Thus $z = c_1e^{-t} + c_2e^t,\ z' = -c_1e^{-t} + c_2e^t$. From the boundary conditions, we have $c_1 + c_2 - c_1 + c_2 = 0$ and $\frac{1}{2}c_1 + 2c_2 - \frac{1}{2}c_1 + 2c_2 = 0$. Solving these simultaneous equations yields $c_2 = 0,\ c_1$ arbitrary. Therefore, there is no unique solution to the given problem.

21 (e). $y = c_1e^{-t} + c_2e^t + 4,\ y' = -c_1e^{-t} + c_2e^t$. From the boundary conditions, we have $c_1 + c_2 + 4 - c_1 + c_2 = 5$ and $\frac{1}{2}c_1 + 2c_2 + 4 - \frac{1}{2}c_1 + 2c_2 = 8$. There is no solution to these simultaneous equations, and so the problem has no solution.

22 (a). Theorem 13.2 guarantees a unique solution.

22 (b). $c_1 + c_2 + 4 + c_1 - c_2 = 0,\ \frac{1}{2}c_1 + 2c_2 + 4 - \frac{1}{2}c_1 + 2c_2 = 12 \Rightarrow y = -2e^{-t} + 2e^t + 4$.

23 (a). Theorem 13.2 guarantees a unique solution.

23 (b). $y = c_1e^{-t} + c_2e^t + 4,\ y' = -c_1e^{-t} + c_2e^t$. From the boundary conditions, we have $c_1 + c_2 + 4 = 11$ and $-\frac{1}{2}c_1 + 2c_2 = 4$. Solving these simultaneous equations yields $c_1 = 4,\ c_2 = 3$, and so $y(t) = 4e^{-t} + 3e^t + 4$.

24 (a). Since $q(t) > 0$, neither theorem can guarantee a unique solution.

24 (c). $z'' + z = 0,\ z(0) + z'(0) = 0,\ z(\pi) + z'(\pi) = 0.\ z = c_1\cos t + c_2\sin t,\ z' = -c_1\sin t + c_2\cos t$. $c_1 + c_2 = 0$. $-c_1 - c_2 = 0$. $c_1 = -c_2,\ c_2$ arbitrary. There is no unique solution to the given problem.

24 (e). $y = c_1\cos t + c_2\sin t + 2,\ y' = -c_1\sin t + c_2\cos t.\ c_1 + 2 + c_2 = 7,\ -c_1 + 2 - c_2 = -3$. $y = (5 - c_2)\cos t + c_2\sin t + 2$.

25 (a). Since $q(t) > 0$, neither theorem can guarantee a unique solution.

25 (c). $z'' + z = 0,\ z(0) + z'(0) = 0,\ z(\pi) + z'(\pi) = 0$. Thus $z = c_1\cos t + c_2\sin t,\ z' = -c_1\sin t + c_2\cos t$. From the boundary conditions, we have $c_1 + c_2 = 0$ and $-c_1 - c_2 = 0$. Solving these simultaneous equations yields $c_1 = -c_2,\ c_2$ arbitrary. Therefore, there is no unique solution to the given problem.

25 (e). $y = c_1\cos t + c_2\sin t + 2,\ y' = -c_1\sin t + c_2\cos t$. From the boundary conditions, we have $c_1 + 2 + c_2 = 7,\ -c_1 + 2 - c_2 = 3$. There is no solution to these simultaneous equations, and so the problem has no solution.

26 (a). Since $q(t) > 0$, neither theorem can guarantee a unique solution.

26 (c). $z''+z=0,\ z(0)=0,\ z(\pi)=0$. $z=c_1\cos t+c_2\sin t,\ z'=-c_1\sin t+c_2\cos t$. $c_1=0$. $-c_1=0$. $c_1=0,\ c_2$ arbitrary. There is no unique solution to the given problem.

26 (e). $y=c_1\cos t+c_2\sin t+2$. $c_1+2=7,\ -c_1+2=3$. There is no solution to the given problem.

27 (a). Since $q(t)>0$, neither theorem can guarantee a unique solution.

27 (c). $z''+z=0,\ z(0)=0,\ z(\pi)+z'(\pi)=0$. Thus $z=c_1\cos t+c_2\sin t,\ z'=-c_1\sin t+c_2\cos t$. From the boundary conditions, we have $c_1=0$ and $-c_1-c_2=0$. Thus $c_1=c_2=0$, and so the problem has a unique solution.

27 (d). $y=c_1\cos t+c_2\sin t+2,\ y'=-c_1\sin t+c_2\cos t$. From the boundary conditions, we have $c_1+2=8,\ -c_1+2-c_2=5$. Therefore, $y(t)=6\cos t-9\sin t+2$.

28 (a) and (c). same as (26)

28 (e). $y=c_1\cos t+c_2\sin t+2$. $c_1+2=8,\ -c_1+2=-4$. $y=6\cos t+c_2\sin t+2$

29 (a). $y_1(t)$ is a nonzero solution if $g(t)\neq 0$ and/or $\alpha\neq 0$. Since c_0 and c_1 are not both zero, $\alpha\neq 0$ ensures nontrivial initial conditions. $y_2(t)$ is a nontrivial solution since a_0 and a_1 are not both zero.

31 (b). Choose $c_1=1,\ c_0=0$. Then $a_0c_1-a_1c_0=1$. Then we have $t^2y_1''-ty_1'+y_1=2,\ y_1(1)=3,\ y_1'(1)=0$. Thus $y_1=c_1t+c_2t\ln t+2$, and with the boundary conditions we have $c_1+2=3,\ c_1+c_2[\ln t+1]=0$. Therefore, $c_1=1,\ c_2=-1$, and so $y_1(t)=t-t\ln t+2$. Then, $t^2y_2''-ty_2'+y_2=0,\ y_2(1)=0,\ y_2'(1)=-1$. Thus $y_2=c_1t+c_2t\ln t$, and with the boundary conditions we have $c_1=0,\ c_1+c_2=-1$. Therefore, $c_1=0,\ c_2=-1$, and so $y_2(t)=-t\ln t$.

$y_s=y_1+sy_2=t-t\ln t+2-st\ln t$, $y_s'(2)=1-\ln 2-1+s(-\ln 2-1)=0\Rightarrow s=\dfrac{-\ln 2}{1+\ln 2}$. Finally,

$y(t)=t-t\ln t+2+\dfrac{t\ln t\cdot\ln 2}{1+\ln 2}$.

32 (b). Choose $c_1=1,\ c_0=0$. $y_1''+4y_1'=3\sin t,\ y_1(0)=3,\ y_1'(0)=0$. $y_1=c_1\cos 2t+c_2\sin 2t+\sin t$, $c_1=3,\ 2c_2+1=0$. $y_1(t)=3\cos 2t-\dfrac{1}{2}\sin 2t+\sin t$. $y_2=c_1\cos 2t+c_2\sin 2t$. $c_1=1,\ 2c_2=-1$.

$y_2(t)=\cos 2t-\dfrac{1}{2}\sin 2t$. $y_s=y_1+sy_2$.

$$y_1\left(\frac{\pi}{4}\right)+sy_2\left(\frac{\pi}{4}\right)+y_1'\left(\frac{\pi}{4}\right)+sy_2'\left(\frac{\pi}{4}\right)=8\Rightarrow s=-\frac{2}{5}\left(\frac{29}{2}-\sqrt{2}\right).$$

$$y=3\cos 2t-\frac{1}{2}\sin 2t+\sin t+\left(\frac{-29+2\sqrt{2}}{5}\right)\left(\cos 2t-\frac{1}{2}\sin 2t\right).$$

33 (b). Choose $c_1=1,\ c_0=0$. Then we have $y_1''-t^2y_1=1,\ y_1(0)=0,\ y_1'(0)=0$; $y_2''-t^2y_2=0,\ y_2(0)=0,\ y_2'(0)=-1$. Thus $y_1(1)+sy_2(1)=1\Rightarrow s=\dfrac{1-y_1(1)}{y_2(1)}$. finally,

$r(t)=y_1+sy_2=y_1+\dfrac{1-y_1(1)}{y_2(1)}y_2$.

34 (b). $g=0,\ \alpha=0,\ \beta=1$. $y_1''+ty_1'-y_1=0,\ y_1(0)=0,\ y_1'(0)=0\Rightarrow y_1=0$;

$y_2''+ty_2'-y_2=0,\ y_2(0)=0,\ y_2'(0)=-1$.

$y_1+sy_2=sy_2,\ y(1)=sy_2(1)=1\Rightarrow s=\dfrac{1}{y_2(1)}\Rightarrow y(t)=\dfrac{y_2(t)}{y_2(1)}$.

Section 13.2

1 (b). $y'' = -f(x) \Rightarrow y' = -\int_0^x f(\lambda)d\lambda + c_1$. Using the second boundary condition, we have $c_1 = \int_0^1 f(\lambda)d\lambda \Rightarrow y(x) = -\int_0^x \int_0^\lambda f(\sigma)d\sigma d\lambda + x\int_0^1 f(\lambda)d\lambda$. Integration by parts and some simplification give us $y(x) = \int_x^1 xf(\lambda)d\lambda + \int_0^x \lambda f(\lambda)d\lambda = \int_0^1 G(x,\lambda)f(\lambda)d\lambda$ with $G(x,\lambda) = \begin{cases} \lambda, & 0 \le \lambda \le x \\ x, & x < \lambda \le 1 \end{cases}$.

1 (c). $\int_x^1 x6\lambda d\lambda + \int_0^x 6\lambda^2 d\lambda = 3x - x^3$

2 (b). $y' = -\int_0^x f(\lambda)d\lambda + c_1$. $c_1 = y'(0) = 0 \Rightarrow y(x) = -\int_0^x \int_0^\lambda f(\sigma)d\sigma d\lambda + c_2$.
$y(1) + y'(1) = 0 \Rightarrow c_2 = \int_0^1 \int_0^\lambda f(\sigma)d\sigma d\lambda + \int_0^1 f(\lambda)d\lambda$.
$y(x) = \int_0^x (2-x)f(\lambda)d\lambda + \int_x^1 (2-\lambda)f(\lambda)d\lambda = \int_0^1 G(x,\lambda)f(\lambda)d\lambda$. $G(x,\lambda) = \begin{cases} 2-x, & 0 \le \lambda \le x \\ 2-\lambda, & x < \lambda \le 1 \end{cases}$.

2 (c). $\int_x^1 (2-\lambda)6\lambda d\lambda + \int_0^x (2-x)6\lambda d\lambda = 4 - x^3$

3 (b). $y' = -\int_0^x f(\lambda)d\lambda + c_1 \Rightarrow y(x) = -\int_0^x \int_0^\lambda f(\sigma)d\sigma d\lambda + c_1 x + c_2$. Using the boundary conditions, we have $c_2 - 2c_1 = 0$ and $-\int_0^1 \int_0^\lambda f(\sigma)d\sigma d\lambda + c_1 + c_2 = 0$. Thus $c_1 = \frac{1}{3}\int_0^1 \int_0^\lambda f(\sigma)d\sigma d\lambda,\ c_2 = 2c_1$ and so $y(x) = -\int_0^x \int_0^\lambda f(\sigma)d\sigma d\lambda + \frac{x+2}{3}\int_0^1 \int_0^\lambda f(\sigma)d\sigma d\lambda$. Integration by parts and some simplification give us
$y(x) = \int_0^x \frac{(\lambda+2)}{3}(1-x)f(\lambda)d\lambda + \int_x^1 \frac{(x+2)}{3}(1-\lambda)f(\lambda)d\lambda = \int_0^1 G(x,\lambda)f(\lambda)d\lambda$ with
$G(x,\lambda) = \begin{cases} \frac{(\lambda+2)(1-x)}{3}, & 0 \le \lambda \le x \\ \frac{(x+2)(1-\lambda)}{3}, & x < \lambda \le 1 \end{cases}$.

3 (c). $\frac{1-x}{3}\int_0^x (\lambda+2)6\lambda d\lambda + \frac{x+2}{3}\int_x^1 (1-\lambda)6\lambda d\lambda = \frac{1}{3}(x+2) - x^3$

4 (b). $y' = -\int_0^x f(\lambda)d\lambda + c_1$. $y(x) = -\int_0^x \int_0^\lambda f(\sigma)d\sigma d\lambda + c_1 x + c_2$.
$y(0) - y'(0) = c_2 - c_1 = 0 \Rightarrow c_1 = c_2 = \int_0^1 \int_0^\lambda f(\sigma)d\sigma d\lambda - \int_0^1 f(\lambda)d\lambda$.
$y(x) = \int_0^x (-x)(\lambda+1)f(\lambda)d\lambda - \int_x^1 (x+1)\lambda f(\lambda)d\lambda = \int_0^1 G(x,\lambda)f(\lambda)d\lambda$.
$G(x,\lambda) = \begin{cases} -x(\lambda+1), & 0 \le \lambda \le x \\ -\lambda(x+1), & x < \lambda \le 1 \end{cases}$.

4 (c). $-(x+1)\int_x^1 \lambda 6\lambda d\lambda - x\int_0^x (\lambda+1)6\lambda d\lambda = -2 - 2x - x^3$

6 (c). $y_0 = \cos 2t + \sin t,\ y_1 = \frac{1}{2}\cos 2t - \frac{1}{4}\sin 2t,\ y_2 = \frac{1}{2}\cos 2t + \frac{1}{4}\sin 2t$

7 (i). $y = \frac{1}{9}y_0 + 2y_1 - y_2 = \frac{11}{18}\cos 2t - \frac{3}{4}\sin 2t + \frac{1}{9}\sin t$

7 (ii). $y = -2y_0 + \frac{1}{2}y_1 = -\frac{7}{4}\cos 2t - \frac{1}{8}\sin 2t - 2\sin t$

7 (iii). $y = -4y_1 - 2y_2 = -3\cos 2t + \frac{1}{2}\sin 2t$

8 (b). $y_0 = -18t^{-1} - 1 + \frac{t^2}{3},\ y_1 = \frac{t}{2} - 9t^{-1},\ y_2 = \frac{9}{2}t^{-1}$

9 (i). $y = 3y_0 - \frac{1}{2}y_1 + y_2 = -45t^{-1} - 3 - \frac{t}{4} + t^2$

9 (ii). $y = \frac{1}{3}y_0 - y_1 + 3y_2 = \frac{33}{2}t^{-1} - \frac{1}{3} + \frac{t^2}{9} - \frac{t}{2}$

10. $\mu = e^{2t},\ \left(e^{2t}y'\right)' + e^{2t}y = e^{2t}\sin t$

11. $\mu = e^{-t/2},\ \left(e^{-t/2}y'\right)' = \frac{1}{2}e^{-3t/2}$

12. $\mu = e^{-t^2/4},\ \left(e^{-t^2/4}y'\right)' + 2e^{-t^2/4}y = \frac{t^2}{2}e^{-t^2/4}$

13. $\mu = t^{-2},\ \left(t^{-2}y'\right)' + t^{-3}y = t^{-3}e^t$

14. $\mu = t^2,\ \left(t^2y'\right)' - \sin 2t(y) = 3$

15. $y'' + \tan ty' + 2\sec ty = t^2\sec t \Rightarrow \mu = \sec t,\ \left(\sec ty'\right)' + 2\sec^2 ty = t^2\sec^2 t$

16. $\mu = t,\ \left(ty'\right)' + t^{-1/2}e^t y = 2t^{1/2}$

17. First, we rewrite the equation as follows: $y'' - y' - 2y = 0$. Then we have $(\lambda - 2)(\lambda + 1) = 0 \Rightarrow y(t) = c_1e^{-t} + c_2e^{2t}$.

18. $\lambda^2 - 3\lambda + 2 = (\lambda - 2)(\lambda - 1) = 0 \Rightarrow y(t) = c_1e^t + c_2e^{2t}$.

19. $\lambda(\lambda - 1) + 2\lambda - 2 = (\lambda + 2)(\lambda - 1) = 0 \Rightarrow y(t) = c_1t + c_2t^{-2},\ t > 0$.

20. $\lambda_2 + \lambda - 6 = (\lambda - 2)(\lambda + 3) = 0 \Rightarrow y(t) = c_1t_2 + c_2t^{-3},\ t > 0$.

Section 13.3

1 (b). $p(t) = 1,\ f(t) = -\sin t$

1 (c). $\phi(t) = c_1t + c_2,\ \phi(0) - \phi'(0) = c_2 - c_1 = 0 \Rightarrow \phi = t + 1$,
$\psi(t) = c_1t + c_2,\ \psi(2) - \psi'(2) = c_1 + c_2 = 0 \Rightarrow \psi = t - 1$,
$W_0 = p(s)W(s) = 1\cdot\left(\psi(s)\phi'(s) - \phi(s)\psi'(s)\right) = -2$. From (8), then,

$$G(t,s) = \begin{cases} -\frac{1}{2}(t+1)(s-1), & 0 \le t \le s \\ -\frac{1}{2}(s+1)(t-1), & s < t \le 2 \end{cases}.$$

1 (d). $y(t) = \int_0^2 G(t,s)f(s)ds = \frac{1}{2}(t-1)\int_0^t (s+1)\sin s ds + \frac{1}{2}(t+1)\int_t^2 (s-1)\sin s ds$

2 (b). $p(t) = 1,\ f(t) = -e^{-t}$

2 (c). $\phi(t)=c_1\cos t+c_2\sin t,\ \phi(0)=c_1=0\Rightarrow\phi(t)=\sin t$, $\psi(t)=\cos t$,
$W_0=p(s)W(s)=1\cdot(\psi(s)\phi'(s)-\phi(s)\psi'(s))=1$. $G(t,s)=\begin{cases}\sin t\cos s, & 0\le t\le s\\ \sin s\cos t, & s<t\le \pi/2\end{cases}$.

2 (d). $y(t)=\int_0^{\pi/2}G(t,s)f(s)ds=-\cos t\int_0^t e^{-s}\sin s ds-\sin t\int_t^{\pi/2}e^{-s}\cos s ds$

3 (b). $p(t)=1,\ f(t)=-e^t$

3 (c). $\phi(t)=c_1\cos t+c_2\sin t,\ \phi(0)+\phi'(0)=c_1+c_2=0\Rightarrow\phi=\cos t-\sin t$,
$\psi(t)=c_1\cos t+c_2\sin t,\ \psi'(1)=-c_1\sin 1+c_2\cos 1=0\Rightarrow\psi=\cos t\cos 1+\sin t\sin 1=\cos(t-1)$,
$W_0=p(s)W(s)=1\cdot(\psi(s)\phi'(s)-\phi(s)\psi'(s))=-\cos 1-\sin 1$. From (8), then,
$$G(t,s)=\begin{cases}\dfrac{-(\cos t-\sin t)\cos(s-1)}{\cos 1+\sin 1}, & 0\le t\le s\\ \dfrac{-(\cos s-\sin s)\cos(t-1)}{\cos 1+\sin 1}, & s<t\le 1\end{cases}.$$

3 (d). $y(t)=\int_0^1 G(t,s)f(s)ds=\dfrac{\cos(t-1)}{\cos 1+\sin 1}\int_0^t(\cos s-\sin s)e^s ds+\dfrac{\cos t-\sin t}{\cos 1+\sin 1}\int_t^1\cos(s-1)e^s ds$

4 (b). $p(t)=1,\ f(t)=-e^t$

4 (c). $\phi(t)=\cosh 2t,\ \psi(t)=\sinh 2(t-1)$, $W_0=p(s)W(s)=1\cdot(\psi(s)\phi'(s)-\phi(s)\psi'(s))=-2\cosh 2$.
$$G(t,s)=\begin{cases}-\dfrac{\cosh 2t\sinh 2(s-1)}{2\cosh 2}, & 0\le t\le s\\ -\dfrac{\cosh 2s\sinh 2(t-1)}{2\cosh 2}, & s<t\le 1\end{cases}.$$

4 (d). $y(t)=\int_0^1 G(t,s)f(s)ds=\dfrac{\sinh 2(t-1)}{2\cosh 2}\int_0^t\cosh 2s e^s ds+\dfrac{\cosh 2t}{2\cosh 2}\int_t^1\sinh 2(s-1)e^s ds$

5 (b). $p(t)=1,\ f(t)=e^{-t^2}$

5 (c). $\phi=\sinh t$, $\psi=\sinh(t-1)$, $W_0=p(s)W(s)=1\cdot(\psi(s)\phi'(s)-\phi(s)\psi'(s))=-\sinh 1$. From (8), then,
$$G(t,s)=\begin{cases}\dfrac{-\sinh t\sinh(s-1)}{\sinh 1}, & 0\le t\le s\\ \dfrac{-\sinh s\sinh(t-1)}{\sinh 1}, & s<t\le 1\end{cases}.$$

5 (d). $y(t)=\int_0^1 G(t,s)f(s)ds=\dfrac{\sinh(t-1)}{\sinh 1}\int_0^t\sinh(s)s^2 ds+\dfrac{\sinh t}{\sinh 1}\int_t^1\sinh(s-1)s^2 ds$

6 (b). $p(t)=e^{4t},\ f(t)=-e^{4t}\sin t$

6 (c). $\phi(t)=e^{-2t}\sin t$, $\psi(t)=e^{-2(t-2)}\sin(t-2)$,
$W_0=p(s)W(s)=1\cdot(\psi(s)\phi'(s)-\phi(s)\psi'(s))=-e^{2t+2s}\sin 2$.
$$G(t,s)=\begin{cases}\dfrac{-e^{-2(t+s)}\sin t\sin(s-2)}{\sin 2}, & 0\le t\le s\\ \dfrac{-e^{-2(t+s)}\sin s\sin(t-2)}{\sin 2}, & s<t\le 2\end{cases}.$$

6 (d). $y(t)=\int_0^2 G(t,s)f(s)ds=\dfrac{e^{-2t}\sin(t-2)}{\sin 2}\int_0^t e^{2s}\sin^2 s ds+\dfrac{e^{-2t}\sin t}{\sin 2}\int_t^2 e^{2s}\sin(s-2)\sin s ds$

7 (b). $p(t)=e^{2t},\ f(t)=-t^3e^{2t}$

7 (c). $\phi(t)=c_1+c_2e^{-2t},\ \phi(-1)=c_1+c_2e^2=0\Rightarrow\phi=1-e^{-2(t+1)}$,
$\psi(t)=c_1+c_2e^{-2t},\ \psi'(1)=-2c_2=0\Rightarrow\psi=1$,

$W_0 = p(s)W(s) = e^{2t} \cdot \left(\psi(s)\phi'(s) - \phi(s)\psi'(s)\right) = e^{2t}\left(2e^{-2(s+1)} - 0\right)$. From (8), then,

$$G(t,s) = \begin{cases} \frac{1}{2}e^2\left(1-e^{-2(t+1)}\right), & -1 \le t \le s \\ \frac{1}{2}e^2\left(1-e^{-2(s+1)}\right), & s < t \le 1 \end{cases}.$$

7 (d). $y(t) = \int_0^1 G(t,s)f(s)ds = -\frac{1}{2}e^2\int_{-1}^t \left(1-e^{-2(s+1)}\right)\left(s^3e^{2s}\right)ds - \frac{1}{2}e^2\left(1-e^{-2(t+1)}\right)\int_t^1 s^3e^{2s}ds$

8 (b). $p(t) = t^{-1},\ f(t) = -1$

8 (c). $\phi(t) = t^2 - 1\ \ \psi(t) = t^2 - 4$, $W_0 = p(s)W(s) = t^{-1} \cdot \left(\psi(s)\phi'(s) - \phi(s)\psi'(s)\right) = -6$.

$$G(t,s) = \begin{cases} -\frac{\left(t^2-1\right)\left(s^2-4\right)}{6}, & 1 \le t \le s \\ -\frac{\left(s^2-1\right)\left(t^2-4\right)}{6}, & s < t \le 2 \end{cases}.$$

8 (d). $y(t) = \int_1^2 G(t,s)f(s)ds = \frac{\left(t^2-4\right)}{6}\int_1^t \left(s^2-1\right)ds + \frac{\left(t^2-1\right)}{6}\int_t^2 \left(s^2-4\right)ds$

9 (b). $p(t) = e^{-3t},\ f(t) = -te^{-3t}$

9 (c). $\phi(t) = c_1e^t + c_2e^{2t},\ \phi(0) = c_1 + c_2 = 0 \Rightarrow \phi = e^t - e^{2t}$,

$\psi(t) = c_1e^{t-1} + c_2e^{2t-2},\ \psi(1) = c_1 + c_2 = 0 \Rightarrow \psi = e^{t-1} - e^{2t-2}$,

$W_0 = p(s)W(s) = e^{-3t}\left(\psi(s)\phi'(s) - \phi(s)\psi'(s)\right) = -e^{-1} + e^{-2}$. From (8), then,

$$G(t,s) = \begin{cases} \frac{-\left(e^t - e^{2t}\right)\left(e^{s-1} - e^{2s-2}\right)}{e^{-1} - e^{-2}}, & 0 \le t \le s \\ \frac{-\left(e^s - e^{2s}\right)\left(e^{t-1} - e^{2t-2}\right)}{e^{-1} - e^{-2}}, & s < t \le 1 \end{cases}.$$

9 (d). $y(t) = \int_0^1 G(t,s)f(s)ds = \frac{e^{t-1} - e^{2t-2}}{e^{-1} - e^{-2}}\int_0^t \left(e^s - e^{2s}\right)\left(se^{-3s}\right)ds + \frac{e^t - e^{2t}}{e^{-1} - e^{-2}}\int_t^1 \left(e^{s-1} - e^{2s-2}\right)\left(se^{-3s}\right)ds$

10 (b). $p(t) = e^{-3t},\ f(t) = -te^{-3t}$

10 (c). $\phi(t) = e^t - \frac{1}{2}e^{2t},\ \ \psi(t) = e^{t-1} - \frac{1}{2}e^{2(t-1)}$,

$W_0 = p(s)W(s) = e^{-3t} \cdot \left(\psi(s)\phi'(s) - \phi(s)\psi'(s)\right) = -\frac{e^{-1} - e^{-2}}{2}$.

$$G(t,s) = \begin{cases} -\frac{2\left(e^t - \frac{1}{2}e^{2t}\right)\left(e^{s-1} - \frac{1}{2}e^{2(s-1)}\right)}{e^{-1} - e^{-2}}, & 0 \le t \le s \\ -\frac{2\left(e^s - \frac{1}{2}e^{2s}\right)\left(e^{t-1} - \frac{1}{2}e^{2(t-1)}\right)}{e^{-1} - e^{-2}}, & s < t \le 1 \end{cases}.$$

10 (d). $y(t) = \int_0^1 G(t,s)f(s)ds$

$$= \frac{2\left(e^{t-1} - \frac{1}{2}e^{2t-2}\right)}{e^{-1} - e^{-2}}\int_0^t \left(e^s - \frac{1}{2}e^{2s}\right)se^{-3s}ds + \frac{2\left(e^t - \frac{1}{2}e^{2t}\right)}{e^{-1} - e^{-2}}\int_t^1 \left(e^{s-1} - \frac{1}{2}e^{2s-2}\right)se^{-3s}ds$$

11. $p(t) = 1,\ q(t) = 0,\ a_0 = 0,\ a_1 = 1,\ b_0 = 1,\ b_1 = 1$

12. $p(t) = 1,\ q(t) = 0,\ a_0 = 1,\ a_1 = 1,\ b_0 = 1,\ b_1 = -\frac{1}{2}$

13. $p(t) = 1,\ q(t) = 1,\ a_0 = 1,\ a_1 = 0,\ b_0 = 0,\ b_1 = 1$

14. $p(t)=1,\ q(t)=-4,\ a_0=1,\ a_1=0,\ b_0=1,\ b_1=0$

16 (a). $G(t,s)=\begin{cases} k\cos t\cos(1-s), & 0\le t\le s \\ k\cos s\cos(1-t), & s<t\le 1\end{cases}$. $k\cos s\sin(1-s)+k\sin s\cos(1-s)=-1,\ k=-\dfrac{1}{\sin 1}$.

16 (b). $\gamma=1,\ a_0=0,\ a_1=1,\ b_0=0,\ b_1=1$.

Section 13.4

1. After differentiating and forming the matrices and vectors, we have $\vec{y}'=\begin{bmatrix}0 & 1\\ -1 & 2\end{bmatrix}\vec{y}+\begin{bmatrix}0\\ \cos 2t\end{bmatrix}$.

From the boundary conditions, we form $\begin{bmatrix}2 & -1\\ 0 & 0\end{bmatrix}\vec{y}(0)+\begin{bmatrix}0 & 0\\ 1 & 1\end{bmatrix}\vec{y}(1)=\begin{bmatrix}-1\\ 2\end{bmatrix}$.

2. $\vec{y}'=\begin{bmatrix}0 & 1\\ 0 & -\frac{1}{2}\end{bmatrix}\vec{y}+\begin{bmatrix}0\\ \frac{1}{2}\left(t^2+1\right)\end{bmatrix},\ \begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\vec{y}(0)+\begin{bmatrix}0 & 0\\ 1 & -1\end{bmatrix}\vec{y}(2)=\begin{bmatrix}0\\ 0\end{bmatrix}$

3. After differentiating and forming the matrices and vectors, we have $\vec{y}'=\begin{bmatrix}0 & 1\\ -t^{-1}e^t & -t^{-1}\end{bmatrix}\vec{y}+\begin{bmatrix}0\\ 2t^{-1}\end{bmatrix}$. From the boundary conditions, we form $\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}\vec{y}(1)+\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}\vec{y}(2)=\begin{bmatrix}-3\\ 1\end{bmatrix}$.

4. $\vec{y}'=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -1 & -3 & 3\end{bmatrix}\vec{y}+\begin{bmatrix}0\\ 0\\ e^{-t}+\sin t\end{bmatrix},\ \begin{bmatrix}1 & 0 & -1\\ 2 & 1 & 0\\ 0 & 0 & 0\end{bmatrix}\vec{y}(0)+\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 1 & -3\end{bmatrix}\vec{y}(2)=\begin{bmatrix}2\\ -1\\ 3\end{bmatrix}$

5. After differentiating and forming the matrices and vectors, we have $\vec{y}'=\begin{bmatrix}0 & 1 & 0\\ 0 & 0 & 1\\ -2t^{-3} & 2t^{-2} & 0\end{bmatrix}\vec{y}+\begin{bmatrix}0\\ 0\\ 3t^{-3}\sin t\end{bmatrix}$. From the boundary conditions, we form $\begin{bmatrix}1 & 1 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\vec{y}(-2)+\begin{bmatrix}0 & 0 & 0\\ -1 & 2 & -1\\ 0 & 1 & 3\end{bmatrix}\vec{y}(-1)=\begin{bmatrix}1\\ 4\\ 2\end{bmatrix}$.

6. $P^{[0]}=\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix},\ P^{[1]}=\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix},\ D=\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix}+\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}e & e^3\\ -e & e^3\end{bmatrix}=\begin{bmatrix}1 & 1\\ -e & e^3\end{bmatrix}$. The determinant of D is nonzero, and so there is a unique solution for every $\vec{g}(t)$ and $\vec{\alpha}$.

7. $P^{[0]}=\begin{bmatrix}1 & -1\\ 0 & 0\end{bmatrix},\ P^{[1]}=\begin{bmatrix}0 & 0\\ 1 & -1\end{bmatrix},\ D=\begin{bmatrix}1 & -1\\ 0 & 0\end{bmatrix}\begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix}+\begin{bmatrix}0 & 0\\ 1 & -1\end{bmatrix}\begin{bmatrix}e & e^3\\ -e & e^3\end{bmatrix}=\begin{bmatrix}2 & 0\\ 2e & 0\end{bmatrix}$. The determinant of D is zero, and so there is not a unique solution for every $\vec{g}(t)$ and $\vec{\alpha}$.

8. $P^{[0]}=\begin{bmatrix}1 & -1\\ 0 & 0\end{bmatrix}$, $P^{[1]}=\begin{bmatrix}0 & 0\\ 1 & 1\end{bmatrix}$, $D=\begin{bmatrix}1 & -1\\ 0 & 0\end{bmatrix}\begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix}+\begin{bmatrix}0 & 0\\ 1 & 1\end{bmatrix}\begin{bmatrix}e & e^3\\ -e & e^3\end{bmatrix}=\begin{bmatrix}2 & 0\\ 0 & 2e^3\end{bmatrix}$. The determinant of D is nonzero, and so there is a unique solution for every $\vec{g}(t)$ and $\vec{\alpha}$.

9. $P^{[0]}=\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}$, $P^{[1]}=\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}$, $D=\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}\begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix}+\begin{bmatrix}0 & 0\\ 1 & 0\end{bmatrix}\begin{bmatrix}e & e^3\\ -e & e^3\end{bmatrix}=\begin{bmatrix}1 & 1\\ e & e^3\end{bmatrix}$. The determinant of D is nonzero, and so there is a unique solution for every $\vec{g}(t)$ and $\vec{\alpha}$.

10 (b). $\begin{bmatrix}3 & -1\\ 0 & 2e^3\end{bmatrix}\vec{c}=\begin{bmatrix}1\\ -1\end{bmatrix}\Rightarrow \vec{c}=\frac{1}{6e^3}\begin{bmatrix}2e^3-1\\ -3\end{bmatrix}$. $\vec{y}(t)=\Psi\vec{c}=\frac{1}{6e^3}\begin{bmatrix}(2e^3-1)e^{-t}-3e^{3t}\\ (2e^3-1)e^{-t}+3e^{3t}\end{bmatrix}$.

11 (b). $\begin{bmatrix}1 & 1\\ e^{-1} & -e^3\end{bmatrix}\vec{c}=\begin{bmatrix}1\\ 0\end{bmatrix}\Rightarrow \vec{c}=\frac{1}{-e^3-e^{-1}}\begin{bmatrix}-e^3\\ -e^{-1}\end{bmatrix}$. Then we have

$$\vec{y}(t)=\Psi\vec{c}=\frac{1}{e^3+e^{-1}}\begin{bmatrix}e^{-t} & e^{3t}\\ e^{-t} & -e^{3t}\end{bmatrix}\begin{bmatrix}e^3\\ e^{-1}\end{bmatrix}=\frac{1}{e^3+e^{-1}}\begin{bmatrix}e^{-t+3} & e^{3t-1}\\ e^{-t+3} & -e^{3t-1}\end{bmatrix}.$$

12 (b). $\vec{f}=-\begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}\begin{bmatrix}e^{-1} & e^3\\ e^{-1} & -e^{-3}\end{bmatrix}\begin{bmatrix}\frac{1}{2}(e-1)\\ \frac{1}{6}(1-e^{-3})\end{bmatrix}=\begin{bmatrix}0\\ -\frac{2}{3}+\frac{1}{2e}+\frac{1}{6}e^3\end{bmatrix}$.

$$\vec{c}=D^{-1}\vec{f}=-\frac{1}{e^3+e^{-1}}\begin{bmatrix}-e^3 & -1\\ -e^{-1} & 1\end{bmatrix}\begin{bmatrix}0\\ -\frac{2}{3}+\frac{1}{2e}+\frac{1}{6}e^3\end{bmatrix}\Rightarrow c_2=-c_1.$$

$$\vec{y}(t)=\begin{bmatrix}e^{-t} & e^{3t}\\ e^{-t} & -e^{3t}\end{bmatrix}\begin{bmatrix}c_1+\frac{1}{2}(e^t-1)\\ -c_1+\frac{1}{6}(1-e^{-3t})\end{bmatrix}=\begin{bmatrix}c_1(e^{-t}-e^{3t})+\frac{1}{2}(1-e^{-t})+\frac{1}{6}(e^{3t}-1)\\ c_1(e^{-t}+e^{3t})+\frac{1}{2}(1-e^{-t})-\frac{1}{6}(e^{3t}-1)\end{bmatrix}.$$

13. $\vec{y}(t)=\Psi\vec{c}$; $D=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 1\\ 0 & -1 & 1\end{bmatrix}+\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1\end{bmatrix}\begin{bmatrix}e^{-2} & 0 & 0\\ 0 & e & e^3\\ 0 & -e & e^3\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 1\\ 0 & -e & e^3\end{bmatrix}$ (see exercises 6-9). $D\vec{c}=\vec{\alpha}=\begin{bmatrix}1\\ 0\\ 1\end{bmatrix}\Rightarrow \vec{c}=D^{-1}\vec{\alpha}=\begin{bmatrix}1 & 0 & 0\\ 0 & \frac{e^3}{e^3+e} & \frac{-1}{e^3+e}\\ 0 & \frac{e}{e^3+e} & \frac{1}{e^3+e}\end{bmatrix}\begin{bmatrix}1\\ 0\\ 1\end{bmatrix}=\begin{bmatrix}1\\ \frac{-1}{e^3-e}\\ \frac{1}{e^3-e}\end{bmatrix}$. Therefore,

$$\vec{y}(t)=\begin{bmatrix}e^{-2t} & 0 & 0\\ 0 & e^t & e^{3t}\\ 0 & -e^t & e^{3t}\end{bmatrix}\begin{bmatrix}1\\ \frac{-1}{e^3+e}\\ \frac{1}{e^3+e}\end{bmatrix}=\begin{bmatrix}e^{-2t}\\ \frac{e^{3t}-e^t}{e^3+e}\\ \frac{e^{3t}+e^t}{e^3+e}\end{bmatrix}.$$

18 (a). $\Psi=\begin{bmatrix}1 & x\\ 1 & x+\beta^{-1}\end{bmatrix}$. $P^{[0]}=\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix}$, $P^{[\ell]}=\begin{bmatrix}0 & 0\\ -\Gamma & 1\end{bmatrix}$, $\vec{\alpha}=\begin{bmatrix}I^{inc}\\ 0\end{bmatrix}$.

$$\left(P^{[0]}\begin{bmatrix}1 & 0\\ 1 & \beta^{-1}\end{bmatrix}+P^{[\ell]}\begin{bmatrix}1 & \ell\\ 1 & \ell+\beta^{-1}\end{bmatrix}\right)\vec{c}=\begin{bmatrix}1 & 0\\ 1-\Gamma & (1-\Gamma)\ell+\beta^{-1}\end{bmatrix}\vec{c}=\vec{\alpha}$$

$$\Rightarrow \vec{c}=\begin{bmatrix}I^{inc}\\ \dfrac{-(1-\Gamma)I^{inc}}{(1-\Gamma)\ell+\beta^{-1}}\end{bmatrix}\Rightarrow\begin{bmatrix}I^{+}\\ I^{-}\end{bmatrix}=\begin{bmatrix}\dfrac{1+(1-\Gamma)\beta(\ell-x)}{1+(1-\Gamma)\beta\ell}\\ \dfrac{\Gamma+(1-\Gamma)\beta(\ell-x)}{1+(1-\Gamma)\beta\ell}\end{bmatrix}I^{inc}.$$

18 (b). $\dfrac{dI^{-}}{d\Gamma}(0)=\dfrac{\left(1+(1-\Gamma)\beta\ell\right)(1-\beta\ell)-\left(\Gamma+(1-\Gamma)\beta\ell\right)(-\beta\ell)}{\left(1+(1-\Gamma)\beta\ell\right)^2}=\dfrac{1}{\left(1+(1-\Gamma)\beta\ell\right)^2}>0$.

19 (b). $\displaystyle\int\frac{dR}{(R-1)^2}=-\frac{1}{R-1}=-\beta x+C$. Imposing $R(\ell)=0$, we have $1=-\beta\ell+C\Rightarrow C=1+\beta\ell$.

Therefore, $\dfrac{-1}{R-1}=\beta(\ell-x)+1\Rightarrow R=\dfrac{\beta(\ell-x)}{1+\beta(\ell-x)}$. From (11),

$I^{+}=\left[\dfrac{1+\beta(\ell-x)}{\beta\ell+1}\right]I^{inc}$, $I^{-}=\left[\dfrac{\beta(\ell-x)}{\beta\ell+1}\right]I^{inc}\Rightarrow\dfrac{I^{-}}{I^{+}}=R$.

Section 13.5

4 (a). $p=1,\ q=0,\ r=1$

4 (b). $u=c_1\cos\sqrt{\lambda}x+c_2\sin\sqrt{\lambda}x,\ u'=-\sqrt{\lambda}c_1\sin\sqrt{\lambda}x+\sqrt{\lambda}c_2\cos\sqrt{\lambda}x$. From the boundary conditions, we have $u'(0)=\sqrt{\lambda}c_2=0$ and $u(1)=c_1\cos\sqrt{\lambda}+c_2\sin\sqrt{\lambda}=0$. From $\sqrt{\lambda}c_2=0,\ \lambda=0$ and / or $c_2=0$. If $\lambda=0$, then $c_1=0$. We thus conclude that zero is not an eigenvalue since $u=0$ in that case. If $\lambda\neq 0,\ c_2=0$ and $c_1\cos\sqrt{\lambda}=0$. Thus $\sqrt{\lambda_n}=\dfrac{(2n-1)\pi}{2}\Rightarrow\lambda_n=\left(n-\dfrac{1}{2}\right)^2\pi^2,\ u_n=\cos\left(\left(n-\dfrac{1}{2}\right)\pi x\right)$.

5 (a). $p=1,\ q=0,\ r=1$

5 (b). $u=c_1\cos\sqrt{\lambda}x+c_2\sin\sqrt{\lambda}x,\ u'=-\sqrt{\lambda}c_1\sin\sqrt{\lambda}x+\sqrt{\lambda}c_2\cos\sqrt{\lambda}x$. From the boundary conditions, we have $u(0)=c_1=0\Rightarrow u=c_2\sin\sqrt{\lambda}x$. From $u'(1)=c_2\sqrt{\lambda}\cos\sqrt{\lambda}=0,\ \lambda=0$ and / or $\cos\sqrt{\lambda}=0$. If $\lambda=0$, then $u=0$. We thus conclude that zero is not an eigenvalue. $\cos\sqrt{\lambda}=0\Rightarrow$ $\sqrt{\lambda_n}=\left(n-\dfrac{1}{2}\right)\pi\Rightarrow\lambda_n=\left(n-\dfrac{1}{2}\right)^2\pi^2,\ u_n=\sin\left(\left(n-\dfrac{1}{2}\right)\pi x\right)$.

6 (a). $p=1,\ q=-1,\ r=1$

6 (b). $u=c_1\cos\sqrt{\lambda+1}x+c_2\sin\sqrt{\lambda+1}x,\ u'=-\sqrt{\lambda+1}c_1\sin\sqrt{\lambda+1}x+\sqrt{\lambda+1}c_2\cos\sqrt{\lambda+1}x$. From the boundary conditions, we have $u'(0)=c_2\sqrt{\lambda+1}=0\Rightarrow c_2=0$ and / or $\sqrt{\lambda+1}=0$. Therefore, $u=c_1\cos\sqrt{\lambda+1}x$. $u'(1)=-c_1\sqrt{\lambda+1}\sin\sqrt{\lambda+1}=0\Rightarrow\sqrt{\lambda_n+1}=n\pi$. Thus $\lambda_n=-1+(n\pi)^2,\ u_n=\cos(n\pi x)$.

7 (a). $p=1,\ q=1,\ r=1$

7 (b). $u=c_1\cos\sqrt{\lambda-1}x+c_2\sin\sqrt{\lambda-1}x,\ u'=-\sqrt{\lambda-1}c_1\sin\sqrt{\lambda-1}x+\sqrt{\lambda-1}c_2\cos\sqrt{\lambda-1}x$. From the boundary conditions, we have $u(0)=c_1=0\Rightarrow u=c_2\sin\sqrt{\lambda-1}x$. From $u(2)=c_2\sin\left(\sqrt{\lambda-1}\cdot 2\right)=0,\ 2\sqrt{\lambda_n-1}=n\pi$. Thus $\lambda_n=1+\left(\frac{n\pi}{2}\right)^2,\ u_n=\sin\left(\frac{n\pi x}{2}\right)$.

8 (a). $p=1,\ q=0,\ r=1$

8 (b). $u=c_1\cos\sqrt{\lambda}x+c_2\sin\sqrt{\lambda}x,\ u'=-\sqrt{\lambda}c_1\sin\sqrt{\lambda}x+\sqrt{\lambda}c_2\cos\sqrt{\lambda}x$. From the boundary conditions, we have $u(0)=c_1=0\Rightarrow u=c_2\sin\sqrt{\lambda}x$. From $u(1)+u'(1)=c_2\left[\sin\sqrt{\lambda}+\sqrt{\lambda}\cos\sqrt{\lambda}\right]=0,\ \sqrt{\lambda_n}=-\tan\sqrt{\lambda_n},\ u_n=\sin\sqrt{\lambda_n}x$ (or u would be zero). From root-finding software, we obtain $\lambda_1=4.11586,\ \lambda_2=24.1393,\ \lambda_3=63.6591$.

9 (a). $p=1,\ q=-4,\ r=2$

9 (b). $u=c_1\cos\sqrt{2\lambda+4}x+c_2\sin\sqrt{2\lambda+4}x$. From the boundary conditions, we have $u(0)=c_1=0\Rightarrow u=c_2\sin\sqrt{2\lambda+4}x$. From $u(3)=c_2\sin\left(\sqrt{2\lambda+4}\cdot 3\right)=0,\ 3\sqrt{2\lambda_n+4}=n\pi$. Thus $\lambda_n=\frac{1}{2}\left(\frac{n\pi}{3}\right)^2-2,\ u_n=\sin\left(\frac{n\pi x}{3}\right)$.

10 (a). $p=e^{2x},\ q=-2e^{2x},\ r=e^{2x}$

10 (b). $r^2+2r+(\lambda+2)=0\Rightarrow r=\frac{-2\pm\sqrt{4-4(\lambda+2)}}{2}=-1\pm i\sqrt{\lambda-1}$

$\Rightarrow u=c_1e^{-x}\cos\left(\sqrt{\lambda-1}x\right)+c_2e^{-x}\sin\left(\sqrt{\lambda-1}x\right)$. From the boundary conditions, we have $u(0)=c_1=0,\ u(1)=c_2e^{-1}\sin\sqrt{\lambda-1}=0\Rightarrow\sqrt{\lambda_n-1}=n\pi\Rightarrow\lambda_n=1+(n\pi)^2,\ u_n=e^{-x}\sin n\pi x$.

11 (a). $p=e^{x},\ q=-e^{x},\ r=e^{x}$

11 (b). $r^2+r+(\lambda+1)=0\Rightarrow r=\frac{-1\pm\sqrt{1-4(\lambda+1)}}{2}=-\frac{1}{2}\pm i\sqrt{\lambda+\frac{3}{4}}$

$\Rightarrow u=c_1e^{-x/2}\cos\left(\sqrt{\lambda+3/4}\,x\right)+c_2e^{-x/2}\sin\left(\sqrt{\lambda+3/4}\,x\right)$. From the boundary conditions, we have

$u(0)=c_1=0\Rightarrow u=c_2e^{-x/2}\sin\left(\sqrt{\lambda+3/4}\,x\right)$,

$u'=c_2e^{-x/2}\left[-\frac{1}{2}\sin\left(\sqrt{\lambda+3/4}\,x\right)+\sqrt{\lambda+3/4}\cos\left(\sqrt{\lambda+3/4}\,x\right)\right]$.

Also,

$u(1)+2u'(1)=c_2e^{-1/2}\sin\sqrt{\lambda+3/4}+2c_2e^{-1/2}\left[-\frac{1}{2}\sin\left(\sqrt{\lambda+3/4}\right)+\sqrt{\lambda+3/4}\cos\left(\sqrt{\lambda+3/4}\right)\right]=0$.

Therefore, $\sqrt{\lambda_n+3/4}=\left(n-\frac{1}{2}\right)\pi\Rightarrow\lambda_n=-\frac{3}{4}+\left(n-\frac{1}{2}\right)^2\pi^2\ \ u_n=e^{-x/2}\sin\left(\left(n-\frac{1}{2}\right)\pi x\right)$.

12 (a). $p=e^{-x},\ q=0,\ r=e^{-x}$

12 (b). $r^2-r+\lambda=0\Rightarrow r=\frac{1\pm\sqrt{1-4\lambda}}{2}=\frac{1}{2}\pm i\sqrt{\lambda-1/4}$

$\Rightarrow u=c_1e^{x/2}\cos\left(\sqrt{\lambda-1/4}\,x\right)+c_2e^{x/2}\sin\left(\sqrt{\lambda-1/4}\,x\right)$.

From the boundary conditions, we have

$u(0)=c_1=0,\ u(2)=c_2 e\sin\left(2\sqrt{\lambda-1/4}\right)=0 \Rightarrow \sqrt{\lambda_n-1/4}=\dfrac{n\pi}{2} \Rightarrow \lambda_n=\dfrac{1}{4}+\left(\dfrac{n\pi}{2}\right)^2,$

$u_n=e^{x/2}\sin\left(\dfrac{n\pi x}{2}\right).$

13 (a). $p=e^x,\ q=0,\ r=e^x$

13 (b). $r^2+r+\lambda=0 \Rightarrow r=\dfrac{-1\pm\sqrt{1-4\lambda}}{2}=-\dfrac{1}{2}\pm i\sqrt{\lambda-1/4}$

$\Rightarrow u=c_1e^{-1/2(x-1)}\cos\left(\sqrt{\lambda-1/4}(x-1)\right)+c_2e^{-1/2(x-1)}\sin\left(\sqrt{\lambda-1/4}(x-1)\right)$. From the boundary conditions, we have

$u(1)=c_1=0,\ u(2)=c_2e^{-1/2}\sin\left(\sqrt{\lambda-1/4}\right)=0 \Rightarrow \sqrt{\lambda_n-1/4}=n\pi \Rightarrow \lambda_n=\dfrac{1}{4}+(n\pi)^2,$

$u_n=e^{-1/2(x-1)}\sin\left(n\pi(x-1)\right).$

14 (a). $p=x,\ q=0,\ r=\dfrac{1}{x}$

14 (b). First, let $u=x^r$. Now, $r(r-1)+r+\lambda=r^2+\lambda=0 \Rightarrow r=\pm i\sqrt{\lambda}$. Thus $u=c_1\cos\left(\sqrt{\lambda}\ln x\right)+c_2\sin\left(\sqrt{\lambda}\ln x\right)$. From the boundary conditions, we have

$u(1)=c_1=0,\ u(4)=c_2\sin\left(\sqrt{\lambda}\ln 4\right)=0 \Rightarrow \sqrt{\lambda_n}\ln 4=n\pi \Rightarrow \lambda_n=\left(\dfrac{n\pi}{\ln 4}\right)^2,\ u_n=\sin\left(\dfrac{n\pi\ln x}{\ln 4}\right).$

15 (a). From Problem 14, $p=x,\ q=0,\ r=\dfrac{1}{x}$

15 (b). $u=c_1\cos\left(\sqrt{\lambda}\ln x\right)+c_2\sin\left(\sqrt{\lambda}\ln x\right)$. From the boundary conditions, we have

$u'(1)=c_2\sqrt{\lambda}=0,\ u'(3)=-c_1\dfrac{\sqrt{\lambda}}{3}\sin\left(\sqrt{\lambda}\ln 3\right)=0 \Rightarrow \sqrt{\lambda_n}\ln 3=n\pi \Rightarrow \lambda_n=\left(\dfrac{n\pi}{\ln 3}\right)^2,$

$u_n=\cos\left(\dfrac{n\pi\ln x}{\ln 3}\right).$

18. $p=1,\ q=-1,\ r=1 \Rightarrow \theta'=(\lambda+1)\sin^2\theta+\cos^2\theta=1+\lambda\sin^2\theta$. $R'=-\lambda R\sin\theta\cos\theta$.
$u(0)=R(0)\sin\theta(0)=0,\ u(1)=R(1)\sin\theta(1)=0,\ \sin\theta(0)=\sin\theta(1)=0$.

19. $p=1,\ q=2,\ r=3 \Rightarrow \theta'=(3\lambda-2)\sin^2\theta+\cos^2\theta$.
$R'=(1-3\lambda+2)R\sin\theta\cos\theta=-3(\lambda-1)R\sin\theta\cos\theta$.
$u'(0)=\dfrac{1}{p(0)}R(0)\cos\theta(0)=0,\ u(1)=R(1)\sin\theta(1)=0,\ \cos\theta(0)=0,\ \sin\theta(1)=0$.

20. $p=e^{-2x},\ q=0,\ r=e^{-2x} \Rightarrow \theta'=\lambda e^{-2x}\sin^2\theta+e^{2x}\cos^2\theta$. $R'=\left(e^{2x}-\lambda e^{-2x}\right)R\sin\theta\cos\theta$.
$u(0)=u(2)=0$. Therefore, $\sin\theta(0)=0,\ \sin\theta(2)=0$

21. $p=e^{-x^2},\ q=-e^{-x^2},\ r=e^{-x^2} \Rightarrow \theta'=e^{-x^2}(\lambda+1)\sin^2\theta+e^{x^2}\cos^2\theta$.
$R'=\left(e^{x^2}-(\lambda+1)e^{-x^2}\right)R\sin\theta\cos\theta$. $u(0)=R(0)\sin\theta(0)=0,\ u'(1)=\dfrac{1}{p(1)}R(1)\cos\theta(1)=0$.
Therefore, $\sin\theta(0)=0,\ \cos\theta(1)=0$