MA208 Solutions Week 2 Sem B 2000

Groups 4.2 page 19

1. a) t b) s c) e d) q e) t f) e g) p
2. $p(st) = pp = q$ and $(ps)t = rt = q$ hence $p(st) = (ps)t$
3. e
4. q
5. i) a) z b) x c) y d) e e) x f) x

ii)

.	e	x	y	z
e	e	x	y	z
x	x	e	z	y
y	y	z	e	x
z	z	y	x	e

iii) the table is closed, the identity is e, and x, y and z are all self inverse. Composition of permutations is associative. Hence $\{e, x, y, z\}(.)$ form a group

Groups 5.2 page 24

1. $\Rightarrow$ If G is abelian then $ab = ba$ for all $a, b, \in$ G

So $(ab)^2 = abab = aabb = a^2b^2$ "

$\Leftarrow$ If $(ab)^2 = a^2b^2$ "

then $abab = aabb$ "

so $a^{-1}ababb^{-1} = a^{-1}aabbb^{-1}$ "

hence $ba = ab$ " ie G is abelian

2. $x^2 = x \Rightarrow x^{-1}x^2 = x^{-1}x \Rightarrow x = 1$

So x is idempotent $\Rightarrow$ x is the identity.

Since there is only one identity, there is only one idempotent element in G.

3. If $(ab)^{-1} = a^{-1}b^{-1}$ for all $a, b \in$ G

$\Rightarrow$ $(ab)\,(ab)^{-1} = (ab)a^{-1}b^{-1}$ for all $a, b \in$ G

$\Rightarrow$ $1 = aba^{-1}b^{-1}$ for all $a, b \in$ G

$\Rightarrow$ $b = aba^{-1}b^{-1}b$ for all $a, b \in$ G

$\Rightarrow$ $b = aba^{-1}.1$ for all $a, b \in$ G

$\Rightarrow$ $ba = aba^{-1}a$ for all $a, b \in$ G

$\Rightarrow$ $ba = ab$ for all $a, b \in$ G Thus G is abelian

4. If $x^2 = 1 \Rightarrow xy^{-1}yx = 1 \Rightarrow yxy^{-1}yx = y.1 \Rightarrow yxy^{-1}yxy^{-1} = y.y^{-1} \Rightarrow (yxy^{-1})^2 = 1$

If $(yxy^{-1})^2 = 1 \Rightarrow yxy^{-1}yxy^{-1} = 1 \Rightarrow yxxy^{-1} = 1 \Rightarrow yx^2y^{-1}y = 1.y$

$\Rightarrow yx^2 = y \Rightarrow y^{-1}yx^2 = y^{-1}y = 1 \Rightarrow x^2 = 1$

Hence $x^2 = 1$ if and only if $(yxy^{-1})^2 = 1$

5. Given some group G(.) where $a^3 b^3 = (ab)^3$ for all $a, b \in$ G.

Prove i) $a^2 b^2 = (ba)^2$

$a^3 b^3 = (ab)^3 \Rightarrow aaabbb = ababab$

$\Rightarrow aabbb = babab$ (left cancellation, theorem 5.2.1)

$\Rightarrow aabb = baba$ (right cancellation, theorem 5.2.1)

$\Rightarrow a^2 b^2 = (ba)^2$

and hence deduce ii) $a^4 b^4 = (ab)^4$

LHS $= a^4 b^4 = (a^2)^2(b^2)^2 = (b^2a^2)^2$ (result i – which holds for all $a, b \in$ G.)

$= ((ab)^2)^2$ (result i – which holds for all $a, b \in$ G.) $= (ab)^4$ = RHS

Exercise 5.3A page 24

1. In $\{\mathbf{Z}_5 - [0]\}(\odot)$ with element [2] we have the pattern
   [2], [4], [3], [1], [2], [4], [3], [1], . . . (repeats every 4)

2. In $\{\mathbf{Z}_5 - [0]\}(\odot)$ with element [4] we have the pattern
   [4], [1], [4], [1], . . . (repeats every 2)

3. In {1,-1,i,-i} with element i we have
   i, -1, -i, 1, i, -1, -i, 1, . . . (repeats every 4)

4. In {1,-1,i,-i} with element -i we have
   -i, -1, i, 1, -i, -1, i, 1, . . . (repeats every 4)

5. In $\mathbf{Z}_6(\oplus)$ with element [2] we have
   [2], [4], [0], [2], [4], . . . (repeats every 3)

6. In $\mathbf{Z}_6(\oplus)$ with element [3] we have
   [3], [0], [3], [0], [3], . . . (repeats every 2)

7. In $\mathbf{Z}_6(\oplus)$ with element [5] we have
   [5], [4], [3], [2], [1], [0], [5] , [4] . . . (repeats every 6)

8. In S_3 with composition of permutations and element $\begin{pmatrix}1 & 2 & 3\\ 2 & 3 & 1\end{pmatrix}$ we have

$$\begin{pmatrix}1 & 2 & 3\\ 2 & 3 & 1\end{pmatrix}\begin{pmatrix}1 & 2 & 3\\ 3 & 1 & 2\end{pmatrix}\begin{pmatrix}1 & 2 & 3\\ 1 & 2 & 3\end{pmatrix}\begin{pmatrix}1 & 2 & 3\\ 2 & 3 & 1\end{pmatrix}\begin{pmatrix}1 & 2 & 3\\ 3 & 1 & 2\end{pmatrix}\begin{pmatrix}1 & 2 & 3\\ 1 & 2 & 3\end{pmatrix}\begin{pmatrix}1 & 2 & 3\\ 2 & 3 & 1\end{pmatrix}\ldots\ldots\ldots$$

   (repeats every 3- see also notes page 26)

9. In $\mathbf{Z}(+)$ with element 3 we have
   3, 6, 9, 12, 15, 18, 21, . . . (no repetition)

10. In R-{0}, (.) with element 2 we have
    2, 4, 8, 16, 32, . . . (no repetition)

Exercise 5.3B page 28

1.

	[0]	[1]	[2]	[3]	[4]	[5]	[6]	[7]	[8]	[9]	[10]	[11]
order	1	12	6	4	3	12	2	12	3	4	6	12

2. a) $\begin{pmatrix}1 & 2 & 3 & 4\\ 2 & 3 & 4 & 1\end{pmatrix}$ has order 4

b)

$$\begin{pmatrix}1 & 2 & 3 & 4\\ 1 & 4 & 2 & 3\end{pmatrix} = \begin{pmatrix}1 & 2 & 3 & 4\\ 1 & 4 & 2 & 3\end{pmatrix}$$

$$\begin{pmatrix}1 & 2 & 3 & 4\\ 1 & 4 & 2 & 3\end{pmatrix}\begin{pmatrix}1 & 2 & 3 & 4\\ 1 & 4 & 2 & 3\end{pmatrix} = \begin{pmatrix}1 & 2 & 3 & 4\\ 1 & 3 & 4 & 2\end{pmatrix}$$

$$\begin{pmatrix}1 & 2 & 3 & 4\\ 1 & 4 & 2 & 3\end{pmatrix}\begin{pmatrix}1 & 2 & 3 & 4\\ 1 & 4 & 2 & 3\end{pmatrix}\begin{pmatrix}1 & 2 & 3 & 4\\ 1 & 4 & 2 & 3\end{pmatrix} = \begin{pmatrix}1 & 2 & 3 & 4\\ 1 & 3 & 4 & 2\end{pmatrix}\begin{pmatrix}1 & 2 & 3 & 4\\ 1 & 4 & 2 & 3\end{pmatrix} = \begin{pmatrix}1 & 2 & 3 & 4\\ 1 & 2 & 3 & 4\end{pmatrix}$$

so the element has order 3

3. $\{\mathbf{Z}_7 - [0]\}(\odot)$

a) [2] has order 3

b) [5] has order 6 since:

[5]	
$[5] \odot [5]$	$= [25] = [4]$
$[5] \odot [5] \odot [5]$	$= [4] \odot [5] = [20] = [6]$
$[5] \odot [5] \odot [5] \odot [5]$	$= [6] \odot [5] = [30] = [2]$
$[5] \odot [5] \odot [5] \odot [5] \odot [5]$	$= [2] \odot [5] = [10] = [3]$
$[5] \odot [5] \odot [5] \odot [5] \odot [5] \odot [5]$	$= [3] \odot [5] = [15] = [1]$

4. a)

	1	*a*	*b*	*ab*	*ba*	*aba*
1	1	*a*	*b*	*ab*	*ba*	*aba*
a	*a*	1	*ab*	*b*	*aba*	*ba*
b	*b*	*ba*	1	*aba*	*a*	*ab*
a	*ab*	*aba*	*a*	*ba*	1	*b*
ba	*ba*	*b*	*aba*	1	*ab*	*a*
aba	*aba*	*ab*	*ba*	*a*	*b*	1

b) 1 has order 1, a, b, aba have order 2, ab, ba have order 3

c) a, b, aba are self inverse, ab has inverse ba, ba has inverse ab

5. 1 has order 1, c, c^3 have order 4, all other elements have order 2

6. a) Let a have order n and a^{-1} have order m.

So $a^n = 1$, multiplying by a^{-1} $\Rightarrow a^{-1} a^n = a^{-1} 1 \Rightarrow a^{n-1} = a^{-1}$

multiplying again by a^{-1} $\Rightarrow a^{-1} a^{n-1} = a^{-1} a^{-1} \Rightarrow a^{n-2} = (a^{-1})^2$

… … … $\Rightarrow 1 = (a^{-1})^n$

$\Rightarrow$ m the order of a^{-1} is $\leq$ n

Similarly $(a^{-1})^m = 1$

multiplying by a $\Rightarrow a\,(a^{-1})^m = a. \Rightarrow (a^{-1})^{m-1} = a \Rightarrow$ … … … $\Rightarrow 1 = a^m$

$\Rightarrow$ n the order of a is $\leq$ m

So m $\leq$ n and n $\leq$ m and hence n = m

b). Suppose ab has order m and ba has order n. Then

$(ab)^m = 1$ and so $ababab \ldots ab = 1$ (m times)

Hence $a^{-1}(ababab \ldots ab)a = a^{-1}\, 1\, a = 1$ and so $(ba)^m = 1$.

Hence n the order of $ba \leq$ m.

Similarly

$(ba)^n = 1$ and so $bababa \ldots ba = 1$ (n times)

Hence $b^{-1}(bababa \ldots ba)b = b^{-1}\, 1\, b = 1$ and so $(ab)^n = 1$.

Hence m the order of $ab \leq$ n.

So m $\leq$ n and n $\leq$ m and hence n = m