MA208 Solutions Week 3 Sem B 2003

Exercise 6.1 page 30

1 a) $\langle [1] \rangle = \langle [2] \rangle = \langle [3] \rangle = \langle [4] \rangle = \langle [5] \rangle = \langle [6] \rangle = \mathbf{Z}_7(\oplus)$

b) $\langle [1] \rangle = \langle [3] \rangle = \langle [5] \rangle = \langle [7] \rangle = \mathbf{Z}_8(\oplus)$

c) $\langle [2] \rangle = \langle [3] \rangle = \{\mathbf{Z}_5 - [0]\} (\odot)$

d) $\langle [3] \rangle = \langle [5] \rangle = \{\mathbf{Z}_7 - [0]\} (\odot)$

2 a) $\langle i, j \rangle = \langle -i, j \rangle = \langle i, -j \rangle = \langle -i, -j \rangle = \langle j, k \rangle = \langle -j, k \rangle = \langle j, -k \rangle = \langle -j, -k \rangle$
$= \langle i, k \rangle = \langle -i, k \rangle = \langle i, -k \rangle = \langle -i, -k \rangle$

e) $\langle ([1], [1]) \rangle = \langle ([1], [2]) \rangle$

f) Apart from $\{a, a^2\}$ any pair of group elements neither of which is 1 forms a generating set for the group.

3. $S_3 = \{e, p, q, r, s, t\}$ (.) where

$e = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$ $p = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$ $q = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$ $r = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$ $s = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$ $t = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$

the generating sets of $S_3 = \langle p, r \rangle = \langle p, s \rangle = \langle p, t \rangle = \langle q, r \rangle = \langle q, s \rangle = \langle q, t \rangle$

Exercise 6.2 page 32

1. Find all the generating elements of the following cyclic groups

a) $\mathbf{Z}_5 (\oplus) = \langle [1] \rangle = \langle [2] \rangle = \langle [3] \rangle = \langle [4] \rangle$

b) $\mathbf{Z}_4 (\oplus) = \langle [1] \rangle = \langle [3] \rangle$

c) $\{\mathbf{Z}_{11} - [0]\} (\odot) = \langle [2] \rangle = \langle [6] \rangle = \langle [7] \rangle = \langle [8] \rangle$

d) $\{\mathbf{Z}_7 - [0]\} (\odot) \rangle = \langle [3] \rangle = \langle [5] \rangle$

2. If the cyclic group has prime order m, then there are m–1 single generators, ie every element except the identity will generate the group.

3. $\mathbf{Z}_2 \times \mathbf{Z}_3$ (*) has elements ([0], [0]), ([1], [0]), ([0], [1]), ([1], [1]), ([0], [2]), ([1],
[2]),
which can all be generated by either ([1], [1]) or ([1], [2]) from 6.1, 2b) above. Since the whole group can be generated from a single element, it is cyclic.

4. $\mathbf{Z}_2 \times \mathbf{Z}_2$ (*) has elements ([0], [0]), ([1], [0]), ([0], [1]), ([1], [1])

orders: 1 2 2 2

Since there is no element of order 4, no single element could generate the group and hence the group is not cyclic.

5. **Error, t**he question does not give a cyclic group so change G as shown
G(.) = $\{1, x, y, y^2, y^3, \mathbf{y^4}, xy, xy^2, xy^3, \mathbf{xy^4}\}$ where $x^2 = \mathbf{y^5} = 1$ and $xy = yx$.
Prove that G is cyclic.

To do this we need to find one element that will generate the whole group
Consider xy (clearly we need to chose an element that contains both x and y

xy	$= xy$
$xy.xy = xxyy = x^2 y^2 = 1.\ y^2$	$= y^2$
$(xy)^3 = y^2 . xy$	$= xy^3$
$(xy)^4 = xy^3 . xy = x^2 y^4 = 1\ y^4$	$= y^4$
$(xy)^5 = y^4 . xy = xy^5 = x.1$	$= x$
$(xy)^6 = x . xy = x^2 y = 1\ y$	$= y$
$(xy)^7 = y . xy$	$= xy^2$
$(xy)^8 = xy^2 . xy = x^2 y^3 = 1\ y^3$	$= y^3$
$(xy)^9 = y^3 . xy = xy^4$	$= xy^4$
$(xy)^{10} = xy^4 . xy = x^2 y^5 = 1 . 1$	$= 1$

So G(.) = $\langle xy \rangle$

In a similar way you could show that $\langle xy^2 \rangle = \langle xy^3 \rangle = \langle xy^4 \rangle$