MA208 – Solutions Week 4 Semester B 2005

Exercises 7.1

1.

⊕	0	6
0	0	6
6	6	0

⊕	0	4	8
0	0	4	8
4	4	8	0
8	8	0	4

⊕	0	3	6	9
0	0	3	6	9
3	3	6	9	0
6	6	9	0	3
9	9	0	3	6

⊕	0	2	4	6	8	10
0	0	2	4	6	8	10
2	2	4	6	8	10	0
4	4	6	8	10	0	2
6	6	8	10	0	2	4
8	8	10	0	2	4	6
10	10	0	2	4	6	8

•	1	12
1	1	12
12	12	1

•	1	3	9
1	1	3	9
3	3	9	1
9	9	1	3

•	1	5	8	12
1	1	5	8	12
5	5	12	1	8
8	8	1	12	5
12	12	8	5	1

•	1	3	4	9	10	12
1	1	3	4	9	10	12
3	3	9	12	1	4	10
4	4	12	3	10	1	9
9	9	1	10	3	12	4
10	10	4	1	12	9	3
12	12	10	9	4	3	1

2. a) Subgroups are {0}, {0,4}, {0, 2, 4, 6} and whole set.

   b) Subgroups are {1}, {1, 10}, {1, 3, 4, 5, 9} and whole set.

   c) Only subgroups are {0} and the whole set.

   d) Subgroups are {1}, {1, 18}, {1,7,11}, {1, 7, 8, 11, 12, 18}, {1, 4, 5, 6, 7, 9, 11, 16, 17} and the whole set.

3. The number of elements in a subgroup divides the number of elements in a group.

4. If H is a subgroup of G and a, b $\in$ H then ab = ba since both a and b are elements of G and G is abelian.

Exercises 7.2

1. a)
   {1, a, a^2, a^3, a^4, a^5, a^6}
   {1, a^2, a^4}
   {1, a^3}
   {1, b}
   {1, ab, a^2, a^3b, a^4, a^5b}
   {1, a^2b, a^4, b, a^2, a^4b}
   {1, a^3b}
   {1, a^3, b, a^3b}
   {1}, whole set

   b)
   {1, a, a^2, a^3, a^4, a^5, a^6}
   {1, a^2, a^4}
   {1, a^3}
   {1, b}
   {1, ab}
   {1, a^2b}
   {1, a^3b}
   {1, a^4b}
   {1, a^5b}
   {1, a^3, ab, a^4b}
   {1, a^3, a^2b, a^5b}
   {1, a^3, b, a^3b}
   {1, a^2, a^4, b, a^2b, a^4b}
   {1, a^2, a^4, ab, a^3b, a^5b}
   {1} and whole set

   c)
   {1, a, a^2, a^3}
   {1, a^2}
   {1, b, a^2, a^2b}
   {1, ab, a^2, a^3b}
   {1} and whole set

   d)
   {1, a}
   {1, b, b^2}
   {1, ab, abab}
   {1, ba, ab^2}
   {1, aba, bab}
   {1, b^2ab}
   {1, bab^2}
   {1, a, b^2ab, bab^2}
   {1} and the whole set

2. If $e = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$, $p = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$, $q = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix}$, $r = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$,

$s = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$, $t = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}$ then the subgroups are

{e}, {e, p, q}, {e, r}, {e, s}, {e, t} and {e, p, q, r, s, t}.

3. If $e = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix}$, $p = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$, $q = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$,

$r = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 3 \end{pmatrix}$, $s = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 2 & 1 & 4 \end{pmatrix}$, $t = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \end{pmatrix}$

$u = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix}$, $v = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 3 & 2 & 1 \end{pmatrix}$ then the subgroups are

{e}, {e, p, q, r}, {e, q}, {e,s}, {e, t}, {e, u}, {e,v}, {e, q, s, t}, {e, q, u, v} and {e, p, q, r, s, t, u, v}.

Exercises 7.3

The set of odd integers does not form a subgroup with respect to addition as it doesn't contain the identity element 0.

The set of even integers is a subgroup since if a = 2k and b = 2l are even integers then a – b = 2k – 2l = 2(k-l) is an even integer. Hence $ab^{-1} \in H$ and H is a subgroup.

1. The set H does not form a subgroup as it is not closed. If z_1 and z_2 are elements of H then $|z_1z_2| = |z_1||z_2| = 4$. Hence $z_1z_2 \notin H$.

2. a) If a, b $\in$ H then a = $3k_1$, b = $3k_2$ for some k_1, k_2. But then

$ab^{-1} = a - b = 3(k-l)$ so ab^{-1} is an element of H so H is a subgroup.

b) If a, b $\in$ H then $a = 2^m$, $b = 2^n$ for some m, n. But then

$ab^{-1} = 2^m/2^n = 2^{m-n}$ so ab^{-1} is an element of H so H is a subgroup.

c) If a, b $\in$ H then $gag^{-1} = a$, $gbg^{-1} = b$. But then

$(gbg^{-1})^{-1} = b^{-1}$ and so $gb^{-1}g^{-1} = b^{-1}$

Hence $gab^{-1}g^{-1} = gag^{-1}gb^{-1}g^{-1} = ab^{-1}$ so ab^{-1} is an element of H.
So H is a subgroup.

d) If a, b $\in$ H then $a^n = 1$, $b^n = 1$. But then

$$\begin{aligned}(ab^{-1})^n &= a^nb^{-n} \quad \text{(as G is abelian)} \\ &= 1_G\, 1_G = 1_G\end{aligned}$$

So ab^{-1} is an element of H and so H is a subgroup.

e) If $a, b \in H$ then $ka = ak$ for all k in K, $bk = kb$ for all k in K. But then

$$b^{-1}bkb^{-1} = b^{-1}kbb^{-1} \text{ and so } kb^{-1} = b^{-1}k.$$

Hence $k(ab^{-1}) = (ka)b^{-1} = (ak)b^{-1} = a(kb^{-1}) = a(b^{-1}k) = (ab^{-1})k$ for all k in K. Hence ab^{-1} is in H and so H is a subgroup.

3. Suppose a and b belong to $H \cap K$. Then $a \in H$, $a \in K$, $b \in H$, $b \in K$. But H is a subgroup so $ab^{-1} \in H$. Similarly $ab^{-1} \in K$. Hence $ab^{-1} \in H \cap K$ and so $H \cap K$ is a subgroup.

5. $H \cup K$ is not always a subgroup. For example take $G = \{1, a, a^2, a^3, a^4, a^5\}$ where $a^6 = 1$, $H = \{1, a^3\}$, $K = \{1, a^2, a^4\}$.

6. a) $\{1, a^2\}$

b) $\{1\}$

c) $\{1\}$

Suppose a, b belong to the centre of G. Then $ga = ag$, $gb = bg$ for all g in G. But then

$$b^{-1}gbb^{-1} = b^{-1}bgb^{-1} \text{ and so } b^{-1}g = gb^{-1}$$

for every g in G. Hence

$$g(ab^{-1}) = (ga)b^{-1} = (ag)b^{-1} = a(gb^{-1}) = a(b^{-1}g) = (ab^{-1})g$$

and so ab^{-1} is in the centre of G as it commutes with every element.

If we have an operation table for our group then an element x belongs to the centre provided the row and column labelled x contain the same elements in the same order.