MA208 Solutions Week 5 Semester B

Exercises 9.1 – Solutions

1. $0 + H = \{3k : k \in Z\}$
   $1 + H = \{1+3k : k \in Z\}$
   $2 + H = \{2+3k : k \in Z\}$

2. For example $\{1, a^2, a^4\}$ in C_6 has this property. Any subgroup of an abelian group will have identical left and right cosets. It is possible for subgroups of non-abelian groups to have this property – for example $\{1, a, a^2, a^3\}$ in D_4.

3. a) $H1 = Ha^2 = \{ 1, a^2\} = 1H = a^2H$
   $Ha = Ha^3 = \{a, a^3\} = aH = a^3H$
   $Hb = Ha^2b = \{b, a^2b\} = bH = a^2bH$
   $Hab = Ha^3b = \{ab, a^3b\} = abH = a^3bH$

   b) $H1 = Hb = \{1, b\}$
   $Ha = Ha^3b = \{a, a^3b\}$
   $Ha^2 = Ha^2b = \{a^2, a^2b\}$
   $Ha^3 = Hab = \{a^3, ab\}$

   $1H = bH = \{1,b\}$
   $aH = abH = \{a, ab\}$
   $a^2H = a^2bH = \{a^2, a^2b\}$
   $a^3H = a^3bH = \{a^3, a^3b\}$

   c) $H1 = Ha = Ha^2 = Ha^3 = \{1, a, a^2, a^3\}$
   $Hb = Hab = Ha^2b = Ha^3b = \{b, ab, a^2b, a^3b\}$

   $1H = aH = a^2H = a^3H = \{1, a, a^2, a^3\}$
   $bH = abH = a^2bH = a^3bH = \{b, ab, a^2b, a^3b\}$

   d) $H1 = Hb = Hb^2 = \{1, b, b^2\}$
   $Ha = Hba = Habab = \{a, ba, abab\}$
   $Hab = Hbab = Hb^2ab = \{ab, bab, b^2ab\}$
   $Haba = Hbab = Hbab^2 = \{aba, bab, bab^2\}$

   $1H = bH = b^2H = \{1, b, b^2\}$
   $aH = abH = ab^2H = \{a, ab, ab^2\}$
   $baH = babH = bab^2H = \{ba, bab, bab^2\}$
   $abaH = ababH = b^2abH = \{aba, abab, b^2ab\}$

   e) $H1 = Ha = Ha^2 = Ha^3 = \{1, a, a^2, a^3\}$
   $Hb = Hab = Ha^2b = Ha^3b = \{b, ab, a^2b, a^3b\}$

   $1H = aH = a^2H = a^3H = \{1, a, a^2, a^3\}$
   $bH = abH = a^2bH = a^3bH = \{b, ab, a^2b, a^3b\}$

5. a) True. See proof in the notes. Essentially all cosets contain exactly the same number of elements as H does.

b) False. For example let G = set of all non-zero rational numbers with operation multiplication. Then H = {1, -1} is a subgroup and all the cosets of H will contain exactly 2 elements. (Cosets are of the form {a, -a}).

c) False. See any of the examples in question 1. There are cosets that don't contain the identity in each of these.

d) False. Consider 1 e) above. Left cosets and right cosets are the same but G is not abelian.

e) True. In an abelian group xy = yx for all x and y; but then

$$\begin{aligned} x \in Ha &\Leftrightarrow x = ha \text{ for some a in H} \\ &\Leftrightarrow x = ah \ \ (\text{as } ha = ah) \\ &\Leftrightarrow x \in aH \end{aligned}$$

f) True. See proof in the notes.

Exercises 9.3

2. The centre of $Q_6 = \{1, a^3\}$.

$H1 = Ha^3 = \{1, a^3\} = 1H = a^3H$
$Ha = Ha^4 = \{a, a^4\} = aH = a^4H$
$Ha^2 = Ha^5 = \{a^2, a^5\} = a^2H = a^5H$
$Hb = Ha^3b = \{a, a^3b\} = bH = a^3bH$
$Hab = Ha^4b = \{ab, a^4b\} = abH = a^4bH$
$Ha^2b = Ha^5b = \{a^2b, a^5b\} = a^2bH = a^5bH$

3. The centre of $S_3 = \{1\}$. Hence

$H1 = \{1\} = 1H$
$Ha = \{a\} = aH$
$Ha^2 = \{a^2\} = a^2H$
$Hb = \{b\} = bH$
$Hab = \{ab\} = abH$
$Ha^2b = \{a^2b\} = a^2bH$

The centre of Q_4 is $\{1, a^2\}$.

$H1 = Ha^2 = \{1, a^2\} = 1H = a^2H$
$Ha = Ha^3 = \{a, a^3\} = aH = a^3H$
$Hb = Ha^2b = \{b, a^2b\} = bH = a^2bH$
$Hab = Ha^3b = \{ab, a^3b\} = abH = a^3bH$

The centre of D_4 is $\{1, a^2\}$.

$H1 = Ha^2 = \{1, a^2\} = 1H = a^2H$
$Ha = Ha^3 = \{a, a^3\} = aH = a^3H$
$Hb = Ha^2b = \{b, a^2b\} = bH = a^2bH$
$Hab = Ha^3b = \{ab, a^3b\} = abH = a^3bH$

The centre of A_4 is {1} and so Hg = {g} = gH for every g in A_4.

If H is the centre of G then left and right cosets are the same.

4. $H1 = Hab = Hba = \{1, ab, ba\} = 1H = abH = baH$
$Ha = Hb = Haba = \{a, b, aba\} = aH = bH = abaH$

5. $H+0 = \{4k : k \in Z\} = 0+H$
$H+1 = \{4k+1 : k \in Z\} = 1+H$
$H+2 = \{4k+2 : k \in Z\} = 2+H$
$H+3 = \{4k+3 : k \in Z\} = 3+H$

6. $H = \left\{\begin{pmatrix} a & 0 \\ 0 & 1/a \end{pmatrix} : a \in R-\{0\}\right\}$. Hence if $A = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ then

$$HA = \left\{\begin{pmatrix} a & 0 \\ 0 & 1/a \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} : a \in R-\{0\}\right\}$$

$$= \left\{\begin{pmatrix} a & 0 \\ 0 & -1/a \end{pmatrix} : a \in R-\{0\}\right\}$$

7. i) Suppose $Ha = Hb$. Then $a \in Ha$ and so $a \in Hb$. Hence $a = hb$ for some h in H. Hence $h = ab^{-1}$ and so $ab^{-1} \in H$ as required.

ii) Suppose $ab^{-1} \in H$ and that $x \in Ha$. Then $x = h_1a$. But we also know that $a = (ab^{-1})b$. Hence $x = h_1(ab^{-1})b$ and so $x \in Hb$ as $h_1(ab^{-1})$ is an element of H as H is a subgroup (and so is closed).

Similarly if $x \in Hb$ then $x = h_2b$ for some h_2. But $ab^{-1} \in H \Rightarrow (ab^{-1})^{-1} = ba^{-1} \in H$. Hence $x = h_2(ba^{-1})a$ and so $x \in Ha$ as $h_2ba^{-1} \in H$.

8. Suppose $a \in Hb$. Then $a = h_1b$ for some h_1 in H.

Suppose $x \in Ha$. Then $x = h_2a = h_2h_1b$ and so $x \in Hb$ as h_1h_2 is an element of H. Now suppose $y \in Hb$. Then $y = h_3b$ for some h_3 in H and so $y = h_3h_1^{-1}a$ and so $y \in Ha$.

9. Suppose a has order k. Then $a^k = 1$ and k is the smallest such power. But then

$(x^{-1}ax)^k = (x^{-1}ax)(x^{-1}ax)...(x^{-1}ax) = x^{-1}a(xx^{-1})a(xx^{-1})...(xx^{-1})ax = x^{-1}a^kx = x^{-1}x = 1$
If $(x^{-1}ax)^n = 1$ then as above $x^{-1}a^nx = 1 \Rightarrow xx^{-1}a^nxx^{-1} = x1x^{-1} = 1 \Rightarrow a^n = 1 \Rightarrow n \geq k$.

Hence the order of $x^{-1}ax$ is k.

Hence if b has order 2 then so does $g^{-1}bg$ for every g in G. But there is only one element of order 2 so $g^{-1}bg = b$ for every g and so $gb = bg$.

10. If g has order 2^k then (by Lagrange's Theorem) the order of any subgroup must divide 2^k. But the only divisors of 2^k are numbers of the form 2^j where $0 \leq j \leq k$. All such numbers are even except $j = 0$ which corresponds to the improper subgroup whose only element is the identity.

11. Any subgroup of order 2^k will do by question 10. E.g. C_8, K_4, Q_4, D_4.