10 Normal Subgroups and Quotient Groups

We have seen that in general left cosets and right cosets are not identical. In some cases, though they are the same and this allows us to prove some very important theorems.

Definition 10.1.1

If N is a subgroup of G such that gN = Ng for every g we say that N is a *normal* subgroup.

Equivalently N is a normal subgroup if $gng^{-1} \in N$ for every $g \in G$, $n \in N$.

Example 10.1.2

If G = {1, a, b, ab, ba, aba} where $a^2 = b^2 = 1$, aba = bab and H = {1,b} we see that H is not a normal subgroup since

aH = {a, ab}, Ha = {a, ba}

Example 10.1.3

If G is as above and H = {1, ab, ba} then H is normal as there are two right cosets

H1 = {1, ab, ba} and Ha = {a, b, aba}

There are also two left cosets

1H = {1, ab, ba} and aH = {a, b, aba}.

Note:

In general just because aH = Ha does not mean that ah = ha for every h in H. It merely says that if $H = \{h_1, h_2, \ldots, h_k\}$ then the sets $\{h_1a, h_2a, \ldots, h_ka\}$ and $\{ah_1, ah_2, \ldots, ah_k\}$ are equal as sets (i.e. they contain the same elements though possibly in a different order).

Of course there are some situations where ah = ha for every h. In particular if G is an abelian group then every subgroup is normal.

In addition there are other situations where we can immediately conclude that a subgroup is normal:

Theorem 10.1.4

Suppose G is a group and that H is a subgroup of index 2. Then H is a normal subgroup.

Proof:

If H has index 2 there are two right cosets. One of these must be H (= $H1_G$) and, since cosets are distinct, the other must be G-H = $\{x \in G: x \notin H\}$. But H is also a left coset (= 1_GH) and so the other left coset is also G-H. Hence left cosets and right cosets are identical and H is normal.

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Theorem 10.1.5

Suppose G is a group and that H is the centre of G. Then H is a normal subgroup.

Proof

Suppose $a \in G$. Then $aH = \{ah : h \in H\}$. But H is the centre of G and so ah = ha for every $a \in G$, $h \in H$. But this means that aH = Ha and so H is normal in G.

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Exercises 10.1

1. Find all normal subgroups of

   i) Q_4
   ii) A_4
   iii) D_6
   iv) $Z_2 \times Z_6$

2. Suppose A and B are normal subgroups of G. Prove that $A \cap B$ is also normal.

3. Suppose G is the group of all non-singular 2x2 matrices wrt multiplication and suppose H is the set of all matrices of the form $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$: $a,b \neq 0$. Show that H is a subgroup of G. Is H a normal subgroup?

   [Hint: Use the alternative definition of a normal subgroup.]

10.2 Operations on Cosets

In this section we attempt to define an operation on cosets and we will see that the operation we define 'makes sense' (or is *well-defined*) precisely in those circumstances where our subgroup is normal. Consider the following example in the quaternion group Q_4.

Example 10.2.1:

In Q_4 we have eight elements $\{1, a, a^2, a^3, b, ab, a^2b, a^3b\}$ where $a^4 = 1$, $a^2 = b^2$, $ba = a^3b$. We consider the subgroup $H = \{1, a^2\}$ which partitions Q_4 into four right cosets:

$$\alpha = \{1, a^2\},\ \ \beta = \{a, a^3\},\ \rho = \{b, a^2b\},\ \ \sigma = \{ab, a^3b\}.$$

If we now take our operation table for Q_4 :

		α		β		ρ		σ	
		1	a^2	a	a^3	b	a^2b	ab	a^3b
α	1	1	a^2	a	a^3	b	a^2b	ab	a^3b
	a^2	a^2	1	a^3	a	a^2b	b	a^3b	ab
β	a	a	a^3	a^2	1	ab	a^3b	a^2b	b
	a^3	a^3	a	1	a^2	a^3b	ab	b	a^2b
ρ	b	b	a^2b	a^3b	ab	1	a^2	a^3	a
	a^2b	a^2b	b	ab	a^3b	a^2	1	a	a^3
σ	ab	ab	a^3b	b	a^2b	a	a^3	1	a^2
	a^3b	a^3b	ab	a^2b	b	a^3	a	a^2	1

We see that it divides into cells all of which contain elements from just one coset. We can therefore rewrite the table to indicate the coset in which the element lies. This gives:

		α		β		ρ		σ	
		1	a^2	a	a^3	b	a^2b	ab	a^3b
α	1	α	α	β	β	ρ	ρ	σ	σ
	a^2	α	α	β	β	ρ	ρ	σ	σ
β	a	β	β	α	α	σ	σ	ρ	ρ
	a^3	β	β	α	α	σ	σ	ρ	ρ
ρ	b	ρ	ρ	σ	σ	α	α	β	β
	a^2b	ρ	ρ	σ	σ	α	α	β	β
σ	ab	σ	σ	ρ	ρ	β	β	α	α
	a^3b	σ	σ	ρ	ρ	β	β	α	α

which we can then reduce to an operation table on cosets as below

	α	β	ρ	σ
α	α	β	ρ	σ
β	β	α	σ	ρ
ρ	ρ	σ	α	β
σ	σ	ρ	β	α

We notice that this operation table is not only well-defined but that it also has the structure of a group operation table (in this case the Klein 4-group). We will see that provided our subgroup is normal then this always happens. We call the 4 x 4 operation table derived above, the *operation table for the induced operation.*

Definition 10.2.2:

Suppose G is a group and H a subgroup of G. We say that

$$HaHb = \{h_1ah_2b : h_1, h_2 \in H\}$$

Theorem 10.2.3

Suppose H is a normal subgroup of a group G. Then

$$HaHb = Hab \quad \text{for all } a,b \in G.$$

(i.e. if H is a normal subgroup we have a well-defined operation on cosets just as we saw in the above example).

Proof:

We shall prove i) $HaHb \subseteq Hab$ ii) $Hab \subseteq HaHb$

i) Suppose h_1ah_2b is an element of HaHb. Now $ah_2 \in aH$; but H is normal in G so $aH = Ha$. Hence $ah_2 = h_3a$ for some $h_3 \in H$. Thus

$$h_1ah_2b = h_1h_3ab \in Hab$$

since H is a subgroup and so $h_1, h_3 \in H \Rightarrow h_1h_3 \in H$.

ii) Now suppose $hab \in Hab$. Then

$$hab = h(a1_G)b = (ha)(1_Gb)$$

But $1_G \in H$ as H is a subgroup and so $1_Gb \in Hb$.

Hence $hab \in HaHb$ as required.

So we have shown that every element of HaHb is also an element of Hab while every element of Hab is also an element of HaHb. Hence they are equal as sets.

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This theorem shows that it make sense to define an operation on cosets of a normal subgroup. We **cannot** define an operation unless H is normal in G. We saw in our earlier example that the set of cosets can be turned into a group with respect to this induced operation. The following theorem tells us that we can always do this when we have a normal subgroup.

Definition 10.2.4:

Suppose H is a normal subgroup of a group G. The set of cosets of H in G with respect to the induced operation is written G/H and is called the *quotient group* of H in G.

Theorem 10.2.5:

If H is a normal subgroup of a group G then G/H forms a group with respect to the operation * on cosets defined by Ha * Hb = Hab.

Proof:

We need to demonstrate that the four group axioms hold for our given operation.

i) We have already seen that Ha*Hb = Hab and so our operation is closed on the set of cosets.

ii) (Ha*Hb)*(Hc) = (Hab)*Hc = H(ab)c = Ha(bc) = Ha*Hbc = Ha*(Hb*Hc)
Hence our operation is associative.

iii) Ha * H1 = Ha1 = H = H1a = H1* Ha
Hence we have an identity element H1 = H.

iv) Ha * Ha^{-1} = Haa^{-1} = H1 = $Ha^{-1}a$ = Ha^{-1}*Ha
Hence the coset Ha has an inverse element in G/H, namely Ha^{-1}.

Exercises 10.2:

1. Go back to Exercises 10.1 Question 1. For every normal subgroup that you found, try to write down an operation table for the induced operation on the set of cosets.

2. Mark the following statements true or false. Where you think a statement is true try to give an explanation, where you think a statement is false provide a counter example :

    a) if G is an abelian group, every quotient group of G is also abelian.

    b) if G is a non-abelian group, every quotient group is non-abelian.

    c) if G is a finite group, every quotient group is finite.

d) if G is an infinite group, every quotient group is infinite.

d) If x has order n in the group G then Hx has order n in G/H (H a normal subgroup of G).

3. A student attempts to prove that statement a) above is true. Their answer begins as follows:

"Suppose G is abelian and suppose a and b are elements of G/H. We aim to show that ab = ba."

Why would we expect the rest of this proof to be incorrect?