13 Homomorphisms and Quotient Groups

Homomorphisms of groups are very closely connected to the idea of quotient groups as we shall see shortly. To establish the connection we need a couple of definitions:

13.1.1 Definition

Suppose $\theta: G \to G'$ is a homomorphism. We define

i) the *kernel* of θ to be the set

$$\text{Ker}(\theta) = \{g \in G : g\theta = 1_{G'}\}$$

(i.e. the set of all elements that get mapped to the identity element)

ii) the *image* of θ is the set

$$\text{Im}(\theta) = \{x \in G' : x = g\theta \text{ for some } g \in G\}$$

(i.e. the set of all elements of G' that are the image of something in G).

Theorem 13.1.2

Suppose θ is a homomorphism from G to G'. Then Ker(θ) is a normal subgroup of G.

Proof:

We begin by showing that Ker(θ) is a subgroup. Suppose g_1, g_2 belong to the kernel. Then

$$g_1\theta = 1_{G'},\ g_2\theta = 1_{G'}$$

But θ is a homomorphism and so by Theorem 11.3.1 (which we know can be applied to homomorphisms)

$$(g_2^{-1})\theta = (g_2\theta)^{-1} = (1_{G'})^{-1} = 1_{G'}$$

But then

$$g_1g_2^{-1}\theta = (g_1\theta)(g_2^{-1}\theta) = 1_{G'}1_{G'} = 1_{G'}$$

Hence $g_1g_2^{-1}$ belongs to the kernel and so by Theorem 7.3.1 Ker(θ) is a subgroup.

We aim to show that ker(θ)is a normal subgroup by showing that

$$x \in \ker(\theta) \Rightarrow g^{-1}xg \in \ker(\theta) \text{ for every } g \text{ in } G.$$

If $x \in \ker(\theta)$ and $g \in G$ then

$$(g^{-1}xg)\theta = (g^{-1}\theta)(x\theta)(g\theta) = (g^{-1}\theta)\, 1_{G'}(g\theta) = (g\theta)^{-1}(g\theta) = 1_{G'}$$

Hence $g^{-1}xg \in \ker(\theta)$ so we have a normal subgroup.

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Corollary 13.1.2

If θ is an isomorphism then $\ker(\theta) = \{1_G\}$.

Proof

θ is a one-one and onto function and so exactly one element is mapped to the identity of G'. But we know (Theorem 11.3.1) that the identity element of G is mapped to $1_{G'}$ and so this is the only element in the kernel.

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Theorem 13.1.3

$\mathrm{Im}(\theta)$ is a subgroup of G'.

Proof

Suppose x and y belong to $\mathrm{Im}(\theta)$ with $x = g_1\theta$, $y = g_2\theta$; we aim to show that xy^{-1} is also in the image.

But $xy^{-1} = (g_1\theta)(g_2\theta)^{-1} = (g_1\theta)(g_2^{-1}\theta) = (g_1g_2^{-1})\theta$ and so $xy^{-1} \in \mathrm{Im}(\theta)$ as required.

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13.2 The Fundamental Theorem of Homomorphisms

We complete our study of homomorphisms with one of the most important theorems of group theory and one which connects the idea of homomorphism with the idea of quotient groups. We know that $\ker(\theta)$ is a normal subgroup and so we can form quotient groups. The structure of these quotient groups is described as follows:

Theorem 13.2.1

Suppose $\theta : G \to G'$ is a homomorphism with kernel K. Then

$$G/K \cong \mathrm{Im}(\theta)$$

Proof

To show that two structures are isomorphic we need to find an isomorphism. We define

$$\varphi : G/K \to \mathrm{Im}(\theta) \quad \text{by} \quad (aK)\varphi = a\theta$$

and because we are mapping cosets by choosing coset representatives we have to convince ourselves that the mapping is well-defined (i.e is independent of coset representative). We do this by supposing $b \in aK$ so that $b = ak_1$ for some k_1 in the kernel (i.e. b is another possible coset representative). But then

$a^{-1}b = k_1$ and so $1_{G'} = k_1\theta = (a^{-1}b)\theta = (a^{-1}\theta)(b\theta) = (a\theta)^{-1}(b\theta)$

and so $(a\theta) = (b\theta)$. Hence whatever coset representative we choose we still end up with the same image under the operation φ. In other words φ is well-defined.

We now aim to show that φ is one-one:

$$(aK)\varphi = bK\varphi \Rightarrow a\theta = b\theta.$$

But then

$$1_{G'} = (a\theta)^{-1}(b\theta) = (a^{-1}b)\theta$$

and so $a^{-1}b$ is in the kernel. But $a^{-1}b \in K \Rightarrow b \in aK$ and so $bK = aK$. Hence our cosets were the same thing all along and φ is one-one.

Clearly φ is onto by definition and so we only have to check the homomorphism property:

$$[(aK)(bK)]\varphi = (abK)\varphi = (ab)\theta = (a\theta)(b\theta) = [(aK)\varphi][(bK)\varphi]$$

This is precisely what we need and so φ is an isomorphism and the structures G/K and $\mathrm{Im}\theta$ are identical.

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