9 Cosets and Lagrange's Theorem

9.1 Cosets

9.1.1 Definition

If H is a subgroup of a group G and x is any element of G we call

$\mathrm{H}x = \{hx : h \in \mathrm{H}\}$ a right coset of H in G

$x\mathrm{H} = \{xh : h \in \mathrm{H}\}$ a left coset of H in G.

9.1.2 Example

Suppose $G = \{1, a, b, ab, ba, aba\}$ where $a^2 = b^2 = 1$, $aba = bab$ and let H be the subgroup $\{1,b\}$. Then we have, for example,

$\mathrm{H}ba = \{ba, bba\} = \{ba, a\}$.

Working through all the possible elements we find that there are three distinct right cosets and three distinct left cosets. The right cosets are

$\mathrm{H}1 = \{1, b\} = \mathrm{H}b$
$\mathrm{H}a = \{a, ba\} = \mathrm{H}ba$
$\mathrm{H}ab = \{ab, aba\} = \mathrm{H}aba$

The left cosets are

$1\mathrm{H} = \{1, b\} = b\mathrm{H}$
$a\mathrm{H} = \{a, ab\} = ab\mathrm{H}$
$ba\mathrm{H} = \{ba, aba\} = aba\mathrm{H}$

We note the following properties:

- In general left cosets are not the same as right cosets; (sometimes they are the same - if this happens many useful theorems can be derived as we will see later).
- Every element belongs to its own coset. (i.e. $ab \in ab\mathrm{H}$, $ab \in \mathrm{H}ab$).
- if $g \in \mathrm{H}$ then $g\mathrm{H} = \mathrm{H} = \mathrm{H}g$.

9.1.3 Example

We can also have cosets of subgroups of infinite groups. Suppose we have the subgroup H of **Z** (with operation addition) and suppose $\mathrm{H} = \{3x : x \in \mathrm{Z}\}$. Then there are three distinct right cosets:

$$\mathrm{H} + 0 = \mathrm{H} = \{3x : x \in \mathrm{Z}\},\ \mathrm{H} + 1 = \{3x+1 : x \in \mathrm{Z}\},\ \mathrm{H}+2 = \{3x+2 : x \in \mathrm{Z}\}.$$

Exercises 9.1

1. Let G = Z and let H be as in Example 9.1.3. What are the left cosets of H?

2. Write down two examples of subgroups where the left cosets and right cosets are the same. What type of group always has identical left and right cosets.

3. Let G and H be as described below. Write down all the left and right cosets of H in G: (the group elements and defining relations are as described in the table in section 8)

   a) $G = D_4$, $H = \{1, a^2\}$,
   b) $G = D_4$, $H = \{1, b\}$,
   c) $G = Q_4$, $H = \langle a \rangle$,
   d) $G = A_4$, $H = \langle b \rangle$,
   e) $G = Z_4 \times Z_2$, $H = \langle a \rangle$.

4. Write down any properties of cosets that you have observed.

5. Mark the following statements *true* or *false*. When a statement is true try to explain why you think it is true; when you think a statement is false try to produce a counter example.

   a) all right cosets contain the same number of elements
   b) if G is infinite then all cosets will contain an infinite number of elements
   c) every coset contains the identity element
   d) if left cosets are always the same as right cosets then G is an abelian group
   e) if G is an abelian group then left cosets are always the same as right cosets
   f) $Hx \cap Hy = \varnothing$ unless $Hx = Hy$.

9.2 Properties of Cosets

Having seen examples of subgroups and their cosets we now attempt to prove some properties of cosets. Our aim is to prove the following statements:

- the right (left) cosets of a subgroup H partition G into a number of distinct subsets with no overlap of elements.

- all right (left) cosets contain the same number of elements.

The results below are stated for right cosets; it is an easy exercise for you to rewrite them so that they apply to left cosets.

Theorem 9.2.1

If x and y are any elements of G and H is a subgroup of G then

EITHER $Hx = Hy$ **OR** $Hx \cap Hy = \varnothing$

Proof

The statement of the theorem allows two possibilities. Our strategy is to assume that $Hx \cap Hy \neq \varnothing$ and to show that this means that the cosets are identical. We do this by showing that $Hx \subseteq Hy$ and $Hy \subseteq Hx$.

If $Hx \cap Hy \neq \varnothing$ then Hx and Hy have an element z in common. Thus

$h_1x = z = h_2y$ for some elements $h_1, h_2 \in H$.

Hence $h_1x = h_2y$ so $x = h_1^{-1}h_2y$, $y = h_2^{-1}h_1x$.

1. We now aim to show that $Hx \subseteq Hy$

Suppose we have an element h_3x of Hx. Then

$h_3x = h_3h_1^{-1}h_2y$.

But H is a subgroup so $h_3h_1^{-1}h_2 \in H$ as each of the $h_i \in H$. But then $h_3x \in Hy$ and so $Hx \subseteq Hy$.

2. We now aim to show that $Hy \subseteq Hx$.

Suppose we have an element h_4y of Hy. Then

$h_4y = h_4h_2^{-1}h_1x$.

But H is a subgroup so $h_4h_2^{-1}h_1 \in H$ as each of the $h_i \in H$. But then $h_4y \in Hx$ and so $Hy \subseteq Hx$.

Hence we have shown that if the cosets have any elements in common then they must be identical.

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[Note: This uses a standard mathematical technique; we aim to show that two sets X and Y are equal by showing that $X \subseteq Y$ and $Y \subseteq X$.]

Theorem 9.2.2

Suppose H is a subgroup of a group G and suppose that H contains a finite number of elements. Every right coset contains exactly the same number of elements; each coset has |H| elements.

Proof

Suppose $H = \{h_1, h_2, \ldots, h_k\}$. Then

$$Hx = \{h_1x, h_2x, \ldots, h_kx\}$$

and so Hx will contain |H| elements provided the h_ix are distinct. But

$$h_ix = h_jx \Rightarrow h_i = h_j.$$

But the elements of H are distinct and so $h_i \neq h_j$. Hence the elements of Hx are distinct.

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The two results above lead to one of the most important theorems in group theory and one which is very useful when we attempt to find possible subgroups of a group G.

Theorem 9.2.3 (Lagrange's Theorem)

If H is a subgroup of a group G then the order of H divides the order of G.

Proof

Every element of G lies in exactly one coset (Theorem 9.2.1) while each coset contains the same number, |H|, of elements (Theorem 9.2.2). Hence we can write G as the union of distinct right cosets:

$$G = H \cup Ha_1 \cup Ha_2 \cup \ldots \cup Ha_q$$

Since all the cosets are distinct and contain |H| elements it follows that $|G| = (q+1)|H|$.

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Lagrange's theorem allows us to determine the subgroup structure of particular groups. For example D_4 (the dihedral group) has order 8 (see Section 8.3). Hence any proper subgroup must have order 2 or 4 and these can be easily determined:

1. We have a subgroup $\{1, x\}$ of order 2 whenever x has order 2. In D_4 we have five elements of order 2 - a^2, b, a^2b, ab, a^3b; these give rise to 5 subgroups of order 2.

2. We have a cyclic subgroup $\{1, x, x^2, x^3\}$ of order 4 whenever x has order 4. There is one subgroup of this type in D_4, generated by a.

3. There may be subgroups of order 4 which have the structure of the Klein 4-group. These have two generating elements (x and y say) each of order 2 which commute with each other (xy = yx). By inspection we see that a^2 commutes with both b (giving rise to a subgroup $\{1, a^2, b, a^2b\}$) and with ab (giving rise to a subgroup $\{1, ab, a^2, a^3b\}$). Since these are the only pairs of elements of order 2 that commute we know that there are no more subgroups of this type.

These are the only possible subgroups since Lagrange's theorem tells us that no subgroups of order 3 (for example) can possibly exist.

Definition 9.2.4:

Suppose that G is a group and that H is a subgroup of G. If H has k right (or left) cosets in G we say that H is a subgroup of *index k*.

For example if G has order 12 and H has order 3 then H is a subgroup of index 4.

9.3 Important Corollaries of Lagrange's Theorem

Corollary 9.3.1

A group of prime order has no proper subgroups.

Proof

If G has order p for some prime p then any subgroup must have order dividing p. But the only divisors of p are 1 (where the subgroup is the trivial group) or p itself (where the subgroup is G itself). Thus both possibilities can only give rise to improper subgroups.

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Corollary 9.3.2

If G is a finite group then the order of any element divides the order of the group.

Proof

If x is an element of order n then x will generate the subgroup

$$<x> = \{1, x, x^2, . . . , x^{n-1}\}$$

which has order n. By corollary 9.3.1 above the order of this subgroup (which is n) must divide the order of G.

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Corollary 9.3.3

Any group of prime order must be cyclic.

Proof

Suppose that x is a non-identity element of G and that G has order p. Then the order of x is not equal to 1 (as it is non-identity) and so it must be p (as these are the only divisors of p). But then <x> contains p elements and so it must be the whole of G. Hence G is generated by the single element x and so is cyclic.

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The theorems and corollaries of this section give us help when we want to construct all possible right (or left cosets) of a subgroup H. We illustrate by finding all right cosets of the subgroup $H = \{1, a^3, b, a^3b\}$ in the group

$D_6 = \{1, a, a^2, a^3, a^4, a^5, b, ab, a^2b, a^3b, a^4b, a^5b\}$ where $a^6 = b^2 = 1$ $ba = a^5b$.

1. We know that H is itself a coset and so one coset is

   $H1 = \{1, a^3, b, a^3b\}$

2. We know that every element is in its own coset and also that cosets are either identical or completely disjoint. Hence we have

   $Ha^3 = Hb = Ha^3b = \{1, a^3, b, a^3b\}$.

3. We now choose an element of G that does not appear in any coset we have already written and find its right coset. For example, choosing a gives

   $Ha = \{a, a^4, ba, a^3ba\} = \{a, a^4, a^5b, a^2b\}$

   But, as above this tells us that $Ha = Ha^4 = Ha^5b = Ha^2b = \{a, a^4, a^5b, a^2b\}$.

4. We know that every coset has four elements and that different cosets have no elements in common. Hence the four elements we have yet to write down must form the third coset. These are a^2, a^5, ab, a^4b and so we have

   $Ha^2 = Ha^5 = Hab = Ha^4b = \{a^2, a^5, ab, a^4b\}$.

We have therefore found the cosets of all twelve elements but have only had to calculate one of these directly; the others all follow immediately from the theorems.

Exercises 9.3

1. Rewrite theorems 9.2.1 and 9.2.2 and their proofs so that all references to right cosets are replaced by left cosets.

2. Suppose $G = Q_6$ and let H be the centre of G. Find all

   i) right cosets
   ii) left cosets

   of H in G.

3. Repeat question 2 when G is

   i) S_3,
   ii) Q_4,
   iii) D_4,
   iv) A_4.

   What can you say about the left and right cosets of the centre of a group?

4. Suppose $G = \{1, a, b, ab, ba, aba\}$ where $a^2 = b^2 = 1$, $aba = bab$, $H = \{1, ab, ba\}$. Write down all left cosets and right cosets of H in G.

5. Suppose $H = \{4k : k \in Z\}$ is a subgroup of $Z(+)$. What are the left and right cosets of H in G?

6. Suppose $G = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} : a, b \in R - \{0\} \right\}$ with operation matrix multiplication and let H be the subgroup consisting of all elements of G with determinant 1. Describe the right coset of the element $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

7. Suppose H is a subgroup of a group G and that a and b are elements of G. Prove that

   $$Ha = Hb \Leftrightarrow ab^{-1} \in H$$

   [Hint: Prove i) if $Ha = Hb$ then $ab^{-1} \in H$

   ii) If $ab^{-1} \in H$ then $Ha \subseteq Hb$ and $Hb \subseteq Ha$.]

8. H is a subgroup of a group G and a and b are elements of G. Prove that

   $$a \in Hb \Rightarrow Ha = Hb$$

   [Once again prove that $Ha \subseteq Hb$ and $Hb \subseteq Ha$].

9. Suppose a is an element of a group G and that a has order k. Prove that $x^{-1}ax$ also has order k for every element x. Hence prove that if G has exactly one element b of order 2 the gb = bg for every g in G (i.e. prove that b belongs to the centre of G).

10. Suppose G has order 2^k for some k. Show that G contains no proper subgroups of odd order.

11. Write down four groups of order ≥ 4 that have no subgroups of order 3.