① (i) closure : let $x = \frac{3^{2k_1}}{4^{k_1}}$ and $y = \frac{3^{2k_2}}{4^{k_2}}$ then

$$x \cdot y = \frac{3^{2(k_1+k_2)}}{4^{(k_1+k_2)}} \in G \qquad 1$$

(iii) $$e = \frac{3^{2\times 0}}{4^{0}} = 1 \in G \qquad 1$$

(iv) $$x^{-1} = \frac{3^{2(-k)}}{4^{-k}} \in G \qquad 2$$

$\therefore$ G - group

② $$(a^{-2})^2 = a^{-4} = 1 \quad \text{order}(a^{-2}) = 2 \quad 1$$

$$(b^3)^4 = b^{12} = 1 \quad \text{order}(b^3) = 4 \quad 1$$

$$(ab)^{12} \underset{G \text{ abelian}}{=} a^{12} b^{12} = (a^4)^3 b^{12} = 1 \qquad 3$$

order $(ab) = 12$

③ $y^N = 1$. Then $(x^{-1} y x)^N = \underbrace{x^{-1} y x \ldots x^{-1} y x}_{N} =$

$$= x^{-1} y^N x = x^{-1} 1 x = x^{-1} x = 1$$

5

④

Subgroups:

$\{1\}$ not a, b, not c 1

G not a, not b, not c 1

$\{1, y\}, \{1, xy\}, \{1, yx\}$ a, b, c

$\{1, x, x^2\}$ a, b, c

6

Q5/ 6

$g\theta h\theta = g^{-1}h^{-1} = (hg)^{-1} = (gh)^{-1} = (gh)\theta$

3

$\ker(\theta) = \{1_G\}$ since

$g\theta = 1$ $g \in \ker\theta$ and

3

$g\theta = g^{-1}$ then only $1\theta = 1$

Q6/

(1) • Contains $\begin{pmatrix}0\\0\\0\end{pmatrix}$, when $a=b=0$

(1) • If $v=\begin{pmatrix}0\\-3b\\a+2b\end{pmatrix}$ then $\lambda v=\begin{pmatrix}0\\-3(\lambda b)\\\lambda a+2(\lambda b)\end{pmatrix}\in S$

(1) • If $v=\begin{pmatrix}0\\-3b\\a+2b\end{pmatrix}$, $w=\begin{pmatrix}0\\-3b'\\a'+2b'\end{pmatrix}$ then

$$v+w=\begin{pmatrix}0\\-3(b+b')\\(a+a')+2(b+b')\end{pmatrix}\in S$$

(1) $$S=\left\{a\begin{pmatrix}0\\0\\1\end{pmatrix}+b\begin{pmatrix}0\\-3\\2\end{pmatrix}\right\}=\left\langle\begin{pmatrix}0\\0\\1\end{pmatrix},\begin{pmatrix}0\\-3\\2\end{pmatrix}\right\rangle$$

Lin. independent

so dim $=2$ with basis $\begin{pmatrix}0\\0\\1\end{pmatrix},\begin{pmatrix}0\\-3\\2\end{pmatrix}$

(1) (1)

/6

Q7/

2

$$\begin{array}{c} r_1\\ r_2\\ r_3\\ r_4\end{array}\begin{pmatrix}1&1&1&1\\3&2&2&3\\1&2&1&2\\4&2&3&3\end{pmatrix} \sim \begin{array}{c} r_1'=r_1\\ r_2'=r_2-3r_1\\ r_3'=r_3-r_1\\ r_4'=r_4-4r_1\end{array}\begin{pmatrix}1&1&1&1\\0&-1&-1&0\\0&1&0&1\\0&-2&-1&-1\end{pmatrix} \sim \begin{array}{c}\\ \\ r_3''=r_3'+r_2'\\ r_4''=r_4'-2r_2'\end{array}\begin{pmatrix}1&1&1&1\\0&-1&-1&0\\0&0&-1&1\\0&0&1&-1\end{pmatrix}\begin{array}{l}\\ \\ \leftarrow \text{lin.}\\ \leftarrow \text{dep}^{t}\end{array}$$

2

$$r_3''+r_4''=0 \Rightarrow r_3'+r_2'+r_4'-2r_2'=0$$

$$\Rightarrow (r_3-r_1)-(r_2-3r_1)+(r_4-4r_1)=0$$

$$\Rightarrow -2r_1-r_2+r_3+r_4=0$$

1

i.e $-2\begin{pmatrix}1&1\\1&1\end{pmatrix}-\begin{pmatrix}3&2\\2&3\end{pmatrix}+\begin{pmatrix}1&2\\\cancel{2}1&2\end{pmatrix}+\begin{pmatrix}4&2\\3&3\end{pmatrix}=0$

/5

Q8/ Several methods possible, e.g. reduced row echelon forms identical,

OR:

$$1+2x+3x^2 = (2x+4x^2)+(1-x^2) \in W$$
$$4+5x+6x^2 = (5+7x+9x^2)-(1+2x+3x^2) \in W$$

$$2x+4x^2 = -\tfrac{2}{3}\left(4+5x+6x^2-4(1+2x+3x^2)\right) \in V$$
$$1-x^2 = \tfrac{1}{3}\left(2(4+5x+6x^2)-5(1+2x+3x^2)\right) \in V$$
$$5+7x+9x^2 = 1+2x+3x^2 + 4+5x+6x^2 \in V$$

/5

Q9/ A linear map $f: V \to W$ is a function satisfying

- $f(v+u) = f(v) + f(u)$ $\forall u, v \in V$
- $f(\lambda v) = \lambda . f(v)$ $\forall \lambda \in \mathbb{R}$ $v \in V$

(2)

a) Yes.

- $f\left(\begin{pmatrix} a \\ b \end{pmatrix}\right) + f\left(\begin{pmatrix} c \\ d \end{pmatrix}\right) = a + 2b + c + 2d = (a+c) + 2(b+d) = f\begin{pmatrix} a+c \\ b+d \end{pmatrix}$
- $f\left(\lambda \begin{pmatrix} a \\ b \end{pmatrix}\right) = f\begin{pmatrix} \lambda a \\ \lambda b \end{pmatrix} = \lambda a + 2\lambda b = \lambda(a+2b) = \lambda f\begin{pmatrix} a \\ b \end{pmatrix}$

(2)

b) Yes

- $f(p+q) = (x-1)(p+q) = (x-1)p + (x-1)q = f(p) + f(q)$
- $f(\lambda p) = (x-1)\lambda p = \lambda f(p)$

$\forall p, q \in P_2$

(2)

c) No

$\left.\begin{array}{l} f(0.1) = f(0) = -1 \\ 0. f(1) = 0.0 = 0 \end{array}\right\}$ not equal.

(2)

/8

Q10/

$$f(1) = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix} \quad f(x) = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad f(x^2) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad f(x^3) = \begin{pmatrix} 0 & 1 \\ 0 & 1 \end{pmatrix}$$

Rep^n is
$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

3

Echelon form

$$\begin{array}{c} \\ \\ r_3 - r_1 \\ \\ \end{array} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix} \sim \begin{array}{c} \\ \\ r_3 + r_2 \rightarrow \\ r_4 - r_2 \rightarrow \end{array} \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

3

rank is 3

rank + nullity = dim $(P_3) = 4$

2

$\Rightarrow$ dim ker$(f) = 4 - 3 = 1$

/8

Section B

11 (a) $g, h \in H \Rightarrow gh^{-1} \in H$

$g = x^{-1}gx \quad \forall x \in H$

$h = x^{-1}hx$

$gh^{-1} = x^{-1}gx(x^{-1}hx)^{-1} = x^{-1}gxx^{-1}h^{-1}x = x^{-1}(gh^{-1})x$

$\therefore gh^{-1} \in H.$

$xh = hx \Rightarrow x = hxh^{-1}$

$\Rightarrow h^{-1}x = xh^{-1}$ so if $h \in H$ then $h^{-1} \in H$

$x(gh^{-1}) = (xg)h^{-1} = g(xh^{-1}) = (gh^{-1})x$

$\therefore gh^{-1} \in H$

6

OR

(b) column x = row x then x in $Z(G)$ 2

(c) $H = \{1, a^2\}$ 2

(d) (i) $(H \subseteq G$ and $gH = Hg,\ \forall g \in G) \Rightarrow H \triangleleft G$

$(ghg^{-1} \in H \quad \forall g \in G, \forall h \in H) \Rightarrow H \triangleleft G$

(ii) $H = \{x \in G : gx = xg \ .\ \forall g \in G\}$ 1

2

let $H = \{x_1, x_2, \dots, x_n\}$

then $gH = \{gx_1, gx_2, \dots, gx_n\} =$

$= \{x_1g, x_2g, \dots x_ng\} = Hg$ 2

$\therefore$ H normal

(e) (i) $|G/H| = 4$ 2

(ii) H itself 2

12

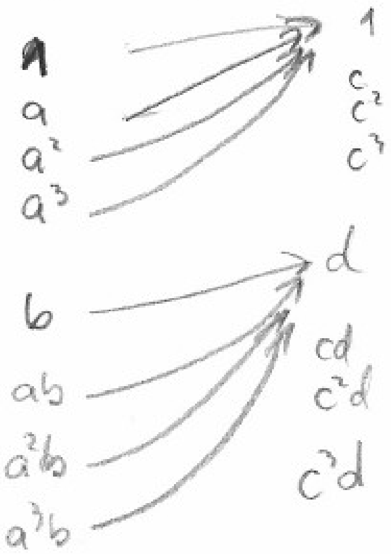
$\langle a, b \rangle = G$

$a^4 = 1 \quad , \quad b^4 = 1$

$a\theta = 1$

$b\theta = d$

2



3

relations.

$a^4 = 1$

$a^4\theta = (a\theta)^4 = (1_{G'})^4 = 1_{G'} = 1_G\theta$ ✓ 1

$a^2 = b^2$

$a^2\theta = a\theta a\theta = 1_{G'} 1_{G'} = 1_{G'}$ ✓ 1

$b^2\theta = b\theta b\theta = d\,d = d^2 = 1_{G'}$

$ba = a^3 b$

$(ba)\theta = b\theta\, a\theta = d \cdot 1 = d$ ✓ 1

$(a^3 b)\theta = (a\theta)^3 (b\theta) = 1 d = d$

12b) $\mathcal{B} = 1, x, x^2, x^3$

$\frac{1}{2}$

i) $C = \begin{pmatrix} 2 & 0 & -3 & 0 \\ 0 & 2 & -1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$

2

ii) $p(\lambda) = (2-\lambda)(2-\lambda)(1-\lambda)(2-\lambda)$

e'vals $\lambda = 1, \lambda = 2$ (3 times)

2

iii) $\underline{\lambda = 1}$ $(C - I)\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 1 & 0 & -3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = 0$

$\Leftrightarrow$ $a = 3c, \; b = c, \; d = 0$

eg. if $c = 1$, $\begin{pmatrix} 3 \\ 1 \\ 1 \\ 0 \end{pmatrix}$ is e'vec.

$\underline{\lambda = 2}$ $(C - 2I)\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = \begin{pmatrix} 0 & 0 & -3 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix} = 0$.

e'vecs eg. $\begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$ $\Leftrightarrow$ $c = 0$

4

$\frac{1}{2}$ So $\mathcal{E} = \underline{3 + x + x^2, \; 1, \; x, \; x^3}$.

1 vi) $P = \begin{pmatrix} 3 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

1 v) $D = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}$

/12

1 vi) $CP - PD = 0$.

Q13 a) $\cdot\ f(p+p') = \begin{pmatrix} (p+p')(+2) \\ (p+p')(-2) \end{pmatrix} = \begin{pmatrix} p(+2) + p'(+2) \\ p(-2) + p'(-2) \end{pmatrix}$

2 $\quad = \begin{pmatrix} p(2) \\ p(-2) \end{pmatrix} + \begin{pmatrix} p'(2) \\ p'(-2) \end{pmatrix} = f(p) + f(p')$

$\cdot\ f(\lambda p) = \begin{pmatrix} \lambda p(2) \\ \lambda p(-2) \end{pmatrix} = \lambda f(p) \Rightarrow$ LINEAR MAP

2

$A = \begin{pmatrix} 1 & 2 & 4 & 8 \\ +1 & -2 & +4 & -8 \end{pmatrix}$

b) rows are independent $\Rightarrow$ rank $r = 2$ (Note: $A \sim \begin{pmatrix} 1 & 2 & 4 & 8 \\ 0 & -2 & 0 & 8 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 & 4 & 0 \\ 0 & 2 & 0 & 8 \end{pmatrix}$)

2 $\quad r + k = \dim(P_3) = 4 \Rightarrow k = 2$

$A\begin{pmatrix} 4 \\ 0 \\ -1 \\ 0 \end{pmatrix} = 0, \quad A\begin{pmatrix} 0 \\ 4 \\ 0 \\ -1 \end{pmatrix} = 0$

2 $\quad \Rightarrow$ basis for kernel is $p_1(x) = 4 - x^2$, $p_2(x) = 4x - x^3$

2 $\quad Q = \begin{pmatrix} 4 & 0 & 1 & 0 \\ 0 & 4 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$ 2 Let $p_3(x) = 1$, $p_4(x) = x$. Then p_1, p_2, p_3, p_4 independent $\Rightarrow$ basis. (OTHER ANS. POSSIBLE) FOR THIS ANS...

c) $f(1) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad f(x) = \begin{pmatrix} 2 \\ -2 \end{pmatrix}$

2 $\quad$ independent $\Rightarrow$ basis for $\mathbb{R}^2$

2 $\quad P = \begin{pmatrix} 1 & 2 \\ 1 & -2 \end{pmatrix}$

2 $\quad$ d) $f(p_1) = 0,\ f(p_2) = 0 \Rightarrow B = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

2 $\quad AQ = \begin{pmatrix} 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & -2 \end{pmatrix} = PB$