MA2008 $\quad$ **$2\frac{1}{2}$ hours** $\quad$ **May 2006**

SECTION A — Answer all ALL questions — 50 marks

1. A binary operation $*$ is defined on the set $\mathbb{R}^2$ of all pairs of real numbers by

$$(a,b)*(c,d) \quad = \quad (c+ad, bd).$$

   (a) Find the identity with respect to $*$.

   (b) Determine whether all elements of $\mathbb{R}^2$ have inverses with respect to $*$. Explain your reasoning carefully.

(4 marks)

2. Let x and y be elements of a group G, and let n be a positive integer. Prove that if $(xy)^n = 1$ then $(yx)^n = 1$.

(4 marks)

3. Let G be the group $\{1, a, b, ab, ba, aba\}$ where $a^2 = b^2 = 1$ and $aba = bab$. Find all the subgroups of G.

(6 marks)

4. (a) State Lagrange's theorem for a finite group G.

   (b) Suppose that G has a prime number p of elements and $g \in G$. Prove that either g is the identity or

$$G \quad = \quad \{\ 1,\ g,\ g^2,\ \ldots,\ g^{p-1}\ \}\,.$$

(6 marks)

5. Let G be the following group of conjugacy classes modulo 10 under multiplication,

$$G \quad = \quad \{\ [1]_{10},\ [3]_{10},\ [7]_{10},\ [9]_{10}\ \}\,.$$

   Prove that G is isomorphic to the group $G' = \{1, c, c^2, c^3\}$ where $c^4 = 1$.

(4 marks)

6. Prove that the set of matrices

$$W = \left\{ \begin{pmatrix} 0 & a+b & a-b \\ 0 & 2a & 2c \end{pmatrix} \; : \; a, b, c \in \mathbb{R} \right\}$$

is a vector subspace of $M_{2\times 3}(\mathbb{R})$.

Calculate the dimension of W.

(5 marks)

7. Prove that the following polynomials in $\mathcal{P}_4$ are linearly dependent and give a linear relation between them:

$$x + 2x^2 - x^4, \qquad 1 - x + x^4, \qquad 1 + x + x^2, \qquad x - x^2.$$

(4 marks)

8. Consider the following set of polynomials in $\mathcal{P}_2$:

$$S = \{1 + x, \quad 1 + x^2, \quad x - x^2, \quad 1 + x + x^2\}.$$

Prove that $\text{span}(S) = \mathcal{P}_2$ and find a subset of S which is a basis for $\mathcal{P}_2$.

(5 marks)

9. Explain carefully which of the following functions define homomorphisms of vector spaces:

   (a) $f : \mathbb{R} \to \mathbb{R}$ defined by $f(a) = 0$,

   (b) $f : \mathbb{R}^2 \to \mathbb{R}$ defined by $f\begin{pmatrix} a \\ b \end{pmatrix} = a + b + 1$,

   (c) $f : M_{2\times 2}(\mathbb{R}) \to M_{2\times 2}(\mathbb{R})$ defined by $f\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 9 & 4 \\ 6 & 8 \end{pmatrix}$.

(6 marks)

10. Consider the linear map $f : \mathbb{R}^3 \to \mathbb{R}^3$ defined by

$$f\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} x - y + z \\ 2x - 3z \\ 2y - 5z \end{pmatrix}.$$

Give the matrix representation of f with respect to the standard basis.

Hence find a basis for the image (or *rangespace*) of f.

Use the rank–nullity theorem to write down the dimension of the kernel of f.

(6 marks)

SECTION B — Answer TWO questions — 25 marks each

11. Suppose G is a group and H is a subgroup of G.

(a) Explain what is meant by saying that H is a *normal* subgroup of G.

(2 marks)

(b) If H is a normal subgroup there is a well-defined binary operation on the set of cosets G/H, where

$$Ha * Hb \quad = \quad Hab$$

Prove that the set of cosets forms a group with respect to this operation.

(8 marks)

(c) Let G be the group $\{1, b, b^2, b^3, a, ab, ba, b^2a\}$ where $a^2 = b^4 = 1$ and $b^3a = ab$.

i. Construct the left and right cosets of the subgroup $H = \{1, b^2\}$ in G. Hence show that H is normal.

(7 marks)

ii. Copy and complete the following multiplication for G/H

$*$	$H1$	Ha	Hb	Hab
$H1$				
Ha				
Hb				
Hab				

(5 marks)

iii. Is G/H isomorphic to the cyclic group of order 4, or to the Klein 4-group?

(3 marks)

12. (a) Consider the groups

$$\begin{aligned} G &= \{1, a, a^2, a^3, b, ab, a^2b, ba\} \text{ where } a^4 = b^2 = 1,\ a^3b = ba, \\ G' &= \{1, c, c^2, c^3, d, cd, c^2d, dc\} \text{ where } c^4 = 1,\ c^2 = d^2,\ c^3d = dc. \end{aligned}$$

Construct a non-trivial homomorphism θ from G to G'.

Find the kernel and the image of your homomorphism θ.

(12 marks)

(b) i. Consider the linear map of vector spaces $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined on the standard basis by

$$f\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \qquad f\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}.$$

Write down the matrix A which represents f with respect to the standard basis.

Find the characteristic polynomial and two linearly independent eigenvectors v_1, v_2 of f.

Give the change of basis matrix P from $\{v_1, v_2\}$ to the standard basis, and write down the matrix $P^{-1}AP$.

ii. Find all the eigenvalues of the linear map $f : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ 0 \end{pmatrix}.$$

Explain whether or not f is diagonalisable.

(13 marks)

13. Consider the function

$$f\colon M_{2\times 2}(\mathbb{R}) \longrightarrow \mathcal{P}_3$$

defined by

$$f\begin{pmatrix} a & b \\ c & d \end{pmatrix} \;=\; (b-d)+(a+b-2d)x-3cx^2+(a-d)x^3.$$

(a) Prove that f is a homomorphism of vector spaces.

(2 marks)

(b) Write down the standard bases for $M_{2\times 2}(\mathbb{R})$ and $\mathcal{P}_3$.
Find the matrix representation of f with respect to these bases.

(7 marks)

(c) Find a basis for the nullspace of f, and extend this to a basis $\mathcal{B}_1$ for $M_{2\times 2}(\mathbb{R})$.

(6 marks)

(d) Calculate the elements of the set

$$\{f(A) \;:\; A \in \mathcal{B}_1,\; f(A) \neq 0\}.$$

Extend this set to a basis $\mathcal{B}_2$ for $\mathcal{P}_3$.

(7 marks)

(e) Write down the matrix representation of f with respect to the bases $\mathcal{B}_1$ and $\mathcal{B}_2$.

(3 marks)