Chapter 2

Number of ways to sample r items from n

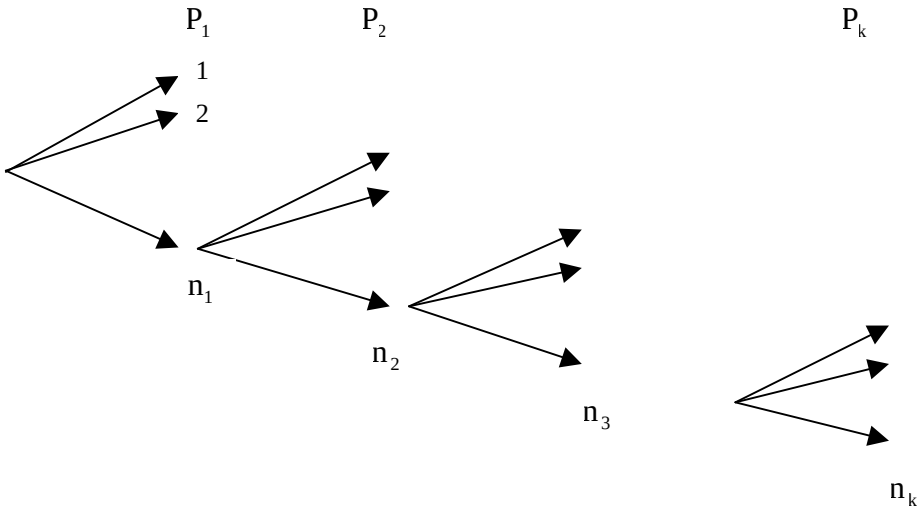
Summary

	ORDER IMPORTANT ?	
REPLACEMENT ?	YES	NO
YES	n^r	C_r^{n+r-1} but not equally likely
NO	$P_r^n = \frac{n!}{(n-r)!}$	$C_r^N = \frac{n!}{r!(n-r)!}$

General multiplication rule for finding n(S)

An experiment E comprises k procedures, $P_1, P_2, P_3, \ldots, P_k$ each with $n_1, n_2, n_3, \ldots, n_k$ possible outcomes respectively, then the total number of possible outcomes of the experiment E is

$$n(S) = n_1 * n_2 * \ldots * n_k$$



$$n(S) = n_1 * n_2 * n_3 * \ldots\ldots * n_k$$

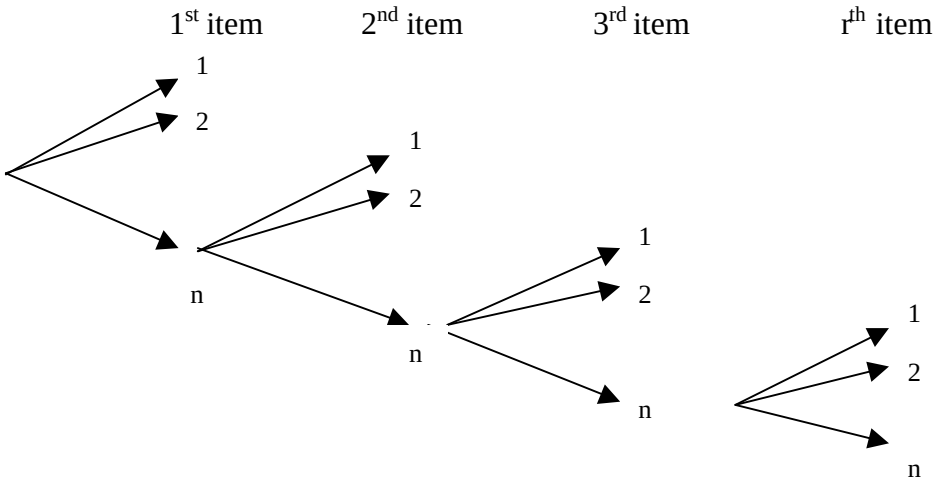
2.1 Sampling r items from n :

replacement YES and order important YES

Result 2.1 The number of ways to sample r items from n **with replacement** and where the **order is important**
i.e. replacement – **YES**, order important -**YES**

$$n(S) = n^r$$

Proof 2.1



n(S) = n * n * n * ……* n

$$n(S) = n^r$$

Example 2.1 How many different ways are there to allocate 3 different prizes (e.g. one for Maths, one for Stats and one for Computing) to a class of 20 students?

i.e. sample 3 students from 20 **with replacement** where the **order is important**,
replacement – **YES**, order important -**YES**

$$n(S) = 20^3 = 8000$$

2.2 Permutations

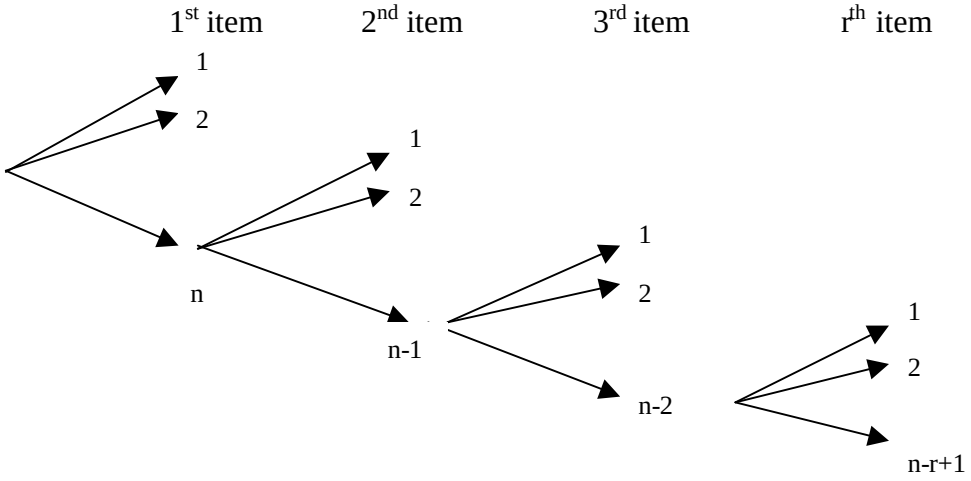
i.e. Sampling r items from n :

replacement NO and order important YES

Result 2.2 The number of ways to sample r items from n **without replacement** and where the **order is important**
i.e. replacement – **NO**, order important -**YES**

$$n(S) = P_r^n = \frac{n!}{(n-r)!}$$

Proof 2.2



n(S) = n * (n-1) * (n-2) * ……* (n-r+1)

Example 2.2 How many different ways are there to allocate 3 different prizes (1st, 2nd and 3rd prize for Statistics) to a class of 20 students?
i.e. sample 3 students from 20 **without replacement** where **order is important**,
Replacement – **NO**, order important -**YES**

$$n(S) = P_3^{20} = \frac{20!}{(20-3)!} = \frac{20!}{17!} = 20*19*18 = 6840$$

2.3 Combinations

i.e. Sampling r items from n :

replacement NO and order important NO

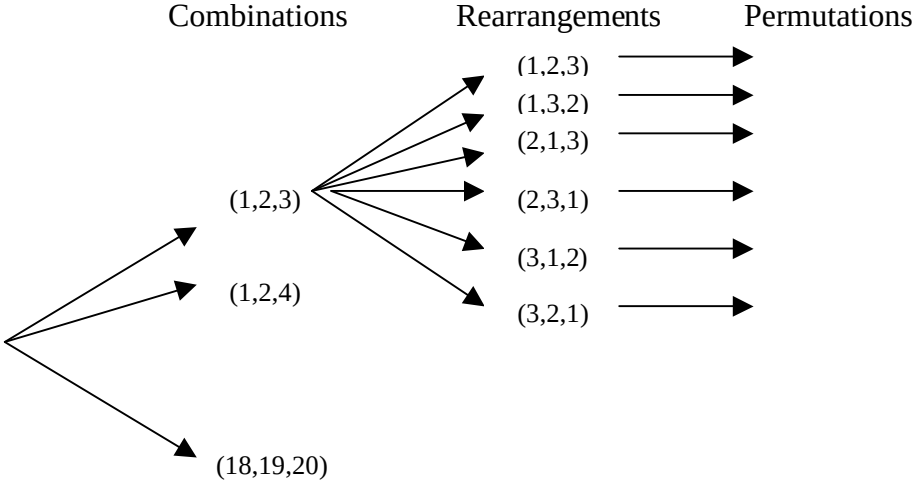
Result 2.3 The number of ways to sample r items from n **without replacement** and where the **order is not important**
i.e. replacement – **NO**, order important –**NO**

$$n(S) = C_r^n = \frac{n!}{r!(n-r)!}$$

Example How many different ways are there to allocate 3 identical prizes (e.g. 3 prizes for punctuality) to a class of 20 students ?
i.e. sample 3 students from 20 **without replacement** where **order is not important**,
replacement – **NO**, order important **-NO**

$$n(S) = C_3^{20} = \frac{20!}{3!(20-3)!} = \frac{20!}{3!*17!} = \frac{20*19*18}{3*2*1} = 1140$$

Proof of example 2.3



Combinations * Rearrangements = Permutations

$$C_3^{20} \quad * \quad 3! \quad = \quad P_3^{20}$$

$$C_r^n \quad * \quad r! \quad = \quad P_r^n$$

Hence :

$$Combinations \quad = \quad \frac{Permutations}{\text{Re}\, arrangements}$$

$$C_3^{20} = \frac{P_3^{20}}{3!} = \frac{20!}{3!*(20-3)!}$$

In general

$$C_r^n = \frac{P_r^n}{r!} = \frac{n!}{r!(n-r)!}$$

2.4 Combining permutations or combinations

Example How many different ways are there to choose a student committee of 2 males and 3 females from a class of 20 males and 30 females?

n(S) = ways to choose 2 males from 20 * ways to choose 3 females from 30

$$\text{n(S)} \quad = \quad C_2^{20} \quad * \quad C_3^{30} \quad = \quad \frac{20*19}{2*1} * \frac{30*29*28}{3*2*1} = 190*4060 = 771{,}400$$

2.5 Calculating probabilities of events using permutations or combinations

Provided all ways of selecting r items from n are equally likely, then for any event A concerning r items,

$$p(A) = \frac{n(A)}{n(S)} = \begin{cases} \dfrac{no.\ of\ \ permutations\ in\ A}{no.\ of\ \ permutations\ in\ S} \\ \dfrac{no.\ of\ \ combinations\ in\ A}{no.\ of\ \ cmbinations\ in\ S} \end{cases}$$

Do **NOT** mix permutations and combinations in the same probability calculation.

EXERCISE 3

PERMUTATIONS and COMBINATIONS

Q1 Evaluate each of the following

$$P_2^6, \quad P_6^6, \quad C_2^6, \quad C_6^6$$

Q2 a) A lock comprises a sequence of 3 digits (each of which may be any number from 0 to 9 inclusive).
How many possible lock numbers are there?

b) A lock comprises a sequence of 3 digits (0 to 9) followed by 3 letters (A to Z).
How many possible lock numbers are there?

c) Car number plates comprise a letter (A to Z but excluding the letters I, O, Q and U),
followed by a sequence of 3 digits (each 0 to 9),
followed by 3 letters (each A to Z but each excluding the letters I and O).
How many possible different car number plates are there?

d) Telephone numbers in London have code either 0207 or 0208
followed by a sequence of 7 digits (each 0 to 9).
How many possible different London telephone numbers are there?

Q3 a) How many different ways can the letters in PROBLEM be rearranged?

b) How many different ways can the letters in EXERCISE be rearranged?

c) How many different sequences of 3 digit codes (each digit 0 to 9) are there if no repetitions of a digit are allowed?

d) How many different sequences of 3 letters are there if no repetitions of a letter are allowed?

e) A travel company is preparing an itinerary to visit 6 major cities.
How many different itineraries are possible?

f) In a horse race involving 30 horses, how many different results (for the first 3 places) are possible?

g) A company wishes to fill 3 vice-presidential positions (responsible for sales, finance and production) respectively from 24 managers. How many different ways can the positions be filled?

Q4 a) How many different ways are there to choose a class committee of 5 students from a class of 50 students?

b) How many different ways are there to choose a set of 5 cards from a pack of 52?

c) On a football pools form, how many ways are there to choose 8 from 60 football matches?

Q5 How many different ways are there to choose a committee of 3 managers and 5 employees from a company of 8 managers and 50 employees?

Q6 U.K. Lottery

In the UK lottery gamblers select 6 numbers from 49.
The winning 6 numbers are then selected randomly from balls live on television (together with a 'bonus ball number').

a) How many different selections of 6 numbers from 49 can be made?
What is the probability of selecting ALL 6 winning numbers?

b) A prize of £10 is given if the selection includes exactly 3 of the 6 winning numbers.
How many ways are there to choose exactly 3 winning numbers (and hence also exactly 3 non-winning numbers)?
What is the probability of winning the £10 prize?

c) Find the probability of choosing exactly 4 winning numbers?

d) Find the probability of choosing exactly 5 winning numbers?

e) The second highest prize is awarded to selections which have 5 of the winning numbers and the 6^{th} number exactly the same as the 'bonus ball number'.
What is the probability of winning this number?

Q7 The Birthday problem

[Assume there are 365 days in the year and that all 365 days are equally likely as birthdays, and that birthdays for 30 students in a class are independent of each other.]

a) If a sample space comprises all possible ordered lists of birthdays for a class list of 30 students in alphabetical order,
how many possible lists of birthdays are there, i.e. what is the size of the sample space?

b) How many lists are there in which the 30 students in the class list all have different birthdays?
Under the assumptions above, what is the probability that the 30 students in a class all have different birthdays?

c) Hence state the probability that at least 2 of the 30 students have the same birthday?

Q8 CARD GAMES : 5 card poker

A pack of 52 cards comprises cards 1 to 13 in each of 4 suits (i.e. 4 different designs : Clubs, Diamonds, Hearts and Spades).

	A ace										J jack	Q queen	K king
SUIT	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**	**13**
C													
D													
H													
S													

a) How many different selections of 5 cards from 52 are possible?
b) What is the probability that the 5 cards comprise
   i) 4 Aces and one other card?
   ii) 'A Royal Flush', i.e. A,K,Q,J,10 of the same suit?
   iii) any 'Running Flush', e.g. 4,5,6,7,8 of the same suit?
   iv) any 'Run', e.g. 4,5,6,7,8 not necessarily of the same suit?
   v) any 'Flush' i.e. 5 cards of the same suit?