Numerical Differentiation

31.3

Introduction

In this Section we will look at ways in which derivatives of a function may be approximated numerically.

Prerequisites

Before starting this Section you should ...

- review previous material concerning differentiation

Learning Outcomes

On completion you should be able to ...

- obtain numerical approximations to the first and second derivatives of certain functions

1. Numerical differentiation

This Section deals with ways of numerically approximating derivatives of functions. One reason for dealing with this now is that we will use it briefly in the next Section. But as we shall see in these next few pages, the technique is useful in itself.

2. First derivatives

Our aim is to approximate the slope of a curve f at a particular point $x = a$ in terms of $f(a)$ and the value of f at a nearby point where $x = a + h$. The shorter broken line Figure 11 may be thought of as giving a reasonable approximation to the required slope (shown by the longer broken line), if h is small enough.

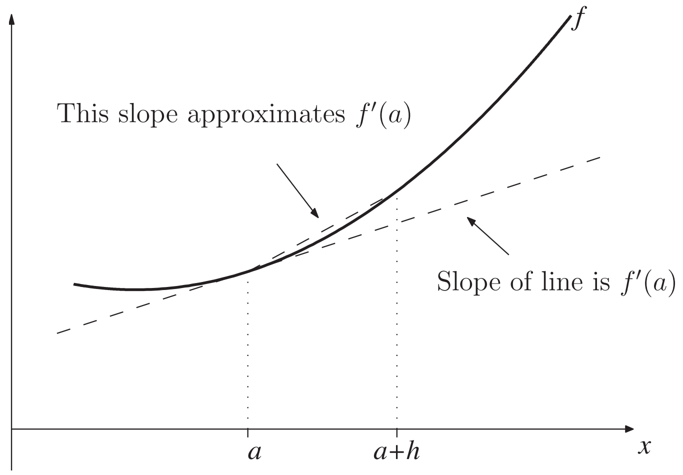

Figure 11

So we might approximate

$$f'(a) \approx \text{slope of short broken line} = \frac{\text{difference in the } y\text{-values}}{\text{difference in the } x\text{-values}} = \frac{f(a+h) - f(a)}{h}.$$

This is called a **one-sided difference** or **forward difference** approximation to the derivative of f.

A second version of this arises on considering a point to the left of a, rather than to the right as we did above. In this case we obtain the approximation

$$f'(a) \approx \frac{f(a) - f(a-h)}{h}$$

This is another **one-sided difference**, called a **backward difference**, approximation to $f'(a)$.

A third method for approximating the first derivative of f can be seen in Figure 12.

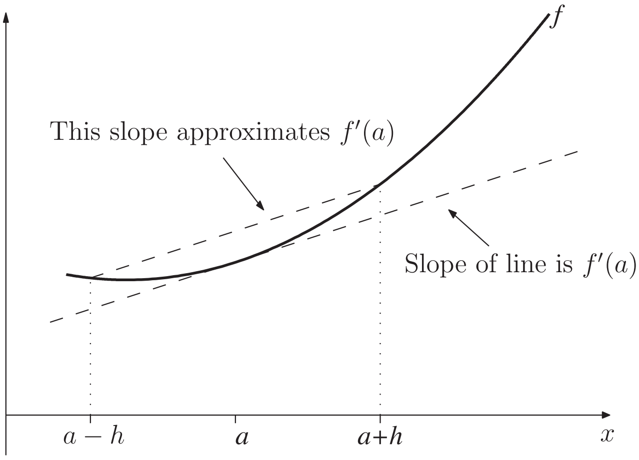

Figure 12

Here we approximate as follows

$$f'(a) \approx \text{slope of short broken line} \;=\; \frac{\text{difference in the } y\text{-values}}{\text{difference in the } x\text{-values}} \;=\; \frac{f(x+h)-f(x-h)}{2h}$$

This is called a **central difference** approximation to $f'(a)$.

Key Point 11

First Derivative Approximations

Three approximations to the derivative $f'(a)$ are

1. the one-sided (forward) difference $\qquad \dfrac{f(a+h)-f(a)}{h}$
2. the one-sided (backward) difference $\qquad \dfrac{f(a)-f(a-h)}{h}$
3. the central difference $\qquad \dfrac{f(a+h)-f(a-h)}{2h}$

In practice, the central difference formula is the most accurate.

These first, rather artificial, examples will help fix our ideas before we move on to more realistic applications.

Example 18

Use a forward difference, and the values of h shown, to approximate the derivative of $\cos(x)$ at $x = \pi/3$.

(a) $h = 0.1$ (b) $h = 0.01$ (c) $h = 0.001$ (d) $h = 0.0001$

Work to 8 decimal places throughout.

Solution

(a) $f'(a) \approx \dfrac{\cos(a+h) - \cos(a)}{h} = \dfrac{0.41104381 - 0.5}{0.1} = -0.88956192$

(b) $f'(a) \approx \dfrac{\cos(a+h) - \cos(a)}{h} = \dfrac{0.49131489 - 0.5}{0.01} = -0.86851095$

(c) $f'(a) \approx \dfrac{\cos(a+h) - \cos(a)}{h} = \dfrac{0.49913372 - 0.5}{0.001} = -0.86627526$

(d) $f'(a) \approx \dfrac{\cos(a+h) - \cos(a)}{h} = \dfrac{0.49991339 - 0.5}{0.0001} = -0.86605040$

One advantage of doing a simple example first is that we can compare these approximations with the 'exact' value which is

$$f'(a) = -\sin(\pi/3) = -\frac{\sqrt{3}}{2} = -0.86602540 \quad \text{to 8 d.p.}$$

Note that the accuracy levels of the four approximations in Example 15 are:

(a) 1 d.p. (b) 2 d.p. (c) 3 d.p. (d) 3 d.p. (almost 4 d.p.)

The errors to 6 d.p. are:

(a) 0.023537 (b) 0.002486 (c) 0.000250 (d) 0.000025

Notice that the errors reduce by about a factor of 10 each time.

Example 19

Use a central difference, and the value of h shown, to approximate the derivative of $\cos(x)$ at $x = \pi/3$.

(a) $h = 0.1$ (b) $h = 0.01$ (c) $h = 0.001$ (d) $h = 0.0001$

Work to 8 decimal places throughout.

Solution

(a) $f'(a) \approx \dfrac{\cos(a+h)-\cos(a-h)}{2h} = \dfrac{0.41104381 - 0.58396036}{0.2} = -0.86458275$

(b) $f'(a) \approx \dfrac{\cos(a+h)-\cos(a-h)}{2h} = \dfrac{0.49131489 - 0.50863511}{0.02} = -0.86601097$

(c) $f'(a) \approx \dfrac{\cos(a+h)-\cos(a-h)}{2h} = \dfrac{0.49913372 - 0.50086578}{0.002} = -0.86602526$

(d) $f'(a) \approx \dfrac{\cos(a+h)-\cos(a-h)}{2h} = \dfrac{0.49991339 - 0.50008660}{0.0002} = -0.86602540$

This time successive approximations generally have *two* extra accurate decimal places indicating a superior formula. This is illustrated again in the following Task.

Task

Let $f(x) = \ln(x)$ and $a = 3$. Using both a forward difference and a central difference, and working to 8 decimal places, approximate $f'(a)$ using $h = 0.1$ and $h = 0.01$.
(Note that this is another example where we can work out the exact answer, which in this case is $\frac{1}{3}$.)

Your solution

Answer

Using the forward difference we find, for $h = 0.1$

$$f'(a) \approx \frac{\ln(a+h) - \ln(a)}{h} = \frac{1.13140211 - 1.09861229}{0.1} = 0.32789823$$

and for $h = 0.01$ we obtain

$$f'(a) \approx \frac{\ln(a+h) - \ln(a)}{h} = \frac{1.10194008 - 1.09861229}{0.01} = 0.33277901$$

Using central differences the two approximations to $f'(a)$ are

$$f'(a) \approx \frac{\ln(a+h) - \ln(a-h)}{2h} = \frac{1.13140211 - 1.06471074}{0.2} = 0.33345687$$

and

$$f'(a) \approx \frac{\ln(a+h) - \ln(a-h)}{2h} = \frac{1.10194008 - 1.09527339}{0.02} = 0.33333457$$

The accurate answer is, of course, 0.33333333

There is clearly little point in studying this technique if all we ever do is approximate quantities we could find exactly in another way. The following example is one in which this so-called **differencing method** is the best approach.

Example 20

The distance x of a runner from a fixed point is measured (in metres) at intervals of half a second. The data obtained are

t	0.0	0.5	1.0	1.5	2.0
x	0.00	3.65	6.80	9.90	12.15

Use central differences to approximate the runner's velocity at times $t = 0.5$ s and $t = 1.25$ s.

Solution

Our aim here is to approximate $x'(t)$. The choice of h is dictated by the available data given in the table.

Using data with $t = 0.5$ s at its centre we obtain

$$x'(0.5) \approx \frac{x(1.0) - x(0.0)}{2 \times 0.5} = 6.80 \text{ m s}^{-1}.$$

Data centred at $t = 1.25$ s gives us the approximation

$$x'(1.25) \approx \frac{x(1.5) - x(1.0)}{2 \times 0.25} = 6.20 \text{ m s}^{-1}.$$

Note the value of h used.

Task

The velocity v (in m s^{-1}) of a rocket measured at half second intervals is

t	0.0	0.5	1.0	1.5	2.0
v	0.000	11.860	26.335	41.075	59.051

Use central differences to approximate the acceleration of the rocket at times $t = 1.0$ s and $t = 1.75$ s.

Your solution

Answer

Using data with $t = 1.0$ s at its centre we obtain

$$v'(1.0) \approx \frac{v(1.5) - v(0.5)}{1.0} = 29.215 \text{ m s}^{-2}.$$

Data centred at $t = 1.75$ s gives us the approximation

$$v'(1.75) \approx \frac{v(2.0) - v(1.5)}{0.5} = 35.952 \text{ m s}^{-2}.$$

3. Second derivatives

An approach which has been found to work well for second derivatives involves applying the notion of a central difference three times. We begin with

$$f''(a) \approx \frac{f'(a + \frac{1}{2}h) - f'(a - \frac{1}{2}h)}{h}.$$

Next we approximate the two derivatives in the numerator of this expression using central differences as follows:

$$f'(a + \tfrac{1}{2}h) \approx \frac{f(a+h) - f(a)}{h} \qquad \text{and} \qquad f'(a - \tfrac{1}{2}h) \approx \frac{f(a) - f(a-h)}{h}.$$

Combining these three results gives

$$\begin{aligned} f''(a) &\approx \frac{f'(a+\frac{1}{2}h) - f'(a-\frac{1}{2}h)}{h} \\ &\approx \frac{1}{h}\left\{\left(\frac{f(a+h)-f(a)}{h}\right) - \left(\frac{f(a)-f(a-h)}{h}\right)\right\} \\ &= \frac{f(a+h)-2f(a)+f(a-h)}{h^2} \end{aligned}$$

Key Point 12

Second Derivative Approximation

A central difference approximation to the second derivative $f''(a)$ is

$$f''(a) \approx \frac{f(a+h)-2f(a)+f(a-h)}{h^2}$$

Example 21

The distance x of a runner from a fixed point is measured (in metres) at intervals of half a second. The data obtained are

t	0.0	0.5	1.0	1.5	2.0
x	0.00	3.65	6.80	9.90	12.15

Use a central difference to approximate the runner's acceleration at $t = 1.5$ s.

Solution

Our aim here is to approximate $x''(t)$.
Using data with $t = 1.5$ s at its centre we obtain

$$\begin{aligned} x''(1.5) &\approx \frac{x(2.0)-2x(1.5)+x(1.0)}{0.5^2} \\ &= -3.40 \text{ m s}^{-2}, \end{aligned}$$

from which we see that the runner is slowing down.

Exercises

1. Let $f(x) = \cosh(x)$ and $a = 2$. Let $h = 0.01$ and approximate $f'(a)$ using forward, backward and central differences. Work to 8 decimal places and compare your answers with the exact result, which is $\sinh(2)$.

2. The distance x, measured in metres, of a downhill skier from a fixed point is measured at intervals of 0.25 s. The data gathered are

t	0	0.25	0.5	0.75	1	1.25	1.5
x	0	4.3	10.2	17.2	26.2	33.1	39.1

Use a central difference to approximate the skier's velocity and acceleration at the times t =0.25 s, 0.75 s and 1.25 s. Give your answers to 1 decimal place.

Answers

1. Forward: $f'(a) \approx \dfrac{\cosh(a+h) - \cosh(a)}{h} = \dfrac{3.79865301 - 3.76219569}{0.01} = 3.64573199$

   Backward: $f'(a) \approx \dfrac{\cosh(a) - \cosh(a-h)}{h} = \dfrac{3.76219569 - 3.72611459}{0.01} = 3.60810972$

   Central: $f'(a) \approx \dfrac{\cosh(a+h) - \cosh(a-h)}{2h} = \dfrac{3.79865301 - 3.72611459}{0.02} = 3.62692086$

   The accurate result is $\sinh(2) = 3.62686041$.

2. Velocities at the given times approximated by a central difference are:

   20.4 m s^{-1}, 32.0 m s^{-1} and 25.8 m s^{-1}.

   Accelerations at these times approximated by a central difference are:

   25.6 m s^{-2}, 32.0 m s^{-2}and -14.4 m s^{-2}.