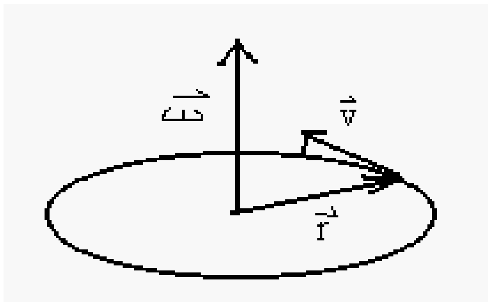
Chapter 1: Describing the universe

1. **Circular motion**. A particle is moving around a circle with angular velocity $\vec{\omega}$. Write its velocity vector $\vec{\mathbf{v}}$ as a vector product of $\vec{\omega}$ and the position vector $\mathbf{r}$ with respect to the center of the circle. Justify your expression. Differentiate your relation, and hence derive the angular form of Newton's second law ($\vec{\tau} = I\vec{\alpha}$) from the standard form (equation 1.8).



The direction of the velocity is perpendicular to $\vec{\omega}$ and also to the radius vector $\mathbf{r}$, and is given by putting your right thumb along the vector $\vec{\omega}$: your fingers then curl in the direction of the velocity. The speed is $v = \omega r$. Thus the vector relation we want is:

$$\vec{\mathbf{v}} = \vec{\omega} \times \vec{\mathbf{r}}$$

Differentiating, we get:

$$\begin{aligned}\vec{\mathbf{a}} &= \frac{d\vec{\mathbf{v}}}{dt} = \frac{d\vec{\omega}}{dt} \times \vec{\mathbf{r}} + \vec{\omega} \times \frac{d\vec{\mathbf{r}}}{dt} \\ &= \vec{\alpha} \times \vec{\mathbf{r}} + \vec{\omega} \times \vec{\mathbf{v}} \\ &= \vec{\alpha} \times \vec{\mathbf{r}} + \vec{\omega} \times \left(\vec{\omega} \times \vec{\mathbf{r}}\right) \\ &= \vec{\alpha} \times \vec{\mathbf{r}} + \vec{\omega}\left(\vec{\omega} \cdot \vec{\mathbf{r}}\right) - \omega^2 \vec{\mathbf{r}} \\ &= \vec{\alpha} \times \vec{\mathbf{r}} - \omega^2 \vec{\mathbf{r}}\end{aligned}$$

since $\vec{\omega}$ is perpendicular to $\mathbf{r}$. The second term is the usual centripetal term. Then

$$\vec{\mathbf{F}} = m\vec{\mathbf{a}}$$

and

$$\begin{aligned}\vec{\tau} &= \vec{\mathbf{r}} \times \vec{\mathbf{F}} = \vec{\mathbf{r}} \times m\left(\vec{\alpha} \times \vec{\mathbf{r}} - \omega^2 \vec{\mathbf{r}}\right) \\ &= m\left(\vec{\alpha} r^2 - \vec{\mathbf{r}}\left(\vec{\alpha} \cdot \vec{\mathbf{r}}\right)\right) \\ &= m r^2 \vec{\alpha} = I\vec{\alpha}\end{aligned}$$

since $\vec{\alpha}$ is perpendicular to $\mathbf{r}$, and for a particle $I = mr^2$.

2. Find two vectors, each perpendicular to the vector $\vec{u} = (1, 2, 2)$ and perpendicular to each other. *Hint:* Use dot and cross products. Determine the transformation matrix a that allows you to transform to a new coordinate system with x' −axis along $\vec{u}$ and y' − and z' −axes along your other two vectors.

We can find a vector $\vec{v}$ perpendicular to $\vec{u}$ by requiring that $\vec{u} \cdot \vec{v} = 0.$ A vector satsifying this is:

$$\vec{v} = (0, 1, -1)$$

Now to find the third vector we choose

$$\vec{w} = \vec{u} \times \vec{v} = \begin{pmatrix} 1, & 2, & 2 \end{pmatrix} \times \begin{pmatrix} 0, & 1, & -1 \end{pmatrix} = \begin{pmatrix} -4, & 1, & 1 \end{pmatrix}$$

To find the transformation matrix, first we find the magnitude of each vector and the corresponding unit vectors:

$$\hat{u} = \frac{\begin{pmatrix} 1, & 2, & 2 \end{pmatrix}}{\sqrt{1+4+4}} = \frac{\begin{pmatrix} 1, & 2, & 2 \end{pmatrix}}{3}$$

$$\hat{v} = \frac{\begin{pmatrix} 0, & 1, & -1 \end{pmatrix}}{\sqrt{2}}$$

and

$$\hat{w} = \frac{\begin{pmatrix} -4, & 1, & 1 \end{pmatrix}}{\sqrt{16+1+1}} = \frac{\begin{pmatrix} -4, & 1, & 1 \end{pmatrix}}{3\sqrt{2}}$$

The elements of the transformation matrix are given by the dot products of the unit vectors along the old and new axes (equation 1.21)

$$A = \frac{1}{3\sqrt{2}} \begin{pmatrix} \sqrt{2} & 2\sqrt{2} & 2\sqrt{2} \\ 0 & 3 & -3 \\ -4 & 1 & 1 \end{pmatrix}$$

To check, we evaluate:

$$A\hat{u} = \frac{1}{9\sqrt{2}} \begin{pmatrix} \sqrt{2} & 2\sqrt{2} & 2\sqrt{2} \\ 0 & 3 & -3 \\ -4 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} = \frac{1}{9\sqrt{2}} \begin{pmatrix} 9\sqrt{2} \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \hat{u}'$$

as required. Similarly

$$A\hat{v} = \frac{1}{6}\begin{pmatrix} \sqrt{2} & 2\sqrt{2} & 2\sqrt{2} \\ 0 & 3 & -3 \\ -4 & 1 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = \frac{1}{6}\begin{pmatrix} 0 \\ 6 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$$

and finally:

$$A\hat{w} = \frac{1}{18}\begin{pmatrix} \sqrt{2} & 2\sqrt{2} & 2\sqrt{2} \\ 0 & 3 & -3 \\ -4 & 1 & 1 \end{pmatrix}\begin{pmatrix} -4 \\ 1 \\ 1 \end{pmatrix} = \frac{1}{18}\begin{pmatrix} 0 \\ 0 \\ 18 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

3. Show that the vectors $\vec{u} =$ (15, 12, 16), $\vec{v} =$ (-20, 9, 12) and $\vec{w} =$ (0,-4, 3) are mutually orthogonal and right handed. Determine the transformation matrix that transforms from the original (x, y, z) cordinate system, to a system with x' $-$axis along $\vec{u}$, y' $-$axis along $\vec{v}$ and z' $-$axis along $\vec{w}$. Apply the transformation to find components of the vectors $\vec{a} = (1, 1, 1)$, $\vec{b} = (3, 2, 1)$ and $\vec{c} = (-2, 1, -2)$ in the prime system. Discuss the result for vector $\vec{c}$.

Two vectors are orthogonal if their dot product is zero.

$$\vec{u} \cdot \vec{v} = \begin{pmatrix} 15, & 12, & 16 \end{pmatrix} \cdot \begin{pmatrix} -20, & 9, & 12 \end{pmatrix} = -300 + 108 + 192 = 0$$

and

$$\vec{v} \cdot \vec{w} = \begin{pmatrix} -20, & 9, & 12 \end{pmatrix} \cdot \begin{pmatrix} 0, & -4, & 3 \end{pmatrix} = -36 + 36 = 0$$

Finally

$$\vec{u} \cdot \vec{w} = \begin{pmatrix} 15, & 12, & 16 \end{pmatrix} \cdot \begin{pmatrix} 0, & -4, & 3 \end{pmatrix} = -48 + 48 = 0$$

So the vectors are mutually orthogonal. In addition

$$\vec{u} \times \vec{v} = \begin{pmatrix} 15, & 12, & 16 \end{pmatrix} \times \begin{pmatrix} -20, & 9, & 12 \end{pmatrix} = \begin{pmatrix} 0 & -500 & 375 \end{pmatrix}$$
$$= 125\vec{w}$$

So the vectors form a right-handed set.

To find the transformation matrix, first we find the magnitude of each vector and the corresponding unit vectors.

$$|\vec{u}|^2 = 15^2 + 12^2 + 16^2 = 625 \Rightarrow |\vec{u}| = \sqrt{625} = 25$$

So

$$\hat{u} = \frac{1}{25}\begin{pmatrix} 15, & 12, & 16 \end{pmatrix}$$

Similarly

$$\hat{\mathbf{v}} = \frac{\begin{pmatrix} -20, & 9, & 12 \end{pmatrix}}{25}$$

and

$$\hat{\mathbf{w}} = \frac{\begin{pmatrix} 0, & -4, & 3 \end{pmatrix}}{5}$$

The elements of the transformation matrix are given by the dot products of the unit vectors along the old and new axes (equation 1.21)

$$A_{ij} = \hat{\mathbf{x}}_i' \cdot \hat{\mathbf{x}}_j$$

Thus the matrix is:

$$A = \frac{1}{25}\begin{pmatrix} 15 & 12 & 16 \\ -20 & 9 & 12 \\ 0 & -20 & 15 \end{pmatrix}$$

Check:

$$A\vec{\mathbf{u}} = \frac{1}{25}\begin{pmatrix} 15 & 12 & 16 \\ -20 & 9 & 12 \\ 0 & -20 & 15 \end{pmatrix}\begin{pmatrix} 15 \\ 12 \\ 16 \end{pmatrix} = \frac{1}{25}\begin{pmatrix} 625 \\ 0 \\ 0 \end{pmatrix} = 25\hat{\mathbf{x}}'$$

as required.

Then:

$$\vec{\mathbf{a}}' = \frac{1}{25}\begin{pmatrix} 15 & 12 & 16 \\ -20 & 9 & 12 \\ 0 & -20 & 15 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \frac{1}{25}\begin{pmatrix} 43 \\ 1 \\ -5 \end{pmatrix}$$

and

$$\vec{\mathbf{b}}' = \frac{1}{25}\begin{pmatrix} 15 & 12 & 16 \\ -20 & 9 & 12 \\ 0 & -20 & 15 \end{pmatrix}\begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} = \frac{1}{25}\begin{pmatrix} 85 \\ -30 \\ -25 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} 17 \\ -6 \\ -5 \end{pmatrix}$$

$$\vec{\mathbf{c}}' = \frac{1}{25}\begin{pmatrix} 15 & 12 & 16 \\ -20 & 9 & 12 \\ 0 & -20 & 15 \end{pmatrix}\begin{pmatrix} -2 \\ 1 \\ -2 \end{pmatrix} = \frac{1}{25}\begin{pmatrix} -50 \\ 25 \\ -50 \end{pmatrix} = \begin{pmatrix} -2 \\ 1 \\ -2 \end{pmatrix} = \vec{\mathbf{c}}$$

Since the components of the vector $\vec{c}$ remain unchanged, this vector must lie along the rotation axis.

4. A particle moves under the influence of electric and magnetic fields $\vec{E}$ and $\vec{B}$. Show that a particle moving with initial velocity $\vec{v}_0 = \frac{1}{B^2}\vec{E}\times\vec{B}$ is not accelerated if $\vec{E}$ is perpendicular to $\vec{B}$.

A particle reaches the origin with a velocity $\vec{v} = \vec{v}_0 + \varepsilon\hat{e}$, where $\hat{e}$ is a unit vector in the direction of $\vec{E}$ and $\varepsilon \ll v_0$. If $\vec{E} = E_0(1,1,1)$ and $\vec{B} = B_0(1,-2,1)$, set up a new coordinate system with $x'-$axis along $\vec{E}\times\vec{B}$ and $y'-$axis along $\vec{E}$. Determine the particle's position after a short time t. Determine the components of $\vec{v}(t)$ and $\vec{x}(t)$ in both the original and the new system. Give a criterion for ``short time''.

$$\vec{F} = q\left(\vec{E} + \vec{v}\times\vec{B}\right) = q\left(\vec{E} + \left(\frac{1}{B^2}\vec{E}\times\vec{B}\right)\times\vec{B}\right)$$
$$= q\left(\vec{E} + \frac{1}{B^2}\left[\vec{B}\left(\vec{E}\cdot\vec{B}\right) - \vec{E}B^2\right]\right)$$

But if $\vec{E}$ is perpendicular to $\vec{B}$, then $\vec{E}\cdot\vec{B} = 0$, so:

$$\vec{F} = q\left(\vec{E} - \vec{E}\right) = 0$$

and if there is no force, then the particle does not accelerate.

With the given vectors for $\vec{E}$ and $\vec{B}$, then

$$\vec{E}\times\vec{B} = E_0B_0(1+2, 1-1, -2-1) = E_0B_0(3,0,-3)$$

Then , since $\left|\vec{B}\right| = \sqrt{6}B_0$

$$\vec{v}_0 = \frac{3}{6}\frac{E_0}{B_0}(1,0,-1) = \frac{1}{2}\frac{E_0}{B_0}(1,0,-1)$$

Now we want to create a new coordinate system with $x'-$ axis along the direction of $\vec{v}_0$. Then we can put the y'-axis along $\vec{E}_0$ and the $z'-$axis along $\vec{B}_0$. The components in the original system of unit vectors along the new axes are the rows of the transformation matrix. Thus the transformation matrix is:

$$\mathbb{A} = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{pmatrix}$$

and the new components of $\vec{\mathbf{v}}_0$ are

$$\vec{\mathbf{v}}_0' = \frac{\sqrt{2}}{2}\frac{E_0}{B_0}(1,0,0)$$

Let's check that the matrix we found actually does this:

$$\vec{\mathbf{v}}_0' = A\vec{\mathbf{v}}_0 = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}\frac{E_0}{2B_0}$$

$$= \frac{E_0}{2B_0}\begin{pmatrix} \sqrt{2} \\ 0 \\ 0 \end{pmatrix}$$

as required.

Now let $\vec{\mathbf{v}} = \vec{\mathbf{v}}_0 + \varepsilon\hat{\mathbf{y}}'$. Then

$$\vec{\mathbf{F}} = q\left(\vec{\mathbf{E}} + [\vec{\mathbf{v}}_0 + \varepsilon\hat{\mathbf{y}}'] \times \vec{\mathbf{B}}\right) = q\varepsilon\hat{\mathbf{y}}' \times \vec{\mathbf{B}}$$

in the new system, the components of $\vec{\mathbf{B}}$ are:

$$\vec{\mathbf{B}}' = B_0\begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{pmatrix}\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} = B_0\begin{pmatrix} 0 \\ 0 \\ \sqrt{6} \end{pmatrix} = \sqrt{6}B_0\hat{\mathbf{z}}'$$

and so

$$\vec{\mathbf{F}}' = \sqrt{6}q\varepsilon B_0\hat{\mathbf{x}}' = m\vec{\mathbf{a}}$$

Since the initial velocity is $\vec{\mathbf{v}} = v_0\hat{\mathbf{x}}' + \varepsilon\hat{\mathbf{y}}'$ the particle's velocity at time t is:

$$\vec{\mathbf{v}}' = \left(v_0 + \frac{\sqrt{6}q\varepsilon B_0}{m}t\right)\hat{\mathbf{x}}' + \varepsilon\hat{\mathbf{y}}'$$

and the path is intially parabolic:

$$\vec{\mathbf{r}}' = \varepsilon t \hat{\mathbf{y}}' + \left(v_0 t + \frac{\sqrt{6}}{2} \frac{q \varepsilon B_0}{m} t^2 \right) \hat{\mathbf{x}}'$$

This result is valid so long as the initial velocity has not changed appreciably, so that the acceleration is approximately constant. That is:

$$t \ll \frac{v_0}{\varepsilon} \frac{2m}{\sqrt{6}\, q B_0} = \frac{v_0}{\varepsilon} \frac{2}{\omega_c}$$

or v_0/ε times (the cyclotron period divided by π). The time may be quite long if ε is small. Now we convert back to the original coordinates:

$$\begin{aligned}
\vec{\mathbf{r}} &= \mathbb{A}^{-1}\vec{\mathbf{r}}' = \mathbb{A}^{T}\vec{\mathbf{r}}' \\
&= \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \\ 0 & \frac{1}{\sqrt{3}} & -\frac{2}{\sqrt{6}} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{6}} \end{pmatrix} \begin{pmatrix} v_0 t + \frac{\sqrt{6}}{2} \frac{q \varepsilon B_0}{m} t^2 \\ \varepsilon t \\ 0 \end{pmatrix} \\
&= \begin{pmatrix} \frac{v_0 t}{\sqrt{2}} + \sqrt{3}\,\varepsilon t \left(\frac{1}{2} \frac{q B_0}{m} t + \frac{1}{3} \right) \\ \frac{\sqrt{3}}{3} \varepsilon t \\ -\frac{v_0 t}{\sqrt{2}} - \sqrt{3}\,\varepsilon t \left(\frac{1}{2} \frac{q B_0}{m} t - \frac{1}{3} \right) \end{pmatrix}
\end{aligned}$$

5. A solid body rotates with angular velocity $\vec{\boldsymbol{\omega}}$. Using cylindrical coordinates with $z-$axis along the rotation axis, find the components of the velocity vector $\vec{\mathbf{v}}$ at an arbitrary point within the body. Use the expression for curl in cylindrical coordinates to evaluate $\vec{\nabla} \times \vec{\mathbf{v}}$. Comment on your answer.

The velocity has only a $\phi-$component.

$$\vec{\mathbf{v}} = (0, \rho\omega, 0)$$

Then the curl is given by:

$$\begin{aligned}
\vec{\nabla} \times \vec{\mathbf{v}} &= \hat{\boldsymbol{\rho}} \left(\frac{1}{\rho} \frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z} \right) + \hat{\boldsymbol{\phi}} \left(\frac{\partial v_\rho}{\partial z} - \frac{\partial v_z}{\partial \rho} \right) + \hat{\mathbf{z}} \frac{1}{\rho} \left[\frac{\partial}{\partial \rho}(\rho v_\phi) - \frac{\partial v_\rho}{\partial \phi} \right] \\
&= \hat{\mathbf{z}} \frac{1}{\rho} \left[\frac{\partial}{\partial \rho} \left(\rho^2 \omega \right) \right] = \hat{\mathbf{z}} \frac{1}{\rho} (2\rho\omega) = 2\omega \hat{\mathbf{z}} = 2\vec{\boldsymbol{\omega}}
\end{aligned}$$

Thus the curl of the velocity equals twice the angular velocity- this seems logical for an operator called curl.

6. Starting from conservation of mass in a fixed volume $V,$ use the divergence theorem to derive the continuity equation for fluid flow:

$$\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{\mathbf{v}}) = 0$$

where ρ is the fluid density and $\vec{\mathbf{v}}$ its velocity.

The mass inside the volume can change only if fluid flows in or out across the boundary. Thus:

$$\frac{dM}{dt} = -\int_S \rho\vec{\mathbf{v}}\cdot\hat{\mathbf{n}}dA$$

where flow outward ($\vec{\mathbf{v}}\cdot\hat{\mathbf{n}} > 0$) decreases the mass. Now if the volume is fixed, then:

$$\frac{dM}{dt} = \frac{d}{dt}\int\rho dV = \int\frac{\partial\rho}{\partial t}dV = -\int_S \rho\vec{\mathbf{v}}\cdot\hat{\mathbf{n}}dA$$

Then from the divergence theorem:

$$\int\frac{\partial\rho}{\partial t}dV = -\int\vec{\nabla}\cdot(\rho\vec{\mathbf{v}})dV$$

and since this must be true for *any* volume $V,$ then

$$\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot(\rho\vec{\mathbf{v}})=0$$

7. Find the matrix that represents the transformation obtained by (a) rotating about the $x-$axis by 45° counterclockwise, and then (b) rotating about the $y'-$axis by 30° clockwise. What are the components of a unit vector along the original $z-$axis in the new (double-prime) system?

The first rotation is represented by the matrix

$$\mathbb{A}_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos 45° & \sin 45° \\ 0 & -\sin 45° & \cos 45° \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ 0 & -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix}$$

The second rotation is:

$$\mathbb{A}_2 = \begin{pmatrix} \cos(-30°) & 0 & \sin(-30°) \\ 0 & 1 & 0 \\ -\sin(-30°) & 0 & \cos(-30°) \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{3}}{2} & 0 & -\frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{\sqrt{3}}{2} \end{pmatrix}$$

And the result of the two rotations is:

$$\mathbb{A}_2\mathbb{A}_1 = \begin{pmatrix} \frac{\sqrt{3}}{2} & 0 & -\frac{1}{2} \\ 0 & 1 & 0 \\ \frac{1}{2} & 0 & \frac{\sqrt{3}}{2} \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ 0 & -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{pmatrix} = \begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} \\ 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{1}{2} & -\frac{\sqrt{6}}{4} & \frac{\sqrt{6}}{4} \end{pmatrix}$$

The new components of the orignal $z-$axis are:

$$\begin{pmatrix} \frac{\sqrt{3}}{2} & \frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} \\ 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ \frac{1}{2} & -\frac{\sqrt{6}}{4} & \frac{\sqrt{6}}{4} \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -\frac{\sqrt{2}}{4} \\ \frac{\sqrt{2}}{2} \\ \frac{\sqrt{6}}{4} \end{pmatrix} = \frac{\sqrt{2}}{2}\begin{pmatrix} -\frac{1}{2} \\ 1 \\ \frac{\sqrt{3}}{2} \end{pmatrix}$$

8. Does the matrix

$$\begin{pmatrix} \cos\theta & \sin\theta & 0 \\ \sin\theta & -\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

represent a rotation of the coordinate axes? If not, what transformation does it represent? Draw a diagram showing the old and new coordinate axes, and comment.

The determinant of this matrix is:

$$\begin{vmatrix} \cos\theta & \sin\theta & 0 \\ \sin\theta & -\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = -\cos^2\theta - \sin^2\theta = -1$$

Thus this transformation cannot be a rotation since a rotation matrix has determinant $+1$.
Let's see where the axes go:

$$\mathbb{A}\hat{\mathbf{x}} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ \sin\theta & -\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix}$$

and

$$\mathbb{A}\hat{\mathbf{y}} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ \sin\theta & -\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} \sin\theta \\ -\cos\theta \\ 0 \end{pmatrix}$$

while

$$\mathbb{A}\hat{\mathbf{z}} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

These are the components of the original $x-$ and y axes in the new system. The new $x'-$ and y' axes have the following components in the original system:

$$\hat{\mathbf{u}} = \mathbb{A}^{-1} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

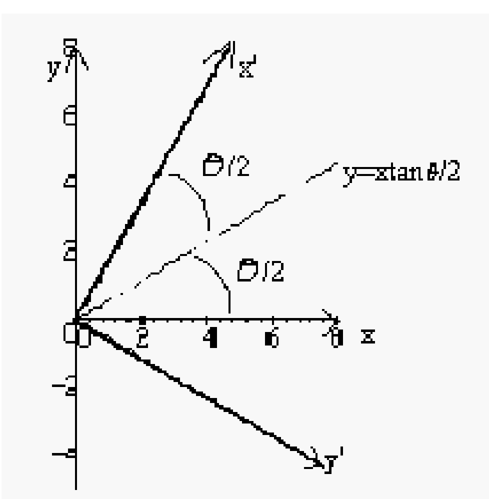
where

$$\mathbb{A}^{-1} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ \sin\theta & -\cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Thus:

$$\hat{\mathbf{u}} = \begin{pmatrix} \cos\theta \\ \sin\theta \\ 0 \end{pmatrix}$$

The picture looks like this:



Problem 8: $\theta = \pi/3$

The matrix represents a reflection of the $x-$and $y-$axes about the line $y = x\tan(\theta/2)$

9. Represent the following transformation using a matrix: (a) a rotation about the $z-$axis through an angle $\pi/3,$ followed by (b) a reflection in the line through the origin and in the $x-y$-plane, at an angle 2$\pi/3$ to the *original* $x-$axis, where both angles are measured counter-clockwise from the positive $x-$axis. Express your answer as a single matrix. You should be able to recognize the matrix either as a rotation about the $z-$axis through an angle $\alpha,$ or as a reflection in a line through the origin at an angle α to the $x-$axis. Decide whether this transformation is a reflection or a rotation, and give the value of $\alpha.$ (*Note*: For the purposes of this problem, reflection in a line in the $x-y$ plane leaves the $z-$axis unchanged.)

Since only the x and $y-$components are transformed, we may work with 2×2 matrices. The rotation matrix is:

$$\begin{pmatrix} \cos\frac{\pi}{3} & \sin\frac{\pi}{3} \\ -\sin\frac{\pi}{3} & \cos\frac{\pi}{3} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{1}{2}\sqrt{3} \\ -\frac{1}{2}\sqrt{3} & \frac{1}{2} \end{pmatrix}$$

The line in which we reflect is at 2$\pi/3$ to the original $x-$axis, and thus at $\pi/3$ to the new $x-$axis. Thus the matrix we want is (see Problem 8 above):

$$\begin{pmatrix} \cos\frac{2\pi}{3} & \sin\frac{2\pi}{3} \\ \sin\frac{2\pi}{3} & -\cos\frac{2\pi}{3} \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2}\sqrt{3} \\ \frac{1}{2}\sqrt{3} & \frac{1}{2} \end{pmatrix}$$

Thus the complete transformation is described by the matrix:

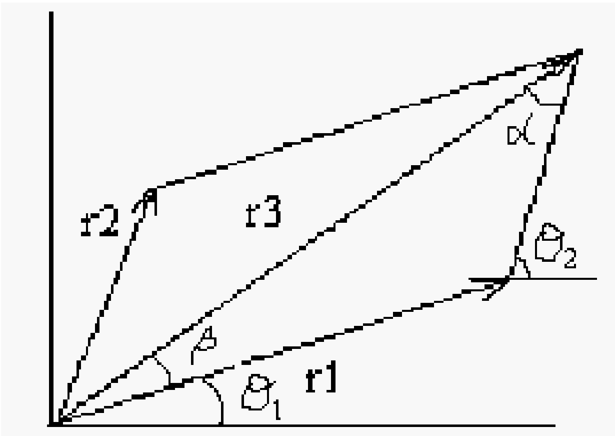
$$A = \begin{pmatrix} -\frac{1}{2} & \frac{1}{2}\sqrt{3} \\ \frac{1}{2}\sqrt{3} & \frac{1}{2} \end{pmatrix}\begin{pmatrix} \frac{1}{2} & \frac{1}{2}\sqrt{3} \\ -\frac{1}{2}\sqrt{3} & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

The determinant of this matrix is -1, and so the transformation is a reflection. It sends x to $-x$ and y to $y,$ so it is a reflection in the $y-$axis ($\alpha = \pi/2$).

10. Using polar coordinates, write the *components* of the position vectors of two points in a plane: P_1 with coordinates r_1 and $\theta_1,$ and P_2 with coordinates r_2 and $\theta_2.$ (That is, write each vector in the form $\vec{\mathbf{v}} = v_r\hat{\mathbf{r}} + v_\theta\hat{\boldsymbol{\theta}}.$) What are the coordinates r_3 and θ_3 of the point P_3 whose position vector is

$$\vec{\mathbf{r}}_3 = \vec{\mathbf{r}}_1 + \vec{\mathbf{r}}_2?$$

Hint: Start by drawing the position vectors.



Problem 10

The position vector has only a single component: the r −component. Thus the vectors are:

$$\vec{\mathbf{r}}_1 = (r_1, 0)$$

and

$$\vec{\mathbf{r}}_2 = (r_2, 0)$$

The sum also only has a single component:

$$\vec{\mathbf{r}}_3 = (r_3, 0)$$

where, from the diagram $\alpha + \beta = \theta_2 - \theta_1$, and:

$$r_3^2 = r_2^2 + r_1^2 - 2r_1 r_2 \cos(\pi - \alpha - \beta) = r_2^2 + r_1^2 + 2r_1 r_2 \cos(\theta_2 - \theta_1)$$

Thus P_3 has coordinates $r_3 = \sqrt{r_2^2 + r_1^2 + 2r_1 r_2 \cos(\theta_2 - \theta_1)}$, $\theta_3 = \theta_1 + \beta$

where

$$\frac{\sin\beta}{r_2} = \frac{\sin\alpha}{r_1} = \frac{\sin(\theta_2 - \theta_1 - \beta)}{r_1}$$
$$= \frac{1}{r_1}(\sin(\theta_2 - \theta_1)\cos\beta - \cos(\theta_2 - \theta_1)\sin\beta)$$

and thus

$$\tan\beta = \frac{\sin(\theta_2 - \theta_1)}{r_1/r_2 + \cos(\theta_2 - \theta_1)}$$

We can check this in the special case $r_1 = r_2$, $\beta = (\theta_2 - \theta_1)/2.$ Then

$$\tan\beta = \frac{2\sin(\theta_2 - \theta_1)/2\cos(\theta_2 - \theta_1)/2}{1 + 2\cos^2(\theta_2 - \theta_1)/2 - 1} = \tan\frac{(\theta_2 - \theta_1)}{2}$$

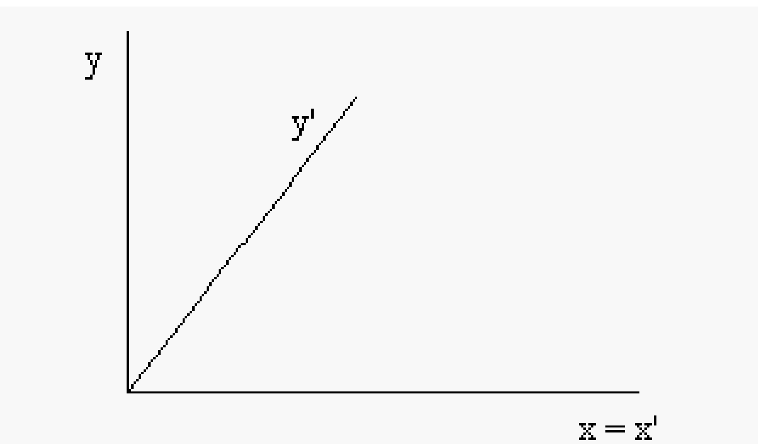
as required.

11. A skew (non-orthogonal) coordinate system in a plane has $x' -$ axis along the $x -$axis and $y' -$axis at an angle θ to the $x -$axis, where $\theta < \pi/2.$

(a) Write the transformation matrix that transforms vector components from the Cartesian $x - y$ system to the skew system.

(b) Write an expression for the distance between two neighboring points in the skew system. Comment on the differences between your expression and the standard Cartesian expression.

(c) Write the equation for a circle of radius $a,$ with center at the origin, in the skew system.



Problem 1.11

(**a**) The new coordinates are:

$$x' = x - \frac{y}{\tan\theta}$$

and

$$y' = \frac{y}{\sin\theta}$$

Thus the transformation matrix is:

$$A = \begin{pmatrix} 1 & -\cot\theta \\ 0 & \frac{1}{\sin\theta} \end{pmatrix} = \frac{1}{\sin\theta}\begin{pmatrix} \sin\theta & -\cos\theta \\ 0 & 1 \end{pmatrix}$$

Compare this result with equation 1.21. Here the components are given by

$$A_{ij} \det \mathbb{A} = \cos\theta_{ij}$$

(**b**)

$$\begin{aligned} ds^2 &= \left(dx'\,\hat{\mathbf{i}}' + dy'\,\hat{\mathbf{j}}'\right)\cdot\left(dx'\,\hat{\mathbf{i}}' + dy'\,\hat{\mathbf{j}}'\right) \\ &= \left(dx'\right)^2 + 2dx'dy'\,\hat{\mathbf{i}}'\cdot\hat{\mathbf{j}}' + \left(dy'\right)^2 \\ &= \left(dx'\right)^2 + 2dx'dy'\cos\theta + \left(dy'\right)^2 \end{aligned}$$

The cross term indicates that the system is not orthogonal. We could also have obtained this result from the cosine rule.

(**c)** The circle is described by the equation

$$\begin{aligned} \vec{\mathbf{r}}\cdot\vec{\mathbf{r}} = a^2 &= \left(x'\,\hat{\mathbf{i}}' + y'\,\hat{\mathbf{j}}'\right)\cdot\left(x'\,\hat{\mathbf{i}}' + y'\,\hat{\mathbf{j}}'\right) \\ &= \left(x'\right)^2 + 2x'y'\cos\theta + \left(y'\right)^2 \end{aligned}$$

a result that could also be obtained by applying the cosine rule to find the radius of the circle in terms of the coordinates x' and y'.

12. Prove the Jacobi identity:

$$\vec{\mathbf{a}}\times\left(\vec{\mathbf{b}}\times\vec{\mathbf{c}}\right) + \vec{\mathbf{b}}\times\left(\vec{\mathbf{c}}\times\vec{\mathbf{a}}\right) + \vec{\mathbf{c}}\times\left(\vec{\mathbf{a}}\times\vec{\mathbf{b}}\right) = 0$$

The triple cross product is

$$\vec{\mathbf{a}}\times\left(\vec{\mathbf{b}}\times\vec{\mathbf{c}}\right) = \vec{\mathbf{b}}\left(\vec{\mathbf{a}}\cdot\vec{\mathbf{c}}\right) - \vec{\mathbf{c}}\left(\vec{\mathbf{a}}\cdot\vec{\mathbf{b}}\right)$$

and thus

$$\begin{aligned} &\vec{\mathbf{a}}\times\left(\vec{\mathbf{b}}\times\vec{\mathbf{c}}\right) + \vec{\mathbf{b}}\times\left(\vec{\mathbf{c}}\times\vec{\mathbf{a}}\right) + \vec{\mathbf{c}}\times\left(\vec{\mathbf{a}}\times\vec{\mathbf{b}}\right) \\ &= \vec{\mathbf{b}}\left(\vec{\mathbf{a}}\cdot\vec{\mathbf{c}}\right) - \vec{\mathbf{c}}\left(\vec{\mathbf{a}}\cdot\vec{\mathbf{b}}\right) + \vec{\mathbf{c}}\left(\vec{\mathbf{a}}\cdot\vec{\mathbf{b}}\right) - \vec{\mathbf{a}}\left(\vec{\mathbf{b}}\cdot\vec{\mathbf{c}}\right) + \vec{\mathbf{a}}\left(\vec{\mathbf{b}}\cdot\vec{\mathbf{c}}\right) - \vec{\mathbf{b}}\left(\vec{\mathbf{a}}\cdot\vec{\mathbf{c}}\right) \\ &= 0 \end{aligned}$$

Since the dot product is commutative, the result is zero, as required.

13. Evaluate the vector product

$$\left(\vec{\mathbf{a}}\times\vec{\mathbf{b}}\right)\times\left(\vec{\mathbf{c}}\times\vec{\mathbf{d}}\right)$$

in terms of triple scalar products. What is the result if all four vectors lie in a single plane? What is the result if $\vec{\mathbf{a}}$, $\vec{\mathbf{b}}$ and $\vec{\mathbf{c}}$ are mutually perpendicular? What is the result if $\vec{\mathbf{b}} = \vec{\mathbf{d}}$?

We can start with the bac-cab rule:

$$\left(\vec{a}\times\vec{b}\right)\times\left(\vec{c}\times\vec{d}\right) = \vec{b}\left\{\vec{a}\cdot\left(\vec{c}\times\vec{d}\right)\right\} - \vec{a}\left\{\vec{b}\cdot\left(\vec{c}\times\vec{d}\right)\right\}$$
$$= \vec{b}\left[\vec{a},\vec{c},\vec{d}\right] - \vec{a}\left[\vec{b},\vec{c},\vec{d}\right]$$

Equivalently, we may write:

$$\left(\vec{a}\times\vec{b}\right)\times\left(\vec{c}\times\vec{d}\right) = \vec{c}\left\{\left(\vec{a}\times\vec{b}\right)\cdot\vec{d}\right\} - \vec{d}\left\{\left(\vec{a}\times\vec{b}\right)\cdot\vec{c}\right\}$$
$$= \vec{c}\left[\vec{a},\vec{b},\vec{d}\right] - \vec{d}\left[\vec{a},\vec{b},\vec{c}\right]$$

If all four vectors lie in a single plane, then each of the triple scalar products is zero, and therefore the final result is also zero.

If $\vec{a}$, $\vec{b}$ and $\vec{c}$ are mutually perpendicular

$$\left(\vec{a}\times\vec{b}\right) = ab\hat{c}$$

$$\left(\vec{a}\times\vec{b}\right)\cdot\vec{c} = \pm abc$$

where the plus sign applies if the vectors form a right-handed set, and

$$\left(\vec{a}\times\vec{b}\right)\times\left(\vec{c}\times\vec{d}\right) = \vec{c}\left\{\pm ab\hat{c}\cdot\vec{d}\right\} - \vec{d}abc$$
$$= \pm abc\left(\hat{c}\left(\hat{c}\cdot\vec{d}\right)-\vec{d}\right)$$

If $\vec{b} = \vec{d}$, then $\left(\vec{a}\times\vec{b}\right)\cdot\vec{d} = 0$, and

$$\left(\vec{a}\times\vec{b}\right)\times\left(\vec{c}\times\vec{d}\right) = -\vec{d}\left\{\left(\vec{a}\times\vec{b}\right)\cdot\vec{c}\right\} = -\vec{b}\left[\vec{a},\vec{b},\vec{c}\right]$$

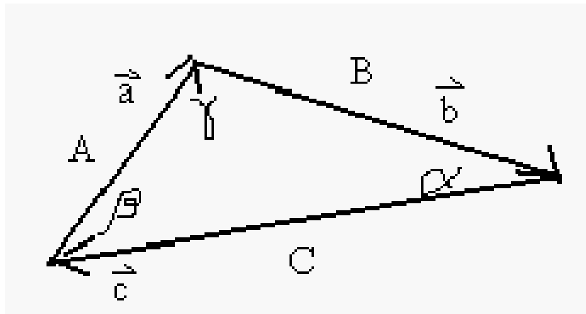
14. Evaluate the product $\left(\vec{a}\times\vec{b}\right)\cdot\left(\vec{c}\times\vec{d}\right)$ in terms of dot products of $\vec{a}$, $\vec{b}$, $\vec{c}$ and $\vec{d}$.

$$\left(\vec{a}\times\vec{b}\right)\cdot\left(\vec{c}\times\vec{d}\right) = \varepsilon_{ijk}a_j b_k \varepsilon_{ilm} c_l d_m$$
$$= \left(\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}\right)a_j b_k c_l d_m$$
$$= a_j b_k c_j d_k - a_j b_k c_k d_j$$
$$= \left(\vec{a}\cdot\vec{c}\right)\left(\vec{b}\cdot\vec{d}\right) - \left(\vec{a}\cdot\vec{d}\right)\left(\vec{b}\cdot\vec{c}\right)$$

15. Use the vector cross product to express the area of a triangle in three different ways. Hence prove the sine rule:

$$\frac{\sin\alpha}{A} = \frac{\sin\beta}{B} = \frac{\sin\gamma}{C}.$$

First we define the vectors $\vec{a}$, $\vec{b}$, and $\vec{c}$ that lie along the sides of the triangle, as shown in the diagram.



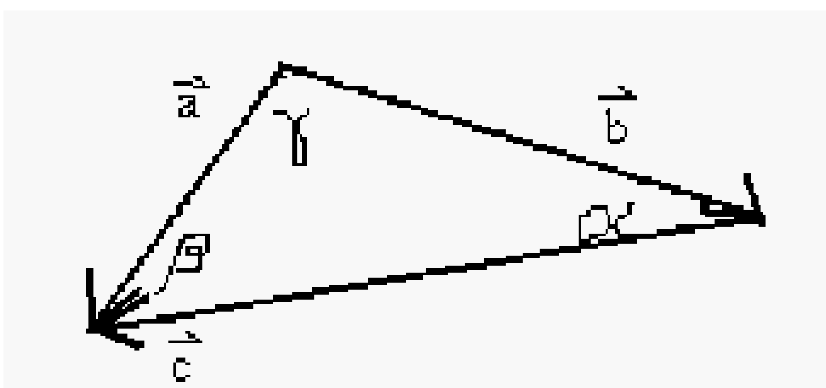
Then the area equals the magnitude of $\vec{a}\times\vec{b}$ or of $\vec{b}\times\vec{c}$ or of $\vec{a}\times\vec{c}$. Hence

$$AB\sin\gamma = BC\sin\alpha = AC\sin\beta$$

Dividing through by the product ABC, we obtain the desired result.

16. Use the dot product $(\vec{a}-\vec{b})\cdot(\vec{a}-\vec{b})$ to prove the cosine rule for a triangle:

$$c^2 = a^2 + b^2 - 2ab\cos\gamma$$



With the vectors defined as in the diagram above,

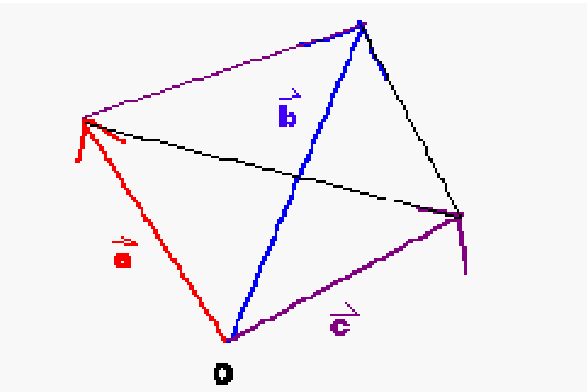
$$(\vec{a}-\vec{b})\cdot(\vec{a}-\vec{b}) = a^2 + b^2 - 2\vec{a}\cdot\vec{b}$$

But if $\vec{a}$ and $\vec{b}$ lie along two sides of a triangle s shown, then the third side $\vec{c} = \vec{a}-\vec{b}$. Thus

$$c^2 = a^2 + b^2 - 2ab\cos\gamma$$

as required.

17. A tetrahedron has its apex at the origin and its edges defined by the vectors $\vec{\mathbf{a}}$, $\vec{\mathbf{b}}$, and $\vec{\mathbf{c}}$, each of which has its tail at the origin (see figure). Defining the normal to each face to be outward from the interior of the tetrahedron, determine the total vector area of the four faces of the tetrahedron. Find the volume of the tetrahedron.



Problem 1.17

With direction along the outward normal, the area of one face is

$$\vec{\mathbf{A}}_1 = \frac{1}{2}\vec{\mathbf{b}} \times \vec{\mathbf{a}}$$

The total area is given by:

$$2\vec{\mathbf{A}} = \vec{\mathbf{b}} \times \vec{\mathbf{a}} + \vec{\mathbf{c}} \times \vec{\mathbf{b}} + \vec{\mathbf{a}} \times \vec{\mathbf{c}} + \left(\vec{\mathbf{b}} - \vec{\mathbf{a}}\right) \times \left(\vec{\mathbf{c}} - \vec{\mathbf{b}}\right)$$

Expanding out the last product, and using the result that $\vec{\mathbf{b}} \times \vec{\mathbf{b}} = 0$:

$$\begin{aligned} 2\vec{\mathbf{A}} &= \vec{\mathbf{b}} \times \vec{\mathbf{a}} + \vec{\mathbf{c}} \times \vec{\mathbf{b}} + \vec{\mathbf{a}} \times \vec{\mathbf{c}} + \vec{\mathbf{b}} \times \vec{\mathbf{c}} - \vec{\mathbf{a}} \times \vec{\mathbf{c}} + \vec{\mathbf{a}} \times \vec{\mathbf{b}} \\ &= \vec{\mathbf{b}} \times \vec{\mathbf{a}} + \vec{\mathbf{a}} \times \vec{\mathbf{b}} + \vec{\mathbf{c}} \times \vec{\mathbf{b}} + \vec{\mathbf{b}} \times \vec{\mathbf{c}} = \mathbf{0} \end{aligned}$$

since $\vec{\mathbf{b}} \times \vec{\mathbf{a}} = -\vec{\mathbf{a}} \times \vec{\mathbf{b}}$.

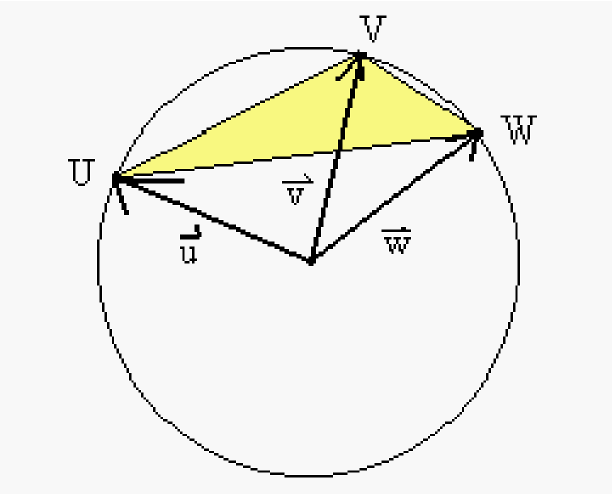
The volume is 1/6 of the parallelopiped formed by the three vectors, (or 1/3 base times height of tetrahedron) and so $V = \frac{1}{6}\left[\vec{\mathbf{a}}, \vec{\mathbf{b}}, \vec{\mathbf{c}}\right]$.

18. A sphere of unit radius is centered at the origin. Points U, V and W on the surface of the sphere have position vectors $\vec{\mathbf{u}}$, $\vec{\mathbf{v}}$ and $\vec{\mathbf{w}}$. Show that points P and Q

on the sphere, located on a diameter perpendicular to the plane containing the points U, V and W, have position vectors given by

$$\vec{r} = \pm \frac{\vec{u}\times\vec{v} + \vec{v}\times\vec{w} + \vec{w}\times\vec{u}}{[\vec{u},\vec{v},\vec{w}]}\cos\theta$$

where θ is the angle between the vectors $\mathbf{r}$ and $\vec{u}$.



Problem 1.18

The triangle UVW has sides given by the vectors $\vec{u}-\vec{v}$, $\vec{v}-\vec{w}$, and $\vec{w}-\vec{u}$. The plane of the triangle may thus be described by the vector

$$\begin{aligned}\vec{a} &= (\vec{u}-\vec{v})\times(\vec{v}-\vec{w}) = \vec{u}\times\vec{v} + \vec{v}\times\vec{w} - \vec{u}\times\vec{w} \\ &= \vec{u}\times\vec{v} + \vec{v}\times\vec{w} + \vec{w}\times\vec{u}\end{aligned}$$

This vector is normal to the plane. The vector $\mathbf{r}$ is a unit vector, as are the vectors $\vec{u}$, $\vec{v}$ and $\vec{w}$, since the sphere has unit radius. Thus we may write $\mathbf{r} = \vec{a}/a$, and

$$\begin{aligned}\vec{r}\cdot\vec{u} = \cos\theta &= (\vec{u}\times\vec{v} + \vec{v}\times\vec{w} + \vec{w}\times\vec{u})\cdot\frac{\vec{u}}{a} \\ &= \frac{[\vec{u},\vec{v},\vec{w}]}{a}\end{aligned}$$

Thus

$$a = \frac{[\vec{u},\vec{v},\vec{w}]}{\cos\theta}$$

and thus

$$\vec{r} = \frac{\vec{u}\times\vec{v} + \vec{v}\times\vec{w} + \vec{w}\times\vec{u}}{[\vec{u},\vec{v},\vec{w}]}\cos\theta$$

To obtain both ends of the diameter, we need to add the $\pm$ sign, as given in the problem statement.

19. Show that

$$\vec{\nabla}\times\left(\vec{\nabla}\Phi\right) = 0$$

for any scalar field Φ.

$$\begin{aligned}\vec{\nabla}\times\left(\vec{\nabla}\phi\right) &= \left[\frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial z}\right) - \frac{\partial}{\partial z}\left(\frac{\partial\phi}{\partial y}\right)\right]\hat{x} + \text{two similar terms}\\ &= 0\end{aligned}$$

because the order of the partial derivatives is irrelevant.

20. Find an expression for $\vec{\nabla}\times\left(\vec{a}\times\vec{b}\right)$ in terms of derivatives of $\vec{a}$ and $\vec{b}$.

$$\begin{aligned}\vec{\nabla}\times\left(\vec{a}\times\vec{b}\right)_i &= \varepsilon_{ijk}\partial_j\varepsilon_{klm}a_lb_m\\ &= \left(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}\right)\partial_ja_lb_m\\ &= \partial_ma_ib_m - \partial_la_lb_i\end{aligned}$$

Now remember that the differential operator operates on everything to its right, so, expanding the derivatives of the products, we have:

$$\vec{\nabla}\times\left(\vec{a}\times\vec{b}\right) = \vec{a}\left(\vec{\nabla}\cdot\vec{b}\right) + \left(\vec{b}\cdot\vec{\nabla}\right)\vec{a} - \vec{b}\left(\vec{\nabla}\cdot\vec{a}\right) - \left(\vec{a}\cdot\vec{\nabla}\right)\vec{b}$$

21. Prove the identity:

$$\vec{\nabla}\left(\vec{\mathbf{a}}\cdot\vec{\mathbf{b}}\right) = \left(\vec{\mathbf{a}}\cdot\vec{\nabla}\right)\vec{\mathbf{b}} + \left(\vec{\mathbf{b}}\cdot\vec{\nabla}\right)\vec{\mathbf{a}} + \vec{\mathbf{a}}\times\left(\vec{\nabla}\times\vec{\mathbf{b}}\right) + \vec{\mathbf{b}}\times\left(\vec{\nabla}\times\vec{\mathbf{a}}\right)$$

Hint: start with the last two terms on the right hand side**.**

We expand the third term, being careful to keep the differential operator operating on $\vec{\mathbf{b}}$ but not $\vec{\mathbf{a}}$. The ith component is:

$$\begin{aligned}\vec{\mathbf{a}}\times\left(\vec{\nabla}\times\vec{\mathbf{b}}\right)_i &= \varepsilon_{ijk}a_j\varepsilon_{klm}\partial_l b_m \\ &= \left(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}\right)a_j\partial_l b_m \\ &= a_m\partial_i b_m - a_l\partial_l b_i\end{aligned}$$

Thus

$$\vec{\mathbf{a}}\times\left(\vec{\nabla}\times\vec{\mathbf{b}}\right)_i + \vec{\mathbf{b}}\times\left(\vec{\nabla}\times\vec{\mathbf{a}}\right)_i = a_m\partial_i b_m - \left(\vec{\mathbf{a}}\cdot\vec{\nabla}\right)b_i + b_m\partial_i a_m - \left(\vec{\mathbf{b}}\cdot\vec{\nabla}\right)a_i$$

Combining terms:

$$\vec{\mathbf{a}}\times\left(\vec{\nabla}\times\vec{\mathbf{b}}\right) + \vec{\mathbf{b}}\times\left(\vec{\nabla}\times\vec{\mathbf{a}}\right) = \vec{\nabla}\left(\vec{\mathbf{a}}\cdot\vec{\mathbf{b}}\right) - \left(\vec{\mathbf{a}}\cdot\vec{\nabla}\right)\vec{\mathbf{b}} - \left(\vec{\mathbf{b}}\cdot\vec{\nabla}\right)\vec{\mathbf{a}}$$

and so

$$\vec{\nabla}\left(\vec{\mathbf{a}}\cdot\vec{\mathbf{b}}\right) = \left(\vec{\mathbf{a}}\cdot\vec{\nabla}\right)\vec{\mathbf{b}} + \left(\vec{\mathbf{b}}\cdot\vec{\nabla}\right)\vec{\mathbf{a}} + \vec{\mathbf{a}}\times\left(\vec{\nabla}\times\vec{\mathbf{b}}\right) + \vec{\mathbf{b}}\times\left(\vec{\nabla}\times\vec{\mathbf{a}}\right)$$

as required.

22. Compute $\vec{\nabla}\cdot\left(\vec{\mathbf{a}}\times\vec{\mathbf{b}}\right)$ in terms of curl $\vec{\mathbf{a}}$ and curl $\vec{\mathbf{b}}$.

$$\begin{aligned}\left[\vec{\nabla}\cdot\left(\vec{\mathbf{a}}\times\vec{\mathbf{b}}\right)\right]_i &= \partial_i\left(\varepsilon_{ijk}a_j b_k\right) = \varepsilon_{ijk}[(\partial_i a_j)b_k + a_j\partial_i b_k] \\ &= \varepsilon_{ijk}(\partial_i a_j)b_k - \varepsilon_{ikj}a_j\partial_i b_k\end{aligned}$$

and so

$$\vec{\nabla}\cdot\left(\vec{\mathbf{a}}\times\vec{\mathbf{b}}\right) = \vec{\mathbf{b}}\cdot\left(\vec{\nabla}\times\vec{\mathbf{a}}\right) - \vec{\mathbf{a}}\cdot\left(\vec{\nabla}\times\vec{\mathbf{b}}\right)$$

23. Obtain an expression for $\vec{\nabla}\times\left(\phi\vec{\mathbf{u}}\right)$ and hence show that $\vec{\nabla}\times\left(\phi\vec{\nabla}\phi\right) = 0$.

$$\begin{aligned}\varepsilon_{ijk}\partial_j(\phi u_k) &= \varepsilon_{ijk}(u_k\partial_j\phi + \phi\partial_j u_k) \\ &= \vec{\nabla}\phi\times\vec{\mathbf{u}} + \phi\left(\vec{\nabla}\times\vec{\mathbf{u}}\right)\end{aligned}$$

Now with $\vec{\mathbf{u}} = \vec{\nabla}\phi$, the first term is the cross product of a vector with itself, and so is zero, while the second is zero beacuse the curl of a gradient is zero.

24. The equation of motion for a fluid may be written

$$\rho\left(\frac{\partial\vec{\mathbf{v}}}{\partial t}+\left(\vec{\mathbf{v}}\cdot\vec{\nabla}\right)\vec{\mathbf{v}}\right)=-\vec{\nabla}P+\rho\vec{\mathbf{g}}$$

where $\vec{\mathbf{v}}$ is the fluid velocity at a point, ρ its density and P the pressure. the acceleration due to gravity is $\vec{\mathbf{g}}$. Use the result of Problem 21 to show that for fluid flow that is incompressible (ρ = constant) and steady ($\frac{\partial}{\partial t}\equiv 0$), Bernoulli's law holds:

$$P+\frac{1}{2}\rho v^2+\rho gh=\text{ constant along a streamline}$$

Hint: express the statement "constant along a streamline" as a directional derivative being equal to zero.

Use the result of problem 21 with $\vec{\mathbf{a}}=\vec{\mathbf{b}}=\vec{\mathbf{v}}$:

$$\vec{\nabla}\left(v^2\right)=2\left(\vec{\mathbf{v}}\cdot\vec{\nabla}\right)\vec{\mathbf{v}}+2\vec{\mathbf{v}}\times\left(\vec{\nabla}\times\vec{\mathbf{v}}\right)$$

Write $\vec{\mathbf{g}}$ as the gradient of the gravitational potential, gh, and dot the equation with $\vec{\mathbf{v}}$:

$$\frac{\rho}{2}\vec{\mathbf{v}}\cdot\left(\vec{\nabla}\left(v^2\right)-\vec{\mathbf{v}}\times\left(\vec{\nabla}\times\vec{\mathbf{v}}\right)\right)=-\vec{\mathbf{v}}\cdot\vec{\nabla}P-\rho\vec{\mathbf{v}}\cdot\vec{\nabla}gh$$

Since $\vec{\mathbf{v}}\times\left(\vec{\nabla}\times\vec{\mathbf{v}}\right)$ is perpendicular to $\vec{\mathbf{v}}$, its dot product with $\vec{\mathbf{v}}$ is zero, and we may move the constant ρ inside the derivative to get:

$$\vec{\mathbf{v}}\cdot\vec{\nabla}\left(\frac{1}{2}\rho v^2+P+\rho gh\right)$$

as required.

Under what conditions is $P+\frac{1}{2}\rho v^2+\rho gh$ equal to an absolute constant, the same throughout the fluid?

If the flow is irrotational ($\vec{\nabla}\times\vec{\mathbf{v}}=0$), then $\left(\vec{\mathbf{v}}\cdot\vec{\nabla}\right)\vec{\mathbf{v}}=\frac{1}{2}\vec{\nabla}\left(v^2\right)$ and we may simplify immediately to get

$$\vec{\nabla}\left(\frac{1}{2}\rho v^2+P+\rho gh\right)=0$$

in which case the constant of integration is the same throughout the fluid.

25. Evaluate the integral

$$\oint_C\vec{\mathbf{u}}\cdot d\vec{\mathbf{l}}$$

where **(a)** C is the unit circle in the $x-y-$plane and centered at the origin

$$\vec{\mathbf{u}}=x^2y\hat{\mathbf{x}}-xy^2\hat{\mathbf{y}}$$

We can use Stokes theorem:

$$\oint_c \vec{\mathbf{u}} \cdot d\vec{\mathbf{l}} = \int_S (\nabla \times \vec{\mathbf{u}}) \cdot \hat{\mathbf{n}} dA$$

Here the surface is in the $x-y$ plane, and the $z-$component of the curl is:

$$\nabla \times \vec{\mathbf{u}}\Big|_{z\text{-comp}} = \frac{\partial}{\partial x}\left(-xy^2\right) - \frac{\partial}{\partial y}\left(x^2 y\right) = \left(-y^2 - x^2\right) = -r^2$$

and so the integral is

$$\int_S (\nabla \times \vec{\mathbf{u}}) \cdot \hat{\mathbf{n}} dA = -\int_0^1 r^2 (2\pi r dr) = -2\pi \frac{r^4}{4}\Big|_0^1 = -\frac{\pi}{2}$$

(**b**) C is a semicircle of radius a with the flat side along the $x-$axis, the center of the circle at the origin, and

$$\vec{\mathbf{u}} = xy^2 \hat{\mathbf{x}} + yx^2 \hat{\mathbf{y}}$$

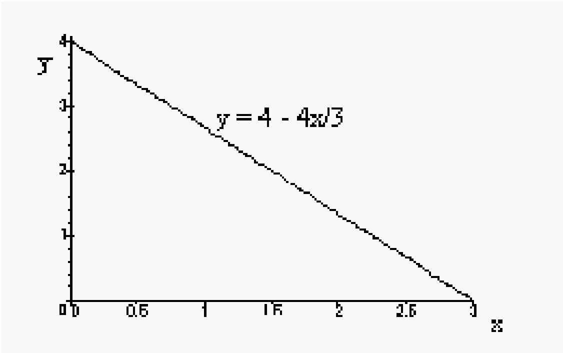
We need only the $z-$component of the curl.

$$\nabla \times \vec{\mathbf{u}}\Big|_{z\text{-comp}} = \frac{\partial}{\partial x}\left(yx^2\right) - \frac{\partial}{\partial y}\left(xy^2\right) = 2xy - 2xy = 0$$

and so the integral is zero.

(**c**) C is a 3-4-5 right-angled triangle with the sides of length 3 and 4 along the $x-$ and $y-$axes respectively, and

$$\vec{\mathbf{u}} = x^2 \hat{\mathbf{x}} + xy \hat{\mathbf{y}}$$



Using Stoke's theorem:

$$\oint_c \vec{\mathbf{u}} \cdot d\vec{\mathbf{l}} = \int_S (\nabla \times \vec{\mathbf{u}}) \cdot \hat{\mathbf{n}} dA$$

with the $z-$component of the curl being:

$$\frac{\partial}{\partial x}(xy) - \frac{\partial}{\partial y}\left(x^2\right) = y$$

we have

$$\int_0^3 dx \int_0^{4-4x/3} y\,dy = \int_0^3 dx \frac{4^2}{2}\left(1 - \frac{x}{3}\right)^2$$
$$= 8\int_1^0 u^2(-3du) = -8u^3\,\Big|_0^1 = 8$$

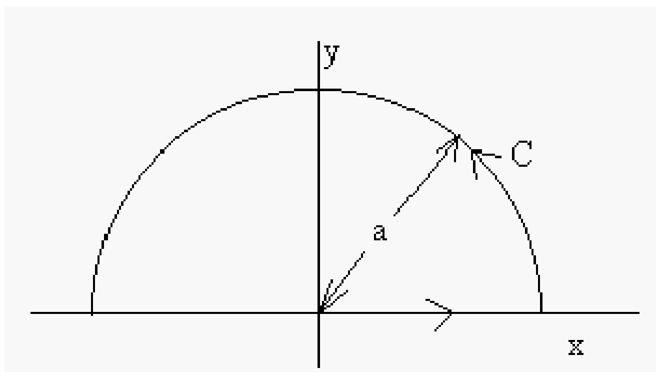
Or, doing the line integral:

$$\oint_c \vec{\mathbf{u}} \cdot d\vec{\mathbf{l}} = \int_0^3 x^2 dx + \int_{3,0}^{0,4}\left(x^2\hat{\mathbf{x}} \cdot d\vec{\mathbf{l}} + xy\hat{\mathbf{y}} \cdot d\vec{\mathbf{l}}\right) + \int_4^0 (0)dy$$
$$= \int_0^3 x^2 dx + \int_3^0 x^2 dx + \int_0^4 xy\,dy$$
$$= \int_3^0 x\left(4 - \frac{4}{3}x\right)\left(-\frac{4}{3}dx\right)$$
$$= \frac{16}{3}\left(\frac{x^2}{2} - \frac{x^3}{9}\right)\Big|_0^3 = \frac{16}{3}\left(\frac{9}{2} - 3\right) = 8$$

The same result, as we expected, but the calculation is more difficult.

(**d**) C is a semicircle of radius a with the flat side along the $x-$axis, the center of the circle at the origin, and

$$\vec{\mathbf{u}} = \left(2x - y^3\right)\hat{\mathbf{x}} - \left(3y^2 + x^3\right)\hat{\mathbf{y}}$$



$$\left(\vec{\nabla} \times \vec{\mathbf{u}}\right)_z = \frac{\partial}{\partial x}\left(3y^2 + x^3\right) - \frac{\partial}{\partial y}\left(2x - y^3\right) = 3x^2 + 3y^2 = 3\rho^2$$

Thus the integral is

$$\int_0^\pi \int_0^a 3\rho^3 d\rho d\theta = \frac{3}{4}\pi a^4$$

26. Evaluate the integral

$$\int_S \vec{\mathbf{v}} \cdot d\vec{\mathbf{A}}$$

where (**a**) S is a sphere of radius 2 centered on the origin, and

$$\vec{\mathbf{v}} = x^3\hat{\mathbf{x}} + 3yz^2\hat{\mathbf{y}} + 3y^2 z\hat{\mathbf{z}}$$

We use the divergence theorem:

$$\int_S \vec{\mathbf{v}} \cdot d\vec{\mathbf{A}} = \int_V \vec{\nabla} \cdot \vec{\mathbf{v}}\,dV$$

Here

$$\vec{\nabla}\cdot\vec{v} = \frac{\partial}{\partial x}\left(x^3\right) + \frac{\partial}{\partial y}\left(3yz^2\right) + \frac{\partial}{\partial z}\left(3y^2z\right) = 3\left(x^2 + z^2 + y^2\right) = 3r^2$$

and so

$$\int_S \vec{v}\cdot d\vec{A} = \int_0^2 3r^2 4\pi r^2 dr = 12\pi\frac{r^5}{5}\Big|_0^2 = 12\pi\frac{32}{5} = \frac{384}{5}\pi$$

(**b**) S is a hemisphere of radius 1, with the center of the sphere at the origin, the flat side in the $x-y-$plane, and

$$\vec{v} = x^2yz(\hat{y} + \hat{z})$$

$$\vec{\nabla}\cdot\vec{v} = \frac{\partial}{\partial x}(0) + \frac{\partial}{\partial y}\left(x^2yz\right) + \frac{\partial}{\partial z}\left(x^2yz\right) = x^2(y+z) = r^3\sin^2\theta\cos^2\phi(\sin\theta\sin\phi + \cos\theta)$$

Integrating over the hemisphere, we get:

$$I = \int_0^1\int_0^{\pi/2}\int_0^{2\pi} r^3\sin^2\theta\cos^2\phi(\sin\theta\sin\phi + \cos\theta)r^2\sin\theta\, dr\, d\theta\, d\phi$$

Doing the integral over ϕ first, the first term is zero, and we have:

$$I = \int_0^1\int_0^{\pi/2} r^5\sin^2\theta(\pi\cos\theta)\sin\theta\, dr\, d\theta = \pi\frac{r^6}{6}\Big|_0^1\int_0^1\left(1-\mu^2\right)\mu\, d\mu$$

$$= \frac{\pi}{6}\left(\frac{\mu^2}{2} - \frac{\mu^4}{4}\right)\Big|_0^1 = \frac{\pi}{24}$$

27. Show that the vector

$$\vec{u} = x\hat{x} + y\hat{y} - 2z\hat{z}$$

has zero divergence (it is solenoidal) and zero curl (it is irrotational). Find a scalar function ϕ such that

$$\vec{u} = \vec{\nabla}\phi$$

and a vector $\vec{A}$ such that

$$\vec{u} = \vec{\nabla}\times\vec{A}$$

$$\vec{\nabla}\cdot\vec{u} = 1 + 1 - 2 = 0$$

and

$$\left(\vec{\nabla}\times\vec{u}\right)_x = \frac{\partial}{\partial y}(-2z) - \frac{\partial}{\partial z}(y) = 0$$

and similarly for the other components.

If $\vec{\mathbf{u}} = \vec{\nabla}\phi$, then $\frac{\partial\phi}{\partial x} = x \Rightarrow \phi = \frac{x^2}{2} + f(y,z)$. Similarly, we obtain $\phi = \frac{y^2}{2} + g(x,z)$ and $\phi = -z^2 + h(x,y)$. Thus

$$\phi = \frac{x^2 + y^2}{2} - z^2$$

will do the trick. The curl is a bit harder. We have:

$$\begin{aligned} \frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z} &= x \\ \frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x} &= y \\ \frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y} &= -2z \end{aligned}$$

Then:

$$A_z = xy + \int \frac{\partial A_y}{\partial z} dy + f(x,z)$$

from the first equation, and

$$A_z = -yx + \int \frac{\partial A_x}{\partial z} dx$$

from the second. Thus we can take $\frac{\partial A_y}{\partial z} = -2x$ and $\frac{\partial A_x}{\partial z} = 0$. This gives

$$A_y = -2xz \text{ and } A_x = 0$$

which also satisfies the last equation, and we are done:

$$\vec{\mathbf{A}} = -2xz\hat{\mathbf{y}} - xy\hat{\mathbf{z}}$$

28. Show that the vector

$$\vec{\mathbf{v}} = \frac{\hat{\mathbf{r}}}{r^2}$$

has zero divergence (it is solenoidal) and zero curl (it is irrotational) for $r \neq 0$. Find a scalar function ϕ such that

$$\vec{\mathbf{v}} = \vec{\nabla}\phi$$

and a vector $\vec{\mathbf{A}}$ such that

$$\vec{\mathbf{v}} = \vec{\nabla} \times \vec{\mathbf{A}}$$

In spherical coordinates:

$$\vec{\nabla} \cdot \vec{\mathbf{v}} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{1}{r^2}\right) = 0$$

and

$$\vec{\nabla} \times \vec{\mathbf{v}} = \hat{\boldsymbol{\theta}}\left(\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\frac{1}{r}\right) - \hat{\boldsymbol{\phi}}\left(\frac{1}{r}\frac{\partial}{\partial\theta}\frac{1}{r}\right) = 0$$

Then

$$\vec{\mathbf{v}} = \vec{\nabla}\left(-\frac{1}{r}\right)$$

and $\vec{\mathbf{v}} = \vec{\nabla} \times \vec{\mathbf{A}}$ has only an $r-$component provided that $A_r \equiv 0$, $A_\phi \equiv 0$ and rA_θ is independent of r. Then

$$\vec{\mathbf{v}} = \vec{\nabla} \times \vec{\mathbf{A}} = \frac{1}{r\sin\theta}\left[-\frac{\partial}{\partial\phi}A_\theta\right]\hat{\mathbf{r}} = \frac{\hat{\mathbf{r}}}{r^2}$$

is satisfied provided

$$\vec{\mathbf{A}} = \left(-\frac{\sin\theta}{r}\phi\hat{\boldsymbol{\theta}}\right)$$

satisfies all the constraints.

29. A surface S is bounded by a curve C. The solid angle subtended by the surface S at a point P, where P is in the vicinity of but not **on** the curve, is given by

$$\Omega = \int_S \frac{da_\perp}{R^2}$$

Here $da_\perp$ is an element of area of the loop projected perpendicular to the vector $\vec{\mathbf{R}} = \vec{\mathbf{x}} - \vec{\mathbf{x}}'$, $\vec{\mathbf{x}}$ is the position vector of the point P with respect to some chosen origin O, and $\vec{\mathbf{x}}'$ is a vector that labels an arbitrary point on the surface or the curve. Now let the curve be rigidly displaced by a small amount $d\vec{s}$. Express the resulting change in solid angle $d\Omega$ as an integral around the curve. Hence show that $\vec{\nabla}\Omega = -\vec{\nabla} \times \oint \frac{d\vec{l}}{R}$.

The solid angle subtended at P by an area element da is

$$d\Omega = \vec{\nabla}\Omega \cdot d\vec{s} = \frac{da_\perp}{R^2} = \left(\frac{d\vec{l} \times d\vec{s}}{R^2}\right) \cdot \hat{\mathbf{R}}$$

where $da_\perp$ is the element of surface area projected perpendicular to the vector $\mathbf{r}$ from the origin to that element. The total change in solid angle due to the displacement of the loop is thus

$$\begin{aligned} \Delta\Omega &= \oint d\Omega = \oint\left(\frac{d\vec{l} \times d\vec{s}}{R^2}\right) \cdot \hat{\mathbf{r}} = \oint\left(\frac{\hat{\mathbf{r}} \times d\vec{l}}{R^2}\right) \cdot d\vec{s} \\ &= -\left(\oint \vec{\nabla}\frac{1}{R} \times d\vec{l}\right) \cdot d\vec{s} \end{aligned}$$

and so

$$\begin{aligned} \vec{\nabla}\Omega &= -\left(\oint \vec{\nabla}\frac{1}{R} \times d\vec{l}\right) \\ &= -\vec{\nabla} \times \oint \frac{1}{R}d\vec{l} \end{aligned}$$

30. Prove the theorems **(a)**

$$\int_V \vec{\nabla}\Phi dV = \oint_S \Phi \hat{\mathbf{n}} dA$$

We begin by proving the result for a differential cube. Start with the right hand side:

$$\oint_S \Phi \hat{\mathbf{n}} dA = (\Phi(x+dx)dydz - \Phi(x)dydz)\hat{\mathbf{x}} + (\Phi(y+dy)dxdz - \Phi(y)dxdz)\hat{\mathbf{y}}$$
$$+(\Phi(z+dz)dxdy - \Phi(z)dxdy)\hat{\mathbf{z}}$$
$$= \left(\frac{\partial\Phi}{\partial x}\hat{\mathbf{x}} + \frac{\partial\Phi}{\partial y}\hat{\mathbf{y}} + \frac{\partial\Phi}{\partial z}\hat{\mathbf{z}}\right)dxdydz = \vec{\nabla}\Phi dV$$

and since the result is true for one differential cube, and we can make up an arbitrary volume from differential cubes as in the proof of the divergence theorem (Chapter 1 §1.2.3), it is true in general.

b. We use the same method:

$$\int \vec{\nabla} \times \vec{\mathbf{u}} dV = \oint_S (\hat{\mathbf{n}} \times \vec{\mathbf{u}}) dA$$

On the right hand side, the first pair of faces gives:

$$\oint_S (\hat{\mathbf{n}} \times \vec{\mathbf{u}}) dA = (\hat{\mathbf{x}} \times \vec{\mathbf{u}}(x+dx)dydz - \hat{\mathbf{x}} \times \vec{\mathbf{u}}(x)dydz)$$
$$= \hat{\mathbf{x}} \times \frac{\partial \vec{\mathbf{u}}}{\partial x} dxdydz = \left(-\frac{\partial u_3}{\partial x}\hat{\mathbf{y}} + \frac{\partial u_2}{\partial x}\hat{\mathbf{z}}\right)dxdydz$$

Including all the 6 sides we have:

$$\oint_S (\hat{\mathbf{n}} \times \vec{\mathbf{u}}) dA = \left(\hat{\mathbf{x}} \times \frac{\partial \vec{\mathbf{u}}}{\partial x} + \hat{\mathbf{y}} \times \frac{\partial \vec{\mathbf{u}}}{\partial y} + \hat{\mathbf{z}} \times \frac{\partial \vec{\mathbf{u}}}{\partial z}\right)dxdydz$$
$$= \left(-\frac{\partial u_3}{\partial x}\hat{\mathbf{y}} + \frac{\partial u_2}{\partial x}\hat{\mathbf{z}} + \frac{\partial u_3}{\partial y}\hat{\mathbf{x}} - \frac{\partial u_1}{\partial y}\hat{\mathbf{z}} - \hat{\mathbf{x}}\frac{\partial u_2}{\partial z} + \hat{\mathbf{y}}\frac{\partial u_1}{\partial z}\right)dxdydz$$
$$= \left(\left(\frac{\partial u_1}{\partial z} - \frac{\partial u_3}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y}\right)\hat{\mathbf{z}} + \hat{\mathbf{x}}\left(\frac{\partial u_3}{\partial y} - \frac{\partial u_2}{\partial z}\right)\right)dxdydz$$
$$= \vec{\nabla} \times \vec{\mathbf{u}} dV$$

and since the result is true for one differential cube, and we can make up an arbitrary volume from differential cubes as in the proof of the divergence theorem (Chapter 1 §1.2.3), it is true in general.

31. Prove that (**a**)

$$\oint_C \Phi d\vec{\mathbf{l}} = \int_S \hat{\mathbf{n}} \times \vec{\nabla} \Phi dA$$

We use the general technique used for Stokes' theorem in the chapter. We integrate around a differential rectangle in the $x-y-$plane. Then

$$\begin{aligned}\oint_{\text{diff rect}} \Phi d\vec{\mathbf{l}} &= \Phi(x,y)dx\hat{\mathbf{x}} + \Phi(x+dx,y)dy\hat{\mathbf{y}} + \Phi(x,y+dy)dx(-\hat{\mathbf{x}}) + \Phi(x,y)dy(-\hat{\mathbf{y}}) \\ &= -\Phi\{(x,y+dy) - \Phi(x,y)\}dx\hat{\mathbf{x}} + \{\Phi(x+dx,y) - \Phi(x,y)\}dy\hat{\mathbf{y}} \\ &= \left\{-\frac{\partial \Phi}{\partial y}\hat{\mathbf{x}} + \frac{\partial \Phi}{\partial x}\hat{\mathbf{y}}\right\}dxdy = \hat{\mathbf{z}} \times \nabla \Phi dA\end{aligned}$$

But for our curve and the area spanning it, $\hat{\mathbf{n}} = \hat{\mathbf{z}}$, so

$$\oint_{\text{diff rect}} \Phi d\vec{\mathbf{l}} = \int \hat{\mathbf{n}} \times \nabla \Phi dA$$

Now we sum up over all the differential rectangles making up our arbitrary curve, to show

$$\oint_C \Phi d\vec{\mathbf{l}} = \int_S \hat{\mathbf{n}} \times \vec{\nabla} \Phi dA$$

as required.

(**b**)

$$\int_S \left(\hat{\mathbf{n}} \times \vec{\nabla}\right) \times \vec{\mathbf{u}} dA = \oint_C d\vec{\mathbf{l}} \times \vec{\mathbf{u}}$$

Again we begin with a differential rectangle in the $x-y$ plane.

$$\begin{aligned}\oint_{\text{diff rect}} d\vec{\mathbf{l}} \times \vec{\mathbf{u}} &= -u_y(x,y)dx\hat{\mathbf{z}} + u_z(x,y)dx\hat{\mathbf{y}} + u_x(x+dx,y)dy\hat{\mathbf{z}} + u_z(x+dx,y)dy\hat{\mathbf{x}} \\ &\quad + u_y(x,y+dy)dx\hat{\mathbf{z}} - u_z(x,y+dy)dx\hat{\mathbf{y}} - u_x(x,y)dy\hat{\mathbf{z}} - u_z(x,y)dy\hat{\mathbf{x}} \\ &= dxdy\left\{\frac{\partial u_y}{\partial y}\hat{\mathbf{z}} - \frac{\partial u_z}{\partial x}\hat{\mathbf{x}} + \frac{\partial u_x}{\partial x}\hat{\mathbf{z}} - \frac{\partial u_z}{\partial y}\hat{\mathbf{y}} + \frac{\partial u_z}{\partial z}\hat{\mathbf{z}} - \frac{\partial u_z}{\partial z}\hat{\mathbf{z}}\right\} \\ &= dA\left\{\left(\vec{\nabla} \cdot \vec{\mathbf{u}}\right)\hat{\mathbf{n}} - \vec{\nabla}(\vec{\mathbf{u}} \cdot \hat{\mathbf{n}})\right\} \\ &= \int_S \left(\hat{\mathbf{n}} \times \vec{\nabla}\right) \times \vec{\mathbf{u}} dA\end{aligned}$$

32. Derive the expressions for gradient, divergence, curl and the Laplacian in spherical coordinates.

The line element in spherical coordinates (equation 1.7) gives us the metric coefficients:

$$
\begin{aligned}
h_1 &= 1 \\
h_2 &= r \\
h_3 &= r\sin\theta
\end{aligned}
$$

Thus we have:

$$\vec{\nabla}\Phi = \frac{\partial\Phi}{\partial r} + \frac{1}{r}\frac{\partial\Phi}{\partial\theta} + \frac{1}{r\sin\theta}\frac{\partial\Phi}{\partial\phi}$$

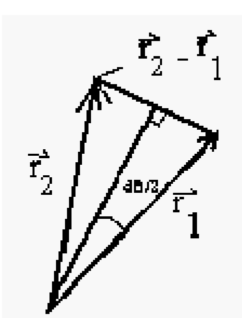
$$
\begin{aligned}
\vec{\nabla}\cdot\vec{u} &= \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}\left(r^2\sin\theta u_r\right) + \frac{\partial}{\partial\theta}(r\sin\theta u_\theta) + \frac{\partial}{\partial\phi}(r u_\phi)\right] \\
&= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 u_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta u_\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(u_\phi)
\end{aligned}
$$

$$
\begin{aligned}
\vec{\nabla}\times\vec{u} &= \frac{1}{r^2\sin\theta}\left(\frac{\partial}{\partial\theta}r\sin\theta u_\phi - \frac{\partial}{\partial\phi}r u_\theta\right)\hat{r} + \frac{1}{r\sin\theta}\left(\frac{\partial}{\partial\phi}u_r - \frac{\partial}{\partial r}r\sin\theta u_\phi\right) \\
&\quad + \frac{1}{r}\left(\frac{\partial}{\partial r}r u_\theta - \frac{\partial}{\partial\theta}u_r\right)\hat{\phi} \\
&= \frac{1}{r\sin\theta}\left(\frac{\partial}{\partial\theta}\sin\theta u_\phi - \frac{\partial}{\partial\phi}u_\theta\right)\hat{r} + \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}u_r - \frac{\partial}{\partial r}r u_\phi\right)\hat{\theta} \\
&\quad + \frac{1}{r}\left(\frac{\partial}{\partial r}r u_\theta - \frac{\partial}{\partial\theta}u_r\right)\hat{\phi}
\end{aligned}
$$

and finally:

$$
\begin{aligned}
\nabla^2\Phi &= \vec{\nabla}\cdot\left(\vec{\nabla}\Phi\right) \\
&= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{1}{r}\frac{\partial\Phi}{\partial\theta}\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\left(\frac{1}{r\sin\theta}\frac{\partial\Phi}{\partial\phi}\right) \\
&= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\Phi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\Phi}{\partial\phi^2}
\end{aligned}
$$

33. In polar coordinates in a plane the unit vectors $\hat{r}$ and $\hat{\theta}$ are functions of position. Draw a diagram showing the vectors $\hat{r}$ at two neighboring points with angular coordinates θ and $\theta + d\theta$. Use your diagram to find the difference $\Delta\vec{r}$ and hence find the derivative $\partial\hat{r}/\partial\theta$.



Problem 1.33

$$d\hat{\mathbf{r}} = \hat{\mathbf{r}}_2 - \hat{\mathbf{r}}_1$$

has magnitude

$$dr = 2|\hat{\mathbf{r}}|\sin\frac{d\theta}{2} = d\theta$$

and in the limit $d\theta \to 0,$ it is perpendicular to $\hat{\mathbf{r}},$ so

$$d\hat{\mathbf{r}} = d\theta\hat{\boldsymbol{\theta}}$$

and thus

$$\frac{\partial\hat{\mathbf{r}}}{\partial\theta} = \hat{\boldsymbol{\theta}}$$

34. The vector operator $\vec{\mathbf{L}} = \frac{1}{i}$ appears in physics as the angular momentum operator. (Here $i = \sqrt{-1}$ and $\mathbf{r}$ is the position vector.) Prove the identity:

$$\vec{\nabla}(\vec{\mathbf{r}}\cdot\vec{\mathbf{u}}) = \vec{\mathbf{u}} + \vec{\mathbf{r}}(\vec{\nabla}\cdot\vec{\mathbf{u}}) + i(\vec{\mathbf{L}}\times\vec{\mathbf{u}})$$

for an arbitrary vector $\vec{\mathbf{u}}.$

Begin with the result of problem 21:

$$\vec{\nabla}(\vec{\mathbf{r}}\cdot\vec{\mathbf{u}}) = (\vec{\mathbf{r}}\cdot\vec{\nabla})\vec{\mathbf{u}} + (\vec{\mathbf{u}}\cdot\vec{\nabla})\vec{\mathbf{r}} + \vec{\mathbf{r}}\times(\vec{\nabla}\times\vec{\mathbf{u}}) + \vec{\mathbf{u}}\times(\vec{\nabla}\times\vec{\mathbf{r}})$$

Working on these terms one at a time:

$$\vec{\nabla}\times\vec{\mathbf{r}} = \left(\frac{\partial z}{\partial y} - \frac{\partial y}{\partial z}\right)\hat{\mathbf{x}} \text{ plus two similar terms } \equiv 0$$

and

$$(\vec{\mathbf{u}}\cdot\vec{\nabla})\vec{\mathbf{r}} = \left(u_x\frac{\partial}{\partial x} + u_y\frac{\partial}{\partial y} + u_z\frac{\partial}{\partial z}\right)(x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}})$$
$$= u_x\hat{\mathbf{x}} + u_y\hat{\mathbf{y}} + u_z\hat{\mathbf{z}} = \vec{\mathbf{u}}$$

Now we are left with

$$\vec{\nabla}(\vec{\mathbf{r}}\cdot\vec{\mathbf{u}}) = (\vec{\mathbf{r}}\cdot\vec{\nabla})\vec{\mathbf{u}} + \vec{\mathbf{u}} + \vec{\mathbf{r}}\times(\vec{\nabla}\times\vec{\mathbf{u}})$$

Now look at

$$i(\vec{\mathbf{L}}\times\vec{\mathbf{u}}) = (\vec{\mathbf{r}}\times\vec{\nabla})\times\vec{\mathbf{u}}$$

The ith component is

$$\varepsilon_{ijk}(\varepsilon_{jpq}r_p\partial_q)u_k = \varepsilon_{jki}\varepsilon_{jpq}r_p\partial_q u_k = (\delta_{kp}\delta_{iq} - \delta_{kq}\delta_{ip})r_p\partial_q u_k$$
$$i(\vec{\mathbf{L}}\times\vec{\mathbf{u}})_i = r_k\partial_i u_k - r_i\partial_k u_k = r_k\partial_i u_k - r_i(\vec{\nabla}\cdot\vec{\mathbf{u}})$$

while

$$\{\vec{\mathbf{r}}\times(\vec{\nabla}\times\vec{\mathbf{u}})\}_i = \varepsilon_{ijk}r_j(\varepsilon_{kpq}\partial_p u_q) = \varepsilon_{kij}\varepsilon_{kpq}r_j\partial_p u_q = (\delta_{ip}\delta_{jq} - \delta_{iq}\delta_{jp})r_j\partial_p u_q$$
$$= r_j\partial_i u_j - r_j\partial_j u_i = r_j\partial_i u_j - (\vec{\mathbf{r}}\cdot\vec{\nabla})u_i$$

Substituting into our result (1.1) above:

$$\left[\vec{\nabla}(\vec{r}\cdot\vec{u})\right]_i = (\vec{r}\cdot\vec{\nabla})u_i + u_i + r_j\partial_i u_j - (\vec{r}\cdot\vec{\nabla})u_i$$

Using equation (1.2) to evaluate $r_j\partial_i u_j$, we have

$$\vec{\nabla}(\vec{r}\cdot\vec{u}) = \vec{u} + \vec{r}(\vec{\nabla}\cdot\vec{u}) + i(\vec{L}\times\vec{u})$$

as required.

35. Can you express the vector $\vec{a} = (1,2,3)$ as a linear combination of the vectors $\vec{u}_1 = (1,1,1)$, $\vec{u}_2 = (1,0,-1)$ and $\vec{u}_3 = (2,1,0)$? Can you express the vector $\vec{b} = (1,3,2)$ as a linear combination of the vectors $\vec{u}_1$, $\vec{u}_2$ and $\vec{u}_3$? Explain your answers geometrically.

Let

$$\begin{aligned}\vec{a} &= \alpha\vec{u}_1 + \beta\vec{u}_2 + \gamma\vec{u}_3\\ (1,2,3) &= (\alpha+\beta+2\gamma, \alpha+\gamma, \alpha-\beta)\end{aligned}$$

Thus we have the three equations:

$$\begin{aligned}1 &= \alpha+\beta+2\gamma\\ 2 &= \alpha+\gamma\\ 3 &= \alpha-\beta\end{aligned}$$

From the third equation

$$\beta = \alpha - 3$$

and from the second:

$$\gamma = 2 - \alpha$$

and so from the first:

$$1 = \alpha + (\alpha-3) + 2(2-\alpha) = 1$$

which is true no matter what the value of α. Thus we can find a solution for any α. For example, with $\alpha = 1$:

$$\vec{a} = \vec{u}_1 - 2\vec{u}_2 + \vec{u}_3$$

For the vector $\vec{b}$ we would have:

$$\begin{aligned}1 &= \alpha+\beta+2\gamma\\ 3 &= \alpha+\gamma\\ 2 &= \alpha-\beta\end{aligned}$$

or

$$1 = \alpha + (\alpha-2) + 2(3-\alpha) = 4$$

which cannot be true for any value of α. Thus no combination of the three $\vec{u}_k$ can equal $\vec{b}$.

Geometrically, the three $\vec{u}$ vectors all lie in a single plane, and $\vec{a}$ lies in the same plane. But $\vec{b}$ lies out of the plane. Note that the cross products:

$$\begin{pmatrix} 1, & 1, & 1 \end{pmatrix} \times \begin{pmatrix} 1, & 2, & 3 \end{pmatrix} = \begin{pmatrix} 1, & -2, & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1, & 1, & 1 \end{pmatrix} \times \begin{pmatrix} 1, & 0, & -1 \end{pmatrix} = \begin{pmatrix} -1, & 2, & -1 \end{pmatrix}$$

$$\begin{pmatrix} 1, & 1, & 1 \end{pmatrix} \times \begin{pmatrix} 2, & 1, & 0 \end{pmatrix} = \begin{pmatrix} -1, & 2, & -1 \end{pmatrix}$$

are all multiples of the same vector, indicating that all four vectors are coplanar. However, $\begin{pmatrix} 1, & 1, & 1 \end{pmatrix} \times \begin{pmatrix} 1, & 3, & 2 \end{pmatrix} = \begin{pmatrix} -1, & -1, & 2 \end{pmatrix}$ is not a multiple of $(1, -2, 1)$, indicating that $\vec{b}$ lies out of that plane.

36. Show that an antisymmetric 3×3 matrix has only three independent elements. How many independent elements does a symmetric 3×3 matrix have? Extend these results to an $N \times N$ matrix.

If $a_{ij} = -a_{ji}$, then $a_{ii} = -a_{ii}$ and so all the diagonal elements are zero. There are three elements above the diagonal. The elements below the diagonal are the negative of these three, which are the three independent elements.

A symmetric matrix can have non-zero elements along the diagonal. There are only three independent off-diagonal elements, giving a total of 6 independent elements.

An $N \times N$ matrix has N elements along the diagonal, so an antisymmetric matrix has

$$\left(N^2 - N\right)/2 = N(N-1)/2$$

independent elements. A symmetric matrix has

$$\left(N^2 - N\right)/2 + N = \left(N^2 + N\right)/2 = N(N+1)/2$$

independent elements.

37. Show that if any two rows of a matrix are equal, its determinant is zero.

To demonstrate the result for a 3×3 matrix, we form the determinant by taking the cofactors of the elements in the non-repeated row. Then the cofactors are the determinants of 2×2 matrices of the form $\begin{pmatrix} a & b \\ a & b \end{pmatrix}$. The determinant equals $ab - ab = 0.$ If each cofactor is zero, then the

determinant is zero. For a $N \times N$ matrix, we can always reduce to 3×3 using the Laplace development, and those determinants are zero as we have just shown.

38. Prove that a matrix with one row of zeros has a determinant equal to zero. Also show that if a matrix is multiplied by a constant $c,$ its determinant is multiplied by $c^N.$

Use the Laplace development, with the row of zeros as the row of chosen elements, and the result follows immediately.

Since each product in equation (1.71) in the text has three factors, the result is clearly true for a 3×3 matrix. But then, from the Laplace development, each product in a 4×4 determinant is one factor times a 3×3 determinant, and so is c^4 times the original. Continuing in this way, we obtain the general result.

39. Prove that a matrix and its transpose have the same determinant.

Using equation 1.72 in the text (first part)

$$\det \mathbb{A} = \sum_i a_{ij} A_{ij}$$

Now if $b_{ij} = a_{ji}$ is the transpose of $\mathbb{A}$, then

$$\det \mathbb{A} = \sum_i b_{ji} B_{ji} = \det \mathbb{B}$$

by the second part of equation 1.72.

40. Prove that the trace of a matrix is invariant under change of basis, that is,

$$Tr(\mathbb{A}') = Tr(\mathbb{C}\mathbb{A}\mathbb{C}^{-1}) = Tr(\mathbb{A})$$

$$Tr(\mathbb{C}\mathbb{A}\mathbb{C}^{-1}) = \sum_{i,j,k} c_{ij} a_{jk} (\mathbb{C}^{-1})_{ki} = \sum_{i,j,k} (\mathbb{C}^{-1})_{ki} c_{ij} a_{jk}$$
$$= \sum_{j,k} \delta_{kj} a_{jk} = Tr(\mathbb{A})$$

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41. Show that the determinant of a matrix is invariant under change of basis, i.e. det $(\mathbb{A}') = \det(\mathbb{A})$. Hence show that the determinant of a real, symmetric matrix equals the product of its eigenvalues.

$$\det(\mathbb{A}') = \det\left(\mathbb{C}\mathbb{A}\mathbb{C}^{-1}\right) = \det(\mathbb{C})\det(\mathbb{A})\det\left(\mathbb{C}^{-1}\right)$$
$$= \det(\mathbb{C})\det(\mathbb{A})\frac{1}{\det(\mathbb{C})} = \det(\mathbb{A})$$

For a diagonalized matrix,

$$\det(\mathbb{A}') = \lambda_1\lambda_2\lambda_3$$

QED.

42. If the product of two matrices is zero, it is not necessary that either one be zero. In particular, show that a 2×2 matrix whose square is zero may be written in terms of two parameters a and b, and find the general form of the matrix.

$$\mathbb{A}^2 = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \mathbf{0}$$
$$\begin{pmatrix} a^2 + bc & ab + bd \\ ca + dc & bc + d^2 \end{pmatrix} = \mathbf{0}$$

Thus either $b = 0$ and $c = 0$, or $d = -a$. If b and c are both zero, then a and d are also zero, and $\mathbb{A} = 0$ But if $d = -a$, then also $bc = -a^2$. Thus the matrix may be expressed in terms of the two parameters a and b:

$$\mathbb{A} = \begin{pmatrix} a & b \\ -\frac{a^2}{b} & -a \end{pmatrix}$$

43. If the product of the matrix $\mathbb{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and another non-zero matrix $\mathbb{B}$ is zero, find the elements of $\mathbb{B}$.

You may find it necessary to impose some conditions on matrix $\mathbb{A}$. If so, state what they are.

We know that det$(\mathbb{B}) = 0$, so if $\mathbb{B} = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$, then $eh - fg = 0$, so $h = fg/e$. Thus the product is:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} e & f \\ g & fg/e \end{pmatrix} = 0$$
$$\begin{pmatrix} ae + bg & f\frac{ae+bg}{e} \\ ce + dg & f\frac{ce+dg}{e} \end{pmatrix} = 0$$

Thus

$$g = -ae/b$$

$$\begin{pmatrix} 0 & 0 \\ e\frac{bc-ad}{b} & f\left(c - \frac{ad}{b}\right) \end{pmatrix} = 0$$

which can be satisfied if $e = f = 0,$ in which case matrix $\mathbb{B} = 0$, or $ad - bc = \det(\mathbb{A}) = 0.$ In the latter case, matrix $\mathbb{B}$ is specified in terms of arbitrary values e and f as

$$\mathbb{B} = \begin{pmatrix} e & f \\ -\frac{ae}{b} & -\frac{fa}{b} \end{pmatrix} = \frac{1}{b}\begin{pmatrix} eb & fb \\ -ea & -fa \end{pmatrix}$$

44. Diagonalize the matrix: $\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

We solve the equation

$$\det\begin{pmatrix} 1-\lambda & 1 & 1 \\ 1 & -\lambda & 0 \\ 1 & 0 & -\lambda \end{pmatrix} = 0$$

$$\lambda^2(1-\lambda) + 2\lambda = 0 = \lambda\left(-\lambda^2 + \lambda + 2\right) = -\lambda(\lambda+1)(\lambda-2)$$

Thus the eigenvalues are: $0, 2, -1.$ The corresponding eigenvectors satisfy the equation

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} u \\ v \\ w \end{pmatrix} = \lambda\begin{pmatrix} u \\ v \\ w \end{pmatrix}$$

For $\lambda = 0,$ we have

$$\begin{aligned} u + v + w &= 0 \\ u &= 0 \end{aligned}$$

and similarly for the other two values. So the eigenvectors are

eigenvectors: $\left\{\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}\right\} \leftrightarrow 0, \left\{\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}\right\} \leftrightarrow -1, \left\{\begin{pmatrix} 2 \\ 1 \\ 1 \end{pmatrix}\right\} \leftrightarrow 2$,

45. Show that a real symmetric matrix with one or more eigenvalues equal to zero has no inverse (it is *singular).*

Since the determinant equals the product of the eigenvalues, (Problem 41), the determinant equals zero, and thus the matrix is singular.

46. Diagonalize the matrix $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, and find the eigenvectors. Are the eigenvectors orthogonal?

$$\det\begin{pmatrix} 1-\lambda & 2 \\ 3 & 4-\lambda \end{pmatrix} = -2 - 5\lambda + \lambda^2 = 0$$

The eigenvalues are: $\frac{5}{2} + \frac{1}{2}\sqrt{33}$, $\frac{5}{2} - \frac{1}{2}\sqrt{33}$ and we find the eigenvectors from the equation

$$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} u \\ v \end{pmatrix} = \lambda\begin{pmatrix} u \\ v \end{pmatrix}$$

Thus

$$\begin{aligned} u + 2v &= \lambda u \\ 3u + 4v &= \lambda v \end{aligned}$$

So

$$v = \frac{\lambda - 1}{2}u$$

and then

$$\begin{aligned} 3u &= (\lambda - 4)\frac{\lambda - 1}{2}u = \left(-\frac{3}{2} \pm \frac{1}{2}\sqrt{33}\right)\left(\frac{3}{2} \pm \frac{1}{2}\sqrt{33}\right)\frac{u}{2} \\ &= -(-6)\frac{u}{2} = 3u \end{aligned}$$

Thus we may pick any value for u. Choose $u = 1$. Then

$$v = \left(\frac{3}{2} \pm \frac{1}{2}\sqrt{33}\right)\frac{1}{2} = \frac{1}{4}\left(3 \pm \sqrt{33}\right)$$

and the eigenvectors are: $\begin{pmatrix} 1 \\ \frac{1}{4}\left(3 \pm \sqrt{33}\right) \end{pmatrix}$

The inner product is

$$\begin{aligned} 1 + \frac{1}{16}\left(3 + \sqrt{33}\right)\left(3 - \sqrt{33}\right) &= 1 + \frac{1}{16}(-24) \\ &= -\frac{1}{2} \end{aligned}$$

Since the product is not zero, the vectors are not orthogonal. Since the matrix is not symmetric, the eigenvectors need not be orthogonal.

47. What condition must be imposed on the matrix $\mathbb{A}$ in order that $\mathbb{AB} = \mathbb{AC}$ with $\mathbb{B} \neq \mathbb{C}$. If $\mathbb{B} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ and $\mathbb{C} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$, find a matrix $\mathbb{A}$ such that $\mathbb{AB} = \mathbb{AC}$.

We must have $\det(\mathbb{A}) = 0$. So write $\mathbb{A} = \begin{pmatrix} a & b \\ c & bc/a \end{pmatrix}$. Then

$$\mathbb{AB} = \begin{pmatrix} a & b \\ c & bc/a \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} a & 2b \\ c & 2b\frac{c}{a} \end{pmatrix}$$

and

$$\mathbb{AC} = \begin{pmatrix} a & b \\ c & bc/a \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & a+b \\ c & c\frac{a+b}{a} \end{pmatrix}$$

We can make the two answers equal if $a = b$. Then

$$\mathbb{A} = \begin{pmatrix} a & a \\ c & c \end{pmatrix}$$

and

$$\mathbb{AB} = \mathbb{AC} = \begin{pmatrix} a & 2a \\ c & 2c \end{pmatrix}$$

48. Show that if $\mathbb{A}$ is a real symmetric matrix and $\mathbb{C}$ is orthogonal, then $\mathbb{A}' = \mathbb{CAC}^{-1}$ is also symmetric.

If a matrix is orthogonal, then its inverse equals its transpose, so $\mathbb{C}^{-1} = \mathbb{C}^T$. Then:

$$(\mathbb{A}')^T = (\mathbb{CAC}^T)^T = \mathbb{CA}^T\mathbb{C}^T = \mathbb{CAC}^T = \mathbb{A}'$$

and so $\mathbb{A}'$ is symmetric if $\mathbb{A}$ is.

49. Show that $\mathbb{AB} = \mathbb{BA}$ if both $\mathbb{A}$ and $\mathbb{B}$ are diagonal matrices.

If $\mathbb{A}$ and $\mathbb{B}$ are both diagonal, then

$$\mathbb{AB} = \begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix}\begin{pmatrix} b_{11} & 0 & 0 \\ 0 & b_{22} & 0 \\ 0 & 0 & b_{33} \end{pmatrix} = \begin{pmatrix} a_{11}b_{11} & 0 & 0 \\ 0 & a_{22}b_{22} & 0 \\ 0 & 0 & a_{33}b_{33} \end{pmatrix}$$

is also diagonal. Then

$$\mathbb{AB} = \begin{pmatrix} b_{11}a_{11} & 0 & 0 \\ 0 & b_{22}a_{22} & 0 \\ 0 & 0 & b_{33}a_{33} \end{pmatrix} = \mathbb{BA}$$

and the matrices commute.

50. Let $\vec{\mathbf{a}} * \vec{\mathbf{b}} = \sum a_i b_i.$ Now let $b_i' = c_{ij} b_j,$ similarly for $a,$ and compute the product

$$\begin{aligned} \vec{\mathbf{a}} \clubsuit \vec{\mathbf{b}} &= \sum a_i' b_i' \\ &= \sum_i \sum_j \sum_k c_{ik} c_{ij} a_j b_k \\ &= \sum_i \sum_j \sum_k \mathbb{C}^T_{ki} c_{ij} a_j b_k \end{aligned}$$

Now if the matrix $\mathbb{C}$ is orthogonal, then $\mathbb{C}^T = \mathbb{C}^{-1},$ and so in this case

$$\vec{\mathbf{a}} \clubsuit \vec{\mathbf{b}} = \sum_j \sum_k \delta_{kj} a_j b_k = \sum_j a_j b_j = \vec{\mathbf{a}} * \vec{\mathbf{b}}$$

and the inner product is invariant.

51. A quadratic expression of the form $\alpha x^2 + 2\beta xy + \gamma y^2 = 1$ represents a curve in the $x - y$ plane. (a) Write this expression in matrix form. (b) Diagonalize the matrix, and hence identify the form of the curve and find its symmetry axes. Determine how the shape of the curve depends on the values of α, β and $\gamma.$ Draw the curve in the case $\alpha = \beta = 2,$ $\gamma = 3.$

(a)

$$\mathbf{x}^T \mathbb{A} \mathbf{x} = 1$$

where the vector $\mathbf{x}$ has components (x, y) and the matrix

$$\mathbb{A} = \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix}$$

Check:

$$(x,y)\begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = (x,y)\begin{pmatrix} \alpha x + \beta y \\ \beta x + \gamma y \end{pmatrix} = \alpha x^2 + \beta xy + \beta yx + \gamma y^2 = 1$$

Now we diagonalize:

$$\begin{aligned} \begin{vmatrix} \alpha - \lambda & \beta \\ \beta & \gamma - \lambda \end{vmatrix} &= 0 \\ (\alpha - \lambda)(\gamma - \lambda) - \beta^2 &= 0 \\ \alpha\gamma - \lambda(\alpha + \gamma) + \lambda^2 - \beta^2 &= 0 \end{aligned}$$

Thus the eigenvalues are:

$$\lambda = \frac{\alpha + \gamma \pm \sqrt{(\alpha + \gamma)^2 - 4\left(\alpha\gamma - \beta^2\right)}}{2}$$
$$= \frac{\alpha + \gamma \pm \sqrt{(\gamma - \alpha)^2 + 4\beta^2}}{2}$$

The eigenvectors are given by:

$$\begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \lambda\begin{pmatrix} x \\ y \end{pmatrix}$$
$$\begin{pmatrix} \alpha x + \beta y \\ \beta x + \gamma y \end{pmatrix} = \lambda\begin{pmatrix} x \\ y \end{pmatrix}$$

or

$$y = \frac{(\lambda - \alpha)}{\beta}x = \frac{1}{2\beta}\left(\gamma - \alpha \pm \sqrt{(\gamma - \alpha)^2 + 4\beta^2}\right) = \frac{\beta x}{\lambda - \gamma}$$

The new equation is

$$\mathbf{x}'^T \mathbb{A}' \mathbf{x} = 0$$
$$(x,y)\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = (x,y)\begin{pmatrix} \lambda_1 x \\ \lambda_2 y \end{pmatrix} = \lambda_1 x^2 + \lambda_2 y^2 = 1$$

If λ_1 and λ_2 are both positive, the equation is an ellipse. This happens when

$$\sqrt{(\gamma - \alpha)^2 + 4\beta^2} < \alpha + \gamma$$
$$(\gamma - \alpha)^2 + 4\beta^2 < (\alpha + \gamma)^2$$
$$\beta^2 < \alpha\gamma$$

But if $\beta^2 > \alpha\gamma,$ then λ_2 is negative, and the curve is an hyberbola. For the ellipse, the eigenvectors found above give the direction of the major (minus sign in λ) and minor axes.

For the case $\alpha = \beta = 2,$ $\gamma = 3,$ we have $\beta^2 = 4 < \alpha\gamma = 6,$ so the curve is an ellipse.

$\lambda = \frac{5 \pm \sqrt{1+16}}{2}$

$\lambda_1 = \frac{5}{2} + \frac{1}{2}\sqrt{17} = 4.5616$ so $b = \frac{1}{\sqrt{4.5616}} = 0.46821$

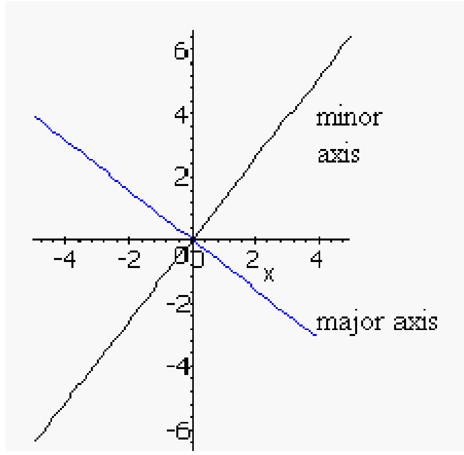
$\lambda_2 = \frac{5}{2} - \frac{1}{2}\sqrt{17} = 0.43845$ so $a = \frac{1}{\sqrt{.43845}} = 1.5102$

The equation of the minor axis is:

$$y = \frac{(\lambda - \alpha)}{\beta}x = \frac{2.56}{2}x = 1.28x$$

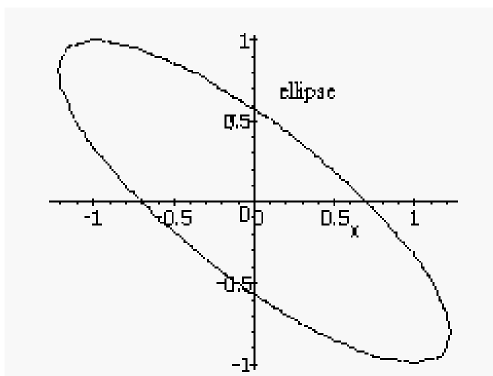
while for the major axis:

$$y = \frac{0.43845 - 2}{2}x = -.78078x$$



The equation of the ellipse is:

$$2x^2 + 4xy + 3y^2 = 1$$



52. Two small objects, each of mass m, are joined by a spring of relaxed length ℓ and spring constant k. Identical springs hold each mass to a wall. The walls are separated by a distance 3ℓ. Write the Lagrangian for the system, find the normal modes and the oscillation frequency for each mode.

Let x_1 and x_2 be the rightward displacement of each object from equilibrium. Then the kinetic energy is

$$K = \frac{1}{2}m\left(\frac{dx_1}{dt}\right)^2 + \frac{1}{2}m\left(\frac{dx_2}{dt}\right)^2$$

and the potential energy is

$$V = \frac{1}{2}k\left(x_1^2 + x_2^2 + (x_2 - x_1)^2\right)$$
$$= \frac{1}{2}k\left(2x_1^2 + 2x_2^2 - 2x_2x_1\right)$$

Thus the Lagrangian is:

$$L = K - V = \frac{1}{2}m\left\{\frac{d}{dt}\mathbf{x}^T\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\frac{d}{dt}\mathbf{x} + \frac{k}{m}\mathbf{x}^T\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}\mathbf{x}\right\}$$

Thus the normal mode frequencies are given by the characteristic equation:

$$\begin{vmatrix} 2-\lambda & -1 \\ -1 & 2-\lambda \end{vmatrix} = 0$$

$$3 - 4\lambda + \lambda^2 = 0$$

$$\lambda = 3, 1$$

Thus the frequencies are $\sqrt{k/m}$ and $\sqrt{3k/m}$. The eigenvectors are given by:

$$2x_1 - x_2 = \lambda x_1$$

$$x_2 = (2-\lambda)x_1 = -x_1 \text{ or } x_1$$

Thus the two objects either move together, or exactly opposite each other. When moving together, the middle spring is not stretched or compressed. The outer two springs both pull or push the system in the same direction. The frequency is $\sqrt{k/m}$, the same as for a single object-on-spring system. When they move opposite each other, all three springs are distorted and each exerts an equal force on the system. The frequency is thus $\sqrt{3k/m}$.

Finally we check the transformation matrix:

$$\mathbb{T} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

As expected, the matrix is orthogonal.The transformed potential energy matrix is:

$$V' = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}$$

53. Find the normal modes of a jointed pendulum system. Two point objects, each of mass m, are linked by stiff but massless rods each of length l. The upper rod is attached to a pivot. The system is in equilibrium when both rods hang vertically below the pivot. The diagram shows the system when displaced from equilibrium.

The system is most easily analyzed using Lagrangian methods. The kinetic energy is:

$$K = \frac{1}{2}m\left(l\frac{d\theta_1}{dt}\right)^2 + \frac{1}{2}ml^2\left(\left(\frac{d\theta_1}{dt}\right)^2 + \left(\frac{d\theta_2}{dt}\right)^2 + 2\cos(\theta_1 - \theta_2)\frac{d\theta_1}{dt}\frac{d\theta_2}{dt}\right)$$

Taking the reference level at the pivot, the potential energy is:

$$V = -mgl\cos\theta_1 - mg(l\cos\theta_1 + l\cos\theta_2)$$

Now if the displacement from equilibrium remains small, $\theta_1 \ll 1$ and $\theta_2 \ll 1$, we can approximate the cosines in the expression for V by Taylor series, truncated after the second term. Then:

$$V \simeq -mgl\left(1 - \frac{\theta_1^2}{2}\right) - mgl\left\{\left(1 - \frac{\theta_1^2}{2}\right) + \left(1 - \frac{\theta_2^2}{2}\right)\right\}$$

$$= -3gml + \frac{1}{2}gml\left(2\theta_1^2 + \theta_2^2\right)$$

Thus the Lagrangian is:

$$
\begin{aligned}
\mathcal{L} &= K - V \\
&= ml^2\left(\frac{d\theta_1}{dt}\right)^2 + ml^2\frac{d\theta_1}{dt}\frac{d\theta_2}{dt}\left(1 - \frac{(\theta_1-\theta_2)^2}{2}\right) + \frac{m}{2}l^2\left(\frac{d\theta_2}{dt}\right)^2 \\
&\quad + 3gml - \frac{1}{2}gml\left(2\theta_1^2 + \theta_2^2\right) \\
&= ml^2\left\{\left(\frac{d\theta_1}{dt}\right)^2 + \frac{d\theta_1}{dt}\frac{d\theta_2}{dt} + \frac{1}{2}\left(\frac{d\theta_2}{dt}\right)^2 + 3\frac{g}{l} - \frac{1}{2}\frac{g}{l}\left(2\theta_1^2 + \theta_2^2\right)\right\}
\end{aligned}
$$

to 2nd order in small quantities. Lagrange's equations are:

$$
\begin{aligned}
\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot\theta_i} - \frac{\partial \mathcal{L}}{\partial \theta_i} &= 0 \\
2\frac{d^2\theta_1}{dt^2} + \frac{d^2\theta_2}{dt^2} - \frac{g}{l}2\theta_1 &= 0 \\
\frac{d^2\theta_2}{dt^2} + \frac{d^2\theta_1}{dt^2} - \frac{g}{l}\theta_2 &= 0
\end{aligned}
$$

Here the coupling is in the derivative terms: it is called dynamic coupling.

$$
L = ml^2\left\{\begin{pmatrix} \frac{d\theta_1}{dt} & \frac{d\theta_2}{dt} \end{pmatrix}\begin{pmatrix} 1 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}\begin{pmatrix} \frac{d\theta_1}{dt} \\ \frac{d\theta_2}{dt} \end{pmatrix} - \frac{g}{l}\begin{pmatrix} \theta_1 & \theta_2 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}\begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} + 3\frac{g}{l}\right\}
$$

This time we need to simultaneously diagonalize both matrices.

$$
\begin{aligned}
\begin{vmatrix} \frac{g}{l} - \lambda & -\lambda/2 \\ -\lambda/2 & \frac{g}{2l} - \lambda/2 \end{vmatrix} &= 0 \\
\left(2\frac{g}{l} - 2\lambda\right)\left(\frac{g}{l} - \lambda\right) - \lambda^2 &= 0 \\
2\left(\frac{g}{l}\right)^2 - 4\lambda\frac{g}{l} + \lambda^2 &= 0 \\
\lambda &= \left(4 \pm \sqrt{16-8}\right)\frac{g}{2l} \\
&= \left(2 \pm \sqrt{2}\right)\frac{g}{l}
\end{aligned}
$$

The eigenvectors are found next:

$$
\begin{aligned}
\lambda\begin{pmatrix} 1 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} &= \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\
\lambda\begin{pmatrix} x_1 + \frac{1}{2}x_2 \\ \frac{1}{2}x_1 + \frac{1}{2}x_2 \end{pmatrix} &= \begin{pmatrix} x_1 \\ \frac{1}{2}x_2 \end{pmatrix}
\end{aligned}
$$

Thus

$$
x_2 = 2\frac{(1-\lambda)}{\lambda}x_1
$$

So with our λ, we get:

$$\begin{aligned}x_2 &= 2\frac{\left(1-2\mp\sqrt{2}\right)}{2\pm\sqrt{2}}x_1 = 2\frac{\left(-1\mp\sqrt{2}\right)}{2\pm\sqrt{2}}x_1\\ &= 2\frac{\left(-1\mp\sqrt{2}\right)\left(2\mp\sqrt{2}\right)}{\left(2\pm\sqrt{2}\right)\left(2\mp\sqrt{2}\right)}x_1\\ &= -\sqrt{2}x_1,\ \sqrt{2}x_1\end{aligned}$$

So the eigenvectors are:

$$\begin{pmatrix} 1 \\ -\sqrt{2} \end{pmatrix} \text{and} \begin{pmatrix} 1 \\ \sqrt{2} \end{pmatrix}$$

The matrix that effects the transformation is given by

$$\mathbb{T} = \begin{pmatrix} 1 & 1 \\ -\sqrt{2} & \sqrt{2} \end{pmatrix}$$

Then

$$\begin{aligned}\mathbb{K}' = \mathbb{T}^T\mathbb{K}\mathbb{T} &= \begin{pmatrix} 1 & -\sqrt{2} \\ 1 & \sqrt{2} \end{pmatrix}\begin{pmatrix} 1 & 1/2 \\ 1/2 & 1/2 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -\sqrt{2} & \sqrt{2} \end{pmatrix}\\ &\begin{pmatrix} 1 & -\sqrt{2} \\ 1 & \sqrt{2} \end{pmatrix}\begin{pmatrix} 1-\frac{1}{2}\sqrt{2} & 1+\frac{1}{2}\sqrt{2} \\ \frac{1}{2}-\frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2}+\frac{1}{2} \end{pmatrix}\\ &= \begin{pmatrix} 2-\sqrt{2} & 0 \\ 0 & 2+\sqrt{2} \end{pmatrix}\end{aligned}$$

and

$$\begin{aligned}\mathbb{V}' = \mathbb{T}^T\mathbb{V}\mathbb{T} &= \begin{pmatrix} 1 & -\sqrt{2} \\ 1 & \sqrt{2} \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{2} \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -\sqrt{2} & \sqrt{2} \end{pmatrix}\\ &= \begin{pmatrix} 1 & -\sqrt{2} \\ 1 & \sqrt{2} \end{pmatrix}\begin{pmatrix} 1 & 1 \\ -\frac{1}{2}\sqrt{2} & \frac{1}{2}\sqrt{2} \end{pmatrix}\\ &= \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}\end{aligned}$$

Both are diagonal. Notice that the transformation is not orthogonal in this case.

Chapter 2: Complex variables

1. If $z_1 = 5 + 2i$ and $z_2 = 3 - 4i$, find z_1/z_2 and $z_1 \times z_2$.

$$\frac{z_1}{z_2} = \frac{5+2i}{3-4i} = \left(\frac{5+2i}{3-4i}\right)\left(\frac{3+4i}{3+4i}\right) = \frac{15+26i+8i^2}{9+16} = \frac{7}{25} + \frac{26}{25}i$$

$$z_1 z_2 = (5+2i)(3-4i) = 15 - 14i - 8i^2 = 23 - 14i$$

2. Use the polar representation of z to write an expression for z^3 in terms of r and θ. Use your result to express $\cos 3\theta$ and $\sin 3\theta$ in terms of cos θ and $\sin\theta$.

$$z^3 = \left(re^{i\theta}\right)^3 = r^3 e^{3i\theta}$$
$$r^3(\cos\theta + i\sin\theta)^3 = r^3(\cos 3\theta + i\sin 3\theta)$$
$$\cos^3\theta + 3i\cos^2\theta\sin\theta + 3i^2\cos\theta\sin^2\theta + i^3\sin^3\theta = \cos 3\theta + i\sin 3\theta$$

The real part gives:

$$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta = \cos\theta\left(\cos^2\theta - 3\sin^2\theta\right) = \cos\theta\left(4\cos^2\theta - 3\right)$$

and from the imaginary part:

$$\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta = \sin\theta\left(3\cos^2\theta - \sin^2\theta\right) = \sin\theta\left(3 - 4\sin^2\theta\right)$$

3. Prove De Moivre's theorem: $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta.$

$$(\cos\theta + i\sin\theta)^n = \left(e^{i\theta}\right)^n = e^{in\theta} = \cos n\theta + i\sin n\theta.$$

4. The equation $(y - y_0)^2 = 4a(x - x_0)$ describes a parabola. Write this equation in terms of $z = x + iy$. Hint: use the geometric definition of the parabola.

The parabola is a curve such that for any point on the curve the distance from a point is equal to the distance to a line. In this case the point is at $\left(x_0 + a,\ y_0\right)$. The distance from the point is d where:

$$d = \sqrt{(x - x_0 - a)^2 + (y - y_0)^2}$$
$$d^2 = (x - x_0)^2 - 2a(x - x_0) + a^2 + (y - y_0)^2$$

Using the equation of the parabola:

$$d^2 = (x - x_0)^2 + 2a(x - x_0) + a^2 = (x - x_0 + a)^2 = s^2$$

where

$$s = x - (x_0 - a)$$

is the distance from the vertical line at $x = x_0 - a$.

Now we can express these ideas using complex numbers. The distance from the point $z_0 = (x_0 + a,\ y_0)$ is $|z - z_0|$ and the distance from the line is $\mathrm{Re}(z - (x_0 - a))$. Thus the equation we want is:

$$|z - z_0| = \mathrm{Re}(z - (x_0 - a))$$

5. Show that the equation

$$|z - c| + |z - d| = \alpha$$

represents an ellipse in the complex plane, where c and d are complex constants, and α is a real constant. Use geometrical arguments to determine the position of the center of the ellipse and its semi-major and semi-minor axes.

The absolute value $|z - c|$ is the distance between a point P in the Argand diagram described by $z = x + iy$ and the point C described by the number c. Thus the equation describes a curve such that the sum of the distances of P from the points C and D is a constant (α). This is the definition of an ellipse. The points C and D are the foci of the ellipse, so its center is half way between them, at $z = \frac{1}{2}(c + d)$.

When P is at the end of the semi-major axis, then $|z - c| = a(1 - e)$ and $|z - d| = a(1 + e)$ so $\alpha = 2a$, and the semi-major axis is

$$a = \alpha/2$$

Then also

$$|d - c| = 2ae \Rightarrow e = \frac{|d - c|}{\alpha}$$

and so

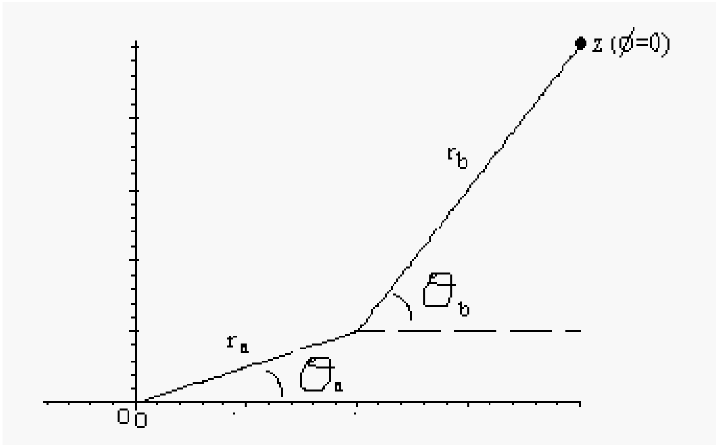
$$b = a\sqrt{1 - e^2} = \frac{\alpha}{2}\sqrt{1 - \frac{|d - c|^2}{\alpha^2}} = \frac{1}{2}\sqrt{\alpha^2 - |d - c|^2}$$

6. Show that the equation

$$z = ae^{i\phi} + be^{-i\phi}$$

represents an ellipse in the complex plane, where a and b are complex constants and ϕ is a real variable. Determine the position of the center of the ellipse and its semi-major and semi-minor axes.

First recall that multiplication by $e^{i\phi}$ corresponds to rotation counter-clockwise by an angle ϕ (Figure 2.3c) . Thus if $a = r_a e^{i\theta_a}$ and $b = r_b e^{i\theta_b}$, then z is represented as follows:



Now as ϕ increases, the lower line rotates counterclockwise, while the upper line rotates clockwise. The two lines align when:

$$\theta_a + \phi = \theta_b - \phi$$

or

$$\phi = \frac{\theta_b - \theta_a}{2}$$

which is the direction of the major axis. The length of the major axis is $r_a + r_b = |a| + |b|.$ The smallest value of $|z|$ occurs when the two ``vectors'' are in opposite directions, i.e.

$$\theta_a + \phi = \theta_b - \phi + \pi$$

or

$$\phi = \frac{\theta_b - \theta_a}{2} + \frac{\pi}{2}$$

Thus the minor axis, of length $||a| - |b||$ is perpendicular to the major axis, as expected.

The angle that the major axis makes with the $x -$ axis is $\theta_a + \left(\frac{\theta_b - \theta_a}{2}\right) = \frac{\theta_b + \theta_a}{2}.$ Let x' and y' be the coordinates of z with axes coincident with the major and minor axes of the ellipse. Then :

$$\begin{aligned} z &= Ae^{i\phi} + Be^{-i\phi} = r_a \exp(i(\theta_a + \phi)) + r_b \exp(i(\theta_b - \phi)) \\ &= \exp\left(i\frac{(\theta_a + \theta_b)}{2}\right)\left[r_a \exp\left(i\left(\frac{(\theta_a - \theta_b)}{2} + \phi\right)\right) + r_b \exp\left(-i\left(\frac{(\theta_a - \theta_b)}{2} + \phi\right)\right)\right] \\ &= \exp\left(i\frac{(\theta_a + \theta_b)}{2}\right) z' \end{aligned}$$

Again we note that the factor $\exp\left(i\frac{(\theta_a + \theta_b)}{2}\right)$ rotates the number in square brackets (z') by an angle $\frac{(\theta_a + \theta_b)}{2}$ counter-clockwise. Thus:

$$z' = r_a e^{i\alpha} + r_b e^{-i\alpha} = (r_a + r_b)\cos\alpha + i(r_a - r_b)\sin\alpha$$

where $\alpha = \frac{(\theta_a - \theta_b)}{2} + \phi.$ Thus we have

$$\frac{(x')^2}{(r_a + r_b)^2} + \frac{(y')^2}{(r_a - r_b)^2} = 1$$

which is the equation of an ellipse with semi-major axis $|a| + |b|$ and semi-minor axis $||a| - |b||.$ The center of the ellipse is at the origin.

7. Find all solutions of the equations (**a**) $z^5 = -1.$

Write z in polar form:

$$r^5 e^{5i\theta} = -1 = 1\exp(i\pi + 2n\pi i)$$

for $0 \le n \le 4.$ Thus the solutions are

$$z = 1\exp\left(i\frac{\pi}{5} + i\frac{2n\pi}{5}\right)$$

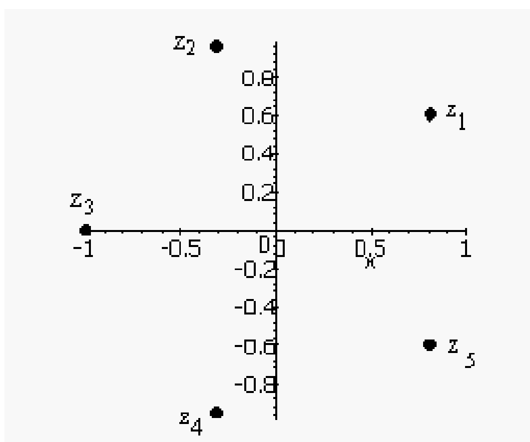
$$z_1 = \cos\left(\frac{\pi}{5}\right) + i\sin\left(\frac{\pi}{5}\right) = 0.80902 + 0.58779i$$

$$z_2 = \cos\left(\frac{\pi}{5} + \frac{2\pi}{5}\right) + i\sin\left(\frac{\pi}{5} + \frac{2\pi}{5}\right) = \cos\left(\frac{3}{5}\pi\right) + i\sin\left(\frac{3\pi}{5}\right)$$
$$= -0.30902 + 0.95106i$$

$$z_3 = \cos(\pi) + i\sin(\pi) = -1$$

$$z_4 = \cos\left(\frac{7}{5}\pi\right) + i\sin\left(\frac{7\pi}{5}\right) = -0.30902 - 0.95106i$$

$$z_5 = \cos\left(\frac{9}{5}\pi\right) + i\sin\left(\frac{9\pi}{5}\right) = 0.80902 - 0.58779i$$



(**b**) $z^4 = 16.$ The roots are $2(1)^{1/4} = 2\left(e^{2\pi i n/4}\right),$ $n = 0, 1, 2, 3$

$$z_n = 2e^{in\pi/2}$$

These points are at the corners of a square: $z_0 = 2$ (on the real axis) $z_1 = 2i$ (on the imaginary axis), $z_2 = -2,$ $z_3 = -2i.$

8. Find all solutions of the equation (**a)** $\cos z = 100.$

Write $z = x + iy$, where x and y are real, and expand the cosine:

$$\cos x \cos iy - \sin x \sin iy = 100$$
$$\cos x \cosh y - i \sin x \sinh y = 100$$

Writing the real and imaginary parts separately, we have:

$$\cos x \cosh y = 100 \quad \text{and} \quad \sin x \sinh y = 0$$

We can solve the second equation with either $y = 0$, or $x = 0$, $n\pi$. But with $y = 0$ the first equation becomes $\cos x = 100$, which has no solutions. (Remember that x is real.) So we must choose $x = n\pi$, where n is any positive or negative integer, or zero. Then:

$$\cos n\pi \cosh y = 100$$

Now the hyperbolic cosine is always positive if y is real, so we must choose n to be even, or zero. Then $\cos n\pi = +1$ and:

$$e^y + e^{-y} = 200$$
$$(e^y)^2 - 200e^y + 1 = 0$$
$$e^y = \frac{200 \pm \sqrt{200^2 - 4}}{2}$$
$$= 100 \pm \sqrt{9999} = 199.99,\ 5.0001 \times 10^{-3}$$

and thus

$$y = \ln(199.99) = 5.2983$$

or

$$y = \ln\left(5.0001 \times 10^{-3}\right) = -5.2983$$

Both values give the same value for the cosh. Then

$$\boxed{z = 2m\pi \pm 5.2983i}$$

(**b**) $\sin z = 6$

$$\sin(x + iy) = \sin x \cos iy + \cos x \sin iy$$
$$= \sin x \cosh y + i \cos x \sinh y = 6$$

Equating real and imaginary parts:

$$\sin x \cosh y = 6$$
$$\cos x \sinh y = 0$$

Clearly $y = 0$ is not a viable solution, so we need

$$\cos x = 0 \Rightarrow x = \left(n + \frac{1}{2}\right)\pi$$

Then

$$\sin\left(n + \frac{1}{2}\right)\pi \cosh y = (-1)^n \cosh y = 6$$

Since cosh y is always positive (y is real) then n must be even, and

$$e^y + e^{-y} = 12$$
$$e^{2y} - 12e^y + 1 = 0$$
$$e^y = \frac{12 \pm \sqrt{144-4}}{2} = 6 \pm \sqrt{35}$$

Thus

$$y = \ln\left(6 \pm \sqrt{35}\right) = 2.4779 \text{ or } -2.4779$$

Thus

$$z = \left(2n + \frac{1}{2}\right)\pi \pm 2.4779i$$

9. Find all solutions of the equation $\cosh z = -5.$

$\cosh z = \cosh(x+iy) = \cosh x \cosh iy + \sinh x \sinh iy = \cosh x \cos y + i \sinh x \sin y = -5.$ The imaginary part must be zero, so we must have $x = 0$ or $y = n\pi.$ The real part would be $\cos y$ or $(-1)^n \cosh x$ in the two cases. Since $\cos y$ can never equal $-5,$ we must choose $y = n\pi$ with n odd, and then setting the real part equal to -5 we need

$$\cosh x = 5$$

and the solution is : $x = \pm 2.2924.$ Thus $z = \pm 2.2924 + (2n+1)\pi,$ where n is any positive or negative integer.

10. Find all numbers z such that $z = \ln(-5).$

$$z = \ln\left(5e^{i\pi + 2\pi i n}\right) = \ln 5 + i\pi(2n+1) = 1.6094 + i\pi(2n+1)$$

11. Investigate the function $w = 1/\sqrt{z}.$ Find the functions $u(r,\theta)$ and $v(r,\theta)$ where $w = u + iv.$ How many branches does this function have? Find the image of the unit circle under this mapping.

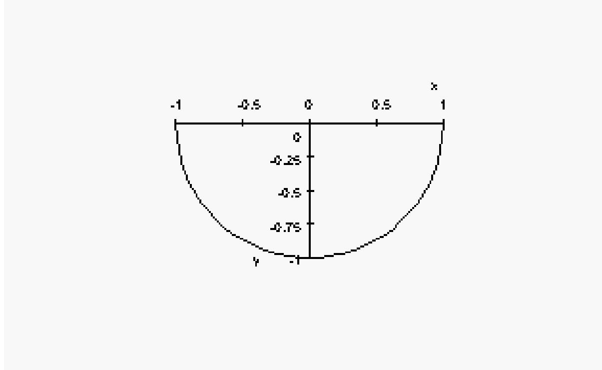
$$\frac{1}{\sqrt{z}} = \frac{1}{\sqrt{r}}e^{-i\theta/2} = \frac{1}{\sqrt{r}}\left(\cos\frac{\theta}{2} - i\sin\frac{\theta}{2}\right)$$

Thus

$$u = \rho\cos\phi = \frac{1}{\sqrt{r}}\cos\frac{\theta}{2}; \quad v = \rho\sin\phi = -\frac{1}{\sqrt{r}}\sin\frac{\theta}{2}$$

The function has a branch point at $z = 0,$ and it has two branches. Two circuits of the $z-$ plane give the whole $w-$ plane.

The unit circle is defined by $|z| = r = 1, \ \theta \le \theta < \pi.$ Then in the $w-$ plane we get a piece of the unit circle: $|w| = \sqrt{r} = 1,$ and, for the principal branch, $\phi = -\theta/2.$ So $0 \ge \phi > -\pi.$



12. The function $w(z) = z^{1/4}$. Find the functions $u(r,\theta)$ and $v(r,\theta)$ where $w = u + iv$. How many branches does this function have? Find the image under this mapping of a square of side 1 centered at the origin .

$$w = \left(re^{i\theta}\right)^{1/4} = r^{1/4}e^{i\theta/4} = r^{1/4}\left(\cos\frac{\theta}{4} + i\sin\frac{\theta}{4}\right)$$

Thus

$$u = r^{1/4}\cos\frac{\theta}{4} \text{ and } v = r^{1/4}\sin\frac{\theta}{4}$$

The function has four branches since we have to go around the original plane four times to get the whole $w-$ plane.

The line $x = \frac{1}{2},\ y = 0$ to $+\frac{1}{2}$ ($r = \frac{1}{2}\sec\theta, 0 \le \theta \le \pi/4)$ is mapped to

$$w = \left(\frac{1}{2}\sec\theta\right)^{1/4}\exp\left(i\frac{\theta}{4}\right)$$

The top side at $y = 1/2$ ($r = \frac{1}{2\sin\theta}$ $\pi/4 < \theta < 3\pi/4)$ maps to

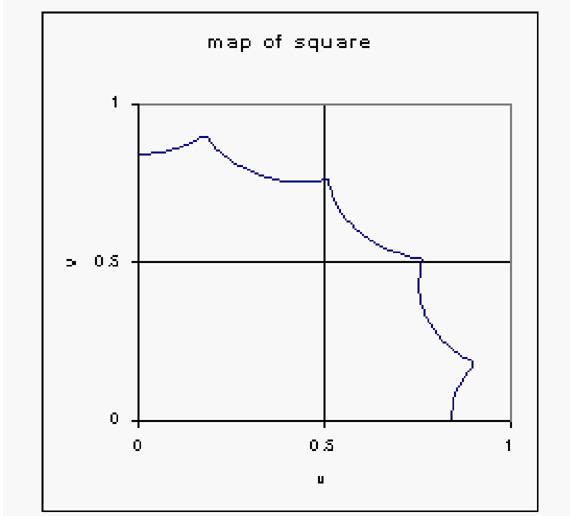
$$w = \left(\frac{1}{2\sin\theta}\right)^{1/4}\exp\left(i\frac{\theta}{4}\right)$$

The left side at $x = -1/2$ ($r = \frac{1}{2}\sec\theta, 3\pi/4 \le \theta \le 5\pi/4)$ is mapped to:

$$\left(\frac{1}{2}\sec(\theta - \pi)\right)^{1/4}\exp\left(i\frac{\theta}{4}\right)$$

The bottom at $y = -1/2$ ($r = \frac{1}{2\sin(\theta-\pi)}$ $5\pi/4 < \theta < 7\pi/4)$ maps to

$$w = \left(\frac{1}{2\sin(\theta - \pi)}\right)^{1/4}\exp\left(i\frac{\theta}{4}\right)$$



The entire square has mapped into the first quadrant and has been deformed into a curvy polygon. The other four branches of the function would close the polygon by completing the other three quadrants.

13. Oblate spheroidal coordinates u, v, w are defined in terms of cylindrical coordinates ρ, ϕ, z by the relations:

$$\rho + iz = c\cosh(u + iv), \;\; w = \phi$$

Show that the surfaces of constant u and constant v are ellipsoids and hyperboloids, respectively. What values of u and v correspond to the $z-$ axis and the $z = 0$ plane?

$$\rho + iz = c(\cosh u \cos v + i \sinh u \sin v)$$

Equating real and imaginary parts, we have:

$$\rho = c\cosh u \cos v \text{ and } z = c \sinh u \sin v$$

We want to find the shape of the constant u and constant v surfaces. First eliminate v :

$$\cos v = \frac{\rho}{c\cosh u} \qquad \text{and} \qquad \sin v = \frac{z}{c\sinh u}$$

Thus

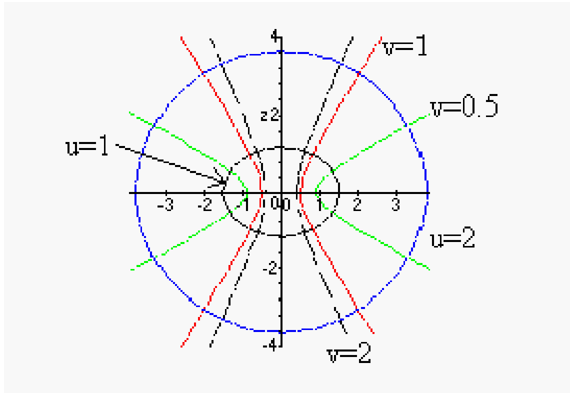
$$1 = \cos^2 v + \sin^2 v = \left(\frac{\rho}{c\cosh u}\right)^2 + \left(\frac{z}{c\sinh u}\right)^2$$

Thus the surfaces of constant u are ellipsoids with semi-major axis c cosh u and semi-minor axis $c\sinh u.$ Similarly, by solving for $\cosh u$ and $\sinh u,$ squaring and subtracting, we find:

$$1 = \cosh^2 u - \sinh^2 u = \left(\frac{\rho}{c\cos v}\right)^2 - \left(\frac{z}{c\sin v}\right)^2$$

so the constant v surfaces are hyperboloids.

The $z-$ axis is described by $\cos v = 0,$ i.e. $v = \pm\frac{\pi}{2}.$ Then $z = \pm c\sinh u$ which ranges from $-\infty$ to $+\infty$ as u does. The $z = 0$ plane is described by $u = 0$ or $v = 0$ or $v = \pi.$ These choices correspond to different regions for $\rho.$ But ρ is always positive, so we don't need $v = \pi.$ Thus $-\pi/2 \le v \le +\pi/2$, $0 \le u \le \infty$ and $0 \le w < 2\pi$ describes all of space.



This plot shows surfaces of constant u and constant v for

$c = 1.$

14. An AC circuit contains a capacitor C in series with a coil with resistance R and inductance $L.$ The circuit is driven by an AC power supply with emf $\varepsilon = \varepsilon_0 \cos\omega t.$

(a) Use Kirchhoff's rules to write equations for the steady-state current in the circuit.

Loop rule:

$$\varepsilon_0 \cos\omega t = IR + L\frac{dI}{dt} + \frac{Q}{C}$$

Charge conservation:

$$I = \frac{dQ}{dt}$$

(b) Using the fact that cos $\omega t = \mathrm{Re}\left(e^{i\omega t}\right)$, find the current through the power supply in the form:

$$I = \mathrm{Re}\left(\frac{\varepsilon_0}{Z}e^{i\omega t}\right)$$

where Z is the complex impedance of the circuit.

First write cos $\omega t = \mathrm{Re}\left(e^{i\omega t}\right)$ so the first equation becomes:

$$\mathrm{Re}\,\varepsilon_0 e^{i\omega t} = IR + L\frac{dI}{dt} + \frac{Q}{C}$$

Now let $I = \mathrm{Re}\left(Ie^{i\omega t}\right)$. Then diffferentiate the loop equation with respect to time:

$$\begin{aligned} i\omega\varepsilon_0 &= R\frac{dI}{dt} + L\frac{d^2I}{dt^2} + \frac{1}{C}\frac{dQ}{dt} \\ i\omega\varepsilon_0 &= i\omega IR - \omega^2 LI + \frac{I}{C} \\ &= I\left(i\omega R - \omega^2 L + 1/C\right) \end{aligned}$$

Thus

$$I = \frac{\varepsilon_0}{R + i\omega L + \frac{1}{i\omega C}} = \frac{\varepsilon_0}{Z}$$

The complex impedance is:

$$Z = R + i\omega L + \frac{1}{i\omega C}$$

(c) Use the result of (b) to find the amplitude and phase shift of the current. How much power is provided by the power supply? (Your answer should be the time-averaged power.)

Multiply top and bottom by the complex conjugate:

$$\begin{aligned}
I &= \mathrm{Re}\left(\varepsilon_0 \frac{R - i\left(\omega L - \frac{1}{\omega C}\right)}{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2} e^{i\omega t}\right) \\
&= \frac{\varepsilon_0}{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}\left(R\cos\omega t + \left(\omega L - \frac{1}{\omega C}\right)\sin\omega t\right) \\
&= \frac{\varepsilon_0}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}\left(\frac{R}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}\cos\omega t + \frac{\left(\omega L - \frac{1}{\omega C}\right)}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}\sin\omega t\right) \\
&= \frac{\varepsilon_0}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}(\cos\phi\cos\omega t + \sin\phi\sin\omega t) \\
&= \frac{\varepsilon_0}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}\cos(\omega t - \phi)
\end{aligned}$$

Thus the amplitude is

$$I_0 = \frac{\varepsilon_0}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}$$

and the phase shift is:

$$\phi = \tan^{-1}\frac{\left(\omega L - \frac{1}{\omega C}\right)}{R}$$

The time-averaged power is:

$$\begin{aligned}
P &= \langle I\varepsilon \rangle = \left\langle \frac{\varepsilon_0}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}(\cos\phi\cos\omega t + \sin\phi\sin\omega t)\varepsilon_0\cos\omega t \right\rangle \\
&= \frac{\varepsilon_0^2}{\sqrt{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}}\frac{\cos\phi}{2} \\
&= \frac{1}{2}\frac{R\varepsilon_0^2}{R^2 + \left(\omega L - \frac{1}{\omega C}\right)^2}
\end{aligned}$$

(d) Show that the power is given by

$$P = \frac{1}{2}\mathrm{Re}(I\varepsilon^*)$$

$$P = \frac{1}{2}\mathrm{Re}(Z)\frac{\varepsilon_0^2}{|Z|^2} = \frac{1}{2}\mathrm{Re}\frac{Z|\varepsilon|^2}{|Z|^2} = \frac{1}{2}\mathrm{Re}\frac{Z\varepsilon\varepsilon^*}{ZZ^*} = \frac{1}{2}\mathrm{Re}\frac{Z^*\varepsilon\varepsilon^*}{ZZ^*}$$

since $\mathrm{Re}\,Z = \mathrm{Re}\,Z^*.$ Then

$$P = \frac{1}{2}\mathrm{Re}\frac{\varepsilon^*}{Z^*}\varepsilon = \frac{1}{2}\mathrm{Re}\frac{\varepsilon}{Z}\varepsilon^*$$
$$= \frac{1}{2}\mathrm{Re}\,I^*\varepsilon = \frac{1}{2}\mathrm{Re}\,I\varepsilon^*$$

15. Small amplitude waves in a plasma are described by the relations

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial x}(n_0 v) = 0$$
$$\varepsilon_0\frac{\partial E}{\partial x} = -en$$
$$\text{and } m\frac{\partial v}{\partial t} = -eE - m\nu v$$

where $n_0,$ $e,$ $m,$ ν and ε_0 are constants. The constant ν is the collision frequency. Assume that $n,$ E and v are all proportional to $\exp(ikx - i\omega t).$ Solve the equations for non-zero n, E and v to show that ω satisfies the equation:

$$\omega^2 + i\nu\omega = \frac{n_0 e^2}{m\varepsilon_0} \equiv \omega_p^2$$

and hence show that collisions damp the waves.

Putting in the exponential form, the equations become:

$$-i\omega n + ikn_0 v = 0$$
$$ik\varepsilon_o E = -en$$
$$\text{and } -i\omega m v = -eE - m\nu v$$

Use the second equation to eliminate E from the last:

$$-i\omega m v = -e\left(\frac{-en}{ik\varepsilon_0}\right) - m\nu v$$

and then use the first equation to eliminate n :

$$-i\omega m v = \left(\frac{e^2}{ik\varepsilon_0}\right)\left(\frac{kn_0 v}{\omega}\right) - m\nu v$$

Now we have an equation with v in every term. Either $v = 0,$ a solution we are told to discard, or else:

$$-i\omega m = \frac{e^2}{i\varepsilon_0}\frac{n_0}{\omega} - m\nu$$
$$\omega^2 + i\omega\nu = \frac{n_0 e^2}{m\varepsilon_0} = \omega_p^2$$

which is the desired result. Now we solve this quadratic for ω

$$\omega = \frac{-i\nu \pm \sqrt{-\nu^2 + 4\omega_p^2}}{2}$$

With no collisions, $\nu = 0$, the solution is $\omega = \pm\omega_p.$ With collisions, the real part of the frequency is slightly altered, but the important difference is the addition of the imaginary part $-i\nu/2.$ The wave then has the form

$$
\begin{aligned}
\exp\left(ikx - it\left[\omega_p\sqrt{1-\frac{\nu^2}{4\omega_p^2}} - i\frac{\nu}{2}\right]\right) &= \exp\left(ikx - it\omega_p\sqrt{1-\frac{\nu^2}{4\omega_p^2}}\right)\exp\left(i^2\frac{\nu}{2}t\right) \\
&= \exp\left(ikx - it\omega_p\sqrt{1-\frac{\nu^2}{4\omega_p^2}}\right)\exp\left(-\frac{\nu}{2}t\right)
\end{aligned}
$$

The real exponential shows that the wave amplitude decreases in time.

Chapter 2: Complex variables

16. Write the real and imaginary parts u and v of the complex functions(a) $f = z^2 \sin z$ and (b) $f = \frac{1}{1+z}$. In each case, show that u and v obey the Cauchy-Riemann relations. Find the derivative df/dz first in terms of x and y, and then express the answer in terms of z. Is the result what you expected?

$$\begin{aligned} f &= (x+iy)^2 \sin(x+iy) \\ &= \left(x^2 + 2ixy + (iy)^2\right)(\sin x \cos iy + \cos x \sin iy) \\ &= \left(x^2 - y^2 + 2ixy\right)(\sin x \cosh y + i \cos x \sinh y) \end{aligned}$$

since

$$\sin iy = \frac{e^{i(iy)} - e^{-i(iy)}}{2i} = \frac{e^{-y} - e^{y}}{2i} = -\frac{\sinh y}{i} = i \sinh y$$

Thus

$$f = \left(x^2 - y^2\right) \sin x \cosh y - 2xy \cos x \sinh y + i\left(2xy \sin x \cosh y + \left(x^2 - y^2\right) \cos x \sinh y\right)$$

Thus

$$u = \left(x^2 - y^2\right) \sin x \cosh y - 2xy \cos x \sinh y$$

and

$$v = 2xy \sin x \cosh y + \left(x^2 - y^2\right) \cos x \sinh y$$

Then

$$\frac{\partial u}{\partial x} = 2x \sin x \cosh y + \left(x^2 - y^2\right) \cos x \cosh y - 2y \cos x \sinh y + 2xy \sin x \sinh y$$

while

$$\begin{aligned} \frac{\partial v}{\partial y} &= 2x \sin x \cosh y + 2xy \sin x \sinh y - 2y \cos x \sinh y + \left(x^2 - y^2\right) \cos x \cosh y \\ &= \frac{\partial u}{\partial x} \end{aligned}$$

So the first relation is satisfied.

Then

$$\frac{\partial u}{\partial y} = -2y \sin x \cosh y + \left(x^2 - y^2\right) \sin x \sinh y - 2x \cos x \sinh y - 2xy \cos x \cosh y$$

while

$$\begin{aligned} \frac{\partial v}{\partial x} &= 2y \sin x \cosh y + 2xy \cos x \cosh y + 2x \cos x \sinh y - \left(x^2 - y^2\right) \sin x \sinh y \\ &= -\frac{\partial u}{\partial y} \end{aligned}$$

and the second relation is also satisfied.

The derivative is

$$\begin{aligned} \frac{df}{dz} &= \frac{\partial u}{\partial x} + i \frac{\partial v}{\partial x} \\ &= 2x \sin x \cosh y + (x^2 - y^2) \cos x \cosh y - 2y \cos x \sinh y + 2xy \sin x \sinh y \\ &\quad + i(2y \sin x \cosh y + 2xy \cos x \cosh y + 2x \cos x \sinh y - (x^2 - y^2) \sin x \sinh y) \\ &= 2(x+iy)(\sin x \cosh y + i \cos x \sinh y) + (x^2 - y^2 + 2ixy)(\cos x \cosh y - i \sin x \sinh y) \\ &= 2z \sin z + z^2 \cos z \end{aligned}$$

as expected.

(b) $f = \frac{1}{1+z} = \frac{1+x-iy}{(x+1)^2+y^2}$. Thus

$$u = \frac{1+x}{(x+1)^2+y^2};\ v = \frac{-y}{(x+1)^2+y^2}$$

Then

$$\frac{\partial u}{\partial x} = \frac{1}{(x+1)^2+y^2} - \frac{2(1+x)^2}{\left((x+1)^2+y^2\right)^2} = \frac{(x+1)^2+y^2-2(1+x)^2}{\left((x+1)^2+y^2\right)^2}$$
$$= \frac{y^2-(1+x)^2}{\left((x+1)^2+y^2\right)^2}$$

$$\frac{\partial u}{\partial y} = \frac{-2y(1+x)}{\left((x+1)^2+y^2\right)^2}$$

$$\frac{\partial v}{\partial x} = \frac{2(1+x)y}{\left((x+1)^2+y^2\right)^2} = -\frac{\partial u}{\partial y}$$

and

$$\frac{\partial v}{\partial y} = \frac{-1}{(x+1)^2+y^2} + \frac{2y^2}{\left((x+1)^2+y^2\right)^2} = \frac{2y^2-\left((x+1)^2+y^2\right)}{\left((x+1)^2+y^2\right)^2}$$
$$= \frac{y^2-(1+x)^2}{\left((x+1)^2+y^2\right)^2} = \frac{\partial u}{\partial x}$$

So the CR relations are satisfied. Then the derivative is:

$$\frac{df}{dz} = \frac{y^2-(1+x)^2}{\left((x+1)^2+y^2\right)^2} + i\frac{2y(1+x)}{\left((x+1)^2+y^2\right)^2} = \frac{(y+i(1+x))^2}{\left((x+1)^2+y^2\right)^2}$$
$$= \frac{(iz^*+i)^2}{|z+1|^4} = -\frac{(z^*+1)^2}{(z+1)^2(z^*+1)^2} = -\frac{1}{(z+1)^2}$$

which is the expected result.

17. The variables x and y in a complex number $z = x+iy$ may be expressed in terms of z and its complex conjugate z^*:

$$x = \frac{1}{2}(z+z^*)$$
$$y = \frac{1}{2i}(z-z^*)$$

Show that the Cauchy-Riemann relations are equivalent to the condition

$$\frac{\partial f}{\partial z^*} \equiv 0.$$

We rewrite the derivatives using the chain rule. Suppose that $f = f(z,z^*)$. Then:

$$\frac{\partial f}{\partial z^*} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial z^*} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial z^*} = \left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\frac{1}{2} + \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\left(-\frac{1}{2i}\right)$$
$$= \frac{1}{2}\left[\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right] + \frac{i}{2}\left[\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right]$$

If the Cauchy-Riemann relations are satisfied, both terms in square brackets are zero, and hence

$\frac{\partial f}{\partial z^*} \equiv 0$, as required. This means that the function $f = f(z)$, and z^* does not appear.

18. One of the functions $u_1 = 2(x-y)^2$ and $u_2 = \frac{x^3}{3} - xy^2$ is the real part of an analytic function $w(z) = u + iv$. Which is it? Find the function $v(x,y)$ and write w as a function of z.

Both the real and imaginary parts of an analytic fucntion satisfy the equation

$$\nabla^2 u = 0$$

so let's test the two functions:

$$\nabla^2 u_1 = 4\frac{\partial}{\partial x}(x-y) - 4\frac{\partial}{\partial y}(x-y) = 4 + 4 = 8 \neq 0$$

and

$$\nabla^2 u_2 = \frac{\partial}{\partial x}x^2 - \frac{\partial}{\partial y}2yx = 2x - 2x = 0$$

So the correct function is u_2.

Then from the C-R relations:

$$\frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} = x^2 - y^2 \Rightarrow v = x^2 y - \frac{y^3}{3} + f(x)$$

and

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} = 2xy \Rightarrow v = x^2 y + f(y)$$

Thus

$$v = x^2 y - \frac{y^3}{3}$$

Then

$$f = \frac{x^3}{3} + ix^2 y - xy^2 - i\frac{y^3}{3} = \frac{1}{3}(x+iy)^3 = \frac{z^3}{3}$$

19. A cylinder of radius a has potential V on one half and $-V$ on the other half. The potential inside the cylinder may be written as a series:

$$\Phi(r,\theta) = \frac{4V}{\pi}\sum_{n=0}^{\infty}\left(\frac{r^2}{a^2}\right)^{2n+1}\frac{\sin[2(2n+1)\theta]}{2n+1}$$

Express each term in the sum as the imaginary part of a complex number, and hence sum the series. Show that the result may be expressed in terms of an inverse tangent.

$$\begin{aligned}\Phi(r,\theta) &= \frac{4V}{\pi}\sum_{n=0}^{\infty}\operatorname{Im}\left(\frac{z^2}{a^2}\right)^{2n+1}\frac{1}{2n+1}\\ &= \frac{4V}{\pi a^2}\sum_{n=0}^{\infty}\operatorname{Im}\int_0^{z^2}\left(\frac{w}{a^2}\right)^{2n}dw\\ &= \frac{4V}{\pi a^2}\operatorname{Im}\int_0^{z^2}\sum_{n=0}^{\infty}\left(\frac{w}{a^2}\right)^{2n}dw\end{aligned}$$

The sum may be recognized as the geometric series (2.43)

$$\Phi(r,\theta) = \frac{4V}{\pi a^2}\operatorname{Im}\int_0^{z^2}\frac{1}{1-\left(w/a^2\right)^2}dw$$

To do the integral, let $w/a^2 = \sin\phi;\ dw/a^2 = \cos\phi\, d\phi.$

$$\begin{aligned}\Phi(r,\theta) &= \frac{4V}{\pi}\,\mathrm{Im}\int_0^{\sin^{-1}z^2/a^2}\frac{\cos\phi}{1-\sin^2\phi}d\phi\\ &= \frac{4V}{\pi}\,\mathrm{Im}\int_0^{\sin^{-1}z^2/a^2}\sec\phi\, d\phi\\ &= \frac{4V}{\pi}\,\mathrm{Im}\left[\ln(\sec\phi+\tan\phi)\big|_0^{\sin^{-1}z^2/a^2}\right]\\ &= \frac{4V}{\pi}\,\mathrm{Im}\left\{\ln\left(\frac{1+z^2/a^2}{\sqrt{1-z^4/a^4}}\right)\right\}\\ &= \frac{4V}{\pi}\,\mathrm{Im}\left\{\ln\sqrt{\frac{1+z^2/a^2}{1-z^2/a^2}}\right\}\end{aligned}$$

Now the logarithm is

$$\ln\frac{1+z^2/a^2}{1-z^2/a^2} = \ln\left|\frac{1+z^2/a^2}{1-z^2/a^2}\right| + i\arg\left(\frac{1+z^2/a^2}{1-z^2/a^2}\right)$$

and thus

$$\Phi(r,\theta) = \frac{2V}{\pi}\arg\frac{1+z^2/a^2}{1-z^2/a^2}$$

Next we find the argument:

$$\begin{aligned}\frac{1+z^2/a^2}{1-z^2/a^2} &= \frac{a^2+r^2\cos 2\theta+ir^2\sin 2\theta}{a^2-r^2\cos 2\theta-ir^2\sin 2\theta}\\ &= \frac{\left(a^2+r^2\cos 2\theta+ir^2\sin 2\theta\right)\left(a^2-r^2\cos 2\theta+ir^2\sin 2\theta\right)}{\left(a^2-r^2\cos 2\theta\right)^2+r^4\sin^2 2\theta}\\ &= \frac{a^4-r^4+2ia^2r^2\sin 2\theta}{a^4-2a^2r^2\cos 2\theta+r^4} = Ae^{i\alpha}\end{aligned}$$

where

$$\tan\alpha = \frac{2a^2r^2\sin 2\theta}{a^4-r^4}$$

and thus

$$\Phi(r,\theta) = \frac{2V}{\pi}\tan^{-1}\frac{2a^2r^2\sin 2\theta}{a^4-r^4}$$

20. The function $f = \sin\left(z^2\right)$ (cf Example 2.10) also has a zero at $z = \sqrt{\pi}$. What is its order?

To find the order of the zero, we write the Taylor series centered at $\sqrt{\pi}$.

$$\left.\frac{df}{dz}\right|_{\sqrt{\pi}} = 2z\cos z^2\Big|_{\sqrt{\pi}} = 2\sqrt{\pi}\cos\pi = -2\sqrt{\pi}$$

Thus the series is

$$f(z) = -2\sqrt{\pi}\,(z-\sqrt{\pi}) + \cdots$$

and the zero is of order 1.

21. Find the Taylor series for the following functions about the point specified:

(a) $z\cos z$ about $z = 0$

The series is z times the cosine series, i.e.

$$z\cos z = z\left(1 - \frac{z^2}{2} + \frac{z^4}{4!} + \cdots\right)$$
$$= z - \frac{z^3}{2} + \frac{z^5}{5!} + \cdots$$

(b) $\ln(1+z)$ about $z = 0$

At $z = 0,\ f(z) =$ $\ln(1) = 0$

The derivative is

$$\frac{df}{dz} = \frac{1}{1+z} = 1 \text{ at } z = 0$$

The 2nd derivative is

$$\frac{d^2f}{dz^2} = -\frac{1}{(1+z)^2} = -1 \text{ at } z = 0$$

The 3nd derivative is

$$\frac{d^3f}{dz^3} = \frac{2}{(1+z)^3} = 2 \text{ at } z = 0$$

So the series is:

$$\ln(1+z) = z - \frac{z^2}{2!} + \frac{2}{3!}z^3 + \cdots$$
$$= z - \frac{z^2}{2} + \frac{z^3}{3} + \cdots$$

The radius of convergence is $1,$ since $\ln(1+z)$ has a branch point at $z = -1.$

(c) $\frac{\sin z}{z}$ about $z = \pi/2$

$$f\left(\frac{\pi}{2}\right) = \frac{1}{\pi/2} = \frac{2}{\pi}$$

The derivative is

$$\frac{df}{dz} = \frac{\cos z}{z} - \frac{\sin z}{z^2} = -\left(\frac{2}{\pi}\right)^2 \text{at } z = \frac{\pi}{2}$$

The 2nd derivative is

$$\frac{d^2f}{dz^2} = \frac{-\sin z}{z} - 2\frac{\cos z}{z^2} + 2\frac{\sin z}{z^3} = -\left(\frac{2}{\pi}\right) + 2\left(\frac{2}{\pi}\right)^3 \text{at } z = \frac{\pi}{2}$$

So the series is:

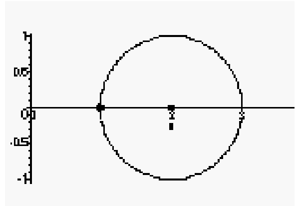
$$\frac{\sin z}{z} = \frac{2}{\pi} - \left(\frac{2}{\pi}\right)^2\left(z - \frac{\pi}{2}\right) + \frac{1}{\pi}\left[2\left(\frac{2}{\pi}\right)^2 - 1\right]\left(z - \frac{\pi}{2}\right)^2 + \cdots$$

The radius of convergence is $\infty,$ since the function has no singularities (other than the removable singularity at $z = 0.$)

(d) $\frac{1}{z^2-1}$ about $z = 2.$

First factor the denominator:

$$\frac{1}{z^2-1} = \frac{1}{(z-1)(z+1)} = \frac{1}{2}\left(\frac{1}{z-1} - \frac{1}{z+1}\right)$$



There are poles at $z = \pm 1$. Now let $w = z - 2$:

$$f(z) = \frac{1}{2}\left(\frac{1}{w+1} - \frac{1}{w+3}\right)$$

Expand each term in a geometric series:

$$\begin{aligned} f(z) &= \frac{1}{2}\left(1 - w + w^2 - w^3 + \cdots - \frac{1}{3}\left(\frac{1}{1+\frac{w}{3}}\right)\right) \\ &= \frac{1}{2}\left(1 - w + w^2 - w^3 + \cdots - \frac{1}{3}\left(1 - \frac{w}{3} + \left(\frac{w}{3}\right)^2 + \cdots\right)\right) \\ &= \frac{1}{2}\left(1 - w + w^2 - w^3 + \cdots - \frac{1}{3} + \frac{w}{9} - \frac{w^2}{3^3} + \cdots\right) \\ &= \frac{1}{2}\left(\frac{2}{3} - \frac{8w}{9} + w^2\left(1 - \frac{1}{3^3}\right) - w^3\left(1 - \frac{1}{3^4}\right) + \cdots\right) \\ &= \frac{1}{3} - \frac{4}{9}(z-2) + \frac{1}{2}\left(1 - \frac{1}{3^3}\right)(z-2)^2 - \frac{1}{2}\left(1 - \frac{1}{3^4}\right)(z-2)^3 + \cdots \end{aligned}$$

The radius of convergence is 1, since $f(z)$ has a pole at $z = 1$.

22. Determine the Taylor or Laurent series for each of the following functions about the point specified:

(a) $\frac{\cos z}{z-1}$ about $z = 1$

The function has a pole at $z = 1$, so the series is a Laurent series.

First find the Taylor series for $\cos z$:

$$\cos z = \cos 1 - \sin 1(z-1) - \frac{\cos 1}{2}(z-1)^2 + \cdots$$

The general term is

$$\begin{aligned} \frac{d^m}{dz^m}\cos z\bigg|_{z=1}\frac{(z-1)^m}{m!} &= (-1)^{m/2}\frac{(z-1)^m}{m!}\cos 1 \text{ for } m \text{ even} \\ &= (-1)^{(m+1)/2}\frac{(z-1)^m}{m!}\sin 1 \text{ for } m \text{ odd} \end{aligned}$$

and thus

$$\frac{\cos z}{z-1} = \cos 1\sum_{m=0}^{\infty}(-1)^m\frac{(z-1)^{2m}}{2m!} + \sin 1\sum_{m=0}^{\infty}(-1)^{m+1}\frac{(z-1)^{2m+1}}{(2m+1)!}$$

The radius of convergence is infinite, since the function has no other poles or singularities.

(b) $\frac{\sin z^2}{z}$ about $z = 0$

The function is analytic at

$z = 0$ (there is a removable singularity) so the series is a Taylor series. We start with the series for $\sin z^2$:

$$\frac{\sin z^2}{z} = \frac{1}{z}\left(z^2 - \frac{z^6}{3!} + \frac{z^{10}}{5!} + \cdots\right)$$
$$= z - \frac{z^5}{3!} + \frac{z^9}{5!} + \cdots$$

The radius of convergence is infinite, since the function has no poles or other singularities

(c) $\frac{e^z}{z - i\pi}$ about $z = i\pi$

There is a simple pole at $z = i\pi$: the series is a Laurent series:

$$\frac{e^z}{z - i\pi} = e^{i\pi}\frac{e^{z - i\pi}}{z - i\pi}$$
$$= -\frac{e^w}{w} = -\frac{1}{w}\left(1 + w + \frac{w^2}{2!} + \frac{w^3}{3!} + \cdots\right)$$
$$= -\frac{1}{w} - 1 - \frac{w}{2} - \frac{w^2}{6} - \cdots \text{where } w = z - i\pi$$

The radius of convergence is infinite, since the function has no other poles or singularities.

(d) $\frac{\ln z}{z-1}$ about $z = 1$

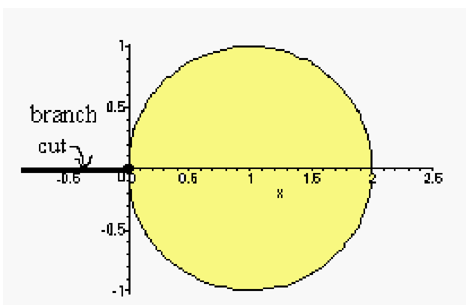
Th function has a branch point at $z = 0$. The singularity at $z = 1$ is removable, since $\ln z$ has a zero at $z = 1$. We should be able to find a Taylor series valid for $0 < |z - 1| < 1$.

First find the Taylor series for $\ln z$. Let $w = z - 1$

$$\ln(1 + w) = w - \frac{w^2}{2} + \frac{w^3}{3} + \cdots$$

So

$$\frac{\ln z}{z - 1} = \frac{1}{z - 1}\left(z - 1 - \frac{(z - 1)^2}{2} + \frac{(z - 1)^3}{3} + \cdots\right)$$
$$= 1 - \frac{z - 1}{2} + \frac{(z - 1)^3}{3} + \cdots$$



(e) $\tan^{-1}(z) = w$

$$z = \tan w = \frac{e^{iw} - e^{-iw}}{i\left(e^{iw} + e^{-iw}\right)}$$

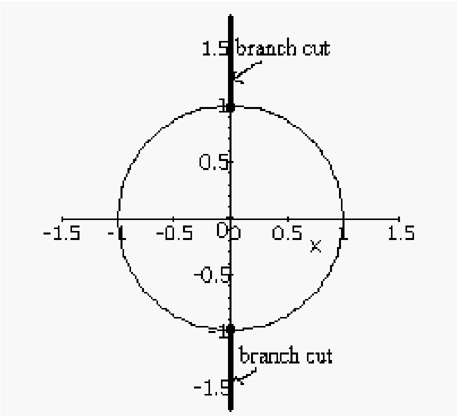
$$iz\left(e^{iw} + e^{-iw}\right) = e^{iw} - e^{-iw}$$
$$e^{2iw}(iz - 1) + iz + 1 = 0$$

So

$$w = \frac{1}{2i}\ln\left(\frac{1+iz}{1-iz}\right) = \frac{1}{2i}\ln\left(\frac{i-z}{i+z}\right)$$

There are branch points at $z = \pm i.$ There is a Taylor series valid for $|z| < 1.$

$$\begin{aligned} w &= \frac{1}{2i}\left[iz - \frac{1}{2}\left(-z^2\right) + \cdots - (z \to -z)\right] \\ &= z + \frac{2}{3i}(iz)^3 + \frac{2}{5i}(iz)^5 + \cdots \\ &= z - \frac{1}{3}z^3 + \frac{1}{5}z^5 + \cdots \end{aligned}$$



Problem 22

23. Determine *all* Taylor or Laurent series about the specified point for each of the following functions.

(a) $\frac{e^z}{z^2+1}$ about the origin.

The function is analytic about the origin, so there is a Taylor series. The function has poles at

$z = \pm i,$ so the Taylor series is valid for $|z| < 1.$ There is a Laurent series valid for $|z| > 1.$

Taylor series:

$$\begin{aligned} \frac{e^z}{z^2+1} &= \left(1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \frac{z^4}{4!} + \frac{z^5}{5!}\cdots\right)\left(1 - z^2 + z^4 - z^6 + \cdots\right) \\ &= 1 + z - \frac{1}{2}z^2 - \frac{5}{6}z^3 + \frac{13}{24}z^4 + \frac{101}{120}z^5 - \frac{389}{720}z^6 + \cdots \text{ for } |z| < 1 \\ &= = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty}(-1)^m \frac{z^{n+2m}}{n!} \end{aligned}$$

Laurent series:

$$\begin{aligned} \frac{e^z}{z^2+1} &= \frac{e^z}{z^2\left(1 + \frac{1}{z^2}\right)} = \frac{1}{z^2}\sum_{n=0}^{\infty}\frac{z^n}{n!}\sum_{m=0}^{\infty}(-1)^m\frac{1}{z^{2m}} \\ &= \frac{1}{z^2}\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\frac{z^{n-2m}}{n!}(-1)^m \end{aligned}$$

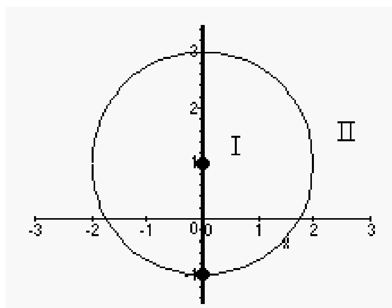
We may simplify the negative powers as follows:

$$\begin{aligned}\frac{e^z}{z^2+1} &= \frac{e^z}{z^2\left(1+\frac{1}{z^2}\right)} = \frac{1}{z^2}\left(1+z+\frac{z^2}{2!}+\frac{z^3}{3!}+\frac{z^4}{4!}+\frac{z^5}{5!}\cdots\right)\left(1-\frac{1}{z^2}+\frac{1}{z^4}-\frac{1}{z^6}+\cdots\right)\\ &= \cdots\frac{1}{z^5}\left(1-\frac{1}{3!}+\cdots\right)+\frac{1}{z^4}\left(-1+\frac{1}{2}+-\frac{1}{4!}\cdots\right)+\frac{1}{z^3}\left(1-\frac{1}{3!}+\frac{1}{5!}+\cdots\right)\\ &\quad -\frac{1}{z^2}\left(1-\frac{1}{2}+\frac{1}{4!}+\cdots\right)\\ &\quad -\frac{1}{z}\left(1-\frac{1}{3!}+\cdots\right)+\frac{1}{2}-\frac{1}{4!}+\cdots+z\left(\frac{1}{3!}-\frac{1}{5!}+\right)+z^2\left(\frac{1}{4!}+\cdots\right)+z^3\left(\frac{1}{5!}+\cdots\right)+\cdots\\ &= \cos 1\sum_{n=1}^{\infty}\frac{(-1)^n}{z^{2n}}+\sin 1\sum_{n=0}^{\infty}\frac{(-1)^n}{z^{2n+1}}+\sum_{n=2m+2}^{\infty}\sum_{m=0}^{\infty}(-1)^m\frac{z^{n-2m-2}}{n!}\end{aligned}$$

valid for $|z| > 1.$

(b) $\frac{1}{z^2+1}$ about $z = i$

The function has simple poles at $z = \pm i,$ so we can find a Laurent series valid for $0 < |z-i| < 2$ and another valid for $|z-i| > 2$.



$$\frac{1}{z^2+1} = \frac{1}{2i}\left(\frac{1}{z-i}-\frac{1}{z+i}\right)$$

Let $w = z - i$

$$\frac{1}{z^2+1} = \frac{1}{2i}\left(\frac{1}{w}-\frac{1}{w+2i}\right)$$

In Region I, expand the second term in a geometric series:

$$\begin{aligned}\frac{1}{z^2+1} &= \frac{1}{2i}\frac{1}{w}-\frac{1}{(2i)^2}\frac{1}{(1+w/2i)}\\ &= \frac{1}{2iw}+\frac{1}{4}\left(1-\frac{w}{2i}+\left(\frac{w}{2i}\right)^2-\left(\frac{w}{2i}\right)^3+\cdots\right)\\ &= -\frac{i}{2w}+\frac{1}{4}\left(1+\frac{iw}{2}-\frac{w^2}{4}-\frac{w^3}{8}i+\cdots\right)\\ &= \frac{1}{4}\left(1-\frac{(z-i)^2}{4}+\cdots\right)-\frac{i}{2}\left(\frac{1}{z-i}+\frac{z-i}{4}-\frac{(z-i)^3}{16}+\cdots\right)\end{aligned}$$

which is valid for $0 < |z-i| < 2.$

In the outer region (II) we expand the other way:

$$\begin{aligned}\frac{1}{z^2+1} &= \frac{1}{2i}\left(\frac{1}{w} - \frac{1}{w}\frac{1}{(1+2i/w)}\right)\\ &= \frac{1}{2iw}\left[1 - \left(1 - \frac{2i}{w} + \left(\frac{2i}{w}\right)^2 - \left(\frac{2i}{w}\right)^3 + \cdots\right)\right]\\ &= \frac{1}{2iw}\left[\frac{2i}{w} - \left(\frac{2i}{w}\right)^2 + \left(\frac{2i}{w}\right)^3 + \cdots\right]\\ &= \frac{1}{w^2} - \frac{2i}{w^3} - \frac{4}{w^4} + \cdots\\ &= \frac{1}{(z-i)^2} - \frac{2i}{(z-i)^3} - \frac{4}{(z-i)^4} + \cdots\end{aligned}$$

which is valid for $|z-i| > 2.$

(c) $\frac{z}{z^2-9}$ about $z = 3$

The function has poles at $z = \pm 3.$ We should be able to find a Laurent series valid for $0 < |z-3| < 6$ and another for $|z-3| > 6.$

$$\frac{z}{z^2-9} = \frac{z}{(z-3)(z+3)} = \frac{w+3}{w(w+6)}$$

where $w = z - 3.$ Then for $|w| < 6$ we have:

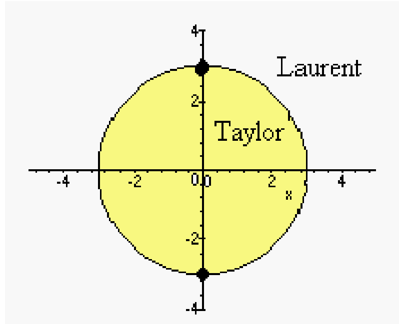
$$\begin{aligned}\frac{z}{z^2-9} &= \frac{w+3}{w}\frac{1}{6\left(1+\frac{w}{6}\right)}\\ &= \frac{w+3}{6w}\left(1 - \frac{w}{6} + \frac{w^2}{6^2} - \frac{w^3}{6^3} + \cdots\right)\\ &= \left(\frac{1}{6} + \frac{1}{2w}\right)\left(1 - \frac{w}{6} + \frac{w^2}{6^2} - \frac{w^3}{6^3} + \cdots\right)\\ &= \frac{1}{2w} + \frac{1}{12} - \frac{1}{72}w + \frac{1}{432}w^2 - \frac{1}{1296}w^3 + \cdots\\ &= \frac{1}{2(z-3)} + \frac{1}{12} - \frac{(z-3)}{72} + \frac{(z-3)^2}{432} + \cdots\end{aligned}$$

while for $|w| > 6$

$$\begin{aligned}\frac{z}{z^2-9} &= \frac{w+3}{w}\frac{1}{w\left(1+\frac{6}{w}\right)}\\ &= \frac{w+3}{w^2}\left(1 - \frac{6}{w} + \frac{6^2}{w^2} - \frac{6^3}{w^3} + \cdots\right)\\ &= \frac{1}{w} - \frac{3}{w^2} + \frac{18}{w^3} - \frac{108}{w^4} + \cdots\\ &= \frac{1}{z-3} - \frac{3}{(z-3)^2} + \frac{18}{(z-3)^3} - \frac{108}{(z-3)^4} + \cdots\end{aligned}$$

(d) $\frac{1}{z^2+9}$ about the origin.

The function has poles at $z = \pm 3i,$ so there is a Taylor series valid for $|z| < 3$ and a Laurent series valid for $|z| > 3.$



$$\begin{aligned}\frac{1}{z^2+9} &= \frac{1}{6i}\left(\frac{1}{z-3i} - \frac{1}{z+3i}\right)\\ &= -\frac{1}{6i}\,\frac{1}{3i}\left(\frac{1}{1-z/3i} + \frac{1}{1+z/3i}\right)\\ &= \frac{1}{18}\left(1+\frac{z}{3i}+\left(\frac{z}{3i}\right)^2+\left(\frac{z}{3i}\right)^3+\cdots+1-\frac{z}{3i}+\left(\frac{z}{3i}\right)^2-\left(\frac{z}{3i}\right)^3+\cdots\right)\\ &= \frac{1}{9}\left(1-\frac{z^2}{9}+\frac{z^4}{3^4}-\cdots\right)\end{aligned}$$

while for $|z| > 3$

$$\begin{aligned}\frac{1}{z^2+9} &= \frac{1}{6iz}\left(\frac{1}{1-3i/z} - \frac{1}{1+3i/z}\right)\\ &= \frac{1}{6iz}\left(1+\frac{3i}{z}+\left(\frac{3i}{z}\right)^2+\left(\frac{3i}{z}\right)^3+\cdots-\left(1-\frac{3i}{z}+\left(\frac{3i}{z}\right)^2-\left(\frac{3i}{z}\right)^3+\cdots\right)\right)\\ &= \frac{1}{3iz}\left(\frac{3i}{z}+\left(\frac{3i}{z}\right)^3+\left(\frac{3i}{z}\right)^5+\cdots\right)\\ &= \frac{1}{z^2}-\frac{9}{z^4}+\frac{3^4}{z^6}+\cdots\end{aligned}$$

24. Find all the singularities of each of the following functions, and describe each of them completely.

(a) $\frac{e^z}{z} - \sin\frac{1}{z}$

Expand out each term in a series:

$$\begin{aligned}\frac{e^z}{z} - \sin\frac{1}{z} &= \frac{1+z+z^2/2+\cdots}{z} - \left(\frac{1}{z} - \frac{1}{3!}\left(\frac{1}{z}\right)^3+\cdots\right)\\ &= 1+\frac{z}{2}+\frac{z^2}{3!}+\cdots+\frac{1}{3!z^3}-\frac{1}{5!z^5}+\cdots\end{aligned}$$

This is a Laurent series with infinitely many negative powers, and it is valid up to the singularity at $z = 0$, so the function has an essential singularity at $z = 0$.

(b) $\frac{\cos z}{z} - \frac{\sin z}{z^2}$

Let's look at the series for this function about the origin:

$$\begin{aligned}\frac{\cos z}{z} - \frac{\sin z}{z^2} &= \frac{1-z^2/2+z^4/4!+\cdots}{z} - \frac{z-z^3/3!+z^5/5!-\cdots}{z^2}\\ &= -\frac{z}{2}-\frac{z}{6}+\frac{z^3}{4!}-\frac{z^3}{5!}+\cdots\\ &= -\frac{2}{3}z+\frac{z^3}{30}+\cdots\end{aligned}$$

This is a Taylor series valid for all z. Thus the function has a removable singularity at $z = 0$.

(c) $\frac{\tanh z}{z}$

The function has a removable singularity at $z = 0$:

$$\lim_{z\to 0}\frac{\tanh z}{z} = \lim_{z\to 0}\frac{\operatorname{sech}^2 z}{1} = 1$$

But the tanh function also has singularities regularly spaced along the imaginary axis.

$$\tanh iy = \frac{e^{-y} - e^{y}}{i(e^{-y} + e^{y})} = i\tan y$$

and $\tan y$ has singularities at $y = (2n + 1)\pi/2.$ The singularities are all simple poles. For example

$$\begin{aligned}\cosh iy &= \cos y = -\sin(y - \pi/2)\\ &= -\left((y - \pi/2) - \frac{1}{3!}(y - \pi/2)^3 + \cdots\right)\end{aligned}$$

$$\begin{aligned}\lim_{z\to i\pi/2}\left(z - i\frac{\pi}{2}\right)\frac{\tanh z}{z} &= \lim_{z\to i\pi/2}\frac{\left(z - i\frac{\pi}{2}\right)\sinh z}{z\cosh z}\\ &= \lim_{z\to i\pi/2}\frac{\left(z - i\frac{\pi}{2}\right)\sinh z}{-z\left(-i(z - i\pi/2) - \frac{i}{3!}(z - i\pi/2)^3 + \cdots\right)}\\ &= \lim_{z\to i\pi/2}\frac{\sinh z}{iz\left(1 + \frac{1}{3!}(z - i\pi/2)^2 + \cdots\right)}\\ &= \frac{i}{i(i\pi/2)} = -\frac{2}{\pi}i\end{aligned}$$

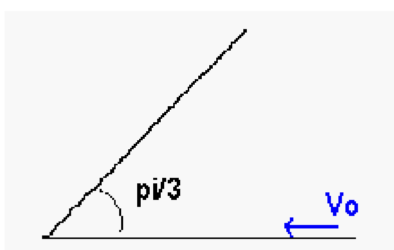
Since the limit exists, the pole is simple.

(d) $\ln\left(1 + z^2\right)$

The function has a branch point where $1 + z^2 = 0,$ or $z = \pm i.$

25. Incompressible fluid flows over a thin sheet from a distance

X_0 into a corner as shown in the diagram. The angle between the barriers is $\pi/3,$ and at $x = X_0,$ $\vec{\mathbf{v}} = V_0\hat{\mathbf{x}}.$ Assuming that the flow is as simple as possible, determine the streamlines of the flow. What is the velocity at $r = X_0/3, \theta = \pi/6?$



The velocity potential satsifies

$$\nabla^2\phi = 0$$

and thus we may look for a complex potential $\Phi = \phi + i\psi.$ Φ must be an analytic function in the region $\leq \theta \leq \pi/3$, and at $x = X_0$ we need

$$-\vec{\nabla}\phi = -V_0\hat{\mathbf{x}}$$

The streamline function must be a constant on the surfaces $\theta = 0$ and

$\theta = \pi/3.$ We may take this constant to be zero, and then the function $\sin 3\theta$ does the job. (The function $\sin 3n\theta$ would also work, but would lead to more complicated flow.) This suggests that we look at the analytic function

$kz^3 = kr^3 e^{3i\theta} = kr^3(\cos 3\theta + i \sin 3\theta).$ The imaginary part of this function satisfies the boundary conditions at the two surfaces. Thus the streamlines are given by

$$\psi = kr^3 \sin 3\theta = \text{ constant}$$

and the velocity is given by

$$\begin{aligned}\vec{\mathbf{v}} &= -\vec{\nabla}\phi = -\vec{\nabla}\left(kr^3 \cos 3\theta\right)\\ &= -3kr^2 \cos 3\theta\hat{\mathbf{r}} + 3kr^2 \sin 3\theta\hat{\boldsymbol{\theta}}\\ &= -3kr^2[\cos 3\theta(\hat{\mathbf{x}}\cos\theta + \hat{\mathbf{y}}\sin\theta) + \sin 3\theta(-\hat{\mathbf{x}}\sin\theta + \hat{\mathbf{y}}\cos\theta)]\\ &= 3kr^2[-\hat{\mathbf{x}}(\sin 3\theta\sin\theta + \cos 3\theta\cos\theta) - \hat{\mathbf{y}}(\cos 3\theta\sin\theta - \sin 3\theta\cos\theta)]\\ &= 3kr^2[-\hat{\mathbf{x}}(\cos 2\theta) + \hat{\mathbf{y}}(\sin 2\theta)]\end{aligned}$$

Thus at $\theta = 0$ we have

$$\vec{\mathbf{v}} = -3kr^2\hat{\mathbf{x}}$$

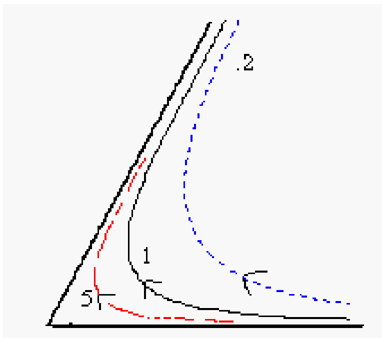
and so

$$3kX_0^2 = V_0$$

Thus the streamlines are given by

$$\begin{aligned}\psi &= \frac{V_0 r^3}{3X_0^2}\sin 3\theta = \text{ constant } = C\\ r &= \left(\frac{3X_0^2 C}{V_0 \sin 3\theta}\right)^{1/3}\end{aligned}$$

See Figure. $\frac{3X_0^2 C}{V_0} = 1$ (solid line), 5 (dashes), and 1/5 (dots).



The velocity is

$$\frac{\vec{\mathbf{v}}}{V_0} = \frac{r^2}{3X_0^2}(\hat{\mathbf{x}}\cos 2\theta + \hat{\mathbf{y}}\sin 2\theta)$$

and so at $r = X_0/2,$ $\theta = \pi/6$ we have

$$\begin{aligned}\frac{\vec{\mathbf{v}}}{V_0} &= \hat{\mathbf{x}}\frac{1}{12}\sin\frac{\pi}{3} + \hat{\mathbf{y}}\frac{1}{12}\cos\frac{\pi}{3}\\ \vec{\mathbf{v}} &= \frac{V_0}{24}\left(\sqrt{3}\,\hat{\mathbf{x}} + \hat{\mathbf{y}}\right)\end{aligned}$$

Chapter 2: Complex variables

26. Prove the Schwarz reflection principle: If a function $f(z)$ is analytic in a region including the real axis, and $f(x)$ is real when x is real,

$$f^*(z) = f(z^*)$$

Show that the result may be extended to functions that posess a Laurent series about the origin with real coefficients.

Verify the result for the functions (a) $f(z) = \cos z$ and (b) $f(z) = \tan^{-1}(z)$.

(c) Show that the result does not hold for all z if $f(z) = \ln(z)$ (the principal branch is assumed).

If the function is analytic, it may be expanded in a Taylor series about a point x_0 on the real axis:

$$f(z) = \sum_{n=0}^{\infty} a_n (z - x_0)^n$$

and since

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n$$

is real, then each of the a_n must be real. Then

$$\begin{aligned} f^*(z) &= \sum_n a_n^* (z - x_0)^n = \sum_n a_n \left(r^n e^{in\theta}\right)^* \\ &= \sum_n a_n r^n e^{-in\theta} = \sum_n a_n \left(re^{-i\theta}\right)^n = f(z^*) \end{aligned}$$

The proof extends trivially to the case where the series is a Laurent series with real coefficients.

$$\cos z = \frac{e^{iz} + e^{-iz}}{2} = \frac{e^{ix} e^{-y} + e^{-ix} e^{y}}{2}$$

So

$$(\cos z)^* = \frac{e^{-ix} e^{-y} + e^{ix} e^{y}}{2} = \frac{e^{-i(x-iy)} + e^{i(x-iy)}}{2} = \cos(z^*)$$

The function $w = \tan^{-1}(z)$ is trickier.

$$\tan w = z = \frac{1}{i} \frac{e^{iw} - e^{-iw}}{e^{iw} + e^{-iw}} = \frac{1}{i} \frac{e^{2iw} - 1}{e^{2iw} + 1}$$

Thus

$$\begin{aligned} iz\left(e^{2iw} + 1\right) &= e^{2iw} - 1 \\ e^{2iw} &= \frac{1 + iz}{1 - iz} \end{aligned}$$

and, choosing the principal branch of the logarithm,

$$\begin{aligned} w &= \frac{1}{2i}\ln\left(\frac{1+iz}{1-iz}\right) = \frac{1}{2i}[\ln(1+iz) - \ln(1-iz)] \\ &= \frac{1}{2i}\left(\ln\sqrt{(1-y)^2+x^2} + i\tan^{-1}\frac{x}{1-y} - \ln\sqrt{(1+y)^2+x^2} - i\tan^{-1}\frac{-x}{1+y}\right) \\ &= \frac{1}{2}\left(-i\ln\sqrt{(1-y)^2+x^2} + \tan^{-1}\frac{x}{1-y} + i\ln\sqrt{(1+y)^2+x^2} + \tan^{-1}\frac{x}{1+y}\right) \end{aligned}$$

Then

$$f^*(z) = \frac{1}{2}\left(i\ln\sqrt{(1-y)^2+x^2} + \tan^{-1}\frac{x}{1-y} - i\ln\sqrt{(1+y)^2+x^2} + \tan^{-1}\frac{x}{1+y}\right)$$

and

$$f(z^*) = \frac{1}{2}\left(-i\ln\sqrt{(1+y)^2+x^2} + \tan^{-1}\frac{x}{1+y} + i\ln\sqrt{(1-y)^2+x^2} + \tan^{-1}\frac{x}{1-y}\right)$$

and the two expressions are the same.

Note that this function has branch points at $z = \pm i$, but it is analytic on the real axis.

(c)

$$\ln z = \ln r + i\theta$$

We proceed by showing that the relation fails at one point, $z = -1$. At $z = -1$, on the real axis,

$$\ln z = i\pi$$

Then

$$(\ln z)^* = -i\pi$$

but

$$\ln(z^*) = \ln(z) = i\pi$$

27. Find the residues of each of the following functions at the point specified.

(a) $\frac{z-2}{z^2-1}$ at $z = 1$

First factor the function:

$$\frac{z-2}{z^2-1} = \frac{z-2}{(z+1)(z-1)}$$

The function has a simple pole at $z = 1$ and the residue is:

$$\lim_{z\to 1}(z-1)\frac{z-2}{(z+1)(z-1)} = \lim_{z\to 1}\frac{z-2}{z+1} = \boxed{-\frac{1}{2}}$$

(b) $\exp\left(\frac{1}{z} - 1\right)$ at $z = 0$

First rewrite the function:

$$\exp\left(\frac{1}{z} - 1\right) = e^{-1}\exp\left(\frac{1}{z}\right)$$

and then expand in a Laurent series:

$$f(z) = \frac{1}{e}\left(1 + \frac{1}{z} + \frac{1}{2z^2} + \frac{1}{3!z^3} + \cdots\right)$$

Now we can pick out the residue: it is the coefficient of $1/z$. The residue is

$1/e$

(c) $\frac{\sin z}{z^2}$ at the origin

The easiest method here is to find the Laurent series:

$$\begin{aligned}\frac{\sin z}{z^2} &= \frac{1}{z^2}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots\right) \\ &= \frac{1}{z} - \frac{z}{3!} + \frac{z^3}{5!} + \cdots\end{aligned}$$

and thus the residue is

1

.

(d) $\frac{\cos z}{1/2 - \sin z}$ at $z = \pi/6$

Since the denominator is a function $h(z) = 1/2 - \sin z$ that has a simple zero at $z = \pi/6$, we can use method 4. The derivative is

$$h'(z) = -\cos z$$

and so

$$\text{Res} = \lim_{z \to \pi/6} \frac{\cos z}{-\cos z} = \boxed{-1}$$

28. Evaluate the following integrals:

(a) $\oint_C \frac{\cos z}{z} dz$ where C is a circle of radius 2 centered at the origin.

The integrand has a simple pole at $z = 0$, which is inside the circle. The residue there is:

$$\lim_{z \to 0} \cos z = 1$$

and thus

$$\oint_C \frac{\cos z}{z} dz = 2\pi i$$

(b) $\oint_C \frac{\sinh z}{z-1} dz$ where C is a square of side 4 centered at the origin.

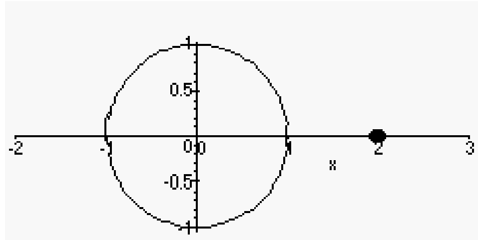
The integrand has a simple pole at $z = 1$, which is inside the square. The residue there is:

$$\lim_{z \to 1} \sinh z = \sinh 1$$

and thus

$$\oint_C \frac{\sinh z}{z - 1} dz = 2\pi i \sinh 1 = \pi i\left(e - \frac{1}{e}\right)$$

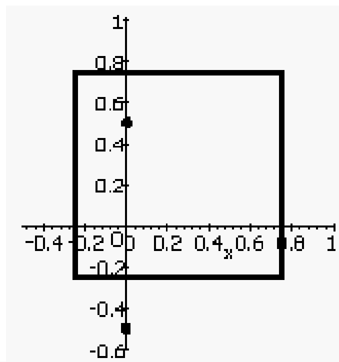
(c) $\oint_C \frac{z-1}{z+2}dz$ where C is a circle of radius 1 centered at the origin.



The integrand has a pole at $z = -2$, which is outside the circle. Thus:

$$\oint_C \frac{z-1}{z+2}dz = 0$$

(d) $\oint_C \frac{z}{4z^2+1}dz$ where C is a square of side 1 centered at the point $z = (1+i)/4$.



The integrand has two simple poles, at $z = \pm i/2$. Only one, at $z = i/2$, is inside the square. The residue at $z = i/2$ is

$$\lim_{z\to i/2}\left(z - \frac{i}{2}\right)\frac{z}{(2z-i)(2z+i)} = \lim_{z\to i/2}\frac{z}{2(2z+i)} = \frac{i}{4(2i)} = \frac{1}{8}$$

and so

$$\oint_C \frac{z}{4z^2+1} = 2\pi i\frac{1}{8} = \boxed{\frac{\pi i}{4}}$$

29. Evaluate the following integrals:

(a) $\int_0^{2\pi} \frac{1+\cos\theta}{2-\sin\theta}d\theta$

We evaluate as an ingtegral around the unit circle. Let $z = e^{i\theta}$. Then

$$\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right)$$

and

$$\sin\theta = \frac{1}{2i}\left(z - \frac{1}{z}\right)$$

and

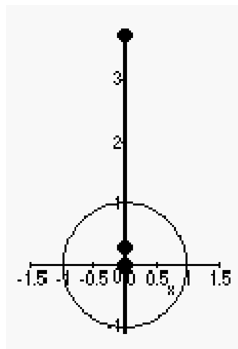
$$dz = ie^{i\theta}d\theta \Rightarrow d\theta = \frac{dz}{iz}$$

Then

$$\begin{aligned}\int_0^{2\pi}\frac{1+\cos\theta}{2-\sin\theta}d\theta &= \oint_C \frac{1+\frac{1}{2}\left(z+\frac{1}{z}\right)}{2-\frac{1}{2i}\left(z-\frac{1}{z}\right)}\frac{dz}{iz}\\ &= \oint_C \frac{2+\left(z+\frac{1}{z}\right)}{4i-\left(z-\frac{1}{z}\right)}\frac{dz}{z}\\ &= \oint_C \frac{2z+\left(z^2+1\right)}{4iz-\left(z^2-1\right)}\frac{dz}{z}\\ &= -\oint_C \frac{z^2+2z+1}{z^2-4iz-1}\frac{dz}{z}\end{aligned}$$

The integrand has poles at $z = 0$ and

$$z = \frac{4i \pm \sqrt{-16+4}}{2} = 2i \pm i\sqrt{3}$$



Only the poles at $z = 0$ and $z = \left(2-\sqrt{3}\right)i$ are inside the circle. The residues at these poles are -1 and

$$\begin{aligned}\lim_{z\to\left(2-\sqrt{3}\right)i}\frac{(z+1)^2}{z}\frac{1}{z-\left(2+\sqrt{3}\right)i} &= \frac{\left(\left(2-\sqrt{3}\right)i+1\right)^2}{\left(2-\sqrt{3}\right)i}\frac{1}{\left(2-\sqrt{3}\right)i-\left(2+\sqrt{3}\right)i}\\ &= \frac{3-2\sqrt{3}-\left(2-\sqrt{3}\right)i}{\left(2-\sqrt{3}\right)i}\frac{1}{i\sqrt{3}}\\ &= 1+i\frac{\sqrt{3}}{3}\end{aligned}$$

So the integral is:

$$\int_0^{2\pi}\frac{1+\cos\theta}{2-\sin\theta}d\theta = -2\pi i\left(-1+1+i\frac{\sqrt{3}}{3}\right) = \frac{2\sqrt{3}}{3}\pi$$

(b) $\int_0^{\pi}\frac{\sin^2\theta}{1+\cos^2\theta}d\theta$

Let $2\theta = \phi$. Then: $\cos 2\theta = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta$, so

$$\int_0^{\pi}\frac{\frac{1}{2}(1-\cos 2\theta)}{1+\frac{1}{2}(\cos 2\theta+1)}d\theta = \int_0^{2\pi}\frac{1-\cos\phi}{3+\cos\phi}\frac{d\phi}{2}$$

$$\frac{1}{2}\int_0^{2\pi}\frac{1-\cos\phi}{3+\cos\phi}d\phi = \frac{1}{2}\oint_{\text{unit circle}}\frac{1-\frac{1}{2}\left(z+\frac{1}{z}\right)}{3+\frac{1}{2}\left(z+\frac{1}{z}\right)}\frac{dz}{iz}$$
$$= \frac{1}{2i}\oint_{\text{unit circle}}\frac{2z-\left(z^2+1\right)}{6z+\left(z^2+1\right)}\frac{dz}{z}$$

The integrand has poles where

$$z = \frac{-6\pm\sqrt{36-4}}{2}$$
$$= -3\pm\sqrt{8} = -3\pm 2\sqrt{2} = -.17157, -5.8284$$

Only one of these poles is inside the unit circle. There is an additional pole at $z = 0.$ The residues are -1 and:

$$\lim_{z\to -3+2\sqrt{2}}\frac{-(z-1)^2}{z\left(z-\left(-3-2\sqrt{2}\right)\right)} = \frac{-\left(-3+2\sqrt{2}-1\right)^2}{\left(-3+2\sqrt{2}\right)\left(-3+2\sqrt{2}-\left(-3-2\sqrt{2}\right)\right)}$$
$$= \frac{-\left(-4+2\sqrt{2}\right)^2}{\left(-3+2\sqrt{2}\right)\left(4\sqrt{2}\right)} = \frac{-2\left(3-2\sqrt{2}\right)}{\left(-3+2\sqrt{2}\right)\sqrt{2}}$$
$$= \sqrt{2}$$

Thus the integral is:

$$\int_0^{\pi}\frac{\sin^2\theta}{1+\cos^2\theta}d\theta = \frac{1}{2i}2\pi i\left(-1+\sqrt{2}\right) = \pi\left(\sqrt{2}-1\right) = 1.3013$$

(c) $\int_0^{2\pi}\frac{1}{1+\sin^2\theta}d\theta$

$$\int_0^{2\pi}\frac{1}{1+\sin^2\theta}d\theta = \oint_{\text{unit circle}}\frac{1}{\left(1-\frac{1}{4}\left(z-\frac{1}{z}\right)^2\right)}\frac{dz}{iz}$$
$$= \oint_{\text{unit circle}}\frac{z}{\left(z^2-\frac{1}{4}\left(z^2-1\right)^2\right)i}dz$$

The integrand has poles where

$$z = \pm\frac{1}{2}\left(z^2-1\right)$$
$$z^2-1\mp 2z = 0$$
$$z = \frac{\pm 2\pm\sqrt{4+4}}{2} = \pm 1\pm\sqrt{2}$$

Of these 4 poles only 2 are inside the circle, at $z = 1-\sqrt{2}$ and $z = -1+\sqrt{2}.$ The residues are:

$$\lim_{z\to 1-\sqrt{2}}\frac{\left(z-1+\sqrt{2}\right)z}{\left(z^2-\frac{1}{4}\left(z^2-1\right)^2\right)i} = \lim_{z\to 1-\sqrt{2}}\frac{4iz}{\left(z-1-\sqrt{2}\right)\left(z^2+2z-1\right)}$$
$$= -\frac{1}{4}\sqrt{2}i$$

and

$$\lim_{z\to -1+\sqrt{2}} \frac{\left(z+1-\sqrt{2}\right)z}{\left(z^2-\frac{1}{4}\left(z^2-1\right)^2\right)i} = \lim_{z\to -1+\sqrt{2}} \frac{4iz}{\left(z^2-2z-1\right)\left(z+1+\sqrt{2}\right)}$$
$$= -\frac{1}{4}i\sqrt{2}$$

Thus the integral is:

$$\int_0^{2\pi} \frac{1}{1+\sin^2\theta}d\theta = 2\pi i\left(-\frac{1}{4}\sqrt{2}i\right)2 = \pi\sqrt{2}$$

(d) $\int_0^{\pi} \sin^{2n}\theta d\theta$

Since $\sin^2(-\theta) = \sin^2\theta$, we may rewrite the integral:

$$\int_0^{\pi} \sin^{2n}\theta d\theta = \frac{1}{2}\int_{-\pi}^{\pi} \sin^{2n}\theta d\theta = \frac{1}{2}\oint_{\text{unit circle}} \left[\frac{1}{2i}\left(z-\frac{1}{z}\right)\right]^{2n}\frac{dz}{iz}$$
$$= \frac{1}{2^{2n+1}}\frac{(-1)^n}{i}\oint_{\text{unit circle}} \left(z^2-1\right)^{2n}\frac{dz}{z^{2n+1}}$$

The integrand has a pole of order $2n+1$ at $z=0$. The residue is:

$$\lim_{z\to 0}\frac{1}{(2n)!}\frac{d^{2n}}{dz^{2n}}\left(z^2-1\right)^{2n} = \lim_{z\to 0}\frac{1}{(2n)!}\frac{d^{2n}}{dz^{2n}}\left[\left(z^2\right)^{2n} - 2n\left(z^2\right)^{2n-1} + \frac{(2n)(2n-1)}{2}\left(z^2\right)^{2n-2} + \cdots\right]$$

All the terms in powers $> 2n$ are zero in the limit, and all the terms in powers $< 2n$ differentiate away. Thus

$$\lim_{z\to 0}\frac{1}{(2n)!}\frac{d^{2n}}{dz^{2n}}(z^2-1)^{2n} = \lim_{z\to 0}\frac{1}{(2n)!}\frac{d^{2n}}{dz^{2n}}\left[\frac{(2n)(2n-1)\ldots(2n-n+1)}{2.3.4\ldots n}(z^2)^n\right]$$
$$= \frac{(-1)^n}{(2n)!}\frac{(2n)(2n-1)\ldots(n+1)}{2.3.4\ldots n}(2n)!$$
$$= \frac{(-1)^n(2n)!}{(n)!(n)!}$$

Thus the integral is:

$$\int_0^{\pi} \sin^{2n}\theta d\theta = \frac{1}{2^{2n+1}}\frac{(-1)^n}{i}2\pi i\frac{(-1)^n(2n)!}{[(n)!]^2}$$
$$= \frac{\pi}{2^{2n}}\frac{(2n)!}{[(n)!]^2}$$

30. Evaluate each of the following integrals:

(a) $\int_{-\infty}^{+\infty} \frac{1}{x^2+2}dx$

We close the contour with a big semicircle at infinity. The integral over the semicircle is:

$$\left|\int_{\text{semicircle}} \frac{1}{z^2+2}dz\right| \le \pi R\max\left|\frac{1}{z^2+2}\right|$$
$$= \pi R\max\frac{1}{R^2}\left|\frac{1}{1+2/z^2}\right| \le \frac{\pi}{R}\frac{1}{\left|1-2/|z|^2\right|}$$
$$\to 0 \text{ as } R = |z| \to \infty$$

The poles of the integrand are at $z = \pm i\sqrt{2}$. Only the pole at $z = +i\sqrt{2}$ is inside the contour. the residue is:

$$\lim_{z \to i\sqrt{2}} \frac{1}{z + i\sqrt{2}} = \frac{1}{2i\sqrt{2}}$$

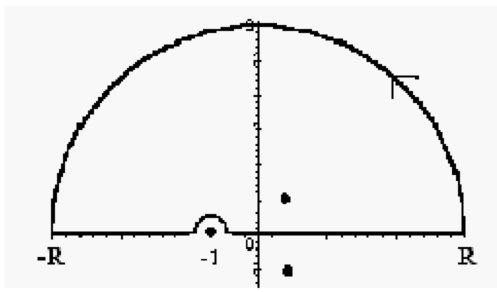
and the integral is:

$$\int_{-\infty}^{+\infty} \frac{1}{x^2 + 2} dx = 2\pi i \left(\frac{1}{2i\sqrt{2}} \right) = \frac{\sqrt{2}}{2}\pi$$

(b) $P\int_{-\infty}^{+\infty} \frac{x}{x^3+1} dx$

We close the contour with a big semicircle at infinity. The integral over the semicircle is:

$$\left| \int_{\text{semicircle}} \frac{z}{z^3 + 1} dz \right| \leq \pi R \max \left| \frac{z}{z^3 + 1} \right|$$
$$= \pi R \max \frac{R}{R^3} \left| \frac{1}{1 + 1/z^3} \right| \leq \frac{\pi}{R} \frac{1}{|1 - 1/|z|^3|}$$
$$\to 0 \text{ as } R = |z| \to \infty$$



The integrand has poles at

$$z = (-1)^{1/3} = e^{i\pi/3}, -1, e^{i5\pi/3}$$
$$= \frac{1}{2} + \frac{\sqrt{3}}{2}i, -1, \frac{1}{2} - \frac{\sqrt{3}}{2}i$$

The first of these is inside the contour and the second is on it. We'll evaluate the principal value. The residue at $z = \frac{1}{2} + \frac{\sqrt{3}}{2}i$ is:

$$\lim_{z \to \frac{1}{2} + \frac{\sqrt{3}}{2}i} \left(z - \frac{1}{2} - \frac{\sqrt{3}}{2}i \right) \frac{z}{z^3 + 1} = \lim_{z \to \frac{1}{2} + \frac{\sqrt{3}}{2}i} \frac{z}{(z+1)\left(z - \frac{1}{2} + \frac{\sqrt{3}}{2}i\right)}$$
$$= \frac{\frac{1}{2} + \frac{\sqrt{3}}{2}i}{\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i + 1\right)\left(\frac{1}{2} + \frac{\sqrt{3}}{2}i - \frac{1}{2} + \frac{\sqrt{3}}{2}i\right)}$$
$$= \frac{\frac{1}{2} + \frac{\sqrt{3}}{2}i}{\left(\frac{3}{2} + \frac{1}{2}i\sqrt{3}\right)\left(i\sqrt{3}\right)}$$
$$= -\frac{1}{6}i\left(\sqrt{3} + i\right)$$

The integral around the little semicircle where $z = -1 + \varepsilon e^{i\theta}$ is:

$$\lim_{\varepsilon\to 0}\int_{\pi}^{0}\frac{-1+\varepsilon e^{i\theta}}{(-1+\varepsilon e^{i\theta})^3+1}\,i\varepsilon e^{i\theta}d\theta = \lim_{\varepsilon\to 0}\int_{\pi}^{0}\frac{-1+\varepsilon e^{i\theta}}{-1+3\varepsilon e^{i\theta}-3\varepsilon^2 e^{2(i\theta)}+\varepsilon^3 e^{3(i\theta)}+1}\,i\varepsilon e^{i\theta}d\theta$$

$$= \lim_{\varepsilon\to 0} i\int_{\pi}^{0}\frac{-1+\varepsilon e^{i\theta}}{3e^{i\theta}-3\varepsilon e^{2(i\theta)}+\varepsilon^2 e^{3(i\theta)}}\,e^{i\theta}d\theta$$

$$= i\int_{\pi}^{0}\frac{-1}{3}d\theta = \frac{\pi}{3}i$$

Thus

$$P\int_{-\infty}^{+\infty}\frac{x}{x^3+1}dx + \frac{\pi}{3}i = 2\pi i\left(-\frac{1}{6}i\left(\sqrt{3}+i\right)\right) = \frac{1}{3}\pi\sqrt{3} + \frac{1}{3}i\pi$$

and so

$$P\int_{-\infty}^{+\infty}\frac{x}{x^3+1}dx = \frac{\sqrt{3}}{3}\pi$$

(c) $\int_{-\infty}^{+\infty}\frac{\cos\omega x}{x^2+9}dx$

There are no poles on the real axis, so we may assume that the integral is real. Then we may evaluate:

$$\int_{-\infty}^{+\infty}\frac{\cos\omega x}{x^2+9}dx = \mathrm{Re}\int_{-\infty}^{+\infty}\frac{\exp(i\omega x)}{x^2+9}dx$$

Close the contour with a big semicircle in the upper half plane. The integral along the semicircle is zero by Jordan's lemma. The poles are at $z = \pm 3i,$ but only the pole at $z = +3i$ is inside the contour. The residue is:

$$\lim_{z\to 3i}\frac{e^{i\omega z}}{z+3i} = \frac{e^{i\omega(3i)}}{6i} = \frac{e^{-3\omega}}{6i}$$

Then:

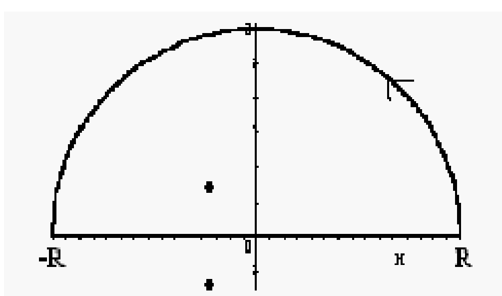
$$\int_{-\infty}^{+\infty}\frac{\cos\omega x}{x^2+9}dx = \mathrm{Re}\left(2\pi i\frac{e^{-3\omega}}{6i}\right) = \frac{\pi}{3}e^{-3\omega}$$

(d) $\int_{-\infty}^{+\infty}\frac{x\sin x}{x^2+2x+2}dx$

The poles of the integrand are where

$$z^2+2z+2 = 0$$

$$z = \frac{-2\pm\sqrt{4-8}}{2} = -1\pm i$$



None are on the real axis. Thus we may take:

$$\int_{-\infty}^{+\infty}\frac{x\sin x}{x^2+2x+2}dx = \mathrm{Im}\int_{-\infty}^{+\infty}\frac{xe^{ix}}{x^2+2x+2}dx$$

and close the contour with a big semicircle in the upper half plane. The integral along the semicircle is zero by Jordan's lemma. Only the pole at $z = -1+i$ is inside the contour. The residue is:

$$\lim_{z\to -1+i} \frac{ze^{iz}}{z-(-1-i)} = \frac{(-1+i)\exp(i(-1+i))}{2i} = \frac{(-1+i)\exp(-1-i)}{2i}$$

Thus the integral is:

$$\begin{aligned}\int_{-\infty}^{+\infty} \frac{x\sin x}{x^2+2x+2}dx &= \operatorname{Im}\left(2\pi i\frac{(-1+i)\exp(-1-i)}{2i}\right)\\ &= \operatorname{Im}\left(\pi e^{-1}(\sin 1-\cos 1)+i\pi e^{-1}(\cos 1+\sin 1)\right)\\ &= \pi e^{-1}(\cos 1+\sin 1)\\ &= 1.597\end{aligned}$$

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31. Use a rectangular contour to evaluate the integrals:

(a) $\int_{-\infty}^{+\infty} \frac{e^{ax}}{1+e^{bx}} dx$ $0 < \text{Re}\, a < b$

The upper side of the rectangle should be at $y = 2\pi/b$ (for real b). Then on the upper side:

$$\int_{-\infty+i2\pi/b}^{+\infty+i2\pi/b} \frac{e^{az}}{1+e^{bz}} dz = \int_{-\infty}^{+\infty} \frac{e^{ax} e^{ia2\pi/b}}{1+e^{bx} e^{i2\pi}} dx = e^{ia2\pi/b} \int_{-\infty}^{+\infty} \frac{e^{ax}}{1+e^{bx}} dx$$

Then around the whole rectangle:

$$\begin{aligned}\oint_{\text{rectangle}} \frac{e^{az}}{1+e^{bz}} dz &= \lim_{R\to\infty} \int_{-R}^{+R} \frac{e^{ax}}{1+e^{bx}} dx + \int_{R}^{R+i2\pi/b} \frac{e^{az}}{1+e^{bz}} dz \\ &\quad + \int_{R+i2\pi/b}^{-R+i2\pi/b} \frac{e^{az}}{1+e^{bz}} dz + \int_{-R+i2\pi/b}^{-R} \frac{e^{az}}{1+e^{bz}} dz \\ &= \lim_{R\to\infty} \left(1 - e^{ia2\pi/b}\right) \int_{-R}^{+R} \frac{e^{ax}}{1+e^{bx}} dx + \int_{R}^{R+i2\pi/b} \frac{e^{az}}{1+e^{bz}} dz \\ &\quad + \int_{-R+i2\pi/b}^{-R} \frac{e^{az}}{1+e^{bz}} dz\end{aligned}$$

Along the end at $x = R$, with $a = \alpha + i\gamma$

$$\int_{R}^{R+i2\pi/b} \frac{e^{az}}{1+e^{bz}} dz = \int_{0}^{2\pi/b} \frac{e^{\alpha R - \gamma y} e^{i\alpha y + i\gamma R}}{1+e^{bR} e^{iby}} i dy = e^{(a-b)R} e^{i\gamma R} \int_{0}^{2\pi/b} \frac{e^{i\alpha y} e^{-\gamma y}}{e^{-bR} + e^{iby}} i dy$$

$$\left| \int_{R}^{R+i2\pi/b} \frac{e^{az}}{1+e^{bz}} dz \right| \le e^{\text{Re}(a-b)R} \max \left| \frac{e^{i\alpha y} e^{-\gamma y}}{e^{-bR} + e^{iby}} \right| \frac{2\pi}{b} \le e^{\text{Re}(a-b)R} \frac{1}{\left|e^{iby}\right| - e^{-bR}} \frac{2\pi}{b}$$

$$= e^{\text{Re}(a-b)R} \frac{1}{1 - e^{-bR}} \frac{2\pi}{b} \to 0 \text{ as } R \to \infty$$

provided that $\text{Re}\, a < b$.

Along the end at $x = -R$:

$$\int_{-R+i2\pi/b}^{-R} \frac{e^{az}}{1+e^{bz}} dz = \int_{2\pi/b}^{0} \frac{e^{-\alpha R - \gamma y} e^{i\alpha y - i\gamma R}}{1+e^{-bR} e^{iby}} i dy$$

$$\left| \int_{-R+i2\pi/b}^{-R} \frac{e^{az}}{1+e^{bz}} dz \right| \le e^{-\text{Re}\, aR} \max \left| \frac{e^{i\alpha y} e^{-\gamma y}}{1+e^{-bR} e^{iby}} \right| \frac{2\pi}{b}$$

$$= e^{-\text{Re}\, aR} \frac{1}{1 - e^{-bR}} \frac{2\pi}{b} \to 0 \text{ as } R \to \infty$$

provided that $\text{Re}\, a > 0$. Thus we have:

$$\oint_{\text{rectangle}} \frac{e^{az}}{1+e^{bz}} dz = \left(1 - e^{ia2\pi/b}\right) \int_{-\infty}^{+\infty} \frac{e^{ax}}{1+e^{bx}} dx$$

Now the integrand has a pole where

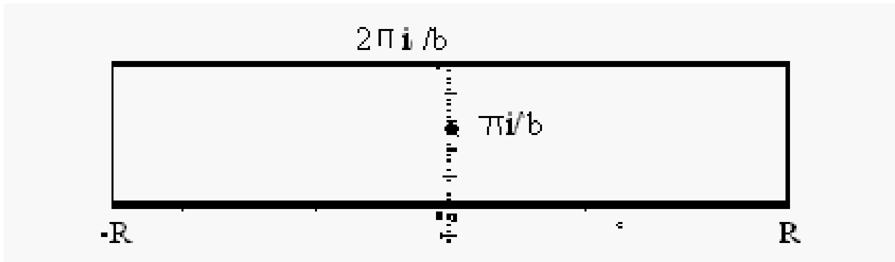
$$1 + e^{bz} = 0$$

or

$$z = i\pi/b$$

which is inside the contour. The residue there may be found from method 4:

$$\text{residue} = \lim_{z \to i\pi/b} \frac{e^{az}}{be^{bz}} = \frac{1}{b}\frac{e^{ai\pi/b}}{-1}$$



and so the integral is:

$$\oint_{\text{rectangle}} \frac{e^{az}}{1+e^{bz}} dz = 2\pi i\left(\frac{-e^{i\pi a/b}}{b}\right)$$

and thus

$$\int_{-\infty}^{+\infty} \frac{e^{ax}}{1+e^{bx}} dx = \frac{2\pi i}{b}\frac{-e^{i\pi a/b}}{1-e^{ia2\pi/b}} = \frac{2\pi i}{b}\frac{1}{\left(e^{i\pi a/b} - e^{-i\pi a/b}\right)}$$
$$= \frac{\pi}{b}\frac{1}{\sin \pi a/b}$$

The result is real, as expected.

(b) $\int_{-\infty}^{+\infty} \frac{\sinh ax}{\sinh 4ax} dx$

We put the top of the rectangle at $z = i\pi/a$. Then:

$$\oint_{\text{rectangle}} \frac{\sinh az}{\sinh 4az} dx = \int_{-\infty}^{+\infty} \frac{\sinh ax}{\sinh 4ax} dx + \int_{R}^{R+i\pi/a} \frac{\sinh az}{\sinh 4az} dz$$
$$+ \int_{R+i\pi/a}^{-R+i\pi/a} \frac{\sinh az}{\sinh 4az} + \int_{-R+i\pi/a}^{-R} \frac{\sinh az}{\sinh 4az}$$
$$= \int_{-\infty}^{+\infty} \frac{\sinh ax}{\sinh 4ax} dx + \int_{-\infty}^{+\infty} \frac{\sinh ax}{\sinh 4ax} dx = 2\int_{-\infty}^{+\infty} \frac{\sinh ax}{\sinh 4ax} dx$$

There are poles inside at $z = in\pi/4a$, $n = 1, 2,$ and 3. The residues are:

$$\lim_{z \to in\pi/4a} \frac{\sinh az}{4a\cosh 4az} = \frac{\sinh in\pi/4}{4a\cosh in\pi} = \frac{i\sin\frac{1}{4}n\pi}{4a\cos n\pi} = \frac{i}{4a}\frac{\sin\frac{n\pi}{4}}{(-1)^n}$$

Thus summing the 3 residues, we get:

$$\int_{-\infty}^{+\infty} \frac{\sinh ax}{\sinh 4ax} dx = \pi i\frac{i}{4a}\left(-\sin\frac{\pi}{4} + \sin\frac{\pi}{2} - \sin\frac{3\pi}{4}\right) = -\frac{1}{4}\frac{\pi}{a}\left(-\sqrt{2}+1\right) = \frac{\pi}{a}\frac{\sqrt{2}-1}{4}$$

Note: the singularities on the top line at $z = i\pi/a$ and on the x-axis at $x = 0$ are removable.

(c) $\int_{-\infty}^{+\infty} \frac{x^2}{\cosh ax} dx$

Again we want the integral along the upper side of the contour to be a multiple of that along the lower. Here we find there is an additional integral that we have already evaluated. We can make use of the results

$$\cosh(az) = \cosh(a(x+iy)) = \cosh ax \cos ay + i \sinh ax \sin ay$$

So we can take $y = \pi/a$ on the upper side of the rectangle, so that $\cos ay = \cos\pi = -1$ and $\sin ay = \sin\pi = 0$. Then:

$$\oint_{\text{rectangle}} \frac{z^2}{\cosh az}dz = \lim_{R\to\infty}\left[\int_{-R}^{+R}\frac{x^2}{\cosh ax}dx + \int_{R}^{R+i\frac{\pi}{a}}\frac{z^2}{\cosh az}dz\right] + \lim_{R\to\infty}\left[\int_{R+i\frac{\pi}{a}}^{-R+i\frac{\pi}{a}}\frac{z^2}{\cosh az}dz + \int_{-R+i\frac{\pi}{a}}^{-R}\frac{z^2}{\cosh az}dz\right]$$

On the top side:

$$\int_{R+i\frac{\pi}{a}}^{-R+i\frac{\pi}{a}}\frac{z^2}{\cosh az}dz = \int_{R}^{-R}\frac{(x+i\pi/a)^2}{\cosh(ax+i\pi)}dx = \int_{-R}^{+R}\frac{x^2}{\cosh ax}dx + i\frac{2\pi}{a}\int_{-R}^{+R}\frac{x}{\cosh ax}dx - \frac{\pi^2}{a^2}\int_{-R}^{+R}\frac{1}{\cosh ax}dx$$

TThe second integral is zero because the integrand is odd and there are no poles on the real axis. The third integral was evaluated in § 2.7.3, Example 2.22. The result is $\pi/a.$ Thus:

$$\int_{R+i\frac{\pi}{a}}^{-R+i\frac{\pi}{a}}\frac{z^2}{\cosh az}dz = \int_{-R}^{+R}\frac{x^2}{\cosh ax}dx - \left(\frac{\pi}{a}\right)^3$$

Now at the two ends, we have:

$$\left|\int_{R}^{R+i\frac{\pi}{a}}\frac{z^2}{\cosh az}dz\right| = \left|\int_{0}^{i\frac{\pi}{a}}\frac{(R+iy)^2}{e^{aR}e^{iay} - e^{-aR}e^{-iay}}idy\right| \le \frac{\pi}{a}e^{-aR}\left|\frac{2\left(R^2+\pi^2/a^2\right)}{1-e^{-4aR}}\right| \to 0 \text{ as } R\to\infty$$

for $a > 0.$ A similar proof works for Re$a < 0$: just factor out e^{-aR} in the denominator.

Now we have:

$$\oint_{\text{rectangle}}\frac{z^2}{\cosh az}dz = 2\lim_{R\to\infty}\int_{-R}^{+R}\frac{x^2}{\cosh ax}dx - \left(\frac{\pi}{a}\right)^3 = 2\pi i\sum(\text{residues})$$

There is a pole where

$$\cosh az = 0$$

i.e. at

$$z = i\frac{\pi}{2a}$$

and the residue there is:

$$\lim_{z\to i\pi/2a}\frac{z^2}{a\sinh z} = \frac{(i\pi/2a)^2}{ia} = -\frac{\pi^2}{4ia^3}$$

and therefore

$$\int_{-\infty}^{+\infty} \frac{x^2}{\cosh ax} dx = \frac{1}{2}\left(\frac{\pi}{a}\right)^3 - \pi i \frac{\pi^2}{4ia^3}$$
$$= \frac{1}{2}\left(\frac{\pi}{a}\right)^3 - \frac{1}{4}\frac{\pi^3}{a^3} = \frac{1}{4}\frac{\pi^3}{a^3}$$

32. Evaluate the integrals

(a)

$$\int_0^{+\infty} \frac{x^{3/2}}{1+x^3} dx$$

The integrand has a branch point at the origin and a branch cut, which we may take along the positive real axis. Let's evaluate

$$\oint_C \frac{z^{3/2}}{1+z^3} dz$$

where C is the keyhole contour in Figure 2.36.

Along the bottom of the branch cut:

$$\int_{0,\text{bottom}}^{\infty} \frac{z^{3/2}}{1+z^3} dz = \int_{0,\text{bottom}}^{\infty} \frac{r^{3/2} e^{3\pi i}}{1+r^3} dr = -\int_0^{\infty} \frac{r^{3/2}}{1+r^3} dr$$

Now along the big circle, we have:

$$\left|\int_{\text{circle}} \frac{z^{3/2}}{1+z^3} dz\right| = \lim_{R\to\infty}\left|\int_0^{2\pi} \frac{R^{3/2}}{1+R^3 e^{3i\theta}} iR\, e^{i\theta} d\theta\right|$$
$$\le \frac{2\pi R^{5/2}}{R^3-1} \to \frac{2\pi}{R^{1/2}} \to 0 \text{ as } R \to \infty$$

The integrand has poles at $z = (-1)^{1/3} = e^{i\pi/3} = \frac{1}{2} + \frac{\sqrt{3}}{2}i$, $e^{3\pi i/3} = -1$, $e^{5\pi i/3} = \frac{1}{2} - \frac{\sqrt{3}}{2}i.$ All three are inside the contour. the residues are:

$$\text{Res}\left(e^{i\pi/3}\right) = \lim_{z\to -1} \frac{z^{3/2}}{3z^2} = \lim_{z\to -1} \frac{1}{3z^{1/2}} = \frac{1}{3e^{i\pi/6}} = \frac{\sqrt{3}-i}{6}$$

$$\text{Res}(-1) = \frac{1}{3e^{i\pi/2}} = -\frac{i}{3}$$

and

$$\text{Res}\left(e^{5\pi i/3}\right) = \frac{1}{3e^{5\pi i/6}} = \frac{-\sqrt{3}-i}{6}$$

So

$$\oint_C \frac{z^{3/2}}{1+z^3} dz = 2\int_0^{+\infty} \frac{x^{3/2}}{1+x^3} dx = 2\pi i\left(\frac{\sqrt{3}-i}{6} - \frac{i}{3} - \frac{\sqrt{3}+i}{6}\right)$$
$$\int_0^{+\infty} \frac{x^{3/2}}{1+x^3} dx = \frac{2}{3}\pi$$

(b)

$$\int_0^\infty \frac{x^{1/3}}{x^2+1}dx$$

Use the keyhole contour. There are two poles inside, at $z = \pm i$, that is, $z = e^{i\pi/2}$ and $e^{i3\pi/2}$.

$$\oint_C \frac{z^{1/3}}{z^2+1}dz = \int_0^\infty \frac{r^{1/3}}{r^2+1}dr + \int_\infty^0 \frac{r^{1/3}e^{2\pi i/3}}{r^2+1}dr = 2\pi i\left(\frac{e^{i\pi/6}}{2i} + \frac{e^{i3\pi/6}}{-2i}\right)$$

$$\left(1 - e^{2\pi i/3}\right)I = \pi\left(e^{i\pi/6} - e^{i3\pi/6}\right)$$

$$I = \frac{\pi\left(e^{i\pi/6} - e^{i3\pi/6}\right)}{\left(1 - e^{2\pi i/3}\right)} = \pi\frac{e^{i\pi/6}\left(1 - e^{2\pi i/6}\right)}{\left(1 - e^{2\pi i/3}\right)}$$

$$= \pi\frac{e^{i\pi/6}\left(1 - e^{i\pi/3}\right)}{\left(1 - e^{-\pi i/3}\right)} = \frac{\pi}{2}\frac{\left(\sqrt{3}+i\right)\left(1 - i\sqrt{3}\right)}{\left(3 - i\sqrt{3}\right)}$$

$$= \frac{\pi}{6}\frac{\left(\sqrt{3}+i\right)\left(1 - i\sqrt{3}\right)}{\left(1 - i\sqrt{3}/3\right)}\frac{\left(1 + i\sqrt{3}/3\right)}{\left(1 + i\sqrt{3}/3\right)}$$

$$= \pi\frac{\sqrt{3}}{3}$$

Check that the integral around the small circle goes to zero:

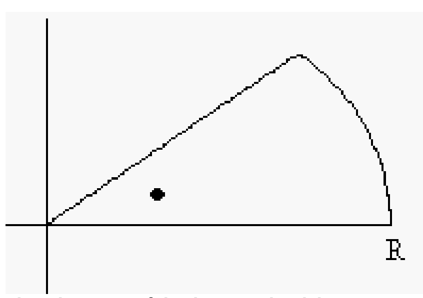
$$\int_0^{2\pi} \frac{\varepsilon^{1/3}e^{i\theta/3}\varepsilon e^{i\theta}}{1+\varepsilon^2 e^{2i\theta}}id\theta \to 0 \text{ as } \varepsilon \to 0$$

33. Evaluate the integral

$$\int_0^\infty \frac{dx}{1+x^{2N}}$$

by integrating over a pie-slice contour with sides at $\phi = 0$ and at $\phi = \pi/N$, $0 \le r < \infty$.

We evaluate $\int \frac{dz}{1+z^{2N}}$ over the suggested contour.



On the curved part of the contour, the integral is bounded by

$$\left|\int \frac{dz}{1+z^{2N}}\right| \le \frac{\pi R}{N}\frac{2}{R^{2N}} \to 0 \text{ as } R \to \infty$$

On the straight line at $\phi = \pi/N$, $z = re^{i\pi/N}$ and we have

$$\int_\infty^0 \frac{dre^{i\pi/N}}{1+r^{2N}e^{i2\pi}} = -e^{i\pi/N}\int_0^\infty \frac{dr}{1+r^{2N}}$$

Thus

$$\int_0^\infty \frac{dx}{1+x^{2N}} = \frac{1}{1-e^{i\pi/N}}\oint \frac{dz}{1+z^{2N}}$$

The integrand has a pole where

$$1+z^{2N} = 0$$

or

$$z_p = e^{i\frac{\pi}{2N}}$$

(the other roots are outside the contour) and the residue there is

$$\begin{aligned}\lim_{z\to z_p} \frac{1}{2Nz^{2N-1}} &= \frac{1}{2N}\exp\left(-i\frac{\pi}{2N}(2N-1)\right) = \frac{1}{2N}\exp\left(-i\pi + i\frac{\pi}{2N}\right)\\ &= -\frac{1}{2N}\exp\left(i\frac{\pi}{2N}\right)\end{aligned}$$

Thus

$$\begin{aligned}\int_0^\infty \frac{dx}{1+x^{2N}} &= \frac{1}{1-e^{i\pi/N}}2\pi i\left(-\frac{1}{2N}\exp\left(i\frac{\pi}{2N}\right)\right)\\ &= -\frac{\pi}{2N}\frac{2i}{\exp\left(-i\frac{\pi}{2N}\right)-\exp\left(i\frac{\pi}{2N}\right)}\\ &= \frac{\pi}{2N\sin(\pi/2N)}\end{aligned}$$

34. Evaluate the integral

$$\int_0^\infty e^{ix^2}dx$$

along the positive real axis by making the change of variable $u^2 = -ix^2$. Take care to discuss the path of integration for the $u-$integral. Use the Cauchy theorem to show that the resulting $u-$integral may be reduced to a known integral along the real axis. Hence show that $\int_0^\infty \sin x^2 dx = \int_0^\infty \cos x^2 dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}$. (The result has numerous applications in physics, for example in signal propagation.)**.**

$$\sin x^2 = \operatorname{Im} e^{ix^2}$$

and letting $u = \sqrt{-i}\,x = e^{-i\pi/4}x,$ then

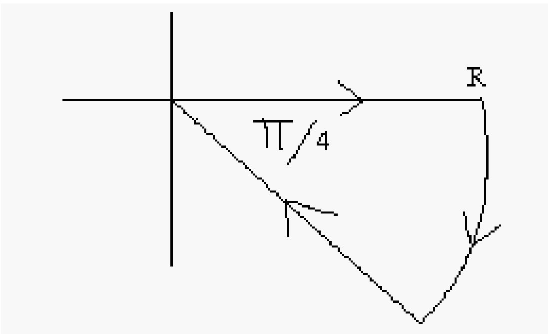
$$\int_0^\infty e^{ix^2}dx = \int_0^\infty e^{-u^2}e^{i\pi/4}du$$

The path of integration is moved off the real axis: when $x = R\to\infty,$ then $u = e^{-i\pi/4}R\to\infty$ along a line making an angle $-\pi/4$ with the real axis. But the integral around the closed contour formed by the real axis, this line, and the arc at infinity is zero because there are no poles of the integrand inside, and the integral along the arc $\to 0$:

$$\int_{arc} = \int_0^{-\pi/4} e^{-R^2 e^{2i\theta}} Rie^{i\theta} d\theta = iR \int_0^{-\pi/4} e^{-R^2(\cos 2\theta + i \sin 2\theta)} e^{i\theta} d\theta$$

and cos2θ is positive throughout the range $-\pi/4 \leq \theta \leq 0,$ so the integral $\to 0.$ Thus the integral along the sloping line equals the integral along the real axis. Thus

$$\int_0^\infty e^{ix^2} dx = e^{i\pi/4} \int_0^\infty e^{-u^2} du = e^{i\pi/4} \frac{\sqrt{\pi}}{2} = \frac{1}{\sqrt{2}}(1+i)\frac{\sqrt{\pi}}{2} = \frac{1}{2}\sqrt{\frac{\pi}{2}}(1+i)$$



Problem 34.

Thus

$$\int_0^\infty \sin x^2 dx = \int_0^\infty \cos x^2 dx = \frac{1}{2}\sqrt{\frac{\pi}{2}}$$

35. The power radiated per unit solid angle by a charge undergoing simple harmonic motion is

$$\frac{dP}{d\Omega} = K \sin^2\theta \frac{\cos^2 \omega t}{(1 + \beta \cos\theta \sin \omega t)^5}$$

where the constant $K = e^2 c\beta^4 / 4\pi a^2$ and $\beta = a\omega/c$ is the speed amplitude$/c$ (see.e.g, Jackson p 701). Using methods from section 7.2.1, perform the time average over one period to show that

$$\left\langle \frac{dP}{d\Omega} \right\rangle = \frac{K}{8} \sin^2\theta \frac{4 + \beta^2\cos^2\theta}{\left(1 - \beta^2\cos^2\theta\right)^{7/2}}$$

Write $\phi = \omega t.$ Then the time average is:

$$\left\langle \frac{dP}{d\Omega} \right\rangle = \frac{K\sin^2\theta}{2\pi} \int_0^{2\pi} \frac{\cos^2\phi}{(1 + \beta\cos\theta\sin\phi)^5} d\phi$$

We can simplify by doing one integration by parts:

$$\left\langle \frac{dP}{d\Omega} \right\rangle = \frac{K\sin^2\theta}{(-4\beta\cos\theta)2\pi} \left[\left. \frac{\cos\phi}{(1+\beta\cos\theta\sin\phi)^4} \right|_0^{2\pi} - \int_0^{2\pi} \frac{-\sin\phi}{(1+\beta\cos\theta\sin\phi)^4} d\phi \right]$$

$$= \frac{K\sin^2\theta}{8\pi\beta\cos\theta} \left[\int_0^{2\pi} \frac{-\sin\phi}{(1+\beta\cos\theta\sin\phi)^4} d\phi \right]$$

We convert to an integral over the unit circle in the $z-$plane and write $\cos\phi = \frac{1}{2}\left(z + \frac{1}{z}\right)$ and

$\sin\phi = \frac{1}{2i}\left(z - \frac{1}{z}\right).$ Also $dz = e^{i\phi} i\, d\phi$, so $d\phi = dz/iz.$ With $A = \beta\cos\theta,$ the integral is

$$\int_0^{2\pi} \frac{-\sin\phi}{(1 + A\sin\phi)^4} d\phi = -\frac{1}{2i}\oint_{\text{unit circle}} \frac{\left(z - \frac{1}{z}\right)}{\left(1 + \frac{A}{2i}\left(z - \frac{1}{z}\right)\right)^4}\frac{dz}{iz}$$
$$= \frac{1}{2}\oint_{\text{unit circle}} \frac{z^2\left(z^2 - 1\right)}{\left(z - iB\left(z^2 - 1\right)\right)^4} dz$$

where $B = A/2.$ The denominator is

$$\left(z - iBz^2 + iB\right)^4 = (-iB)^5\left(z^2 - \frac{z}{iB} - 1\right)^4 = (-iB)^4\left(z^2 + i\frac{z}{B} - 1\right)^4$$

and there are 2 fourth-order poles at:

$$z = \frac{-i/B \pm \sqrt{-1/B^2 + 4}}{2} = \pm\sqrt{1 - 1/\left(4B^2\right)} - \frac{i}{2B} = \frac{\pm\sqrt{A^2 - 1} - i}{A}$$

where $A < 1$ and so the square root is imaginary:

$$z = \frac{i}{A}\left[-1 \pm \sqrt{1 - A^2}\right]$$

Only one of the two poles is inside the unit circle:

$$z_p = \frac{i}{A}\left[-1 + \sqrt{1 - A^2}\right]$$

Now we find the residue using method 3:

$$\operatorname{Res} f(z_p) = \lim_{z\to z_p} \frac{1}{3!}\frac{d^3}{dz^3}(z - z_p)^4 \frac{z^2\left(z^2 - 1\right)}{((z - z_p)(z - z_0))^4}$$
$$= \lim_{z\to z_p} \frac{1}{3!}\frac{d^3}{dz^3}\frac{\left(z^4 - z^2\right)}{(z - z_0)^4}$$

where

$$z_0 = \frac{i}{A}\left[-1 - \sqrt{1 - A^2}\right]$$

Thus

$$\operatorname{Res} f(z_p) = \lim_{z\to z_p} \frac{1}{3!}\frac{d^2}{dz^2}\left[\frac{4z^3 - 2z}{(z - z_0)^4} - 4\frac{z^4 - z^2}{(z - z_0)^5}\right]$$
$$= \lim_{z\to z_p} \frac{1}{3}\frac{d}{dz}\left(\frac{6z^2 - 1}{(z - z_0)^4} - 8z\frac{2z^2 - 1}{(z - z_0)^5} + 10\frac{z^4 - z^2}{(z - z_0)^6}\right)$$
$$= \lim_{z\to z_p} 4\left(\frac{z}{(z - z_0)^4} - \frac{6z^2 - 1}{(z - z_0)^5} + 5z\frac{2z^2 - 1}{(z - z_0)^6} - 5\frac{z^2(z^2 - 1)}{(z - z_0)^7}\right)$$

Now

$$z_p - z_0 = 2i\frac{\sqrt{1 - A^2}}{A} = \frac{2iC}{A}$$

and

$$z_p^2 - 1 = \frac{z_p}{iB} = \frac{2}{iA}\frac{i}{A}\left[-1 + \sqrt{1 - A^2}\,\right] = \frac{2}{A^2}\left[-1 + \sqrt{1 - A^2}\,\right]$$
$$= \frac{2}{A^2}(C - 1) \text{ where } C = \sqrt{1 - A^2} = \sqrt{1 - \beta^2\cos^2\theta}$$

So

$$z_p = i\left(\frac{C-1}{A}\right) \text{ and } z_p^2 = -\frac{(C-1)^2}{A^2}$$

Thus

$$\operatorname{Res} f(z_p) = 4\left(\frac{z}{(z-z_0)^4} - \frac{6z^2-1}{(z-z_0)^5} + 5z\frac{2z^2-1}{(z-z_0)^6} - 5\frac{z^2\left(z^2-1\right)}{(z-z_0)^7}\right)$$
$$= 4\left(\frac{A}{2iC}\right)^4\left(\begin{array}{c} i\left(\frac{C-1}{A}\right) - \frac{-6\frac{(C-1)^2}{A^2}-1}{\frac{2iC}{A}} + \\ 5i\left(\frac{C-1}{A}\right)\frac{-2\frac{(C-1)^2}{A^2}-1}{\left(\frac{2iC}{A}\right)^2} + 5\frac{(C-1)^2}{A^2}\frac{\frac{2}{A^2}(C-1)}{\left(\frac{2iC}{A}\right)^3}\end{array}\right)$$
$$= \frac{1}{4}\frac{A^4}{C^4}\left(\begin{array}{c} i\left(\frac{C-1}{A}\right) - i\frac{6(C-1)^2+A^2}{2AC} + \\ \frac{5}{4}i\left(\frac{C-1}{A}\right)\frac{2(C-1)^2+A^2}{C^2} + i\frac{5}{4}\frac{(C-1)^3}{AC^3}\end{array}\right)$$
$$= \frac{i}{16}\frac{A^3}{C^7}\left(\begin{array}{c} 4(C-1)C^3 - 2\left(6(C-1)^2 + A^2\right)C^2 \\ +5C(C-1)\left(2(C-1)^2 + A^2\right) + 5(C-1)^3\end{array}\right)$$
$$= \frac{i}{16}\frac{A^3}{C^7}\left(2C^4 - 5C^3 + 3C^2 + 3C^2A^2 + 5C - 5A^2C - 5\right)$$
$$= -\frac{i}{16}\frac{A^5}{C^7}\left(4 + A^2\right)$$
$$= \frac{-i\left(\beta^5\cos^5\theta\right)}{16}\frac{4 + \beta^2\cos^2\theta}{\left(1 - \beta^2\cos^2\theta\right)^{7/2}}$$

and thus the integral is:

$$\frac{1}{2}\oint_{\text{unit circle}}\frac{z\left(z^2-1\right)}{\left(z - iB\left(z^2-1\right)\right)^4}dz = \frac{2\pi i}{(-iB)^4}\frac{K\sin^2\theta}{8\pi\beta\cos\theta}\left(\frac{\beta^4\cos^4\theta}{16}\right)\frac{1 + 4\beta^2\cos^2\theta}{\left(1 - \beta^2\cos^2\theta\right)^{7/2}}$$
$$= 2\pi\beta\cos\theta\frac{4 + \beta^2\cos^2\theta}{\left(1 - \beta^2\cos^2\theta\right)^{7/2}}$$

and finally

$$\left\langle \frac{dP}{d\Omega} \right\rangle = \frac{K\sin^2\theta}{8\pi\beta\cos\theta}2\pi\beta\cos\theta\frac{4 + \beta^2\cos^2\theta}{\left(1 - \beta^2\cos^2\theta\right)^{7/2}}$$
$$= \frac{1}{4}K\sin^2\theta\frac{4 + \beta^2\cos^2\theta}{\left(1 - \beta^2\cos^2\theta\right)^{7/2}}$$

as required.

36. Langmuir waves. The equation for the Langmuir wave dispersion relation takes the form:

$$0 = 1 + \frac{\omega_p^2}{k}\int_{-\infty}^{+\infty}\frac{\partial f(v)/\partial v}{\omega - kv}dv$$

where ω_p is the plasma frequency $ne^2/\varepsilon_0 m$ and $f(v)$ is the 1-dimensional Maxwellian

$$f(v) = \sqrt{\frac{m}{2\pi k_B T}}\exp\left(-\frac{mv^2}{2k_B T}\right)$$

Notice that the integrand has a singularity at $v = \omega/k.$ Landau showed that the integral is to be regarded as an integral along the real axis in the complex $v-$ plane, and that the correct integration path passes around and **under** the pole.

(a) Show that the integral may be expressed as:

$$\int_{-\infty}^{+\infty}\frac{\partial f(v)/\partial v}{\omega - kv}dv = P\int_{-\infty}^{+\infty}\frac{\partial f(v)/\partial v}{\omega - kv}dv - \frac{i\pi}{k}\frac{\partial f}{\partial v}\bigg|_{v=\omega/k}$$

(cf Section 7.3.5)

The principal value is defined in the section referred to

$$P\int_{-\infty}^{+\infty}\frac{\partial f(v)/\partial v}{\omega - kv}dv = \lim_{\varepsilon\to 0}\left(\int_{-\infty}^{\omega/k-\varepsilon} + \int_{\omega/k+\varepsilon}^{+\infty}\right)\frac{\partial f(v)/\partial v}{\omega - kv}dv$$

We need to add to this the integral around the small semicircle that passes beneath the pole. On this path, $v = \omega/k + \varepsilon e^{i\theta}$, and the integral is

$$\frac{1}{k}\frac{\partial f}{\partial v}\bigg|_{v=\omega/k}\int_{-\pi}^{0}\frac{\varepsilon i e^{i\theta}}{-\varepsilon e^{i\theta}}d\theta = \frac{1}{k}\frac{\partial f}{\partial v}\bigg|_{v=\omega/k} i\left(-\theta|_{-\pi}^{0}\right) = \frac{1}{k}\frac{\partial f}{\partial v}\bigg|_{v=\omega/k} i(-\pi)$$

which is the required result.

(b) Evaluate the principal value approximately, assuming $\omega/k \gg v_T = \sqrt{k_B T/m}$ and hence find the frequency ω as a function of $k.$ What is the effect of the pole at $\omega/k?$

First we integrate by parts:

$$P\int_{-\infty}^{+\infty}\frac{\partial f(v)/\partial v}{\omega - kv}dv = P\left[\frac{f}{\omega - kv}\bigg|_{-\infty}^{+\infty} - k\int_{-\infty}^{+\infty}\frac{f}{(\omega - kv)^2}dv\right]$$

$$= 0 - kP\int_{-\infty}^{+\infty}\frac{f}{(\omega - kv)^2}dv$$

Because of the exponential in the Maxwellian, the numerator is very small except when $v \leq v_t \ll \omega/k.$ Thus we expand the denominator:

$$\frac{1}{(\omega - kv)^2} = -\frac{1}{\omega^2(kv/\omega - 1)^2} = -\frac{1}{\omega^2}\left(1 + 2\frac{kv}{\omega} + 3\left(\frac{kv}{\omega}\right)^2 + \cdots\right)$$

and thus, integrating by parts

$$P\int_{-\infty}^{+\infty} \frac{\partial f(v)/\partial v}{\omega - kv} dv = -\frac{k}{\omega^2}\int_{-\infty}^{+\infty} f\left(1 + \frac{kv}{\omega} + \left(\frac{kv}{\omega}\right)^2 + \cdots\right) dv$$
$$= -\frac{k}{\omega^2}\left(1 + 0 + 3\frac{k^2}{\omega^2}v_t^2 + \cdots\right)$$

Finally, the pole on the real axis contributes a term:

$$-\frac{i\pi}{k}\frac{\partial f}{\partial v}\bigg|_{v=\omega/k} = -i\frac{\pi}{kv_t}\frac{1}{\sqrt{2\pi}}\left(-\frac{v}{v_t^2}\right)\exp\left(-\frac{v^2}{2v_t^2}\right)\bigg|_{v=\omega/k}$$
$$= i\sqrt{\frac{\pi}{2}}\left(\frac{\omega}{k^2 v_t^3}\right)\exp\left(-\frac{\omega^2}{2k^2v_t^2}\right)$$

This term is small because the exponent is large, so let's neglect it for the moment. Then:

$$0 = 1 - \frac{\omega_p^2}{k}\frac{k}{\omega^2}\left(1 + 3\frac{k^2}{\omega^2}v_t^2\right) = 1 - \frac{\omega_p^2}{\omega^2}\left(1 + 3\frac{k^2}{\omega^2}v_t^2\right)$$

To zeroth order the result is $\omega = \omega_p$. The first order correction gives:

$$\omega^2 = \omega_p^2 + 3k^2v_t^2$$

the Langmuir wave dispersion relation. Now we add in the small imaginary part:

$$\omega^2 = \omega_p^2 + 3k^2v_t^2 - i\sqrt{\frac{\pi}{2}}\left(\frac{\omega_p^3}{k^2v_t^3}\right)\exp\left(-\frac{\omega_p^2}{2k^2v_t^2}\right)$$

Thus ω must have an imaginary part, $\omega = \omega_r + i\gamma$, and thus $\omega^2 \simeq \omega_r^2 + 2i\omega_r\gamma$, with

$$\gamma = -\frac{1}{2}\sqrt{\frac{\pi}{2}}\left(\frac{\omega_p^2}{k^2v_t^3}\right)\exp\left(-\frac{\omega_p^2}{2k^2v_t^2}\right)$$

The wave form

exp $(ikx - i\omega t) = \exp(ikx - i\omega_r t - i(i\gamma t)) = \exp(ikx - i\omega_r t)\exp\gamma t$ shows that with

anegative γ, the wave is damped.

(c) How would the result change if the path of integration passed over, rather than under, the pole? The contribution from the pole would change sign, and we would predict growth of the waves rather than damping. This is contradicted by experiment.

37. Is the mapping $w = z^2$ conformal? Find the image in the $w-$ plane of the circle $|z - i| = 1$ in the $z-$ plane, and plot it.

The function $w = z^2$ is analytic. The derivative

$$\frac{dw}{dz} = 2z$$

is not zero except at the origin. Thus the mapping is conformal except at the origin.

The circle is described by

$$(z - i)(z^* + i) = 1$$

or

$$zz^* + i(z - z^*) = 0$$

which maps to

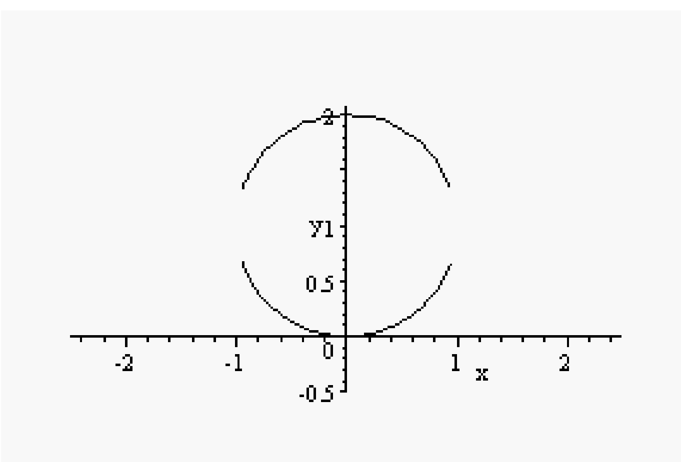
$$\sqrt{w}\left(\sqrt{w}\right)^* + i\left(\sqrt{w} - \sqrt{w}^*\right) = 0$$

and if $w = \rho e^{i\phi}$,

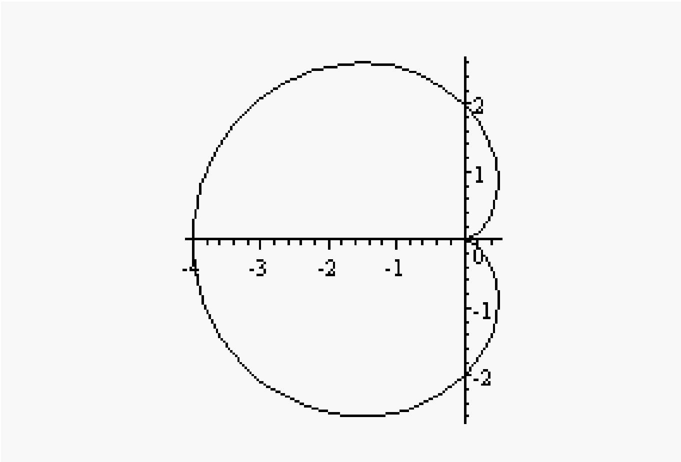
$$\rho + i\sqrt{\rho}\left(e^{i\phi/2} - e^{-i\phi/2}\right) = 0$$

$$\sqrt{\rho} = 2\sin\frac{\phi}{2}$$

Here's the plot:



$z-$ plane



$w-$ plane

Invariance of angles breaks down at $z = 0,$ where the mapping is not conformal.

38. Is the mapping $w = z + \frac{1}{z}$ conformal? Find the image in the $w-$ plane of (a) the $x-$ axis , (b) the $y-$ axis, and (c) the unit circle in the $z-$ plane.

The function $w = z + \frac{1}{z}$ is analytic except at $z = 0$ and at infinity. The derivative is

$$\frac{dw}{dz} = 1 - \frac{1}{z^2}$$

which is zero at $z = \pm 1.$ Thus the mapping is not conformal at these two points.

(a) The real axis maps to

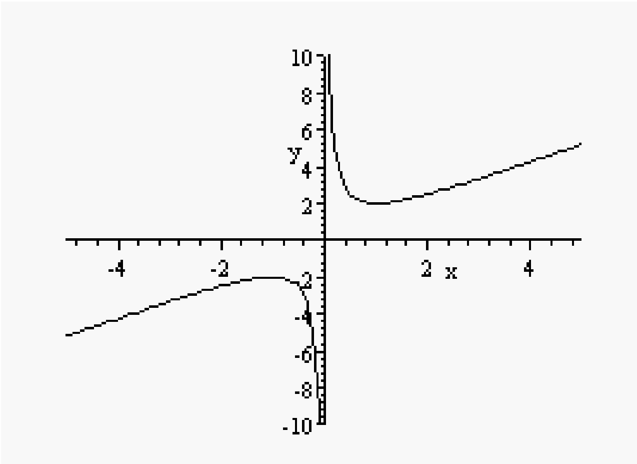
$$w = x + \frac{1}{x}$$

The origin maps to infinity, the positive $x-$ axis maps to the positive $u-$ axis with $u > 2$, and the negative $x-$ axis maps to the negative $u-$ axis with $u < -2.$

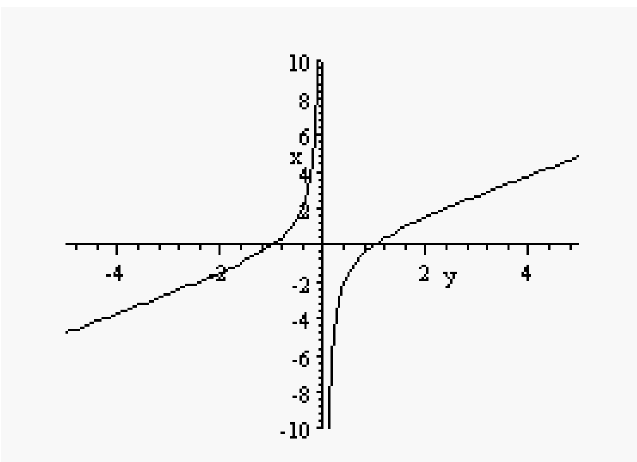
(b) The imaginary axis maps to

$$w = iy + \frac{1}{iy} = i\left(y - \frac{1}{y}\right)$$

Thus the points $y = \pm 1$ map to the origin. Points with $0 < y < 1$ map to negative $v,$ while points with $y > 1$ map to positive $v.$



u versus
x



v versus
y

(c)The unit circle $z = e^{i\theta}$ maps to

$$w = e^{i\theta} + e^{-i\theta} = 2\cos\theta$$

---a chunk of the real- w axis between $u = -2$ and $u = +2.$

A capacitor plate has a cylindrical bump of radius $a.$ on it. The second plate is a distance $d \gg a$ away. One plate is maintained at potential $V,$ and the other is grounded. Find the potential everywhere between the plates.

We want to convert to a coordinate system with $a = 1,$ so let $x' = x/a,$ $y' = y/a$. Then the cylinder has radius $r' = 1.$ Now we map to the $w -$ plane using the mapping $w = z' + \frac{1}{z'}$. This maps the cylinder plus x' axis to the u' axis. The second plate has coordinate $y' = d/a \gg 1$. It maps to the line $v = y' - \frac{1}{y'} \simeq d/a$. In this plane the potential is

$$\phi = V\frac{a}{d}v$$

whic is zero for $v = 0$ and V for $v = d/a$. The complex potential is then:

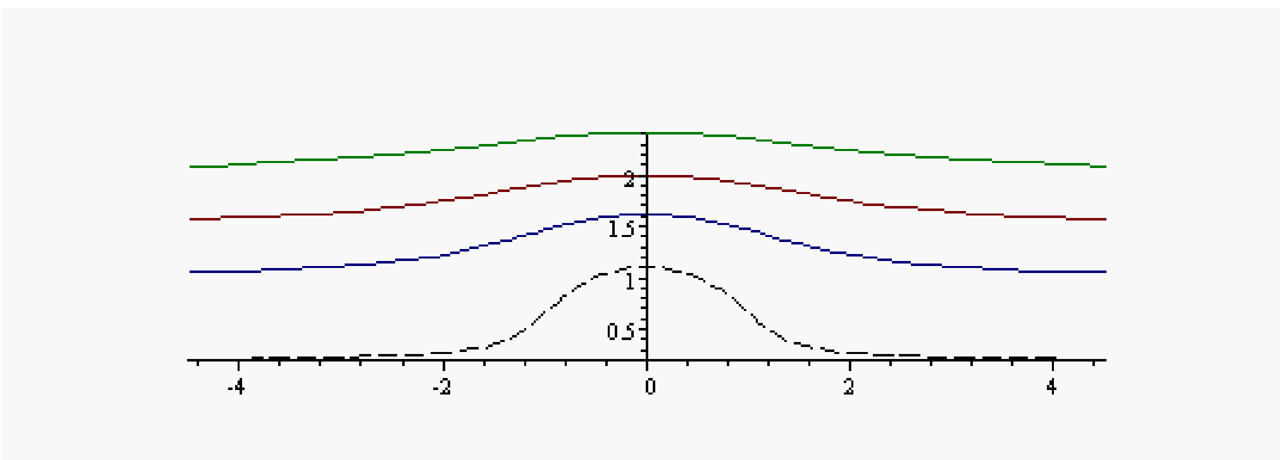
$$\Phi = V\frac{a}{d}w = \psi + i\phi$$

Mapping back, the potential in the $z' -$ plane is:

$$\begin{aligned}\Phi &= V\frac{a}{d}\left(z' + \frac{1}{z'}\right) \\ &= V\frac{a}{d}\left(r'e^{i\theta} + \frac{1}{r'}e^{-i\theta}\right)\end{aligned}$$

so the electric potential is:

$$\begin{aligned}\phi &= V\frac{a}{d}\sin\theta\left(r' - \frac{1}{r'}\right) \\ &= V\frac{a}{d}\sin\theta\left(\frac{r}{a} - \frac{a}{r}\right)\end{aligned}$$



Equipotentials for $\phi/V = 0.1$ (dashed), 0.5 (solid blue), 0.75 (red) and 1 (green).

39. Show that the mapping $z = w + e^w$ is conformal except at a finite set of points.

A parallel plate capacitor has plates that extend from $x = -1$ to $x = -\infty$. Find an appropriate scaling that allows you to place the plates at $y = \pm\pi.$ Show that the given transformation maps the plates to the lines $v = \pm\pi.$ Solve for the potential between the plates in the $w-$ plane, map to the $z-$ plane, and hence find the equipotential surfaces at the ends of the capacitor. Sketch the field lines. This is the so-called fringing field.

Choose $y = 2s\pi/d$ where s is a coordinate measured perpendicular to the plates, and d is the plate separation. The function $w + e^w$ is analytic everywhere, and the derivative is

$$\frac{dz}{dw} = 1 + e^w$$

It is non-zero except at the points

$$e^w = -1$$
$$w = \pm i\pi, \quad \pm 3i\pi \text{ etc}$$

or, equivalently,

$$z = -1 \pm i(2n+1)\pi.$$

The mapping takes the form:

$$x + iy = u + iv + e^u e^{iv} = u + e^u \cos v + i(v + e^u \sin v)$$

Then for $v = 0$ $(\text{the real axis in the } u-\text{plane})$ x ranges from $-\infty$ to $+\infty,$ i.e. we get the whole real axis in the $z-$ plane. The line $\omega = i\pi$ maps to $x = u - e^u,$ $y = \pi.$ x ranges from $-\infty$ at $u = \infty$ to -1 at $u = 0.$ This is the top plate of the capacitor. Similarly $w = -i\pi$ maps to the lower plate.

The mapping $w = f(z)$ has a branch point at each of the points $z = -1 \pm i(2n+1)\pi.$ Each 2 $\pi-$ wide strip of the $w-$ plane maps to the whole $z-$ plane. For each branch there are two points in the $z-$ plane at which the mapping is not conformal.

In the $w-$ plane we can write the potential as $\phi = vV/2\pi,$ giving a complex potential $\Phi = wV/2\pi,$ with the complex part being the physical potential. Equipotentials correspond to $v =$ const $= v_0.$ The corresponding curves in the $z-$ plane are:

$$x = u + e^{u}\cos v_0$$
$$y = v_0 + e^{u}\sin v_0$$

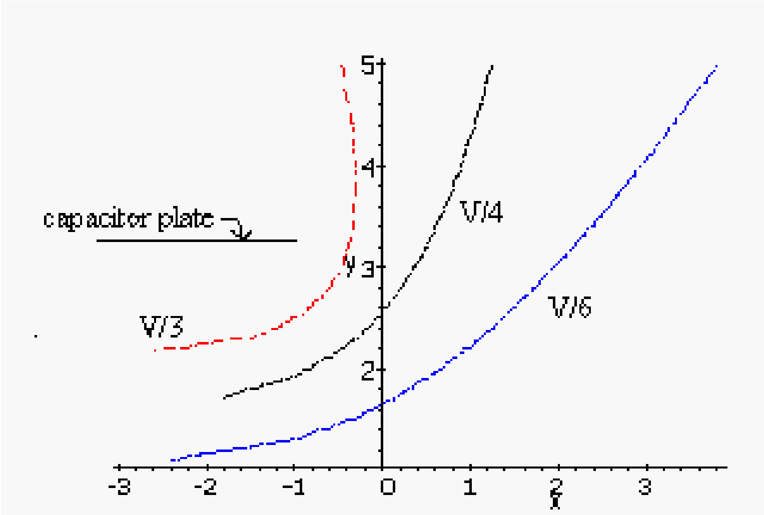
Thus

$$e^{u} = \frac{y - v_0}{\sin v_0}$$

and

$$x = (y - v_0)\cot v_0 + \ln\left(\frac{y - v_0}{\sin v_0}\right)$$

The equipotenials are shown in the figure.



40. Two conducting cylinders, each of radius $a,$ are touching. An insulating strip lies along the line at which they touch. One cylinder is grounded and the other is at potential $V.$ Use one of the mappings from the chapter to solve for the potential outside the cylinders.

. The transformation $z = 2a/w$ maps each of the cylinders to a straight line in the $w-$ plane. For a circle centered at $z = \pm ia$ with radius a we may write a point on the circle as

$$z = \pm ia + ae^{i\phi}$$

which maps to

$$w = \frac{2a}{\pm ia + ae^{i\phi}} = \frac{2a}{\left(\pm ia + ae^{i\phi}\right)\left(\mp ia + ae^{-i\phi}\right)}\left(\mp ia + ae^{-i\phi}\right)$$
$$= \frac{1}{(1 \pm \sin\phi)}(\mp i + \cos\phi - i\sin\phi)$$
$$= \frac{\cos\phi}{1 \pm \sin\phi} \mp i$$

As ϕ varies, u takes on all real values and w falls on the lines $\mp i$.

In the $w-$ plane, the potential is $\phi = (v+1)V/2.$ So we can write a complex potential

$$\Phi = (w+i)V/2$$

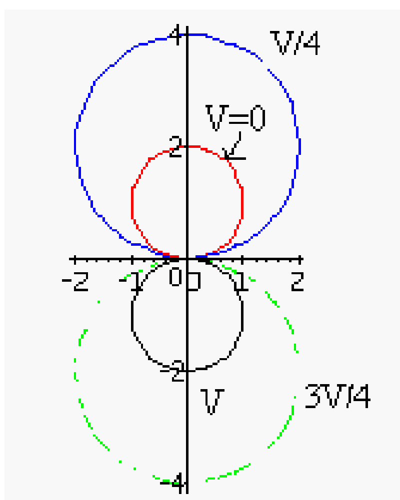
where the physical potential is the imaginary part. In the $z-$ plane we have:

$$\Phi = \left(\frac{2a}{z} + i\right)\frac{V}{2}$$

$$\Phi = \left(\frac{2a(x-iy)}{x^2+y^2} + i\right)\frac{V}{2} = \left(\frac{2a}{r}e^{-i\theta} + i\right)\frac{V}{2}$$

The imaginary part is:

$$\phi = -V\frac{a}{r}\sin\theta + \frac{V}{2} = \frac{-Vay}{x^2+y^2} + \frac{V}{2}$$



Problem 40. Equipotentials for $\Phi = 0, V/4,$ 3 V /4 and $V.$

The equipotentials are given by

$$r = a\frac{\sin\theta}{\left(\frac{1}{2} - \frac{\phi}{V}\right)}$$

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Chapter 2: Complex variables.

41. Show that the mapping $w = 1/(z-2)$ maps the arcs (a) $|z-4| = 2$ with end points at $z = 3 \pm \sqrt{3}i$ and (b) $|z-(2+i)| = 1$ with end points at $z = 3+i$ and $z = 1+i$ to straight line segments.

(a) The arc is described by the equation

$$\begin{aligned} (z-4)(z^*-4) &= 4 \\ zz^* - 4(z+z^*) + 12 &= 0 \end{aligned}$$

Using the transformation:

$$z = 2 + \frac{1}{w}$$

and thus

$$\begin{aligned} \left(2+\frac{1}{w}\right)\left(2+\frac{1}{w}\right)^* - 4\left(2+\frac{1}{w}+\left(2+\frac{1}{w}\right)^*\right) + 12 &= 0 \\ 4 + 2\left(\frac{1}{w}+\frac{1}{w^*}\right) + \frac{1}{ww^*} - 16 - 4\left(\frac{1}{w}+\frac{1}{w^*}\right) + 12 &= 0 \\ \frac{1}{ww^*} - 2\left(\frac{1}{w}+\frac{1}{w^*}\right) &= 0 \end{aligned}$$

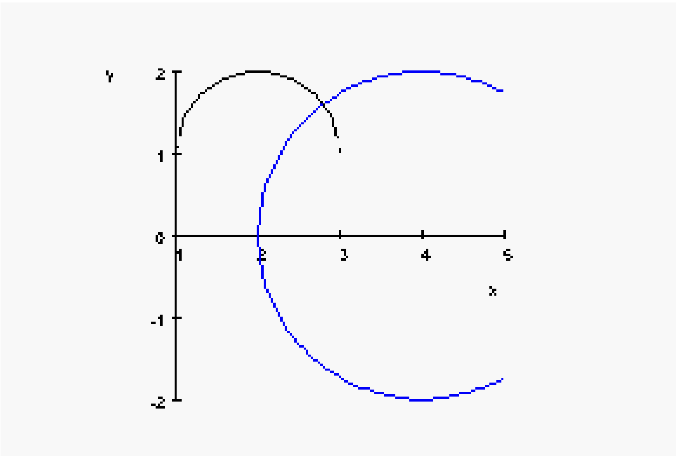
Thus

$$1 - 2(w + w^*) = 0$$

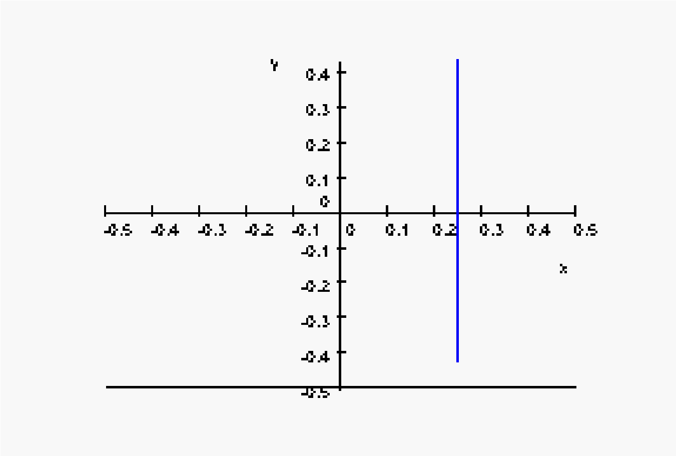
This a straight line parallel to the $v-$axis: $u = 1/4.$ With $z = 3 \pm \sqrt{3}i,$

$$w = \frac{1}{1 \pm \sqrt{3}i} = \frac{1 \mp \sqrt{3}i}{4}$$

The line extends from $v = \sqrt{3}/4$ to $v = -\sqrt{3}/4.$



The arc in the $z-$plane. (a) blue (b) black



The line in the $w-$plane

(b) The circle is

$$\begin{aligned}
[z-(2+i)][z-(2+i)]^* &= 1 \\
\left(\frac{1}{w}-i\right)\left(\frac{1}{w^*}+i\right) &= 1 \\
(1-iw)(1+iw^*) &= ww^* \\
1-i(w-w^*) &= 0 \\
1 &= i(2iv) \\
v &= -\frac{1}{2}
\end{aligned}$$

This is a straight line parallel to the $u-$axis. It extends from $w=\frac{1}{-1+i}=-\frac{1}{2}-\frac{i}{2}$ to $w=\frac{1}{1+i}=\frac{1}{2}-\frac{i}{2}$.

42. Show that $\Gamma(x)<0$ for $-1<x<0.$

If $-1<x<0,$ then we can write

$$\Gamma(x)=\frac{\pi}{\sin\pi x\,\Gamma(1-x)}=\frac{\pi}{-|\sin\pi x|\Gamma(y)}$$

where $1<y<2.$ Then $\Gamma(y)$ is positive and hence $\Gamma(x)$ is negative.

43. Prove *Cauchy's inequality*: If $f(z)$ is analytic and bounded in a region $R:$ $|z-z_0|<R,$ and $|f(z)|<M$ on the circle $|z-z_0|=r<R,$ then the coefficients in the Taylor series expansion of f about z_0 $\left(\text{eqn } 44\right)$ satisfy the inequality

$$|c_n| \le \frac{M}{r^n}$$

Hence prove *Liouville's theorem:*

If $f(z)$ is analytic and bounded in the entire complex plane, then it is a constant.

Using expression (45) with Γ equal to the circle of radius r,

$$|c_n| = \frac{1}{2\pi}\left|\oint_{\text{circle}} \frac{f(z)}{(z-z_0)^{n+1}}dz\right| \le \frac{1}{2\pi}M\oint_{\text{circle}} \frac{1}{r^{n+1}}r\,d\theta = \frac{M}{r^n}$$

as required.

To Prove Liouville's theorem, we let R and $r \to \infty$. Then $c_n \to 0$ for all $n > 0$. Thus $f(z) = c_0$, a constant.

44. A function $f(z)$ is analytic except for well-separated simple poles at $z = z_n$, $n = 1 - N$, $z_n \ne 0$. Show that the function may be expanded in a series

$$f(z) = f(0) + \sum_{n=1}^{N} a_n\left(\frac{1}{z_n} + \frac{1}{z - z_n}\right)$$

where a_n is the residue of f at z_n. Is the result valid for $N \to \infty$? Why or why not?

Hint: Evaluate the integral

$$I_N = \frac{1}{2\pi i}\int_{C_N} \frac{f(w)}{w(w-z)}dw$$

where C_N is a circle of radius R_N about the origin that contains the N poles. You may assume that $|f(z)| < \varepsilon R_N$ on C_N for ε a small positive constant.

The integrand has simple poles at the origin, at z, and at z_n, $n \le N$. Near one of the poles z_n, the integrand has the form

$$\frac{a_n}{w(w-z)(w-z_n)} + \sum_{k=0}^{\infty} \frac{c_k}{w(w-z)}(w-z_n)^k$$

The denominator of the first term has a simple zero at z_n and the sum is analytic at z_n, so the residue at z_n is

$$\frac{a_n}{\frac{d}{dw}[w(w-z)(w-z_n)]\Big|_{w=z_n}} = \frac{a_n}{[(w-z)(w-z_n)+w(w-z_n)+w(w-z)]|_{w=z_n}}$$
$$= \frac{a_n}{z_n(z_n-z)}$$

Thus

$$I_N = \frac{f(0)}{-z} + \frac{f(z)}{z} + \sum_{n=1}^{N} \frac{a_n}{z_n(z_n-z)}$$

But also

$$|I_N| \le \frac{1}{2\pi}\frac{\varepsilon R_N}{R_M(R_N-|z|)}2\pi R_N \le \frac{\varepsilon}{1-|z|/R_N}$$

Thus $I_n \to 0$ as $R_N \to \infty$ and so

$$f(z) = f(0) + \sum_{n=1}^{N} \frac{zb_n}{z_n(z-z_n)} = f(0) + \sum_{n=1}^{N} b_n\left(\frac{1}{z_n} + \frac{1}{z-z_n}\right)$$

as required.

The residue theorem holds when there are a finite number of poles inside the contour, so this proof is limited to finite N.

See also Jeffreys and Jeffreys 11.175.

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Chapter 3: Differential equations

1. A vehicle moves under the influence of a constant force $\vec{F}$ and air resistance proportional to velocity (equation 3.5 with $\vec{F}$ replacing the gravitational force.) Find the speed of the vehicle as a function of time if it starts from rest at $t = 0$.

Choose $x-$axis along the direction of $\vec{F}$. The equation of motion is then:

$$m\frac{dv}{dt} + \alpha v = F$$

and the solution to the inhomogeneous equation is:

$$v = \frac{F}{\alpha}$$

The solution to the homogeneous equation is of the form e^{at} where

$$ma + \alpha = 0 \Rightarrow a = -\frac{\alpha}{m}$$

Thus the complete solution is of the form:

$$v = \frac{F}{\alpha} + Ae^{-\alpha t/m}$$

Now we apply the initial conditions: $v(0) = 0$

$$0 = A + \frac{F}{\alpha}$$

So the solution is:

$$v = \frac{F}{\alpha}\left(1 - e^{-\alpha t/m}\right)$$

The vehicle reaches a terminal velocity $v_t = F/\alpha$ as $t \to \infty$.

2. Find the general solution to the differential equation

$$y''' - 3y'' + 3y' - y = 0$$

Hint: Extend the result for a double root from section 3.1.1.

The solution is of the form e^{sx} where

$$s^3 - 3s^2 + 3s - 1 = 0$$
$$(s-1)^3 = 0$$

or $s = 1$, a root repeated 3 times. Extending the result from the chapter, we guess that the two additional solutions are xe^x and x^2e^x. Let's check. With $y = ve^{sx}$, $y' = v'e^{sx} + sve^{sx}$, $y'' = \left(v'' + 2sv' + s^2v\right)e^{sx}$ and $y''' = \left(v''' + 3sv'' + 3s^2v' + s^3v\right)e^{sx}$. Substituting into the differential equation:

$$\left[v''' + 3sv'' + 3s^2v' + s^3v - 3\left(v'' + 2sv' + s^2v\right) + 3\left(v' + sv\right) - v\right]e^{sx} = 0$$
$$v''' + v''(3s - 3) + 3v'\left(s^2 - 2s + 1\right) + v\left(s^3 - 3s^2 + 3s - 1\right) = 0$$

The coefficients of v, v', and v'' are each zero, so this equation simplifies to $v''' = 0$, which has solution

$$v = Ax^2 + Bx + C$$

as expected. Thus the general solution is:

$$Y = \left(Ax^2 + Bx + C\right)e^{sx}$$

3. A capacitor C, inductor L, and resistor R are connected in series with a switch. The capacitor is charged by connecting it across a battery with emf ε. The battery is disconnected, and the switch is closed. Find the current in the circuit as a function of time after the switch is closed.

The differential equation is:

$$L\frac{dI}{dt} + IR + \frac{Q}{C} = L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = 0$$

and the intial condition is $Q = \varepsilon C$ at $t = 0$. The inductor prevents the current from changing immediately after the switch is closed, so we also have $I = \frac{dQ}{dt} = 0$ at $t = 0$. The solution is (§3.1.1)

$$Q = (A\cos\omega t + B\sin\omega t)e^{-\alpha t}$$

with $\alpha = R/2L$ and $\omega^2 = 1/LC - \alpha^2$. Differentiating, we find

$$I = -\alpha(A\cos\omega t + B\sin\omega t)e^{-\alpha t} + \omega(-A\sin\omega t + B\cos\omega t)e^{-\alpha t}$$
$$= ([\omega B - \alpha A]\cos\omega t - [\alpha B + \omega A]\sin\omega t)e^{-\alpha t}$$

Applying the initial conditions, we have:

$$A = \varepsilon C$$

and

$$-\alpha A + \omega B = 0 \Rightarrow B = \frac{\alpha}{\omega}A = \frac{\alpha}{\omega}\varepsilon C$$

Thus

$$I(t) = -\frac{\varepsilon C}{\omega}\left(\alpha^2 + \omega^2\right)\sin\omega t e^{-\alpha t}$$
$$= -\frac{\varepsilon}{\omega L}\sin\omega t e^{-\alpha t}$$

Notice that I is negative for small t, implying that the capacitor is discharging.

4. The Airy differential equation is:

$$y'' - xy = 0$$

Find the two solutions of this equation as power series in x.

The point $x = 0$ is a regular point of this equation, so we may write

$$y = \sum a_n x^n$$

and

$$y'' = \sum n(n-1)a_n x^{n-2}$$

Then the equation is

$$\sum n(n-1)a_n x^{n-2} - \sum a_n x^{n+1} = 0$$

The lowest power that appears in this equation is x^0 and its coefficient is:

$$a_2 = 0$$

Then for x^1 we have:

$$3(2)a_3 - a_0 = 0 \Rightarrow a_3 = \frac{a_0}{6}$$

and for x^m :

$$(m+2)(m+1)a_{m+2} - a_{m-1} = 0 \Rightarrow a_{m+2} = \frac{a_{m-1}}{(m+2)(m+1)}$$

Thus the recursion relation skips two. One solution starts with a_1 :

$$y = a_1\left(x + \frac{x^4}{4\times 3} + \frac{x^7}{7\times 6\times 4\times 3} + \cdots\right)$$

and the other starts with a_0 :

$$y = a_0\left(1 + \frac{x^3}{6} + \frac{x^6}{6\times 5\times 3\times 2} + \cdots\right)$$

5. Solve the equation $xy'' + 2y = 0$ (§3.3.2) using the Frobenius method. Show that $y(0)$ cannot equal any non-zero constant, as discussed in §3.3.2.

$$x\sum(n+p)(n+p-1)a_n x^{n+p-2} + 2\sum a_n x^{n+p} = 0$$

The lowest power of x is x^{p-1} with coefficient

$$p(p-1)a_0 = 0$$

and solutions $p = 0,$ $p = 1.$ Then for x^p we have

$$(p+1)(p)a_1 + 2a_0 = 0 \Rightarrow a_1 = a_0\frac{-2}{p(p+1)}$$

For $p = 1,$

$$a_1 = -a_0$$

and for $p = 0,$

$$a_0 = 0$$

The general recursion relation is:

$$(p+n)(p+n-1)a_n + 2a_{n-1} = 0 \Rightarrow a_n = a_{n-1}\frac{-2}{(p+n)(p+n-1)}$$

Thus for $p = 1$:

$$\begin{aligned} a_n &= a_{n-1}\frac{-2}{n(n+1)} = a_{n-2}\frac{-2}{n(n-1)}\frac{-2}{n(n+1)} = a_{n-2}\frac{(-2)^2}{(n+1)n^2(n-1)} \\ &= a_{n-3}\frac{(-2)^3}{(n-1)(n-2)(n+1)n^2(n-1)} = a_{n-3}\frac{(-2)^3}{(n+1)n^2(n-1)^2(n-2)} \\ &= a_0\frac{(-2)^n}{(n+1)!n!} \end{aligned}$$

Thus the first solution is:

$$\begin{aligned} y_1 &= \sum \frac{(-2)^n}{(n+1)!n!}x^{n+1} = x\sum_{n=0}^{\infty}\frac{(-2x)^n}{(n+1)!n!} \\ &= x - \frac{x^2}{3} + \frac{2^3x^3}{4!3!} - \frac{2^4x^4}{5!4!} + \cdots \\ &= x - \frac{x^2}{3} + \frac{1}{18}x^3 - \frac{1}{180}x^4 + \cdots \end{aligned}$$

Check by differentiating

$$\begin{aligned} y_1' &= \sum_{n=0}^{\infty}\frac{(-2x)^n}{n!n!} \\ y_1'' &= \sum_{n=1}^{\infty}\frac{(-2)^n x^{n-1}}{n!(n-1)!} = -2\sum_{n=0}^{\infty}\frac{(-2)^n x^n}{(n+1)!n!} = -2\frac{y_1}{x} \end{aligned}$$

as required.

For $p = 0$

$$2a_2 + 2a_1 = 0 \Rightarrow a_2 = -a_1$$

$$a_n = a_{n-1}\frac{-2}{n(n-1)} = a_{n-2}\frac{(-2)^2}{n(n-1)^2(n-2)}$$

This generates the same solution as $p = 0$.

This solution shows explicitly that the regular solution $y_1 \to 0$ and $y_1/x \to$const as $x \to 0$. Thus $y(0)$ cannot equal any non-zero constant, as discussed in the text.

The second solution is found using equation 3.37:

$$y_2 = y_1 \ln x + \sum_{n=0}^{\infty} b_n x^{n+p}$$

Differentiating:

$$y_2' = y_1' \ln x + \frac{y_1}{x} + \sum_{n=0}^{\infty} b_n (n+p) x^{n+p-1}$$

$$y_2'' = y_1'' \ln x + 2\frac{y_1'}{x} - \frac{y_1}{x^2} + \sum_{n=0}^{\infty} b_n (n+p)(n+p-1) x^{n+p-2}$$

Stuffing this into the differential equation, we get:

$$y_1'' x \ln x + 2y_1' - \frac{y_1}{x} + \sum_{n=0}^{\infty} b_n (n+p)(n+p-1) x^{n+p-1} + 2\left(y_1 \ln x + \sum_{n=0}^{\infty} b_n x^{n+p} \right) = 0$$

$$\ln x (x y_1'' + 2y_1) + 2y_1' - \frac{y_1}{x} + \sum_{n=0}^{\infty} b_n (n+p)(n+p-1) x^{n+p-1} + 2\sum_{n=0}^{\infty} b_n x^{n+p} = 0$$

$$2y_1' - \frac{y_1}{x} + \sum_{n=0}^{\infty} b_n (n+p)(n+p-1) x^{n+p-1} + 2\sum_{n=0}^{\infty} b_n x^{n+p} = 0$$

Using equation () our equation becomes:

$$2\sum_{n=0}^{\infty} \frac{(-2x)^n}{n!n!} - \sum_{n=0}^{\infty} \frac{(-2x)^n}{(n+1)!n!} + \sum_{n=0}^{\infty} b_n (n+p)(n+p-1) x^{n+p-1} + 2\sum_{n=0}^{\infty} b_n x^{n+p} = 0$$

The lowest power of x in the first two terms is x^0. Thus p must be an integer. With $p = 0$, we find:

$$\sum_{n=0}^{\infty} \frac{(-2x)^n}{(n!)^2} \left(\frac{2n+1}{n+1} \right) + \sum_{n=0}^{\infty} b_n n(n-1) x^{n-1} + 2\sum_{n=0}^{\infty} b_n x^n = 0$$

The coefficient of x^0 is:

$$1 + 2b_0 = 0 \Rightarrow b_0 = -\frac{1}{2}$$

x^1 :

$$-\frac{2 \times 3}{2} + 2b_2 + 2b_1 = 0 \Rightarrow b_2 = \frac{3}{2} - b_1$$

x^2 :

$$0 = \frac{2^2 \times 5}{2^2 \times 3} + 6b_3 + 2b_2 \Rightarrow b_3 = -\frac{b_2}{3} - \frac{5}{3 \times 6} = -\frac{1}{3}\left(\frac{3}{2} - b_1 \right) - \frac{5}{18} = -\frac{7}{9} + \frac{b_1}{3}$$

x^3 :

$$0 = \frac{2^3}{(3!)^2}\frac{7}{4} + 12b_4 + 2b_3 \Rightarrow b_4 = -\frac{b_3}{6} - \frac{7}{18} = -\frac{1}{6}\left(-\frac{7}{9} + \frac{b_1}{3} \right) - \frac{7}{18} = -\frac{7}{27} - \frac{1}{18}b_1$$

The terms in b_1 are just y_1 again. Thus the general solution is:

$$y = a_0 x \sum_{n=0}^{\infty} \frac{(-2x)^n}{(n+1)!n!} + b_0 \left\{ x \ln x \sum_{n=0}^{\infty} \frac{(-2x)^n}{(n+1)!n!} - \frac{1}{2} + \frac{3x^2}{2} - \frac{7x^3}{9} - \frac{7x^4}{27} + \cdots \right\}$$

The second solution $y_2 \to -b_0/2$ as $x \to 0,$ but cannot be expressed as a Taylor series. The first derivative is

$$y_2' = \sum_{n=0}^{\infty} \frac{(-2x)^n}{(n+1)!n!} + \ln x \sum_{n=0}^{\infty} \frac{(-2x)^n}{n!n!} + 3x - \frac{7x^2}{3} - \frac{28x^3}{27} + \cdots$$

which diverges at the origin.

6. Find a solution of Laguerre's differential equation:

$$xy'' + (1-x)y' + \alpha y = 0$$

that is regular at the origin. Show that if α is an integer $k,$ then this solution is a polynomial of degree $k.$

$x = 0$ is a singular point of the differential equation, so we may write the solution in the form:

$$y = \sum a_n x^{n+p}$$

Then

$$y' = \sum (n+p) a_n x^{n+p-1}$$

and

$$y'' = \sum (n+p)(n+p-1) a_n x^{n+p-2}$$

The differential equation becomes:

$$\sum (n+p)(n+p-1) a_n x^{n+p-1} + \sum (n+p) a_n x^{n+p-1} - \sum (n+p) a_n x^{n+p} + \alpha \sum a_n x^{n+p} = 0$$

The lowest power is $x^{p-1}.$ Its coefficient is:

$$p(p-1)a_0 + p a_0 = 0 \Rightarrow p = 0,$$

a repeated root. The coefficient of x^m is:

$$(m+1)^2 a_{m+1} - m a_m + \alpha a_m = 0$$

So

$$a_{m+1} = \frac{(m-\alpha)}{(m+1)^2} a_m = \frac{(m-\alpha)(m-1-\alpha)}{(m+1)^2 m^2} a_{m-1}$$

Thus

$$a_m = (-1)^m \frac{\alpha(\alpha-1)\cdots(\alpha-m+1)}{(m!)^2} a_0$$

Thus one solution is:

$$y = a_0 \sum_{m=0}^{\infty} (-1)^m \frac{\alpha(\alpha-1)\cdots(\alpha-m+1)}{(m!)^2} x^m$$

Now if $\alpha = k,$ an integer, then the series will terminate with $m = k,$ and this solution becomes a polynomial or order k.

$$y_k = a_0 \sum_{m=0}^{k} (-1)^m \frac{k(k-1)\cdots(k-m+1)}{(m!)^2} x^m$$

The first few are:

$k = 1$

$$y_1 = a_0(1 - x)$$

$k = 2$

$$y_2 = a_0\left(1 - 2x + \frac{x^2}{2}\right)$$

and so on.

The second solution is found by introducing the logarithm:

$$y = y_1 \ln x + \sum a_n x^{n+p}$$

and inserting into the de.

7. Solve the Bessel equation:

$$4x^2y'' + 4xy' + \left(4x^2 - 1\right)y = 0$$

as a Frobenius series in powers of $x.$ Sum the series to obtain closed-form expressions for the two solutions.

The differential equation has a singular point at $x = 0,$ so we write:

$$y = x^p \sum a_n x^n$$

$$y' = \sum (n+p) a_n x^{n+p-1}$$

and

$$y'' = \sum (n+p)(n+p-1) a_n x^{n+p-2}$$

The differential equation becomes:

$$4\sum (n+p)(n+p-1)a_n x^{n+p} + 4\sum (n+p)a_n x^{n+p} + 4\sum a_n x^{n+p+2} - \sum a_n x^{n+p} = 0$$

The lowest power is x^p and its coefficient is:

$$(4p(p-1) + 4p - 1)a_0 = 0$$

so we have the indicial equation:

$$4p^2 - 1 = 0$$

with solutions

$$p = \pm\frac{1}{2}$$

The coefficient of x^{p+1} is:

$$4(p+1)p + 4(p+1) - 1 = 0$$

which gives $p + 1 = \pm\frac{1}{2}$, leading to the same two series. Thus we need only consider $p = \pm\frac{1}{2}$.

The recursion relation is obtained by looking at the coefficient of x^{m+p} :

$$\{4(m+p)(m+p-1) + 4(m+p) - 1\}a_m + 4a_{m-2} = 0$$

So

$$\begin{aligned} a_m &= -\frac{a_{m-2}}{(m+p)^2 - 1/4} \\ &= -\frac{a_{m-2}}{\left(m \pm \frac{1}{2}\right)^2 - 1/4} \\ &= -\frac{a_{m-2}}{m(m \pm 1)} \end{aligned}$$

So with $p = +1/2$,

$$\begin{aligned} a_m &= -\frac{a_{m-2}}{m(m+1)} = (-1)^2 \frac{a_{m-4}}{(m+1)m(m-1)(m-2)} \\ a_{2r} &= (-1)^r \frac{a_0}{(2r+1)!} \end{aligned}$$

and the solution is:

$$y = a_0\sqrt{x}\sum_{r=0}^{\infty} \frac{(-1)^r}{(2r+1)!}x^{2r}$$

The second solution has $p = -1/2$:

$$\begin{aligned} a_m &= -\frac{a_{m-2}}{m(m-1)} = (-1)^2 \frac{a_{m-4}}{m(m-1)(m-2)(m-3)} \\ a_{2r} &= (-1)^r \frac{a_0}{(2r)!} \end{aligned}$$

and the solution is:

$$y = \frac{a_0}{\sqrt{x}}\sum_{r=0}^{\infty} \frac{(-1)^r}{(2r)!}x^{2r}$$

The first of these may be written:

$$y = \frac{a_0}{\sqrt{x}}\sum_{r=0}^{\infty} \frac{(-1)^r}{(2r+1)!}x^{2r+1} = a_0\frac{\sin x}{\sqrt{x}}$$

while the second is:

$$y = a_0 \frac{\cos x}{\sqrt{x}}$$

8. Solve the hypergeometric equation

$$(x^2 - x)y'' + (3x - 1/2)y' + y = 0$$

as a series (a) in powers of x and (b) in powers of $x - 1$.

(a) The equation has singular points at $x = 0$ and at $x = 1$, so we write a Frobenius series:

$$y = x^p \sum a_n x^n$$

$$y' = \sum (n+p) a_n x^{n+p-1}$$

and

$$y'' = \sum (n+p)(n+p-1) a_n x^{n+p-2}$$

Then the differential equation becomes:

$$0 = \sum (n+p)(n+p-1) a_n x^{n+p} - \sum (n+p)(n+p-1) a_n x^{n+p-1} + 3\sum (n+p) a_n x^{n+p} - \frac{1}{2}\sum (n+p) a_n x^{n+p-1} + x^p \sum a_n x^n$$

The lowest power is x^{p-1} and its coefficient is:

$$p(p-1)a_0 + \frac{1}{2} p a_0 = 0$$

So the indicial equation is

$$p\left(p - \frac{1}{2}\right) = 0$$

with solutions $p = 0$ and $p = 1/2$. So one solution is regular at the origin and one is not. The recursion relation is found by looking at the $m + p$ power:

$$[(m+p)(m+p-1) + 3(m+p) + 1]a_m - \left[(m+p)(m+p+1) + \frac{1}{2}(m+p)\right]a_{m+1} = 0$$

Thus

$$a_{m+1} = a_m \frac{(m+p)(m+p+2) + 1}{(m+p+1)\left(m+p+\frac{1}{2}\right)}$$

So with $p = 0$ we have:

$$a_{m+1} = a_m \frac{m^2 + 2m + 1}{(m+1)(m+1/2)} = 2a_m \frac{(m+1)^2}{(m+1)(2m+1)} = 2a_m \frac{m+1}{2m+1}$$

and thus the solution is:

$$y = a_0\left(1 + 2x + 2^2 \frac{2}{3} x^2 + 2^3 \frac{3}{5}\frac{2}{3} x^3 + \cdots + 2^n \frac{n!}{(2n-1)!!} x^n + \cdots\right)$$

The second solution has the recursion relation:

$$a_{m+1} = a_m \frac{\left(m + \frac{1}{2}\right)\left(m + \frac{5}{2}\right) + 1}{\left(m + \frac{3}{2}\right)(m+1)}$$
$$= a_m \frac{\left(m + \frac{3}{2}\right)}{(m+1)}$$

and so the solution is:

$$y = a_0 \sqrt{x}\left(1 + \frac{3}{2}x + \frac{3}{2}\frac{5}{2}\frac{x^2}{2} + \cdots + \frac{(2n+1)!!}{2^n}\frac{x^n}{n!} + \cdots\right)$$

(b) Now let $w = x - 1.$ The equation becomes:

$$w(w+1)y'' + (3(w+1) - 1/2)y' + y = 0$$
$$\left(w^2 + w\right)y'' + \left(3w + \frac{5}{2}\right)y' + y = 0$$

Now look for a series solution in w :

$$0 = \sum (n+p)(n+p-1)a_n w^{n+p} + \sum (n+p)(n+p-1)a_n w^{n+p-1}$$
$$+ 3\sum (n+p)a_n w^{n+p} + \frac{5}{2}\sum (n+p)a_n w^{n+p-1} + \sum a_n w^{n+p}$$

The indicial equation is:

$$p(p-1) + \frac{5}{2}p = 0$$
$$p\left(p + \frac{3}{2}\right) = 0$$

So $p = 0,$ or $p = -3/2.$ The recursion relation is:

$$0 = [(m+p)(m+p-1) + 3(m+p) + 1]a_m$$
$$+ \left[(m+p)(m+p+1) + \frac{5}{2}(m+p+1)\right]a_{m+1}$$

So

$$a_{m+1} = -a_m \frac{(m+p)(m+p+2) + 1}{\left(m + p + \frac{5}{2}\right)(m+p+1)}$$

With $p = 0,$ we get:

$$a_{m+1} = -a_m \frac{(m+1)}{\left(m + \frac{5}{2}\right)}$$

So the solution is:

$$y = a_0\left(1 - \frac{2}{5}w + \frac{2}{\frac{5}{2}\frac{7}{2}}w^2 + \cdots + \frac{2^n}{(2n+3)!!}n!w^n + \cdots\right)$$

while with $p = -3/2,$ we have:

$$a_{m+1} = -a_m \frac{\left(m-\frac{3}{2}\right)\left(m+\frac{1}{2}\right)+1}{(m+1)\left(m-\frac{1}{2}\right)} = -a_m \frac{\left(m-\frac{1}{2}\right)}{(m+1)}$$

and the second solution is:

$$y = a_0\left(1+\frac{1}{2}w+\frac{1}{2^2 2}w^2-\frac{3}{2}\frac{1}{2^2}\frac{1}{3!}w^3+\cdots+(-1)^n\frac{(2n-3)!!}{2^n}\frac{w^n}{n!}+\cdots\right)$$

9. Find two solutions of the Bessel equation

$$x^2y''+xy'+\left(x^2-\frac{9}{4}\right)y = 0$$

as series in x. Verify that your solutions agree with the standard forms $\sqrt{\frac{2}{\pi x}}\left(\frac{\sin x}{x}-\cos x\right)$ and $-\sqrt{\frac{2}{\pi x}}\left(\frac{\cos x}{x}+\sin x\right)$

The equation has a singular point at $x = 0$, so we use a Frobenius series.

$$\sum(n+p)(n+p-1)a_nx^{n+p}+\sum(n+p)a_nx^{n+p}+\left(x^2-\frac{9}{4}\right)\sum a_nx^{n+p} = 0$$

The indicial equation is:

$$p(p-1)+p-\frac{9}{4} = 0$$

with solutions $p = \pm\frac{3}{2}$. The recursion relation is:

$$a_m\left((m+p)^2-\frac{9}{4}\right)+a_{m-2} = 0$$

So with $p = +3/2$:

$$a_m = -\frac{a_{m-2}}{\left(m+\frac{3}{2}\right)^2-\frac{9}{4}} = -\frac{a_{m-2}}{m(m+3)}$$

and the solution is:

$$y = a_0x^{3/2}\left(1-\frac{x^2}{5.2}+\frac{x^4}{7.5.4.2}+\cdots+(-1)^n\frac{x^{2n}}{(2n+3)!!2^n n!}+\cdots\right)$$
$$= a_0\sqrt{x}\left(x-\frac{x^3}{5.2}+\frac{x^5}{7.5.4.2}+\cdots+(-1)^n\frac{x^{2n+1}}{(2n+3)!!2^n n!}+\cdots\right)$$

But $\frac{\sin x}{x}-\cos x = \frac{1}{3}x^2-\frac{1}{30}x^4+\frac{1}{840}x^6-\cdots = \frac{1}{3}\left(x^2-\frac{1}{10}x^4+\frac{1}{280}x^6-\cdots\right)$ so

$$y = \frac{3a_0}{\sqrt{x}}\left(\frac{\sin x}{x}-\cos x\right)$$

as required.

The second solution has:

$$a_m = -\frac{a_{m-2}}{m(m-3)}$$

and therefore

$$\begin{aligned} y &= a_0 x^{-3/2}\left(1 + \frac{x^2}{2} - \frac{x^4}{4.2} + \cdots + (-1)^{n+1}\frac{x^{2n}}{(2n-3)!!(2n)!!} + \cdots\right) \\ &= \frac{a_0}{\sqrt{x}}\left(\frac{1}{x} + \frac{x}{2} - \frac{x^3}{4.2} + \cdots + (-1)^{n+1}\frac{x^{2n-1}}{(2n-3)!!(2n)!!} + \cdots\right) \end{aligned}$$

Then $\sin x + \frac{\cos x}{x} = x^{-1} + \frac{1}{2}x - \frac{1}{8}x^3 + \frac{1}{144}x^5 - \cdots$

So the second solution is:

$$y = \frac{a_0}{\sqrt{x}}\left(\sin x + \frac{\cos x}{x}\right)$$

10. Consider a linear differential equation of the form:

$$x^2 y'' + xfy' + gy = 0$$

Expand the functions $f(x)$ and $g(x)$ in power series of the form

$$f(x) = f_0 + f_1 x + f_2 x^2 + \cdots$$

and similarly for $g.$ Find the indicial equation. What is the condition on f_0 and g_0 if there is only one root? What is the value of the root in that case? Use the method of variation of parameters to show that the second solution of the differential equation is given by equation (3.37). *Hint:* show that the equation for v may be reduced to the form:

$$\frac{d}{dx}(\ln v') = -\frac{1}{x} + h(x)$$

where $h(x)$ is a series of positive powers of $x.$ Integrate this equation twice to obtain equation (3.37).

. With $y = \sum a_n x^{n+p}$ we have

$$\begin{aligned} 0 = &\sum a_n (n+p)(n+p-1)x^{n+p-2} + (f_0 + f_1 x + \cdots)\sum a_n (n+p)x^{n+p-2} \\ &+ (g_0 + g_1 x + \cdots)\sum a_n x^{n+p-2} \end{aligned}$$

The lowest power is $x^{p-2},$ giving the indicial equation:

$$p(p-1) + f_0 p + g_0 = 0$$

with roots:

$$p = \frac{1 - f_0 \pm \sqrt{(1-f_0)^2 - 4g_0}}{2}$$

There is only one root if

$$(1-f_0)^2 = 4g_0$$

and then

$$p = \frac{1-f_0}{2}$$

Then assuming

$$y_2 = vy_1$$

and

$$\begin{aligned} x^2y_2'' + fxy_2' + gy_2 &= 0 \\ \left(v''y_1 + 2v'y_1' + vy_1''\right)x^2 + fx\left(v'y_1 + vy_1'\right) + gvy_1 &= 0 \\ \left(v''x^2 + fxv'\right)y_1 + 2v'y_1'x^2 + v\left(x^2y_1'' + fxy_1' + gy_1\right) &= 0 \end{aligned}$$

The term multiplying v is zero, because y_1 satisfies the original differential equation. Thus

$$\begin{aligned} v''x^2y_1 &= -v'\left(fxy_1 + 2x^2y_1'\right) \\ \frac{v''}{v'} &= -\frac{f}{x} - 2\frac{y_1'}{y_1} = -\frac{f_0}{x} - \frac{2p}{x} + \cdots \\ &= -\frac{1}{x} + \cdots \end{aligned}$$

since $2p + f_0 = 1.$

To obtain this result, we used

$$\begin{aligned} \frac{y_1'}{y_1} &= \frac{pa_0x^{p-1} + a_1(p+1)x^p + \cdots}{a_0x^p + a_1x^{p+1} + \cdots} \\ &= \frac{p}{x}\left(\frac{1 + \frac{a_1}{a_0}\frac{p+1}{p}x + \cdots}{1 + \frac{a_1}{a_0}x + \cdots}\right) = \frac{p}{x} + h(x) \end{aligned}$$

where $h(x)$ contains only positive powers of $x.$

Integrating, we get

$$\ln v' = -\ln x + \left(\text{positive powers of } x\right)$$

and hence

$$v' = \frac{1}{x}\exp\left(\text{stuff}\right) = \frac{1}{x} + \left(\text{positive powers of } x\right)$$

integrating again, we have

$$v = \ln x + \text{ positive powers of } x$$

so that the second solution has a logarithmic term.

$$y_2 = y_1\left(\ln x + \text{ series of positive powers }\right)$$

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Chapter 3: Differential equations

11. For a linear differential equation of the form $x^2y'' + xfy' + gy = 0,$ where the functions $f(x)$ and $g(x)$ are analytic, the indicial equation may be written as $h(p) = 0$ where $h(p)$ is a quadratic function. Show that in determining the recursion relation, the coefficient of the c_n term is $h(p+n).$ Hence argue that the method fails to provide two solutions if the solutions of the equation $h(p) = 0$ differ by an integer.

First we insert the series into the differential equation:

$$\sum(n+p)(n+p-1)a_n x^{n+p} + \sum(n+p)a_n x^{n+p} f(x) + \sum a_n x^{n+p} g(x) = 0$$

To isolate the lowest power (x^p) in this equation, we write the functions f and g using Taylor series about the origin: Then

$$p(p-1) + pf(0) + g(0) = 0 = h(p)$$

Now we look at the power x^{p+m} :

$$0 = a_m(m+p)(m+p-1) + (m+p)a_m f(0) + (m-1+p)a_{m-1} f'(0) + \cdots$$
$$+ a_m g(0) + a_{m-1} g'(0) + \cdots$$

The coefficient of a_m is

$$(m+p)(m+p-1) + (m+p)f(0) + g(0) = h(p+m)$$

Now if the solutions of $h(p) = 0$ are p_1 and $p_2 = p_1 + N,$ then we will not be able to obtain a solution for c_N because its coefficient will be zero, and the method fails.

When does this argument fail? If the differential operator $x^2 d^2/dx^2 + xfd/dx + g$ is even, then the solution $y(x)$ is purely even or purely odd. The recursion relation relates a_m to a_{m-2}. If the roots of the indicial equation differ by unity, we will have two linearly independent solutions, one even solution and one odd solution, given by the two different roots.

12. Solve the equation $y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 0$ (equation 3.19 in the chapter) by writing it in the form

$$\frac{y''}{y'} = -\frac{y'}{y}$$

and integrating twice.

$$\frac{y''}{y'} = -\frac{y'}{y}$$

$$\frac{d}{dx}\left(\ln y'\right) = -\frac{d}{dx}(\ln y) \Rightarrow y' = \frac{A}{2y} \text{ (factor of 2 inserted for later convenience)}$$

$$\frac{d}{dx}\left(y^2\right) = A$$

$$y^2 = Ax + C = Ax + y_0^2$$

$$y' = \frac{A}{2\sqrt{Ax + y_0^2}} \Rightarrow y'(0) = \frac{A}{2y_0} \Rightarrow A = 2y_0 y_0'$$

$$y = \sqrt{y_0\left(2y_0' x + y_0\right)}$$

13. Find the two solutions of the equation

$$y'' - y' + \frac{y}{x} = 0$$

The equation has a singular point at $x = 0$, so we write a solution of the Frobenius type:

$$y = \sum a_n x^{n+p}$$

Then, multiplying by x, the differential equation becomes:

$$\sum (n+p)(n+p-1)a_n x^{n+p-1} - \sum (n+p)a_n x^{n+p} + \sum a_n x^{n+p} = 0$$

The lowest power that appears is x^{p-1} and its coefficient is:

$$(p-1)pa_0 = 0$$

which has solutions $p = 0$, $p = 1$. Inspection of the equation shows that $y = x$ is the complete solution. Or, we can write the recursion relation by looking at the coefficient of x^{p+m} :

With $p = 0$ we get:

$$(m+1)ma_{m+1} - ma_m + a_m = 0$$
$$(m+1)ma_{m+1} - (m-1)a_m = 0$$

So

$$a_{m+1} = \frac{(m-1)a_m}{m(m+1)}$$

if we start with a_0 we get an immediate problem, so we must conclude that $a_0 = 0$ and this is not a valid solution.

Starting with $p = 1$, we get the recursion relation:

$$(m+2)(m+1)a_{m+1} - (m+1)a_m + a_m = 0 \Rightarrow a_{m+1} = m\frac{a_m}{(m+1)(m+2)}$$

which gives a_1 and all succeeding terms zero. This is the solution $y = x$ that we guessed above.

For the second solution we choose a logarithm:

$$y = x\ln x + \sum a_n x^{n+p}$$

$$y' = \ln x + 1 + \sum (n+p) a_n x^{n+p-1}$$

and

$$y'' = \frac{1}{x} + \sum (n+p)(n+p-1) a_n x^{n+p-2}$$

Stuffing into the differential equation, we have:

$$1 + \sum (n+p)(n+p-1) a_n x^{n+p-1} - x\ln x - x - \sum (n+p) a_n x^{n+p} + x\ln x + \sum a_n x^{n+p} = 0$$

$$1 + \sum (n+p)(n+p-1) a_n x^{n+p-1} - x - \sum (n+p) a_n x^{n+p} + \sum a_n x^{n+p} = 0$$

The lowest power is x^0, so we take $p = 0$.

$$1 + \sum_{n=2}^{\infty} n(n-1) a_n x^{n-1} - x - \sum_{n=1}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} a_n x^n = 0$$

Then for the x^0 term we have:

$$1 + a_0 = 0 \Rightarrow a_0 = -1$$

For the x^1 term:

$$2a_2 - 1 - a_1 + a_1 = 0 \Rightarrow a_2 = \frac{1}{2}$$

For the 2nd power:

$$3 \times 2a_3 - 2a_2 + a_2 = 0 \Rightarrow a_3 = \frac{a_2}{3 \times 2} = \frac{1}{2 \times 3!}$$

and for the mth power:

$$(m+1)m a_{m+1} - m a_m + a_m = 0 \Rightarrow a_{m+1} = a_m \frac{(m-1)}{m(m+1)}$$

Then we can step down:

$$\begin{aligned}
a_m &= \frac{(m-2)}{m(m-1)}a_{m-1} \\
&= \frac{(m-2)}{m(m-1)}\frac{(m-3)}{(m-1)(m-2)}a_{m-2} = \frac{(m-3)}{m(m-1)^2}a_{m-2} \\
&= \frac{(m-3)}{m(m-1)^2}\frac{(m-4)}{(m-2)(m-3)}a_{m-3} = \frac{(m-4)}{m(m-1)^2(m-2)}a_{m-3} \\
&= \frac{(m-4)}{m(m-1)^2(m-2)}\frac{(m-5)}{(m-3)(m-4)}a_{m-4} = \frac{m-5}{m(m-1)^2(m-2)(m-3)}a_{m-4} \\
&= \frac{1}{(m-1)m(m-1)(m-2)(m-3)\cdots 3}a_2 = \frac{1}{(m-1)m!}2a_2 \\
&= \frac{1}{(m-1)m!}
\end{aligned}$$

Thus the solution is:

$$y = x\ln x - 1 + \frac{x^2}{2} + \frac{x^3}{2(3!)} + \frac{x^4}{3(4!)} + \cdots \frac{x^m}{(m-1)m!} + \cdots$$

14. Determine a solution of the equation

$$(1+x)y'' + (3+2x)y' + (2+x)y = 0$$

at large x. Hence determine the solution for all x.

At large x the equation simplifies, since $x \gg 1$:

$$y'' + 2y' + y = 0$$

The solution is an exponential e^{mx} where

$$m^2 + 2m + 1 = (m+1)^2 = 0$$

with only one solution, $m = -1$. Thus at large x, $y \sim e^{-x}$.

To determine the solution for all x, we look for a solution of the form $y = u(x)e^{-x}$. Then:

$$y' = u'e^{-x} - ue^{-x}$$

and

$$y'' = u''e^{-x} - 2u'e^{-x} + ue^{-x}$$

and stuffing in, we get:

$$(1+x)\left(u''e^{-x} - 2u'e^{-x} + ue^{-x}\right) + (3+2x)\left(u'e^{-x} - ue^{-x}\right) + (2+x)ue^{-x} = 0$$

Since each term contains e^{-x}, we cancel it:

$$(1+x)u'' + u' + (0)u = 0$$

Thus

$$\frac{u''}{u'} = \frac{-1}{1+x}$$

Integrating once, we get

$$\ln u' = -\ln(1+x) + \text{Const}$$

so

$$u' = \frac{A}{1+x}$$

integrating again:

$$u = A\ln(1+x) + B$$

Thus the complete solution to the original differential equation is:

$$y = e^{-x}[A\ln(1+x) + B]$$

15. Determine the large argument expansion of the Legendre function Q_1 by finding a solution of the equation

$$(1-x^2)y'' - 2xy' + 2y = 0$$

as a series in powers of $1/x.$

First we let $w = 1/x.$ Then

$$\frac{dy}{dx} = \frac{dy}{dw}\frac{dw}{dx} = -\frac{1}{x^2}\frac{dy}{dw} = -w^2\frac{dy}{dw}$$

and

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{d}{dw}\left(\frac{dy}{dx}\right)\frac{dw}{dx} = -\frac{1}{x^2}\frac{d}{dw}\left(-w^2\frac{dy}{dw}\right)\\ &= -w^2\left(-2w\frac{dy}{dw} - w^2\frac{d^2y}{dw^2}\right)\\ &= 2w^3\frac{dy}{dw} + w^4\frac{d^2y}{dw^2}\end{aligned}$$

So the differential equation becomes:

$$\begin{aligned}\left(1-\frac{1}{w^2}\right)\left(2w^3\frac{dy}{dw} + w^4\frac{d^2y}{dw^2}\right) - \frac{2}{w}\left(-w^2\frac{dy}{dw}\right) + 2y = 0\\ w^2(w^2-1)\frac{d^2y}{dw^2} + 2w^3\frac{dy}{dw} + 2y = 0\end{aligned}$$

The equation has a singular point at $w = 0,$ so the solution may be written as a Frobenius series:

$$y = \sum a_n w^{n+p}$$

Then we have:

$$\begin{aligned}\sum(n+p)(n+p-1)a_n\left(w^{n+p+2} - w^{n+p}\right) + 2\sum(n+p)a_n w^{n+p+2} + 2\sum a_n w^{n+p} = 0\\ \sum(n+p)(n+p+1)a_n w^{n+p+2} + \sum[2-(n+p)(n+p-1)]a_n w^{n+p} = 0\end{aligned}$$

The lowest power is w^p and its coefficient is:

$$-p(p-1)a_0 + 2a_0 = 0$$

and so, with $a_0 \neq 0$, the indicial equation is:

$$p^2 - p - 2 = 0$$
$$(p-2)(p+1) = 0$$

with solutions $p = 2,\ -1.$ The two values differ by an integer. One value gives a solution that is well behaved at $w = 0\ (x \to \infty)$, and this is the solution that we want.

The $(p+1)$ power has coefficient:

$$-(p+1)pa_1 + 2a_1 = 0$$
$$-a_1(p+2)(p-1) = 0$$

which gives $a_1 \neq 0$ and $p = 1$ or -2. These values give the same two solutions as before.

The $p+m$ power for $m \geq 2$ gives:

$$(p+m-2)(p+m-1)a_{m-2} + [2-(p+m)(p+m-1)]a_m = 0$$

So

$$a_m = \frac{(p+m-2)(p+m-1)}{(p+m)(p+m-1)-2}a_{m-2}$$

With $p = 2$, we have:

$$a_m = \frac{m(m+1)}{(m+2)(m+1)-2}a_{m-2} = \frac{m(m+1)}{m^2+3m}a_{m-2} = \frac{m+1}{m+3}a_{m-2}$$

and so

$$a_m = \frac{3}{m+3}a_0$$

Specifically:

$$a_2 = \frac{3}{5}a_0$$

$$a_4 = \frac{5}{7}a_2 = \frac{5\times 3}{7\times 5}a_0 = \frac{3}{7}a_0$$

and so the solution is:

$$y = a_0 w^2\left(1 + \frac{3}{5}w^2 + \frac{3}{7}w^4 + \cdots\right)$$
$$= A\frac{1}{x^2}\left(\frac{1}{3} + \frac{1}{5x^2} + \frac{1}{7x^4} + \cdots\right)$$
$$= \frac{A}{x^2}\sum_{n=0}^{\infty}\frac{1}{2n+3}\frac{1}{x^{2n}}$$

where $A = 3a_0$ is an arbitrary constant.

16. Solve the equation

$$x^2y'' - 4xy' + \left(6 + x^2\right)y = 0$$

The equation has a regular singular point at $x = 0$, so we use a Frobenius series.

$$\sum(n+p)(n+p-1)a_nx^{n+p} - 4\sum(n+p)a_nx^{n+p} + 6\sum a_nx^{n+p} + \sum a_nx^{n+p+2} = 0$$

The lowest power is x^p and its coefficient is:

$$p(p-1)a_0 - 4pa_0 + 6a_0 = 0$$

So the indicial equation is:

$$p^2 - 5p + 6 = 0$$
$$(p-3)(p-2) = 0$$

So the solutions are $p = 3$ and $p = 2$, which differ by an integer. Thus we may find only one series. Let's see.

The coefficient of x^{p+1} is:

$$[(p+1)p - 4(p+1) + 6]a_1 = 0$$

which gives $p + 1 = 3,\ 2,$ with $a_1 \neq 0,$ and these values will give the same solutions as $p = 3,$ 2 with $a_0 \neq 0.$

The recursion relation is:

$$(m+p)(m+p-1)a_m - 4(m+p)a_m + 6a_m + a_{m-2} = 0$$

So

$$a_m = -\frac{a_{m-2}}{(m+p)(m+p-5)+6}$$

and with $p = 2,$ we get

$$a_m = -\frac{a_{m-2}}{(m+2)(m-3)+6}$$

The first few terms are:

$$a_2 = -\frac{a_0}{2}$$

$$a_4 = -\frac{a_2}{6\times 1+6} = -\frac{a_2}{12} = \frac{a_0}{4!}$$

$$a_6 = -\frac{a_4}{8\times 3+6} = -\frac{a_4}{6\times 5} = -\frac{a_0}{6!}$$

So the solution is:

$$y = a_0 x^2 \left(1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^5}{5!} + \cdots\right)$$
$$= a_0 x^2 \cos x$$

The second value of p, $p = 3$, gives:

$$a_m = -\frac{a_{m-2}}{(m+3)(m-2)+6}$$

So

$$a_2 = -\frac{a_0}{6}$$

$$a_4 = -\frac{a_2}{7 \times 2 + 6} = \frac{a_0}{6(20)} = \frac{a_0}{5 \times 4 \times 3 \times 2} = \frac{a_0}{5!}$$

$$a_6 = -\frac{a_4}{9 \times 4 + 6} = -\frac{a_4}{7 \times 6} = -\frac{a_0}{7!}$$

So the solution is:

$$y = x^3 \left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots\right)$$
$$= x^2 \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right)$$
$$= x^2 \sin x$$

Thus the two solutions are $x^2 \cos x$ and $x^2 \sin x$.

Check:

$$\frac{d^2}{dx^2}\left(x^2 \cos x\right) = 2\cos x - 4x \sin x - x^2 \cos x$$

$$\frac{d}{dx}\left(x^2 \cos x\right) = 2x \cos x - x^2 \sin x$$

$$x^2\left(2\cos x - 4x\sin x - x^2\cos x\right) - 4x\left(2x\cos x - x^2 \sin x\right) + \left(6 + x^2\right)x^2\cos x = 0$$

as required.

17. Solve the equation

$$xy'' - y' + 4x^3 y = 0$$

The equation has a regular singular point at $x = 0$, so we use a Frobenius series.

$$\sum (n+p)(n+p-1)a_n x^{n+p-1} - \sum (n+p)a_n x^{n+p-1} + 4\sum a_n x^{n+p+3} = 0$$

The lowest power is x^{p-1} and its coefficient is:

$$p(p-1)a_0 - pa_0 = 0$$

So the indicial equation is:

$$p(p-2) = 0$$

with solutions $p = 0$ and $p = 2.$ The recursion relation is:

$$(n+p)(n+p-1)a_n - (n+p)a_n + 4a_{n-4} = 0$$

Thus:

$$a_n = -\frac{4a_{n-4}}{(n+p)(n+p-2)}$$

So each series has only every 4th power. With $p = 0,$ we get

$$a_n = -\frac{4a_{n-4}}{(n)(n-2)}$$

So

$$a_4 = -\frac{4a_0}{4 \times 2} = -\frac{a_0}{2}$$

$$a_8 = -\frac{4a_4}{8 \times 6} = -\frac{a_4}{4 \times 3} = \frac{a_0}{4!}$$

and the solution is:

$$y = a_0\left(1 - \frac{x^4}{2!} + \frac{x^8}{4!} + \cdots\right) = a_0 \cos x^2$$

With $p = 2$ we get

$$a_n = -\frac{4a_{n-4}}{(n+2)(n)}$$

So

$$a_4 = -\frac{4a_0}{6 \times 4} = -\frac{a_0}{3!}$$

$$a_8 = -\frac{4a_4}{10 \times 8} = \frac{-a_4}{5 \times 4} = \frac{a_0}{5!}$$

and the solution is

$$y = a_0 x^2\left(1 - \frac{x^4}{3!} + \frac{x^8}{5!} + \cdots\right) = a_0\left(x^2 - \frac{x^6}{3!} + \frac{x^{10}}{5!} + \cdots\right)$$
$$= a_0 \sin x^2$$

Check:

$$\frac{d}{dx}\left(\sin x^2\right) = 2\left(\cos x^2\right)x$$

$$\frac{d}{dx}\left(2\left(\cos x^2\right)x\right) = -4\left(\sin x^2\right)x^2 + 2\cos x^2$$

$$x\left(-4\left(\sin x^2\right)x^2 + 2\cos x^2\right) - 2\left(\cos x^2\right)x + 4x^3 \sin x^2 = 0$$

as required.

18. The conical functions are Legendre functions with $l = -\frac{1}{2} + i\lambda.$

(a) Starting from the Legendre equation (cf Example 3.7), find the differential equation satisfied by the conical functions $P_{-\frac{1}{2}+i\lambda}(x)$ and $Q_{-\frac{1}{2}+i\lambda}(x)$

(b) Show that one solution is analytic at the point $x = 1,$ and determine a series expansion for the conical function $P_{-\frac{1}{2}+i\lambda}(\cos\theta)$ in powers of $\sin\frac{\theta}{2}$. Hence show that this conical function is real.

(a) The differential equation is

$$\left(1 - x^2\right)y'' - 2xy' + l(l+1)y = 0$$

where $x = \cos\theta.$ Now substitute $l = -\frac{1}{2} + i\lambda.$

$$\begin{aligned}\left(1 - x^2\right)y'' - 2xy' + \left(-\frac{1}{2} + i\lambda\right)\left(\frac{1}{2} + i\lambda\right)y &= 0\\ \left(1 - x^2\right)y'' - 2xy' - \left(\frac{1}{4} + \lambda^2\right)y &= 0\end{aligned}$$

(b) Since $\cos\theta = 1 - 2\sin^2\theta/2,$ we can let $u = \sin\theta/2,$ and then $x = 1$ corresponds to $u = 0.$ Further:

$$x = 1 - 2u^2 \Rightarrow \frac{dx}{du} = -4u$$

So

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = \frac{dy}{du}\left(\frac{1}{-4u}\right)$$

and

$$\begin{aligned}\frac{d^2y}{dx^2} &= \frac{d}{du}\left(\frac{dy}{dx}\right)\frac{du}{dx} = \frac{d}{du}\left(\frac{-1}{4u}\frac{dy}{du}\right)\frac{-1}{4u}\\ &= \frac{1}{(4u)^2}\frac{d^2y}{du^2} - \frac{1}{16u^3}\frac{dy}{du}\end{aligned}$$

Also

$$1 - x^2 = (1-x)(1+x) = 2u^2 2\left(1 - u^2\right) = 4u^2\left(1 - u^2\right)$$

So the equation becomes:

$$4u^2(1-u^2)\left(\frac{1}{(4u)^2}\frac{d^2y}{du^2}-\frac{1}{16u^3}\frac{dy}{du}\right)+2(1-2u^2)\frac{1}{4u}\frac{dy}{du}-\left(\frac{1}{4}+\lambda^2\right)y=0$$

$$\frac{(1-u^2)}{4}\frac{d^2y}{du^2}-\frac{(1-u^2)}{4u}\frac{dy}{du}+\left(\frac{1}{2u}-u\right)\frac{dy}{du}-\left(\frac{1}{4}+\lambda^2\right)y=0$$

$$(1-u^2)\frac{d^2y}{du^2}+\left(\frac{1}{u}-3u\right)\frac{dy}{du}-(1+4\lambda^2)y=0$$

$u = 0$ is a singular point of the equation, so look for a solution of the form

$$y=\sum a_n u^{n+p}$$

Then:

$$0=\sum(n+p)(n+p-1)a_n u^{n+p-2}-\sum(n+p)(n+p-1)a_n u^{n+p}$$
$$+\sum(n+p)a_n u^{n+p-2}-3\sum(n+p)a_n u^{n+p}-\left(1+4\lambda^2\right)\sum a_n u^{n+p}$$

The lowest power is u^{p-2} and its coefficient is:

$$p(p-1)a_0+pa_0=0$$

So the indicial equation is

$$p^2=0$$

with the solution $p = 0$. Thus one solution is analytic at the origin. The recursion relation is:

$$0=(m+p+2)(m+p+1)a_{m+2}-(m+p)(m+p-1)a_m$$
$$+(m+p+2)a_{m+2}-3(m+p)a_m-\left(1+4\lambda^2\right)a_m$$

And so

$$a_{m+2}=a_m\frac{(m+p)(m+p+2)+1+4\lambda^2}{(m+p+2)^2}$$

So with $p = 0$ we get:

$$a_{m+2}=a_m\frac{m(m+2)+1+4\lambda^2}{(m+2)^2}$$

So

$$a_2=a_0\frac{4\lambda^2+1}{2^2}$$

$$a_4=a_2\frac{2(4)+1+4\lambda^2}{4^2}=a_0\frac{\left(4\lambda^2+1\right)}{2^2}\frac{\left(4\lambda^2+3^2\right)}{4^2}$$

$$a_6=a_4\frac{4(6)+1+4\lambda^2}{6^2}=a_0\frac{\left(4\lambda^2+1\right)}{2^2}\frac{\left(4\lambda^2+3^2\right)}{4^2}\frac{\left(4\lambda^2+5^2\right)}{6^2}$$

and thus the solution is:

$$y(\theta) = a_0\left(\begin{array}{l} 1 + \frac{4\lambda^2+1}{2^2}\sin^2\frac{\theta}{2} + \frac{(4\lambda^2+1)}{2^2}\frac{(4\lambda^2+3^2)}{4^2}\sin^4\frac{\theta}{2} + \\ \cdots\frac{(4\lambda^2+1)}{2^2}\frac{(4\lambda^2+3^2)}{4^2}\cdots\frac{(4\lambda^2+(2n-1)^2)}{(2n)^2}\sin^{2n}\frac{\theta}{2} + \cdots \end{array}\right)$$

which is clearly real.

19. Write the equation $x^4y'' + y = 0$ in standard form, and use Fuch's theorem to show that the Frobenius method may not give two series-type solutions about $x = 0$. Change to the new variable $u = 1/x$ (cf Example 3.10) and show that the new equation can be solved by the Frobenius method. Obtain the two solutions.

In standard form, the equation is:

$$y'' + \frac{y}{x^4} = 0$$

Then the function $g(x) = 1/x^4$, and $x^2g(x) = 1/x^2$ has a second order pole at $x = 0$. Thus by Fuch's theorem, the Frobenius method will not give two series solutions.

With $u = 1/x$, $dy/dx = -u^2du/dx$ and $d^2y/dx^2 = u^4d^2y/du^2 + 2u^3dy/du$. Then the equation becomes:

$$u^4\frac{d^2y}{du^2} + 2u^3\frac{dy}{du} + u^4y = 0$$
$$y'' + \frac{2}{u}y' + y = 0$$

This equation has $f(u) = 2/u$ and $g(u) = 1$, so both $uf(u)$ and $u^2g(u)$ are analytic at $u = 0$. Thus we look for a solution of the form:

$$y = u^p\sum_{n=0}^{\infty}a_nu^n$$

The equation becomes:

$$\sum_{n=0}^{\infty}(p+n)(p+n-1)a_nu^{n+p-2} + 2\sum_{n=0}^{\infty}(n+p)a_nu^{n+p-2} + \sum_{n=0}^{\infty}a_nu^{n+p} = 0$$

The lowest power of u is u^{p-2}, and its coefficient is:

$$[p(p-1) + 2p]a_0 = 0$$
$$p^2 + p = 0$$

which has the solutions

$$p = 0, -1$$

The recursion relation is:

$$(p+n)(p+n+1)a_n + a_{n-2} = 0$$

or

$$a_n = -\frac{a_{n-2}}{(p+n+1)(p+n)}$$

With $p = 0$:

$$a_n = -\frac{a_{n-2}}{(n+1)n} = (-1)^2\frac{a_{n-4}}{(n+1)(n)(n-1)(n-2)} = (-1)^{n/2}\frac{a_0}{(n+1)!}$$

With $p = -1$

$$a_n = -\frac{a_{n-2}}{n(n-1)} = (-1)^{n/2}\frac{a_0}{n!}$$

The recursion relation skips powers, so the series will have only even n terms:

$$a_{2m} = (-1)^m\frac{a_0}{(2m)!}$$

Thus the general solution is:

$$\begin{aligned} y &= \alpha\sum_{m=0}^{\infty}(-1)^m\frac{u^{2m}}{(2m+1)!} + \frac{\beta}{u}\sum_{m=0}^{\infty}(-1)^m\frac{u^{2m}}{(2m)!} \\ &= \frac{\alpha}{u}\sum_{m=0}^{\infty}(-1)^m\frac{u^{2m+1}}{(2m+1)!} + \frac{\beta}{u}\sum_{m=0}^{\infty}(-1)^m\frac{u^{2m}}{(2m)!} \\ &= \frac{\alpha}{u}\sin u + \frac{\beta}{u}\cos u \\ &= x\left(\alpha\sin\frac{1}{x} + \beta\cos\frac{1}{x}\right) \end{aligned}$$

In this case the two roots of the indicial equation do provide two independent solutions, even though they differ by an integer.

Check:

$$\begin{aligned} y' &= \alpha\sin\frac{1}{x} + \beta\cos\frac{1}{x} + x\left(-\frac{\alpha}{x^2}\cos\frac{1}{x} + \frac{\beta}{x^2}\sin\frac{1}{x}\right) \\ &= \left(\alpha + \frac{\beta}{x}\right)\sin\frac{1}{x} + \left(\beta - \frac{\alpha}{x}\right)\cos\frac{1}{x} \end{aligned}$$

$$\begin{aligned} y'' &= -\frac{\beta}{x^2}\sin\frac{1}{x} + \frac{\alpha}{x^2}\cos\frac{1}{x} + \left(\alpha + \frac{\beta}{x}\right)\left(\frac{-1}{x^2}\right)\cos\frac{1}{x} + \left(\beta - \frac{\alpha}{x}\right)\left(\frac{1}{x^2}\right)\sin\frac{1}{x} \\ &= -\frac{\alpha}{x^3}\sin\frac{1}{x} - \frac{\beta}{x^3}\cos\frac{1}{x} \end{aligned}$$

$$\begin{aligned} x^4y'' + y &= x^4\left(-\frac{\alpha}{x^3}\sin\frac{1}{x} - \frac{\beta}{x^3}\cos\frac{1}{x}\right) + x\left(\alpha\sin\frac{1}{x} + \beta\cos\frac{1}{x}\right) \\ &= -\alpha x\sin\frac{1}{x} - \beta x\cos\frac{1}{x} + x\left(\alpha\sin\frac{1}{x} + \beta\cos\frac{1}{x}\right) = 0 \end{aligned}$$

as required.

20. Solve the equation

$$y'' + y\cosh x = 0$$

Hint: first expand the hyperbolic cosine in a series, then use a power series method.

$$y'' + y\cosh x = 0$$
$$y'' + y\left(1 + \frac{x^2}{2!} + \cdots\right) = 0$$
$$y'' + y\sum_{m=0}^{\infty} \frac{x^{2m}}{(2m)!} = 0$$

Now look for a series solution:

$$\sum_n a_n n(n-1)x^{n-2} + \sum_n a_n x^n \sum_{m=0}^{\infty} \frac{x^{2m}}{(2m)!} = 0$$

The lowest power is x^0 :

$$2a_2 + a_0 = 0 \Rightarrow a_2 = -\frac{a_0}{2}$$

Now look at x^1 :

$$a_3 \times 3 \times 2 + a_1 = 0 \Rightarrow a_3 = -\frac{a_1}{6}$$

x^2 :

$$a_4 \times 4 \times 3 + a_2 + \frac{a_0}{2!} = 0 \Rightarrow a_4 = -\frac{2a_2 + a_0}{4!} = 0$$

x^p, p odd, $= 2q + 1$:

$$a_{2q+3}(2q+3)(2q+2) + \frac{a_1}{(2q)!} + \frac{a_3}{(2q-2)!} + \cdots a_{2q+1} = 0$$

$$a_{2q+1} = -\frac{(2q-1)a_1}{(2q+1)!} - \frac{a_3}{(2q+1)(2q)(2q-4)!} - \cdots \frac{a_{2q-1}}{(2q+1)(2q)}$$

x^{2q} :

$$a_{2q+2}(2q+2)(2q+1) + \frac{a_0}{(2q)!} + \frac{a_2}{(2q-2)!} + \cdots a_{2q} = 0$$

$$a_{2q} = -\frac{a_0}{(2q)!} - \frac{a_2}{2q(2q-1)(2q-4)!} - \cdots \frac{a_{2q-2}}{2q(2q-1)}$$

There are two solutions, one with even powers and one with odd powers. Let's look for a pattern:

$$a_6 = -\frac{a_0}{6!} - \frac{a_2}{6 \times 5 \times 2} - \frac{a_4}{6 \times 5} = -\frac{a_0}{6!} + \frac{a_0}{6 \times 5 \times 4} = \frac{a_0}{6!}(6-1) = \frac{5a_0}{6!}$$

$$
\begin{aligned}
a_8 &= -\frac{a_0}{8!} - \frac{a_2}{8\times 7\times 4!} - \frac{a_4}{8\times 7\times 2} - \frac{a_6}{8\times 7} \\
&= -\frac{a_0}{8!} + \frac{a_0}{8\times 7\times 4!\times 2} - \frac{5a_0}{8\times 7\times 6!} \\
&= \frac{a_0}{8!}[-1 + 5\times 3 - 5] = \frac{9a_0}{8!}
\end{aligned}
$$

$$
\begin{aligned}
a_{10} &= -\frac{a_0}{10!} - \frac{a_2}{10\times 9\times 6!} - \frac{a_4}{10\times 9\times 4!} - \frac{a_6}{10\times 9\times 2!} - \frac{a_8}{10\times 9} \\
&= -\frac{a_0}{10!} + \frac{a_0/2}{10\times 9\times 6!} - \frac{5a_0/2}{10\times 9\times 6!} - \frac{9a_0}{10\times 9\times 8!} \\
&= \frac{a_0}{10!}(-1 + 4\times 7 - 4\times 7\times 5 - 9) = -122\frac{a_0}{10!}
\end{aligned}
$$

$$
a_5 = -\frac{a_1}{5\times 4\times 2!} - \frac{a_3}{5\times 4} = -\frac{3a_1}{5!} + \frac{a_1}{5!} = -\frac{2a_1}{5!}
$$

$$
\begin{aligned}
a_7 &= -\frac{a_1}{7\times 6\times 4!} - \frac{a_3}{7\times 6\times 2!} - \frac{a_5}{7\times 6} \\
&= -\frac{5a_1}{7!} - \frac{-a_1}{7\times 6\times 3\times 2\times 2!} - \frac{-2a_1}{7\times 6\times 5!} \\
&= \frac{a_1}{7!}(-5 + 5\times 2 + 2) = \frac{a_1}{7!}(7) = \frac{a_1}{6!}
\end{aligned}
$$

There is no obvious pattern. The solutions are:

$$
y_1 = a_0\left(1 - \frac{x^2}{2} + \frac{5x^6}{6!} - \frac{9x^8}{8!} - 122\frac{x^{10}}{10!} + \cdots\right)
$$

and

$$
y_2 = a_1 x\left(1 - \frac{x^2}{3!} - \frac{2x^4}{5!} + \frac{x^6}{6!} + \cdots\right)
$$

Chapter 3: Differential equations

21. The Stark effect describes the energy shift of atomic energy levels due to applied electric fields. The differential equation describing this effect may be written:

$$xy'' + y' + \left(k - \frac{1}{4}Ex^2 + \frac{Ux}{2} - \frac{m^2}{4x}\right)y = 0$$

where the term $\frac{1}{4}Ex^2$ is the perturbation due to the electric field. Obtain a power series solution for y and obtain explicit expressions for the first four non-zero terms. How many terms are needed before any effect of the electric field is included?

There is a singular point at $x = 0$, so we use the Frobenius method.

$$0 = \sum_n a_n(n+p)(n+p-1)x^{n+p-1} + \sum_n (n+p)a_n x^{n+p-1} + \left(\frac{Ux}{2} + k - \frac{1}{4}Ex^2 - \frac{m^2}{4x}\right)\sum_n a_n x^{n+p}$$

The lowest power appearing is x^{p-1}, and its coefficient is:

$$a_0 p(p-1) + a_0 p - \frac{m^2}{4}a_0 = 0$$

$$p^2 - \frac{m^2}{4} = 0$$

with solutions

$$p = \pm\frac{m}{2}$$

Then we look at successively higher powers:

x^p :

$$a_1\left[(p+1)^2 - \frac{m^2}{4}\right] = -ka_0$$

$$a_1 = -\frac{k}{(p+1)^2 - \frac{m^2}{4}}a_0 = -\frac{k}{m+1}a_0$$

where we took $p = m/2$ in the last step.

x^{p+1} :

$$a_2\left[(p+2)^2 - \frac{m^2}{4}\right] = -ka_1 - \frac{U}{2}a_0 = \left(\frac{k^2}{m+1} - \frac{U}{2}\right)a_0$$

$$a_2 = \left(\frac{k^2}{m+1} - \frac{U}{2}\right)a_0 \frac{1}{(p+2)^2 - \frac{m^2}{4}} = \left(\frac{k^2}{m+1} - \frac{U}{2}\right)a_0 \frac{1}{2(m+2)}$$

x^{p+2} :

$$a_3\left[(3+p)^2 - \frac{m^2}{4}\right] = -\left(\frac{U}{2}a_1 + ka_2 - \frac{1}{4}Ea_0\right)$$
$$= -\left(-\frac{U}{2}\frac{k}{m+1} + k\left(\frac{k^2}{m+1} - \frac{U}{2}\right)\frac{1}{2(m+2)} - \frac{1}{4}E\right)a_0$$

$$a_3 = \left[\frac{1}{12}Uk\frac{3m+5}{(m+3)(m+2)(m+1)} - \frac{1}{6(m+3)}\frac{k^3}{(m+2)(m+1)} + \frac{1}{12(3+m)}E\right]a_0$$

The general recursion relation is obtained from the $p+q$ power:

$$a_{q+1}(p+q+1)^2 + \frac{U}{2}a_{q-1} + ka_q - \frac{E}{4}a_{q-2} - \frac{m^2}{4}a_{q+1} = 0$$

$$a_{q+1} = \frac{\frac{E}{4}a_{q-2} - \frac{U}{2}a_{q-1} - ka_q}{(p+q+1)^2 - \frac{m^2}{4}}$$

Choosing $p = m/2,$ we obtain the relation:

$$a_{q+1} = \frac{\frac{E}{4}a_{q-2} - \frac{U}{2}a_{q-1} - ka_q}{(q+1)(q+m+1)}$$

The first four terms are:

$$y = a_0x^{m/2}\left\{\begin{array}{c} 1 - \frac{kx}{m+1} + \left(\frac{k^2x^2}{2(m+2)(m+1)} - \frac{Ux^2}{4(m+2)}\right) \\ +\frac{1}{12}Ukx^3\frac{3m+5}{(m+3)(m+2)(m+1)} - \frac{1}{6(m+3)}\frac{(kx)^3}{(m+2)(m+1)} + \frac{x^3}{12(3+m)}E \end{array}\right\}$$
$$= a_0x^{m/2}\left\{\begin{array}{c} 1 - \frac{kx}{m+1} + \frac{k^2x^2}{2(m+2)(m+1)} - \frac{(kx)^3}{3!(m+3)(m+2)(m+1)} \\ -\frac{Ux^2}{4(m+2)} + \frac{1}{12}Ukx^3\frac{3m+5}{(m+3)(m+2)(m+1)} + \frac{x^3}{12(3+m)}E \end{array}\right\}$$

The effect of the electric field does not show up until the cubic term.

22. Show that the indicial equation for the Bessel equation

$$\frac{d}{dx}\left(x\frac{dy}{dx}\right) + xy = 0$$

has a repeated root. Show that this root leads to only one solution Find the second solution using equation 3.37. Try to get at least the first three terms in the series.

Expanding the derivative, we get

$$xy'' + y' + xy = 0$$

so there is a singular point at the origin.

$$\sum_n a_n(n+p)(n+p-1)x^{n+p-1} + \sum_n a_n(n+p)x^{n+p-1} + \sum_n a_nx^{n+p+1} = 0$$
$$\sum_n a_n(n+p)^2x^{n+p-1} + \sum_n a_nx^{n+p+1} = 0$$

The indicial equation is

$$p^2 = 0,\ p = 0$$

Then we have:

$$\sum_n a_n n^2 x^{n-1} + \sum_n a_n x^{n+1} = 0$$

$$a_1 = -a_0$$

$$a_m = -\frac{a_{m-2}}{m^2} = \frac{(-1)^{m/2}}{m^2(m-2)^2 \cdots 2^2}$$
$$a_{2k} = \frac{(-1)^k}{2^{2k}(k!)^2}$$

The solution is

$$y_1 = a_0 \sum \frac{(-1)^k}{2^{2k}(k!)^2} x^{2k}$$

Now we look for a second solution of the form:

$$y_2 = y_1 \ln x + \sum_n a_n x^{n+p}$$

$$\frac{dy}{dx} = \frac{y_1}{x} + y_1' \ln x + \sum_n (n+p) a_n x^{n+p-1}$$
$$x\frac{dy}{dx} = y_1 + y_1' x \ln x + \sum_n (n+p) a_n x^{n+p}$$
$$\frac{d}{dx}\left(x\frac{dy}{dx}\right) = y_1' + y_1' \ln x + y_1' + y_1'' x \ln x + \sum_n (n+p)(n+p-1) a_n x^{n+p-1}$$

Stuffing in, we get:

$$\frac{d}{dx}\left(x\frac{dy}{dx}\right) + xy = 0$$
$$y_1'' x \ln x + y_1'(2 + \ln x)) + y_1 x \ln x + \sum_n a_n (n+p)^2 x^{n+p-1} + \sum_n a_n x^{n+p+1} = 0$$
$$\ln x \left(y_1'' x + y_1' + y_1 x\right) + 2y_1' + \sum_n a_n (n+p)^2 x^{n+p-1} + \sum_n a_n x^{n+p+1} = 0$$
$$2\sum_{k=1}^{\infty} \frac{(-1)^k 2k}{2^{2k}(k!)^2} x^{2k-1} + \sum_n a_n (n+p)^2 x^{n+p-1} + \sum_n a_n x^{n+p+1} = 0$$
$$\sum_{k=1}^{\infty} \frac{(-1)^k}{2^{2(k-1)} k!(k-1)!} x^{2k-1} + \sum_n a_n (n+p)^2 x^{n+p-1} + \sum_n a_n x^{n+p+1} = 0$$
$$-\sum_{k=0}^{\infty} \frac{(-1)^k}{2^{2k} k!(k+1)!} x^{2k+1} + \sum_n a_n (n+p)^2 x^{n+p-1} + \sum_n a_n x^{n+p+1} = 0$$

Since the first series has only odd powers of x, $n+p$ must be even. The lowest power in the first series is x^1 so we can take $p = 0$ and n even to get:

$$-\sum_{k=0}^{\infty} \frac{(-1)^k}{2^{2k} k!(k+1)!} x^{2k+1} + \sum_{n=2}^{\infty} a_n n^2 x^{n-1} + \sum_{n=0}^{\infty} a_n x^{n+1} = 0$$

Then x^1

$$-1 + 2^2 a_2 + a_0 = 0 \Rightarrow a_2 = \frac{1 - a_0}{2^2}$$

x^3

$$-\frac{(-1)}{2^2 2!} + a_4 4^2 + a_2 = 0 \Rightarrow a_4 = -\frac{1}{4^2}\left\{\frac{1}{2^2 2!} + a_2\right\}$$

$$a_4 = -\frac{1}{4^2}\left\{\frac{1}{2^2 2!} + \frac{1 - a_0}{2^2}\right\}$$

$$= \frac{1}{2^4 (2!)^2}\left(a_0 - \frac{3}{2}\right)$$

x^5

$$-\frac{(-1)^2}{2^4 2!3!} + 6^2 a_6 + a_4 = 0 \Rightarrow a_6 = -\frac{1}{6^2}\left\{\frac{1}{2^4 2!3!} + a_4\right\}$$

$$a_6 = -\frac{1}{6^2}\left\{\frac{1}{2^4 2!3!} + \frac{1}{2^4 (2!)^2}\left(a_0 - \frac{3}{2}\right)\right\}$$

$$= -\frac{1}{2^2 3^2}\left(\frac{1}{2^4 2!3!} - \frac{3}{2^5 (2!)^2} + \frac{1}{2^4 (2!)^2} a_0\right)$$

$$= -\frac{1}{2^2 3^2}\left(-\frac{7}{2^7 3} + \frac{1}{2^4 (2!)^2} a_0\right)$$

$$= \frac{7}{2^9 3^2} - \frac{1}{2^6 (3!)^2} a_0$$

x^7

$$-\frac{(-1)^3}{2^6 3!4!} + a_8 8^2 + a_6 = 0$$

$$a_8 = -\frac{1}{8^2}\left(\frac{1}{2^6 3!4!} + a_6\right)$$

$$= -\frac{1}{8^2}\left(\frac{1}{2^6 3!4!} + \frac{7}{2^9 3^2} - \frac{1}{2^6 (3!)^2} a_0\right)$$

$$= -\frac{5}{2^{16} 3} + \frac{1}{2^8 (4!)^2} a_0$$

We can see that the function multiplying a_0 is just our first solution. The second solution is:

$$y_2 = y_1 \ln x + 1 + \left(\frac{x}{2}\right)^2 + \frac{3}{2^3}\left(\frac{x}{2}\right)^4 + \frac{7}{2^3 3^2}\left(\frac{x}{2}\right)^6 - \frac{5}{2^8 3}\left(\frac{x}{2}\right)^8 \cdots$$

23. Attempt to solve the equation

$$x^2 y'' + y' = 0$$

using the Frobenius method. Show that the resulting series does not converge for any value of x.

$$\sum_n a_n (n+p)(n+p-1) x^{n+p} + \sum_n a_n (n+p) x^{n+p-1} = 0$$

The lowest power is x^{p-1}

$$a_0 p = 0$$

Thus we need $p = 0$ or $a_0 = 0.$ The next power is x^p

$$a_0 p(p-1) + a_1(p+1) = 0$$

With $p = 0$ we get

$$a_1 = 0$$

The general recursion relation is:

$$a_m(m+p)(m+p-1) + a_{m+1}(m+p+1) = 0$$
$$a_{m+1} = \frac{-m(m-1)}{m+1} a_m$$

From the ratio test, the ratio of two successive terms is

$$\left|\frac{u_{m+1}}{u_m}\right| = \frac{m(m-1)}{m+1}|x| \to m|x| \text{ as } m \to \infty$$

This ratio is > 1 for any finite value of x for $m > 1/|x|,$ and thus the series diverges.

24. Weber's equation is

$$y'' + \left(m + \frac{1}{2} - \frac{x^2}{4}\right)y = 0$$

Show that the substitution $y = \exp\left(-\frac{x^2}{4}\right)v(x)$ simplifies this equation. Find two solutions for $v(x)$ as power series in x.

$$y = \exp\left(-\frac{x^2}{4}\right)v$$
$$y' = -\frac{x}{2}\exp\left(-\frac{x^2}{4}\right)v + \exp\left(-\frac{x^2}{4}\right)v'$$
$$y'' = -\frac{1}{2}\exp\left(-\frac{x^2}{4}\right)v + \frac{x^2}{4}\exp\left(-\frac{x^2}{4}\right)v - x\exp\left(-\frac{x^2}{4}\right)v' + \exp\left(-\frac{x^2}{4}\right)v''$$

Substituting in:

$$-\frac{1}{2}\exp\left(-\frac{x^2}{4}\right)v + \frac{x^2}{4}\exp\left(-\frac{x^2}{4}\right)v - x\exp\left(-\frac{x^2}{4}\right)v'$$
$$+ \exp\left(-\frac{x^2}{4}\right)v'' + \left(m + \frac{1}{2} - \frac{x^2}{4}\right)\exp\left(-\frac{x^2}{4}\right)v$$
$$= -x\exp\left(-\frac{x^2}{4}\right)v' + \exp\left(-\frac{x^2}{4}\right)v'' + m\exp\left(-\frac{x^2}{4}\right)v = 0$$

Thus the equation for v is:

$$v'' - xv' + mv = 0$$

$x = 0$ is a regular point, so

$$\sum n(n-1)a_n x^{n-2} - \sum n a_n x^n + m\sum a_n x^n = 0$$

Starting with $x^0,$ we have:

$$2 \times 1 \times a_2 + ma_0 = 0 \Rightarrow a_2 = \frac{-m}{2}a_0$$

From x^1 we get

$$3 \times 2 \times a_3 - a_1 + ma_1 = 0 \Rightarrow a_3 = -\frac{m-1}{3!}a_1$$

and in general, from x^p

$$(p+2)(p+1)a_{p+2} - pa_p + ma_p = 0 \Rightarrow a_{p+2} = -\frac{m-p}{(p+2)(p+1)}a_p$$

Thus

$$\begin{aligned} a_p &= (-1)^2 \frac{(m-p+2)}{p(p-1)} \frac{(m-p+4)}{(p-2)(p-3)} a_{p-4} \\ &= \frac{(m-p+2)(m-p+4)\cdots m}{p!} a_0 \\ &= \frac{m!!}{p!(m-p)!!} a_0 \end{aligned}$$

for p even, and

$$a_p = \frac{(m-p+2)(m-p+4)\cdots(m-1)}{p!}a_1$$

for p odd. Thus the solutions are

$$y = \exp\left(-\frac{x^2}{4}\right)a_0\left(1 - \frac{m}{2}x^2 + \frac{m(m-2)}{4!}x^4 + \cdots \frac{m!!}{p!(m-p)!!}x^p + \cdots\right)$$

and

$$y = \exp\left(-\frac{x^2}{4}\right)a_1\left(\begin{array}{c} x - \frac{m-1}{2}x^3 + \frac{(m-3)(m-1)}{4!}x^5 + \cdots \\ \frac{(m-1)!!}{p!(m-p)!!}x^p + \cdots \end{array}\right)$$

25. The Schrödinger equation in one dimension has the form

$$\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + (E - V)\psi = 0$$

Develop a series solution for ψ in the case that V is the potential due to the interaction of two nucleons:

$$V = C\frac{e^{-\alpha x}}{x}$$

Obtain at least the first three non-zero terms.

$$\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + \left(E - C\frac{e^{-\alpha x}}{x}\right)\psi = 0$$

Let $e = 2Em/\hbar^2$ and $c = 2EC/\hbar^2$. Then

$$\psi'' + \left(e - \frac{ce^{-\alpha x}}{x}\right)\psi = 0$$

The equation has a singular point at $x = 0$, so we use a Frobenius series.

$$\sum_n a_n(n+p)(n+p-1)x^{n+p-2} + \left(e - \frac{ce^{-\alpha x}}{x}\right)\sum_n a_n x^{n+p} = 0$$

Now expand the exponential in a series:

$$\sum_n a_n(n+p)(n+p-1)x^{n+p-2} + \left(e - \frac{c}{x}\sum_{m=0}^{\infty}\frac{(-\alpha x)^m}{m!}\right)\sum_n a_n x^{n+p} = 0$$

$$\sum_n a_n(n+p)(n+p-1)x^{n+p-2} + e\sum_n a_n x^{n+p} - c\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} a_n x^{n+p+m-1}\frac{(-\alpha)^m}{m!} = 0$$

The lowest power is x^{p-2}:

$$a_0 p(p-1) = 0 \Rightarrow p = 0, 1$$

The next power is x^{p-1}

$$a_1(p+1)p - ca_0 = 0 \Rightarrow a_1 = \frac{ca_0}{p(p+1)}$$

If $p = 0$ we would get $a_0 = 0.$ So take $p = 1.$ Then

$$a_1 = \frac{ca_0}{2}$$

The next power is x^p

$$a_2(p+2)(p+1) + ea_0 - c(-\alpha a_0 + a_1) = 0$$

$$a_2 = \frac{-(c\alpha + e)a_0 + ca_1}{3 \times 2} = \frac{-2(c\alpha + e) + c^2}{12}a_0$$

With $p = 1$ the differential equation becomes:

$$\sum_n a_n(n+1)nx^{n-1} + e\sum_n a_n x^{n+1} - c\sum_{m=0}^{\infty}\sum_{n=0}^{\infty} a_n x^{n+m}\frac{(-\alpha)^m}{m!} = 0$$

for x^{s-1} we get

$$a_s s(s+1) + ea_{s-2} - c\left(a_0\frac{(-\alpha)^{s-1}}{(s-1)!} + a_1\frac{(-\alpha)^{s-2}}{(s-2)!} + \cdots\right) = 0$$

$$a_s = \frac{1}{s(s+1)}\left(-ea_{s-2} + \sum_{m=0}^{s-1}\frac{(-\alpha)^m}{m!}a_{s-m-1}\right)$$

Thus

$$\psi(x) = a_0\left(x - \frac{cx^2}{2} + \frac{c^2 - 2(c\alpha + e)}{12}x^3 + \cdots\right)$$

26. The Kompaneets equation describes the evolution of the photon spectrum in a scattering atmosphere.

$$\frac{\partial n}{\partial t} = n_e \sigma_T c \frac{kT}{mc^2}\frac{1}{x^2}\frac{\partial}{\partial x}\left(x^4\left(n' + n + n^2\right)\right)$$

Here n is the photon number density, x is the dimensionless frequency, and σ_T is the Thomson scattering cross section. We may find a steady state solution ($\partial/\partial t = 0$) when photons are produced by a source $q(x)$ and subsequently escape from the cloud. When n remains $\ll 1$, the Kompaneets equation becomes a linear equation:

$$\frac{1}{x^2}\frac{\partial}{\partial x}\left(x^4\left(n' + n\right)\right) + q(x) - \frac{4n}{y} = 0$$

where y is the Compton "y" parameter, equal to (fractional energy change per scattering)$\times$(mean # of scatterings). Assume that $q(x) \simeq 0$ except for $x \ll 1$.

(a) Show that for $x \gg 1$ the solution is an exponential. This is the Wien law.

For $x \gg 1$, we may ignore both q and $4n/y$. Then the equation simplifies to

$$n' + n = 0$$
$$n' = -n$$
$$n = e^{-nx}$$

(b) Show that in the special case $y = 1$ the solution is a power law in x .

If $y = 1$, negelecting q the equation becomes

$$\frac{1}{x^2}\frac{\partial}{\partial x}\left(x^4\left(n' + n\right)\right) - 4n = 0$$

Look for a power law solution, $n = x^p$. Then:

$$\frac{1}{x^2}\frac{\partial}{\partial x}\left(x^4\left(px^{p-1} + x^p\right)\right) - 4x^p = 0$$
$$\frac{1}{x^2}\frac{\partial}{\partial x}\left(px^{p+3} + x^{p+4}\right) - 4x^p = 0$$
$$\frac{1}{x^2}\left(p(p+3)x^{p+2} + (p+4)x^{p+3}\right) - 4x^p = 0$$
$$p(p+3)x^p + (p+4)x^{p+1} - 4x^p = 0$$

Negelecting the larger power of x in the limit $x \ll 1$, we have a solution with

$$p^2 + 3p - 4 = 0$$
$$p = -4, 1$$

The positive power does not make sense physically, so the solution is x^{-4}. Notice that with this solution the term we neglected is zero anyway, so this is an *exact* solution, not valid only for small x.

(c) Verify your answers to (a) and (b) by letting $n = e^{-x}v$ and finding a power series solution for v.

$$\frac{1}{x^2}\frac{\partial}{\partial x}\left(x^4(n' + n)\right) - \frac{4}{y}n = 0$$

$$\frac{1}{x^2}\frac{\partial}{\partial x}\left(x^4(-e^{-x}v + e^{-x}v' + e^{-x}v)\right) - \frac{4}{y}e^{-x}v = 0$$

$$\frac{1}{x^2}\frac{\partial}{\partial x}\left(x^4 e^{-x}v'\right) - \frac{4}{y}e^{-x}v = 0$$

$$\frac{1}{x^2}\left(4x^3e^{-x}v' - x^4e^{-x}v' + x^4e^{-x}v''\right) - \frac{4}{y}e^{-x}v = 0$$

Now divide out the factor e^{-x} to obtain:

$$x^2v'' + \left(4x - x^2\right)v' - \frac{4}{y}v = 0$$

There is a singular point at $x = 0$, so use a Frobenius series:

$$0 = \sum_n a_n(n+p-1)(n+p)x^{n+p-2} + 4\sum_n a_n(n+p)x^{n+p-2} - \sum_n a_n(n+p)x^{n+p-1} - \frac{4}{y}\sum_n a_n x^{n+p-2}$$

The lowest power is x^{n+p-2} and its coefficient is:

$$a_0p(p-1) + 4a_0p - \frac{4}{y}a_0 = 0$$

$$p^2 + 3p - \frac{4}{y} = 0 \Rightarrow p = \frac{-3 \pm \sqrt{9 + 16/y}}{2}$$

Note: as $y \to \infty$, $p \to 0$ and we get a regular power series.

The recursion relation is found by looking at the coefficient of x^{m+p-2}:

$$a_m(m+p-1)(m+p) + 4a_m(m+p) - a_{m-1}(m-1+p) - \frac{4}{y}a_m = 0$$

$$a_m = \frac{(m+p-1)}{(m+p)(m+p+3) - \frac{4}{y}}a_{m-1} = \frac{(m+p-1)}{\left(m^2 + 2mp + 3m + p^2 + 3p\right) - \frac{4}{y}}a_{m-1}$$

and using the indicial equation, this simplifies to:

$$a_m = \frac{(m+p-1)}{\left(m^2 + 2mp + 3m\right)}a_{m-1} = \frac{\left(2m - 5 \pm \sqrt{(9 + 16/y)}\right)}{2m\left(m \pm \sqrt{\frac{9y+16}{y}}\right)}a_{m-1}$$

Thus the solution is

$$n = e^{-x}a_0\left(1 - \frac{x\left(-3 \pm \sqrt{(9 + 16/y)}\right)}{2\left(1 \pm \sqrt{(9 + 16/y)}\right)} + \frac{x^2\left(-1 \pm \sqrt{(9 + 16/y)}\right)\left(-3 \pm \sqrt{(9 + 16/y)}\right)}{8\left(2 \pm \sqrt{(9 + 16/y)}\right)\left(1 \pm \sqrt{(9 + 16/y)}\right)} + \cdots\right)$$

When $y = 1$, the square root equals 5 and we get:

$$a_m = \frac{2m - 5 \pm 5}{2m(m \pm 5)}a_{m-1} = \frac{1}{(m+5)}a_{m-1} \text{ or } \frac{1}{m}a_{m-1}$$

corresponding to $p = 1$ and -4 respectively. Thus

$$a_m = \frac{a_0}{m!} \text{ or } a_m = \frac{5!}{(m+5)!}a_0$$

$$n = e^{-x}a_0x^{-4}e^x = a_0x^{-4}$$

as we found in (b), or

$$n = e^{-x}a_0x5!\left(\frac{1}{5!} + \frac{1}{6!}x + \frac{1}{7!}x^2 + \cdots\right) = e^{-x}a_0\frac{5!}{x^4}\left(e^x - 1 - \frac{x^2}{2!} - \frac{x^3}{3!} - \frac{x^4}{4!}\right)$$
$$= a_0\frac{5!}{x^4}\left(1 - e^{-x}\left(1 + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!}\right)\right)$$

27. A particle falls a distance d under gravity. Air resistance is proportional to the square of the particle's speed. Write the differential equation that describes the particle's position as a function of time.

We start with Newton's 2nd law:

$$F_x = ma_x$$
$$mg - k\left(\frac{dx}{dt}\right)^2 = m\frac{d^2x}{dt^2}$$

Choose dimensionless variables, and show that the equation may be put into the form:

$$y'' + \alpha y'^2 - \beta = 0$$

Divide by mg:

$$\frac{1}{g}\frac{d^2x}{dt^2} + \frac{k}{mg}\left(\frac{dx}{dt}\right)^2 - 1 = 0$$

Each term in this equation is dimensionless. Express the distance travelled as a fraction of the total distance d: $y = x/d$ and define a dimensionless time $\tau = t\sqrt{g/d}.$ Then the equation becomes:

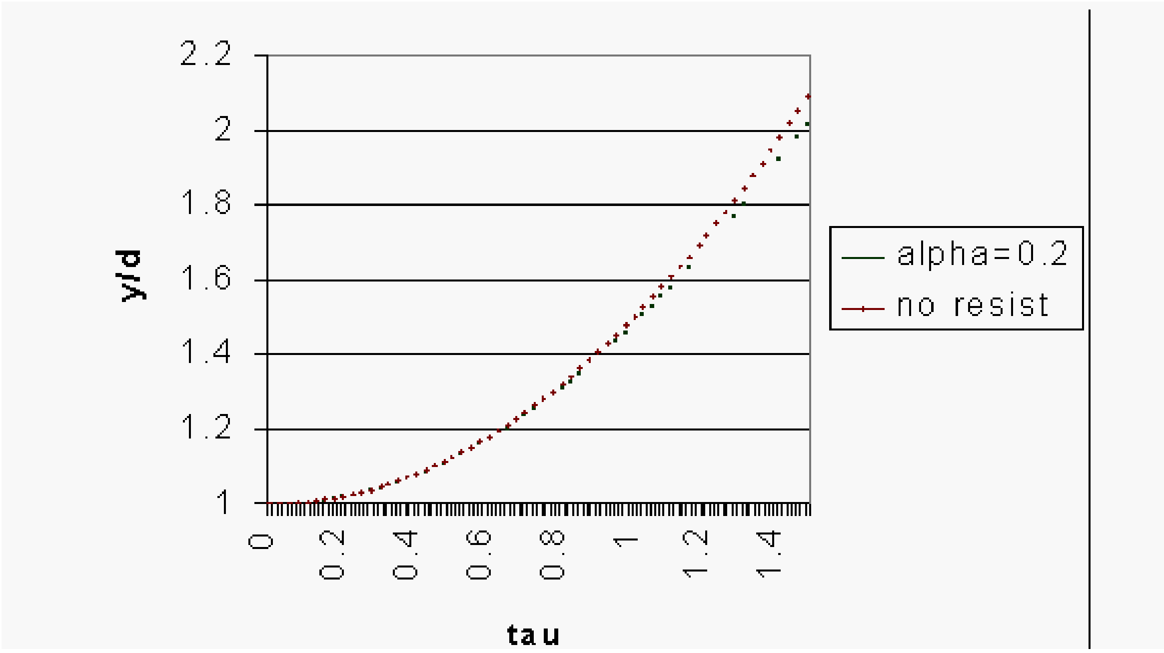
$$\frac{d}{g}\frac{d^2y}{dt^2} + \frac{kd^2}{mg}\left(\frac{dy}{dt}\right)^2 - 1 = 0$$
$$\frac{d^2y}{d\tau^2} + \frac{kd}{m}\left(\frac{dy}{d\tau}\right)^2 - 1 = 0$$

which is of the desired form with $\alpha = kd/m$ and $\beta = 1.$ Using a spreadsheet to compute the coordinates, starting at $y = 1$ we reach $y = 2$ at $\tau = 1.45,$ corresponding to $t = 1.46$ s. With no air resistance we would have:

$$t = \sqrt{2d/g} = \sqrt{\frac{2 \times 10}{9.8}}\ \text{s} = 1.43\ \text{s}$$

Air resistance increases the time by 0.02 s or $1.4\%.$

With $\alpha = .2,$ $\tau = 1.465,$ an increase of $2.4\%.$



28. In astrophysics, the Lane-Emden equation decribes the structure of a star with equation of state $P = K\rho^{(n+1)/n}$. Defining $\rho = \lambda\phi^n$, the equation of hydrostatic equilibrium becomes:

$$\frac{1}{x^2}\frac{d}{dx}\left(x^2\frac{d\phi}{dx}\right) + \phi^n = 0$$

where x is a dimensionless distance variable. This is the Lane-Emden equation.

(a) Find a series solution for ϕ in the case $n = 1$.

For $n = 1$ the equation is linear, with a singular point at $x = 0$ so we may solve with a Frobenius series:

$$\phi'' + \frac{2}{x}\phi' + \phi = 0$$

$$\sum_n a_n(n+p-1)(n+p)x^{n+p-2} + 2\sum_n a_n(n+p)x^{n+p-2} + \sum_n a_n x^{n+p} = 0$$

The indicial equation is found from the coefficient of x^{p-2}:

$$a_0[(p-1)p + 2p] = 0$$

So for $a_0 \neq 0$,

$$p(p+1) = 0$$
$$p = 0, \text{ or } p = -1$$

The coefficient of x^{p-1} is:

$$a_1(p+2)(p+1) = 0$$

So if $a_1 \neq 0$, $p = -2$ or -1. Thus these two solutions duplicate the first two.

With $p = 0$,

$$2a_2 + 2\times 2a_2 + a_0 = 0 \Rightarrow a_2 = -\frac{a_0}{2\times 3}$$

and for $m > 2$:

$$
\begin{aligned}
a_m m(m-1) + 2a_m m + a_{m-2} &= 0 \\
a_m &= -\frac{a_{m-2}}{m(m+1)} = \frac{a_{m-4}}{(m+1)(m)(m-1)(m-2)} \\
a_{2n} &= \frac{(-1)^n}{(2n+1)!} a_0
\end{aligned}
$$

Thus the solution is

$$\phi = a_0 \left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} + \cdots \frac{(-1)^n}{(2n+1)!} x^{2n} \right)$$

This function is the spherical Bessel function $j_0(x)$ (Chapter 8), and this solution is regular at the origin.

The second solution is found from $p = -1$.

$$
\begin{aligned}
a_{m+2}(m+2)(m+1) + a_m &= 0 \\
a_{m+2} &= -\frac{a_m}{(m+2)(m+1)}
\end{aligned}
$$

$$a_{2k} = \frac{(-1)^k}{(2k)!} a_0$$

and so the solution is

$$\frac{a_0}{x} \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} x^{2k} = a_0 \frac{\cos x}{x}$$

This solution is not well behaved at the origin, and so can be ruled out on physical grounds. Note this is one of the rare cases in which we get two independent solutions even though the roots of the indicial equation differ by an integer.

(b) Find the first 3 non-zero terms in a series solution for ϕ for arbitrary n. Verify that your result agrees with the result of part (a) when $n = 1$ (*Hint:* begin by arguing that the solution contains only even powers of x.)

If we replace x with $-x$, the equation is unchanged. Thus the solution must be even in x, (or purely odd, but that would give zero density at the center of the star, which is clearly unphysical), and thus the first 3 non-zero terms are:

$$\phi = a_0 + a_2 x^2 + a_4 x^4$$

and we may choose $a_0 = 1$. Then

$$
\begin{aligned}
\phi^n &= \left(1 + a_2 x^2 + a_4 x^4\right)^n \\
&= 1 + n a_2 x^2 + n a_4 x^4 + \frac{n(n-1)}{2} \left(a_2 x^2\right)^2 + \mathcal{O}\left(x^6\right) \\
&= 1 + n a_2 x^2 + n a_4 x^4 + \frac{n(n-1)}{2} a_2^2 x^4 + \mathcal{O}\left(x^6\right)
\end{aligned}
$$

Now we stuff into the DE

$$\frac{1}{x^2}\frac{d}{dx}\left[x^2\left(2a_2x+4a_4x^3\right)\right]+1+na_2x^2+na_4x^4+\frac{n(n-1)}{2}a_2^2x^4=0$$

$$\frac{1}{x^2}\left(6x^2a_2+20x^4a_4\right)+1+na_2x^2+\left(na_4+\frac{n(n-1)}{2}a_2^2\right)x^4=0$$

$$6a_2+20x^2a_4+1+na_2x^2+\cdots=0$$

There are additional terms in x^4 from the derivatives of the next term in the series, so we cannot use the terms beyond x^2. Thus, equating terms of equal power,

$$6a_2+1=0\Rightarrow a_2=-\frac{1}{6}$$

and

$$20a_4+na_2=0\Rightarrow a_4=-\frac{na_2}{20}=\frac{n}{120}$$

Thus

$$\phi=1-\frac{x^2}{6}+\frac{nx^4}{120}+\cdots$$

The series converges quite fast for $x<1,$, and agrees with the previous result for $n=1.$

(c) Solve the equation numerically for $n=2,$ $\phi(0)=1$ and $\phi'(0)=0.$ At what value of x does $\phi(x)$ first equal zero? (This corresponds to the surface of the star.)

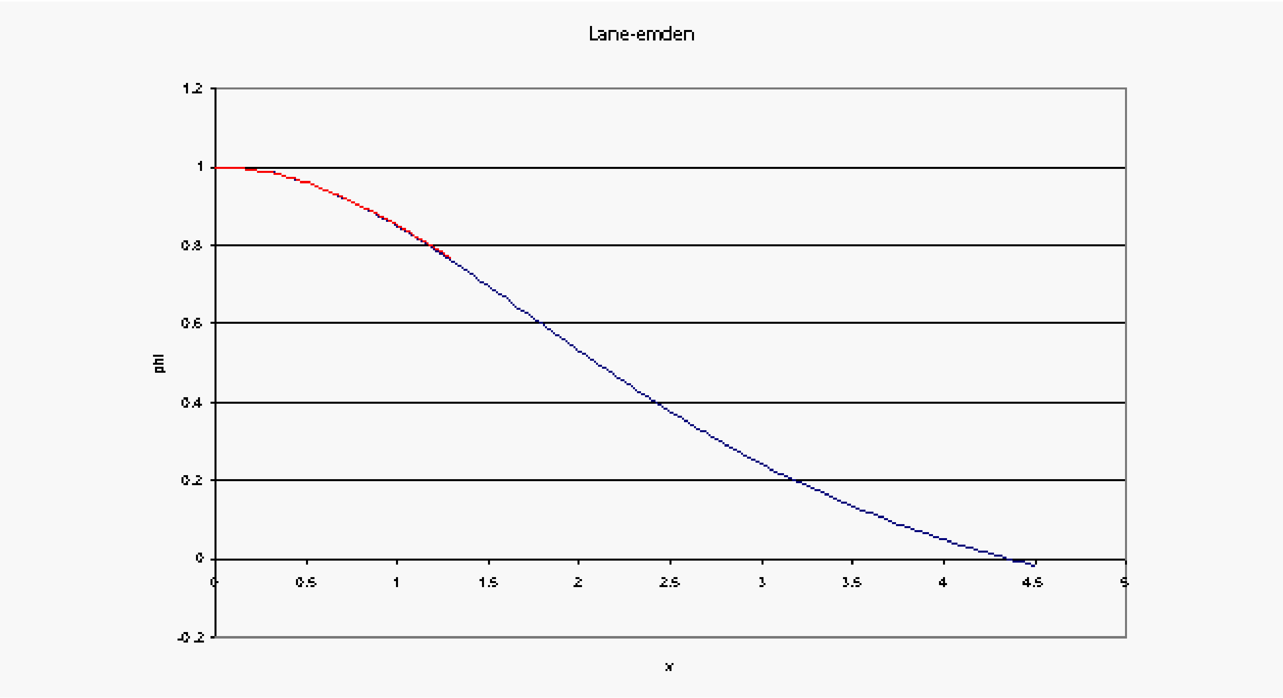
We'll use the Runge-Kutta method. The result is $\phi(x)=0$ at $x=4.35.$ Here are the spreadsheet formulae:

			u''+2u'/x+u^n=0		
n=2	v'+2v/x+u^2=0		v=u'	v'=g=-2v/x-u^2	
h	0.05				
x	u	v=u'	v'=u''	eta1	m1
0	1	0	=-B5^2	=$B$3*C5	=-$B$3*B5^2
=A5+$B$3	=B5+M5	=C5+N5	=-2*C6/A6-B6^2	=$B$3*C6	=-(2*C6/A6+B6^2)*$B$3
=A6+$B$3	=B6+M6	=C6+N6	=-2*C7/A7-B7^2	=$B$3*C7	=-(2*C7/A7+B7^2)*$B$3

eta2	m2	eta3
=$B$3*(C5+F5/2)	=-(2*(C5+F5/2)/(A5+$B$3/2)+(B5+E5/2)^2)*$B$3	=$B$3*(C5+H5/2)
=$B$3*(C6+F6/2)	=-(2*(C6+F6/2)/(A6+$B$3/2)+(B6+E6/2)^2)*$B$3	=$B$3*(C6+H6/2)
=$B$3*(C7+F7/2)	=-(2*(C7+F7/2)/(A7+$B$3/2)+(B7+E7/2)^2)*$B$3	=$B$3*(C7+H7/2)

m3	eta4
=-(2*(C5+H5/2)/(A5+$B$3/2)+(B5+G5/2)^2)*$B$3	=$B$3*(C5+J5)
=-(2*(C6+H6/2)/(A6+$B$3/2)+(B6+G6/2)^2)*$B$3	=$B$3*(C6+J6)
=-(2*(C7+H7/2)/(A7+$B$3/2)+(B7+G7/2)^2)*$B$3	=$B$3*(C7+J7)

m4	eta	m
=-(2*(C5+J5)/(A5+$B$3)+(B5+I5)^2)*$B$3	=(E5+2*G5+2*I5+K5)/6	=(F5+2*H5+2*J5+L5)/6
=-(2*(C6+J6)/(A6+$B$3)+(B6+I6)^2)*$B$3	=(E6+2*G6+2*I6+K6)/6	=(F6+2*H6+2*J6+L6)/6
=-(2*(C7+J7)/(A7+$B$3)+(B7+I7)^2)*$B$3	=(E7+2*G7+2*I7+K7)/6	=(F7+2*H7+2*J7+L7)/6



Solution for $n = 2.$ The numerical solution agrees well with the series (red line) from part (b) up to $x = 1.3.$ The first three terms of the series are not sufficient for larger values of $x.$

29. Investigate the effect of air resistance on the range of a projectile launched with speed v_0. Assume that air resistance is proportional to velocity: $\vec{\mathbf{F}}_{\text{res}} = -\alpha\vec{\mathbf{v}}$. Write the equations for the $x-$ and $y-$coordinates in dimensionless form. Scale the coordinates with the maximum range $R = v_0^2/g.$ What is the dimensionless air-resistance parameter? Determine the dimensionless range for values of the air resistance parameter equal to 0, 0.1, 0.2 , 0.4 and 0.5. Determine how the maximum range changes, and also determine how the launch angle for maximum range changes as air resistance increases. *Hint:* if there is no air resistance, you can obtain exact expressions for the increments in position and velocity in a time interval $\Delta t.$ Use the same expressions when $\alpha \neq 0,$ but with acceleration computed from the value of $\vec{\mathbf{v}}$ at the beginning of your time interval.

$$F_y = m\frac{d^2y}{dt^2} = -g - \alpha\frac{dy}{dt}$$

and

$$F_x = m\frac{d^2x}{dt^2} = -\alpha\frac{dx}{dt}$$

Dividing by $mg,$ we get:

$$\frac{1}{g}\frac{d^2y}{dt^2} = -1 - \frac{\alpha}{mg}\frac{dy}{dt}$$

and

$$\frac{1}{g}\frac{d^2x}{dt^2} = -\frac{\alpha}{mg}\frac{dx}{dt}$$

We can make this dimensionless by expressing x in terms of the maximum range without air resistance:

$R = v_0^2/g$. Then: $u = x/R = ug/v_0^2$

$$\frac{R}{g}\frac{d^2u}{dt^2} = -\frac{\alpha R}{mg}\frac{du}{dt}$$

and with $w = y/R$

$$\frac{v_0^2}{g^2}\frac{d^2w}{dt^2} = -1 - \frac{\alpha v_0^2}{mg^2}\frac{dw}{dt}$$

Then let $\tau = tg/v_0$. The equations become:

$$\frac{d^2w}{d\tau^2} = -1 - \frac{\alpha v_0}{mg}\frac{dw}{d\tau}$$

and:

$$\frac{d^2u}{d\tau^2} = -\frac{\alpha v_0}{mg}\frac{du}{d\tau}$$

Now let $\beta = \alpha v_0/mg$. Then integrating once, we get:

$$\frac{du}{d\tau} = -\beta u + U_0$$

where U_0 is the dimensionless initial velocity in the x $-$direction

$$u_0 = \frac{v_0 \cos\theta}{R}\frac{v_0}{g} = \cos\theta$$

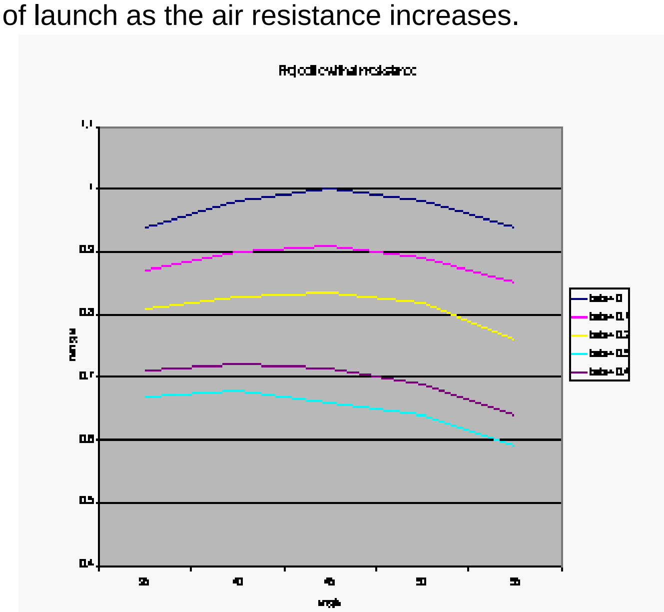
Now we set up a numerical scheme to integrate this set of equations. Notice that if $\alpha = 0$, then

$$w = w_0 + \left.\frac{dw}{d\tau}\right|_0 \tau + \frac{1}{2}\tau^2$$

solves the equation exactly. So let's try basing our numerical scheme on this. In each time step compute the increment in w by

$$\Delta w = \left.\frac{dw}{d\tau}\right|_0 \tau + \frac{1}{2}\left.\frac{d^2w}{d\tau^2}\right|_0 \tau^2$$

and similarly for x. Setting up a spreadsheet, we find that the range decreases and the maximum range moves to a smaller angle of launch as the air resistance increases.



30. The equation that describes the motion of a pendulum is

$$y'' = -\frac{g}{\ell}\sin y$$

(a) When y remains small, the equation may be reduced to the harmonic oscillator equation. Solve this equation to obtain the solution $y(t)$.

(b) With the initial conditions $y(0) = \pi/3$, $y'(0) = 0$, solve the non-linear equation numerically to obtain the period. By how much does the period differ from your result in (a)?

Pendulum equation

$$y'' = -\frac{g}{\ell}\sin y$$

For dimensionless variables, let time scale be $\sqrt{\ell/g}$. Then

When y remains small, then

$$y'' = -\frac{g}{\ell}y$$

with solution

$$y = y_0 \cos\sqrt{\frac{g}{\ell}}t = y_0 \cos\tau$$

where τ is the dimensionless time variable defined below. When y is not always small, we solve numerically.

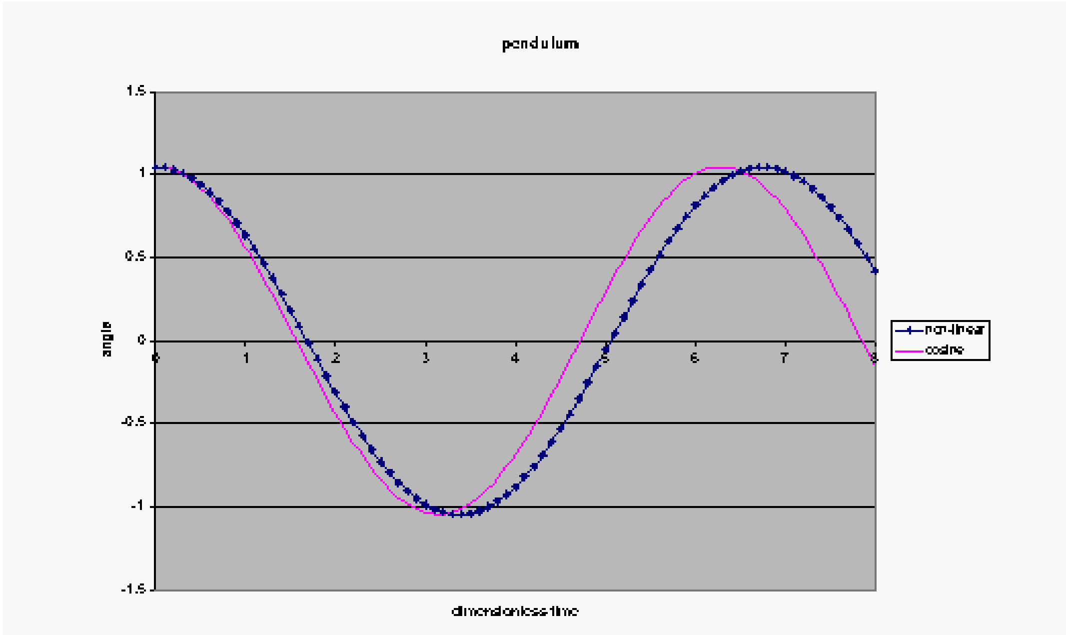
$$u = y'$$
$$u' = -\sin y$$

$y(0) = \frac{\pi}{3}$; $y'(0) = 0$. Thus $u'(0) = -\sin\frac{\pi}{2} = -\frac{\sqrt{3}}{2}$ and $u(0) = 0$.

See spreadsheet for solution.

The period is 6.75 seconds. For the exact cosine solution, the period is 2π. The % difference is

$$\frac{6.75 - 2\pi}{2\pi} = 7.4\%$$



31. Bessel's equation of order ν has the form:

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + \left(1 - \frac{\nu^2}{x^2}\right)y = 0$$

Show that the differential equation

$$\frac{d^2f}{dz^2} + z^r f = 0$$

may be converted to Bessel's equation through the relations

$$f = \sqrt{z}\,y$$

and

$$x = \frac{2}{r+2}z^{1+r/2}$$

What is the order of the resulting Bessel's equation? (The solutions are given in Chapter 8.)

First change f to y.

$$\frac{df}{dz} = \frac{1}{2}\frac{y}{\sqrt{z}} + \sqrt{z}\frac{dy}{dz}$$

and

$$\frac{d^2f}{dz^2} = -\frac{1}{4}\frac{y}{z^{3/2}} + \frac{1}{\sqrt{z}}\frac{dy}{dz} + \sqrt{z}\frac{d^2y}{dz^2}$$

Next change z to x. For ease of computation, let $s = 1 + r/2$. Then $x = z^s/s$, and $dx = z^{s-1}dz$

$$\frac{dy}{dz} = \frac{dy}{dx}\frac{dx}{dz} = \frac{dy}{dx}z^{s-1} = \frac{dy}{dx}(sx)^{(s-1)/s}$$

and

$$\frac{d^2y}{dz^2} = \frac{d}{dz}\left(\frac{dy}{dx}z^{s-1}\right) = (s-1)z^{s-2}\frac{dy}{dx} + z^{s-1}\frac{d^2y}{dx^2}\frac{dx}{dz}$$
$$= \frac{d^2y}{dx^2}(sx)^{2(s-1)/s} + (s-1)(sx)^{(s-2)/s}\frac{dy}{dx}$$

Thus the differential equation takes the form:

$$\frac{d^2 f}{dz^2} + z^r f = 0$$

$$-\frac{1}{4}\frac{y}{z^{3/2}} + \frac{1}{\sqrt{z}}\frac{dy}{dz} + \sqrt{z}\frac{d^2 y}{dz^2} + \sqrt{z} z^r y = 0$$

Divide by $\sqrt{z}$ and use the expressions for the derivatives:

$$\frac{d^2 y}{dx^2}(sx)^{2(s-1)/s} + (s-1)(sx)^{(s-2)/s}\frac{dy}{dx} + \frac{1}{(xs)^{1/s}}\frac{dy}{dx}(sx)^{(s-1)/s} - \frac{1}{4}\frac{y}{(xs)^{2/s}} + (xs)^{2(s-1)/s} y = 0$$

Dividing by $(sx)^{2(s-1)/s} = (sx)^{2-2/s}$ and rearranging:

$$\frac{d^2 y}{dx^2} + \frac{1}{x}\frac{dy}{dx} - \frac{y}{4(xs)^2} + y = 0$$

This is Bessel's equation with $\nu = 1/2s = 1/(2+r)$.

The equation

$$\frac{d^2 f}{dz^2} - z^r f = 0$$

becomes the modified Bessel equation 3.38.

32. Show that the equation

$$u'' + u' + \frac{k}{x^2}u = 0$$

has a solution of the form

$$u = \sqrt{x}\, e^{-x/2} K_\nu\left(\frac{x}{2}\right)$$

and find the order ν of the modified Bessel function.

First let $z = x/2$ so $\frac{d}{dz} = 2\frac{d}{dx}$

$$\frac{1}{4}\frac{d^2 u}{dz^2} + \frac{1}{2}\frac{du}{dz} + \frac{k}{4z^2}u = 0$$

$$u'' + 2u' + \frac{k}{z^2}u = 0$$

Now let $u = \sqrt{z}\, e^{-z} y$.

$$u' = \frac{1}{2\sqrt{z}}e^{-z}y - \sqrt{z}e^{-z}y + \sqrt{z}e^{-z}y'$$

$$u'' = -\frac{1}{4z^{3/2}}e^{-z}y - \frac{1}{\sqrt{z}}e^{-z}y + \frac{1}{\sqrt{z}}e^{-z}y' + \sqrt{z}e^{-z}y - 2\sqrt{z}e^{-z}y' + \sqrt{z}e^{-z}y''$$

Thus

$$0 = -\frac{1}{4z^{3/2}}e^{-z}y - \frac{1}{\sqrt{z}}e^{-z}y + \frac{1}{\sqrt{z}}e^{-z}y' + \sqrt{z}e^{-z}y - 2\sqrt{z}e^{-z}y' + \sqrt{z}e^{-z}y''$$

$$+ \frac{1}{\sqrt{z}}e^{-z}y - 2\sqrt{z}e^{-z}y + 2\sqrt{z}e^{-z}y' + \frac{k}{z^2}\sqrt{z}e^{-z}y$$

$$y'' + \frac{1}{z}y' - y\left(1 + \frac{1/4 - k}{z^2}\right) = 0$$

This is the modified Bessel equation (3.38) with $\nu^2 = \frac{1}{4} - k$ or $k = 1/4 - \nu^2$.

Chapter 4: Fourier Series

1. Show that the Fourier series (equation 4.1) for a function $f(x)$ may be written:

$$f(x) = \sum_{n=0}^{\infty} k_n \cos(nx + \phi_n)$$

and find expressions for k_n and ϕ_n.

Expand the cosine to obtain:

$$f(x) = \sum_{n=0}^{\infty} k_n (\cos nx \cos \phi_n - \sin nx \sin \phi_n)$$

Comparing with equation 4.1, we have:

$$a_n = -k_n \sin \phi_n; \quad b_n = k_n \cos \phi_n$$

Thus

$$k_n = \sqrt{a_n^2 + b_n^2}$$

and

$$\tan \phi_n = -\frac{a_n}{b_n}$$

where a_n and b_n are given by equations 4.7 and 4.8.

We may also work from the exponential series:

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx} = \sum_{n=-\infty}^{\infty} r_n e^{i\theta_n} e^{inx}$$
$$= \sum_{n=-\infty}^{\infty} r_n [\cos(nx + \theta_n) + i \sin(nx + \theta_n)]$$

In this formulation c_n may be complex, but r_n is real. Thus if $f(x)$ is real, the imaginary terms must combine to give zero, leaving:

$$f(x) = \sum_{n=-\infty}^{\infty} r_n \cos(nx + \theta_n)$$

where

$$r_n = |c_n| = \left| \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-inx} dx \right|$$

and

$$\phi_n = \theta_n = \arg(c_n) = \arg\left\{ \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-inx} dx \right\}$$

Note that

$$\phi_{-n} = -\phi_n$$

and

$$r_n = r_{-n}$$

and so the sine terms sum to zero in pairs, as required, and the cosines combine to give a sum over positive n only.

$$
\begin{aligned}
f(x) &= \sum_{n=-\infty}^{\infty} r_n \cos(nx+\phi_n) = \sum_{n=-1}^{-\infty} r_n \cos(nx+\phi_n) + r_0\cos\phi_0 + \sum_{n=1}^{\infty} r_n\cos(nx+\phi_n) \\
&= r_0\cos\phi_0 + \sum_{n=1}^{\infty} r_{-n}\cos(-nx+\phi_{-n}) + \sum_{n=1}^{\infty} r_n\cos(nx+\phi_n) \\
&= r_0\cos\phi_0 + 2\sum_{n=1}^{\infty} r_n\cos(nx+\phi_n)
\end{aligned}
$$

which is of the required form, with $k_0 = r_0$ and $k_n = 2r_n$ for $n > 0$.

2. Develop the Fourier series for the function $f(x) = x$

(a) over the range $0 \le x \le 1$

(b) over the range $-1 \le x \le 1$

(c) Make a plot showing the original function and the sum of the first 3 non-zero terms in each series.

Comment on the similarities and differences between the two series.

We choose a variable $z = 2\pi x$ that varies from 0 to 2π as x varies from 0 to 1. Then we write the series as

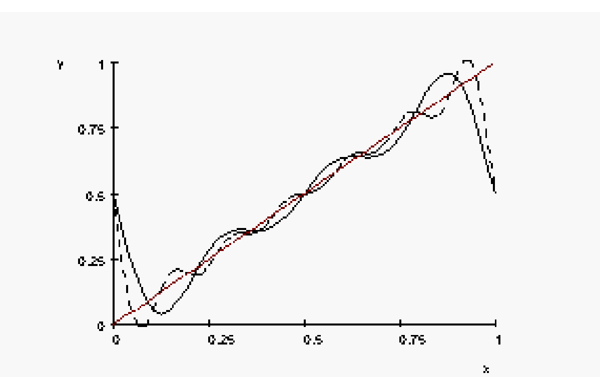
$$f(x) = x = \sum a_n \sin 2\pi nx + b_n \cos 2\pi nx$$

The coeffcients are:

$$
\begin{aligned}
a_n &= 2\int_0^1 x\sin 2\pi nx\,dx \\
&= 2\left[x\left(-\frac{\cos 2\pi nx}{2\pi n}\right)\bigg|_0^1 - \int_0^1 -\frac{\cos 2\pi nx}{2\pi n}dx\right] \\
&= \frac{1}{\pi n}\left[-1 + \frac{\sin 2\pi nx}{2\pi n}\bigg|_0^1\right] = -\frac{1}{\pi n}
\end{aligned}
$$

and

$$
\begin{aligned}
b_n &= 2\int_0^1 x\cos 2\pi nx\,dx \\
&= 2\left[x\left(\frac{\sin 2\pi nx}{2\pi n}\right)\bigg|_0^1 - \int_0^1 \frac{\sin 2\pi nx}{2\pi n}dx\right] \\
&= \frac{1}{\pi n}\left[0 - \frac{-\cos 2\pi nx}{2\pi n}\bigg|_0^1\right] = 0
\end{aligned}
$$



Solid- 3 terms. Dashed- 6 terms

We need to calculate b_0 separately:

$$b_0 = \int_0^1 x dx = \frac{1}{2}$$

So the series is:

$$x = \frac{1}{2} - \sum_{n=1}^{\infty} \frac{\sin 2\pi n x}{\pi n}$$

(b) Over the second range we choose the variable $u = \pi x$ which ranges from $-\pi$ to $+\pi$. Then we have:

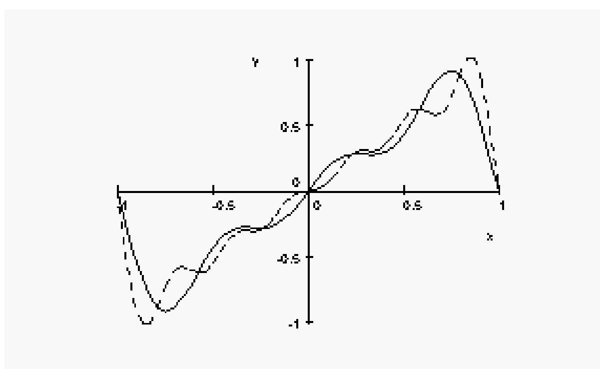
$$f(x) = x = \sum a_n \sin \pi n x + b_n \cos \pi n x$$

And the coefficents are:

$$\begin{aligned} a_n &= \int_{-1}^{1} x \sin \pi n x dx \\ &= x\left(-\frac{\cos \pi n x}{\pi n}\right)\Big|_{-1}^{1} - \int_{-1}^{1} -\frac{\cos \pi n x}{\pi n} dx \\ &= \frac{1}{\pi n}\left[-2(-1)^n + \frac{\sin \pi n x}{\pi n}\Big|_{1}^{1}\right] = -\frac{2(-1)^n}{\pi n} \end{aligned}$$

and

$$\begin{aligned} b_n &= \int_{-1}^{1} x \cos \pi n x dx \\ &= x\left(\frac{\sin \pi n x}{\pi n}\right)\Big|_{-1}^{1} - \int_{-1}^{1} \frac{\sin \pi n x}{\pi n} dx \\ &= \frac{1}{\pi n}\left[0 - \frac{-\cos \pi n x}{\pi n}\Big|_{-1}^{1}\right] = 0 \end{aligned}$$



which is expected since the function is odd over this interval. Finally:

$$b_0 = \int_{-1}^{1} x dx = 0$$

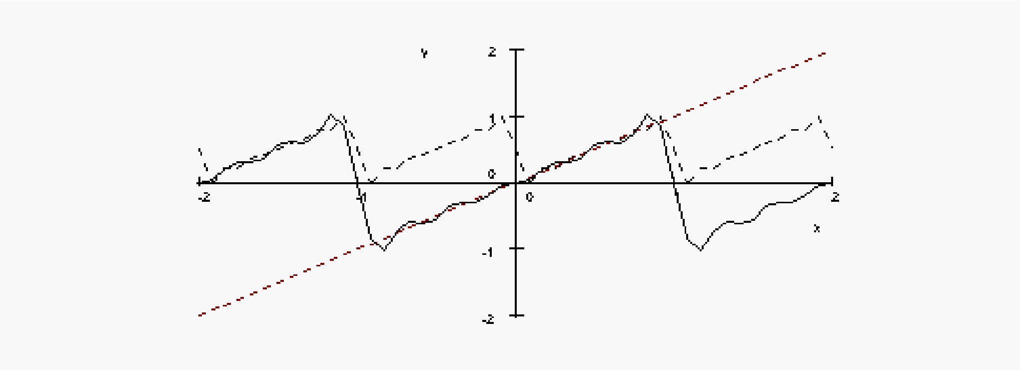
So the series is:

$$x = -\frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin n \pi x$$

(c) The two series are:

$$x = \frac{1}{2} - \sum_{n=1}^{\infty} \frac{\sin 2\pi n x}{\pi n} = -\frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin n \pi x$$

Both series have sine terms have coefficients that decrease as $1/n$. The second series has coefficients that alternate in sign, while in the first series the signs remain constant. The diagrams show that the series appear to converge about equally well to the original function $f(x) = x$. Of course they differ outside the range $(0, 1)$:



Solid- series on (-1.1). Dashed- (0,1)

3. Develop the full Fourier series for the function $f(x) = x^2$ over the range $0 \le x \le 1.$

The series has the form:

$$f(x) = x^2 = \sum a_n \sin 2\pi n x + b_n \cos 2\pi n x$$

where the argument $2\pi x$ of the sines and cosines varies from 0 to 2π as x varies from 0 to 1. Then

$$\begin{aligned}
a_n &= 2\int_0^1 x^2 \sin 2\pi n x dx \\
&= 2\left[x^2\left(-\frac{\cos 2\pi n x}{2\pi n}\right)\Big|_0^1 - \int_0^1 -2x\frac{\cos 2\pi n x}{2\pi n}dx\right] \\
&= \frac{1}{\pi n}\left(-1 + 2\left[x\left(\frac{\sin 2\pi n x}{2\pi n}\right)\Big|_0^1 - \int_0^1 \frac{\sin 2\pi n x}{2\pi n}dx\right]\right) \\
&= \frac{1}{\pi n}\left(-1 + \frac{1}{\pi n}\left[0 - \frac{-\cos 2\pi n x}{2\pi n}\Big|_0^1\right]\right) \\
&= \frac{-1}{\pi n}
\end{aligned}$$

and

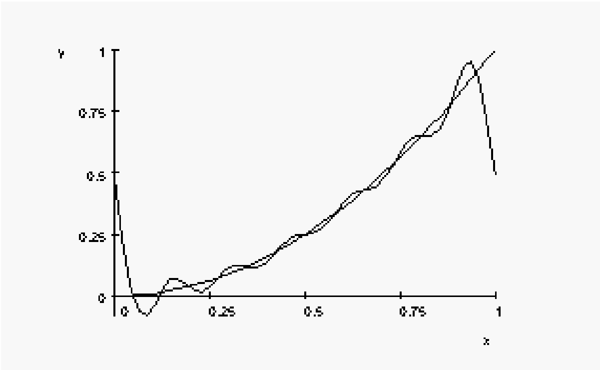
$$\begin{aligned}
b_n &= 2\int_0^1 x^2 \cos 2\pi n x dx \\
&= 2\left[x^2\left(\frac{\sin 2\pi n x}{2\pi n}\right)\Big|_0^1 - \int_0^1 2x\frac{\sin 2\pi n x}{2\pi n}dx\right] \\
&= -\frac{2}{\pi n}\left[0 + x\frac{-\cos 2\pi n x}{2\pi n}\Big|_0^1 - \int_0^1 \frac{-\cos 2\pi n x}{2\pi n}dx\right] \\
&= \frac{1}{(\pi n)^2}\left[1 - \frac{\sin 2\pi n x}{2\pi n}\Big|_0^1\right] = \frac{1}{(\pi n)^2}
\end{aligned}$$

Finally:

$$b_0 = \int_0^1 x^2 dx = \frac{1}{3}$$

Thus the series is:

$$x^2 = \frac{1}{3} + \sum_{n=1}^{\infty}\left(-\frac{1}{\pi n}\sin 2\pi n x + \frac{1}{(\pi n)^2}\cos 2\pi n x\right)$$

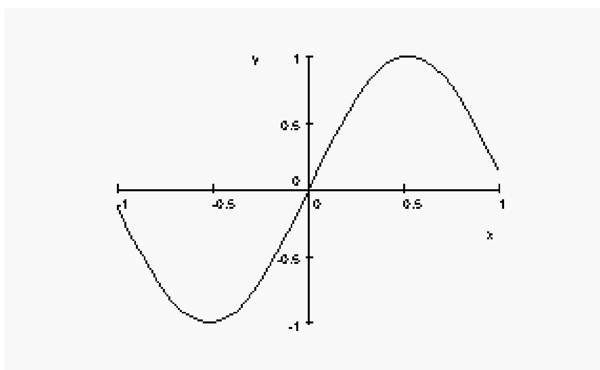


Sum of first 6 terms

4 An odd function $f(x)$ has the additional property that $f(x+L) = -f(x)$.

(a) Make a sketch showing the important features of this function.

Since $f(x+L) = -f(x)$, then $f(x+L/2) = f(x-L/2+L) = -f(x-L/2) = f(L/2-x)$, the function is even about the point $x = L/2$. Thus it looks like this:



(b) Which kind of Fourier series (sine series, cosine series, or full series) represents this function on the range $-L \le x \le L$?

Since the function is odd it is represented by a sine series.

(c) Show that the series has only terms of odd order ($n = 2m+1$), and find a formula for the coefficients as an integral over the range $0 \le x \le L/2$.

$$f(x) = \sum a_n \sin \frac{n\pi x}{L}$$

with

$$\begin{aligned} a_n &= \frac{1}{L}\int_{-L}^{+L} f(x) \sin \frac{n\pi x}{L} dx \\ &= \frac{2}{L}\int_0^L f(x) \sin \frac{n\pi x}{L} dx \end{aligned}$$

since both $f(x)$ and the sine are odd functions of x, their product is even. Now we divide the range again:

$$\begin{aligned}a_n &= \frac{2}{L}\left(\int_0^{L/2} + \int_{L/2}^{L}\right) f(x)\sin\frac{n\pi x}{L}dx \\ &= \frac{2}{L}\left(\int_0^{L/2} f(x)\sin\frac{n\pi x}{L}dx + \int_{-L/2}^{0} f(x+L)\sin\frac{n\pi(x+L)}{L}dx\right) \\ &= \frac{2}{L}\left(\int_0^{L/2} f(x)\sin\frac{n\pi x}{L}dx - (-1)^n\int_{-L/2}^{0} f(x)\sin\frac{n\pi x}{L}dx\right) \\ &= \frac{4}{L}\int_0^{L/2} f(x)\sin\frac{n\pi x}{L}dx \text{ if } n \text{ is odd}\end{aligned}$$

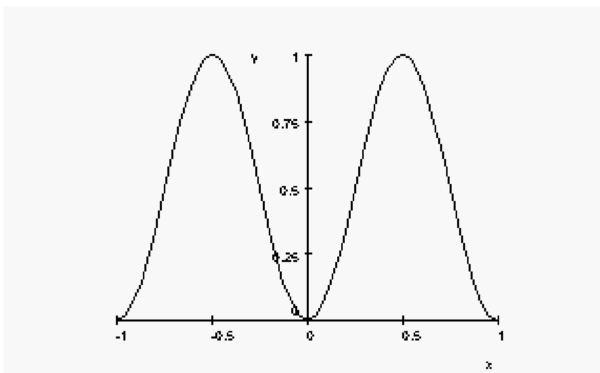
where we used the evenness of $f(x)\sin\frac{n\pi x}{L}$ again in the last step. The result is zero for n even, as expected.

If instead $f(x+L) = +f(x)$, then we will have only n even terms.

Even function, $f(x+L) = f(x)$

(a) Make a sketch showing the important features of this function.

Since the function is even, $f(-x) = f(x)$, so the function has mirror symmetry about the $y-$axis. Since $f(x+L) = f(x)$, the function has period L rather than $2L$. Thus it looks like:



(b) Which kind of Fourier series (sine series, cosine series, or full series) represents this function on the range $-L \le x \le L$?

Since the function is even it is represented by a cosine series.

(c) Show that the series has only terms of even order ($n = 2m$), and find a formula for the coefficients as an integral over the range $0 \le x \le L/2$.

With a range $2L$ we choose the variable $\frac{\pi x}{L}$ which varies from $-\pi$ to π as x varies from $-L$ to $+L$. But since the function has period L, only even n terms should appear in the sum. The arguments of the cosines are multiples of $2\pi x/L$, so that the resulting sum has period L. Then:

$$f(x) = \sum a_n \cos\frac{n\pi x}{L}$$

with

$$\begin{aligned}a_n &= \frac{1}{L}\int_{-L}^{+L} f(x)\cos\frac{n\pi x}{L}dx \\ &= \frac{2}{L}\int_0^{L} f(x)\cos\frac{n\pi x}{L}dx\end{aligned}$$

since both $f(x)$ and the cosine are even functions of x. Now we divide the range again:

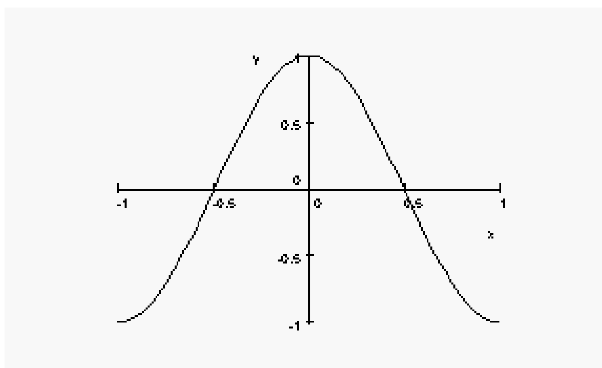
$$\begin{aligned} a_n &= \frac{2}{L}\left(\int_0^{L/2} + \int_{L/2}^{L}\right) f(x)\cos\frac{n\pi x}{L}dx \\ &= \frac{2}{L}\left(\int_0^{L/2} f(x)\cos\frac{n\pi x}{L}dx + \int_{-L/2}^{0} f(x+L)\cos\frac{n\pi(x+L)}{L}dx\right) \\ &= \frac{2}{L}\left(\int_0^{L/2} f(x)\cos\frac{n\pi x}{L}dx + (-1)^n\int_{-L/2}^{0} f(x)\cos\frac{n\pi x}{L}dx\right) \\ &= \frac{4}{L}\int_0^{L/2} f(x)\cos\frac{n\pi x}{L}dx \text{ if } n \text{ is even} \end{aligned}$$

where we used the evenness of f again in the last step. The result is zero for n odd, as expected.

(d) If $f(x+L) = -f(x)$ we must amend the derivation of a_n :

$$\begin{aligned} a_n &= \frac{2}{L}\left(\int_0^{L/2} + \int_{L/2}^{L}\right) f(x)\cos\frac{n\pi x}{L}dx \\ &= \frac{2}{L}\left(\int_0^{L/2} f(x)\cos\frac{n\pi x}{L}dx + \int_{-L/2}^{0} f(x+L)\cos\frac{n\pi(x+L)}{L}dx\right) \\ &= \frac{2}{L}\left(\int_0^{L/2} f(x)\cos\frac{n\pi x}{L}dx + (-1)^n\int_{-L/2}^{0} -f(x)\cos\frac{n\pi x}{L}dx\right) \\ &= \frac{4}{L}\int_0^{L/2} f(x)\cos\frac{n\pi x}{L}dx \text{ if } n \text{ is odd} \end{aligned}$$

All the a_n with n even are zero. The function looks like:



5. Which series, the sine series or the cosine series, do you expect will converge more rapidly to the function $f(x) = x^3$ on the range $0 < x < 1$? Give reasons for your answer. Evaluate the first four terms in the optimum series. How large is the fractional deviation $\left|\frac{S_4 - f(x)}{f(x)}\right|$ at $x = 0.5$ and $x = 1$?

The sine series gives an odd extension of the function $f(x) = x^3$ on the range $(-1, 1)$, and has a discontinuity of 2 at $x = 1$. The cosine series gives an even extension, and its periodic repetition is continuous at $x = 1$. Thus the cosine series will converge faster.

Let

$$x^3 = \sum a_n \cos\pi n x$$

Then

$$a_0 = \int_0^1 x^3 dx = \frac{x^4}{4} = \frac{1}{4}$$

while

$$
\begin{aligned}
a_n &= 2\int_0^1 x^3 \cos \pi n x dx \\
&= 2\left[-x^3 \frac{\sin \pi n x}{\pi n}\Big|_0^1 + \int_0^1 3x^2 \sin \frac{\pi n x}{\pi n} dx\right] \\
&= \frac{6}{\pi n}\left[x^2 \cos \frac{\pi n x}{\pi n}\Big|_0^1 - \int_0^1 2x \frac{\cos \pi n x}{\pi n} dx\right] \\
&= \frac{6}{\pi n}\left[\frac{(-1)^n}{\pi n} - \frac{2}{\pi n}\left\{-x \frac{\sin \pi n x}{\pi n}\Big|_0^1 + \int_0^1 \frac{\sin \pi n x}{\pi n} dx\right\}\right] \\
&= \frac{6}{\pi n}\left[\frac{(-1)^n}{\pi n} - \frac{2}{(\pi n)^2} \frac{\cos \pi n x}{\pi n}\Big|_0^1\right] = \frac{6}{(\pi n)^2}\left(1 - \frac{2[(-1)^n - 1]}{(\pi n)^2}\right)
\end{aligned}
$$

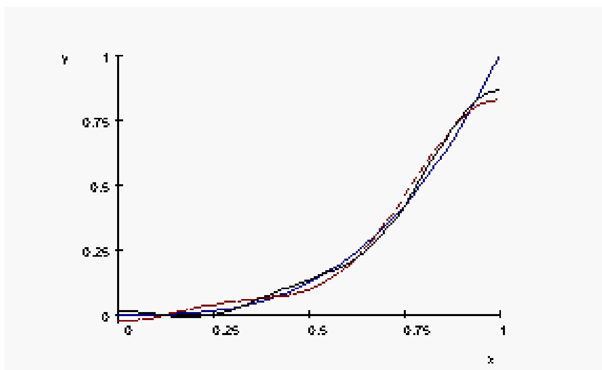
Thus the series is

$$x^3 = \frac{1}{4} + \frac{6}{\pi^2}\sum_{n=1}^{\infty} \frac{\cos \pi n x}{n^2}\left((-1)^n + \frac{2}{(\pi n)^2}(1 - (-1)^n)\right)$$

The first four terms are

$$\frac{1}{4} + \frac{6}{\pi^2}\sum_{n=1}^{3} \frac{\cos \pi n x}{n^2}\left((-1)^n + \frac{2}{(\pi n)^2}(1 - (-1)^n)\right) = \frac{1}{4} + \frac{3}{2\pi^2}\left(\cos 2\pi x + \frac{1}{4}\cos 4\pi x + \frac{1}{9}\cos 9\pi x\right)$$

x^3

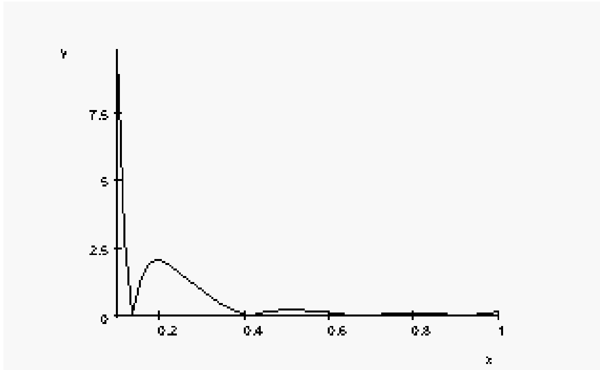


The fit "looks" pretty good between about 0.1 and 0.9. (x^3 – solid blue line. 3 terms, red line dashed, 4 terms black solid.)

Deviation is

$$
\begin{aligned}
D(x) &= \left|\frac{x^3 - \frac{1}{4} - \frac{6}{\pi^2}\sum_{n=1}^{3} \frac{\cos \pi n x}{n^2}\left((-1)^n + \frac{2}{(\pi n)^2}(1 - (-1)^n)\right)}{x^3}\right| \\
&= \left|1 - \frac{1}{4x^3} - \frac{6}{\pi^2 x^3}\sum_{n=1}^{3} \frac{\cos \pi n x}{n^2}\left((-1)^n + \frac{2}{(\pi n)^2}(1 - (-1)^n)\right)\right|
\end{aligned}
$$

and is plotted below



The percent deviation is less than 25% for $.3 < x < 1.$ (approximately).

In particular, at $x = 0.5$

$$\left|1 - \frac{1}{4(0.5)^3} - \frac{6}{\pi^2(0.5)^3}\sum_{n=1}^{3}\frac{\cos \pi n/2}{n^2}\left((-1)^n + \frac{2}{(\pi n)^2}(1-(-1)^n)\right)\right| = 0.21585$$

and at $x = 1$

$$\left|1 - \frac{1}{4} - \frac{6}{\pi^2}\sum_{n=1}^{3}\frac{\cos \pi n}{n^2}\left((-1)^n + \frac{2}{(\pi n)^2}(1-(-1)^n)\right)\right| = 0.17197$$

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6. Find the Fourier series on the range $0 \le x \le 2\pi$ for the function $f(x) = \sin\alpha x,$ where α is *not* an integer. Check your result by checking the limit $\alpha \to n.$ With the value $\alpha = 0.7,$ plot the original function and the first 3 terms of your series on the range $0 \le x \le 2\pi.$ Comment.

The series may be written:

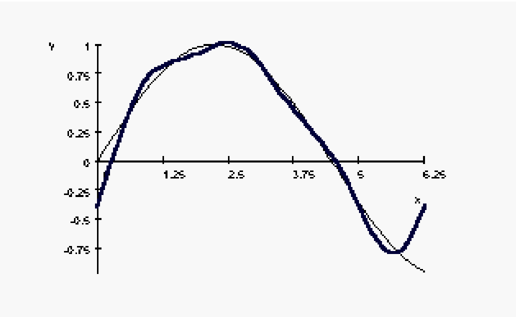
$$\sin\alpha x = \sum a_n \sin nx + b_n \cos nx$$

where

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_0^{2\pi} \sin\alpha x \sin nx dx \\ &= \frac{1}{2\pi}\int_0^{2\pi} (\cos(\alpha - n)x - \cos(\alpha + n)x) dx \\ &= \frac{1}{2\pi}\left(\frac{\sin(\alpha - n)x}{(\alpha - n)} - \frac{\sin(\alpha + n)x}{(\alpha + n)}\right)\Bigg|_0^{2\pi} \\ &= \frac{1}{2\pi}\left(\frac{\sin 2\pi(\alpha - n)}{(\alpha - n)} - \frac{\sin 2\pi(\alpha + n)}{(\alpha + n)}\right) \\ &= \frac{1}{2\pi}\left(\frac{\sin 2\pi\alpha}{(\alpha - n)} - \frac{\sin 2\pi\alpha}{(\alpha + n)}\right) \\ &= \frac{\sin 2\pi\alpha}{2\pi}\left(\frac{2n}{\alpha^2 - n^2}\right) \end{aligned}$$

and

$$\begin{aligned} b_n &= \frac{1}{\pi}\int_0^{2\pi} \sin\alpha x \cos nx dx \\ &= \frac{1}{2\pi}\int_0^{2\pi} (\sin(\alpha + n)x + \sin(\alpha - n)x) dx \\ &= \frac{-1}{2\pi}\left(\frac{\cos(\alpha - n)x}{(\alpha - n)} + \frac{\cos(\alpha + n)x}{(\alpha + n)}\right)\Bigg|_0^{2\pi} \\ &= \frac{-1}{2\pi}\left(\frac{\cos 2\pi(\alpha - n) - 1}{(\alpha - n)} + \frac{\cos 2\pi(\alpha + n) - 1}{(\alpha + n)}\right) \\ &= \frac{-1}{2\pi}\left(\frac{\cos 2\pi\alpha - 1}{(\alpha - n)} + \frac{\cos 2\pi\alpha - 1}{(\alpha + n)}\right) \\ &= \frac{1 - \cos 2\pi\alpha}{2\pi}\,\frac{2\alpha}{\alpha^2 - n^2} \end{aligned}$$



Finally

$$b_0 = \frac{1}{2\pi}\int_0^{2\pi} \sin\alpha x dx = -\frac{1}{2\pi}\frac{\cos\alpha x}{\alpha}\Big|_0^{2\pi} = \frac{1 - \cos 2\pi\alpha}{2\pi\alpha}$$

and so the series is:

$$\sin\alpha x = \frac{1 - \cos 2\pi\alpha}{2\pi\alpha} + \frac{1}{\pi}\sum_{n=1}^{\infty}\left(\frac{n}{\alpha^2 - n^2}\right)\sin 2\pi\alpha \sin nx + \frac{\alpha}{\alpha^2 - n^2}(1 - \cos 2\pi\alpha)\cos nx$$

Limit as $\alpha \to m$:

We let $\alpha = m + \varepsilon$ and let $\varepsilon \to 0.$

$$\frac{1-\cos 2\pi(m+\varepsilon)}{2\pi(m+\varepsilon)} + \frac{1}{\pi}\sum_{n=1}^{\infty}\left(\frac{n}{(m+\varepsilon)^2-n^2}\right)\sin 2\pi(m+\varepsilon)\sin nx + \frac{(m+\varepsilon)}{(m+\varepsilon)^2-n^2}(1-\cos 2\pi(m+\varepsilon))\cos nx$$
$$\to \frac{1-\cos 2\pi\varepsilon}{2\pi(m+\varepsilon)} + \frac{1}{\pi}\sum_{n=1}^{\infty}\left(\frac{n\sin nx}{m^2+2m\varepsilon+\varepsilon^2-n^2}\right)\sin 2\pi\varepsilon + \frac{(m+\varepsilon)\cos nx}{(m+\varepsilon)^2-n^2}(1-\cos 2\pi\varepsilon)$$
$$\to \frac{(2\pi\varepsilon)^2}{4\pi(m+\varepsilon)} + \frac{1}{\pi}\sum_{n=1}^{\infty}\left(\frac{n\sin nx}{m^2+2m\varepsilon+\varepsilon^2-n^2}\right)2\pi\varepsilon + \frac{(m+\varepsilon)\cos nx}{(m+\varepsilon)^2-n^2}\left(\frac{(2\pi\varepsilon)^2}{2}\right)$$

Each term $\to 0$ as $\varepsilon \to 0$, except for the one term with $n = m$. The surviving term is:

$$\frac{1}{\pi}\left(\frac{m\sin mx}{2m\varepsilon}\right)2\pi\varepsilon + \frac{m\cos mx}{2m\varepsilon}\left(\frac{(2\pi\varepsilon)^2}{2}\right) \to \frac{1}{\pi}\left(\frac{m\sin mx}{2m\varepsilon}\right)2\pi\varepsilon = \sin mx$$

as required.

7. Find an exponential Fourier series for the function $\sinh \alpha x$ on the range $0 \le x \le 2\pi$. By combining terms, rewrite your answer as a series in sines and cosines.

We write the series as

$$\sinh \alpha x = \sum_{n=-\infty}^{\infty} c_n e^{inx}$$

where

$$c_n = \frac{1}{2\pi}\int_0^{2\pi}\sinh \alpha x e^{-inx}dx = \frac{1}{4\pi}\int_0^{2\pi}(e^{\alpha x}-e^{-\alpha x})e^{-inx}dx$$
$$= \frac{1}{4\pi}\left(\frac{e^{(\alpha-in)x}}{\alpha-in} - \frac{e^{-(\alpha+in)x}}{-(\alpha+in)}\right)\Bigg|_0^{2\pi}$$
$$= \frac{1}{4\pi}\left(\frac{e^{(\alpha-in)2\pi}-1}{\alpha-in} - \frac{e^{-(\alpha+in)2\pi}-1}{-(\alpha+in)}\right)$$
$$= \frac{1}{4\pi}\left(\frac{e^{2\pi\alpha}-1}{\alpha-in} + \frac{e^{-2\pi\alpha}-1}{(\alpha+in)}\right)$$
$$= \frac{1}{4\pi}\frac{(\alpha+in)(e^{2\pi\alpha}-1)+(\alpha-in)(e^{-2\pi\alpha}-1)}{\alpha^2+n^2}$$
$$= \frac{1}{2\pi}\frac{\alpha\cosh 2\pi\alpha - \alpha + in\sinh 2\pi\alpha}{\alpha^2+n^2}$$

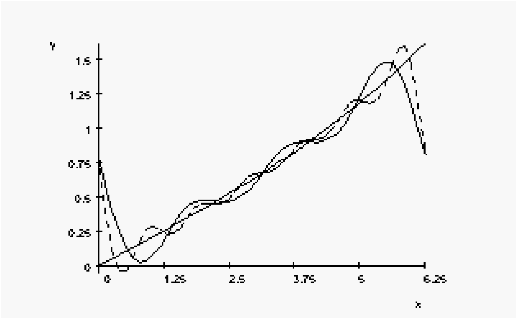
And so the series is:

$$\sinh \alpha x = \frac{1}{2\pi}\sum_{n=-\infty}^{\infty}\frac{\alpha\cosh 2\pi\alpha - \alpha + in\sinh 2\pi\alpha}{\alpha^2+n^2}e^{inx}$$

Notice that the real part of c_n is an even function of n, while the imaginary part is an odd function. Thus we can rewrite the series as:

$$\sinh \alpha x = \frac{1}{2\pi}\left(\frac{\cosh 2\pi\alpha - 1}{\alpha} + \sum_{n=1}^{\infty}\frac{\alpha(\cosh 2\pi\alpha - 1)(e^{inx}+e^{-inx}) + in\sinh 2\pi\alpha(e^{inx}-e^{-inx})}{\alpha^2+n^2}\right)$$
$$= \frac{1}{2\pi}\left(\frac{\cosh 2\pi\alpha - 1}{\alpha} + 2\sum_{n=1}^{\infty}\frac{\alpha(\cosh 2\pi\alpha - 1)\cos nx - n\sinh 2\pi\alpha\sin nx}{\alpha^2+n^2}\right)$$

Note that if we let $\alpha = i\beta$, then $\sinh(i\beta x) = i\sin\beta x$ and we get back the result of problem 6.



solid-3 terms. Dashed - 6 terms

8. Obtain the first four non-zero terms in a Fourier series for the function $\tan x$ on the range $-\pi/4 < x < \pi/4$

The function is odd and so we need a sine series with period $\pi/2$. Thus:

$$\tan x = \sum_{n=1}^{\infty} a_n \sin 4nx$$

where

$$a_n = \frac{8}{\pi}\int_0^{\pi/4} \tan x \sin 4nx\,dx$$

The first four terms are:

$$\begin{aligned}
a_1 &= \frac{8}{\pi}\int_0^{\pi/4} \tan x \sin(4x)dx = \frac{8}{\pi}\int_0^{\pi/4} \frac{\sin x}{\cos x} 2\sin(2x)\cos 2x dx \\
&= \frac{16}{\pi}\int_0^{\pi/4} \frac{\sin x}{\cos x} 2\sin x \cos x \cos 2x dx \\
&= \frac{32}{\pi}\int_0^{\pi/4} \sin^2 x \cos 2x dx = \frac{32}{\pi}\int_0^{\pi/4} \frac{(1-\cos 2x)}{2} \cos 2x dx \\
&= \frac{16}{\pi}\int_0^{\pi/4} \cos 2x - \left(\frac{\cos 4x + 1}{2}\right)dx \\
&= \frac{16}{\pi}\left[\frac{\sin 2x}{2} - \left(\frac{\sin 4x}{8} + \frac{x}{2}\right)\right]\Big|_0^{\pi/4} \\
&= \frac{16}{\pi}\left[\frac{1}{2} - \frac{\pi}{8}\right] = \frac{8}{\pi} - 2 = 0.54648
\end{aligned}$$

Similarly

$$a_2 = \frac{8}{\pi}\int_0^{\pi/4} \tan x \sin 8x dx = \frac{8}{\pi}\left(\frac{2}{3} - \frac{1}{4}\pi\right) = -0.30235$$

$$a_3 = \frac{8}{\pi}\int_0^{\pi/4} \tan x \sin 12x dx = \frac{8}{\pi}\left(\frac{13}{15} - \frac{1}{4}\pi\right) = 0.20695$$

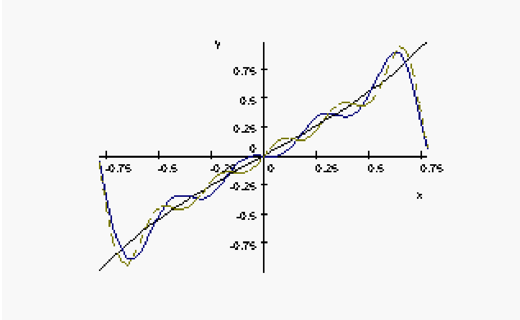
$$a_4 = \frac{8}{\pi}\int_0^{\pi/4} \tan x \sin 16x dx = \frac{8}{\pi}\left(\frac{76}{105} - \frac{1}{4}\pi\right) = -0.15683$$

Thus

$$\tan x = 0.54648 \sin 4x - 0.30235 \sin 8x + 0.20695 \sin 12x - 0.15683 \sin 16x + \cdots$$

The series converges rather slowly. The next term is:

$$a_5 = \frac{8}{\pi}\int_0^{\pi/4} \tan x \sin 20x dx = \frac{8}{\pi}\left(-\frac{1}{4}\pi + \frac{263}{315}\right) = 0.12611$$



Solid curve -four terms; dashed curve- five terms.

9. Use numerical integration to find the first 10 terms in a fourier series for the function $\sin x^2$ on the range $0 < x < \pi.$ What is the maximum % error between your series and the function $\sin x^2$ over the given range?

We may choose an even extension of the function to the range $-\pi$ to π and expand in a cosine series. Thus

$$\sin x^2 = \sum_{n=0}^{\infty} a_n \cos nx$$

where

$$a_n = \frac{2}{\pi}\int_0^{\pi} \sin x^2 \cos nx dx$$

and

$$a_0 = \frac{1}{\pi}\int_0^{\pi} \sin x^2 dx = 0.24594$$

The first few terms are:

$$a_1 = \frac{2}{\pi}\int_0^{\pi} \sin x^2 \cos x dx = .19245$$

$$a_2 = \frac{2}{\pi}\int_0^{\pi} \sin x^2 \cos 2x dx = -1.6751 \times 10^{-2}$$

$$a_3 = \frac{2}{\pi}\int_0^{\pi} \sin x^2 \cos 3x dx = -.68035$$

$$a_4 = \frac{2}{\pi}\int_0^{\pi} \sin x^2 \cos 4x dx = .18925$$

$$a_5 = \frac{2}{\pi}\int_0^{\pi} \sin x^2 \cos 5x dx = .21467$$

$$a_6 = \frac{2}{\pi}\int_0^{\pi} \sin x^2 \cos 6x dx = -.25927$$

$$a_7 = \frac{2}{\pi}\int_0^{\pi} \sin x^2 \cos 7x dx + .18982$$

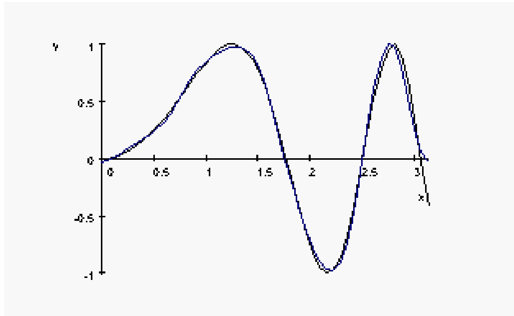
$$a_8 = \frac{2}{\pi}\int_0^{\pi} \sin x^2 \cos 8x dx = -.12607$$

$$a_9 = \frac{2}{\pi}\int_0^{\pi} \sin x^2 \cos 9x dx = .08531$$

$$a_{10} = \frac{2}{\pi}\int_0^{\pi} \sin x^2 \cos 10x dx = -6.0703 \times 10^{-2}$$

Thus

$$\begin{aligned}\sin x^2 = 0.24594 + 0.19245\cos x - 1.6751 \times 10^{-2}\cos 2x \\ - 0.6803518\cos 3x + 0.18925\cos 4x + 0.21467\cos 5x \\ - 0.25927\cos 6x + 0.18982\cos 7x - 0.12607\cos 8x + 0.08531\cos 9x \\ - 6.0703 \times 10^{-2}\cos 10x\end{aligned}$$



The greatest percent difference is at the end points, and at the other points where $\sin x^2 = 0$ ($\sqrt{\pi} = 1.7725$ and $\sqrt{2\pi} = 2.5066$). Other than at these places, the greatest percent difference is <10%. The greatest difference is 0.4 at $x = \pi$.

$0.24594 - 0.19245 - 1.6751 \times 10^{-2} + 0.6803518 + 0.18925 - 0.21467 - 0.25927 - 0.18982 - 0.12607 - 0.08531 = 3.1201 \times 10^{-2}$

10. Find the full Fourier series for the ramp function

$$f(x) = \begin{cases} x & \text{if } 0 < x < 1 \\ 1 & \text{if } 1 < x < 2 \end{cases}$$

on the interval $0 < x < 2$.

The argument of our harmonic function must be 2π when $x = 2$, and so it is πx. Thus the series has the form:

$$f(x) = \sum_{n=0}^{\infty} a_n \sin n\pi x + b_n \cos n\pi x$$

where

$$\begin{aligned}a_n &= \int_0^1 x \sin n\pi x dx + \int_1^2 \sin n\pi x dx \\ &= \frac{\sin n\pi - n\pi\cos n\pi}{n^2\pi^2} - \frac{\cos 2n\pi - \cos n\pi}{n\pi} \\ &= \frac{(-1)^{n+1}}{n\pi} - \frac{1-(-1)^n}{n\pi} = -\frac{1}{n\pi}\end{aligned}$$

and

$$\begin{aligned}b_n &= \int_0^1 x \cos n\pi x dx + \int_1^2 \cos n\pi x dx \\ &= \frac{\cos n\pi + n\pi \sin n\pi - 1}{n^2\pi^2} + \frac{\sin 2n\pi - \sin n\pi}{n\pi} \\ &= \frac{(-1)^n - 1}{n^2\pi^2}\end{aligned}$$

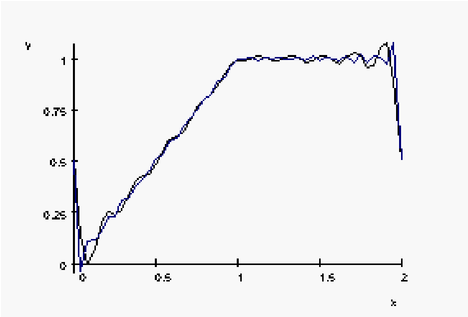
Finally

$$b_0 = \frac{1}{2}\left[\int_0^1 x dx + \int_1^2 dx\right] = \frac{1}{2}\left[\frac{1}{2} + 1\right] = \frac{3}{4}$$

Thus

$$f(x) = \frac{3}{4} - \frac{1}{\pi}\sum_{n=1}^{\infty}\left(\frac{\sin n\pi x}{n} + \frac{1-(-1)^n}{n^2\pi}\cos n\pi x\right)$$

Notice that only odd n contribute to the cosine terms.



Black $N = 10$ Blue $N = 20$

Note the Gibbs phenomenon at the edge of the flat plateau.

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11. An electric circuit contains a 1.5 mH inductor, a 5 μF capacitor and a 200 Ω resistor in series with a power supply that supplies a rectified sine wave voltage with amplitude 110 V and period 2 ms. Determine the capacitor voltage as a Fourier series.

The equation satisified by the circuit is:

$$L\frac{dI}{dt} + IR + \frac{Q}{C} = \varepsilon(t)$$

where

$$\varepsilon(t) = \varepsilon_0 \begin{cases} \sin\omega t & \text{if} \quad 0 < t < \pi/\omega = T/2 \\ -\sin\omega t & \pi/\omega < t < 2\pi/\omega = T = 2\text{ ms} \end{cases}$$

First we find the Fourier series of the emf. Since the differential equation has 1st and 2nd derivatives, we'll use the exponential series:

$$\varepsilon(t) = \sum c_n e^{in\omega t}$$

where

$$\begin{aligned} c_n &= \frac{\varepsilon_0}{T}\int_0^T \varepsilon(t)e^{-in\omega t}dt = \frac{1}{T}\left(\int_0^{T/2} \sin\omega t e^{-in\omega t}dt - \int_{T/2}^T \sin\omega t e^{-in\omega t}dt\right) \\ &= \frac{\varepsilon_0}{T}\left(\int_0^{T/2} \frac{e^{i\omega t} - e^{-i\omega t}}{2i}e^{-in\omega t}dt - \int_{T/2}^T \frac{e^{i\omega t} - e^{-i\omega t}}{2i}e^{-in\omega t}dt\right) \\ &= \frac{\varepsilon_0}{2iT}\left(\frac{e^{i(1-n)\omega t}}{i(1-n)\omega} - \frac{e^{-i(1+n)\omega t}}{-i(1+n)\omega}\Bigg|_0^{T/2} - \left(\frac{e^{i(1-n)\omega t}}{i(1-n)\omega} - \frac{e^{-i(1+n)\omega t}}{-i(1+n)\omega}\right)\Bigg|_{T/2}^T\right) \\ &= \frac{\varepsilon_0}{2iT}\left(\frac{e^{i(1-n)\pi} - 1}{i(1-n)\omega} - \frac{e^{-i(1+n)\pi} - 1}{-i(1+n)\omega} - \left(\frac{1 - e^{i(1-n)\pi}}{i(1-n)\omega} - \frac{1 - e^{-i(1+n)\pi}}{-i(1+n)\omega}\right)\right) \\ &= \frac{\varepsilon_0}{iT}\left(\frac{e^{i(1-n)\pi} - 1}{i(1-n)\omega} + \frac{e^{-i(1+n)\pi} - 1}{i(1+n)\omega}\right) \\ &= \frac{\varepsilon_0}{iT}\left(\frac{(-1)^{(1-n)} - 1}{i(1-n)\omega} + \frac{(-1)^{(1+n)} - 1}{i(1+n)\omega}\right) \end{aligned}$$

We could obtain the same result from the first "bump" in the function, repeating with period $T/2$.

The result is zero if n is odd. If n is even, we get:

$$c_n = \frac{2\varepsilon_0}{T}\left(\frac{1}{(1-n)\omega} + \frac{1}{(1+n)\omega}\right) = -\frac{2\varepsilon_0}{\pi(n-1)(1+n)}$$

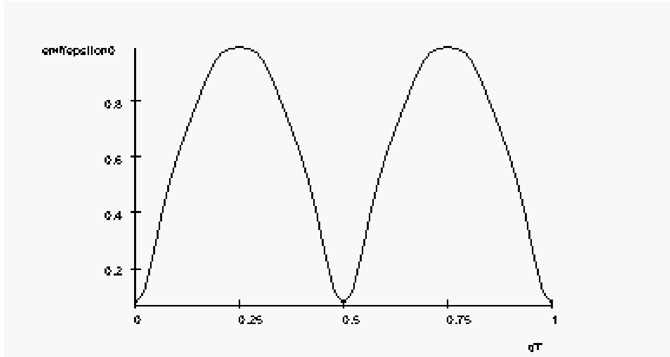
We have to evaluate the $n = 1$ and $n = -1$ terms separately:

$$\begin{aligned} c_1 &= \frac{\varepsilon_0}{T}\left(\int_0^{T/2} \frac{e^{i\omega t} - e^{-i\omega t}}{2i}e^{-i\omega t}dt - \int_{T/2}^T \frac{e^{i\omega t} - e^{-i\omega t}}{2i}e^{-i\omega t}dt\right) \\ &= \frac{\varepsilon_0}{T}\left(\int_0^{T/2} \frac{1 - e^{-2i\omega t}}{2i}dt - \int_{T/2}^T \frac{1 - e^{-2i\omega t}}{2i}dt\right) \\ &= \frac{\varepsilon_0}{2iT}\left(t - \frac{e^{-2i\omega t}}{-2i\omega}\Bigg|_0^{T/2} - \left(t - \frac{e^{-2i\omega t}}{-2i\omega}\right)\Bigg|_{T/2}^T\right) \\ &= \frac{\varepsilon_0}{2iT}\left(\frac{T}{2} - \frac{e^{-2\pi i} - 1}{-2i\omega}\Bigg|_0^{T/2} - \left(\frac{T}{2} - \frac{e^{-4\pi i} - e^{-2\pi i}}{-2i\omega}\right)\Bigg|_{T/2}^T\right) = 0 \end{aligned}$$

and similarly for c_{-1}. Thus the series is

$$\varepsilon(t) = -\frac{2\varepsilon_0}{\pi}\sum_{m=-\infty}^{\infty}\frac{e^{i2m\omega t}}{(4m^2 - 1)}$$

We can plot this to check the result:



Now the differential equation becomes:

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{Q}{C} = -\frac{2\varepsilon_0}{\pi}\sum_{m=-\infty}^{\infty}\frac{e^{i2m\omega t}}{\left(4m^2-1\right)}$$

Now express Q as a Fourier series:

$$Q = \sum Q_n e^{in\omega t}$$

Because of the orthogonality of the exponentials, we have an equation for each coefficient Q_n in terms of c_n. Thus only even n survive, and:

$$-n^2\omega^2 L Q_n + in\omega R Q_n + \frac{Q_n}{C} = -\frac{2\varepsilon_0}{\pi\left(n^2-1\right)}$$

and so

$$Q_n = -\frac{2\varepsilon_0}{\pi\left(n^2-1\right)\left(-n^2\omega^2 L + in\omega R + \frac{1}{C}\right)}$$

Now as usual let

$$\alpha = \frac{R}{2L} = \frac{200\ \Omega}{3\times 10^{-3}\ \text{H}} = 6.6667\times 10^4\ /\text{s}$$

and

$$\omega_0^2 = \frac{1}{LC} = \frac{1}{\left(1.5\times 10^{-3}\ \text{H}\right)\left(5\times 10^{-6}\ \text{F}\right)} = 1.3333\times 10^8\ /\text{s}^2$$

Then

$$Q_n = \frac{2\varepsilon_0 C}{\pi\left(n^2-1\right)\left(n^2(\omega/\omega_0)^2 - 2in\alpha\omega/\omega_0^2 - 1\right)}$$

and thus the capacitor voltage is:

$$\begin{aligned} V_C = \frac{Q}{C} &= \sum_{n=2m}\frac{2\varepsilon_0}{\pi\left(n^2-1\right)\left(n^2(\omega/\omega_0)^2 - 2in\alpha\omega/\omega_0^2 - 1\right)}e^{in\omega t} \\ &= \sum_{n=2m}\frac{2\varepsilon_0\omega_0^2\left(n^2\omega^2-\omega_0^2+2in\alpha\omega\right)}{\pi\left(n^2-1\right)\left[\left(n^2\omega^2-\omega_0^2\right)^2+(2n\alpha\omega)^2\right]}e^{in\omega t} \end{aligned}$$

The real terms are even in n while the imaginary terms are odd, so:

$$V_C = \frac{4\varepsilon_0}{\pi}\left[\frac{1}{2} + \omega_0^2\sum_{n=2m\neq 0}\frac{\left(n^2\omega^2-\omega_0^2\right)\cos n\omega t - 2n\alpha\omega\sin n\omega t}{\left(n^2-1\right)\left[\left(n^2\omega^2-\omega_0^2\right)^2+(2n\alpha\omega)^2\right]}\right]$$

Alternatively, we may simplify by writing

$$V_C = \sum_{n=2m}\frac{2\varepsilon_0 e^{-i\phi_n}}{\pi\left(n^2-1\right)\sqrt{\left(n^2(\omega/\omega_0)^2-1\right)^2+\left(2n\alpha\omega/\omega_0^2\right)^2}}e^{in\omega t}$$

where

$$\tan\phi_n = \frac{-2n\alpha\omega}{n^2\omega^2 - \omega_0^2}$$

Both expressions give the real result:

$$V_C = \frac{4\varepsilon_0}{\pi}\left[\frac{1}{2} + \sum_{n=2m\neq 0} \frac{\cos(n\omega t - \phi_n)}{\left(n^2-1\right)\sqrt{\left(n^2\omega^2/\omega_0^2 - 1\right)^2 + \left(2n\alpha\omega/\omega_0^2\right)^2}}\right]$$

The Fourier component may be large for any n close to $\frac{\omega_0}{\omega}$. Now

$$\omega = \frac{2\pi}{2\times 10^{-3}\text{ s}} = 1000\pi\text{ /s}$$

so

$$\frac{\omega_0}{\omega} = \frac{\sqrt{1.3333\times 10^8}}{1000\pi} = \frac{11.547}{\pi} = 3.6755$$

and so c_4 could be large if α/ω_0 is small. However, it is not:

$$2\frac{\alpha\omega}{\omega_0^2} = 2\frac{\left(6.6667\times 10^4\text{ /s}\right)\left(1000\pi\text{ /s}\right)}{1.3333\times 10^8\text{ /s}^2} = \pi$$

so

$$V_C = 140\text{ V}\left[\frac{1}{2} + \sum_{n=2m\neq 0} \frac{\cos(n\omega t - \phi_n)}{\left(n^2-1\right)\sqrt{\left(n^2/(3.6755)^2 - 1\right)^2 + \pi^2 n^2}}\right]$$

and

$$\tan\phi_n = -\frac{2n\alpha}{\omega\left(n^2 - (\omega_0/\omega)^2\right)} = -\frac{2n\left(6.6667\times 10^4\text{ /s}\right)}{\left(1000\pi\text{ /s}\right)\left(n^2 - 3.6755^2\right)}$$
$$= -\frac{133.33n}{\pi\left(n^2 - 13.509\right)} = -\frac{42.44n}{\left(n^2 - 13.509\right)}$$

The first few terms are:

$$V_C = 140\text{ V}\left(\frac{1}{2} + \frac{\cos(2\omega t - \phi_2)}{18.967} + \frac{\cos(4\omega t - \phi_4)}{188.52} + \cdots\right)$$

where

$$\tan\phi_2 = \frac{-42.44\times 2}{\left(2^2 - 13.509\right)} = 8.9263$$
$$\Rightarrow \phi_2 = \tan^{-1}(8.9263) = 1.4592 + \pi = 4.6008$$

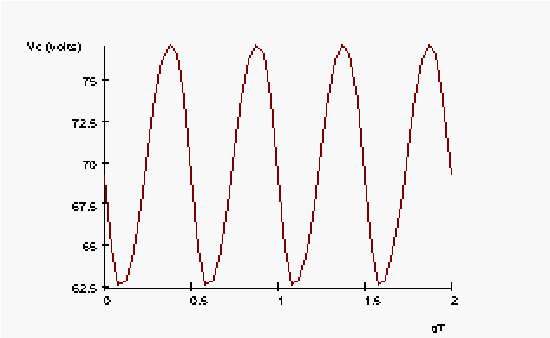
Notice we want an angle whose sine is negative and cosine is also negative, so the correct quadrant is the third:

$\sin 4.6008 = -.99378;$ $\cos 4.6008 = -.11136$

$$\tan\phi_4 = -\frac{42.44\times 4}{\left(4^2 - 13.509\right)} = -68.149 \Rightarrow \phi_4 = \tan^{-1}(-68.149) = -1.5561$$

Here we want a negative sine but positive cosine. Thus

$$V_c = 140\text{ V}\left(\frac{1}{2} + \frac{\cos(2\omega t - 4.6008)}{18.967} + \frac{\cos(4\omega t + 1.5561)}{188.52} + \frac{\cos(6\omega t + 1.4827)}{662.3}\cdots\right)$$



Notice that the variation is not very large: about 15 V. The circuit smooths out the variations in the power supply voltage.

12. A single loop, series LRC circuit has resistance $R = 15\ \Omega$, inductance $L = 10$ mH and capacitance $C = 1.5$ μF

A rectified sine wave power supply (see Figure) with Period

$T = 1.57 \times 10^{-3}$ s is attached to the circuit. Find the voltage across the capacitor as a Fourier series in time once the circuit has reached a steady state.

The equation is:

$$L\frac{dI}{dt} + RI + \frac{Q}{C} = E(t)$$

Write Q as a Fourier series:

$$Q = \sum_{n=-\infty}^{+\infty} Q_n e^{in\omega t}$$

where $\omega = 2\pi/T$. Then

$$E(t) = \sum_{n=-\infty}^{+\infty} E_n e^{in\omega t}$$

and the de is

$$-Ln^2\omega^2 Q_n + in\omega R Q_n + \frac{Q_n}{C} = E_n$$

and so

$$Q_n = \frac{E_n C}{-LCn^2\omega^2 + in\omega RC + 1} = \frac{E_n}{L}\frac{1}{\omega_0^2 + i2n\alpha\omega - n^2\omega^2}$$

where

$$\omega_0^2 = \frac{1}{LC} \text{ and } \alpha = \frac{R}{2L}$$

Half-rectified sine wave:

Series:

$$\varepsilon_{hr}(t) = \sum E_n e^{in\omega t}$$

where

$$
\begin{aligned}
E_n &= \frac{1}{T}\int_0^{T/2} \sin\omega t e^{-in\omega t}dt = \frac{1}{2iT}\int_0^{T/2}\left(e^{i\omega t} - e^{-i\omega t}\right)e^{-in\omega t}dt \\
&= \frac{1}{2iT}\left(\frac{e^{i\omega t}}{i\omega(1-n)} - \frac{e^{-i\omega t}}{-i\omega(1+n)}\right)e^{in\omega t}\Big|_0^{T/2} \\
&= \frac{1}{2iT}\left(\frac{e^{i\pi}e^{in\pi} - 1}{i\omega(1-n)} - \frac{e^{-i\pi}e^{in\pi} - 1}{-i\omega(1+n)}\right) \\
&= \frac{(-1)^{n+1} - 1}{-2\omega T}\left(\frac{1}{(1-n)} + \frac{1}{(1+n)}\right) \\
&= \frac{(-1)^n + 1}{2\omega T}\left(\frac{2}{1-n^2}\right) = \frac{(-1)^n + 1}{2\pi}\left(\frac{1}{1-n^2}\right) \\
&= \frac{1}{\pi\left(1-n^2\right)} \quad n \text{ even}
\end{aligned}
$$

We must do the integral differently if $n = \pm 1$:

$$
\begin{aligned}
E_1 &= \frac{1}{T}\int_0^{T/2} \sin\omega t e^{-i\omega t}dt = \frac{1}{2iT}\int_0^{T/2}\left(e^{i\omega t} - e^{-i\omega t}\right)e^{-i\omega t}dt \\
&= \frac{1}{2iT}\int_0^{T/2}\left(1 - e^{-i2\omega t}\right)dt = \frac{1}{4i} - \frac{e^{-2i\omega t}}{2iT(-2i\omega)}\Big|_0^{T/2} \\
&= \frac{1}{4i} - \frac{e^{-2i\pi} - 1}{4T\omega} = \frac{1}{4i}
\end{aligned}
$$

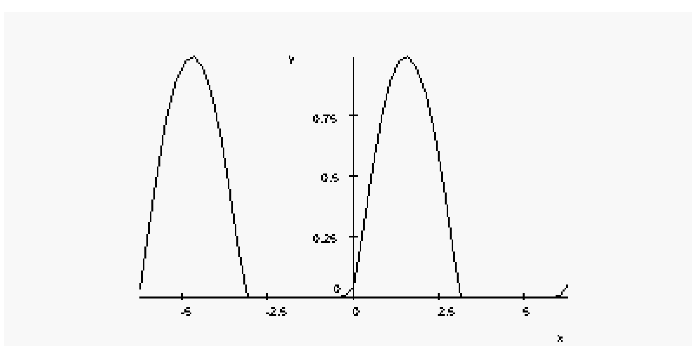
Similarly, for $n = -1$

$$
\begin{aligned}
E_{-1} &= \frac{1}{T}\int_0^{T/2} \sin\omega t e^{i\omega t}dt = \frac{1}{2iT}\int_0^{T/2}\left(e^{2i\omega t} - 1\right)dt \\
&= -\frac{1}{4i}
\end{aligned}
$$

Thus

$$
\begin{aligned}
\varepsilon_{hr}(t) &= \sum_{n=-\infty}^{+\infty} \frac{1}{\pi\left(1-4n^2\right)}e^{i2n\omega t} + \frac{1}{4i}\left(e^{i\omega t} - e^{-i\omega t}\right) \\
&= \frac{1}{\pi} + \sum_{n=1}^{+\infty} \frac{2}{\pi\left(1-4n^2\right)}\cos 2n\omega t + \frac{1}{2}\sin\omega t
\end{aligned}
$$

We test this result by plotting it:



Then:

$$
\begin{aligned}
Q_n &= \frac{E_n C}{-LCn^2\omega^2 + in\omega RC + 1} = \frac{E_n}{L}\frac{1}{\omega_0^2 + i2n\alpha\omega - n^2\omega^2} \\
&= \frac{1}{\pi\left(1-n^2\right)}\frac{1}{L}\frac{1}{\omega_0^2 + i2n\alpha\omega - n^2\omega^2} \quad n \text{ even}
\end{aligned}
$$

and

$$
Q_{\pm 1} = \frac{E_{\pm 1}}{L}\frac{1}{\omega_0^2 \pm i2\alpha\omega - \omega^2} = \frac{1}{L}\frac{\pm 1}{4i}\frac{1}{\omega_0^2 \pm i2\alpha\omega - \omega^2}
$$

Thus

$$V_C = \frac{E}{\pi LC}\sum_{n,\text{ even}} \frac{1}{\left(1-n^2\right)} \frac{1}{\omega_0^2 + i2n\alpha\omega - n^2\omega^2} e^{in\omega t}$$

$$+ \frac{E}{4iLC}\left(\frac{e^{i\omega t}}{\omega_0^2 + i2\alpha\omega - \omega^2} - \frac{e^{-i\omega t}}{\omega_0^2 - i2\alpha\omega - \omega^2}\right)$$

$$= E\omega_0^2 \left(\frac{1}{\pi}\sum_{n,\text{ even}} \frac{2}{\left(1-n^2\right)} \frac{\left(\omega_0^2 - n^2\omega^2\right)\cos n\omega t + 2n\alpha\omega \sin n\omega t}{\left(\omega_0^2 - n^2\omega^2\right)^2 + (2n\alpha\omega)^2} + \frac{1}{2}\frac{\left(\omega_0^2-\omega^2\right)\sin \omega t + 2\alpha\omega \cos \omega t}{\left(\omega_0^2-\omega^2\right)^2 + (2\alpha\omega)^2} \right)$$

$$= E\left(\frac{1}{\pi} + \frac{1}{\pi}\sum_{n,\text{ even}} \frac{2}{\left(1-n^2\right)} \frac{\left(1-n^2\omega^2/\omega_0^2\right)\cos n\omega t + \frac{2n\alpha\omega}{\omega_0^2}\sin n\omega t}{\left(1-n^2\omega^2/\omega_0^2\right)^2 + \left(2n\alpha\omega/\omega_0^2\right)^2} + \frac{1}{2}\frac{\left(1-\omega^2/\omega_0^2\right)\sin\omega t + 2\frac{\alpha\omega}{\omega_0^2}\cos\omega t}{\left(1-\omega^2/\omega_0^2\right)^2 + \left(2\alpha\omega/\omega_0^2\right)^2} \right)$$

Now we put in the numbers:

$$\omega_0^2 = \frac{1}{LC} = \frac{1}{10\times 10^{-3}\times 1.5\times 10^{-6}} = 6.6667\times 10^7\ \text{s}^{-2}$$
$$\omega_0 = \sqrt{6.6667\times 10^7}\ \text{s}^{-1} = 8.165\times 10^3\ \text{s}^{-1}$$

and

$$\omega = \frac{2\pi}{\left(1.57\times 10^{-3}\ \text{s}\right)} = 4.0\times 10^3\ \text{s}^{-1}$$
$$\omega^2 = 1.6\times 10^7\ \text{s}^{-2}$$

and

$$\alpha = \frac{R}{2L} = \frac{15\ \Omega}{2\left(12\times 10^{-3}\ \text{H}\right)} = 625\ \text{s}^{-1}$$

So

$$\frac{\alpha}{\omega_0} = \frac{625}{8.165\times 10^3} = 7.6546\times 10^{-2}$$

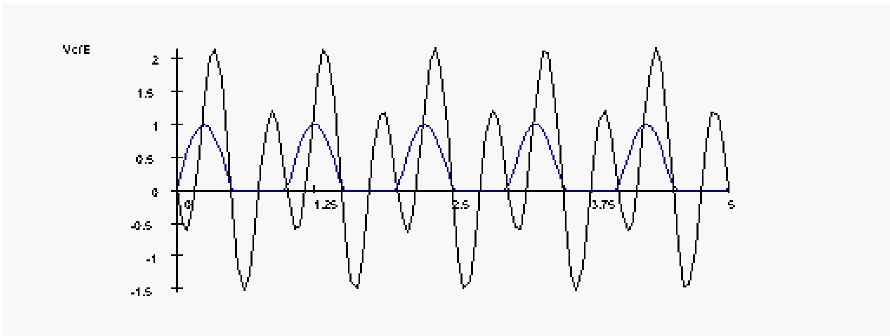
and

$$\frac{\omega}{\omega_0} = \frac{4.0\times 10^3}{8.165\times 10^3} = 0.4899$$

$$\left(\frac{\omega}{\omega_0}\right)^2 = 0.4899^2 = 0.24$$

$$\frac{2\alpha\omega}{\omega_0^2} = 2\times 7.6546\times 10^{-2}\times 0.4899 = 0.075$$

$$V_C = E\left(\frac{1}{\pi} + \frac{1}{\pi}\sum_{n,\text{ even}} \frac{2}{(1-n^2)} \frac{(1-.24n^2)\cos n\omega t + 0.075n\sin n\omega t}{(1-.24n^2)^2 + (0.075n)^2} + \frac{1}{2}\frac{.76\sin\omega t + 0.075\cos\omega t}{(.76)^2 + 0.075^2}\right)$$
$$= E\left(\frac{1}{\pi} + \frac{1}{\pi}\sum_{n,\text{ even}} \frac{2}{(1-n^2)} \frac{(1-.24n^2)\cos n\omega t + 0.075n\sin n\omega t}{(1-.24n^2)^2 + (0.075n)^2} + \frac{.76\sin\omega t + 0.075\cos\omega t}{1.1665}\right)$$
$$= E\left(\begin{array}{c} .31831 + .65152\sin\omega t + 6.4295\times 10^{-2}\cos\omega t - 1.1065\cos 2\omega t - 4.1494\sin 2\omega t \\ +.04643\cos 4\omega t - 2.4523\times 10^{-3}\sin 4\omega t + \cdots \end{array}\right)$$



Blue- input. Black- output

Notice the major contribution from the resonance at the second harmonic ($n = 2$).

13. A spring-and-dashpot system satisfies the equation

$$\frac{d^2x}{dt^2} + \alpha\frac{dx}{dt} + k^2x = f(t)$$

The system is driven by a periodic driving force with period T:

$$f(t) = \begin{cases} at & \text{if } 0 < t \le T/2 \\ a(T-t) & \text{if } T/2 \le t < T \end{cases}$$

Find the response of the system $x(t)$ as a Fourier series.

We begin by finding the Fourier series for

$f(t)$. Since the de has 1st and 2nd derivatives, we'll use the exponential series:

$$f(t) = \sum c_n e^{in2\pi t/T}$$

with

$$c_n = \frac{1}{T}\int_0^T f(t)e^{-in2\pi t/T}dt = \frac{1}{T}\left(\int_0^{T/2} ate^{-in2\pi t/T}dt + \int_{T/2}^T a(T-t)e^{-in2\pi t/T}dt\right)$$
$$= \frac{a}{T}\left(\frac{te^{-in2\pi t/T}}{-in2\pi/T}\Big|_0^{T/2} - \int_0^{T/2}\frac{e^{-in2\pi t/T}}{-in2\pi/T}dt + T\frac{e^{-in2\pi t/T}}{-in2\pi/T}\Big|_{T/2}^T - \frac{te^{-in2\pi t/T}}{-in2\pi/T}\Big|_{T/2}^T + \int_{T/2}^T\frac{e^{-in2\pi t/T}}{-in2\pi/T}dt\right.$$
$$= \frac{a}{-i2n\pi}\left(\frac{Te^{-in\pi}}{2} - \frac{e^{-in2\pi t/T}}{-in2\pi/T}\Big|_0^{T/2} + T\left(e^{-in2\pi} - e^{-in\pi}\right) - \left(Te^{-in2\pi} - \frac{Te^{-in\pi}}{2}\right) + \frac{e^{-in2\pi t/T}}{-in2\pi/T}\Big|_{T/2}^T\right.$$
$$= \frac{a}{-i2n\pi}\left(\frac{e^{-in\pi}-1}{in2\pi/T} + \frac{e^{-in2\pi} - e^{-in\pi}}{-in2\pi/T}\right)$$
$$= \frac{aT}{2n^2\pi^2}\left(e^{-in\pi} - 1\right) = \frac{aT}{2n^2\pi^2}((-1)^n - 1)$$

The result is zero if n is even and

$$c_n = -\frac{aT}{2n^2\pi^2}$$

if n is odd.

Now we write the displacement of the spring system as a similar series:

$$x = \sum x_n e^{in2\pi t/T}$$

and stuff into the de. By orthogonality of the exponentials, we can evaluate each term in the sum separately:

$$-n^2\left(\frac{2\pi}{T}\right)^2 x_n + \alpha in\frac{2\pi}{T}x_n + k^2 x_n = -\frac{aT}{2n^2\pi^2}$$

and thus

$$x_n = -\frac{aT}{2n^2\pi^2}\frac{1}{k^2 - n^2\left(\frac{2\pi}{T}\right)^2 + \alpha in\frac{2\pi}{T}}$$

Let $\omega_0 = 2\pi/T.$ Then:

$$x(t) = \sum \frac{a}{n^2\pi\omega_0}\frac{1}{n^2\omega_0^2 - k^2 - \alpha in\omega_0}e^{in\omega_0 t}$$

We can group together the terms with $n = N$ and $n = -N$ to get:

$$x(t) = \sum \frac{a}{n^2\pi\omega_0}\frac{\left(n^2\omega_0^2 - k^2\right)\cos n\omega_0 t - 2\alpha n\omega_0 \sin n\omega_0 t}{\left(n^2\omega_0^2 - k^2\right)^2 + (\alpha n\omega_0)^2}$$

If $k = n\omega_0$ for some $n,$ there is a resonance at that value of $n,$ and the response of the system is large at the harmonic, particularly if the damping is small $(\alpha/\omega_0 \ll 1).$

14. A simply supported beam of length L bears a load W that is uniformly distributed over the first 1/4 of its length. Determine the deflection of the beam as a Fourier series. Make plots showing the first 1, 2, and 3 terms of your answer. How many terms are needed to obtain a result accurate to 1%? (The differential equation satisfied by the beam deflection is equation 3.10, and the displacement is zero at the two ends.)

The deflection of the beam is given by:

$$\frac{d^4y}{dx^4} = \frac{1}{EI}q(x)$$

where

$$q(x) = \begin{cases} \frac{4W}{L} & \text{if } x < L/4 \\ 0 & \text{if } x > L/4 \end{cases}$$

Since the deflection is zero at $y = 0$ and $y = L,$ we should be able to express the deflection y as a Fourier sine series of the form:

$$y(x) = \sum y_n \sin\frac{n\pi x}{L}$$

Let's express q as a similar series:

$$q(x) = \sum q_n \sin\frac{n\pi x}{L}$$

where

$$\begin{aligned} q_n &= \frac{2}{L}\int_0^L q(x)\sin\frac{n\pi x}{L}dx \\ &= \frac{2}{L}\int_0^{L/4}\frac{4W}{L}\sin\frac{n\pi x}{L}dx \\ &= \frac{8W}{L^2}\left(-\frac{L}{n\pi}\cos\frac{n\pi x}{L}\Big|_0^{L/4}\right) \\ &= -\frac{8W}{L\pi n}\left(\cos\frac{n\pi}{4} - 1\right) \end{aligned}$$

where

$$\cos\frac{n\pi}{4} = \begin{cases} \frac{\sqrt{2}}{2} & if \;\; n = 1,7,9,15\cdots \\ 0 & if \;\; n = 2,6,10\cdots \\ -\frac{\sqrt{2}}{2} & if \;\; n = 3,5,11,13\cdots \\ -1 & if \;\; n = 4,12,\cdots \\ 1 & if \;\; n = 8,16,\cdots \end{cases}$$

Stuffing the series into the de, we get:

$$\left(\frac{n\pi}{L}\right)^4 y_n = \frac{1}{EI}q_n = -\frac{8W}{EIL\pi n}\left(\cos\frac{n\pi}{4} - 1\right)$$

and thus:

$$y_n = -\left(\frac{L}{n\pi}\right)^4 \frac{8W}{EIL\pi n}\left(\cos\frac{n\pi}{4} - 1\right)$$

and so

$$y = \frac{8WL^3}{EI\pi^5}\sum \frac{1 - \cos\frac{n\pi}{4}}{n^5}\sin\frac{n\pi x}{L}$$

Because of the very strong dependence of y_n on n, the series converges very rapidly. The first few terms are:

$$\begin{aligned} y &= \frac{8WL^3}{EI\pi^5}\left(\left(1 - \frac{\sqrt{2}}{2}\right)\sin\frac{\pi x}{L} + \frac{1}{2^5}\sin\frac{2\pi x}{L} + \frac{1}{3^5}\left(1 + \frac{\sqrt{2}}{2}\right)\sin\frac{3\pi x}{L} + \frac{2}{4^5}\sin\frac{4\pi x}{L} + \cdots\right) \\ &= \frac{8WL^3}{EI\pi^5}\left(0.29289\sin\frac{\pi x}{L} + 0.03125\sin\frac{2\pi x}{L} + 7.0251\times10^{-3}\sin\frac{3\pi x}{L} + 1.9531\times10^{-3}\sin\frac{4\pi x}{L} + \cdots\right) \end{aligned}$$

The first term alone should give 1% accuracy. Let's check by directly integrating the differential equation:

$$\frac{d^4y}{dx^4} = \frac{1}{EI}q(x) = \frac{1}{EI}\begin{cases} \frac{4W}{L} & \text{if} \;\; x < L/4 \\ 0 & \text{if} \;\; x > L/4 \end{cases}$$

Integrating once, we get:

$$y''' = \frac{4Wx}{LEI} + A$$

if $x < L/4$ and

$$y''' = B$$

if $x > L/4$.

We can use the discussion in Chapter 3 to determine appropriate boundary conditions.

$$y'' = -\frac{1}{EI}m(x)$$

where m is the net cc torque of all forces to the right of x, and

$$\frac{dm}{dx} = t(x)$$

where t is the net vertical force to the right of x. Thus

$$y''' = \frac{-1}{EI}t(x)$$

Now we can find the support force at the right support ($x = L$) by computing torques about the left end:

$$\tau = \int_0^{L/4}\frac{4W}{L}x\,dx - F_S L = 0$$

$$\frac{4W}{L}\frac{1}{2}\left(\frac{L}{4}\right)^2 = F_s L$$

$$F_s = \frac{W}{8}$$

Thus

$$y'''(L) = \frac{1}{EI}\frac{W}{8} = B$$

Continuity of y''' at $x = L/4$ requires:

$$y'''\left(\frac{L}{4}\right) = \frac{W}{EI} + A = B$$

and thus

$$A = B - \frac{W}{EI} = \frac{1}{EI}\frac{W}{8} - \frac{W}{EI} = -\frac{7}{8}\frac{W}{EI}$$

Integrating again, we find:

$$y'' = \frac{2Wx^2}{LEI} - \frac{7}{8}\frac{W}{EI}x + C$$

for $x < L/4$ and

$$y'' = \frac{1}{EI}\frac{W}{8}x + D$$

otherwise. Again the boundary conditions require $C = 0$ and continuity at $x = L/4$ gives:

$$\frac{2WL^2}{16LEI} - \frac{7}{8}\frac{W}{EI}\frac{L}{4} = \frac{1}{EI}\frac{W}{8}\frac{L}{4} + D \Rightarrow D = -\frac{1}{8}\frac{WL}{EI}$$

Integrate again:

$$y' = \frac{2Wx^3}{3LEI} - \frac{7}{8}\frac{W}{EI}\frac{x^2}{2} + F$$

for $x < L/4$ and

$$y' = \frac{1}{EI}\frac{W}{8}\frac{x^2}{2} - \frac{WL}{8EI}x + G$$

otherwise. Continuity at $x = L/4$ requires:

$$\frac{2WL^3}{4^3 3LEI} - \frac{7}{8}\frac{W}{EI}\frac{L^2}{32} + F = \frac{1}{EI}\frac{W}{8}\frac{L^2}{32} - \frac{WL}{8EI}\frac{L}{4} + G$$
$$\frac{1}{96}W\frac{L^2}{EI} + F = G$$

Then the final integration gives:

$$y = \frac{Wx^4}{6LEI} - \frac{7}{48}\frac{W}{EI}x^3 + Fx + J$$

and

$$y = \frac{1}{48}\frac{W}{EI}x^3 - \frac{WL}{16EI}x^2 + \left(\frac{1}{96}\frac{WL^2}{EI} + F\right)x + K$$

in the two regions. Since $y = 0$ at the two ends, $J = 0$ and:

$$\frac{1}{48}\frac{W}{EI}L^3 - \frac{WL}{16EI}L^2 + \left(\frac{1}{96}\frac{WL^3}{LEI} + F\right)L + K = 0$$
$$\frac{1}{32}\frac{W}{EI}L^3 - LF = K$$

Continuity at $x = L/4$ requires:

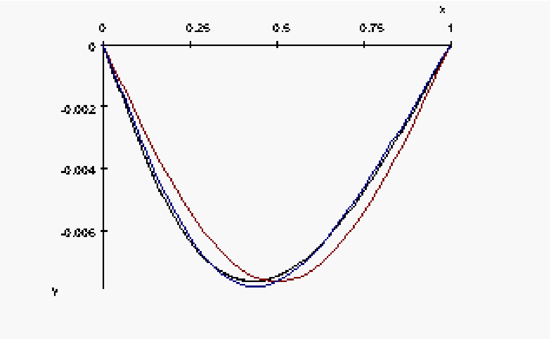
$$\frac{WL^4}{4^4 \times 6LEI} - \frac{7}{48}\frac{W}{EI}\frac{L^3}{4^3} + F\frac{L}{4} = \left(\frac{W}{EI}\right)\frac{L^3}{4^3 \times 48} - \frac{WL}{16^2 EI}L^2 + \left(\frac{1}{96}\frac{WL^2}{EI} + F\right)\frac{L}{4} + K$$
$$-\frac{1}{1536}\frac{W}{EI}L^3 = K$$

Then

$$F = \frac{49}{1536}W\frac{L^2}{EI}$$

and the full solution is:

$$y = \frac{W}{EI}\begin{cases} \frac{x^4}{6L} - \frac{7}{48}x^3 + \frac{49}{1536}L^2x & \text{if } x < L/4 \\ \frac{x^3}{48} - \frac{L}{16}x^2 + \frac{65}{1536}L^2x - \frac{1}{1536}L^3 & \text{if } x > L/4 \end{cases}$$



Red- 1st term only. Blue- 2 terms

The first 4 terms of the series give a curve that is indistinguishable from the exact solution. One term does not give the correct off-center peak of the deflection, but the first two terms give a result that is very close to the exact result.

x/L	exact	1 term	error (%)	2 terms	error (%)	3 terms	error (%)
.25	6.3476×10^{-3}	5.414×10^{-3}	14.7	6.231×10^{-3}	1.835	6.361×10^{-3}	−0.211
.5	7.487×10^{-3}	7.657×10^{-3}	−2.268	7.657×10^{-3}	−2.268	7.473×10^{-3}	0.186
.75	.00472	5.414×10^{-3}	−14.7	4.597×10^{-3}	2.60	4.727×10^{-3}	−0.15

Thus two terms of the series gives an error of about 2%, while three terms gives about 2 tenths of a percent error.

15. A beam rests on supports at its ends, $x = 0$ and $x = L.$ The load $q(x)$ varies linearly along the beam: $q = ax.$ What are the boundary conditions? Find the displacement of the beam as a Fourier series. Plot your results, and comment.

$$\frac{d^4y}{dx^4} = \frac{1}{EI}q(x) = \frac{1}{EI}ax$$

with $y(0) = y(L) = 0.$ Thus we can write the solution as a Fourier sine series of the form:

$$y = \sum_{n=1}^{\infty} y_n \sin\left(\frac{n\pi x}{L}\right)$$

The de becomes:

$$\sum_{n=1}^{\infty} y_n \left(\frac{n\pi}{L}\right)^4 \sin\left(\frac{n\pi x}{L}\right) = \frac{1}{EI}ax = \frac{a}{EI}\sum_{n=1}^{\infty} q_n$$

where

$$\begin{aligned} q_n &= \frac{2}{L}\int_0^L x \sin\frac{n\pi x}{L}dx = \frac{2}{L}\left(-x\frac{L}{n\pi}\cos\frac{n\pi x}{L}\Big|_0^L + \frac{L}{n\pi}\int_0^L \cos\frac{n\pi x}{L}dx\right) \\ &= \frac{2}{n\pi}\left(-L\cos n\pi + \frac{L}{n\pi}\sin\frac{n\pi x}{L}\Big|_0^L\right) = \frac{2L}{n\pi}(-1)^{n+1} \end{aligned}$$

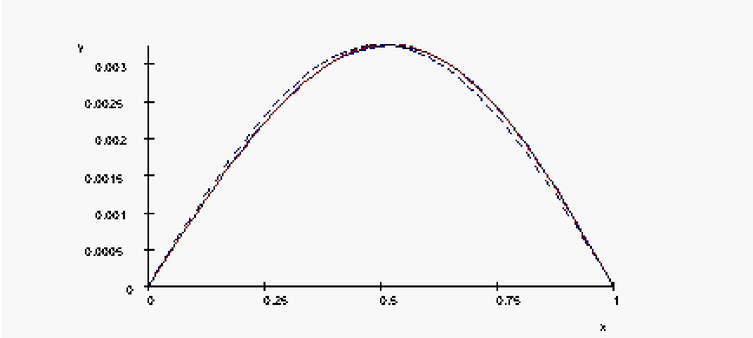
and so, from equation (),

$$y_n = \frac{2a}{EI}\left(\frac{L}{n\pi}\right)^5(-1)^{n+1}$$

and so:

$$y = \frac{2a}{EI}\sum_{n=1}^{\infty}\left(\frac{L}{n\pi}\right)^5(-1)^{n+1}\sin\left(\frac{n\pi x}{L}\right)$$

This series converges very fast. Here's a plot of the first five terms:



Black 5 terms -- Blue 3 terms Red 2 terms Navy 1 term

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16. A guitar string of length $L = 65$ cm is plucked by pulling it to the shape:

$$y(x,0) = \begin{cases} ax^2 & \text{if } 0 < x < L/3 \\ \frac{a}{4}(L-x)^2 & \text{if } L/3 < x < L \end{cases}$$

and then letting go. Determine the subsequent motion of the string. Which harmonics are excited?

Since the string is fixed at each end, we can express the displacement as a Fourier sine series:

$$y(x,t) = \sum a_n(t) \sin \frac{n\pi x}{L}$$

Stuff this into the wave equation:

$$\frac{\partial^2 y}{\partial t^2} = v^2 \frac{\partial^2 y}{\partial x^2}$$

where $v^2 = T/\mu.$ Then we have:

$$\frac{d^2 a_n}{dt^2} = -v^2 \left(\frac{n\pi}{L}\right)^2 a_n$$

and so

$$a_n = A_n \cos \frac{n\pi v t}{L} + B_n \sin \frac{n\pi v t}{L}$$

Since the string has a non-zero displacement but zero velocity at $t = 0,$ we need the cosine:

$$y(x,t) = \sum A_n \cos \frac{n\pi v t}{L} \sin \frac{n\pi x}{L}$$

Now at $t = 0,$ the shape of the string is:

$$y(x,0) = \sum A_n \sin \frac{n\pi x}{L} = \begin{cases} ax^2 & \text{if } 0 < x < L/3 \\ \frac{a}{4}(L-x)^2 & \text{if } L/3 < x < L \end{cases}$$

and thus

$$\begin{aligned} A_n &= \frac{2}{L}\left(\int_0^{L/3} ax^2 \sin \frac{n\pi x}{L} dx + \int_{L/3}^{L} \frac{a}{4}(L-x)^2 \sin \frac{n\pi x}{L} dx\right) \\ &= \frac{2a}{L}\left(\int_0^{L/3} x^2 \sin \frac{n\pi x}{L} dx - \int_{2L/3}^{0} \frac{1}{4} u^2 \sin \frac{n\pi(L-u)}{L} du\right) \\ &= \frac{2a}{L}\left(\int_0^{L/3} x^2 \sin \frac{n\pi x}{L} dx - \frac{(-1)^n}{4} \int_0^{2L/3} u^2 \sin \frac{n\pi u}{L} du\right) \end{aligned}$$

Now

$$\begin{aligned} \int x^2 \sin \frac{n\pi x}{L} dx &= \frac{L}{n\pi}\left(-x^2 \cos \frac{n\pi x}{L} + \int 2x \cos \frac{n\pi x}{L} dx\right) \\ &= \frac{L}{n\pi}\left(-x^2 \cos \frac{n\pi x}{L} + \frac{L}{n\pi} 2x \sin \frac{n\pi x}{L} - \int 2\frac{L}{n\pi} \sin \frac{n\pi x}{L} dx\right) \\ &= \frac{L}{n\pi}\left(-x^2 \cos \frac{n\pi x}{L} + \frac{2xL}{n\pi} \sin \frac{n\pi x}{L} + 2\frac{L^2}{\pi^2 n^2} \cos \frac{n\pi x}{L}\right) \end{aligned}$$

and so:

$$A_n = \frac{2a}{L}\left(\begin{array}{c} \frac{L}{n\pi}\left(-\frac{L^2}{9}\cos\frac{n\pi}{3} + \frac{2L^2}{3n\pi}\sin\frac{n\pi}{3} + 2\frac{L^2}{\pi^2 n^2}\left(\cos\frac{n\pi}{3} - 1\right)\right) \\ -\frac{(-1)^n}{4}\frac{L}{n\pi}\left(-\frac{4L^2}{9}\cos\frac{2n\pi}{3} + \frac{4L^2}{3n\pi}\sin\frac{2n\pi}{3} + 2\frac{L^2}{\pi^2 n^2}\left(\cos\frac{2n\pi}{3} - 1\right)\right) \end{array}\right)$$

$$= \frac{2a}{n\pi}\left(\begin{array}{c} -\frac{L^2}{9}\left(\cos\frac{n\pi}{3} - (-1)^n\cos\frac{2n\pi}{3}\right) + \frac{2L^2}{3n\pi}\left(\sin\frac{n\pi}{3} - \frac{(-1)^n}{2}\sin\frac{2n\pi}{3}\right) \\ +2\frac{L^2}{\pi^2 n^2}\left(\cos\frac{n\pi}{3} - 1 - \frac{(-1)^n}{4}\left(\cos\frac{2n\pi}{3} - 1\right)\right) \end{array}\right)$$

$$= \frac{2a}{n\pi}\left(\begin{array}{c} -\frac{L^2}{9}\left(\cos\frac{n\pi}{3} - (-1)^n\cos\frac{2n\pi}{3}\right) + \frac{2L^2}{3n\pi}\left(\sin\frac{n\pi}{3} - \frac{(-1)^n}{2}\sin\frac{2n\pi}{3}\right) \\ +2\frac{L^2}{\pi^2 n^2}\left(\cos\frac{n\pi}{3} - 1 - \frac{(-1)^n}{4}\left(\cos\frac{2n\pi}{3} - 1\right)\right) \end{array}\right)$$

$2n\pi/3 = 3n\pi/3 - n\pi/3$ so:

$$\cos\frac{2n\pi}{3} = (-1)^n\cos\frac{n\pi}{3}$$

and

$$\sin\frac{2n\pi}{3} = -(-1)^n\sin\frac{n\pi}{3}$$

so we can simplify:

$$A_n = \frac{2a}{n\pi}\left(\begin{array}{c} -\frac{L^2}{9}(0) + \frac{2L^2}{3n\pi}\sin\frac{n\pi}{3}\left(1 + \frac{(-1)^n}{2}(-1)^n\right) \\ +2\frac{L^2}{\pi^2 n^2}\left(\cos\frac{n\pi}{3}\left(1 - \frac{1}{4}\right) - 1 + \frac{1}{4}(-1)^n\right) \end{array}\right)$$

$$= \frac{2aL^2}{(n\pi)^2}\left(\sin\frac{n\pi}{3} + \frac{2}{\pi n}\left(\frac{3}{4}\cos\frac{n\pi}{3} - 1 + \frac{1}{4}(-1)^n\right)\right)$$

The first few terms are:

$n = 1$:

$$A_1 = \frac{2aL^2}{\pi^2}\left(\sin\frac{\pi}{3} + \frac{2}{\pi}\left(\frac{3}{4}\cos\frac{\pi}{3} - 1 + \frac{1}{4}(-1)^1\right)\right)$$

$$= \frac{2aL^2}{\pi^2}(0.30898) = \frac{aL^2}{\pi^2}(0.61796)$$

$n = 2$:

$$A_2 = \frac{2aL^2}{(2\pi)^2}\left(\sin\left(\frac{2\pi}{3}\right) + \frac{1}{\pi}\left(\frac{3}{4}\cos\left(\frac{2\pi}{3}\right) - 1 + \frac{1}{4}(-1)^2\right)\right)$$

$$= 0.25396a\frac{L^2}{\pi^2}$$

$n = 3$:

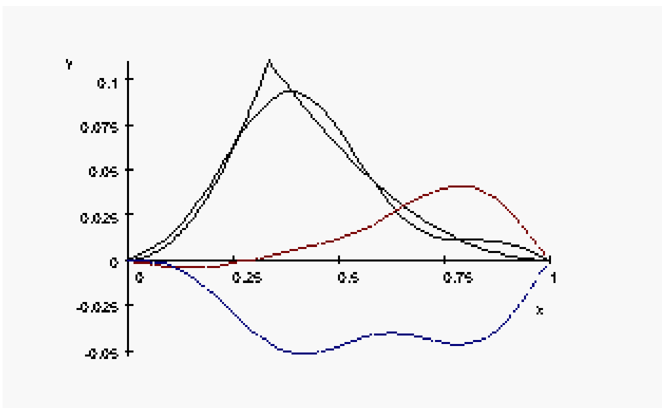
$$A_3 = \frac{2aL^2}{(3\pi)^2}\left(\sin\left(\frac{3\pi}{3}\right) + \frac{2}{\pi 3}\left(\frac{3}{4}\cos\left(\frac{3\pi}{3}\right) - 1 + \frac{1}{4}(-1)^3\right)\right)$$
$$= -9.4314\times 10^{-2} a\frac{L^2}{\pi^2}$$

$n = 4$

$$A_4 = \frac{2aL^2}{(4\pi)^2}\left(\sin\left(\frac{4\pi}{3}\right) + \frac{1}{2\pi}\left(\frac{3}{4}\cos\left(\frac{4\pi}{3}\right) - 1 + \frac{1}{4}(-1)^4\right)\right)$$
$$= -0.13063 a\frac{L^2}{\pi^2}$$

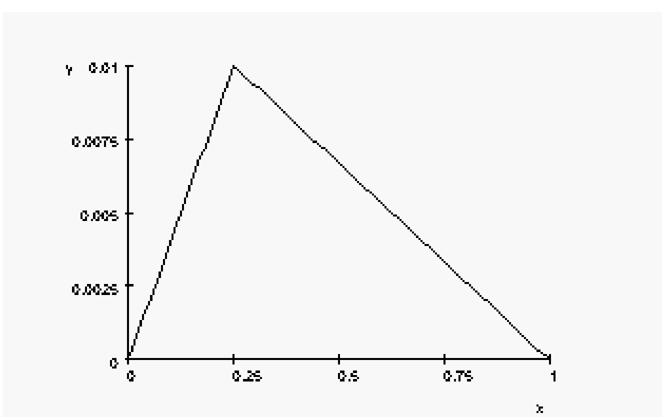
and thus the solution is:

$$y = \frac{aL^2}{\pi^2}\left(\begin{array}{c} 0.61796\sin\frac{\pi x}{L}\cos\frac{\pi vt}{L} + 0.25396\sin\frac{2\pi x}{L}\cos\frac{2\pi vt}{L} \\ -9.4314\times 10^{-2}\sin\frac{3\pi x}{L}\cos\frac{3\pi vt}{L} - 0.13063\sin\frac{4\pi x}{L}\cos\frac{4\pi vt}{L} + \cdots \end{array}\right)$$



Red: $vt/L = 0.4$ Blue $vt/L = 0.8$

17. A violin string is plucked to a triangle shape as shown in the figure, and then let go. Find the displacement of the string at later times.



The initial displacement function is:

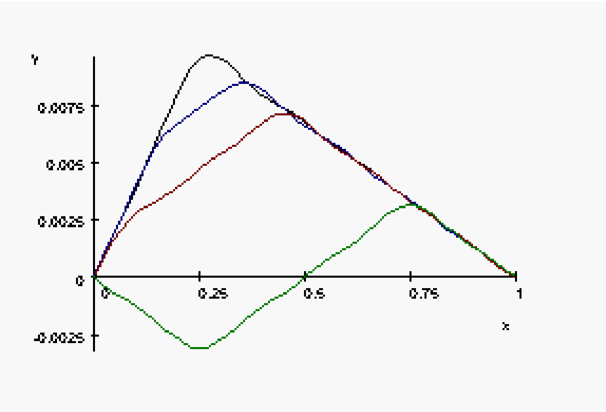
$$y(x,0) = \begin{cases} .04x & \text{if } 0 < x < L/4 \\ \frac{.04}{3}(L-x) & \text{if } L/4 < x < L \end{cases}$$

The solution is

$$y(x,t) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{L}\cos\frac{n\pi v t}{L}$$

The coefficients a_n are given by:

$$\begin{aligned} a_n &= \frac{2}{L}\int_0^L y(x,0)\sin\frac{n\pi x}{L}\,dx \\ &= \frac{2}{L}\int_0^{L/4} .04x\sin\frac{n\pi x}{L}\,dx + \frac{2}{L}\int_{L/4}^{L}\frac{.04}{3}(L-x)\sin\frac{n\pi x}{L}\,dx \\ &= \frac{.08}{L}\left\{-\frac{1}{4}L^2\frac{-4\sin\frac{1}{4}n\pi + n\pi\cos\frac{1}{4}n\pi}{n^2\pi^2} + \frac{1}{3}\left(\frac{1}{4}L^2\frac{3n\pi\cos\frac{1}{4}n\pi + 4\sin\frac{1}{4}n\pi}{n^2\pi^2}\right)\right\} \\ &= \frac{.32}{3}L\frac{\sin\frac{1}{4}n\pi}{n^2\pi^2} \end{aligned}$$



Blue: $t = L/10v$; red $t = L/5v$; green $t = L/2v$

Every fourth ($n = 4m$) coefficient is zero. These harmonics have nodes at $x = L/4$ and so are inconsistent with the given initial condition. The resulting displacement is

$$y(x,t) = \frac{.32}{3}\frac{L}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\sin\frac{n\pi}{4}\sin\frac{n\pi x}{L}\cos\frac{n\pi v t}{L}$$

18. A piano string of length L is hit by a hammer of length $\ell = L/10$. The hammer is centered at $x = L/4$ and the impulse it imparts is I. Determine the subsequent displacement of the string as a function of x and t. Which harmonics are excited?

The solution may be written as a Fourier sine series, since $y(0) = y(L) = 0$ at all times. Then

$$y(x,t) = \sum a_n(t)\sin\frac{n\pi x}{L}$$

Stuff this into the wave equation:

$$\frac{\partial^2 y}{\partial t^2} = v^2 \frac{\partial^2 y}{\partial x^2}$$

where $v^2 = T/\mu.$ Then we have:

$$\frac{d^2 a_n}{dt^2} = -v^2 \left(\frac{n\pi}{L}\right)^2 a_n$$

and so

$$a_n = A_n \cos\frac{n\pi v t}{L} + B_n \sin\frac{n\pi v t}{L}$$

Now since the string displacement is zero at $t = 0,$ we need the sine function. Thus:

$$y(x,t) = \sum B_n \sin\frac{n\pi v t}{L} \sin\frac{n\pi x}{L}$$

Immediately after the hammer hits, the string velocity is not zero. Using the impulse momentum theorem:

$$I = \Delta p = \mu \ell \frac{\partial y}{\partial t} \text{ for } \frac{L}{4} - \frac{\ell}{2} < x < \frac{L}{4} + \frac{\ell}{2}$$

Thus

$$\left.\frac{\partial y}{\partial t}\right|_{t=0} = \begin{cases} \frac{I}{\mu\ell} & \text{if} \quad \frac{1}{5}L < x < \frac{3}{10}L \\ 0 & \text{otherwise} \end{cases}$$

The left hand side is:

$$\begin{aligned} \left.\frac{\partial y}{\partial t}\right|_{t=0} &= \sum B_n \frac{n\pi v}{L} \cos\frac{n\pi v t}{L} \sin\frac{n\pi x}{L} \bigg|_{t=0} \\ &= \sum B_n \frac{n\pi v}{L} \sin\frac{n\pi x}{L} \end{aligned}$$

Thus

$$\begin{aligned} B_n \frac{n\pi v}{L} &= \frac{I}{\mu\ell} \frac{2}{L} \int_{L/5}^{3L/10} \sin\frac{n\pi x}{L} dx \\ &= \frac{20I}{\mu L^2} \frac{L}{n\pi} \left(-\cos\frac{n\pi x}{L}\right) \bigg|_{L/5}^{3L/10} \\ &= \frac{20I}{\mu L n\pi} \left(\cos\frac{n\pi}{5} - \cos\frac{3n\pi}{10}\right) \\ &= \frac{20I}{\mu L n\pi} 2 \sin\frac{n\pi}{4} \sin\frac{n\pi}{20} \end{aligned}$$

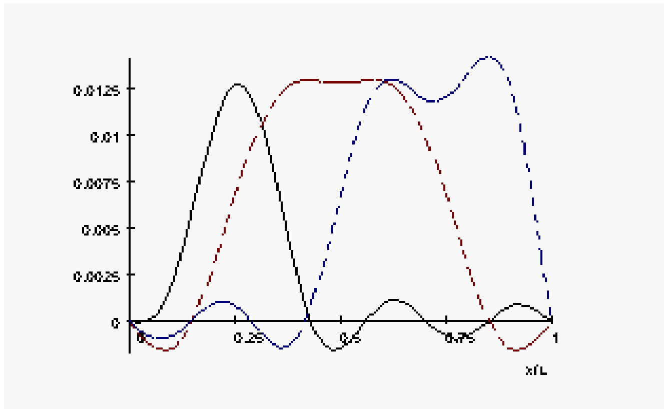
Thus:

$$\begin{aligned} B_n &= \frac{L}{n\pi v} \frac{40I}{\mu L n\pi} \sin\frac{n\pi}{4} \sin\frac{n\pi}{20} \\ &= \frac{40}{n^2\pi^2} \frac{I}{\mu v} \sin\frac{n\pi}{4} \sin\frac{n\pi}{20} \end{aligned}$$

and the displacement of the string is:

$$y(x,t) = \frac{40I}{\mu v} \sum_{n=1}^{\infty} \frac{1}{n^2\pi^2} \sin\frac{n\pi}{4} \sin\frac{n\pi}{20} \sin\frac{n\pi v t}{L} \sin\frac{n\pi x}{L}$$

Since $\sin \frac{n\pi}{4} = 10$ for $n = 4m,$ these harmonics are all missing. The second sine is zero when $n = 20m,$ but these values are included in our first set.



vt/L = 0.1 (black), 0.5 (red) and 0.75 (blue). The vertical axis is the dimensionless variable $\mu vy/40I.$

19. Fourier series may be used to evaluate certain series of integers**.** To illustrate the method, develop the Fourier series for the function x^2 on the range $-\pi$ to $\pi.$ Set $x = 0$ and hence evaluate:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2} = 1 - \frac{1}{4} + \frac{1}{9} + \cdots$$

Which sum do you obtain by setting $n = \pi?$ Finally, use Parseval's theorem to evaluate

$$\sum_{n=1}^{\infty} \frac{1}{n^4}$$

The function is even on this range, and so the series has the form:

$$f(x) = x^2 = \sum a_n \cos nx$$

where the argument x of the cosines varies from $-\pi$ to π . Then

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} x^2 \cos nx\,dx \\ &= \frac{1}{\pi}\left[x^2 \frac{\sin nx}{n}\Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} 2x \frac{\sin nx}{n}dx\right] \\ &= -\frac{2}{\pi n}\left[0 + x\frac{-\cos nx}{n}\Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi}\frac{-\cos nx}{n}dx\right] \\ &= \frac{2}{\pi n^2}\left[2\pi(-1)^n + \frac{\sin nx}{n}\Big|_{-\pi}^{\pi}\right] = \frac{4}{n^2}(-1)^n \end{aligned}$$

Finally:

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} x^2 dx = \frac{1}{\pi}\frac{x^3}{6}\Big|_{-\pi}^{\pi} = \frac{\pi^2}{3}$$

Thus the series is:

$$x^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4}{n^2}(-1)^n \cos nx$$

Setting $x = 0$ gives:

$$0 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$$

and hence

$$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^2} = \frac{\pi^2}{12}$$

Setting $x = \pi$ we obtain:

$$\pi^2 = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4}{n^2}(-1)^n(-1)^n$$

and so

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{1}{4}\frac{2\pi^2}{3} = \frac{\pi^2}{6}$$

Writing the series in exponential form, we have:

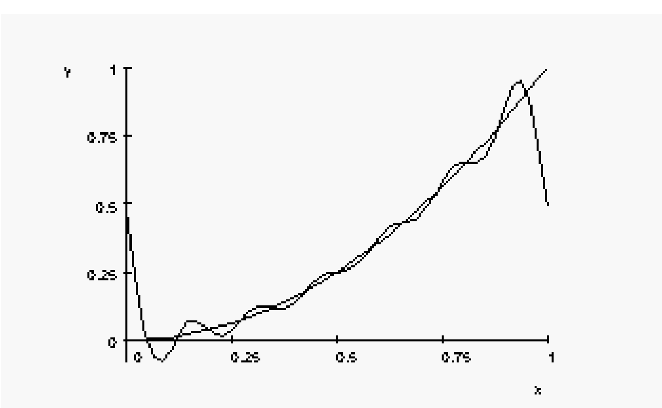
$$\begin{aligned} x^2 &= \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4}{n^2}(-1)^n\frac{e^{inx}+e^{-inx}}{2} \\ &= \frac{\pi^2}{3} + \left(\sum_{n=-\infty}^{-1} + \sum_{n=1}^{\infty}\right)\frac{2}{n^2}(-1)^n e^{inx} \end{aligned}$$

We have

$$\begin{aligned} \frac{1}{2\pi}\int_{-\pi}^{\pi}x^4\,dx &= \sum_{n=-\infty}^{\infty}|c_n|^2 \\ \frac{1}{2\pi}\frac{x^5}{5}\bigg|_{-\pi}^{\pi} &= \left(\frac{\pi^2}{3}\right)^2 + 2\sum_{n=1}^{\infty}\frac{4}{n^4} \\ \frac{\pi^4}{5} &= \frac{\pi^4}{9} + 8\sum_{n=1}^{\infty}\frac{1}{n^4} \end{aligned}$$

and thus

$$\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{1}{8}\left(\frac{\pi^4}{5} - \frac{\pi^4}{9}\right) = \frac{\pi^4}{90}$$



Sum of first 6 terms

20. Use the Fourier series for the step function to evaluate the sum

$$\sum_{m=0}^{\infty} \frac{(-1)^m}{2m+1}$$

Use Parseval's theorem applied to the same series to obtain the sum

$$\sum_{m=0}^{\infty} \frac{1}{(2m+1)^2}$$

The series we need is equation 4.10 (or 4.13). Setting $x = \frac{1}{4}$, we get:

$$0 = \frac{1}{2} - \frac{2}{\pi} \sum_{m=0}^{\infty} \frac{\sin(2m+1)\pi/2}{2m+1}$$

Thus

$$\frac{1}{2} = \frac{2}{\pi} \sum_{m=0}^{\infty} \frac{(-1)^m}{2m+1}$$

and thus

$$\sum_{m=0}^{\infty} \frac{(-1)^m}{2m+1} = \frac{\pi}{4}$$

Using Parseval's theorem applied to equation 4.13:

$$\int_0^1 f(x)^2 dx = \int_{1/2}^1 1dx = \frac{1}{2} = \sum_{n=-\infty}^{\infty} |c_n|^2 = \frac{1}{4} + \frac{1}{\pi^2} \sum_{n=-\infty}^{\infty} \frac{1}{(2n+1)^2}$$

and so

$$\sum_{n=0}^{\infty} \frac{1}{(2n+1)^2} = \frac{1}{4}\frac{\pi^2}{2} = \frac{\pi^2}{8}$$

Chapter 4: Fourier Series

21. A function $f(x)$ is represented by the Fourier series:

$$f(x) = \sum_{n=0}^{\infty}\left(a_n \sin\frac{n\pi x}{L} + b_n \cos\frac{n\pi x}{L}\right)$$

on the range $(-L, L)$. Derive a form of Parseval's Theorem (equation 4.19) applicable to this series, that is, express $\int_{-L}^{+L} f(x)^2 dx$ in terms of the coefficients a_n and b_n.

$$\begin{aligned}\frac{1}{L}\int_0^{2L} f(x)^2 dx &= \frac{1}{L}\int_0^{2L}\sum_{n=0}^{\infty}\left(a_n \sin\frac{n\pi x}{L} + b_n \cos\frac{n\pi x}{L}\right) \\ &\quad \times \sum_{m=0}^{\infty}\left(a_m \sin\frac{m\pi x}{L} + b_m \cos\frac{m\pi x}{L}\right) dx \\ &= 2b_0^2 + \sum_{n=1}^{\infty}\left(a_n^2 + b_n^2\right)\end{aligned}$$

where we used orthogonality of the sines and cosines to evaluate the integrals on the right hand side.

22. If $f(x)$ is represented by the series $\sum f_n e^{inx}$ over the interval $0 < x < 2\pi$, and $g(x) = \sum g_n e^{inx}$ over the same range, prove the generalized Parseval theorem:

$$\frac{1}{2\pi}\int_0^{2\pi} f(x)g(x)dx = \sum_n f_n g_{-n} = \sum_n f_n g_n^*$$

where the second expression applies when the function $g(x)$ is real.

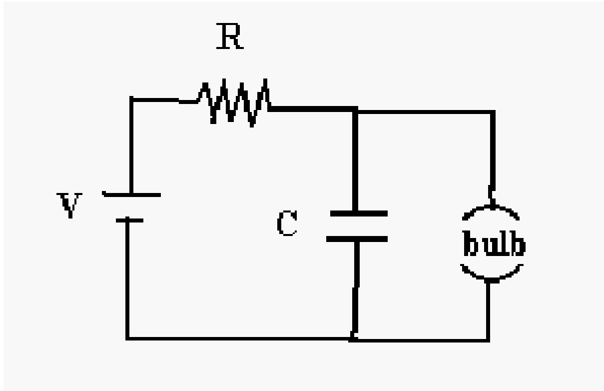
The integral on the right hand side may be writtem in terms of the two series:

$$\begin{aligned}\frac{1}{2\pi}\int_0^{2\pi} f(x)g(x)dx &= \frac{1}{2\pi}\int_0^{2\pi}\sum_n f_n e^{inx}\sum_m g_m e^{imx} dx \\ &= \frac{1}{2\pi}\sum_n\sum_m g_m f_n \int_0^{2\pi} e^{inx} e^{imx} dx \\ &= \frac{1}{2\pi}\sum_n\sum_m g_m f_n 2\pi\delta_{n,-m} \\ &= \sum_n f_n g_{-n}\end{aligned}$$

23. The capacitor shown in the figure is charged by the battery, and discharges through the bulb when the potential across it equals $0.9V$. Assuming that the capacitor discharges very rapidly, show that the potential across the capacitor as a function of time is:

$$V_C = V\left(1 - e^{-t/RC}\right) \quad 0 < t < RC\ln 10$$

and repeats periodically with period $T = RC\ln 10$. Find a Fourier series with period T that represents this function.



To find the period we first find when the capacitor voltage reaches 0.9V.
$0.9 = 1 - e^{-t/RC} \Rightarrow .1 = e^{-t/RC} \Rightarrow 10 = e^{t/RC} \Rightarrow t = RC\ln 10.$

$$V(t) = V\left(1 - e^{-t/RC}\right) = V\sum_{n=-\infty}^{\infty} c_n e^{in2\pi t/T}$$

The coefficients are:

$$\begin{aligned}
c_n &= \frac{1}{T}\int_0^T\left(1 - \exp\left(-\frac{t\ln 10}{T}\right)\right)e^{-in2\pi t/T}dt \\
&= \frac{1}{T}\left\{\frac{e^{-in2\pi t/T}}{-in2\pi/T} - \frac{\exp\left(-\frac{t[\ln 10 + in2\pi]}{T}\right)}{-[\ln 10 + in2\pi]/T}\right\}\Bigg|_0^T \\
&= \frac{e^{-in2\pi} - 1}{-in2\pi} - \frac{\exp(-[\ln 10 + in2\pi]) - 1}{-[\ln 10 + in2\pi]} \\
&= \frac{1/10 - 1}{\ln 10 + in2\pi} = -\frac{9/10}{\ln 10 + in2\pi} = -\frac{9}{10}\frac{\ln 10 - in2\pi}{(\ln 10)^2 + 4\pi^2 n^2}
\end{aligned}$$

When $n = 0$, we have

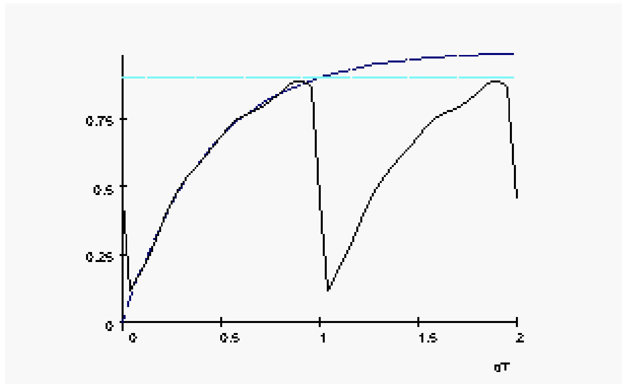
$$\begin{aligned}
c_0 &= \frac{1}{T}\int_0^T\left(1 - \exp\left(-\frac{t\ln 10}{T}\right)\right)dt = \frac{1}{T}\left(T - \frac{\exp\left(-\frac{t\ln 10}{T}\right)}{-(\ln 10)/T}\Bigg|_0^T\right) \\
&= 1 + \frac{e^{-\ln 10} - 1}{\ln 10} = 1 + \frac{1/10 - 1}{\ln 10} = 1 - \frac{9}{10\ln 10}
\end{aligned}$$

Thus

$$\frac{V(t)}{V} = 1 - \frac{9}{10\ln 10} + \frac{9}{10}\sum_{\substack{n=-\infty \\ n\neq 0}}^{\infty}\frac{in2\pi - \ln 10}{(\ln 10)^2 + 4\pi^2 n^2}e^{in2\pi t/T}$$

Now we combine terms to get a real series:

$$\begin{aligned}
V(t) &= \frac{9}{10}V\left\{\frac{10}{9} - \frac{1}{\ln 10} + \sum_{n=1}^{\infty}\frac{in2\pi(e^{in2\pi t/T} - e^{-in2\pi t/T})}{(\ln 10)^2 + 4\pi^2 n^2} - \frac{\ln 10(e^{in2\pi t/T} + e^{-in2\pi t/T})}{(\ln 10)^2 + 4\pi^2 n^2}\right\} \\
&= \frac{9}{10}V\left\{\frac{10}{9} - \frac{1}{\ln 10} + \sum_{n=1}^{\infty}\frac{in2\pi}{(\ln 10)^2 + 4\pi^2 n^2}2i\sin\frac{2\pi nt}{T} - \frac{\ln 10}{(\ln 10)^2 + 4\pi^2 n^2}2\cos\frac{2\pi nt}{T}\right\} \\
&= \frac{9}{10}V\left\{\frac{10}{9} - \frac{1}{\ln 10} - \sum_{n=1}^{\infty}\frac{n4\pi\sin\frac{2\pi nt}{T} + 2\ln 10\cos\frac{2\pi nt}{T}}{(\ln 10)^2 + 4\pi^2 n^2}\right\}
\end{aligned}$$



The plot shows the first twenty terms in the series solution, as well as the function $V(t)/V = 1 - e^{-t/RC}$.

24. A rectangular box of dimensions $a \times b \times a$ has conducting walls. All the walls are grounded, except for the one at $y = b$. This wall is separated from the others by a thin insulating strip, and it is at potential V. Using the method illustrated in Chapter 3 Example 3.15, find the potential everywhere inside the box.

Following Example 3.15, the solution is

$$\Phi(x,y,z) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} a_n \sin\frac{n\pi x}{a}\sin\frac{m\pi z}{a}\sinh\left(\frac{\sqrt{n^2+m^2}\,\pi y}{a}\right)$$

Evauating this expression at $y = b$, we have:

$$\Phi(x,b,z) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} a_n \sin\frac{n\pi x}{a}\sin\frac{m\pi z}{a}\sinh\left(\frac{\sqrt{n^2+m^2}\,\pi b}{a}\right) = V$$

and thus the coefficients are:

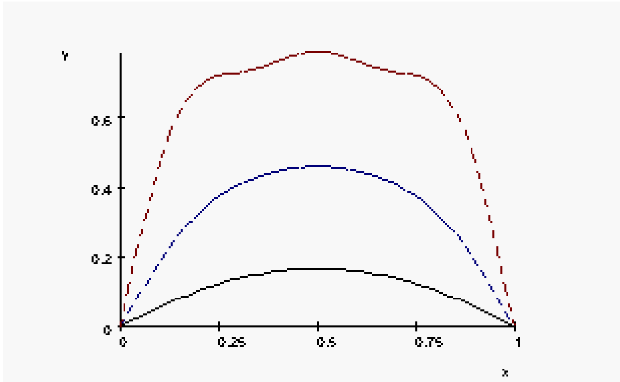
$$a_n \sinh\left(\frac{\sqrt{n^2+m^2}\,\pi b}{a}\right) = \frac{2}{a}V\int_0^a \sin\frac{n\pi x}{a}dx\,\frac{2}{a}\int_0^a \sin\frac{m\pi z}{a}dz$$
$$= \frac{4}{a^2}a^2 V\frac{1-\cos n\pi}{n\pi}\,\frac{1-\cos m\pi}{m\pi} = 4V\left(\frac{1-(-1)^n}{n\pi}\right)\left(\frac{1-(-1)^m}{m\pi}\right)$$

Thus only odd n and odd m terms contribute

$$\Phi(x,y,z) = \frac{16V}{\pi^2}\sum_{n=1,\text{ odd}}^{\infty}\ \sum_{m=1,\text{ odd}}^{\infty} \sin\frac{n\pi x}{a}\sin\frac{m\pi z}{a}\frac{\sinh\left(\frac{\sqrt{n^2+m^2}\,\pi y}{a}\right)}{nm\sinh\frac{\sqrt{n^2+m^2}\,\pi b}{a}}$$

At $z = a/2$, the solution is

$$\Phi\left(x,y,\frac{a}{2}\right) = \frac{16V}{\pi^2}\sum_{n=1,\text{ odd}}^{\infty}\ \sum_{m=1,\text{ odd}}^{\infty} \sin\frac{n\pi x}{a}\sin\frac{m\pi}{2}\frac{\sinh\left(\frac{\sqrt{n^2+m^2}\,\pi y}{a}\right)}{nm\sinh\frac{\sqrt{n^2+m^2}\,\pi b}{a}}$$
$$= \frac{16V}{\pi^2}\sum_{n=1,\text{ odd}}^{\infty}\ \sum_{p=0}^{\infty} \sin\frac{n\pi x}{a}(-1)^p\frac{\sinh\left(\sqrt{n^2+(2p+1)^2}\,\frac{y}{a}\right)}{n(2p+1)\sinh\left(\sqrt{n^2+(2p+1)^2}\,\frac{\pi b}{a}\right)}$$



With $b = a$, Solid line $y = a/2$, dashed blue line $y = 3a/4$, dotted red line $y = 9a/10$. The plot shows the first three nonzero values of n and m.

25. A rectangular box measuring $a \times b \times c$ has all its walls at temperature T_1 except for the one at $z = c$, which is held at temperature T_2. When the box comes to equilibrium, the temperature function $T(x,y)$ satisfies equation 3.14

$$\frac{\partial T}{\partial t} = D\nabla^2 T$$

with the time derivative on the left equal to zero. Use the method of Chapter 3 Example 15 to find the temperature T in the box in the form

$$T(x,y,z) = T_1 + \tau(x,y,z)$$

where τ is expressed in a Fourier series

$$\tau(x,y) = \sum_{n,m} a_{nm} \sin\frac{n\pi x}{a} \sin\frac{m\pi y}{b} f(z)$$

Find the function $f(z)$ and the coefficients a_{mn}.

Since the differential equation reduces to

$$\tau(x,y,z) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} a_{nm} \sin\frac{n\pi x}{a} \sin\frac{m\pi y}{b} \sinh\left(\sqrt{\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2}\, z\right)$$

and on the side at $z = c$,

$$T_2 - T_1 = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} a_{nm} \sin\frac{n\pi x}{a} \sin\frac{m\pi y}{b} \sinh\left(\sqrt{\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2}\, c\right)$$

Thus

$$\begin{aligned} a_{nm} \sinh\left(\sqrt{\left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2}\, c\right) &= \frac{2}{a}\frac{2}{b}\int_0^a\int_0^b \sin\frac{n\pi x}{a} \sin\frac{m\pi y}{b}(T_2 - T_1)\,dx\,dy \\ &= \frac{4}{ab}(T_2 - T_1)a\frac{1-\cos n\pi}{n\pi} b\frac{1-\cos m\pi}{m\pi} \\ &= \frac{4(T_2 - T_1)}{nm\pi^2}(1-(-1)^n)(1-(-1)^m) \end{aligned}$$

Thus only odd values of n and n have non-zero coefficients. The temperature function is:

$$T(x,y,z) = T_1 + (T_2 - T_1)\frac{16}{\pi^2}\sum_{n=1,\text{ odd}}^{\infty}\sum_{m=1,\text{ odd}}^{\infty}\frac{\sin\frac{n\pi x}{a}\sin\frac{m\pi y}{b}}{nm}\frac{\sinh\left(\sqrt{\left(\frac{n\pi}{a}\right)^2+\left(\frac{m\pi}{b}\right)^2}\,z\right)}{\sinh\left(\sqrt{\left(\frac{n\pi}{a}\right)^2+\left(\frac{m\pi}{b}\right)^2}\,c\right)}$$

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26. A infinitely long, conducting tube with circular cross section of radius a is divided into four pieces by insulating strips running along its length. One of the four pieces is at potential V, and the other three are grounded. Solve Laplace's equation in two dimensions using the method of Example 3.15. Evaluate the solution at $\rho = a$ and show that the result is a Fourier series. Determine the coefficients, and hence find the potential everywhere inside the tube.

The equation is (equation 1.44)

$$\nabla^2\Phi = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\Phi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\Phi}{\partial\phi^2} = 0$$

Using separation of variables, let $\Phi = R(\rho)W(\phi)$

$$\frac{\rho}{R}\frac{\partial}{\partial\rho}\left(\rho R'\right) + \frac{W''}{W} = 0$$

Thus the solution for $W = e^{im\phi}$. We choose this function because it is periodic with period ϕ, and thus the potential at ϕ is the same as at $\phi + 2\pi$, as it must be, since both values represent the same physical point. Then the equation for R is:

$$\rho\frac{\partial}{\partial\rho}\left(\rho R'\right) - m^2R = 0$$

The solution is a power, $R = \rho^p$, where

$$\rho\frac{\partial}{\partial\rho}(p\rho^p) = p^2\rho^p = m^2\rho^p$$

Thus

$$p = \pm m$$

and the general solution is of the form

$$\Phi = \sum_m (a_m\rho^m + b_m\rho^{-m})e^{im\phi}$$

We must eliminate the negative powers because the solution is finite at $\rho = 0$. Thus

$$\Phi(\rho,\phi) = \sum_m \rho^m\left(a_m e^{im\phi} + b_m e^{-im\phi}\right)$$

and at $\rho = a$

$$\Phi(a,\phi) = \sum_m a_m a^m e^{im\phi}$$

a Fourier series for Φ.

Equivalently, we may write the series in the form:

$$\Phi(\rho,\phi) = \sum_m \rho^m (c_m \cos m\phi + d_m \sin m\phi)$$

To find the coefficients we use the given values for Φ

$$\sum_m a_m a^m e^{im\phi} = \begin{cases} V & if \;\; 0 < \phi < \pi/2 \\ 0 & if \;\; \pi/2 < \phi < 2\pi \end{cases}$$

Thus

$$\begin{aligned} a_m &= \frac{1}{2\pi a^m} \int_0^{\pi/2} V e^{-im\phi} d\phi = \frac{V}{2\pi a^m} \frac{e^{-im\phi}}{-im} \Big|_0^{\pi/2} \\ &= \frac{V}{2\pi a^m} \frac{e^{-im\pi/2} - 1}{-im} = \frac{V}{2\pi a^m} \frac{(-i)^m - 1}{-im} \end{aligned}$$

If m is even, $m = 2n$, we have

$$a_{2n} = \frac{V}{4\pi n a^{2n}} \frac{(-1)^n - 1}{-i} = \frac{V}{2\pi n a^{2n} i} \begin{cases} 0 & \text{if } \; n \text{ is even} \\ 1 & \text{if } \; n \text{ is odd} \end{cases}$$

If m is odd, we have

$$\begin{aligned} a_{2n+1} &= \frac{V}{2\pi a^{2n+1}} \frac{(-i)^{2n+1} - 1}{-i(2n+1)} = \frac{V}{2\pi a^{2n+1}} \frac{(-i)(-1)^n - 1}{-i(2n+1)} \\ &= \frac{V}{2\pi a^{2n+1}} \frac{(-1)^n + i}{(2n+1)} \end{aligned}$$

While for $m = 0$:

$$a_m = \frac{1}{2\pi} \int_0^{\pi/2} V d\phi = \frac{V}{4}$$

Thus

$$\begin{aligned} \Phi(\rho,\phi) &= \frac{V}{4} + \frac{V}{2\pi} \left(\begin{array}{c} \sum_{n=0}^{\infty} \left(\frac{\rho}{a}\right)^{2(2n+1)} \left(\frac{e^{i2(2n+1)\phi} - e^{-i2(2n+1)\phi}}{2(2n+1)i} \right) \\ + \sum_{n=0}^{\infty} \left(\frac{\rho}{a}\right)^{2n+1} \left\{ \frac{(-1)^n + i}{(2n+1)} e^{i(2n+1)\phi} - \frac{-(-1)^n + i}{(2n+1)} e^{-i(2n+1)\phi} \right\} \end{array} \right) \\ &= \frac{V}{4} + \frac{V}{2\pi} \left(\begin{array}{c} \sum_{n=0}^{\infty} \left(\frac{\rho}{a}\right)^{2(2n+1)} \frac{\sin 2(2n+1)\phi}{2n+1} \\ + \sum_{n=0}^{\infty} \left(\frac{\rho}{a}\right)^{2n+1} 2 \left\{ \frac{(-1)^n \cos(2n+1)\phi - \sin(2n+1)\phi}{(2n+1)} \right\} \end{array} \right) \end{aligned}$$

Alternatively, using the series in sines and cosines, we have

$$c_m = \frac{1}{\pi a^m}\int_0^{\pi/2}\cos m\phi\, d\phi = \frac{1}{\pi}\frac{\sin m\phi}{m}\Big|_0^{\pi/2}$$

$$= \frac{1}{m\pi}\left(\sin\frac{m\pi}{2}\right) = \frac{1}{m\pi}\begin{cases} 0 & \text{if } m \text{ is even} \\ (-1)^{(m-1)/2} & \text{if } m \text{ is odd}\end{cases}$$

and

$$d_m = \frac{1}{\pi a^m}\int_0^{\pi/2}\sin m\phi\, d\phi = \frac{-1}{\pi}\frac{\cos m\phi}{m}\Big|_0^{\pi/2}$$

$$= \frac{1}{m\pi}\left(1 - \cos\frac{m\pi}{2}\right) = \frac{1}{m\pi}\begin{cases} 1-(-1)^{m/2} & \text{if } m \text{ is even} \\ 1 & \text{if } m \text{ is odd}\end{cases}$$

Finally,

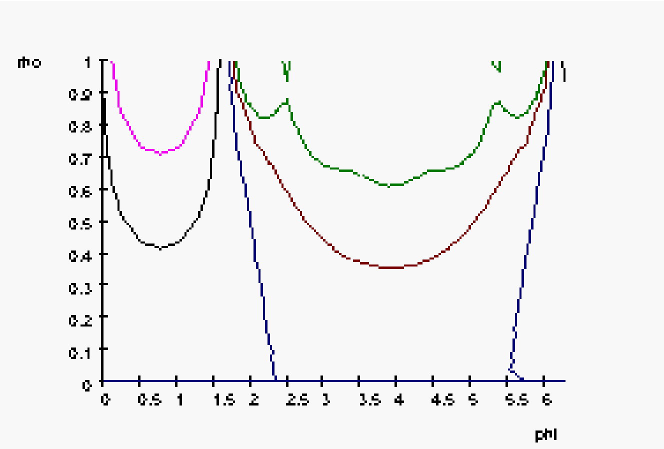
$$c_0 = \frac{1}{2\pi}\left(V\frac{\pi}{2}\right) = \frac{V}{4}$$

Thus

$$\Phi(\rho,\phi) = \frac{V}{4} + \frac{V}{\pi}\sum_{n=0}^{\infty}\left(\frac{\rho}{a}\right)^{2n+1}\frac{(-1)^n\cos(2n+1)\phi}{2n+1}$$
$$+ \frac{V}{\pi}\sum_{n=0}^{\infty}\frac{2}{2(2n+1)}\left(\frac{\rho}{a}\right)^{2(2n+1)}\sin 2(2n+1)\phi$$
$$+ \frac{V}{\pi}\sum_{n=0}^{\infty}\frac{1}{(2n+1)}\left(\frac{\rho}{a}\right)^{2n+1}\sin(2n+1)\phi$$
$$= \frac{V}{4} + \frac{V}{\pi}\sum_{n=0}^{\infty}\left(\frac{\rho}{a}\right)^{2n+1}\frac{(-1)^n\cos(2n+1)\phi + \sin(2n+1)\phi}{2n+1}$$
$$+ \left(\frac{\rho}{a}\right)^{2(2n+1)}\frac{\sin 2(2n+1)\phi}{2n+1}$$

Both solutions are the same, of course.

The plot shows contours of constant potential.



black 1/2 blue 1/4 red 1/8 green 1/16 purple

27. A Fourier series of the form

$$f(x) = \sum c_n e^{inx}$$

may be expressed as a power series

$$f(x) = \sum c_n z^n$$

where $z = \lim_{r\to 1} re^{ix}$. By identifying the power series, the function $f(x)$ may be identified. Use this technique to sum the Fourier series

$$\sum_{n=1}^{\infty} \frac{\sin nx}{n}$$

where $0 < x < \pi.$ Check your result by evaluating the Fourier sine series of the function you found.

Let $z_1 = \lim_{r\to 1} re^{ix}$ and $z_2 = \lim_{r\to 1} re^{-ix}$

$$f(x) = \sum_{n=1}^{\infty} \frac{e^{inx} - e^{-inx}}{2in} = \frac{1}{2i} \sum_{n=1}^{\infty} \frac{1}{n}(z_1^n - z_2^n)$$

For $r < 1$ we may sum the series as follows:

$$\begin{aligned} 2if(x) &= \int_0^{z_1} \sum_{n=0}^{\infty} z^n dz - \int_0^{z_2} \sum_{n=0}^{\infty} z^n dz = \int_0^{z_1} \left(\frac{1}{1-z} - 1\right) dz - \int_0^{z_2} \left(\frac{1}{1-z} - 1\right) dz \\ &= -\ln(1 - z_1) + \ln(1 - z_2) \\ &= \ln\left(\frac{1 - re^{-ix}}{1 - re^{ix}}\right) \end{aligned}$$

Now take the limit $r \to 1$ from below.

$$f(x) = \frac{1}{2i} \ln \frac{1 - e^{-ix}}{1 - e^{ix}}$$

We want to write the argument of the log in the form

$$z = \rho e^{i\phi}$$

where

$$z = \frac{1 - e^{-ix}}{1 - e^{ix}} \frac{1 - e^{-ix}}{1 - e^{-ix}} = \frac{1 - 2e^{-ix} + e^{-2ix}}{2 - 2\cos x} = \frac{1 - 2(\cos x - i\sin x) + \cos 2x - i\sin 2x}{2 - 2\cos x}$$

Thus

$$\rho = |z| = \sqrt{zz^*} = \frac{1-e^{-ix}}{1-e^{ix}}\frac{1-e^{ix}}{1-e^{-ix}} = 1$$

and

$$\begin{aligned}\tan\phi &= \frac{\operatorname{Im} z}{\operatorname{Re} z} = \frac{2\sin x - \sin 2x}{1 - 2\cos x + \cos 2x} \\ &= \frac{2\sin x - 2\sin x\cos x}{1 - 2\cos x + 2\cos^2 x - 1} \\ &= \frac{2\sin x(1-\cos x)}{-2\cos x(1-\cos x)} \\ &= -\tan x\end{aligned}$$

One solution of this equation is $\phi = -x,$ but there are also other possibilities. Since $\sin(\pi \pm x) = \mp\sin x$ and $\cos(\pi \pm x) = -\cos x,$ we might choose $\phi = \pi - x$ or $\phi = -(\pi + x).$ If $0 < x \leq \pi/2,$ $\sin x$ is positive and $\cos x$ is also positive. Since $1 - \cos x \geq 0$ for all $x,$ z has a positive imaginary part and a negative real part, so ϕ must be in the second quadrant, and $\phi = \pi - x.$ If $\pi/2 < x \leq \pi,$ then $\sin x$ is positive but $\cos x$ is negative, so z has a positive imaginary part and a positive real part, so ϕ must be in the first quadrant. Again the appropriate value is $\phi = \pi - x.$

Putting all this together, we get

$$\begin{aligned}f(x) &= \frac{1}{2i}\ln\left(\rho e^{i\phi}\right) = \frac{1}{2i}[\ln 1 + i\phi] \\ &= \frac{\phi}{2} = \frac{\pi - x}{2} \quad \text{for } 0 < x < \pi\end{aligned}$$

The function is odd in x, since the series is a sine series.

Check by evaluating the Fourier coefficients of our function $f(x).$

$$\begin{aligned}f(x) &= \sum a_n \sin nx \\ a_n &= \frac{2}{\pi}\int_0^\pi \frac{\pi - x}{2}\sin nx\,dx \\ &= \int_0^\pi \left(1 - \frac{x}{\pi}\right)\sin nx\,dx \\ &= \frac{-\cos nx}{n}\Big|_0^\pi - \frac{1}{\pi}\left[-x\frac{\cos nx}{n}\Big|_0^\pi + \int_0^\pi \frac{\cos nx}{n}dx\right] \\ &= \frac{1-(-1)^n}{n} + \frac{(-1)^n}{n} = \frac{1}{n}\end{aligned}$$

as required.

Chapter 5: Laplace Transforms

1. Show that the following functions are of exponential order, and find their Laplace transforms.

(a) $f(t) = \sinh \alpha t$

Since the function is a linear combination of exponentials, it is of exponential order α.

$$\begin{aligned} F(s) &= \int_0^\infty \sinh \alpha t e^{-st} dt \\ &= \int_0^\infty \frac{e^{\alpha t} - e^{-\alpha t}}{2} e^{-st} dt \\ &= \frac{1}{2}\left(\frac{e^{-(s-\alpha)t}}{-(s-\alpha)} - \frac{e^{-(s+\alpha)t}}{-(s+\alpha)} \right) \Big|_0^\infty \\ &= \frac{1}{2}\left(\frac{1}{(s-\alpha)} - \frac{1}{(s+\alpha)} \right) \text{ for } s > \alpha \\ &= \frac{\alpha}{s^2 - \alpha^2} \end{aligned}$$

(b) $f(t) = \tanh \alpha t$

$$\begin{aligned} \lim_{t\to\infty} e^{-pt} \tanh \alpha t &= \lim_{t\to\infty} e^{-pt} \frac{e^{\alpha t} - e^{-\alpha t}}{e^{\alpha t} + e^{-\alpha t}} \\ &= \lim_{t\to\infty} e^{-pt} \frac{1 - e^{-2\alpha t}}{1 + e^{-2\alpha t}} \to 0 \text{ for } p > 0 \end{aligned}$$

Thus $\tanh \alpha t$ is of exponential order 0.

$$\begin{aligned} F(s) &= \int_0^\infty \tanh \alpha t e^{-st} dt \\ &= \int_0^\infty \frac{e^{\alpha t} - e^{-\alpha t}}{e^{\alpha t} + e^{-\alpha t}} e^{-st} dt \\ &= \int_0^\infty \frac{1 - e^{-2\alpha t}}{1 - e^{-2\alpha t}} e^{-\frac{s}{\alpha}\alpha t} dt \end{aligned}$$

Now let $e^{-2\alpha t} = u$. Then $du = -2\alpha e^{-2\alpha t} dt = -2\alpha u dt$

$$\begin{aligned} F(s) &= -\int_1^0 \frac{1-u}{1+u} u^{s/2\alpha} \frac{du}{2\alpha u} \\ &= \frac{1}{2\alpha} \int_0^1 u^{s/2\alpha - 1} (1-u)\left(1 - u + u^2 + \cdots (-1)^n u^n + \cdots\right) du \\ &= \frac{1}{2\alpha} \int_0^1 u^{s/2\alpha - 1} \left(1 - 2u + 2u^2 - 2u^3 + \cdots 2(-1)^n u^n + \cdots\right) du \\ &= \frac{1}{2\alpha} \left(\frac{u^{s/2\alpha}}{\frac{s}{2\alpha}} - 2\frac{u^{s/2\alpha+1}}{s/2\alpha + 1} + \cdots 2(-1)^n \frac{u^{s/2\alpha+n}}{\frac{s}{2\alpha} + n} + \cdots \right) \Big|_0^1 \\ &= \frac{1}{2\alpha} \left(\frac{2\alpha}{s} - 2\frac{1}{\frac{s}{2\alpha} + 1} + \cdots 2(-1)^n \frac{1}{\frac{s}{2\alpha} + n} + \cdots \right) \\ &= \frac{1}{s} - \frac{2}{s + 2\alpha} + \cdots 2(-1)^n \frac{1}{s + 2\alpha n} + \cdots \end{aligned}$$

Check: Inverting gives

$$\begin{aligned} f(t) &= 1 - e^{-2\alpha t} + \cdots 2(-1)^n e^{-2\alpha n t} + \cdots \\ &= \frac{\left(1 - e^{-2\alpha t}\right)}{1 + e^{-2\alpha t}} = \tanh \alpha t \end{aligned}$$

as required.

(c) $f(t) = \sin \sqrt{\alpha t}$

$$\left|\sin\sqrt{at}\right| \le 1 \text{ for all real } a \text{ and } t$$

Thus $f(t)$ is of exponential order 0.

$$F(s) = \int_0^\infty \sin\sqrt{at}\, e^{-st} dt$$

Let $\sqrt{at} = u$ Then $at = u^2$ and $adt = 2udu$

$$\begin{aligned} F(s) &= \int_0^\infty \sin u e^{-su^2/a} \frac{2u}{a} du \\ &= \frac{1}{s}\int_0^\infty \sin u d\left(-e^{-su^2/a}\right) \\ &= \frac{1}{s}\left(-\sin u e^{-su^2/a}\Big|_0^\infty + \int_0^\infty \cos u e^{-su^2/a} du\right) \\ &= \frac{1}{2s}\int_0^\infty \left(e^{iu} + e^{-iu}\right) e^{-su^2/a} du \end{aligned}$$

Now we complete the square:

$$\begin{aligned} -\frac{s}{a}u^2 + iu &= -\frac{s}{a}\left(u^2 - \frac{ia}{s}u\right) = -\frac{s}{a}\left(u^2 - \frac{ia}{s}u + \left(\frac{ia}{2s}\right)^2 - \left(\frac{ia}{2s}\right)^2\right) \\ &= -\frac{s}{a}\left(\left(u - \frac{ia}{2s}\right)^2 + \left(\frac{a}{2s}\right)^2\right) \end{aligned}$$

and similarly for the other term. Thus:

$$\begin{aligned} F(s) &= \frac{1}{2s}\int_0^\infty \left[\exp\left(-\frac{s}{a}\left(\left(u - \frac{ia}{2s}\right)^2 + \left(\frac{a}{2s}\right)^2\right)\right) + \exp\left(-\frac{s}{a}\left(\left(u + \frac{ia}{2s}\right)^2 + \left(\frac{a}{2s}\right)^2\right)\right)\right] du \\ &= \frac{1}{2s}\exp\left(-\frac{a}{4s}\right)\left[\int_{0-\frac{ia}{2s}}^{\infty-\frac{ia}{2s}} \exp\left(-\frac{s}{a}x^2\right)dx + \int_{0+\frac{ia}{2s}}^{\infty+\frac{ia}{2s}} \exp\left(-\frac{s}{a}x^2\right)dx\right] \\ &= \frac{1}{s}\exp\left(-\frac{a}{4s}\right)\frac{\sqrt{\pi}}{2}\sqrt{\frac{a}{s}} = \frac{\sqrt{\pi a}}{2s^{3/2}}\exp\left(-\frac{a}{4s}\right) \end{aligned}$$

valid for $\mathrm{Re}(s) > 0$.

(d) $f(t) = e^{-at^2}$

$$\left|e^{-\alpha t^2}\right| \le 1 \text{ for all real } \alpha \text{ and real } t$$

Thus $f(t)$ is of exponential order 0.

$$F(s) = \int_0^\infty e^{-at^2} e^{-st} dt$$

Complete the square of the exponent:

$$\begin{aligned} -at^2 - st &= -a\left(t^2 + \frac{s}{a}t\right) = -a\left(t^2 + \frac{s}{a}t + \left(\frac{s}{2a}\right)^2 - \left(\frac{s}{2a}\right)^2\right) \\ &= -a\left(\left(t + \frac{s}{2a}\right)^2 - \left(\frac{s}{2a}\right)^2\right) \end{aligned}$$

and thus

$$\begin{aligned} F(s) &= \exp\left(\frac{s^2}{4a}\right)\int_0^\infty \exp\left[-a\left(t + \frac{s}{2a}\right)^2\right]dt \\ &= \exp\left(\frac{s^2}{4a}\right)\int_{s/2\sqrt{a}}^\infty \exp\left(-u^2\right)\frac{du}{\sqrt{a}} \\ &= \frac{1}{2}\sqrt{\frac{\pi}{a}}\exp\left(\frac{s^2}{4a}\right)\mathrm{erfc}\left(s/2\sqrt{a}\right) \end{aligned}$$

(e) $f(t) = te^{\sqrt{t}}$

$$f(t)e^{-t} = te^{\sqrt{t}}e^{-t} = t\exp\left\{-\sqrt{t}\left(\sqrt{t} - 1\right)\right\}$$

If $t < 1$, then

$$f(t)e^{-t} = t\exp\left\{\sqrt{t}\left(1 - \sqrt{t}\right)\right\} \le te^{\sqrt{t}} \le e$$

while if $t > 1$

$$f(t)e^{-t} = t\exp\left\{-\sqrt{t}\left(\sqrt{t} - 1\right)\right\} \le te^{-\sqrt{t}}$$

Now

$$x^2e^{-x} = \frac{x^2}{e^x} = \frac{x^2}{1 + x + \frac{x^2}{2} + \ldots} \le \frac{x^2}{x^2/2} = 2$$

Thus

$$f(t)e^{-t} \le te^{-\sqrt{t}} \le 2 < e$$

for $t > 1$. Thus $f(t)$ is of exponential order 1.

The transform is:

$$F(s) = \int_0^\infty te^{\sqrt{t}}e^{-st}dt$$

Let $\sqrt{t} = u$ Then $dt = 2udu$

$$\begin{aligned} F(s) &= 2\int_0^\infty u^3e^{-su^2+u}du \\ &= 2\int_0^\infty u^3\exp\left(-s\left(u^2 - \frac{u}{s}\right)\right)du \\ &= 2\int_0^\infty \exp\left(-s\left(u^2 - \frac{u}{s} + \frac{1}{4s^2} - \frac{1}{4s^2}\right)\right)du \\ &= 2\exp\left(\frac{1}{4s}\right)\int_0^\infty u^3\exp\left(-s\left(u - \frac{1}{2s}\right)^2\right)du \end{aligned}$$

Now change variables to $v = \sqrt{s}\left(u - \frac{1}{2s}\right)$, to obtain

$$\begin{aligned} F(s) &= 2\exp\left(\frac{1}{4s}\right)\int_{-1/2\sqrt{s}}^\infty \left(\frac{v}{\sqrt{s}} + \frac{1}{2s}\right)^3 \exp(-v^2)\frac{dv}{\sqrt{s}} \\ &= \frac{2}{s^2}\exp\left(\frac{1}{4s}\right)\int_{-1/2\sqrt{s}}^\infty \left(v + \frac{1}{2\sqrt{s}}\right)^3 \exp(-v^2)dv \\ &= \frac{2}{s^2}\exp\left(\frac{1}{4s}\right)\int_{-1/2\sqrt{s}}^\infty \left(v^3 + \frac{3}{2}\frac{v^2}{\sqrt{s}} + \frac{3}{4}\frac{v}{s} + \frac{1}{8s^{3/2}}\right)\exp(-v^2)dv \end{aligned}$$

Use the following results:

$$\int_{-1/2\sqrt{s}}^\infty e^{-v^2}dv = \frac{\sqrt{\pi}}{2}\left[1 + \operatorname{erf}\left(\frac{1}{2\sqrt{s}}\right)\right]$$

(See Appendix 9.)

$$\int_{-1/2\sqrt{s}}^\infty ve^{-v^2}dv = \frac{1}{2}\left(-e^{-v^2}\right)\Big|_{-1/2\sqrt{s}}^\infty = \frac{1}{2}\exp\left(-\frac{1}{4s}\right)$$

$$\begin{aligned} \int_{-1/2\sqrt{s}}^\infty v^2e^{-v^2}dv &= \frac{1}{2}\int_{-1/2\sqrt{s}}^\infty v2ve^{-v^2}dv = \frac{1}{2}\left(-ve^{-v^2}\Big|_{-1/2\sqrt{s}}^\infty + \int_{-1/2\sqrt{s}}^\infty e^{-v^2}dv\right) \\ &= \frac{1}{2}\left(\frac{-1}{2\sqrt{s}}\exp\left(-\frac{1}{4s}\right) + \frac{\sqrt{\pi}}{2}\left[1 + \operatorname{erf}\left(\frac{1}{2\sqrt{s}}\right)\right]\right) \\ &= \frac{-1}{4\sqrt{s}}\exp\left(-\frac{1}{4s}\right) + \frac{\sqrt{\pi}}{4}\left[1 + \operatorname{erf}\left(\frac{1}{2\sqrt{s}}\right)\right] \end{aligned}$$

and

$$\begin{aligned}\int_{-1/2\sqrt{s}}^{\infty} v^3 e^{-v^2} dv &= \frac{1}{2}\int_{-1/2\sqrt{s}}^{\infty} v^2 2v e^{-v^2} dv = \frac{1}{2}\left(-v^2 e^{-v^2}\Big|_{-1/2\sqrt{s}}^{\infty} + \int_{-1/2\sqrt{s}}^{\infty} 2v e^{-v^2} dv\right) \\ &= \frac{1}{2}\left(\frac{1}{4s}\exp\left(-\frac{1}{4s}\right) + 2\frac{1}{2}\exp\left(-\frac{1}{4s}\right)\right) \\ &= \frac{1}{2}\left(1+\frac{1}{4s}\right)\exp\left(-\frac{1}{4s}\right)\end{aligned}$$

Thus

$$\frac{2}{s^2}\exp\left(\frac{1}{4s}\right)\int_{-1/2\sqrt{s}}^{\infty}\left(v^3 + \frac{3}{2}\frac{v^2}{\sqrt{s}} + \frac{3}{4}\frac{v}{s} + \frac{1}{8s^{3/2}}\right)\exp\left(-v^2\right)dv$$

$$\begin{aligned}F(s) &= \frac{2}{s^2}\exp\left(\frac{1}{4s}\right)\left[\begin{array}{c}\frac{1}{2}\left(1+\frac{1}{4s}\right)\exp\left(-\frac{1}{4s}\right) + \frac{3}{2\sqrt{s}}\left(\frac{-1}{4\sqrt{s}}\exp\left(-\frac{1}{4s}\right) + \frac{\sqrt{\pi}}{4}\left[1+\operatorname{erf}\left(\frac{1}{2\sqrt{s}}\right)\right]\right) \\ +\frac{3}{4s}\frac{1}{2}\exp\left(-\frac{1}{4s}\right) + \frac{1}{8s^{3/2}}\frac{\sqrt{\pi}}{2}\left[1+\operatorname{erf}\left(\frac{1}{2\sqrt{s}}\right)\right]\end{array}\right] \\ &= \frac{1}{s^2}\left[\left(1+\frac{1}{4s}-\frac{3}{4s}+\frac{3}{4s}\right) + \frac{\sqrt{\pi}}{4\sqrt{s}}\exp\left(\frac{1}{4s}\right)\left[1+\operatorname{erf}\left(\frac{1}{2\sqrt{s}}\right)\right]\left(3+\frac{1}{2s}\right)\right] \\ &= \frac{1}{s^2}\left[\left(1+\frac{1}{4s}\right) + \frac{1}{8}\sqrt{\frac{\pi}{s}}\left(6+\frac{1}{s}\right)\exp\left(\frac{1}{4s}\right)\left[1+\operatorname{erf}\left(\frac{1}{2\sqrt{s}}\right)\right]\right]\end{aligned}$$

Another method: Using equation 12:

$$\begin{aligned}\mathcal{L}\left(e^{\sqrt{t}}\right) &= \int_0^{\infty}\exp\left(\sqrt{t}-st\right)dt \\ &= \int_0^{\infty}\exp\left(-s\left(t-\frac{\sqrt{t}}{s}\right)\right)dt \\ &= \int_0^{\infty}\exp\left(-s\left(\sqrt{t}-\frac{1}{2s}\right)^2+\frac{1}{4s}\right)dt\end{aligned}$$

Let $\sqrt{t} - 1/2s = u;$ $du = \frac{1}{2\sqrt{t}}dt;$ $dt = 2du\left(u+\frac{1}{2s}\right)$ So

$$\begin{aligned}\mathcal{L}\left(e^{\sqrt{t}}\right) &= \exp\left(\frac{1}{4s}\right)\int_{-1/2s}^{\infty} e^{-su^2}\left(2u+\frac{1}{s}\right)du \\ &= \exp\left(\frac{1}{4s}\right)\left(\frac{-e^{-su^2}}{s}\Bigg|_{-1/2s}^{\infty} + \frac{1}{s^{3/2}}\frac{\sqrt{\pi}}{2}\left(1+\operatorname{erf}\left(\frac{1}{2\sqrt{s}}\right)\right)\right) \\ &= \exp\left(\frac{1}{4s}\right)\left(\frac{e^{-1/4s}}{s} + \frac{1}{s^{3/2}}\frac{\sqrt{\pi}}{2}\left(1+\operatorname{erf}\left(\frac{1}{2\sqrt{s}}\right)\right)\right) \\ &= \frac{1}{s} + \frac{1}{s^{3/2}}\exp\left(\frac{1}{4s}\right)\frac{\sqrt{\pi}}{2}\left(1+\operatorname{erf}\left(\frac{1}{2\sqrt{s}}\right)\right)\end{aligned}$$

and then

$$\begin{aligned}\mathcal{L}\left(te^{\sqrt{t}}\right) &= -\frac{d}{ds}\left\{\frac{1}{s} + \frac{1}{s^{3/2}}\exp\left(\frac{1}{4s}\right)\frac{\sqrt{\pi}}{2}\left(1+\operatorname{erf}\left(\frac{1}{2\sqrt{s}}\right)\right)\right\} \\ &= \frac{1}{s^2} + \frac{3}{2}\frac{1}{s^{5/2}}\exp\left(\frac{1}{4s}\right)\frac{\sqrt{\pi}}{2}\left(1+\operatorname{erf}\left(\frac{1}{2\sqrt{s}}\right)\right) \\ &\quad - \frac{1}{s^{3/2}}\left(-\frac{1}{4s^2}\right)\exp\frac{1}{4s}\frac{\sqrt{\pi}}{2}\left(1+\operatorname{erf}\left(\frac{1}{2\sqrt{s}}\right)\right) \\ &\quad - \frac{1}{s^{3/2}}\exp\left(\frac{1}{4s}\right)\left(\frac{-1}{4s^{3/2}}\right)\exp\left(-\frac{1}{4s}\right) \\ &= \frac{1}{s^2} + \frac{1}{4s^3} + \frac{1+6s}{4s^{7/2}}\exp\left(\frac{1}{4s}\right)\frac{\sqrt{\pi}}{2}\left(1+\operatorname{erf}\left(\frac{1}{2\sqrt{s}}\right)\right)\end{aligned}$$

as before.

(f) $f(t) = \sin(\omega t + \phi_0)$.

This function is of exponential order 0 because the two complex exponentials $e^{i(\omega t+\phi_0)}$ and $e^{-i(\omega t+\phi_0)}$ are. (See text.)

First write

$$\sin(\omega t + \phi_0) = \sin\omega t \cos\phi_0 + \cos\omega t \sin\phi_0$$

Then use the linearity of the transform to obtain:

$$\begin{aligned} F(s) &= \cos\phi_0 \frac{\omega}{s^2+\omega^2} + \sin\phi_0 \frac{s}{s^2+\omega^2} \\ &= \frac{\omega\cos\phi_0 + s\sin\phi_0}{s^2+\omega^2} = \frac{\cos(\phi_0 - \xi)}{\sqrt{s^2+\omega^2}} \end{aligned}$$

where $\tan\xi = s/\omega$.

(g) The ramp function $f(t) = \begin{cases} t & \text{if } 0 < x < t_0 \\ t_0 & \text{if } t > t_0 \end{cases}$

This function is of exponential order because t is (see text) and $f(t) \leq t$ for all t.

$$\begin{aligned} F(s) &= \int_0^{t_0} te^{-st}dt + \int_{t_0}^{\infty} t_0 e^{-st}dt \\ &= -e^{-st_0}\frac{t_0}{s} - \frac{e^{-st_0}-1}{s^2} + \frac{t_0}{s}e^{-st_0} \\ &= \frac{1-e^{-st_0}}{s^2} \end{aligned}$$

2. Using the shifting property, or otherwise, find the Laplace transform of the function

$$f(t) = \begin{cases} 0 & \text{if } t < 2 \\ (t-2)^3 e^{-at} & \text{if } t \geq 2 \end{cases}$$

The factor e^{-at} suggests that we also use the attenuation property

. Then with the shifting property and the transform of t^3 from Table 5.1 :

$$\mathcal{L}(t-2)^3 = e^{-2s}\mathcal{L}\left(t^3\right) = e^{-2s}\frac{3!}{s^4}$$

Now use the attenuation property:

$$F(s) = e^{-2(s+a)}\frac{3!}{(s+a)^4}$$

3. Find the Laplace transform of the function $t\cosh\alpha t$.

First note that the transform of $g(t) = \cosh\alpha t$ is

$$\begin{aligned} G(s) &= \frac{1}{2}\int_0^{\infty}\left(e^{\alpha t} + e^{-\alpha t}\right)e^{-st}dt \\ &= \frac{1}{2}\left(\frac{1}{s-\alpha} + \frac{1}{s+\alpha}\right) = \frac{s}{\left(s^2-\alpha^2\right)} \end{aligned}$$

Then using equation (12) with $n = 1$:

$$\mathcal{L}(t\cosh\alpha t) = -\frac{d}{ds}G(s) = -\frac{d}{ds}\frac{s}{\left(s^2 - \alpha^2\right)}$$
$$= -\left[\frac{1}{\left(s^2-\alpha^2\right)} - \frac{2s^2}{\left(s^2-\alpha^2\right)^2}\right]$$
$$= \frac{2s^2 - \left(s^2-\alpha^2\right)}{\left(s^2-\alpha^2\right)^2} = \frac{s^2+\alpha^2}{\left(s^2-\alpha^2\right)^2}$$

4. Find the Laplace transform of the function $f(t) = \frac{\sinh \alpha t}{t}$.

Let $g(t) = \sinh\alpha t$. Then

$$G(s) = \frac{1}{2}\int_0^\infty \left(e^{\alpha t} - e^{-\alpha t}\right)e^{-st}dt$$
$$= \frac{1}{2}\left(\frac{1}{s-\alpha} - \frac{1}{s+\alpha}\right) = \frac{\alpha}{\left(s^2-\alpha^2\right)}$$

Then using equation (5.14):

$$F(s) = \int_s^\infty G(\sigma)d\sigma = \int_s^\infty \frac{\alpha}{\left(\sigma^2-\alpha^2\right)}d\sigma$$

Let $\sigma = \alpha\coth x$. Then $d\sigma = \alpha\frac{d}{dx}\coth x dx = \alpha\left(1-\coth^2 x\right)dx$ and

$$F(s) = \alpha\int\frac{1}{\alpha^2\left(\coth^2 x - 1\right)}\alpha\left(1-\coth^2 x\right)dx$$
$$= -x = -\coth^{-1}\sigma/\alpha\Big|_s^\infty$$

Now

$$\coth x = \frac{e^x + e^{-x}}{e^x - e^{-x}} = u$$

So

$$\coth^{-1} u = x$$

where

$$\left(e^{2x}-1\right)u = \left(e^{2x}+1\right)$$

So

$$e^{2x}(u-1) = 1+u$$

and thus

$$x = \frac{1}{2}\ln\left(\frac{u+1}{u-1}\right)$$

Thus:

$$F(s) = -\frac{1}{2}\ln\left(\frac{\sigma/\alpha+1}{\sigma/\alpha-1}\right)\Big|_s^\infty = \frac{1}{2}\ln\frac{s+\alpha}{s-\alpha}$$

5. (a) Find the Laplace transform of the triangle wave function with period T:

$$f(t) = \begin{cases} a(t-nT) & \text{if } nT < t < nT + T/2 \\ a((n+1)T - t) & \text{if } nT + T/2 < t < (n+1)T \end{cases}$$

First we find the transform of the first period:

$$g(t) = \begin{cases} at & \text{if} \quad 0 < t < T/2 \\ a(T-t) & \text{if} \quad T/2 < t < T \\ 0 & \text{otherwise} \end{cases}$$

Then

$$\begin{aligned} G(s) &= \int_0^{T/2} ate^{-st}dt + \int_{T/2}^{T} a(T-t)e^{-st}dt \\ &= a\left(\frac{te^{-st}}{-s}\Big|_0^{T/2} - \int_0^{T/2}\frac{e^{-st}}{-s}\right) + aT\frac{e^{-st}}{-s}\Big|_{T/2}^{T} - a\left(\frac{te^{-st}}{-s}\Big|_{T/2}^{T} - \int_{T/2}^{T}\frac{e^{-st}}{-s}\right) \\ &= a\left(\frac{Te^{-sT/2}}{-2s} + \frac{1}{s}\frac{e^{-st}}{-s}\Big|_0^{T/2}\right) + aT\frac{e^{-sT} - e^{-sT/2}}{-s} - a\left(\frac{Te^{-sT} - Te^{-sT/2}/2}{-s} + \frac{1}{s}\frac{e^{-st}}{-s}\Big|_{T/2}^{T}\right) \\ &= a\left(\frac{Te^{-sT/2}}{-2s} - \frac{e^{-sT/2} - 1}{s^2}\right) - aT\frac{e^{-sT} - e^{-sT/2}}{s} - a\left(\frac{Te^{-sT} - Te^{-sT/2}/2}{-s} - \frac{e^{-sT} - e^{-sT/2}}{s^2}\right) \\ &= \frac{a}{s^2}\left(0 + 1 + e^{-sT} - 2e^{-sT/2}\right) = \frac{a}{s^2}\left(1 - e^{-sT/2}\right)^2 \end{aligned}$$

Thus the transform we want is:

$$\begin{aligned} F(s) &= \frac{G(s)}{1 - e^{-sT}} = \frac{a}{s^2}\frac{\left(1 - e^{-sT/2}\right)^2}{\left(1 - e^{-sT/2}\right)\left(1 + e^{-sT/2}\right)} \\ &= \frac{a}{s^2}\frac{\left(1 - e^{-sT/2}\right)}{\left(1 + e^{-sT/2}\right)} = \frac{a}{s^2}\tanh\left(\frac{sT}{4}\right) \end{aligned}$$

(b) the sawtooth function: $f(t) = \quad a(t - nT) \quad \text{if} \quad nT < t < (n+1)T$

The function is periodic, so first we find the transform of $g(t) = at$:

$$\begin{aligned} G(s) &= \int_0^T ate^{-st}dt = a\left(t\frac{e^{-st}}{-s}\Big|_0^T - \int_0^T \frac{e^{-st}}{-s}dt\right) \\ &= a\left(-\frac{T}{s}e^{-sT} - \frac{e^{-st}}{s^2}\Big|_0^T\right) = a\left(-\frac{T}{s}e^{-sT} - \frac{e^{-sT} - 1}{s^2}\right) \\ &= \frac{a}{s^2}\left(1 - e^{-sT}(sT + 1)\right) \end{aligned}$$

Then applying the periodic signature:

$$F(s) = \frac{a}{s^2}\frac{\left(1 - e^{-sT}(sT+1)\right)}{\left(1 - e^{-sT}\right)} = \frac{a}{s^2}\left(1 - \frac{sT}{e^{sT} - 1}\right)$$

6. Use the Mellin Inversion integral to invert the following transforms:

(a) $F(s) = s/\left(s^2 + 2s + 3\right)$

The transform has poles where its denominator has zeroes, ie at

$$s = \frac{1}{2}\left(-2 \pm \sqrt{4 - 12}\right) = -1 \pm i\sqrt{2}$$

Thus we must place the contour to the right of $x = -1.$ A line along the imaginary axis will do. Then we close to the left, enclosing both of the poles. The integral along the big semicircle is zero, since $F(s) \to 0$ uniformly as $s \to \infty.$ Then

$$\begin{aligned} f(t) &= \frac{1}{2\pi i}\int_{-i\infty}^{+i\infty} \frac{s}{s^2+2s+3} e^{st} ds = \frac{-1+i\sqrt{2}}{2i\sqrt{2}} e^{-t} e^{i\sqrt{2}t} + \frac{-1-i\sqrt{2}}{-2i\sqrt{2}} e^{-t} e^{-i\sqrt{2}t} \\ &= \frac{e^{-t}}{2i\sqrt{2}}\left(e^{-i\sqrt{2}t} - e^{i\sqrt{2}t} + i\sqrt{2}\left(e^{i\sqrt{2}t} + e^{-i\sqrt{2}t}\right)\right) \\ &= e^{-t}\left(\cos\sqrt{2}\,t - \frac{\sqrt{2}}{2}\sin\sqrt{2}\,t\right) \end{aligned}$$

(b) $F(s) = \frac{1}{\left(s^2+a^2\right)^2}$

The function has 2nd order poles at $s = \pm ia.$ Thus we must choose $\gamma > 0.$

$$f(t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{e^{st}}{\left(s^2 + a^2\right)^2} ds$$

We close the contour to the left, enclosing both poles. The integral along the big semi-circle goes to zero as $R \to \infty.$ The residues are:

$$\begin{aligned} \lim_{s\to ia} \frac{d}{ds}(s - ia)^2 \frac{e^{st}}{\left(s^2 + a^2\right)^2} &= \lim_{s\to ia} \frac{d}{ds} \frac{e^{st}}{(s + ia)^2} \\ &= \lim_{s\to ia} \frac{te^{st}}{(s + ia)^2} - 2\frac{e^{st}}{(s + ia)^3} \\ &= \frac{2iat - 2}{(2ia)^3} e^{iat} = \frac{1 - iat}{4ia^3} e^{iat} \end{aligned}$$

and similarly

$$\begin{aligned}\lim_{s\to -ia}\frac{d}{ds}(s+ia)^2\frac{e^{st}}{(s^2+a^2)^2} &= \lim_{s\to -ia}\frac{d}{ds}\frac{e^{st}}{(s-ia)^2}\\ &= \lim_{s\to -ia}\left(\frac{-2e^{st}}{(s-ia)^3}+\frac{te^{st}}{(s-ia)^2}\right)\\ &= \frac{-2iat-2}{(-2ia)^3}e^{-iat}\\ &= -\frac{iat+1}{4ia^3}e^{-iat}\end{aligned}$$

Thus the function is:

$$\begin{aligned}f(t) &= \frac{1}{2\pi i}(2\pi i)\left(\frac{1-iat}{4ia^3}e^{iat}-\frac{iat+1}{4ia^3}e^{-iat}\right)\\ &= \frac{1}{4ia^3}(2i\sin at-2iat\cos at)\\ &= \frac{1}{2a^3}(\sin at-at\cos at)\end{aligned}$$

(c) $F(s) = e^{-\sqrt{s}}/s$

The function has a branch point at the origin. We choose $\gamma > 0$ and put the branch cut along the negative real axis. The function has no other singularities. Then:

$$f(t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{e^{-\sqrt{s}}e^{st}}{s}ds$$

The integral along the big semicircle goes to zero and we have:

$$\int_C = \int_{\gamma-i\infty}^{\gamma+i\infty}+\int_{\text{top of branch cut}}+\int_{\text{little circle}}+\int_{\text{bottom of branch cut}} = 0$$

and thus:

$$\int_{\gamma-i\infty}^{\gamma+i\infty} = -\int_{\text{top of branch cut}}-\int_{\text{little circle}}-\int_{\text{bottom of branch cut}}$$

Along the little circle:

$$\int_{\text{little circle}} = \lim_{\varepsilon\to 0}\int_{\pi}^{-\pi}\frac{e^{-\sqrt{\varepsilon}e^{i\theta/2}}e^{\varepsilon t e^{i\theta}}}{\varepsilon e^{i\theta}}i\varepsilon e^{i\theta}d\theta = -2\pi i$$

On the top of the branch cut $s = re^{i\pi}$ and

$$\int_{\text{top of branch cut}} = \int_{\infty}^{0}e^{-\sqrt{r}e^{i\pi/2}}e^{-rt}\frac{(-dr)}{-r} = -\int_0^{\infty}e^{-i\sqrt{r}}e^{-rt}\frac{dr}{r}$$

while along the bottom $s = re^{-i\pi}$ and therefore:

$$\int_{\text{bottom of branch cut}} = \int_0^{\infty}e^{i\sqrt{r}}e^{-rt}\frac{dr}{r}$$

Putting it all together, we get:

$$\begin{aligned}f(t) &= \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\frac{e^{\sqrt{s}}e^{st}}{s}ds = \frac{-1}{2\pi i}\left(-2\pi i+2i\int_0^{\infty}\sin\sqrt{r}\,e^{-rt}\frac{dr}{r}\right)\\ &= 1-\frac{1}{\pi}\int_0^{\infty}\sin\sqrt{r}\,e^{-rt}\frac{dr}{r}\end{aligned}$$

To evaluate the integral, we can complete the square. First let $u^2 = r$

$$\begin{aligned}\frac{1}{\pi}\int_0^\infty \sin\sqrt{r}\,e^{-rt}\frac{dr}{r} &= \frac{1}{\pi}\int_0^\infty \sin u e^{-tu^2}\frac{2udu}{u^2}\\ &= \frac{2}{\pi}\int_0^\infty \sin u e^{-tu^2}\frac{du}{u}\\ &= \frac{2}{\pi}\int_0^\infty\int_0^1 \cos yu dy e^{-tu^2}du\end{aligned}$$

Then

$$\begin{aligned}\exp\left(-tu^2 \pm iyu\right) &= \exp\left(-t\left(u^2 \mp \frac{iyu}{t}\right)\right) = \exp\left(-t\left(u^2 \mp \frac{iyu}{t} + \left(\frac{iy}{2t}\right)^2 - \left(\frac{iy}{2t}\right)^2\right)\right)\\ &= \exp\left(-t\left(\left(u \mp \frac{iy}{2t}\right)^2\right)\right)\exp\left(-\frac{y^2}{4t}\right)\end{aligned}$$

So

$$\begin{aligned}\frac{2}{\pi}\int_0^1\int_0^\infty \cos yu e^{-tu^2}dudy &= \frac{2}{\pi}\int_0^1 \exp\left(-\frac{y^2}{4t}\right)\frac{1}{\sqrt{t}}\frac{\sqrt{\pi}}{2}dy\\ &= \frac{2}{\sqrt{\pi}}\int_0^{1/2\sqrt{t}} e^{-w^2}dw\\ &= \operatorname{erf}\left(\frac{1}{2\sqrt{t}}\right)\end{aligned}$$

and thus

$$f(t) = 1 - \operatorname{erf}\left(\frac{1}{2\sqrt{t}}\right) = \operatorname{erfc}\left(\frac{1}{2\sqrt{t}}\right)$$

(d) $F(s) = -\frac{\ln(1+s)}{s}$ (Express the answer in terms of the exponential integral $\operatorname{Ei}(x) = \int_{-\infty}^x \frac{e^w dw}{w}$, where $x < 0$)

$$f(t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} -\frac{\ln(1+s)}{s}e^{st}ds$$

The integrand has a branch point at $s = -1$, and also a removable singularity at $s = 0$. Thus we need $\gamma > 0$. We close to the left, and go around the branch cut which lies along the negative x −axis. Then

$$0 = \frac{1}{2\pi i}\left(\int_{\gamma-i\infty}^{\gamma+i\infty} + \int_{\text{big semicircle}} + \int_{\text{top of br. cut}} + \int_{\text{bott. br. cut}} + \int_{\text{little circle}}\right) - \frac{\ln(1+s)}{s}e^{st}ds$$

The integral around the big circle goes to zero as $R \to \infty$. The integral around the little circle $\left(s = -1 + \varepsilon e^{i\theta}\right)$ is:

$$\begin{aligned}\int_{\text{little circle}} -\frac{\ln(1+s)}{s}e^{st}ds &= -\int_{\pi}^{-\pi}\frac{\ln\left(\varepsilon e^{i\theta}\right)}{-1+\varepsilon e^{i\theta}}e^{\left(-1+\varepsilon e^{i\theta}\right)t}i\varepsilon e^{i\theta}d\theta\\ &= -\int_{\pi}^{-\pi}\frac{\ln\varepsilon + i\theta}{-1+\varepsilon e^{i\theta}}e^{\left(-1+\varepsilon e^{i\theta}\right)t}i\varepsilon e^{i\theta}d\theta \to 0\end{aligned}$$

as $\varepsilon \to 0$. Thus:

$$\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} -\frac{\ln(1+s)}{s}e^{st}ds = \frac{1}{2\pi i}\left(\int_{\text{top of br. cut}} + \int_{\text{bott. br. cut}}\right)\frac{\ln(1+s)}{s}e^{st}ds$$

On the top of the cut, $s = -1 + re^{i\pi}$, so

$$\int_{\text{top of br. cut}} \frac{\ln(1+s)}{s} e^{st} ds = \int_\infty^0 \frac{\ln\left(re^{i\pi}\right) e^{\left(-1+re^{i\pi}\right)t} e^{i\pi} dr}{-1 + re^{i\pi}}$$
$$= \int_\infty^0 \frac{(\ln r + i\pi) e^{-(1+r)t} dr}{r+1}$$

while on the bottom, $s = -1 + re^{-i\pi}$:

$$\int_{\text{bott. br. cut}} \frac{\ln(1+s)}{s} e^{st} ds = \int_0^\infty \frac{\ln\left(re^{-i\pi}\right) e^{\left(-1+re^{-i\pi}\right)t} e^{-i\pi} dr}{-1 + re^{-i\pi}}$$
$$= \int_0^\infty \frac{(\ln r - i\pi) e^{-(1+r)t} dr}{r+1}$$

So

$$\frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} - \frac{\ln(1+s)}{s} e^{st} ds = \frac{1}{2\pi i} \left(\int_0^\infty \frac{(\ln r - i\pi) e^{-(1+r)t} dr}{r+1} - \int_0^\infty \frac{(\ln r + i\pi) e^{-(1+r)t} dr}{r+1} \right)$$
$$= -\int_0^\infty \frac{e^{-(1+r)t} dr}{r+1}$$
$$= -\int_1^\infty \frac{e^{-ut} du}{u}$$

where $u = 1 + r$. Now let $w = -ut$. Then:

$$f(t) = -\int_{-t}^{-\infty} \frac{e^w dw}{w} = \int_{-\infty}^{-t} \frac{e^w dw}{w} = \mathrm{Ei}(-t)$$

7. $F(s) = \frac{1}{\sqrt{s^2 - a^2}}$

(a)

$$F(s) = \frac{1}{s} \frac{1}{\sqrt{1 - \frac{a^2}{s^2}}} = \frac{1}{s}\left(1 + \frac{1}{2}\frac{a^2}{s^2} + \frac{3}{2^2}\frac{1}{2}\frac{a^4}{s^4} + \dots \frac{(2n-1)!!}{2n!!}\frac{a^{2n}}{s^{2n}} + \dots\right)$$

Each power inverts to a power, so we obtain a power series for $f(t)$

$$f(t) = 1 + \frac{1}{2}\frac{(at)^2}{2!} + \frac{3}{8}\frac{(at)^4}{4!} + \dots \frac{(2n-1)!!}{2n!!}\frac{(at)^{2n}}{2n!} + \dots$$
$$= 1 + \frac{(at)^2}{2^2} + \frac{(at)^4}{8^2} + \dots \frac{(at)^{2n}}{(2n!!)^2} + \dots$$
$$= 1 + \frac{(at)^2}{2^2} + \frac{(at)^4}{8^2} + \dots \frac{(at)^{2n}}{2^{2n}(n!)^2} + \dots$$

(b)

$$f(t) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} \frac{e^{st}}{\sqrt{s^2 - a^2}} ds$$

The function has branch points at $s = \pm a$, so we can run the branch cut along a line between these two points. The contour must be placed with $\gamma > a$. Then we must run the contour along the real axis, around the branch line, and back along the bottom of the real axis, so as to exclude the non-isolated singularities at $x = \pm a$. There are no other poles inside the contour, so

$$f(t) = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} \frac{e^{st}}{\sqrt{s^2 - a^2}} ds = \frac{1}{2\pi i}\left(\int_{\text{top of real axis}} + \int_{\text{botom of real axis}}\right)\frac{e^{st}}{\sqrt{s^2 - a^2}} ds$$

The function is continuous except over the branch cut. Now we change variables to z where

$$s = \frac{a}{2}\left(z + \frac{1}{z}\right)$$
$$\left(z + \frac{1}{z}\right) = \frac{2s}{a}$$

Then

$$ds = \frac{a}{2}\left(1 - \frac{1}{z^2}\right) dz$$

From problem 2-38, the branch cut maps to the unit circle, so

$$f(t) = \frac{1}{2\pi i}\int_{\gamma - i\infty}^{\gamma + i\infty} \frac{e^{st}}{\sqrt{s^2 - a^2}} ds = \frac{1}{2\pi i}\oint_{\text{unit circle}} \frac{\exp \frac{at}{2}\left(z + \frac{1}{z}\right)}{\sqrt{\frac{a^2}{4}\left(z + \frac{1}{z}\right)^2 - a^2}} \frac{a}{2}\left(1 - \frac{1}{z^2}\right) dz$$
$$= \frac{1}{2\pi i}\oint_{\text{unit circle}} \frac{1}{z}\exp \frac{at}{2}\left(z + \frac{1}{z}\right) dz = \frac{1}{2\pi i}\oint_{\text{unit circle}} \frac{1}{z}\exp \frac{a}{2} tz \exp \frac{a}{2}\frac{t}{z} dz$$

The integrand has a pole at the origin. To find the residue there, look at the series:

$$\frac{1}{z}\exp \frac{a}{2} tz \exp \frac{a}{2}\frac{t}{z} = \frac{1}{z}\left(\sum_n \frac{1}{n!}\left(\frac{atz}{2}\right)^n\right)\sum_m \frac{1}{m!}\left(\frac{at}{2z}\right)^m$$
$$= \sum_n \sum_m \left(\frac{at}{2}\right)^{n+m} \frac{z^{n-m-1}}{n!m!}$$

Thus when $z^{n-m-1} = z^{-1}$, we must have $n = m$

$$\text{Res} = \sum_n \left(\frac{at}{2}\right)^{2n} \frac{1}{(n!)^2}$$

Thus

$$f(t) = \sum_n \left(\frac{at}{2}\right)^{2n} \frac{1}{(n!)^2}$$

which is the result obtained in (a).

8. Use the convolution theorem (eqn 17) to evaluate the inverse transform of

$$F(s) = \frac{\omega s}{\left(s^4 - \omega^4\right)}$$

We can write the transform as the product of the two transforms:

$$F_1 = \frac{\omega}{s^2 - \omega^2}$$

and

$$F_2 = \frac{s}{s^2 + \omega^2}$$

The first may be easily inverted:

$$F_1 = \frac{1}{2}\left(\frac{1}{s - \omega} - \frac{1}{s + \omega}\right) \Rightarrow f_1(t) = \frac{1}{2}\left(e^{\omega t} - e^{-\omega t}\right) = \sinh(\omega t)$$

The second function is in table 5.1:

$$f_2(t) = \cos\omega t$$

Thus:

$$\begin{aligned} f(t) &= \int_0^t f_2(\tau) f_1(t-\tau)\,d\tau \\ &= \int_0^t \cos\omega\tau \sinh\omega(t-\tau)\,d\tau \end{aligned}$$

We can write the integrand in terms of exponentials:

$$\begin{aligned} f(t) &= \frac{1}{4}\int_0^t \left(e^{i\omega\tau} + e^{-i\omega\tau}\right)\left(e^{\omega(t-\tau)} - e^{-\omega(t-\tau)}\right)d\tau \\ &= \frac{1}{4}\int_0^t e^{(i-1)\omega\tau}e^{\omega t} - e^{(i+1)\omega\tau}e^{-\omega t} + e^{-(i+1)\omega\tau}e^{\omega t} - e^{-\omega t}e^{(1-i)\omega\tau}\,d\tau \\ &= \frac{1}{4\omega}\left[\frac{\left(e^{(i-1)\omega t} - 1\right)e^{\omega t}}{i-1} - \frac{e^{(i+1)\omega t} - 1}{i+1}e^{-\omega t} + \frac{e^{-(i+1)\omega t} - 1}{-(i+1)}e^{\omega t} - e^{-\omega t}\frac{e^{(1-i)\omega t} - 1}{1-i}\right] \\ &= \frac{1}{4\omega}\left(\frac{e^{i\omega t} - e^{\omega t}}{i-1} - \frac{e^{i\omega t} - e^{-\omega t}}{i+1} + \frac{e^{-i\omega t} - e^{\omega t}}{-(i+1)} - \frac{e^{-i\omega t} - e^{-\omega t}}{1-i}\right) \\ &= \frac{1}{2\omega}(-\cos\omega t + \cosh\omega t) \end{aligned}$$

Check using inversion integral:

$$f(t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{\omega s}{\left(s^4 - \omega^4\right)} e^{st}\,dt$$

The integrand has simple poles at $s = \pm\omega$, $s = \pm i\omega$. Thus we take $\gamma > \omega$, and close to the left. The integral around the big semicircle $\to 0$, and the residues are:

$$\frac{\omega^2 e^{\omega t}}{2\omega\left(2\omega^2\right)} = \frac{1}{4\omega}e^{\omega t}$$

$$\frac{-\omega^2 e^{-\omega t}}{(-2\omega)\left(2\omega^2\right)} = \frac{1}{4\omega}e^{-\omega t}$$

$$\frac{i\omega^2 e^{i\omega t}}{-2\omega^2(2i\omega)} = -\frac{1}{4\omega}e^{i\omega t}$$

and

$$\frac{-i\omega^2 e^{-i\omega t}}{-2\omega^2(-2i\omega)} = -\frac{1}{4\omega}e^{-i\omega t}$$

Thus

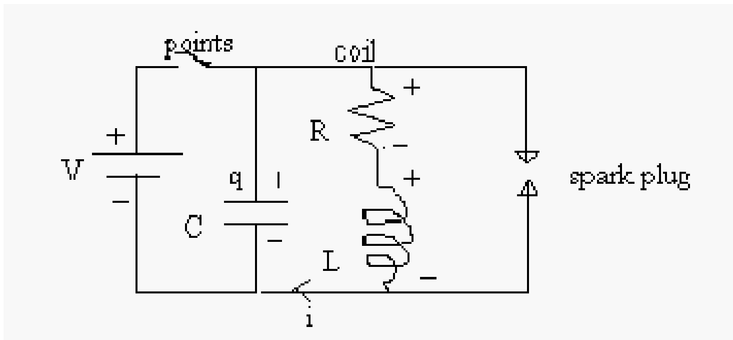
$$\begin{aligned} f(t) &= \frac{1}{4\omega}\left(e^{\omega t} + e^{-\omega t} - e^{i\omega t} - e^{-i\omega t}\right) \\ &= \frac{1}{2\omega}(\cosh\omega t - \cos\omega t) \end{aligned}$$

9. Use the integration rule (equation 5.14) and Table 5.1 to derive the result of Example 5.89 for the transform of $1/\sqrt{t}$.

$1/t^{1/2} = \frac{1}{t}\sqrt{t}$. The transform of $\sqrt{t}$ is $\Gamma(3/2)/s^{3/2} = \frac{1}{2}\Gamma(1/2)/s^{3/2}$. Thus, using relation (14), we have

$$\mathcal{L}\left(\frac{1}{t^{1/2}}\right) = \frac{1}{2}\int_s^\infty \sqrt{\frac{\pi}{s^{3/2}}}\,ds = -\frac{\sqrt{\pi}}{2}2s^{-1/2}\Big|_s^\infty$$
$$= \sqrt{\frac{\pi}{s}}$$

10. The diagram shows a simplified version of an automobile spark coil circuit. The spark plug itself acts like an open circuit until the potential across it reaches the breakdown voltage for air. Thus you may ignore that branch of the circuit until the end of the problem (part (e)).The battery voltage $V = 12$ V, $C = 0.1\ \mu$F, $R = 10\ \Omega$, and $L = 10$ mH.



(a) How long a ``long time'' is necessary? Write down expressions for the charge on the capacitor and the current through the coil at $t = 0$.

With the switch closed, the circuit reaches a final equilibrium with the capacitor fully charged ($Q = \varepsilon C = (12\text{ V})(0.1\ \mu\text{F}) = 1.2\ \mu$C) and a steady state current of $i = \varepsilon/R = (12\text{ V})/(10\ \Omega) = 1.2$ A. Applying Kirchhoff's loop rule to the left loop, we can see how the circuit reaches this steady state:

$$\varepsilon = L\frac{di_1}{dt} + i_1 R$$

Thus the current i_1 reaches its final value exponentially with a time constant $\tau = L/R = (10\text{ mH})/(10\ \Omega) = 1$ ms. Thus a few miliseconds is the "long time" necessary.

(b) At $t = 0$ the points open. Qualitatively discuss the circuit behavior. What is the expected long time solution for the charge and current?

When the points open, the capacitor begins to discharge. The inductor resists immediate change in the current. The charge and current will oscillate as they decay towards the final values ($q = 0,\ i = 0$). During the oscillation, the potential difference across the capacitor (and hence across the plug) will reach a value >12 V, hence allowing the plug to spark.

(c) Use a Laplace transform method to solve for the potential difference across the spark plug as a function of time.

The current variable i amd the charge variable q are defined in the diagram. The equations describing the circuit behavior are found from Kirchhoff's loop and junction rules:

$$i = -\frac{dq}{dt}$$

$$\frac{q}{C} - L\frac{di}{dt} - iR = 0$$

The initial conditions are:

$$i(0) = \varepsilon/R, \; q(0) = \varepsilon C$$

Applying the Laplace transform, we have:

$$I = -(sQ - q(0)) = -sQ + \varepsilon C$$

and

$$\frac{Q}{C} - L(sI - i(0)) - IR = 0$$
$$\frac{Q}{C} - I(Ls + R) + L\frac{\varepsilon}{R} = 0$$

Combining equations and , we get:

$$\frac{Q}{C} - (-sQ + \varepsilon C)(Ls + R) + L\frac{\varepsilon}{R} = 0$$
$$Q(1 + s(LCs + RC)) = \varepsilon C(LCs + RC) - LC\frac{\varepsilon}{R}$$

Thus the transform is:

$$Q = \varepsilon\frac{C(s + R/L) - 1/R}{s^2 + Rs/L + 1/LC}$$

or, writing $1/LC = \omega_0^2$, $RC = \tau_C$, and $L/R = \tau_L$, we have:

$$Q = \varepsilon C\frac{(s + 1/\tau_L) - 1/\tau_C}{s^2 + s/\tau_L + \omega_0^2}$$

We can complete the square in the denominator:

$$s^2 + \tau_L s + \omega_0^2 = (s + 1/2\tau_L)^2 + \omega_0^2 - 1/4\tau_L^2$$

and then we may write Q as:

$$Q = \varepsilon C\frac{(s + 1/2\tau_L) + 1/2\tau_L - 1/\tau_C}{(s + 1/2\tau_L)^2 + \omega_0^2 - 1/4\tau_L^2}$$

Now compare with the standard results:

$$\mathcal{L}(\cos kt) = \frac{s}{s^2 + k^2}$$

$$\mathcal{L}(\sin kt) = \frac{k}{s^2 + k^2}$$

together with the shifting property:

$$\mathcal{L}\left(e^{-at}f(t)\right) = F(s + a)$$

So defining $\omega^2 \equiv \omega_0^2 - 1/4\tau_L^2$, we find the solution for q:

$$q(t) = \varepsilon C\exp(-t/2\tau_L)\left(\cos\omega t + \frac{1/2\tau_L - 1/\tau_C}{\omega}\sin\omega t\right)$$

and the potential difference across the plug is:

$$\Delta V = \frac{q}{C} = \varepsilon \exp(-t/2\tau_L)\left(\cos(\omega t) + \frac{1/2\tau_L - 1/\tau_C}{\omega}\sin(\omega t)\right)$$

Putting in the numbers, we have:

$$\tau_L = L/R = (10\text{ mH})/(10\ \Omega) = 1\text{ ms}$$

$$\tau_C = RC = (10\ \Omega)(0.1\ \mu\text{F}) = 1.0\ \mu\text{s}$$

and

$$\omega_0^2 = \sqrt{\frac{1}{LC}} = \sqrt{\frac{1}{(10\text{ mH})(0.1\ \mu\text{F})}} = \sqrt{1.0\times 10^9}\ (\text{rad/s})^2$$
$$= 3.1623\times 10^4\text{ rad/s}$$

Then:

$$\omega = \sqrt{\omega_0^2 - 1/4\tau_L^2} = \sqrt{1.0\times 10^9 - \frac{1}{4\times 10^{-6}}}\text{ rad/s} = 3.16\times 10^4\text{ rad/s}$$

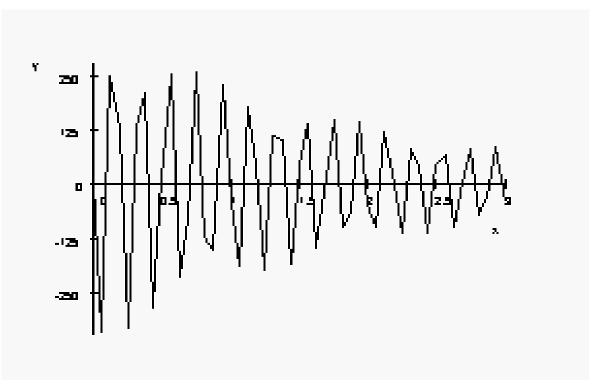
and

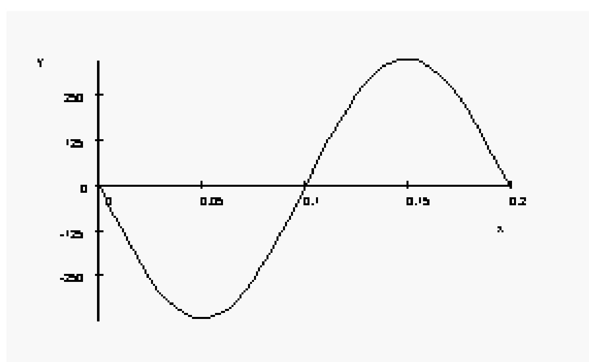
$$\frac{1/2\tau_L - 1/\tau_C}{\omega} = \frac{1/2\text{ ms} - 1/1\ \mu\text{s}}{3.16\times 10^4\text{ rad/s}} = -31.63$$

Thus:

$$\Delta V = (12\text{ V})\exp(-t/2\text{ ms})$$
$$\times\left(\cos\left((3.16\times 10^4\text{ rad/s})t\right) - 31.63\sin\left((3.16\times 10^4\text{ rad/s})t\right)\right)$$

(d) Plot your solution. What is the maximum potential difference achieved?





The function we plot is: $(12)\exp(-t/2)\left(\cos(31.6\,t) - 31.63\sin(31.6t)\right)$ where t is in ms. The plot shows potential difference in volts versus time in miliseconds. The plot on the right is an enlarged view. The maximum potential difference is achieved where

$$\frac{d}{dt}\Delta V = \frac{d}{dt}\varepsilon\exp(-t/2\tau_L)(\cos(\omega t) - 31.63\sin(\omega t)) = 0$$

$$-\frac{1}{2}\frac{\varepsilon}{\tau_L}\exp(-t/2\tau_L)(\cos\omega t-31.63\sin\omega t)-\varepsilon\omega\exp(-t/2\tau_L)((\sin\omega t)+31.63(\cos\omega t)) = 0$$

$$\exp(-t/2\tau_L)\left(\left(-\frac{1}{2\tau_L}-31.63\omega\right)\cos\omega t+\left(\frac{31.63}{2\tau_L}-\omega\right)\sin\omega t\right) = 0$$

or

$$\tan\omega t = \frac{\frac{1}{2\tau_L}+31.63\omega}{\frac{31.63}{2\tau_L}-\omega} = \frac{1+63.26\omega\tau_L}{15.815-\omega\tau_L}$$

$$= \frac{1+63.26\left(3.16\times10^4\text{ rad/s}\right)\left(1\times10^{-3}\text{ s}\right)}{15.815-\left(3.16\times10^4\text{ rad/s}\right)\left(1\times10^{-3}\text{ s}\right)}$$

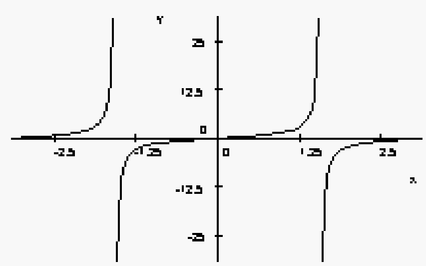
$$= \frac{1+63.26(31.6)}{15.815-31.6} = -126.7$$

Thus

$$t = \frac{1}{\omega}\tan^{-1}(-126.7) = \frac{1}{3.16\times10^4\text{ rad/s}}(-1.5629+\pi) = 4.9959\times10^{-5}\text{ s}$$

$$= 0.050\text{ ms}$$

where we have added π to get into the correct quadrant (see graph of tan function below)



Then the maximum potential difference is:

$$\Delta V = \left(12\text{ V}\right)\exp\left(-0.050/2\right)\left(\cos\left(31.6\times0.050\right)-31.63\sin\left(31.6\times0.050\right)\right)$$

$$= \boxed{-370.28\text{ V}}$$

(e) If the breakdown voltage of air is 3 MV/m, what spark plug gap would be required with this circuit? Remember that you would like the engine to start even if the battery is a bit low!

To get a field of 3 MV/m, we need a gap of:

$$d = \frac{|\Delta V|}{E} = \frac{370.28\text{ V}}{3\times10^6\text{ V/m}} = 1.2343\times10^{-4}\text{ m} = 0.12\text{ mm}$$

In order to allow for a margin of error, we ought to make the gap about 0.01 mm. This is pretty small. A transformer in the circuit increases the potential difference yet more, allowing for a larger gap.

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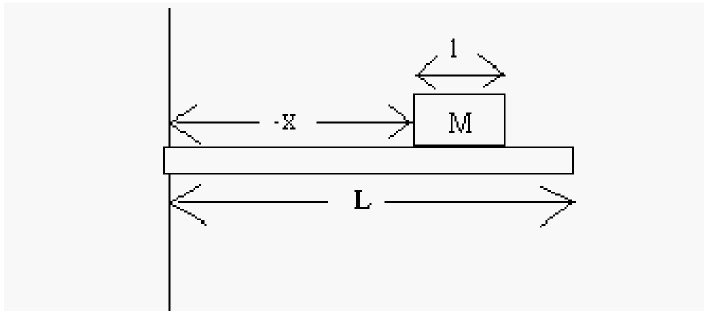
11. A beam is supported at one end, as shown in the diagram. A block of mass M and length l is placed on the beam, as shown. Write down the known conditions at $x = 0$. Use the Laplace transform to solve for the beam displacement.



The equation satisfied by the beam is:

$$\frac{d^4y}{dx^4} = \frac{q(x)}{EI}$$

where

$$q(x) = \begin{cases} 0 & \text{if } x < x_0 \\ Mg/l & \text{if } x_0 < x < x_0 + l \\ 0 & \text{if } x > x_0 + l \end{cases}$$

The initial conditions are: $y(0) = y'(0) = 0$. The second derivative is given by equation 3.8:

$$y''(0) = -\frac{1}{EI}m(0) = \frac{1}{EI}Mg\left(x_0 + \frac{l}{2}\right)$$

and then the third derivative, from equation 3.9, is:

$$y'''(0) = -\frac{1}{EI}t(0) = -\frac{1}{EI}Mg$$

Now we have enough information to solve the problem. First transform the whole equation:

$$s^4Y - s^3y(0) - s^2y'(0) - sy''(0) - y'''(0) = \frac{Q(s)}{EI}$$

where

$$\begin{aligned} Q(s) &= \int_0^\infty q(x)e^{-sx}dx = \int_{x_0}^{x_0+l} \frac{Mg}{l}e^{-sx}dx \\ &= \frac{Mg}{ls}e^{-sx_0}\left(1 - e^{-sl}\right) \end{aligned}$$

Thus:

$$s^4Y - s\frac{Mg}{EI}\left(x_0 + \frac{l}{2}\right) + \frac{Mg}{EI} = \frac{Mg}{EIls}e^{-sx_0}\left(1 - e^{-sl}\right)$$

and so

$$Y = \frac{Mg}{ls^5}e^{-sx_0}\left(1 - e^{-sl}\right) - \frac{Mg}{EIs^4} + \frac{Mg}{s^3EI}\left(x_0 + \frac{l}{2}\right)$$

The last two terms can be inverted immediately using Table 5.1: We get:

$$-\frac{Mg}{EI}\frac{x^3}{3!} + \frac{Mg}{EI}\left(x_0 + \frac{l}{2}\right)\frac{x^2}{2!}$$

The first term may be evaluated using the shifting property:

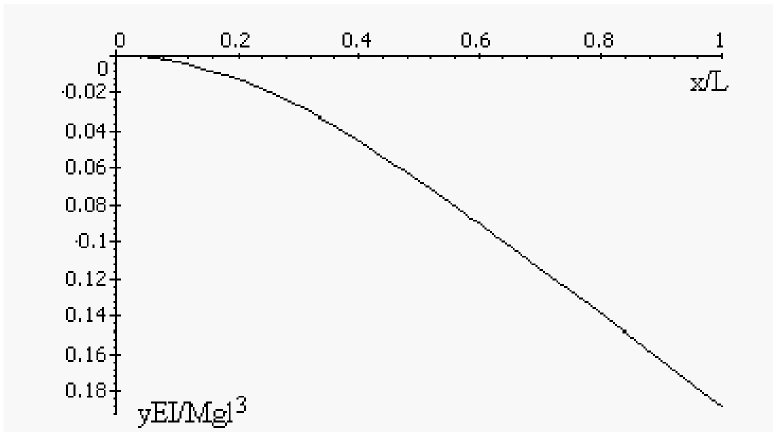
$$\frac{Mg}{4!EIl}\left(S(x - x_0)(x - x_0)^4 - S(x - x_0 - l)(x - x_0 - l)^4\right)$$

Thus the solution is:

$$y(x) = \frac{Mg}{EI}\left[\frac{1}{4!l}\left(S(x-x_0)(x-x_0)^4 - S(x-x_0-l)(x-x_0-l)^4\right) - \frac{x^3}{3!} + \left(x_0 + \frac{l}{2}\right)\frac{x^2}{2!}\right]$$

The quantity is square brackets is:

$$\begin{cases} -\frac{x^3}{3!} + \left(x_0 + \frac{l}{2}\right)\frac{x^2}{2!} & \text{if } x < x_0 \\ \frac{1}{4!l}\left((x-x_0)^4\right) - \frac{x^3}{3!} + \left(x_0 + \frac{l}{2}\right)\frac{x^2}{2!} & \text{if } x_0 < x < x_0 + l \\ \frac{1}{4!}\left((2(x-x_0)-l)\left(2(x-x_0)^2 - l(2(x-x_0)-l)\right)\right) - \frac{x^3}{3!} + \left(x_0 + \frac{l}{2}\right)\frac{x^2}{2!} & \text{if } x > x_0 + l \end{cases}$$



12. Technetium is used in medical procedures as a diagnostic tool. The technetium is obtained as the decay product of ^{99}Mo, which decays to ^{99m}Tc with a half-life of 66.02 h. The technetium in turn decays with a half life of 6.02 h. A medical radiology department receives a source containing 100mCi of ^{99}Mo at 9.00am on MOnday morning. Find the amount of Technetium present in the sample as a function of time after 9.00 am. When is the amount of ^{99m}Tc a maximum?

The amounts of the three elements (Mo, Tc and the decay product of Tc) are described by the equations:

$$\frac{dn_M}{dt} = -\lambda_1 n_m$$

$$\frac{dn_T}{dt} = \lambda_1 n_m - \lambda_2 n_T$$

and

$$\frac{dn_3}{dt} = \lambda_2 n_T$$

where λ_i are the decay rates. The decay rates are related to the half-lives. Let's transform each of the equations:

$$sN_m - n_m(0) = -\lambda_1 N_m$$

$$sN_T - n_T(0) = \lambda_1 N_m - \lambda_2 N_T$$

and

$$sN_3 - n_3(0) = \lambda_2 N_T$$

Solving, we find:

$$N_m = \frac{n_m(0)}{s + \lambda_1}$$

$$N_T = \frac{n_T(0) + \lambda_1 N_M}{s + \lambda_2} = \frac{n_T(0)}{s + \lambda_2} + \left(\frac{\lambda_1}{s + \lambda_2}\right)\left(\frac{n_m(0)}{s + \lambda_1}\right)$$

and similarly for N_3. Inverting the first equation gives:

$$n_m = n_m(0) e^{-\lambda_1 t}$$

Thus n_m decreases to one half of its initial value at time $t_{1/2}$ where:

$$\frac{1}{2} = e^{-\lambda_1 t_{1/2}}$$

and thus

$$\lambda_1 t_{1/2} = \ln 2$$

and thus

$$\lambda_1 = \frac{\ln 2}{t_{1/2}}$$

A similar relation holds for each species. Thus the decay rates are:

$$\lambda_1 = \frac{\ln 2}{66.02 \text{ h}}$$

and

$$\lambda_2 = \frac{\ln 2}{6.02 \text{ h}}$$

In the second equation, we set $n_T(0) = 0$ to get:

$$N_T = \left(\frac{\lambda_1}{s + \lambda_2}\right)\left(\frac{n_m(0)}{s + \lambda_1}\right) = \frac{\lambda_1 n_m(0)}{\lambda_1 - \lambda_2}\left(\frac{1}{s + \lambda_2} - \frac{1}{s + \lambda_1}\right)$$

and inverting, we get:

$$n_T(t) = \frac{\lambda_1 n_m(0)}{\lambda_1 - \lambda_2}\left(e^{-\lambda_2 t} - e^{-\lambda_1 t}\right)$$

Thus the maximum amount occurs at time t where

$$\frac{dn_T}{dt} = \frac{\lambda_1 n_m(0)}{\lambda_1 - \lambda_2}\left(-\lambda_2 e^{-\lambda_2 t} + \lambda_1 e^{-\lambda_1 t}\right) = 0$$

or

$$\begin{aligned}
e^{(\lambda_2 - \lambda_1)t} &= \frac{\lambda_2}{\lambda_1} \\
(\lambda_2 - \lambda_1)t &= \ln \frac{\lambda_2}{\lambda_1} \\
t &= \frac{1}{(\lambda_2 - \lambda_1)} \ln \frac{\lambda_2}{\lambda_1} \\
&= \frac{1}{\frac{\ln 2}{6.02 \text{ h}} - \frac{\ln 2}{66.02 \text{ h}}} \ln \frac{\ln 2}{6.02 \text{ h}} \frac{66.02 \text{ h}}{\ln 2} \\
&= \frac{(6.02 \text{ h})(66.02 \text{ h})}{(66.02 \text{ h} - 6.02 \text{ h}) \ln 2} \ln \frac{66.02}{6.02} \\
&= 22.887 \text{ h}
\end{aligned}$$

The maximum Technetium occurs at 9.00 am Monday +22.9 h = 7 h 54. m on Tuesday morning.

13. An overdamped harmonic oscillator satisfies the equation

$$\frac{d^2x}{dt^2} + 2\alpha\frac{dx}{dt} + \omega_0^2 x = f(t)$$

where $\alpha^2 > \omega_0^2$ and the driving force is a square wave of period T. Find the displacement $x(t)$ if the intial conditions are $x(0) = \frac{dx}{dt}\Big|_{t=0} = 0.$ Plot the result for $\alpha = 2\omega_0$ and $\alpha T = 1$, and $0 \le t \le 3T/2.$

First we transform the equation:

$$s^2 X + 2\alpha s X + \omega_0^2 X = F$$

and solve for the transform X:

$$X = \frac{F}{s^2 + 2\alpha s + \omega_0^2}$$

where

$$F = \frac{1}{s\left(1 + e^{-sT/2}\right)}$$

(Example 5.6). Thus:

$$\begin{aligned} X &= \frac{1}{s\left(1 + e^{-sT/2}\right)\left(s^2 + 2\alpha s + \omega_0^2\right)} \\ &= \sum_{n=0}^{\infty}(-1)^n e^{-snT/2}\frac{1}{s\left(s^2 + 2\alpha s + \omega_0^2\right)} \end{aligned}$$

We can use the shifting property to evaluate each term in the sum, so let's work on the multiplier

$$Y(s) = \frac{1}{s\left(s^2 + 2\alpha s + \omega_0^2\right)} = \frac{1}{s\left((s+\alpha)^2 + \omega_0^2 - \alpha^2\right)} = \frac{1}{s\left((s+\alpha)^2 - \omega_0^2\right)}$$

The factor $1/s$ tells us that the result is an integral, and we can invert the other factor using the attenuation property. Thus

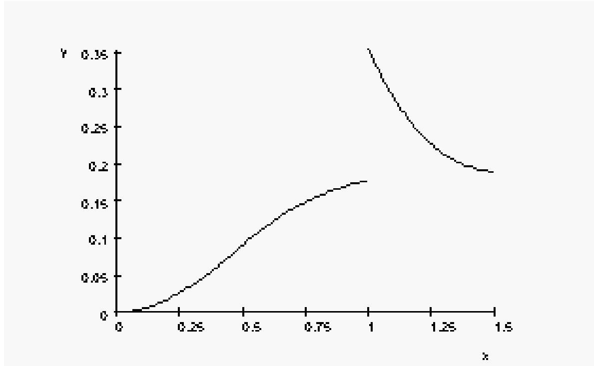
$$\begin{aligned} y(t) &= \frac{1}{2\omega_0}\int_0^t e^{-\alpha t}\left(e^{\omega_0 t} - e^{-\omega_0 t}\right)dt \\ &= \frac{1}{2\omega_0}e^{-\alpha t}\left(\frac{e^{\omega_0 t}}{\omega_0 - \alpha} + \frac{e^{-\omega_0 t}}{\omega_0 + \alpha}\right)\Bigg|_0^t \\ &= \frac{1}{2\omega_0}e^{-\alpha t}\left(\frac{e^{\omega_0 t}}{\omega_0 - \alpha} + \frac{e^{-\omega_0 t}}{\omega_0 + \alpha}\right) - \frac{1}{2\omega_0}\left(\frac{1}{\omega_0 - \alpha} + \frac{1}{\omega_0 + \alpha}\right) \\ &= \frac{1}{2\omega_0}e^{-\alpha t}\left(\frac{e^{\omega_0 t}}{\omega_0 - \alpha} + \frac{e^{-\omega_0 t}}{\omega_0 + \alpha}\right) + \frac{1}{\alpha^2 - \omega_0^2} \\ &= \frac{1}{2\omega_0}e^{-\alpha t}\left(\frac{(\omega_0 + \alpha)e^{\omega_0 t} + (\omega_0 - \alpha)e^{-\omega_0 t}}{\omega_0^2 - \alpha^2}\right) + \frac{1}{\alpha^2 - \omega_0^2} \\ &= \frac{1}{\alpha^2 - \omega_0^2} - e^{-\alpha t}\left(\frac{\cosh\omega_0 t + \frac{\alpha}{\omega_0}\sinh\omega_0 t}{\alpha^2 - \omega_0^2}\right) \end{aligned}$$

Thus:

$$x(t) = \sum_{n=0}^{\infty}(-1)^n S\left(t - n\frac{T}{2}\right)\left[\frac{1}{\alpha^2 - \omega_0^2} - e^{-\alpha\left(t - n\frac{T}{2}\right)}\left(\frac{\cosh\omega_0\left(t - n\frac{T}{2}\right) + \frac{\alpha}{\omega_0}\sinh\omega_0\left(t - n\frac{T}{2}\right)}{\alpha^2 - \omega_0^2}\right)\right]$$

Then with $\alpha = 2\omega_0$ and $\alpha T = 1 = 2\omega_0 T$, we get:

$$x(t) = \frac{4}{3}T^2\sum_{n=0}^{\infty}(-1)^n S\left(t - n\frac{T}{2}\right)\left[1 - \exp\left(-\frac{t}{T} + \frac{n}{2}\right)\left(\cosh\left(\frac{t}{2T} - \frac{1}{4}n\right) + 2\sinh\left(\frac{t}{2T} - \frac{1}{4}n\right)\right)\right]$$



14. (a) A harmonic oscillator with resonant frequency ω_0 is driven by a sinusoidal force $F(t) = F_0 \sin\omega t$. If the initial conditions are $x(0) = 0$, $dx/dt = 0$ at $t = 0$, find $x(t)$. What happens if $\omega = \omega_0$?

The equation is

$$\frac{d^2x}{dt^2} + \omega_0^2 x = \frac{F_0}{m}\sin\omega t$$

Transform both sides:

$$s^2 X + \omega_0^2 X = A\frac{\omega}{\omega^2 + s^2}$$

Thus

$$\begin{aligned} X(s) &= A\frac{\omega}{\left(s^2+\omega^2\right)\left(s^2+\omega_0^2\right)} \\ &= A\frac{\omega}{\left(\omega_0^2-\omega^2\right)}\left(\frac{1}{\left(s^2+\omega^2\right)} - \frac{1}{\left(s^2+\omega_0^2\right)}\right) \end{aligned}$$

Inverting, we get

$$x(t) = \frac{A}{\left(\omega_0^2-\omega^2\right)}\left(\sin\omega t - \frac{\omega}{\omega_0}\sin\omega_0 t\right)$$

If $\omega = \omega_0$, we back up to

$$X(s) = A\frac{\omega_0}{\left(s^2+\omega_0^2\right)^2}$$

We can invert this using the Mellin integral

$$x(t) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} A\frac{\omega_0}{\left(s^2+\omega_0^2\right)^2}e^{st}dt$$

There are 2 second order poles at $s = \pm i\omega_0$. The residues are:

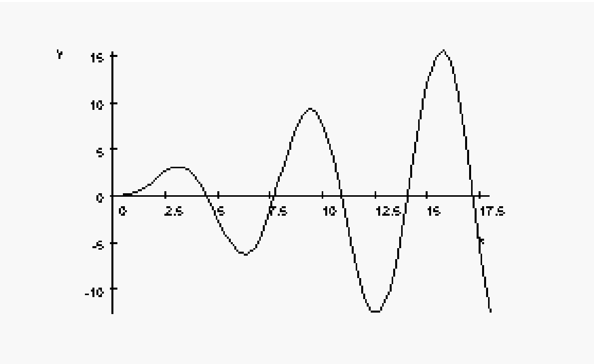
$$\begin{aligned} \lim_{s\to\pm i\omega_0}\frac{d}{ds}(s\mp i\omega_0)^2 A\frac{\omega_0}{\left(s^2+\omega_0^2\right)^2}e^{st} &= \lim_{s\to\pm i\omega_0}\frac{d}{ds}A\frac{\omega_0}{(s\pm i\omega_0)^2}e^{st} \\ &= A\omega_0\lim_{s\to\pm i\omega_0}\frac{-2}{(s\pm i\omega_0)^3}e^{st} + \frac{t}{(s\pm i\omega_0)^2}e^{st} \\ &= A\omega_0\left(\frac{-2}{(\pm 2i\omega_0)^3}e^{\pm i\omega_0 t} + \frac{t}{(\pm 2i\omega_0)^2}e^{\pm i\omega_0 t}\right) \\ &= A\omega_0\frac{-2\pm 2i\omega_0 t}{\mp 8i\omega_0^3}e^{\pm i\omega_0 t} \\ &= A\frac{-1\pm i\omega_0 t}{\mp 4i\omega_0^2}e^{\pm i\omega_0 t} \end{aligned}$$

Thus

$$x(t) = \frac{A}{2}\frac{(-1+i\omega_0 t)e^{i\omega_0 t} - (-1-i\omega_0 t)e^{-i\omega_0 t}}{-2i\omega_0^2}$$
$$= \frac{F_0}{2m\omega_0^2}(\sin\omega_0 t - \omega_0 t\cos\omega_0 t)$$

Check the derivative at $t = 0$

$$\frac{d}{dt}(\sin\omega_0 t - \omega_0 t\cos\omega_0 t) = \omega_0^2 t\sin\omega_0 t = 0 \text{ at } t = 0$$



$2m\omega_0^2 x/F_0$ versus $\omega_0 t$

Notice that this solution has an amplitude that increases linearly with t, indicating the resonance.

(b) If $x'(0) = a$, then

$$s^2X - a + \omega_0^2 X = A\frac{\omega}{\omega^2+s^2}$$
$$X = A\frac{\omega}{\omega^2+s^2}\frac{1}{s^2+\omega_0^2} + \frac{a}{s^2+\omega_0^2}$$

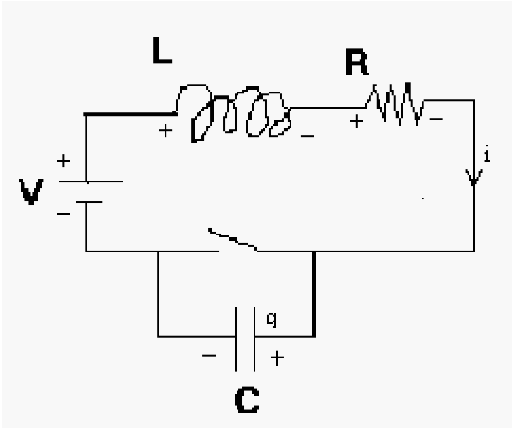
Thus

$$x(t) = \frac{A}{\left(\omega_0^2-\omega^2\right)}\left(\sin\omega t - \frac{\omega}{\omega_0}\sin\omega_0 t\right) + \frac{a}{\omega_0}\sin\omega_0 t$$

When $\omega = \omega_0$

$$\frac{A}{2\omega_0^2}\left(\left(1+\frac{a\omega_0}{A}\right)\sin\omega_0 t - \omega_0 t\cos\omega_0 t\right)$$

15. The two circuits shown in diagrams (a) and (b) show how we might use a capacitor to prevent sparks across a switch when the switch is opened. Assume that the switch has been closed for a long time, and is opened at $t = 0$. For each circuit, use Kirchhoff's rules to solve for the current through the inductor and the charge on the capacitor as a function of time after the switch is opened. Discuss the merits of each of the circuit designs.



Problem 15a

$$V = iR + L\frac{di}{dt} + \frac{q}{C}$$

$$i = \frac{dq}{dt}$$

Initial conditions:

$$i(0) = \frac{V}{R},\ \ q(0) = 0$$

Transform:

$$\frac{V}{s} = IR + sLI - L\frac{V}{R} + \frac{Q}{C}$$

and

$$I = sQ$$

Thus

$$I\left(R + sL + \frac{1}{sC}\right) = \frac{V}{s} + L\frac{V}{R}$$

Multiply by s and divide by L

$$I\left(s\frac{R}{L} + s^2 + \frac{1}{LC}\right) = \frac{V}{L} + s\frac{V}{R}$$

Now let $\omega_0^2 = 1/LC$ and $\alpha = R/2L$:

$$\begin{aligned} I\left(s^2 + 2\alpha s + \omega_0^2\right) &= \frac{V}{R}(2\alpha + s) \\ I &= \frac{V}{R}\frac{(2\alpha + s)}{\left(s^2 + 2\alpha s + \omega_0^2\right)} \\ &= \frac{V}{R}\frac{(s+\alpha) + \alpha}{(s+\alpha)^2 + \Omega^2} \end{aligned}$$

where

$$\Omega^2 = \omega_0^2 - \alpha^2$$

Now invert:

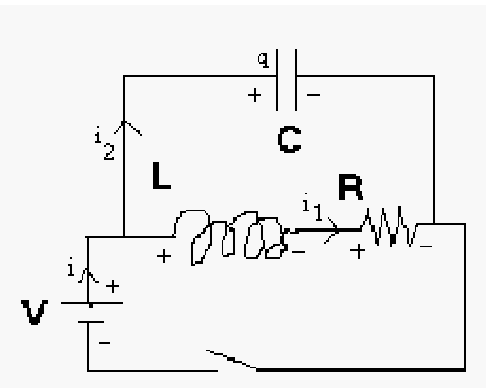
$$i(t) = \frac{V}{R}e^{-\alpha t}\left(\frac{\alpha}{\Omega}\sin\Omega t + \cos\Omega t\right)$$

The charge on the capacitor is:

$$
\begin{aligned}
q(t) &= \int_0^t i(t)dt = \int_0^t \frac{V}{R}e^{-\alpha t}\left(\frac{\alpha}{\Omega}\sin\Omega t + \cos\Omega t\right)dt \\
&= \frac{V}{R}\left(\frac{\alpha}{2i\Omega}\left[\frac{e^{-\alpha t+i\Omega t}-1}{-\alpha+i\Omega} - \frac{e^{-\alpha t-i\Omega t}-1}{-\alpha-i\Omega}\right] + \frac{1}{2}\left[\frac{e^{-\alpha t+i\Omega t}-1}{-\alpha+i\Omega} + \frac{e^{-\alpha t-i\Omega t}-1}{-\alpha-i\Omega}\right]\right) \\
&= \frac{V}{R}\left(\begin{array}{l} \frac{\alpha}{2i\Omega}\frac{(e^{-\alpha t+i\Omega t}-1)(-\alpha-i\Omega)-(-\alpha+i\Omega)(e^{-\alpha t-i\Omega t}-1)}{\alpha^2+\Omega^2} \\ +\frac{1}{2}\frac{(e^{-\alpha t+i\Omega t}-1)(-\alpha-i\Omega)+(-\alpha+i\Omega)(e^{-\alpha t-i\Omega t}-1)}{\alpha^2+\Omega^2} \end{array}\right) \\
&= \frac{V}{R}\left(\frac{\alpha}{\Omega}\frac{-\alpha e^{-\alpha t}\sin\Omega t - \Omega(e^{-\alpha t}\cos\Omega t - 1)}{\omega_0^2} + \frac{\alpha(1-e^{-\alpha t}\cos\Omega t) + \Omega e^{-\alpha t}\sin\Omega t}{\omega_0^2}\right) \\
&= \frac{V}{R}LC\left(2\alpha(1-e^{-\alpha t}\cos\Omega t) + \frac{\Omega^2-\alpha^2}{\Omega}e^{-\alpha t}\sin\Omega t\right) \\
&= VC\left((1-e^{-\alpha t}\cos\Omega t) + \frac{\omega_0^2-2\alpha^2}{2\alpha\Omega}e^{-\alpha t}\sin\Omega t\right)
\end{aligned}
$$

The charge $q(t) \to VC$ as $t \to \infty$, as expected.

(b)



Problem 15b

$$L\frac{di_1}{dt} + i_1R - \frac{q}{C} = 0$$

$$i_2 = \frac{dq}{dt}$$

Initial conditions:

$$i_1(0) = \frac{V}{R}$$

$$q(0) = CV$$

Now transform everything:

$$I_1R + sLI_1 - L\frac{V}{R} - \frac{Q}{C} = 0$$

$$I = I_1 + I_2 = 0$$

$$I_2 = sQ - CV$$

Substitute

$$(sL+R)I_1 - L\frac{V}{R} = \frac{I_2+CV}{sC} = -\frac{I_1}{sC} + \frac{V}{s}$$

$$\left(sL+R+\frac{1}{sC}\right)I_1 = L\frac{V}{R} + \frac{V}{s}$$

$$I_1 = \frac{LV/R + V/s}{\left(sL+R+\frac{1}{sC}\right)}$$

$$= \frac{V}{R}\frac{s+2\alpha}{s^2+2\alpha s+\omega_0^2}$$

From here the solution is almost the same as to part (a), and i_1 equals the i in part (a). (Compare equations and .)

When finding the charge, there is a sign change, and we must remember that $q(0) = CV$ so there is a result from the lower limit on the LHS. Thus:

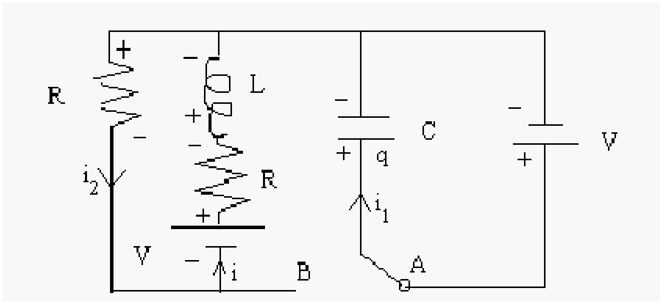
$$q(t) = VC\left(e^{-\alpha t}\cos\Omega t - \frac{\omega_0^2 - 2\alpha^2}{2\alpha\Omega}e^{-\alpha t}\sin\Omega t\right)$$

We find that $q \to 0$ as $t \to \infty$. This system is safer than (a) because the capacitor is uncharged in the "off" state.

Chapter 5: Laplace Transforms

16. The switch has been in position

A in the circuit shown for a long time. What is the charge on the capacitor and the current
i through the inductor? At time $t = 0$ the switch is moved to position
$B.$ What is the charge on the capacitor and the current a long time later? Find the charge on the capacitor as a function of time for
$t > 0.$ Give your answer in terms of ω_0 where $\omega_0^2 = 1/LC,$ $\alpha = R/L$ and
$\beta = 1/RC.$ You may also find it useful to define $\gamma = (\alpha + \beta)/2$ and $\Omega = \sqrt{2\alpha\beta - \gamma^2}.$ You may assume that
Ω is real.



Initially the capacitor charge is $q = CV$ and the current through the inductor is a constant and equals $V/2R.$

A long time after the switch is moved the current has the same value $V/2R$ and the capacitor charge is
$(V - iR)C = \frac{1}{2}VC.$ The opposite plate of the capacitor is positively charged compared with the initial state.

After the switch is moved Kirchhoff's loop and junction rules give:

$$i + i_1 = i_2$$

$$i_1 = \frac{dq}{dt}$$

$$V - iR - L\frac{di}{dt} + \frac{q}{C} = 0$$

and

$$i_2 R + \frac{q}{C} = 0$$

Now transform everything:

$$I + I_1 = I_2$$

$$I_1 = sQ - q(0) = sQ - VC$$

$$\frac{V}{s} - IR - sLI - Li(0) + \frac{Q}{C} = 0$$
$$\frac{V}{s} - IR - sLI - \frac{LV}{2R} + \frac{Q}{C} = 0$$

and

$$I_2 R + \frac{Q}{C} = 0$$

Eliminate I_1 and I_2 :

$$I = I_2 - I_1 = \frac{-Q}{RC} - sQ + VC = VC - Q(s + \beta)$$

Then:

$$\frac{V}{s} - (R + sL)(VC - Q(s+\beta)) - \frac{LV}{2R} + \frac{Q}{C} = 0$$

So

$$Q\left(\beta + (s+\beta)\left(1 + \frac{s}{\alpha}\right)\right) + \frac{V}{Rs} - VC\left(1 + \frac{s}{\alpha}\right) - \frac{V}{2R\alpha} = 0$$

and

$$\begin{aligned} Q &= VC\frac{-\frac{\alpha\beta}{s} + \alpha + s + \frac{\beta}{2}}{\alpha\beta + (s+\beta)(\alpha+s)} \\ &= \frac{VC}{s}\frac{-s^2 + \left(\alpha + \frac{\beta}{2}\right)s + \alpha\beta}{s^2 + s(\alpha+\beta) + 2\alpha\beta} \\ &= \frac{VC}{s}\frac{-s^2 + \left(\alpha + \frac{\beta}{2}\right)s + \alpha\beta}{s^2 + 2\gamma s + 2\alpha\beta} \\ &= \frac{VC}{s}\frac{-s^2 + \left(\alpha + \frac{\beta}{2}\right)s + \alpha\beta}{s^2 + 2\gamma s + \gamma^2 + 2\alpha\beta - \gamma^2} \\ &= \frac{VC}{s}\frac{-s^2 + \left(2\gamma - \frac{\beta}{2}\right)s + \alpha\beta}{(s+\gamma)^2 + \Omega^2} \end{aligned}$$

The denominator factors:

$$\begin{aligned} Q &= VC\left(-\frac{\alpha\beta}{s} + s + \gamma + \frac{\alpha}{2}\right)\frac{1}{(s+\gamma+i\Omega)(s+\gamma-i\Omega)} \\ &= VC\left(-\frac{\alpha\beta}{2i\Omega s}\left(\frac{1}{(s+\gamma-i\Omega)} - \frac{1}{(s+\gamma+i\Omega)}\right) + \frac{s+\gamma+\frac{\alpha}{2}}{(s+\gamma)^2+\Omega^2}\right) \end{aligned}$$

We invert using the attenuation property, and the integration rule:

$$\begin{aligned} q(t) &= VC\left(-\frac{\alpha\beta}{2i\Omega}\int_0^t e^{(i\Omega-\gamma)t} - e^{-(\gamma+i\Omega)t}dt + e^{-\gamma t}\left(\cos\Omega t + \frac{\gamma + \alpha/2}{\Omega}\sin\Omega t\right)\right) \\ &= VC\left(-\frac{\alpha\beta}{2i\Omega}\left[\frac{e^{(i\Omega-\gamma)t}}{i\Omega-\gamma} - \frac{e^{-(\gamma+i\Omega)t}}{-(\gamma+i\Omega)}\right]\Bigg|_0^t + e^{-\gamma t}\left(\cos\Omega t + \frac{\gamma+\alpha/2}{\Omega}\sin\Omega t\right)\right) \\ &= VC\left(-\frac{\alpha\beta}{2i\Omega}\left[\frac{e^{(i\Omega-\gamma)t}-1}{i\Omega-\gamma} + \frac{e^{-(\gamma+i\Omega)t}-1}{\gamma+i\Omega}\right] + e^{-\gamma t}\left(\cos\Omega t + \frac{\gamma+\alpha/2}{\Omega}\sin\Omega t\right)\right) \\ &= VC\left(\frac{\alpha\beta}{-2i\Omega}\left[\frac{e^{-(\gamma+i\Omega)t}\gamma - e^{-(\gamma-i\Omega)t}\gamma - i\Omega\left(e^{-(\gamma-i\Omega)t} + e^{-(\gamma+i\Omega)t}\right) + 2i\Omega}{\Omega^2+\gamma^2}\right] + e^{-\gamma t}\left(\cos\Omega t + \frac{\gamma+\alpha/2}{\Omega}\sin\Omega t\right)\right) \\ &= VC\left(-\frac{\alpha\beta}{\Omega}\left[\frac{-\gamma e^{-\gamma t}\sin\Omega t - \Omega e^{-\gamma t}\cos\Omega t + \Omega}{\Omega^2+\gamma^2}\right] + e^{-\gamma t}\left(\cos\Omega t + \frac{\gamma+\alpha/2}{\Omega}\sin\Omega t\right)\right) \\ &= VC\left(\frac{-\alpha\beta}{\Omega^2+\gamma^2} + e^{-\gamma t}\left(\cos\Omega t\left(1 + \frac{\alpha\beta}{\Omega^2+\gamma^2}\right) + \left(\frac{\gamma+\alpha/2}{\Omega} + \frac{\gamma\alpha\beta}{\Omega\left(\Omega^2+\gamma^2\right)}\right)\sin\Omega t\right)\right) \end{aligned}$$

But

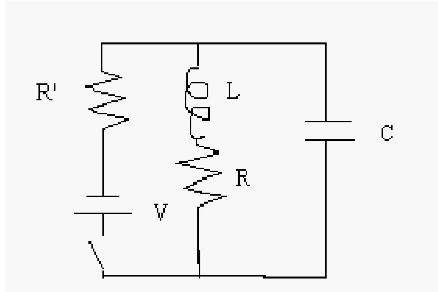
$$\Omega^2 + \gamma^2 = 2\alpha\beta - \gamma^2 + \gamma^2 = 2\alpha\beta$$

So:

$$q(t) = \frac{VC}{2}\left(-1 + e^{-\gamma t}\left(3\cos\Omega t + \frac{3\gamma+\alpha}{\Omega}\sin\Omega t\right)\right)$$

Check the limits: At $t = 0$, $q(t) = VC$ while as $t \to \infty$, $q \to -\frac{1}{2}VC$, as required.

17. The switch has been closed in the circuit shown for a long time, and a constant current flows. What is the charge on the capacitor? At time $t = 0$ the switch is opened. What is the charge on the capacitor and the current through the inductor a long time later? Find the current through the inductor as a function of time for $t > 0$. Give your answer in terms of ω_0 and α where $\omega_0^2 = 1/LC$ and $\alpha = R/2L$.



The constant curent is $V/(R+R')$ and the charge on the capacitor is $CVR/(R+R')$. A long time after opening the switch, the current and the charge on the capacitor are both zero.

The equations satsified by the charge and current result from an application of Kirchhoff's rules:

$$i_2 + i = i_1$$

$$i_2 = \frac{dq}{dt}$$

and after the switch is opened, $i_1 = 0$, so $i_2 = -i$. The loop rule gives:

$$\frac{q}{C} - iR - L\frac{di}{dt} = 0$$

Now transform everything:

$$I_2 = sQ - q(0) = sQ - \frac{VCR}{R+R'} = -I$$

$$\frac{Q}{C} - IR - sLI + Li(0) = 0$$
$$\frac{Q}{C} - IR - sLI + L\frac{V}{R+R'} = 0$$

Eliminate Q:

$$Q = \frac{1}{s}\left(\frac{VCR}{R+R'} - I\right)$$

Then

$$\frac{1}{sC}\left(\frac{VCR}{R+R'} - I\right) - IR - sLI + L\frac{V}{R+R'} = 0$$
$$I\left(-\frac{1}{sC} - R - sL\right) = -L\frac{V}{R+R'} - \frac{VR}{s(R+R')}$$

Thus

$$I = \frac{VC}{R+R'}\left(\frac{Ls+R}{1+RCs+s^2LC}\right)$$
$$= \frac{V}{R+R'}\left(\frac{s+2\alpha}{-\omega_0^2 - 2\alpha s - s^2}\right) = \frac{V}{R+R'}\left(\frac{(s+\alpha)+\alpha}{(s+\alpha)^2+\omega_0^2-\alpha^2}\right)$$

We may invert using the attentuation property and the known transforms of sine and cosine:

$$i(t) = \frac{V}{R+R'}e^{-\alpha t}\left(\cos\sqrt{\omega_0^2-\alpha^2}\,t + \frac{\alpha}{\sqrt{\omega_0^2-\alpha^2}}\sin\sqrt{\omega_0^2-\alpha^2}\,t\right)$$

Check: At $t = 0$, $i = \frac{V}{R+R'}$, while $i \to 0$ as $t \to \infty$.

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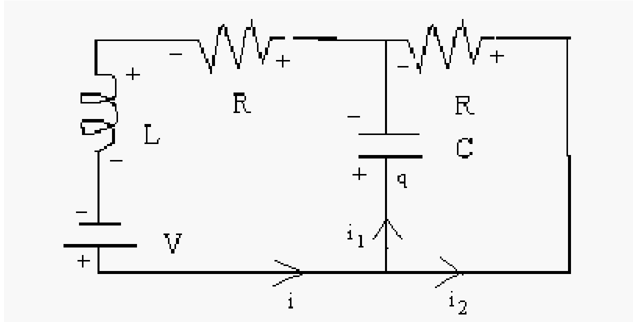
. The switch has been open in the circuit shown for a long time. At

$t = 0$ the switch is closed. Find the current through the inductor and the charge on the capacitor as functions of time for

$t > 0$. Give your answer in terms of ω_0, α and β where $\omega_0^2 = 1/LC$, $\alpha = R/2L$ and $\beta = \omega_0^2/4\alpha = \frac{1}{2RC}$.



With the switch open for a long time, the current $i = i_2 = V/2R$ and the charge on the capacitor is $VC/2$.

When the switch is closed, we define the variables as shown in the diagram, and apply Kirchhoff's rules:

$$i = i_1 + i_2$$

$$i_1 = \frac{dq}{dt}$$

$$V - \frac{q}{C} - iR - L\frac{di}{dt} = 0$$

and

$$\frac{q}{C} - i_2 R = 0$$

Transform everything:

$$I = I_1 + I_2$$

$$I_1 = sQ - q(0) = sQ$$

$$\frac{V}{s} - \frac{Q}{C} - IR - sLI + Li(0) = 0$$
$$\frac{V}{s} - \frac{Q}{C} - IR - sLI + L\frac{V}{2R} = 0$$

and

$$\frac{Q}{C} - I_2 R = 0$$

Eliminate I_1 and I_2:

$$I = sQ + \frac{Q}{RC} = Q\left(s + \frac{L}{RLC}\right) = Q\left(s + \frac{\omega_0^2}{2\alpha}\right) = Q(s + 2\beta)$$

Then:

$$\frac{V}{s} - \frac{Q}{C} - (R + sL)Q(s + 2\beta) + \frac{V}{4\alpha} = 0$$

Thus

$$\begin{aligned} Q &= \frac{\frac{V}{s} + \frac{V}{4\alpha}}{\frac{1}{C} + L(s+2\alpha)(s+2\beta)} \\ &= \frac{V}{L}\frac{1/s + 1/4\alpha}{(s+2\alpha)(s+2\beta) + \omega_0^2} \\ &= \frac{V}{L}\frac{1/s + 1/4\alpha}{s^2 + 2s(\beta+\alpha) + 4\alpha\beta + \omega_0^2} \\ &= \frac{V}{L}\frac{1/s + 1/4\alpha}{s^2 + 2s(\beta+\alpha) + 2\omega_0^2} \\ &= \frac{V}{4\alpha L}\frac{4\alpha/s + 1}{(s+\beta+\alpha)^2 - (\beta+\alpha)^2 + 2\omega_0^2} \\ &= \frac{V}{2R}\frac{4\alpha/s + 1}{(s+\beta+\alpha)^2 - \beta^2 - \alpha^2 + \frac{3}{2}\omega_0^2} \end{aligned}$$

Now define $\Omega = \sqrt{\frac{3}{2}\omega_0^2 - \alpha^2 - \beta^2}$. Then:

$$Q = \frac{V}{2R}\frac{4\alpha/s + 1}{(s+\beta+\alpha)^2 + \Omega^2}$$

and inverting gives:

$$q(t) = \frac{V}{2R}\left\{\int_0^t \frac{4\alpha}{\Omega}\exp\left(-(\alpha+\beta)t'\right)\sin\Omega t' dt' + \frac{1}{\Omega}\exp(-(\alpha+\beta)t)\sin\Omega t\right\}$$

The integral is:

$$\begin{aligned}\frac{1}{2i}\int_0^t e^{(-(\alpha+\beta)+i\Omega)t} - e^{(-(\alpha+\beta)-i\Omega)t}dt &= \frac{1}{2i}\frac{e^{(-(\alpha+\beta)+i\Omega)t}}{-(\alpha+\beta)+i\Omega} - \frac{e^{(-(\alpha+\beta)-i\Omega)t}}{-(\alpha+\beta)-i\Omega}\Bigg|_0^t \\ &= \frac{1}{2i}\left(\frac{e^{(-(\alpha+\beta)+i\Omega)t}-1}{-(\alpha+\beta)+i\Omega} + \frac{e^{(-(\alpha+\beta)-i\Omega)t}-1}{(\alpha+\beta)+i\Omega}\right)\end{aligned}$$

We may simplify the right hand side as follows:

$$\begin{aligned}&e^{-(\alpha+\beta)t}\frac{1}{2i}\left(\frac{-\left(e^{i\Omega t}-e^{-i\Omega t}\right)(\alpha+\beta) - i\Omega\left(e^{i\Omega t}+e^{-i\Omega t}\right)}{(\alpha+\beta)^2+\Omega^2}\right) + \frac{\Omega}{(\alpha+\beta)^2+\Omega^2} \\ &= -e^{-(\alpha+\beta)t}\frac{(\alpha+\beta)\sin\Omega t + \Omega\cos\Omega t}{(\alpha+\beta)^2+\Omega^2} + \frac{\Omega}{(\alpha+\beta)^2+\Omega^2}\end{aligned}$$

The denominator is:

$$\begin{aligned}(\alpha+\beta)^2+\Omega^2 &= \alpha^2 + 2\alpha\beta + \beta^2 + \frac{3}{2}\omega_0^2 - \alpha^2 - \beta^2 \\ &= 2\alpha\beta + \frac{3}{2}\omega_0^2 = 2\alpha\frac{\omega_0^2}{4\alpha} + \frac{3}{2}\omega_0^2 = 2\omega_0^2\end{aligned}$$

Thus:

$$\begin{aligned}q(t) &= \frac{V}{2R\Omega}\left\{\frac{2\alpha}{\omega_0^2}\left(\Omega - e^{-(\alpha+\beta)t}[(\alpha+\beta)\sin\Omega t + \Omega\cos\Omega t]\right) + e^{-(\alpha+\beta)t}\sin\Omega t\right\} \\ &= \frac{V}{2R\Omega}\left(\left(e^{-(\alpha+\beta)t}\left(\frac{3\omega_0^2/4 - \alpha^2}{\omega_0^2}\right)\sin\Omega t + \frac{\alpha\Omega}{\omega_0^2}\left(1 - e^{-(\alpha+\beta)t}\cos\Omega t\right)\right)\right)\end{aligned}$$

Thus at $t = 0$, $q(0) = 0$ and as $t \to \infty$,

$$q \to \frac{V}{2R\Omega}\frac{\alpha\Omega}{\omega_0^2} = \frac{V}{R}\frac{\alpha}{\omega_0^2} = \frac{V}{R}\frac{R}{2L}LC = \frac{1}{2}VC,$$

as required.

Now

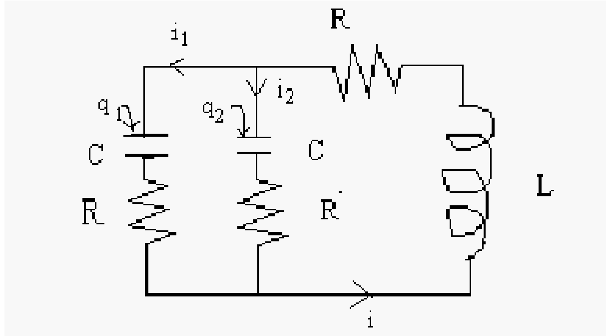
$$\begin{aligned}I = Q(s+2\beta) &= \frac{V}{2R}\frac{4\alpha/s+1}{(s+\beta+\alpha)^2+\Omega^2}(s+2\beta) \\ &= \frac{V}{2R}\frac{2\frac{\omega_0^2}{s} + s + 4\alpha + 2\beta}{(s+\beta+\alpha)^2+\Omega^2}\end{aligned}$$

We have already done most of the work to invert this. Thus

$$\begin{aligned}i(t) &= \frac{V}{2R}\left[\begin{array}{c}\frac{2\omega_0^2}{\Omega}\left(\frac{\Omega}{2\omega_0^2} - e^{-(\alpha+\beta)t}\frac{(\alpha+\beta)\sin\Omega t + \Omega\cos\Omega t}{2\omega_0^2}\right) \\ +2\frac{(2\alpha+\beta)}{\Omega}e^{-(\alpha+\beta)t}\sin\Omega t + e^{-(\alpha+\beta)t}\cos\Omega t\end{array}\right] \\ &= \frac{V}{2R}\left[\begin{array}{c}1 - e^{-(\alpha+\beta)t}\left\{\left(\frac{\alpha+\beta}{\Omega}\right)\sin\Omega t + \cos\Omega t\right\} \\ +2\frac{(2\alpha+\beta)}{\Omega}e^{-(\alpha+\beta)t}\sin\Omega t + e^{-(\alpha+\beta)t}\cos\Omega t\end{array}\right] \\ &= \frac{V}{2R}\left[1 - e^{-(\alpha+\beta)t}\left(\frac{3\alpha+\beta}{\Omega}\right)\sin\Omega t\right]\end{aligned}$$

The current is $V/2R$ both at $t = 0$ and as $t \to \infty$, again as expected.

19. In the figure shown, capacitor C_1 has charge Q and capacitor C_2 is uncharged. At $t = 0$ the switch is closed. The two capacitances are equal. Find the voltage across each capacitor as a function of time for $t > 0$.



For $t > 0$, we define the currents as shown. Then we apply Kirchhoff's loop and junction rules to get:

$$i = i_1 + i_2$$

$$i_1 = \frac{dq_1}{dt}$$

$$i_2 = \frac{dq_2}{dt}$$

$$\frac{q_1}{C} + i_1 R - i_2 R - \frac{q_2}{C} = 0$$

$$\frac{q_1}{C} + i_1 R + L\frac{di}{dt} + iR = 0$$

The initial conditions are:

$$q_1(0) = Q$$

$$q_2(0) = 0$$

$$i(0) = 0$$

Now we transform all the equations:

$$I = I_1 + I_2$$

$$I_1 = sQ_1 - q_1(0) = sQ_1 - Q$$

$$I_2 = sQ_2 - q_2(0) = sQ_2$$

$$\frac{Q_1}{C} + I_1 R - I_2 R - \frac{Q_2}{C} = 0$$

$$\frac{Q_1}{C} + I_1 R + sLI - Li(0) + IR = 0$$

Next we eliminate the currents:

$$I = s(Q_1 + Q_2) - Q$$

$$\frac{Q_1}{C} + (sQ_1 - Q)R + (sL + R)(s(Q_1 + Q_2) - Q) = 0$$

Now let $\alpha = \frac{1}{RC}$, $\frac{R}{L} = \beta,$ and $\omega_0^2 = \frac{1}{LC}$. We may rewrite the equations as:

$$\frac{Q_1}{RC} + (sQ_1 - Q) - sQ_2 - \frac{Q_2}{RC} = 0$$
$$(Q_1 - Q_2)(\alpha + s) = Q$$

and

$$\frac{Q_1}{RC} + (sQ_1 - Q) + \left(s\frac{L}{R} + 1\right)(s(Q_1 + Q_2) - Q) = 0$$
$$(\alpha + s)Q_1 - Q\left(2 + \frac{s}{\beta}\right) + s\left(\frac{s}{\beta} + 1\right)(Q_1 + Q_2) = 0$$
$$Q_1\left(s^2 + 2s\beta + \alpha\beta\right) + sQ_2(s + \beta) - Q(2\beta + s) = 0$$

Now eliminate Q_2 :

$$Q_2 = Q_1 - \frac{Q}{s + \alpha}$$

Then:

$$Q_1\left(s^2 + 2s\beta + \alpha\beta\right) + s\left(Q_1 - \frac{Q}{s + \alpha}\right)(s + \beta) - Q(2\beta + s) = 0$$
$$Q_1\left(s(s + \beta) + s^2 + 2s\beta + \alpha\beta\right) - Q\left(2\beta + s + \frac{s(s + \beta)}{s + \alpha}\right) = 0$$

$$\begin{aligned} Q_1 &= Q\frac{(2\beta + s) + s(s + \beta)/(s + \alpha)}{2s^2 + 3s\beta + \alpha\beta} \\ &= Q\frac{2s^2 + 3s\beta + 2\alpha\beta + s\alpha}{\left(2s^2 + 3s\beta + \alpha\beta\right)(s + \alpha)} \\ &= \frac{Q}{2}\left(\frac{1}{s + \alpha} + \frac{3\beta + 2s}{2s^2 + 3s\beta + \alpha\beta}\right) \\ &= \frac{Q}{2}\left(\frac{1}{s + \alpha} + \frac{1}{2}\frac{2s + 3\beta}{\left(s^2 + \frac{3}{2}s\beta + \frac{\alpha\beta}{2}\right)}\right) \end{aligned}$$

The first term inverts easily, to give

$$q_1^{\text{1st term}} = \frac{Q}{2}e^{-\alpha t}$$

For the second we use the attenuation property and Table 5.1:

$$\begin{aligned} Q_1\left(\text{2nd term}\right) &= \frac{Q}{4}\frac{2s + 3\beta}{\left(s^2 + \frac{3}{2}s\beta + \frac{9}{4}\beta^2 + \frac{\alpha\beta}{2} - \frac{9}{4}\beta^2\right)} \\ &= \frac{Q}{2}\frac{s + \frac{3}{4}\beta + \frac{3}{4}\beta}{\left(\left(s + \frac{3}{4}\beta\right)^2 + \frac{\alpha\beta}{2} - \frac{9}{16}\beta^2\right)} \end{aligned}$$

and thus

$$q_1^{\text{2nd term}} = \frac{Q}{2}e^{-3\beta t/4}\left(\cos\Omega t + \frac{3\beta}{4\Omega}\sin\Omega t\right)$$

where

$$\begin{aligned} \Omega^2 &= \frac{\alpha\beta}{2} - \frac{9}{16}\beta^2 = \frac{1}{2}\frac{R}{L}\left(\frac{1}{RC} - \frac{9}{8}\frac{R}{L}\right) \\ &= \frac{1}{2LC} - \left(\frac{3}{4}\frac{R}{L}\right)^2 \end{aligned}$$

This result can be understood as the appropriate frequency for an LRC circuit with the two capacitors in parallel ($C_{eq} = 2C$) and the two resistors in parallel, and that combination in series with the third resistor ($R_{eq} = \frac{3}{2}R$).

The final result is:

$$V_1(t) = Cq_1(t) = \frac{CQ}{2}e^{-\alpha t} + \frac{CQ}{2}e^{-3\beta t/4}\left(\cos\Omega t + \frac{3\beta}{4\Omega}\sin\Omega t\right)$$

This solution has the correct limiting forms at $t = 0$ and as $t \to \infty$. For Q_2 we get:

$$Q_2 = Q_1 - \frac{Q}{s + \alpha}$$

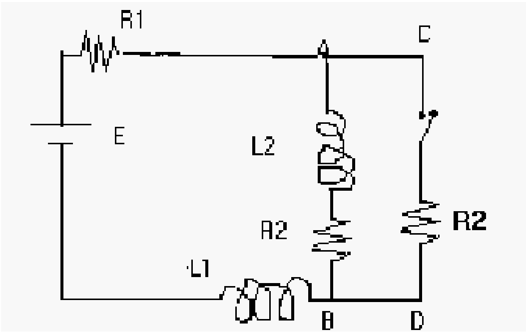
and thus

$$\begin{aligned} V_2 &= Cq_2 = C\left(q_1 - Qe^{-\alpha t}\right) \\ &= \frac{CQ}{2}e^{-3\beta t/4}\left(\cos\Omega t + \frac{3\beta}{4\Omega}\sin\Omega t\right) - \frac{CQ}{2}e^{-\alpha t} \end{aligned}$$

which also has the correct limiting forms.

20. The switch in the circuit shown in the Figure has been closed for a long time, and is opened at

$t = 0$. Find the currents in the circuit as a function of time for $t > 0$.



Circuit for Problem 20

We define the current variables i_1 through the battery, i_2 through the arm AB, and i_3 through the arm CD. Kirchhoff's junction rule gives:

$$i_1 = i_2 + i_3$$

Kirchhoff's loop rule gives the differential equation satisfied by the currents for $t > 0$:

$$L_1 \frac{di_1}{dt} + L_2 \frac{di_2}{dt} + R_1 i_1 + R_2 i_2 = E$$

The open switch imposes the condition $i_3 = 0$ for $t > 0$.

With the switch closed for a long time, the inductor plays no role and the currents are in a steady state. The resistors R_2 form a parallel combination of equivalent resistance $R_2/2$. The total resistance in the circuit is then $R_1 + R_2/2$. The initial conditions are $i_1(0) = E/(R_1 + R_2/2) = i_0$, and $i_2(0) = i_3(0) = i_0/2 = E/(2R_1 + R_2)$.

Transforming the equations, we get for $t > 0$

$$I_1 = I_2 + I_3 = I_2$$

$$L_1 \left(sI_1 - \frac{E}{R_1} \right) + L_2 s I_2 + R_1 I_1 + R_2 I_2 = \frac{E}{s}$$

which has the solution

$$I_1 = I_2 = \frac{\frac{E}{s} + L_1 i_0}{(L_1 + L_2)s + (R_1 + R_2)} = \frac{E\left(\frac{1}{s} + \frac{L_1}{R_1 + R_2/2} \right)}{(L_1 + L_2)\left(s + \frac{(R_1 + R_2)}{(L_1 + L_2)} \right)}$$

Inverting, we get:

$$\begin{aligned}
i_1(t) = i_2(t) &= \frac{E}{R_1 + R_2/2} \left(\frac{L_1}{L_1 + L_2} \right) \exp\left(-\frac{(R_1 + R_2)}{(L_1 + L_2)} t \right) + \frac{E}{(L_1 + L_2)} \int_0^t \exp\left(-\frac{(R_1 + R_2)}{(L_1 + L_2)} t \right) dt \\
&= \frac{E}{R_1 + R_2/2} \left(\frac{L_1}{L_1 + L_2} \right) \exp\left(-\frac{(R_1 + R_2)}{(L_1 + L_2)} t \right) - \frac{E}{(R_1 + R_2)} \left(\exp\left(-\frac{(R_1 + R_2)}{(L_1 + L_2)} t \right) - 1 \right) \\
&= E\left(\frac{1}{R_1 + R_2/2} \left(\frac{L_1}{L_1 + L_2} \right) - \frac{1}{(R_1 + R_2)} \right) \exp\left(-\frac{(R_1 + R_2)}{(L_1 + L_2)} t \right) + \frac{E}{(R_1 + R_2)} \\
&= E\left(\frac{R_2(L_1 - L_2) - 2R_1 L_2}{(2R_1 + R_2)(L_1 + L_2)(R_1 + R_2)} \right) \exp\left(-\frac{(R_1 + R_2)}{(L_1 + L_2)} t \right) + \frac{E}{(R_1 + R_2)} \\
&= \frac{E}{(R_1 + R_2)} \left\{ \frac{R_2(L_1 - L_2) - 2R_1 L_2}{(2R_1 + R_2)(L_1 + L_2)} \exp\left(-\frac{(R_1 + R_2)}{(L_1 + L_2)} t \right) + 1 \right\}
\end{aligned}$$

At $t = 0$,

$$\begin{aligned}
i_1(0) = i_2(0) &= \frac{E}{(R_1 + R_2)} \left\{ \frac{R_2(L_1 - L_2) - 2R_1 L_2}{(2R_1 + R_2)(L_1 + L_2)} + 1 \right\} \\
&= 2E \frac{L_1}{(2R_1 + R_2)(L_1 + L_2)}
\end{aligned}$$

in conflict with the known initial conditions $i_1(0) = E/(R_1 + R_2/2)$ and $i_2(0) = E/(2R_1 + R_2)$. that we used in equation (). The currents have to make an instantaneous jump at

$t = 0.$ The inductor will not let this happen. Instead, a spark will jump across the switch as it is opened. The solution we obtained does not correctly predict the behavior of the circuit because our system of equations fails to model the physical system correctly for a short time after the switch is opened. In practice, a capacitor is usually placed across the switch to absorb the current and avoid the spark.

2nd part

Rework the problem leaving the initial value of the current

i_2 in arm AB as an unknown to be found. Find the solution for the current i_1 through R_1 and require that it satisfy $i_1(0) = E/(R_1 + R_2/2)$. What value of $i_2(0)$ is required? Give a physical explanation of this result. If $R_1 = R_2$ and $L_1 = L_2$, plot both solutions. Plot current in units of E/R versus time in units of L/R. How long is it before both solutions give the same result to within 1%?

$$L_1\left(sI_1 - \frac{E}{R_1 + R_2/2}\right) + L_2(sI_2 - i_2(0)) + R_1 I_1 + R_2 I_2 = \frac{E}{s}$$

$$I_1 = I_2 = \frac{E/s + L_1 E/(R_1 + R_2/2) + L_2 i_2(0)}{(L_1 + L_2)s + (R_1 + R_2)}$$

Thus

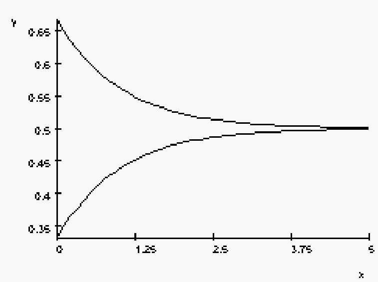
$$i_1 = i_2 = \left(\frac{L_1 E}{R_1 + R_2/2} + L_2 i_2(0)\right)\frac{1}{L_1 + L_2}\exp\left(-\frac{R_1 + R_2}{L_1 + L_2}t\right) - \frac{E}{R_1 + R_2}\left(\exp\left(-\frac{R_1 + R_2}{L_1 + L_2}t\right) - 1\right)$$

We need $i_1(0) = E/(R_1 + R_2/2)$, and we can get it by choosing $i_2(0) = E/(R_1 + R_2/2)$ too.

This means the current jumps from arm CD to arm AB instantaneously at $t = 0.$

If $R_1 = R_2$ and $L_1 = L_2$, plot both solutions. How long is it before both solutions give the same result to within 1%?

Plot i in units of E/R and t in units of $L/R.$



Values are within 1% after $t = 4.2(L/R)$

The difference is $\frac{1}{3}e^{-x}$

$\frac{e^{-x}}{3/2 - e^{-x}/2} = .01$, Solution is : $\{x = 4.2047\}$

$\frac{e^{-x}}{3/2 + e^{-x}/2} = .01$, Solution is : $\{x = 4.1947\}$

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21. The radioactive series that begins with Neptunium 93 contains the following decays:

Decay	type	half-life
$^{237}Np \to {}^{233}Pa$	α	2.14×10^6 y
$^{233}Pa \to {}^{233}U$	β	27.0 d
$^{233}U \to {}^{229}Th$	α	1.6×10^5 y
$^{229}Th \to {}^{225}Ra$	α	7340 y
$^{225}Ra \to {}^{225}Ac$	β	14.8 d
$^{225}Ac \to {}^{221}Fr$	α	10.0 d
$^{221}Fr \to {}^{217}At$	α	4.8 min
$^{217}At \to {}^{213}Bi$	α	0.032 s
$^{213}Bi \to {}^{213}Po$ (98%)	β	47 min
$^{213}Bi \to {}^{209}Tl$ (2%)	α	
$^{213}Po \to {}^{209}Pb$	α	4.2 μs
$^{209}Tl \to {}^{209}Pb$	β	2.2 min
$^{209}Pb \to {}^{209}Bi$	β	3.3 h

If we regard any decay that takes less than one year to be essentially instantaneous, then the chain simplifies to:

$$^{237}Np \to {}^{233}U \to {}^{229}Th \to {}^{209}Bi$$

Write a series of differential equations that describes this decay chain. Apply the Laplace transform to find the fraction of the original ^{237}Np that is in the form of Uranium, Thorium and Bismuth after (a) 10^5 and (b) 10^6 years.

Let the numbers be n_N, n_U etc. Then

$$\frac{dn_N}{dt} = -\lambda_1 n_N \to sN_N - n_N(0) = -\lambda_1 N_N$$

The solution is

$$N_N = \frac{n_N(0)}{s + \lambda_1}$$

Next

$$\frac{dn_U}{dt} = \lambda_1 n_N - \lambda_2 n_U \to sN_U = \lambda_1 N_N - \lambda_2 N_U$$

Thus

$$N_U = \frac{\lambda_1 N_N}{s + \lambda_2} = \left(\frac{\lambda_1}{s + \lambda_2}\right)\left(\frac{n_N(0)}{s + \lambda_1}\right)$$

Next

$$\frac{dn_T}{dt} = \lambda_2 n_U - \lambda_3 n_T \to sN_T = \lambda_2 N_U - \lambda_3 N_T$$

$$N_T = \frac{\lambda_2 N_U}{s + \lambda_3} = \left(\frac{\lambda_2}{s + \lambda_3}\right)\left(\frac{\lambda_1}{s + \lambda_2}\right)\left(\frac{n_N(0)}{s + \lambda_1}\right)$$

and finally

$$\frac{dn_B}{dt} = \lambda_3 n_T \to sN_B = \lambda_3 N_T$$

So

$$N_B = \frac{\lambda_3 N_T}{s} = \frac{\lambda_3}{s}\left(\frac{\lambda_2}{s + \lambda_3}\right)\left(\frac{\lambda_1}{s + \lambda_2}\right)\left(\frac{n_N(0)}{s + \lambda_1}\right)$$

Now we invert this using the Mellin integral. There are four poles, so we get

$$\frac{n_B(t)}{n_N(0)} = \lambda_1 \lambda_2 \lambda_3 \left\{ \begin{array}{c} \frac{1}{\lambda_3 \lambda_2 \lambda_1} - \frac{e^{-\lambda_3 t}}{\lambda_3(\lambda_2 - \lambda_3)(\lambda_1 - \lambda_3)} \\ - \frac{e^{-\lambda_2 t}}{\lambda_2(\lambda_3 - \lambda_2)(\lambda_1 - \lambda_2)} - \frac{e^{-\lambda_1 t}}{\lambda_1(\lambda_3 - \lambda_1)(\lambda_2 - \lambda_1)} \end{array} \right\}$$

Check this result at $t = 0$:

$$\begin{aligned}
\frac{n_B(0)}{n_M(0)} &= \lambda_1\lambda_2\lambda_3\left\{\begin{matrix} \frac{1}{\lambda_3\lambda_2\lambda_1} - \frac{e^{-\lambda_3 t}}{\lambda_3(\lambda_2-\lambda_3)(\lambda_1-\lambda_3)} \\ -\frac{e^{-\lambda_2 t}}{\lambda_2(\lambda_3-\lambda_2)(\lambda_1-\lambda_2)} - \frac{e^{-\lambda_1 t}}{\lambda_1(\lambda_3-\lambda_1)(\lambda_2-\lambda_1)} \end{matrix}\right\} \\
&= 1 - \frac{\lambda_1\lambda_2\lambda_3}{\lambda_3(\lambda_2-\lambda_3)(\lambda_1-\lambda_3)} - \frac{\lambda_1\lambda_2\lambda_3}{\lambda_2(\lambda_3-\lambda_2)(\lambda_1-\lambda_2)} - \frac{\lambda_1\lambda_2\lambda_3}{\lambda_1(\lambda_3-\lambda_1)(\lambda_2-\lambda_1)} \\
&= 1 - \frac{\lambda_1\lambda_2\lambda_3}{\lambda_3(\lambda_2-\lambda_3)(\lambda_1-\lambda_3)} - (-\lambda_2+\lambda_3-\lambda_1)\frac{\lambda_3}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)} \\
&= 1 - \frac{\lambda_2\lambda_1 + (-\lambda_2+\lambda_3-\lambda_1)\lambda_3}{(\lambda_2-\lambda_3)(\lambda_1-\lambda_3)} = 1 - \frac{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)}{(\lambda_2-\lambda_3)(\lambda_1-\lambda_3)} = 0
\end{aligned}$$

as required.

Similarly

$$\frac{n_T(t)}{n_M(0)} = \lambda_2\lambda_1\left\{\frac{e^{-\lambda_3 t}}{(\lambda_2-\lambda_3)(\lambda_1-\lambda_3)} + \frac{e^{-\lambda_2 t}}{(\lambda_3-\lambda_2)(\lambda_1-\lambda_2)} + \frac{e^{-\lambda_1 t}}{(\lambda_3-\lambda_1)(\lambda_2-\lambda_1)}\right\}$$

At $t = 0$

$$\begin{aligned}
\frac{n_T(0)}{n_M(0)} &= \lambda_2\lambda_1\left\{\frac{1}{(\lambda_2-\lambda_3)(\lambda_1-\lambda_3)} + \frac{1}{(\lambda_3-\lambda_2)(\lambda_1-\lambda_2)} + \frac{1}{(\lambda_3-\lambda_1)(\lambda_2-\lambda_1)}\right\} \\
&= \lambda_2\lambda_1\left(\frac{1}{(\lambda_2-\lambda_3)(\lambda_1-\lambda_3)} - \frac{1}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)}\right) = 0
\end{aligned}$$

as required.

$$\frac{n_U(t)}{n_M(0)} = \lambda_1\left\{\frac{e^{-\lambda_2 t} - e^{-\lambda_1 t}}{(\lambda_1-\lambda_2)}\right\}$$

The decay rates are:

$$\lambda = \frac{\ln 2}{t_{1/2}}$$

So

$$e^{-\lambda t} = e^{-t(\ln 2)/t_{1/2}} = \left(e^{-\ln 2}\right)^{t/t_{1/2}} = \left(\frac{1}{2}\right)^{t/t_{1/2}}$$

So at 10^5 y,

$$\begin{aligned}
\frac{n_U(t)}{n_M(0)} &= \frac{\lambda_1}{(\lambda_1-\lambda_2)}\left\{e^{-\lambda_2 t} - e^{-\lambda_1 t}\right\} = \frac{1}{(1-\lambda_2/\lambda_1)}\left\{e^{-\lambda_2 t} - e^{-\lambda_1 t}\right\} \\
&= \frac{1}{1-t_1/t_2}\left\{\left(\frac{1}{2}\right)^{t/1.6\times10^5\text{ y}} - \left(\frac{1}{2}\right)^{t/2.14\times10^6\text{ y}}\right\} \\
&= \frac{1}{1-2.14\times10^6/1.6\times10^5}\left\{\left(\frac{1}{2}\right)^{t/1.6\times10^5\text{ y}} - \left(\frac{1}{2}\right)^{t/2.14\times10^6\text{ y}}\right\} \\
&= -8.0808\times10^{-2}\left\{\left(\frac{1}{2}\right)^{t/1.6\times10^5\text{ y}} - \left(\frac{1}{2}\right)^{t/2.14\times10^6\text{ y}}\right\} \\
&= -8.0808\times10^{-2}\left(\left(\frac{1}{2}\right)^{1/1.6} - \left(\frac{1}{2}\right)^{1/2.14\times10}\right) \\
&= -8.0808\times10^{-2}(.64842 - .96813) = 2.5835\times10^{-2}
\end{aligned}$$

At 10^6 y

$$\frac{n_U(t)}{n_M(0)} = -8.0808\times10^{-2}\left(\left(\frac{1}{2}\right)^{10/1.6} - \left(\frac{1}{2}\right)^{1/2.14}\right) = 5.7388\times10^{-2}$$

and at 10^4 y

$$\frac{n_U(t)}{n_M(0)} = -8.0808\times10^{-2}\left(\left(\frac{1}{2}\right)^{1/16} - \left(\frac{1}{2}\right)^{1/214}\right) = 3.1647\times10^{-3}$$

Next we look at n_T

$$\begin{aligned}\frac{n_T(t)}{n_M(0)} &= \lambda_2\lambda_1\left\{\frac{e^{-\lambda_3 t}}{(\lambda_2-\lambda_3)(\lambda_1-\lambda_3)}+\frac{e^{-\lambda_2 t}}{(\lambda_3-\lambda_2)(\lambda_1-\lambda_2)}+\frac{e^{-\lambda_1 t}}{(\lambda_3-\lambda_1)(\lambda_2-\lambda_1)}\right\}\\
&= \frac{1}{t_2t_1}\left\{\frac{(1/2)^{t/t_3}}{(1/t_2-1/t_3)(1/t_1-1/t_3)}+\frac{(1/2)^{t/t_2}}{(1/t_3-1/t_2)(1/t_1-1/t_2)}+\frac{(1/2)^{t/t_1}}{(1/t_3-1/t_1)(1/t_2-1/t_1)}\right\}\\
&= \frac{(1/2)^{t/t_3}}{(1-t_2/t_3)(1-t_1/t_3)}+\frac{(1/2)^{t/t_2}}{(t_2/t_3-1)(1-t_1/t_2)}+\frac{(1/2)^{t/t_1}}{(t_1/t_3-1)(1-t_2/t_1)}\\
&= \frac{(1/2)^{t/7340\text{ y}}}{\left(1-1.6\times10^5/7340\right)\left(1-2.14\times10^6/7340\right)}+\frac{(1/2)^{t/1.6\times10^5\text{ y}}}{\left(1.6\times10^5/7340-1\right)\left(1-2.14\times10^6/1.6\times10^5\right)}\\
&\quad+\frac{(1/2)^{t/2.14\times10^6\text{ y}}}{\left(2.14\times10^6/7340-1\right)\left(1-1.6\times10^5/2.14\times10^6\right)}\\
&= \frac{(1/2)^{t/7340\text{ y}}}{6.043\times10^3}+\frac{(1/2)^{t/1.6\times10^5\text{ y}}}{-257.38}+\frac{(1/2)^{t/2.14\times10^6\text{ y}}}{268.83}\end{aligned}$$

So at 10^4 y

$$\frac{n_T(t)}{n_M(0)} = \frac{(1/2)^{10^4/7340}}{6.043\times10^3}+\frac{(1/2)^{1/16}}{-257.38}+\frac{(1/2)^{1/214}}{268.83} = 5.1574\times10^{-5}$$

at 10^5 y

$$\frac{n_T(t)}{n_M(0)} = \frac{(1/2)^{10^5/7340}}{6.043\times10^3}+\frac{(1/2)^{1/1.6}}{-257.38}+\frac{(1/2)^{1/21.4}}{268.83} = 1.082\times10^{-3}$$

and at $10^6 y$

$$\frac{n_T(t)}{n_M(0)} = \frac{(1/2)^{10^6/7340}}{6.043\times10^3}+\frac{(1/2)^{1/.16}}{-257.38}+\frac{(1/2)^{1/2.14}}{268.83} = 2.6396\times10^{-3}$$

For Bi we get

$$\begin{aligned}\frac{n_B(t)}{n_M(0)} &= \lambda_1\lambda_2\lambda_3\left\{\begin{array}{c}\frac{1}{\lambda_3\lambda_2\lambda_1}-\frac{e^{-\lambda_3 t}}{\lambda_3(\lambda_2-\lambda_3)(\lambda_1-\lambda_3)}\\ -\frac{e^{-\lambda_2 t}}{\lambda_2(\lambda_3-\lambda_2)(\lambda_1-\lambda_2)}-\frac{e^{-\lambda_1 t}}{\lambda_1(\lambda_3-\lambda_1)(\lambda_2-\lambda_1)}\end{array}\right\}\\
&= 1-\left(\frac{1}{2}\right)^{t/t_3}\frac{1}{(1-t_2/t_3)(1-t_1/t_3)}-\left(\frac{1}{2}\right)^{t/t_2}\frac{1}{(1-t_3/t_2)(1-t_1/t_2)}\\
&\quad-\left(\frac{1}{2}\right)^{t/t_1}\frac{1}{(1-t_3/t_1)(1-t_2/t_1)}\\
&= 1-\frac{(1/2)^{t/7340}}{\left(1-1.6\times10^5/7340\right)\left(1-2.14\times10^6/7340\right)}-\frac{(1/2)^{t/1.6\times10^5}}{\left(1-7340/1.6\times10^5\right)\left(1-2.14\times10^6/1.6\times10^5\right)}\\
&\quad-\left(\frac{1}{2}\right)^{t/2.14\times10^6}\frac{1}{\left(1-7340/2.14\times10^6\right)\left(1-1.6\times10^5/2.14\times10^6\right)}\\
&= 1-\left(\frac{1}{2}\right)^{t/7340}\frac{1}{6.0429\times10^3}+\left(\frac{1}{2}\right)^{t/1.6\times10^5}\frac{1}{11.807}-\left(\frac{1}{2}\right)^{t/2.14\times10^6}\frac{1}{.92206}\end{aligned}$$

$$\begin{aligned}1-\frac{1}{6.0429\times10^3}+\frac{1}{11.807}-\frac{1}{.92206} &= 1-1.6548\times10^{-4}+8.4696\times10^{-2}-1.0845\\ &= .99983-.9998 = 1.914\times10^{-6}\end{aligned}$$

So at 10^4 y

$$\begin{aligned}\frac{n_B(t)}{n_M(0)} &= 1-\left(\frac{1}{2}\right)^{10^4/7340}\frac{1}{6.043\times10^3}+\left(\frac{1}{2}\right)^{1/16}\frac{1}{11.807}-\left(\frac{1}{2}\right)^{1/214}\frac{1}{.92206}\\ &= 1.9334\times10^{-5}\end{aligned}$$

At 10^5

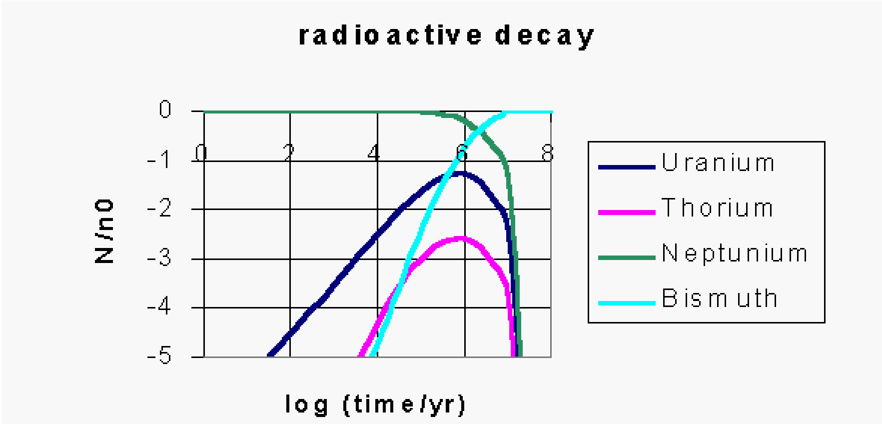
$$\begin{aligned}\frac{n_B(t)}{n_M(0)} &= 1-\left(\frac{1}{2}\right)^{10^5/7340}\frac{1}{6.043\times10^3}+\left(\frac{1}{2}\right)^{1/1.6}\frac{1}{11.807}-\left(\frac{1}{2}\right)^{1/21.4}\frac{1}{.92206}\\ &= 4.9552\times10^{-3}\end{aligned}$$

and at 10^6

$$\begin{aligned}\frac{n_B(t)}{n_M(0)} &= 1-\left(\frac{1}{2}\right)^{10^6/7340}\frac{1}{6.043\times10^3}+\left(\frac{1}{2}\right)^{1/.16}\frac{1}{11.807}-\left(\frac{1}{2}\right)^{1/2.14}\frac{1}{.92206}\\ &= .21665\end{aligned}$$

and of course

$$n_M(t) = n_M(0)e^{-\lambda_1 t} = \frac{n_M(0)}{2^{t/t_1}} = \frac{n_M(0)}{2^{t/2.14\times10^6\text{ y}}}$$



22. Find the Laplace transform of the function $g(t) = t\frac{df}{dt}$. Express the result in terms of the transform $F(s)$ of the function $f(t)$. Use the result to solve the differential equation

$$ty' + y = e^{-t}$$

subject to the intial conditions $y(0) = 1$

From equation 12:

$$\begin{aligned} G &= -\frac{d}{ds}\mathcal{L}\left(\frac{df}{dt}\right) = -\frac{d}{ds}(sF - f(0)) \\ &= -F - s\frac{dF}{ds} \end{aligned}$$

Transforming the differential equation:

$$\left(-Y - s\frac{dY}{ds}\right) + Y = \frac{1}{1+s}$$

Simplifying:

$$\begin{aligned} s\frac{dY}{ds} &= -\frac{1}{1+s} \\ \frac{dY}{ds} &= -\frac{1}{s(1+s)} \\ Y(s) - Y_\infty &= \int_s^\infty \left(\frac{1}{\sigma} - \frac{1}{1+\sigma}\right) d\sigma \end{aligned}$$

The left hand side = $Y(s)$ because $Y \to 0$ as $s \to \infty$. Then using the integration property (§4.2) and the fact that the inverse transforms of the two terms are e^{-t} and 1, we have:

$$y(t) = \frac{1 - e^{-t}}{t}$$

Check that this has the right initial condition:

$$\lim_{t \to 0} = \frac{1 - (1 - t)}{t} = 1$$

Let's stuff into the equation to test the solution:

$$ty' = t\left(-\frac{1 - e^{-t}}{t^2} + \frac{e^{-t}}{t}\right) = e^{-t} - \frac{1 - e^{-t}}{t}$$

Thus

$$ty' + y = e^{-t} - \frac{1 - e^{-t}}{t} + \frac{1 - e^{-t}}{t} = e^{-t}$$

as required.

23. Apply the Laplace transform to the differential equation:

$$y'' - t^2 y = t^2$$

Does the Laplace transform offer any advantages in solving this equation? Using any method of your choice solve the original equation or the transformed equation subject to the initial conditions $y(0) = 1$ and $y'(0) = 0$, and comment.

The transformed equation is:

$$- Y'' + s^2 Y - sy(0) - y'(0) = \frac{2}{s^3}$$
$$- Y'' + s^2 Y = \frac{2}{s^3} + s$$

and this equation is more complicated than the original.

Series method: Original equation

$$\sum n(n-1)a_n t^{n-2} - \sum a_n t^{n+2} = t^2$$

lowest power is t^0

$$2a_2 = 0$$
$$3 \times 2a_3 = 0$$
$$4 \times 3a_4 - a_0 = 1$$
$$a_4 = \frac{1 + a_0}{12}$$

$$5 \times 4a_5 - a_1 = 0$$
$$a_5 = \frac{a_1}{20}$$

$$(m+2)(m+1)a_{m+2} - a_{m-2} = 0$$
$$a_{m+2} = \frac{a_{m-2}}{(m+2)(m+1)}$$
$$a_n = \frac{a_{n-4}}{n(n-1)}$$

Thus the solution is

$$y(t) = a_0 + (1 + a_0)\left(\frac{t^4}{12} + \frac{t^8}{8.7.4.3} + \cdots\right) + a_1 t\left(1 + \frac{t^4}{20} + \frac{t^8}{9.8.5.4} + \cdots\right)$$

With the intial conditions, we have $a_1 = 0$ and $a_0 = 1$

$$y(t) = 1 + 2\left(\frac{t^4}{12} + \frac{t^8}{8.7.4.3} + \cdots\right)$$

Check

$$y' = 2\left(\frac{t^3}{3} + \frac{t^7}{7.4.3} + \cdots\right)$$
$$y'' = 2\left(t^2 + \frac{t^6}{4.3} + \cdots\right)$$
$$t^2 y = t^2 + 2\left(\frac{t^6}{12} + \frac{t^{10}}{8.7.4.3} + \cdots\right)$$

So

$$y'' - t^2 y = t^2$$

as required.

Solving the transformed equation is much more difficult.

We can transform the solution to get:

$$y(t) = 1 + 2\left(\frac{t^4}{12} + \frac{t^8}{8.7.4.3} + \cdots\right)$$
$$Y(s) = \frac{1}{s} + \frac{4}{s^5} + \frac{6.5.4}{s^9} + \cdots$$

Check this in the de

$$Y' = -\frac{1}{s^2} - \frac{5.4}{s^6} - \frac{9.6.5.4}{s^{10}} + \cdots$$
$$Y'' = \frac{2}{s^3} + \frac{6.5.4}{s^7} + \frac{10.9.6.5.4}{s^{11}} + \cdots$$

$$- Y'' + s^2 Y = \frac{2}{s^3} + s$$
$$- \frac{2}{s^3} - \frac{6.5.4}{s^7} - \frac{10.9.6.5.4}{s^{11}} + \cdots + s^2\left(\frac{1}{s} + \frac{4}{s^5} + \frac{6.5.4}{s^9} + \cdots\right) = s + \frac{2}{s^3} + \cdots$$

as required.

Look for a solution

$$\begin{aligned} Y &= \sum \frac{a_n}{s^n} \\ Y' &= \sum -n\frac{a_n}{s^{n+1}} \\ Y'' &= \sum n(n+1)\frac{a_n}{s^{n+2}} \end{aligned}$$

$$Y'' - s^2 Y = \sum n(n+1)\frac{a_n}{s^{n+2}} - \sum \frac{a_n}{s^{n-2}} = -s - \frac{2}{s^3}$$

The successive powers are:

s^2 :

$$a_0 = 0$$

s^1

$$a_1 = 1$$

s^0

$$a_2 = 0$$

s^{-1}

$$a_3 = 0$$

s^{-2}

$$a_4 = 0$$

s^{-3}

$$\begin{aligned} 2a_1 - a_5 &= -2 \\ a_5 &= 2 + 2 = 4 \end{aligned}$$

s^{-p}, $p > 5$

$$\begin{aligned} (p-2)(p-1)a_{p-2} - a_{p+2} &= 0 \\ a_n &= (n-4)(n-3)a_{n-4} \end{aligned}$$

Thus

$$Y = \frac{1}{s} + \frac{4}{s^5} + \frac{6.5.4}{s^9}\ldots$$

The transform doesn't help!

24. Take the Laplace transform of the Bessel equation of order zero

$$y'' + \frac{1}{x}y' + y = 0$$

and show that

$$\left(s^2+1\right)Y'(s) + sY(s) = 0$$

Solve for $Y(s)$ with the initial condition $y(0) = 1$ and hence find an integral expression for $J_0(x)$.

The tricky term here is y'/x. So multiply the whole equation by x :

$$xy'' + y' + xy = 0$$

Now use equation 5.12

$$-\frac{d}{ds}\left(s^2Y - sy(0) - y'(0)\right) + sY - y(0) - \frac{d}{ds}(Y) = 0$$

The unknown $y'(0)$ disappears:

$$\begin{aligned} -2sY - s^2Y' + y(0) + sY - y(0) - Y' &= 0 \\ \left(s^2+1\right)Y' + sY &= 0 \end{aligned}$$

The initial value $y(0)$ also disappears. Then

$$\frac{Y'}{Y} = -\frac{s}{s^2+1}$$

Integrating

$$\ln Y = -\frac{1}{2}\ln\left(s^2+1\right) + C$$

or

$$Y = \frac{A}{\sqrt{s^2+1}}$$

Inverting the transform,

$$J_0(x) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{A}{\sqrt{s^2+1}} e^{sx} ds$$

where $\gamma > 0.$

$$J_0(x) = \frac{A}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} \frac{e^{sx}}{\sqrt{s^2+1}} ds$$

There are branch points on the imaginary axis at $\pm i.$

(Ref: Jeffreys &Jeffreys pg 581, Morse and Feshbach pg 619-624, Gradshtyen and Ryzhik). By choosing different branches of the integrand, and moving the contour appropriately, we can obtain expressions for the Hankel functions as well as for

$J.$ (See Chapter 8.)

By choosing the branch cuts to run from $-i$ to i $\left(cf \text{ Ch } 2\,\S 2.2.1\right)$, we obtain the form

$$J_0(x) = \frac{A}{\pi i}\int_{-i}^{+i} \frac{e^{sx}}{\sqrt{s^2+1}} ds$$

Evaluate this at $x = 0$

$$J_0(0) = \frac{A}{\pi i}\int_{-1}^{+1} \frac{d(iu)}{\sqrt{1-u^2}}$$

Let $u = \sin\theta$

$$J_0(0) = \frac{A}{\pi}\int_{-\pi/2}^{\pi/2} \frac{\cos\theta d\theta}{\cos\theta} = A = 1$$

and thus $A = 1,$ giving

$$J_0(x) = \frac{1}{\pi i}\int_{-i}^{+i} \frac{e^{sx}}{\sqrt{s^2+1}} ds$$

Chapter 6: Generalized functions in physics

1. Show that the following sequences of functions are delta-sequences:

(a)

$$\phi_n(x) = \frac{n}{2} e^{-n|x|}$$

$$\begin{aligned}\int_{-\infty}^{+\infty} \phi_n(x) f(x) dx &= \int_{-\infty}^{+\infty} \frac{n}{2} e^{-n|x|} f(x) dx \\ &= \left(\int_{-\infty}^{-1/\sqrt{n}} + \int_{-1/\sqrt{n}}^{+1/\sqrt{n}} + \int_{+1/\sqrt{n}}^{+\infty} \right) \frac{n}{2} e^{-n|x|} f(x) dx \\ &= I_1 + I_2 + I_3\end{aligned}$$

Then since $\phi_n(x) \geq 0$ for all values of x, the mean value theorem gives:

$$\begin{aligned}I_1 &= f(\xi) \int_{-\infty}^{-1/\sqrt{n}} \frac{n}{2} e^{-n|x|} dx = f(\xi) \int_{-\infty}^{-1/\sqrt{n}} \frac{n}{2} e^{nx} dx \\ &= \frac{f(\xi)}{2} n \frac{e^{nx}}{n} \Big|_{-\infty}^{-1/\sqrt{n}} = \frac{f(\xi)}{2} e^{-\sqrt{n}}\end{aligned}$$

where $-\infty \leq \xi \leq -1/\sqrt{n}$. Thus

$$I_1 \to 0 \text{ as } n \to \infty$$

An almost identical proof shows that $I_3 \to 0$ as $n \to \infty$.

Finally for I_2 we have:

$$I_2 = \int_{-1/\sqrt{n}}^{+1/\sqrt{n}} \frac{n}{2} e^{-n|x|} f(x) dx = f(\xi) \int_{-1/\sqrt{n}}^{+1/\sqrt{n}} \frac{n}{2} e^{-n|x|} f(x) dx$$

where $1/\sqrt{n} \leq \xi \leq -1/\sqrt{n}$. Thus

$$\begin{aligned}I_2 &= f(\xi) \frac{n}{2} \left(\int_{-1/\sqrt{n}}^{0} e^{nx} dx + \int_{0}^{+1/\sqrt{n}} e^{-nx} dx \right) \\ &= f(\xi) \frac{n}{2} \left(\frac{e^{nx}}{n} \Big|_{-1/\sqrt{n}}^{0} + \frac{e^{-nx}}{-n} \Big|_{0}^{1/\sqrt{n}} \right) \\ &= f(\xi) \frac{n}{2} \left(\frac{1 - e^{-\sqrt{n}}}{n} + \frac{e^{-\sqrt{n}} - 1}{-n} \right) \\ &= f(\xi) \frac{2}{2} \left(1 - e^{-\sqrt{n}} \right) \to f(0) \text{ as } n \to \infty\end{aligned}$$

Thus this ϕ_n is a delta sequence.

(b)

$$\phi_n(x) = \frac{n}{\pi} \frac{1}{1 + n^2 x^2}$$

Hint: Use contour integration.

Multiply by a test function that is analytic in the UHP and integrate by closing the contour in the UHP:

$$I = \int_{-\infty}^{+\infty} f(x) \frac{n}{\pi} \frac{1}{1 + n^2 x^2} dx = \oint \frac{n}{\pi} \frac{f(z)}{1 + n^2 z^2} dz$$

The poles are at $z = \pm i/n$. Only the pole at $+i/n$ is inside the contour, so:

$$I = \frac{n}{\pi}\frac{2\pi i}{n^2}\left(\frac{f(i/n)}{2i/n}\right) = f(i/n)$$

Thus in the limit:

$$\lim_{n\to\infty}\int_{-\infty}^{+\infty}\phi_n(x)dx = \lim_{n\to\infty}f(i/n) = f(0)$$

which is the sifting property. Thus ϕ_n is a delta-sequence.

(c)

$$\phi_n(x) = \frac{1-\cos nx}{n\pi x^2}$$

We evaluate

$$\int_{-\infty}^{+\infty}\frac{1-\cos nx}{n\pi x^2}f(x)dx$$

We evaluate along the real axis in the complex plane. We'll displace the path of integration downward so that it lies beneath the (removable) singularity at $x = 0$. Then we write the integral as:

$$I = \int_{-\infty}^{+\infty}\frac{1-\cos nx}{n\pi x^2}f(x)dx = \int_{\text{real axis}}\frac{2-e^{inz}-e^{-inz}}{2n\pi z^2}f(z)dz$$

Then provided that $f(z)$ is analytic except for a set of poles, we have:

$$I = I_1 + I_2 = \int_{\text{real axis}}\frac{1-e^{inz}}{2n\pi z^2}f(z)dz + \int_{\text{real axis}}\frac{1-e^{-inz}}{2n\pi z^2}f(z)dz$$

We close the contour upward for the first integral and downward for the second. In both cases the integral along the big semi-circle goes to zero, and we can use the residue theorem. For the first integral, the pole at $z = 0$ is inside the contour, and

$$I_1 = 2\pi i\left(\frac{-in}{2n\pi}f(0) + \sum_p \operatorname{Res}\left(f\frac{1-e^{inz}}{2n\pi z^2}\right)_{z_p}\right)$$

(Note: the exponential is bounded because $y_p > 0$ for poles in the upper-half plane.) Then:

$$\lim_{n\to\infty} I_1 = f(0)$$

For the second integral, the pole at $z = 0$ is not inside the contour and the integral is zero in the limit. Thus

$$\int_{-\infty}^{+\infty}\frac{1-\cos nx}{n\pi x^2}f(x)dx = f(0)$$

and the sifting property holds. Thus ϕ_n is a delta sequence.

2. Find a Fourier series representation of the delta function $\delta(x)$ in the range $(-L,+L)$ in two ways.

(a) Start with the fourier series for a step function (cf equations 4.11 or 4.20) and differentiate.

The series (4.11) is a step function with a downward step at $x = 0$. We modify it to give an upward step at $x = 0$, and a full period of $2L$. Then the series we want is:

$$\Theta(x) = 1 - \left(\frac{1}{2} - \frac{2}{\pi}\sum_{n=1,\,\text{odd}}^{\infty}\frac{\sin n\pi x/L}{n}\right) = \frac{1}{2} + \frac{2}{\pi}\sum_{n=1,\,\text{odd}}^{\infty}\frac{\sin n\pi x/L}{n}$$

Now we take the derivative:

$$\delta(x) = \frac{d\Theta}{dx} = \frac{2}{\pi}\sum_{n=1,\text{odd}}^{\infty}\frac{n\pi}{L}\frac{\cos n\pi x/L}{n} = \frac{2}{L}\sum_{n=1,\text{odd}}^{\infty}\cos n\pi x/L$$

which is a series with constant coefficients, as we have come to expect.

(b) Start with the block functions (equation 6.2) and form the Fourier series. Take the limit as $n \to \infty$.

Using the block functions:

$$\phi_n = \begin{cases} \frac{n}{2} & \text{if} \quad -\frac{1}{n} < x < \frac{1}{n} \\ 0 & \text{otherwise} \end{cases} = \sum_{m=0}^{\infty} a_m \cos\frac{m\pi x}{L}$$

where

$$a_m = \frac{1}{L}\int_{-1/n}^{1/n}\frac{n}{2}\cos\frac{m\pi x}{L} = \frac{n}{2L}\left(\frac{L}{m\pi}\sin\frac{m\pi x}{L}\right)\Bigg|_{-1/n}^{1/n}$$
$$= \frac{n}{L}\frac{L}{m\pi}\sin\frac{m\pi}{Ln},\quad m \neq 0$$

and

$$a_0 = \frac{1}{2L}\int_{-1/n}^{1/n}\frac{n}{2}dx = \frac{n}{4L}x\Big|_{-1/n}^{1/n} = \frac{n}{4L}\frac{2}{n} = \frac{1}{2L}$$

Now we take the limit as $n \to \infty$:

$$\delta(x) = \lim_{n\to\infty}\phi_n(x) = \lim_{n\to\infty}\left(\frac{1}{2L} + \sum_{m=1}^{\infty}\frac{1}{L}\frac{Ln}{m\pi}\sin\frac{m\pi}{Ln}\cos\frac{m\pi x}{L}\right)$$
$$= \frac{1}{2L} + \sum_{m=1}^{\infty}\frac{1}{L}\cos\frac{m\pi x}{L}$$

Are the results the same? If not, why not? Give a quantitative as well as a qualitative account of any discrepancy.

The results are not the same. The first result is a sum over only odd values of n, but its amplitude is twice that of the second result. We should not expect the results to be the same, because the first result gives a periodic set of both positive and negative $\delta -$functions, positive at $x = 0$, $2L$, $4L$ etc and negative at $-L$, L, $3L$ etc. The second result gives a periodic repetition of positive $\delta -$functions at $x = -2L$, 0, $2L$ etc.

To verify this, let's take the second series, shift it by L, multiply by -1 and add it to the original series. This should give the first series:

$$\frac{2}{L} + \sum_{m=1}^{\infty}\frac{1}{L}\cos\frac{m\pi x}{L} - \left(\frac{2}{L} + \sum_{m=1}^{\infty}\frac{1}{L}\cos\frac{m\pi(x-L)}{L}\right) = \sum_{m=1}^{\infty}\frac{1}{L}\left(\cos\frac{m\pi x}{L} - (-1)^m\cos\frac{m\pi x}{L}\right)$$
$$= \sum_{m=1,\,\text{odd}}^{\infty}\frac{2}{L}\cos\frac{m\pi x}{L}$$

which is the first series, as expected.

3. A point load Mg is placed on a beam of length L at a distance $L/3$ from the left hand end. Find the displacement of the beam:

(a) if the beam is supported at one end, as in problem 5.11.

The relevant differential equation is

$$\frac{d^4y}{dx^4} = \frac{1}{EI}q(x) = \frac{Mg}{EI}\delta\left(x - \frac{L}{3}\right)$$

The boundary conditions are $y(0) = y'(0) = 0.$ We can solve this problem using the Laplace transform method. We can obtain additional boundary conditions at $x = 0$ using equations 3.7 and 3.8:

$$y'''(0) = -\frac{1}{EI}t(0) = -\frac{Mg}{EI}$$

and

$$y''(0) = \frac{1}{EI}Mg\frac{L}{3}$$

Now we transform. The transform of the delta-function is:.

$$\int_0^\infty \delta(x - L/3)e^{-sx}dx = e^{-sL/3}$$

Thus

$$s^4Y - y'''(0) - sy''(0) - s^2y'(0) - s^3y(0) = \frac{Mg}{EI}e^{-sL/3}$$

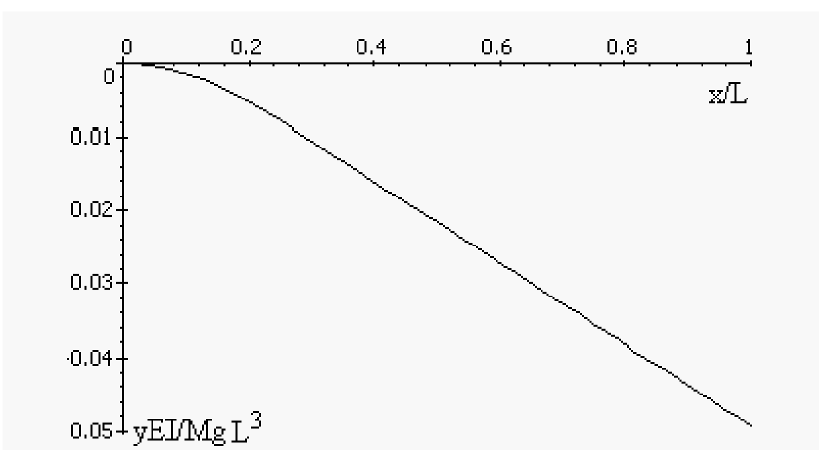
Thus:

$$Y = \frac{Mg}{s^4EI}e^{-sL/3} - \frac{Mg}{s^4EI} + \frac{Mg}{s^3EI}\frac{L}{3}$$

Now we invert. The exponential tells us that the function is shifted:

$$y(x) = \frac{Mg}{EI}\left(\frac{S\left(x - \frac{L}{3}\right)\left(x - \frac{L}{3}\right)^3}{3!} - \frac{x^3}{3!} + \frac{L}{3}\frac{x^2}{2!}\right)$$
$$= \frac{Mg}{6EI}\begin{cases} -x^3 + Lx^2 & \text{if } x < L/3 \\ \frac{1}{3}xL^2 - \frac{1}{27}L^3 & \text{if } x > L/3 \end{cases}$$

The solution looks like:



(b) if the beam rests on supports at each end, as in Example 5.2.

We have the same differential equation but different boundary conditions. Now $y(0) = y(L) = 0.$ Let's use a Fourier series:

$$y(x) = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi x}{L}$$

Then

$$\sum_{n=1}^{\infty} a_n \left(\frac{n\pi}{L}\right)^4 \sin \frac{n\pi x}{L} = \frac{Mg}{EI} \delta\left(x - \frac{L}{3}\right)$$

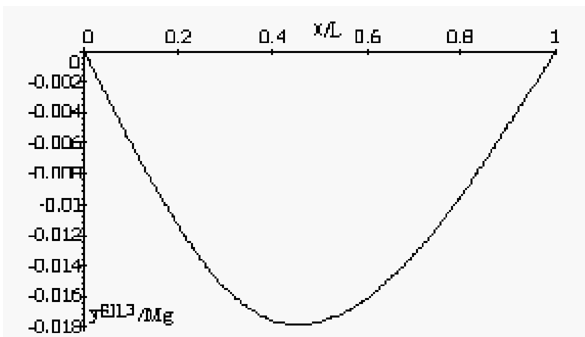
Thus the coefficients are given by:

$$a_n \left(\frac{n\pi}{L}\right)^4 = \frac{2}{L} \int_0^L \frac{Mg}{EI} \delta\left(x - \frac{L}{3}\right) \sin \frac{n\pi x}{L} dx = \frac{2}{L} \frac{Mg}{EI} \sin \frac{n\pi}{3}$$

and so

$$y(x) = \frac{2}{L} \frac{Mg}{EI} \sum_{n=1}^{\infty} \left(\frac{L}{n\pi}\right)^4 \sin \frac{n\pi}{3} \sin \frac{n\pi x}{L}$$

The series converges very fast because of the factor n^4 in the denominator. Every third term is missing ($\sin \frac{n\pi}{3} = 0$ for $n = 3, 6,$ etc). The solution looks like:



4. A damped harmonic oscillator (cf Problem 4.13) has initial condistions $x(0) = x_0$ and $\left.\frac{dx}{dt}\right|_{t=0} = v_0.$ An impulse I is applied at $t = t_0.$ Find the motion of the oscillator for $t > 0$.

$$\frac{d^2x}{dt^2} + \alpha \frac{dx}{dt} + k^2 x = \frac{I}{m} \delta(t - t_0)$$

We'll solve this using a Laplace transform:

$$s^2 X - x'(0) - sx(0) + \alpha[sX - x(0)] + k^2 X = \frac{I}{m} e^{-t_0 s}$$

Thus

$$X = \frac{\frac{I}{m} e^{-t_0 s} + (s + \alpha) x_0 + v_0}{s^2 + \alpha s + k^2}$$

Let's factor the denominator, and use the attenuation property:

$$s^2 + \alpha s + k^2 = \left(s + \frac{\alpha}{2}\right)^2 + k^2 - \frac{\alpha^2}{4}$$

Then, setting $\omega = \sqrt{k^2 - \frac{\alpha^2}{4}}$:

$$X = \frac{\frac{I}{m}e^{-t_0 s} + \left(s + \frac{\alpha}{2} + \frac{\alpha}{2}\right)x_0 + v_0}{\left(s + \frac{\alpha}{2}\right)^2 + \omega^2}$$

Inverting gives:

$$x(t) = \frac{I}{m\omega}S(t - t_0)e^{-\alpha(t-t_0)/2}\sin\omega(t - t_0) + \frac{\left(\frac{\alpha}{2}x_0 + v_0\right)}{\omega}e^{-\alpha t}\sin\omega t + x_0 e^{-\alpha t/2}\cos\omega t$$

Thus for $t < t_0$:

$$x(t) = \frac{\left(\frac{\alpha}{2}x_0 + v_0\right)}{\omega}e^{-\alpha t/2}\sin\omega t + x_0 e^{-\alpha t/2}\cos\omega t$$

while for $t > t_0$:

$$\begin{aligned} x(t) &= \frac{I}{m\omega}e^{-\alpha(t-t_0)/2}\sin\omega(t - t_0) + \frac{\left(\frac{\alpha}{2}x_0 + v_0\right)}{\omega}e^{-\alpha t/2}\sin\omega t + x_0 e^{-\alpha t/2}\cos\omega t \\ &= e^{-\alpha t/2}\left\{\frac{I}{m\omega}e^{\alpha t_0/2}(\sin\omega t\cot\omega t_0 - \cos\omega t\sin\omega t_0) + \frac{\left(\frac{\alpha}{2}x_0 + v_0\right)}{\omega}\sin\omega t + x_0\cos\omega t\right\} \\ &= e^{-\alpha t/2}\left\{\frac{\sin\omega t}{\omega}\left[\frac{I}{m}e^{\alpha t_0/2}\cot\omega t_0 + \frac{\alpha}{2}x_0 + v_0\right] + \cos\omega t\left[\frac{I}{m\omega}e^{\alpha t_0/2}\sin\omega t_0 + x_0\right]\right\} \end{aligned}$$

5. Distributions may be multiplied by infinitely differentiable functions. Do you expect the product

$$\psi(x) = \frac{\delta(x - a)}{x - a}$$

to be a valid distribution? Why or why not?

Not necessarily, because the function $\frac{1}{x-a}$ is not differentiable at $x = a$. (Distributions may be multiplied by infinitely differentiable functions.)

Investigate the properties of this quantity by evaluating the integral

$$\int_{-\infty}^{\infty}\frac{\phi_n(x - a)}{x - a}f(x)dx$$

where $\phi_n(x)$ is a delta sequence of your choice, and $f(x)$ is a test function. In particular, determine the result for functions that have the property $f(a) = 0.$ Is $\psi(x)$ a valid distribution in this case? Can you identify it?

We can evaluate the integral by using the Taylor series for $f(x)$ about $x = a$:

$$\begin{aligned} \int_{-\infty}^{\infty}\frac{\phi_n(x - a)}{x - a}f(x)dx &= \int_{-\infty}^{\infty}\frac{\phi_n(x - a)}{x - a}\left[f(a) + (x - a)f'(a) + (x - a)^2 f''(a) + \cdots\right]dx \\ &= \int_{-\infty}^{\infty}\phi_n(x - a)\left[\frac{f(a)}{x - a} + f'(a) + (x - a)f''(a) + \cdots\right]dx \end{aligned}$$

All terms after the first two are zero in the limit $n \to \infty.$ The second gives $f'(a).$ The first is indeterminate unless a is a zero of $f : f(a) = 0.$ In that case we may write:

$$\psi(x) = \frac{\delta(x - a)}{x - a} = -\delta'(x - a)$$

Alternative proof: Use the block function delta-sequence:

$$\int_{a-1/n}^{a+1/n} \frac{f(x)}{x-a}dx$$

Now let $u = x - a$

$$\int_{-1/n}^{1/n} \frac{f(u+a)}{u}du = \frac{f(\xi+a)}{\xi} \text{ where } -\frac{1}{n} < \xi < \frac{1}{n}$$

Now if $f(a) = 0,$ then we may add it:

$$\int_{-\infty}^{\infty} \frac{\phi_n(x-a)}{x-a}f(x)dx = \frac{f(\xi+a)-f(a)}{\xi} \to f'(a) \text{ as } n \to \infty$$

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Chapter 6: Generalized functions in physics

6. Evaluate

$$\int_{-\infty}^{\infty} e^{-|x|}\delta\left(x^2+2x-3\right)dx$$

Since $x^2+2x-3 = (x+3)(x-1)$, then

$$\begin{aligned}\delta\left(x^2+2x-3\right) &= \frac{\delta(x+3)}{|2(x+1)||_{x=-3}} + \frac{\delta(x-1)}{|2(x+1)||_{x=1}}\\ &= \frac{\delta(x+3)}{|-4|} + \frac{\delta(x-1)}{|4|}\\ &= \frac{1}{4}(\delta(x+3)+\delta(x-1))\end{aligned}$$

Thus

$$\begin{aligned}\int_{-\infty}^{\infty} e^{-|x|}\delta\left(x^2+2x-3\right)dx &= \int_{-\infty}^{\infty} e^{-|x|}\frac{1}{4}(\delta(x+3)+\delta(x-1))dx\\ &= \frac{1}{4}\left(e^{-|-3|}+e^{-|1|}\right)\\ &= \frac{1}{4}\left(e^{-3}+e^{-1}\right)\\ &= 0.10442\end{aligned}$$

(b)

$$\int_{-\infty}^{+\infty} e^{-x^2}\delta\left(x^2+x-6\right)dx$$

The delta function may be written:

$$\delta\left(x^2+x-6\right) = \delta[(x+3)(x-2)] = \frac{\delta(x+3)}{\left|2x+1|_{x=-3}\right|} + \frac{\delta(x-2)}{\left|2x+1|_{x=2}\right|}$$

and thus

$$\begin{aligned}\int_{-\infty}^{+\infty} e^{-x^2}\delta\left(x^2+x-12\right)dx &= \frac{e^{-3^2}}{5} + \frac{e^{-2^2}}{5} = \frac{1}{5}\left(e^{-9}+e^{-4}\right)\\ &= \frac{1}{5}\left(1.2341\times10^{-4}+1.8316\times10^{-2}\right)\\ &= 3.6879\times10^{-3}\end{aligned}$$

7. A string of length L with tension T and mass per unit length μ is hit simultaneously at $t=0$ at the two points $x=L/3$ and $x=2L/3$. The impulse delivered at each point is I. Find the subsequent displacement of the string.

The initial conditions are $y(x,0)=0$ and

$$\left.\frac{\partial y}{\partial t}\right|_{t=0} = \frac{I}{\mu}[\delta(x-L/3)+\delta(x-2L/3)]$$

The solution is then of the form

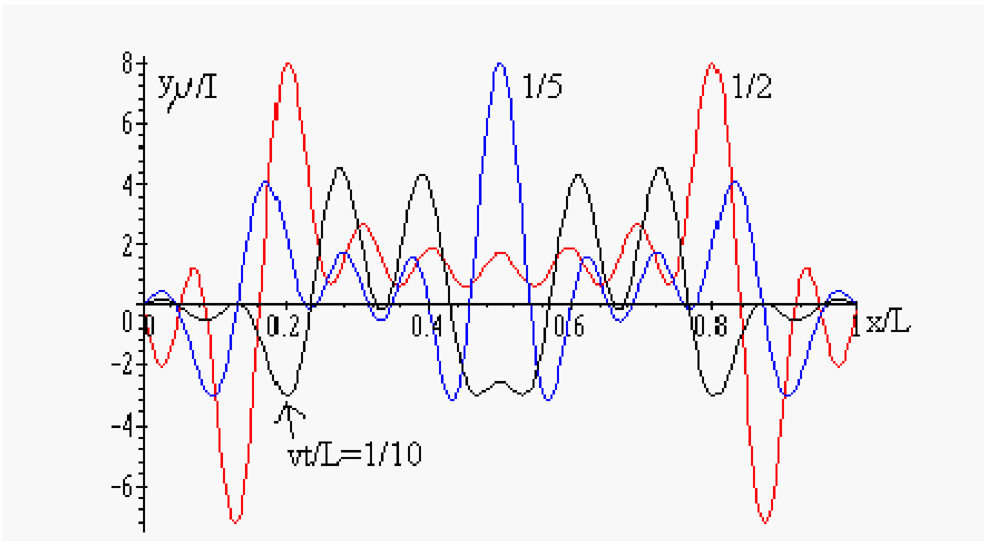
$$y(x,t) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{L} \sin\frac{n\pi vt}{L}$$

where

$$\frac{I}{\mu}[\delta(x - L/3) + \delta(x - 2L/3)] = \sum_{n=1}^{\infty} a_n \sin\left(\frac{n\pi x}{L}\right)\frac{n\pi v}{L}$$

The coefficients are given by:

$$\begin{aligned} a_n\frac{n\pi v}{L} &= \int_0^L \sin\left(\frac{n\pi x}{L}\right)\frac{I}{\mu}[\delta(x - L/3) + \delta(x - 2L/3)]dx \\ &= \frac{I}{\mu}\left(\sin\frac{n\pi}{3} + \sin\frac{2n\pi}{3}\right) \end{aligned}$$



Thus:

$$y(x,t) = \frac{I}{\mu}\sum_{n=1}^{\infty}\left(\sin\frac{n\pi}{3} + \sin\frac{2n\pi}{3}\right)\sin\frac{n\pi x}{L}\sin\frac{n\pi vt}{L}$$

Every third harmonic is suppressed. The plot show the string displacement at $vt/L = 1/10$, (black) 1/5 (blue) and 1/2 (red).

8. Using a general curvilinear coordinate system (cf Chapter 1 section 3) with coordinates u, v, w, find the charge density due to a point charge q placed at the point $u = u_0$, $v = v_0$, $w = w_0$. *Hint:* start with the delta sequence (6.25) and note that as $n \to \infty$, only a differential line element ds^2 is needed in the exponent. Then make use of equation 1.61.

$$\begin{aligned} \rho(\vec{r}) &= q\delta(\vec{r}) \\ &= \lim_{n\to\infty} q\left(\frac{n}{\sqrt{\pi}}\right)^3 \exp\left(-n^2 dl^2\right) \end{aligned}$$

where dl is the differential length element between $P\ (u_0, v_0, w_0)$ and a neighboring point with coordinates (u, v, w). Thus

$$\begin{aligned}\rho(\vec{\mathbf{r}}) &= \lim_{n\to\infty} q\left(\frac{n}{\sqrt{\pi}}\right)^3 \exp\left(-n^2\left(h_1^2(u-u_0)^2 + h_2^2(v-v_0)^2 + h_3^2(w-w_0)^2\right)\right) \\ &= \lim_{n\to\infty} \frac{q}{h_1h_2h_3}\left(\frac{nh_1}{\sqrt{\pi}}\right)\exp\left(-n^2h_1^2(u-u_0)^2\right)\left(\frac{nh_2}{\sqrt{\pi}}\right)\times \\ &\quad \exp\left(-n^2h_2^2(v-v_0)^2\right)\left(\frac{nh_3}{\sqrt{\pi}}\right)\exp\left(-n^2h_3^2(w-w_0)^2\right) \\ &= \frac{q}{h_1h_2h_3}\delta(u-u_0)\delta(v-v_0)\delta(w-w_0)\end{aligned}$$

where we used the result

$$\begin{aligned}\lim_{n\to\infty}\left(\frac{nh_1}{\sqrt{\pi}}\right)\exp\left(-n^2h_1^2(u-u_0)^2\right) &= \lim_{x\to\infty}\left(\frac{x}{\sqrt{\pi}}\right)\exp\left(-x^2(u-u_0)^2\right) \\ &= \delta(u-u_0)\end{aligned}$$

(equation 6.19) and similarly for the delta functions in v and w.

9. We must show the existence of

$$\lim_{n\to\infty}\int_{-\infty}^{+\infty} f_n(x)g(x)dx = \lim_{n\to\infty}\int_{-\infty}^{+\infty}\frac{n}{2a\sqrt{\pi}}\int_{-a}^{+a}\exp\left(-n^2\left(x-x'\right)^2\right)dx'g(x)dx$$

Changing the order of integration, and writing $g(x)$ as a Taylor series about x', we have:

$$\begin{aligned}&\lim_{n\to\infty}\frac{n}{2a\sqrt{\pi}}\int_{-a}^{+a}\int_{-\infty}^{+\infty}\exp\left(-n^2(x-x')^2\right)[g(x') + (x-x')g'(x') + \cdots]dxdx' \\ &= \lim_{n\to\infty}\frac{n}{2a\sqrt{\pi}}\int_{-a}^{+a}\int_{-\infty}^{+\infty}\exp\left(-n^2(x-x')^2\right)\sum_{m=0}^{\infty}\frac{(x-x')^m}{m!}\left.\frac{d^mg}{dx^m}\right|_{x=x'}dxdx'\end{aligned}$$

All terms in the Taylor series with odd m integrate to zero. For even m, we change variables to $u = n^2\left(x-x'\right)^2$, $du = n^2 2\left(x-x'\right)dx = 2n\sqrt{u}\,dx$ to obtain

$$\begin{aligned}&\lim_{n\to\infty}\frac{n}{2a\sqrt{\pi}}\int_{-a}^{+a}2\int_{0}^{+\infty}e^{-u}\sum_{m=0}^{\infty}\frac{u^{m/2}}{n^m m!}\left.\frac{d^mg}{dx^m}\right|_{x=x'}\frac{du}{2n\sqrt{u}}dx' \\ &= \lim_{n\to\infty}\frac{1}{2a\sqrt{\pi}}\sum_{m=0}^{\infty}\frac{1}{n^m m!}\int_{-a}^{+a}\left.\frac{d^mg}{dx^m}\right|_{x=x'}dx'\int_{-\infty}^{+\infty}e^{-u}u^{(m-1)/2}du \\ &= \lim_{n\to\infty}\frac{1}{2a\sqrt{\pi}}\sum_{m=0}^{\infty}\frac{1}{n^m m!}\int_{-a}^{+a}\left.\frac{d^mg}{dx^m}\right|_{x=x'}dx'\,\Gamma\left(\frac{m+1}{2}\right)\end{aligned}$$

(See equation 2.75 for the gamma function.) In the limit, only the $m = 0$ term survives, and we have

$$\lim_{n\to\infty}\int_{-\infty}^{+\infty}f_n(x)g(x)dx = \frac{1}{2a}\int_{-a}^{+a}g\left(x'\right)dx' = \langle g \rangle$$

as required.

10. Show that the sequence of functions

$$f_n(x) = \frac{n}{2\cosh^2 nx}$$

converges weakly to the delta function.

We investigate the integral:

$$I_n = \int_{-\infty}^{+\infty} \frac{n}{2\cosh^2 nx} g(x)dx$$

Using the mean value theorem,

$$\begin{aligned} I_n &= g(\xi)\int_{-\infty}^{+\infty} \frac{n}{2\cosh^2 nx}dx \\ &= g(\xi)\left(\frac{\tanh(nx)}{2}\right)\Bigg|_{-\infty}^{+\infty} = g(\xi) \text{ where } -\infty < \xi < \infty \end{aligned}$$

We need to zero in on the value of ξ, so divide the range of integration up into pieces:

$$I_n = \left(\int_{-\infty}^{-1/\sqrt{n}} + \int_{-1/\sqrt{n}}^{+1/\sqrt{n}} + \int_{1/\sqrt{n}}^{\infty}\right) \frac{n}{2\cosh^2 nx} g(x)dx$$

In the first integral:

$$|I_{n1}| \le M_1 \frac{1}{2}\left|\tanh nx\right|_{-\infty}^{-1/\sqrt{n}}\Big| = \frac{M_1}{2}\left|\left(\tanh\left(-\sqrt{n}\right) - (-1)\right)\right|$$

where M_1 is the upper bound of $|g(x)|$ on the interval $(-\infty, -1/\sqrt{n})$. Now

$$\begin{aligned} \tanh x &= \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{1 - e^{-2x}}{1 + e^{-2x}} = \left(1 - e^{-2x}\right)\left(1 - e^{-2x} + \cdots\right) \\ &= 1 - 2e^{-2x} + \cdots \text{ for } x > 0 \\ &= \frac{e^{2x} - 1}{e^{2x} + 1} = -1 + 2e^{2x} + \cdots \text{ for } x < 0 \end{aligned}$$

So

$$|I_{n1}| \le M_1\left(e^{-2\sqrt{n}} + \cdots\right) \to 0 \text{ as } n \to \infty$$

A similar argument shows that the third integral goes to zero. The middle integral is

$$\begin{aligned} \frac{g(\xi)}{2}\left(\tanh\left(\sqrt{n}\right) - \tanh\left(-\sqrt{n}\right)\right) &= g(\xi)\tanh\left(\sqrt{n}\right) \text{ for } -\frac{1}{\sqrt{n}} < \xi < \frac{1}{\sqrt{n}} \\ &\to g(0) \text{ as } n \to \infty \end{aligned}$$

Thus $f_n(x)$ converges weakly to the delta function.

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Chapter 6: Generalized functions in physics

11. According to the properties of distributions in section 6.3, $e^{-x}\delta'(x)$ is a distribution. Which distribution is it?

$$\begin{aligned}\int[e^{-x}\delta'(x)]g(x)dx &= \int\delta'(x)(e^{-x}g(x))dx = -\frac{d}{dx}(e^{-x}g(x))\Big|_0 \\ &= -e^{-x}g(x) + e^{-x}g'(x)|_0 = -g(0) + g'(0) \\ &= -\int\delta(x)g(x)dx - \int\delta'(x)g(x)\end{aligned}$$

Thus

$$e^{-x}\delta'(x) = -\delta(x) - \delta'(x)$$

12. Starting with the integral (6.16), show that

$$\delta(x) = \frac{1}{\pi}\int_0^{\infty}\cos kx dk$$

$$\begin{aligned}\frac{1}{2\pi}\int_{-\infty}^{+\infty}e^{ikx}dx &= \frac{1}{2\pi}\left(\int_0^{\infty}e^{ikx}dx + \int_{\infty}^{0}e^{-iku}(-du)\right) \\ &= \frac{1}{2\pi}\int_0^{\infty}2\cos kx dx = \frac{1}{\pi}\int_0^{\infty}\cos kx dk\end{aligned}$$

13. Show that, for x and a both positive,

$$\delta(x-a) = \frac{2}{\pi}\int_0^{\infty}\cos kx\cos ka dk$$

and obtain a similar expression as an integral over sines.

Is this result consistent with problem 12? Discuss.

$$\begin{aligned}I(x,a) &= \frac{2}{\pi}\int_0^{\infty}\cos kx\cos ka dk \\ &= \frac{2}{\pi}\int_0^{\infty}\left(\frac{e^{ikx}+e^{-ikx}}{2}\right)\left(\frac{e^{ika}+e^{-ika}}{2}\right)dk \\ &= \frac{1}{2\pi}\int_0^{\infty}\left(e^{ikx+ika}+e^{-ikx+ika}+e^{ikx-ika}+e^{-ikx-ika}\right)dk \\ &= \frac{1}{2\pi}\int_0^{\infty}\left(e^{ikx+ika}+e^{ikx-ika}\right)dk - \frac{1}{2\pi}\int_0^{-\infty}(e^{i\omega x+i\omega a}+e^{i\omega x-i\omega a})d\omega\end{aligned}$$

where in the second term we set $\omega = -k.$ Thus:

$$I(x,a) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left(e^{ikx+ika} + e^{ikx-ika}\right)dk$$
$$= \delta(x+a) + \delta(x-a)$$

But if x and a are both positive, x can never equal $-a$, so

$$\frac{2}{\pi}\int_0^{\infty}\cos kx\cos ka\,dk = \delta(x-a)$$

as required.

Start with the relation

$$\delta(x-a) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}e^{ik(x-a)}dk = \frac{1}{2\pi}\int_{-\infty}^{+\infty}e^{ikx}e^{-ika}dk$$
$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty}(\cos kx + i\sin kx)(\cos ka - i\sin ka)dk$$
$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty}(\cos kx\cos ka + \sin kx\sin ka)dk$$

The other terms have odd integrands and so integrate to zero. Then

$$\delta(x-a) = \frac{1}{\pi}\int_0^{+\infty}(\cos kx\cos ka + \sin kx\sin ka)dk$$
$$= \frac{1}{2}\delta(x-a) + \frac{1}{\pi}\int_0^{+\infty}\sin kx\sin ka\,dk$$

where we have used the relation already proved, and thus

$$\delta(x-a) = \frac{2}{\pi}\int_0^{+\infty}\sin kx\sin ka\,dk$$

As $a \to 0,$ neither of these results seems consistent with problem 12. The expression in terms of sines goes to zero, while the expression in terms of cosines is twice the result of problem 12. This happens because we constrained x and a to be strictly positive. Thus the sifting property for these expressions is

$$f(a) = \int_0^{\infty}f(x)\delta(x-a)dx = \int_0^{\infty}f(x)\frac{2}{\pi}\int_0^{+\infty}\sin kx\sin ka\,dk\,dx$$
$$= \int_0^{\infty}f(x)\frac{2}{\pi}\int_0^{+\infty}\cos kx\cos ka\,dk\,dx$$

whereas, using the result of problem 12,

$$f(0) = \int_{-\infty}^{+\infty}f(x)\frac{1}{\pi}\int_0^{\infty}\cos kx\,dk\,dx$$

Now if $f(x)$ is even, we may rewrite this integral as

$$2\int_0^{+\infty}f(x)\frac{1}{\pi}\int_0^{\infty}\cos kx\,dk\,dx$$

which is consistent with problem 13. Similarly, for odd functions we find equivalence for the sine expression.

14. Find the Laplace transform of $\delta(t-a)$. Express the inverse as an integral using equation 5.19 and demonstrate that this integral possesses the sifting property**.**

$$\mathcal{L}(\delta(t-a)) = \int_0^\infty e^{-st}\delta(t-a)dt = e^{-sa}$$

Thus

$$\delta(t-a) = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{-sa}e^{st}ds$$

So we check for the sifting property:

$$\int_{-\infty}^{+\infty} f(t)\frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} e^{-sa}e^{st}dsdt = \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty}\int_{-\infty}^{+\infty} f(t)e^{st}dte^{-sa}ds$$
$$= \frac{1}{2\pi i}\int_{\gamma-i\infty}^{\gamma+i\infty} F(s)e^{-sa}ds = f(a)$$

as required.

15. A disk of radius a and mass M lies in the $x-y$ plane. Express the density in terms of delta functions

(a) in rectangular Cartesian coordinates

In Cartesian coordinates, the disk is located at $z = 0$. We have:

$$\rho(\vec{\mathbf{x}}) = \frac{M}{\pi a^2}\delta(z)$$

if $\sqrt{x^2+y^2} < a$, and zero otherwise. We can also express this in terms of step functions:

$$\rho(\vec{\mathbf{x}}) = \frac{M}{\pi a^2}\delta(z)\Theta\left(a - \sqrt{x^2+y^2}\right)$$

(b) in cylindrical coordinates

In cylindrical coordinates, the step function looks nicer:

$$\rho(\vec{\mathbf{x}}) = \frac{M}{\pi a^2}\delta(z)\Theta(a-\rho)$$

(c) in spherical coordinates.

The disk is at $\theta = \pi/2$, so the density looks like:

$$\begin{aligned}\rho(\vec{\mathbf{x}}) &= \frac{M}{\pi a^2}\delta(r\cos\theta)\Theta(a-r)\\ &= \frac{M}{\pi a^2 r}\delta(\cos\theta)\Theta(a-r)\\ &= \frac{M}{\pi a^2 r}\frac{\delta(\theta-\pi/2)}{|-\sin\theta|_{\theta=\pi/2}}\Theta(a-r)\\ &= \frac{M}{\pi a^2 r}\delta(\theta-\pi/2)\Theta(a-r)\end{aligned}$$

16. A rod of length ℓ and mass M lies along the x −axis with one end at the origin. Determine the density using delta functions

(a) in rectangular Cartesian coordinates

The rod is restricted to $y = 0$ and $z = 0$, so we may write:

$$\rho(\vec{\mathbf{x}}) = \begin{cases} \frac{M}{\ell}\delta(y)\delta(z) & \text{if} \quad 0 \le x \le \ell \\ 0 & \text{otherwise} \end{cases}$$

We may also write the result using step functions:

$$\rho(\vec{\mathbf{x}}) = \frac{M}{\ell}\delta(y)\delta(z)[\Theta(x) - \Theta(x-\ell)]$$

(b) in cylindrical coordinates

The rod is restricted to $z = 0$ and $\phi = 0$. We start with the result from part (a) and convert to cylindrical coordinates. In particular:

$$\delta(y) = \delta(\rho\sin\phi) = \frac{1}{\rho}\delta(\sin\phi) = \frac{1}{\rho}\frac{\delta(\phi)}{|\cos\phi|_{\phi=0}} = \frac{\delta(\phi)}{\rho}$$

Thus:

$$\rho(\vec{\mathbf{x}}) = \frac{M}{\ell}\frac{\delta(\phi)}{\rho}\delta(z)[\Theta(\rho) - \Theta(\rho-\ell)]$$

Check the dimensions of the result!

(c) in spherical coordinates.

In spherical coordinates, the rod is at $\theta = \pi/2$ and $\phi = 0$. The result must look like:

$$\rho(\vec{\mathbf{x}}) = A\delta(\theta-\pi/2)\delta(\phi)[\Theta(r) - \Theta(r-\ell)]$$

We need to find the function A. We integrate over a spherical shell with inner radius r and outer radius $r+dr$. The amount of charge enclosed is $dm = \frac{M}{\ell}dr$. Thus:

$$\begin{aligned}\frac{M}{\ell}dr &= dm = \int_0^\pi\int_0^{2\pi}\rho(\vec{\mathbf{x}})r^2\sin\theta d\theta d\phi dr \\ &= \int_0^\pi\int_0^{2\pi}A\delta(\theta-\pi/2)\delta(\phi)[\Theta(r)-\Theta(r-\ell)]r^2\sin\theta d\theta d\phi dr\end{aligned}$$

Now if r is between 0 and ℓ, we have:

$$\begin{aligned}\frac{M}{\ell}dr &= \int_0^\pi\int_0^{2\pi}A\delta(\theta-\pi/2)\delta(\phi)r^2\sin\theta d\theta d\phi dr = \int_0^\pi A\delta(\theta-\pi/2)r^2\sin\theta d\theta dr \\ &= Ar^2\sin\frac{\pi}{2}dr = Ar^2 dr\end{aligned}$$

Thus

$$A = \frac{M}{\ell r^2}$$

and thus

$$\rho(\vec{\mathbf{x}}) = \frac{M}{\ell r^2}\delta(\theta-\pi/2)\delta(\phi)[\Theta(r)-\Theta(r-\ell)]$$

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Chapter 6: Generalized functions in physics

17. A line of charge with uniform line charge density λ lies along the $z-$axis. Find the volume charge density (a) in cylindrical coordinates and (b) in spherical coordinates.

(a)

$$\rho(\vec{x}) = f(\rho,z)\lambda\delta(\rho)$$

The system has azimuthal symmetry, so there is no dependence on ϕ. To find f, integrate over a cylindrical slice of height dz

$$dq = \lambda dz = \lambda\int f\delta(\rho)\rho d\phi dz d\rho$$

If f did not depend on ρ, the result would be zero. Thus we must have $f \propto 1/\rho$. Then

$$\lambda dz = \lambda 2\pi\int \frac{A}{\rho}\delta(\rho)\rho dz d\rho = \lambda 2\pi A dz$$

Thus $A = 1/2\pi$, and

$$\rho(\vec{x}) = \frac{\lambda}{2\pi\rho}\delta(\rho)$$

(b) In spherical coordinates the charge exists only at $\theta = 0$ and at $\theta = \pi$. Again there is no dependence on ϕ.

$$\rho(\vec{x}) = f(r,\theta)[\delta(\theta) + \delta(\theta - \pi)]$$

This time we integrate over a spherical shell of radius r. This shell cuts the line at two places, so the charge enclosed is $2\lambda dr$.

$$\begin{aligned} dq &= 2\lambda dr = \int_0^\pi\int_0^{2\pi} f(r,\theta)[\delta(\theta) + \delta(\theta-\pi)]r^2 dr d\phi \sin\theta d\theta \\ &= 2\pi\int_0^\pi f(r,\theta)[\delta(\theta) + \delta(\theta-\pi)]r^2 dr \sin\theta d\theta \end{aligned}$$

Here f must have a factor $1/\sin\theta$ in order that the result of the integration be non-zero. Thus

$$2\lambda dr = 2\pi\int_0^\pi \frac{A}{\sin\theta}[\delta(\theta) + \delta(\theta-\pi)]r^2 dr \sin\theta d\theta = 4\pi A r^2 dr$$

Thus

$$A = \frac{\lambda}{2\pi r^2}$$

and

$$\rho(\vec{x}) = \frac{\lambda}{2\pi r^2 \sin\theta}[\delta(\theta) + \delta(\theta-\pi)]$$

18. A disk of charge with radius a and surface charge density $\sigma(r) = \sigma_0 r/a$ lies in the $x-y$ plane with center at the origin. Find the volume charge density (a) in cylindrical coordinates and (b) in spherical coordinates.

(a)

$$\rho(\vec{x}) = f(\rho,z)\sigma_0\Theta(a-\rho)\delta(z)$$

Integrating over a wedge of a cylindrical shell at $\rho < a$

$$\begin{aligned} dq &= \sigma_0\frac{\rho^2}{a}d\rho d\phi = \int_{-\infty}^{+\infty} f(\rho,z)\sigma_0\Theta(a-\rho)\delta(z)\rho d\rho d\phi dz \\ &= f(\rho,0)\sigma_0\rho d\rho d\phi \end{aligned}$$

Thus $f = \frac{\rho}{a}$ and

$$\rho(\vec{x}) = \sigma_0 \frac{\rho}{a} \Theta(a-\rho)\delta(z)$$

(b)

$$\rho(\vec{x}) = f(r,\theta)\sigma_0 \Theta(a-r)\delta(\theta-\pi/2)$$

and integrating over an orange wedge shell at $r < a$ we get

$$\begin{aligned} \sigma_0 \frac{r^2}{a} dr d\phi &= \int_0^\pi f(r,\theta)\sigma_0 \Theta(a-r)\delta(\theta-\pi/2) r^2 dr d\phi \sin\theta d\theta \\ &= f(r,\pi/2)\sigma_0 r^2 dr d\phi \end{aligned}$$

Thus

$$f = \frac{1}{a}$$

and

$$\rho(\vec{x}) = \frac{\sigma_0}{a} \Theta(a-r)\delta(\theta-\pi/2)$$

19. Current I flows in a loop of radius a lying in the $x-y-$plane with its center at the origin. Find an expression for the current density (a) in cylindrical coordinates and (b) in spherical coordinates.

(a)

$$\vec{\mathbf{j}} = I\delta(\rho-a)\delta(z)\hat{\phi}$$

(b) The current exists only at $r = a$ and $z = 0.$ Thus

$$\begin{aligned} \vec{\mathbf{j}} &= I\delta(r-a)\delta(z)\hat{\boldsymbol{\phi}} = I\delta(r-a)\delta(r\cos\theta)\hat{\boldsymbol{\phi}} \\ &= I\frac{\delta(r-a)}{r}\delta(\cos\theta)\hat{\boldsymbol{\phi}} = I\frac{\delta(r-a)}{r}\frac{\delta\left(\theta-\frac{\pi}{2}\right)}{\left|-\sin\theta\right|_{\theta=\pi/2}}\hat{\boldsymbol{\phi}} \\ &= I\frac{\delta(r-a)}{r}\delta\left(\theta-\frac{\pi}{2}\right)\hat{\boldsymbol{\phi}} \end{aligned}$$

20. Prove the relation (equation 6.27)

$$\nabla^2 \ln\frac{\rho}{a} = 2\pi\delta(\vec{\boldsymbol{\rho}})$$

where ρ is the radial coordinate in a cylindrical coordinate system, and $\vec{\boldsymbol{\rho}}$ is the position vector in a plane.

For $\rho \neq 0$

$$\begin{aligned} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\ln\left(\frac{\sqrt{x^2+y^2}}{a}\right) &= \frac{\partial}{\partial x}\frac{1}{2}\frac{2x}{(x^2+y^2)} + \frac{\partial}{\partial y}\frac{1}{2}\frac{2y}{(x^2+y^2)} \\ &= \frac{2}{x^2+y^2} - \frac{2x^2}{(x^2+y^2)^{3/2}} - \frac{2y^2}{(x^2+y^2)^{3/2}} \\ &= \frac{2}{\sqrt{x^2+y^2}} - 2\frac{x^2+y^2}{(x^2+y^2)^{3/2}} = 0 \end{aligned}$$

For $\rho = 0$ the derivatives cannot be computed. So we check for the sifting property:

$$\int_{\text{all 2-D space}} f(\vec{\mathbf{r}})\nabla^2 \ln\frac{\rho}{a}dA = \int_{\text{circle of radius }\varepsilon}\left\{\vec{\nabla}\cdot\left(f(\vec{\mathbf{r}})\vec{\nabla}\ln\frac{\rho}{a}\right) - \vec{\nabla}\ln\frac{\rho}{a}\cdot\vec{\nabla}f\right\}dA$$
$$= \int_{\text{circumference}} \hat{\boldsymbol{\rho}}\cdot\left(f\vec{\nabla}\ln\frac{\rho}{a}\right)d\ell - \int_{\text{circle of radius }\varepsilon}\frac{\hat{\boldsymbol{\rho}}\cdot\vec{\nabla}f}{\rho}\rho d\rho d\phi$$
$$= \int_{\text{circumference}}\frac{f}{\rho}\rho d\phi - \int_{\text{circle of radius }\varepsilon}\frac{\partial f}{\partial\rho}d\rho d\phi$$
$$= \int_0^{2\pi} f(\varepsilon,\phi)d\phi - \int_0^{2\pi}(f(\varepsilon,\phi) - f(0))d\phi$$
$$= 2\pi f(0)$$

as required.

Use the result to find the potential due to a line charge λ running parallel to the $z-$axis at $x = a,\ y = b.$

First we put the z' axis along the line occupied by the charge. The charge density in these coordinates is:

$$\rho(\vec{\mathbf{x}}) = \lambda\delta(\vec{\rho}\,')$$

Then the equation satisfied by the potential is

$$\nabla^2\Phi = -\frac{\lambda\delta(\vec{\rho}\,')}{\varepsilon_0}$$

Using the result above, we thus conclude that:

$$\Phi = -\frac{\lambda}{2\pi\varepsilon_0}\ln\frac{\rho'}{R}$$

where here R is the distance from the origin to the reference point where we choose Φ to be zero. Then converting back to the original coordinates:

$$\Phi = -\frac{\lambda}{4\pi\varepsilon_0}\ln\left[\frac{(x-a)^2+(y-b)^2}{R}\right]$$

21. A circuit contains a resistor, a capacitor, and a square wave power supply with period $T.$ Use Kirchhoff's loop rule to write an equation for the current in the circuit in terms of delta-functions, and solve it to find the current as a function of time.

$$E(t) = IR + \frac{Q}{C}$$

Differentiate to get:

$$\frac{dE}{dt} = \frac{dI}{dt}R + \frac{I}{C}$$

The term on the left is a sequence of $\delta-$functions, up at $t = 0,\ T$ and down at $t = T/2$ etc Using equation 6.13

$$\mathcal{E}(t) = \frac{V_0}{T}\sum_n e^{in2\pi t/T} - \frac{V_0}{T}\sum_n e^{in\pi(2t-T)/T}$$
$$= \frac{V_0}{T}\sum_n e^{in2\pi t/T} - \frac{V_0}{T}\sum_n(-1)^n e^{in\pi 2t/T}$$
$$= \frac{2V_0}{T}\sum_{n\text{ odd}} e^{in2\pi t/T}$$

Then applying Kirchhoff's loop rule, we have

$$\frac{2V_0}{T}\sum_{n\text{ odd}} e^{in2\pi t/T} = \frac{dI}{dt}R + \frac{I}{C}$$

So express I as a Fourier series

$$I = \sum I_n e^{in2\pi t/T}$$

Then

$$\frac{2V_0}{T} = in\frac{2\pi}{T}I_n R + \frac{I_n}{C}$$

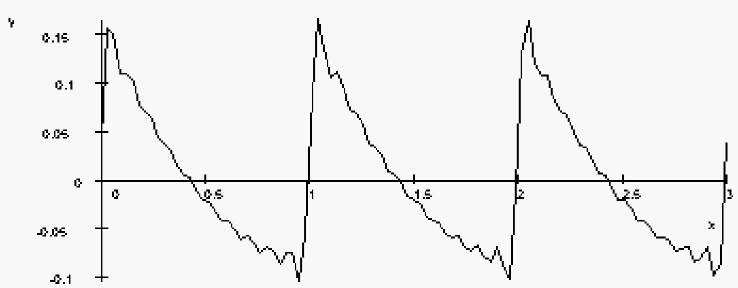
for n odd, and zero otherwise. Thus

$$I_n = \frac{2V_0}{(in2\pi R + T/C)}$$

and thus

$$\begin{aligned} I &= \frac{2V_0}{R}\sum_{n\text{ odd}}\frac{e^{in2\pi t/T}}{in2\pi + T/RC} = \frac{2V_0}{R}\sum_{n\text{ odd}}\frac{T/RC - in2\pi}{4\pi^2 n^2 + T^2/(RC)^2}e^{in2\pi t/T} \\ &= 4\frac{V_0}{R}\sum_{n\text{ odd}}\frac{T/RC}{4\pi^2 n^2 + T^2/(RC)^2}\cos\frac{2\pi nt}{T} + \frac{n2\pi}{4\pi^2 n^2 + T^2/(RC)^2}\sin\frac{2\pi nt}{T} \end{aligned}$$

The plot shows IR/V_0 versus t/T in the case that $RC = 2T$:



22. Starting with the result

$$\Phi = \frac{\vec{\mathbf{p}}\cdot\vec{\mathbf{r}}}{r^3}$$

for the electric potential due to a dipole placed at the origin (cf Example 6.1), calculate the electric field everywhere, including *at* the origin. Use a method similar to that used in §6.5 to prove relation 6.26.

The electric field is the gradient of the potential.

$$\begin{aligned} \vec{\mathbf{E}} &= -\vec{\nabla}\Phi = -\vec{\nabla}\left(\frac{\vec{\mathbf{p}}\cdot\vec{\mathbf{r}}}{r^3}\right) \\ &= \left(\vec{\mathbf{p}}\cdot\vec{\nabla}\right)\frac{\vec{\mathbf{r}}}{r^3} + \vec{\mathbf{p}}\times\left(\vec{\nabla}\times\frac{\vec{\mathbf{r}}}{r^3}\right) = \left(\vec{\mathbf{p}}\cdot\vec{\nabla}\right)\frac{\vec{\mathbf{r}}}{r^3} \end{aligned}$$

The curl is zero because $\mathbf{r}/r^3 = -\vec{\nabla}(1/r)$ and the curl of a gradient is zero. So we are left with:

$$\vec{\mathbf{E}} = -\left(\vec{\mathbf{p}}\cdot\vec{\nabla}\right)\vec{\nabla}\frac{1}{r}$$

Now away from the origin $(r \neq 0)$, and taking the x −axis along $\vec{\mathbf{p}}$:

$$\vec{\mathbf{E}} = -\quad p\frac{\partial}{\partial x}\frac{\vec{\mathbf{r}}}{r^3} = -p\left(\frac{\hat{\mathbf{x}}}{r^3} + \vec{\mathbf{r}}\left(-\frac{3x}{r^5}\right)\right)$$
$$= -\frac{\vec{\mathbf{p}}}{r^3} + 3\vec{\mathbf{r}}\frac{\vec{\mathbf{r}}\cdot\vec{\mathbf{p}}}{r^5} \qquad \text{(6.1 solutions)}$$

However, at the origin, the derivatives cannot be computed in the usual way. Since we already know that

$\nabla^2\frac{1}{r} = -4\pi\delta(\vec{\mathbf{x}})$, we should suspect the presence of a delta-function in this case too. We can check this assertion by testing for the sifting property. We integrate over a small sphere of radius ε surrounding the origin:

$$\int_{\text{sphere}} -\vec{\nabla}\left(\frac{\vec{\mathbf{p}}\cdot\vec{\mathbf{r}}}{r^3}\right)dV = -\int_{\text{surface of sphere}} \frac{\vec{\mathbf{p}}\cdot\vec{\mathbf{r}}}{r^3}\hat{\mathbf{r}}r^2 d\Omega$$
$$= -\int_{\text{surface of sphere}} (\vec{\mathbf{p}}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}d\Omega$$
$$= -p\int_{\text{surface of sphere}} \sin\theta\cos\phi(\hat{\mathbf{x}}\sin\theta\cos\phi + \hat{\mathbf{y}}\sin\theta\sin\phi + \hat{\mathbf{z}}\cos\theta)\sin\theta d\theta d\phi$$
$$= -p\int_{-1}^{1}(1-\mu^2)d\mu\int_0^{2\pi}\cos^2\phi d\phi\hat{\mathbf{x}}$$

All other components are zero. Then:

$$\int_{\text{sphere}} \vec{\mathbf{E}}dV = -p\left(1 - \frac{\mu^3}{3}\right)\Bigg|_{-1}^{+1}\pi = -\frac{4}{3}\pi\vec{\mathbf{p}}$$

whereas, using the explicit form (6.1 solutions) we get:

$$\int_{\text{sphere}}\left(-\frac{\vec{\mathbf{p}}}{r^3} + 3\vec{\mathbf{r}}\frac{\vec{\mathbf{r}}\cdot\vec{\mathbf{p}}}{r^5}\right)dV$$
$$= \int_0^{\varepsilon}\int_{-1}^{+1}\int_0^{2\pi}\left\{-\frac{\vec{\mathbf{p}}}{r^3} + 3p\frac{\sin\theta\cos\phi}{r^3}(\hat{\mathbf{x}}\sin\theta\cos\phi + \hat{\mathbf{y}}\sin\theta\sin\phi + \hat{\mathbf{z}}\cos\phi)\right\}r^2 d\mu d\phi$$
$$= \vec{\mathbf{p}}\int_0^{\varepsilon}\frac{-4\pi + 3\pi(4/3)}{r}dr = \vec{\mathbf{p}}\int_0^{\varepsilon}0dr = 0$$

To make these two results consistent, we must add a delta-function:

$$\vec{\mathbf{E}} = 3\vec{\mathbf{r}}\frac{\vec{\mathbf{r}}\cdot\vec{\mathbf{p}}}{r^5} - \frac{\vec{\mathbf{p}}}{r^3} - \frac{4}{3}\pi\vec{\mathbf{p}}\delta(\vec{\mathbf{x}})$$

23. Using a delta-sequence of your choice, show that the limit

$$\lim_{\ell\to 0}\left[\frac{\delta(x) - \delta(x-\ell)}{\ell}\right]$$

exhibits the sifting property of $\delta'(x)$.

$$\lim_{\ell\to 0}\frac{1}{\ell}\int(\phi_n(x) - \phi_n(x-\ell))f(x)dx = \lim_{\ell\to 0}\frac{1}{\ell}(f(0) - f(\ell))$$
$$= -f'(0)$$

which is the sifting property of $\delta'(x)$.

24. Use the derivative property 6.20 to show that, for distributions, the Laplace transform of the derivative $\phi'(x)$ equals s times the Laplace transform of ϕ. Show that the Laplace transform of $\ln t$ is $-(\gamma + \ln s)/s$ where γ is Euler's constant, $-\int_0^\infty e^{-x}\ln x dx = 0.5772.$ Hence show that the Laplace transform of $1/t$ ($t > 0$), condsidered as a distribution, is $-\gamma - \ln s.$

Here is Maple verifying the value of γ by numerical integration: $\int_0^\infty e^{-x}\ln x dx = -0.57722$

Now

$$\mathcal{L}\left(\phi'(t)\right) = \int_0^\infty e^{-st}\phi'(t)dt = -\int_0^\infty \phi(t)\frac{d}{dt}\left(e^{-st}\right)dt = s\Phi(s)$$

$$\mathcal{L}(\ln t) = \int_0^\infty \ln t e^{-st}dt = \int_0^\infty \ln(x/s)e^{-x}d\frac{x}{s} = \frac{1}{s}\int_0^\infty(\ln x - \ln s)e^{-x}dx = \frac{1}{s}(-\gamma - \ln s(-e^{-x})|_0^\infty)$$
$$= \frac{-\gamma - \ln s}{s} \text{ as required.}$$

Then by the derivative rule, $1/t = \frac{d}{dt}(\ln t)$ and thus

$$\mathcal{L}\left(\frac{1}{t}\right) = s\mathcal{L}(\ln t) = -\gamma - \ln s$$

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Chapter 6: Generalized functions in physics

25. Starting from equation 6.16, show that

$$\lim_{R\to\infty} \frac{\sin Rx}{\pi x} = \delta(x)$$

Confirm your result by demonstrating the sifting property. Use contour integration to do the integral.

Similarly, show that

$$\lim_{R\to\infty} \frac{\cos Rx}{x} = 0$$

if the integral

$$\int_{-\infty}^{+\infty} f(x)\frac{\cos Rx}{x}dx$$

is taken to be the principal value.

Demonstrate the plausibility of the results by evaluating $I_1 = \frac{2}{\pi}\int_{\varepsilon}^{\infty} \frac{\sin Nx}{x}dx$ and $I_2 = \int_{\varepsilon}^{\infty} \frac{\cos Nx}{x}dx$ numerically for a set of values of $\varepsilon \ll 1$ and $N \gg 1$. Show that as N increases, $I_1 \to 1$ and I_2 decreases toward zero.

$$\lim_{R\to\infty} \frac{1}{2\pi}\int_{-R}^{R} e^{ixr}dr = \delta(x)$$

$$\lim_{R\to\infty} \frac{e^{ixR} - e^{-ixR}}{2ix\pi} = \lim_{R\to\infty} \frac{1}{\pi}\frac{\sin xR}{x} = \delta(x)$$

To check for the sifting property, we multiply by a test function $f(x)$ that has a set of simple poles in the upper-half plane, and is analytic in the lower half plane. Then integrate.

$$I = \int_{-\infty}^{+\infty} f(x)\frac{\sin Rx}{\pi x}dx = \frac{1}{2\pi i}\int_{-\infty}^{+\infty} f(x)\frac{e^{iRx} - e^{-iRx}}{x}dx$$

Close the contour upward for the first term and downward for the second term. There is a pole at

$x = 0.$ Putting a little semicircle under the pole, we get

$$I = f(0) + \sum_p \mathrm{Res}(f)_{z_p}\frac{e^{iRz_p}}{z_p} - 0$$

Then as $R \to \infty$,

$$e^{iRz_p} = e^{iRx_p}e^{-Ry_p} \to 0,$$

because y_p is positive. Thus

$$\lim_{R\to\infty} \int_{-\infty}^{+\infty} f(x)\frac{\sin Rx}{\pi x}dx = f(0)$$

The result is the same if we put the semicicrle over the pole at the origin, as it must be, since

$\sin Rx/x$ has a removable singularity at $x = 0$.

Replacing the sine with a cosine, the pole is not removable, and the result depends on how the path is chosen. The principal value is zero. Putting the semicircle under the pole:

$$P\int_{-\infty}^{+\infty} f(x)\,\frac{\cos Rx}{x}\,dx + \int_{\text{semicircle}} f(z)\,\frac{\cos Rz}{z}\,dz = \frac{2\pi i}{2}\left[f(0) + \sum_p \operatorname{Res}(f)_{z_p}\,\frac{e^{iRz_p}}{z_p} \right]$$

$$P\int_{-\infty}^{+\infty} f(x)\,\frac{\cos Rx}{x}\,dx + \lim_{\varepsilon\to 0}\int_{-\pi}^{0} f(\varepsilon e^{i\theta})\cos(R\varepsilon e^{i\theta})\,i\,d\theta = \pi i\left[f(0) + \sum_p \operatorname{Res}(f)_{z_p}\,\frac{e^{iRz_p}}{z_p} \right]$$

$$\lim_{R\to\infty} P\int_{-\infty}^{+\infty} f(x)\,\frac{\cos Rx}{x}\,dx + i\pi f(0) = i\pi f(0)$$

Thus

$$\lim_{R\to\infty} P\int_{-\infty}^{+\infty} f(x)\,\frac{\cos Rx}{x}\,dx = 0$$

Now for some numerical integration: First the sine.

$$\frac{2}{\pi}\int_{.001}^{10} \frac{\sin 10x}{x}\,dx = 0.98816$$

$$\frac{2}{\pi}\int_{.0001}^{10} \frac{\sin 10x}{x}\,dx = 0.99389$$

$$\frac{2}{\pi}\int_{.0001}^{10} \frac{\sin 20x}{x}\,dx = 0.99719$$

$$\frac{2}{\pi}\int_{.0001}^{10} \frac{\sin 30x}{x}\,dx = 0.99814$$

The results approach 1.

For the cosine, we have:

$$\int_{.0001}^{10} \frac{\cos 10x}{x}\,dx = 6.3254$$

$$\int_{.00001}^{10} \frac{\cos 10x}{x}\,dx = 8.628$$

$$\int_{.00001}^{10} \frac{\cos 20x}{x}\,dx = 7.9356$$

$$\int_{.0001}^{20} \frac{\cos 30x}{x}\,dx = 5.232$$

$$\int_{.00001}^{10} \frac{\cos 30x}{x}\,dx = 7.5312$$

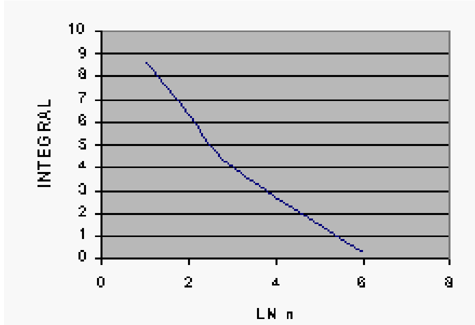
$$\int_{.00001}^{10} \frac{\cos 40x}{x}\,dx = 7.2447$$

$$\int_{.00001}^{10} \frac{\cos 100x}{x}\,dx = 6.3314$$

$$\int_{.00001}^{10} \frac{\cos 1000x}{x}\,dx = 4.0279$$

$$\int_{.00001}^{10} \frac{\cos 10^6 x}{x}\,dx = 0.21121$$

The plot shows $\int_{.00001}^{10} \frac{\cos Nx}{x}\,dx$ versus $\log N$. The integral approaches zero as N increases.



The result is needed in Appendix 8.

26. Show that $\frac{d}{dx}\text{sign}(x) = 2\delta(x)$ where $\text{sign}(x) = \frac{x}{|x|}$.

First note that the function $g(x) = |x| = x\ \text{sign}(x)$ is continuous and thus, by the smudging theorem, we can find an equivalent distribution. Then the derivative $g'(x)$ is also a distribution whose value is $\text{sign}(x)$, and the second derivative is also a distribution with the property

$$\begin{aligned}\int_{-\infty}^{+\infty} \frac{d}{dx}\text{sign}(x)f(x)dx &= -\int_{-\infty}^{+\infty} \text{sign}(x)f'(x)dx \\ &= \int_{-\infty}^{0} f'(x)dx - \int_{0}^{\infty} f'(x)dx \\ &= f(0) - (-f(0)) = 2f(0) \\ &= \int_{-\infty}^{\infty} 2\delta(x)f(x)dx\end{aligned}$$

and thus

$$\frac{d}{dx}\text{sign}(x) = 2\delta(x)$$

27. Show that

$$x^m\delta^{(n)}(x) = \begin{cases} 0 & \text{if } n < m \\ (-1)^m\frac{n!}{(n-m)!}\delta^{(n-m)}(x) & \text{if } n \geq m \end{cases}$$

First note that the result is true in the following cases:

$m = 0$, any n –This is an identity.

$m = 0$, . (Equation 6.7)

$m = 1$, $n = 1$ (Example 6.4)

To prove the general result, we integrate by parts:

$$\begin{aligned}\int_{-\infty}^{+\infty} x^m\delta^{(n)}(x)f(x)dx &= -\int_{-\infty}^{+\infty} \frac{d}{dx}(x^m f(x))\delta^{(n-1)}(x)dx \\ &= -\int_{-\infty}^{+\infty}(x^m f'(x) + mx^{m-1}f(x))\delta^{(n-1)}(x)dx \\ &= (-1)^2\int_{-\infty}^{+\infty} \frac{d}{dx}(x^m f'(x) + mx^{m-1}f(x))\delta^{(n-2)}(x)dx \\ &= (-1)^2\int_{-\infty}^{+\infty}(x^m f^{(2)}(x) + 2mx^{m-1}f'(x) + m(m-1)x^{m-2}f(x))\delta^{(n-2)}(x)dx\end{aligned}$$

If $n < m$, we proceed this way until we have completed n integrations by parts, leaving $\delta(x)$ in the integrand. Each term multiplying the delta function has a positive power of x, and the result is zero.

Now assume the result is true for some value $n \geq m$. Then

$$\int_{-\infty}^{+\infty} x^m \delta^{(n+1)}(x) f(x)\,dx = -\int_{-\infty}^{+\infty} \frac{d}{dx}(x^m f(x))\delta^{(n)}(x)\,dx$$
$$= -\int_{-\infty}^{+\infty} \left(x^m f'(x) + m x^{m-1} f(x)\right)\delta^{(n)}(x)\,dx$$

Use the given result for n.

$$\int_{-\infty}^{+\infty} x^m \delta^{(n+1)}(x) f(x)\,dx = -\int_{-\infty}^{+\infty}\left[(-1)^m \frac{n!}{(n-m)!}\delta^{(n-m)}(x) f'(x) + m(-1)^{m-1}\frac{n!}{(n-m+1)!}\delta^{(n-m+1)}(x) f(x)\right]dx$$

Integrate the first term by parts to get

$$\int_{-\infty}^{+\infty}(-1)^{m+1}\frac{n!}{(n-m)!}\delta^{(n-m+1)}(x) f(x)\,dx$$

and then combining the terms:

$$\int_{-\infty}^{+\infty} x^m \delta^{(n+1)}(x) f(x)\,dx = (-1)^m \frac{n!}{(n-m+1)!}\int_{-\infty}^{+\infty}(n-m+1+m)\delta^{(n-m+1)}(x) f(x)\,dx$$
$$= (-1)^m \frac{(n+1)!}{(n-m+1)!}\int_{-\infty}^{+\infty}\delta^{(n-m+1)}(x) f(x)\,dx$$

and so the result is true for $n+1$ if it is true for n. But the result is true for $n = 1$, with $m = 1$ or 0. Thus it is true for all $n > 1$ if $m = 1$.

Next we increase m. Assume the result is true for some $m < n$. Then

$$\int_{-\infty}^{+\infty} x^{m+1}\delta^{(n)}(x) f(x)\,dx = \int_{-\infty}^{+\infty} x^m \delta^{(n)}(x)[x f(x)]\,dx$$
$$\int_{-\infty}^{+\infty}(-1)^m \frac{n!}{(n-m)!}\delta^{(n-m)}(x)[x f(x)]\,dx$$
$$= -\int_{-\infty}^{+\infty}(-1)^m \frac{n!}{(n-m)!}\delta^{(n-m-1)}(x)[f(x) + x f'(x)]\,dx$$

where we used this result:

$$\int \delta^{(n)}(x) f(x)\,dx = (-1)^n f^{(n)}(0) = -(-1)^{n-1}\frac{d^{n-1}}{dx^{n-1}} f'(x)\bigg|_0 = -\int \delta^{(n-1)}(x) f'(x)\,dx$$

Work on the second term, assuming $m + 1 < n$.

$$\int_{-\infty}^{+\infty} x\,\delta^{(n-m-1)}(x) f'(x)\,dx = -\int_{-\infty}^{+\infty}\frac{(n-m-1)!}{(n-m-2)!}\delta^{(n-m-2)}(x) f'(x)\,dx$$
$$= -\int_{-\infty}^{+\infty}(n-m-1)\delta^{(n-m-1)} f(x)\,dx$$

Thus

$$\int_{-\infty}^{+\infty} x^{m+1}\delta^{(n)}(x) f(x)\,dx = -(-1)^m \frac{n!}{(n-m)!}\int_{-\infty}^{+\infty}\left[\delta^{(n-m-1)}(x) + (n-m-1)\delta^{(n-m-1)}\right] f(x)\,dx$$
$$= (-1)^{m+1}\frac{n!}{(n-m)!}\int_{-\infty}^{+\infty}[n-m]\delta^{(n-m-1)}(x) f(x)\,dx$$
$$= (-1)^{m+1}\frac{n!}{(n-m-1)!}\int_{-\infty}^{+\infty}\delta^{(n-m-1)}(x) f(x)\,dx$$

Thus the result is true for $m + 1 < n$ if it is true for $m < n$.

Putting these results together, we conclude that the result is true for all n and m.

28. The integral $\int_0^{\infty} x^{\alpha} f(x)\,dx = \int_{-\infty}^{+\infty} x^{\alpha}\Theta(x) f(x)\,dx$ where $\alpha < 0$ may be integrated if x^{α} is interpreted as a distribution. First show that

$$x^{\alpha}\Theta(x) = \frac{1}{(\alpha+1)(\alpha+2)\cdots(\alpha+n)}\frac{d^n}{dx^n}[x^{\alpha+n}\Theta(x)]$$

where $\alpha + n > 0$ and n is an integer. Use the result to evaluate the integral

$$\int_0^\infty x^{-3/2}e^{-x}dx$$

Start with the RHS

$$\begin{aligned}\int_{-\infty}^{+\infty}\frac{d}{dx}(x^{\alpha}\Theta(x))f(x)dx &= -\int_{-\infty}^{+\infty}x^{\alpha}\Theta(x)f'(x)dx = -\int_0^{+\infty}x^{\alpha}f'(x)dx\\ &= x^{\alpha}f(x)\big|_{x=0} + \int_0^\infty \alpha x^{\alpha-1}f(x)dx\\ &= \int_{-\infty}^{\infty}\alpha x^{\alpha-1}\Theta(x)f(x)dx \text{ for } \alpha > 0\end{aligned}$$

Thus

$$x^{\alpha}\Theta(x) = \frac{1}{\alpha+1}\frac{d}{dx}\left(x^{\alpha+1}\Theta(x)\right) \text{ for } \alpha > -1$$

Now we can repeat the process $n-1$ times to obtain

$$x^{\alpha}\Theta(x) = \frac{1}{(\alpha+n)(\alpha+n-1)\cdots(\alpha+1)}\frac{d^n}{dx^n}(x^{\alpha+n}\Theta(x)) \text{ for } \alpha + n > 0$$

Thus

$$\int_0^\infty x^{\alpha}f(x)dx = \frac{(-1)^n}{(\alpha+1)(\alpha+2)\cdots(\alpha+n)}\int_0^\infty x^{\alpha+n}f^{(n)}(x)dx$$

Now we apply this result. Choose $n = 2.$

$$\int_0^\infty x^{-3/2}e^{-x}dx = \frac{1}{(-1/2)(1/2)}\int_0^\infty x^{1/2}e^{-x}dx$$

The integral is $\Gamma(3/2) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}/2.$ Thus

$$\int_0^\infty x^{-3/2}e^{-x}dx = -2\sqrt{\pi}$$

This is the Hadamard "finite part".

29. A material absorbs light at frequency

ν_I due to an atomic transition. The imaginary part of the dielectric constant may be approximated as $\sigma_0\delta(\nu - \nu_I).$ Use the Kramers-Kronig relations (Chapter 2 Example 2.24) to determine the behavior of the refractive index $n = \sqrt{\mathrm{Re}(\varepsilon/\varepsilon_0)}$ as a function of frequency. Comment.

Using the results of Example 2.24, and writing $\mathrm{Re}\,\varepsilon/\varepsilon_0 = n^2,$ we have

$$\begin{aligned}n(\omega_0)^2 - 1 &= \frac{2}{\pi\varepsilon_0}P\int_0^\infty \frac{\omega\sigma_0\delta\left(\frac{\omega-\omega_I}{2\pi}\right)}{\omega^2-\omega_0^2}d\omega\\ &= \frac{4\sigma_0}{\varepsilon_0}P\int_0^\infty\frac{\omega\delta(\omega-\omega_I)}{\omega^2-\omega_0^2}d\omega\\ &= \frac{4\sigma_0}{\varepsilon_0}\frac{\omega_I}{\omega_I^2-\omega_0^2}\end{aligned}$$

Thus

$$n(\nu) = \sqrt{1 + \frac{2\sigma_0}{\pi\varepsilon_0}\frac{\nu_L}{\nu_L^2 - \nu^2}}$$

The refractive index approaches a constant for $\nu \ll \nu_L$ and approaches 1, the vacuum result, for $\nu \gg \nu_L$.

$$n \simeq \left(1 - \frac{\sigma_0}{\pi\varepsilon_0}\frac{\nu_L}{\nu^2}\right)$$

This result is consistent with the fact that, for most materials, the refractive index is greater in the blue than in the red. (For $\nu \gg \nu_L$, $\frac{dn}{d\nu} = \frac{2\sigma_0}{\pi\sqrt{\varepsilon_0}}\frac{\nu_L}{\nu^3}$ is positive. This frequency range includes the visible if ν_L is in the infrared.). Near the line center,

$$n \simeq \sqrt{\frac{2\sigma_0}{\pi}\frac{\nu_L}{\nu_L^2 - \nu^2}}$$

We have answered the question posed, but we should check that the second relation in Example 2.24 is also satisfied.

$$\nu(\omega_1) = -\frac{2\sigma_0}{\pi^2} P\int_0^\infty \frac{\omega_1}{\left(\omega^2 - \omega_1^2\right)}\frac{\omega_L}{\left(\omega_L^2 - \omega^2\right)}d\omega$$

The integrand is even in ω so we may rewrite it as

$$-\frac{2\sigma_0}{\pi^2} P\int_{-\infty}^\infty \frac{\omega_1}{\left(\omega^2 - \omega_1^2\right)}\frac{\omega_L}{\left(\omega_L^2 - \omega^2\right)}d\omega$$

We evaluate the integral by putting semicircles over the poles at $\pm\omega_1$, $\pm\omega_L$. The poles are simple. We may close the contour with a big semicircle in the upper half plane. The integral around the closed contour is zero, because there are no singularities inside. Thus

$$P\int_{-\infty}^\infty \frac{\omega_1}{\left(\omega^2 - \omega_1^2\right)}\frac{\omega_L}{\left(\omega_L^2 - \omega^2\right)}d\omega + \sum\left(\text{integrals around semicircles}\right) = 0$$

The integral around each semicircle is similar. For the semicircle around ω_1, write $\omega = \omega_1 + \varepsilon e^{i\theta}$

$$\lim_{\varepsilon\to 0}\int_\pi^0 \frac{\omega_1}{\left(2\omega_1 + \varepsilon e^{i\theta}\right)\varepsilon e^{i\theta}}\frac{\omega_L}{\left(\omega_L^2 - \left[\omega_1 + \varepsilon e^{i\theta}\right]^2\right)}\varepsilon e^{i\theta} i d\theta = \int_\pi^0 \frac{\omega_1}{(2\omega_1)}\frac{\omega_L}{\left(\omega_L^2 - \omega_1^2\right)} i d\theta$$
$$= -i\frac{\pi}{2}\frac{\omega_L}{\left(\omega_L^2 - \omega_1^2\right)}$$

For the semicircle around $-\omega_1$, write $\omega = -\omega_1 + \varepsilon e^{i\theta}$

$$\lim_{\varepsilon\to 0}\int_\pi^0 \frac{\omega_1}{\left(-2\omega_1 - \varepsilon e^{i\theta}\right)\varepsilon e^{i\theta}}\frac{\omega_L}{\left(\omega_L^2 - \left[-\omega_1 + \varepsilon e^{i\theta}\right]^2\right)}\varepsilon e^{i\theta} i d\theta = i\frac{\pi}{2}\frac{\omega_L}{\left(\omega_L^2 - \omega_1^2\right)}$$

and the sum of these two terms is zero. The sum of the integrals over the poles at $\pm\omega_L$ is also zero.

The result is different if $\omega_1 = \omega_L$. In this case there are second-order poles at $\pm\omega_L$. The integrals are

$$\lim_{\varepsilon\to 0}\int_\pi^0 \frac{\omega_L^2}{\left(\omega_L^2 - [\pm\omega_L + \varepsilon e^{i\theta}]^2\right)^2}\varepsilon e^{i\theta} i d\theta = \lim_{\varepsilon\to 0}\int_\pi^0 \frac{\omega_L^2}{(\mp 2\varepsilon\omega_L e^{i\theta})^2}\varepsilon e^{i\theta} i d\theta$$
$$= \lim_{\varepsilon\to 0}\int_\pi^0 \frac{1}{4\varepsilon}e^{-i\theta} i d\theta$$
$$= \lim_{\varepsilon\to 0}\frac{i}{4\varepsilon}\frac{e^{-i\theta}}{-i}\Bigg|_\pi^0$$
$$= \lim_{\varepsilon\to 0}\frac{-1}{2\varepsilon}$$

Thus the result is zero unless $\omega_1 = \omega_L$, when it is infinite. We have regained the delta-function type behavior.

To be sure, we should check for the sifting property. I leave that for another day.

You might wonder why we obtain an infinite result for the integral around the semicircle, whereas the integral around an entire circle (the residue) is finite. This happens because for a pole or order greater than one, the contribution of each segment of the path to the residue is not the same. This is clear from the integral above: for the entire circle (limits 0 and 2π) the integral is zero, but for both halves the result is (positive or negative) infinite.

30. Demonstrate the sifting property of the delta sequence (6.5), $\phi_n(x) = \frac{1}{n\pi}\frac{\sin^2 nx}{x^2}$, in the case that $f(x)$ has a second order pole at $z = z_p$ in the upper-half plane. Can you extend the result to a pole of order m?

We can borrow the result from the chapter, changing only the evaluation of the residues.

The residue at z_p, by method 3, is:

$$\text{Res}(z_p) = \frac{1}{\pi n}\lim_{z\to z_p}\frac{d}{dz}\left((z-z_p)^2 f(z)\frac{1-e^{2inz}}{z^2}\right)$$

Now if $f(z)$ has a second order pole, then its Laurent series is of the form

$$f(z) = \frac{a_{-2}}{(z-z_p)^2} + \sum_{m=-1}^{\infty} a_m(z-z_p)^m$$

and thus

$$\begin{aligned}\frac{d}{dz}\left((z-z_p)^2 f(z)\frac{1-e^{2inz}}{z^2}\right) &= \frac{d}{dz}\left(\left[a_{-2} + \sum_{m=-1}^{\infty} a_m(z-z_p)^{m+2}\right]\frac{1-e^{2inz}}{z^2}\right)\\ &= \left[\sum_{m=-1}^{\infty} a_m(m+2)(z-z_p)^{m+1}\right]\frac{1-e^{2inz}}{z^2}\\ &\quad + \left[a_{-2} + \sum_{m=-1}^{\infty} a_m(z-z_p)^{m+2}\right]\left(\frac{-2ine^{2inz}}{z^2} - 2\frac{1-e^{2inz}}{z^3}\right)\end{aligned}$$

and thus

$$\begin{aligned}\text{Res}(z_p) &= \frac{1}{n\pi}\left[\left(a_{-1}\frac{1-e^{2inz_p}}{z_p^2}\right) + a_{-2}\left(\frac{-2ine^{2inz_p}}{z_p^2} - 2\frac{1-e^{2inz_p}}{z_p^3}\right)\right] \quad \text{eqn 1 P 6.30}\\ &\to 0 \text{ as } n\to\infty\end{aligned}$$

since

$$e^{2inz_p} = e^{2inx_p}e^{-2ny_p}$$

and $y_p > 0.$

The first term in equation (eqn1 P 6.30) is the same as the result in the chapter. The remaining terms arise because this pole is of higher order. For a pole of order m, we would have to differentiate m times, leading to a term of the form $n^m e^{2inz_p}$. But this term also $\to 0$ as $n\to\infty$ for any m. Thus the result is unchanged.

Chapter 7: Fourier Transforms

1. Find the Fourier transform of the following functions, and verify your results by computing the inverse transform.

(a) $1/\left(x^2 + 4x + 13\right)$

$$F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{e^{-ikx}}{x^2 + 4x + 13} dx$$

We do the integral by completing the contour with a big semicircle, and rewriting the denominator:

$$F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{e^{-ikx}}{(x + 2 - 3i)(x + 2 + 3i)} dx$$

There are poles at $x = -2 \pm 3i$. For $k < 0$ we close the contour upward. Only the pole at $x = -2 + 3i$ is inside, and we obtain:

$$F(k) = \frac{1}{\sqrt{2\pi}} (2\pi i) \frac{\exp(-ik(-2 + 3i))}{6i} = \frac{\sqrt{2\pi}}{6} e^{(3+2i)k}$$

For $k > 0$ we close the contour downward. Only the pole at $x = -2 - 3i$ is inside, and we obtain:

$$F(k) = \frac{1}{\sqrt{2\pi}} (-2\pi i) \frac{\exp(-ik(-2 - 3i))}{-6i} = \frac{\sqrt{2\pi}}{6} e^{(-3+2i)k}$$

In both cases we have:

$$F(k) = \frac{\sqrt{2\pi}}{6} e^{2ik} e^{-3|k|}$$

To invert, we evaluate:

$$\begin{aligned}
f(x) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{\sqrt{2\pi}}{6} e^{2ik} e^{-3|k|} e^{ikx} dk = \frac{1}{6} \int_{-\infty}^{+\infty} e^{2ik} e^{-3|k|} e^{ikx} dk \\
&= \frac{1}{6} \int_{-\infty}^{0} e^{2ik} e^{3k} e^{ikx} dk + \frac{1}{6} \int_{0}^{+\infty} e^{2ik} e^{-3k} e^{ikx} dk \\
&= \frac{1}{6} \left(\frac{e^{ik(2+x)+3k}}{(2+x)i + 3} \Bigg|_{-\infty}^{0} + \frac{e^{ik(2+x)-3k}}{(2+x)i - 3} \Bigg|_{0}^{\infty} \right) \\
&= \frac{1}{6} \left(\frac{1}{(2+x)i + 3} + \frac{-1}{(2+x)i - 3} \right) \\
&= \frac{1}{6} \left(\frac{6}{9 + (2+x)^2} \right) = \frac{1}{x^2 + 4x + 13}
\end{aligned}$$

as required.

(b) $e^{-\alpha x^2} \cos \beta x$

$$F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\alpha x^2} \cos\beta x e^{-ikx} dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\alpha x^2} \frac{e^{i\beta x} + e^{-i\beta x}}{2} e^{-ikx} dx$$

So there are 2 integrals of the form

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-\alpha x^2} e^{-i\gamma x} dx$$

with $\gamma = k \pm \beta.$ To do each we complete the square:

$$-\alpha x^2 - i\gamma x = -\alpha\left(x^2 + i\frac{\gamma}{\alpha}x\right) = -\alpha\left(x + i\frac{\gamma}{2\alpha}\right)^2 - \alpha\frac{\gamma^2}{4\alpha^2}$$

and the integral is

$$\exp\left(-\frac{\gamma^2}{4\alpha}\right)\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \exp\left(-\alpha\left(x + i\frac{\gamma}{2\alpha}\right)^2\right)dx = \exp\left(-\frac{\gamma^2}{4\alpha}\right)\frac{1}{\sqrt{2\pi}} \int_{-\infty+i\gamma/2\sqrt{\alpha}}^{+\infty+i\gamma/2\sqrt{\alpha}} e^{-u^2} \frac{du}{\sqrt{\alpha}}$$

$$= \exp\left(-\frac{\gamma^2}{4\alpha}\right)\frac{1}{\sqrt{2\alpha}}$$

where $u = \sqrt{\alpha}\left(x + i\frac{\gamma}{2\alpha}\right).$ Thus:

$$F(k) = \frac{1}{2\sqrt{2\alpha}}\left(\exp\left(-\frac{(k+\beta)^2}{4\alpha}\right) + \exp\left(-\frac{(k-\beta)^2}{4\alpha}\right)\right)$$

We may invert the two terms separately:

$$f_\pm(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \exp\left(-\frac{(k\pm\beta)^2}{4\alpha}\right) e^{ikx} dk$$

Again we complete the square:

$$\begin{aligned}
-\frac{(k\pm\beta)^2}{4\alpha} + ikx &= -\frac{1}{4\alpha}(k^2 \pm 2k\beta + \beta^2 + 4ik\alpha x) \\
&= -\frac{\beta^2}{4\alpha} - \frac{1}{4\alpha}(k^2 + 2k(2i\alpha x \pm \beta)) \\
&= -\frac{\beta^2}{4\alpha} - \frac{1}{4\alpha}\left(k^2 + 2k(2i\alpha x \pm \beta) + (2i\alpha x \pm \beta)^2 - (2i\alpha x \pm \beta)^2\right) \\
&= -\frac{\beta^2}{4\alpha} - \frac{1}{4\alpha}\left((2i\alpha x + \beta + k)^2 - (2i\alpha x \pm \beta)^2\right)
\end{aligned}$$

Thus

$$\begin{aligned}
f_\pm(x) &= \exp\left(-\frac{\beta^2}{4\alpha} + \frac{(2i\alpha x \pm \beta)^2}{4\alpha}\right)\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \exp\left(-\frac{1}{4\alpha}(2i\alpha x \pm \beta + k)^2\right)dk \\
&= \exp(-\alpha x^2 \pm ix\beta)\frac{2\sqrt{\alpha}}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \exp\left(-\frac{1}{4\alpha}(2i\alpha x \pm \beta + k)^2\right)\frac{dk}{2\sqrt{\alpha}} \\
&= \exp(-\alpha x^2 \pm ix\beta)\frac{2\sqrt{\alpha}}{\sqrt{2}}
\end{aligned}$$

and so

$$f(x) = \frac{1}{2\sqrt{2\alpha}}(f_+(x) + f_-(x)) = \frac{1}{2\sqrt{2\alpha}}\frac{2\sqrt{\alpha}}{\sqrt{2}}\left(\exp\left(-\alpha x^2 + ix\beta\right) + \exp\left(-\alpha x^2 - ix\beta\right)\right)$$
$$= \frac{1}{2}e^{-\alpha x^2}(\exp(ix\beta) + \exp(-ix\beta)) = e^{-\alpha x^2}\cos\beta x$$

as required.

(c) $f(x) = \begin{cases} x & \text{if} \quad 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$

$$\begin{aligned}
F(k) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} f(x)e^{-ikx}dx = \frac{1}{\sqrt{2\pi}}\int_0^1 xe^{-ikx}dx \\
&= \frac{1}{\sqrt{2\pi}}\left(x\frac{e^{-ikx}}{-ik}\Big|_0^1 - \int_0^1 \frac{e^{-ikx}}{-ik}dx\right) \\
&= \frac{1}{\sqrt{2\pi}}\left(\frac{e^{-ik}}{-ik} + \frac{1}{ik}\frac{e^{-ikx}}{-ik}\Big|_0^1\right) \\
&= \frac{1}{\sqrt{2\pi}}\left(\frac{e^{-ik}}{-ik} + \frac{1}{ik}\left(\frac{e^{-ik}-1}{-ik}\right)\right) \\
&= \frac{1}{\sqrt{2\pi}}\left(\frac{ie^{-ik}}{k} + \frac{e^{-ik}-1}{k^2}\right) \\
&= \frac{1}{\sqrt{2\pi}}\left(\frac{e^{-ik}(ik+1)-1}{k^2}\right)
\end{aligned}$$

Inverting:

$$f(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} F(k)e^{ikx}dk = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\left(\frac{ie^{-ik}}{k} + \frac{e^{-ik}-1}{k^2}\right)e^{ikx}dk$$
$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{\exp(ik(x-1))}{k^2}(ik+1)dk - \frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{e^{ikx}}{k^2}dk$$

We invert by completing the contour with a big semi-circle. Because of the terms $\exp(ik(x-1))$ in the first integral, we complete upward for $x > 1$ and downward for $x < 1$. There is a pole at $k = 0$, so we choose to put the path of integration slightly below the axis. Then for $x > 1$ the pole is inside the contour. To find the residue we evaluate the Laurent series:

$$\frac{\exp(ik(x-1))}{k^2}(ik+1) = \frac{\left(1 + ik(x-1) + (ik(x-1))^2/2 + \cdots\right)}{k^2}(ik+1)$$
$$= \frac{1}{k^2} + \frac{i}{k}x + \frac{1}{2} - \frac{1}{2}x^2 + \cdots$$

and so the residue is ix, and the integral is:

$$I_1 = i(ix) = -x$$

For $x < 1$ there is no pole inside the contour and so the integral is zero.

For the second integral we close up for $x > 0$ and down for $x < 0$. There is a second order pole at $k = 0$:

$$\frac{e^{ikx}}{k^2} = \frac{1 + ikx + (ikx)^2/2 + \cdots}{k^2} = \frac{1}{k^2} + \frac{i}{k}x - \frac{1}{2}x^2 + \cdots$$

and so the residue is ix. For $x > 0$, the integral is:

$$I_2 = i(ix) = -x$$

while for $x < 0$ it is zero. Thus our result is:

$$I = I_1 - I_2 = \begin{cases} 0 - 0 = 0 & \text{if } x < 0 \\ 0 - (-x) = x & \text{if } 0 < x < 1 \\ -x - (-x) = 0 & \text{if } x > 1 \end{cases}$$

as expected.

(d) $\frac{1}{\cosh ax}$

$$F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{e^{-ikx}}{\cosh ax} dx$$

We close the contour with a rectangle with its top side at $y = \pi/a$. Then on the top side:

$$\int_{R+i\pi/a}^{-R+i\pi/a} \frac{e^{-ikz}}{\cosh az} dz = \int_R^{-R} \frac{e^{-ikx} e^{k\frac{\pi}{a}}}{-\cosh ax} dx = e^{k\frac{\pi}{a}} \int_{-R}^{R} \frac{e^{-ikx}}{\cosh ax} dx$$

The integral along the two vertical sides goes to zero as $R \to \infty$:

$$\int_R^{R+i\pi/a} \frac{e^{-ikz}}{\cosh az} dz = e^{-ikR} \int_0^{\pi/a} \frac{e^{ky}}{\cosh aR \cos ay + i \sinh aR \sin ay} dy$$
$$= \frac{e^{-ikR}}{\cosh aR} \int_0^{\pi/a} \frac{e^{ky}}{\cos ay + i \tanh aR \sin ay} dy \to 0 \text{ as } R \to \infty$$

since the integral $\to \int_0^{\pi/a} \frac{e^{ky}}{\cos ay + i \sin ay} dy$ which is bounded and the factor in front goes to zero.

There is one pole inside the contour, at $z = i\pi/2a$. The residue is (method 4):

$$\lim_{z \to i\pi/2a} \frac{e^{-ikz}}{a \sinh az} = \frac{e^{k\pi/2a}}{a \sinh \frac{1}{2} i\pi} = \frac{e^{k\pi/2a}}{ia}$$

and thus the integral is:

$$\int_{\text{rectangle}} = \left(1 + e^{k\frac{\pi}{a}}\right) \int_{-\infty}^{+\infty} \frac{e^{-ikx}}{\cosh ax} dx = 2\pi i \left(\frac{e^{k\pi/2a}}{ia}\right)$$

and thus

$$F(k) = \frac{\sqrt{2\pi}}{a}\frac{e^{k\pi/2a}}{1+e^{k\frac{\pi}{a}}} = \frac{\sqrt{2\pi}}{a}\frac{1}{e^{-k\pi/2a}+e^{k\pi/2a}} = \sqrt{\frac{\pi}{2}}\frac{1}{a\cosh(k\pi/2a)}$$

So the transform of sech is another sech! The inverse is then:

$$\sqrt{\frac{\pi}{2}}\frac{1}{a}\sqrt{\frac{\pi}{2}}\frac{1}{\frac{\pi}{2a}\cosh\left(x\frac{\pi}{2}\frac{2a}{\pi}\right)} = \frac{1}{\cosh ax}$$

as expected.

(e) te^{-t}

$$\begin{aligned} F(\omega) &= \frac{1}{\sqrt{2\pi}}\int_0^\infty te^{-at}e^{i\omega t}dt \\ &= \frac{1}{\sqrt{2\pi}}\left(\frac{te^{(i\omega-a)t}}{i\omega-a}\bigg|_0^\infty - \int_0^\infty \frac{e^{(i\omega-a)t}}{i\omega-a}d\omega\right) \\ &= \frac{1}{\sqrt{2\pi}}\left(0-\frac{e^{(i\omega-a)t}}{(i\omega-a)^2}\bigg|_0^\infty\right) \\ &= \frac{1}{\sqrt{2\pi}}\left(\frac{1}{(i\omega-a)^2}\right) \end{aligned}$$

To invert:

$$\begin{aligned} f(t) &= \frac{1}{\sqrt{2\pi}}\int_0^\infty \frac{1}{\sqrt{2\pi}}\left(\frac{1}{(i\omega-a)^2}\right)e^{-i\omega t}d\omega \\ &= -\frac{1}{2\pi}\int_0^\infty \frac{e^{-i\omega t}d\omega}{(\omega+ia)^2} \end{aligned}$$

There is a second order pole at $\omega = -ia$. For $t < 0$ we close upward. The contour encloses no poles, and the result is zero. For $t > 0$, we close downward. the residue at the pole is:

$$\lim_{\omega\to -ia}\frac{d}{d\omega}(\omega+ia)^2\frac{e^{-i\omega t}}{(\omega+ia)^2} = -ite^{-it(-ia)} = -ite^{-at}$$

Thus the inverse is:

$$f(t) = -2\pi i\left(-\frac{1}{2\pi}\right)\left(-ite^{-at}\right) = te^{-at}$$

as required.

(f) $x/\left(x^2+a^2\right)$

$$F(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\frac{x}{x^2+a^2}e^{-ikx}dx$$

We complete the contour upward for $k < 0$ and downward for $k > 0$. There are simple poles at $x = \pm ia$. The result is:

$$F(k) = \frac{1}{\sqrt{2\pi}} 2\pi i \frac{ia}{2ia} e^{-ik(ia)} = i\sqrt{\frac{\pi}{2}} e^{ka} \text{ for } k < 0$$
$$= \frac{1}{\sqrt{2\pi}} (-2\pi i) \frac{-ia}{-2ia} e^{-ik(-ia)} = -i\sqrt{\frac{\pi}{2}} e^{-ka} \text{ for } k > 0$$

$$F(k) = -i\frac{k}{|k|}\sqrt{\frac{\pi}{2}} e^{-|ka|}$$

Inverting, we get:

$$f(x) = \frac{1}{\sqrt{2\pi}} i\sqrt{\frac{\pi}{2}} \left\{ \int_{-\infty}^{0} e^{ka} e^{ikx} dk - \int_{0}^{\infty} e^{-ka} e^{ikx} dk \right\}$$
$$= \frac{i}{2} \left\{ \frac{e^{k(a+ix)}}{(a+ix)} \Bigg|_{-\infty}^{0} - \frac{e^{k(ix-a)}}{(ix-a)} \Bigg|_{0}^{\infty} \right\}$$
$$= \frac{i}{2} \left(\frac{1}{a+ix} - \frac{-1}{ix-a} \right) = \frac{x}{a^2+x^2}$$

as required.

2.(a) $F(k) = \frac{1-2ik}{1+4k^2}$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{1-2ik}{1+4k^2} e^{ikx} dk$$

There are poles at $k = \pm i/2$

$$f(x) = \sqrt{2\pi}\,\frac{i}{4} \left(\frac{1-2i(i/2)}{i} e^{-x/2} \right) \text{ for } x > 0$$
$$= \sqrt{\frac{\pi}{2}} e^{-x/2} \text{ for } x > 0$$

and for $x < 0$

$$f(x) = -\sqrt{2\pi}\,\frac{i}{4} \frac{1-2i(-i/2)}{-i} e^{-x/2} = 0$$

(b) $F(k) = \frac{1}{1+ik^3}$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{1}{1+ik^3} e^{ikx} dk$$

The poles are the cube roots of $-1/i = i$. These have the values

$$k_n = e^{i(\pi/2+2\pi n)/3} = e^{i(\pi/6+2\pi n/3)}$$

for $n = 0, 1, 2.$

$$k_0 = e^{i\pi/6} = \frac{1}{2}\sqrt{3} + \frac{1}{2}i$$

$$k_1 = e^{i5\pi/6} = -\frac{1}{2}\sqrt{3} + \frac{1}{2}i$$

and

$$k_2 = e^{i3\pi/2} = -i$$

For $x > 0$ we close upward, enclosing the first two poles:

$$\begin{aligned} f(x) &= \frac{1}{\sqrt{2\pi}} 2\pi \left(\frac{\exp\left(\sqrt{3}i/2 - 1/2\right)x}{3\left(\frac{1}{2}\sqrt{3} + \frac{1}{2}i\right)^2} + \frac{\exp\left(-\sqrt{3}i/2 - 1/2\right)x}{3\left(-\frac{1}{2}\sqrt{3} + \frac{1}{2}i\right)^2} \right) \\ &= \sqrt{2\pi}\, e^{-x/2} \left(\frac{\exp\left(\sqrt{3}ix/2\right)}{\frac{3}{2} + \frac{3}{2}i\sqrt{3}} + \frac{\exp\left(-\sqrt{3}ix/2\right)}{\frac{3}{2} - \frac{3}{2}i\sqrt{3}} \right) \\ &= \sqrt{2\pi}\, e^{-x/2} \frac{\exp\left(\sqrt{3}ix/2\right)\left(\frac{3}{2} - \frac{3}{2}i\sqrt{3}\right) + \left(\frac{3}{2} + \frac{3}{2}i\sqrt{3}\right)\exp\left(-\sqrt{3}ix/2\right)}{9} \\ &= \sqrt{2\pi}\, e^{-x/2} \frac{\cos\left(\sqrt{3}x/2\right) + \sqrt{3}\sin\left(\sqrt{3}x/2\right)}{3} \end{aligned}$$

For $x < 0$ we close downward, enclosing the pole at $k = -i$:

$$\begin{aligned} f(x) &= -\frac{1}{\sqrt{2\pi}} 2\pi \left(\frac{e^x}{3(-i)^2} \right) \\ &= \frac{\sqrt{2\pi}}{3} e^x \end{aligned}$$

(c) $\frac{1}{i \sinh ak}$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{e^{ikx}}{i \sinh ak} dk$$

We close the contour with a rectangle with its top side at $y = \pi/a.$ Then on the top side:

$$\begin{aligned} \int_{R+i\pi/a}^{-R+i\pi/a} \frac{e^{ikx}}{i \sinh ak} dk &= \int_{R}^{-R} \frac{e^{ikx} e^{-x\frac{\pi}{a}}}{i \sinh(ka + i\pi)} dk = e^{-x\frac{\pi}{a}} \int_{-R}^{R} \frac{e^{ikx}}{i \sinh ka} dk \\ &= e^{-x\frac{\pi}{a}} f(x) \sqrt{2\pi} \end{aligned}$$

The integral along the two vertical sides goes to zero as $R \to \infty$:

$$\begin{aligned} \int_{R}^{R+i\pi/a} \frac{e^{ikz}}{\sinh az} dz &= e^{ikR} \int_0^{\pi/a} \frac{e^{-ky}}{\sinh aR \cos ay + i \cosh aR \sin ay} dy \\ &= \frac{e^{ikR}}{\sinh aR} \int_0^{\pi/a} \frac{e^{-ky}}{\cos ay + i \coth aR \sin ay} dy \to 0 \text{ as } R \to \infty \end{aligned}$$

since the integral $\to \int_0^{\pi/a} \frac{e^{-ky}}{\cos ay + i \sin ay} dy$ which is bounded and the factor in front goes to zero.

We know that the pole of the transform will be in the upper half plane, so we put the path of integration under the pole at $k = 0.$ Treating the integral along the upper contour similarly excludes the pole at $k = i\pi/a.$ There is one pole inside the contour and the residue is (method 4):

$$\lim_{k\to 0} \frac{e^{ikx}}{ia\cosh ak} = \frac{1}{ia}$$

and thus the integral is:

$$\int_{\text{rectangle}} = \left(1 + e^{-x\frac{\pi}{a}}\right)\int_{-\infty}^{+\infty} \frac{e^{-ikx}}{\cosh ax}dx = 2\pi i\left(\frac{1}{ia}\right)$$

and thus

$$f(x) = \frac{\sqrt{2\pi}}{a\left(1 + e^{-x\frac{\pi}{a}}\right)}$$

Verify:

$$F(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} \frac{\sqrt{2\pi}}{a\left(1 + e^{-x\frac{\pi}{a}}\right)}e^{-ikx}dx = \int_{-\infty}^{+\infty} \frac{1}{ae^{-x\pi/2a}2\cosh(x\pi/2a)}e^{-ikx}dx$$

We use a similar method.

$$\int_{R+2ia}^{-R+2ia} \frac{1}{ae^{-x\pi/2a}2\cosh(x\pi/2a)}e^{-ikx}dx = \int_{R}^{-R} \frac{e^{-ikx}e^{2ka}e^{x\pi/2a}e^{-i\pi}}{2a\cosh(x\pi/2a + i\pi)}dx$$
$$= -\frac{e^{-2ka}}{2a}\int_{-R}^{+R} \frac{e^{x\pi/2a}e^{-ikx}}{\cosh x\frac{\pi}{2a}}dx$$

Thus

$$F(k) = \frac{1}{2a\left(1 - e^{2ka}\right)}2\pi i\left(\frac{e^{i\pi/2}e^{ka}}{\frac{\pi}{2a}\sinh i\pi/2}\right)$$
$$= 2\frac{i}{e^{-ka} - e^{ka}} = \frac{1}{i\sinh ka}$$

as required.

3. If $F(k)$ is the Fourier trasnform of $f(x)$, show that idF/dk is the transform of $xf(x)$. What conditions must $F(k)$ satisfy for this result to hold?

Let $g(x)$ be the inverse transform of idF/dk. Then

$$g(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} i\frac{dF}{dk}e^{ikx}dk = \frac{i}{\sqrt{2\pi}}\left(e^{ikx}F(k)\Big|_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} ixFe^{ikx}dk\right)$$
$$= -i^2x\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} Fe^{ikx}dk = xf(x)$$

provided that $F(k) \to 0$ as $k \to \pm\infty$.

4. Verify Parseval's theorem in the form of equation 7.10 by evaluating the transforms of the functions $f(x) = \cos\beta x$ and $g(x) = e^{-\alpha x^2}$ and evaluating the two integrals in equation 7.10.

Since

$$f(x) = \cos\beta x = \frac{1}{2}\left(e^{i\beta x} + e^{-i\beta x}\right)$$

Then

$$F(k) = \frac{1}{2}(\delta(k-\beta) + \delta(k+\beta))$$

and from Example 7.2

$$G(k) = \frac{1}{\alpha\sqrt{2}} \exp\left(-\frac{k^2}{4\alpha^2}\right)$$

Then

$$\int_{-\infty}^{+\infty} F(k)G^*(k)dk = \int_{-\infty}^{+\infty} \frac{1}{2}(\delta(k-\beta) + \delta(k+\beta))\frac{1}{\sqrt{2}\alpha} \exp\left(-\frac{k^2}{4\alpha^2}\right)dk$$
$$= \frac{1}{2\sqrt{2}\alpha}\left[\exp\left(-\frac{\beta^2}{4\alpha^2}\right) + \exp\left(-\frac{\beta^2}{4\alpha^2}\right)\right] = \frac{1}{\sqrt{2}\alpha} \exp\left(-\frac{\beta^2}{4\alpha^2}\right)$$

We also need

$$\int_{-\infty}^{+\infty} f(x)g(x)dx = \int_{-\infty}^{+\infty} \cos\beta x e^{-\alpha^2 x^2} dx = \int_{-\infty}^{+\infty} \frac{1}{2}\left(e^{i\beta x} + e^{-i\beta x}\right)e^{-\alpha^2 x^2} dx$$

and then from the result of Example 7.2:

$$\int_{-\infty}^{+\infty} f(x)g(x)dx = \frac{1}{2}\frac{1}{\sqrt{2}\alpha}\left[\exp\left(-\frac{\beta^2}{4\alpha^2}\right) + \exp\left(-\frac{\beta^2}{4\alpha^2}\right)\right] = \frac{1}{\sqrt{2}\alpha} \exp\left(-\frac{\beta^2}{4\alpha^2}\right)$$

and the two integrals are equal, as required.

5. Verify Parseval's theorem in the form of equation 7.11 by evaluating the transform of

$$f(x) = \begin{cases} 1 \text{ if} & -1 \le x \le 1 \\ 0 \text{ otherwise} & \end{cases}$$

and evaluating the integrals of $f(x)^2$ and $|F(k)|^2$.

$$F(k) = \frac{1}{\sqrt{2\pi}} \int_{-1}^{+1} e^{-ikx} dx = \frac{1}{\sqrt{2\pi}} \frac{e^{-ikx}}{-ik}\bigg|_{-1}^{+1} = \frac{1}{\sqrt{2\pi}} \frac{e^{-ik} - e^{ik}}{-ik}$$
$$= \sqrt{\frac{2}{\pi}} \frac{\sin k}{k}$$

Then

$$\int f(x)^2 dx = \int_{-1}^{+1} 1 dx = x|_{-1}^{+1} = 2$$

and

$$\int F(k)^2 dk = \frac{2}{\pi} \int_{-\infty}^{+\infty} \frac{\sin^2 k}{k^2} dk = \frac{2}{\pi} \int_{-\infty}^{+\infty} \frac{1}{k^2}\left(\frac{e^{ik} - e^{-ik}}{2i}\right)^2 dk$$
$$= \frac{2}{\pi} \int_{-\infty}^{+\infty} \frac{1}{k^2}\left(-\frac{1}{4}\right)\left(e^{2ik} - 2 + e^{-2ik}\right)dk$$

We can most easily do the integrals by dividing into 2 pieces and closing the contour separately for each piece:

$$\int F(k)^2 dk = -\frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{e^{2ik} - 1}{k^2} dk - \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{e^{-2ik} - 1}{k^2} dk$$

Each integrand then has a first order pole at $k = 0.$ We displace the contour down by a small amount so as to pass beneath the pole. Then:

$$\int F(k)^2 dk = -\frac{1}{2\pi} 2\pi i(2i) = 2$$

and Parseval's theorem is verified for this function.

Chapter 7: Fourier Transforms

6. Show that if $F(k)$ is the transform of $f(x)$, then $\frac{1}{a}$ $F(k/a)$ is the transform of $f(ax)$. Show that the result is consistent with Parseval's theorem.

$$\begin{aligned} F_a(k) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(ax) e^{-ikx} dx \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(u) e^{-iku/a} du/a \\ &= \frac{1}{a} F\left(\frac{k}{a}\right) \end{aligned}$$

Parseval's theorem states:

$$\int_{-\infty}^{+\infty} |f(x)|^2 dx = \int_{-\infty}^{+\infty} |F(k)|^2 dk$$

Now for $f(ax)$, we have:

$$\int_{-\infty}^{+\infty} |f(ax)|^2 dx = \int_{-\infty}^{+\infty} |F_a(k)|^2 dk = \frac{1}{a^2} \int_{-\infty}^{+\infty} |F(k/a)|^2 dk$$

where $F_a(x)$ is the transform of $f(ax)$. On the left hand side, change variables to $u = ax$ and on the right change variables to $\kappa = k/a$

$$\int_{-\infty}^{+\infty} |f(ax)|^2 dx = \int_{-\infty}^{+\infty} |f(u)|^2 \frac{du}{a} = \frac{1}{a^2} \int_{-\infty}^{+\infty} |F(\kappa)|^2 a d\kappa$$

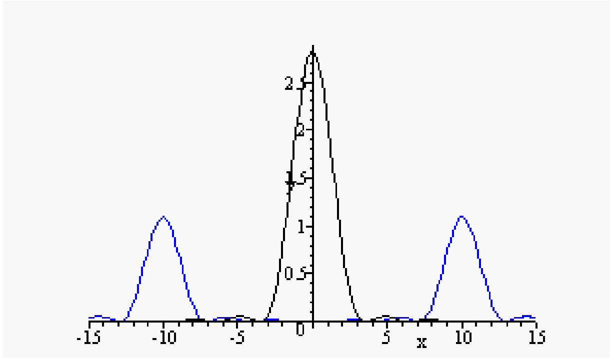
Thus

$$\int_{-\infty}^{+\infty} |f(a)|^2 du = \int_{-\infty}^{+\infty} |F(\kappa)|^2 d\kappa$$

as required.

7. Find the Fourier transform of the function $f(t) = \begin{cases} A\cos\omega_0 t & \text{if} \quad -T < t < T \\ 0 & \text{otherwise} \end{cases}$ that represents a finite train of data. Plot the Fourier power spectrum $|F(\omega)|^2$ as a function of ωT for the two cases $\omega_0 T = 1$ and $\omega_0 T = 10$, and comment. What happens as T increases toward infinity?

$$\begin{aligned} F(\omega) &= \frac{1}{\sqrt{2\pi}} \int_{-T}^{T} A\cos\omega_0 t e^{i\omega t} dt = \frac{1}{\sqrt{2\pi}} \int_{-T}^{T} A \frac{e^{i\omega_0 t} + e^{-i\omega_0 t}}{2} e^{i\omega t} dt \\ &= \frac{A}{2\sqrt{2\pi}} \left(\frac{e^{i(\omega_0+\omega)t}}{i(\omega_0 + \omega)} + \frac{e^{i(\omega-\omega_0)t}}{i(\omega - \omega_0)} \right) \Bigg|_{-T}^{T} \\ &= \frac{A}{\sqrt{2\pi}} \left(\frac{\sin(\omega_0 + \omega)T}{(\omega_0 + \omega)} + \frac{\sin(\omega - \omega_0)T}{(\omega - \omega_0)} \right) \end{aligned}$$



The Fourier power spectrum (see Figure--Black: $\omega_0 T = 10.$ Blue $\omega_0 T = 10$) is

$$|F(\omega)|^2 = \frac{A^2}{2\pi}\left(\frac{\sin^2(\omega_0+\omega)T}{(\omega_0+\omega)^2} + \frac{\sin^2(\omega-\omega_0)T}{(\omega-\omega_0)^2} + 2\frac{\sin(\omega_0+\omega)T}{(\omega_0+\omega)}\frac{\sin(\omega-\omega_0)T}{(\omega-\omega_0)}\right)$$

As T increases the spectrum becomes more concentrated around $\omega = \pm\omega_0,$ ultimately becoming a delta-function spike at each frequency. The finite length of the data train introduces additional frequency components.

8. Find the Fourier transform of:
$$f(t) = \begin{cases} 1 - \frac{|t|}{T} & \text{if} \quad -T < t < T \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{aligned}
F(\omega) &= \frac{1}{\sqrt{2\pi}}\int_{-T}^{T}\left(1-\frac{|t|}{T}\right)e^{i\omega t}dt \\
&= \sqrt{\frac{1}{1\pi}}\left[\int_0^T\left(1-\frac{t}{T}\right)e^{i\omega t}dt + \int_{-T}^{0}\left(1+\frac{t}{T}\right)e^{i\omega t}dt\right] \\
&= \sqrt{\frac{1}{2\pi}}\left(\frac{e^{i\omega t}}{i\omega}\Bigg|_{-T}^{T} + \frac{1}{T}\left(-t\frac{e^{i\omega t}}{i\omega}\Bigg|_0^T + \int_0^T\frac{e^{i\omega t}}{i\omega}dt + t\frac{e^{i\omega t}}{i\omega}\Bigg|_{-T}^{0} - \int_{-T}^{0}\frac{e^{i\omega t}}{i\omega}dt\right)\right) \\
&= \sqrt{\frac{1}{1\pi}}\left(\frac{e^{i\omega T}-e^{-i\omega T}}{i\omega} + \frac{1}{T}\left(\frac{-Te^{i\omega T}}{i\omega} + \frac{e^{i\omega t}}{(i\omega)^2}\Bigg|_0^T - \frac{(-T)e^{-i\omega T}}{i\omega} - \frac{e^{i\omega t}}{(i\omega)^2}\Bigg|_{-T}^{0}\right)\right) \\
&= \sqrt{\frac{1}{2\pi}}\left(\frac{2\sin\omega T}{\omega} + \frac{1}{T}\left(-T\frac{e^{i\omega T}-e^{-i\omega T}}{i\omega} + \frac{e^{i\omega T}-1}{(i\omega)^2} - \frac{1-e^{-i\omega T}}{(i\omega)^2}\right)\right) \\
&= \sqrt{\frac{2}{\pi}}\frac{1-\cos\omega T}{T\omega^2} = \sqrt{\frac{2}{\pi}}\,2\frac{\sin^2\omega T/2}{T\omega^2} = T\sqrt{\frac{1}{2\pi}}\frac{\sin^2\omega T/2}{(T\omega/2)^2}
\end{aligned}$$

Hence find the transform of the function
$$g(t) = \begin{cases} 1 & \text{if} \quad -T < t < 0 \\ -1 & \text{if} \quad 0 < t < T \\ 0 & \text{otherwise} \end{cases}$$

Notice that $g(t) = df/dt,$ and thus

$$\begin{aligned}
G(\omega) &= -i\omega F(\omega) = -i\omega T\sqrt{\frac{1}{2\pi}}\frac{\sin^2\omega T/2}{(T\omega/2)^2} \\
&= -i\sqrt{\frac{2}{\pi}}\frac{\sin^2\omega T/2}{(T\omega/2)}
\end{aligned}$$

9. Show that the square deviation between two functions,

$$D = \int_{-\infty}^{+\infty} |f(x) - g(x)|^2 dx$$

equals the square deviation between the transforms:

$$D = \int_{-\infty}^{+\infty} |F(k) - G(k)|^2 dk$$

$$\begin{aligned} D &= \int_{-\infty}^{+\infty} |f(x) - g(x)|^2 dx \\ &= \int_{-\infty}^{+\infty} \left\{ [f(x)]^2 - 2f(x)g(x) + [g(x)]^2 \right\} dx \\ &= \int_{-\infty}^{+\infty} |F(k)|^2 - 2F(k)G(k)^* + |G(k)|^2 dk \end{aligned}$$

by Parseval's theorem. Thus

$$D = \int_{-\infty}^{+\infty} |F(k) - G(k)|^2 dk$$

10. A spring- and -dashpot system satisfies the equation

$$\frac{d^2x}{dt^2} + 2\alpha\frac{dx}{dt} + \omega_0^2 x = f(t)$$

with $\omega_0 > \alpha.$ The driving force per unit mass $f(t)$ equals $e^{-\alpha t}\sin\Omega t.$. Find $x(t)$ for $t > 0$ and verify that your method gives $x = 0$ for $t < 0.$

Transforming the equation, we get:

$$-\omega^2 X - 2i\omega\alpha X + \omega_0^2 X = F(\omega)$$

where

$$\begin{aligned} F(\omega) &= \frac{1}{\sqrt{2\pi}} \int_0^{\infty} e^{-\alpha t} \sin\Omega t e^{i\omega t} dt \\ &= \frac{1}{2i\sqrt{2\pi}} \int_0^{\infty} \left(e^{i\Omega t} - e^{-i\Omega t}\right) e^{-\alpha t} e^{i\omega t} dt \\ &= \frac{1}{2i\sqrt{2\pi}} \left(\frac{-1}{i(\Omega + \omega) - \alpha} - \frac{-1}{i(\omega - \Omega) - \alpha} \right) \end{aligned}$$

Using this result in the transformed equation, we solve for X:

$$X = \frac{1}{2i\sqrt{2\pi}} \left(\frac{-1}{i(\Omega + \omega) - \alpha} + \frac{1}{i(\omega - \Omega) - \alpha} \right) \frac{-1}{\left(\omega^2 + 2i\omega\alpha - \omega_0^2\right)}$$

Thus inverting, we get:

$$\begin{aligned} x(t) &= \frac{1}{4\pi i} \int_{-\infty}^{+\infty} \left(\frac{1}{i(\Omega + \omega) - \alpha} - \frac{1}{i(\omega - \Omega) - \alpha} \right) \frac{1}{\left(\omega^2 + 2i\omega\alpha - \omega_0^2\right)} e^{-i\omega t} d\omega \\ &= \frac{1}{4\pi} \int_{-\infty}^{+\infty} \left(\frac{1}{(\omega - \Omega) + i\alpha} - \frac{1}{(\Omega + \omega) + i\alpha} \right) \frac{1}{\left(\omega^2 + 2i\omega\alpha - \omega_0^2\right)} e^{-i\omega t} d\omega \end{aligned}$$

It will be simpler to do the two terms separately. The integrand has first order poles at $\omega = -i\alpha \pm \sqrt{\omega_0^2 - \alpha^2}$, $\omega = \pm\Omega - i\alpha.$ All the poles are in the lower half-plane, so $x(t) = 0$ for $t < 0.$

For $t > 0$ we must close the contour downward. To simplify the solution, set $\gamma = \sqrt{\omega_0^2 - \alpha^2}.$

First term: The contour contains all three poles of the integrand, at $\omega = \Omega - i\alpha, \; \pm\gamma - i\alpha$. The residues are:

$$\frac{1}{\left((\Omega - i\alpha)^2 + 2i(\Omega - i\alpha)\alpha - \omega_0^2\right)} \exp(-it(\Omega - i\alpha)) = \frac{1}{\Omega^2 + \alpha^2 - \omega_0^2} \exp(-i\Omega - \alpha)t,$$

$$\frac{1}{(\gamma - \Omega)(2\gamma)} \exp(-it(\gamma - i\alpha)) = \frac{1}{(\gamma - \Omega)(2\gamma)} \exp(-i\gamma - \alpha)t$$

and

$$\frac{1}{(-\Omega - \gamma)(-2\gamma)} \exp(i\gamma - \alpha)t = \frac{1}{(\Omega + \gamma)(2\gamma)} \exp(i\gamma - \alpha)t$$

and using the residue theorem, we have:

$$x_1(t) = \frac{(-2\pi i)}{4\pi} \left\{ \frac{\exp(-i\Omega - \alpha)t}{\Omega^2 + \alpha^2 - \omega_0^2} + \frac{\exp(-i\gamma - \alpha)t}{(\gamma - \Omega)(2\gamma)} + \frac{\exp(i\gamma - \alpha)t}{(\Omega + \gamma)(2\gamma)} \right\}$$

Combining the last two terms, we get:

$$\begin{aligned} &\frac{e^{-\alpha t}}{2\gamma} \frac{(\Omega + \gamma)e^{-i\gamma t} + (\gamma - \Omega)e^{i\gamma t}}{\gamma^2 - \Omega^2} \\ &= \frac{e^{-\alpha t}}{2\gamma} \frac{\Omega\left(e^{-i\gamma t} - e^{i\gamma t}\right) + \gamma\left(e^{i\gamma t} + e^{-i\gamma t}\right)}{\gamma^2 - \Omega^2} \\ &= \frac{e^{-\alpha t}}{\gamma} \frac{-\Omega i \sin\gamma t + \gamma\cos\gamma t}{\omega_0^2 - \alpha^2 - \Omega^2} \end{aligned}$$

So:

$$x_1(t) = -\frac{i}{2} e^{-\alpha t} \left\{ \frac{\cos\gamma t - \frac{\Omega}{\gamma} i \sin\gamma t - e^{-i\Omega t}}{\omega_0^2 - \alpha^2 - \Omega^2} \right\}$$

For the second term, we get the same result with $\Omega \to -\Omega$:

$$x_2(t) = -\frac{i}{2} e^{-\alpha t} \left\{ \frac{\cos\gamma t + \frac{\Omega}{\gamma} i \sin\gamma t - e^{i\Omega t}}{\omega_0^2 - \alpha^2 - \Omega^2} \right\}$$

Now we combine both terms, to get:

$$\begin{aligned} x(t) &= x_1(t) - x_2(t) \\ &= -\frac{i}{2} e^{-\alpha t} \left\{ \frac{\cos\gamma t - \frac{\Omega}{\gamma} i \sin\gamma t - e^{-i\Omega t}}{\omega_0^2 - \alpha^2 - \Omega^2} - \frac{\frac{\Omega}{\gamma} i \sin\gamma t + \cos\gamma t - e^{i\Omega t}}{\omega_0^2 - \alpha^2 - \Omega^2} \right\} \\ &= \frac{1}{2} e^{-\alpha t} \left\{ \frac{-2\frac{\Omega}{\gamma} \sin\gamma t - i(e^{i\Omega t} - e^{-i\Omega t})}{\omega_0^2 - \alpha^2 - \Omega^2} \right\} \\ &= e^{-\alpha t} \frac{\sin\Omega t - \frac{\Omega}{\gamma} \sin\gamma t}{\omega_0^2 - \alpha^2 - \Omega^2} \end{aligned}$$

The result is zero at $t = 0$ as expected.

Let's check the result at $\Omega = \gamma$. The limit as $\Omega \to \gamma$ may be computed using l'Hospital's rule:

$$x(t) = \lim_{\Omega\to\gamma} e^{-\alpha t}\frac{t\cos\Omega t - \frac{1}{\gamma}\sin\gamma t}{-2\Omega} = e^{-\alpha t}\frac{\gamma t\cos\gamma t - \sin\gamma t}{-2\gamma^2}$$

which is finite, as expected for a damped oscillator.

11. An electron in an atom may be modelled classically as a damped harmonic oscillator (cf problem 10 above.) The electron is driven by an electric field $E(t) = E_0\frac{\sin\Omega t}{\Omega t}$. What is the appropriate $f(t)$ for this problem? Solve for the transform $x(\omega)$ of the electron's position.

$$\frac{d^2x}{dt^2} + 2\alpha\frac{dx}{dt} + \omega_0^2 x = f(t)$$

with

$$f(t) = -\frac{eE(t)}{m}$$

Transforming the RHS, we get:

$$\begin{aligned} E(\omega) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} E_0\frac{\sin\Omega t}{\Omega t}e^{i\omega t}dt \\ &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty} E_0\frac{e^{i\Omega t} - e^{-i\Omega t}}{2i\Omega t}e^{i\omega t}dt \\ &= \frac{E_0}{2i\sqrt{2\pi}}\int_{-\infty}^{+\infty}\frac{e^{i(\Omega+\omega)t} - e^{i(\omega-\Omega)t}}{\Omega t}dt \end{aligned}$$

We can do this integral by moving the contour slightly downward. Then for the first term: we close upward for $\omega > -\Omega$. The pole at the origin is enclosed. Thus::

$$\int_{-\infty}^{+\infty}\frac{e^{i(\Omega+\omega)t}}{t}dt = 2\pi i\,\mathrm{Res}(0) = 2\pi i$$

For $\omega < -\Omega$ we close downward. No poles are enclosed and the result is zero. For the second term, we close upward if $\omega > \Omega$, downward if $\omega < \Omega$. Then:

$$\int_{-\infty}^{+\infty}\frac{e^{-i(\Omega+\omega)t}}{t}dt = \begin{cases} 0 & if\ \ \omega < \Omega \\ 2\pi i & if\ \ \omega > \Omega \end{cases}$$

$$E(\omega) = \frac{E_0}{2i\Omega\sqrt{2\pi}}\begin{cases} 0 & if\ \ \omega > \Omega \\ 2\pi i & if\ \ -\Omega < \omega < \Omega \\ 0 & if\ \ \omega < -\Omega \end{cases}$$

And then the transformed equation is:

$$-\omega^2 X - 2i\alpha\omega X + \omega_0^2 X = \frac{E_0}{\Omega}\sqrt{\frac{\pi}{2}}\begin{cases} 0 & if\ \ \omega > \Omega \\ 1 & if\ \ -\Omega < \omega < \Omega \\ 0 & if\ \ \omega < -\Omega \end{cases}$$

and so

$$X(\omega) = \frac{E_0}{\Omega}\sqrt{\frac{\pi}{2}}\frac{1}{\omega_0^2 - \omega^2 - 2i\alpha\omega}\begin{cases} 0 & if\ \ \omega > \Omega \\ 1 & if\ \ -\Omega < \omega < \Omega \\ 0 & if\ \ \omega < -\Omega \end{cases}$$

Use the results of section 7.6 to determine the power spectrum of the radiated energy. Plot your results in the case $\alpha = \omega_0/10$, $\Omega = 2\omega_0$ and comment.

The power spectrum is:

$$\begin{aligned} \frac{dW}{d\omega} &= \frac{e^2}{c^3(4\pi)^2\varepsilon_0}\omega^4|X(\omega)|^2 \\ &= \frac{e^2}{c^3(4\pi)^2\varepsilon_0}\frac{\omega^4}{\Omega^2}E_0^2\frac{\pi}{2}\frac{1}{\left(\omega_0^2 - \omega^2\right)^2 + 4\alpha^2\omega^2}\begin{cases} 0 & if\ \ \omega > \Omega \\ 1 & if\ \ -\Omega < \omega < \Omega \\ 0 & if\ \ \omega < -\Omega \end{cases} \\ &= \frac{e^2 E_0^2}{32\pi c^3\Omega^2\varepsilon_0}\frac{\omega^4}{\left(\omega_0^2 - \omega^2\right)^2 + 4\alpha^2\omega^2}\begin{cases} 0 & if\ \ \omega > \Omega \\ 1 & if\ \ -\Omega < \omega < \Omega \\ 0 & if\ \ \omega < -\Omega \end{cases} \end{aligned}$$

Now put in the values $\alpha = \omega_0/10$, $\Omega = 2\omega_0$.

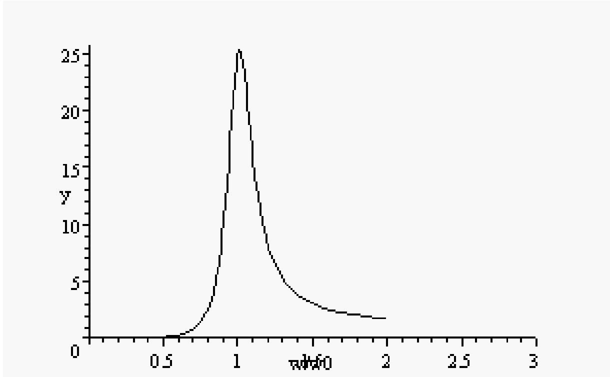
$$\frac{dW}{d\omega} = \frac{e^2 E_0^2}{32\pi c^3\Omega^2\varepsilon_0}\frac{\omega^4}{\left(\omega_0^2 - \omega^2\right)^2 + 0.04\omega_0^2\omega^2}\left\{\begin{cases} 0 & if\ \ \omega > 2\omega_0 \\ 1 & if\ \ -2\omega_0 < \omega < 2\omega_0 \\ 0 & if\ \ \omega < -2\omega_0 \end{cases}\right.$$

The spectrum looks like:

$$P(\omega) = \begin{cases} 0 & if\ \ \omega > 2 \\ \frac{\omega^4}{\left(1-\omega^2\right)^2 + 0.04\omega^2} & if\ \ -2 < \omega < 2 \\ 0 & if\ \ \omega < -2 \end{cases}$$

Since the negative frequency has the same physical meaning as the positive frequency, it is usual to look only at the positive values. Then:

$$\frac{dW}{d\omega} = \frac{e^2 E_0^2}{16\pi c^3\Omega^2\varepsilon_0}\frac{\omega^4}{\left(\omega_0^2 - \omega^2\right)^2 + 0.04\omega_0^2\omega^2}\left\{\begin{cases} 1 & if\ \ 0 < \omega < 2\omega_0 \\ 0 & if\ \ \omega > 2\omega_0 \end{cases}\right.$$



Notice the peak at the resonant frequency $\omega = \omega_0$.

12. The electric displacement $\vec{D}$ is related to the electric field $\vec{E}$ by the dielectric constant ϵ . In general, ϵ is a function of frequency, so that the relationship is one between the Fourier transforms of $\vec{D}$ and $\vec{E}$:

$$\vec{D}(x,\omega)=\epsilon(\omega)\vec{E}(x,\omega)$$

a) Show that the relationship between $\vec{D}(x,t)$ and $\vec{E}(x,t)$ is:

$$\vec{D}(x,t) = \vec{E}(x,t) + \int_{-\infty}^{\infty} G(\tau)\vec{E}(x,t-\tau)d\tau$$

and determine an expression for $G(\tau)$ in terms of $\epsilon(\omega)$.

First define the function $G(\omega)$ in terms of the dielectric constant $\epsilon(\omega)$: $\epsilon(\omega) \equiv 1 + G(\omega)$. Then:

$$\vec{D}(x,\omega)=(1 + G(\omega))\vec{E}(x,\omega)$$

We compute the inverse using the convolution theorem:

$$\vec{D}(x,t) = \vec{E}(x,t) + \int_{-\infty}^{\infty} G(\tau)\vec{E}(x,t-\tau)d\tau$$

as required.

b) Find $G(t)$ for the one-resonance model

$$\epsilon(\omega) = 1 + \frac{\omega_p^2}{\omega^2 - \omega_0^2 - i\gamma\omega}$$

where ω_p , ω_0, and γ are real, positive constants, and $\gamma < \omega_0$.

$$G(t) = \int_{-\infty}^{+\infty} \frac{\omega_p^2}{\omega_0^2 - \omega^2 - i\gamma\omega} e^{-i\omega t} d\omega$$

The integrand has two simple poles, at

$$\omega = \frac{-i\gamma \pm \sqrt{-\gamma^2 + 4\omega_0^2}}{2} = -i\frac{\gamma}{2} \pm \sqrt{\omega_0^2 - \left(\frac{\gamma}{2}\right)^2}$$

Both are in the lower half plane. We close the contour downward for $t > 0$, enclosing both poles. Then

$$G(t) = -2\pi i\omega_p^2\left(\frac{\exp\left(-it\left(-i\frac{\gamma}{2}+\sqrt{\omega_0^2-\left(\frac{\gamma}{2}\right)^2}\right)\right)}{-\left(2\sqrt{\omega_0^2-\left(\frac{\gamma}{2}\right)^2}\right)}+\frac{\exp\left(-it\left(-i\frac{\gamma}{2}-\sqrt{\omega_0^2-\left(\frac{\gamma}{2}\right)^2}\right)\right)}{-\left(-2\sqrt{\omega_0^2-\left(\frac{\gamma}{2}\right)^2}\right)}\right)$$

$$= 2\pi i\omega_p^2\frac{e^{-t\gamma/2}}{2\sqrt{\omega_0^2-\left(\frac{\gamma}{2}\right)^2}}\left(\exp\left(-it\sqrt{\omega_0^2-\left(\frac{\gamma}{2}\right)^2}\right)-\exp\left(it\sqrt{\omega_0^2-\left(\frac{\gamma}{2}\right)^2}\right)\right)$$

$$= 2\pi\omega_p^2\frac{e^{-t\gamma/2}}{\sqrt{\omega_0^2-\left(\frac{\gamma}{2}\right)^2}}\sin\sqrt{\omega_0^2-\left(\frac{\gamma}{2}\right)^2}\,t$$

c) Discuss the physical meaning of your result. Be specific!

The integral expression for $\vec{D}$ shows that $\vec{D}$ depends on the electric field in the past, but the form of G shows that we need only look a short time into the past. ($\Delta t \sim 2/\gamma$).

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13. An electron in an atom may be represented by a damped harmonic oscillator with frequency ω_0 and damping rate Γ. An external electric field $\vec{\mathbf{E}}(t)$ acts on the electron. Find the Fourier transform $\vec{\mathbf{x}}(\omega)$ of the electron position as a function of time. If the electron loses energy at a rate $P = \vec{\mathbf{j}} \cdot \vec{\mathbf{E}} = -e\vec{\mathbf{v}} \cdot \vec{\mathbf{E}}$, use Parseval's Theorem to show that the total energy loss is:

$$\triangle U = \frac{e^2}{m}\int_{-\infty}^{\infty} \frac{\omega^2\Gamma\left|\vec{\mathbf{E}}(\omega)\right|^2}{\left(\omega_0^2 - \omega^2\right)^2 + \omega^2\Gamma^2} d\omega$$

Note that the integrand is sharply peaked at $\omega \simeq \omega_0$, while $\vec{\mathbf{E}}(\omega)$ is a slowly varying function, and thus the integral may be approximated as:

$$\triangle U = \frac{e^2}{m}\left|\vec{\mathbf{E}}(\omega_0)\right|^2 \int_{-\infty}^{\infty} \frac{\omega^2\Gamma}{\left(\omega_0^2 - \omega^2\right)^2 + \omega^2\Gamma^2} d\omega$$

Evaluate the integral by contour integration to show that $\triangle U$ is independent of Γ and hence find $\triangle U$. (In this expression, ω_0 and $\Gamma$are real positive constants, and $\omega_0 > \Gamma$.)

The equation satsified by the oscillator is:

$$m\frac{d^2\vec{\mathbf{x}}}{dt^2} + \Gamma m\frac{d\vec{\mathbf{x}}}{dt} + m\omega_0^2\vec{\mathbf{x}} = -e\vec{\mathbf{E}}(t)$$

Transforming, we get:

$$-\omega^2\vec{\mathbf{x}} - i\omega\Gamma\vec{\mathbf{x}} + \omega_0^2\vec{\mathbf{x}} = -\frac{e}{m}\vec{\mathbf{E}}(\omega)$$

and thus

$$\vec{\mathbf{x}}(\omega) = \frac{e}{m}\frac{\vec{\mathbf{E}}(\omega)}{\omega^2 + i\omega\Gamma - \omega_0^2}$$

The total energy lost is the time integral of the power:

$$W = \int_{-\infty}^{+\infty} P(t)dt = \int_{-\infty}^{+\infty} -e\vec{\mathbf{v}}(t)\cdot\vec{\mathbf{E}}(t)dt$$

Now we use Parseval's theorem to write this in terms of the transforms:

$$W = -e\int_{-\infty}^{+\infty} \vec{\mathbf{v}}(\omega)\cdot\vec{\mathbf{E}}(-\omega)d\omega$$

and since $\vec{\mathbf{v}}(t) = d\vec{\mathbf{x}}/dt$, then:

$$\begin{aligned} W &= e\int_{-\infty}^{+\infty} i\omega\vec{\mathbf{x}}(\omega)\cdot\vec{\mathbf{E}}(-\omega)d\omega \\ &= e\int_{-\infty}^{+\infty} i\omega\frac{e}{m}\frac{\vec{\mathbf{E}}(\omega)}{\omega^2 + i\omega\Gamma - \omega_0^2}\cdot\vec{\mathbf{E}}(-\omega)d\omega \\ &= i\frac{e^2}{m}\int_{-\infty}^{+\infty} \omega\frac{\left|\vec{\mathbf{E}}(\omega)\right|^2\left(\omega^2 - i\omega\Gamma - \omega_0^2\right)}{\left(\omega^2 - \omega_0^2\right)^2 + (\omega\Gamma)^2}d\omega \\ &= \frac{e^2}{m}\int_{-\infty}^{+\infty} \frac{\left|\vec{\mathbf{E}}(\omega)\right|^2\left(\omega^2\Gamma + i\omega\left(\omega^2 - \omega_0^2\right)\right)}{\left(\omega^2 - \omega_0^2\right)^2 + (\omega\Gamma)^2}d\omega \end{aligned}$$

Since the energy loss is purely real, we may take the real part of this expression. (Note also that the imaginary part is an odd function integrated over an even interval, and thus integrates to zero.) Thus:

$$\Delta U = \frac{e^2}{m}\int_{-\infty}^{+\infty} \frac{\left|\vec{\mathbf{E}}(\omega)\right|^2\omega^2\Gamma}{\left(\omega^2 - \omega_0^2\right)^2 + (\omega\Gamma)^2}d\omega$$

as required.

Now the integrand is sharply peaked at $\omega \approx \omega_0$,so we may approximate as:

$$\Delta U = \frac{e^2}{m}\left|\vec{\mathbf{E}}(\omega_0)\right|^2\int_{-\infty}^{+\infty} \frac{\omega^2\Gamma}{\left(\omega^2 - \omega_0^2\right)^2 + (\omega\Gamma)^2}d\omega$$

The integrand has 4 poles, given by:

$$\omega^2 - \omega_0^2 = \pm i\omega\Gamma$$

and thus

$$\omega = \frac{\pm i\Gamma \pm \sqrt{-\Gamma^2 + 4\omega_0^2}}{2}$$

Two are in the upper-half plane and two are in the lower half plane. We may close the contour either way. Let's close it upward. Then the integral is:

$$\begin{aligned}\int_{-\infty}^{+\infty} \frac{\omega^2\Gamma}{\left(\omega^2 - \omega_0^2\right)^2 + (\omega\Gamma)^2} d\omega &= \Gamma 2\pi i \left(\lim_{\omega\to\omega_1} + \lim_{\omega\to\omega_2}\right) \frac{\omega^2}{2(2\omega)\left(\omega^2 - \omega_0^2\right) + 2\omega\Gamma^2} \\ &= \pi i\Gamma \left(\lim_{\omega\to\omega_1} + \lim_{\omega\to\omega_2}\right) \frac{\omega}{2(i\omega\Gamma) + \Gamma^2} \\ &= \pi i \left(\lim_{\omega\to\omega_1} + \lim_{\omega\to\omega_2}\right) \frac{\omega}{\Gamma + 2i\omega} \\ &= \frac{\pi i}{2}\left(\frac{i\Gamma + \sqrt{4\omega_0^2 - \Gamma^2}}{i\sqrt{4\omega_0^2 - \Gamma^2}} - \frac{i\Gamma - \sqrt{4\omega_0^2 - \Gamma^2}}{i\sqrt{4\omega_0^2 - \Gamma^2}}\right) \\ &= \pi\end{aligned}$$

Thus

$$\Delta U = \pi \frac{e^2}{m} \left|\vec{\mathbf{E}}(\omega_0)\right|^2$$

and is independent of Γ.

.

14. The Radon problem. Radon diffuses from the ground into the atmosphere at a rate

r. Model the atmosphere as a semi-infinite medium with boundary (the ground) at $y = 0$. Then the density

$\rho(y,t)$ of atmospheric radon is described by the equation:

$$\frac{\partial\rho}{\partial t} = D\frac{\partial^2\rho}{\partial y^2} - \lambda\rho$$

where D is the appropriate diffusion coeffcient and

λ is the decay rate for radon. The boundary condition at the ground is

$$\left.\frac{\partial\rho}{\partial y}\right|_{y=0} = \text{const} = -\alpha$$

What is the boundary condition at $y \to \infty$? Use the Fourier cosine transform in

y to derive an integral expression for $\rho(y,t)$in the case that $\rho(x,0) = 0$. Evaluate $\frac{\partial\rho}{\partial t}$ at $t = 0$ and hence determine

α in terms of r and D.

The boundary condition at ∞ is $\rho(y,t) \to 0$ as $y \to \infty$.

Applying the cosine transform:

$$R(k,t) = \sqrt{\frac{2}{\pi}} \int_0^\infty \rho(y,t)\cos ky dy$$

we have

$$\begin{aligned}\frac{\partial R}{\partial t} &= D\left(\sqrt{\frac{2}{\pi}}\alpha - k^2 R\right) - \lambda R \\ &= -\left(k^2 D + \lambda\right)R + \sqrt{\frac{2}{\pi}}\alpha D\end{aligned}$$

which we may integrate to obtain:

$$R - \sqrt{\frac{2}{\pi}}\frac{\alpha D}{\left(k^2 D + \lambda\right)} = R_0 \exp\left\{-\left(k^2 D + \lambda\right)t\right\}$$

But $R(k,0) = 0$, so

$$-\sqrt{\frac{2}{\pi}}\frac{\alpha D}{\left(k^2 D + \lambda\right)} = R_0$$

and thus

$$\rho(y,t) = \sqrt{\frac{2}{\pi}}\int_0^\infty \sqrt{\frac{2}{\pi}}\frac{\alpha D}{\left(k^2D+\lambda\right)}\left(1-\exp\left\{-\left(k^2D+\lambda\right)t\right\}\right)\cos ky\,dk$$
$$= \frac{2\alpha D}{\pi}\int_0^\infty \frac{1-\exp\left\{-\left(k^2D+\lambda\right)t\right\}}{\left(k^2D+\lambda\right)}\cos ky\,dk \qquad \text{Radon eqn 1}$$

Now differentiating, we get

$$\frac{\partial\rho}{\partial t} = \frac{2\alpha D}{\pi}\int_0^\infty \exp\left\{-\left(k^2D+\lambda\right)t\right\}\cos ky\,dk$$
$$\left.\frac{\partial\rho}{\partial t}\right|_{t=0} = \frac{\alpha D}{\pi}\int_0^\infty\left(e^{iky}+e^{-iky}\right)dk = \frac{\alpha D}{\pi}\int_{-\infty}^\infty e^{iky}dk$$
$$= \alpha D\delta(y) = r\delta(y)$$

Thus $\alpha = r/D$

$$\left.\frac{\partial\rho}{\partial t}\right|_{y=0} = \frac{2r}{\pi}\int_0^\infty \exp\left\{-\left(k^2D+\lambda\right)t\right\}dk$$
$$= \frac{2r}{\pi}e^{-\lambda t}\frac{1}{\sqrt{Dt}}\frac{\sqrt{\pi}}{2} = r\frac{e^{-\lambda t}}{\sqrt{\pi Dt}}$$

$$\rho(0,t) = 2\frac{r}{\pi}\int_0^\infty dk\int_0^t d\tau\exp\left\{-\left(k^2D+\lambda\right)\tau\right\}$$
$$= \frac{r}{\pi}\int_0^t d\tau e^{-\lambda\tau}\sqrt{\frac{\pi}{D\tau}} = \frac{r}{\sqrt{\pi D}}\int_0^t\frac{d\tau e^{-\lambda\tau}}{\sqrt{\tau}}$$

Let $\lambda\tau = u^2$. Then $d\tau = 2udu/\lambda$ and

$$\rho(0,t) = \frac{r}{\sqrt{\pi\lambda D}}\int_0^{\sqrt{\lambda t}}\frac{2udue^{-u^2}}{u} = \frac{2r}{\sqrt{\pi\lambda D}}\int_0^{\sqrt{\lambda t}}due^{-u^2} = \frac{r}{\sqrt{\lambda D}}\Phi\left(\sqrt{\lambda t}\right)$$

where Φ is the error function.

Also note

$$\frac{\partial\rho}{\partial t} = \frac{2r}{\pi}\int_0^\infty\exp\left\{-\left(k^2D+\lambda\right)t\right\}\cos ky\,dk$$
$$= \frac{r}{\pi}\int_0^\infty\exp\left\{-\left(k^2D+\lambda\right)t\right\}\left(e^{iky}+e^{-iky}\right)dk$$
$$= r\frac{e^{-\lambda t}}{\sqrt{\pi Dt}}\exp\left(-\frac{y^2}{4Dt}\right)$$

Then:

$$\rho\left(y,t'\right) = r\int_0^{t'}\frac{e^{-\lambda t}}{\sqrt{\pi Dt}}\exp\left(-\frac{y^2}{4Dt}\right)dt$$

After a long time: (G&R 3.471#9 with $\nu = 1/2,$ $x = t,$ $\gamma = \lambda,$ $\beta = y^2/4D,$

$$\rho(y,\infty) = \frac{r}{\sqrt{\pi D}}2\left(\frac{y^2}{4\lambda D}\right)^{1/4}K_{1/2}\left(\sqrt{\frac{y^2\lambda}{D}}\right)$$
$$= \frac{r}{\sqrt{\pi}}\frac{\sqrt{2y}}{\left(\lambda D^3\right)^{1/4}}K_{1/2}\left(y\sqrt{\frac{\lambda}{D}}\right)$$

and for large y this takes the form

$$\rho(y,\infty) = \frac{r}{\sqrt{\pi}}\frac{\sqrt{2y}}{\left(\lambda D^3\right)^{1/4}}\sqrt{\frac{\pi}{2y}}\left(\frac{D}{\lambda}\right)^{1/4}\exp\left(-y\sqrt{\frac{\lambda}{D}}\right)$$
$$= \frac{r}{\sqrt{\lambda D}}\exp\left(-y\sqrt{\frac{\lambda}{D}}\right)$$

We can also use equation (Radon eqn 1) to get the long time solution:

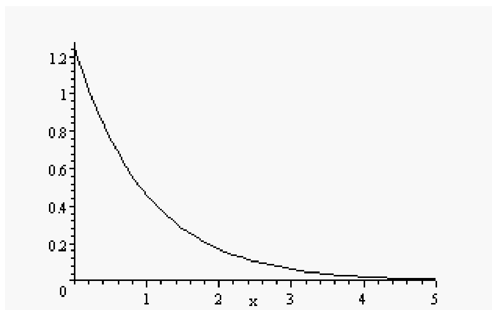
$$\rho(y,t \to \infty) = \frac{2\alpha D}{\pi}\int_0^\infty \frac{1}{\left(k^2 D + \lambda\right)}\cos ky\,dk$$
$$= \frac{\alpha D}{\pi}\int_0^\infty \frac{e^{iky} + e^{-iky}}{\left(k^2 D + \lambda\right)}dk$$
$$= \frac{\alpha}{\pi}\int_{-\infty}^\infty \frac{e^{iky}}{\left(k^2 + \lambda/D\right)}dk$$

There are poles at $k = \pm i\sqrt{\lambda/D}$. We close upward for $y > 0$, enclosing the pole at $+i\sqrt{\lambda/D}$. The integral is then:

$$\rho(y,t \to \infty) = \frac{\alpha}{\pi}2\pi i\frac{\exp\left(-y\sqrt{\lambda/D}\right)}{2i\sqrt{\lambda/D}} = \frac{r}{D}\sqrt{\frac{D}{\lambda}}\exp\left(-y\sqrt{\frac{\lambda}{D}}\right)$$
$$= \frac{r}{\sqrt{D\lambda}}\exp\left(-y\sqrt{\frac{\lambda}{D}}\right)$$

as before.

Long-time distribution.



15. A long copper rod of cross sectional area $A = 1$ cm^2 is initially at 15°C. At time t, one end (at $x = 0$) is placed into a vat of hot oil at 300C.

(i) Refer to Chapter 3 §2.5. Write the equation that describes the change of temperature at position x along the rod at time t.

(ii) Write an expression for the temperature $T(x)$ of the rod immediately after the end is placed in the oil.

(iii) Discuss the use of Fourier and/or Laplace transforms in solving this equation. What determines the best choice of transform for this problem?

(iv) Find the temperature of the rod as a function of position and time for $t > 0$.

(v) Given the following data for copper, plot the temperature along the first 5 m of the rod at times $t = 0.5$ s, 1.5 s, 3.0 s, 6.0 s. Thermal conductivity: 400 W/mK Specific heat: 385 J/kgK Density 8.96 kg/m^3.

(i)

$$mc\frac{\partial T}{\partial t} = -kA\frac{\partial^2 T}{\partial x^2}$$

where $m = \rho A$.

(ii) $T(0,0) = 300°C$. $T(x,0) = 15°C,\ x > 0$. So we may solve for $T - 15°C = \tau$ with $\tau(0,t) = 285$ and $\tau(x,0) = 0$ for $x > 0$.

(iii) The Laplace transform is best suited to the time variable since this is an initial value problem. We could also use the Laplace transform in space, but we do not have enough conditions at $x = 0$. The sine transform may work well since we know $T(0,t)$. Let's try it.

(iv) Taking the sine transform, we'll call the transform variable α to avoid confusion with the thermal conductivity.

$$mc\frac{\partial}{\partial t}F_s(\alpha) = -kA\left(\alpha\sqrt{\frac{2}{\pi}}T_0 - \alpha^2 F_s\right)$$

$$\frac{\partial}{\partial t}F_s(\alpha) = \frac{kA}{mc}\alpha^2 F_s - \frac{kA}{mc}\alpha\sqrt{\frac{2}{\pi}}T_0$$

$$= \frac{kA}{mc}\alpha^2\left(F_s - \sqrt{\frac{2}{\pi}}\frac{T_0}{\alpha}\right)$$

which may be integrated to give:

$$F_s = \sqrt{\frac{2}{\pi}}\frac{T_0}{\alpha} + C\exp\left(\frac{kA}{mc}\alpha^2 t\right)$$

Now at $t = 0$,

$$F_s(\alpha, 0) = 0$$

so

$$0 = \sqrt{\frac{2}{\pi}}\frac{T_0}{\alpha} + C$$

and thus

$$F_s = \sqrt{\frac{2}{\pi}}\frac{T_0}{\alpha}\left\{1 - \exp\left(\frac{kA}{mc}\alpha^2 t\right)\right\}$$

Thus

$$\tau(x,t) = \sqrt{\frac{2}{\pi}}\int_0^\infty \sqrt{\frac{2}{\pi}}\frac{T_0}{\alpha}\left\{1 - \exp\left(\frac{kA}{mc}\alpha^2 t\right)\right\}\sin\alpha x d\alpha$$

$$= \frac{2T_0}{\pi}\int_0^\infty d\alpha\int_0^x\left\{1 - \exp\left(\frac{kAt}{mc}\alpha^2\right)\right\}\cos\alpha u du d\alpha$$

$$= \frac{2T_0}{\pi}\int_0^x\left\{\pi\delta(u) - \frac{\sqrt{\pi}}{2}\sqrt{\frac{mc}{kAt}}\exp\left(-u^2\frac{mc}{4kAt}\right)\right\}du$$

We can evaluate the integral of the delta function by using one of our delta-sequences.

$$\int_0^x \delta(u)du = \lim_{n\to\infty}\int_0^{1/n}\frac{n}{2}du = \lim_{n\to\infty}\frac{1}{2} = \frac{1}{2}$$

Thus

$$\tau(x,t) = T_0 - T_0\,\mathrm{erf}\left(x\sqrt{\frac{mc}{4kAt}}\right) = T_0\,\mathrm{erfc}\left(x\sqrt{\frac{mc}{4kAt}}\right)$$

and so

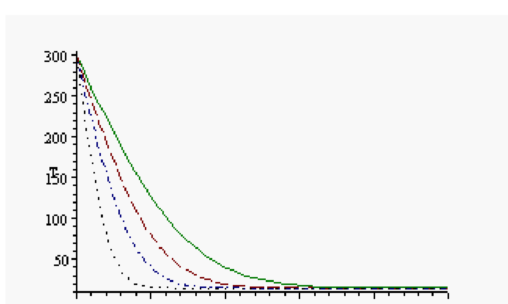
$$T(x,t) = 15^\circ C + 285^\circ C\,\mathrm{erfc}\left(x\sqrt{\frac{mc}{4kAt}}\right)$$

With the given numbers $k =$ 400 W/m·K $c =$ 385 J/kg·K $m = \rho A$, so $m/A = \rho = (8.96\ \mathrm{kg/m^3})$. Then

$$\sqrt{\frac{mc}{4kA}} = \sqrt{\frac{(8.96\ \mathrm{kg/m^3})(385\ \mathrm{J/kg\cdot K})}{4(400\ \mathrm{W/m\cdot K})}} = 1.4683\ \mathrm{s^{1/2}/m}$$

Thus $T(x,t) = 15^\circ C + 285^\circ C\,\mathrm{erfc}\left(\frac{x}{\sqrt{t}}1.4683\ \mathrm{s^{1/2}/m}\right) = 15^\circ C + 285^\circ C\left(1 - \mathrm{erf}\left(\frac{x}{\sqrt{t}}1.4683\ \mathrm{s^{1/2}/m}\right)\right)$

$$= 300^\circ C - 285^\circ C\,\mathrm{erf}\left(\frac{x}{\sqrt{t}}1.4683\ \mathrm{s^{1/2}/m}\right)$$



The plot show T versus x in meters at times

$t = 0.5$ s (black, dotted line), 1.5 s (blue,dot-dash line), 3 s (red dashed line) and 6 s (green line).

16. A long beam is resting on an elastic foundation. The equation satisfied by the beam displacement is:

$$EI\frac{d^2y}{dx^4} = q(x) - \alpha y(x)$$

where $q(x)$ is the load and

α is a constant describing the elastic properties of the foundation. If the load is concentrated toward the center of the beam, then we may assume that $y \to 0$ as $x \to \infty$. Transform the equation, and find $Y(k)$ in terms of $Q(k)$. Solve for the beam displacement if

(a) $q(x) = Mg\delta(x-a)$ and

(b) $q(x) = \frac{Mg}{L}\left\{S\left(x+\frac{L}{2}\right) - S\left(x-\frac{L}{2}\right)\right\}$

$$k^4 EIY = Q - \alpha Y$$

thus

$$Y = \frac{Q}{k^4 EI + \alpha}$$

(a)

$$Q = \frac{Mg}{\sqrt{2\pi}}\int \delta(x-a)e^{-ikx}dx = \frac{Mg}{\sqrt{2\pi}}e^{-ika}$$

Then

$$Y = \frac{Mg}{\sqrt{2\pi}}\frac{e^{-ika}}{k^4 EI + \alpha}$$

and

$$y(x) = \frac{Mg}{2\pi}\int_{-\infty}^{+\infty}\frac{e^{-ika}}{k^4 EI + \alpha}e^{ikx}dk$$

We integrate by completing the contour (upward for $x > a$ and downward for $x < a$) There are poles at

$$k = \left(\frac{\alpha}{EI}\right)^{1/4} e^{i\pi/4 + 2\pi i n/4},\ n = 0, 1, 2, 3$$

and two poles lie inside each contour. Write

$$\beta = \left(\frac{\alpha}{EI}\right)^{1/4}$$

Then for $x > a$ we close upward, enclosing the poles at $k_0 = \beta e^{i\pi/4}, k_1 = \beta e^{3\pi i/4}$. The residues are:

$$\begin{aligned}\mathrm{Res}(k_0) &= \lim_{k\to k_0}\frac{(k-k_0)}{k^4 EI + \alpha}\exp(ik(x-a)) = \lim_{k\to k_0}\frac{1 + i(x-a)(k-k_0)}{4k^3 EI}\exp(ik(x-a)) \\ &= \frac{1}{4\beta^3 EIe^{3\pi i/4}}\exp(i\beta e^{i\pi/4}(x-a)) = \frac{e^{-3\pi i/4}}{4\beta^3 EI}\exp\left(i\beta\frac{\sqrt{2}}{2}(1+i)(x-a)\right) \\ &= -\frac{\sqrt{2}}{2}\frac{1+i}{4\beta^3 EI}\exp\left(i\beta\frac{\sqrt{2}}{2}(1+i)(x-a)\right)\end{aligned}$$

and

$$\begin{aligned}\mathrm{Res}(k_1) &= \frac{1}{4\beta^3 EIe^{9\pi i/4}}\exp\left(i\beta e^{i3\pi/4}(x-a)\right) \\ &= \frac{e^{-9\pi i/4}}{4\beta^3 EI}\exp\left(i\beta\frac{\sqrt{2}}{2}(-1+i)(x-a)\right) \\ &= \frac{\sqrt{2}}{2}\frac{1-i}{4\beta^3 EI}\exp\left(i\beta\frac{\sqrt{2}}{2}(-1+i)(x-a)\right)\end{aligned}$$

Thus for $x > a$:

$$\begin{aligned} y(x) &= \frac{Mg}{2\pi}(2\pi i)\left(\frac{\sqrt{2}}{2}\frac{1}{4\beta^3 EI}\right)\left(-(1+i)e^{i\beta\sqrt{2}(x-a)/2} + (1-i)e^{-i\beta\sqrt{2}(x-a)/2}\right)e^{-\beta\sqrt{2}(x-a)/2} \\ &= \frac{\sqrt{2}}{8}\frac{Mg}{\beta^3 EI}\left(-(i-1)e^{i\beta\sqrt{2}(x-a)/2} + (i+1)e^{-i\beta\sqrt{2}(x-a)/2}\right)e^{-\beta\sqrt{2}(x-a)/2} \\ &= \frac{\sqrt{2}}{8}\frac{Mg}{\beta^3 EI}\left(\frac{1}{i}\left(e^{i\beta\sqrt{2}(x-a)/2} - e^{-i\beta\sqrt{2}(x-a)/2}\right) + e^{i\beta\sqrt{2}(x-a)/2} + e^{-i\beta\sqrt{2}(x-a)/2}\right)e^{-\beta\sqrt{2}(x-a)/2} \\ &= \frac{\sqrt{2}}{2}\frac{Mg}{\beta^3 EI}\left(\cos\left(\beta\frac{\sqrt{2}}{2}(x-a)\right) + \sin\left(\beta\frac{\sqrt{2}}{2}(x-a)\right)\right)e^{-\beta\sqrt{2}(x-a)/2} \end{aligned}$$

Now for $x > a$ we must close downward, enclosing the poles k_2 and k_3. The residues are:

$$\begin{aligned} \text{res}(k_2) &= \frac{1}{4\beta^3 EI e^{15\pi i/4}}\exp\left(i\beta e^{i5\pi/4}(x-a)\right) \\ &= \frac{\sqrt{2}}{2}\frac{1+i}{4\beta^3 EI}\exp\left(-i\beta\frac{\sqrt{2}}{2}(1+i)(x-a)\right) \end{aligned}$$

and

$$\begin{aligned} \text{res}(k_3) &= \frac{1}{4\beta^3 EI e^{21\pi i/4}}\exp\left(i\beta e^{i7\pi/4}(x-a)\right) \\ &= \frac{\sqrt{2}}{2}\frac{-1+i}{4\beta^3 EI}\exp\left(i\beta\frac{\sqrt{2}}{2}(1-i)(x-a)\right) \end{aligned}$$

and thus for $x < a$:

$$\begin{aligned} y(x) &= \frac{Mg}{2\pi}(-2\pi i)\left(\frac{\sqrt{2}}{2}\frac{1}{4\beta^3 EI}\right)\left((1+i)e^{-i\beta\sqrt{2}(x-a)/2} + (-1+i)e^{i\beta\sqrt{2}(x-a)/2}\right)e^{\beta\sqrt{2}(x-a)/2} \\ &= -\frac{\sqrt{2}}{8}\frac{Mg}{\beta^3 EI}\left((i-1)e^{-i\beta\sqrt{2}(x-a)/2} - (i+1)e^{i\beta\sqrt{2}(x-a)/2}\right)e^{\beta\sqrt{2}(x-a)/2} \\ &= \frac{\sqrt{2}}{4}\frac{Mg}{\beta^3 EI}\left(\cos\left(\beta\frac{\sqrt{2}}{2}(x-a)\right) - \sin\left(\beta\frac{\sqrt{2}}{2}(x-a)\right)\right)e^{\beta\sqrt{2}(x-a)/2} \end{aligned}$$

Putting the results together, we get:

$$y(x) = \frac{\sqrt{2}}{4}\frac{Mg}{\beta^3 EI}\left(\cos\left(\beta\frac{\sqrt{2}}{2}(x-a)\right) + \sin\left(\beta\frac{\sqrt{2}}{2}|x-a|\right)\right)e^{-\beta\sqrt{2}|x-a|/2}$$

(b) $\frac{Mg}{L}\left\{S\left(x+\frac{L}{2}\right) - S\left(x-\frac{L}{2}\right)\right\}$

$$\begin{aligned} Q(k) &= \sqrt{\frac{1}{2\pi}}\frac{Mg}{L}\int_{-L/2}^{L/2} e^{-ikx}dx = \sqrt{\frac{1}{2\pi}}\frac{Mg}{L}\frac{e^{-ikx}}{-ik}\Bigg|_{L/2}^{L/2} \\ &= \sqrt{\frac{1}{2\pi}}\frac{Mg}{L}\frac{e^{-ikL/2} - e^{ikL/2}}{-ik} = \sqrt{\frac{2}{\pi}}\frac{Mg}{L}\frac{\sin kL/2}{k} \end{aligned}$$

and so

$$Y(k) = Y = \sqrt{\frac{2}{\pi}}\frac{Mg}{L}\frac{\sin kL/2}{k}\frac{1}{k^4 EI + \alpha}$$

and inverting we get:

$$\begin{aligned} y(x) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\sqrt{\frac{2}{\pi}}\frac{Mg}{L}\frac{\sin kL/2}{k}\frac{1}{k^4 EI + \alpha}e^{ikx}dk \\ &= \frac{Mg}{2i\pi L}\int_{-\infty}^{+\infty}\frac{e^{ikL/2} - e^{-ikL/2}}{k}\frac{1}{k^4 EI + \alpha}e^{ikx}dk \\ &= \frac{Mg}{2i\pi L}\int_{-\infty}^{+\infty}\frac{\exp\left(\frac{1}{2}ik(L+2x)\right) - \exp\left(\frac{1}{2}ik(-L+2x)\right)}{k\left(k^4 EI + \alpha\right)}dk \end{aligned}$$

The integrand has poles at:

$$k_n = \left(\frac{\alpha}{EI}\right)^{1/4} e^{i\pi/4+2\pi in/4},\ n = 0, 1, 2, 3 \text{ and } k_4 = 0$$

We evaluate the integral by putting the contour slightly below the real axis.

First term:

For $L + 2x > 0$, ie $x > -L/2$, we close upward enclosing the poles at k_0, k_1, and k_4. By method 4, the residues are:

$$R_0 = \lim_{k \to k_0} \frac{\exp\left(\frac{1}{2}ik(L+2x)\right)}{5k^4EI + \alpha} = \frac{\exp\left(\frac{\sqrt{2}}{4}i\beta(1+i)(L+2x)\right)}{5(-1)EI + \alpha}$$
$$= \frac{\exp\left(\frac{\sqrt{2}}{2}\beta(i-1)\left(x + \frac{L}{2}\right)\right)}{\alpha - 5EI}$$

where $\beta = \left(\frac{\alpha}{EI}\right)^{1/4}$

$$R_1 = \lim_{k \to k_1} \frac{\exp\left(\frac{1}{2}ik(L+2x)\right)}{5k^4EI + \alpha} = \frac{\exp\left(\frac{\sqrt{2}}{4}i\beta(-1+i)(L+2x)\right)}{5(-1)EI + \alpha}$$
$$= \frac{\exp\left(-\frac{\sqrt{2}}{2}\beta(i+1)\left(x + \frac{L}{2}\right)\right)}{\alpha - 5EI}$$

and

$$R_4 = \lim_{k \to 0} \frac{\exp\left(\frac{1}{2}ik(L+2x)\right)}{5k^4EI + \alpha} = \frac{1}{\alpha}$$

and the first term is:

$$T_{1>} = \int_{-\infty}^{+\infty} \frac{\exp\left(\frac{1}{2}ik(L+2x)\right)}{k\left(k^4EI + \alpha\right)} dk$$
$$= 2\pi i \left(\frac{\exp\left(-\frac{\sqrt{2}}{2}\beta\left(x + \frac{L}{2}\right)\right)}{\alpha - 5EI} \left(\exp\left(\frac{\sqrt{2}}{2}\beta i \left(x + \frac{L}{2}\right)\right) + \exp\left(-\frac{\sqrt{2}}{2}\beta i\left(x + \frac{L}{2}\right)\right)\right) + \frac{1}{\alpha} \right)$$
$$= 2\pi i \left(\frac{\exp\left(-\frac{\sqrt{2}}{2}\beta\left(x + \frac{L}{2}\right)\right)}{\alpha - 5EI} 2\cos\left(\frac{\sqrt{2}}{2}\beta\left(x + \frac{L}{2}\right)\right) + \frac{1}{\alpha} \right)$$

When $x < -L/2$, we close downward enclosing the poles at k_2, and k_3.

The residues are:

$$R_2 = \frac{\exp\left(-\frac{\sqrt{2}}{2}\beta i(i+1)\left(x + \frac{L}{2}\right)\right)}{\alpha - 5EI} = \frac{\exp\left(\frac{\sqrt{2}}{2}\beta(1-i)\left(x + \frac{L}{2}\right)\right)}{\alpha - 5EI}$$

and:

$$R_3 = \frac{\exp\left(\frac{\sqrt{2}}{2}\beta i(1-i)\left(x + \frac{L}{2} - a\right)\right)}{\alpha - 5EI} = \frac{\exp\left(\frac{\sqrt{2}}{2}\beta(1+i)\left(x + \frac{L}{2} - a\right)\right)}{\alpha - 5EI}$$

giving

$$T_{1<} = \int_{-\infty}^{+\infty} \frac{\exp\left(\frac{1}{2}ik(L - 2a + 2x)\right)}{k\left(k^4EI + \alpha\right)} dk$$
$$= -2\pi i \left(\frac{\exp\left(\frac{\sqrt{2}}{2}\beta\left(x + \frac{L}{2}\right)\right)}{\alpha - 5EI} 2\cos\left(\frac{\sqrt{2}}{2}\beta\left(x + \frac{L}{2}\right)\right) \right)$$

The second term is evaluated similarly, giving:

$$T_{2>} = \int_{-\infty}^{+\infty} \frac{\exp\left(\frac{1}{2}ik(-L+2x)\right)}{k\left(k^4EI+\alpha\right)}dk$$
$$= 2\pi i\left(\frac{\exp\left(-\frac{\sqrt{2}}{2}\beta\left(x-\frac{L}{2}\right)\right)}{\alpha-5EI}2\cos\left(\frac{\sqrt{2}}{2}\beta\left(x-\frac{L}{2}\right)\right)+\frac{1}{\alpha}\right)$$

for $x > L/2$, and

$$T_{2<} = \int_{-\infty}^{+\infty} \frac{\exp\left(\frac{1}{2}ik(-L+2x)\right)}{k\left(k^4EI+\alpha\right)}dk$$
$$= -2\pi i\left(\frac{\exp\left(\frac{\sqrt{2}}{2}\beta\left(x-\frac{L}{2}\right)\right)}{\alpha-5EI}2\cos\left(\frac{\sqrt{2}}{2}\beta\left(x-\frac{L}{2}\right)\right)\right)$$

for $x < L/2$.

Thus we have:

For $x > L/2$

$$y(x) = \frac{Mg}{2i\pi L}(T_{1>} - T_{2>})$$
$$= \frac{Mg}{L}\left(\begin{array}{c}\frac{\exp\left(-\frac{\sqrt{2}}{2}\beta\left(x+\frac{L}{2}\right)\right)}{\alpha-5EI}2\cos\left(\frac{\sqrt{2}}{2}\beta\left(x+\frac{L}{2}\right)\right)+\frac{1}{\alpha}\\ -\left(\frac{\exp\left(-\frac{\sqrt{2}}{2}\beta\left(x-\frac{L}{2}\right)\right)}{\alpha-5EI}2\cos\left(\frac{\sqrt{2}}{2}\beta\left(x-\frac{L}{2}\right)\right)+\frac{1}{\alpha}\right)\end{array}\right)$$
$$= 2\frac{Mg}{L}\frac{\exp\left(-\frac{\sqrt{2}}{2}\beta(x)\right)}{\alpha-5EI}\left\{\begin{array}{c}\cos\left[\frac{\sqrt{2}\beta}{2}\left(x+\frac{L}{2}\right)\right]\exp\left(-\frac{\sqrt{2}}{4}\beta L\right)\\ -\cos\left[\frac{\sqrt{2}\beta}{2}\left(x-\frac{L}{2}\right)\right]\exp\left(\frac{\sqrt{2}}{4}\beta L\right)\end{array}\right\}$$
$$= 2\frac{Mg}{L}\frac{\exp\left(-\frac{\sqrt{2}}{2}\beta x\right)}{\alpha-5EI}\left(\begin{array}{c}\left\{\cos\left(\frac{\sqrt{2}}{2}\beta x\right)\cos\frac{\sqrt{2}}{4}\beta L-\sin\left(\frac{\sqrt{2}}{2}\beta x\right)\sin\frac{\sqrt{2}}{4}\beta L\right\}\exp\left(-\frac{\sqrt{2}}{4}\beta L\right)\\ -\left\{\cos\left(\frac{\sqrt{2}}{2}\beta x\right)\cos\frac{\sqrt{2}}{4}\beta L+\sin\left(\frac{\sqrt{2}}{2}\beta x\right)\sin\frac{\sqrt{2}}{4}\beta L\right\}\exp\left(\frac{\sqrt{2}}{4}\beta L\right)\end{array}\right)$$
$$= 2\frac{Mg}{L}\frac{\exp\left(-\frac{\sqrt{2}}{2}\beta x\right)}{5EI-\alpha}\left\{\begin{array}{c}\cos\left(\frac{\sqrt{2}}{2}\beta x\right)\cos\frac{\sqrt{2}}{4}\beta L\sinh\frac{\sqrt{2}}{4}\beta L\\ +\sin\left(\frac{\sqrt{2}}{2}\beta x\right)\sin\frac{\sqrt{2}}{4}\beta L\cosh\frac{\sqrt{2}}{4}\beta L\end{array}\right\}$$

For $L/2 > x > -L/2$:

$$y(x) = \frac{Mg}{2i\pi L}(T_{1>} - T_{2<})$$
$$= \frac{Mg}{L}\left(\begin{array}{c}\frac{\exp\left(-\frac{\sqrt{2}}{2}\beta\left(x+\frac{L}{2}\right)\right)}{\alpha-5EI}2\cos\left(\frac{\sqrt{2}}{2}\beta\left(x+\frac{L}{2}\right)\right)+\frac{1}{\alpha}\\ +\frac{\exp\left(\frac{\sqrt{2}}{2}\beta\left(x-\frac{L}{2}\right)\right)}{\alpha-5EI}2\cos\left(\frac{\sqrt{2}}{2}\beta\left(x-\frac{L}{2}\right)\right)\end{array}\right)$$
$$= 2\frac{Mg}{L}\left(\frac{1}{\alpha}+\frac{\exp\left(-\frac{\sqrt{2}}{4}\beta L\right)}{\alpha-5EI}\left\{\begin{array}{c}\exp\left(-\frac{\sqrt{2}}{2}\beta x\right)\left\{\cos\left(\frac{\sqrt{2}}{2}\beta x\right)\cos\frac{\sqrt{2}}{4}\beta L-\sin\left(\frac{\sqrt{2}}{2}\beta x\right)\sin\frac{\sqrt{2}}{4}\beta L\right\}\\ +\exp\left(\frac{\sqrt{2}}{2}\beta x\right)\cos\left(\frac{\sqrt{2}}{2}\beta x\right)\cos\frac{\sqrt{2}}{4}\beta L+\sin\left(\frac{\sqrt{2}}{2}\beta x\right)\sin\frac{\sqrt{2}}{4}\beta L\end{array}\right\}\right)$$
$$= 2\frac{Mg}{L}\left(\frac{1}{\alpha}+\frac{\exp\left(-\frac{\sqrt{2}}{4}\beta L\right)}{\alpha-5EI}\left\{\cos\frac{\sqrt{2}}{2}\beta x\cos\frac{\sqrt{2}}{4}\beta L\cosh\frac{\sqrt{2}}{2}\beta x-\sin\frac{\sqrt{2}}{2}\beta x\sin\frac{\sqrt{2}}{4}\beta L\sinh\frac{\sqrt{2}}{2}\beta x\right\}\right)$$

Finally for $x < -L/2$

$$y(x) = \frac{Mg}{2i\pi L}(T_{1<} - T_{2<})$$
$$= 2\frac{Mg}{L}\frac{\exp\left(\frac{\sqrt{2}}{2}\beta x\right)}{5EI-\alpha}\left(\cos\frac{\sqrt{2}}{2}\beta x\cos\frac{\sqrt{2}}{4}\beta L\sinh\frac{\sqrt{2}}{4}\beta L-\sin\frac{\sqrt{2}}{2}\beta x\sin\frac{\sqrt{2}}{4}\beta L\cosh\frac{\sqrt{2}}{4}\beta L\right.$$

17. Find the Fourier sine transform of For the function

$e^{-x}\sin x$, find (**a**) the Fourier sine transform and (**b**) the Fourier cosine transform.

(**a**) The sine transform is

$$\begin{aligned}
&\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-x}\sin x\sin kx\,dx \\
&= \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-x}\left(\frac{e^{ix}-e^{-ix}}{2i}\right)\left(\frac{e^{ikx}-e^{-ikx}}{2i}\right)dx \\
&= \frac{-1}{4}\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-x+ix+ikx}-e^{-x+ix-ikx}-e^{-x-ix+ikx}+e^{-x-ix-ikx}dx \\
&= \frac{-1}{4}\sqrt{\frac{2}{\pi}}\left(\frac{e^{-x+ix+ikx}}{-1+i(k+1)}-\frac{e^{-x+ix-ikx}}{-1+i(1-k)}-\frac{e^{-x-ix+ikx}}{-1+i(k-1)}+\frac{e^{-x-ix-ikx}}{-1-i(1+k)}\right)\Bigg|_0^\infty \\
&= \frac{1}{4}\sqrt{\frac{2}{\pi}}\left(\frac{1}{-1+i(k+1)}-\frac{1}{-1+i(1-k)}-\frac{1}{-1+i(k-1)}+\frac{1}{-1-i(1+k)}\right) \\
&= \frac{1}{4}\sqrt{\frac{2}{\pi}}\left(8\frac{k}{4+k^4}\right)=\sqrt{\frac{2}{\pi}}\frac{2k}{k^4+4}
\end{aligned}$$

(**b**) The cosine transform is:

$$\begin{aligned}
&\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-x}\sin x\cos kx\,dx \\
&= \sqrt{\frac{2}{\pi}}\int_0^\infty e^{-x}\left(\frac{e^{ix}-e^{-ix}}{2i}\right)\left(\frac{e^{ikx}+e^{-ikx}}{2}\right)dx \\
&= \frac{1}{4i}\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-x+ix+ikx}+e^{-x+ix-ikx}-e^{-x-ix+ikx}-e^{-x-ix-ikx}dx \\
&= \frac{1}{4i}\sqrt{\frac{2}{\pi}}\left(\frac{e^{-x+ix+ikx}}{-1+i(k+1)}+\frac{e^{-x+ix-ikx}}{-1+i(1-k)}-\frac{e^{-x-ix+ikx}}{-1+i(k-1)}-\frac{e^{-x-ix-ikx}}{-1-i(1+k)}\right)\Bigg|_0^\infty \\
&= \frac{-1}{4i}\sqrt{\frac{2}{\pi}}\left(\frac{1}{-1+i(k+1)}+\frac{1}{-1+i(1-k)}-\frac{1}{-1+i(k-1)}-\frac{1}{-1-i(1+k)}\right) \\
&= \frac{-1}{4i}\sqrt{\frac{2}{\pi}}\left(4i\frac{k^2-2}{4+k^4}\right)=\sqrt{\frac{2}{\pi}}\frac{2-k^2}{k^4+4}
\end{aligned}$$

18. For the function $xe^{-\alpha x}$, find (**a**) the Fourier sine transform and (**b**) the Fourier cosine transform

(**a**) The sine transform is

$$\begin{aligned}
\sqrt{\frac{2}{\pi}}\int_0^\infty xe^{-\alpha x}\sin kx\,dx &= \sqrt{\frac{2}{\pi}}\int_0^\infty xe^{-\alpha x}\left(\frac{e^{ikx}-e^{-ikx}}{2i}\right)dx \\
&= \frac{1}{2i}\sqrt{\frac{2}{\pi}}\int_0^\infty x\left(e^{-\alpha x+ikx}-e^{-\alpha x-ikx}\right)dx \\
&= \frac{1}{2i}\left(x\frac{e^{-\alpha x+ikx}}{-\alpha+ik}\Bigg|_0^\infty-\int_0^\infty\frac{e^{-\alpha x+ikx}}{-\alpha+ik}dx\right) \\
&\quad -\frac{1}{2i}\left(x\frac{e^{-\alpha x-ikx}}{-\alpha-ik}\Bigg|_0^\infty-\int_0^\infty\frac{e^{-\alpha x-ikx}}{-\alpha-ik}dx\right)
\end{aligned}$$

The integrated terms vanish, leaving

$$\begin{aligned}
\sqrt{\frac{2}{\pi}}\int_0^\infty xe^{-\alpha x}\sin kx\,dx &= \frac{1}{2i}\sqrt{\frac{2}{\pi}}\left(-\frac{e^{-\alpha x+ikx}}{(-\alpha+ik)^2}\Bigg|_0^\infty+\frac{e^{-\alpha x-ikx}}{(-\alpha-ik)^2}\Bigg|_0^\infty\right) \\
&= \frac{1}{2i}\sqrt{\frac{2}{\pi}}\left(\frac{1}{(-\alpha+ik)^2}-\frac{1}{(-\alpha-ik)^2}\right) \\
&= \sqrt{\frac{2}{\pi}}\frac{2k\alpha}{(\alpha^2+k^2)^2}
\end{aligned}$$

(**b**) The cosine transform is:

$$\sqrt{\frac{2}{\pi}}\int_0^\infty xe^{-\alpha x}\sin kx\,dx = \sqrt{\frac{2}{\pi}}\int_0^\infty xe^{-\alpha x}\left(\frac{e^{ikx}+e^{-ikx}}{2}\right)dx$$
$$= \frac{1}{2}\sqrt{\frac{2}{\pi}}\int_0^\infty x\left(e^{-\alpha x+ikx}+e^{-\alpha x-ikx}\right)dx$$
$$= \frac{1}{2}\sqrt{\frac{2}{\pi}}\left(x\frac{e^{-\alpha x+ikx}}{-\alpha+ik}\bigg|_0^\infty - \int_0^\infty \frac{e^{-\alpha x+ikx}}{-\alpha+ik}dx + x\frac{e^{-\alpha x-ikx}}{-\alpha-ik}\bigg|_0^\infty - \int_0^\infty \frac{e^{-\alpha x-ikx}}{-\alpha-ik}dx\right)$$

The integrated terms vanish, leaving

$$\sqrt{\frac{2}{\pi}}\int_0^\infty xe^{-\alpha x}\sin kx\,dx = \frac{1}{2}\sqrt{\frac{2}{\pi}}\left(-\frac{e^{-\alpha x+ikx}}{(-\alpha+ik)^2}\bigg|_0^\infty - \frac{e^{-\alpha x-ikx}}{(-\alpha-ik)^2}\bigg|_0^\infty\right)$$
$$= \frac{1}{2}\sqrt{\frac{2}{\pi}}\left(\frac{1}{(-\alpha+ik)^2}+\frac{1}{(-\alpha-ik)^2}\right)$$
$$= \sqrt{\frac{2}{\pi}}\frac{\alpha^2-k^2}{(\alpha^2+k^2)^2}$$

19. Show that the Fourier cosine transform of the function x^{p-1} for $0 < p < 1$ is

$\sqrt{\frac{2}{\pi}}\frac{1}{k^p}\cos\frac{p\pi}{2}\Gamma(p).$ Hence show that the function

$1/\sqrt{x}$ is its own cosine transform. Obtain similar results for the sine transform. (The results of Chapter 2 §9 may prove useful.)

$$\sqrt{\frac{2}{\pi}}\int_0^\infty x^{p-1}\cos kx\,dx = \frac{1}{\sqrt{2\pi}}\int_0^\infty x^{p-1}\left(e^{ikx}+e^{-ikx}\right)dx$$

In the first term let $u = -ikx$ and in the second let $u = ikx$

$$F_c(k) = \frac{1}{\sqrt{2\pi}}\left[\int_0^{-i\infty}\left(\frac{u}{-ik}\right)^{p-1}e^{-u}\frac{du}{-ik} + \int_0^{i\infty}\left(\frac{u}{ik}\right)^{p-1}e^{-u}\frac{du}{ik}\right]$$
$$= \frac{1}{\sqrt{2\pi}}\left[\left(\frac{e^{i\pi/2}}{k}\right)^p\int_0^\infty u^{p-1}e^{-u}du + \left(\frac{e^{-i\pi/2}}{k}\right)^p\int_0^\infty u^{p-1}e^{-u}du\right]$$
$$= \frac{1}{\sqrt{2\pi}}\frac{1}{k^p}\left(e^{-ip\pi/2}+e^{ip\pi/2}\right)\int_0^\infty u^{p-1}e^{-u}du$$
$$= \sqrt{\frac{2}{\pi}}\frac{1}{k^p}\cos\frac{p\pi}{2}\Gamma(p)$$

where we can move both integrals to the real axis since there are no poles between the imagnary axis and the real axis.

In the special case $p = 1/2,$ w e find

$$F_c(k) = \sqrt{\frac{2}{\pi k}}\frac{1}{\sqrt{2}}\sqrt{\pi} = \frac{1}{\sqrt{k}}$$

so this function is its own transform.

The sine transform is

$$\sqrt{\frac{2}{\pi}}\int_0^\infty x^{p-1}\cos kx\,dx = \frac{1}{\sqrt{2\pi}\,i}\int_0^\infty x^{p-1}\left(e^{ikx}-e^{-ikx}\right)dx$$
$$= \frac{1}{\sqrt{2\pi}\,i}\left[\int_0^{-i\infty}\left(\frac{u}{-ik}\right)^{p-1}e^{-u}\frac{du}{-ik} - \int_0^{i\infty}\left(\frac{u}{ik}\right)^{p-1}e^{-u}\frac{du}{ik}\right]$$
$$= \frac{1}{\sqrt{2\pi}\,i}\frac{1}{k^p}\left(e^{-ip\pi/2}-e^{ip\pi/2}\right)\int_0^\infty u^{p-1}e^{-u}du$$
$$= \sqrt{\frac{2}{\pi}}\frac{1}{k^p}\sin\frac{p\pi}{2}\Gamma(p)$$

so that with $p = 1/2$ we obtain the same result as for the cosine transform.

20. Determine the form of Parseval's theorem (equation 7.10 and 7.11) that applies to the cosine transform.

Following the method in section 7.3.5, and remembering that the functions are defined only for positive x:

$$\begin{aligned}\int_0^{+\infty} f(x)g(x)dx &= \frac{2}{\pi}\int_0^{\infty} dx \int_0^{\infty} dkF(k)\cos kx dk \int_0^{\infty} d\omega G(\omega)\cos\omega x d\omega \\ &= \int_0^{\infty} dk \int_0^{\infty} d\omega F(k)G(\omega)\frac{2}{\pi}\int_0^{\infty} \cos kx \cos\omega x dx\end{aligned}$$

Then from the result of Problem 6.13,

$$\begin{aligned}\int_0^{+\infty} f(x)g(x)dx &= \int_0^{\infty} dk \int_0^{\infty} d\omega F(k)G(\omega)\delta(k-\omega) \\ &= \int_0^{\infty} F(k)G(k)dk\end{aligned}$$

The same expression holds for the sine transform.

The equivalent relation to 7.11 is thus:

$$\int_0^{\infty} [f(x)]^2 dx = \int_0^{\infty} [F(k)]^2 dk$$

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Chapter 7: Fourier Transforms

21. The magnetic field in a conducting medium diffuses away according to the equation:

$$\frac{\partial^2 H(x,t)}{\partial x^2} = \mu\sigma \frac{\partial H(x,t)}{\partial t}$$

Solve this equation by taking the Fourier transform in space and the Laplace transform in time. Find $H(x,t)$ if the magnetic field at $t = 0$ is a step function:

$$H(x,0) = \begin{cases} H_0 & \text{if} \quad -d/2 < x < d/2 \\ 0 & \text{otherwise} \end{cases}$$

Express your answer in terms of the error function (Appendix IX).

$$\frac{\partial \tilde{H}}{\partial t} = -\frac{k^2}{\mu\sigma}\tilde{H}$$

Thus

$$\tilde{H} = A \exp\left(\frac{-k^2 t}{\mu\sigma}\right)$$

At $t = 0$,

$$\tilde{H}(k,0) = \sqrt{\frac{2}{\pi}} H_0 \frac{\sin kd/2}{k}$$

Thus

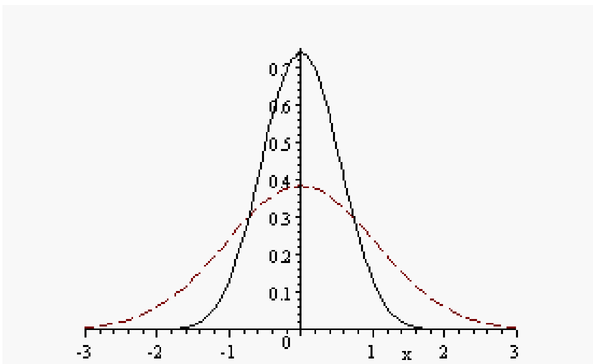
$$\tilde{H}(k,t) = \sqrt{\frac{2}{\pi}} H_0 \frac{\sin kd/2}{k} e^{-k^2 t/\mu\sigma}$$

We may invert using convolution and the results of problem 5 and Example 7.2:

$$H(x,t) = \frac{1}{\sqrt{2\pi}} H_0 \sqrt{\frac{\mu\sigma}{2t}} \int_{-d/2}^{d/2} e^{-(x-u)^2\mu\sigma/4t} du$$

Let $y = \frac{(x-u)}{2}\sqrt{\frac{\mu\sigma}{t}}$. Then

$$\begin{aligned} H(x,t) &= \frac{H_0}{\sqrt{\pi}} \int_{(x+d/2)\sqrt{\mu\sigma}/2\sqrt{t}}^{(x-d/2)\sqrt{\mu\sigma}/2\sqrt{t}} e^{-y^2}(-dy) \\ &= \frac{1}{2} H_0 \left(\operatorname{erf}\left(\frac{(x+d/2)}{2}\sqrt{\frac{\mu\sigma}{t}}\right) - \operatorname{erf}\left(\frac{(x-d/2)}{2}\sqrt{\frac{\mu\sigma}{t}}\right)\right) \end{aligned}$$



The plot shows H/H_0 versus x/d for $t/\mu\sigma$ = 0.1 (black line) and 0.5 (red, dashed line).

22. This solution is similar to that for problem 21. The transform is:

$$\tilde{\rho}(k,t) = Ae^{-k^2Dt}$$

where

$$\tilde{\rho}(k,0) = A = \frac{\rho_0}{\sqrt{2}}a\exp\left(-\frac{k^2a^2}{4}\right)$$

(Example 7.2 with $\alpha = 1/a$). Thus

$$\tilde{\rho}(k,t) = \frac{\rho_0}{\sqrt{2}}a\exp\left(-\frac{k^2a^2}{4}\right)e^{-k^2Dt}$$

and thus

$$\begin{aligned}\rho(x,t) &= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}\frac{\rho_0}{\sqrt{2}}a\exp\left(-k^2\left(\frac{a^2}{4}+Dt\right)\right)e^{ikx}dk \\ &= \rho_0\frac{a}{\sqrt{\left(a^2+4Dt\right)}}\exp\left(-\frac{x^2}{a^2+4Dt}\right)\end{aligned}$$

23. Develop a three-dimensional version of the convolution theorem. Use the result to obtain the solution of Poisson's equation

$$\nabla^2\Phi = -\frac{\rho(\vec{\mathbf{r}})}{\varepsilon_0}$$

Evaluate the resulting integral explicitly if $\rho(\vec{\mathbf{r}}) = q\delta(\vec{\mathbf{r}})$.

If the transform is of the form $F(\vec{\mathbf{k}})G(\vec{\mathbf{k}})$ then

$$\begin{aligned}\frac{1}{(2\pi)^{3/2}}\int_{-\infty}^{+\infty}F(\vec{\mathbf{k}})G(\vec{\mathbf{k}})e^{i\vec{k}\cdot\vec{x}}d^3\vec{\mathbf{k}} &= \frac{1}{(2\pi)^{3/2}}\int_{-\infty}^{+\infty}F(\vec{\mathbf{k}})\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{+\infty}g(\vec{\mathbf{u}})e^{-i\vec{k}\cdot\vec{u}}d^3\vec{\mathbf{u}}e^{i\vec{k}\cdot\vec{x}}d^3\vec{\mathbf{k}} \\ &= \frac{1}{(2\pi)^{3/2}}\int_{-\infty}^{+\infty}g(\vec{\mathbf{u}})f(\vec{\mathbf{x}}-\vec{\mathbf{u}})d^3\vec{\mathbf{u}}\end{aligned}$$

The transformed equation is:

$$\begin{aligned}-k^2\Phi &= -\frac{R(\vec{\mathbf{k}})}{\varepsilon_0} \\ \Phi(\vec{\mathbf{k}}) &= \frac{R(\vec{\mathbf{k}})}{k^2\varepsilon_0}\end{aligned}$$

Now the function $1/k^2$ inverts to

$$\begin{aligned}\frac{1}{(2\pi)^{3/2}}\int_{-\infty}^{+\infty}\frac{e^{i\vec{\mathbf{k}}\cdot\vec{\mathbf{x}}}}{k^2}d^3\vec{\mathbf{k}} &= \frac{1}{(2\pi)^{3/2}}\int_0^{+\infty}\int_{-1}^{+1}\frac{e^{ikr\mu}}{k^2}k^2dkd\mu 2\pi \\ &= \frac{1}{\sqrt{2\pi}}\int_0^{+\infty}\frac{e^{ikr\mu}}{ikr}\bigg|_{-1}^{+1}dk \\ &= \frac{1}{\sqrt{2\pi}\,ir}\int_0^{+\infty}\frac{e^{ikr}-e^{-ikr}}{k}dk \\ &= \frac{1}{\sqrt{2\pi}\,ir}\int_{-\infty}^{+\infty}\frac{e^{ikr}}{k}dk\end{aligned}$$

The integral has a simple pole at the origin. With

r positive, we close upward with a small semicircle over the pole and a big semicircle at ∞. Then

$$\oint = P\int_{-\infty}^{\infty} + \int_{\text{semicircle radius }\varepsilon} + \int_{\text{semicircle radius }R} = 0$$

The integral around the closed curve is zero by Cauchy's theorem ,and so the principle value is the negative of the integral around the little semicircle:

$$\frac{1}{\sqrt{2\pi}\, ir}(\pi i) = \sqrt{\frac{\pi}{2}}\,\frac{1}{r}$$

and so

$$\Phi(\vec{\mathbf{x}}) = \frac{1}{(2\pi)^{3/2}}\sqrt{\frac{\pi}{2}}\int \frac{\rho(\vec{\mathbf{u}})}{\varepsilon_0}\frac{1}{|\vec{\mathbf{x}}-\vec{\mathbf{u}}|}d^3\vec{\mathbf{u}} = \frac{1}{4\pi\varepsilon_0}\int \frac{\rho(\vec{\mathbf{u}})}{|\vec{\mathbf{x}}-\vec{\mathbf{u}}|}d^3\vec{\mathbf{u}}$$

as expected.

Now if $\rho = q\delta(\vec{\mathbf{r}})$ then we have:

$$\Phi(\vec{\mathbf{x}}) = \frac{1}{4\pi\varepsilon_0}\int \frac{q\delta(\vec{\mathbf{u}})}{|\vec{\mathbf{x}}-\vec{\mathbf{u}}|}d^3\vec{\mathbf{u}} = \frac{q}{4\pi\varepsilon_0 r}$$

as expected.

24. Find the three-dimensional Fourier transform of the charge distribution

$$\rho(\vec{\mathbf{r}}) = \frac{e^{-r/a}}{4\pi r}$$

$$\begin{aligned}
R(\vec{\mathbf{k}}) &= \frac{1}{(2\pi)^{3/2}}\int \frac{e^{-r/a}}{4\pi r}e^{-i\vec{\mathbf{k}}\cdot\vec{\mathbf{r}}}d^3\vec{\mathbf{r}} \\
&= \frac{1}{(2\pi)^{3/2}}\int_0^\infty\int_{-1}^{+1}\int_0^{2\pi} r^2\frac{e^{-r/a}}{4\pi r}e^{-ikr\mu}d\mu\, d\phi\, dr \\
&= \frac{1}{2(2\pi)^{3/2}}\int_0^\infty r e^{-r/a}\frac{e^{-ikr\mu}}{-ikr}\Bigg|_{-1}^{+1}dr \\
&= \frac{1}{2(2\pi)^{3/2}}\int_0^\infty e^{-r/a}\frac{e^{-ikr}-e^{ikr}}{-ik}dr \\
&= \frac{1}{-2ik(2\pi)^{3/2}}\left(\frac{e^{-ikr-r/a}}{-ik-r/a}-\frac{e^{ikr-r/a}}{ikr-r/a}\right)\Bigg|_0^\infty \\
&= \frac{1}{2ik(2\pi)^{3/2}}\left(\frac{1}{-ik-1/a}-\frac{1}{ik-1/a}\right) \\
&= \frac{1}{(2\pi)^{3/2}}\left(\frac{a^2}{k^2a^2+1}\right)
\end{aligned}$$

25. Take the Fourier transform of the three-dimensional wave equation

$$\frac{d^2s}{dt^2} - \nu^2\nabla^2 s = f(\vec{\mathbf{x}},t)$$

and solve for the transform $S(\vec{\mathbf{k}},\omega)$. Show that the introduction of a damping force (through the addition of a term $\alpha\frac{ds}{dt}$ on the left hand side) moves the poles off the real axis. Invert the transform in the case $\alpha \to 0$ for the case $f(\vec{\mathbf{x}},t) = e^{-r/a}\delta(t)$.

$$\begin{aligned}
(\nu^2k^2-\omega^2)S &= S(\vec{\mathbf{k}},\omega) \\
S &= \frac{F}{\nu^2k^2-\omega^2}
\end{aligned}$$

Thus

$$s(\vec{\mathbf{x}},t) = \frac{1}{(2\pi)^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{F}{v^2k^2-\omega^2}e^{i\vec{\mathbf{k}}\cdot\vec{\mathbf{x}}-i\omega t}d\vec{\mathbf{k}}d\omega$$

The integrand has poles at $\omega = \pm vk,$ on the real axis, provided by the differential equation, as well as poles of the transform S. The damping term modifies the transformed equation:

$$\left(v^2k^2-\omega^2-i\omega\alpha\right)S = F\left(\vec{\mathbf{k}},\omega\right)$$

and the poles are then at

$$\omega = \frac{-i\alpha \pm \sqrt{4v^2k^2-\alpha^2}}{2}$$

and both are in the lower half plane. The integration path along the real axis passes above these poles. Thus in the limit $\alpha \to 0,$ we keep the path of integration above the poles.

Now we look at the function $f(\vec{\mathbf{x}},t) = e^{-r/a}\delta(t)$

$$\begin{aligned}
F\left(\vec{\mathbf{k}},\omega\right) &= \frac{1}{(2\pi)^{3/2}}\int_0^{2\pi}d\phi\int_{-1}^{+1}d\mu\int_0^{\infty}r^2dr\int_{-\infty}^{+\infty}dt\,e^{-r/a}\delta(t)e^{-ikr\mu}e^{+i\omega t}\\
&= \frac{1}{(2\pi)^{3/2}}\int_0^{2\pi}d\phi\int_0^{\infty}r^2e^{-r/a}dr\,\frac{e^{-ikr\mu}}{-ikr}\Bigg|_{-1}^{+1}\\
&= \frac{1}{\sqrt{2\pi}\,ik}\int_0^{\infty}re^{-r/a}\left(e^{ikr}-e^{-ikr}\right)dr\\
&= \frac{1}{\sqrt{2\pi}\,ik}\left(r\frac{e^{r(-1/a+ik)}}{-1/a+ik}\Bigg|_0^{\infty}-\int_0^{\infty}\frac{e^{r(-1/a+ik)}}{-1/a+ik}dr-r\frac{e^{r(-1/a-ik)}}{-1/a-ik}\Bigg|_0^{\infty}+\int_0^{\infty}\frac{e^{r(-1/a-ik)}}{-1/a-ik}dr\right)\\
&= -\frac{1}{\sqrt{2\pi}\,ik}\left[\frac{e^{r(-1/a+ik)}}{(-1/a+ik)^2}\Bigg|_0^{\infty}-\frac{e^{r(-1/a-ik)}}{(-1/a-ik)^2}\Bigg|_0^{\infty}\right]\\
&= \frac{1}{\sqrt{2\pi}\,ik}\left[\frac{1}{(-1/a+ik)^2}-\frac{1}{(-1/a-ik)^2}\right] = \frac{1}{\sqrt{2\pi}}\frac{4a^3}{\left(1+k^2a^2\right)^2}
\end{aligned}$$

$$s(\vec{\mathbf{x}},t) = \frac{1}{(2\pi)^2}\int_0^{\infty}k^2dk\int_0^{2\pi}d\phi_k\int_{-1}^{+1}d\mu_k\int_{-\infty}^{\infty}\frac{1}{\sqrt{2\pi}}\frac{4a^3}{\left(1+k^2a^2\right)^2}\frac{1}{v^2k^2-\omega^2}e^{i\vec{\mathbf{k}}\cdot\vec{\mathbf{x}}-i\omega t}d\omega$$

For $t < 0$ we close upward and the result is zero. For $t > 0$ we close downward, enclosing the poles at $\omega = \pm vk.$

$$\begin{aligned}
s(\vec{\mathbf{x}},t) &= \frac{i4a^3}{(2\pi)^{3/2}}\int_0^{\infty}k^2dk\int_0^{2\pi}d\phi_k\int_{-1}^{+1}d\mu_k\frac{1}{(1+k^2a^2)^2}\frac{1}{2vk}\left(e^{ikr\mu-ivkt}-e^{ikr\mu+ivkt}\right)\\
&= \frac{4\pi ia^2}{v(2\pi)^{3/2}ir}\int_0^{\infty}dk\frac{1-k^2a^2}{(1+k^2a^2)^2}\left(e^{ikr-ivkt}-e^{-ikr-ikvt}-e^{ikr+ivkt}+e^{-ikr+ikvt}\right)\\
&= \frac{2}{\sqrt{2\pi}}\frac{a^2}{vr}\int_{-\infty}^{\infty}dk\frac{1-k^2a^2}{(1+k^2a^2)^2}\left(e^{ikr-ivkt}-e^{ikr+ivkt}\right)
\end{aligned}$$

The integrand has two second order poles at $ka = \pm i.$ For $r > vt$ we close upward for the first term, enclosing the pole at $ka = +i.$ Similarly, for $r < vt$ we close downward, and the pole at $ka = -i$ is inside. Then the residues are

$$\lim_{k \to \pm i/a} \frac{d}{dk}(k \mp i/a)^2 \frac{1-k^2a^2}{\left(1+k^2a^2\right)^2} e^{ikr-ivkt}$$

$$= \lim_{k \to \pm i/a} \frac{d}{dk} \frac{1-k^2a^2}{a^4(k \pm i/a)^2} e^{ikr-ivkt}$$

$$= \frac{1}{a^4} \lim_{k \to \pm i/a} \left(\frac{-2ka^2}{(k \pm i/a)^2} - 2\frac{1-k^2a^2}{(k \pm i/a)^3} + \frac{1-k^2a^2}{a^4(k \pm i/a)^2} i(r-vt) \right) e^{ikr-ikvt}$$

$$= \frac{1}{a^4} \left(\frac{\mp 2ia}{(2i/a)^2} - 2\frac{2}{(\pm 2i/a)^3} + \frac{2}{a^4(2i/a)^2} i(r-vt) \right) e^{\mp(r-vt)/a}$$

$$= -\frac{1}{2a^6} i(r-vt) e^{\mp(r-vt)/a}$$

The result is similar for $r > vt$.

Thus for $r < vt$

$$s(\vec{\mathbf{x}},t) = \sqrt{\frac{2}{\pi}} \frac{a^2}{vr}(2\pi i) \frac{1}{2a^6} i\left((r-vt)e^{+(r-vt)/a} - (r+vt)e^{-(r+vt)/a}\right)$$

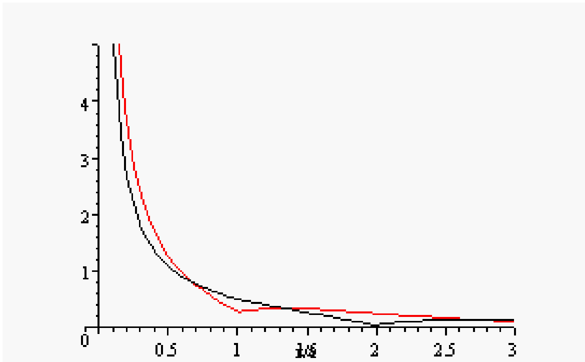
$$= \frac{\sqrt{2\pi}}{a^4 vr}\left((r+vt)e^{-(r+vt)/a} - (r-vt)e^{+(r-vt)/a}\right)$$

$$= \frac{2\pi}{vra^4} e^{-vt/a}\left((r+vt)e^{-r/a} - (r-vt)e^{r/a}\right)$$

whle for $r > vt$:

$$s(\vec{\mathbf{x}},t) = \frac{2\pi}{vra^4} e^{-r/a}\left((r-vt)e^{vt/a} + (r+vt)e^{-vt/a}\right)$$

At $r = vt$ both solutions give the same result.



Red $vt/a = 1$; Black: $vt/a = 2$

(**b**)

$$s(\vec{\mathbf{x}},t) = \frac{1}{(2\pi)^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{F}{v^2k^2 - \omega^2} e^{i\vec{\mathbf{k}}\cdot\vec{\mathbf{x}} - i\omega t} d\vec{\mathbf{k}} d\omega$$

where

$$F\left(\vec{k},\omega\right) = \frac{1}{(2\pi)^2} \int e^{-i\vec{k}\cdot\vec{x} + i\omega t} \delta\left(\vec{x}\right) \delta(t) d^3\vec{x} dt = \frac{1}{(2\pi)^2}$$

$$\begin{aligned} s(\vec{\mathbf{x}},t) &= \frac{1}{(2\pi)^4}\int_0^\infty k^2dk\int_0^{2\pi}d\phi_k\int_{-1}^{+1}d\mu_k\int_{-\infty}^{\infty}\frac{1}{v^2k^2-\omega^2}e^{i\vec{\mathbf{k}}\cdot\vec{\mathbf{x}}-i\omega t}d\omega \\ &= -\frac{i}{(2\pi)^2}\int_0^\infty k^2dk\int_{-1}^{+1}d\mu_k\exp(ikr\mu)\frac{e^{ikvt}-e^{-ikvt}}{2kv} \\ &= -\frac{i}{(2\pi)^2}\int_0^\infty kdk\frac{e^{ikr}-e^{-ikr}}{ikr}\,\frac{e^{ikvt}-e^{-ikvt}}{2v} \\ &= -\frac{1}{2(2\pi)^2vr}\int_0^\infty e^{ik(r+vt)}+e^{-ik(r+vt)}-e^{ik(r-vt)}-e^{-ik(r-vt)}dk \\ &= \frac{1}{4\pi rv}[\delta(r-vt)-\delta(r+vt)] \end{aligned}$$

Since both r and t are positive, we may drop the second delta function, to get

$$s(\vec{\mathbf{x}},t) = \frac{\delta(r-vt)}{4\pi rv}$$

26. At $t = 0$ the distribution of salt in a pipe of fresh water is given by

$$\rho(x,0) = \rho_0\left(\frac{\sin\alpha x}{\alpha x} + \frac{1}{4}\right)$$

Solve the diffusion equation to find the salt distribution at $t > 0$ in terms of the diffusion coefficient D.

The transform of this function is a step function plus a delta-function (see, eg, Problem 3 where $\alpha = 1$).

$$\begin{aligned} \rho\left(\vec{\mathbf{k}},0\right) &= \rho_0\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left(\frac{\sin\alpha x}{\alpha x}+\frac{1}{4}\right)e^{-ikx}dx \\ &= \rho_0\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\left(\frac{\left(e^{i\alpha x}-e^{-i\alpha x}\right)}{2i\alpha x}+\frac{1}{4}\right)e^{-ikx}dx \\ &= \rho_0\frac{1}{\sqrt{2\pi}}\left\{\int_{-\infty}^{\infty}\frac{e^{i(\alpha-k)x}-e^{-i(\alpha+k)x}}{2i\alpha x}dx+\frac{2\pi}{4}\delta(k)\right\} \end{aligned}$$

Each term in the integrand has a simple pole at the origin, although the original function has a removable singularity there. Thus the result of the integration should not depend on the method we choose to avoid the pole. We choose to put the path under the pole.

In the first term we close up if $k-\alpha < 0$ and down if $k-\alpha > 0$. The pole at the origin is displaced slightly downward so it is included in the lower contour. Thus we obtain zero for $k-\alpha < 0$ and $-2\pi i$ for $k-\alpha > 0$. The second term is treated similarly, to obtain

$$\rho\left(\vec{\mathbf{k}},0\right) = \frac{\rho_0}{\alpha}\sqrt{\frac{\pi}{2}}[S(k+\alpha)-S(k-\alpha)] + \frac{\rho_0}{4}\sqrt{2\pi}\,\delta(k)$$

This function is a box that extends from $-\alpha$ to $+\alpha$ and thus

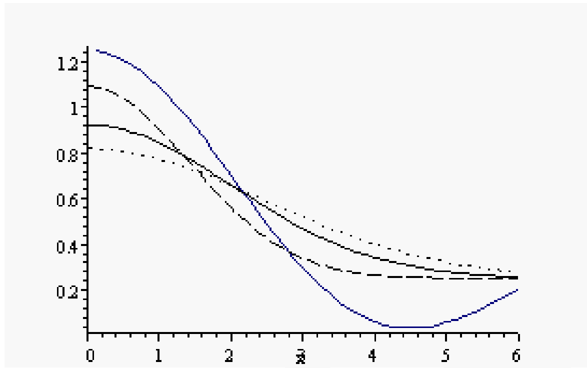
$$\rho\left(\vec{\mathbf{x}},t\right) = \frac{1}{\alpha\sqrt{2\pi}}\int_{-\alpha}^{+\alpha}\rho_0\sqrt{\frac{\pi}{2}}e^{-k^2Dt}e^{ikx}dk + \frac{\rho_0}{4}\int_{-\infty}^{+\infty}\delta(k)e^{ikx}e^{-k^2Dt}dk$$

We integrate the first term by completing the square:

$$-k^2Dt + ikx = -\left(k\sqrt{Dt} - \frac{ix}{2\sqrt{Dt}}\right)^2 - \frac{x^2}{4Dt}$$

and the second using the sifting property, giving:

$$\rho(\vec{\mathbf{x}},t) = \frac{\rho_0}{2\alpha\sqrt{Dt}} e^{-x^2/4Dt}\left(\int_{-\alpha\sqrt{Dt}}^{\alpha\sqrt{Dt}} \exp\left(-\left(k\sqrt{Dt} - \frac{ix}{2\sqrt{Dt}}\right)^2\right) dk\sqrt{Dt}\right) + \frac{\rho_0}{4}$$
$$= = \frac{\rho_0}{\alpha\sqrt{Dt}} e^{-x^2/4Dt} \operatorname{erf}\left(\alpha\sqrt{Dt}\right) + \frac{\rho_0}{4}$$



Distribution of salt at various times. The horizontal axis is $\alpha x.$ Dashed: $\alpha^2 Dt = 1;$ solid, 2; dots, 3. Blue: $t = 0.$

27. Sum the series

$$\sum_{p=0}^{\infty} (-1)^p \frac{2p+1}{x^2 + (2p+1)^2}$$

by taking the Fourier transform of each term, summing the series in the transform space, and then transforming back.

First we transform the function:

$$F_n(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \frac{n}{x^2 + n^2} e^{-ikx} dx$$

The integrand has poles at $x = \pm in$ $(n = 2p + 1).$ For
$k > 0$ we close the contour downward, and we close upward for $k < 0.$ Thus:

$$F_n(k) = \frac{n}{\sqrt{2\pi}} (-2\pi i) \frac{e^{-ik(-in)}}{-2in} \text{ for } k > 0$$
$$= \sqrt{\frac{\pi}{2}} e^{-kn}$$

while for $k < 0$

$$F_n(k) = \frac{n}{\sqrt{2\pi}} (2\pi i) \frac{e^{-ik(in)}}{2in} = \sqrt{\frac{\pi}{2}} e^{kn}$$

Thus

$$F_n(k) = \sqrt{\frac{\pi}{2}} e^{-|k|n}$$

Thus in the transform space, the sum is:

$$F(k) = \sum_{p=0}^{\infty} (-1)^p \sqrt{\frac{\pi}{2}} e^{-|k|(2p+1)} = \sqrt{\frac{\pi}{2}} e^{-|k|} \sum_p (-1)^p \left(e^{-2|k|}\right)^p$$
$$= \sqrt{\frac{\pi}{2}} \frac{e^{-|k|}}{1 + e^{-2|k|}} = \sqrt{\frac{\pi}{2}} \frac{1}{2\cosh|k|} = \sqrt{\frac{\pi}{2}} \frac{1}{2\cosh k}$$

where we recognized the sum as the geometric series (equation 2.43) with $z = -e^{-|k|}$ and $|z| < 1.$

Now we transform back:

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \sqrt{\frac{\pi}{2}} \frac{1}{2\cosh k} e^{ikx} dk$$

The integral was computed in problem 1d.. The result is:

$$f(x) = \frac{\pi}{4\cosh(\pi x/2)}$$

28. Use the derivative rule (7.6) and the symmetry property of Fourier transforms to evaluate the transform of x^n.

We already know that the transform of the function 1 is

$\sqrt{2\pi}\,\delta(k)$, and so by the symmetry property and the derivative rule:

$$\mathcal{F}((ix)^n) = \sqrt{2\pi}\,(-1)^n \frac{d^n}{dk^n}\delta(-k)$$

Thus

$$\mathcal{F}(x^n) = \sqrt{2\pi}\,(i)^n \frac{d^n}{dk^n}\delta(k)$$

We can check our result by inverting:

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \sqrt{2\pi}\,(i)^n \frac{d^n}{dk^n}\delta(k) e^{ikx} dk$$
$$= i^n(-1)^n(ix)^n e^{ikx}\Big|_{k=0}$$

by the sifting property of derivatives (equation 6.7). Thus

$$f(x) = i^{2n}(-1)^n x^n = x^n$$

as required.

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Chapter 8: Sturm-Liouville Theory

1. Find the eigenfunctions for the Helmoltz equation

$$\frac{d^2y}{dx^2} + k^2y = 0$$

subject to the boundary conditions

$$y = 0$$

at $x = 0$ and

$$y' = 0$$

at $x = L$.

The functions that solve the equation are

$$y_1 = \sin kx$$

and

$$y_2 = \cos kx$$

To satisfy the first boundary condition we must choose the sine. Then to satisfy the second we have:

$$kL = \left(\frac{2n+1}{2}\right)\pi$$

so the eigenvalues are:

$$k = \left(\frac{2n+1}{2}\right)\frac{\pi}{L}$$

and the eigenfunctions are:

$$y_n = \sin\left(\left(\frac{2n+1}{2}\right)\frac{\pi x}{L}\right)$$

2. Find the eigenfunctions for the Helmoltz equation

$$\frac{d^2y}{dx^2} + k^2y = 0$$

subject to the boundary conditions

$$ay + by' = 0$$

at $x = 0$ and

$$\alpha y + \beta y' = 0$$

at $x = L$.

The general solution of the differential equation is:

$$y = A\sin kx + B\cos kx$$

Now we apply the boundary conditions:

$$y' = kA\cos kx - kB\sin kx$$

So at $x = 0$:

$$aB + bkA = 0$$

while at $x = L$

$$\alpha(A\sin kL + B\cos kL) + \beta k(A\cos kL - B\sin kL) = 0$$
$$B(\alpha\cos kL - \beta k\sin kL) + A(\alpha\sin kL + \beta k\cos kL) = 0$$

The two equations for A and B have a non-zero solution only if the determinant of the coefficients is zero:

$$\begin{vmatrix} a & bk \\ \alpha\cos kL - \beta k\sin kL & \alpha\sin kL + \beta k\cos kL \end{vmatrix} = 0$$

or

$$a(\alpha\sin kL + \beta k\cos kL) - bk(\alpha\cos kL - \beta k\sin kL) = 0$$
$$\sin kL\left(a\alpha + b\beta k^2\right) + \cos kL(a\beta k - b\alpha k) = 0$$

or

$$\tan kL = \frac{k(b\alpha - a\beta)}{a\alpha + b\beta k^2}$$

This equation gives the set of eigenvalues. Then:

$$B = -\frac{b}{a}kA$$

and the eigenfunctions are:

$$y = A\left(\sin kx - \frac{b}{a}k\cos kx\right)$$

If $a = \alpha$ and $b = \beta$, then we get:

$$\tan kL = \frac{k(b\alpha - a\beta)}{a\alpha + b\beta k^2} = 0 \Rightarrow kL = n\pi$$

The eigenfunctions are:

$$y = A\left(\sin\frac{n\pi x}{L} - \frac{b}{a}\frac{n\pi}{L}\cos\frac{n\pi x}{L}\right)$$

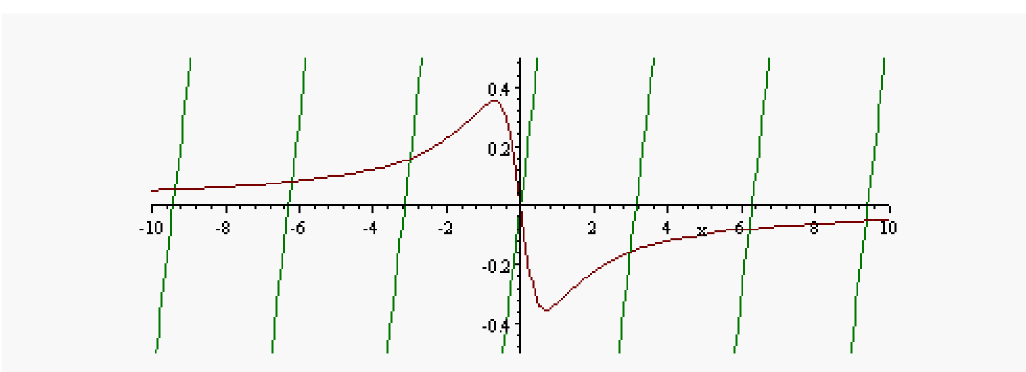
Check: at $x = L$

$$ay + by' = A\left(-b\frac{n\pi}{L}(-1)^n + b\frac{n\pi}{L}(-1)^n\right) = 0$$

as required.

Now let $aL = b$ and $\beta = 2\alpha L$. Then:

$$\tan kL = \frac{k(b\alpha - a\beta)}{a\alpha + b\beta k^2} = \frac{kL(a\alpha - a2\alpha)}{a\alpha + a2\alpha k^2L^2} = \frac{-kL}{1 + 2k^2L^2}$$



$\tan x = -\frac{x}{1+2x^2}$, Solutions are: $x = \pm 2.9843, \pm 6.2038$ etc.

3. The displacement of a square, vibrating membrane of side L satisfies the two-dimensional Helmholtz equation

$$\frac{\partial^2 s}{\partial x^2} + \frac{\partial^2 s}{\partial y^2} + k^2 s = 0$$

where $k = \omega/v$, ω is the frequency and v is the speed of waves on the membrane. Suppose the membrane is fixed at its edges at $x = 0, L$ and $y = 0, L.$ Separate variables and solve for the eigenfunctions $s(x,y).$ Show that the system exhibits degeneracy, that is, there is more than one eigenfunction corresponding to a given eigenvalue $k^2.$ In the particular, show that there are two eigenfunctions s_1 and s_2 that correspond to the eigenvalue $k^2 = 5\pi^2/L^2$. What symmetry of the physical system causes this degeneracy? (*Hints:* (a) where are the nodal lines for the two modes? (b) what happens if one side of the membrane is slightly shorter, equal to $L - \varepsilon$?) Any linear combination of the two eigenfunctions is also a solution. Find some of the nodal lines for combinations of the modes, eg $s_1 + s_2.$ How do these modes reflect the symmetry of the system? Can you find an eigenvalue that has three-fold degeneracy? If so, what do those modes look like?

Separating variables, we have $s = XY$ where:

$$\frac{1}{X}\frac{\partial^2 X}{\partial x^2} + \frac{1}{Y}\frac{\partial^2 Y}{\partial y^2} + k^2 = 0$$

The separated equations are

$$X'' = -\alpha^2 X \text{ and } Y'' = -\left(k^2 - \alpha^2\right) Y$$

To satisfy the boundary conditions we need the solution $X = \sin \alpha x$ and the eigenvalue $\alpha = m\pi/L.$ Similarly $Y = \sin(n\pi y/L)$ and thus $k^2 = \left(m^2 + n^2\right)\pi^2/L^2$ The eigenfunctions have the form

$$s_{mn} = \sin\frac{n\pi x}{L}\sin\frac{m\pi y}{L}$$

Thus the two eigenvalues s_{mn} and s_{nm} have the same eigenvalue. This is a reflection of the fact that the system may be rotated by $\pi/2$ without change. This rotation changes s_{mn} to $-s_{nm}.$ Changing the length of one of the sides destroys the symmtery and removes the degeneracy. The nodal lines for the mode s_{mn} are at $y = pL/m,$ $0 < p < m$ and at $x = qL/n,\ 0 < q < n.$ Rotation by $\pi/2$ sends one set of lines to the other. The linear combinations look like:

$$s = s_{mn} + Cs_{nm}$$

With $m = 2,\ n = 1$ we get

$$s = \sin\frac{\pi x}{L}\sin\frac{2\pi y}{L} + C\sin\frac{2\pi x}{L}\sin\frac{\pi y}{L}$$

With $C = 1,$ the nodal lines are given by

$$\sin\frac{\pi x}{L}\sin\frac{2\pi y}{L} + \sin\frac{2\pi x}{L}\sin\frac{\pi y}{L} = 0$$
$$2\sin\frac{\pi x}{L}\sin\frac{\pi y}{L}\left(\cos\frac{\pi y}{L} + \cos\frac{\pi x}{L}\right) = 0$$

or

$$\cos\frac{\pi y}{L} = -\cos\frac{\pi x}{L}$$

that is

$$\frac{\pi y}{L} = \pm\frac{\pi x}{L} \pm \pi$$
$$y = \pm x \pm L$$

The solution that we need, with $0 \le x, y \le L$ is

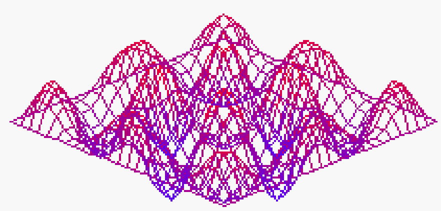
$$y = L - x$$

a diagonal line. The square has reflection symmetry about this line.

If we can find a number k^2 that has more than one set of values of m, n that satisfy the relation $k^2 = \left(m^2 + n^2\right)\pi^2/L^2$, we will have a greater level of degeneracy. For example, $50 = 5^2 + 5^2 = 7^2 + 1$. Thus the eigenfunctions $g_{5,5}$, $g_{7,1}$ and $g_{1,7}$ all have the same eigenvalue- a three-fold degeneracy. The combination

$$\sin\frac{5\pi x}{L}\sin\frac{5\pi y}{L} + \sin\frac{7\pi x}{L}\sin\frac{\pi y}{L} + \sin\frac{\pi x}{L}\sin\frac{7\pi y}{L}$$

looks like this:



These functions also share the rotational symmetry of the square.

4 A set of eigenfunctions $y_n(x)$ satisifies the Sturm-Liouville equation (8.1) with boundary conditions (8.2) The function $g \equiv 0$. Show that the derivatives $u_n(x) = y'_n(x)$ are also orthogonal functions. Determine the weighting function $w(x)$ for these functions. What boundary conditions are required for orthogonality?

The differential equation is:

$$\frac{d}{dx}\left(f\frac{dy}{dx}\right) + \lambda w y = 0$$

Multiply by y_n/λ_m:

$$\frac{y_n}{\lambda_m}\frac{d}{dx}\left(fy'_m\right) + w y_m y_n = 0$$

Now, as we did before, multiply the equation for y_n by y_m/λ_n, subtract and integrate:

$$\int_a^b\left(\frac{y_n}{\lambda_m}\frac{d}{dx}\left(fy'_m\right) + w y_m y_n - \left[\frac{y_m}{\lambda_n}\frac{d}{dx}\left(fy'_n\right) + w y_n y_m\right]\right) = 0$$

Integrate by parts:

$$f\left(\frac{y_n}{\lambda_m}y'_m - \frac{y_m}{\lambda_n}y'_n\right)\Big|_a^b - \left(\frac{1}{\lambda_m} - \frac{1}{\lambda_n}\right)\int_a^b fy'_n y'_m d\mu = 0$$

Thus the integral

$$\int_a^b f y_n' y_m' d\mu = 0$$

unless $n = m$, provided that:

$$f\left(\frac{y_n}{\lambda_m} y_m' - \frac{y_m}{\lambda_n} y_n'\right)\Big|_a^b = 0$$

This will be the case if, for example,

(a) $f(a) = f(b) = 0$, as in the case of Legendre functions.

(b) $y(a) = y(b) = 0$

(c) $y(a) = 0$ and $y'(b) = 0$

(d) $y'(a) = y'(b) = 0$

(e) $y'(a) = y(b) = 0$

etc.

5. Use the recursion relations to show that the derivatives $P_l'(\mu)$ of the Legendre polynomials are orthogonal on the range (-1,1) with weighting function $(1 - \mu^2)$, in agreement with the results of problem 4.

Using the ladder relations, we have

$$P_l' = \frac{l}{1 - \mu^2}(P_{l-1} - \mu P_l)$$

Thus:

$$\int_{-1}^{1}(1 - \mu^2)P_l' P_m' d\mu = lm \int_{-1}^{+1}(P_{l-1} - \mu P_l)P_m' d\mu$$

Integrating by parts on the right hand side:

$$\int_{-1}^{+1} P_{l-1} P_m' d\mu = P_{l-1}P_m\Big|_{-1}^{+1} - \int P_{l-1}' P_m d\mu$$

and

$$\begin{aligned}\int_{-1}^{+1} \mu P_l P_m' d\mu &= \mu P_l P_m\Big|_{-1}^{+1} - \int\left(P_l + \mu P_l'\right)P_m d\mu \\ &= 1 + (-1)^{l+m} - \int\left(P_{l-1}' + (l+1)P_l\right)P_m d\mu\end{aligned}$$

Thus the right hand side is:

$$\begin{aligned}1 - (-1)^{l+m+1} - \int P_{l-1}' P_m d\mu - \left\{1 + (-1)^{l+m} - \int\left(P_{l-1}' + (l+1)P_l\right)P_m d\mu\right\} &= (l+1)\int P_l P_m d\mu \\ &= \frac{2(l+1)}{2l+1}\delta_{lm}\end{aligned}$$

which demonstrates that the P_l' are orthogonal with weighting function $(1 - \mu^2)$.

6. To obtain Fourier-Legendre series we often need to evaluate integrals of the form:

$$I_l^m = \int_0^1 \mu^n P_l(\mu) d\mu$$

(a) Start by evaluating I_l^0 and I_0^m.

$$I_l^0 = \int_0^1 \mu^0 P_l(\mu)d\mu$$

This integral was evaluated in the text (Example 8.2). We found:

$$I_l^0 = -\frac{P_{l+1}(0)}{l} = -\frac{(-1)^{\frac{l-1}{2}}(l+1)!}{l[(l+1)!!]^2} = \frac{(-1)^{\frac{l-1}{2}}(l+1)(l-1)!}{[(l+1)!!]^2}$$
$$= \frac{(-1)^{\frac{l-1}{2}}(l-1)!}{(l+1)!!(l-1)!!} = (-1)^{\frac{l-1}{2}}\frac{(l-2)!!}{(l+1)!!}$$

for l odd, and zero for l even (except $l = 0$). $I_0^0 = 1.$

We do the next integral by parts, using the recursion relation (8.39):

$$I_l^1 = \int_0^1 \mu P_l(\mu)d\mu = \mu\left(\frac{P_{l+1} - P_{l-1}}{2l+1}\right)\Bigg|_0^1 - \int_0^1 \frac{P_{l+1} - P_{l-1}}{2l+1}d\mu$$

The integrated term is zero, and we can use our first result to do the remaining integrals. The result is zero if l is odd, and for l even we get:

$$I_l^1 = \frac{1}{2l+1}\left(I_{l-1}^0 - I_{l+1}^0\right)$$
$$= \frac{1}{2l+1}\frac{(-1)^{(l-2)/2}}{l!!}\left((l-3)!! - \frac{(-1)(l-1)!!}{(l+2)}\right)$$
$$= \frac{(-1)^{(l-2)/2}}{2l+1}\frac{(l-3)!!}{(l+2)!!}(l+2+l-1)$$
$$= (-1)^{(l-2)/2}\frac{(l-3)!!}{(l+2)!!}$$

.Next:

$$I_0^n = \int_0^1 \mu^n P_0(\mu)d\mu = \int_0^1 \mu^n d\mu = \frac{\mu^{n+1}}{n+1}\Bigg|_0^1 = \frac{1}{n+1}$$

and

$$I_1^n = \int_0^1 \mu^n P_1(\mu)d\mu = \int_0^1 \mu^{n+1}d\mu = \frac{\mu^{n+2}}{n+2}\Bigg|_0^1 = \frac{1}{n+2}$$

(b) Now working from equation 8.37 we get:

$$lI_l^n = \int_0^1 \mu^{n+1}P_l'd\mu - \int_0^1 \mu^n P_{l-1}'d\mu$$
$$= \mu^{n+1}P_l\Big|_0^1 - \int_0^1 (n+1)\mu^n P_l d\mu - \mu^n P_{l-1}\Big|_0^1 + \int_0^1 n\mu^{n-1}P_{l-1}d\mu$$
$$= 1 - (n+1)I_l^n - 1 + nI_{l-1}^{n-1}$$

So:

$$I_l^n = \frac{n}{n+l+1}I_{l-1}^{n-1}$$

(c) Use these results to "step down" until you can use your results from (a) to obtain an explicit expression for I_l^n.

First consider the case $n \geq l.$ We step down using the result above to get:

$$\begin{aligned} I_l^n &= \frac{n}{n+l+1} I_{l-1}^{n-1} = \frac{n(n-1)}{(n+l+1)(n+l-1)} I_{l-2}^{n-2} \\ &= \frac{n(n-1)\cdots(n-l+1)}{(n+l+1)(n+l-1)\cdots(n-l+3)} I_0^{n-l} \\ &= \frac{n(n-1)\cdots(n-l+1)}{(n+l+1)(n+l-1)\cdots(n-l+3)} \left(\frac{1}{n-l+1}\right) \\ &= \frac{n(n-1)\cdots(n-l+2)}{(n+l+1)(n+l-1)\cdots(n-l+3)} \\ &= \frac{n!}{(n-l+1)!} \frac{(n-l+1)!!}{(n+l+1)!!} = \frac{n!}{(n-l)!!(n+l+1)!!} \end{aligned}$$

Let's check this against our previous result for I_1^n:

$$I_1^n = \frac{n!}{(n)!} \frac{n!!}{(n+2)!!} = \frac{1}{n+2}$$

which checks OK.

Now if $n < l,$ we get:

$$I_l^n = \frac{n(n-1)\cdots 1}{(n+l+1)(n+l-1)\cdots(l-n+3)} I_{l-n}^0$$

which is zero if $l-n$ is even and for $l-n$ odd we get:

$$\begin{aligned} I_l^n &= \frac{n!}{(l+n+1)(l+n-1)\cdots(l-n+3)} (-1)^{\frac{l-n-1}{2}} \frac{(l-n-2)!!}{(l-n+1)!!} \\ &= (-1)^{\frac{l-n-1}{2}} n! \frac{(l-n-2)!!}{(n+l+1)!!} \end{aligned}$$

Check against the result for $I_l^1.$ We got zero for l odd (i.e. $l-1$ even, as required.)

$$I_l^1 = (-1)^{\frac{l-2}{2}} \frac{(l-3)!!}{(l+2)!!}$$

which checks with the result calculated in part (a).

If $l = n,$ Then

$$I_l^n = \frac{n(n-1)\cdots 1}{(2n+1)(2n-1)\cdots(3)} I_0^0 = \frac{n!}{(2n+1)!!}$$

7. We have already verified in the text that the formula gives the correct normalization for $P_1(x)$ and $P_2(x).$ Then:

$$\begin{aligned} \frac{d^l}{dx^l}(x-1)^l(x+1)^l &= (x-1)^l \frac{d^l}{dx^l}(x+1)^l + l\frac{d}{dx}(x-1)^l \frac{d^{l-1}}{dx^{l-1}}(x+1)^l + \cdots \left[\frac{d^l}{dx^l}(x-1)^l\right](x+1)^l \\ &= (x-1)^l \frac{d^l}{dx^l}(x+1)^l + l^2(x-1)^{l-1} \frac{d^{l-1}}{dx^{l-1}}(x+1)^l + \cdots + l!(x+1)^l \end{aligned}$$

Now evaluate this at $x = 1.$ All terms except the last are zero, and thus:

$$\left.\frac{d^l}{dx^l}\left(x^2-1\right)^l\right|_{x=1} = l!2^l$$

and thus

$$\frac{1}{2^l l!}\frac{d^l}{dx^l}\left(x^2-1\right)^l\bigg|_{x=1} = 1$$

as required.

8. Evaluate the integral

$$\int_{-1}^{+1}\frac{P_l(x)}{\sqrt{1-x^2}}dx$$

and hence obtain a Fourier-Legendre series for the function $1/\sqrt{1-x^2}$.

Start with $l = 0$ and $l = 1$. Then:

$$I_0 = \int_{-1}^{+1}\frac{1}{\sqrt{1-x^2}}dx$$

Making the change of variable $x = \sin\theta,\ dx = \cos\theta d\theta$

$$\int\frac{dx}{\sqrt{1-x^2}} = \int\frac{\cos\theta d\theta}{\cos\theta} = \int d\theta = \theta = \sin^{-1}(x)$$

and so

$$I_0 = \sin^{-1}(x)\Big|_{-1}^{+1} = 2\sin^{-1}1 = \pi$$

and

$$I_1 = \int_{-1}^{+1}\frac{x}{\sqrt{1-x^2}}dx = 0$$

Indeed the result is zero for all odd l because in that case the integrand is odd.

Next use the ladder relation:

$$I_l = \int_{-1}^{+1}\frac{P_l(x)}{\sqrt{1-x^2}}dx = \int_{-1}^{+1}\frac{xP_{l-1}(x) - \frac{(1-x^2)}{l}P'_{l-1}}{\sqrt{1-x^2}}dx$$

and integrate by parts. The first term is:

$$-P_{l-1}\sqrt{1-x^2}\,\Big|_{-1}^{+1} + \int_{-1}^{+1}P'_{l-1}\sqrt{1-x^2}\,dx = 0 + \int_{-1}^{+1}P'_{l-1}\sqrt{1-x^2}\,dx$$

Thus

$$I_l = \left(1-\frac{1}{l}\right)\int_{-1}^{+1}P'_{l-1}\sqrt{1-x^2}\,dx = \frac{l-1}{l}\int_{-1}^{+1}P'_{l-1}\sqrt{1-x^2}\,dx$$

and using the ladder operator again, we get

$$I_l = \frac{(l-1)^2}{l}\int_{-1}^{+1}\left(\frac{P_{l-2}-xP_{l-1}}{1-x^2}\right)\sqrt{1-x^2}\,dx$$
$$= \frac{(l-1)^2}{l}\left[I_{l-2} - \int_{-1}^{+1}\frac{xP_{l-1}(x)}{\sqrt{1-x^2}}dx\right]$$

In the second term we use the pure recursion relation:

$$\int_{-1}^{+1}\frac{xP_{l-1}(x)}{\sqrt{1-x^2}}dx = \frac{1}{2(l-1)+1}\int_{-1}^{+1}\frac{(l-1)P_{l-2}(x)+lP_l}{\sqrt{1-x^2}}dx$$

and thus

$$I_l = \frac{(l-1)^2}{l} I_{l-2}\left(1 - \frac{l-1}{2l-1}\right) - \frac{(l-1)^2}{l}\frac{l}{2l-1} I_l$$

Rearranging, we get:

$$\begin{aligned} I_l\left((2l-1) + (l-1)^2\right) &= (l-1)^2 I_{l-2} \\ I_l &= \left(\frac{l-1}{l}\right)^2 I_{l-2} \end{aligned}$$

Or

$$I_{2n} = \left(\frac{2n-1}{2n}\right)^2 I_{2(n-1)}$$

Stepping down, we get:

$$\begin{aligned} I_{2n} &= \left(\frac{2n-1}{2n}\right)^2 \left(\frac{2n-3}{2(n-1)}\right)^2 I_{2(n-2)} \\ &= \left(\frac{(2n-1)!!}{2^n n!}\right)^2 I_0 = \left(\frac{(2n-1)!!}{2^n n!}\right)^2 \pi \\ &= \left(\frac{(2n-1)!!}{(2n)!!}\right)^2 \pi \end{aligned}$$

Thus for the series we want:

$$\frac{1}{\sqrt{1-x^2}} = \sum a_n P_n(x)$$

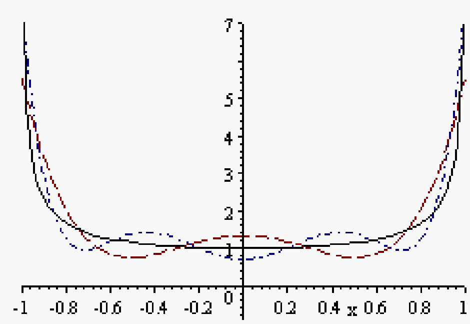
we find

$$a_n = \frac{2n+1}{2}\int_{-1}^{+1} \frac{P_n(x)}{\sqrt{1-x^2}} dx = \frac{2n+1}{2} I_n$$

and thus

$$\frac{1}{\sqrt{1-x^2}} = \frac{\pi}{2}\sum_n (4n+1)\left(\frac{(2n-1)!!}{2^n n!}\right)^2 P_{2n}(x)$$

We can verify the result by plotting the first few terms along with the function $1/\sqrt{1-x^2}$ (black line). .



Red dashed line, 3 terms. Blue dot-dash line, 4 terms.

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9. Write Laplaces's equation in oblate spheroidal coordinates (cf Chapter 2 problem 13), separate variables, and hence show that the solution requires Legendre functions in both the coordinates u and v. Argue that the solution exterior to an oblate shperoidal boundary requires the use of the Legendre function of the second kind, Q.

Oblate spheroidal coordinates are defined by:

$$\rho + iz = c\cosh(u + iv) = c\cosh u\cos v + ic\sinh u\sin v$$

We want to find the shape of the constant u and constant v surfaces. First eliminate v :

$$\cos v = \frac{\rho}{c\cosh u} \quad \text{and} \quad \sin v = \frac{z}{c\sinh u}$$

Thus

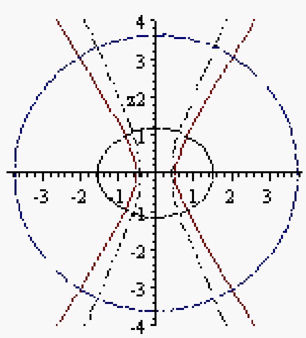
$$1 = \cos^2 v + \sin^2 v = \left(\frac{\rho}{c\cosh u}\right)^2 + \left(\frac{z}{c\sinh u}\right)^2$$

Thus the surfaces of constant u are ellipsoids with semi-major axis $c\cosh u$ and semi-minor axis $c\sinh u$. Similarly, by solving for $\cosh u$ and $\sinh u$, squaring and subtracting, we find:

$$1 = \cosh^2 u - \sinh^2 u = \left(\frac{\rho}{c\cos v}\right)^2 - \left(\frac{z}{c\sin v}\right)^2$$

so the constant v surfaces are hyperboloids.

The z −axis is described by $\cos v = 0$, i.e. $v = \pm\frac{\pi}{2}$. Then $z = \pm c\sinh u$ which ranges from $-\infty$ to $+\infty$ as u does. The $z = 0$ plane is described by $u = 0$ or $v = 0$ or $v = \pi$. These choices correspond to different regions for ρ. But ρ is always positive, so we don't need $v = \pi$. Here $-\pi/2 \le v \le +\pi/2$ and $0 \le u \le \infty$.



Next we look at the line element:

Note:

$$d\rho = c\sinh u\cos v\,du - c\cosh u\sin v\,dv$$

and

$$dz = c\cosh u\sin v\,du + c\sinh u\cos v\,dv$$

$$
\begin{aligned}
ds^2 &= dx^2 + dy^2 + dz^2 = d\rho^2 + \rho^2 d\phi^2 + dz^2 \\
&= (c\sinh u\cos v\,du - c\cosh u\sin v\,dv)^2 + (c\cosh u\sin v\,du + c\sinh u\cos v\,dv)^2 \\
&\quad + (c\cosh u\cos v)^2 dw^2 \\
&= c^2\left(\begin{array}{c}\sinh^2 u\cos^2 v\,du^2 + \cosh^2 u\sin^2 v\,dv^2 + \cosh^2 u\sin^2 v\,du^2 \\ + \sinh^2 u\cos^2 v\,dv^2 + \cosh^2 u\cos^2 v\,dw^2\end{array}\right) \\
&= c^2\left(\left(du^2 + dv^2\right)\left(\cosh^2 u\sin^2 v + \sinh^2 u\cos^2 v\right) + \cosh^2 u\cos^2 v\,dw^2\right) \\
&= c^2\left(\left(du^2 + dv^2\right)\left(\cosh^2 u\left(1 - \cos^2 v\right) + \left(\cosh^2 u - 1\right)\cos^2 v\right) + \cosh^2 u\cos^2 v\,dw^2\right) \\
&= c^2\left(\left(du^2 + dv^2\right)\left(\cosh^2 u - \cos^2 v\right) + \cosh^2 u\cos^2 v\,dw^2\right) \\
&= h_1^2 du^2 + h_2^2 dv^2 + h_3^2 dw^2
\end{aligned}
$$

Thus

$$h_1 = h_2 = c\sqrt{\cosh^2 u - \cos^2 v}$$

and

$$h_3 = c\cosh u\cos v$$

Now we are ready to write the ∇^2 operator:

$$
\begin{aligned}
\nabla^2\Phi &= \frac{1}{h_1 h_2 h_3}\left\{\frac{\partial}{\partial u}\left(\frac{h_2 h_3}{h_1}\frac{\partial\Phi}{\partial u}\right) + \frac{\partial}{\partial v}\left(\frac{h_3 h_1}{h_2}\frac{\partial\Phi}{\partial v}\right) + \frac{\partial}{\partial w}\left(\frac{h_1 h_2}{h_3}\frac{\partial\Phi}{\partial w}\right)\right\} \\
&= \frac{1}{\left(\cosh^2 u - \cos^2 v\right)\cosh u\cos v} \times \\
&\quad \left\{\frac{\partial}{\partial u}\left(\cosh u\cos v\frac{\partial\Phi}{\partial u}\right) + \frac{\partial}{\partial v}\left(\cosh u\cos v\frac{\partial\Phi}{\partial v}\right) + \frac{\partial}{\partial w}\left(\frac{\cosh^2 u - \cos^2 v}{\cosh u\cos v}\frac{\partial\Phi}{\partial w}\right)\right\} \\
&= \frac{1}{\left(\cosh^2 u - \cos^2 v\right)}\left\{\frac{1}{\cosh u}\frac{\partial}{\partial u}\left(\cosh u\frac{\partial\Phi}{\partial u}\right) + \frac{1}{\cos v}\frac{\partial}{\partial v}\left(\cos v\frac{\partial\Phi}{\partial v}\right)\right\} \\
&\quad + \frac{1}{(\cosh u\cos v)^2}\frac{\partial^2\Phi}{\partial w^2}
\end{aligned}
$$

Laplace's equation is $\nabla^2\Phi = 0.$ Next we separate variables:

$$\Phi = U(u)V(v)W(w)$$

$$
\begin{aligned}
\nabla^2\Phi = 0 &= \frac{W}{\left(\cosh^2 u - \cos^2 v\right)}\left\{\frac{V}{\cosh u}\frac{\partial}{\partial u}\left(\cosh u\frac{\partial U}{\partial u}\right) + \frac{U}{\cos v}\frac{\partial}{\partial v}\left(\cos v\frac{\partial V}{\partial v}\right)\right\} \\
&\quad + \frac{UV}{(\cosh u\cos v)^2}\frac{\partial^2 W}{\partial w^2}
\end{aligned}
$$

Now divide through by $UVW,$ and multiply by $(\cosh u\cos v)^2$:

$$0 = \frac{(\cosh u\cos v)^2}{\left(\cosh^2 u - \cos^2 v\right)}\left\{\frac{1}{U\cosh u}\frac{\partial}{\partial u}\left(\cosh u\frac{\partial U}{\partial u}\right) + \frac{1}{V\cos v}\frac{\partial}{\partial v}\left(\cos v\frac{\partial V}{\partial v}\right)\right\} + \frac{\partial^2 W}{W\partial w^2}$$

The final term has separated out: it is a function of w only while the other two terms are functions of u and v only. Since w is our old friend $\phi,$ we choose separation constant $-m^2$ so that the solutions are

$$W = e^{\pm imw}$$

Then:

$$0 = \frac{(\cosh u \cos v)^2}{\left(\cosh^2 u - \cos^2 v\right)} \left\{ \frac{1}{U\cosh u} \frac{\partial}{\partial u}\left(\cosh u \frac{\partial U}{\partial u}\right) + \frac{1}{V\cos v}\frac{\partial}{\partial v}\left(\cos v \frac{\partial V}{\partial v}\right)\right\} - m^2$$

Now multiply through by $\left(\cosh^2 u - \cos^2 v\right)/(\cosh u \cos v)^2$

$$0 = \frac{1}{U\cosh u} \frac{\partial}{\partial u}\left(\cosh u \frac{\partial U}{\partial u}\right) + \frac{1}{V\cos v}\frac{\partial}{\partial v}\left(\cos v \frac{\partial V}{\partial v}\right) - m^2 \frac{\left(\cosh^2 u - \cos^2 v\right)}{(\cosh u \cos v)^2}$$
$$= \frac{1}{U\cosh u} \frac{\partial}{\partial u}\left(\cosh u \frac{\partial U}{\partial u}\right) + \frac{m^2}{\cosh^2 u} + \frac{1}{V\cos v}\frac{\partial}{\partial v}\left(\cos v \frac{\partial V}{\partial v}\right) - \frac{m^2}{\cos^2 v}$$

Now we have separated again. We can recognize each of the pieces as Legendre's equation, so the separation constant is $l(l+1)$:

$$\frac{1}{V\cos v}\frac{\partial}{\partial v}\left(\cos v \frac{\partial V}{\partial v}\right) - \frac{m^2}{\cos^2 v} + l(l+1) = 0$$

and

$$\frac{1}{U\cosh u} \frac{\partial}{\partial u}\left(\cosh u \frac{\partial U}{\partial u}\right) + \frac{m^2}{\cosh^2 u} - l(l+1) = 0$$

In the V equation, let $\mu = \sin v,\ d\mu = \cos v\, dv$

$$\frac{d}{d\mu}\left[\left(1-\mu^2\right)V'\right] + l(l+1)V - \frac{m^2}{1-\mu^2}V = 0$$

The solutions are:

$$V = P_l^m(\mu) = P_l^m(\sin v),\ \text{or } Q_l^m(\sin v)$$

In the U equation, let $\xi = i\sinh u,\ d\xi = i\cosh u\, du$, $\cosh^2 u - \sinh^2 u = 1$, so $\cosh^2 u = 1 - \xi^2$

$$-\frac{d}{d\xi}\left[\left(1-\xi^2\right)U'\right] - l(l+1)U + \frac{m^2}{1-\xi^2} = 0$$

with solution

$$U = P_l^m(i\sinh u),\ Q_l^m(i\sinh u)$$

Thus the eigenfunctions are of the form:

$$e^{\pm imw}\left(P_l^m(\sin v),\ Q_l^m(\sin v)\right)\left(P_l^m(i\sinh u),\ Q_l^m(i\sinh u)\right)$$

We cannot eliminate the Qs here because the argument $i\sinh u$ can become large.

10. Expand the Legendre function $Q_0(x)$ for large values of the argument, and show that your result agrees with the asymptotic form in equation (8.28), modulo a constant.

$$Q_0(x) = \frac{1}{2}\ln\left(\frac{1+x}{1-x}\right) = \frac{1}{2}\ln\left(-\frac{1+1/x}{1-1/x}\right) = \frac{i\pi}{2} + \frac{1}{2}\left(\frac{2}{x}\right) = \frac{i\pi}{2} + \frac{1}{x}$$

Compare

$$\sqrt{\pi}\frac{\Gamma(l+1)}{\Gamma\left(l+\frac{3}{2}\right)}\frac{1}{(2x)^{l+1}} = \sqrt{\pi}\frac{1}{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}\frac{1}{2x} = \frac{2\sqrt{\pi}}{\sqrt{\pi}}\frac{1}{2x} = \frac{1}{x}$$

11. Rewrite the Legendre equation

$$\frac{d}{dx}\left(\left(1-x^2\right)\frac{dQ_l}{dx}\right)+l(l+1)Q_l=0$$

in terms of the variable $u=1/x$ and obtain a solution as a series in u. Show that for large x, $Q_l(x)$ goes to zero as $1/x^{l+1}$. Show that for $l=0$ the solution $Q_0(x)$ may be written as in equation (8.28) but with $x-1$ in the denominator instead of $1-x$.

We use the method of Chapter 3 section 3.3.5.

$$\frac{dQ}{dx}=\frac{dQ}{du}\frac{du}{dx}=-u^2\frac{dQ}{du}$$

Thus the equation becomes:

$$u^2\frac{d}{du}\left[\left(1-\frac{1}{u^2}\right)u^2\frac{dQ_l}{du}\right]+l(l+1)Q_l=0$$

The equation has singular points at $u=0$, and at $u=\pm 1$. We look for a solution of the form

$$Q=\sum a_n u^{n+p}$$

$$\begin{aligned}
u^2\frac{d}{du}\left[\left(u^2-1\right)\sum(n+p)a_n u^{n+p-1}\right]+l(l+1)\sum a_n u^{n+p}&=0\\
u^2\left(\sum(n+p)(n+p+1)a_n u^{n+p}-\sum(n+p)(n+p-1)a_n u^{n+p-2}\right)+l(l+1)\sum a_n u^{n+p}&=0\\
\sum(n+p)(n+p+1)a_n u^{n+p+2}-\sum(n+p)(n+p-1)a_n u^{n+p}+l(l+1)\sum a_n u^{n+p}&=0
\end{aligned}$$

The lowest power is p which gives the indicial equation:

$$-p(p-1)+l(l+1)=0$$

with solutions $p=-l$ and $p=l+1$. Looking at the power u^{m+p} we obtain the recursion relation:

$$(m+p-2)(m+p-1)a_{m-2}-(m+p)(m+p-1)a_m+l(l+1)a_m=0$$

or

$$a_m=\frac{(m+p-2)(m+p-1)a_{m-2}}{(m+p)(m+p-1)-l(l+1)}$$

With $p=l+1$, we have

$$a_m=\frac{(m+l-1)(m+l)a_{m-2}}{(m+l+1)(m+l)-l(l+1)}$$

which is valid for $m>2$, leading to the series:

$$Q_l(x)=\frac{a_0}{x^{l+1}}\sum_n\frac{(2n+l)!}{[(2n+l+1)(2n+l)-l(l+1)][(2n+l-1)(2n+l-2)-l(l+1)]\cdots}x^{-2n}$$

The series with $p=-l$ is not regular at infinity; in fact this is the Legendre polynomial. The recursion relation

$$a_m=\frac{(m-l-2)(m-l-1)a_{m-2}}{(m-l)(m-l-1)-l(l+1)}$$

blows up for $m=l$, so coefficients beyond a_{l-2} cannot be found.

For $l=0$, we obtain

$$Q_0(x) = \frac{a_0}{x}\sum_n \frac{(2n+l)!}{(2n+1)2n(2n-1)(2n-2)\cdots}x^{-2n} = \frac{a_0}{x}\sum_n \frac{1}{x^{2n}}$$
$$= \frac{a_0}{2}\ln\left(\frac{1+1/x}{1-1/x}\right) = \frac{a_0}{2}\ln\left(\frac{x+1}{x-1}\right)$$

as required.

12. In a steady state, the time derivative of the charge density is zero, and so

$$0 = \vec{\nabla}\cdot\vec{\mathbf{j}} = \vec{\nabla}\cdot\left(\sigma\vec{\mathbf{E}}\right)$$

So if the conductivity is uniform, we can pull it through the divergence to get:

$$\sigma\vec{\nabla}\cdot\vec{\mathbf{E}} = -\sigma\nabla^2\Phi = 0$$

and thus Φ satisfies Laplace's equation.

In polar coordinates:

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\Phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\Phi}{\partial\theta^2} = 0$$

Separate variables:

$$\frac{r}{R}\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right) + \frac{1}{W}\frac{\partial^2 W}{\partial\theta^2} = 0$$

As usual, we choose

$$\frac{1}{W}\frac{\partial^2 W}{\partial\theta^2} = -k^2$$

so that

$$W = A\cos k\theta + B\sin k\theta$$

Then

$$\frac{r}{R}\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right) = k^2$$

The solution is a power:

$$R = r^p$$

$$r\frac{\partial}{\partial r}\left(r\frac{\partial r^p}{\partial r}\right) = r\frac{\partial}{\partial r}(pr^p) = p^2 r^p$$

So $p = \pm k.$ Thus the eigenfunctions are:

$$(A\cos k\theta + B\sin k\theta)\left(\alpha r^k + \beta r^{-k}\right)$$

and the solution may be written:

$$\Phi = \sum_k (A_k\cos k\theta + B_k\sin k\theta)\left(r^k + \beta_k r^{-k}\right)$$

First we set $\beta_k = 0$ because we want our solution to be finite at $r = 0.$ (With $k = 0$ we obtain the solution $A_0\ln r.$ We also eliminate this solution because it is not finite at $r = 0.$) Now our boundary conditions at $r = a$ are:

$$j_r = -\sigma\frac{\partial\Phi}{\partial r} = 0$$

except

$$j_r = \frac{I}{a\gamma t} \text{ for } -\frac{\gamma}{2} < \theta < \frac{\gamma}{2}$$

and

$$j_r = -\frac{I}{a\gamma t} \text{ for } \pi - \frac{\gamma}{2} < \theta < \pi + \frac{\gamma}{2}$$

This is a Neumann problem. Inserting the solution for Φ, we have

$$-\sigma \sum_k k(A_k \cos k\theta + B_k \sin k\theta) a^{k-1} = \text{above function of } \theta$$

Now we make use of the orthogonality of the trig functions. Multiply both sides by $\cos n\theta$ and integrate. Only the one cosine term with $k = n$ survives the integration:

$$\begin{aligned} -\sigma A_n \pi n a^{n-1} &= \left(\int_{-\gamma/2}^{\gamma/2} - \int_{\pi-\gamma/2}^{\pi+\gamma/2}\right) \frac{I}{a\gamma t} \cos n\theta d\theta \\ &= \frac{I}{a\gamma t n}\left(\sin n\theta|_{-\gamma/2}^{\gamma/2} - \sin n\theta|_{\pi-\gamma/2}^{\pi+\gamma/2}\right) \\ &= \frac{I}{a\gamma t n}\left(2\sin\frac{n\gamma}{2} - 2(-1)^n \sin\frac{n\gamma}{2}\right) \\ &= \begin{cases} 0 & \text{if } n \text{ is even} \\ \frac{4I}{a\gamma t n}\sin\frac{n\gamma}{2} & \text{if } n \text{ is odd} \end{cases} \end{aligned}$$

while for the sine terms we get:

$$\begin{aligned} -\sigma B_n \pi n a^{n-1} &= \left(\int_{-\gamma/2}^{\gamma/2} - \int_{\pi-\gamma/2}^{\pi+\gamma/2}\right) \frac{I}{a\gamma t} \sin n\theta d\theta \\ &= -\frac{I}{a\gamma t n}\left(\cos n\theta|_{-\gamma/2}^{\gamma/2} - \cos n\theta|_{\pi-\gamma/2}^{\pi+\gamma/2}\right) \\ &= -\frac{I}{a\gamma t n}(0 - 0) = 0 \end{aligned}$$

Thus:

$$A_n = \frac{-4I}{a^n \sigma\gamma t n^2 \pi} \sin\frac{n\gamma}{2}$$

and

$$\Phi = \frac{-4I}{\pi\sigma\gamma t} \sum_{n,\text{ odd}} \frac{\sin n\gamma/2}{n^2}\left(\frac{r}{a}\right)^n \cos n\theta$$

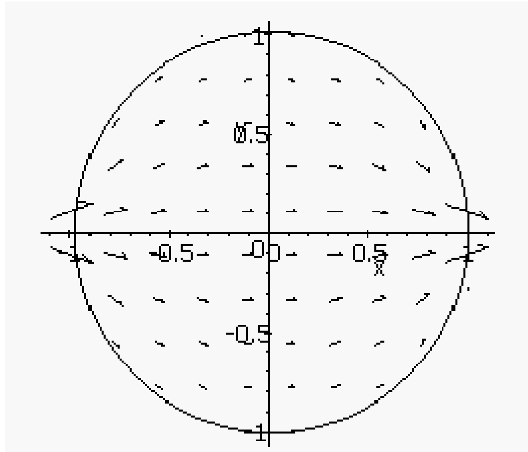
Then

$$\begin{aligned} \vec{\mathbf{j}} &= \sigma \frac{4I}{\pi\sigma\gamma t} \sum_{n=1,\, n \text{ odd}}^{\infty} \frac{\sin n\gamma/2}{n^2}\left[\hat{\mathbf{r}} \frac{\partial}{\partial r}\left(\frac{r}{a}\right)^n \cos n\theta + \frac{\hat{\boldsymbol{\theta}}}{r}\left(\frac{r}{a}\right)^n \frac{\partial}{\partial\theta} \cos n\theta \right. \\ &= \frac{4I}{\pi\gamma t} \sum_{n=1,\, n \text{ odd}}^{\infty} \frac{\sin n\gamma/2}{n^2}\left(n\frac{r^{n-1}}{a^n} \cos n\theta\, \hat{\mathbf{r}} - n\frac{r^{n-1}}{a^n} \sin n\theta\, \hat{\boldsymbol{\theta}}\right) \\ &= \frac{4I}{\pi\gamma t a} \sum_{n=1,\, n \text{ odd}}^{\infty} \frac{\sin n\gamma/2}{n} \frac{r^{n-1}}{a^{n-1}}\left(\cos n\theta\, \hat{\mathbf{r}} - \sin n\theta\, \hat{\boldsymbol{\theta}}\right) \end{aligned}$$

The first term is:

$$\frac{4I}{\pi\gamma t a} \sin\frac{\gamma}{2} \hat{\mathbf{x}}$$

and is constant. At $\theta = \pi/2$, only the $\hat{\theta}$ component is non-zero: at $\theta = 0$ and π, only the $\hat{\mathbf{r}}$ component survives.



13. A solid sphere of radius a is immersed in a vat of fluid at temperature T_0. Heat is conducted into the sphere according to equation 3.14. If the temperature at the boundary is fixed at T_0, and the initial temperature of the sphere is T_1, find the temperature within the sphere as a function of time.

$$D\nabla^2 T = \frac{\partial T}{\partial t}$$

Look for a solution with $T - T_0 = f(t)R(r)$ (We expect no dependence on the angles because the boundary conditions are spherically symmetric.) Then we have

$$D\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right)f = R\frac{\partial f}{\partial t}$$

or

$$D\frac{1}{r^2 R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) = \frac{1}{f}\frac{\partial f}{\partial t} = \text{constant}$$

Let the constant be λ, so that the solution for f is

$$f = e^{\lambda t}$$

Then the r equation is:

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) - \frac{\lambda}{D}r^2 R = 0$$

The secret here is to rewrite the differential operator in the form:

$$\frac{d^2}{dr^2}(rR) = \frac{\lambda}{D}rR$$

We choose $\lambda = -\mu^2 D$ to be negative, and the solution is of the form

$$R = \frac{1}{r}\begin{cases}\sin\mu r\\ \cos\mu r\end{cases}$$

Next we choose the sine solution with eigenvalue $\mu = n\pi/a$ so that $R(a) = 0$. Thus the solution is of the form

$$T = T_0 + \sum_n \frac{A_n}{r}\sin\frac{n\pi r}{a}e^{-(n\pi/a)^2 Dt} \quad r < a$$

Finally, at $t = 0$

$$T_1 - T_0 = \sum_n \frac{A_n}{r} \sin \frac{n\pi r}{a}$$

So

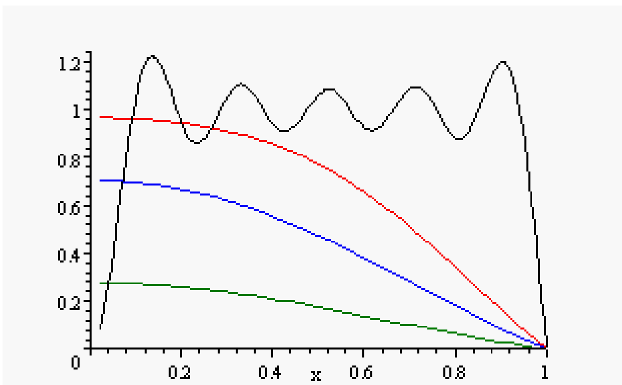
$$\begin{aligned}
A_n &= \frac{2}{a} \int_0^a r(T_1 - T_0) \sin \frac{n\pi r}{a} dr \\
&= \frac{2}{a}(T_1 - T_0)\left(-r \frac{a}{n\pi} \cos \frac{n\pi r}{a} \Big|_0^a + \int_0^a \frac{a}{n\pi} \cos \frac{n\pi r}{a} dr\right) \\
&= \frac{2}{n\pi}(T_1 - T_0)\left(-a(-1)^n + \int_0^a \frac{a}{n\pi} \sin \frac{n\pi r}{a} dr\right) \\
&= \frac{2}{n\pi}(T_1 - T_0)\left(-a(-1)^n + 2\left(\frac{a}{n\pi}\right) \sin \frac{n\pi r}{a} \Big|_0^a\right) \\
&= \frac{2}{n\pi}(T_1 - T_0)(-a(-1)^n + 0)
\end{aligned}$$

So we get:

$$A_n = \frac{2a}{n\pi}(T_0 - T_1)(-1)^n$$

So

$$T = T_0 - 2\frac{(T_1 - T_0)}{\pi} \frac{a}{r} \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin \frac{n\pi r}{a} e^{-(n\pi/a)^2 Dt}$$



The plot shows $(T - T_0)/(T_1 - T_0)DT$ versus r for $Dt/a^2 =$ 1/20 (red), 1/10 (blue), and 1/5(green). The black line shows the original temperature distribution (first ten terms)

14. Use the Cauchy formula together with the Rodrigues formula to write $P_l(\mu)$ as a contour integral in the complex plane. Take the contour to be a circle of radius $\sqrt{x^2 - 1}$ and hence obtain the integral expression

$$P_l(x) = \frac{1}{\pi} \int_0^{\pi} \left[x + \sqrt{x^2 - 1} \cos \phi\right]^l d\phi$$

$$P_l(x) = \frac{1}{2^l l!} \frac{d^n}{dx^n} \left(x^2 - 1\right)^l = \frac{1}{2^l l!} \frac{1}{2\pi i} l! \oint \frac{\left(z^2 - 1\right)^l}{(z - x)^{l+1}} dz$$

where the integral is along a closed curve enclosing the point $z = x$. In particular, choosing the curve to be a circle of radius $\sqrt{x^2 - 1}$ centered at $z = x$, we have

$$
\begin{aligned}
P_l(x) &= \frac{1}{2^{l+1}\pi i}\int_0^{2\pi}\frac{\left[\left(x+\sqrt{x^2-1}\,e^{i\phi}\right)^2-1\right]^l}{\left(\sqrt{x^2-1}\,e^{i\phi}\right)^{l+1}}\sqrt{x^2-1}\,ie^{i\phi}d\phi \\
&= \frac{1}{2^{l+1}\pi}\int_0^{2\pi}\frac{\left[x^2+2x\sqrt{x^2-1}\,e^{i\phi}+\left(x^2-1\right)e^{2i\phi}-1\right]^l}{\left(x^2-1\right)^{l/2}e^{il\phi}}d\phi \\
&= \frac{1}{2^{l+1}\pi}\int_0^{2\pi}\frac{\left[2x\sqrt{x^2-1}\,e^{i\phi}+\left(x^2-1\right)\left(1+e^{2i\phi}\right)\right]^l}{\left(x^2-1\right)^{l/2}e^{il\phi}}d\phi \\
&= \frac{1}{2\pi}\int_0^{2\pi}\left[x+\sqrt{x^2-1}\left(\frac{e^{-i\phi}+e^{i\phi}}{2}\right)\right]^l d\phi \\
&= \frac{1}{\pi}\int_0^{\pi}\left[x+\sqrt{x^2-1}\cos\phi\right]^l d\phi
\end{aligned}
$$

15. Starting from the relations in section 8.3.5, derive the following recursion relations for the Associated Legendre functions:

(**a**)

$$(l-m+1)\sqrt{1-\mu^2}\,P_l^{m-1} = P_{l-1}^m - \mu P_l^m$$

Starting with equation 8.37, we differentiate:

$$l\frac{d}{d\mu}P_l = P_l' + \mu P_l'' - P_{l-1}''$$
$$(l-1)\frac{d}{d\mu}P_l = \mu\frac{d^2}{d\mu^2}P_l - \frac{d^2}{d\mu^2}P_{l-1}$$

Differentiating again

$$(l-1)\frac{d^2}{d\mu^2}P_l = \frac{d^2}{d\mu^2}P_l + \mu\frac{d^3}{d\mu^3}P_l - \frac{d^3}{d\mu^3}P_{l-1}$$
$$(l-2)\frac{d^2}{d\mu^2}P_l = \mu\frac{d^3}{d\mu^3}P_l - \frac{d^3}{d\mu^3}P_{l-1}$$

Continuing, we get

$$(l-m+1)\frac{d^{m-1}}{d\mu^{m-1}}P_l = \mu\frac{d^m}{d\mu^m}P_l - \frac{d^m}{d\mu^m}P_{l-1}$$

Multiply by $(-1)^m\left(1-\mu^2\right)^{m/2}$

$$-(l-m+1)\sqrt{1-\mu^2}\,P_l^{m-1} = \mu P_l^m - P_{l-1}^m$$

which is the first relation.

(**b**)

$$(2l+1)\sqrt{1-\mu^2}\,P_l^{m-1} = P_{l-1}^m - P_{l+1}^m$$

Similarly, starting with the pure recursion relation (8.34)

$$l\frac{d}{d\mu}P_{l-1} - (2l+1)\frac{d}{d\mu}(\mu P_l) + (l+1)\frac{d}{d\mu}P_{l+1} = 0$$
$$l\frac{d}{d\mu}P_{l-1} - (2l+1)\left(\mu\frac{d}{d\mu}P_l + P_l\right) + (l+1)\frac{d}{d\mu}P_{l+1} = 0$$

Do it again:

$$l\frac{d^2}{d\mu^2}P_{l-1} - (2l+1)\left(\frac{d}{d\mu}\left(\mu\frac{d}{d\mu}P_l\right) + \frac{d}{d\mu}P_l\right) + (l+1)\frac{d^2}{d\mu^2}P_{l+1} = 0$$
$$l\frac{d^2}{d\mu^2}P_{l-1} - (2l+1)\left(\mu\frac{d^2}{d\mu^2}P_l + 2\frac{d}{d\mu}P_l\right) + (l+1)\frac{d^2}{d\mu^2}P_{l+1} = 0$$

Continuing

$$l\frac{d^m}{d\mu^m}P_{l-1} - (2l+1)\left(\mu\frac{d^m}{d\mu^m}P_l + m\frac{d^{m-1}}{d\mu^{m-1}}P_l\right) + (l+1)\frac{d^m}{d\mu^m}P_{l+1} = 0$$

Now multiply by $(-1)^m\left(1-\mu^2\right)^{m/2}$

$$lP_{l-1}^m - (2l+1)\mu P_l^m - (2l+1)m(-1)\sqrt{1-\mu^2}\,P_l^{m-1} + (l+1)P_{l+1}^m = 0$$

Using the first relation to eliminate μP_l^m,

$$lP_{l-1}^m - (2l+1)\left(P_{l-1}^m - (l-m+1)\sqrt{1-\mu^2}\,P_l^{m-1}\right) + (2l+1)m\sqrt{1-\mu^2}\,P_l^{m-1} + (l+1)P_{l+1}^m = 0$$
$$[-l-1]P_{l-1}^m + (2l+1)(l+1)\sqrt{1-\mu^2}\,P_l^{m-1} + (l+1)P_{l+1}^m = 0$$

Thus:

$$(2l+1)\sqrt{1-\mu^2}\,P_l^{m-1} = P_{l-1}^m - P_{l+1}^m$$

From these two relations derive the following:

(**c)**

$$(2l+1)\mu P_l^m(\mu) = (l-m+1)P_{l+1}^m + (l+m)P_{l-1}^m$$

We want to eliminate the square root:

$$-(l-m+1)(P_{l-1}^m - P_{l+1}^m) = (2l+1)(\mu P_l^m - P_{l-1}^m)$$

Thus

$$\begin{aligned}(2l+1)\mu P_l^m(\mu) &= (l-m+1)P_{l+1}^m - (l-m+1-2l-1)P_{l-1}^m \\ &= (l-m+1)P_{l+1}^m + (m+l)P_{l-1}^m\end{aligned}$$

QED.

16. Starting from the definition (8.53), obtain the m-raising recursion relation:

$$P_l^{m+1} = -m\frac{\mu}{\sqrt{1-\mu^2}}P_l^m - \sqrt{1-\mu^2}\,\frac{d}{d\mu}P_l^m$$

Solution:

$$\begin{aligned}\frac{d}{d\mu}P_l^m &= (-1)^m\frac{d}{d\mu}(1-\mu^2)^{m/2}\frac{d^m}{d\mu^m}P_l \\ &= (-1)^m\left[-m\mu(1-\mu^2)^{m/2-1}\frac{d^m}{d\mu^m}P_l + (1-\mu^2)^{m/2}\frac{d^{m+1}}{d\mu^{m+1}}P_l\right] \\ &= -\frac{m\mu}{1-\mu^2}P_l^m - \frac{1}{\sqrt{1-\mu^2}}P_l^{m+1}\end{aligned}$$

Thus

$$P_l^{m+1} = -m\frac{\mu}{\sqrt{1-\mu^2}}P_l^m - \sqrt{1-\mu^2}\,\frac{d}{d\mu}P_l^m$$

Combine this result with equation (8.59) to obtain the m –lowering relation

$$(l+m)(l-m+1)P_l^{m-1} = \sqrt{1-\mu^2}\,\frac{d}{d\mu}P_l^m - m\frac{\mu}{\sqrt{1-\mu^2}}P_l^m$$

Use equation 8.59 with $m \to m-1$

$$0 = P_l^{m+1} + 2m\frac{\mu}{\sqrt{1-\mu^2}}P_l^m + [l(l+1) - m(m-1)]P_l^{m-1}$$

Use the previous relation to eliminate P_l^{m+1}:

$$m\frac{\mu}{\sqrt{1-\mu^2}}P_l^m + \sqrt{1-\mu^2}\,\frac{d}{d\mu}P_l^m = 2m\frac{\mu}{\sqrt{1-\mu^2}}P_l^m + [l(l+1) - m(m-1)]P_l^{m-1}$$

Thus

$$(m+l)(l+1-m)P_l^{m-1} = \sqrt{1-\mu^2}\,\frac{d}{d\mu}P_l^m - m\frac{\mu}{\sqrt{1-\mu^2}}P_l^m$$

17. Use the results of problem 15 to show that, for $l+m$ even,

$$P_l^m(0) = (-1)^{(l+m)/2}\frac{(l+m-1)!!}{(l-m)!!}$$

From 15(a)

$$(l-m+1)P_l^{m-1}(0) = P_{l-1}^m(0)$$

and from 15 (c)

$$(l-m+1)P_{l+1}^m(0) = -(l+m)P_{l-1}^m(0)$$

Thus

$$P_{l+1}^m(0) = -\frac{l+m}{l-m+1}P_{l-1}^m(0) = -\frac{l+m}{l-m+1}(l-m+1)P_l^{m-1}(0)$$

and so

$$P_l^m(0) = -(l+m-1)P_{l-1}^{m-1}(0)$$

Now step down:

$$\begin{aligned} P_l^m(0) &= (-1)^2(l+m-1)(l+m-3)P_{l-2}^{m-2}(0) \\ &= (-1)^m(l+m-1)(l+m-3)\ldots(l-m+1)P_{l-m}(0) \end{aligned}$$

Now use equation 8.47 for $P_{l-m}(0)$:

$$\begin{aligned} P_l^m(0) &= (-1)^m\frac{(l+m-1)!!}{(l-m-1)!!}(-1)^{(l-m)/2}\frac{(l-m)!}{[(l-m)!!]^2} \\ &= (-1)^{(l+m)/2}\frac{(l+m-1)!!}{(l-m-1)!!}\frac{(l-m-1)!!}{(l-m)!!} \\ &= (-1)^{(l+m)/2}\frac{(l+m-1)!!}{(l-m)!!} \end{aligned}$$

as required.

18. Show by direct substitution into equation (8.15) that

$P_m^m(\theta) \propto \sin^m\theta.$ Use the value of the orthogonality integral (8.55) together with the result

$\int_0^{\pi/2}\sin^{2m+1}\theta d\theta = \frac{(2m)!!}{(2m+1)!!}$ (eg Gradshteyn and Ryzhik formula 3.621#4) to show that

$$P_m^m(\theta) = \frac{(2m)!}{2^m m!}\sin^m\theta$$

Stuffing in:

$$\begin{aligned} &\frac{d}{d\theta}\left(\sin\theta\frac{d}{d\theta}\sin^m\theta\right) - \frac{m^2}{\sin\theta}\sin^m\theta + m(m+1)\sin\theta\sin^m\theta \\ &= \frac{d}{d\theta}\left(\sin\theta m\sin^{m-1}\theta\cos\theta\right) - m^2\sin^{m-1}\theta\cos^2\theta + m\sin^{m+1}\theta \end{aligned}$$

The derivative is

$$m^2\sin^{m-1}\theta\cos^2\theta - m\sin^{m+1}\theta$$

and thus the equation is satisfied.

If $P_m^m = A\sin^m\theta,$ then

$$A^2\int_0^\pi \sin^{2m+1}\theta d\theta = \frac{2(2m)!}{2m+1} = 2A^2\frac{(2m)!!}{(2m+1)!!}$$

Thus

$$A^2 = \frac{(2m)!}{(2m)!!}(2m-1)!! = \frac{(2m)!}{(2m)!!}\frac{(2m)!}{(2m)!!} = \left[\frac{(2m)!}{2^m m!}\right]^2$$

Finally note that for $m = 1,$

$$P_1^1 = (-1)\sqrt{1-\mu^2}\frac{d}{d\mu}\mu = -\sin\theta$$

and for $m = 2$:

$$P_2^2 = 3\left(1-\mu^2\right) = 3\sin^2\theta$$

So we also need a factor $(-1)^m$. Thus

$$P_m^m(\theta) = (-1)^m \frac{(2m)!}{2^m m!}\sin^m\theta$$

QED

19 The integral

$$\int_{-1}^{+1} \frac{[P_l^m(\mu)]^2}{1-\mu^2}d\mu = \frac{1}{m}\frac{(l+m)!}{(l-m)!}$$

Verify this result for (a) $l = m = 1$, (b) $l = 2,\ m = 1$ and (c) $l = m$. (d) Stepping down in m, use proof by induction to show that the result is true in general.

$P_1^1 = -\sqrt{1-\mu^2}$ so

$$I_{11} = \int_{-1}^{+1} 1 d\mu = 2\int_0^1 d\mu = 2 = \frac{2!}{1(0!)}$$

as required.

$P_2^1 = (-1)^2\sqrt{1-\mu^2}\,\frac{d}{d\mu}\left(\frac{3\mu^2-1}{2}\right) - \sqrt{1-\mu^2}\,3\mu$ and thus

$$I_{22} = \int_{-1}^{+1} 9\mu^2 d\mu = 3\mu^3\Big|_{-1}^{+1} = 6 = \frac{3!}{1(1!)}$$

and finally

$$\begin{aligned}
I_{mm} &= 2\int_0^{+\pi/2}\left[\frac{(2m)!}{2^m m!}\right]^2 \frac{\sin^{2m}\theta}{\sin^2\theta}\sin\theta d\theta = 2\left[\frac{(2m)!}{2^m m!}\right]^2 \frac{(2(m-1))!!}{(2m-1)!!} \\
&= 2\left[\frac{(2m)!}{2^m m!}\right]^2 \frac{[(2m-2)!!]^2}{(2m-1)!} = 2\left[\frac{(2m)!}{2^m m!}\right]^2 \frac{\left[2^{m-1}(m-1)!\right]^2}{(2m-1)!} \\
&= \frac{(2m)!}{m^2}m = \frac{(2m)!}{m} \qquad \text{P19 equation 1}
\end{aligned}$$

Now we want to show that

$$\int_{-1}^{+1} \frac{[P_l^m(\mu)]^2}{1-\mu^2}d\mu = \frac{1}{m}\frac{(l+m)!}{(l-m)!} \quad m \geq 1 \qquad \text{P 19 equation 2}$$

for $l < m$. First assume that the result is true for some m and $l \leq m$. Then

$$\begin{aligned}
I_{l,m-1} &= \frac{1}{(l+m)(l-m+1)}\int_{-1}^{+1} \frac{\left[\sqrt{1-\mu^2}\,(P_l^m)' - \frac{m\mu}{\sqrt{1-\mu^2}}P_l^m\right]P_l^{m-1}}{1-\mu^2}d\mu \\
&= \frac{1}{(l+m)(l-m+1)}\int_{-1}^{+1} \frac{(P_l^m)'P_l^{m-1}}{\sqrt{1-\mu^2}} + \frac{m}{m-1}\frac{\left(P_l^m + \sqrt{1-\mu^2}\,\frac{d}{d\mu}P_l^{m-1}\right)P_l^m}{1-\mu^2}d\mu \\
&= \frac{m}{m-1}\frac{I_{lm}}{(l+m)(l-m+1)} + \int_{-1}^{+1} \frac{(P_l^m)'P_l^{m-1}}{\sqrt{1-\mu^2}} + \frac{m}{m-1}\frac{\left(\frac{d}{d\mu}P_l^{m-1}\right)P_l^m}{\sqrt{1-\mu^2}}d\mu
\end{aligned}$$

where we used the result of Problem 16. The first term is:

$$\frac{m}{m-1}\frac{1}{(l+m)(l-m+1)}\frac{1}{m}\frac{(l+m)!}{(l-m)!} = \frac{1}{m-1}\frac{(l+m-1)!}{(l-m+1)!}$$

which is the result we want. The other terms are:

$$\int_{-1}^{+1} \frac{(P_l^m)' P_l^{m-1}}{\sqrt{1-\mu^2}} + \frac{m}{m-1} \frac{\left(\frac{d}{d\mu} P_l^{m-1}\right) P_l^m}{\sqrt{1-\mu^2}} d\mu = \frac{1}{m-1} \int_{-1}^{+1} \left(\frac{m}{\sqrt{1-\mu^2}} \frac{d}{d\mu} \left(P_l^m P_l^{m-1}\right) - \frac{(P_l^m)' P_l^{m-1}}{\sqrt{1-\mu^2}} \right) d\mu$$

We integrate the first term in the integrandby parts:

$$\frac{1}{\sqrt{1-\mu^2}} \left(P_l^m P_l^{m-1}\right) \Bigg|_{-1}^{+1} - \int_{-1}^{+1} \frac{\mu}{\left(1-\mu^2\right)^{3/2}} P_l^m P_l^{m-1} d\mu$$

The integrated term is zero provided that $m \geq 1$. Then we have

$$\int_{-1}^{+1} \left(\frac{-m\mu}{(1-\mu^2)^{3/2}} P_l^m P_l^{m-1} - \frac{(P_l^m)' P_l^{m-1}}{\sqrt{1-\mu^2}} \right) d\mu = \int_{-1}^{+1} \frac{1}{1-\mu^2} \left(\sqrt{1-\mu^2} (P_l^m)' + P_l^{m+1} \right) P_l^{m-1} - \frac{(P_l^m)' P_l^{m-1}}{\sqrt{1-\mu^2}} d\mu$$
$$= \int_{-1}^{+1} \frac{1}{1-\mu^2} P_l^{m+1} P_l^{m-1} d\mu = 0$$

by orthogonality relation 8.50. Thus the relation is true for $m-1$ if it is true for m. But we have already shown it is true for $m = l$ (P19 equation1 above) and thus it is true for all m, where $l \geq m \geq 1$.

20. Using the generating function $G(x,\mu)$ (equation 8.32) and the addition theorem (8.65), derive the expansion

$$\frac{1}{|\vec{x} - \vec{x}'|} = \sum_{l=0}^{\infty} \sum_{m} \frac{4\pi}{2l+1} \frac{r_<^l}{r_>^{l+1}} Y_{lm}(\theta,\phi) Y_{lm}^*(\theta',\phi')$$

We begin with the result from

$$\frac{1}{|\vec{x} - \vec{x}'|} = \sum_{l=0}^{\infty} \frac{r_<^l}{r_>^{l+1}} P_l(\cos\gamma)$$

and then use the addition theorem

$$\frac{1}{|\vec{x} - \vec{x}'|} = \sum_{l=0}^{\infty} \frac{r_<^l}{r_>^{l+1}} \sum_{m=-l}^{+l} \frac{4\pi}{2l+1} Y_{lm}(\theta,\phi) Y_{lm}^*(\theta',\phi')$$

as required.

Hence find the magnetic vector potential due to a circular loop of wire with radius a and carrying current I.

$$\vec{\mathbf{A}} = \frac{\mu_0}{4\pi} \int \frac{\vec{\mathbf{j}}(\vec{x}')}{|\vec{x} - \vec{x}'|} d^3x'$$
$$= \frac{\mu_0}{4\pi} \int I \frac{\delta(r'-a)}{a} \delta(\mu') \hat{\boldsymbol{\phi}} \sum_{l=0}^{\infty} \frac{r_<^l}{r_>^{l+1}} \sum_{m=-l}^{+l} \frac{4\pi}{2l+1} Y_{lm}(\theta,\phi) Y_{lm}^*(\theta',\phi') (r')^2 dr' d\mu' d\phi'$$
$$= \frac{\mu_0 a I}{4\pi} \sum_{l=0}^{\infty} \frac{r_<^l}{r_>^{l+1}} \sum_{m=-l}^{+l} \frac{4\pi}{2l+1} Y_{lm}(\theta,\phi) \int \delta(\mu')(-\hat{\mathbf{x}} \sin\phi' + \hat{\mathbf{y}} \cos\phi') Y_{lm}^*(\theta',\phi') d\mu' d\phi'$$

where in the last step we changed the meaning of $r_<$ to be the lesser of r and a, and similarly for $r_>$. We now use orthogonality of the $e^{im\phi}$ to argue that only terms with $m = \pm 1$ survive the integration over ϕ'.

$$\int_0^{2\pi} \sin\phi' e^{-im\phi'} d\phi' = \int_0^{2\pi} \frac{e^{i\phi'} - e^{-i\phi'}}{2i} e^{-im\phi'} d\phi' = \frac{2\pi}{2i} (\delta_{m1} - \delta_{m,-1})$$

and

$$\int_0^{2\pi} \cos\phi' e^{-im\phi'} d\phi' = \int_0^{2\pi} \frac{e^{i\phi'} + e^{-i\phi'}}{2} e^{-im\phi'} d\phi' = \frac{2\pi}{2} (\delta_{m1} + \delta_{m,-1})$$

Thus

$$\vec{\mathbf{A}} = \frac{\mu_0 a I}{4\pi}\sum_{l=0}^{\infty}\frac{r_<^l}{r_>^{l+1}}\sum_{m=-l}^{+l}\frac{4\pi}{2l+1}Y_{lm}(\theta,\phi)\pi\sqrt{\frac{(2l+1)(l-m)!}{4\pi(l+m)!}}$$
$$\times\left(-\hat{\mathbf{x}}\frac{(\delta_{m1}-\delta_{m,-1})}{i}+\hat{\mathbf{y}}(\delta_{m1}+\delta_{m,-1})\right)P_l^m\left(\frac{\pi}{2}\right)$$
$$= \frac{\mu_0 a I}{4\pi}\sum_{l=0}^{\infty}\frac{r_<^l}{r_>^{l+1}}\sum_{m=-l}^{+l}P_l^m(\theta)e^{im\phi}\pi\frac{(l-m)!}{(l+m)!}\left(-\hat{\mathbf{x}}\frac{(\delta_{m1}-\delta_{m,-1})}{i}+\hat{\mathbf{y}}(\delta_{m1}+\delta_{m,-1})\right)P_l^m(0)$$
$$= \frac{\mu_0 a I}{4}\sum_{l=1}^{\infty}\frac{r_<^l}{r_>^{l+1}}\left\{P_l^1(\theta)P_l^1(0)e^{i\phi}\frac{(l-1)!}{(l+1)!}(\hat{\mathbf{y}}+i\hat{\mathbf{x}})+\frac{(l+1)!}{(l-1)!}P_l^{-1}(\theta)P_l^{-1}(0)e^{-i\phi}(\hat{\mathbf{y}}-i\hat{\mathbf{x}})\right\}$$
$$= \frac{\mu_0 a I}{4}\sum_{l=1}^{\infty}\frac{r_<^l}{r_>^{l+1}}\left\{P_l^1(\theta)P_l^1(0)\frac{(l-1)!}{(l+1)!}\left(\hat{\mathbf{y}}\left(e^{i\phi}+e^{-i\phi}\right)+i\hat{\mathbf{x}}\left(e^{i\phi}-e^{-i\phi}\right)\right)\right\}$$
$$= \frac{\mu_0 a I}{2}\sum_{l=1}^{\infty}\frac{r_<^l}{r_>^{l+1}}\left\{P_l^1(\theta)P_l^1(0)\frac{(l-1)!}{(l+1)!}(\hat{\mathbf{y}}\cos\phi-\hat{\mathbf{x}}\sin\phi)\right\}$$
$$= \frac{\mu_0 a I}{2}\sum_{l=1}^{\infty}\frac{r_<^l}{r_>^{l+1}}P_l^1(\theta)P_l^1(0)\frac{(l-1)!}{(l+1)!}\hat{\boldsymbol{\phi}}$$

We can simplify a bit by inserting the value of $P_l^1(0).$ First note that P_l is even if l is even, and odd if l is odd. Since P_l^m is the mth derivative, P_l^m will be odd if $l+m$ is odd and even if $l+m$ is even. So $P_l^m(0) = 0$ unless $l+m$ is even, or, in this case, l is odd.

Now we can use the recursion relation (8.37)

$$lP_l(\mu) = \mu P_l'(\mu) - P_{l-1}'(\mu)$$
$$lP_l(0) = -P_{l-1}'(0) = P_{l-1}^1(0)$$

Now we use equation 8.47 for $P_l(0)$ to get:

$$P_l^1(0) = (l+1)P_{l+1}(0) = (l+1)(-1)^{(l+1)/2}\frac{(l+1)!}{[(l+1)!!]^2}$$
$$P_{2n+1}^1 = (-1)^{n+1}(2n+2)\frac{(2n+1)!!}{(2n+2)!!} = (-1)^{n+1}2(n+1)\frac{(2n+1)!!}{2^{n+1}(n+1)!}$$
$$= (-1)^{n+1}\frac{(2n+1)!!}{2^n n!}$$

Thus:

$$\vec{\mathbf{A}} = \frac{\mu_0 a}{2}I\sum_{n=0}^{\infty}\frac{r_<^{2n+1}}{r_>^{2n+2}}\frac{(2n)!}{(2n+2)!}(-1)^{n+1}\frac{(2n+1)!!}{2^n n!}P_{2n+1}^1(\mu)\hat{\boldsymbol{\phi}}$$
$$= \frac{\mu_0 a}{2}I\sum_{n=0}^{\infty}\frac{r_<^{2n+1}}{r_>^{2n+2}}\frac{(2n-1)!!}{2^{n+1}(n+1)!}(-1)^{n+1}P_{2n+1}^1(\mu)\hat{\boldsymbol{\phi}}$$

21. Verify the result (8.67)

$$I_{lm} = \int_{-1}^{+1}P_l^m(\mu)d\mu = 2\int_0^1 P_l^m(\mu)d\mu = \pi P_{l+1}(0)P_l^m(0)\frac{m}{l}(-1)^{(m+1)/2} \qquad l,m \text{ odd}$$

First we evaluate the Legendre functions:

$$I_{lm} = \pi(-1)^{(l+1)/2}\frac{(l)!!}{(l+1)!!}(-1)^{(l+m)/2}\frac{(l+m-1)!!}{(l-m)!!}\frac{m}{l}(-1)^{(m+1)/2} = -m\pi\frac{(l-2)!!}{(l+1)!!}\frac{(l+m-1)!!}{(l-m)!!}$$

for $l > 2.$ If $l = m = 1,$

$$I_{11} = -\frac{\pi}{2}$$

If $l = m$ we should get

$$I_{mm} = -\pi \frac{(m)!!(2m-1)!!}{(m+1)!!}$$

$$I_{mm} = -\frac{1}{2}\int_0^{2\pi} \frac{(2m)!}{2^m m!} \sin^m\theta \sin\theta d\theta = -\frac{(2m)!}{2^{m+1} m!(2i)^{m+1}} \oint_{\text{unit circle}} \left(z - \frac{1}{z}\right)^{m+1} \frac{dz}{iz}$$

There is a pole of order $m+2$ at the origin. The Taylor series is:

$$\frac{1}{z^{m+2}}\left(z^{2(m+1)} - (m+1)z^{2m} + \cdots + (-1)^p \frac{(m+1)!}{p!(m-p+1)!} z^{2(m-p+1)} + \cdots (-1)^{m+1}\right)$$
$$= \left(z^m \cdots + (-1)^p \frac{(m+1)!}{p!(m-p+1)!} z^{m-2p} + \cdots \frac{(-1)^{m+1}}{z^{m+2}}\right)$$

The $1/z$ term is the $m-2p=-1$ or $p=(m+1)/2$ term. Thus the residue is

$$(-1)^{(m+1)/2} \frac{(m+1)!}{\left(\frac{m+1}{2}\right)!\left(m-\frac{m+1}{2}+1\right)!} = (-1)^{(m+1)/2} \frac{(m+1)!}{\left[\left(\frac{m+1}{2}\right)!\right]^2}$$

Thus

$$I_{mm} = -(-1)^{(m+1)/2} \frac{(2m)!}{2^{m+1} m! 2^{m+1}} 2\pi (-1)^{(m+1)/2} \frac{(m+1)!}{\left[\left(\frac{m+1}{2}\right)!\right]^2}$$
$$= -\frac{2\pi}{2^{m+1}} \frac{(2m)!(m+1)}{[(m+1)!!]^2} = -\frac{\pi}{2^m} \frac{(2m)!}{(m+1)!!(m-1)!!}$$

Compare

$$-\pi \frac{(m)!!(2m-1)!!}{(m+1)!!} = -\pi \frac{m!!(2m)!}{(m+1)!!(2m)!!} = -\pi \frac{(2m)!m!!}{(m+1)!!2^m(m)!} = -\frac{\pi}{2^m} \frac{(2m)!}{(m+1)!!(m-1)!!}$$

If $m=1$, but l is any odd integer, then:

$$I_{l1} = \pi P_{l+1}(0) P_l^1(0) \frac{1}{l}(-1)$$

$$I_{l1} = \int_{-1}^{+1} P_l^1(\mu) d\mu = \int_{-1}^{+1} (-1)\left(1-\mu^2\right)^{1/2} \frac{d}{d\mu} P_l(\mu) d\mu$$
$$= -\left[\left(1-\mu^2\right)^{1/2} P_l(\mu)\Big|_0^1 - \int_{-1}^{+1} \frac{-\mu}{\left(1-\mu^2\right)^{1/2}} P_l(\mu) d\mu\right]$$
$$= -\int_{-1}^{+1} \frac{1}{2^l l!} \frac{\mu}{\left(1-\mu^2\right)^{1/2}} \frac{d^l}{d\mu^l}\left(\mu^2-1\right)^l d\mu$$

From the soln to Problem 8:

$$I_{l1} = \int_{-1}^{+1} \frac{x P_{l-1}(x)}{\sqrt{1-x^2}} dx = \frac{1}{(2l+1)} \int_{-1}^{+1} \frac{l P_{l-1}(x) + (l+1) P_{l+1}}{\sqrt{1-x^2}} dx$$
$$= \frac{\pi}{2l+1}\left[l\left(\frac{(l-2)!!}{(l-1)!!}\right)^2 + (l+1)\left(\frac{(l)!!}{(l+1)!!}\right)^2\right]$$
$$= \frac{\pi}{2l+1}\left[\frac{l!!(l-2)!!}{((l-1)!!)^2} + \frac{(l!!)^2}{(l+1)!!(l-1)!!}\right]$$
$$= \frac{\pi(l!!)^2}{(2l+1)[(l-1)!!]^2}\left[\frac{1}{l} + \frac{1}{l+1}\right]$$
$$= \frac{\pi l!!(l-2)!!}{(l-1)!!(l+1)!!}$$
$$= -\pi P_{l-1}(0) P_{l+1}(0) = -\frac{\pi}{l} P_l^1(0) P_{l+1}(0)$$

which is what we want!!

Now for the proof by induction. We step down in m using eqn 8.59:

$$\int_{-1}^{+1} P_l^{m-2} d\mu = \frac{-1}{l(l+1)-(m-2)(m-1)}\left[\int_{-1}^{+1} P_l^m d\mu + 2(m-1)\int \frac{\mu}{\sqrt{1-\mu^2}} P_l^{m-1}\right]$$

Then using the result of Problem 16:

$$\begin{aligned} I_{l,m-2} &= \frac{-1}{l(l+1)-(m-2)(m-1)}\left\{I_{lm} - 2\int\left[P_l^m + \sqrt{1-\mu^2}\left(P_l^{m-1}\right)'\right]d\mu\right\} \\ &= \left[\frac{-1}{l(l+1)-(m-2)(m-1)}\right]\left\{-I_{lm} - 2\left[\sqrt{1-\mu^2}\left(P_l^{m-1}\right)\Big|_{-1}^{+1} - \int -\frac{\mu}{\sqrt{1-\mu^2}} P_l^{m-1}\right]\right\} \\ &= \left[\frac{-1}{l(l+1)-(m-2)(m-1)}\right]\left\{-I_{lm} - 2\int \frac{\mu}{\sqrt{1-\mu^2}} P_l^{m-1}\right\} \end{aligned}$$

So

$$-2(m-1)J_{lm} = 2I_{lm} + 2J_{lm}$$

Thus

$$J_{lm} = \frac{I_{lm}}{-(m-1)-1} = -\frac{I_{lm}}{m}$$

and so

$$\begin{aligned} I_{l,m-2} &= \frac{1}{l(l+1)-(m-2)(m-1)}\left\{I_{lm} - 2\frac{(m-1)}{m} I_{lm}\right\} \\ &= \left(\frac{1}{(l+m-1)(l-m+2)}\right)[-m+2]\pi\frac{(l-2)!!}{(l+1)!!}\frac{(l+m-1)!!}{(l-m)!!} \\ &= -(m-2)\pi\frac{(l-2)!!}{(l+1)!!}\frac{(l+m-3)!!}{(l-m+2)!!} \end{aligned}$$

So result is true for $m-2$ if it is true for $m.$ Since we have shown it is true for $m = l,$ it is true for all odd $m.$ for any odd $l.$

22. Find the electrostatic potential inside a hemisphere of radius a with potential $\Phi = 0$ on the flat side and $\Phi = V$ on the curved part.

The solution is of the form

$$\Phi = \sum_{l,m} A_{lm} r^l Y_{lm}(\theta,\phi)$$

at $r = a,$ Φ must be zero at $\phi = 0, \pi,$ thus we need sine functions $\sin m\phi.$ Next we evaluate on the curved part:

$$\Phi(a,\phi,\phi) = \sum_l \sum_{m=1}^{l} B_{lm} a^l P_l^m(\cos\theta)\sin m\phi = V$$

Thus

$$\begin{aligned} \int_0^{\pi} V \sin m\phi \, d\phi &= \frac{\pi}{2}\sum_l B_{lm} a^l P_l^m(\cos\theta) = \frac{V}{m}(1-(-1)^m) \\ &= \frac{2V}{m} \text{ for } m \text{ odd and zero for } m \text{ even.} \end{aligned}$$

$$\frac{2V}{m}\int_{-1}^{1} P_{l'}^m(\mu)d\mu = \frac{\pi}{2}\sum_l B_{lm} a^l \int_{-1}^{1} P_l^m(\cos\theta) P_{l'}^m(\mu) d\mu$$

The result is zero unless $l' + m$ is even, so we need l' odd. Dropping primes, and using the result of Problem 21:

$$\frac{2V}{m}\left(-m\pi\frac{(l-2)!!}{(l+1)!!}\frac{(l+m-1)!!}{(l-m)!!}\right) = \frac{\pi}{2} B_{lm} a^l \frac{(l+m)!}{(l-m)!}\frac{2}{2l+1}$$

Thus

$$\begin{aligned} B_{lm} &= -2V\frac{(l-2)!!}{(l+1)!!}\frac{(l+m-1)!!}{(l-m)!!}\frac{(2l+1)}{l}\frac{(l-m)!}{(l+m)!}\frac{1}{a^l} \\ &= -2V\frac{(l-2)!!}{(l+1)!!}\frac{(2l+1)}{l}\frac{(l-m-1)!!}{(l+m)!!}\frac{1}{a^l} \end{aligned}$$

for $l > 2$. For $l = m = 1$:

$$B_{11} = -\frac{V}{a}\left(\frac{3}{2}\right) = -\frac{3}{2}\frac{V}{a}$$

and the potential is

$$\frac{\Phi(r,\phi,\phi)}{V} = \frac{3}{2}\frac{r}{a}\sin\phi\sin\theta - 2\sum_{l=3,\, l\text{ odd}}^{\infty}\frac{r^l}{a^l}\frac{(2l+1)}{l}\frac{(l-2)!!}{(l+1)!!}\sum_{m=1,\, m\text{ odd}}^{l}\frac{(l-m-1)!!}{(l+m)!!}P_l^m(\cos\theta)\sin m\phi$$

The first few terms are:

$$\begin{aligned} \frac{\Phi(r,\theta,\phi)}{V} &= \frac{3}{2}\frac{r}{a}\sin\phi\sin\theta - \frac{7}{4}\frac{r^3}{a^3}\left(\frac{1}{8}P_3^1(\cos\theta)\sin\phi + \frac{1}{6\times 4\times 2}P_3^3(\cos\theta)\sin 3\phi\right) \\ &= \frac{3}{2}\frac{r}{a}\sin\phi\sin\theta + \frac{7}{64}\frac{r^3}{a^3}\sin\theta\left(3\left(5\cos^2\theta - 1\right)\sin\phi + 5\sin^2\theta\sin 3\phi\right) \end{aligned}$$

The $l = 5$ term is

$$\frac{11}{5}\frac{3}{48}\left(\frac{3}{48}P_5^1\sin\phi + \frac{1}{48\times 8}P_5^3\sin 3\phi + \frac{1}{48\times 8\times 10}P_5^5\sin 5\phi\right)$$

To calculate the P_5^m, we start with P_5. Do a series expansion of the generating function using Maple:

$$\frac{1}{\sqrt{1-2xa+x^2}} = 1 + ax + \left(-\frac{1}{2} + \frac{3}{2}a^2\right)x^2 + \left(-\frac{3}{2}a + \frac{5}{2}a^3\right)x^3 + \left(\frac{3}{8} - \frac{15}{4}a^2 + \frac{35}{8}a^4\right)x^4$$

$$+\left(\frac{15}{8}a - \frac{35}{4}a^3 + \frac{63}{8}a^5\right)x^5 + \cdots$$

Thus $P_5(x) = \left(\frac{1}{8}x\left(15 - 70x^2 + 63x^4\right)\right)$.

Next use equation 8.53.

$$(-1)\frac{\sin\theta}{8}\frac{d}{dx}\left(15x - 70x^3 + 63x^5\right) = -\frac{1}{8}(\sin\theta)\left(15 - 210x^2 + 315x^4\right)$$

$$= -\frac{15}{8}(\sin\theta)\left(1 - 14x^2 + 21x^4\right) = P_5^1$$

$$(-1)^2\frac{\sin^2\theta}{8}\frac{d}{dx}\left(15 - 210x^2 + 315x^4\right) = \frac{1}{8}\left(\sin^2\theta\right)\left(-420x + 1260x^3\right)$$

$$= \frac{105}{2}\left(\sin^2\theta\right)x\left(3x^2 - 1\right) = P_5^2$$

$$(-1)\sin^3\theta\frac{d}{dx}\left(-420x + 1260x^3\right) = 420\sin^3\theta\left(1 - 9x^2\right) = P_5^3$$

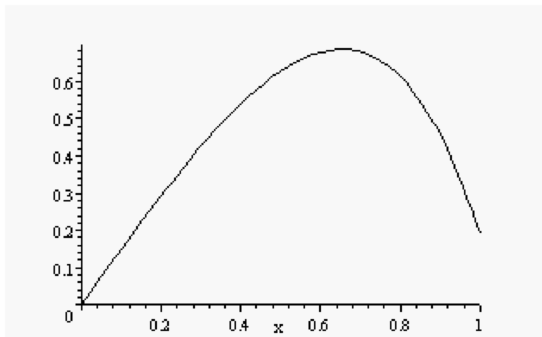
$$\sin^5(\theta)\frac{d^2}{dx^2}\left(420 - 3780x^2\right) = -7560\sin^5\theta = P_5^5$$

Then the $l = 5$ term is:

$$\frac{11}{80}\left(\begin{array}{c}\frac{3}{48}\left(-\frac{11}{8}(\sin\theta)(1-14\cos^2\theta+21\cos^4\theta)\right)\sin\phi \\ +\frac{1}{48\times 8}(420\sin^3\theta(1-9\cos^2\theta))\sin 3\phi-\frac{1}{48\times 8\times 10}7560\sin^5\theta\sin 5\phi\end{array}\right)$$
$$=\frac{11}{80}\left(\begin{array}{c}-\frac{11}{128}(\sin\theta)(1-14\cos^2\theta+21\cos^4\theta)\sin\phi \\ +\frac{35}{32}\sin^3\theta(1-9\cos^2\theta)\sin 3\phi-\frac{63}{32}\sin^5\theta\sin 5\phi\end{array}\right)$$

At $\theta = \pi/2$, $\phi = \pi/2$ we have

$$\frac{\Phi(r,\pi/2,\pi/2)}{V} = \frac{3}{2}\frac{r}{a} - \frac{7}{8}\frac{r^3}{a^3} - \frac{4433}{10240}\frac{r^5}{a^5}$$

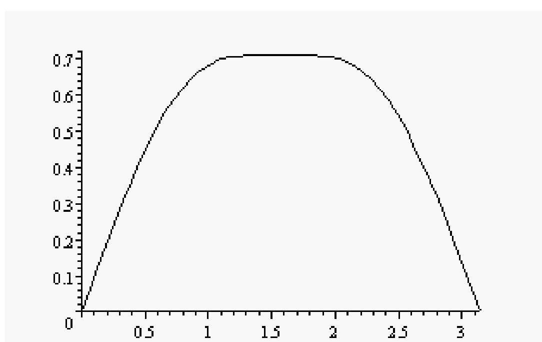


This expansion to $l = 5$ appears to be good to about $r = 0.6a$.

At $r/a = 0.5$, $\theta = \pi/2$

$$\frac{\Phi\left(0.5,\frac{\pi}{2},\phi\right)}{V} = \frac{3}{4}\sin\phi + \frac{7}{512}(3\sin\phi + 5\sin 3\phi) + \frac{11}{2560}\left(-\frac{11}{128}\sin\phi + \frac{35}{32}\sin 3\phi - \frac{63}{32}\sin 5\phi\right)$$

The plot shows Φ/V versus ϕ.



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Chapter 8: Sturm-Liouville Theory

23. Quantum mechanical treatment of the harmonic oscillator results in the Hermite differential equation

$$y'' - 2xy' + \lambda y = 0$$

Write this equation in standard Sturm-Liouville form. If the boundary conditions are $y(x) \to 0$ as $x \to \pm\infty$, show that the solutions are orthogonal on the range $(-\infty, +\infty)$, and find the weight function $w(x)$. Solve the equation to find a series expansion for the Hermite functions. What value of the eigenvalue λ is required for the functions to remain bounded throughout the interval, including $x \to \pm\infty$? (*Hint:* experience with Legendre functions should prove useful.) Normalize the solutions by choosing the coefficient of the highest power x^n to be 2^n, and hence determine the first three eigenfunctions.

We want to write the equation in the form (8.1). Expanding out the differential operator, we get:

$$fy'' + f'y' - gy + \lambda wy = 0$$

Comparing with Hermite's equation, we can multiply by a function $h(x)$ such that

$$\begin{aligned} -2xh(x) &= h'(x) \\ \ln(h) &= -x^2 + \text{ constant} \end{aligned}$$

Thus $h = e^{-x^2}$ and the standard form is:

$$\frac{d}{dx}\left(e^{-x^2}y'\right) + \lambda e^{-x^2}y = 0$$

Then the weight function is $w(x) = e^{-x^2}$ and

$$\int_{-\infty}^{+\infty} e^{-x^2} y_n y_m dx = 0 \text{ unless } n = m$$

For the series solution, we use the original form of the differential equation and let $y = \sum a_n x^n$. Then

$$\sum a_n n(n-1)x^{n-2} - 2\sum a_n n x^n + \lambda \sum a_n x^n = 0$$

x^0 :

$$a_2 2 + \lambda a_0 = 0 \Rightarrow a_2 = \frac{-\lambda}{2}a_0$$

x^p, $p > 0$:

$$\begin{aligned} a_{p+2}(p+2)(p+1) - 2a_p p + \lambda a_p &= 0 \\ a_{p+2} &= a_p \frac{(2p - \lambda)}{(p+2)(p+1)} \end{aligned}$$

The series converges for $|x| < \infty$, but for large p we get

$$a_{p+2} = a_p \frac{2}{p}$$

and the ratio of successive terms is

$$\frac{a_{p+2}x^{p+2}}{a_p x^p} \simeq \frac{2x^2}{p}$$

This ratio is greater than 1 for $x^2 > p/2$, so that the series diverges as $x \to \pm\infty$. We can avoid this problem by choosing $\lambda = 2n$ for some integer n. Then $a_{n+2} \equiv 0$ and the series terminates with the x^n term. As with Legendre functions, only one of the two solutions to the differential equation terminates. That solution is the Hermite polynomial. Then we find:

$$a_{n-2} = a_n \frac{(n-1)n}{2(n-2)-2n} = 2^n \frac{n(n-1)}{-2^2} = (-1)2^{n-2}n(n-1)$$

$$a_{n-4} = a_{n-2}\frac{(n-3)(n-2)}{2(n-4)-2n} = (-1)2^{n-2}n(n-1)\frac{(n-3)(n-2)}{-2^3} = (-1)^2 2^{n-4}\frac{n!}{2(n-4)!}$$

or, in general, the coefficient a_m of x^m in the nth polynomial is

$$a_{m,n} = (-1)^{\frac{n-m}{2}} 2^m \frac{n!}{m!\left(\frac{n-m}{2}\right)!}$$

Let's find the first 3.

$n = 0$: There is only one term, and it is a constant equal to $2^0 = 1 = H_0(x)$

$n = 1$: There is again only one term, and the solution is $2x = H_1(x)$

$n = 2$: There are two terms:

$$a_2 = -2a_0 = 2^2 = 4$$

So

$$a_0 = -2$$

and

$$H_2(x) = 4x^2 - 2$$

24. The generating function for Hermite polynomials is

$$G(x,t) = e^{-t^2+2xt} = \sum_{n=0}^{\infty}\frac{t^n}{n!}H_n(x)$$

Use this generating function to establish a pure recursion relation for Hermite polynomials(analogous to equation 8.34 for Legendre polynomials). Also obtain the derivative dH_n/dx in terms of the H_n (analogous to equations 8.40 and 8.41).

Differentiate $G(x,t)$ with respect to t a total of n times to obtain the Rodrigues-type formula

$$H_n(x) = (-1)^n e^{x^2}\frac{d^n}{dx^n}e^{-x^2}$$

As with the Legendre functions, we differentiate with respect to t :

$$\frac{\partial G}{\partial t} = (-2t + 2x)G$$

$$\sum_{n=1}^{\infty} \frac{t^{n-1}}{(n-1)!} H_n(x) = -2\sum_{n=0}^{\infty} \frac{t^{n+1}}{n!} H_n(x) + 2x \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x)$$

Then equating coefficients of powers of t^m for $m > 0,$ we get:

$$\frac{H_{m+1}}{m!} = -2\frac{H_{m-1}}{(m-1)!} + 2x\frac{H_m(x)}{m!}$$

$$H_{m+1} + 2mH_{m-1} = 2xH_m(x)$$

Now differentiating with respect to x :

$$\frac{\partial G}{\partial x} = 2tG$$

$$\sum_{n=0}^{\infty} \frac{t^n}{n!} \frac{dH_n(x)}{dx} = 2\sum_{n=0}^{\infty} \frac{t^{n+1}}{n!} H_n(x)$$

$$\frac{dH_n(x)}{dx} = 2nH_{n-1}(x)$$

valid for $n \geq 1.$

Begin by writing G in the form:

$$G = e^{-(t-x)^2} e^{x^2} = \sum_{n=0}^{\infty} \frac{t^n}{n!} H_n(x)$$

Then

$$\frac{\partial^n G}{\partial t^n} = e^{x^2}(-1)^n \frac{\partial^n e^{-(t-x)^2}}{\partial x^n}$$

and also, from the right hand side,

$$\frac{\partial^n G}{\partial t^n} = \sum_{m=n}^{\infty} \frac{t^{m-n}}{(m-n)!} H_n(x)$$

Now set $t = 0$. Only the first term in the sum remains, and we get:

$$H_n(x) = \left.\frac{\partial^n G}{\partial t^n}\right|_{t=0} = (-1)^n e^{x^2} \frac{\partial^n}{\partial x^n} e^{-x^2}$$

25. By using the Rodrigues formula (Problem 24) for the Hermite polynomials, or otherwise, obtain the normalization integral:

$$\int_{-\infty}^{+\infty} e^{-x^2}[H_n(x)]^2 dx = \int_{-\infty}^{+\infty} e^{-x^2} H_n(x)(-1)^n e^{x^2} \frac{\partial^n}{\partial x^n} e^{-x^2} dx$$

$$= \int_{-\infty}^{+\infty} H_n(x)(-1)^n \frac{\partial^n}{\partial x^n} e^{-x^2} dx$$

Now integrate by parts:

$$
\begin{aligned}
(-1)^n I_n &= H_n(x)\frac{d^{n-1}}{dx^{n-1}}e^{-x^2}\Big|_{-\infty}^{+\infty} - \int \frac{dH_n}{dx}\frac{d^{n-1}}{dx^{n-1}}e^{-x^2}dx \\
&= H_n(x)(-1)^{n-1}e^{-x^2}H_{n-1}(x)\Big|_{-\infty}^{+\infty} - \int 2nH_{n-1}(x)\frac{d^{n-1}}{dx^{n-1}}e^{-x^2}dx \\
&= 0-(-1)^{n-1}2nI_{n-1}
\end{aligned}
$$

Thus

$$I_n = 2nI_{n-1} = 2^n n! I_0$$

But

$$I_0 = \int_{-\infty}^{+\infty} e^{-x^2}dx = \sqrt{\pi}$$

Thus

$$I_n = 2^n n! \sqrt{\pi}$$

26. Starting with the relation (8.86) derive the Rodrigues-type formula for Bessel functions:

$$J_n(x) = x^n\left(-\frac{1}{x}\frac{d}{dx}\right)^n J_0(x)$$

$$
\begin{aligned}
J_n(x) &= x^{n-1}\frac{d}{dx}\left(\frac{J_{n-1}}{x^{n-1}}\right) \\
&= x^{n-1}\frac{d}{dx}\left(\frac{x^{n-2}}{x^{n-1}}\frac{d}{dx}\frac{J_{n-2}}{x^{n-2}}\right) \\
&= x^n\left(\frac{1}{x}\frac{d}{dx}\left(\frac{1}{x}\frac{d}{dx}\left(\frac{J_{n-2}}{x^{n-2}}\right)\right)\right) \\
&= x^n\left(\frac{1}{x}\frac{d}{dx}\right)^2\left(\frac{J_{n-2}}{x^{n-2}}\right)
\end{aligned}
$$

Continuing in this way, we get

$$J_n(x) = x^n\left(\frac{1}{x}\frac{d}{dx}\right)^n J_0(x)$$

as required.

27. A drumhead is a circular membrane of radius a. When it is struck, waves propagate across the drumhead. The membrane vibrates with displacement ξ where $\xi(r,\theta,t) = \eta(r,\theta)e^{-i\omega t}$ and $\eta(r,\theta)$ satisifes the Helmholtz equation

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\eta}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\eta}{\partial\theta^2} + k^2\eta = 0$$

where $k^2 = \omega^2/v^2$ and v is the speed with which waves propagate across the drumhead. (The speed v depends on the tension in the drumhead, among other things.) The boundary condition is that $\eta = 0$ at $r = a$. Separate variables, and find the eigenfunctions. Determine the first three allowable frequencies ω in terms of the drum parameters v and a.

Let $\eta = R\Theta$ Then

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right)\Theta + \frac{R}{r^2}\frac{\partial^2\Theta}{\partial\theta^2} + k^2 R\Theta = 0$$

$$\frac{1}{Rr}\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right) + k^2 + \frac{1}{r^2\Theta}\frac{\partial^2\Theta}{\partial\theta^2} = 0$$

Multiply through by r^2 to get:

$$\frac{r}{R}\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right) + k^2 r^2 + \frac{1}{\Theta}\frac{\partial^2\Theta}{\partial\theta^2} = 0$$

Once again we choose

$$\frac{1}{\Theta}\frac{\partial^2\Theta}{\partial\theta^2} = -m^2 \Rightarrow \theta = \begin{cases} \cos m\theta \\ \sin m\theta \end{cases}$$

and then the equation for R is Bessel's equation:

$$r\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right) + \left(k^2 r^2 - m^2\right)R = 0$$

We need a solution that is finite at the origin (the center of the drumhead), i.e. $J_m(kr).$ Then we choose the eigenvalues to make η zero at $r = a.$ Thus

$$ka = x_{mn}$$

the nth zero of $J_m(x).$ The solutions are of the form

$$J_m\left(x_{mn}\frac{r}{a}\right)(A_m \cos m\theta + B_m \sin m\theta)$$

and the allowed frequencies are

$$\omega_{mn} = k_{mn}v = x_{mn}\frac{v}{a}$$

The zeros are x_{0n} =2.4048, 5.5201 and $8.654.$

$x_{1n} = 3.8317,$ 7.0156

$x_{2n} = 5.1356$

Thus the first three frequencies are $\omega = 2.4048v/a,$ 3.8317v/a and 5.1356$v/a.$

28. Sound waves propagating through a tube may be described by a velocity potential (cf Chapter 2 §2.4) that satisfies the Helmholtz equation

$$\left(\nabla^2 + \frac{\omega^2}{c_s^2}\right)\Phi = 0$$

where c_s is the sound speed in the tube. Now assume that for propagation along the length of the tube (in the $+z$-direction) the potential may be written:

$$\Phi = \Phi_t e^{ikz}$$

where Φ_t is a function of the transverse coordinates $\left(x \text{ and } y \text{ or } r \text{ and } \theta\right)$. Because the air cannot move perpendicular to the walls of the tube, the boundary condition is

$$\hat{\mathbf{n}} \cdot \vec{\nabla}\Phi = \frac{\partial \Phi}{\partial n} = 0 \text{ on the boundary surface}$$

Write the differential equation and boundary conditions satisfied by Φ_t, and hence find the eigenvalues and the set of allowed frequencies ω if

(**a**) the tube has a rectangular cross section measuring $a \times b$, or

(**b**) the tube has a circular cross section of radius a.

In each case show that there is a minimum frequency for waves that propagate along the tube with Φ_t not constant.

We may write $\nabla^2 = \nabla_t^2 + \frac{\partial^2}{\partial z^2}$, and thus the differential equation is:

$$\left(\nabla_t^2 - k^2 + \frac{\omega^2}{c_s^2}\right)\Phi_t = 0 = \left(\nabla_t^2 + \lambda\right)\Phi_t$$

where the eigenvalue $\lambda = \frac{\omega^2}{c_s^2} - k^2$.

(a) With a rectangular cross section, we have

$$\frac{\partial^2 \Phi_t}{\partial x^2} + \frac{\partial^2 \Phi_t}{\partial y^2} + \lambda\Phi_t = 0$$

with

$$\frac{\partial \Phi_t}{\partial x} = 0 \text{ at } x = 0,\ x = a$$

and

$$\frac{\partial \Phi_t}{\partial y} = 0 \text{ at } y = 0,\ y = b$$

Now separate by looking for a solution of the form $\Phi_t = X(x)Y(y)$. We have

$$\frac{X''}{X} + \frac{Y''}{Y} + \lambda = 0$$

Both X and Y satisfy equations of the same form:

$$X'' = -\alpha^2 X;\ Y'' = -\beta^2 Y$$

with

$$\alpha^2 + \beta^2 = \lambda$$

and the boundary conditions are also of the same form:

$$X' = 0 \text{ at } x = 0, a;\ Y' = 0 \text{ at } y = 0, b$$

Thus the solution must be of the form:

$$X = \cos \alpha x$$

so that $X' = -\alpha \sin \alpha x = 0$ at $x = 0$. Then we choose the eigenvalue α so that $\sin \alpha a = 0$, i.e. $\alpha = n\pi/a$. Doing the same thing for Y we find that the eigenfunctions are of the form:

$$\Phi_t = \cos \frac{n\pi}{a} x \cos \frac{m\pi}{b} y$$

and the eigenvalues are

$$\lambda_{nm} = \left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2$$

Thus the allowed frequencies are given by:

$$\omega_{mn}^2 = \left[k^2 + \left(\frac{n\pi}{a}\right)^2 + \left(\frac{m\pi}{b}\right)^2\right] c_s^2$$

The solution with $n = m = 0$ corresponds to a constant value of Φ. With Φ constant, the air velocity is identically zero, and so there is no wave. Otherwise,

$$k^2 = \frac{\omega_{mn}^2}{c_s^2} - \left(\frac{n\pi}{a}\right)^2 - \left(\frac{m\pi}{b}\right)^2$$

The minimum frequency occurs for $k \to 0$, and either m or $n = 1$. (For lesser frequencies k becomes imaginary and the wave does not propagate.) If the larger dimension of the tube is $a > b$, then the minimum frequency is:

$$\omega_{\min} = \omega_{01} = c_s \frac{\pi}{a}$$

(b) With $\Phi_t = R(r)\Theta(t)$ the differential equation is

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right)\Theta + \frac{R}{r^2}\frac{\partial^2 \Theta}{\partial \theta^2} + \lambda R\Theta = 0$$

which separates to give

$$\Theta'' = -m^2\Theta; \quad \Theta = \begin{cases} \sin m\theta \\ \cos m\theta \end{cases}$$

and

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial R}{\partial r}\right) - m^2\frac{R}{r^2} + \lambda R = 0$$

which is Bessel's equation, with solution

$$J_m\left(\sqrt{\lambda} r\right)$$

The boundary condition $\frac{\partial R}{\partial r} = 0$ at $r = a$ requires that we choose $\sqrt{\lambda} a$ to be one one of the roots x'_{mn} of J'. (See eg Abramowitz and Stegun page 411) Then the eigenvalue $\lambda = (x'_{mn})^2/a$. Then the frequency is

$$\omega = c_s \sqrt{k^2 + \left(\frac{x'_{mn}}{a}\right)^2}$$

With $m = 0$ and $x'_{01} = 0$, Φ is a constant and there is no wave. Otherwise, the smallest value of x'_{mn} is $x'_{11} = 1.84118$ giving a cut-off frequency of

$$\omega_c = 1.84118\frac{c_s}{a}$$

Compare this with the rectangular guide: the closest equivalent dimension to the radius is one half the dimension a, and the cutoff is

$$\frac{\pi}{2\text{“radius”}}c_s = 1.5708\frac{c_s}{\text{“radius”}}$$

which is comparable to that for the circular guide.

Writing $A = \pi a^2$ or $A = a^2$ in the case of circular or square guides, we have

$$\frac{\omega}{c_s} = 1.84118\sqrt{\frac{\pi}{A}} = \frac{3.2634}{\sqrt{A}}$$

and

$$\frac{\omega}{c_s} = \frac{\pi}{\sqrt{A}} = \frac{3.1416}{\sqrt{A}}$$

repectively.

29. If γ_{mn} is the nth zero of $J'_m(x)$, show that the Bessel functions satisfy the orthogonality relation:

$$\int_0^a J_m\left(\gamma_{mn}\frac{\rho}{a}\right)J_m\left(\gamma_{mk}\frac{\rho}{a}\right)\rho d\rho = \frac{a^2}{2}\left(1-\frac{m^2}{\gamma_{mn}^2}\right)[J_m(\gamma_{mn})]^2\delta_{mn}$$

We start with the general orthogonality relation (equation 8.6)

$$\left(\frac{\gamma_{mn}^2}{a^2}-\frac{\gamma^2}{a^2}\right)\int_0^a \rho J_m\left(\gamma_{mn}\frac{\rho}{a}\right)J_m\left(\gamma\frac{\rho}{a}\right)d\rho = \int_0^a\left[\begin{array}{c} J_m(\gamma_{mn}\frac{\rho}{a})\frac{d}{d\rho}\left(\rho\frac{d}{d\rho}J_m(\gamma\frac{\rho}{a})\right) \\ -J_m(\gamma\frac{\rho}{a})\frac{d}{d\rho}\left(\rho\frac{d}{d\rho}J_m(\gamma_{mn}\frac{\rho}{a})\right)\end{array}\right]\rho d\rho$$

Integrating by parts, the first term on the right hand side is:

$$J_m\left(\gamma_{mn}\frac{\rho}{a}\right)\rho\frac{d}{d\rho}J_m\left(\gamma\frac{\rho}{a}\right)\Big|_0^a - \int_0^a \rho\frac{d}{d\rho}J_m\left(\gamma_{mn}\frac{\rho}{a}\right)\frac{d}{d\rho}J_m\left(\gamma\frac{\rho}{a}\right)d\rho$$

Subtracting the second term, the integrals cancel. Taking $\gamma = \gamma_{mk}$, another zero of J'_m , the integrated terms are both zero, and so we have

$$\left(\frac{\gamma_{mn}^2}{a^2}-\frac{\gamma_{mk}^2}{a^2}\right)\int_0^a \rho J_m\left(\gamma_{mn}\frac{\rho}{a}\right)J_m\left(\gamma_{mk}\frac{\rho}{a}\right)d\rho = 0 \text{ if } n \neq k.$$

To obtain the value of the integral if $n = k$, we differentiate with respect to γ :

$$\left(\frac{\gamma_{mn}^2}{a^2}-\frac{\gamma^2}{a^2}\right)\int_0^a \rho J_m\left(\gamma_{mn}\frac{\rho}{a}\right)\frac{d}{d\gamma}J_m\left(\gamma\frac{\rho}{a}\right)d\rho - \frac{2\gamma}{a^2}\int_0^a \rho J_m\left(\gamma_{mn}\frac{\rho}{a}\right)J_m\left(\gamma\frac{\rho}{a}\right)d\rho$$
$$= J_m\left(\gamma_{mn}\frac{\rho}{a}\right)\rho\frac{d}{d\gamma}\frac{d}{d\rho}J_m\left(\gamma\frac{\rho}{a}\right)\Big|_0^a$$

On the right hand side, we make use of the differential equation to get:

$$J_m\left(\gamma_{mn}\frac{\rho}{a}\right)\rho\frac{d}{d\rho}\left(\frac{\rho}{a}J_m'\left(\frac{\gamma\rho}{a}\right)\right) = J_m\left(\gamma_{mn}\frac{\rho}{a}\right)\rho\frac{d}{d\rho}\left(\frac{\rho}{a}\frac{a}{\gamma}\frac{\partial}{\partial\rho}J_m\left(\frac{\gamma\rho}{a}\right)\right)$$
$$= \frac{1}{\gamma}\left(m^2 - \frac{\gamma^2}{a^2}\rho^2\right)J_m\left(\frac{\gamma\rho}{a}\right)J_m\left(\gamma_{mn}\frac{\rho}{a}\right)$$

Now let $\gamma \to \gamma_{mn}.$ The first term on the left goes to zero, and we have:

$$\int_0^a \rho\left[J_m\left(\gamma_{mn}\frac{\rho}{a}\right)\right]^2 d\rho = -\frac{a^2}{2\gamma_{mn}^2}\left(m^2 - \frac{\gamma_{mn}^2}{a^2}\rho^2\right)\left[J_m\left(\gamma_{mn}\frac{\rho}{a}\right)\right]^2\Big|_0^a$$

The contribution from the lower limit is zero for $m > 0$ since $J_m(0) = 0$ in that case, and is zero for $m = 0$ because then the factor multiplying $J_0(0)^2$ is $\propto \rho^2 = 0.$ Thus

$$\int_0^a \rho\left[J_m\left(\gamma_{mn}\frac{\rho}{a}\right)\right]^2 d\rho = \frac{a^2}{2}\left(1 - \frac{m^2}{\gamma_{mn}^2}\right)[J_m'(\gamma_{mn})]^2$$

which is the required result.

30. Use the generating function (8.93) to show that

$$\sin x = 2\sum_{n=0}^{\infty} J_{2n+1}(x)$$

We start with

$$\exp\left(\frac{kr}{2}\left(t - \frac{1}{t}\right)\right) = \sum_{m=-\infty}^{+\infty} J_m(kr)t^m$$

Let $t = i.$ Then

$$\exp\left(\frac{kr}{2}\left(i - \frac{1}{i}\right)\right) = \exp(ikr) = \sum_{m=-\infty}^{+\infty} J_m(kr)i^m$$

Now let $t = -i$:

$$\exp\left(\frac{kr}{2}\left(-i + \frac{1}{i}\right)\right) = \exp(-ikr) = \sum_{m=-\infty}^{+\infty} J_m(kr)(-i)^m$$

Now subtract the two results:

$$2i\sin kr = \sum_{m=-\infty}^{+\infty} J_m(kr)[i^m - (-i)^m]$$

For the even terms:

$$i^{2p} - (-i)^{2p} = 0$$

while for the odd terms

$$i^{2p+1} - (-i)^{2p+1} = i(-1)^p\left(1 - (-1)^{2p+1}\right) = 2i(-1)^p$$

and

$$i^{-(2p+1)} - (-i)^{2p+1} = \frac{1}{i}\left(i^{-2p} - (-i)^{-2p}\right) = 2\frac{(-1)^p}{i}$$

Then since

$$J_{-m}(x) = (-1)^m J_m(x) = -J_m(x)$$

for odd $m,$ we may combine the positive and negative terms to get:

$$\sin kr = 2\sum_{p=0}^{\infty}(-1)^n J_{2p+1}(kr)$$

(**b**)

$$1 = J_0(x) + 2\sum_{n=1}^{\infty} J_{2n}(x)$$

This time we set $t = 1.$ Then

$$e^0 = 1 = \sum_{m=-\infty}^{+\infty} J_m(kr) = J_0(kr) + \sum_{m=1}^{\infty} J_m(kr)(1 + (-1)^m)$$

The odd terms in the sum are zero, leaving

$$1 = J_0(kr) + 2\sum_{n=1}^{\infty} J_{2n}(kr)$$

as required.

Chapter 8: Sturm-Liouville Theory

31. Show that

$$J_n(x+y) = \sum_{m=-\infty}^{+\infty} J_m(x)J_{n-m}(y)$$

Start with:

$$\exp(ix\sin\theta) = \sum_{m=-\infty}^{+\infty} J_m(x)e^{im\theta}$$

and

$$\exp(iy\sin\theta) = \sum_{k=-\infty}^{+\infty} J_k(y)e^{ik\theta}$$

Multiply these results together:

$$\exp(i(x+y)\sin\theta) = \sum_{m=-\infty}^{+\infty} J_m(x)e^{im\theta}\sum_{k=-\infty}^{+\infty} J_k(y)e^{ik\theta}$$
$$\sum_{p=-\infty}^{+\infty} J_p(x+y)e^{ip\theta} = \sum_{m=-\infty}^{+\infty}\sum_{k=-\infty}^{+\infty} J_m(x)J_k(y)e^{i(m+k)\theta}$$

Now multiply both sides by $e^{-in\theta}$ and integrate over the range 0 to 2 π. By the orthogonality of the exponential functions, only the term with $p = n$ survives on the left, and the terms with $m + k = n$ on the right:

$$J_n(x+y) = \sum_{m=-\infty}^{+\infty} J_m(x)J_{n-m}(y)$$

as required.

Now let $x = y$ and $n = 0$ to get:

$$J_0(2x) = \sum_{m=-\infty}^{+\infty} J_m(x)J_{-m}(x) = J_0^2(x) + \sum_{m=-\infty}^{\infty}(-1)^m J_m^2(x)$$
$$= J_0^2(x) + 2\sum_{m=1}^{\infty}(-1)^m J_m^2(x)$$

32. Show that

$$\int_0^\infty \frac{J_m(x)J_n(x)}{x}dx = \frac{2}{\pi}\frac{\sin(m-n)\frac{\pi}{2}}{m^2-n^2} \text{ for } m+n > 0$$

We use the technique from Appendix VIII.

$$\left(m^2-n^2\right)\int_0^\infty \frac{J_mJ_n}{x}dx = \int_0^\infty\left\{\left[\frac{d}{dx}\left(x\frac{dJ_m}{dx}\right)+xJ_m\right]J_n - \left[\frac{d}{dx}\left(x\frac{dJ_n}{dx}\right)+xJ_n\right]J_m\right\}dx$$
$$= \int_0^\infty\left\{\left[\frac{d}{dx}\left(x\frac{dJ_m}{dx}\right)\right]J_n - \left[\frac{d}{dx}\left(x\frac{dJ_n}{dx}\right)\right]J_m\right\}dx$$
$$= J_n(xJ_m') - J_m(xJ_n')\Big|_0^\infty - \int_0^\infty[xJ_m'J_n' - xJ_n'J_m']dx$$

Since the series for J_m starts with x^m, the integrated term is zero at $x = 0$ provided that $n + m > 0$, and the integrand vanishes, so, using the large argument expansion of the Bessel functions, we are left with:

$$(m^2-n^2)\int_0^\infty \frac{J_m J_n}{x}dx = \lim_{x\to\infty} x\sqrt{\frac{2}{\pi x}}\cos\left(x-\frac{n\pi}{2}-\frac{\pi}{4}\right)\frac{d}{dx}\left(\sqrt{\frac{2}{\pi x}}\cos\left(x-\frac{m\pi}{2}-\frac{\pi}{4}\right)\right)-(m\leftrightarrow n)$$
$$= \lim_{x\to\infty} x\sqrt{\frac{2}{\pi x}}\cos\left(x-\frac{n\pi}{2}-\frac{\pi}{4}\right)\left(\begin{array}{c} -\frac{1}{2}\sqrt{\frac{2}{\pi x^3}}\cos\left(x-\frac{m\pi}{2}-\frac{\pi}{4}\right) \\ -\sqrt{\frac{2}{\pi x}}\sin\left(x-\frac{m\pi}{2}-\frac{\pi}{4}\right)\end{array}\right)-(m\leftrightarrow n)$$

The first term $\to 0$ as $x\to\infty$ because of the factor $1/x$. Thus we are left with:

$$(m^2-n^2)\int_0^\infty \frac{J_m J_n}{x}dx = \lim_{x\to\infty}\frac{2}{\pi}\left(\begin{array}{c} \cos\left(x-\frac{m\pi}{2}-\frac{\pi}{4}\right)\sin\left(x-\frac{n\pi}{2}-\frac{\pi}{4}\right) \\ -\cos\left(x-\frac{n\pi}{2}-\frac{\pi}{4}\right)\sin\left(x-\frac{m\pi}{2}-\frac{\pi}{4}\right)\end{array}\right)$$
$$=\frac{2}{\pi}\sin\left[\frac{1}{2}(m-n)\pi\right]$$

and thus

$$\int_0^\infty \frac{J_m J_n}{x}dx = \frac{2}{\pi}\frac{\sin\frac{(m-n)\pi}{2}}{m^2-n^2}$$

Then as $m\to n$ we get:

$$\lim_{\varepsilon\to 0}\frac{2}{\pi}\left(\frac{\varepsilon\pi}{2}\right)\frac{1}{\varepsilon(2n+\varepsilon)}\to\frac{1}{2n}$$

This agrees with GR 6.538#2 with $\nu = 0.$

Note that the Bessel equation may be considered as an eigenvalue problem with eigenvalue m:

$$\frac{d}{dx}\left(x\frac{dJ_m}{dx}\right)+xJ_m-m^2\frac{J_m}{x}=0$$

with $f(x)=x,\ g(x)=-x$ and the weighting function $w(x)=-\frac{1}{x}.$ Our proof shows that

$$\int_0^\infty \frac{J_m(x)J_n(x)}{x}dx = 0$$

if m is not equal to n and
$m-n$ is an even integer. Thus the *even* order Bessel functions or *odd* order Bessel functions satisfy a second orthogonality relation. This is somewhat analogous to the orthogonality of sines or cosines on the half range $[0,\pi]$.

33. At time $t=0,$ the surface of the water in a pond has the form

$$s(\rho,\phi,0)=hJ_0(\alpha\rho)$$

and

$$\left.\frac{\partial s}{\partial t}\right|_{t=0}=0$$

By taking the Fourier transform of the wave equation with two spatial dimensions, find the displacement $s(\rho,\phi,t)$ at later times.

$$\left(-k^2v^2+\omega^2\right)S=0$$

Thus

$$S\left(\vec{\mathbf{k}},\omega\right)=A\delta(\omega-kv)+B\delta(\omega+kv)$$

and thus

$$s(\rho,\phi,t)=\frac{1}{(2\pi)^{3/2}}\int_0^\infty k\,dk\int_0^{2\pi}d\chi\left(Ae^{ikvt}+Be^{-ikvt}\right)e^{ik\rho\cos\chi}$$

where (a)

$$s(\rho,\phi,0) = \frac{1}{(2\pi)^{3/2}}\int_0^\infty kdk\int_0^{2\pi} d\chi(A+B)e^{ik\rho\cos\chi} = hJ_0(\alpha\rho)$$

and

$$\left.\frac{\partial s}{\partial t}\right|_{t=0} = 0 = \frac{1}{(2\pi)^{3/2}}\int_0^\infty kdk\int_0^{2\pi} d\chi ikv(A-B)e^{ik\rho\cos\chi}$$

Thus

$$A = B$$

$$\frac{1}{(2\pi)^{3/2}}\int_0^\infty kdk\int_0^{2\pi} d\chi 2Ae^{ik\rho\cos\chi} = hJ_0(\alpha\rho)$$

We may make use of the integral expression (8.92) for the Bessel function to obtain:

$$\frac{2}{(2\pi)^{1/2}}\int_0^\infty kdkA(k)J_0(k\rho) = hJ_0(\alpha\rho)$$

Now multiply both sides by $\rho J_0\left(k'\rho\right)$ and integrate

$$\frac{2}{(2\pi)^{1/2}}\int_0^\infty kdkA(k)\int_0^\infty \rho J_0(k\rho)J_0(k'\rho)d\rho = h\int_0^\infty \rho J_0(\alpha\rho)J_0(k'\rho)d\rho$$

$$\frac{2}{(2\pi)^{1/2}}\int_0^\infty kdkA(k)\frac{\delta(k-k')}{k} = h\frac{\delta(\alpha-k')}{\alpha}$$

$$\frac{2}{(2\pi)^{1/2}}A(k') = h\frac{\delta(\alpha-k')}{\alpha}$$

Thus

$$s(\rho,\phi,t) = \frac{1}{(2\pi)^{3/2}}\int_0^\infty kdk\int_0^{2\pi} d\chi h\sqrt{\frac{\pi}{2}}\frac{\delta(\alpha-k)}{\alpha}\left(e^{ikvt}+e^{-ikvt}\right)e^{ik\rho\cos\chi}$$

$$= \frac{1}{4\pi}\int_0^{2\pi} d\chi h\left(e^{i\alpha vt}+e^{-i\alpha vt}\right)e^{i\alpha\rho\cos\chi}$$

$$= h\cos(\alpha vt)J_0(\alpha\rho)$$

The initial disturbance oscillates in time.

34. Evaluate the integral

$$\int_0^\infty e^{-ax}J_m(x)dx = \frac{1}{2\pi}\int_0^\infty e^{-ax}\int_0^{2\pi} e^{ix\sin\phi - im\phi}d\phi$$

$$= \frac{1}{2\pi}\int_0^{2\pi} e^{-im\phi}\left.\frac{e^{x(-a+i\sin\phi)}}{-a+i\sin\phi}\right|_0^\infty d\phi$$

$$= \frac{1}{2\pi}\int_0^{2\pi} e^{-im\phi}\frac{1}{a-i\sin\phi}d\phi$$

Now we let $z = e^{i\phi}$, $dz = ie^{i\phi}d\phi$, and use the methods of Chapter 2 §2.7.2:

$$\int_0^\infty e^{-ax}J_m(x)dx = \frac{1}{2\pi}\oint_{\text{unit circle}}\frac{1}{z^m}\frac{1}{a-i\frac{1}{2i}\left(z-\frac{1}{z}\right)}\frac{dz}{iz}$$

$$= \frac{1}{\pi i}\oint_{\text{unit circle}}\frac{1}{z^m}\frac{1}{2az-\left(z^2-1\right)}dz$$

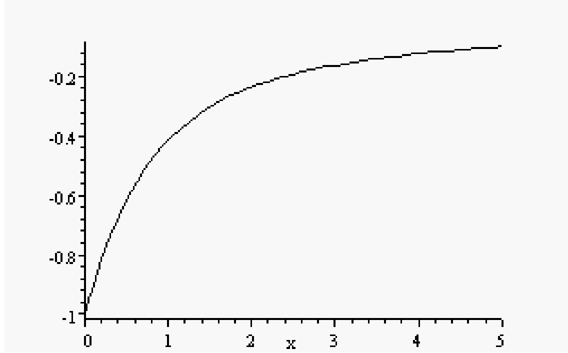
$$= -\frac{1}{\pi i}\oint_{\text{unit circle}}\frac{1}{z^m}\frac{1}{z^2-1-2az}dz$$

The integrand has poles at $z = 0$ (for $m > 0$) and

$$z = a \pm \sqrt{a^2+1}$$

Of these, the poles at $z = 0$ (order m) and at $a - \sqrt{a^2+1}$ are inside the circle.

(The plot is a graph of $x - \sqrt{x^2+1}$ versus x. All values are between -1 and 0.)



Writing the integrand as

$$\frac{1}{z^m}\frac{1}{\left(z-a-\sqrt{a^2+1}\right)}\frac{1}{\left(z-a+\sqrt{a^2+1}\right)}$$

The residue at $a-\sqrt{a^2+1}$ is

$$\frac{1}{\left(a-\sqrt{a^2+1}\right)^m\left(-2\sqrt{a^2+1}\right)}$$

To find the residue at zero we expand in a series,

$$\frac{1}{z^m}\frac{1}{\left(-a-\sqrt{a^2+1}\right)\left(1-\frac{z}{a+\sqrt{a^2+1}}\right)}\frac{1}{\left(-a+\sqrt{a^2+1}\right)}\frac{1}{\left(1-\frac{z}{a-\sqrt{a^2+1}}\right)}$$

$$=-\frac{1}{z^m}\sum_{n=0}^{\infty}\left(\frac{z}{a+\sqrt{a^2+1}}\right)^n\sum_{p=0}^{\infty}\left(\frac{z}{a-\sqrt{a^2+1}}\right)^p$$

$$=-\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\frac{z^{n+p-m}}{\left(a+\sqrt{a^2+1}\right)^n\left(\sqrt{a^2+1}-a\right)^p}$$

The residue is the coefficient of z^{-1} in this series, that is, we need $n+p-m=-1$ or $p=m-n-1$

$$c_{-1}=-\sum_{n=0}^{m-1}\frac{1}{\left(a+\sqrt{a^2+1}\right)^n\left(a-\sqrt{a^2+1}\right)^{m-n-1}}$$

$$=-\frac{1}{\left(a-\sqrt{a^2+1}\right)^{m-1}}\sum_{n=0}^{m-1}\left(\frac{a-\sqrt{a^2+1}}{a+\sqrt{a^2+1}}\right)^n$$

$$=-\frac{1}{\left(a-\sqrt{a^2+1}\right)^{m-1}}\sum_{n=0}^{m-1}(-1)^n\left(a-\sqrt{a^2+1}\right)^{2n}$$

Thus our integral is:

$$\int_0^\infty e^{-ax}J_m(x)dx=2\pi i\frac{-1}{\pi i}\left\{\begin{array}{c}\frac{1}{\left(a-\sqrt{a^2+1}\right)^m\left(-2\sqrt{a^2+1}\right)}\\ -\frac{1}{\left(a-\sqrt{a^2+1}\right)^{m-1}}\sum_{n=0}^{m-1}(-1)^n\left(a-\sqrt{a^2+1}\right)^{2n}\end{array}\right\}$$

$$=\frac{1}{\left(a-\sqrt{a^2+1}\right)^m}\left[\frac{1}{\sqrt{a^2+1}}+2\sum_{n=0}^{m-1}(-1)^n\left(a-\sqrt{a^2+1}\right)^{2n+1}\right]$$

The sum may be evaluated using equation 2.42:

$$\sum_{n=0}^{m-1}(-1)^n\left(a-\sqrt{a^2+1}\right)^{2n} = \frac{1-(-1)^m\left(a-\sqrt{a^2+1}\right)^{2m}}{1+\left(a-\sqrt{a^2+1}\right)^2}$$
$$= \frac{1}{2}\,\frac{1-(-1)^m\left(a-\sqrt{a^2+1}\right)^{2m}}{1+a^2-a\sqrt{a^2+1}}$$
$$= \frac{1}{2}\,\frac{1-(-1)^m\left(a-\sqrt{a^2+1}\right)^{2m}}{\sqrt{a^2+1}\left(\sqrt{1+a^2}-a\right)}$$

Thus

$$\int_0^\infty e^{-ax}J_m(x)dx = \frac{1}{\left(a-\sqrt{a^2+1}\right)^m}\left[\frac{1}{\sqrt{a^2+1}} - \frac{1-(-1)^m\left(a-\sqrt{a^2+1}\right)^{2m}}{\sqrt{a^2+1}}\right]$$
$$= \frac{(-1)^m\left(a-\sqrt{a^2+1}\right)^m}{\sqrt{a^2+1}}$$
$$= \frac{\left(\sqrt{a^2+1}-a\right)^m}{\sqrt{a^2+1}}$$

For $m = 0$ we have the nice result

$$\int_0^\infty e^{-ax}J_0(x)dx = \frac{1}{\sqrt{a^2+1}}$$

35.

$$\int_0^1 x^3J_0(ax)dx = \int_0^1 x^2\frac{d}{dx}(xJ_1(ax))dx$$
$$= x^3J_1(ax)\Big|_0^1 - \int_0^1 2x^2J_1(ax)dx$$
$$= J_1(a) - 2\int_0^1 \frac{d}{dx}(x^2J_2(ax))dx$$
$$= J_1(a) - 2(x^2J_2(ax))\Big|_0^1$$
$$= J_1(a) - 2J_2(a)$$
$$= J_1(a) - 2\left(\frac{2}{a}J_1(a) - J_0(a)\right)$$
$$= 2J_0(a) + \left(1-\frac{4}{a}\right)J_1(a)$$

The result is

$$\int_0^1 x^3J_0(ax)dx = \left(1-\frac{4}{a}\right)J_1(a)$$

if $J_0(a) = 0$ and

$$\int_0^1 x^3J_0(ax)dx = 2J_0(a)$$

if $J_1(a) = 0.$

36. Show that the first zero (other than zero) of the Bessel function $J_m(x),$ $x_{m,1}$ is an increasing function of $m,$ that is

$$x_{0,1} < x_{1,1} < x_{2,1} < x3, 1$$

and so on.

First note that each Bessel function is positive for sufficiently small values of x, as indicated by the small argument expansion (8.80). Similarly, for $m \geq 1$, the derivative is also positive for sufficiently small x. The function increases from zero, then decreases to the first zero. Thus the first zero occurs at x_m where $J'_m(x_m)$ is negative. Now from relations 8.89 and 8.90:

$$J_{m+1}(x) + J_{m-1}(x) = \frac{2m}{x} J_m(x)$$
$$J_{m+1} - J_{m-1} = -2J'_m$$

So

$$J_{m+1} = \frac{m}{x} J_m - J'_m$$

Evaluate at x_m to get:

$$J_{m+1}(x_m) = -J'_m(x_m)$$

which is therefore positive, and also

$$J_{m-1}(x_m) = J_{m+1}(x_m) + 2J'_m(x_m) = J'_m(x_m)$$

is negative. If $J_{m-1}(x_m)$ is negative then $x_m > x_{m-1}$ for $m - 1 > 0$.

QED

We still need to show that $x_1 > x_0$. We know that $J_1(x) = -J'_0(x)$ and therefore $J'_0(x_1) = 0$. Since $x_1 > 0$ by assumption, x_1 must be the first minimum of J_0 . Furthermore,

$$\frac{d}{dx}(xJ_1) = xJ_0 = xJ'_1 - J_1$$

and evaluating at x_1

$$J_0(x_1) = J'_1(x_1)$$

and is therefore negative. These results show that $x_1 > x_0$.

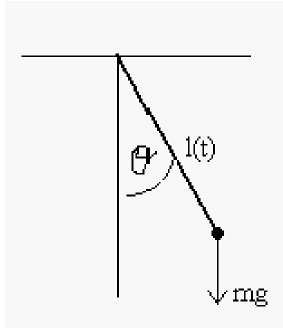
37. A pendulum has steadily increasing length

$l(t) = l_0 + \alpha t$. Show that the equation that describes small oscillations of this pendulum is:

$$l\theta'' + 2\alpha\theta' + g\theta = 0$$

Change variables to $u = \left(1 + \frac{\alpha}{l_0} t\right)^{1/2}$ and

$y = u\theta$, and hence show that the general solution may be expressed in terms of Bessel functions. Find the solution if the pendulum is released from rest at an angle θ_0 at $t = 0$ ($l = l_0$).



The torque about the support is:

$$\vec{\tau} = \vec{r} \times \vec{F} = -mgl(t)\sin\theta\,\hat{\otimes}$$

and the angular momentum with respect to the same origin is

$$\vec{L} = \vec{r} \times \vec{p} = lml\theta'\,\hat{\odot}$$

Thus

$$\vec{\tau} = \frac{d}{dt}\vec{L} \Rightarrow -mgl(t)\sin\theta = \frac{d}{dt}\left(ml^2\theta'\right) = m\left(2l\frac{dl}{dt}\theta' + l^2\theta''\right)$$

So for small oscillations:

$$-g\theta = 2\alpha\theta' + l\theta''$$

as required.

Change variables to $u = \left(1 + \frac{\alpha}{l_0}t\right)^{1/2}$. Then $l = l_0u^2$, and

$$\frac{d}{dt} = \frac{du}{dt}\frac{d}{du} = \frac{1}{2}\frac{\alpha}{l_0}\frac{1}{u}\frac{d}{du}$$

The equation becomes:

$$l_0u^2\frac{1}{2}\frac{\alpha}{l_0}\frac{1}{u}\frac{d}{du}\left(\frac{1}{2}\frac{\alpha}{l_0}\frac{1}{u}\frac{d}{du}\theta\right) + 2\alpha\frac{1}{2}\frac{\alpha}{l_0}\frac{1}{u}\frac{d}{du}\theta + g\theta = 0$$

$$u\frac{1}{4}\frac{\alpha^2}{l_0}\left(-\frac{1}{u^2}\theta' + \frac{1}{u}\theta''\right) + \frac{\alpha^2}{l_0}\frac{1}{u}\theta' + g\theta = 0$$

$$\theta'' + \frac{3}{u}\theta' + 4\frac{gl_0}{\alpha^2}\theta = 0$$

Multiplying by u^2 gives

$$u^2\theta'' + 3u\theta' + 4\frac{gl_0}{\alpha^2}u^2\theta = 0$$

where now prime means d/du. Now let $y = u\theta$, $\theta = y/u$

$$\theta' = \frac{y'}{u} - \frac{y}{u^2}$$

$$\theta'' = -\frac{2y'}{u^2} + \frac{y''}{u} + 2\frac{y}{u^3}$$

$$u^2\left(-\frac{2y'}{u^2} + \frac{y''}{u} + 2\frac{y}{u^3}\right) + 3u\left(\frac{y'}{u} - \frac{y}{u^2}\right) + 4\frac{gl_0}{\alpha^2}uy = 0$$

$$uy'' + y' + y\left(-\frac{1}{u} + 4\frac{gl_0}{\alpha^2}u\right) = 0$$

which is Bessel's equation of order 1 (compare with equation 8.69), with solution:

$$y = AJ_1\left(\frac{2}{\alpha}\sqrt{gl_0}\,u\right) + BN_1\left(\frac{2}{\alpha}\sqrt{gl_0}\,u\right)$$

$$\theta = \sqrt{\frac{l_0}{l}}\left\{AJ_1\left(\frac{2}{\alpha}\sqrt{gl}\right) + BN_1\left(\frac{2}{\alpha}\sqrt{gl}\right)\right\}$$

Here we must include N_1 in the solution because the argument is never zero. If the pendulum starts from rest when $\theta = \theta_0$, then

$$\theta_0 = AJ_1\left(\frac{2}{\alpha}\sqrt{gl_0}\right) + BN_1\left(\frac{2}{\alpha}\sqrt{gl_0}\right)$$

and

$$0 = AJ_1'\left(\frac{2}{\alpha}\sqrt{gl_0}\right) + BN_1'\left(\frac{2}{\alpha}\sqrt{gl_0}\right)$$

So

$$\theta_0 = AJ_1\left(\frac{2}{\alpha}\sqrt{gl_0}\right) - A\frac{J_1'\left(\frac{2}{\alpha}\sqrt{gl_0}\right)}{N_1'\left(\frac{2}{\alpha}\sqrt{gl_0}\right)}N_1\left(\frac{2}{\alpha}\sqrt{gl_0}\right)$$

and thus

$$A = \frac{\theta_0 N_1'\left(\frac{2}{\alpha}\sqrt{gl_0}\right)}{J_1\left(\frac{2}{\alpha}\sqrt{gl_0}\right)N_1'\left(\frac{2}{\alpha}\sqrt{gl_0}\right) - J_1'\left(\frac{2}{\alpha}\sqrt{gl_0}\right)N_1\left(\frac{2}{\alpha}\sqrt{gl_0}\right)}$$

The denominator is the Wronskian of the two solutions, and equals:

$$W(x) = W(x_0)\exp\left(-\int_{x_0}^{x}\frac{1}{x}dx\right) = W(x_0)\exp(-\ln x/x_0) = \frac{W(x_0)x_0}{x}$$
$$W(x) = \frac{C}{x}$$

Using the large arument form of the Bessels, we can evaluate the constant:

$$\begin{aligned}
W(x) &= \sqrt{\frac{2}{\pi x}}\cos\left(x - \frac{m\pi}{2} - \frac{\pi}{4}\right)\frac{d}{dx}\sqrt{\frac{2}{\pi x}}\sin\left(x - \frac{m\pi}{2} - \frac{\pi}{4}\right) \\
&\quad - \sqrt{\frac{2}{\pi x}}\sin\left(x - \frac{m\pi}{2} - \frac{\pi}{4}\right)\frac{d}{dx}\sqrt{\frac{2}{\pi x}}\cos\left(x - \frac{m\pi}{2} - \frac{\pi}{4}\right) \\
&= \frac{2}{\pi x}\left(\cos^2\left(x - \frac{m\pi}{2} - \frac{\pi}{4}\right) + \sin^2\left(x - \frac{m\pi}{2} - \frac{\pi}{4}\right)\right) \\
&= \frac{2}{\pi x}
\end{aligned}$$

Thus

$$A = \theta_0\frac{\pi}{\alpha}\sqrt{gl_0}\,N_1'\left(\frac{2}{\alpha}\sqrt{gl_0}\right)$$

and the solution for θ is

$$\begin{aligned}
\theta &= \theta_0\sqrt{\frac{l_0}{l}}\frac{\pi}{\alpha}\sqrt{gl_0}\,N_1'\left(\frac{2}{\alpha}\sqrt{gl_0}\right)\left\{J_1\left(\frac{2}{\alpha}\sqrt{gl}\right) - \frac{J_1'\left(\frac{2}{\alpha}\sqrt{gl_0}\right)}{N_1'\left(\frac{2}{\alpha}\sqrt{gl_0}\right)}N_1\left(\frac{2}{\alpha}\sqrt{gl}\right)\right\} \\
&= \theta_0\sqrt{\frac{g}{l}}\frac{\pi l_0}{\alpha}\left\{J_1\left(\frac{2}{\alpha}\sqrt{gl}\right)N_1'\left(\frac{2}{\alpha}\sqrt{gl_0}\right) - J_1'\left(\frac{2}{\alpha}\sqrt{gl_0}\right)N_1\left(\frac{2}{\alpha}\sqrt{gl}\right)\right\} \\
&= \theta_0\sqrt{\frac{gl_0}{1+\frac{\alpha}{l_0}t}}\frac{\pi}{\alpha}\left\{J_1\left(\frac{2}{\alpha}\sqrt{gl_0}\sqrt{1+\frac{\alpha}{l_0}t}\right)N_1'\left(\frac{2}{\alpha}\sqrt{gl_0}\right) - J_1'\left(\frac{2}{\alpha}\sqrt{gl_0}\right)N_1\left(\frac{2}{\alpha}\sqrt{gl_0}\sqrt{1+\frac{\alpha}{l_0}t}\right)\right\}
\end{aligned}$$

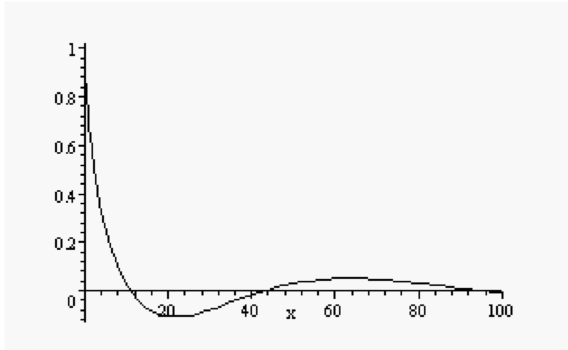
Check the dimensions: αt is a length, and so

$$\left[\frac{\sqrt{gl_0}}{\alpha}\right] = \frac{\sqrt{\frac{L}{T^2}L}}{L/T} = 1 \text{ (dimensionless)}$$

as is required. The plot shows $\theta(\tau)/\theta_0$ with $2\sqrt{gl_0}/\alpha = 1$ and the dimensionless time variable $\tau = \alpha t/l_0$.

We used the recursion relation 8.90 to evaluate the derivatives: $N_1' = \frac{1}{2}(N_0 - N_2)$ and similarly for

J. The numerical values are easily computed using EXCEL (``bessely'' is the Neumann function). We find $N_1'(1) = 0.86947$ and $J_1'(1) = 0.325147$



38. The equation that describes the angular displacement of a vertical pole or column is

$$EI\frac{d^2\theta}{dx^2} + g\lambda x\theta = 0$$

where x increases downward from the top of the pole, E is the Young's modulus,
I is the moment of inertia (see also Chapter 3 §3.2.3) and
λ is the mass per unit length. Make a change of variables to $u = \frac{2}{3}\sqrt{\frac{g\lambda}{EI}x^3}$,
$y = \theta/\sqrt{x}$ and hence show that the solution may be expressed in terms of Bessel functions. Show that there is no solution that fits the boundary conditions $\theta(L) = 0$ and $\theta'(0) = 0$ unless the pole has a minimum length $L_{\min}$. Find an expression for $L_{\min}$ in terms of the physical parameters of the pole.

Let $g\lambda/EI = \mu^2$. Then

$$\theta'' + \mu^2 x\theta = 0$$

Now change variables: $x = \left(\frac{3u}{2\mu}\right)^{2/3}$, and $\theta = \sqrt{x}y$

$$\theta' = \frac{1}{2\sqrt{x}}y + \sqrt{x}y' = \frac{1}{2}\left(\frac{3u}{2\mu}\right)^{-1/3}y + \left(\frac{3u}{2\mu}\right)^{1/3}y'$$

$$\theta'' = -\frac{1}{4}x^{-3/2}y + \frac{y'}{\sqrt{x}} + \sqrt{x}y''$$

$$= -\frac{1}{4}\left(\frac{3u}{2\mu}\right)^{-1}y + \left(\frac{3u}{2\mu}\right)^{-1/3}y' + \left(\frac{3u}{2\mu}\right)^{1/3}y''$$

The differential equation becomes:

$$-\frac{1}{4}\left(\frac{3u}{2\mu}\right)^{-1}y + \left(\frac{3u}{2\mu}\right)^{-1/3}y' + \left(\frac{3u}{2\mu}\right)^{1/3}y'' + \mu^2\left(\frac{3u}{2\mu}\right)y = 0$$

$$y'' + \left(\frac{3u}{2\mu}\right)^{-2/3}y' + \left(\mu^2 - \frac{1}{4}\left(\frac{3u}{2\mu}\right)^{-2}\right)\left(\frac{3u}{2\mu}\right)^{2/3}y = 0$$

$$y'' + \left(\frac{3u}{2\mu}\right)^{-2/3}y' + \left(1 - \frac{1}{9u^2}\right)\mu^2\left(\frac{3u}{2\mu}\right)^{2/3}y = 0$$

Next, with $u = \frac{2}{3}\mu x^{3/2}$:

$$\frac{d}{dx} = \frac{du}{dx}\frac{d}{du} = \frac{2}{3}\mu\frac{3}{2}x^{1/2}\frac{d}{du} = \mu\left(\frac{3u}{2\mu}\right)^{1/3}\frac{d}{du}$$

and

$$\begin{aligned}\frac{d^2}{dx^2} &= \mu\left(\frac{3u}{2\mu}\right)^{1/3}\frac{d}{du}\mu\left(\frac{3u}{2\mu}\right)^{1/3}\frac{d}{du}\\ &= \mu^2\left(\frac{3u}{2\mu}\right)^{2/3}\frac{d^2}{du^2} + \mu^2\left(\frac{3}{2\mu}\right)^{2/3}u^{1/3}\left(\frac{1}{3}u^{-2/3}\right)\frac{d}{du}\\ &= \mu^2\left(\frac{3u}{2\mu}\right)^{2/3}\left(\frac{d^2}{du^2} + \frac{1}{3u}\frac{d}{du}\right)\end{aligned}$$

Thus

$$\begin{aligned}0 &= \mu^2\left(\frac{3u}{2\mu}\right)^{2/3}\left(\frac{d^2y}{du^2} + \frac{1}{3u}\frac{dy}{du}\right) + \left(\frac{3u}{2\mu}\right)^{-2/3}\mu\left(\frac{3u}{2\mu}\right)^{1/3}\frac{dy}{du} + \left(1 - \frac{1}{9u^2}\right)\mu^2\left(\frac{3u}{2\mu}\right)^{2/3}y\\ 0 &= y'' + \frac{1}{3u}y' + \frac{1}{\mu}\left(\frac{2\mu}{3u}\right)y' + \left(1 - \frac{1}{9u^2}\right)y\\ 0 &= uy'' + y' + \left(u - \frac{1}{9u}\right)y\end{aligned}$$

This is Bessel's equation of order 1/3, so the solution is:

$$y = J_{\pm 1/3}(u)$$

or, in the original variables:

$$\theta = \sqrt{x}\left\{A_+ J_{1/3}\left(\frac{2}{3}\sqrt{\frac{g\lambda}{EI}x^3}\right) + A_- J_{-1/3}\left(\frac{2}{3}\sqrt{\frac{g\lambda}{EI}x^3}\right)\right\}$$

Then

$$\begin{aligned}\theta'(x) &= \frac{1}{2\sqrt{x}}\left\{A_+ J_{1/3}\left(\frac{2}{3}\sqrt{\frac{g\lambda}{EI}x^3}\right) + A_- J_{-1/3}\left(\frac{2}{3}\sqrt{\frac{g\lambda}{EI}x^3}\right)\right\}\\ &\quad + \sqrt{x}\left\{A_+\frac{d}{dx}J_{1/3}\left(\frac{2}{3}\sqrt{\frac{g\lambda}{EI}x^3}\right) + A_-\frac{d}{dx}J_{-1/3}\left(\frac{2}{3}\sqrt{\frac{g\lambda}{EI}x^3}\right)\right\}\end{aligned}$$

Using the series 8.73 with $\alpha = \frac{2}{3}\sqrt{g\lambda/EI}$

$$\begin{aligned}J_{\pm 1/3}\left(\alpha x^{3/2}\right) &= \sum_{n=0}^{\infty}\frac{(-1)^n}{n!\Gamma(n+1\pm 1/3)}\left(\frac{\alpha x^{3/2}}{2}\right)^{2n\pm 1/3}\\ &= \sum_{n=0}^{\infty}\frac{(-1)^n}{n!\Gamma(n+1\pm 1/3)}\left(\frac{\alpha}{2}\right)^{2n\pm 1/3}x^{3n\pm 1/2}\end{aligned}$$

and the derivatives are:

$$\begin{aligned}\frac{d}{dx}J_{+1/3}\left(\alpha x^{3/2}\right) &= \sum_{n=0}^{\infty}\frac{(-1)^n(3n+1/2)}{n!\Gamma(n+1+1/3)}\left(\frac{\alpha}{2}\right)^{2n\pm 1/3}x^{3n-1/2}\\ \frac{d}{dx}J_{-1/3}\left(\alpha x^{3/2}\right) &= \sum_{n=0}^{\infty}\frac{(-1)^n(3n-1/2)}{n!\Gamma(n+1-1/3)}\left(\frac{\alpha}{2}\right)^{2n\pm 1/3}x^{3n-3/2}\end{aligned}$$

Thus

$$\begin{aligned}2\sqrt{x}\,\theta'(x) &= A_+\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\Gamma(n+1+1/3)}\left(\frac{\alpha}{2}\right)^{2n+1/3}x^{3n+1/2}(2+6n)\\ &\quad + A_-\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\Gamma(n+1-1/3)}\left(\frac{\alpha}{2}\right)^{2n-1/3}x^{3n-1/2}(6n)\end{aligned}$$

Thus the boundary condition $\theta'(0) = 0$ is satisfied if we choose $A_+ \equiv 0.$ Then:

$$\begin{aligned}\theta'(x) &= \frac{1}{2\sqrt{x}}6A_-\sum_{n=1}^{\infty}\frac{(-1)^n}{(n-1)!\Gamma\left(n+\frac{2}{3}\right)}\left(\frac{\alpha}{2}\right)^{2n-1/3}x^{3n-1/2}\\ &= 3A_-\left(\frac{1}{3}\right)^{-1/3}\left(\frac{g\lambda}{EI}\right)^{-1/6}\sum_{n=1}^{\infty}\frac{(-1)^n}{(n-1)!\Gamma(n+2/3)}\left(\frac{\alpha}{2}\right)^{2n}x^{3n-1}\end{aligned}$$

and so $\theta'(0) = 0.$ Then to satisfy the second condition, we must have

$$J_{-1/3}\left(\frac{2}{3}\sqrt{\frac{g\lambda}{EI}L^3}\right) = 0$$

Thus $\frac{2}{3}\sqrt{\frac{g\lambda}{EI}L^3}$ must be one of the zeroes of $J_{-1/3}.$ Thus

L can take on only a specific set of values, and in particular there is a smallest value corresponding to the first zero of $J_{-1/3}.$ That value is:

$$\frac{2}{3}\sqrt{\frac{g\lambda}{EI}L^3} = 1.8664$$

or

$$L = \left(\frac{3}{2}1.8664\right)^{2/3}\left(\frac{EI}{g\lambda}\right)^{1/3} = 1.9864\left(\frac{EI}{g\lambda}\right)^{1/3}$$

39. Establish the addition theorem for Bessel functions.

We start by expanding the function $1/\left|\vec{\mathbf{x}} - \vec{\mathbf{x}}'\right|,$ where x and x' are vectors in the $x - y$ plane,. in the complete orthogonal set of functions $e^{im\phi}J_m(k\rho)$:

$$\frac{1}{\left|\vec{\mathbf{x}} - \vec{\mathbf{x}}'\right|} = \sum_m \int dk A(k) e^{im\phi} J_m(k\rho)$$

By symmetry in $\vec{\mathbf{x}}, \vec{\mathbf{x}}',$ we have

$$\frac{1}{\left|\vec{\mathbf{x}} - \vec{\mathbf{x}}'\right|} = \sum_{m=-\infty}^{\infty} \int dk\alpha(k) e^{im(\phi-\phi')} J_m(k\rho) J_m(k\rho')$$

Now let $\rho' \to 0.$ The only non-zero Bessel function is $J_0,$ so

$$\frac{1}{\rho} = \int_0^\infty \alpha(k) J_0(k\rho)dk = \int_0^\infty \alpha\left(\frac{u}{\rho}\right) J_0(u)\frac{du}{\rho}$$

From the result of problem 34 in the limit $a \to 0,$ we conclude that $\alpha(k) \equiv 1.$ Thus

$$\frac{1}{\left|\vec{\mathbf{x}} - \vec{\mathbf{x}}'\right|} = \sum_m \int dk e^{im(\phi-\phi')} J_m(k\rho) J_m(k\rho') = \frac{1}{R} = \int_0^\infty J_0(kR)dk$$

Since this result is valid for any value of $R,$ we conclude that

$$\sum_m e^{im\phi} J_m(k\rho) J_m(k\rho') = J_0(kR)$$

Note also that this expression gives correct results in the limit $\phi \to \pi$ (Problem 31) and in the limit $\rho' \to 0.$

40. Starting from the definitions 8.100, 8.76 and 8.78, show that::

$$K_\nu(x) = \frac{\pi}{2}\frac{I_{-\nu}(x) - I_\nu(x)}{\sin\nu\pi}$$

where ν is not an integer.

$$
\begin{aligned}
K_\nu(x) &= \frac{\pi}{2} i^{\nu+1} H_\nu^{(1)}(ix) \\
&= \frac{\pi}{2} i^{\nu+1} \left(J_\nu(ix) + iN_\nu(ix)\right) \\
&= \frac{\pi}{2} i^{\nu+1} \left(J_\nu(ix) + i\frac{J_\nu(ix)\cos\nu\pi - J_{-\nu}(ix)}{\sin\nu\pi} \right) \\
&= \frac{\pi}{2} i^{\nu+1} \left(\frac{J_\nu(ix)(\sin\nu\pi + i\cos\nu\pi) - iJ_{-\nu}(ix)}{\sin\nu\pi} \right) \\
&= \frac{\pi}{2} i^{\nu+1} \left(\frac{i^\nu I_\nu(x) i(-i\sin\nu\pi + \cos\nu\pi) - ii^{-\nu} I_{-\nu}(x)}{\sin\nu\pi} \right) \\
&= \frac{\pi}{2} \left(\frac{-i^{2\nu} I_\nu(x) e^{-i\nu\pi} + I_{-\nu}(x)}{\sin\nu\pi} \right) \\
&= \frac{\pi}{2} \left(\frac{I_{-\nu}(x) - e^{i\pi\nu} I_\nu(x) e^{-i\nu\pi}}{\sin\nu\pi} \right) \\
&= \frac{\pi}{2} \left(\frac{I_{-\nu}(x) - I_\nu(x)}{\sin\nu\pi} \right)
\end{aligned}
$$

as required. Also

$$
K_{-\nu}(x) = \frac{\pi}{2} \left(\frac{I_\nu(x) - I_{-\nu}(x)}{\sin(-\nu\pi)} \right) = \frac{\pi}{2} \left(\frac{-I_\nu(x) + I_{-\nu}(x)}{\sin(\nu\pi)} \right) = K_\nu(x)
$$

Chapter 8: Sturm-Liouville Theory

41. The potential on a plane is V_0 for $\rho < a$ and zero for $r > a.$ Find the potential everywhere.

Since the system is axisymmetric, only $m = 0$ appears in the solution, which has the form

$$\Phi(\rho,z) = \int_0^\infty A_0(k) J_0(k\rho) e^{\pm kz} dk$$

where, to ensure that the potential decreases as we move away from the plane, the minus sign applies for $z > 0$ and the plus sign for $z < 0$.

$$A_0(k) = k\int_0^a V_0 J_0(k\rho)\rho d\rho = \frac{V_0}{k}\int_0^{ka} J_0(x) x dx = \frac{V_0}{k}\int_0^{ka} \frac{d}{dx}[J(x)x] dx$$
$$= V_0 a J_1(ka)$$

Thus

$$\Phi(\rho,\phi,z) = V_0 a \int_0^\infty J_1(ka) J_0(k\rho) e^{-kz} dk$$
$$= V_0 \int_0^\infty J_1(x) J_0\left(\frac{\rho}{a}x\right) \exp\left(-\frac{z}{a}x\right) dx$$

On the $z-$axis

$$\Phi(z) = V_0 \int_0^\infty J_1(x) \exp\left(-\frac{z}{a}x\right) dx = V_0 \frac{\sqrt{1 + z^2/a^2} - \frac{z}{a}}{\sqrt{1 + z^2/a^2}}$$
$$= V_0 \frac{\sqrt{a^2 + z^2} - z}{\sqrt{a^2 + z^2}} = V_0\left[1 - \frac{z}{\sqrt{a^2 + z^2}}\right]$$

where we used GR 6.611#1 or the result of Problem 34b

42. A cylinder of height h and radius a has the top and bottom grounded. The potential on the wall at $\rho = a$ is $V_0.$ Find the potential inside the cylinder.

Again we have axisymmetry, so only $m = 0$ terms contribute. The potential should be finite on the axis at $\rho = 0,$ so we exclude the K function. Thus the potential is of the form (compare with Example 8.5).

$$\Phi(\rho,z) = \sum_n A_n \sin\left(\frac{n\pi z}{h}\right) I_0\left(\frac{n\pi}{h}\rho\right)$$

Now evaluate the potential at $\rho = a$:

$$V_0 = \sum_n A_n \sin\left(\frac{n\pi z}{h}\right) I_0\left(\frac{n\pi}{h}a\right)$$

This is a Fourier sine series, and we find the coefficients in the usual way:

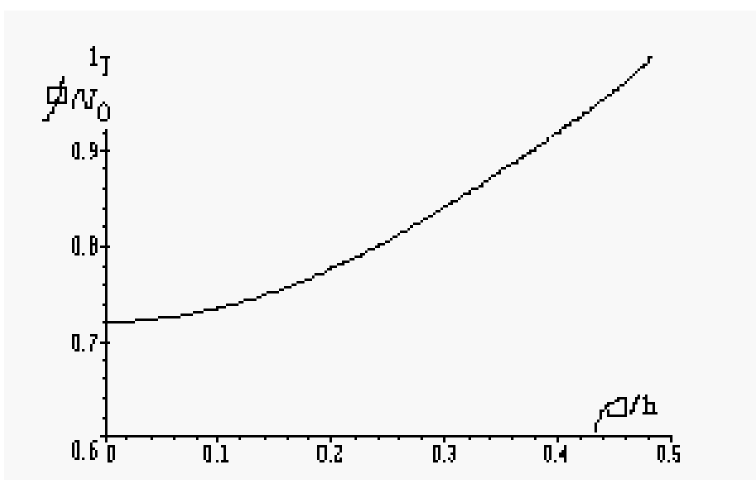
$$A_n \frac{h}{2} = \frac{V_0}{I_0\left(\frac{n\pi}{h}a\right)} \int_0^h \sin\left(\frac{n\pi z}{h}\right) dz = \frac{V_0}{I_0\left(\frac{n\pi}{h}a\right)} \frac{h}{n\pi}(1 - \cos n\pi)$$

$$= \frac{V_0}{I_0\left(\frac{n\pi}{h}a\right)} \frac{2h}{n\pi} \text{ for } n \text{ odd}$$

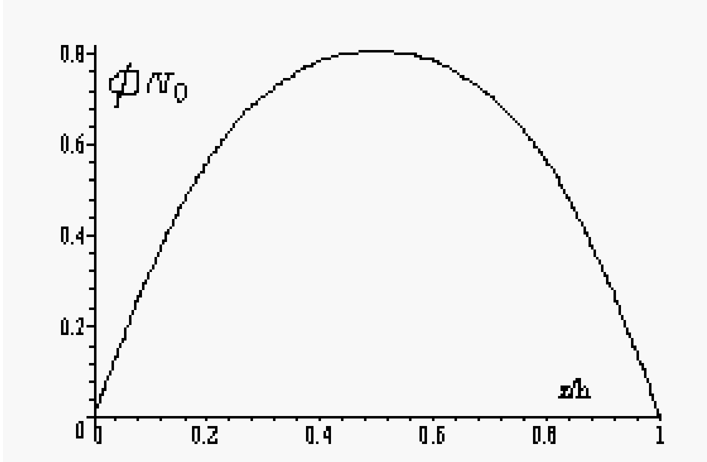
Thus

$$\Phi(\rho,z) = \frac{4V_0}{\pi} \sum_{n \text{ odd}} \frac{\sin\left(\frac{n\pi z}{h}\right)}{n} \frac{I_0\left(\frac{n\pi}{h}\rho\right)}{I_0\left(\frac{n\pi}{h}a\right)}$$

The first plot shows the first 9 terms in the expansion of $\Phi(\rho, h/2)/V_0$ for $a = h/2$.



The second plot shows $\Phi(a/2,z)/V_0$ with $a = h/2$.



43. A cylinder of height h and radius a is grounded except for its base at $z = 0.$ On the base the potential is $\frac{V_0}{\sqrt{1-\rho^2/a^2}}.$ Find the potential inside the cylinder.

Again we have axisymmetry, and we must choose the eigenvalue to make the potential zero at $\rho = a.$ To make the potential zero at $\rho = h$ we use the hyperbolic sine function.

$$\Phi(\rho,z) = \sum_n A_n \sinh\left(x_{0n}\frac{(h-z)}{a}\right) J_0\left(x_{0n}\frac{\rho}{a}\right)$$

where

$$\begin{aligned} A_n \sinh\left(\frac{h}{a}x_{0n}\right)\frac{a^2}{2}\left[J_0'(x_{0n})\right]^2 &= \int_0^a \frac{V_0}{\sqrt{1-\rho^2/a^2}} J_0\left(x_{0n}\frac{\rho}{a}\right)\rho d\rho \\ &= a^2\int_0^1 \frac{V_0}{\sqrt{1-y^2}} J_0(x_{0n}y)ydy \\ &= V_0 a^2 \frac{\sin x_{0n}}{x_{0n}} \end{aligned}$$

We used GR 6.554#2 to evaluate the integral. . Thus

$$\sum_n 2V_0 \frac{\sin x_{0n}}{x_{0n}} \frac{\sinh\left(x_{0n}\frac{(h-z)}{a}\right)}{\sinh\left(\frac{h}{a}x_{0n}\right)} \frac{J_0(x_{0n}\frac{\rho}{a})}{\left[J_0'(x_{0n})\right]^2}$$

44. (a) Use the series for $J_0(x)$ to show that its Laplace transform is

$$\mathcal{L}(J_0(x)) = \frac{1}{\sqrt{1+s^2}}$$

(b) Then use the recursion relation 8.87 to find the Laplace transform of $J_1(x)$. Extend the result to show that

$$\mathcal{L}(J_m(x)) = \frac{\left(\sqrt{s^2+1}-s\right)^m}{\sqrt{s^2+1}}$$

(c) Use the convolution theorem to establish the relation

$$\int_0^x J_0(x-u)J_0(u)du = \sin x$$

$$\begin{aligned}\mathcal{L}(J_0(x)) &= \int_0^\infty e^{-sx}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!n!}\left(\frac{x}{2}\right)^{2n}dx \\ &= \sum_{n=0}^{\infty}\frac{(-1)^n}{n!n!}\frac{1}{2^{2n}}\frac{(2n)!}{s^{2n+1}} \\ &= \sum_{n=0}^{\infty}\frac{(-1)^n}{n!}\frac{1}{2^n}\frac{(2n-1)!!}{s^{2n+1}}\end{aligned}$$

Compare with the series expansion

$$\frac{1}{s\sqrt{1+1/s^2}} = \frac{1}{s}\left(1-\frac{1}{2s^2}+\frac{1\times 3}{2^2 2! s^4}+\cdots(-1)^n\frac{(2n-1)!!}{2^n n! s^{2n}}+\cdots\right)$$

The series are the same, and thus

$$\mathcal{L}(J_0(x)) = \frac{1}{\sqrt{s^2+1}}$$

(b) Also

$$\mathcal{L}(J_1(x)) = \mathcal{L}\left(-J_0'(x)\right)$$

and by the derivative rule:

$$\begin{aligned}\mathcal{L}(J_1(x)) &= -[s\mathcal{L}(J_0(x)) - J_0(0)] \\ &= 1-\frac{s}{\sqrt{s^2+1}} = \frac{\sqrt{s^2+1}-s}{\sqrt{s^2+1}}\end{aligned}$$

First assume the result is true for some value of m. We have just shown it is true for $m = 0$ and $m = 1$). Then using the recurrence relation (8.90)

$$J_{m+1} = J_{m-1} - 2J'_m$$

Taking the transform

$$\begin{aligned}\mathcal{L}(J_{m+1}(x)) &= \mathcal{L}(J_{m-1}(x)) - 2\mathcal{L}\left(J'_m(x)\right) \\ &= \frac{\left(\sqrt{s^2+1} - s\right)^{m-1}}{\sqrt{s^2+1}} - 2s\frac{\left(\sqrt{s^2+1} - s\right)^{m}}{\sqrt{s^2+1}}\end{aligned}$$

since $J_m(0) = 0$ for $m \geq 1$. Simplifying:

$$\begin{aligned}\mathcal{L}(J_{m+1}(x)) &= \frac{\left(\sqrt{s^2+1} - s\right)^{m-1}\left(1 - 2s\left[\sqrt{s^2+1} - s\right]\right)}{\sqrt{s^2+1}} \\ &= \frac{\left(\sqrt{s^2+1} - s\right)^{m-1}\left(1 + 2s^2 - 2s\sqrt{s^2+1}\right)}{\sqrt{s^2+1}} \\ &= \frac{\left(\sqrt{s^2+1} - s\right)^{m-1}\left(s - \sqrt{s^2+1}\right)^2}{\sqrt{s^2+1}} \\ &= \frac{\left(\sqrt{s^2+1} - s\right)^{m+1}}{\sqrt{s^2+1}}\end{aligned}$$

hus the result is true for $m + 1$ if it is true for m. Since it is true for $m = 1$, it is true for all m.

(c) $\int_0^x J_0(x-u)J_0(u)du$ is the convolution of J_0 with itself. The transform is thus

$$\mathcal{L}\left(\int_0^x J_0(x-u)J_0(u)du\right) = [\mathcal{L}(J_0)]^2 = \frac{1}{1+s^2}$$

This is the transform of $\sin x$ (Table 5.1) and thus

$$\int_0^x J_0(x-u)J_0(u)du = \sin x$$

as required.

45. Obtain expression (8.126) for $j_2(x)$ from the expressions for j_0 and j_1 and the recursion relation (8.120).

$$\begin{aligned} j_2(x) &= \frac{3}{x} j_1(x) - j_0(x) \\ &= \frac{3}{x}\left[\frac{\sin x}{x^2} - \frac{\cos x}{x}\right] - \frac{\sin x}{x} \\ &= \sin x\left(\frac{3}{x^3} - \frac{1}{x}\right) - \frac{3}{x^2}\cos x \end{aligned}$$

46. Starting with the recursion relations (8.86) and (8.88), derive the relations:

$$\frac{d}{dx}\left(\frac{j_l(x)}{x^l}\right) = -\frac{j_{l+1}(x)}{x^l}$$

and

$$\frac{d}{dx}\left(x^{l+1} j_l(x)\right) = x^{l+1} j_{l-1}(x)$$

The recursion relation (8.86) is:

$$\frac{d}{dx}\left(\frac{J_m(x)}{x^m}\right) = -\frac{J_{m+1}(x)}{x^m}$$

Let $m = l + 1/2$, then

$$\frac{d}{dx}\left(\frac{\sqrt{x}\, j_l(x)}{x^{l+1/2}}\right) = -\frac{\sqrt{x}\, j_{l+1}(x)}{x^{l+1/2}}$$

$$\frac{d}{dx}\left(\frac{j_l(x)}{x^l}\right) = -\frac{j_{l+1}(x)}{x^l}$$

The second relation is:

$$\frac{d}{dx}\left(x^m J_m(x)\right) = x^m J_{m-1}(x)$$

Again let $m = l + 1/2$:

$$\frac{d}{dx}\left(x^{l+1} j_l(x)\right) = x^{l+1} j_{l-1}(x)$$

47. Use proof by induction (Appendix III) to establish the Rodrigues-type formula

$$j_n(x) = (-1)^n x^n \left(\frac{1}{x}\frac{d}{dx}\right)^n \left(\frac{\sin x}{x}\right)$$

Start with the relation from problem 46:

$$j_1(x) = -\frac{d}{dx} j_0(x) = -x \frac{1}{x}\frac{d}{dx}\left(\frac{\sin x}{x}\right)$$

So the result is true for $n = 1.$ Now assume the result is true for $n = m.$ Then

$$\begin{aligned} j_{m+1}(x) &= -x^m \frac{d}{dx}\left(\frac{j_m}{x^m}\right) = -x^m \frac{d}{dx}\left(\frac{1}{x^m}(-1)^m x^m \left(\frac{1}{x}\frac{d}{dx}\right)^m \frac{\sin x}{x}\right) \\ &= (-1)^{m+1} x^{m+1} \frac{1}{x}\frac{d}{dx}\left(\left(\frac{1}{x}\frac{d}{dx}\right)^m \frac{\sin x}{x}\right) \\ &= (-1)^{m+1} x^{m+1} \left(\frac{1}{x}\frac{d}{dx}\right)^{m+1} \frac{\sin x}{x} \end{aligned}$$

so the result is true for $n = m + 1.$ Thus it is true for all $n.$

48. Use the recursion relations to show that the orthogonality relation (8.130) is equivalent to (8.129).

We start with the recursion relation

$$j_{l+1}(x) + j_{l-1}(x) = \frac{2l+1}{x} j_l(x)$$

So if x is a zero of $j_l,$ then $j_{l+1}(x) = -j_{l-1}(x)$

$$\begin{aligned} j_l'(\alpha_{l+1/2,p}) &= \frac{(l+1)j_{l+1}(\alpha_{l+1/2,p}) - lj_{l-1}(\alpha_{l+1/2,p})}{-(2l+1)} \\ &= -\frac{(l+1)j_{l+1}(\alpha_{l+1/2,p}) + lj_{l+1}(\alpha_{l+1/2,p})}{(2l+1)} \\ &= -\frac{(2l+1)j_{l+1}(\alpha_{l+1/2,p})}{(2l+1)} \\ &= -j_{l+1}(\alpha_{l+1/2,p}) \end{aligned}$$

Upon squaring, the minus sign disappears, and the equivalence is proved.

49. Starting from the definition (8.76), show that

$$n_l(x) = (-1)^{l+1} j_{-(l+1)}(x)$$

Hence show that

$$n_0(x) = -\frac{\cos x}{x}$$

$$N_{l+1/2}(x) = \frac{1}{\sin\left(l+\frac{1}{2}\right)\pi}\left\{\cos\left(l+\frac{1}{2}\right)\pi J_{l+1/2}(x) - J_{-(l+1/2)}(x)\right\}$$

$$= \frac{-J_{-(l+1/2)}(x)}{\cos l\pi} = (-1)^{l+1}J_{-(l+1/2)}(x)$$

Divide through by $\sqrt{x}$ and the result follows.

Using the series expansion for J, we have:

$$n_0(x) = (-1)\sqrt{\frac{\pi}{2x}}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\Gamma(n-1/2+1)}\left(\frac{x}{2}\right)^{2n-1/2}$$

$$= -\frac{\sqrt{\pi}}{2}\sum_{n=0}^{\infty}\frac{(-1)^n}{n!\Gamma(n+1/2)}\left(\frac{x}{2}\right)^{2n-1}$$

As in the text, we can rewrite the gamma function:

$$\Gamma(n+1/2) = \left(n-\frac{1}{2}\right)\Gamma\left(n-\frac{1}{2}\right) = \frac{(2n-1)!!}{2^{n-1}}\sqrt{\pi}$$

$$= \frac{(2n-1)!}{2^{n-1}(2[n-1])!!}\sqrt{\pi} = \frac{(2n-1)!}{2^{2(n-1)}(n-1)!}\sqrt{\pi}$$

and thus

$$n_0(x) = -\frac{1}{x}\sum_{n=0}^{\infty}\frac{(-1)^n 2^{2(n-1)}(n-1)!}{n!(2n-1)!}\left(\frac{x}{2}\right)^{2n}$$

$$= -\frac{1}{x}\sum_{n=0}^{\infty}\frac{(-1)^n}{2n(2n-1)!}x^{2n}$$

$$= -\frac{1}{x}\sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}x^{2n} = -\frac{\cos x}{x}$$

as required.

50. Starting from relations 8.111 and 8.112, establish the recursion relations for the spherical modified Bessel functions $k_l(x) = \sqrt{2/\pi x}\,K_{l+1/2}(x)$.

$$k_{l-1} - k_{l+1} = -\left(\frac{2l+1}{x}\right)k_l$$

and

$$(l+1)k_{l+1} + lk_{l-1} = -(2l+1)\frac{d}{dx}k_l(x)$$

From 8.111 we obtain immediately

$$k_{m-1} - k_{m+1} = -\frac{2(m+1/2)}{x} k_m = -\left(\frac{2m+1}{x}\right) k_m$$

For the derivative, using 8.112 we have:

$$\begin{aligned}
\frac{d}{dx} k_l(x) &= \frac{d}{dx} \frac{1}{\sqrt{x}} K_{l+1/2}(x) = -\frac{1}{2x^{3/2}} K_{l+1/2}(x) + \frac{1}{\sqrt{x}} \frac{d}{dx} K_{l+1/2}(x) \\
&= -\frac{1}{2x} k_l(x) - \frac{1}{2\sqrt{x}} \left(K_{l-1/2} + K_{l+3/2}\right) \\
&= -\frac{1}{2x} k_l(x) - \frac{1}{2} \left(k_{l-1} + k_{l+1}\right) \\
&= \frac{1}{2} \left(\frac{1}{2l+1}\right) \left(k_{l-1} - k_{l+1}\right) - \frac{1}{2} \left(k_{l-1} + k_{l+1}\right) \\
&= \frac{1}{2} \left(\frac{1}{2l+1}\right) \left[-(2l+2) k_{l+1} - 2l k_{l-1}\right] \\
&= -\left(\frac{1}{2l+1}\right) \left[(l+1) k_{l+1} + l k_{l-1}\right]
\end{aligned}$$

51. The Fresnel integrals

$$S(x) = \sqrt{\frac{2}{\pi}} \int_0^x \sin t^2 dt$$

and

$$C(x) = \sqrt{\frac{2}{\pi}} \int_0^x \cos t^2 dt$$

may be expressed as a series of spherical Bessel functions. First show that

$$S(x) = \frac{1}{\sqrt{2\pi}} \int_0^{x^2} \sqrt{u}\, j_0(u) du$$

and obtain a similar expression for

$C(x).$ Use the recursion relation to do the integration, and hence establish the result

$$S(x) = \sqrt{\frac{2}{\pi}}\, x \sum_{n=1}^{\infty} j_{2n-1}\left(x^2\right)$$

Determine a similar expression for $C(x).$

Let $t^2 = u.$ Then $2tdt = du,$ and the integral becomes:

$$S(x) = \sqrt{\frac{2}{\pi}} \int_0^{x^2} \sin u \frac{du}{2\sqrt{u}}$$

Then using the expression for $j_0,$ we have

$$S(x) = \sqrt{\frac{1}{2\pi}} \int_0^{x^2} \sqrt{u}\, j_0(u) du$$

as required.

Next we use the recursion relations. A most useful relation for our purposes here is between equations 8.120 and 8.121.:

$$\begin{aligned}
\sqrt{x} j_0(x) &= \sqrt{x} j_2(x) + 2\frac{d}{dx}\left(\sqrt{x} j_1\right) \\
&= \sqrt{x} j_4(x) + 2\frac{d}{dx}\left(\sqrt{x} j_3\right) + 2\frac{d}{dx}\left(\sqrt{x} j_1\right) \\
&= \sqrt{x} j_{2n}(x) + 2\frac{d}{dx}\sum_{m=1}^{n} \sqrt{x} j_{2m-1}
\end{aligned}$$

Thus

$$\begin{aligned}
S(x) &= \sqrt{\frac{1}{2\pi}} \int_0^{x^2} \left[\sqrt{u} j_{2n}(u) + 2\frac{d}{du}\sum_{m=1}^{n} \sqrt{u} j_{2m-1}(u) \right] du \\
&= \sqrt{\frac{1}{2\pi}} \int_0^{x^2} \sqrt{u} j_{2n}(u) du + \sqrt{\frac{2}{\pi}} \sum_{m=1}^{n} \sqrt{u} j_{2m-1}(u) \Big|_0^{x^2} \\
&= \sqrt{\frac{1}{2\pi}} \int_0^{x^2} \sqrt{u} j_{2n}(u) du + \sqrt{\frac{2}{\pi}} \sum_{m=1}^{n} x j_{2m-1}\left(x^2\right)
\end{aligned}$$

As n increases, the first term approaches zero, as we show below. The second term is the result we want. Thus

$$S(x) = \sqrt{\frac{2}{\pi}}\, x \sum_{m=1}^{n} j_{2m-1}\left(x^2\right)$$

as required.

To show that the first term is zero, we do the integral by inserting the series 8.122 for $j_{2n}.$

$$\int_0^{x^2} \sqrt{u}\, j_{2n}(u)du = \int_0^{x^2} \sqrt{\frac{\pi}{2}} \sum_{p=0}^{\infty} \frac{(-1)^p}{p!\Gamma\left(p + 2n + \frac{3}{2}\right)} \left(\frac{u}{2}\right)^{2n+\frac{1}{2}+2p} du$$
$$= \frac{\sqrt{\pi}}{2} \sum_{p=0}^{\infty} \frac{(-1)^p}{p!\Gamma\left(p + 2n + \frac{3}{2}\right) 2^{2(n+p)}} \frac{\left(x^2\right)^{2n+\frac{3}{2}+2p}}{2(n+p) + 3/2}$$

Using the result above equation (8.124) in the text with $n \to 2n + p$:

$$\Gamma\left(p + 2n + \frac{3}{2}\right) = \sqrt{\pi} \frac{(4n + 2p + 1)!}{2^{2p+4n+1}(p + 2n)!}$$

Thus

$$I_n = \int_0^{x^2} \sqrt{u}\, j_{2n}(u)du$$
$$= \sum_{p=0}^{\infty} \frac{(-1)^p 2^{2n+1}(p + 2n)!}{p!(2p + 4n + 1)!} \frac{x^{4(n+p)+3}}{4(n + p) + 3}$$
$$= x\left(\sqrt{2}x\right)^{4n+2} \sum_{p=0}^{\infty} \frac{(-1)^p (p + 2n)!}{p!(2p + 4n + 1)!} \frac{x^{4p}}{4(n + p) + 3}$$

and so

$$|I_n| \le x\left(\sqrt{2}x\right)^{4n+2} \sum_{p=0}^{\infty} \frac{(p + 2n)!}{(4(n + p) + 3)(2p + 4n + 1)!} \frac{x^{4p}}{p!}$$
$$= 2^{2n+1} x^{4n+3} \sum_{p=0}^{\infty} \frac{1}{(4(n + p) + 3)(2p + 4n + 1)(2p + 4n)\cdots(p + 2n + 1)} \frac{x^{4p}}{p!}$$
$$\le 2^{2n+1} x^{4n+3} \sum_{p=0}^{\infty} \frac{1}{(4n + 3)(4n + 1)(4n)\cdots(2n + 1)} \frac{x^{4p}}{p!}$$
$$\le 2^{2n+1} x^{4n+3} \frac{(2n)!}{(4n + 3)!} \sum_{p=0}^{\infty} \frac{x^{4p}}{p!}$$
$$= 2^{2n+1} x^{4n+3} \frac{(2n)!}{(4n + 3)!} e^{x^4} = f(n)$$

Now we investigate the limit as $n \to \infty$. Let $y = x^4$. Then, using Stirling's formula

$$\ln(p!) = \frac{1}{2}\ln 2\pi + \left(p + \frac{1}{2}\right)\ln p - p + \frac{1}{12p} + \cdots$$

we have

$$\log[f(n)] = \left(2n + \frac{1}{2}\right)\log 2 + \left(n + \frac{3}{4}\right)\log y + \log(2n)! - \log(4n + 3)! + y$$
$$= y + \left(2n + \frac{1}{2}\right)\log 2 + \left(n + \frac{3}{4}\right)\log y + \left(2n + \frac{1}{2}\right)\ln 2n - 2n + \cdots$$
$$- \left(4n + \frac{7}{2}\right)\ln(4n + 3) + 4n + 3 - \cdots$$

where we have dropped terms that decrease as $n \to \infty$. The right hand side is

$$y + \frac{3}{4}\log y + \frac{1}{2}\log 2 + 3 + n(\log y + 2\log 2) - \left(4n + \frac{7}{2}\right)\log(4n + 3)$$
$$+ \left(2n + \frac{1}{2}\right)\log 2n + 2n + \cdots$$
$$= A + n\log 4y - \left(4n + \frac{7}{2}\right)\left[\log n + \log\left(4 + \frac{3}{n}\right)\right]$$
$$+ \left(2n + \frac{1}{2}\right)(\log n + \log 2) + 2n + \cdots$$

where

$$A = y + \frac{3}{4}\log y + \frac{1}{2}\log 2 + 3$$

Continuing to simplify, we have

$$
\begin{aligned}
&A-(2n+3)\log n+2n+n\log 4y-\left(2n+\frac{1}{2}\right)\log 2\\
&\quad-\left(4n+\frac{7}{2}\right)\log\left(4+\frac{3}{n}\right)+\cdots\\
&=A-\frac{9}{2}\log 2-2n\left(\log n-1+5\log 2-\frac{1}{2}\log 4y\right)-2\log n\\
&\quad-\left(4n+\frac{5}{2}\right)\left(\frac{3}{4n}-\frac{1}{2}\left(\frac{3}{4n}\right)^2+\cdots\right)+\cdots
\end{aligned}
$$

As $n\to\infty$, the dominant term in this expression is $-2n\log n$, and thus the expression $\to-\infty$ as $n\to\infty$. Since $\log[f(n)]\to-\infty$ as $n\to\infty$ then $f(n)=\exp(\log(f))\to 0$.

$$
\begin{aligned}
\lim_{n\to\infty}\left[2^{2n+1}y^{n+3/4}\frac{(2n)!}{(4n+3)!}e^{y}\right] &= \exp\left(\log 2^{2n+1}y^{n+3/4}\frac{(2n)!}{(4n+3)!}e^{y}\right)\\
&\to 0 \text{ as } n\to\infty \text{ for any finite } x.
\end{aligned}
$$

Thus

$$|I_n|<f(n)\to 0 \text{ as } n\to\infty$$

A similar argument works for the second integral:

$$
\begin{aligned}
C(x) &= \sqrt{\frac{2}{\pi}}\int_0^x\cos t^2dt = C(x) = \sqrt{\frac{1}{2\pi}}\int_0^{x^2}\frac{\cos u}{\sqrt{u}}du\\
&= \sqrt{\frac{1}{2\pi}}\int_0^{x^2}\frac{\cos u}{\sqrt{u}}du = -\sqrt{\frac{1}{2\pi}}\int_0^{x^2}\sqrt{u}\,n_0(u)du\\
&= \sqrt{\frac{1}{2\pi}}\int_0^{x^2}\sqrt{u}\,j_{-1}(u)du
\end{aligned}
$$

where we used the result of Problem 5. Now we step up using the recursion relations, as before:

$$
\begin{aligned}
C(x) &= \sqrt{\frac{1}{2\pi}}\int_0^{x^2}\sqrt{u}\,j_{2N+1}(u)du+\sqrt{\frac{2}{\pi}}\sum_{m=0}^{N}x j_{2m}\left(x^2\right)\\
&= \sqrt{\frac{2}{\pi}}\,x\sum_{m=0}^{\infty}j_{2m}\left(x^2\right)
\end{aligned}
$$

52. Sound waves in a spherical cavity satisfy the differential equation $\left(\nabla^2+k^2\right)F=0$ for $r<R$ with $\frac{\partial F}{\partial r}=0$ at $r=R$. Find the eigenvalues k_n for the problem and hence find the allowed frequencies $\omega_n=k_n v$ for sound waves inside the cavity.

Following the procedure in the text, we find the solutions to be:

$$F=j_l(kr)Y_{lm}(\theta,\phi)$$

The eigenvalues are determined by the condition:

$$\left.\frac{\partial j_l(kr)}{\partial r}\right|_{r=R}=0$$

Thus kR must be one of the zeros of j_l'.

Using the recursion relations:

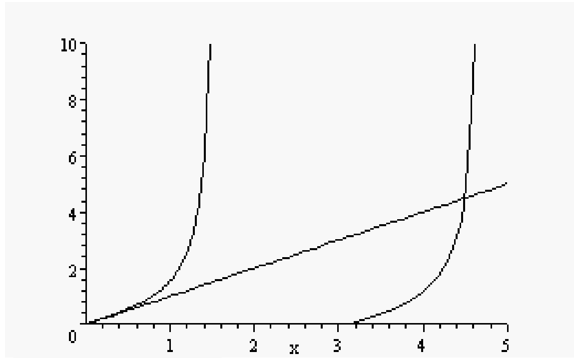
$$-(2l+1)\frac{dj_l(x)}{dx}=(l+1)j_{l+1}(x)-lj_{l-1}(x)=0$$

With $l=0$:

$$-\frac{dj_0}{dx}=j_1(x)=0=\frac{\sin x}{x^2}-\frac{\cos x}{x}$$

The solutions of this equation are the solutions of the equation

$$\tan x = x$$



A simple numerical algorithm gives $x = 4.493409485$, 7.725251837 and 10.90412166 as the first three values.

Thus the first allowed frequency is $\omega_{01} = 4.493409485v/R,$ and so on.

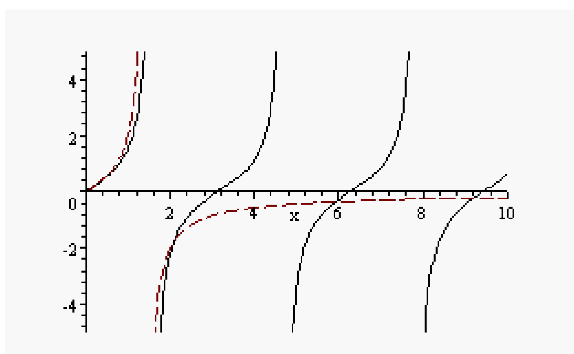
With $l = 1$:

$$2j_2(x) - j_0(x) = 0$$

$$2\left(\frac{3}{x^3} - \frac{1}{x}\right)\sin x - 2\frac{3}{x^2}\cos x - \frac{\sin x}{x} = 0$$

$$\left(\frac{2}{x^3} - \frac{1}{x}\right)\sin x - \frac{2}{x^2}\cos x = 0$$

$$\tan x = \frac{2x}{2 - x^2}$$



The solutions are: $2.081520847,$ 5.94037074 and 9.205840071 (Dashed line = $\frac{2x}{2-x^2}$, solid line = $\tan x.$)

The second freqency is $\omega_{11} = 2.081520847v/R,$ which is lower than the value found from $j_0.$

53. Electromagnetic waves in a spherical cavity may be described by a mathematiccal problem similar to that in problem 52, with $v = c$. The boundary conditions depend on the polarization. Find the allowed frequencies if the boundary condition is $F(R) = 0.$

Following the procedure in the text, we find the solutions to be:

$$F = j_l(kr)Y_{lm}(\theta,\phi)$$

Here we need $j_l(kR) = 0.$ These values are tabulated (eg Abramowitz and Stegun, p468). The first few values are: $kR =$ 1.16556 ($l = 0$), 2.460536 ($l = 1$) 3.632797 ($l = 2$) and 4.604 ($l = 0$). Since $k = \omega/c,$ the frequencies are:

$$\omega = 1.16556\frac{c}{R},\ \ 2.460536\frac{c}{R},\ \ 3.632797\frac{c}{R}\ \text{etc}$$

54. The modified spherical Bessel functions are defined as $i_l(x) = \sqrt{\frac{\pi}{2x}}I_{l+1/2}(x)$ and $k_l(x) = \sqrt{\frac{2}{\pi x}}K_{l+1/2}(x).$ Using the expression 8.99 and the result of problem 8.40, verify the expressions for the modified spherical Bessel functions $i_0(x) = \frac{\sinh x}{x}$ and $k_0(x) = \frac{2}{\pi x}e^{-x}.$

$$\begin{aligned} i_0(x) &= \sqrt{\frac{\pi}{2x}}I_{1/2}(x) = \sqrt{\frac{\pi}{2x}}\sum_{n=0}^{\infty}\frac{1}{n!\Gamma(n+1/2+1)}\left(\frac{x}{2}\right)^{1/2+2n} \\ &= \frac{1}{2}\sqrt{\pi}\sum_{n=0}^{\infty}\frac{1}{n!\Gamma(n+3/2)}\left(\frac{x}{2}\right)^{2n} \\ &= \frac{1}{2}\sqrt{\pi}\sum_{n=0}^{\infty}\frac{1}{n!\Gamma(n+3/2)}\left(\frac{x}{2}\right)^{2n} \end{aligned}$$

As in the text we expand the Γ function.

$$\Gamma\left(n+\frac{3}{2}\right) = \frac{(2n+1)!}{2^n n!}\frac{\sqrt{\pi}}{2^{n+1}}$$

Thus:

$$\begin{aligned} i_0(x) &= \frac{1}{2}\sum_{n=0}^{\infty}\frac{2^{2n+1}}{(2n+1)!}\left(\frac{x}{2}\right)^{2n} \\ &= \sum_{n=0}^{\infty}\frac{1}{(2n+1)!}(x)^{2n} \\ &= \frac{1}{x}\sum_{n=0}^{\infty}\frac{1}{(2n+1)!}x^{2n+1} = \frac{(e^x - e^{-x})}{2x} = \frac{\sinh x}{x} \end{aligned}$$

as required.

Now

$$\begin{aligned} k_0(x) &= \sqrt{\frac{2}{\pi x}}K_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\frac{\pi}{2}\frac{I_{-1/2}(x) - I_{1/2}(x)}{\sin\pi/2} \\ &= \sqrt{\frac{\pi}{2x}}(I_{-1/2}(x) - I_{1/2}(x)) \end{aligned}$$

$$I_{-1/2} = \sum_{n=0}^{\infty}\frac{1}{n!\Gamma(n-1/2+1)}\left(\frac{x}{2}\right)^{-1/2+2n}$$

So

$$
\begin{aligned}
I_{-1/2}(x) - I_{1/2}(x) &= \sum_{n=0}^{\infty} \frac{1}{n!\Gamma(n+1/2)}\left(\frac{x}{2}\right)^{-1/2+2n} - \sum_{n=0}^{\infty} \frac{1}{n!\Gamma(n+3/2)}\left(\frac{x}{2}\right)^{1/2+2n} \\
&= \sum_{n=0}^{\infty} 2^{2n+1}\frac{n+1/2}{(2n+1)!\sqrt{\pi}}\left(\frac{x}{2}\right)^{-1/2+2n} - \sum_{n=0}^{\infty} \frac{2^{2n+1}}{(2n+1)!\sqrt{\pi}}\left(\frac{x}{2}\right)^{-1/2+2n+1} \\
&= \sum_{n=0}^{\infty} 2^{2n}\frac{1}{(2n)!\sqrt{\pi}}\left(\frac{x}{2}\right)^{-1/2+2n} - \sum_{n=0}^{\infty} \frac{2^{2n+1}}{(2n+1)!\sqrt{\pi}}\left(\frac{x}{2}\right)^{-1/2+2n+1} \\
&= \sqrt{\frac{2}{\pi x}}\left(\sum_{n=0}^{\infty} \frac{1}{(2n)!}x^{2n} - \sum_{n=0}^{\infty} \frac{1}{(2n+1)!\sqrt{\pi}}x^{2n+1}\right) \\
&= \sqrt{\frac{2}{\pi x}}\left(\sum_{n=0}^{\infty} \frac{1}{(2n)!}x^{2n} + \sum_{n=0}^{\infty} \frac{1}{(2n+1)!\sqrt{\pi}}(-x)^{2n+1}\right) \\
&= \sqrt{\frac{2}{\pi x}}\sum_{n=0}^{\infty} \frac{1}{(n)!}(-x)^{n} = \sqrt{\frac{2}{\pi x}}e^{-x}
\end{aligned}
$$

Thus

$$k_0(x) = \sqrt{\frac{\pi}{2x}}\sqrt{\frac{2}{\pi x}}e^{-x} = \frac{e^{-x}}{x}$$

QED

Use proof by induction to show that

$$k_l(x) = (-1)^l x^l \left(\frac{1}{x}\frac{d}{dx}\right)^l k_0(x)$$

From the recursion relation in Problem 50 with $l = 0$:

$$k_1(x) = -\frac{d}{dx}k_0(x) = -\frac{d}{dx}\left(\frac{e^{-x}}{x}\right)$$

so the formula works for $l = 1.$ Now assume it works for some value $l.$ Using the recursion relations, we have:

$$
\begin{aligned}
(l+1)k_{l+1} &= -lk_{l-1} - (2l+1)\frac{d}{dx}k_l(x) \\
&= -l\left[k_{l+1} - \frac{2l+1}{x}k_l\right] - (2l+1)\frac{d}{dx}k_l(x) \\
(2l+1)k_{l+1} &= \frac{l(2l+1)}{x}k_l - (2l+1)\frac{d}{dx}k_l(x) \\
k_{l+1} &= \frac{l}{x}(-1)^l x^l\left(\frac{1}{x}\frac{d}{dx}\right)^l\left(\frac{e^{-x}}{x}\right) - \frac{d}{dx}\left[(-1)^l x^l\left(\frac{1}{x}\frac{d}{dx}\right)^l\left(\frac{e^{-x}}{x}\right)\right] \\
&= (-1)^l\left[\left(\frac{d}{dx}x^l\right)\left(\frac{1}{x}\frac{d}{dx}\right)^l\left(\frac{e^{-x}}{x}\right) - \frac{d}{dx}x^l\left(\frac{1}{x}\frac{d}{dx}\right)^l\left(\frac{e^{-x}}{x}\right) - x^{l+1}\left(\frac{1}{x}\frac{d}{dx}\right)^{l+1}\left(\frac{e^{-x}}{x}\right)\right] \\
&= (-1)^{l+1}x^{l+1}\left(\frac{1}{x}\frac{d}{dx}\right)^{l+1}\left(\frac{e^{-x}}{x}\right)
\end{aligned}
$$

So the result is true for $l + 1.$ Thus it is true for all $l.$

55. We may model the force between particles in an atomic nucleus by a 3-D square well potential $V = -V_0,$ $r < R$ with $V = 0$ for $r > R.$ Schrodinger's equation for this system takes the form:

$$\left(\nabla^2 - 2\frac{m}{\hbar^2}V(r)\right)\psi = -2\frac{m}{\hbar^2}E\psi$$

Write the differential operator in spherical coordinates and show that the solution may be written in terms of spherical Bessel functions. Find the energy of the lowest energy level.

The equation is of the form (8.114) in the text and thus the solution is

$$\psi = Y_{lm}(\theta,\phi)j_l(kr)$$

where

$$k = \sqrt{2\frac{m}{\hbar^2}(V_0 + E)}$$

inside the well. Outside the well $ik' = \sqrt{2\frac{m}{\hbar^2}E}$, which is imaginary if E is negative (a bound state), and we need a solution that decays exponentially. That solution is the spherical K function (compare equation 8.105), $k_l(k'r)$. The boundary conditions are continuity of both ψ and ψ'. Thus

$$j_l(kR) = Ak_l(k'R)$$

and

$$kj_l'(kR) = k'Ak_l'(k'R) = k'\frac{j_l(kR)}{k_l(k'R)}k_l'(k'R)$$

This is a transcendental equation for the energy values E. Let $\varepsilon = -E/V_0$ and $2\frac{m}{\hbar^2}V_0R^2 = \alpha$ Then

$$k^2 = -k'^2 + 2\frac{m}{\hbar^2}V_0 = \frac{\alpha}{R^2} - k'^2 = \frac{\alpha}{R^2}(1-\varepsilon)$$

and

$$\frac{k}{k'} = \sqrt{\frac{1-\varepsilon}{\varepsilon}} = \frac{j_l(kR)}{k_l(k'R)}\frac{k_l'(k'R)}{j_l'(kR)}$$

We use the recursion relations to eliminate the derivatives:

$$kk_l(k'R)[lj_{l-1}(kR) - (l+1)j_{l+1}(kR)] = -k'j_l(kR)[(l+1)k_{l+1}(k'R) + lk_{l-1}(k'R)]$$

$$kk_l(k'R)\begin{bmatrix}(l+1)j_{l+1}(kR) - \\ l\left\{\frac{2l+1}{kR}j_l(kR) - j_{l+1}\right\}\end{bmatrix} = k'j_l(kR)\left[(l+1)k_{l+1}(k'R) + l\left\{k_{l+1} - \frac{2l+1}{k'R}k_l(k'R)\right\}\right]$$

A factor of $2l+1$ cancels, leaving

$$kk_l(k'R)\left[j_{l+1}(kR) - \frac{l}{kR}j_l(kR)\right] = k'j_l(kR)\left[k_{l+1}(k'R) - \frac{l}{k'R}k_l(k'R)\right]$$

$$kk_l(k'R)j_{l+1}(kR) - \frac{l}{R}k_l(k'R)j_l(kR) = k'j_l(kR)k_{l+1}(k'R) - \frac{l}{R}j_l(kR)k_l(k'R)$$

$$kk_l(k'R)j_{l+1}(kR) = k'j_l(kR)k_{l+1}(k'R)$$

or

$$\sqrt{1-\varepsilon}\,k_l(\alpha\sqrt{\varepsilon})j_{l+1}\left(\alpha\sqrt{1-\varepsilon}\right) = \sqrt{\varepsilon}\,j_l\left(\alpha\sqrt{1-\varepsilon}\right)k_{l+1}(\alpha\sqrt{\varepsilon})$$

For $l = 0$, we insert the explicit expressions for the Bessel functions, using equations (8.124) and (8.125)

$$\sqrt{1-\varepsilon}\,k_0(\alpha\sqrt{\varepsilon})j_1\left(\alpha\sqrt{1-\varepsilon}\right) = \sqrt{\varepsilon}\,j_0\left(\alpha\sqrt{1-\varepsilon}\right)k_1(\alpha\sqrt{\varepsilon})$$

$$\sqrt{1-\varepsilon}\,k_0(\alpha\sqrt{\varepsilon})\left(\frac{\sin\alpha\sqrt{1-\varepsilon}}{\alpha^2(1-\varepsilon)} - \frac{\cos\alpha\sqrt{1-\varepsilon}}{\alpha\sqrt{1-\varepsilon}}\right) = \sqrt{\varepsilon}\,\frac{\sin\alpha\sqrt{1-\varepsilon}}{\alpha\sqrt{1-\varepsilon}}k_1(\alpha\sqrt{\varepsilon})$$

$$k_0(\alpha\sqrt{\varepsilon})\left(\frac{\sin\left(\alpha\sqrt{1-\varepsilon}\right)}{\alpha\sqrt{1-\varepsilon}} - \cos\left(\alpha\sqrt{1-\varepsilon}\right)\right) = \sqrt{\varepsilon}\,\frac{\sin\left(\alpha\sqrt{1-\varepsilon}\right)}{\sqrt{1-\varepsilon}}k_1(\alpha\sqrt{\varepsilon})$$

Now using the result of problem 54

$$k_0(\alpha\sqrt{\varepsilon}) = \frac{e^{-\alpha\sqrt{\varepsilon}}}{\alpha\sqrt{\varepsilon}}$$

and

$$k_1(x) = -\frac{d}{dx}\frac{e^{-x}}{x} = \frac{e^{-x}}{x} + \frac{e^{-x}}{x^2} = \frac{e^{-x}}{x}\left(1 + \frac{1}{x}\right)$$

Thus the eigenvalue equation becomes:

$$\frac{e^{-\alpha\sqrt{\varepsilon}}}{\alpha\sqrt{\varepsilon}}\left(\frac{\sin\left(\alpha\sqrt{1-\varepsilon}\right)}{\alpha\sqrt{1-\varepsilon}} - \cos\left(\alpha\sqrt{1-\varepsilon}\right)\right) = \sqrt{\varepsilon}\,\frac{\sin\left(\alpha\sqrt{1-\varepsilon}\right)}{\sqrt{1-\varepsilon}}\,\frac{e^{-\alpha\sqrt{\varepsilon}}}{\alpha\sqrt{\varepsilon}}\left[1 + \frac{1}{\alpha\sqrt{\varepsilon}}\right]$$

$$\left(\frac{\sin\left(\alpha\sqrt{1-\varepsilon}\right)}{\alpha\sqrt{1-\varepsilon}} - \cos\left(\alpha\sqrt{1-\varepsilon}\right)\right) = \sqrt{\varepsilon}\,\frac{\sin\left(\alpha\sqrt{1-\varepsilon}\right)}{\sqrt{1-\varepsilon}}\left[1 + \frac{1}{\alpha\sqrt{\varepsilon}}\right]$$

$$\left[1 - \alpha\sqrt{\varepsilon} - 1\right]\sin\left(\alpha\sqrt{1-\varepsilon}\right) = \alpha\sqrt{1-\varepsilon}\cos\left(\alpha\sqrt{1-\varepsilon}\right)$$

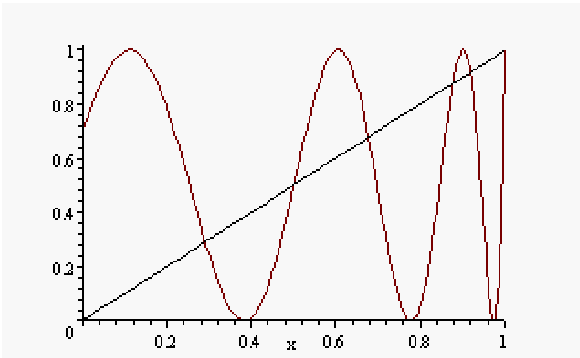
$$\frac{-\sqrt{1-\varepsilon}}{\sqrt{\varepsilon}} = \tan\left(\alpha\sqrt{1-\varepsilon}\right)$$

Rearranging:

$$\frac{1-\varepsilon}{\varepsilon} = \tan^2\alpha\sqrt{1-\varepsilon}$$

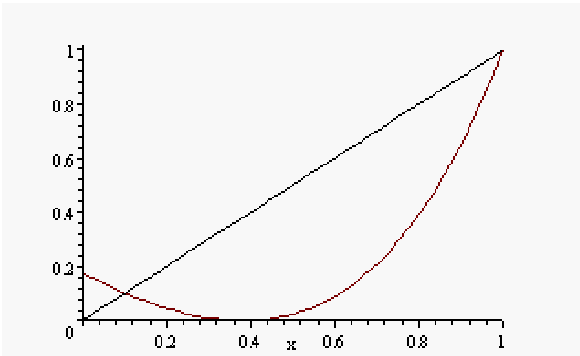
$$\frac{1}{\varepsilon} = 1 + \tan^2\alpha\sqrt{1-\varepsilon} = \sec^2\alpha\sqrt{1-\varepsilon}$$

$$\varepsilon = \cos^2\alpha\sqrt{1-\varepsilon} \qquad \text{equation 1 Pr 55}$$

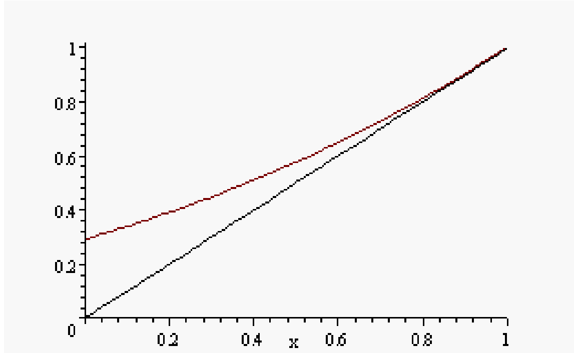


The plot shows the two sides of equation (1 Pr55) for $\alpha = 10.$ The solutions are at $\varepsilon = 0.2905$, 0.5, 0.6775, 0.877565, and $\varepsilon = 0.918641$ Thus the lowest energy level is:

$$E = -0.91864 V_0$$



Plot with $\alpha = 2.$



Plot for $\alpha = 1$.

For a solution, $f(\varepsilon) = \varepsilon - \cos^2\alpha\sqrt{1-\varepsilon}$ must have at least one zero. One solution is always $\varepsilon = 1$, but this is not a physical energy level. To have another zero, the function must have a max or min, i.e.

$$\begin{aligned} f' &= 1 - 2\frac{\alpha}{2}(-1)\frac{1}{\sqrt{1-\varepsilon}}2\cos x(-\sin x) \\ &= 1 - \alpha\frac{1}{\sqrt{1-\varepsilon}}\left(\sin 2\alpha\sqrt{1-\varepsilon}\right) = 0 \end{aligned}$$

for some value of ε, $0 < \varepsilon < 1$. Thus

$$\frac{\sqrt{1-\varepsilon}}{\alpha} = \sin 2\alpha\sqrt{1-\varepsilon}$$

Since the sine lies between -1 and +1, we can find a solution if $\alpha > 1$.

56. The density of neutrons in uranium is described by the equation

$$\frac{\partial n}{\partial t} = D\nabla^2 n + an$$

where the D (the diffusion coefficient) and

a (the net production rate) may be taken to be constants in space and time. Solve the equation using separation of variables. Look for a solution with spherical symmetry that satisfies the boundary condition $n = 0$ at

$r = R$. Show that the density increases exponentially if R exceeds a critical value

R_{crit} and determine that value in terms of D and a.

We look for a solution $n(r,t) = S(r)T(t)$. Then

$$D\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial n}{\partial r}\right) + an = \frac{\partial n}{\partial t}$$

$$\frac{1}{S}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial S}{\partial r}\right) + \frac{a}{D} = \frac{1}{T}\frac{\partial T}{\partial t}$$

Both sides must separately be constant, so

$$\frac{1}{T}\frac{\partial T}{\partial t} = \alpha \Rightarrow T = T_0 e^{\alpha t}$$

and then

$$\frac{1}{S}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial S}{\partial r}\right) + \frac{a}{D} - \alpha = 0$$

The solution of this equation is the spherical Bessel function $j_0\left(\sqrt{\frac{a}{D} - \alpha}\, r\right)$. Since we need $n = 0$ at

$r = R$, we must choose $\sqrt{\frac{a}{D} - \alpha}\, R$ to be one of the zeros of j_0. From equation (8.124),

$$j_0(x) = \frac{\sin x}{x}$$

and thus the first zero is at $x = \pi.$ Thus

$$\sqrt{\frac{a}{D} - \alpha_1}\,R = \pi$$

$$\alpha_1 = \frac{a}{D} - \frac{\pi^2}{R^2}$$

The density increases exponentially if $\alpha_1 > 0,$ or

$$R > R_{\text{crit}} = \pi\sqrt{\frac{D}{a}}$$

Optional topic A: Tensors

1. Determine the velocity of an electron driven by an electric field

$\vec{\mathbf{E}} = \vec{\mathbf{E}}_0 e^{-i\omega t}$ in the presence of a constant, uniform magnetic field $\vec{\mathbf{B}}_0$. Choose the $z-$axis along $\vec{\mathbf{B}}_0$, but do not make any assumptions about the direction of $\vec{\mathbf{E}}_0$. If there are n electrons per unit volume, write the current in the form $j_i = \sigma_{ij} E_j$.

The equation of motion of the electron is:

$$m\frac{d\vec{\mathbf{v}}}{dt} = -e\left(\vec{\mathbf{E}} + \vec{\mathbf{v}} \times \vec{\mathbf{B}}_0\right) = -e\left(\vec{\mathbf{E}}_0 e^{-i\omega t} + \vec{\mathbf{v}} \times \vec{\mathbf{B}}_0\right)$$

The solution will be of the form $\vec{\mathbf{v}} = \vec{\mathbf{v}}_0 e^{-i\omega t}$ where

$$-i\omega m\vec{\mathbf{v}}_0 = -e\left(\vec{\mathbf{E}}_0 + \vec{\mathbf{v}}_0 \times \vec{\mathbf{B}}_0\right)$$

The $z-$component is:

$$-i\omega m v_z = -eE_z \Rightarrow v_z = -i\frac{e}{\omega m}E_z$$

The $x-$ and $y-$components are coupled:

$$-i\omega m v_x = -eE_x - ev_y B_0 \Rightarrow v_x = -i\frac{e}{\omega m}E_x - i\frac{eB_0}{\omega m}v_y$$

and

$$-i\omega m v_y = -eE_y + ev_x B_0 \Rightarrow v_y = -i\frac{e}{\omega m}E_y + i\frac{eB_0}{\omega m}v_x$$

Thus

$$v_x = -i\frac{e}{\omega m}E_x - i\frac{eB_0}{\omega m}\left(-i\frac{e}{\omega m}E_y + i\frac{eB_0}{\omega m}v_x\right)$$

Now write $\Omega = eB_0/m$, so

$$\left(1 - \frac{\Omega^2}{\omega^2}\right)v_x = -i\frac{e}{\omega m}\left(E_x - i\frac{\Omega}{\omega}E_y\right)$$
$$v_x = -i\frac{e}{\omega m}\frac{\left(E_x - i\frac{\Omega}{\omega}E_y\right)}{\left(1 - \frac{\Omega^2}{\omega^2}\right)}$$

and then

$$v_y = -i\frac{e}{\omega m}\left(E_y + i\frac{\Omega}{\omega}\frac{\left(E_x - i\frac{\Omega}{\omega}E_y\right)}{\left(1 - \frac{\Omega^2}{\omega^2}\right)}\right)$$
$$= -i\frac{e}{\omega m}\left(\frac{E_y\left(1 - \frac{\Omega^2}{\omega^2}\right) + i\frac{\Omega}{\omega}\left(E_x - i\frac{\Omega}{\omega}E_y\right)}{\left(1 - \frac{\Omega^2}{\omega^2}\right)}\right)$$
$$= -i\frac{e}{\omega m}\frac{\left(E_y + i\frac{\Omega}{\omega}E_x\right)}{\left(1 - \frac{\Omega^2}{\omega^2}\right)}$$

The the current density is

$$j_i = -nev_i$$

which may be written as a tensor product

$$j_i = \sigma_{ij} E_j$$

where

$$\sigma_{ij} = i\frac{ne^2}{\omega m}\frac{\omega^2}{(\omega^2-\Omega^2)}\begin{pmatrix} 1 & -i\Omega/\omega & 0 \\ i\Omega/\omega & 1 & 0 \\ 0 & 0 & (\omega^2-\Omega^2) \end{pmatrix}$$

$$= i\frac{ne^2}{m}\frac{\omega}{(\omega^2-\Omega^2)}\begin{pmatrix} 1 & -i\Omega/\omega & 0 \\ i\Omega/\omega & 1 & 0 \\ 0 & 0 & (\omega^2-\Omega^2) \end{pmatrix}$$

2. Starting with the expression $\vec{v} = \vec{\omega} \times \vec{r}$ for a particle in circular motion, derive the expression (A.6) for the inertia tensor.

For a element $dm,$ we have

$$\begin{aligned} dL_i &= \varepsilon_{ijk}x_j p_k = \varepsilon_{ijk}x_j\varepsilon_{klm}\omega_l x_m dm \\ &= \left(\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl}\right)x_j x_m \omega_l dm \\ &= (\omega_i x_j x_j - \omega_j x_j x_i)dm \\ &= (x_m x_m \delta_{ij} - x_i x_j)dm\omega_j \end{aligned}$$

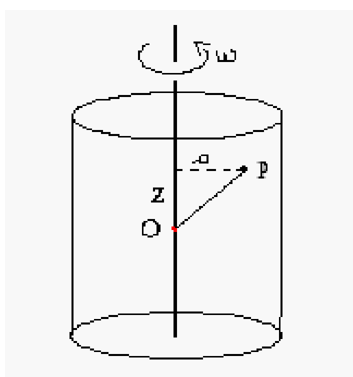
Then integrating over the body, we have:

$$I_{ij} = \int\left(r^2\delta_{ij} - x_i x_j\right)dm$$

3. Compute the inertia tensor for a uniform cylinder of radius R and height $h.$ Hence find its angular momentum when it rotates with angular speed ω about **(a)** an axis through its center and along its length, (**b**) an axis through its center and along a diameter, and (**c)** an axis through its center making a 45° angle with each of the axes in (a) and (b)

$$I_{ij} = \frac{M}{\pi R^2 h}\int\left(r^2\delta_{ij} - x_i x_j\right)dV$$

The inertia tensor is:



$$\begin{aligned} I_{11} &= \frac{M}{\pi R^2 h}\int_{-h/2}^{+h/2}dz\int_0^R \rho d\rho\int_0^{2\pi}d\phi\left(\rho^2 + z^2 - \rho^2\cos^2\phi\right) \\ &= \frac{M}{\pi R^2 h}\int_{-h/2}^{+h/2}dz\int_0^R d\rho\left(2\pi\left(\rho^3 + z^2\rho\right) - \pi\rho^3\right) \\ &= \frac{M}{R^2 h}\int_{-h/2}^{+h/2}dz\left(z^2R^2 + \frac{R^4}{4}\right) \\ &= \frac{M}{R^2 h}\left(\frac{h^3}{4}R^2 + \frac{R^4}{4}h\right) = \frac{M}{4}\left(h^2 + R^2\right) = I_{22} \end{aligned}$$

$$\begin{aligned} I_{33} &= \frac{M}{\pi R^2 h}\int_{-h/2}^{+h/2} dz \int_0^R \rho d\rho \int_0^{2\pi} d\phi \left(\rho^2 + z^2 - z^2\right) \\ &= \frac{2M}{R^2 h}\int_{-h/2}^{+h/2} dz \int_0^R \rho d\rho \rho^2 \\ &= \frac{2M}{R^2 h}\int_{-h/2}^{+h/2} dz \left(\frac{R^4}{4}\right) \\ &= \frac{M}{R^2 h}\left(\frac{R^4}{2}h\right) = \frac{M}{2}R^2 \end{aligned}$$

The off-diagonal terms are:

$$I_{12} = \frac{M}{\pi R^2 h}\int_{-h/2}^{+h/2} dz \int_0^R \rho d\rho \int_0^{2\pi} d\phi \left(-\rho^2 \cos\phi \sin\phi\right) = 0 = I_{21}$$

and

$$I_{13} = \frac{M}{\pi R^2 h}\int_{-h/2}^{+h/2} dz \int_0^R \rho d\rho \int_0^{2\pi} d\phi (-\rho z \cos\phi) = 0 = I_{31} = I_{32} = I_{23}$$

Thus

$$I_{ij} = \frac{M}{4}\begin{pmatrix} \left(h^2 + R^2\right) & 0 & 0 \\ 0 & \left(h^2 + R^2\right) & 0 \\ 0 & 0 & 2R^2 \end{pmatrix}$$

(a) With $\vec{\boldsymbol{\omega}} = \omega \hat{\mathbf{z}}$,

$$L_i = I_{ij}\omega_j = \frac{M}{4}\begin{pmatrix} \left(h^2 + R^2\right) & 0 & 0 \\ 0 & \left(h^2 + R^2\right) & 0 \\ 0 & 0 & 2R^2 \end{pmatrix}\begin{pmatrix} 0 \\ 0 \\ \omega \end{pmatrix} = \frac{MR^2}{2}\begin{pmatrix} 0 \\ 0 \\ \omega \end{pmatrix}$$

(b) With $\vec{\boldsymbol{\omega}} = \omega \hat{\mathbf{x}}$,

$$L_i = I_{ij}\omega_j = \frac{M}{4}\begin{pmatrix} \left(h^2 + R^2\right) & 0 & 0 \\ 0 & \left(h^2 + R^2\right) & 0 \\ 0 & 0 & 2R^2 \end{pmatrix}\begin{pmatrix} \omega \\ 0 \\ 0 \end{pmatrix} = \frac{M\left(h^2 + R^2\right)}{4}\begin{pmatrix} \omega \\ 0 \\ 0 \end{pmatrix}$$

and (c) With $\vec{\boldsymbol{\omega}} = \frac{\omega}{\sqrt{2}}(\hat{\mathbf{x}} + \hat{\mathbf{z}})$,

$$L_i = I_{ij}\omega_j = \frac{M}{4\sqrt{2}}\begin{pmatrix} \left(h^2 + R^2\right) & 0 & 0 \\ 0 & \left(h^2 + R^2\right) & 0 \\ 0 & 0 & 2R^2 \end{pmatrix}\begin{pmatrix} \omega \\ 0 \\ \omega \end{pmatrix} = \frac{M\omega}{4\sqrt{2}}\begin{pmatrix} \left(h^2 + R^2\right) \\ 0 \\ 2R^2 \end{pmatrix}$$

In this case, $\vec{\mathbf{L}}$ is not parallel to $\vec{\boldsymbol{\omega}}$.

4. Show that the outer product $a_i b_j$ of two vectors obeys the transformation law for a rank 2 tensor.

$$\overline{a}_i \overline{b}_j = (A_{ik}a_k)(A_{jm}b_m) = A_{ik}A_{jm}(a_k b_m)$$

which is the correct transformation law for a rank 2 tensor.

5. Show that the inner product

$a_{ijk}b_k$ of a rank 3 tensor and a vector obeys the transformation law for a rank 2 tensor.

$$\begin{aligned}\overline{a}_{ijk}\overline{b}_k &= A_{in}A_{jm}A_{kp}a_{nmp}A_{ks}b_s \\ &= A_{in}A_{jm}A_{kp}A_{ks}a_{nmp}b_s \\ &= A_{in}A_{jm}\delta_{ps}a_{nmp}b_s \\ &= A_{in}A_{jm}a_{nmp}b_m\end{aligned}$$

which is the correct transformation law for a rank 2 tensor.

6. Show that the Kroneker delta tensor δ_{ij} has the same components in every coordinate frame.

We apply the transformation law:

$$\overline{\delta}_{ij} = A_{ik}A_{jm}\delta_{km} = A_{im}A_{jm} = \delta_{ij}$$

7. Show that, if a tensor b_{ij} is symmetric in one frame, i.e.
$b_{ij} = b_{ji}$, then it is symmetric in every frame. Similarly show that the property of anti-symmetry ($b_{ij} = -b_{ji}$) is preserved under coordinate trasnformations.

If b_{ij} is symmetric, then:

$$\overline{b}_{ij} = A_{ik}A_{jm}b_{km} = A_{ik}A_{jm}b_{mk} = A_{jm}A_{ik}b_{mk} = \overline{b}_{ji}$$

so $\overline{b}_{ij}$ is also symmetric. Similarly, for an antisymmteric tensor, we find

$$\overline{b}_{ij} = A_{ik}A_{jm}b_{km} = -A_{ik}A_{jm}b_{mk} = -A_{jm}A_{ik}b_{mk} = -\overline{b}_{ji}$$

8. The following sets of components are defined in 2-dimensional Cartesian space:

(**a)**

$$A_{ij} = \begin{pmatrix} -y^2 & xy \\ xy & -x^2 \end{pmatrix}$$

Does this set of components transform as a tensor? Why or why not?

We may write the tensor components as:

$$A_{ij} = x_i x_j - r^2\delta_{ij}$$

Since $x_i x_j$ is an outer product, r^2 is a scalar and δ_{ij} is a tensor, this set of components transforms as a tensor.

Repeat the problem for the following sets of components:

(**b**)

$$B_{ij} = \begin{pmatrix} -xy & x^2 \\ y^2 & -xy \end{pmatrix}$$

This set of components may be written:

$$B_{ij} = x_i \varepsilon_{3jk} x_k$$

So we need to investigate how the set of components ε_{3jk} transforms.

$$A_{jm}A_{kn}\varepsilon_{3jk} = A_{1m}A_{2n} - A_{2m}A_{1n}$$

where

$$A_{ij} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}$$

Thus

$$A_{1m}A_{2n} = \begin{pmatrix} -\cos\theta\sin\theta & \cos^2\theta \\ -\sin^2\theta & \sin\theta\cos\theta \end{pmatrix}$$

and

$$A_{2m}A_{1n} = \begin{pmatrix} -\cos\theta\sin\theta & -\sin^2\theta \\ \cos^2\theta & \sin\theta\cos\theta \end{pmatrix}$$

Thus

$$A_{1m}A_{2n} - A_{2m}A_{1n} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \varepsilon_{3mn}$$

If instead we have a reflection, we must change the sign of the terms in the bottom row of each matrix. This leads to

$$A_{1m}A_{2n} = \begin{pmatrix} \cos\theta\sin\theta & -\cos^2\theta \\ \sin^2\theta & -\sin\theta\cos\theta \end{pmatrix}$$

and

$$A_{1m}A_{2n} - A_{2m}A_{1n} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = -\varepsilon_{3mn}$$

Thus ε_{3mn}, like ε_{ijk}, transforms as a tensor density. Thus B_{ij} transforms as a tensor density.

We may also form the scalar density

$$B_{ij}u_i v_j = x_i \varepsilon_{3jk} x_k u_i v_j = (\vec{\mathbf{x}} \cdot \vec{\mathbf{u}})(\vec{\mathbf{v}} \times \vec{\mathbf{x}})_{z\text{ comp}}$$

for arbitrary vectors $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ and use the quotient theorem to confirm the previous result.

(**c)**

$$C_{ij} = \begin{pmatrix} x^2 & xy \\ xy & y^2 \end{pmatrix}$$

This set of components may be written:

$$C_{ij} = x_i x_j$$

which is an outer product, and thus this set of components transforms as a rank 2 tensor.

(**d)** In 3-dimensional space,

$$D_{ij} = \begin{pmatrix} 0 & z & -y \\ -z & 0 & x \\ y & -x & 0 \end{pmatrix}$$

This set of components may be written:

$$D_{ij} = \varepsilon_{ijk} x_k,$$

an outer product of the tensor density ε_{ijk} and the vector x_k. Thus this set of components is a rank 2 tensor density.

Alternatively, we may form the set of components

$D_{ij}v_j = \varepsilon_{ijk}v_jx_k = (\vec{v}\times\vec{x})_i$ and since this is a vector density, by the quotient theorem, D_{ij} is a tensor density.

9. The magnetic moment tensor has components:

$$M_{ik} = \int x_i j_k dV$$

where $\mathbf{j}$ is the current density, and the integral is over all space.

(a) Show that M_{ik} is antisymmetric for any steady, finite current distribution.

Note that:

$$\partial_k(x_ix_mj_k) = \delta_{ki}x_mj_k + x_i\delta_{km}j_k + x_ix_m\left(-\frac{\partial\rho}{\partial t}\right)$$
$$= x_mj_i + x_ij_m - x_ix_m\frac{\partial\rho}{\partial t}$$

Now integrate both sides. The integral of the divergence on the LHS converts to a surface integral at infinity, which is zero. The last term on the right is zero if the distribution is steady. Thus

$$0 = \int x_mj_idV + \int x_ij_mdV = M_{mi} + M_{im}$$

$$M_{im} = -M_{mi}$$

and the tensor is antisymmetric.

(b) Show that the corresponding cross product (equation A.7) reduces to the usual magnetic moment vector

$\vec{m} = IA\hat{n}$ in the case of a planar current loop.

For a current loop,

$$M_{ik} = \int x_ij_kdV = \int x_iIdx_k$$

Then

$$m_i = \frac{1}{2}\varepsilon_{imk}M_{mk} = \frac{I}{2}\varepsilon_{imk}\int x_mdx_k = \frac{I}{2}\int(\vec{x}\times d\vec{x})_i = I\int dAn_i$$

where dA is the area element equal to one half the parallelogram formed by $\vec{x}$ and $d\vec{x}$, and $\hat{n}$ is the normal to that area. If the loop is planar, then $\hat{n}$ is constant and

$$\vec{m} = IA\hat{n}$$

10. The electric quadrupole tensor is given by:

$$Q_{ij} = \int\left(3x_ix_j - r^2\delta_{ij}\right)\rho(\vec{x})dV$$

Calculate the quadrupole tensor for a set of four point charges, 2 of charge q and two of charge

$-q$, at the corners of a square of side

a. The charges alternate in sign, so that charges of equal sign are at opposite ends of the diagonals of the square.

The force on a quadrupole charge distribution placed in an electric field is

$$F_i = \frac{1}{2}Q_{jk}\frac{\partial^2E_j}{\partial x_i\partial x_k}$$

Find the force on the square when it is placed in an electric field $\vec{\mathbf{E}} = \alpha\left(2xz, -2yz, x^2 - y^2\right),$ with its normal at angle θ to the $z-$axis and its center at the origin.

The charge density is the sum of four delta-functions. Start with a coordinate system in which the charges lie in the $\overline{x} - \overline{y} -$plane, with coordinate axes along the diagonals of the square. Then:

$$\overline{Q}_{11} = q\int\left(3\overline{x}^2 - (\overline{x}^2 + \overline{y}^2 + \overline{z}^2)\right)\begin{bmatrix} \delta\left(\overline{x} - \frac{a}{\sqrt{2}}\right)\delta(\overline{y}) + \delta\left(\overline{x} + \frac{a}{\sqrt{2}}\right)\delta(\overline{y}) \\ -\delta(\overline{x})\delta\left(\overline{y} - \frac{a}{\sqrt{2}}\right) - \delta(\overline{x})\delta\left(\overline{y} + \frac{a}{\sqrt{2}}\right) \end{bmatrix}\delta(\overline{z})d\overline{x}\,d\overline{y}\,d\overline{z}$$
$$= q\left(2\frac{a^2}{2} + 2\frac{a^2}{2} + \frac{a^2}{2} + \frac{a^2}{2}\right) = 3qa^2 = -\overline{Q}_{22}$$

$$\overline{Q}_{12} = q\int 3\overline{x}\,\overline{y}\begin{bmatrix} \delta\left(\overline{x} - \frac{a}{\sqrt{2}}\right)\delta(\overline{y}) + \delta\left(\overline{x} + \frac{a}{\sqrt{2}}\right)\delta(\overline{y}) \\ -\delta(\overline{x})\delta\left(\overline{y} - \frac{a}{\sqrt{2}}\right) - \delta(\overline{x})\delta\left(\overline{y} + \frac{a}{\sqrt{2}}\right) \end{bmatrix}\delta(\overline{z})d\overline{x}\,d\overline{y}\,d\overline{z}$$
$$= 0 = \overline{Q}_{21} = \overline{Q}_{31} = \overline{Q}_{13}$$

and

$$\overline{Q}_{33} = q\int\left(3\overline{z}^2 - \left(\overline{x}^2 + \overline{y}^2 + \overline{z}^2\right)\right)\begin{bmatrix} \delta\left(\overline{x} - \frac{a}{\sqrt{2}}\right)\delta(\overline{y}) + \delta\left(\overline{x} + \frac{a}{\sqrt{2}}\right)\delta(\overline{y}) \\ -\delta(\overline{x})\delta\left(\overline{y} - \frac{a}{\sqrt{2}}\right) - \delta(\overline{x})\delta\left(\overline{y} + \frac{a}{\sqrt{2}}\right) \end{bmatrix}\delta(\overline{z})d\overline{x}\,d\overline{y}\,d\overline{z}$$
$$= q\int\left(-\left(\overline{x}^2 + \overline{y}^2\right)\right)\begin{bmatrix} \delta\left(\overline{x} - \frac{a}{\sqrt{2}}\right)\delta(\overline{y}) + \delta\left(\overline{x} + \frac{a}{\sqrt{2}}\right)\delta(\overline{y}) \\ -\delta(\overline{x})\delta\left(\overline{y} - \frac{a}{\sqrt{2}}\right) - \delta(\overline{x})\delta\left(\overline{y} + \frac{a}{\sqrt{2}}\right) \end{bmatrix}d\overline{x}\,d\overline{y}$$
$$= 0$$

$$\overline{Q}_{11} = -\overline{Q}_{22} = 3qa^2$$

All the other components are zero.

The electric field is given in a different frame, rotated by θ from the first. So we rotate the quadrupole tensor into this coordinate system:

$$
\begin{aligned}
Q_{ij} &= A_{ik}A_{jl}\overline{Q}_{kl} = A_{ik}\overline{Q}_{kl}A^T_{lj} \\
&= \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} 3qa^2 & 0 & 0 \\ 0 & -3qa^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix} \\
&= \begin{pmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{pmatrix} \begin{pmatrix} 3qa^2\cos\theta & 0 & -3qa^2\sin\theta \\ 0 & -3qa^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 3qa^2\cos^2\theta & 0 & -3qa^2\cos\theta\sin\theta \\ 0 & -3qa^2 & 0 \\ -3qa^2\cos\theta\sin\theta & 0 & 3qa^2\sin^2\theta \end{pmatrix} \\
&= \begin{pmatrix} 3qa^2\cos^2\theta & 0 & -\frac{3qa^2}{2}\sin 2\theta \\ 0 & -3qa^2 & 0 \\ -\frac{3qa^2}{2}\sin 2\theta & 0 & 3qa^2\sin^2\theta \end{pmatrix}
\end{aligned}
$$

The force is given by:

$$
F_i = \frac{1}{6}Q_{jk}\frac{\partial^2 E_j}{\partial x_i \partial x_k} = \frac{1}{6}\frac{\partial}{\partial x_i}\frac{\partial E_j}{\partial x_k}Q_{jk}
$$

First we compute the vector

$$
\begin{aligned}
v_k = E_j Q_{jk} &= 3\alpha q a^2 \begin{pmatrix} 2xz, & -2yx, & x^2-y^2 \end{pmatrix} \begin{pmatrix} \cos^2\theta & 0 & -\frac{1}{2}\sin 2\theta \\ 0 & -1 & 0 \\ -\frac{1}{2}\sin 2\theta & 0 & \sin^2\theta \end{pmatrix} \\
&= 3\alpha q a^2 \begin{pmatrix} 2xz\cos^2\theta - \frac{1}{2}(x^2-y^2)\sin 2\theta & 2yx & -xz\sin 2\theta + (x^2-y^2)\sin^2\theta \end{pmatrix}
\end{aligned}
$$

Then compute the divergence

$$
\begin{aligned}
\frac{\partial v_k}{\partial x_k} &= 3\alpha q a^2 \begin{pmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \end{pmatrix} \begin{pmatrix} 2xz\cos^2\theta - \frac{1}{2}\left(x^2-y^2\right)\sin 2\theta \\ 2yx \\ -xz\sin 2\theta + \left(x^2-y^2\right)\sin^2\theta \end{pmatrix} \\
&= 3\alpha q a^2 \left(2z\cos^2\theta + 2x - 2x\sin 2\theta\right)
\end{aligned}
$$

Finally we take the gradient

$$
\begin{aligned}
F_i &= \frac{1}{6}\frac{\partial}{\partial x_i}3\alpha q a^2\left(2z\cos^2\theta + 2x - x\sin 2\theta\right) \\
&= \frac{\alpha}{2}qa^2 \begin{pmatrix} 2 - 2\sin 2\theta \\ 0 \\ 2\cos^2\theta \end{pmatrix} \\
&= \alpha q a^2 \begin{pmatrix} 1 - \sin 2\theta \\ 0 \\ \cos^2\theta \end{pmatrix}
\end{aligned}
$$

Since there is no dependence on the coordinates, there is no change when we evaluate at the center of the square ($x = y = z = 0$).

Another method:

$$T_{kj} = \frac{\partial E_j}{\partial x_k} = \alpha \begin{pmatrix} \frac{\partial}{\partial x} 2xz & \frac{\partial}{\partial x}(-2yz) & \frac{\partial}{\partial x}(x^2 - y^2) \\ \frac{\partial}{\partial y} 2xz & \frac{\partial}{\partial y}(-2yz) & \frac{\partial}{\partial y}(x^2 - y^2) \\ \frac{\partial}{\partial z} 2xz & \frac{\partial}{\partial z}(-2yz) & \frac{\partial}{\partial z}(x^2 - y^2) \end{pmatrix}$$

$$T_{kj} = \begin{pmatrix} 2z & 0 & 2x \\ 0 & -2z & -2y \\ 2x & -2y & 0 \end{pmatrix}$$

Then

$$Q_{jk} \frac{\partial E_j}{\partial x_k} = Tr\left(\mathbb{T}\mathbb{Q}^T\right)$$

$$Tr\left[\alpha \begin{pmatrix} 2z & 0 & 2x \\ 0 & -2z & -2y \\ 2x & -2y & 0 \end{pmatrix} \begin{pmatrix} 3qa^2\cos^2\theta & 0 & -\frac{3qa^2}{2}\sin 2\theta \\ 0 & -3qa^2 & 0 \\ -\frac{3qa^2}{2}\sin 2\theta & 0 & 3qa^2\sin^2\theta \end{pmatrix} \right]$$

$$= 3qa^2\alpha Tr\left[\begin{pmatrix} 2z\cos^2\theta - x\sin 2\theta & 0 & -z\sin 2\theta + 2x\sin^2\theta \\ y\sin 2\theta & 2z & -2y\sin^2\theta \\ 2x\cos^2\theta & 2yqa^2 & -x\sin 2\theta \end{pmatrix} \right]$$

$$= 6qa^2\alpha\left(z\left(\cos^2\theta + 1\right) - x\sin 2\theta\right)$$

Then

$$\vec{\mathbf{F}} = \frac{1}{6}\vec{\nabla} 6qa^2\left(z\left(\cos^2\theta + 1\right) - x\sin 2\theta\right)$$

$$= qa^2 \begin{pmatrix} -\sin 2\theta \\ 0 \\ 1 + \cos^2\theta \end{pmatrix}$$

Let's check the result. If $\theta = 0$, the charges are at $x = \pm a/\sqrt{2}$, $y = z = 0$ and $x = z = 0$, $y = \pm a/\sqrt{2}$. Thus the forces are

$$\vec{\mathbf{F}} = \sum_{n=1}^{4} q_n \vec{\mathbf{E}}(\vec{\mathbf{x}}_n)$$

$$= 2q\alpha \begin{pmatrix} 0 \\ 0 \\ a^2/2 \end{pmatrix} - 2q\alpha \begin{pmatrix} 0 \\ 0 \\ -a^2/2 \end{pmatrix} = 2qa^2\alpha\hat{\mathbf{z}}$$

which agrees with the result from the quadrupole tensor.

If $\theta = \pi/2$, the charges are at $x = y = 0$, $z = \pm a/\sqrt{2}$, and at $x = z = 0$, $y = \pm a/\sqrt{2}$. Then as expected.

$$\vec{\mathbf{F}} = 2q\alpha \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} - 2q\alpha \begin{pmatrix} 0 \\ 0 \\ -a^2/2 \end{pmatrix} = qa^2\hat{\mathbf{z}}$$

which again agrees with the tensor result.

Optional topic A: Tensors

11. Show that the components of the Levi-Civita symbol

ε_{ijk} transform as a tensor density under coordinate rotations and reflections.

Let us compute the components:

$$f_{lmn} = \frac{\partial x^i}{\partial \bar{x}^l}\frac{\partial x^j}{\partial \bar{x}^m}\frac{\partial x^k}{\partial \bar{x}^n}\varepsilon_{ijk}$$

According to equation 1.71, the right hand side is the determinant of the matrix,

$$\begin{pmatrix} \frac{\partial x^1}{\partial \bar{x}^l} & \frac{\partial x^1}{\partial \bar{x}^m} & \frac{\partial x^1}{\partial \bar{x}^n} \\ \frac{\partial x^2}{\partial \bar{x}^l} & \frac{\partial x^2}{\partial \bar{x}^m} & \frac{\partial x^2}{\partial \bar{x}^n} \\ \frac{\partial x^3}{\partial \bar{x}^l} & \frac{\partial x^3}{\partial \bar{x}^m} & \frac{\partial x^3}{\partial \bar{x}^n} \end{pmatrix}$$

If any two of l, m, n are equal, then two columns are equal and the determinant is zero. If l, m, n equal 1,2,3, we have the determinant of the transformation matrix, which is +1 for rotations and -1 for reflections. If

l, m, n equal 1,3,2 (or any even permutation of this) then the matrix is the transformation matrix with two columns interchanged, and the result changes sign. Thus

$$f_{lmn} = \varepsilon_{lmn} \det \mathbb{A}$$

as required for a tensor density.

12. Starting from the components of the velocity vector in Cartesian coordinates, transform to spherical coordinates to find the components of $\vec{v}$ in the new system, and hence write the velocity vector in spherical coordinates. (c.f. Example 4.)

The metric tensor in spherical coordinates is:

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix}$$

and the transformation matrix has components:

$$B_i^j = \frac{\partial x^j}{\partial \bar{x}^i} = \begin{pmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ r\cos\theta\cos\phi & r\cos\theta\sin\phi & -r\sin\theta \\ -r\sin\theta\sin\phi & r\sin\theta\cos\phi & 0 \end{pmatrix}$$

and thus, since $\mathbb{A} = \left(\mathbb{B}^T\right)^{-1}$

$$A^i{}_j = \begin{pmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \frac{\cos\phi}{r}\cos\theta & \frac{\sin\phi}{r}\cos\theta & -\frac{\sin\theta}{r} \\ -\frac{\sin\phi}{r\sin\theta} & \frac{\cos\phi}{r\sin\theta} & 0 \end{pmatrix}$$

Then the transformed velocity vector has components:

$$
\begin{aligned}
v^i = A^i{}_j v^j &= \begin{pmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \frac{\cos\phi}{r}\cos\theta & \frac{\sin\phi}{r}\cos\theta & -\frac{\sin\theta}{r} \\ -\frac{\sin\phi}{r\sin\theta} & \frac{\cos\phi}{r\sin\theta} & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial x}{\partial t} \\ \frac{\partial y}{\partial t} \\ \frac{\partial z}{\partial t} \end{pmatrix} \\
&= \begin{pmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ \frac{\cos\phi}{r}\cos\theta & \frac{\sin\phi}{r}\cos\theta & -\frac{\sin\theta}{r} \\ -\frac{\sin\phi}{r\sin\theta} & \frac{\cos\phi}{r\sin\theta} & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial r}{\partial t}\sin\theta\cos\phi + r\cos\theta\cos\phi\frac{\partial\theta}{\partial t} - r\sin\theta\sin\phi\frac{\partial\phi}{\partial t} \\ \frac{\partial r}{\partial t}\sin\theta\sin\phi + r\cos\theta\sin\phi\frac{\partial\theta}{\partial t} + r\sin\theta\cos\phi\frac{\partial\phi}{\partial t} \\ \frac{\partial r}{\partial t}\cos\theta - r\sin\theta\frac{\partial\theta}{\partial t} \end{pmatrix} \\
&= \begin{pmatrix} \frac{\partial r}{\partial t} \\ \frac{\partial\theta}{\partial t} \\ \frac{\partial\phi}{\partial t} \end{pmatrix}
\end{aligned}
$$

as we might have expected. Adding back the basis vectors, we get:

$$\mathbf{v} = \hat{\mathbf{r}}\frac{\partial r}{\partial t} + r\frac{\partial\theta}{\partial t}\hat{\boldsymbol{\theta}} + r\sin\theta\frac{\partial\phi}{\partial t}\hat{\boldsymbol{\phi}}$$

13. In a region of space the electric scalar potential has the form $\Phi = -E_0 z$.

(a) Working in Cartesian coordinates, compute the gradient to obtain the electric field components. Transform to a spherical coordinate system using the appropriate transformation law from section 4.

(b) Write the potential in spherical coordinates, and compute the gradient using the operator ∂_i.

Confirm that both methods give the same electric field.

(a) The components of E are

$E_i = -\partial_i\Phi = (0, 0, E_0)$. Now we transform. Note that these are covariant components, so

$$
\begin{aligned}
(E')_i = B_i^j E_j &= \begin{pmatrix} \sin\theta\cos\phi & \sin\theta\sin\phi & \cos\theta \\ r\cos\theta\cos\phi & r\cos\theta\sin\phi & -r\sin\theta \\ -r\sin\theta\sin\phi & r\sin\theta\cos\phi & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ E_0 \end{pmatrix} \\
&= \begin{pmatrix} E_0\cos\theta \\ -E_0 r\sin\theta \\ 0 \end{pmatrix}
\end{aligned}
$$

(b) $\Phi = -E_0 r\cos\theta$. Thus the gradient is

$$-\partial_i\Phi = E_0 \begin{pmatrix} \cos\theta \\ -r\sin\theta \\ 0 \end{pmatrix}$$

which is the same result.

Note that to write this as a vector field, we need to raise indices, and multiply by the unit vectors. This gives

$$
\begin{aligned}
E^i = g^{ij}E_j &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & \frac{1}{r^2\sin^2\theta} \end{pmatrix} E_0 \begin{pmatrix} \cos\theta \\ -r\sin\theta \\ 0 \end{pmatrix} \\
&= E_0 \begin{pmatrix} \cos\theta \\ -\frac{\sin\theta}{r} \\ 0 \end{pmatrix}
\end{aligned}
$$

and thus

$$\vec{\mathbf{E}} = E_0\left(\cos\theta\hat{\mathbf{r}} - \frac{\sin\theta}{r}\left(r\hat{\boldsymbol{\theta}}\right)\right) = E_0\left(\cos\theta\hat{\mathbf{r}} - \sin\theta\hat{\boldsymbol{\theta}}\right)$$

as expected.

14. Write the components of the gradient form in (**a)** cylindrical coordinates and (**b**) spherical coordinates. Use the metric tensor to raise indices, thus mapping to the corresponding vector. Finally multiply by the basis vectors to obtain the conventional expression for $\vec{\nabla}\Phi$.

(a) In cylindrical coordinates, we start with the gradient form:

$$\partial_i = \left(\frac{\partial}{\partial\rho}, \quad \frac{\partial}{\partial\phi}, \quad \frac{\partial}{\partial z} \right)$$

and then raise indices, to obtain the vector components:

$$\partial^i = g^{ij}\partial_j \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\rho^2} & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \frac{\partial}{\partial\rho} \\ \frac{\partial}{\partial\phi} \\ \frac{\partial}{\partial z} \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial\rho} \\ \frac{1}{\rho^2}\frac{\partial}{\partial\phi} \\ \frac{\partial}{\partial z} \end{pmatrix}$$

and thus the gradient vector is:

$$\begin{aligned} \vec{\nabla}\Phi &= \frac{\partial\Phi}{\partial\rho}\hat{\boldsymbol{\rho}} + \frac{1}{\rho^2}\frac{\partial\Phi}{\partial\phi}\rho\hat{\boldsymbol{\phi}} + \frac{\partial\Phi}{\partial z}\hat{\mathbf{z}} \\ &= \frac{\partial\Phi}{\partial\rho}\hat{\boldsymbol{\rho}} + \frac{1}{\rho}\frac{\partial\Phi}{\partial\phi}\hat{\boldsymbol{\phi}} + \frac{\partial\Phi}{\partial z}\hat{\mathbf{z}} \end{aligned}$$

(b) In spherical coordinates, the metric tensor is:

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix}$$

and its inverse is:

$$g^{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & \frac{1}{r^2\sin^2\theta} \end{pmatrix}$$

The gradient form is:

$$\partial_i = \left(\frac{\partial}{\partial r}, \quad \frac{\partial}{\partial\theta}, \quad \frac{\partial}{\partial\phi} \right)$$

and the corresponding vector has components:

$$\partial^i = g^{ij}\partial_j = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{r^2} & 0 \\ 0 & 0 & \frac{1}{r^2\sin^2\theta} \end{pmatrix}\begin{pmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial\theta} \\ \frac{\partial}{\partial\phi} \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial r} \\ \frac{1}{r^2}\frac{\partial}{\partial\theta} \\ \frac{1}{r^2\sin^2\theta}\frac{\partial}{\partial\phi} \end{pmatrix}$$

Multiplying by the basis vectors, we have:

$$\begin{aligned} \vec{\nabla}\Phi &= \frac{\partial\Phi}{\partial r}\hat{\mathbf{r}} + \frac{1}{r^2}\frac{\partial\Phi}{\partial\theta}r\hat{\boldsymbol{\theta}} + \frac{1}{r^2\sin^2\theta}\frac{\partial\Phi}{\partial\phi}r\sin\theta\hat{\boldsymbol{\phi}} \\ &= \frac{\partial\Phi}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial\Phi}{\partial\theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial\Phi}{\partial\phi}\hat{\boldsymbol{\phi}} \end{aligned}$$

15. Use equations (8) and (9) in section 4 to show that the relation $\mathbb{B}^{\mathsf{T}} = \mathbb{A}^{-1}$ is true in general.

$$A^i{}_j B^{\mathsf{T}\,j}{}_k = \frac{\partial\overline{x}^i}{\partial x^j}\frac{\partial x^j}{\partial\overline{x}^k} = \frac{\partial\overline{x}^i}{\partial\overline{x}^k} = \delta^i_k$$

as required.

16. If the tensor A^{ij} is symmetric, $A^{ij} = A^{ji}$, show that $A^i{}_j = A_j{}^i$ and that $A_{ij} = A_{ji}$. Can you find a relation between $A^i{}_j$ and $A^j{}_i$? Why or why not?

$$A^{ij} = A^{ji}$$

$$A^i{}_j = g_{jm}A^{im} = g_{jm}A^{mi} = A_j{}^i$$

$$A_{ij} = g_{im}g_{jn}A^{mn} = g_{im}g_{jn}A^{nm} = A_{ji}$$

We cannot find a relationship between $A^i{}_j$ and $A^j{}_i$ because the indices do not match up properly:

i is up in the first expression and down in the second.

17. Which of the following relations between tensor components could possibly be true? Say what is wrong with the incorrect expressions.

(a) $V^i = \varepsilon^{ijk}U_k$

(b) $T^{ij} = X^{ik}Y_{kj}$

(c) $V^i = X^{ik}U_k + W^i$

(d) $V^i = \varepsilon^{ijk}U_iW_jX_kY^i$

(a) is incorrect: the expression on the left has one free index; i. On the right there are two free indices: i and j.

(b) is incorrect. On the left both indices are up, but on the right one free index (i) is up and one (j) is down.

(c) is possible.

(d) is incorrect. On the right the index i is repeated three times, thus this expression is meaningless.

18. In special relativity, space-time is described by four-component vectors. The position vector has components

(ct, x, y, z) and the metric is

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

The Greek letters signify indices that take on the four values 0,1,2,3. The Lorentz transformation matrix relating two coordinate systems moving with relative speed v along the $x-$axis is:

$$\Lambda^\mu{}_\nu = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

We use the Gaussian unit system in this problem. The electromagnetic potential is described by a 4-vector with components $A^\mu = (\Phi, A_x, A_y, A_z)$ where Φ is the electric scalar potential and

$\vec{A}$ is the magnetic vector potential. The electromagnetic field tensor has components:

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$$

Find the components of the field tensor in terms of $\vec{E}$ and $\vec{B}$.

Two particles, each with charge q and mass m, are moving along lines parallel to the $x-$ axis and a distance d apart. Each particle moves with speed $v \ll c$. Start in a reference frame in which the two particles are at rest. Compute the components of the field tensor in this reference frame, and hence find the force acting on each particle. Now transform the field tensor to the lab frame, and again compute the force on each particle. Verify your result by computing $\vec{E}$ and $\vec{B}$ in the lab frame using Coulomb's law and the Biot-Savart law.

Let's compute the tensor components one at a time. The tensor is antisymmetric, so there are 6 independent components, equal to the number of components in $\vec{E}$ and $\vec{B}$.

$$F^{01} = \partial^0 A^1 - \partial^1 A^0 = \frac{\partial}{\partial ct}A_x + \frac{\partial}{\partial x}\Phi = -E_x$$

We will obtain the other components of $\vec{E}$ from F^{02} and F^{03}. Then:

$$F^{12} = \partial^1 A^2 - \partial^2 A^1 = -\frac{\partial}{\partial x}A_y + \frac{\partial}{\partial y}A_x = -B_z$$

$$F^{13} = \partial^1 A^3 - \partial^3 A^1 = -\frac{\partial}{\partial x}A_z + \frac{\partial}{\partial z}A_x = B_y$$

and

$$F^{23} = \partial^2 A^3 - \partial^3 A^2 = -\frac{\partial}{\partial y}A_z + \frac{\partial}{\partial z}A_y = -B_x$$

Thus the tensor components are:

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}$$

In the rest frame, there is only an electric field. We put the x-axis along the direction of motion, and the y-axis along the line between the particles. Then the field at the position of the second particle has only a $y-$component, of magnitude $E_y = q/d^2$. Thus the field tensor is:

$$\overline{F}^{\mu\nu} = \begin{pmatrix} 0 & 0 & -\frac{q}{d^2} & 0 \\ 0 & 0 & 0 & 0 \\ \frac{q}{d^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and the force between the particles is along the $y-$axis and has magnitude q^2/d^2.

To transform to the lab frame, we use the Lorentz transformation with velocity in the $-x$-direction. Thus:

$$F^{\mu\nu} = \Lambda^\mu{}_\rho \Lambda^\nu{}_\sigma F^{\rho\sigma} = \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & -\frac{q}{d^2} & 0 \\ 0 & 0 & 0 & 0 \\ \frac{q}{d^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & -\frac{q}{d^2} & 0 \\ 0 & 0 & 0 & 0 \\ \frac{q}{d^2}\gamma & \frac{q}{d^2}\gamma\beta & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & -\frac{q}{d^2}\gamma & 0 \\ 0 & 0 & -\frac{q}{d^2}\gamma\beta & 0 \\ \frac{q}{d^2}\gamma & \frac{q}{d^2}\gamma\beta & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Thus in this frame we have an electric field $E_y = \gamma q/d^2$ and a magnetic field $B_z = \gamma\beta q/d^2$. These results are consistent with the Biot-Savart and Coulomb laws in the case $\beta \ll 1$ ($\gamma \simeq 1$). The force between the particles in this case may be computed from:

$$\vec{\mathbf{F}} = q\left(\vec{\mathbf{E}} + \frac{\vec{\mathbf{v}}}{c} \times \vec{\mathbf{B}}\right)$$

$$F_y = q\left(E_y - \frac{v}{c}B_z\right) = q\gamma\frac{q}{d^2}\left(1-\beta^2\right)$$

$$= \frac{q^2}{\gamma d^2} = \frac{q^2}{d^2} \text{ to first order in } \beta$$

19. Using the metric of Lorentz space-time and the electromagnetic field tensor (see problem 18 above) verify that an electromagnetic wave has the same field structure ($\vec{\mathbf{E}} \perp \vec{\mathbf{B}}$, $E = B$) in any inertial frame.

If the wave propagates in the z −direction, then

$$E_x B_x + E_y B_y = 0 \Rightarrow B_y = -\frac{E_x}{E_y} B_x$$

But also

$$\sqrt{E_x^2 + E_y^2} = \sqrt{B_x^2 + B_y^2} = B_x\sqrt{1 + E_x^2/E_y^2}$$

So

$$B_x = E_y$$

Then the field tensor has components:

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & 0 \\ E_x & 0 & 0 & B_y \\ E_y & 0 & 0 & -B_x \\ 0 & -B_y & B_x & 0 \end{pmatrix} = \begin{pmatrix} 0 & -E_x & -E_y & 0 \\ E_x & 0 & 0 & -E_x \\ E_y & 0 & 0 & -E_y \\ 0 & E_x & E_y & 0 \end{pmatrix}$$

Now apply the Lorentz transformation:

$$\overline{F}^{\mu\nu} = \Lambda^{\mu}_{\rho}\Lambda^{\nu}_{\sigma}F^{\rho\sigma} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & -E_x & -E_y & 0 \\ E_x & 0 & 0 & -E_x \\ E_y & 0 & 0 & -E_y \\ 0 & E_x & E_y & 0 \end{pmatrix}\begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} E_x\gamma\beta & -E_x\gamma & -E_y & 0 \\ E_x\gamma & -E_x\gamma\beta & 0 & -E_x \\ E_y\gamma & -E_y\gamma\beta & 0 & -E_y \\ -E_x\gamma\beta & E_x\gamma & E_y & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & -\gamma^2 E_x(1-\beta^2) & -E_y\gamma & E_x\gamma\beta \\ \gamma^2 E_x(1-\beta^2) & 0 & E_y\gamma\beta & -E_x\gamma \\ E_y\gamma & -E_y\gamma\beta & 0 & -E_y \\ -E_x\gamma\beta & E_x\gamma & E_y & 0 \end{pmatrix} = \begin{pmatrix} 0 & -E_x & -E_y\gamma & E_x\gamma\beta \\ E_x & 0 & E_y\gamma\beta & -E_x\gamma \\ E_y\gamma & -E_y\gamma\beta & 0 & -E_y \\ -E_x\gamma\beta & E_x\gamma & E_y & 0 \end{pmatrix}$$

We now have three components to $\vec{\mathbf{E}}$ and $\vec{\mathbf{B}}$. The dot product is:

$$\vec{\mathbf{E}}\cdot\vec{\mathbf{B}} = \overline{E}_x\overline{B}_x + \overline{E}_y\overline{B}_y + \overline{E}_z\overline{B}_z = E_xE_y + \gamma E_y(-\gamma E_x) + \gamma\beta E_x\gamma\beta E_y$$
$$= E_xE_y\left(1-\gamma^2+\gamma^2\beta^2\right) = E_xE_y\left(1-\frac{1}{1-\beta^2}+\frac{\beta^2}{1-\beta^2}\right) = 0$$

So $\vec{\mathbf{E}} \perp \vec{\mathbf{B}}$. The magnitudes are

$$\overline{E}^2 = E_x^2 + \gamma^2E_y^2 + \gamma^2\beta^2E_x^2$$
$$= \left[\left\{1+\gamma^2\beta^2\right\}E_x^2 + \gamma^2E_y^2\right]$$
$$= \gamma^2E^2$$

and

$$\overline{B}^2 = E_y^2 + \gamma^2E_x^2 + \gamma^2\beta^2E_y^2 = E_y^2\left(1+\gamma^2\beta^2\right) + \gamma^2E_x^2$$
$$= \gamma^2E^2 = \overline{E}^2$$

as required.

Now if the wave propagates in the x −direction, we may rotate the axes so that $\vec{\mathbf{E}}$ has only a y-component, in which case $\vec{\mathbf{B}}$ has only a z-component, and $B_z = E_y$. Then we have:

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -E_y & 0 \\ 0 & 0 & -E_y & 0 \\ E_y & E_y & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Performing the Lorentz transformation, we find:

$$\overline{F}^{\mu\nu} = \Lambda^{\mu}_{\ \rho}\Lambda^{\nu}_{\ \sigma}\overline{F}^{\rho\sigma} = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & -E_y & 0 \\ 0 & 0 & -E_y & 0 \\ E_y & E_y & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 0 & -E_y & 0 \\ 0 & 0 & -E_y & 0 \\ E_y\gamma - E_y\gamma\beta & E_y\gamma - E_y\gamma\beta & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & -E_y\gamma + E_y\gamma\beta & 0 \\ 0 & 0 & -E_y\gamma + E_y\gamma\beta & 0 \\ E_y\gamma - E_y\gamma\beta & E_y\gamma - E_y\gamma\beta & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

so the fields have the same components, and only the magnitude has changed, to $\gamma E_y(1-\beta)$

20. What invariants can you form from a tensor

$T^{\mu\nu}$? Compute these invariants for the electromagnetic field tensor in Lorentz space-time (cf Problem 18)

The invariants are $T^{\mu}_{\ \mu}$ and

$T^{\mu\nu}T_{\mu\nu}$. The first is obtained by lowering the index, and taking the trace of the resulting matrix. Since $F^{\mu\nu}$ is antisymmetric, the result is zero. To compute the second, we must lower both indices.

$$F_{\mu\nu} = g_{\mu\rho}g_{\nu\sigma}F^{\rho\sigma} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & E_x & E_y & E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix}$$

Then the invariant $F^{\mu\nu}F_{\mu\nu}$ is the sum of the products of the elements of the two tensors, that is $-E^2 + B^2$. Thus as we change frames, the fields maintain their character as predominantly electric ($E^2 > B^2$), predominantly magnetic ($B^2 > E^2$), or for a wave, $E = B$.

Optional topic A: Tensors

21. In Lorentz space-time, the wave 4-vector has components

$k^\mu = (\frac{\omega}{c}, k_x, k_y, k_z).$ Use the Lorentz transformation matrix

$\Lambda^\mu{}_\nu$ (see Problem 18) to find the components in a second frame moving with velocity $\vec{\mathbf{v}} = v\hat{\mathbf{x}}$ with respect to the first. What is the result if (a) $\vec{\mathbf{k}} = k\hat{\mathbf{x}}$ and

(b) $\vec{\mathbf{k}} = k\hat{\mathbf{y}}.$ Compare with the non-relativistic Doppler shift formula, and comment.

The transformed vector has components

$$\overline{k}^\mu = \Lambda^\mu{}_\nu k^\nu = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{\omega}{c} \\ k_x \\ k_y \\ k_z \end{pmatrix} = \begin{pmatrix} \gamma(\frac{\omega}{c} - \beta k_x) \\ \gamma(k_x - \beta\frac{\omega}{c}) \\ k_y \\ k_z \end{pmatrix}$$

If $k_y = k_z = 0,$ then

$$\varpi = \gamma\left(\frac{\omega}{c} - \beta k_x\right)$$

which is the non-relativistic result multiplied by the factor $\gamma.$ If $k_x = k_z = 0,$ there would be no Doppler shift in the non-relativistic case, but here we have

$$\varpi = \gamma\omega$$

and there is a Doppler shift, called the transverse Doppler shift. It is due to time dilation.

22. Use the metric $g_{\mu\nu}$ for Lorentz space-time (see Problem 18) to compute the line element $ds^2 = g_{\mu\nu}dx^\mu dx^\nu.$ The proper time τ is defined by the relation $d\tau = ds/c.$ Compute the proper time interval $d\tau$ between two neighboring points on the world line of a particle moving at speed $v.$ (Let the points have coordinates t,x,y,z and $t + dt, x + dx, y + dy, z + dz,$ where $dx = v_x dt$ and similarly for dy and $dz.$) Express your result in terms of the time interval $dt,$ $\beta = v/c$ and

$\gamma = 1/\sqrt{1-\beta^2}.$ Compute the components of the 4-velocity $u^\mu = dx^\mu/d\tau$ of a particle and compute the invariant product $u^\mu u_\mu.$ Comment.

$$ds^2 = c^2dt^2 - \left(dx^2 + dy^2 + dz^2\right)$$

and thus

$$d\tau^2 = dt^2 - \frac{dr^2}{c^2} = dt^2(1 - \frac{v^2}{c^2})$$

Thus

$$d\tau = \frac{dt}{\gamma}$$

The velocity has components:

$$u^\mu = \frac{dx^\mu}{d\tau} = \gamma\frac{dx^\mu}{dt} = \gamma\left(c, \vec{\mathbf{v}}\right)$$

The invariant is:

$$u^\mu u_\mu = \gamma^2\left(c^2 - v^2\right) = c^2$$

The result is independent of the particle's 3-velocity components, and is clearly invariant.

23. The set of components $\varepsilon^{\alpha\beta\gamma\delta} =$

$$\varepsilon^{\alpha\beta\gamma\delta} = \begin{cases} 1 & \text{if } \alpha\beta\gamma\delta = 0123 \text{ or an even permutation of this} \\ -1 & \text{if } \alpha\beta\gamma\delta = 1023 \text{ or an even permutation of this} \\ 0 & \text{otherwise} \end{cases}$$ may be used to form the tensor

$\mathcal{F}^{\alpha\beta} = \frac{1}{2}\varepsilon^{\alpha\beta\gamma\delta}F_{\gamma\delta}$ dual to $F.$ (Compare with equation A.7)

(a) Show that the components of $\varepsilon^{\alpha\beta\gamma\delta}$ transform as a tensor.

(b) Find the invariant $\mathcal{F}^{\alpha\beta}F_{\alpha\beta}$ if $F^{\alpha\beta}$ is the electromagnetic field density defined in problem 18. Comment.

In 4-dimensions

$$\begin{aligned} f_{\nu\mu\xi\chi} &= B_\nu{}^\alpha B_\mu{}^\beta B_\xi{}^\gamma B_\chi{}^\delta \varepsilon_{\alpha\beta\gamma\delta} \\ &= \frac{\partial x^\alpha}{\partial \overline{x}^\nu}\frac{\partial x^\beta}{\partial \overline{x}^\mu}\frac{\partial x^\gamma}{\partial \overline{x}^\xi}\frac{\partial x^\delta}{\partial \overline{x}^\chi}\varepsilon_{\alpha\beta\gamma\delta} \end{aligned}$$

In this spacetime,

$\mathbb{B} = \left(\mathbb{A}^{-1}\right)^{\mathsf{T}} = \mathbb{A}(-\beta)$ since the Lorentz transformation matrix is symmetric. Also the elements of $\mathbb{A}(-\beta)$ are the same as $\mathbb{A}(\beta)$ except $A_1^0 = A_0^1$ which changes sign. Using the properties of $\varepsilon_{\alpha\beta\gamma\delta}$, we have:

$$\begin{aligned} f_{\nu\mu\xi\chi} &= A_\nu^0 A_\mu^1 A_\xi^2 A_\chi^3 + A_\nu^1 A_\mu^0 A_\xi^3 A_\chi^2 + 2 \text{ similar terms with even permutations of } 0,1,2,3 \\ &\quad - A_\nu^1 A_\mu^0 A_\xi^2 A_\chi^3 + 3 \text{ similar terms} \end{aligned}$$

Now if any two of the indices μ,ν,ξ,χ are the same, then the terms will cancel in pairs, and the result is zero.

Let $\nu\mu\xi\chi = 0123.$ Then

$$\begin{aligned} f_{0123} &= A_0^0A_1^1A_2^2A_3^3 + A_0^1A_1^0A_2^3A_3^2 + A_0^3A_1^2A_2^1A_3^0 + A_0^2A_1^3A_2^0A_3^1 \\ &\quad - A_0^1A_1^0A_2^2A_3^3 - A_0^0A_1^1A_2^3A_3^2 - A_0^2A_1^3A_2^1A_3^0 - A_0^3A_1^2A_2^1A_3^0 \\ &= \gamma\gamma 11 + (-\gamma\beta)^2(0) + 0 + 0 - (\gamma\beta)^2 - 0 - 0 - 0 \\ &= \gamma^2(1-\beta)^2 = 1 \end{aligned}$$

Similarly:

$$\begin{aligned} f_{0132} &= A_0^0A_1^1A_3^2A_2^3 + A_0^1A_1^0A_3^3A_2^2 + A_0^3A_1^2A_3^1A_2^0 + A_0^2A_1^3A_3^0A_2^1 \\ &\quad - A_0^1A_1^0A_3^2A_2^3 - A_0^0A_1^1A_3^3A_2^2 - A_0^2A_1^3A_3^1A_2^0 - A_0^3A_1^2A_3^1A_2^0 \\ &= 0 + (-\gamma\beta)^2(1) + 0 + 0 - 0 - \gamma^2 - 0 - 0 \\ &= \gamma^2(\beta-1)^2 = -1 \end{aligned}$$

The other components may be computed similarly to show that

$$f_{\alpha\beta\gamma\delta} = \varepsilon_{\alpha\beta\gamma\delta}$$

as required. Thus $\varepsilon_{\alpha\beta\gamma\delta}$ transforms as a tensor.

(b) First note that

$$\begin{aligned} F_{\alpha\beta} = g_{\alpha\mu}F^{\mu\nu}g_{\nu\beta} &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & E_x & E_y & E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix} = \begin{pmatrix} 0 & E_x & E_y & E_z \\ -E_x & 0 & -B_z & B_y \\ -E_y & B_z & 0 & -B_x \\ -E_z & -B_y & B_x & 0 \end{pmatrix} \end{aligned}$$

Then

$$\mathcal{F}^{01} = \frac{1}{2}\varepsilon^{01\gamma\delta}F_{\gamma\delta} = \frac{1}{2}(F_{23} - F_{32}) = F_{23} = -B_x$$

$$\mathcal{F}^{02} = \frac{1}{2}\varepsilon^{02\gamma\delta}F_{\gamma\delta} = \frac{1}{2}(-F_{13} + F_{31}) = F_{31} = -B_y$$

$$\mathcal{F}^{03} = \frac{1}{2}\varepsilon^{03\gamma\delta}F_{\gamma\delta} = \frac{1}{2}(-F_{21} + F_{12}) = F_{12} = -B_z$$

and similarly

$$\mathcal{F}^{12} = \frac{1}{2}\varepsilon^{12\gamma\delta}F_{\gamma\delta} = \frac{1}{2}(F_{03} - F_{30}) = F_{03} = E_z$$

$$\mathcal{F}^{13} = \frac{1}{2}\varepsilon^{13\gamma\delta}F_{\gamma\delta} = \frac{1}{2}(F_{20} - F_{02}) = F_{20} = -E_y$$

and

$$\mathcal{F}^{23} = \frac{1}{2}\varepsilon^{23\gamma\delta}F_{\gamma\delta} = \frac{1}{2}(F_{01} - F_{10}) = F_{01} = E_x$$

Thus

$$\mathcal{F}^{\alpha\beta} = \begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z & -E_y \\ B_y & -E_z & 0 & E_x \\ B_z & E_y & -E_x & 0 \end{pmatrix}$$

The tensor $\mathcal{F}^{\alpha\beta}$ may be obtained from $F^{\alpha\beta}$ by replacing $\vec{E}$ by $\vec{B}$ and $\vec{B}$ by $-\vec{E}$. Thus

$$\mathcal{F}^{\alpha\beta}F_{\alpha\beta} = \mathrm{Tr}\left(\mathcal{F}F^T\right) = \mathrm{Tr}\begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z & -E_y \\ B_y & -E_z & 0 & E_x \\ B_z & E_y & -E_x & 0 \end{pmatrix}\begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix}$$

$$= \mathrm{Tr}\begin{pmatrix} -B_xE_x - B_yE_y - B_zE_z & 0 & 0 & 0 \\ 0 & -B_xE_x - B_yE_y - B_zE_z & 0 & 0 \\ 0 & 0 & -B_xE_x - B_yE_y - B_zE_z & 0 \\ 0 & 0 & 0 & -B_xE_x - B_yE_y - B_zE_z \end{pmatrix}$$

$$= -4\vec{\mathbf{E}}\cdot\vec{\mathbf{B}}$$

Thus

$\vec{\mathbf{E}}\cdot\vec{\mathbf{B}}$ is an invariant under Lorentz transformations. In particular, if this dot product is zero in any frame, it is zero in all frames, that is, the fields are either perpendicular or one of them is zero.

24. Use Gauss' law to find the electric field inside a uniformly charged sphere. Compute the necessary components of

$\Gamma^i{}_{jk}$, and hence find the divergence of this electric field in spherical coordinates. Show that the divergence equals the (uniform) charge density.

The electric field is

$$\vec{\mathbf{E}} = \frac{1}{3\varepsilon_0}\rho r\hat{\mathbf{r}}$$

We start with the metric tensor:

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r^2 & 0 \\ 0 & 0 & r^2\sin^2\theta \end{pmatrix}$$

Then the only non-zero derivatives are $\partial g_{22}/\partial x^1 = 2r$, $\partial g_{33}/\partial x^1 = 2r\sin^2\theta$ and $\partial g_{33}/\partial x^2 = 2r^2\sin\theta\cos\theta$. Then

$$\Gamma^n{}_{kj} = \frac{g^{in}}{2}\left(\frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{ji}}{\partial x^k} - \frac{\partial g_{kj}}{\partial x^i}\right)$$

So

$$\Gamma^1{}_{kj} = \frac{g^{i1}}{2}\left(\frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{ji}}{\partial x^k} - \frac{\partial g_{kj}}{\partial x^i}\right)$$

Since g^{ij} is diagonal, the only non-zero terms have $i = 1$. Thus:

$$\Gamma^1{}_{kj} = \frac{1}{2}\left(\frac{\partial g_{1k}}{\partial x^j} + \frac{\partial g_{j1}}{\partial x^k} - \frac{\partial g_{kj}}{\partial x^1}\right)$$

The only non-zero terms have $kj = 22$ and $kj = 33.$ Thus:

$$\Gamma^1_{22} = \frac{1}{2}(-2r) = -r$$

$$\Gamma^1_{33} = \frac{1}{2}\left(-2r\sin^2\theta\right) = -r\sin^2\theta$$

Then with $n = 2$:

$$\Gamma^2_{kj} = \frac{g^{i2}}{2}\left(\frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{ji}}{\partial x^k} - \frac{\partial g_{kj}}{\partial x^i}\right)$$

The only non-zero terms have $i = 2$:

$$\Gamma^2_{kj} = \frac{1}{2r^2}\left(\frac{\partial g_{2k}}{\partial x^j} + \frac{\partial g_{j2}}{\partial x^k} - \frac{\partial g_{kj}}{\partial x^2}\right)$$

Then the only non-zero values have $k = 2,$ $j = 1$ or $k = j = 3$:

$$\Gamma^2_{21} = \frac{1}{2r^2}(2r) = \frac{1}{r}$$

$$\Gamma^2_{33} = \frac{1}{2r^2}\left(-2r^2\sin\theta\cos\theta\right) = -\sin\theta\cos\theta$$

Finally,

$$\begin{aligned}\Gamma^3_{kj} &= \frac{g^{i3}}{2}\left(\frac{\partial g_{ik}}{\partial x^j} + \frac{\partial g_{ji}}{\partial x^k} - \frac{\partial g_{kj}}{\partial x^i}\right)\\ &= \frac{g^{33}}{2}\left(\frac{\partial g_{3k}}{\partial x^j} + \frac{\partial g_{j3}}{\partial x^k} - \frac{\partial g_{kj}}{\partial x^3}\right)\end{aligned}$$

So:

$$\Gamma^3_{31} = \frac{1}{2r^2\sin^2\theta}\frac{\partial g_{33}}{\partial r} = \frac{2r\sin^2\theta}{2r^2\sin^2\theta} = \frac{1}{r}$$

$$\Gamma^3_{32} = \frac{1}{2r^2\sin^2\theta}\frac{\partial g_{33}}{\partial\theta} = \frac{2r^2\sin\theta\cos\theta}{2r^2\sin^2\theta} = \frac{\cos\theta}{\sin\theta}$$

Then the divergence is:

$$\vec{\nabla}\cdot\vec{\mathbf{E}} = E^i_{\ ;i} = \frac{\partial E^i}{\partial x^i} + \Gamma^i_{\ ki}E^k$$

In this coordinate system, the electric field vector has only one component, so the first term is $\rho/3\varepsilon_0$. Then:

$$\begin{aligned}\vec{\nabla}\cdot\vec{\mathbf{E}} &= \frac{\rho}{3\varepsilon_0}\left(1 + \Gamma^2_{\ 21}r + \Gamma^3_{\ 31}r\right)\\ &= \frac{\rho}{3\varepsilon_0}\left(1 + \frac{1}{r}r + \frac{1}{r}r\right) = \frac{\rho}{\varepsilon_0}\end{aligned}$$

as expected.

25. In two-dimensional flat space described with cylindrical coordinates, a vector

$\vec{\mathbf{u}}$ is in the radial direction. Displace the vector to a neighboring point, and compute the new components in terms of the displacement $(\delta\rho, \delta\theta).$ Compare with relation (A.23) in the text and hence compute the components

$\Gamma^1_{\ ij}$ of the affinity. Perform the same operations with a vector in the θ direction to find the remaining components of

$\Gamma.$ *Hint:* remember that the basis vectors are not unit vectors in this system.

The vector has components $(u, 0).$ When displaced (Figure A.3 in the text), the new ρ $-$component is $u_\rho = u\cos\delta\theta = u$ to first order in $\delta\theta.$ Similarly

$$u_\theta\hat{\boldsymbol{\theta}} = -u\sin\delta\theta\hat{\boldsymbol{\theta}} = -u\delta\theta\hat{\boldsymbol{\theta}} = -\frac{u}{\rho}\delta\theta\hat{\mathbf{e}}_2.$$

(Remember that $\hat{\mathbf{e}}_2 = \rho\hat{\boldsymbol{\theta}}.$) Thus

$$\delta u^i = \left(0, -\frac{u}{\rho}\delta\theta\right) = -\Gamma^i_{\ jk}u^j dx^k.$$

Then

$$0 = -\Gamma^1_{\ jk}u^j dx^k = -\Gamma^1_{\ 12}u\,d\theta - \Gamma^1_{\ 11}u\,d\rho$$

Thus $\Gamma^1_{\ 12} = \Gamma^1_{\ 21} = \Gamma^1_{\ 11} = 0.$

$$-\frac{u}{\rho}\delta\theta = -\Gamma^2{}_{12}ud\theta - \Gamma^2{}_{11}ud\rho$$

So $\Gamma^2{}_{11} = 0$ and $\Gamma^2{}_{12} = \frac{1}{\rho}. = \Gamma^2{}_{21}$.

Optional Topic B: Groups

1. Show that the set of permutations of two elements is a group. What is its order? Write the multiplication table for this group. How many classes are there?

The elemnts are: $\{1\}$ (do nothing - the identity element), and G_{12} = (integrchange the two elements). The order of the group is two, and the multiplication table is

	1	G_{12}
1	1	G_{12}
G_{12}	G_{12}	1

There are two classes, each containing a single element, since

$$G_{12}^{-1} 1 G_{12} = 1$$

and

$$1^{-1} G_{12} 1^{-1} = G_{12}$$

The group is abelian, as you can see from the multiplication table.

2. The symmetry group of a square contains those operations that leave the square unchanged. Show that this group has 8 elements, and write the multiplication table. What are the classes? Are there any subgroups?

The elements are rotations about an axis perpendicular to the square through its center, through the angles: 0, $\pi/2,$ π and 3$\pi/2,$ together with reflections about the symmetry lines (bisectors of the horizontal and vertical sides, and the two diagonals). Let's call these reflections $L_x, L_y,$ L_{d1} and $L_{d2}.$ The multiplication table is:

	1	$\mathbf{R}_{\pi/2}$	$\mathbf{R}_{\pi}$	$\mathbf{R}_{3\pi/2}$	$\mathbf{L}_x$	$\mathbf{L}_y$	$\mathbf{L}_{d1}$	$\mathbf{L}_{d2}$
1	1	$R_{\pi/2}$	R_{π}	$R_{3\pi/2}$	L_x	L_y	L_{d1}	L_{d2}
$\mathbf{R}_{\pi/2}$	$R_{\pi/2}$	R_{π}	$R_{3\pi/2}$	1	L_{d1}	L_{d2}	L_y	L_x
$\mathbf{R}_{\pi}$	R_{π}	$R_{3\pi/2}$	1	$R_{\pi/2}$	L_y	L_x	L_{d2}	L_{d1}
$\mathbf{R}_{3\pi/2}$	$R_{3\pi/2}$	1	$R_{\pi/2}$	R_{π}	L_{d2}	L_{d1}	L_x	L_y
$\mathbf{L}_x$	L_x	L_{d2}	L_y	L_{d1}	1	R_{π}	$R_{3\pi/2}$	$R_{\pi/2}$
$\mathbf{L}_y$	L_y	L_{d1}	L_x	L_{d2}	R_{π}	1	$R_{\pi/2}$	$R_{3\pi/2}$
$\mathbf{L}_{d1}$	L_{d1}	L_x	L_{d2}	L_y	$R_{\pi/2}$	$R_{3\pi/2}$	1	R_{π}
$\mathbf{L}_{d2}$	L_{d2}	L_y	L_{d1}	L_x	$R_{3\pi/2}$	$R_{\pi/2}$	R_{π}	1

The rotations, togther with the identity, form a cyclic subgroup of order 4. There are several subgroups of order 2. Each of the following elements, together with the identity, forms a subgroup: R_{π}, and each of the 4 reflections. Each of the rotations forms a class by itself. For example,

$$R_{\pi/2} R_{\pi} R_{\pi/2}^{-1} = R_{3\pi/2} R_{\pi/2}^{-1} = R_{\pi}$$

and

$$L_x R_\pi L_x^{-1} = R_\pi$$

Now

$$R_{\pi/2} L_x R_{\pi/2}^{-1} = L_{d1} R_{3\pi/2} = L_y$$

$$L_y L_x L_y^{-1} = R_\pi L_y = L_x$$

$$L_{d1} L_x L_{d1}^{-1} = R_{\pi/2} L_{d1} = L_y$$

Thus $\{L_x, L_y\}$ and $\{L_{d1}, L_{d2}\}$ are additional classes.

3. Show that the set $\{1, -1, i, -i\}$ forms a group under algebraic multiplication. Write the multiplication table. How many classes are there? Are there any subgroups?

The Table is:

	1	-1	i	$-i$
1	1	-1	i	$-i$
-1	-1	1	$-i$	i
i	i	$-i$	-1	1
$-i$	$-i$	i	1	-1

Thus the set is a group. The elements form pairs: 1 and -1 are each their own inverse, while i and $-i$ are inverses of each other. Each element forms a class by itself. For example:

$$i^{-1}(-i)i = -i(1) = -i$$

Thus there are four classes. The elements $\{1, -1\}$ form a subgroup, since $(-1)(-1) = 1$ and $(1)(-1) = -1$.

This group has the same structure as the rotation subgroup of the symmetry group in problem 2. We can construct an isomorphism that maps one to the other. An example is:

$$f(R_{n\pi/2}) = e^{in\pi/2}$$

This mapping preserves the operation:

$$f(R_{n\pi/2}) * f(R_{m\pi/2}) = e^{in\pi/2} * e^{im\pi/2} = e^{i(m+n)\pi/2}$$

and

$$f(R_{n\pi/2} * R_{m\pi/2}) = f(R_{(n+m)\pi/2}) = e^{i(n+m)\pi/2}$$

The results are the same.

4. Show that there are two groups of order four and determine their multiplication tables.

We can label the elements $\{1, a, b, c\}$. One possible group is a cyclic group in which $b = a^2$, $c = a^3$

and $a^4 = 1.$ This group can be represented by rotations in a plane, with a equal to a rotation by $\pi/2.$ Thus we can also represent $a = i,$ $b = -1,$ $c = -i$ (cf Problem 3).

A second group can be formed from two distinct elements a and b with $ab = c.$ Then either $ac = b$ or $ac = 1.$ In the first case we would have

$$ab = a(ac) = c \Rightarrow a^2 = 1$$

and

$$ba = (ac)a = a(ab)a = a^2(ba) = ba = c$$

so the group is abelian. Then

$$cb = ab^2 = a \text{ or } 1$$

If $cb = a,$ then $b^2 = 1.$ Finally

$$c^2 = (ab)(ab) = abba = a^2 = 1$$

and this gives a consistent multiplication table:

	1	a	b	c
1	1	a	b	c
a	a	1	c	b
b	b	c	1	a
c	c	b	a	1

This group is called the

Vierergruppe.

If we try to form a group with $ac = 1,$ $\left(c = a^{-1}\right)$ we find

$$ab = a^{-1} \Rightarrow a^2 b = 1$$

so $a^2 = b^{-1}.$ But this element must be either b itself or $c.$ If $b^{-1} = b,$ then $(ab)b = a = a^{-1}$ giving $a^2 = 1 -$ –a contradiction. If $b^{-1} = a^2 = c,$ then $bc = 1$ and thus $a = b,$ also a contradiction. The second case is therefore excluded, and there are only two groups of order 4.

5. Show that any group or order n is isomorphic to a subgroup of $S_n.$

Label the group elements 1 through $n.$ Then multiplication by any element g_i causes each element g_j to convert to another element $g_k -$ we can write this as a permutation of the numbers $1 - n.$ Each g_i is thus identified with a permutation, and the group must be isomorphic to a subgroup of $S_n.$

6. Show that unitary matrices of the form (2) form a group under matrix multiplication.

The unit matrix is unitary and so it is in the set. The product of two elements is:

$$\begin{pmatrix} a_1 & b_1 \\ -b_1^* & a_1^* \end{pmatrix}\begin{pmatrix} a_2 & b_2 \\ -b_2^* & a_2^* \end{pmatrix} = \begin{pmatrix} a_1a_2 - b_1b_2^* & a_1b_2 + a_2^*b_1 \\ -a_2b_1^* - a_1^*b_2^* & -b_1^*b_2 + a_1^*a_2^* \end{pmatrix} = \begin{pmatrix} a_3 & b_3 \\ -b_3^* & a_3^* \end{pmatrix}$$

Furthermore:

$$\begin{aligned} a_3a_3^* + b_3b_3^* &= (a_1a_2 - b_1b_2^*)(-b_1^*b_2 + a_1^*a_2^*) + (a_1b_2 + a_2^*b_1)(a_2b_1^* + a_1^*b_2^*) \\ &= a_1a_1^*a_2a_2^* + b_1b_1^*b_2b_2^* + a_1a_1^*b_2b_2^* + b_1b_1^*a_2a_2^* \\ &= (a_1a_1^* + b_1b_1^*)(a_2a_2^* + b_2b_2^*) = 1 \times 1 = 1 \end{aligned}$$

and so the product is of the correct form. The set is closed under multiplication. The inverse of element 1 above is given by

$$\begin{aligned} a_1a_2 - b_1b_2^* &= 1 \\ a_1b_2 + a_2^*b_1 &= 0 \\ -a_2b_1^* - a_1^*b_2^* &= 0 \\ -b_1^*b_2 + a_1^*a_2^* &= 1 \end{aligned}$$

Thus

$$b_2^* = \frac{a_1a_2 - 1}{b_1} = -\frac{a_2b_1^*}{a_1^*}$$

Thus

$$a_2(a_1a_1^* + b_1b_1^*) = a_1^* \Rightarrow a_2 = a_1^*$$

Then

$$b_2 = -b_1$$

So the inverse is:

$$\begin{pmatrix} a_1^* & -b_1 \\ b_1^* & a_1 \end{pmatrix}$$

Check:

$$\begin{pmatrix} a_1 & b_1 \\ -b_1^* & a_1^* \end{pmatrix}\begin{pmatrix} a_1^* & -b_1 \\ b_1^* & a_1 \end{pmatrix} = \begin{pmatrix} a_1a_1^* + b_1b_1^* & 0 \\ 0 & a_1a_1^* + b_1b_1^* \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

as required. Thus this set is indeed a group.

7. Show that in any Abelian group, each element forms its own class.

In an Abelian group, $ab = ba.$ Thus

$$aba^{-1} = aa^{-1}b = 1b = b$$

for any element $b.$ Thus each element is in its own class.

Since the number of irreps equals the number of classes, and $\sum n_i^2 = m,$ the order of the group, each irrep must have dimension 1.

8. Under addition, any two matrices add to form another matrix of the same form, since the matrix elements add. Thus the sum is also in the set. The identity is the matrix with each element equal to zero, and the inverse of $\begin{pmatrix} x & y \\ -y & x \end{pmatrix}$ is $\begin{pmatrix} -x & -y \\ y & -x \end{pmatrix}$. Thus the set forms a group under this operation. To see the isomorphism, we just map $\begin{pmatrix} x & y \\ -y & x \end{pmatrix}$ to $x + iy$. The mapping is 1-1 and onto.

Under multiplication, we have:

$$\begin{pmatrix} x_1 & y_1 \\ -y_1 & x_1 \end{pmatrix}\begin{pmatrix} x_2 & y_2 \\ -y_2 & x_2 \end{pmatrix} = \begin{pmatrix} x_1x_2 - y_1y_2 & x_1y_2 + y_1x_2 \\ -y_1x_2 - x_1y_2 & x_1x_2 - y_1y_2 \end{pmatrix}$$

which is also in the set. The identity is the unit matrix with $x = 1$ and $y = 0$. The inverse is $\begin{pmatrix} x & y \\ -y & x \end{pmatrix}^{-1} = \begin{pmatrix} \frac{x}{x^2+y^2} & -\frac{y}{x^2+y^2} \\ \frac{y}{x^2+y^2} & \frac{x}{x^2+y^2} \end{pmatrix}$, which is also in the set provided that x and y are not both zero. Now let's compare with the multiplication of complex numbers:

$$(x_1 + iy_1)(x_2 + iy_2) = x_1x_2 - y_1y_2 + i(x_1y_2 + y_1x_2)$$

Thus the mapping $\begin{pmatrix} x & y \\ -y & x \end{pmatrix} \leftrightarrow x + iy$ "preserves the operation" - it is a homorphism. The mapping is 1-1 and onto the set of complex numbers with zero excluded. We have to exclude zero because it has no inverse. Thus the two groups are again isomorphic.

9. The *quaternions* are 4-dimensional complex numbers of the form $q = a + bi + cj + dk$ where a, b, c and d are real numbers, and the quantities i, j and k obey the multiplication rules:

$$i^2 = j^2 = k^2 = -1$$

and

$$ij = k; \quad ji = -k$$

(a) Show that the set $\{\pm 1, \pm i, \pm j, \pm k\}$ forms a group under this multiplication.

To show that the set is a group, we construct the group multiplication table.

	1	-1	i	j	k	$-i$	$-j$	$-k$
1	1	-1	i	j	k	$-i$	$-j$	$-k$
-1	-1	1	$-i$	$-j$	$-k$	i	j	k
i	i	$-i$	-1	k	$i(ij) = i^2 j = -j$	1	$-k$	j
j	j	$-j$	$-k$	-1	$j(-ji) = i$	$-k$	1	$-i$
k	k	$-k$	$(-ji)i = j$	$(ij)j = -i$	-1	$-j$	i	1
$-i$	$-i$	i	1	$-k$	j	-1	k	$-j$
$-j$	$-j$	j	k	1	$-i$	$-k$	-1	i
$-k$	$-k$	k	$-j$	i	1	j	$-i$	-1

Each element has an inverse: The identity 1 and also -1 each form their own inverse. The inverse of i is $-i,$ and so on for each of the elements.

(b) Show that i, j and k may be represented by the matrices:

$$i = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

$$j = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 01 & 0 & 0 & 0 \end{pmatrix}$$

and

$$k = \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}$$

Determine the classes of this group.

Let's check the matrix multiplication:

$$ij = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 01 & 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} = k$$

and

$$i^2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

as required.

Now we find the classes. As usual the identity is its own class, as is $-1.$ The elements conjugate to i are i itself and:

$$j^{-1}ij = -jk = -i$$

$$k^{-1}ik = -k(-j) = -i$$

Similarly the elements conjugate to j are $\pm j,$ and of $k,$ $\pm k.$ Thus there are 5 classes in the group.

(c) Determine the number and dimension of irreps of this group, and find the character table.

(d) Is the representation in (b) reducible? If so, how?

There are also 5 irreps, one being the trivial rep.

$$\sum_{i=2}^{5} n_i^2 = 7$$

Thus $n_i = 1,$ $i = 2-4,$ and $n_5 = 2.$ For the 1-D reps, the characters must obey the multiplication table. Thus

$$\chi_{-1}^2 = \chi_1 = 1 \Rightarrow \chi_{-1} = \pm 1$$

$$(\chi_i)^2 = \chi_{-1}$$

and

$$\chi_i\chi_j = \chi_k;\ \chi_j\chi_k = \chi_i;\ \chi_i\chi_k = \chi_j$$

From these relations we deduce the following character table:

Character table

class	1	-1	$\pm i$	$\pm j$	$\pm k$
rep 1	1	1	1	1	1
rep 2	1	1	-1	-1	1
rep 3	1	1	1	-1	-1
rep 4	1	1	-1	1	-1
rep 5	2	α	β	γ	δ

Now we use orthogonality to find the other relations:

$$2 + \alpha + 2\beta + 2\gamma + 2\delta = 0$$
$$2 + \alpha - 2\beta - 2\gamma + 2\delta = 0$$
$$2 + \alpha + 2\beta - 2\gamma - 2\delta = 0$$
$$2 + \alpha - 2\beta + 2\gamma - 2\delta = 0$$

Adding all four equations, we find $\alpha = -2,$ and thus all the other values are zeroes.

(d) Is the representation in (b) reducible? If so, how?

The character of the 4-rep is $\{4, -4, 0, 0, 0\}$ and so this rep decomposes into $M_2 \oplus M_2.$

10. Show that the integers 1 through 4 form a group under the operation of multiplication mod 5. Write the multiplication table. What is the identity element? How many classes are there? Is the group abelian?

The identity element is 1 and the multiplication table is:

	1	2	3	4
1	1	2	3	4
2	2	4	1	3
3	3	1	4	2
4	4	3	2	1

As usual, the identity element forms its own class. Let's find the others:

$$3^{-1} * 2 * 3 = 2 * 1 = 2$$

$$4^{-1} * 2 * 4 = 4 * 3 = 2$$

So this class contains the single element 2.

$$2^{-1} * 3 * 2 = 3 * 1 = 3$$

and

$$4^{-1} * 3 * 4 = 4 * 2 = 3$$

A similar thing happens with the last element. Thus each element forms its own class. The group is abelian.

This group is not isomorphic to the symmetry group of a square since the group structure is different.

11. Consider the mapping f that maps the group of rationals to the group of integers by

$$f\left(\frac{m}{n}\right) = m + n$$

Is the mapping a homomorphism? Why or why not?

It is not, because

$$f\left(\frac{m}{n}\right)f\left(\frac{1}{1}\right) = (m + n) * (2) = m + n + 2$$

whereas

$$f\left(\frac{m}{n} * \frac{1}{1}\right) = f\left(\frac{m}{n}\right) = m + n$$

The two results are not equal, and so the mapping is not a homomorphism.

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Optional Topic B: Groups

12. Show that all elements in the same class have the same period.

If $a^n = 1$, then

$$\left(gag^{-1}\right)^n = \left(gag^{-1}\right)^{n-2}\left(gag^{-1}\right)\left(gag^{-1}\right) = \left(gag^{-1}\right)^{n-2}\left(ga^2g^{-1}\right)$$

Continuing in this way, we find

$$\left(gag^{-1}\right)^n = \left(ga^ng^{-1}\right) = gg^{-1} = 1$$

Thus the period of gag^{-1} equals the period of a.

13. Show that if a set of elements $\{e_i\}$ forms a class of a group G, then the set $\{e_i^{-1}\}$ of the inverses of $\{e_i\}$ is also a class.

The elements e_i and e_j where

$$ae_ia^{-1} = e_j$$

are in the class $\{e_i\}$ Then

$$e_j^{-1}ae_ia^{-1} = 1$$
$$e_j^{-1}ae_ia^{-1}a = e_j^{-1}ae_i = a$$

So

$$e_j^{-1}ae_ie_i^{-1} = e_j^{-1}a = ae_i^{-1}$$

and thus

$$a^{-1}e_j^{-1}a = e_i^{-1}$$

Thus the elements $\{e_i^{-1}\}$ also form a class.

14. The *center* Z of a group G is the set of elements that commute with every element in the group. Show that the center is an Abelian subgroup of G.

First we show that the center Z is a subgroup. The identity is in Z because the identity commutes with every element. Then if $a, b \in Z$, then for any element g : $ag = ga$ and $bg = gb$. Thus:

$$(ab)g = a(bg) = a(gb) = (ag)b = (ga)b = g(ab)$$

so ab is also in Z. The inverse of a is a^{-1}. Any element $g = ah$ for some h in G and thus

$$a^{-1}g = a^{-1}(ah) = h = h\left(aa^{-1}\right) = (ha)a^{-1} = (ah)a^{-1} = ga^{-1}$$

So a^{-1} also commutes with every g and so is alos in the set. Thus Z is a subgroup.

The subgroup is abelian since every member a commutes with every element in $G,$ and thus with every element in $Z.$

The center always contains at least one element: the identity.

15. A homomorphism f maps group A to group $B.$ The *Kernel* of the homomorphism is the set of all elements of A map to the identity element of $B.$ Show that the Kernel is an invariant subgroup of $A.$

The identity is in $K:$

$$f(g) * f(1) = f(g * 1) = f(g)$$

Thus $f(1)$ is the identity of $B.$ If a_1 and a_2 are in $K,$ then

$$f(a_1 * a_2) = f(a_1) * f(a_2) = 1 * 1 = 1$$

Thus $a_1 a_2$ is also in $K.$ Also

$$f\left(a^{-1}\right) * f(a) = f\left(a^{-1} * a\right) = f(1) = 1$$

But also

$$f\left(a^{-1}\right) f(a) = f\left(a^{-1}\right) * 1 = 1$$

so

$$f\left(a^{-1}\right) = 1$$

and a^{-1} is in $K.$ Thus K is a subgroup.

Now if the subgroup is invariant, then we must also have

$$g^{-1} k g \in K$$

$$f\left(g^{-1} k g\right) = f\left(g^{-1}\right) f(k) f(g) = f\left(g^{-1}\right) f(g) = f\left(g^{-1} g\right) = f(1) = 1$$

and so $g^{-1} k g$ is in K and the subgroup is invariant.

16. A one-dimensional translation operator T_n translates a function along the $x-$ axis by an amount $nd,$ where d is a fixed step length: $T_n f(x) = f(x + nd).$

(a) Show that the set of operators T_n forms a group that may be represented by the complex numbers $T_n \rightleftarrows e^{-iknd}.$ What are the corresponding basis functions?

(b) Work out the orthogonality relation (14) for this representation, and comment. You will have to make some changes to account for the infinite order of the group.

First of all let's check the group properties. There is an identity: $T_0.$ T_{-n} is the inverse of $T_n,$ and

$$T_n T_m f(x) = T_n f(x + md) = f(x + md + nd) = T_{n+m} f(x)$$

Thus the product of any two elements is also an element of the group. This demonstration also shows that the group is abelian. The other group properties, such as associativity, follow trivially.

For an Abelian group, every irrep is 1-dimensional. The given set of numbers is an irrep, because they obey the group multiplication law:

$$e^{-iknd}e^{-ikmd} = e^{-ik(m+n)d}$$

The basis functions are $f_n(x) = e^{-ikx}$. Then

$$T_n f_n(x) = e^{-iknd}e^{-ikx} = e^{-ik(x+nd)}$$

as required. Different irreps have different values of k.

The orthogonality relation is:

$$\sum_n e^{-iknd}e^{ik'nd} \propto \delta_{kk'}$$

Because the order of the group is infinite and k is a continuous variable, we have to change from the Kronecker delta to the delta function:

$$\sum_n e^{-iknd}e^{ik'nd} \propto \delta\left(k - k'\right)$$

which is the expected completeness relation for Fourier series (see, *e.g.*, equation 6.16).

(c) Now let the operator translate by an arbitrary amount x': $T\left(x'\right)f(x) = f\left(x + x'\right)$. What are the generators of this group?

We use the representation found above, with $nd \to x'$. Then $T\left(x'\right) \sim e^{-ikx'}$ and near the identity

$$T\left(dx'\right) = 1 - ikdx'$$

Thus the generators are the values k and $T\left(x'\right)$ is already expressed in terms of its generators in this representation.

17. The operations that preserve the symmetry of this molecule are (i) rotation about its symmetry axis by π, and (ii) Reflection about the symmetry plane. Each of these elements is is its own inverse, $a^2 = b^2 = 1$, and $ab = ba$. Thus the group has order 4.

	1	a	b	ab
1	1	a	b	ab
a	a	1	ab	b
b	b	ab	1	a
ab	ab	b	a	1

Each element forms its own class, thus there are four irreps, each of dimension 1, and the character table is:

	1	a	b	ab
T	1	1	1	1
rep 2	1	1	−1	−1
rep 3	1	−1	1	−1
rep 4	1	−1	−1	1

Now we look for a 3-dim rep to describe the transformation of a vector. The rotation about the symmetry (z) axis is decribed by

$$R^V_{\text{rot}} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

which has trace $-1,$ and the reflection in the $y-z$ plane is described by:

$$R^V_{\text{ref}} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

with trace $+1.$ The product ab is reflection in the $x-z$ plane:

$$R^V_{\text{ab}} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Thus the character is $\{3,-1,+1,+1\}$ and thus the rep decomposes as $T \oplus R_3 \oplus R_4.$ The trivial rep is included and thus we can find an invariant vector. This molecule can support a permanent electric dipole moment.

A pseudo-vector transforms differently under reflection, and the character becomes $\{3,-1,-1,-1\}$ and so the rep decomposes as $R_2 \oplus R_3 \oplus R_4.$ The trivial rep is not included, so this molecule cannot support a permanent magnetic moment.

18. The molecule SbS $_5$ is square-pyramidal. Four S atoms form a square base with the Sb atom at the center. The fifth S atom sits at the top of the pyramid. Determine the symmetry group for this system. What is the order of the group? Work out the multiplication table. How many classes are there? Determine the character table. May this molecule possess a permanent electric dipole moment?

The symmetry group includes

(i) rotations by $\pi/2,$ $\pi,$ 3 $\pi/2$ and 2 $\pi = 0$ about the vertical symmetry axis. We call these operations $a, a^2,$ a^3 and $1.$ These four elements form a cyclic subgroup of $G.$

(ii) Reflections in the two vertical planes perpendicular to the sides of the square. These elements are b, c with $b^2 = c^2 = 1.$

(iii) Reflections in the two vertical planes along diagonals of the square: d, e with $d^2 = e^2 = 1.$

There are eight elements in all and the multiplication table is:

	1	a	a^2	a^3	b	c	d	e
1	1	a	a^2	a^3	b	c	d	e
a	a	a^2	a^3	1	e	d	b	c
a^2	a^2	a^3	1	a	c	b	e	d
a^3	a^3	1	a	a^2	d	e	c	b
b	b	d	c	e	1	a^2	a	a^3
c	c	e	b	d	a^2	1	a^3	a
d	d	c	e	b	a^3	a	1	a^2
e	e	b	d	c	a	a^3	a^2	1

The classes are:

(i) The identity

(ii) $\{a, a^3\}$

(iii) $\{a^2\}$

(iv) $\{b, c\}$

(v) $\{d, e\}$

Since there are 5 classes there are 5 irreps and in order to satisfy the relation $\sum n_i^2 = g$ four of them must be 1- dimensional and one has dimension 2. For the 1-dimensional reps, the characters satisfy the multiplication table. Thus $\chi_{\text{iii}}^2 = \chi_{\text{iv}}^2 = \chi_{\text{v}}^2 = 1 \Rightarrow \chi_{\text{iii}},$ $\chi_{\text{iv}},$ $\chi_{\text{v}} = \pm 1.$ Thus each of the characters is $\pm 1.$ Also $\chi_{\text{ii}}\chi_{\text{iii}} = \chi_{\text{ii}} \Rightarrow \chi_{\text{iii}} = +1.$ The table so far looks like:

class	i	ii	iii	iv	v
rep 1-1 (T)	1	1	1	1	1
rep 1-2	1	−1	+1	−1	+1
rep 1-3	1	−1	+1	+1	−1
rep 1-4	1	+1	+1	−1	−1
rep 2	2	?	?	?	?

Now we use orthogonality. If the character of rep 2 is orthogonal to all the rep-1s, we must have $\chi_{\text{iii}} = -2$ and all the other unknowns are zero.

Now we use a 3-rep that transforms a 3-D vector in space. The matrices representing the a subgroup are rotation matrices. We have

$$M_a = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} M_{a^2} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix};\ M_b = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix};\ M_e = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Thus the character of this rep is $\{3, 1, -1, 1, 1\}$ and so the decomposition is: $R_2 \oplus T$

Since the trivial rep is included, we conclude that a fixed vector remains invariant under the group symmetry, and so an electric dipole moment is possible.

19. The Lorentz group has generators that are 4 $\times 4$ matrices with mostly zero elements. The matrices K_i are given by:

$$K_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and so on. (The non-zero elements of K_i are the i th elements in the top row and the first column, where the first element is labelled with 0 not 1.) Similarly, the generators S_i are given by:

$$S_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}$$

(The non-zero elements of S_i are $\varepsilon_{ijk} a_{jk}$ where a_{jk} is the sub-matrix formed by removing the top row and first column.).

Find the group element generated by K_1 and also by S_1.

$$e^{aK_1} = 1 + aK_1 + \frac{1}{2}a^2K_1^2 + \cdots$$

Now

$$K_1^2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and thus

$$K_1^3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = K_1$$

Thus

$$\begin{aligned} e^{K_1} &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + K_1\left(a + \frac{a^3}{3!} + \cdots\right) + K_1^2\left(1 + \frac{a^2}{2} + \cdots\right) \\ &= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} + K_1 \sinh a + K_1^2 \cosh a \\ &= \begin{pmatrix} \cosh a & \sinh a & 0 & 0 \\ \sinh a & \cosh a & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

This represents a velocity transformation to a frame moving with speed $\beta = \tanh^{-1} a.$ Compute the product of the two group elements e^{aK_1} and $e^{bK_2}.$ Hence show that the elements generated by K_i do not form a subgroup. Do the elements formed by the S_i form a subgroup? Find the class of elements conjugate to $e^{aK_1}.$ For $S_1,$ we have:

$$S_1^2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

and

$$S_1^3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} = -S_1$$

Thus

$$\begin{aligned} e^{\theta S_1} &= 1 + \theta S_1 + \frac{1}{2}\theta^2 S_1^2 + \frac{1}{3!}\theta^3 S_1^3 + \cdots \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + S_1\left(\theta - \frac{\theta^3}{3!} + \cdots\right) + S_1^2\left(-1 + \frac{\theta^2}{2} + \cdots\right) \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + S_1 \sin\theta - S_1^2 \cos\theta \\ &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta \\ 0 & 0 & -\sin\theta & \cos\theta \end{pmatrix} \end{aligned}$$

which represents a rotation by angle θ about the $x-$ axis.

$$\begin{aligned} e^{aK_1} e^{bK_2} &= \begin{pmatrix} \cosh a & \sinh a & 0 & 0 \\ \sinh a & \cosh a & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} \cosh b & 0 & \sinh b & 0 \\ 0 & 1 & 0 & 0 \\ \sinh b & 0 & \cosh b & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} \cosh a \cosh b & \sinh a & \cosh a \sinh b & 0 \\ \sinh a \cosh b & \cosh a & \sinh a \sinh b & 0 \\ \sinh b & 0 & \cosh b & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \end{aligned}$$

This element cannot be generated by the Ks alone, so the Ks do not generate a subgroup. The Ss do however. We can see this by noting the the 3 $\times 3$ matrices form the rotation group, and the extra 1 in the 00 element does not affect this conclusion.

The other subgroups are boosts along a single axis, or rotations about single axis. Each is subgroup is generated by a single K_i or a single S_i. These subgroups are Abelian.

20. Show that the transformations $x' = ax + b$ (where x', x, a and b are real numbers and $a \neq 0$) form a group. Form a two-dimensional representation of this group that acts on the vectors $(x, 1)$.

First let's show it's a group.

The product of two transformations gives a new transformation:

$$x'' = a_2(a_1 x + b_1) + b_2 = a_2 a_1 x + a_2 b_1 + b_2$$

which is of the required form.

$$T(a_2, b_2)T(a_1, b_1) = T(a_2 a_1, a_2 b_1 + b_2)$$

The inverse of the transformation $x' = ax + b$ is the transformation $x' = a^{-1}x + c$ where

$$x'' = a^{-1}(ax + b) + c = x$$
$$x + a^{-1}b + c = x$$
$$c = -a^{-1}b$$

and this element is also of the required form. The identity is the transformation $T(1, 0)$ with $a = 1$ and $b = 0$. The group is not Abelian:

$$T(a_1, b_1)T(a_2, b_2) = T(a_2 a_1, a_1 b_2 + b_1) \neq T(a_2, b_2)T(a_1, b_1)$$

We can construct a 2-rep

$$M(a, b) = \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix}$$

the corresponding vectors have components $(x, 1)$.

$$M(a_2, b_2)M(a_1, b_1) = \begin{pmatrix} a_2 & b_2 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a_1 & b_1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a_2 a_1 & a_2 b_1 + b_2 \\ 0 & 1 \end{pmatrix}$$

as required.

21. A homorphism f maps a group G to a group H. Show that the image $f(G)$ in H is isomorphic to the factor group G/K where K is the kernel of the homorphism (Problem 15).

From problem 15, K is an invariant subgroup. We contruct a mapping F from H to the factor group by $F[f(g)] = gK$, the coset of K. This is an isomorphism because: (a) the mapping preserves the operation.

$$F[f(g)]F[f(h)] = (gK)(hK) = g(Kh)K = (gh)K = F[f(gh)] = F[f(g)f(h)]$$

because K is an invariant subgroup and f is an isomorphism.

(b) The mapping is 1-1. Suppose $F[f(g)] = F[f(h)].$ Then

$$F[f(h)]^{-1}F[f(g)] = 1 = F\left[f\left(h^{-1}\right)\right]F[f(g)] = F\left[f\left(h^{-1}g\right)\right].$$

Thus $h^{-1}g$ must be in $K.$

$$h^{-1}gK = K$$
$$gK = hK$$

Thus the two cosets are the same. Conversely, if $gK = hK,$ then $h^{-1}g$ is in K, $F\left[f\left(h^{-1}g\right)\right] = K,$ the identity of the quotient group, and hence $F[f(h)] = F[f(g)].$

Thus the mapping is an isomorphism, as required.

22. A group G has an invariant subgroup $S.$ If element a of group G has period N where N is prime, and a is not a member of the subgroup $S,$ then element aS of the factor group G/S also has period $N.$

If a has period N then $a^N = 1,$ and $a^NS = 1S = S.$ Then if $N \geq 2,$ since S is invariant,

$$(aS)(aS) = a^2S$$

and continuing in this way we find

$$(aS)^N = a^NS = 1S = S$$

and so aS has period $N.$

We should check that $a^nS \neq S$ for any $n < N.$ But suppose it were, then

$$a^n \in S \text{ for } n < N$$

Then since S is a subgroup, $(a^n)^p$ is in S for every integer $p,$ and $np = mN$ for some $m.$ Thus for every integer p there is an integer m such that $n = mN/p.$

If N is prime, $m \geq p$ and then $n \geq N.$

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Optional Topic C: Green's functions

1. Use the division of region method to find the Green's function for a damped harmonic oscillator. Hence find the response of the oscillator to the input $f(t) = 1 - \frac{t}{T}$ for $0 < t < T$.

The differential equation satisfied by G is

$$\frac{d^2G}{dt^2} + 2\alpha\frac{dG}{dt} + \omega_0^2 G = \delta(t - t')$$

For $t \neq t'$, we have $G = Ae^{\gamma t}$ where

$$\gamma^2 + 2\alpha\gamma + \omega_0^2 = 0$$
$$\gamma = -\alpha \pm i\sqrt{\omega_0^2 - \alpha^2} = -\alpha \pm i\omega$$

We need a solution that is zero as $t \to \pm\infty$, so for $t < t'$, $G = 0$ while for $t > t'$

$$G_{II} = Ae^{-\alpha t}\left(e^{i\omega t} + Be^{-i\omega t}\right)$$

Now we impose the condition $G_I = G_{II}$ at $t = t'$

$$0 = Ae^{-\alpha t'}\left(e^{i\omega t'} + Be^{-i\omega t'}\right) \Rightarrow B = -e^{2i\omega t'}$$

Then

$$G_{II} = Ae^{-\alpha t}\left(e^{i\omega t} - e^{i\omega(2t'-t)}\right)$$

Now integrate the differential equation across the boundary at $t = t'$, and use the result that $G_{II} = G_I$ at $t = t'$.

$$G'_{II} - G'_I = 1$$
$$Ae^{-\alpha t'}e^{i\omega t'}[(-\alpha + i\omega) - (-\alpha - i\omega)] = 1$$
$$Ae^{-\alpha t'}e^{i\omega t'}2i\omega = 1$$

and so

$$A = \frac{\exp(\alpha - i\omega)t'}{2i\omega}$$

Thus

$$\begin{aligned} G_{II}(t,t') &= \frac{\exp(\alpha - i\omega)t'}{2i\omega}e^{-\alpha t}\left(e^{i\omega t} - e^{i\omega(2t'-t)}\right) \\ &= \frac{1}{2i\omega}e^{\alpha(t'-t)}\left(e^{i\omega(t-t')} - e^{i\omega(t'-t)}\right) \\ &= \frac{1}{\omega}e^{\alpha(t'-t)}\sin\omega(t - t') \text{ for } t > t' \end{aligned}$$

Note the required symmetry: $G(t',t) = G(-t,-t')$.

Then with the given input, $x(t) = 0$ for $t < 0$. For $t > 0$:

$$x(t) = \int_0^T \left(1 - \frac{t'}{T}\right) G(t,t')dt'$$

So

$$\begin{aligned} x(t) &= \int_0^t \left(1 - \frac{t'}{T}\right)\frac{1}{\omega}e^{\alpha(t'-t)}\sin\omega(t - t')dt' \\ &= \frac{e^{-\alpha t}}{2i\omega}\int_0^t \left(1 - \frac{t'}{T}\right)\exp(\alpha t')\left(e^{i\omega(t-t')} - e^{-i\omega(t-t')}\right)dt' \\ &= \frac{e^{-\alpha t}}{2i\omega}\left[e^{i\omega t}\int_0^t \left(1 - \frac{t'}{T}\right)\exp(\alpha t' - i\omega t')dt' \right. \\ &\quad \left. -e^{-i\omega t}\int_0^t \left(1 - \frac{t'}{T}\right)\exp(\alpha t' + i\omega t')dt' \right] \end{aligned}$$

for $t < T$, and, for $t > T$

$$
\begin{aligned}
x(t) = \frac{e^{-\alpha t}}{2i\omega}\Bigg[& e^{i\omega t}\int_0^T \left(1 - \frac{t'}{T}\right) \exp\left(\alpha t' - i\omega t'\right) dt' \\
& - e^{-i\omega t}\int_0^T \left(1 - \frac{t'}{T}\right) \exp\left(\alpha t' + i\omega t'\right) dt' \Bigg]
\end{aligned}
$$

The integral contains terms of the form

$$
\begin{aligned}
\int_0^t \left(1 - \frac{t'}{T}\right) e^{\alpha t'} e^{\pm i\omega t'} dt' &= \frac{e^{a_\pm t'}}{a_\pm}\Bigg|_0^t - \frac{1}{T}\left(t' \frac{e^{a_\pm t'}}{a_\pm} - \frac{e^{a_\pm t'}}{a_\pm^2}\right)\Bigg|_0^t \\
&= \frac{e^{a_\pm t} - 1 - t e^{a_\pm t}/T}{a_\pm} + \frac{e^{a_\pm t} - 1}{a_\pm^2 T} \\
&= \frac{e^{a_\pm t}(T - t) - T}{a_\pm T} + \frac{e^{a_\pm t} - 1}{a_\pm^2 T}
\end{aligned}
$$

where

$$
a_\pm = \alpha \pm i\omega
$$

For $t > T,$ we get

$$
\int_0^T \left(1 - \frac{t'}{T}\right) e^{\alpha t'} e^{\pm i\omega t'} dt' = \frac{e^{a_\pm T} - 1}{a_\pm^2 T} - \frac{1}{a_\pm}
$$

Thus, for $t < T,$

$$
\begin{aligned}
x(t) = \frac{e^{-\alpha t}}{2i\omega}\Bigg[& e^{i\omega t}\left(\frac{e^{a_- t}(T - t) - T}{a_- T} + \frac{e^{a_- t} - 1}{a_-^2 T}\right) \\
& - e^{-i\omega t}\left(\frac{e^{a_+ t}(T - t) - T}{a_+ T} + \frac{e^{a_+ t} - 1}{a_+^2 T}\right)\Bigg] \\
= \frac{1}{2i\omega}\Bigg[& \frac{(T - t) - Te^{(-\alpha + i\omega)t}}{(\alpha - i\omega)T} + \frac{1 - e^{(-\alpha + i\omega)t}}{(\alpha - i\omega)^2 T} \\
& - \frac{(T - t) - Te^{-(\alpha + i\omega)t}}{(\alpha + i\omega)T} - \frac{1 - e^{-(\alpha + i\omega)t}}{(\alpha + i\omega)^2 T}\Bigg] \\
= \frac{1}{2i\omega}\Bigg[& 2i\left(1 - \frac{t}{T}\right)\frac{\omega}{\alpha^2 + \omega^2} + e^{-\alpha t}\left(\frac{-2i\omega\cos\omega t - 2i\alpha\sin\omega t}{\alpha^2 + \omega^2}\right) \\
& \frac{4i\alpha\omega}{\left(\alpha^2 + \omega^2\right)^2 T} - e^{-\alpha t}\frac{2i\left(\alpha^2 - \omega^2\right)\sin\omega t + 4i\alpha\omega\cos\omega t}{\left(\alpha^2 + \omega^2\right)^2 T}\Bigg] \\
= & \left(1 - \frac{t}{T}\right)\frac{1}{\alpha^2 + \omega^2} - e^{-\alpha t}\left(\frac{\alpha\sin\omega t + \omega\cos\omega t}{\omega\left(\alpha^2 + \omega^2\right)}\right) \\
& + \frac{2\alpha(1 - e^{-\alpha t}\cos\omega t)}{\left(\alpha^2 + \omega^2\right)^2 T} - e^{-\alpha t}\frac{\left(\alpha^2 - \omega^2\right)\sin\omega t}{\omega\left(\alpha^2 + \omega^2\right)^2 T}
\end{aligned}
$$

which satisfies the initial condition $x(0) = 0.$

For $t > T,$ we have:

$$
\begin{aligned}
x(t) &= \frac{e^{-\alpha t}}{2i\omega}\left[e^{i\omega t}\left(\frac{e^{a_- T}-1}{a_-^2 T}-\frac{1}{a_-}\right)-e^{-i\omega t}\left(\frac{e^{a_+ T}-1}{a_+^2 T}-\frac{1}{a_+}\right)\right]\\
&= \frac{e^{-\alpha t}}{2i\omega}\left[e^{i\omega t}\left(\frac{e^{(\alpha-i\omega)T}-1}{(\alpha-i\omega)^2 T}-\frac{1}{(\alpha-i\omega)}\right)-e^{-i\omega t}\left(\frac{e^{(\alpha+i\omega)T}-1}{(\alpha+i\omega)^2 T}-\frac{1}{(\alpha+i\omega)}\right)\right]\\
&= \frac{1}{2i\omega}\left[e^{i\omega t}\left(\frac{e^{-i\omega T}-e^{-\alpha t}}{(\alpha-i\omega)^2 T}-\frac{e^{-\alpha t}}{(\alpha-i\omega)}\right)-e^{-i\omega t}\left(\frac{e^{i\omega T}-e^{-\alpha t}}{(\alpha+i\omega)^2 T}-\frac{e^{-\alpha t}}{(\alpha+i\omega)}\right)\right]\\
&= e^{-\alpha t}\frac{\left(\alpha^2-\omega^2\right)\left[e^{\alpha T}\sin\omega(t-T)-\sin\omega t\right]+2\alpha\omega\left[e^{\alpha T}\cos\omega(t-T)-\cos\omega t\right]}{\omega\left(\alpha^2+\omega^2\right)^2}\\
&\quad - e^{-\alpha t}\left(\frac{\alpha\sin\omega t+\omega\cos\omega t}{\omega\left(\alpha^2+\omega^2\right)}\right) \qquad \text{Pr 1 Equation 1}
\end{aligned}
$$

As $t \to \infty$ the result is zero, as expected. At $t = T$ we have

$$
x(T) = e^{-\alpha T}\left[\frac{\left(\alpha^2-\omega^2\right)\left[-\sin\omega T\right]+2\alpha\omega\left[e^{\alpha T}-\cos\omega t\right]}{\omega\left(\alpha^2+\omega^2\right)^2}-\left(\frac{\alpha\sin\omega T+\omega\cos\omega T}{\omega\left(\alpha^2+\omega^2\right)}\right)\right]
$$

while from equation (Pr 1 equation 1) we have

$$
\begin{aligned}
x(T) &= -e^{-\alpha T}\left(\frac{\alpha\sin\omega t T+\omega\cos\omega T}{\omega\left(\alpha^2+\omega^2\right)}\right)\\
&\quad + \frac{2\alpha\left(1-e^{-\alpha T}\cos\omega T\right)}{\left(\alpha^2+\omega^2\right)^2 T} - e^{-\alpha T}\frac{\left(\alpha^2-\omega^2\right)\sin\omega T}{\omega\left(\alpha^2+\omega^2\right)^2 T}
\end{aligned}
$$

The results are the same, as expected.

2. Find the Green's function for a beam supported at one end (cf Chapter 5 Problem 11) using a Laplace transform method.

The equation for G is

$$
\frac{d^4 G\left(x,x'\right)}{dx^4} = \frac{1}{EI}\delta\left(x-x'\right)
$$

with boundary conditions $G\left(0,x'\right) = 0$ and $G'\left(0,x'\right) = 0.$ The transformed equation is:

$$
\begin{aligned}
s^4\tilde{G}\left(s,x'\right) - s^3 G(0) - s^2 G'(0) - sG''(0) - G'''(0) &= \frac{1}{EI}e^{-sx'}\\
s^4\tilde{G}\left(s,x'\right) - sG''(0) - G'''(0) &= \frac{e^{-sx'}}{EI}
\end{aligned}
$$

Thus

$$
\tilde{G}\left(s,x'\right) = \frac{e^{-sx'}/EI + G''(0)}{s^4} + \frac{G'''(0)}{s^3}
$$

where, for the moment, $G''(0)$ and $G'''(0)$ are unknown constants. We can find them later using the known boundary conditions at $x = L$: $G''(L) = G'''(L) = 0.$

Inverting the transform using Table 5.1, and using the shifting property in the first term, we get:

$$
G\left(x,x'\right) = \frac{1}{4!}\left(\left(x-x'\right)^3 S\left(x-x'\right) + x^3 G''(0)\right) + \frac{1}{3!}G'''(0)x^2
$$

Thus for $x > x'$:

$$G''(L,x') = \frac{G''(0)}{4}x + \frac{(x-x')}{4} + \frac{G'''(0)}{3}\Bigg|_{x=L} = 0$$

$$= \frac{G''(0)}{4}L + \frac{(L-x')}{4} + \frac{G'''(0)}{3} = 0 \quad \text{Pr 2 equation 1}$$

and

$$G'''(L,x') = \frac{G''(0)}{4} + \frac{1}{4} = 0 \Rightarrow G''(0) = -1$$

and then, from Pr 2 equation 1,

$$G'''(0) = \frac{3}{4}(L - (L - x')) = \frac{3}{4}x'$$

Thus

$$G(x,x') = \frac{1}{4!}\left((x-x')^3 S(x-x') - x^3\right) + \frac{1}{8}x'x^2$$

So we get

$$G(x,x') = \frac{1}{8}\begin{cases} x'x^2 - \frac{x^3}{3} & \text{for } x < x' \\ x(x')^2 - \frac{1}{3}(x')^3 & \text{for } x > x' \end{cases}$$

The result has the form expected for the"division of region method", and also has the expected symmetry ($G(x,x') = G(x',x)$)

3. Find the Green's function for a wave on a string of length L:

$$\frac{d^2G}{dx^2} - \frac{1}{v^2}\frac{d^2G}{dt^2} = \delta(x-x')\delta(t-t')$$

Using the following method:

Fourier transform the equation in time:

$$\frac{d^2G}{dx^2} + \frac{\omega^2}{v^2}G(x,x'\omega,t') = \frac{1}{\sqrt{2\pi}}\delta(x-x')e^{i\omega t'}$$

then use the division of region method to solve the resulting equation in x.

First let $G(x,x',\omega,t') = \frac{g(x,x')}{\sqrt{2\pi}}e^{i\omega t'}$. Then g satisfies the equation:

$$\frac{d^2g}{dx^2} + k^2 g = \delta(x-x')$$

For $x < x'$, the solution is $g = A\sin kx$ while for $x > x'$ the solution is $g = B\sin k(x-L)$. At $x = x'$

$$A\sin kx' = B\sin k(x'-L) \Rightarrow A = B\frac{\sin k(x'-L)}{\sin kx'}$$

Then integrating across the boundary at $x = x'$, we get

$$g'\Big|_{x'-\varepsilon}^{x'+\varepsilon} = 1 = kB\left(\cos k(x'-L) - \frac{\sin k(x'-L)}{\sin kx'}\cos kx'\right)$$

$$= \frac{kB}{\sin kx'}\sin kL$$

Thus

$$B = \frac{\sin kx'}{k\sin kL}$$

Thus

$$G(x,x',\omega,t') = \frac{\sin k(x'-L)\sin kx}{k\sin kL}e^{i\omega t'} \text{ for } x < x'$$
$$= \frac{\sin kx'\sin k(x-L)}{k\sin kL}e^{i\omega t'} \text{ for } x > x'$$

and thus

$$G(x,x',t,t') = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{\sin kx_< \sin k(x_> - L)}{k\sin kL}e^{i\omega t'}e^{-i\omega t}d\omega$$

where $k = \omega/v$.

4. Find the Green's function for the diffusion equation:

$$\frac{\partial G}{\partial t} - D\frac{\partial^2 G}{\partial x^2} = \delta(x-x')\delta(t-t')$$

by taking the Fourier transform in space and using the divsion of space method in time. The boundary conditions are $G = 0$ at $t = 0$ and at $x = \pm\infty$.

Taking the Fourier transform, we get

$$\frac{\partial G(k,x',t,t')}{\partial t} + k^2 DG = \frac{1}{\sqrt{2\pi}}e^{-ikx'}\delta(t-t')$$

For $t \neq t'$, the equation is:

$$\frac{\partial G(k,x',t,t')}{\partial t} + k^2 DG = 0$$

with solution

$$G = Ae^{-k^2 Dt}$$

To satisfy the boundary condition at $t = 0$, we choose $A = 0$ in region I (
$t < t'$). Then integrating the differential equation across the boundary at $t = t'$, we get:

$$G\Big|_{t'-\varepsilon}^{t'+\varepsilon} = \frac{1}{\sqrt{2\pi}}e^{-ikx'}$$
$$Ae^{-k^2 Dt'} - 0 = \frac{1}{\sqrt{2\pi}}e^{-ikx'}$$
$$A = \frac{1}{\sqrt{2\pi}}e^{-ikx'+k^2 Dt'}$$

Thus $G \equiv 0$ for $t < t'$, and for $t > t'$:

$$G(x,x',t,t') = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\exp\left(ik(x-x') - k^2 D(t-t')\right)dk$$

Complete the square:

$$-k^2 D(t-t') + ik(x-x') = -\left(k\sqrt{D(t-t')} - i\frac{(x-x')}{2\sqrt{D(t-t')}}\right)^2 + \left(i\frac{(x-x')}{2\sqrt{D(t-t')}}\right)^2$$
$$= -\left(k\sqrt{D(t-t')} + i\frac{(x-x')}{2\sqrt{D(t-t')}}\right)^2 - \frac{1}{4}\frac{(x-x')^2}{D(t-t')}$$

Thus

$$G(x,x',t>t') = \frac{1}{2\pi}\exp\left(-\frac{1}{4}\frac{(x-x')^2}{D(t-t')}\right)\int_{-\infty+ih}^{+\infty+ih}\exp\left(-u^2\right)\frac{du}{\sqrt{D(t-t')}}$$
$$= \frac{1}{2\sqrt{\pi D(t-t')}}\exp\left(-\frac{1}{4}\frac{(x-x')^2}{D(t-t')}\right)$$

Source $e^{-x^2/a^2}\delta(t)$

First note that $f = 0$ for $t < 0$. For $t > 0$ we get:

$$f(x,t) = \frac{1}{2\sqrt{\pi D t}} \int_{-\infty}^{+\infty} \exp\left(-\frac{1}{4}\frac{(x-x')^2}{Dt} - \frac{x'^2}{a^2} \right) dx'$$

$$= \frac{1}{2\sqrt{\pi D t}} \exp\left(-\frac{x^2}{4Dt} \right) \int_{-\infty}^{+\infty} \exp\left(-\frac{1}{4}\frac{-2xx' + (x')^2}{Dt} - \frac{x'^2}{a^2} \right) dx'$$

$$= \frac{1}{2\sqrt{\pi D t}} \exp\left(-\frac{x^2}{4Dt} \right) \int_{-\infty}^{+\infty} \exp\left(-(x')^2 \left(\frac{1}{a^2} + \frac{1}{4Dt} \right) + \frac{xx'}{2Dt} \right) dx'$$

Complete the square:

$$-(x')^2 \left(\frac{1}{a^2} + \frac{1}{4Dt} \right) + \frac{xx'}{2Dt} = -\left(x' \sqrt{\frac{1}{a^2} + \frac{1}{4Dt}} - \frac{x}{4Dt\sqrt{\frac{1}{a^2} + \frac{1}{4Dt}}} \right)^2 + \frac{1}{4}\frac{a^2 x^2}{Dt\left(4Dt + a^2\right)}$$

Thus

$$f(x,t) = \frac{1}{2\sqrt{\pi D t}} \exp\left(-\frac{x^2}{4Dt} + \frac{1}{4}\frac{a^2 x^2}{Dt(4Dt + a^2)} \right) \frac{\sqrt{\pi}}{\sqrt{\frac{1}{a^2} + \frac{1}{4Dt}}}$$

$$= \exp\left(-\frac{x^2}{4Dt} \left(1 - \frac{a^2}{(4Dt + a^2)} \right) \right) \frac{a}{\sqrt{4Dt + a^2}}$$

$$= \frac{a}{\sqrt{4Dt + a^2}} \exp\left(-\frac{x^2}{4Dt + a^2} \right)$$

Compare this result with problem 7.22.

5. Show that we can use Green's theorem (§5) to obtain a solution for Φ where Φ satisfies the Helmholtz equation

$$\left(\nabla^2 + k^2\right)\Phi(\vec{\mathbf{x}}) = S(\vec{\mathbf{x}})$$

with a source function $S(\vec{\mathbf{x}})$. Determine the solution for Φ in terms of the Green's function when Φ satisfies the Dirichlet boundary conditions $\Phi(\vec{\mathbf{x}}) = F(\vec{\mathbf{x}})$, a known function, on the boundary surface S.

Define Φ as the solution of the equation:

$$\left(\nabla^2 + k^2\right)\Phi = S(\vec{\mathbf{x}})$$

and Ψ similarly:

$$\left(\nabla^2 + k^2\right)\Psi = 4\pi\delta\left(\vec{\mathbf{x}} - \vec{\mathbf{x}}'\right) \quad \text{Pr 5 equation 1}$$

Then

$$\int_V \left(\Phi\nabla^2\Psi - \Psi\nabla^2\Phi\right) dV = \int_S \left(\Phi\vec{\boldsymbol{\nabla}}\Psi - \Psi\vec{\boldsymbol{\nabla}}\Phi\right) \cdot \hat{\mathbf{n}}\, dS$$

$$\int_V \left(\Phi\left(4\pi\delta\left(\vec{\mathbf{x}} - \vec{\mathbf{x}}'\right) - k^2\Psi\right) - \Psi\left(S(\vec{\mathbf{x}}) - k^2\Phi\right)\right) dV = \int_S \left(\Phi\vec{\boldsymbol{\nabla}}\Psi - \Psi\vec{\boldsymbol{\nabla}}\Phi\right) \cdot \hat{\mathbf{n}}\, dS$$

The terms in k^2 cancel, leaving

$$4\pi\Phi\left(\vec{\mathbf{x}}'\right) - \int_V \Psi\left(\vec{\mathbf{x}}, \vec{\mathbf{x}}'\right) S(\vec{\mathbf{x}})\, dV = \int_S \left(\Phi\vec{\boldsymbol{\nabla}}\Psi - \Psi\vec{\boldsymbol{\nabla}}\Phi\right) \cdot \hat{\mathbf{n}}\, dS$$

which is the same relation we had for Poisson's equation. Let the Dirchlet Green's function G_D satisfy equation (Pr 5 equation 1) with boundary conditions $G_D = 0$ on S. Thus the solution for Dirichlet conditions is:

$$\Phi\left(\vec{\mathbf{x}}'\right) = \frac{1}{4\pi}\int_V G_D\left(\vec{\mathbf{x}}, \vec{\mathbf{x}}'\right) S(\vec{\mathbf{x}})\, dV - \frac{1}{4\pi}\int_S F(x)\vec{\boldsymbol{\nabla}} G_D \cdot \hat{\mathbf{n}}\, dS$$

6. Find the Green's function for the one-dimensional Poisson equation

$$\frac{d^2\Phi}{dx^2} = -\rho(x)$$

with boundary conditions $\Phi(x) = 0$ at $x = 0$ and $x = a$. Hence find the solution for Φ when

(a) $\rho(x) = \sin \pi x/a.$

(i) Division of region method:

$$\frac{d^2G}{dx^2} = -4\pi\delta\left(x - x'\right)$$

The equation satisfied by G within each region is

$$\frac{d^2G}{dx^2} = 0$$

with solution

$$G = Ax + B$$

For $x < x'$

$$G = Ax$$

while for $x > x'$

$$G = C(x - a)$$

At $x = x'$

$$Ax' = C\left(x' - a\right) \Rightarrow C = A\frac{x'}{x' - a} = Dx'$$

Thus

$$G_I = Dx\left(x' - a\right) \text{ for } x < x'$$

and

$$G_{II} = Dx'(x - a) \text{ for } x > x'$$

Integrating the differential equation across the boundary, we have

$$\left.\frac{dG}{dx}\right|_{x'-\varepsilon}^{x'+\varepsilon} = -4\pi = Dx' - D\left(x' - a\right) = Da \Rightarrow D = -\frac{4\pi}{a}$$

Thus

$$G\left(x, x'\right) = \frac{4\pi x_<}{a}(a - x_>)$$

Then

$$\begin{aligned}
\Phi(x) &= \frac{1}{\varepsilon_0}\int_0^x \frac{x'}{a}(a - x)\sin\frac{\pi x'}{a}dx' + \int_x^a \left(a - x'\right)\frac{x}{a}\sin\frac{\pi x'}{a}dx' \\
&= \frac{(a - x)}{\varepsilon_0}\frac{1}{\pi}\left[-x'\cos\frac{\pi x'}{a}\Big|_0^x + \int_0^x \cos\frac{\pi x'}{a}dx'\right] + \frac{x}{\varepsilon_0\pi}\left[-\left(a - x'\right)\cos\frac{\pi x'}{a}\Big|_x^a - \int_x^a \cos\frac{\pi x'}{a}dx'\right] \\
&= \frac{(a - x)}{\varepsilon_0}\frac{1}{\pi}\left[-x\cos\frac{\pi x}{a} + \frac{a}{\pi}\sin\frac{\pi x}{a}\right] + \frac{x}{\varepsilon_0\pi}\left[(a - x)\cos\frac{\pi x}{a} + \frac{a}{\pi}\sin\frac{\pi x}{a}\right] \\
&= \frac{1}{\varepsilon_0}\left(\frac{a}{\pi}\right)^2\sin\frac{\pi x}{a}
\end{aligned}$$

Check:

$$\Phi'' = -\frac{1}{\varepsilon_0}\sin\frac{\pi x}{a}$$

as required.

(b) $\rho(x) = x^2\left(a^2 - x^2\right)$

$$\begin{aligned}\Phi(x) &= \frac{1}{\varepsilon_0}\int_0^x \frac{x'}{a}(a-x)x'^2\left(a^2-x'^2\right)dx' + 4\pi\int_x^a\left(a-x'\right)\frac{x}{a}x'^2\left(a^2-x'^2\right)dx' \\ &= \frac{1}{\varepsilon_0}\frac{(a-x)}{a}\int_0^x u^3\left(a^2-u^2\right)du + \frac{1}{\varepsilon_0}\frac{x}{a}\int_x^a(a-u)u^2\left(a^2-u^2\right)du \\ &= \frac{1}{\varepsilon_0}\frac{(a-x)}{a}\left(-\frac{1}{6}x^6+\frac{1}{4}x^4a^2\right) + \frac{1}{\varepsilon_0}\frac{x}{a}\left(\frac{1}{20}a^6-\frac{1}{6}x^6-\frac{1}{3}a^3x^3+\frac{1}{5}ax^5+\frac{1}{4}x^4a^2\right) \\ &= 4\frac{1}{\varepsilon_0}\left[\frac{1}{30}x^6-\frac{1}{12}x^4a^2+\frac{1}{20}xa^5\right] \\ &= \frac{1}{60\varepsilon_0}x(x-a)\left(2x^4+2x^3a-3x^2a^2-3a^3x-3a^4\right)\end{aligned}$$

Check:

$$\Phi''(x) = \frac{1}{\varepsilon_0}x^2(x-a)(x+a)$$

as expected.

(ii) Eigenfunction method:

The normalized eigenfunctions of the Helmholtz equation are:

$$y_n(x) = \sqrt{\frac{2}{a}}\sin\frac{n\pi x}{a}$$

Thus, since the Poisson equation has eigenvalue $\lambda = 0$:

$$G\left(x,x'\right) = \frac{2}{a}4\pi\sum_{n=1}^{\infty}\frac{\sin\frac{n\pi x}{a}\sin\frac{n\pi x'}{a}}{(n\pi/a)^2}$$

In the two cases we have:

(a)

$$\Phi(x) = \frac{1}{4\pi\varepsilon_0}\frac{8\pi}{a}\int_0^a\sum_{n=1}^{\infty}\frac{\sin\frac{n\pi x}{a}\sin\frac{n\pi x'}{a}}{(n\pi/a)^2}\sin\frac{\pi x'}{a}dx' = \frac{1}{\varepsilon_0}\left(\frac{a}{\pi}\right)^2\sin\frac{\pi x}{a}$$

where we used orthogonality of the eigenfunctions to evaluate the integral.

(b)

$$\begin{aligned}\Phi(x) &= \frac{1}{4\pi\varepsilon_0}\frac{8\pi}{a}\int_0^a\sum_{n=1}^{\infty}\frac{\sin\frac{n\pi x}{a}\sin\frac{n\pi x'}{a}}{(n\pi/a)^2}\left(x'\right)^2\left(a^2-\left(x'\right)^2\right)dx' \\ &= \frac{2}{\varepsilon_0 a}\sum_{n=1}^{\infty}\frac{\sin\frac{n\pi x}{a}}{(n\pi/a)^2}\int_0^a\sin\frac{n\pi x'}{a}\left(x'\right)^2\left(a^2-\left(x'\right)^2\right)dx'\end{aligned}$$

Let's look at the integrals:

$$\int_0^a x^2\sin\frac{n\pi x}{a}dx = -a^3\frac{\left(n^2\pi^2-2\right)(-1)^n+2}{n^3\pi^3} = a^3\left(\frac{(-1)^{n+1}}{n\pi}+\frac{2}{n^3\pi^3}((-1)^n-1)\right)$$

and

$$\begin{aligned}\int_0^a x^4\sin\frac{n\pi x}{a}dx &= -a^5\frac{\left(n^4\pi^4+24-12n^2\pi^2\right)(-1)^n-24}{n^5\pi^5} \\ &= a^5\left(\frac{(-1)^{n+1}}{n\pi}+(-1)^n\frac{12}{n^3\pi^3}+\frac{24}{n^5\pi^5}((-1)^n-1)\right)\end{aligned}$$

Thus

$$\begin{aligned}\Phi(x) &= \frac{2}{\varepsilon_0 a}a^5\sum_{n=1}^{\infty}\frac{\sin\frac{n\pi x}{a}}{(n\pi/a)^2}\\ &\quad\times\left[\frac{(-1)^{n+1}}{n\pi}+\frac{2}{n^3\pi^3}((-1)^n-1)-\frac{(-1)^{n+1}}{n\pi}-(-1)^n\frac{12}{n^3\pi^3}-\frac{24}{n^5\pi^5}((-1)^n-1)\right]\\ &= \frac{2}{\varepsilon_0}a^4\sum_{n=1}^{\infty}\frac{\sin\frac{n\pi x}{a}}{(n\pi/a)^2}\left[2\frac{-1-5(-1)^n}{n^3\pi^3}-\frac{24}{n^5\pi^5}((-1)^n-1)\right]\\ &= \frac{4}{\varepsilon_0}a^6\frac{\sin\frac{n\pi x}{a}}{(n\pi)^5}\left[\frac{12}{n^2\pi^2}(1-(-1)^n)-(1+5(-1)^n)\right]\end{aligned}$$

The eigenfunction method works best for source (a) while the division of region method works best for source (b).

7. Use a division of space method to find the Green's function for the one-dimensional Helmholtz equation in the region $0 < x < a$ with $y(x) = 0$ at $x = 0$ and $x = a$. Find the Fourier sine series for $G(x,x')$ and hence show that your result agrees with the result of Example A.2.

$$\frac{d^2y}{dx^2}+k^2y = f(x)$$
$$\frac{d^2G}{dx^2}+k^2G = \delta(x-x')$$

Division of region method. Within each region, the appropriate solution is a sine.

$$G(x,x') = C\sin kx_<\sin k(a-x_>)$$

Integrating across $x = x'$:

$$\begin{aligned}\left.\frac{dG}{dx}\right|_{x'-\varepsilon}^{x'+\varepsilon} = 1 &= Ck\left(-\sin kx'\cos k\left(a-x'\right)-\cos kx'\sin k\left(a-x'\right)\right)\\ &= -Ck\sin ka\end{aligned}$$

Thus

$$G(x,x') = -\frac{\sin kx_<\sin k(a-x_>)}{k\sin ka}$$

From Example 2, the Green's function is

$$G(x,x') = \frac{2a}{\pi^2}\sum_n\frac{\sin\frac{n\pi x}{a}\sin\frac{n\pi x'}{a}}{k^2a^2/\pi^2-n^2}$$

We want to show equivalence between these expressions, so we find the Fourier series for the first expression.

$$G(x,x') = -\frac{\sin kx_<\sin k(a-x_>)}{k\sin ka} = \sum_n a_n\sin\frac{n\pi x}{a}$$

where

$$
\begin{aligned}
a_n &= -\frac{2}{a}\left[\int_0^{x'} \frac{\sin kx \sin k(a-x')}{k \sin ka} \sin\frac{n\pi x}{a}dx + \int_{x'}^{a} \frac{\sin kx' \sin k(a-x)}{k\sin ka}\sin\frac{n\pi x}{a}dx\right] \\
&= -\frac{2}{a}\left[\frac{\sin k(a-x')}{k\sin ka}\int_0^{x'} \sin kx \sin\frac{n\pi x}{a}dx + \frac{\sin kx'}{k\sin ka}\int_{x'}^{a}\sin k(a-x)\sin\frac{n\pi x}{a}dx\right] \\
&= -\frac{1}{a}\frac{\sin k(a-x')}{k\sin ka}\int_0^{x'}\left\{\cos\left[x\left(k-\frac{n\pi}{a}\right)\right] - \cos\left[x\left(k+\frac{n\pi}{a}\right)\right]\right\}dx \\
&\quad - \frac{1}{a}\frac{\sin kx'}{k\sin ka}\int_{x'}^{a}\left[\cos\left(k(a-x)-\frac{n\pi x}{a}\right) - \cos\left(k(a-x)+\frac{n\pi x}{a}\right)\right]dx \\
&= -\frac{\sin k(a-x')}{ka\sin ka}\left[\frac{\sin[x(k-\frac{n\pi}{a})]}{k-\frac{n\pi}{a}} - \frac{\sin[x(k+\frac{n\pi}{a})]}{k+\frac{n\pi}{a}}\right]_0^{x'} - \\
&\quad \frac{\sin kx'}{ka\sin ka}\left[\frac{\sin(k(a-x)-\frac{n\pi x}{a})}{-k-\frac{n\pi}{a}} - \frac{\sin(k(a-x)+\frac{n\pi x}{a})}{-k+\frac{n\pi}{a}}\right]_{x'}^{a} \\
&= -\frac{\sin k(a-x')}{ka\sin ka}\left[\frac{\sin(x'(k-\frac{n\pi}{a}))}{k-\frac{n\pi}{a}} - \frac{\sin(x'(k+\frac{n\pi}{a}))}{k+\frac{n\pi}{a}}\right] + \\
&\quad \frac{\sin kx'}{ka\sin ka}\left[\frac{\sin\left(k(a-x')-\frac{n\pi x'}{a}\right)}{-k-\frac{n\pi}{a}} - \frac{\sin\left(k(a-x')+\frac{n\pi x'}{a}\right)}{-k+\frac{n\pi}{a}}\right] \\
&= -\frac{\sin k(a-x')\sin\left(kx'-\frac{n\pi x'}{a}\right) - \sin kx' \sin\left(k(a-x')+\frac{n\pi x'}{a}\right)}{(k-\frac{n\pi}{a})ka\sin ka} \\
&\quad + \frac{\sin k(a-x')\sin\left(kx'+\frac{n\pi x'}{a}\right) - \sin kx' \sin\left(k(a-x')-\frac{n\pi x'}{a}\right)}{(k+\frac{n\pi}{a})ka\sin ka}
\end{aligned}
$$

We may simplify the two fractions as follows:

$$
\begin{aligned}
&\sin k(a-x')\sin\left(kx'-\frac{n\pi x'}{a}\right) - \sin kx' \sin\left(k(a-x')+\frac{n\pi x'}{a}\right) \\
&= \sin k(a-x')\left(-\cos kx' \sin\frac{n\pi x'}{a}\right) - \sin kx'\left(\cos k(a-x')\sin\frac{n\pi x'}{a}\right) \\
&= -\sin\frac{n\pi x'}{a}\left(\sin k(a-x')\cos kx' + \sin kx' \cos k(a-x')\right) \\
&= -\sin\frac{n\pi x'}{a}\sin ka
\end{aligned}
$$

Thus

$$
\begin{aligned}
a_n &= -\frac{1}{ka}\left(\frac{-\sin\frac{n\pi x'}{a}}{k-\frac{n\pi}{a}} - \frac{\sin\frac{n\pi x'}{a}}{k+\frac{n\pi}{a}}\right) = -\frac{\sin\frac{n\pi x'}{a}}{ka}\left(\frac{1}{\frac{n\pi}{a}-k} - \frac{1}{k+\frac{n\pi}{a}}\right) \\
&= \frac{\sin\frac{n\pi x'}{a}}{a}\left(\frac{2}{k^2-n^2\pi^2/a^2}\right)
\end{aligned}
$$

Thus

$$
\begin{aligned}
G(x,x') &= \frac{2}{a}\sum_n \frac{\sin\frac{n\pi x'}{a}\sin\frac{n\pi x}{a}}{k^2-n^2\pi^2/a^2} \\
&= \frac{2a}{\pi^2}\sum_n \frac{\sin\frac{n\pi x'}{a}\sin\frac{n\pi x}{a}}{k^2a^2/\pi^2-n^2}
\end{aligned}
$$

in agreement with the result of Example 2.

Now with the source and the first version of G, we can find y.

$f(x) = 1$ for $a/4 < x < 3a/4$

There are three regions to consider. For $x < a/4$ we have

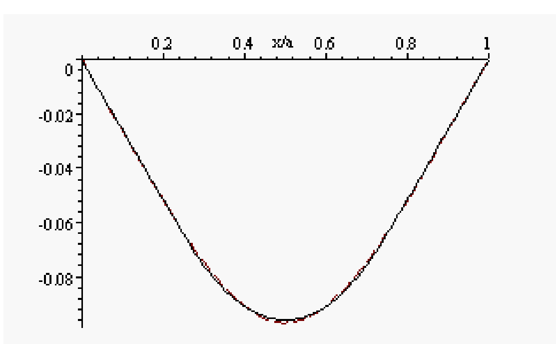
$$
\begin{aligned}
y(x) &= \int G(x,x')f(x')dx' = -\int_{a/4}^{3a/4} \frac{\sin kx_< \sin k(a-x_>)}{k\sin ka}dx' \\
&= -\frac{\sin kx}{k\sin ka}\int_{a/4}^{3a/4} \sin k(a-x')dx' \quad \text{if } x < a/4 \\
&= -\frac{\sin kx}{k\sin ka}\frac{\cos k(a-x')}{k}\Bigg|_{a/4}^{3a/4} = -\frac{\sin kx}{k^2\sin ka}\left(\cos(ka/4) - \cos\frac{3ka}{4}\right) \\
&= -\frac{\sin kx}{k^2\sin ka}\left(2\sin\frac{ka}{2}\sin\frac{ka}{4}\right) \\
&= -\frac{\sin kx}{k^2\cos ka/2}\sin\frac{ka}{4} \qquad \text{if } x < a/4
\end{aligned}
$$

For $x > 3a/4$ the solution is

$$
\begin{aligned}
y(x) &= \int G(x,x')f(x')dx' = -\int_{a/4}^{3a/4} \frac{\sin kx_< \sin k(a-x_>)}{k\sin ka}dx' \\
&= -\frac{\sin k(a-x)}{k\sin ka}\int_{a/4}^{3a/4} \sin kx'dx' \quad \text{if } x > 3a/4 \\
&= -\frac{\sin k(a-x)}{k\sin ka} - \frac{\cos kx'}{k}\Bigg|_{a/4}^{3a/4} \\
&= -\frac{\sin k(a-x)}{k^2\sin ka}\left(-\cos\frac{3}{4}ka + \cos\frac{1}{4}ka\right) = -\frac{\sin k(a-x)}{k^2\sin ka}2\sin\frac{ka}{2}\sin\frac{ka}{4} \\
&= -\frac{\sin k(a-x)}{k^2\cos ka/2}\sin\frac{ka}{4} \quad \text{if } x > 3a/4
\end{aligned}
$$

For $a/4 < x < 3a/4$, we have

$$
\begin{aligned}
y(x) &= \int G(x,x')f(x')dx' = -\int_{a/4}^{3a/4} \frac{\sin kx_< \sin k(a-x_>)}{k\sin ka}dx' \\
&= -\frac{\sin k(a-x)}{k\sin ka}\int_{a/4}^{x} \sin kx'dx' - \frac{\sin kx}{k\sin ka}\int_{x}^{3a/4} \sin k(a-x')dx' \\
&= -\frac{\sin k(a-x)}{k\sin ka} - \frac{\cos kx'}{k}\Bigg|_{a/4}^{x} - \frac{\sin kx}{k\sin ka}\frac{\cos k(a-x')}{k}\Bigg|_{x}^{3a/4} \\
&= \frac{\sin k(a-x)}{k^2\sin ka}\left(\cos(kx) - \cos\frac{ka}{4}\right) - \frac{\sin kx}{k^2\sin ka}\left(\cos\left(\frac{ka}{4}\right) - \cos k(a-x)\right) \\
&= \frac{\sin k(a-x)\cos(kx) - \sin k(a-x)\cos\frac{ka}{4} - \sin kx\cos\frac{ka}{4} + \sin kx\cos k(a-x)}{k^2\sin ka} \\
&= \frac{\sin k(a-x)\cos(kx) + \sin kx\cos k(a-x) - \cos\frac{ka}{4}(\sin k(a-x) + \sin kx)}{k^2\sin ka} \\
&= \frac{\sin ka - 2\cos\frac{ka}{4}\left(\sin\frac{ka}{2}\cos k\left(\frac{a}{2} - x\right)\right)}{k^2\sin ka} \\
&= \frac{1}{k^2}\left(1 - \frac{\cos\frac{ka}{4}}{\cos\frac{ka}{2}}\cos k\left(\frac{a}{2} - x\right)\right) \quad \text{if } a/4 < x < 3a/4
\end{aligned}
$$



The plot shows this solution and the one from Example 2 (the first 2 terms have been plotted) with

$ka = 1/2$. . They are identical.

8. Sometimes we may expand the Green's function as a series of eigenfunctions even if the differential equation is not of Sturm-Liouville form. The governing differential equation for the displacement of a beam is equation 3.11:

$$\frac{d^4y}{dx^4} = \frac{q(x)}{EI}$$

A beam of length L rests on a support at each end so that the boundary conditions are $y(0) = y(L) = 0$. . Show that the Green's function may be expanded in a series of eigenfunctions, and determine the form of the Green's function. Use it to find the beam displacement when it is subjected to a load

$q(x) = ax/L$. Compare with Chapter 4 problem 15.

The governing differential equation is equation 3.11:

$$\frac{d^4y}{dx^4} = \frac{q(x)}{EI}$$

The equation for the Green's function is:

$$\frac{d^4y}{dx^4} = \frac{\delta(x-x')}{EI}$$

The eigenfunctions are solutions of the related equation

$$\frac{d^4y_n}{dx^4} + \lambda_n y_n = 0$$

with $\lambda_n \neq 0$. The solutions to the eigenfunction equation that satisfy the boundary condition at $x = 0$ are:

$$y_n = c_n \sin(-\lambda_n)^{1/4}x$$

To satisfy the second condition at $x = L$ we need:

$$(-\lambda_n)^{1/4}L = n\pi$$

and so the eigenvalues are:

$$\lambda_n = -\left(\frac{n\pi}{L}\right)^4$$

We still need to normalize the functions. We choose the constant c_n so that:

$$\int_0^L y_n(x)y_n(x)dx = 1$$
$$c_n^2\int_0^L \sin^2\left(\frac{n\pi}{L}x\right)dx = 1$$
$$c_n^2\frac{L}{2} = 1$$
$$c_n = \sqrt{\frac{2}{L}}$$

and so the normalized eigenfunctions are:

$$y_n = \sqrt{\frac{2}{L}}\sin\left(\frac{n\pi}{L}x\right)$$

Now let

$$G(x,x') = \sum_n a_n y_n$$

Then

$$\frac{d^4G}{dx^4} = \sum_n a_n\frac{d^4y_n}{dx^4} = -\sum_n a_n\lambda_n y_n$$

Now stuff into the de and use the orthogonality:

$$-\sum_n a_n \lambda_n y_n = \frac{\delta(x - x')}{EI}$$

$$-a_n \lambda_n = \frac{y_n(x')}{EI}$$

and thus

$$G(x, x') = \frac{1}{EI} \sum_n \frac{y_n(x) y_n(x')}{\lambda_n}$$

and the Green's function is (cf equation C.11 with -4π replaced by $\frac{1}{EI}$)

$$G(x, x') = \frac{2}{LEI} \sum_{n=1}^{\infty} \frac{\sin\left(\frac{n\pi}{L}x'\right)\sin\left(\frac{n\pi}{L}x\right)}{\left(\frac{n\pi}{L}\right)^4} = \frac{2L^3}{\pi^4 EI} \sum_{n=1}^{\infty} \frac{\sin\left(\frac{n\pi}{L}x'\right)\sin\left(\frac{n\pi}{L}x\right)}{n^4}$$

Now we use this Green's function to find the displacement with the given load:

$$\begin{aligned} y(x) &= \int_0^L G(x, x') q(x') dx' \\ &= \int_0^L \frac{2L^3}{\pi^4 EI} \sum_{n=0}^{\infty} \frac{\sin\left(\frac{n\pi}{L}x'\right)\sin\left(\frac{n\pi}{L}x\right)}{n^4} \frac{ax'}{L} dx' \\ &= \frac{2aL^2}{\pi^4 EI} \sum_{n=1}^{\infty} \frac{\sin\left(\frac{n\pi}{L}x\right)}{n^4} \int_0^L x' \sin\left(\frac{n\pi}{L}x'\right) dx' \end{aligned}$$

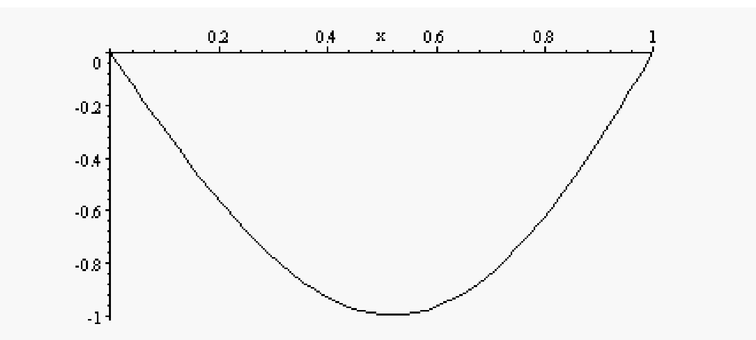
We can do the integral by parts:

$$\begin{aligned} \int_0^L x' \sin\left(\frac{n\pi}{L}x'\right) dx' &= -x' \frac{L}{n\pi} \cos\frac{n\pi x'}{L}\Big|_0^L - \int_0^L -\frac{L}{n\pi} \cos\frac{n\pi x'}{L} dx' \\ &= -\frac{L^2}{n\pi}\cos n\pi + \left(\frac{L}{n\pi}\right)^2 \sin\frac{n\pi x'}{L}\Big|_0^L \\ &= \frac{L^2}{n\pi}(-1)^{n+1} \end{aligned}$$

and so the displacement is:

$$y(x) = \frac{2aL^4}{\pi^5 EI} \sum_{n=1}^{\infty} (-1)^{n+1} \frac{\sin\left(\frac{n\pi}{L}x\right)}{n^5}$$

This series converges very fast.



Problem 8. Displacement of the beam

The solution is the same as that found in Chapter 4.

9. Find the Green's function for heat transfer along a rod with insulated ends. The relevant differential equation is:

$$\frac{\partial T}{\partial t} - D\frac{\partial^2 T}{\partial x^2} = Q(x, t)$$

and the boundary conditions are $\frac{\partial G}{\partial x} = 0$ at $x = 0$ and $x = L$;

$G(x,0) = 0$. Treat the problem as a two-dimensional problem, and use method 1 in section C.4, dividing the region in time.

The differential equation for G is

$$\frac{\partial G}{\partial t} - D\frac{\partial^2 G}{\partial x^2} = \delta(x - x')\delta(t - t')$$

For $x \neq x'$, $t \neq t'$, we have:

$$\frac{\partial G}{\partial t} - D\frac{\partial^2 G}{\partial x^2} = 0$$

Separate variables to find the eigenfunctions:

$$\frac{T'}{T} - D\frac{X''}{X} = 0$$

We choose the separation constant to be $-\gamma.$ Then

$$T = e^{-\gamma t}$$

and

$$-\gamma - D\frac{X''}{X} = 0$$

So

$$\frac{X''}{X} = -\frac{\gamma}{D} \Rightarrow X = \sin\sqrt{\frac{\gamma}{D}}x, \ \cos\sqrt{\frac{\gamma}{D}}x$$

To satisfy the boundary conditions at $x = 0$ and $x = L$ we choose the cosine function, and the eigenvalues are

$$\sqrt{\frac{\gamma}{D}} = n\frac{\pi}{L}$$

Then

$$\gamma = Dn^2\left(\frac{\pi}{L}\right)^2$$

Now we divide space in time. For $t < t'$, $T = 0$. Thus we have:

$$G(x,x',t,t') = \sum_{n=0}^{\infty} g_n(x',t,t')\cos\left[n\pi\frac{x}{L}\right]$$

where

$$g_n(x',t,t') = \gamma_n(x',t')e^{-\gamma t}$$

Then equation () becomes

$$\sum_n \left\{\frac{\partial g_n(x',t,t')}{\partial t} + Dn^2\left(\frac{\pi}{L}\right)^2 g_n(x',t,t')\right\}\cos\left[n\pi\frac{x}{L}\right] = \delta(x - x')\delta(t - t')$$

We multiply by $\frac{2}{L}\cos\left[m\pi\frac{x}{L}\right]$ and integrate along the rod. On the left, only one term survives the integration.

$$\frac{\partial g_m(x',t,t')}{\partial t} + Dm^2\left(\frac{\pi}{L}\right)^2 g_m = \frac{2}{L}\cos\left[m\pi\frac{x'}{L}\right]\delta(t - t')$$

Now we may rewrite g_n as

$$g_n(x',t,t') = \begin{cases} 0 & if \ \ t < t' \\ \tau_n(t')e^{-\gamma t}\frac{2}{L}\cos\left[n\pi\frac{x'}{L}\right] & if \ \ t > t' \end{cases}$$

. and integrate across the boundary at $t = t'$:

$$\tau_n(t')e^{-\gamma t}\Big|_{t'-}^{t'+} = 1$$
$$\tau_n(t')e^{-\gamma t'} - 0 = 1$$

Thus

$$\tau_n(t') = e^{\gamma t'} = \exp\left(D\left(\frac{n\pi}{L}\right)^2 t'\right)$$

and

$$\begin{aligned} G(x, x', t, t') &= \frac{2}{L}\sum_{n=0}^{\infty} \exp\left[-D\left(\frac{n\pi}{L}\right)^2 (t-t')\right] \cos\left[n\pi\frac{x}{L}\right] \cos\left[n\pi\frac{x'}{L}\right] S(t-t') \\ &= \frac{2}{L}\sum_{n=0}^{\infty} \exp\left[-D\left(\frac{n\pi}{L}\right)^2 (t-t')\right] \cos\left[n\pi\frac{x}{L}\right] \cos\left[n\pi\frac{x'}{L}\right] S(t-t') \end{aligned}$$

The

$n = 0$ term is just a constant corresponding to the initial uniform temperature of the rod. The result shows the expected behavior: high frequency spatial variations (large n) are smoothed out faster than low frequency ones.

10. Verify that equation (C.34) gives the correct result $\tau(t)$ in the limit $x \to 0$ from above.

Limit

$$L(t) = \lim_{\substack{x\to 0 \\ x>0}} \int_0^t \frac{x\tau(t')}{\sqrt{4\pi D(t-t')^3}} \exp\left(-\frac{x^2}{4D(t-t')}\right) dt'$$

Because of the exponential, as x approaches zero the integral is dominated by values of t' near t. So expand $\tau(t')$ in a Taylor series:

$$\tau(t') = \tau(t) + (t'-t)\tau'(t) + \cdots$$

Change variables to

$$u = \frac{x}{\sqrt{4D(t-t')}}$$

Then

$$du = \frac{x}{\sqrt{4D}}\left(\frac{1}{2}\right)\frac{1}{(t-t')^{3/2}} dt'$$

Now provided that x remains positive, the upper limit in u becomes infinite. Thus:

$$\begin{aligned} \int_0^t \frac{x\tau(t')}{\sqrt{4\pi D(t-t')^3}} \exp\left(-\frac{x^2}{4D(t-t')}\right) dt' &= \frac{2}{\sqrt{\pi}} \int_{x/\sqrt{4Dt}}^{\infty} \left[\tau(t) - \frac{x^2}{4Du^2}\tau'(t)\right] e^{-u^2} du \\ &= \tau(t)\,\mathrm{erfc}\left(\frac{x}{\sqrt{4Dt}}\right) - \frac{2}{\sqrt{\pi}}\frac{x^2}{4D}\tau'(t)\int_{x/\sqrt{4Dt}}^{\infty} \frac{e^{-u^2}}{u^2} du \\ &\to \tau(t) - 0 \end{aligned}$$

The second (and subsequent) integrals remain finite so long as

x remains positive, so in the limit these terms go to zero.

Optional Topic C: Green's functions

11. Find the Green's function for the wave equation in three space dimensions, using spherical coordinates. The wave equation is:

$$v^2\nabla^2 G - \frac{\partial^2}{\partial t^2}G = \delta\left(\vec{\mathbf{x}} - \vec{\mathbf{x}}', t - t'\right)$$

where $G\left(\vec{\mathbf{x}}, \vec{\mathbf{x}}', t, t'\right)$ is the displacement,

∇^2 is the Laplacian operator in three dimensions. Transform the equation in space and time, and solve for the G.

The transformed equation is

$$\left(-v^2k^2 + \omega^2\right)G = \frac{1}{(2\pi)^2}e^{-i\vec{\mathbf{k}}\cdot\vec{\mathbf{x}}'}e^{i\omega t'}$$

which has the solution

$$G\left(\vec{\mathbf{k}}, \omega\right) = \frac{1}{(2\pi)^2}\frac{e^{-i\vec{\mathbf{k}}\cdot\vec{\mathbf{x}}'}e^{i\omega t'}}{\omega^2 - k^2v^2}$$

as before, but here k is the magnitude of the wave vector $\vec{\mathbf{k}}$. The solution is then:

$$G\left(\vec{\mathbf{x}}, \vec{\mathbf{x}}'; t, t'\right) = \frac{1}{(2\pi)^4}\int\frac{e^{-i\vec{\mathbf{k}}\cdot\vec{\mathbf{x}}'}e^{i\omega t'}}{\omega^2 - k^2v^2}e^{i\vec{\mathbf{k}}\cdot\vec{\mathbf{x}}}e^{-i\omega t}d^3\vec{\mathbf{k}}d\omega$$

The integrand has two poles at $\omega = \pm kv$, on the real axis. We want the result to be zero for $t < t'$, and for $t < t'$ we must close upward, so the poles must be below the path of integration. Thus for $t > t'$ we have:

$$\begin{aligned}
G\left(\vec{\mathbf{x}}, \vec{\mathbf{x}}'; t, t'\right) &= \frac{1}{(2\pi)^4}(-2\pi i)\int\left(\frac{e^{ikv(t'-t)}}{2kv} + \frac{e^{-ikv(t'-t)}}{-2kv}\right)e^{i\vec{\mathbf{k}}\cdot\vec{\mathbf{x}}}e^{-i\vec{\mathbf{k}}\cdot\vec{\mathbf{x}}'}d^3\vec{\mathbf{k}} \\
&= \frac{1}{(2\pi)^2 v}\int_0^\infty kdk\int_{-1}^{+1}\sin kv\left(t - t'\right)\exp ik\mu\left|\vec{\mathbf{x}} - \vec{\mathbf{x}}'\right|dkd\mu \\
&= \frac{1}{(2\pi)^2 v}\int_0^\infty kdk\sin kv\left(t - t'\right)\frac{\exp ik\left|\vec{\mathbf{x}} - \vec{\mathbf{x}}'\right| - \exp\left(-ik\left|\vec{\mathbf{x}} - \vec{\mathbf{x}}'\right|\right)}{ik\left|\vec{\mathbf{x}} - \vec{\mathbf{x}}'\right|}dk \\
&= \frac{2}{(2\pi)^2 v\left|\vec{\mathbf{x}} - \vec{\mathbf{x}}'\right|}\int_0^\infty \sin kv\left(t - t'\right)\sin k\left|\vec{\mathbf{x}} - \vec{\mathbf{x}}'\right|dk \\
&= \frac{1}{4\pi v\left|\vec{\mathbf{x}} - \vec{\mathbf{x}}'\right|}\delta\left(\left|\vec{\mathbf{x}} - \vec{\mathbf{x}}'\right| - v\left(t - t'\right)\right)
\end{aligned}$$

where we used the result of Chapter 6, problem 13.

Then if the source is $h\delta(t)e^{-a^2r^2}$ we get

$$\begin{aligned}
&= h\int\left(\frac{\delta(t')}{a^2 + (r')^2}\right)e^{-a^2(r')^2}\frac{1}{4\pi v\left|\vec{\mathbf{x}} - \vec{\mathbf{x}}'\right|}\delta\left(\left|\vec{\mathbf{x}} - \vec{\mathbf{x}}'\right| - v(t - t')\right)d^3\vec{\mathbf{x}}'dt' \\
&= h\int\left(\frac{1}{a^2 + (r')^2}\right)e^{-a^2(r')^2}\frac{1}{4\pi v\left|\vec{\mathbf{x}} - \vec{\mathbf{x}}'\right|}\delta\left(\left|\vec{\mathbf{x}} - \vec{\mathbf{x}}'\right| - vt\right)d^3\vec{\mathbf{x}}'
\end{aligned}$$

Change variables to $\vec{\mathbf{R}} = \vec{\mathbf{x}} - \vec{\mathbf{x}}'$ with $\vec{\mathbf{x}}$ fixed. Then

$$\vec{\mathbf{x}}' = \vec{\mathbf{x}} - \vec{\mathbf{R}}$$

and

$$r' = \sqrt{r^2 + R^2 - 2rR\mu}$$

we have

$$\begin{aligned}
f &= h\int\exp\left(-a^2\left(R^2 + r^2 - 2rR\mu\right)\right)\frac{1}{4\pi vR}\delta(R - vt)R^2dRd\mu d\phi \\
&= h\frac{t}{2}e^{-a^2\left(r^2+v^2t^2\right)}\int_{-1}^{+1}\exp\left(-2ra^2vt\mu\right)d\mu \\
&= h\frac{t}{2}e^{-a^2\left(r^2+v^2t^2\right)}\frac{\exp\left(-2ra^2vt\right) - \exp\left(2ra^2vt\right)}{-2ra^2vt} \\
&= h\frac{1}{2ra^2v}\exp\left(-a^2\left(r^2 + v^2t^2\right)\right)\cosh\left(2ra^2vt\right)
\end{aligned}$$

As in the one-dimensional case, the displacement never goes negative.

12. Find the Dirichlet Green's function for the Helmholtz equation in a circular region of radius

a. Obtain the result as a double sum over appropriate eigenfunctions.

The equation for G is

$$\left(\nabla^2 + k^2\right)G = -4\pi\delta\left(\vec{\mathbf{x}} - \mathbf{x}'\right)$$

where $\vec{\mathbf{x}}$ is a two-component vector The eigenfunction equation is:

$$\left(\nabla^2 + \alpha^2\right)y = 0$$

As in Chapter 8, we write the operator in polar coordinates to obtain:

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial y}{\partial\rho}\right) + \frac{1}{\rho}\frac{\partial}{\partial\phi}\left(\frac{1}{\rho}\frac{\partial y}{\partial\phi}\right) + \alpha^2 y = 0$$

Separating variables, $y = RW$, we have:

$$\frac{\rho}{R}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial R}{\partial\rho}\right) + \frac{1}{W}\frac{\partial}{\partial\phi}\left(\frac{\partial W}{\partial\phi}\right) + \rho^2\alpha^2 = 0$$

Thus $W = e^{im\phi}$ and

$$\rho\frac{\partial}{\partial\rho}\left(\rho\frac{\partial R}{\partial\rho}\right) + \left(\rho^2\alpha^2 - m^2\right)R = 0$$

This is Bessel's equation (8.56) with solution

$$R = J_m(\alpha\rho)$$

We must choose α so that $J_m(\alpha a) = 0$, that is, $\alpha a = x_{mn}$, the nth zero of J_m. Thus the eigenfunctions we need are

$$y_m(\rho,\phi) = AJ_m\left(x_{mn}\frac{\rho}{a}\right)e^{im\phi}$$

with eigenvalue $\alpha^2 = (x_{mn}/a)^2$. Next we must normalize these functions. Since

$$\int_0^{2\pi} d\phi = 2\pi$$

and

$$\int_0^a\left[J_m\left(x_{mn}\frac{\rho}{a}\right)\right]^2\rho d\rho = \frac{a^2}{2}\left[J_m'(x_{mn})\right]^2$$

(equation 8.96), the normalized eigenfunctions are

$$y_m(\rho,\phi) = \frac{J_m(x_{mn}\frac{\rho}{a})e^{im\phi}}{\sqrt{\pi}\,aJ_m'(x_{mn})}$$

Thus from equation C.11 extended to two dimensions, the Green's function is:

$$\begin{aligned} G(\rho,\rho',\phi,\phi') &= 4\pi\sum_{m,n}\frac{J_m(x_{mn}\frac{\rho}{a})J_m\left(x_{mn}\frac{\rho'}{a}\right)e^{im(\phi-\phi')}}{\pi a^2[J_m'(x_{mn})]^2\left(x_{mn}^2/a^2 - k^2\right)} \\ &= 4\sum_{m,n}\frac{J_m(x_{mn}\frac{\rho}{a})J_m\left(x_{mn}\frac{\rho'}{a}\right)e^{im(\phi-\phi')}}{[J_m'(x_{mn})]^2\left(x_{mn}^2 - k^2a^2\right)} \end{aligned}$$

As in the one-dimensional case, the solution does not exist if k equals a resonant frequency x_{mn}/a. Convince yourself that the result is dimensionally correct.

13. Find the Neumann Green's function for the one-dimensional Poisson equation:

$$\frac{d^2\Phi}{dx^2} = -\frac{\rho(x)}{\varepsilon_0}$$

with boundary conditions $\frac{dG}{dx} = 0$ at $x = 0$ and $x = L$. Express your answer as a series of eigenfunctions. Hence find the potential Φ if $\rho(x) = \rho_0 x/L$ for $0 < x < L/2$ and zero otherwise, and the charge density is zero on the bounding surfaces ($d\Phi/dx = 0$ at $x = 0$ and $x = L$).

The eigenfunctions satisfy the equation:

$$\frac{d^2y}{dx^2} + k^2y = 0$$

with solutions $\sin kx$ and $\cos kx$. To satisfy the boundary conditions we choose the cosine and take

$k = n\pi/L$. We also need to normalize the functions by multiplying by $\sqrt{2/L}$ Thus

$$G_N(x,x') = \frac{2}{L}\sum_{n=1}^{\infty}\frac{\cos\frac{n\pi x}{L}\cos\frac{n\pi x'}{L}}{(n\pi/L)^2}$$

Next we find the potential.

The second term in equation C.38 (the surface integral) is zero. Thus

$$\begin{aligned}
\Phi(x) - \langle\ \Phi\ \rangle &= \int_0^L G_N(x,x')\frac{\rho(x')}{4\pi\varepsilon_0}dx' \\
&= \frac{1}{\varepsilon_0}\int_0^{L/2}\rho_0\frac{x'}{L}\frac{2}{L}\sum_n\frac{\cos\frac{n\pi x}{L}\cos\frac{n\pi x'}{L}}{(n\pi/L)^2}dx' \\
&= \frac{\rho_0}{4\pi\varepsilon_0}2\sum_n\frac{\cos\frac{n\pi x}{L}}{(n\pi)^2}\int_0^{L/2}x'\cos\frac{n\pi x'}{L}dx' \\
&= \frac{\rho_0}{4\pi\varepsilon_0}2\sum_n\frac{\cos\frac{n\pi x}{L}}{(n\pi)^2}\left(x'\frac{L}{n\pi}\sin\frac{n\pi x'}{L}\bigg|_0^{L/2} - \frac{L}{n\pi}\int_0^{L/2}\sin\frac{n\pi x'}{L}dx'\right) \\
&= \frac{\rho_0}{4\pi\varepsilon_0}2\sum_n\frac{\cos\frac{n\pi x}{L}}{(n\pi)^2}\left(\frac{L^2}{2n\pi}\sin\frac{n\pi}{2} - \frac{L^2}{(n\pi)^2}\left(1-\cos\frac{n\pi}{2}\right)\right) \\
&= \frac{\rho_0 L^2}{4\pi\varepsilon_0}\sum_{n=1}^{\infty}\frac{\cos\frac{n\pi x}{L}}{(n\pi)^3}\left(\sin\frac{n\pi}{2} - \frac{2}{n\pi}\left(1-\cos\frac{n\pi}{2}\right)\right)
\end{aligned}$$

Let's check this answer. First note that it is dimensionally correct (charge/(ε_0 ×length). The second derivative is

$$\Phi'' = -\frac{\rho_0}{4\pi\varepsilon_0}\sum_n\frac{\cos\frac{n\pi x}{L}}{n\pi}\left(\sin\frac{n\pi}{2} - \frac{2}{n\pi}\left(1-\cos\frac{n\pi}{2}\right)\right)$$

Compare with the cosine series for the source:

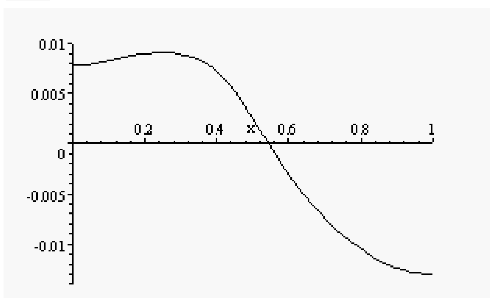
$$\rho = \sum_n a_n\cos\frac{n\pi x}{L}$$

where

$$a_n = \frac{2}{L}\frac{\rho_0}{L}\int_0^{L/2}x\cos\frac{n\pi x}{L}dx = \rho_0\frac{2\left(\cos\frac{n\pi}{2}-1\right)+n\pi\sin\frac{n\pi}{2}}{n^2\pi^2}$$

So the potential we found satisfies Laplace's equation in the volume, as required.

The plot shows the potential $4\pi\varepsilon_0\Phi/\rho_0$ versus x/L.



14. Use a Fourier transform method and cylindrical coordinates to find the Green's function for the wave equation in two dimensions.

$$v^2\nabla^2 G - \frac{\partial^2}{\partial t^2}G = \delta(\vec{x}-\vec{x}')\,\delta(t-t')$$

Transform

$$\left(-v^2k^2+\omega^2\right)G = \frac{1}{(2\pi)^{3/2}}e^{-i\vec{k}\cdot\vec{x}'}e^{i\omega t'}$$

Thus

$$F(\vec{k},\omega) = \frac{1}{(2\pi)^{3/2}}\frac{e^{-i\vec{k}\cdot\vec{x}'}e^{i\omega t'}}{\omega^2-v^2k^2}$$

In polar coordinates,

$$f(\vec{x},t) = \frac{1}{(2\pi)^3}\int \frac{e^{-i\vec{k}\cdot\vec{x}'}e^{i\omega t'}}{\omega^2 - v^2k^2}e^{i\vec{k}\cdot\vec{x}-i\omega t}kdkd\phi d\omega$$
$$= \frac{1}{(2\pi)^3}\int \frac{e^{i\vec{k}\cdot(\vec{x}-\vec{x}')-i\omega(t-t')}}{(\omega - vk)(vk+\omega)}kdkd\phi d\omega$$

With the poles $\pm vk$ in the lower half plane (that is, the path of integration passes over the poles) we find for $t > t'$

$$f(\vec{x},t) = -\frac{2\pi i}{(2\pi)^3}\int_0^\infty\int_0^{2\pi} e^{ik\rho\cos\phi}\left(\frac{e^{-ivk(t-t')}}{2vk} + \frac{e^{ivk(t-t')}}{-2vk}\right)kdkd\phi$$
$$= \frac{1}{2v\pi}\int J_0(k\rho)\sin(kv(t-t'))dk$$
$$= \frac{1}{2v\pi}\left\{\begin{array}{ll} 0 & \text{if } v(t-t') < \rho \\ \frac{1}{\sqrt{v^2t^2-\rho^2}} & \text{if } v(t-t') > \rho \end{array}\right\}$$

where $\rho = |\vec{x} - \vec{x}'|$

(The integral is GR 6.671#7, or see problem 8.34a with $a = \pm iv(t-t')$.)

This agrees with Morse and Feshbach pg 842.

15. Using the division of space method**,** find (a) the Dirichlet and (b) the Neumann Green's function for the interior of a sphere of radius a.

The boundary surface has radius $A = 4\pi a^2$ and so for the Neumann case the boundary condition for G is

$$\frac{\partial G}{\partial r} = -\frac{1}{a^2}$$

Thus we have:

Region I: $0 < r < r'$

$$G_I = \sum_{l,m} A_{lm}r^l Y_{lm}(\theta,\phi)Y^*_{lm}(\theta',\phi')$$

Region II: $r' < r < a$

$$G_{II,D} = \sum_{l,m} B_{lm}\left(r^l - \frac{a^{2l+1}}{r^{l+1}}\right)Y_{lm}(\theta,\phi)Y^*_{lm}(\theta'\phi')$$
$$G_{II,N} = \sum_{l,m}\left(B_{lm}r^l - C_{lm}\frac{a^{2l+1}}{r^{l+1}}\right)Y_{lm}(\theta,\phi)Y^*_{lm}(\theta'\phi')$$

where

$$\sum_{l,m}\left(lB_{lm}a^{l-1} + (l+1)C_{lm}\frac{a^{2l+1}}{a^{l+2}}\right)Y_{lm}(\theta,\phi)Y^*_{lm}(\theta'\phi') = -\frac{1}{a^2}$$

Thus for $l,m \neq 0,$

$$C_{lm} = -\frac{l}{l+1}B_{lm}$$

while for $l = m = 0$

$$C_{00}\frac{1}{4\pi a} = -\frac{1}{a^2}$$

and so

$$C_{00} = -\frac{4\pi}{a}$$

B_{00} is not yet determined. Thus

$$G_{II,N} = \frac{B_{00}}{4\pi} + \frac{1}{r} + \sum_{l=1}^{\infty}\sum_{m=-l}^{l} B_{lm}\left(r^l + \frac{l}{l+1}\frac{a^{2l+1}}{r^{l+1}}\right)Y_{lm}(\theta,\phi)Y^*_{lm}(\theta'\phi')$$

Continuity at $r = r'$:

$$\sum_{l,m} A_{lm}(r')^l Y_{lm}(\theta,\phi)Y^*_{lm}(\theta',\phi') = \sum_{l,m} B_{lm}\left((r')^l - \frac{a^{2l+1}}{(r')^{l+1}} \right) Y_{lm}(\theta,\phi)Y^*_{lm}(\theta'\phi')$$

$$= \frac{B_{00}}{4\pi} + \frac{1}{r} + \sum_{l=1}^{\infty}\sum_{m=-l}^{l} B_{lm}\left((r')^l + \frac{l}{l+1}\,\frac{a^{2l+1}}{(r')^{l+1}} \right) Y_{lm}(\theta,\phi)Y^*_{lm}(\theta'\phi')$$

Making use of the orthogonality of the Y_{lm}, we have in the Dirichlet case:

$$A_{lm} = B_{lm}\left(1 - \left(\frac{a}{r'}\right)^{2l+1}\right) = \beta_{lm}(r')^l\left(1 - \left(\frac{a}{r'}\right)^{2l+1}\right)$$

and thus

$$G_D = \sum_{l,m} \beta_{lm}(r')^l r^l \left(1 - \left(\frac{a}{r_>}\right)^{2l+1}\right) Y_{lm}(\theta,\phi)Y^*_{lm}(\theta'\phi')$$

and for the Neumann case

$$A_{00} = B_{00} + \frac{4\pi}{r'}$$

and for $l,m \neq 0$.

$$A_{lm} = B_{lm}\left(1 + \frac{l}{l+1}\,\frac{a^{2l+1}}{(r')^{2l+1}}\right) = \beta_{lm}(r')^l\left(1 + \frac{l}{l+1}\,\frac{a^{2l+1}}{(r')^{2l+1}}\right)$$

$$G_I = \frac{B_{00}}{4\pi} + \frac{1}{r'} + \sum_{l=1}^{\infty}\sum_{m=-l}^{l} \beta_{lm}(r')^l r^l \left(1 + \frac{l}{l+1}\,\frac{a^{2l+1}}{(r')^{2l+1}}\right) Y_{lm}(\theta,\phi)Y^*_{lm}(\theta',\phi')$$

$$G_{IIN} = \frac{B_{00}}{4\pi} + \frac{1}{r} + \sum_{l=1}^{\infty}\sum_{m=-l}^{l} \beta_{lm}(r')^l r^l \left(1 + \frac{l}{l+1}\,\frac{a^{2l+1}}{r^{2l+1}}\right) Y_{lm}(\theta,\phi)Y^*_{lm}(\theta'\phi')$$

Now we integrate the differential equation across the boundary at $r = r'$:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial G}{\partial r}\right) - l(l+1)G = -4\pi\frac{\delta(r-r')}{r^2}\delta(\mu-\mu')\delta(\phi-\phi')$$

Now write $G = \sum_{lm} g_l(r,r')Y_{lm}(\theta,\phi)Y^*_{lm}(\theta'\phi')$. Multiply both sides by $Y^*_{lm}(\theta,\phi)$ and integrate over the whole sphere:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial g_l}{\partial r}\right) - l(l+1)g_l = -\frac{4\pi}{r^2}\delta(r-r')$$

Next multiply by r^2, and integrate from $r' - \varepsilon$ to $r' + \varepsilon$:

$$r^2\frac{\partial g_l}{\partial r}\Bigg|_{r'-\varepsilon}^{r'+\varepsilon} = -4\pi$$

In the Dirichlet case:

$$(r')^2\beta_{lm}(r')^l\left[l(r')^{l-1} - (l+1)\frac{a^{2l+1}}{(r')^{l+2}} - l(r')^{l-1}\left(1 - \frac{a^{2l+1}}{(r')^{2l+1}}\right)\right] = -4\pi$$

$$\beta_{lm}(r')^{l+2}\left[-(2l+1)\frac{a^{2l+1}}{(r')^{l+2}} \right] = -4\pi$$

and so

$$\beta_{lm} = \frac{4\pi}{2l+1}\,\frac{1}{a^{2l+1}}$$

and

$$G_D = \sum_{l,m}\frac{4\pi}{2l+1}\,\frac{(r')^l r^l}{a^{2l+1}}\left(1 - \left(\frac{a}{r_>}\right)^{2l+1}\right) Y_{lm}(\theta,\phi)Y^*_{lm}(\theta'\phi')$$

For the Neumann case: $l = m = 0$

$$-4\pi = -4\pi$$

while for $l,m \neq 0$,

$$\left[\beta_{lm}(r')^{l+2}\left(l(r')^{l-1} - \frac{l}{l+1}(l+1)\frac{a^{2l+1}}{(r')^{l+2}}\right)\right] - \beta_{lm}l(r')^{2l+1}\left(1 + \frac{l}{l+1}\frac{a^{2l+1}}{(r')^{2l+1}}\right) = -4\pi$$
$$-\beta_{lm}\frac{l}{l+1}(2l+1)a^{2l+1} = -4\pi$$

and thus, choosing $B_{00} = 0$;

$$G_N = \frac{1}{r_>} + \sum_{l=1}^{\infty}\sum_{m=-l}^{l}\frac{4\pi}{2l+1}\frac{(r')^l r^l}{a^{2l+1}}\left(\frac{l+1}{l} + \frac{a^{2l+1}}{(r_>)^{2l+1}}\right)Y_{lm}(\theta,\phi)Y^*_{lm}(\theta',\phi')$$

16. (a) Find the Dirichlet Green's function for the half-space $z > 0$ using Cartesian coordinates.

Dividing space in z, the appropriate eigenfunctions are

$$e^{ikx}e^{iuy}\exp\left(\pm\sqrt{k^2+u^2}\,z\right)$$

Thus in region I, $z < z'$,

$$G_I = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}A(k,u)e^{ikx}e^{iuy}\sinh\left(\sqrt{k^2+u^2}\,z\right)dk\,du$$

and in region 2, $z > z'$,

$$G_{II} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}B(k,u)e^{ikx}e^{iuy}\exp\left(-\sqrt{k^2+u^2}\,z\right)dk\,du$$

Continuity at $z = z'$

$$A(k,u)\sinh\left(\sqrt{k^2+u^2}\,z'\right) = B(k,u)\exp\left(-\sqrt{k^2+u^2}\,z'\right)$$

Thus

$$B = A(k,u)\sinh\left(\sqrt{k^2+u^2}\,z'\right)e^{\sqrt{k^2+u^2}\,z'} \equiv \alpha(\kappa,u)\sinh\left(\sqrt{k^2+u^2}\,z'\right)$$

which defines α. Then"

$$G_I = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\alpha(k,u)e^{-\sqrt{k^2+u^2}\,z'}e^{ikx}e^{iuy}\sinh\left(\sqrt{k^2+u^2}\,z\right)dk\,du$$
$$G_{II} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\alpha(k,u)\sinh\left(\sqrt{k^2+u^2}\,z'\right)e^{ikx}e^{iuy}\exp\left(-\sqrt{k^2+u^2}\,z\right)dk\,du$$

which exhibits the symmetry in z and z'.

Now

$$\nabla^2 G = -4\pi\delta(\vec{x}-\vec{x}')$$
$$\frac{d^2G}{dx^2} + \frac{d^2G}{dy^2} + \frac{d^2G}{dz^2} = -4\pi\delta(x-x')\delta(y-y')\delta(z-z')$$
$$\frac{d^2G}{dz^2} - (k^2+u^2)G = -4\pi\delta(x-x')\delta(y-y')\delta(z-z')$$

Now write $G = \int g(k,u,z,z')e^{ikx}e^{iuy}dk\,du$, multiply by $e^{-ik'x}e^{-iu'y}$ and integrate over x and y :

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\left[\frac{d^2}{dz^2}g(z,z') - (k^2+u^2)g\right](2\pi)^2\delta(k-k')\delta(u-u')\,dk\,du = -4\pi e^{-ik'x'}e^{-iu'y'}\delta(z-z')$$

Performing the integrations, we have:

$$g(k',u',z,z')(2\pi)^2 = -4\pi e^{-ik'x'}e^{-iu'y'}\delta(z-z')$$

Now drop the primes on k' and u', rewrite $\alpha(k,u) = \beta(k,u)e^{-ikx'}e^{-iuy'}$, and integrate across the boundary at $z = z'$:

$$\pi\beta(k,u)\frac{d}{dz}e^{-\sqrt{k^2+u^2}\,z_>}\sinh\left(\sqrt{k^2+u^2}\,z_<\right)\Big|_{z'-}^{z'+} = -1$$
$$\pi\beta\sqrt{k^2+u^2}\{-e^{-\sqrt{k^2+u^2}\,z'}\sinh\sqrt{k^2+u^2}\,z' - e^{-\sqrt{k^2+u^2}\,z'}\cosh\sqrt{k^2+u^2}\,z'\} = -1$$
$$\pi\beta\sqrt{k^2+u^2}\left\{-1 + e^{-2\pi\beta\sqrt{k^2+u^2}\,z'} - 1 - e^{-\pi\beta\sqrt{k^2+u^2}\,z'}\right\} = -1$$

So

$$\beta = \frac{1}{2\pi\sqrt{k^2 + u^2}}$$

Thus

$$G_I = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{1}{\sqrt{k^2+u^2}}e^{-\sqrt{k^2+u^2}z'}e^{ik(x-x')}e^{iu(y-y')}\sinh\left(\sqrt{k^2+u^2}z\right)dkdu$$
$$G_{II} = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{1}{\sqrt{k^2+u^2}}e^{-\sqrt{k^2+u^2}z}e^{ik(x-x')}e^{iu(y-y')}\sinh\left(\sqrt{k^2+u^2}z'\right)dkdu$$

(b) Let $k = r\sin\theta,$ $u = r\cos\theta,$ $Z = z_> - z_<,$ $Z' = z + z',$ $X = x - x',$ $Y = y - y'.$ Then

$$G = \frac{1}{2\pi}\int_0^\infty\int_0^{2\pi}\frac{1}{r}e^{irX\sin\theta + irY\cos\theta}\frac{\left(e^{-rZ} - e^{-rZ'}\right)}{2}rdrd\theta$$

Each integral is of the form:

$$\begin{aligned} I &= \frac{1}{2\pi}\int_0^\infty\int_0^{2\pi}\frac{1}{r}e^{-rZ+irX\sin\theta+irY\cos\theta}rdrd\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi}\frac{e^{-rZ+irX\sin\theta+irY\cos\theta}}{-Z+iX\sin\theta+iY\cos\theta}\Bigg|_0^\infty d\theta \\ &= \frac{1}{2\pi}\int_0^{2\pi}\frac{1}{Z - iX\sin\theta - iY\cos\theta}d\theta \\ &= \frac{1}{2\pi}\oint\frac{1}{Z - \frac{iX}{2i}\left(w - \frac{1}{w}\right) - \frac{iY}{2}\left(w + \frac{1}{w}\right)}\frac{dw}{iw} \end{aligned}$$

where $w = e^{i\theta}$ on the unit circle and $dw = ie^{i\theta}d\theta = iwd\theta.$

$$\begin{aligned} G &= \frac{1}{2\pi i}\oint\frac{1}{Zw - \frac{iX}{2i}\left(w^2 - 1\right) - \frac{iY}{2}\left(w^2 + 1\right)}dw \\ &= \frac{1}{2\pi i}\oint\frac{1}{-\frac{1}{2}(X+iY)w^2 + Zw + \frac{1}{2}(X - iY)}dw \end{aligned}$$

The poles are at

$$w = \frac{-Z \pm\sqrt{Z^2 + 4\frac{1}{4}\left(X^2+Y^2\right)}}{-(X+iY)} = \frac{Z\pm\sqrt{Z^2+X^2+Y^2}}{(X+iY)}$$

and

$$|w| = \frac{\pm Z + \sqrt{Z^2+X^2+Y^2}}{\sqrt{X^2+Y^2}}$$

and

$$|w| < 1 \text{ if } \pm Z + \sqrt{Z^2+X^2+Y^2} < \sqrt{X^2+Y^2}$$

Only the minus sign is possible, in which case

$$Z^2 + X^2 + Y^2 < X^2 + Y^2 + Z^2 + 2Z\sqrt{X^2+Y^2}$$

which is always true for positive $Z.$ Then

$$I = \frac{1}{-\frac{1}{2}(X+iY)}\frac{1}{2\frac{\sqrt{Z^2+X^2+Y^2}}{(X+iY)}} = \frac{1}{\sqrt{Z^2+X^2+Y^2}}$$

and

$$G = \frac{1}{\sqrt{Z^2+X^2+Y^2}} - \frac{1}{\sqrt{\left(Z'\right)^2+X^2+Y^2}}$$

the sum of the potentials due to the point charge and its image, as expected.

17. Find the Dirichlet Green's function for Poisson's equation in the interior of a sphere of radius a as a triple sum over appropriate eigenfunctions.

The eigenfunctions are the soloutions of the Helmholtz equation $\left(\nabla^2 + k^2\right)F = 0$ and thus are

$$F = j_l\left(x_{l,n}\frac{r}{a}\right)Y_{lm}(\theta,\phi)$$

(See Chapter 8 Section 8.5.). The corresponding eigenvalue is

$$k_{l,n,m} = \frac{x_{l,n}}{a}$$

where $x_{l,n}$ is the nth zero of $j_l.$ The Y_{lm} are already orthonormal, and the normlized j_l are (equation 8.130)

$$\sqrt{\frac{2}{a^3}}\frac{j_l(x_{l,n}\frac{r}{a})}{j_{l+1}(x_{l,n})}$$

Thus

$$G(\vec{x},\vec{x}') = 4\pi\sum_{n=1}^{\infty}\sum_{l=0}^{\infty}\sum_{m=-l}^{+l}\frac{2}{a^3}\frac{a^2}{x_{l,n}^2}\frac{j_l(x_{l,n}\frac{r}{a})j_l\left(x_{l,n}\frac{r'}{a}\right)}{[j_{l+1}(x_{l,n})]^2}Y_{lm}(\theta,\phi)Y_{lm}^*(\theta',\phi')$$
$$= \frac{8\pi}{a}\sum_{n=1}^{\infty}\sum_{l=0}^{\infty}\sum_{m=-l}^{+l}\frac{j_l(x_{l,n}\frac{r}{a})j_l\left(x_{l,n}\frac{r'}{a}\right)}{x_{l,n}^2[j_{l+1}(x_{l,n})]^2}Y_{lm}(\theta,\phi)Y_{lm}^*(\theta',\phi')$$

18. Obtain a relation analogous to equation (C.37) for the diffusion equation

$$\frac{\partial f}{\partial t} - D\nabla^2 f = S(x,t) \quad \text{eqn 1 Pr 18}$$

Define the Green's function through the equation

$$\frac{\partial G}{\partial t} - D\nabla^2 G = D\delta(\vec{x}-\vec{x}')\delta(t-t') \quad \text{eqn 2 Pr 18}$$

Apply your result to the example in Section C.5. Apply the sine transform in space to the equation

$$\frac{\partial G}{\partial t} - D\frac{\partial^2 G}{\partial x^2} = D\delta(x-x')\delta(t-t')$$

and obtain the Green's function. Show that the solution (C.34) may be expressed in terms of $\partial G/\partial x'$ and that this solution is consistent with the general result you found above. *Hint:* Note that since $G(t,t') = G(t-t')$, $\frac{\partial G}{\partial t'} = -\frac{\partial G}{\partial t}$.

We follow the method in section C.6, but here we integrate over both space and time to obtain:

$$D\int_0^\infty dt'\int_V\left(\Phi\nabla'^2\Psi - \Psi\nabla'^2\Phi\right)dV' = D\int_0^\infty dt'\int_S\left(\Phi\vec{\nabla}'\Psi - \Psi\vec{\nabla}'\Phi\right)\cdot\hat{\mathbf{n}}dS'$$

Compare with equation C.35. Then letting $\Phi = f$, $\Psi = G$, and using equations 1 Pr 18 and 2 Pr 18, we have

$$\int_0^\infty dt'\int_V\left(f\left[-\frac{\partial G}{\partial t'} - D\delta(\vec{x}-\vec{x}')\delta(t-t')\right] - G\left[\frac{\partial f(\vec{x}',t')}{\partial t'} - S\right]\right)dV' = D\int_0^\infty dt'\int_S\left(f\vec{\nabla}'G\right)\cdot\hat{\mathbf{n}}dS'$$

where we used the fact that $G = 0$ on S. Then performing the integrations on the left:

$$-Df(\vec{x},\vec{t}) - \int_0^\infty\int GSdt'dV' - \int dt'\int_V\frac{\partial}{\partial t'}(fG)dV'$$
$$= -Df(\vec{x},\vec{t}) - \int_0^\infty\int GSdt'dV' - \int_V\left(fG(\vec{x},\vec{x}',t,t')\right)\Big|_{t'=0}^{\infty}dV'$$

Then since $G(t,t')$ is zero for $t < t'$, this reduces to:

$$-Df(\vec{x},\vec{t}) - \int_0^\infty\int GSdt'dV' - \int_V(fG)|_0 dV'$$

Thus combining with the right hand side, we have

$$f(\vec{x},\vec{t}) = \int_0^\infty\int GS(\vec{x}',t')dt'dV' + \int_V\frac{1}{D}(fG)|_{t'=0}dV' - \int dt'\int_S\left(f\vec{\nabla}'G\right)\cdot\hat{\mathbf{n}}dS'$$

which gives the solution for f in terms of the source S, the initial conditions throughout the volume and the boundary condition throughout time.

Now we turn to the problem in the text. Using the sine transform as in the text, we get

$$\frac{\partial G}{\partial t} + k^2DG = D\sqrt{\frac{2}{\pi}}\sin kx'\,\delta(t-t')$$

where $x' > 0$. The solution is (c.f. equation C.32):

$$\tilde{G}(k,x',t,t') = \begin{cases} 0 & if\ \ t < t' \\ D\sqrt{\frac{2}{\pi}}\sin kx'\exp\left(-k^2D(t-t')\right) & if\ \ t > t' \end{cases}$$

and transforming back, we have

$$G(x,x',t,t') = \frac{D2}{\pi}\int_0^\infty\sin kx'\sin kx\exp\left(-k^2D(t-t')\right)dk \text{ if } t > t'$$
$$= \frac{D}{\pi}\int_0^\infty[\cos k(x'-x) - \cos k(x'+x)]\exp\left(-k^2D(t-t')\right)dk$$
$$= \frac{D}{2\pi}\int_{-\infty}^\infty\left[e^{ik(x'-x)} - e^{ik(x'+x)}\right]\exp\left(-k^2D(t-t')\right)dk$$

and zero otherwise. Now complete the square. If $X = x \pm x'$, then

$$-k^2 D(t-t') + ikX = -\left(k\sqrt{D(t-t')} - \frac{iX}{2\sqrt{D(t-t')}}\right)^2 - \frac{1}{4}\frac{X^2}{D(t-t')}$$

Thus

$$G(x,x',t,t') = \frac{D}{2\pi}\left[\exp\left(-\frac{1}{4}\frac{(x'-x)^2}{D(t-t')}\right) - \exp\left(-\frac{1}{4}\frac{(x'+x)^2}{D(t-t')}\right)\right]\frac{\sqrt{\pi}}{\sqrt{D(t-t')}}$$

Then

$$\left.\frac{\partial G}{\partial x'}\right|_{x'=0} = \frac{D}{2\sqrt{\pi D(t-t')}}\frac{x}{D(t-t')}\exp\left(-\frac{1}{4}\frac{x^2}{D(t-t')}\right)$$

and so the solution (C.33) is:

$$T(x,t) = \int_0^t \tau(t')\left.\frac{\partial G}{\partial x'}\right|_{x'=0} dt'$$

Note that the outward normal is in the negative x' direction so this is consistent with our general result with $S = 0$.

19. Find the Dirichlet Green's function for Poisson's equation in the interior of a hemisphere of radius a.

(a) Choose $0 \le \phi \le \pi$

In this case we choose the polar axis as well as the x −axis along the flat side. Then the plane is at $\phi = 0$ and $\phi = \pi$ and so the function of ϕ that we need is $\sin m\phi$. The Green's function is of the form

$$G(\vec{x},\vec{x}') = \sum_{l,m} A_{lm} P_l^m(\mu) P_l^m(\mu') \sin m\phi \sin m\phi' r_<^l \left(\frac{a^{2l+1}}{r_>^{l+1}} - r_>^l\right)$$

Note that the $\sin m\phi$ are orthogonal on $[0,\pi]$. For then:

$$\int_0^\pi \sin m\phi \sin n\phi d\phi = \frac{\pi}{2}\delta_{mn}$$

Write

$$G(\vec{x},\vec{x}') = \sum_{l,m} A_{lm} P_l^m(\mu) P_l^m(\mu') \sin m\phi \sin m\phi' g_l(r,r')$$

We insert this expression into the defining equation, multiply both sides by

$P_{l'}^{m'}(\mu)\sin m'\phi,$ and integrate over the angles. The range of integration for ϕ is 0 to π. On the left hand side, only the terms with $l = l'$ and $m = m'$ survive the integrations, and we have

$$\frac{A_{l'm'}}{2l'+1}\frac{(l'+m')!}{(l'-m')!}\pi P_{l'}^{m'}(\mu')\sin m'\phi'\left[\frac{1}{r^2}\frac{\partial}{\partial r}(r^2 g_{l'}') - l'(l'+1)g_{l'}\right] = -4\pi\frac{\delta(r-r')}{r^2}P_{l'}^{m'}(\mu')\sin m'\phi'$$

As usual, we multiply by r^2 and integrate across the boundary at $r = r'$. Dropping the primes on l and m :

$$\frac{A_{lm}}{2l+1}\frac{(l+m)!}{(l-m)!}\pi\int_{r'-\varepsilon}^{r'+\varepsilon}\left[\frac{\partial}{\partial r}(r^2 g_l') - l(l+1)g_l r^2 = -4\pi\delta(r-r')\right]dr$$

$$A_{lm}\frac{1}{2l+1}\frac{(l+m)!}{(l-m)!}\pi\left(r^2 g_l'\right)\Big|_{r'-\varepsilon}^{r'+\varepsilon} = -4\pi$$

$$\left[-(l+1)a^{2l+1} - l(r')^{2l+1} - l(r')^{l+1}\left(\frac{a^{2l+1}}{(r')^{l+1}} - (r')^l\right)\right]\frac{A_{lm}}{2l+1}\frac{(l+m)!}{(l-m)!}\pi = -4\pi$$

$$-a^{2l+1}(2l+1)\frac{A_{lm}}{2l+1}\frac{(l+m)!}{(l-m)!}\pi = -4\pi$$

$$A_{lm} = \frac{(l-m)!}{(l+m)!}\frac{4}{a^{2l+1}}$$

Thus

$$G(\vec{x},\vec{x}') = \sum_{l,m} \frac{(l-m)!}{(l+m)!} \frac{4}{a^{2l+1}} P_l^m(\mu) P_l^m(\mu') \sin m\phi \sin m\phi' r_<^l \left(\frac{a^{2l+1}}{r_>^{l+1}} - r_>^l \right)$$
$$= 4 \sum_{l,m} \frac{(l-m)!}{(l+m)!} P_l^m(\mu) P_l^m(\mu') \sin m\phi \sin m\phi' r_<^l r_>^l \left(\frac{1}{r_>^{2l+1}} - \frac{1}{a^{2l+1}} \right)$$

(b) Choose $0 \le \theta \le \pi/2$

This time we choose polar axis perpendicular to the plane. Then the plane is at $\mu = 0$ and so we need the odd P_l^ms. (i.e. $l+m$ is odd). The Green's function is of the form

$$G(\vec{x},\vec{x}') = \sum_{l,m} A_{lm} P_l^m(\mu) P_l^m(\mu') e^{im(\phi-\phi')} r_<^l \left(\frac{a^{2l+1}}{r_>^{l+1}} - r_>^l \right) = \sum_{l,m} A_{lm} P_l^m(\mu) P_l^m(\mu') e^{im(\phi-\phi')} g_l(r,r')$$

Note that the P_l^m are orthogonal on $[0,1]$ if $l+m$ is odd. For then:

$$\int_{-1}^0 P_l^m(\mu) P_{l'}^m(\mu) d\mu = \int_1^0 P_l^m(-x) P_{l'}^m(-x)(-dx)$$
$$= (-1)^{l+m}(-1)^{l'+m} \int_0^1 P_l^m(x) P_{l'}^m(x) dx$$

and thus

$$\int_{-1}^1 P_l^m(\mu) P_{l'}^m(\mu) d\mu = \frac{(l+m)!}{(l-m)!} \frac{2}{2l+1} \delta_{ll'} = 2 \int_0^1 P_l^m(\mu) P_{l'}^m(\mu) d\mu$$

Stuff into the differential equation, multiply by

$P_{l'}^{m'}(\mu) e^{-im'\phi}$ and integrate over the angles. The range of integration for μ is 0 to 1.

$$\left[\frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 g' \right) - l(l+1) g \right] A_{lm} P_l^m(\mu') e^{-im\phi'} \frac{(l+m)!}{(l-m)!} \frac{1}{2l+1} 2\pi = -4\pi \frac{\delta(r-r')}{r^2} P_l^m(\mu') e^{-im\phi'}$$

Again we multiply by r^2 and integrate across the boundary at $r = r'$

$$A_{lm} \frac{(l+m)!}{(l-m)!} \frac{2\pi}{2l+1} \int_{r'-\varepsilon}^{r'+\varepsilon} \left[\frac{\partial}{\partial r} \left(r^2 g' \right) - l(l+1) g r^2 \right] dr = -4\pi \int_{r'-\varepsilon}^{r'+\varepsilon} \delta(r-r') dr$$

$$A_{lm} \frac{(l+m)!}{(l-m)!} \frac{2\pi}{2l+1} \left(r^2 g' \right) \Big|_{r'-\varepsilon}^{r'+\varepsilon} = -4\pi$$

$$A_{lm} \frac{(l+m)!}{(l-m)!} \frac{2\pi}{2l+1} \left[-(l+1) a^{2l+1} - l(r')^{2l+1} - l(r')^{l+1} \left(\frac{a^{2l+1}}{(r')^{l+1}} - (r')^l \right) \right] = -4\pi$$

$$-a^{2l+1}(2l+1) A_{lm} \frac{2\pi}{2l+1} \frac{(l+m)!}{(l-m)!} = -4\pi$$

$$A_{lm} = \frac{(l-m)!}{(l+m)!} \frac{2}{a^{2l+1}}$$

Thus

$$G(\vec{x},\vec{x}') = \sum_{l=0}^{\infty} \sum_{m=0,\ l+m \text{ odd}}^{\infty} \frac{(l-m)!}{(l+m)!} \frac{2}{a^{2l+1}} P_l^m(\mu) P_l^m(\mu') e^{im(\phi-\phi')} r_<^l \left(\frac{a^{2l+1}}{r_>^{l+1}} - r_>^l \right)$$

(c) Using one of the two Green's functions, (a) or (b), evaluate the potential inside the hemisphere if $\Phi = 0$ on the spherical surface and $\Phi(r) = V_0(1 - r/a)$ on the flat face.

Using the result of (b), the potential is

$$\Phi(\vec{x}) = -\frac{1}{4\pi} \int \Phi(\vec{x}') \frac{\partial G}{\partial n'} da'$$

where $\hat{n}' = \hat{\theta}'$ and thus

$$\frac{\partial G}{\partial n'} = \hat{\theta}' \cdot \vec{\nabla} G = \frac{1}{r'}\frac{\partial G}{\partial \theta'} = -\frac{\sin\theta'}{r'}\frac{\partial G}{\partial \mu'}\bigg|_{\theta'=\pi/2}$$
$$= -\frac{1}{r'}\sum_{l,m}\frac{(l-m)!}{(l+m)!}\frac{2}{a^{2l+1}}P_l^m(\mu)\frac{dP_l^m(\mu')}{d\mu'}\bigg|_{\mu'=0} e^{im(\phi-\phi')} r_<^l\left(\frac{a^{2l+1}}{r_>^{l+1}} - r_>^l\right)$$

$$\Phi(\vec{x}) = \frac{V_0}{4\pi}\int_0^a\int_0^{2\pi}\frac{a-r'}{a}\frac{1}{r'}\sum_{l,m}\frac{(l-m)!}{(l+m)!}\frac{2}{a^{2l+1}}P_l^m(\mu)\frac{dP_l^m(\mu')}{d\mu'}\bigg|_{\mu'=0} e^{im(\phi-\phi')}r_<^l\left(\frac{a^{2l+1}}{r_>^{l+1}} - r_>^l\right)r'dr'd\phi'$$

The integral over ϕ' gives zero unless $m = 0,$ so

$$\Phi(\vec{x}) = \frac{V_0}{2\pi}2\pi\int_0^a\frac{a-r'}{a}\frac{1}{r'}\sum_{l\text{ odd}}\frac{1}{a^{2l+1}}P_l(\mu)\frac{dP_l(\mu')}{d\mu'}\bigg|_{\mu'=0} r_<^l\left(\frac{a^{2l+1}}{r_>^{l+1}} - r_>^l\right)r'dr'$$

Using the recursion relations for the P_l

$$P_l'(\mu) = (P_{l-1}(\mu) - \mu P_l(\mu))\frac{l}{1-\mu^2}$$

so at $\mu = 0$

$$P_l'(0) = lP_{l-1}(0)$$

which is valid for $l \geq 1.$ ($P_0' \equiv 0.$)

Now we have

$$\Phi(\vec{x}) = V_0\int_0^a\frac{a-r'}{a}\sum_{l=1,\text{ odd}}^{\infty}\frac{l}{a^{2l+1}}P_l(\mu)P_{l-1}(0)r_<^l\left(\frac{a^{2l+1}}{r_>^{l+1}} - r_>^l\right)dr'$$

Now let's do the integral over $r'.$ The first term must be split into two parts. For $l > 1$ we have:

$$\int_0^a\frac{a-r'}{a}r_<^l\left(\frac{a^{2l+1}}{r_>^{l+1}} - r_>^l\right)dr' = -r^l\int_0^a\frac{a-r'}{a}(r')^l dr' + \frac{a^{2l+1}}{r^{l+1}}\int_0^r\frac{a-r'}{a}r'^l dr' + r^l\int_r^a\frac{a-r'}{a}\frac{a^{2l+1}}{(r')^{l+1}}dr'$$
$$= -r^l\left(\frac{a^{l+1}}{l+1} - \frac{a^{l+1}}{l+2}\right) + \frac{a^{2l}}{r^{l+1}}\left(\frac{ar^{l+1}}{l+1} - \frac{r^{l+2}}{l+2}\right) + \frac{r^l a^{2l+1}}{-l}\left(\frac{1}{a^l} - \frac{1}{r^l}\right)$$
$$+ \frac{r^l a^{2l}}{l-1}\left(\frac{1}{a^{l-1}} - \frac{1}{r^{l-1}}\right)$$
$$= -r^l a^{l+1}\left[\frac{1}{(l+1)(l+2)} + \frac{1}{l} - \frac{1}{l-1}\right] + a^{2l+1}\left(\frac{2l+1}{(l+1)l} - \frac{r}{a}\left(\frac{2l+1}{(l+2)(l-1)}\right)\right)$$
$$= -r^l a^{l+1}\left[-2\frac{2l+1}{(l+1)(l+2)l(l-1)}\right] + a^{2l+1}\left(\frac{2l+1}{(l+1)l}\right) - ra^{2l}\frac{2l+1}{(l+2)(l-1)}$$
$$= \frac{(2l+1)}{l}a^{2l+1}\left\{\frac{r^l}{a^l}\frac{2}{(l+1)(l+2)(l-1)} + \frac{1}{(l+1)} - \frac{lr/a}{(l+2)(l-1)}\right\}$$

For $l = 1$

$$\int_0^a\frac{a-r'}{a}r_<\left(\frac{a^3}{r_>^2} - r_>\right)dr' = -r\int_0^a\frac{a-r'}{a}r'dr' + \frac{a^3}{r^2}\int_0^r\frac{a-r'}{a}r'dr' + r\int_r^a\frac{a-r'}{a}\frac{a^3}{(r')^2}dr'$$
$$= -r\left(\frac{a^2}{2} - \frac{a^2}{3}\right) + \frac{a^3}{r^2}\left(\frac{r^2}{2} - \frac{r^3}{3a}\right) + a^3r\left(-\frac{1}{a} + \frac{1}{r} - \frac{1}{a}\ln\left(\frac{a}{r}\right)\right)$$
$$= -r\frac{1}{6}a^2 + \frac{1}{2}a^3 - \frac{1}{3}a^2r + a^3\left(-\frac{r}{a} + 1 - \frac{r}{a}\ln\left(\frac{a}{r}\right)\right)$$
$$= a^3\left[\frac{3}{2}\left(1 - \frac{r}{a}\right) - \frac{r}{a}\ln\left(\frac{a}{r}\right)\right]$$

Thus

$$\frac{\Phi(\vec{x})}{V_0} = \cos\theta\left(\frac{3}{2}\left(1 - \frac{r}{a}\right) - \frac{r}{a}\ln\left(\frac{a}{r}\right)\right)$$
$$+ \sum_{l=3,\text{odd}}^{\infty}P_l(\mu)P_{l-1}(0)(2l+1)\left\{\frac{r^l}{a^l}\frac{2}{(l+1)(l+2)(l-1)} + \frac{1}{(l+1)} - \frac{lr/a}{(l+2)(l-1)}\right\}$$

The first few terms are:

$$\frac{\Phi(\vec{x})}{V_0} = \cos\theta\left(\frac{3}{2}\left(1 - \frac{r}{a}\right) - \frac{r}{a}\ln\left(\frac{a}{r}\right)\right) + \frac{7}{2}\left(-\frac{1}{2}\right)\cos\theta(5\cos^2\theta - 3)\left(\frac{1}{4} - \frac{3}{10}\frac{r}{a} + \frac{1}{20}\frac{r^3}{a^3}\right) + \cdots$$
$$= \cos\theta\left(\frac{3}{2}\left(1 - \frac{r}{a}\right) - \frac{r}{a}\ln\left(\frac{a}{r}\right)\right) - \frac{7}{4}\cos\theta(5\cos^2\theta - 3)\left(\frac{1}{4} - \frac{3}{10}\frac{r}{a} + \frac{1}{20}\frac{r^3}{a^3}\right) + \cdots$$

Using the result of (a), the solution goes as follows.

Then

$$\Phi(\vec{x}) = -\frac{1}{4\pi}\int \Phi(\vec{x}')\frac{\partial G}{\partial n'}dA'$$

The integral reduces to an integral over the flat side, with normal $\hat{n} = -\hat{\phi}$ at $\phi = 0$ and $\hat{n} = \hat{\phi}$ at $\phi = \pi$. Thus

$$\begin{aligned}
\frac{\partial G}{\partial n'} &= \hat{n}'\cdot\vec{\nabla}\Phi \\
-\frac{1}{r'\sin\theta'}\frac{\partial G}{\partial \phi'} &= \frac{-1}{r'\sin\theta'}\sum_{l,m}4m\frac{(l-m)!}{(l+m)!}\frac{r_<^l}{a^{2l+1}}\left(r_>^l - \frac{a^{2l+1}}{r_>^{l+1}}\right)P_l^m(\mu)\sin m\phi P_l^m(\mu')\cos m\phi' \\
&= \frac{4}{r'\sin\theta'}\sum_{l,m}m\frac{(l-m)!}{(l+m)!}\frac{r_<^l}{a^{2l+1}}\left(\frac{a^{2l+1}}{r_>^{l+1}} - r_>^l\right)P_l^m(\mu)\sin m\phi P_l^m(\mu') \text{ at } \phi' = 0
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial G}{\partial n'} &= \frac{1}{r'\sin\theta'}\frac{\partial G}{\partial \phi'} \\
&= \frac{4}{r'\sin\theta'}\sum_{l,m=1}^{\infty}m\frac{(l-m)!}{(l+m)!}\frac{r_<^l}{a^{2l+1}}\left(r_>^l - \frac{a^{2l+1}}{r_>^{l+1}}\right)P_l^m(\mu)\sin m\phi P_l^m(\mu')\cos m\phi' \\
&= \frac{4}{r'\sin\theta'}\sum_{l,m}\frac{(l-m)!}{(l+m)!}(-1)^{m+1}m\frac{r_<^l}{a^{2l+1}}\left(\frac{a^{2l+1}}{r_>^{l+1}} - r_>^l\right)P_l^m(\mu)\sin m\phi P_l^m(\mu') \text{ at } \phi' = \pi
\end{aligned}$$

Then the potential is:

$$\Phi(\vec{x}) = -\frac{V_0}{\pi}\sum_{l,m}P_l^m(\mu)\sin m\phi\int_0^a\int_0^\pi \frac{a-r'}{r'a}m\frac{(l-m)!}{(l+m)!}\frac{(1-(-1)^m)}{\sin\theta'}\frac{r_<^l}{a^{2l+1}}\left(\frac{a^{2l+1}}{r_>^{l+1}} - r_>^l\right)P_l^m(\mu')r'dr'd\theta'$$

Only m odd survives.

$$\Phi(\vec{x}) = -\frac{2V_0}{\pi}\sum_{l,m=1,\, m\text{ odd}}^{\infty}m\frac{(l-m)!}{(l+m)!}P_l^m(\mu)\sin m\phi\int_0^a\frac{a-r'}{a}\frac{r_<^l}{a^{2l+1}}\left(\frac{a^{2l+1}}{r_>^{l+1}} - r_>^l\right)dr'\int_0^\pi P_l^m(\mu')\frac{d\theta'}{\sin\theta'}$$

The integration over r is independent of m, and gives the same result that we have already obtained.

Integrating over θ', we write

$$I_{lm} = \int_0^\pi \frac{P_l^m(\cos\theta')}{\sin\theta'}d\theta'$$

For $l = 1$:

$$I_{11} = \int_0^\pi -\frac{\sin\theta}{\sin\theta}d\theta = -\theta|_0^\pi = -\pi$$

For $l = 2$ the only permissable value of m is $m = 1$.

$$I_{21} = \int_0^\pi -3\cos\theta d\theta = -\frac{3}{2}(-\sin\theta)|_0^\pi = 0$$

For $l = 3$ we have $m = 1$ and $m = 3$.

$$I_{31} = \int_0^\pi -\frac{3}{2}\left(5\cos^2\theta - 1\right)d\theta = -\frac{15}{4}\pi + \frac{3}{2}\pi = -\frac{9}{4}\pi$$

and

$$I_{33} = \int_0^\pi -15\sin^2\theta d\theta = -\frac{15\pi}{2}$$

Thus

$$\Phi(\vec{x}) = 2\frac{V_0}{\pi}\left\{\begin{array}{c} -\pi\left(\frac{3}{2}\frac{r}{a} - \frac{3}{2} + \frac{r}{a}\ln\frac{a}{r}\right)\frac{1}{2!}P_1^1(\mu)\sin\phi + \\ \sum_{l=3}^{\infty}\sum_{m\text{ odd}=1}^{l} m\frac{(l-m)!}{(l+m)!}P_l^m(\mu)\sin m\phi I_{lm}\frac{(2l+1)}{l}\left\{\frac{r^l}{a^l}\frac{2}{(l+1)(l+2)(l-1)} + \frac{1}{(l+1)} - \frac{lr/a}{(l+2)(l-1)}\right\}\end{array}\right\}$$

The first few terms are:

$$\begin{aligned}\frac{\Phi(\vec{x})}{V_0} &= \sin\theta\sin\phi\left(\frac{3}{2}\left(1-\frac{r}{a}\right)-\frac{r}{a}\ln\frac{a}{r}\right)+ \\ &\frac{2}{\pi}\frac{7}{3}\left[-\frac{r^3}{a^3}\frac{1}{20}-\frac{1}{4}+\frac{r}{a}\frac{3}{10}\right]\left[-\frac{9\pi}{4}\frac{2!}{4!}P_3^1(\mu)\sin\phi-3\frac{15\pi}{2}\frac{1}{6!}P_3^3(\mu)\sin 3\phi\right] \\ &= \sin\theta\sin\phi\left(\frac{3}{2}\left(1-\frac{r}{a}\right)-\frac{r}{a}\ln\frac{a}{r}\right)+ \\ &7\left[\frac{r^3}{a^3}\frac{1}{20}+\frac{1}{4}-\frac{r}{a}\frac{3}{10}\right]\left[-\frac{1}{8}\left(-\frac{3}{2}\sin\theta\left(5\cos^2\theta-1\right)\right)\sin\phi-\frac{1}{6\times 4\times 2}\left(-15\sin^3\theta\right)\sin 3\phi\right] \\ &= \sin\theta\sin\phi\left(\frac{3}{2}\left(1-\frac{r}{a}\right)-\frac{r}{a}\ln\frac{a}{r}\right) \\ &+7\left[\frac{r^3}{a^3}\frac{1}{20}+\frac{1}{4}-\frac{r}{a}\frac{3}{10}\right]\left[\frac{3}{16}\sin\theta\left(5\cos^2\theta-1\right)\sin\phi+\frac{5}{16}\sin^3\theta\sin 3\phi\right]+\cdots\end{aligned}$$

To compare with our previous result,

$$\frac{\Phi(\vec{x})}{V_0} = \cos\theta\left(-\frac{3}{2}\frac{r}{a}+\frac{3}{2}-\frac{r}{a}\ln\left(\frac{a}{r}\right)\right)+\frac{7}{4}\cos\theta\left(5\cos^2\theta-3\right)\left(\frac{1}{4}-\frac{3}{10}\frac{r}{a}-\frac{1}{20}\frac{r^3}{a^3}\right)+\cdots$$

we use the addition theorem. (See the discussion following Example 8.3.)

$$\begin{aligned}\Phi(\vec{x}) &= V_0\left\{\begin{array}{c}\sin\theta\sin\phi\left(\frac{3}{2}(1-\frac{r}{a})-\frac{r}{a}\ln(\frac{a}{r})\right)- \\ \frac{7}{2}\left(-\frac{1}{8}\left[3(5\cos^2\theta-1)\sin\theta\sin\phi+5\sin^3\theta\sin 3\phi\right]\right)\left(\frac{1}{4}-\frac{3}{10}\frac{r}{a}-\frac{1}{20}\frac{r^3}{a^3}\right)+\cdots\end{array}\right\} \\ &= V_0\left\{\begin{array}{c}\sin\theta\sin\phi\left(\frac{3}{2}(1-\frac{r}{a})-\frac{r}{a}\ln(\frac{a}{r})\right)+ \\ \frac{7}{16}\left[3(5\cos^2\theta-1)\sin\theta\sin\phi+5\sin^3\theta\sin 3\phi\right]\left(\frac{1}{4}-\frac{3}{10}\frac{r}{a}-\frac{1}{20}\frac{r^3}{a^3}\right)+\cdots\end{array}\right\}\end{aligned}$$

The results are the same.

20. Obtain the Green's function inside a cylindrical tube (Section C.7.4) by dividing space in ρ.

This time we have

$$\begin{aligned}G_{\mathrm{I}} &= \int_0^\infty dk\sum_{-\infty}^{\infty}A_m(k)e^{im(\phi-\phi')}e^{ik(z-z')}I_m(k\rho) \\ G_{\mathrm{II}} &= \int_0^\infty dk\sum_{-\infty}^{\infty}B_m(k)e^{im(\phi-\phi')}e^{ik(z-z')}\left(I_m(ka)K_m(k\rho)-K_m(ka)I_m(k\rho)\right)\end{aligned}$$

At $\rho = \rho'$

$$A_m I_m\left(k\rho'\right) = B_m\left(I_m(ka)K_m\left(k\rho'\right)-K_m(ka)I_m\left(k\rho'\right)\right)$$

$$B_m = A_m\frac{I_m\left(k\rho'\right)}{\left(I_m(ka)K_m\left(k\rho'\right)-K_m(ka)I_m\left(k\rho'\right)\right)}$$

Let

$$\alpha_m = \frac{A_m}{\left(I_m(ka)K_m\left(k\rho'\right)-K_m(ka)I_m\left(k\rho'\right)\right)}$$

Then

$$A_m = \alpha_m\left(I_m(ka)K_m\left(k\rho'\right)-K_m(ka)I_m\left(k\rho'\right)\right)$$

and

$$B_m = \alpha_m I_m\left(k\rho'\right)$$

and

$$\begin{aligned}G &= \int_0^\infty dk\sum_{-\infty}^{\infty}\alpha_m(k)e^{im(\phi-\phi')}e^{ik(z-z')}I_m(k\rho_<)[I_m(ka)K_m(k\rho_>)-I_m(k\rho_>)K_m(ka)] \\ &= \int_0^\infty dk\sum_{-\infty}^{\infty}\alpha_m(k)e^{im(\phi-\phi')}e^{ik(z-z')}g_m(k\rho)\end{aligned}$$

Integrating across the boundary:

$$\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial G}{\partial\rho}\right)-\frac{m^2}{\rho^2}G-k^2G = -\frac{4\pi}{\rho}\delta\left(\rho-\rho'\right)\delta\left(\phi-\phi'\right)\delta\left(z-z'\right)$$

Multiply by the eigenfunctions $e^{-im'\phi}e^{-ik'z}$ and integrate.

$$\alpha_m(k)\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial g_m}{\partial\rho}\right)-\frac{m^2}{\rho^2}g_m-k^2g_m\right]\pi(2\pi)=-\frac{4\pi}{\rho}\delta(\rho-\rho')$$

Mulyiply by ρ and integrate across the boundary

$$\alpha_m(k)\left(\rho\frac{\partial g_m}{\partial\rho}\right)\Big|_{\rho'-}^{\rho'+}=-\frac{2}{\pi}$$

Now stuff in:

$$\alpha_m(k)k\rho'\left[\begin{array}{c} I_m(k\rho')(I_m(ka)K_m'(k\rho')-I_m'(k\rho')K_m(ka)) \\ -I_m'(k\rho')(I_m(ka)K_m(k\rho')-I_m(k\rho')K_m(ka)) \end{array}\right]=-\frac{2}{\pi}$$

$$\alpha_m(k)k\rho'[I_m(k\rho')I_m(ka)K_m'(k\rho')-I_m'(k\rho')I_m(ka)K_m(k\rho')]=-\frac{2}{\pi}$$

$$\alpha_m(k)k\rho' I_m(ka)[I_m(k\rho')K_m'(k\rho')-I_m'(k\rho')K_m(k\rho')]=-\frac{2}{\pi}$$

The term in square brackets is the Wronskian:

$$W=W_0\exp\left[-\int_{x_0}^{x}\frac{1}{x}dx\right]=W_0\exp[-\ln x/x_0]=W_0\frac{x_0}{x}=\pm\frac{1}{x}$$

We can use the asymptotic form of the functions to determine the sign. $K_0'=-K_1$ and $I_0'=I_1$, we have

$$I_0(k\rho')K_0'(k\rho')-I_0'(k\rho')K_0(k\rho')=-I_0K_1-I_1K_0=-\frac{1}{k\rho}$$

and so the sign is negative.

$$\alpha_m(k)k\rho' I_m(ka)\frac{-1}{k\rho'}=-\frac{2}{\pi}$$

$$\alpha_m(k)=\frac{2}{\pi I_m(ka)}$$

and thus

$$G=\frac{2}{\pi}\int_0^\infty dk\sum_{-\infty}^{\infty}e^{im(\phi-\phi')}e^{ik(z-z')}\frac{I_m(k\rho_<)}{I_m(ka)}[I_m(ka)K_m(k\rho_>)-I_m(k\rho_>)K_m(ka)]$$

I

Optional Topic D: Approximate evaluation of integrals

1. Use the method of steepest descent to evaluate the asymptotic form of the Gamma function

$$\Gamma(\xi) = \int_0^\infty t^{\xi-1} e^{-t} dt = \sqrt{2\pi}\, \xi^{\xi-1/2} e^{-\xi}$$

First let $t = \xi z$.

$$\Gamma(\xi) = \xi^{\xi-1} \int_C z^{\xi-1} e^{-\xi z} \xi dz$$

Next we write the integrand as an exponential in ξ:

$$\Gamma(z) = \xi^{\xi} \int_C \frac{1}{z} \exp(-\xi z + \xi \ln z) dz$$

This is of the general form (D.2) with $f(z) = -(z + \ln z)$ and $g(z) = 1/z$. Then

$$\frac{df}{dz} = -\left(1 - \frac{1}{z}\right) = 0 \Rightarrow z_0 = 1$$

and

$$\frac{d^2 f}{dz^2} = -\frac{1}{z^2} = -1 = 1e^{i\pi} \text{ at } z = z_0 = 1$$

Thus

$$\phi_0 = -\frac{\pi}{2} + \frac{\pi}{2} = 0$$

$$f(1) = -(1 + \ln 1) = -1$$

and

$$g(1) = 1$$

Then, applying the general result D.6, we have

$$\Gamma(\xi) = \xi^{\xi} g(z_0) \exp(\xi f(z_0)) e^{i\phi_0} \sqrt{\frac{2\pi}{\xi a}}$$

$$\begin{aligned}\Gamma(\xi) &= \xi^{\xi} e^{-\xi} \sqrt{\frac{2\pi}{\xi}} \\ &= \sqrt{2\pi}\, \xi^{\xi-1/2} e^{-\xi}\end{aligned}$$

This is Stirling's formula.

2. The modified Bessel function $K_\nu(\xi)$ has an integral representation

$$K_\nu(\xi) = \frac{1}{2} \int_0^\infty \exp\left\{-\frac{\xi}{2}\left(s + \frac{1}{s}\right)\right\} \frac{ds}{s^{1-\nu}}$$

Use the method of steepest descent to find the asymptotic form of $K_\nu(\xi)$ as $\xi \to \infty$.

The integral is of the standard form with

$$f(z) = -\frac{1}{2}\left(z + \frac{1}{z}\right) \Rightarrow f'(z) = -\frac{1}{2}\left(1 - \frac{1}{z^2}\right)$$

Thus $z_0 = \pm 1$. Then

$$f''(z_0) = \frac{1}{2}(-2)\frac{1}{z_0^3} = -1 = e^{i\pi}$$

where we chose to put the deformed contour C' through the point $z_0 = +1$, on the original contour. We also have

$$g(z_0) = \frac{1}{z_0^{\nu-1}} = 1$$

and

$$\phi_0 = -\frac{\pi}{2} + \frac{\pi}{2} = 0$$

Thus

$$K_\nu(\xi) = g(z_0)\exp(\xi f(z_0))e^{i\phi_0}\sqrt{\frac{2\pi}{\xi a}}$$
$$= \exp(-\xi)\sqrt{\frac{2\pi}{\xi}}$$

in agreement with Chapter 8 equation 8.105.

3. The Bessel function may be represented by an integral

$$J_\nu(\xi) = \frac{1}{2\pi}\int_C \exp(-i\xi\sin z + i\nu z)dz$$

where the contour C is shown in the figure. Use the method of steepest descent or stationary phase, as appropriate, to derive the asymptotic form (8.83)

For complex z the function $f(z) = -i\sin z$ is analytic and has both real and imaginary parts. So we use the method of steepest descent. First rewrite the integral in the standard form with $g(z) = \exp(i\nu z)$. Then

$$\frac{df}{dz} = -i\cos z = 0 \Rightarrow z_0 = \left(n + \frac{1}{2}\right)\pi$$

Our contour runs through two of these points, at $z_0 = \pm\frac{\pi}{2}$. Then

$$f(z_0) = -i\sin z_0 = \mp i$$

$$f''(z_0) = i\sin z_0 = \pm i = e^{\pm i\pi/2}$$

Thus

$$\phi_0 = \mp\frac{\pi}{4} + (\pm 1)\frac{\pi}{2}$$

If we deform the contour to go through the point $z_0 = -\frac{\pi}{2}$ along the line $\phi = \phi_0 = \pi/4$ (choosing $+\pi/2$ in the expression for ϕ_0) then $\mathrm{Re}(z - z_0)$ goes from negative to positive and $e^{i\pi/4} = \frac{1}{2}\sqrt{2} + \frac{1}{2}i\sqrt{2}$, so r goes from negative to positive. At $z_0 = +\pi/2$, we have to choose the $-$ sign, $e^{-i\pi/4} = \frac{1}{2}\sqrt{2} - \frac{1}{2}i\sqrt{2}$ and again r goes from negative to positive along the line $\phi = \phi_0 = -\pi/4$.

$$g(z_0) = \exp\left(\pm i\nu\frac{\pi}{2}\right)$$

The contribution from each saddle is

$$\exp\left(\pm i\nu\frac{\pi}{2}\right)\exp(\mp i\xi)\exp\left(\pm i\frac{\pi}{4}\right)\sqrt{\frac{2\pi}{\xi}}$$

and adding the two contributions, we get:

$$\begin{aligned} J_\nu(\xi) &\sim \frac{1}{2\pi}\sqrt{\frac{2\pi}{\xi}}\left\{\exp\left(i\nu\frac{\pi}{2} - i\xi + i\frac{\pi}{4}\right) + \exp\left(-i\nu\frac{\pi}{2} + i\xi - i\frac{\pi}{4}\right)\right\} \\ &= \sqrt{\frac{2}{\pi\xi}}\cos\left(\xi - \nu\frac{\pi}{2} - \frac{\pi}{4}\right) \end{aligned}$$

as required.

4. Expand the function $g(x)$ in equation (D.7) in a Taylor series about the stationary point, and show that there is no contribution to the integral from the second (linear) term in the series, if the expansion of the phase ϕ is truncated at the quadratic term, as in equation (D.8).

Equation (D.7) becomes:

$$\begin{aligned} I(x) &= \int_{-\infty}^{+\infty}\left[g(x_s) + (x - x_s)g'(x_s) + \cdots\right]\exp(i\phi(x))dx \\ &= \int_{-\infty}^{+\infty} g(x_s)\exp(i\phi(x))dx + \int_{-\infty}^{+\infty}(x - x_s)g'(x_s)\exp(i\phi(x))dx + \cdots \end{aligned}$$

The first term has already been evaluated in the text. Using the same methods, the second term is:

$$g'(x_s)e^{i\phi(x_s)}\int_{-\infty}^{+\infty}(x - x_s)\exp\left(\frac{i}{2}(x - x_s)^2\phi''(x_s)\right)dx$$

We may do the integral by the change of variable

$$u = (x - x_s)\sqrt{\frac{-i}{2}\phi''(x_s)} = A(x - x_s)e^{-i\pi/4}$$

to obtain

$$g'(x_s)e^{i\phi(x_s)}\frac{e^{i\pi/2}}{A^2}\int_{-\infty}^{+\infty} ue^{-u^2}du$$

The integral over u is zero since the integrand is an odd function. Thus the second term in the Taylor series does not contribute to the result.

5. The function $H_\nu^{(2)}(x)$ has the integral expression

$$H_\nu^{(2)}(\xi) = \frac{1}{\pi i}\int_C \exp\left(\frac{\xi}{2}\left(z - \frac{1}{z}\right)\right)\frac{dz}{z^{\nu+1}}$$

where the path of integration goes from $-\infty$ to zero along a path in the lower-half plane that is the mirror image of the path in Figure 1. Verify the asymptotic form 8.85 for this function.

The solution follows the method in the text, with the following changes. $z_0 = -i,$

$$f''(z_0) = -\frac{1}{z^3}\bigg|_{-i} = i = 1e^{i\pi/2}$$

So $a = 1$ and $\alpha = \pi/2.$ Thus the new path C' must pass through z_0 at an angle $\phi = -\frac{\pi}{4} + \frac{\pi}{2} = \frac{\pi}{4}.$. Before $z_0,$ the difference $z - z_0$ has a negative real part on this path and $e^{i\pi/4} = \frac{1}{2}\sqrt{2} + \frac{1}{2}i\sqrt{2},$ so r is negative, as required.

$$f(z_0) = f(-i) = \frac{1}{2}\left(-i - \frac{1}{-i}\right) = -i$$

and thus

$$H_\nu^{(2)}(\xi) = \frac{1}{\pi i}\frac{1}{(-i)^{\nu+1}}e^{-i\xi}e^{i\pi/4}\sqrt{\frac{2\pi}{\xi}} = \sqrt{\frac{2}{\pi\xi}}\exp\left[-i\left(\xi - \frac{\pi}{4} - \nu\frac{\pi}{2}\right)\right]$$

as required.

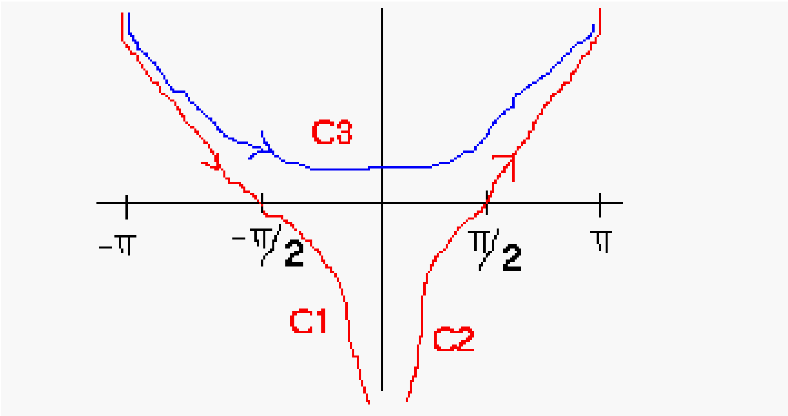
6. An alternative integral expression for the Bessel functions is:

$$F_\nu(x) = k\int_C \exp(i\nu u - ix\sin u)du$$

where

1. for $H^{(1)}$ use contour C_1 and $k = \frac{1}{\pi}$
2. for $H^{(2)}$ use contour C_2 and $k = \frac{1}{\pi}$
3. for J use contour C_3 and $k = \frac{1}{2\pi}$

and the contours are shown in the figure. Evaluate the integrals for large values of x and verify the asymptotic forms in Chapter 8.



Problem 6

$$H_\nu(x) = \frac{1}{\pi}\int_C \exp(i\nu u - ix\sin u)du$$
$$= \frac{1}{\pi}\int_C \exp(i\nu u)\exp(-ix\sin u)du$$

Thus

$$f(u) = -i\sin u$$

and

$$g(u) = e^{i\nu u}$$

$$\frac{df}{du} = -i\cos u = 0 \Rightarrow u = \left(n + \frac{1}{2}\right)\pi$$

Since u must be on or near the path C, we take $n = -1$ for H^1 and $n = 0$ for H^2, so $u_0 = \mp\frac{\pi}{2}$,

$$f(u_0) = \pm i$$

and then

$$f''(u_0) = i\sin u_0 = -i = e^{-i\pi/2} \text{ for } H^1 \text{ and } = i = e^{i\pi/2} \text{ for } H^2$$

so $a = 1$ and $\alpha = \mp\pi/2$. Thus

$$\phi = \pm\frac{\pi}{4} \pm \frac{\pi}{2}$$

Thus the chosen path moves diagonally through the point z_0. The For $H^{(1)}$ the difference $z - z_0$ has a positive imaginary part before z_0 and negative after. Thus $r\sin\left(\frac{\pi}{4} \pm \frac{\pi}{2}\right)$ must be positive when r is negative, so we need the minus sign, and $\phi_0 = -\frac{\pi}{4}$ for H^1. For H^2 the difference $z - z_0$ has a negative imaginary part before z_0 and negative after. Thus $r\sin\left(-\frac{\pi}{4} \pm \frac{\pi}{2}\right)$ must be negative when r is negative. Again we take the positive sign, and $\phi_0 =$

$\frac{\pi}{4}$ for H^2. Finally, then:

$$H_\nu^{(1)}(x) = \frac{1}{\pi}e^{-i\nu\pi/2}\exp(ix)e^{-\pi i/4}\sqrt{\frac{2\pi}{x}} = \sqrt{\frac{2}{\pi x}}\exp\left(ix - i\nu\frac{\pi}{2} - i\frac{\pi}{4}\right)$$

and

$$H_\nu^{(2)}(x) = \frac{1}{\pi}e^{i\nu\pi/2}\exp(-ix)e^{\pi i/4}\sqrt{\frac{2\pi}{x}} = \sqrt{\frac{2}{\pi x}}\exp\left(-ix + i\nu\frac{\pi}{2} + i\frac{\pi}{4}\right)$$

These expressions agree with the results in Chapter 8.

For J we have contributions from both saddles, and we obtain

$$J = \frac{1}{2}\left(H^1 + H^2\right)$$

as required.

7. The Airy integral

$$Ai(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\exp i\left(tx + \frac{t^3}{3}\right)dt$$

arises in the study of diffraction. The path of integration lies slightly *above* the real axis. Use the method of stationary phase to show that

$$Ai(x) \sim \frac{1}{2\sqrt{\pi}\,x^{1/4}}\exp\left(-\frac{2}{3}x^{3/2}\right) \text{ for large, positive } x$$

The phase is

$$\phi(t) = tx + \frac{t^3}{3}$$

$$\frac{d\phi}{dt} = x + t^2 = 0 \text{ at } t_s = \pm i\sqrt{x}$$

The stationary point is off the real axis, so we deform the path to go through one of these points-- the one at $+i\sqrt{x}$. We can do this because the integrand has no singularities between the two paths. (Recall that the path is above the real axis.) Then

$$\phi(t_s) = ix^{3/2} + \frac{i^3x^{3/2}}{3} = \frac{2ix^{3/2}}{3}$$

and

$$\left.\frac{d^2\phi}{dt^2}\right|_{t_s} = 2t_s = 2i\sqrt{x}$$

The asymptotic form is (equation D.9):

$$Ai(x) = \sum_{\pm}\frac{1}{2\pi}\exp\left(i\frac{2ix^{3/2}}{3}\right)e^{i\pi/4}\sqrt{\frac{2\pi}{2i\sqrt{x}}}$$

$$= \frac{1}{2\sqrt{\pi}}\frac{1}{x^{1/4}}\exp\left(-\frac{2z^{3/2}}{3}\right)$$

as required.

8. The amplitude of a signal arriving from a distant source after propagation through a dispersive medium may be wriiten as a Fourier integral of the form:

$$s(x,t) = \int_{-\infty}^{\infty} A(\omega)\exp[ik(\omega)x - i\omega t]d\omega$$

where $k(\omega)$ is the dispersion relation for the medium (see, e.g, Jackson Chapter 7). Use the method of stationary phase to show that the largest amplitude signal is contributed by frequencies with group speed $d\omega/dk = D/t$ where D is the distance from source to receiver, and find an approximate expression for the amplitude at time t.

The stationary phase condition is:

$$\frac{d\phi}{d\omega} = \frac{dk}{d\omega}x - t = 0 \Rightarrow x = \frac{d\omega}{dk}t$$

or $x =$ the distance travelled by the signal at the group speed $d\omega/dk$.

By equation D.9, the amplitude is approximately:

$$s(x,t) = A(\omega_s)e^{ik(\omega_s)-i\omega_s t}e^{i\pi/4}\sqrt{\frac{2\pi}{\frac{d^2k}{d\omega^2}x}}$$

where all derivatives are evaluated at ω_s, and ω_s is a solution of the equation $x = \frac{d\omega}{dk}t$. If there is more than one solution, then we must add the contributions from all the solutions.

Obtain an explicit form for the solution if $ck(\omega) = \sqrt{\omega^2 - \omega_p^2}$ and ω_p is a constant.

$$c^2k^2 = \omega^2 - \omega_p^2$$

Thus

$$2c^2k\frac{dk}{d\omega} = 2\omega \Rightarrow \frac{dk}{d\omega} = \frac{\omega}{c\sqrt{\omega^2 - \omega_p^2}}$$

and the stationary phase condition is

$$\frac{\omega}{\sqrt{\omega^2 - \omega_p^2}}x = ct$$

or

$$\omega^2 = \left(\frac{ct}{x}\right)^2(\omega^2 - \omega_p^2)$$

$$\omega_s = \frac{\omega_p}{\sqrt{1-(x/ct)^2}}$$

There is no solution for $t < x/c$, as expected.

$$\frac{d^2k}{d\omega^2} = \frac{1}{c\sqrt{\omega^2-\omega_p^2}} - \frac{\omega^2}{c\left(\omega^2-\omega_p^2\right)^{3/2}} = \frac{-\omega_p^2}{c\left(\omega^2-\omega_p^2\right)^{3/2}}$$

Evaluating at ω_s :

$$\begin{aligned}\left.\frac{d^2k}{d\omega^2}\right|_{\omega_s} &= \frac{-\omega_p^2}{c(\omega_s x/ct)^3} = \frac{-\omega_p^2(ct)^3}{cx^3\omega_p^3}\left(1-\left(\frac{x}{ct}\right)^2\right)^{3/2} \\ &= -\left(\frac{ct}{x}\right)^3\frac{1}{c\omega_p}\left(1-\left(\frac{x}{ct}\right)^2\right)^{3/2} \\ &= -\frac{1}{c\omega_p}\left(\left(\frac{ct}{x}\right)^2-1\right)^{3/2}\end{aligned}$$

Also

$$ck(\omega_s) = \sqrt{\omega_s^2-\omega_p^2} = \frac{\omega_s x}{ct} = \frac{x}{ct}\frac{\omega_p}{\sqrt{1-(x/ct)^2}} = \frac{\omega_p}{\sqrt{\left(\frac{ct}{x}\right)^2-1}}$$

and

$$\begin{aligned}e^{ik(\omega_s)x-i\omega_s t} &= \exp\left[\frac{ix}{c}\frac{\omega_p}{\sqrt{\left(\frac{ct}{x}\right)^2-1}} - it\frac{\omega_p}{\sqrt{1-(x/ct)^2}}\right] \\ &= \exp\left[\frac{ix}{c}\frac{\omega_p}{\sqrt{\left(\frac{ct}{x}\right)^2-1}} - i\frac{ct^2}{x}\frac{\omega_p}{\sqrt{\left(\frac{ct}{x}\right)^2-1}}\right] \\ &= \exp\left[-i\frac{\omega_p}{c}\sqrt{(ct)^2-x^2}\right]\end{aligned}$$

Putting it all together, we have

$$\begin{aligned}s(x,t) = {} & A\left(\frac{\omega_p}{\sqrt{1-(x/ct)^2}}\right)\exp\left[-i\frac{\omega_p}{c}\sqrt{(ct)^2-x^2}\right]e^{i\pi/4} \\ & \times\sqrt{\frac{2\pi}{-\frac{1}{c\omega_p}\left(\left(\frac{ct}{x}\right)^2-1\right)^{3/2}x}}\end{aligned}$$

Thus the final result is:

$$\begin{aligned}s(x,t) &= A\left(\frac{\omega_p}{\sqrt{1-(x/ct)^2}}\right)\exp\left[-i\frac{\omega_p}{c}\sqrt{(ct)^2-x^2}\right]\frac{e^{-i\pi/4}x\sqrt{2\pi c\omega_p}}{\left((ct)^2-x^2\right)^{3/4}} \quad \text{for } t > x/c \\ &= 0 \text{ otherwise}\end{aligned}$$

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Optional topic E: Calculus of variations

1. The speed of waves in a medium varies with y as $v = v_0(1+y)$. What is the path of a ray? This model describes the propagation of seismic waves through the Earth's outer layers. If the waves start at $y = 0$, $x = 0$ find the value of x at which the waves return to the surface as a function of the initial slope of the ray $\frac{dy}{dx} = m = \tan\theta$. Also determine the total time of travel for each ray.

The time of travel is

$$t = \int_0^D \frac{ds}{v} = \frac{1}{c}\int_0^D \frac{\sqrt{1+(y')^2}}{1+y}dx$$

Here the integrand is independent of x, and we may use equation (E.6).

$$\frac{\sqrt{1+(y')^2}}{1+y} - y'\frac{\partial}{\partial y'}\frac{\sqrt{1+(y')^2}}{1+y} = C$$

$$\frac{\sqrt{1+(y')^2}}{1+y} - \frac{y'}{1+y}\frac{y'}{\sqrt{1+(y')^2}} = C$$

or

$$\frac{1}{1+y} = C\sqrt{1+(y')^2}$$

Thus

$$(y')^2 = \frac{1}{C^2(1+y)^2} - 1$$

Inserting the initial conditions ($y' = m$ at $y = 0$) we have

$$m^2 = \frac{1}{C^2} - 1 \Rightarrow C = \frac{1}{\sqrt{1+\tan^2\theta}} = \cos\theta$$

Then

$$\frac{dy}{dx} = \pm\frac{\sqrt{1-\cos^2\theta(1+y)^2}}{\cos\theta(1+y)}$$

We integrate this expression to obtain x.

$$\pm\int\frac{(1+y)}{\sqrt{\sec^2\theta-(1+y)^2}}dy = x$$

We may integrate immediately, to obtain:

$$\mp\sqrt{\sec^2\theta-(1+y)^2} = x+k$$

and inserting the initial conditions,

$$k = \mp\tan\theta$$

Thus

$$x = \pm\tan\theta\mp\sqrt{\sec^2\theta-(1+y)^2} = \pm\tan\theta\mp\sqrt{\tan^2\theta-2y-y^2}$$

The solution with $x > 0$ that satisfies the condition $x = 0$ at $y = 0$ is

$$x = \tan\theta - \sqrt{\tan^2\theta-2y-y^2}$$

Equivalently,

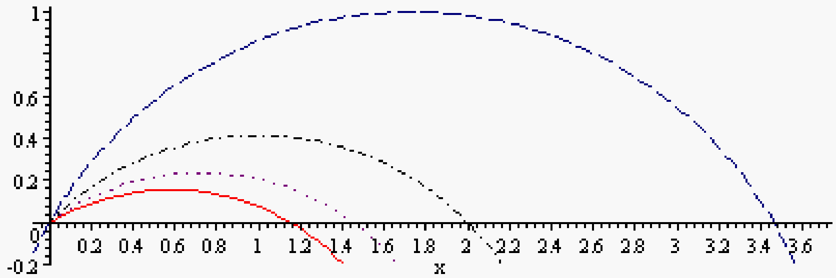
$$\sec^2\theta - (1+y)^2 = (x-\tan\theta)^2$$

Thus

$$y = \sqrt{\sec^2\theta - (x-\tan\theta)^2} - 1$$
$$= \sqrt{1 - x^2 + 2x\tan\theta} - 1$$

The path is a circle with center at $x = \tan\theta$, $y = -1$ and radius $\sec\theta$.

The diagram shows the paths corresponding to different values of θ. Solid red line - $\theta = \pi/6$, purple dotted line $\pi/5$, black dot--dash line, $\pi/4$, blue dashed line $\pi/3$



The waves return to the surface at

$$x = 2\tan\theta$$

in a time

$$t = \frac{2}{c}\int_0^{D/2} \frac{\sqrt{1+(y')^2}}{1+y}dx = \frac{2}{c}\int_0^{\tan\theta} \frac{1}{\cos\theta\left(\sec^2\theta - (x-\tan\theta)^2\right)}dx$$

To do the integral, let $x - \tan\theta = \sec\theta\tanh\chi$

$$t = \frac{1}{c\cos\theta}\int \frac{\sec\theta\,\mathrm{sech}^2\chi\, d\chi}{\sec^2\theta\,\mathrm{sech}^2\chi} = \frac{1}{c}\chi = 2\frac{\tanh^{-1}(\sin\theta)}{c}$$

For $\theta = \pi/3$, we find $t = (2/c)\tanh^{-1}(\sin\pi/3) = 2.6339/c$.

2. Find the path of a light ray through a medium whose refractive index increases linearly with depth: $n = 1 + \frac{x}{a}$. The ray follows the path of minimum time:

$$t = \frac{1}{c}\int_a^b \left(1 + \frac{x}{a}\right)\sqrt{1+(y')^2}\,dx$$

Applying the Euler-Lagrange equation:

$$\frac{d}{dx}\left(\frac{\partial f}{\partial y'}\right) - \frac{\partial f}{\partial y} = 0$$

we have

$$\frac{d}{dx}\left(\frac{\partial(1+\frac{x}{a})\sqrt{1+(y')^2}}{\partial y'}\right) - \frac{\partial(1+\frac{x}{a})\sqrt{1+(y')^2}}{\partial y} = 0$$

This simplifies to

$$\frac{(1+\frac{x}{a})y'}{\sqrt{1+(y')^2}} = C$$

where C is a constant. Thus

$$\left(1+\frac{x}{a}\right)^2 (y')^2 = C^2\left(1+(y')^2\right)$$
$$(y')^2\left(\left(1+\frac{x}{a}\right)^2 - C^2\right) = C^2$$

$$\frac{dy}{dx} = \pm\frac{C}{\sqrt{(1+\frac{x}{a})^2 - C^2}}$$

If $dy/dx = 1$ at $x = 0$, then

$$1 = \frac{C}{\sqrt{1-C^2}} \Rightarrow 1 - C^2 = C^2 \Rightarrow C = \frac{1}{\sqrt{2}}$$

Integrating, we find

$$y = \int \frac{1}{\sqrt{2(1+\frac{x}{a})^2 - 1}}\,dx$$

Let $\sqrt{2}(1+\frac{x}{a}) = \cosh u.$ Then

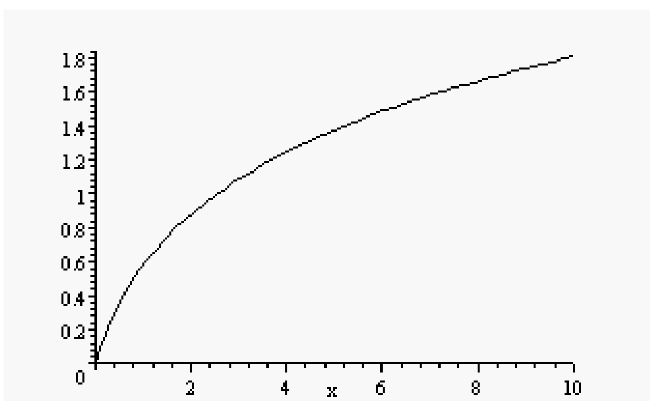
$$y = \int \frac{1}{\sinh u}\frac{a}{\sqrt{2}}\sinh u\,du = \frac{au}{\sqrt{2}} = \frac{a}{\sqrt{2}}\cosh^{-1}\left(\sqrt{2}\left(1+\frac{x}{a}\right)\right) + B$$

If the path starts from the origin, $x = y = 0,$ Then

$$0 = \frac{a}{\sqrt{2}}\cosh^{-1}\sqrt{2} + B$$

and

$$y = \frac{a}{\sqrt{2}}\left[\cosh^{-1}\left(\sqrt{2}\left(1+\frac{x}{a}\right)\right) - \cosh^{-1}\sqrt{2}\right]$$



The path is shown in the figure. The ray bends as it travels.

3. Rework problem 2, reversing the roles of the labels x and $y.$ Is the result the same? Which method is easier?

$$t = \frac{1}{c}\int_0^d \left(1+\frac{y}{a}\right)\sqrt{1+(y')^2}\,dx$$

Here y is horizontal distance and x is vertical distance. Now we use equation 5

$$\left(1+\frac{y}{a}\right)\sqrt{1+(y')^2} - y'\frac{\partial}{\partial y'}\left(1+\frac{y}{a}\right)\sqrt{1+(y')^2} = C$$

$$\left(1+\frac{y}{a}\right)\left(\sqrt{1+(y')^2} - \frac{(y')^2}{\sqrt{1+(y')^2}}\right) = C$$

$$\frac{\left(1+\frac{y}{a}\right)}{\sqrt{1+(y')^2}} = C$$

Put in the initial conditions:

$$C = 1/\sqrt{2}$$

and thus

$$(y')^2 = 2\left(1+\frac{y}{a}\right)^2 - 1$$

$$\int \frac{dy}{\sqrt{2\left(1+\frac{y}{a}\right)^2 - 1}} = \int dx$$

Let $\sqrt{2}(1+y/a) = \coth\chi$. Then $\sqrt{2}\frac{dy}{a} = \frac{d}{d\chi}\frac{\cosh\chi}{\sinh\chi}d\chi = \left(1-\frac{\cosh^2\chi}{\sinh^2\chi}\right)d\chi = -\frac{d\chi}{\sinh^2\chi}$

$$x = \frac{a}{\sqrt{2}}\int\frac{-d\chi}{\sinh\chi} = -\frac{a}{\sqrt{2}}\int\frac{\operatorname{cosech}\chi\left(\coth\chi + \operatorname{cosech}\chi\right)}{\left(\coth\chi + \operatorname{cosech}\chi\right)}d\chi = \frac{a}{\sqrt{2}}\ln\left(\coth\chi + \operatorname{cosech}\chi\right)$$

$$= \frac{a}{\sqrt{2}}\ln\left(\sqrt{2}(1+y/a) + \sqrt{2(1+y/a)^2 - 1}\right) + c$$

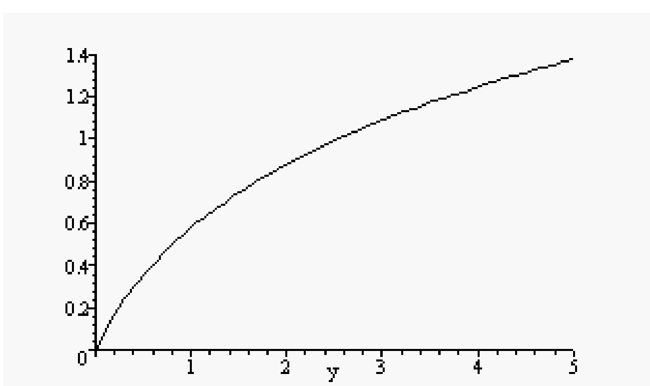
If $x = 0$ at $y = 0$, then

$$0 = \frac{a}{\sqrt{2}}\ln\left(\sqrt{2}+1\right) + c \Rightarrow c = -\frac{a}{\sqrt{2}}\ln\left(\sqrt{2}+1\right)$$

and

$$x = \frac{a}{\sqrt{2}}\ln\left(\frac{\sqrt{2}(1+y/a) + \sqrt{2(1+y/a)^2 - 1}}{\sqrt{2}+1}\right)$$

Here's the path- it is the same.



The methods are equivalent. The integrations are slightly easier in problem 1.

4. Repeat problem 2 with refractive index function $n(x) = n_0 e^x$.

The Euler-Lagrange equation becomes

$$\frac{e^x y'}{\sqrt{1+(y')^2}} = C$$

If $y' = m$ at $x = 0$,

$$C = \frac{m}{\sqrt{1+m^2}}$$

Now let $y' = \tan\theta$, so that

$$\begin{aligned} e^x \tan\theta &= C\sec\theta \\ e^x &= \frac{C}{\sin\theta} \\ x &= \ln C - \ln\sin\theta \end{aligned}$$

Then

$$\frac{dy}{d\theta} = \frac{dy}{dx}\frac{dx}{d\theta} = \tan\theta\left(-\frac{\cos\theta}{\sin\theta}\right) = -1$$

with solution

$$y = \alpha - \theta$$

and hence

$$x = \ln C - \ln\sin(\alpha - y)$$

If $x = 0$ when $y = 0$, then

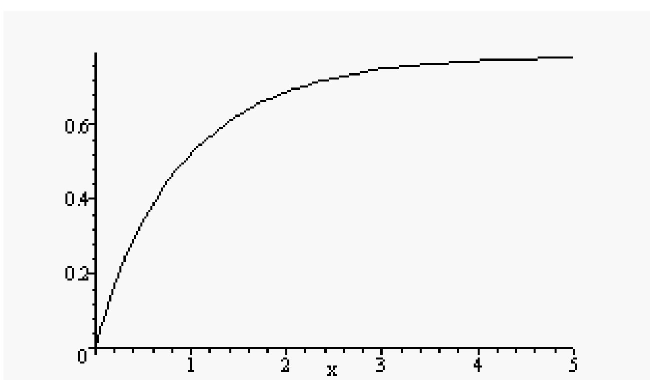
$$\ln C = \ln\sin\alpha$$

and so

$$x = \ln\left[\frac{\sin\alpha}{\sin(\alpha - y)}\right] \Rightarrow e^x = \frac{\sin\alpha}{\sin(\alpha - y)}$$

If $m = 1$, then $\sin\alpha = 1/\sqrt{2}$ and $\alpha = \pi/4$. Then the solution is:

$$\begin{aligned} \sin(\pi/4 - y) &= \frac{e^{-x}}{\sqrt{2}} \\ y &= \frac{\pi}{4} - \sin^{-1}\left(\frac{e^{-x}}{\sqrt{2}}\right) \end{aligned}$$



Check:

$$-\cos(\pi/4 - y)\frac{dy}{dx} = -\frac{e^{-x}}{\sqrt{2}}$$

$$y' = \frac{e^{-x}}{\sqrt{2}}\frac{1}{\sqrt{1-\left(\frac{e^{-x}}{\sqrt{2}}\right)^2}}$$

$$= \frac{1/\sqrt{2}}{\sqrt{e^{2x} - 1/2}}$$

$$\frac{e^x y'}{\sqrt{1+(y')^2}} = C \Rightarrow (y')^2\left(e^{2x} - C^2\right) = C^2$$

They agree.

5. The brachistochrone: A smooth wire runs between two fixed points $A\ (0,0)$ and $B\ (X,Y).$ Find the shape of the wire such that a particle sliding without friction oin the wire reaches point B in minimum time. Assume that the particle starts from point A with speed $v_0.$ *Hint:* set $y' = \tan\theta$ and obtain x and y as functions of $\theta.$ Show that the coordinates of the two fixed points are sufficient to determine the intitial and final values of θ and the two integration constants. Determine an explicit solution in the case $v_0 = 0,\ X = Y = -1.$ Plot the shape of the wire.

We may find the speed of the particle at any time from conservation of energy.

$$\frac{1}{2}mv^2 - mgy = \frac{1}{2}mv_0^2$$

Thus

$$v = \sqrt{v_0^2 + 2gy}$$

and the time taken to go from A to B is

$$t = \int_A^B \frac{dl}{v} = \int_A^B \frac{\sqrt{1+(y')^2}}{\sqrt{v_0^2+2gy}}dx$$

The problem is now in our standard form, and we may apply the Euler-Lagrange equation. Since the integrand does not depend explicitly on $x,$ we may use equation (5):

$$f - y'\frac{\partial f}{\partial y'} = C$$

$$\sqrt{1+(y')^2} - y'\frac{y'}{\sqrt{1+(y')^2}} = C\sqrt{v_0^2+2gy}$$

$$\frac{1}{\sqrt{1+(y')^2}} = C\sqrt{v_0^2+2gy}$$

Squaring both sides, we have:

$$C^2\left(v_0^2+2gy\right) = \frac{1}{1+(y')^2}$$

The trick here is to let $y' = \tan\theta,$ so that

$$C^2\left(v_0^2+2gy\right) = \frac{1}{1+\tan^2\theta} = \cos^2\theta$$

So

$$y = \frac{\cos^2\theta}{2gC} - \frac{v_0^2}{2g} = \frac{\cos 2\theta + 1}{4gC} - \frac{v_0^2}{2g}$$

Thus

$$dy = -\frac{\cos\theta\sin\theta}{gC}d\theta = \tan\theta dx$$

Thus

$$\frac{dx}{d\theta} = -\frac{\cos^2\theta}{gC} = -\frac{1}{2gC}(\cos 2\theta + 1)$$

Thus

$$x = A - \frac{1}{2gC}\left(\frac{\sin 2\theta}{2} + \theta\right)$$

We have four conditions for the 4 unknowns $A, C\ \theta_1$ and θ_2:

$$0 = A - \frac{1}{2gC}\left(\frac{\sin 2\theta_1}{2} + \theta_1\right)$$

$$0 = \frac{\cos 2\theta_1 + 1}{4gC} - \frac{v_0^2}{2g}$$

$$X = A - \frac{1}{2gC}\left(\frac{\sin 2\theta_2}{2} + \theta_2\right)$$

$$Y = \frac{\cos 2\theta_2 + 1}{4gC} - \frac{v_0^2}{2g}$$

Thus in the special case $v_0 = 0$ we can take $\theta_1 = \pi/2,$ and

$$A = \frac{1}{2gC}\left(\frac{\pi}{2}\right) = \frac{\pi}{4gC}$$

$$X = \frac{1}{4gC}(\pi - \sin 2\theta_2 - 2\theta_2)$$

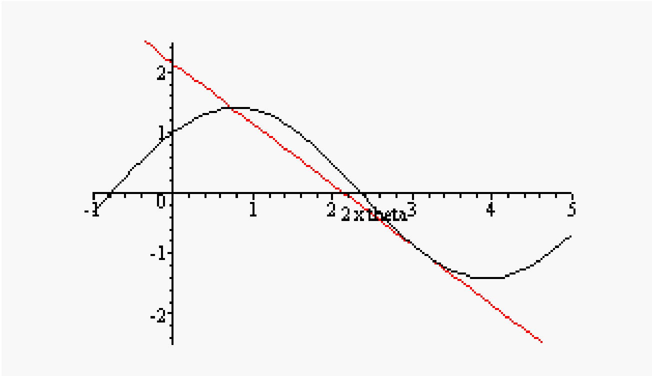
$$Y = \frac{\cos 2\theta_2 + 1}{4gC} \Rightarrow 4gC = \frac{\cos 2\theta_2 + 1}{Y}$$

If $X = Y = -1,$ then

$$\cos 2\theta_2 + 1 = \pi - \sin 2\theta_2 - 2\theta_2$$

$$\cos 2\theta_2 + \sin 2\theta_2 = \pi - 1 - 2\theta_2$$

We can solve this transcendental equation for θ_2 graphically or numerically.



black- LHS; red- RHS

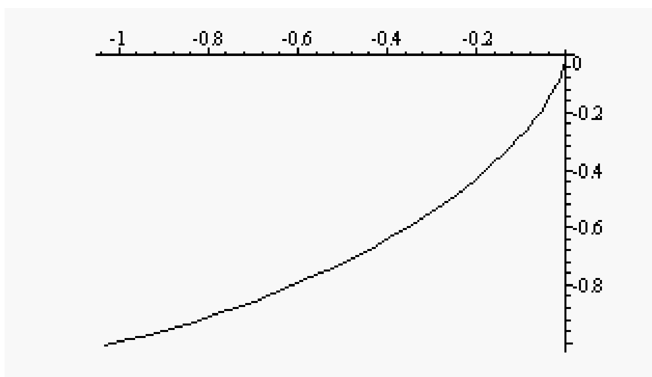
The solutions are 3.$1416 = \pi = 2\theta_2$ and 0.72958151$= 2\theta_2$
Then

$$4gC = \frac{\cos 0.72958151 + 1}{-1} = -1.7455$$

The solution is then:

$$x = -\frac{1}{1.7455}(\pi - \sin 2\theta - 2\theta)$$
$$y = -\frac{\cos 2\theta + 1}{1.7455}$$

Here's the path:



6. Show that the Sturm-Liouville problem (equation 8.1) arises from the problem of finding the extremum of the integral

$$I = \int_a^b \left(f(y')^2 + gy^2\right)dx = \int_a^b F dx$$

subject to the constraint

$$J = \int_a^b wy^2 dx = \int_a^b G dx = \text{constant}$$

with the boundary conditions (8.2).

$$\frac{d}{dx}\left(\frac{\partial(F + \lambda G)}{\partial y'}\right) - \frac{\partial(F + \lambda G)}{\partial y} = 0 = \frac{d}{dx}(2fy') - (2gy + 2\lambda wy)$$
$$0 = \frac{d}{dx}(fy') - gy - \lambda wy$$

which is equation 8.1.

7. Show that the catenary (Example E.3) is symmetric about the midpoint, that is, $y(D - x) = y(x)$.

For $D/2 \le x \le D$, he slope of the cable is negative, and the integration takes the form

$$\int_Y^0 -\frac{dy}{\sqrt{\frac{1}{C^2}(y+\lambda)^2 - 1}} = \int_X^D dx$$

$$-\int_{\cosh^{-1}(Y+\lambda/C)}^{\cosh^{-1}\lambda/C} \frac{C\sinh d\theta}{\sinh\theta} = \int_X^D dx = D - X$$

which gives the solution:

$$C\left[\cosh^{-1}\left(\frac{Y+\lambda}{C}\right) - \cosh^{-1}\frac{\lambda}{C}\right] = D - X \text{ for } D/2 < X < D$$

giving

$$y(x) = C\cosh\left(\frac{D-x}{c} + \cosh^{-1}\frac{\lambda}{C}\right) - \lambda$$

as required.

8. Show that the curve that encloses the greatest area with a fixed perimeter is a circle.

Let the curve be described by the function $y(x)$. Then the area is

$$A = \oint y(x)dx$$

and the perimeter is

$$P = \oint ds = \oint \sqrt{1 + (y')^2}\,dx$$

The integrands do not depend explicitly on x, so the appropriate equation is:

$$f - y'\frac{\partial f}{\partial y'} = C$$

where

$$f = y + \lambda\sqrt{1 + (y')^2}$$

Thus

$$y + \lambda\sqrt{1 + (y')^2} - y'\left(\frac{y'\lambda}{\sqrt{1 + (y')^2}}\right) = C$$

$$y + \frac{\lambda}{\sqrt{1 + (y')^2}} = C$$

Let $y' = \tan\theta$. Then

$$y = C - \lambda\cos\theta$$

So

$$\frac{dy}{d\theta} = \lambda\sin\theta$$

and

$$\frac{dy}{dx} = \tan\theta = \frac{dy}{d\theta}\frac{d\theta}{dx} \Rightarrow \frac{d\theta}{dx} = \frac{\tan\theta}{\lambda\sin\theta}$$

and thus

$$\frac{dx}{d\theta} = \lambda\cos\theta \Rightarrow x = A + \lambda\sin\theta$$

Thus

$$(x-A)^2+(y-C)^2=\lambda^2$$

This curve is a circle.

Optional Topic E: Calculus of variations.

9. Investigate the problem of finding an extremum of the integral

$$I = \int_a^b \psi H \psi^* dx$$

subject to the constraint

$$J = \int_a^b \psi \psi^* dx = 1$$

where H is the Hamiltonian operator

$$H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)$$

and ψ and ψ' vanish at $x = a$ and $x = b.$ Show that the resulting differential equation is the Schrödinger equation.

Hint: first integrate by parts to eliminate the second derivative.

We want to find an extremum of the integral

$$K = I + \lambda J$$

$$K = \int\left[\psi\left(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)\right)\psi^* + \lambda\psi\psi^*\right]dx$$

The first term is:

$$I_1 = -\int \psi \frac{\hbar^2}{2m}\frac{d^2}{dx^2}\psi^* dx = -\frac{\hbar^2}{2m}\left[\psi\frac{d}{dx}\psi^*\Big|_a^b - \int \frac{d\psi}{dx}\frac{d\psi^*}{dx}dx\right]$$

The integrated term is zero, so

$$I = \int\left\{\frac{\hbar^2}{2m}\frac{d\psi}{dx}\frac{d\psi^*}{dx} + [V(x) + \lambda]\psi\psi^*\right\}dx$$

The Euler-Lagrange equation is

$$\frac{d}{dx}\left(\frac{\hbar^2}{2m}\frac{d\psi}{dx}\right) - (V+\lambda)\psi = 0$$

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V\psi = -\lambda\psi$$

which is the Schrödinger equation.

10. Using polar coordinates, write the Lagrangian for a particle moving in the potential $V = -Gmm/r$, and form the Euler-Lagrange equations. Show that the equation in the angular coordinate indicates conservation of angular momentum.

$$L = T - V = \frac{1}{2}m\left(r^2\left(\frac{d\theta}{dt}\right)^2 + \left(\frac{dr}{dt}\right)^2\right) + \frac{GMm}{r}$$

Thus

$$\frac{d}{dt}\left(\frac{\partial L}{\partial(d\theta/dt)}\right) - \frac{\partial L}{\partial\theta} = 0$$

$$\frac{d}{dt}\left[mr^2\frac{d\theta}{dt}\right] - 0 = 0$$

Thus

$$mr^2\frac{d\theta}{dt} = \text{constant}$$

and this is the angular momentum.

The second equation is:

$$\frac{d}{dt}\left(m\frac{dr}{dt}\right) - \frac{GMm}{r^2} = 0$$

This is just the $r-$ component of Newton's law, $\vec{\mathbf{F}} = m\vec{\mathbf{a}}$.

11. A spherical pendulum is a mass free to move on the end of a string of length ℓ. Write the Lagrangian in terms of the spherical angles θ and ϕ, and hence find the equations of motion. Show that one possible motion is the *conical pendulum* with constant θ. What is the value of $d\phi/dt$ in this case?

$$L = T - V = \frac{1}{2}m\left(\ell^2 \sin^2\theta\left(\frac{d\phi}{dt}\right)^2 + \ell^2\left(\frac{d\theta}{dt}\right)^2\right) - (mg\ell(1 - \cos\theta))$$

The Largrange equations are"

$$\frac{d}{dt}\left(\ell^2 \sin^2\theta\left(\frac{d\phi}{dt}\right)\right) = 0$$
$$\frac{d}{dt}\left(\ell^2\left(\frac{d\theta}{dt}\right)\right) - \ell^2 \sin\theta\cos\theta\left(\frac{d\phi}{dt}\right)^2 + g\ell\sin\theta = 0$$

Thus

$$\sin^2\theta\left(\frac{d\phi}{dt}\right) = k = \text{constant}$$

and

$$\frac{d^2\theta}{dt^2} - \sin\theta\cos\theta\left(\frac{d\phi}{dt}\right)^2 + \frac{g}{\ell}\sin\theta = 0$$

With $k = 0$, we retrieve the simple pendulum equations.

Now use the first equation to simplify the second:

$$\frac{d^2\theta}{dt^2} - \sin\theta\cos\theta\frac{k^2}{\sin^4\theta} + \frac{g}{\ell}\sin\theta = 0$$
$$\frac{d^2\theta}{dt^2} - \frac{\cos\theta}{\sin^3\theta}k^2 + \frac{g}{\ell}\sin\theta = 0$$

If $\theta =$ constant, then we have the conical pendulum with

$$k^2 = \frac{g}{\ell}\frac{\sin^4\theta}{\cos\theta}$$

and hence

$$\left(\frac{d\phi}{dt}\right)^2 = \frac{g}{\ell}\frac{1}{\cos\theta}$$

12. Consider the one-dimensional motion of a particle with potential energy

$V(x)$ that is independent of time. Show that The Euler Lagrange equations may be written in the form of equation (5). Give a physical interpretation of this equation.

$$L = T - V = \frac{1}{2}m\left(\frac{dx}{dt}\right)^2 - V(x)$$

Since L does not depend explicitly on time, we may write the Euler-lagrange equations in the form:

$$\frac{1}{2}m\left(\frac{dx}{dt}\right)^2 - V(x) - \left(\frac{dx}{dt}\right)\left(m\frac{dx}{dt}\right) = -V(x) - \frac{1}{2}m\left(\frac{dx}{dt}\right)^2 = \text{constant}$$

Thus this equation states that the total (kinetic plus potential) energy is conserved in this system.

13. The Lagrangian for a vibrating string may be written

$$\mathcal{L} = \int_0^l \left\{\frac{1}{2}\mu\left(\frac{\partial y}{\partial t}\right)^2 - \frac{1}{2}T\left(\frac{\partial y}{\partial x}\right)^2\right\}dx$$

where $y(x,t)$ is the displacement of the string, μ is the mass per unit length, and T is the tension. The first term in the integrand is the kinetic energy and the second is the potential energy. Determine the Euler-Lagrange equations for the system, and comment.

Here we have two independent variables, so:

$$\frac{\partial \mathcal{L}}{\partial(\partial y/\partial t)} = \int_0^l \mu\left(\frac{\partial y}{\partial t}\right)dx; \quad \frac{\partial \mathcal{L}}{\partial(\partial y/\partial x)} = -T\int_0^l \left(\frac{\partial y}{\partial x}\right)dx$$

and the $E - L$ equation is:

$$\int_0^l \left\{\mu\frac{\partial^2 y}{\partial t^2} - T\frac{\partial^2 y}{\partial x^2}\right\}dx = 0$$

Setting the integrand to zero gives the wave equation for the string.

14. As an alternative approach to problem 13, we may expand the displacement $y(x,t)$ as a Fourier series in x $\left(\textit{cf}\text{ Chapter 4 §4.2}\right)$

$$y(x,t) = \sum_{n=0}^{\infty} a_n(t) \sin \frac{n\pi x}{L}$$

Write the Lagragian as a function of the generalized coordinates $a_n(t)$ and the time $t.$ What are the Euler-Lagrange equations now?

$$\frac{\partial y}{\partial x} = \sum_{n=0}^{\infty} a_n(t) \frac{n\pi}{L} \cos \frac{n\pi x}{L}$$

Thus

$$\mathcal{L} = \int_0^L \left\{ \begin{array}{c} \frac{1}{2}\mu \sum_{n=0}^{\infty} \frac{da_n(t)}{dt} \sin \frac{n\pi x}{L} \sum_{m=0}^{\infty} \frac{da_m(t)}{dt} \sin \frac{m\pi x}{L} \\ -\frac{1}{2} T \sum_{n=0}^{\infty} a_n(t) \frac{n\pi}{L} \cos \frac{n\pi x}{L} \sum_{m=0}^{\infty} a_m(t) \frac{m\pi}{L} \cos \frac{m\pi x}{L} \end{array} \right\} dx$$

$$= \sum_{n=0}^{\infty} \left\{ \frac{1}{2}\mu \left(\frac{da_n(t)}{dt} \right)^2 - \frac{1}{2} T \left(\frac{n\pi}{L} \right)^2 a_n^2 \right\}$$

where we used the orthogonality of the sines and cosines to evaluate the integrals. Now L has one independent variable (t) and infinitely many independent variables $a_n.$ Applying Hamilton's principle, the Euler-Lagrange equations take the form:

$$\frac{d}{dt}\left(\frac{\partial \mathcal{L}}{\partial \dot{a}_n} \right) - \frac{\partial \mathcal{L}}{\partial a_n} = 0$$

$$\frac{d}{dt}\left(\mu \frac{da_n(t)}{dt} \right) - T\left(\frac{n\pi}{L} \right)^2 a_n = 0$$

The solution for each a_n must be of the form:

$$a_n(t) = a_{n0} \sin\left(\frac{n\pi}{L} \sqrt{\frac{T}{\mu}}\, t \right) + b_{n0} \cos\left(\frac{n\pi}{L} \sqrt{\frac{T}{\mu}}\, t \right)$$

cf equation 4.28.

15. A volume V is formed by rotating a curve $y(x)$ defined for $-a < x < a$ around the $x-$ axis. Given that the curve is symmetric about the $y-$ axis, $y(0) = R$, $y(a) = y(-a) = 0,$ and $y'(0) = 0,$ show that the curve that gives the maximum volume for a given surface area is a circle and the corresponding volume is a sphere..

The surface area is:

$$A = \int_{-a}^{a} 2\pi y\sqrt{1 + (y')^2}\,dx$$

and the volume is

$$V = \int_{-a}^{a} \pi y^2 dx$$

We can solve our problem by finding an extremum of $V + \lambda A$: The integrand, $\pi y\left(y + 2\lambda\sqrt{1 + (y')^2}\right)$ does not depend on x explicitly, thus equation 5 becomes:

$$y\left(2\lambda\sqrt{1 + (y')^2} + y\right) - y'\frac{\partial}{\partial y'}y\left(2\lambda\sqrt{1 + (y')^2} + y\right) = C$$

$$y\left(2\lambda\sqrt{1 + (y')^2} + y\right) - \left(\frac{2\lambda y(y')^2}{\sqrt{1 + (y')^2}}\right) = C$$

$$y^2 + \frac{2\lambda y}{\sqrt{(1 + (y')^2)}} = C$$

If $y = 0$ at $x = a$, then $C = 0$.

Then

$$1 + (y')^2 = \left(-\frac{2\lambda}{y}\right)^2$$

$$y' = \pm\sqrt{\frac{4\lambda^2}{y^2} - 1}$$

If $y' = 0$ when $y = R$, then $\lambda = R/2$. Then

$$\pm\int \frac{y\,dy}{\sqrt{R^2 - y^2}} = \mp\sqrt{R^2 - y^2} + k = x$$

Now if $y = R$ at $x = 0$, then $k = 0$

$$x = \sqrt{R^2 - y^2}$$

and the curve is a circle.

Note that $y = \sqrt{R^2 - x^2}$ and so $y = 0$ at $x = R$, thus $R = a$.

16. The Lagrangian for a particle moving under the influence of electromagnetic fields is

$$\mathcal{L} = \frac{1}{2}mv^2 - q\phi + q\vec{A} \cdot \vec{v}$$

where ϕ and A are given functions of position. Find the equations of motion, and hence show that the force acting on the particle is the Lorentz force $\vec{F} = q\left(\vec{E} + \vec{v} \times \vec{B}\right)$.

The Euler Lagrange equations are

$$\frac{d}{dt}\left(m\vec{v} + q\vec{A}\right) - q\vec{\nabla}\left(-\phi + \vec{A} \cdot \vec{v}\right) = 0$$

Thus

$$m\frac{d\vec{v}}{dt} = -q\left(\frac{\partial\vec{A}}{\partial t} + \vec{v} \cdot \vec{\nabla}\vec{A}\right) - q\vec{\nabla}\phi + q\left[\left(\vec{v} \cdot \vec{\nabla}\right)\vec{A} + \vec{v} \times \left(\vec{\nabla} \times \vec{A}\right)\right]$$

$$= q\left[-\vec{\nabla}\phi - \frac{\partial\vec{A}}{\partial t}\vec{v} \times \left(\vec{\nabla} \times \vec{A}\right)\right] = q\left(\vec{E} + \vec{v} \times \vec{B}\right)$$

as required.

17. Show that the shortest distance between two points on the surface of a sphere is a great circle. *Hint:* you may place the polar axis through one of the points.

The distance between two neighboring points on the sphere is:

$$dl = \sqrt{(dr)^2 + r^2(d\theta)^2 + r^2\sin^2\theta(d\phi)^2}$$

But on the surface of a sphere, r is fixed, so if the path starts on the polar axis ($\theta = 0$), the distance to a point with coordinates θ_2, ϕ_2 is

$$L = \int_{\theta=0}^{\theta_2,\phi_2} r\sqrt{1+\sin^2\theta\left(\frac{d\phi}{d\theta}\right)^2}\,d\theta$$

where the path is described by $\phi(\theta)$. The Euler-Lagrange equations are:

$$\frac{d}{d\theta}\left(\frac{\frac{1}{2}2\sin^2\theta\frac{d\phi}{d\theta}}{\sqrt{1+\sin^2\theta\left(\frac{d\phi}{d\theta}\right)^2}}\right) - 0 = 0$$

So

$$\sin^2\theta\frac{d\phi}{d\theta} = C\sqrt{1+\sin^2\theta\left(\frac{d\phi}{d\theta}\right)^2}$$

$$\left(\sin^2\theta\frac{d\phi}{d\theta}\right)^2 = C^2\left(1+\sin^2\theta\left(\frac{d\phi}{d\theta}\right)^2\right)$$

Since $\theta = 0$ at the starting point, C must equal zero, and thus

$$\frac{d\phi}{d\theta} = 0$$

and ϕ is constant along the path. This is a great circle.

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