About a hundred years ago David Hilbert, a German mathematician presented twenty-three math puzzles to the International Congress of Mathematicians. Today, only three remain unsolved. Added to those were four more unsolvable problems. The seven famous unsolved math puzzles that have resisted all attempts to solve are listed here: The Birch and Swinnerton-Dyer Conjecture, The Navier-Stokes Equation, The Poincare Conjecture, The Riemann Hypothesis (the oldest and most famous), The P Verses NP Problem, The Hodge Conjecture, Yang-Mills Existence and Mass Gap. Many experts believe that solving these problems would lead to extraordinary advances in physics, medicine and many other unknown areas in the world of math.

The Poincare conjecture

If you stretch a rubber band around the surface of an apple, then we can shrink it down to a point by moving it slowly, without tearing it and without allowing it to leave the surface. But if you were to stretch a rubber band around the surface of a doughnut, then there is no way of shrinking it to a point without breaking either the rubber band or the doughnut. Therefore the surface of an apple is “simply connected,” and the one of the doughnut is not. About a hundred years ago, Poincare knew that a two dimensional sphere is essentially characterized by this property of simple connectivity. He asked the corresponding question for the three dimensional sphere- the set of points in four-dimensional space at unit distance from the origin. As it turns out, this is an extraordinarily difficult question to be answered.

Henri Poincare practically invented topology while trying to understand the set of solutions to a general algebraic equation f(x,y,z)=0, where x,y,z are complex numbers. After trying the analytic approach, he began assigning algebraic invariants to geometric objects as an approach to classifying the objects. Translated into English, Poincare said consider a compact 3-dimensional manifold V without boundary. Is it possible that the fundamental group of V-could be trivial, even though V is not homeomorphic to the 3-dimensional sphere? Since 1904, the hypothesis that every simply connected closed 3-manifold is homeomorphic to the 3-sphere has been known as the Poincare conjecture. Four years earlier he had stated that every compact polyhedral manifold with the homology of an n-dimensional sphere is actually homeomorphic to the n-dimensional sphere. However in 1904 he had constructed a counterexample to this statement by developing the concept of fundamental group. In doing so he basically invented the fundamental group of space. The coset space M cubed=SO(3)/I where I   is the group of rotations which carry a regular icosahedron onto itself. This space has a non-trivial fundamental group   (M) of order 120.

Henry Whitehead made another false theorem in 1934 when he published a proof of the Poincare Conjecture, claiming that every contractible open 3-dimensional manifold is homeomorphic to Euclidean space.   By creating a counterexample to his own theorem he increased our understanding of the topology of manifolds. A contractible manifold which is not simply connected at infinity, the complement S   T is the required Whitehead counterexample.

<u>Whitehead's proof</u>*: Take your simply connected 3-manifold M, and remove a point, to get a non-compact manifold X. If you did this to what you think M is, namely the 3-sphere, you would get R^3. In general, the only thing you can immediately say is the X is contractible; it can be continuously deformed within itself to a point*. He was wrong. About a year later he published a counterexample in the form of an example of a contractible 3-manifold which isn't homeomorphic to R^3.

The discovery that higher dimensional manifolds are easier to work with than 3-dimensional manifolds, in the 1950's and 1960's, was major progress. Stephen Smale announced a proof of the Poincare conjecture in high dimensions in 1960. John Stallings, using a dissimilar method, promptly followed. Soon Andrew Wallace followed, using similar techniques as those of Stallings. Stalling's result has a weak hypotheses and easier proof therefore having a weaker conclusion as well, assuming that the dimension is seven or more. Later, Zeeman extended his argument to dimensions of five and six. **<u>The Stallings-Zeeman Theorem</u>**- (The method of proof consists of pushing all of the difficulties off towards a single point, so that there can be no control near that point.) If M is a finite simplicial complex of dimension n>5 which has the homotopy type of the sphere S and is locally piecewise linearly homeomorphic to the Euclidean space R , then M is homeomorphic to S under a homoeomorphism which is piecewise linear except at a single point. In other words, the complement M \(point) is piecewise linearly homeomorphic to R .

However, the Smale proof and Wallace proof, closely related and given shortly after Smale's, depended on differentiable methods that builded a manifold up inductively starting with an n-dimensional ball, by successively adding handles. **<u>Smale Theorem-</u>** If M is a differentiable homotopy sphere of dimension n>5, then M is homeomorphic to S . In fact M is diffeomorphic to a manifold obtained by gluing together the boundaries of two closed n-balls under a suitable diffeomorphism. Wallace proved this for n>6. Michael Freedman did the much more difficult work, the 4-dimensional case. He used wildly non-differentiable methods to prove it and also to give a complete classification of closed simply connected topological 4-manifolds. **Freedman Theorem-** Two closed simply connected 4-manifolds are homeomorphic if and only if they have the same bilinear form B and the same KirbySiebenmann invariant K. Any B can be realized by such a manifold. If B( ) is odd for some H , then either value of K can be realized also. However, if B( ) is always even, then K is determined by B, being congruent to one eighth of the signature of B. Bottom line: the differentiable methods used by Smale and Wallace and the non-differentiable methods used by Stallings and Zeeman don't work. But Freedman did show that R admits unaccountably many in equivalent differentiable structures using Donaldson's work.

A conjecture by Thurston holds that every three manifold can be cut up along 2-spheres so as to decompose into essentially unique pieces, that each have a simple geometrical structure. There are eight 3-dimensional geometries in Thurston's program. Well understood are six of them. Even thought there has been great advances in the field of geometry of constant negative curvature, the eighth geometry corresponding to constant positive curvature, remains largely untouched. **Thurston Elliptization Conjecture-** Every closed 3-manifold with finite fundamental groups have a metric of constant positive curvature, and hence is homeomorphic to a quotient S / , where SO(4) is a finite group of rotations which acts freely on S .

The idea of creating a counterexample is easy enough: build a 3-manifold whose fundamental group you can compute is trivial (the homology groups then actually come for free) and then try to show that you were lucky enough to build something that isn't a 3-sphere. The last part is the part that nobody could ever figure out so their time was mostly spent trying to find invariants that had a chance of distinguishing a homotopy 3-sphere from the 3-sphere. It's obvious why these puzzles are worth a million dollars. It's amazing that so many people have done this problem wrong after trying for so many years. It really puts our limited studies of mathematics in perspective.