Solution for Chapter 13

(compiled by Guodong Wang)
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1 Problem A.(BT-13.1)

[by Alexander Putilin/00 and Guodong Wang/03]
Recall that in cartesian coordinates

$$\nabla \cdot \mathbf{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y}, \qquad \nabla \times \mathbf{v} = \frac{\partial v_x}{\partial y} - \frac{\partial v_y}{\partial x} \tag{1}$$

and in polar coordinates

$$\nabla \cdot \mathbf{v} = \frac{\partial v_r}{\partial r} + \frac{1}{r}\frac{\partial v_\phi}{\partial \phi} + \frac{1}{r}v_r, \qquad \nabla \times \mathbf{v} = \frac{\partial v_\phi}{\partial r} - \frac{1}{r}\frac{\partial v_r}{\partial \phi} + \frac{1}{r}v_\phi \tag{2}$$

Where $\nabla \times \mathbf{v}$ is the z-component of $\mathbf{v}$'s curl.

1.1 (a)

$$\mathbf{v} = (2xy, x^2) \implies \tag{3}$$

$$\nabla \cdot \mathbf{v} = 2y, \qquad \nabla \times \mathbf{v} = 0 \tag{4}$$

The fluid is compressible and its vorticity is zero.
Streamlines satisfy eqn.:

$$\frac{\partial x}{\partial y} = \frac{v_x}{v_y} = \frac{2y}{x} \tag{5}$$

This gives

$$xdx = 2ydy \qquad \Rightarrow \quad x^2 - 2y^2 = C = const \tag{6}$$

The streamlines are hyperbolas.

1.2 (b)

$$\mathbf{v} = (x^2, -2xy) \tag{7}$$

$$\nabla \cdot \mathbf{v} = 0, \qquad \nabla \times \mathbf{v} = 2y \tag{8}$$

The streamlines are:

$$\frac{\partial x}{\partial y} = \frac{-2y}{x} \;\Rightarrow\; y = \frac{C}{x^2} \tag{9}$$

Figure 1: Part a.

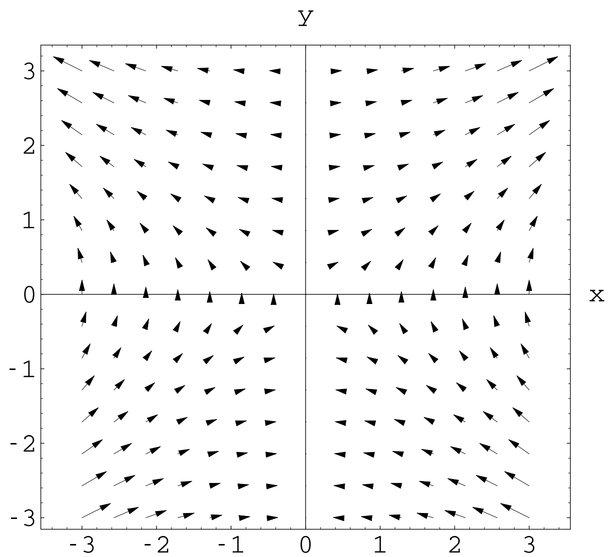


Figure 2: Part b.

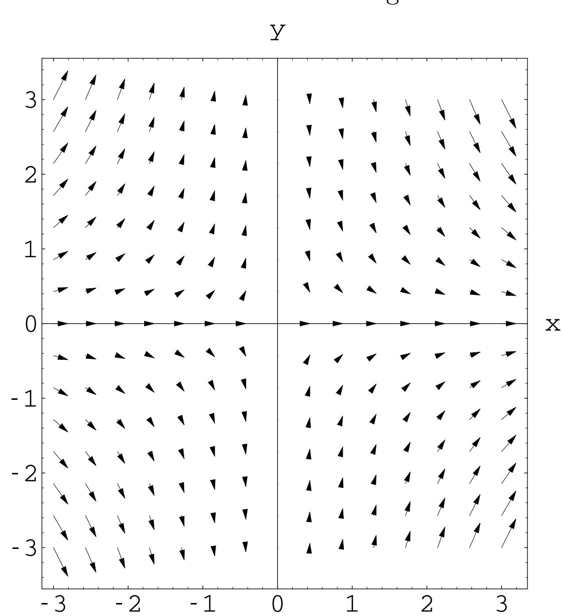
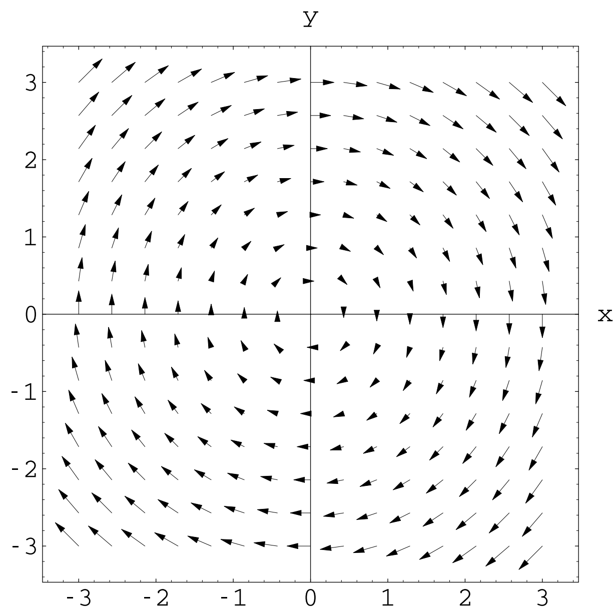


Figure 3: Part c.



1.3 (c)

$$v_r = 0, \;\; v_\phi = r \tag{10}$$

$$\nabla \cdot \mathbf{v} = 0, \qquad \nabla \times \mathbf{v} = 2 \tag{11}$$

The streamlines are $r = const$.

1.4 (d)

$$v_r = 0, \;\; v_\phi = r^{-1} \tag{12}$$

$$\nabla \cdot \mathbf{v} = 0, \qquad \nabla \times \mathbf{v} = 0 \;\text{ if } r \neq 0 \tag{13}$$

Integrate $\nabla \times \mathbf{v}$ around a circle of radius r. We get

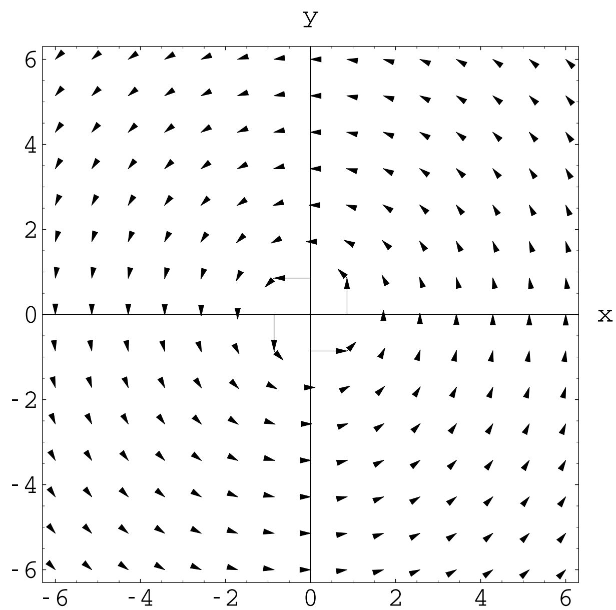
$$\oint d\mathbf{l} \cdot (\nabla \times \mathbf{v}) = \int_0^{2\pi} r d\phi v_\phi = 2\pi \tag{14}$$

It means that $\nabla \times \mathbf{v} = 2\pi\delta(\mathbf{x})$.

The streamlines are $r = const$, as in part (c).

Vector fields are shown on the figures 1-4.

Figure 4: Part d.



2 Problem A.(BT-13.7)

[by Alexander Putilin/00]

$$R = \frac{VL}{\nu}, \tag{15}$$

where V is characteristic velocity, L is characteristic length and ν- viscosity coefficient.

2.1 A hang glider.

$\nu = 10^{-5} m^2/s$ (air), $L \sim$(length of the glider) $\sim 1m$, $V \sim 10m/s$.

$$\Rightarrow \qquad R \sim 10^6 \tag{16}$$

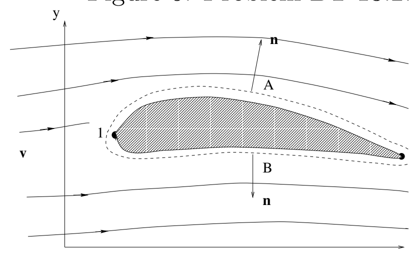
Flow is turbulent.

2.2 Plankton.

$\nu = 10^{-6} m^2/s$ (water), $L \sim 10^{-6} m$. Characteristic velocity of plankton is the velocity of brownian motion. It can be estimated as:

$$\frac{1}{2} m V^2 \sim \frac{3}{2} kT \tag{17}$$

Figure 5: Problem BT-13.2.



$T \approx 300K$ is the temperature, $m \approx \frac{4}{3}\pi L^3 \rho$ is the mass of plankton, $\rho \approx \rho_{water} \approx 1g/cm^3$ is the plankton's density.

$$\Rightarrow \qquad R \sim \frac{3}{2}\frac{1}{\nu}\sqrt{\frac{kT}{\pi \rho L}} \sim 2 \cdot 10^{-3} \tag{18}$$

Flow is laminar.

2.3 Physicist waiving a hand.

$L \sim 0.1m$, $V \sim 1m/s$, $\nu = 10^{-5}m^2/s$.

$$\Rightarrow \qquad R \sim 10^4 \tag{19}$$

Flow is potential except of a thin boundary layer.

3 Problem B.(BT-13.2)

[by Alexander Putilin/00]
Choose the closed loop 1A2B around the airfoil just outside boundary layer. (See figure 5) Since vorticity and gravity are negligable we can write Bernoulli law as

$$P_A + \frac{1}{2}\rho v_A^2 = P_B + \frac{1}{2}\rho v_B^2 \tag{20}$$

$$\Rightarrow \qquad \Delta P = P_B - P_A = \frac{1}{2}\rho(v_B^2 - v_A^2) \approx \rho v \Delta v \tag{21}$$

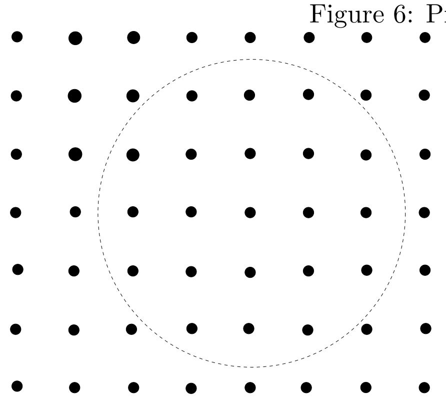
The force on the airfoil is given by

$$\mathbf{F}_L = -\oint L dl \vec{n} P \tag{22}$$

where $\vec{n}$ is a normal unit vector and dl is a length element. The vertical component of the force

$$F_L = -L\oint dl \cdot n_y \cdot P = -L\oint dx \cdot P = -L\int_{1A2} dx P_A + L\int_{1B2} dx P_B =$$

Figure 6: Problem BT-13.3



$$L\int_1^2 dx\Delta P = Lv\rho\int_1^2 dx\Delta v = Lv\rho(\int_{1A2} d\vec{l}\vec{v} - \int_{2B1} d\vec{l}\vec{v}) = \\ = Lv\rho\oint d\vec{l}\vec{v} = Lv\rho\Gamma \tag{23}$$

Conservation of momentum.

The airfoil profile is designed so that outgoing flow of the air is directed slightly downward (See figure). This downward flow is caused by the vortices which trail backward off the wing tips. The change of the momentum of the air has a downward component. It means that there is a net downward force acting on the air and corresponding upward force acting on the airfoil.

4 Problem B.(BT-13.3)

[by Kip Thorne/00]

(i) Consider a circle of radius r that encloses N vortices. (see the fig. 6) The circulation around the circle is:

$$\Gamma = N \cdot \frac{h}{m} = 2\pi r(r\Omega) \tag{24}$$

Where Ω is the angular velocity of the fluid as viewed macroscopically and $r\Omega$ is the macroscopic velocity of the fluid around the circle.

$\Longrightarrow 2\Omega = \frac{N}{\pi r^2}\frac{h}{m}$ = net circulation per unit area.

But the circulation per unit area is the average vorticity inside the circle. So 2Ω = average vorticity.

(ii) Spacing between vortex cores:

$$n^{-\frac{1}{2}} = (\frac{h}{2m\Omega})^{\frac{1}{2}} =\simeq 0.2mm = 200\mu m. \tag{25}$$

Where $m = m_{^4He}, \Omega = 10rpm = 2\pi \times 10/60s$

(iii) Now for a millisecond neutron star,
$m = 2m_{neutron}, \Omega = 2\pi/10^{-3}s$

$$n^{-\frac{1}{2}} = (\frac{h}{2m\Omega})^{\frac{1}{2}} =\simeq 3.6 \times 10^3 nm. \tag{26}$$

5 Problem C.(BT-13.8)

[by Alexander Putilin/00]
The flows are similar if they have the same Reynolds number. It gives the condition:

$$\frac{V_{model} L_{model}}{\nu_{air}} = \frac{V_{car} L_{car}}{\nu_{air}} \tag{27}$$

$$\Rightarrow \qquad V_{model} = \frac{L_{car}}{L_{model}} V_{car} = 8V_{car} = 480\, mph \tag{28}$$

6 Problem D.(BT-13.6)

[by Alexander Putilin/00 and Kip Thorne/00]
We know that the flow should be potential outside thin boundary layer around the cylinder, i.e. $\mathbf{v} = \nabla\psi$. Incompressibility eqn. then gives

$$\nabla \cdot \mathbf{v} = \nabla^2 \psi = 0 \tag{29}$$

The boundary conditions are:

$$\mathbf{v}(r \to \infty) \, = \, \mathbf{V} \tag{30}$$

$$\mathbf{v} \cdot \mathbf{x}(r = R) \, = \, 0 \tag{31}$$

R is the radius of the cylinder. Notice that only normal component of velocity should vanish at $r = R$, since we consider only the region outside boundary layer. Strickly speaking we should impose the second boundary condition at $r = R + \delta$ where δ is thickness of boundary layer, but if δ is small normal component of $\mathbf{v}$ doesn‘t vary significantly (unlike transverse component). Substituting $\psi = \mathbf{V} \cdot \mathbf{x} + f(\mathbf{x})$ we get

$$\nabla^2 f = 0 \tag{32}$$

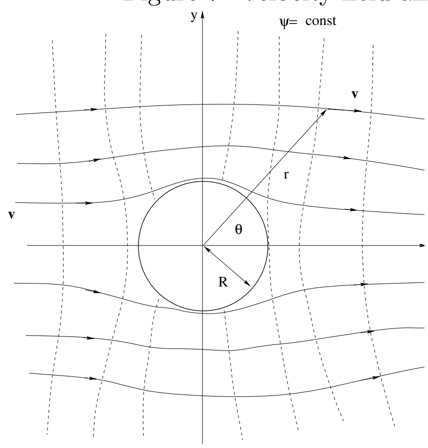
and boundary conditions

$$\nabla f(\infty) = 0, \qquad \nabla f \cdot \frac{\mathbf{x}}{r} = \frac{\partial f}{\partial r}|_{r=R} = -V \cos\theta \tag{33}$$

The general solution of (32) with gradient vanishing at infinity can be written as

$$f = \sum_{n=1}^{\infty} r^{-n} (A_n \cos{(n\theta)} + B_n \sin{(n\theta)}) + A_0 \log r + const \tag{34}$$

Figure 7: Velocity field and equipotentials.(prob. BT-13.6)



We should choose the coefficients A_n, B_n to satisfy the boundary conditions. Obviously we need only the term with $\cos\theta$:

$$f = A\frac{\cos\theta}{r} \tag{35}$$

$$\frac{\partial f}{\partial r}|_{r=R} = -\frac{A\cos\theta}{R^2} = -V\cos\theta \tag{36}$$

$$\Rightarrow \qquad A = VR^2, \qquad f(r,\theta) = \frac{VR^2}{r}\cos\theta \tag{37}$$

$$\Psi = Vr\cos\theta + f(r,\theta) = V\cos\theta\left(r + \frac{R^2}{r}\right) \tag{38}$$

$$\mathbf{v}(\mathbf{x}) = \nabla\Psi = \mathbf{V}(1+\frac{R^2}{r^2}) - \frac{2(\mathbf{V}\cdot\mathbf{x})R^2}{r^4}\mathbf{x} \tag{39}$$

Velocity field and equipotential lines are shown on figure 7

To calculate the pressure use Bernoulli law:

$$\frac{P}{\rho} + \frac{1}{2}v^2 = \frac{P_\infty}{\rho} + \frac{1}{2}V^2 \tag{40}$$

This gives

$$P(\mathbf{x}) = \frac{1}{2}\rho V^2\frac{R^2}{r^2}(4\cos^2\theta - 1) + const \tag{41}$$

The pressure is symmetric under reflections $\theta \to \pi - \theta$ and $\theta \to -\theta$ so the total drag force acting on the cylinder is zero.

$$\mathbf{F}_{drag} = 0 \tag{42}$$

In the real flow the drag force is of course nonzero and directed along flow velocity. To estimate it we should take into account viscous forces acting in

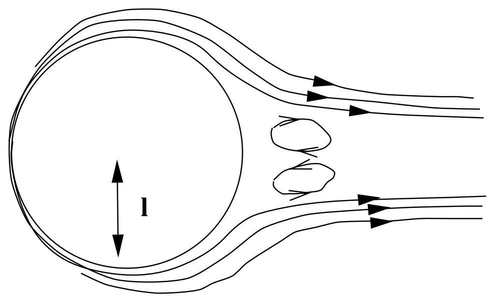


Figure 8: (problem. BT-13.6)

the boundary layer. For large Reynolds numbers ($R_l \sim 10^3 - 10^6$) it's given approximately by (c.f. BT-13.45)

$$\frac{F_{drag}}{\text{unit length}} \propto \frac{\rho V^2 R}{R_l^{1/2}} \tag{43}$$

If $R_l > 10^6$ the flow becomes turbulent and drag force increases.

(Notes): Even for modest Reynolds numbers, say, as low as $R_l \sim 10$, there will be separation of the flow from the back side of the cylinder, which violates our assumption of a thin boundary layer and which leads to a front-to-back asymmetry that gives rise to drag. As $R_l = \frac{Vl}{\nu}$ is increased, the separation occurs around the cylinder, and the eddies on the back side of the cylinder get larger and larger. We shall explore this in the next chapter.

7 Problem D.(BT-13.9)

[by Xinkai Wu/00]
(i) Close to the entrance, $\dot{M} = \rho(\pi a^2) \cdot v_0$

$\Rightarrow v_0 = \frac{\dot{M}}{\rho(\pi a^2)}$

The discussion in the text (13.2.4 Diffusion of Vortex Lines) applies to our case, and we have the following expression for the boundary-layer thickness:

$$\delta(t) \sim (\nu t)^{\frac{1}{2}} \qquad (13.17) \tag{44}$$

Thus letting $\delta(t) = a$, we get $t = a^2/\nu$ and the corresponding travelled distance is

$$l \sim v_0 t = \frac{\dot{M}}{\rho\pi a^2}\frac{a^2}{\nu} = \frac{\dot{M}}{\pi\rho\nu} = \frac{\dot{M}}{\pi\eta} \tag{45}$$

(ii) For $z \gtrsim l$, i.e. after the vorticity has diffused into the center of the flow, the velocity will become z-independent, and Poiseuille formula tells us:

$$v(\varpi) = -\frac{dP}{dz}\frac{a^2 - \varpi^2}{4\eta} = \frac{P_0}{b}\frac{a^2 - \varpi^2}{4\eta} \tag{46}$$

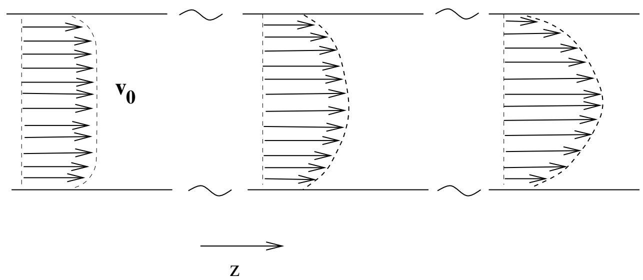


Figure 9: Velocity profile along the pipe. When $z \gtrsim l, v(\varpi = 0) = \frac{P_0}{b}\frac{a^2}{4\eta} = 2v_0$.

and

$$\dot{M} = \frac{\pi \rho a^4}{8\eta}\frac{P_0}{b}. \tag{47}$$

(iii) For this low Reynolds' number laminar flow, after the initial transient segment of the flow in which the bounary layer is growing to encompass the whole flow.

$$\nabla P = \eta \nabla^2 \vec{v} \tag{48}$$

We only have the z-component, i.e.,

$$(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})v = \frac{1}{\eta}\frac{P_0}{b} \tag{49}$$

with boundary condition: v=0 at the wall.

This is a poisson equation that we are familiar with from electrostatics. It can be solved using the appropriate green's function, conformal transformation, etc.. And then the $\dot{M}$ is given by

$$\dot{M} = \rho \int v(x, y) dx dy. \tag{50}$$