CIRCUIT STABILITY

Stability is a property of well-behaved circuits and systems. Typically, stability is discussed in terms of feedback systems. Well-established techniques, such as Nyquist plots, Bode diagrams, and root locus plots are available for studying the stability of feedback systems. Electric circuits can be represented as feedback systems. Nyquist plots, Bode diagrams, and root locus plots can then be used to study the stability of electric circuits.

FEEDBACK SYSTEMS AND STABILITY

Consider a feedback system such as the one shown in Fig. 1. This feedback system consists of three parts: a forward block, sometimes called the "plant," a feedback block, sometimes called the "controller," and a summer. The signals $v_i(t)$ and $v_o(t)$ are the input and output of the feedback system. $A(s)$ is the transfer function of the forward block and $B(s)$ is the transfer function of the feedback block. The summer subtracts the output of the feedback block from $v_i(t)$. The transfer function of the feedback system can be expressed in terms of $A(s)$ and $B(s)$ as

$$T(s) = \frac{V_o(s)}{V_i(s)} = \frac{A(s)}{1 + A(s)B(s)} \tag{1}$$

Suppose that the transfer functions $A(s)$ and $B(s)$ can each be expressed as ratios of polynomials in s. Then

$$A(s) = \frac{N_A(s)}{D_A(s)} \quad \text{and} \quad B(s) = \frac{N_B(s)}{D_B(s)} \tag{2}$$

where $N_A(s)$, $D_A(s)$, $N_B(s)$, and $D_B(s)$ are polynomials in s. Substituting these expressions into Eq. (1) gives

$$T(s) = \frac{\frac{N_A(s)}{D_A(s)}}{1 + \frac{N_A(s)}{D_A(s)}\frac{N_B(s)}{D_B(s)}} = \frac{N_A(s)D_B(s)}{D_A(s)D_B(s) + N_A(s)N_B(s)} = \frac{N(s)}{D(s)} \tag{3}$$

where the numerator and denominator of $T(s)$, $N(s)$ and $D(s)$, are both polynomials in s. The values of s for which $N(s) = 0$ are called the zeros of $T(s)$ and the values of s that satisfy $D(s) = 0$ are called the poles of $T(s)$.

Stability is a property of well-behaved systems. For example, a stable system will produce bounded outputs whenever its input is bounded. Stability can be determined from the poles of a system. The values of the poles of a feedback system will, in general, be complex numbers. A feedback system is stable when all of its poles have negative real parts.

The equation

$$1 + A(s)B(s) = 0 \tag{4}$$

is called the *characteristic equation* of the feedback system. The values of s that satisfy the characteristic equation are poles of the feedback system. The left-hand side of the characteristic equation, $1 + A(s)B(s)$, is called the *return difference* of the feedback system. Figure 2 shows how the return difference can be measured. First, the input, $v_i(t)$, is set to zero. Next, the forward path of the feedback system is broken. Figure 2 shows how a test signal, $V_T(s) = 1$, is applied and the response, $V_R(s) = -A(s)B(s)$, is measured. The difference between the test signal and its response is the return difference.

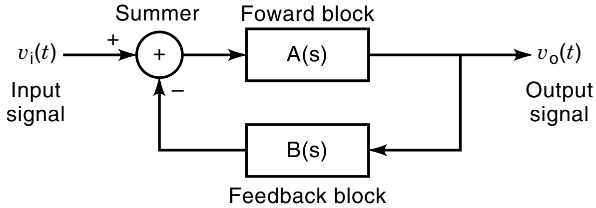

Figure 1. A feedback system.

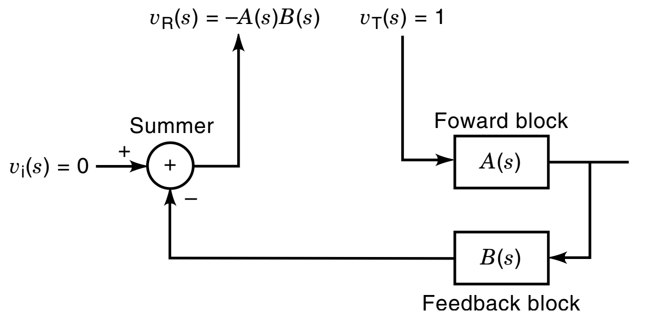

Figure 2. Measuring the return difference: The difference between the test input signal, $V_T(s)$, and the test output signal, $V_R(s)$, is the return difference.

The calculation

$$\text{return difference} = 1 + A(s)B(s) = 1 + \frac{N_A(s)}{D_A(s)}\frac{N_B(s)}{D_B(s)} = \frac{D_A(s)D_B(s) + N_A(s)N_B(s)}{D_A(s)D_B(s)}$$

shows that

1. The zeros of $1 + A(s)B(s)$ are equal to the poles of $T(s)$.
2. The poles of $1 + A(s)B(s)$ are equal to the poles of $A(s)B(s)$.

Consider a feedback system of the form shown in Fig. 1 with

$$A(s) = \frac{s + 5}{s^2 - 4s + 1} \quad \text{and} \quad B(s) = \frac{3s}{s + 3} \tag{5}$$

The poles of the forward block are the values of s that satisfy $s^2 - 4s + 1 = 0$ (that is, $s_1 = 3.73$ and $s_2 = 0.26$). In this case, both poles have real, rather than complex, values. The forward block would be stable if both poles were negative. They are not, so the forward block is itself an unstable system. To see that this unstable system is not well behaved, consider its step response (1,2). The step response of a system is its zero state response to a step input. In other words, suppose the input to the forward block was zero for a very long time. At some particular time, the value of input suddenly becomes equal to 1 and remains equal to 1. The response of the system is called the step response. The step response can be calculated by taking the inverse Laplace transform of $A(s)/s$. In this example, the step response of the forward block is

$$\text{step response} = 5 + 0.675e^{3.73t} - 5.675e^{0.27t}$$

As time increases, the exponential terms of the step response get very, very large. Theoretically, they increase without bound. In practice, they increase until the system saturates or breaks. This is typical of the undesirable behavior of an unstable system.

According to Eq. (3), the transfer function of the whole feedback system is

$$T(s) = \frac{\dfrac{s+5}{s^2-4s+1}}{1+\dfrac{s+5}{s^2-4s+1}\times\dfrac{3s}{s+3}} = \frac{(s+5)(s+3)}{(s^2-4s+1)(s+3)+(s+5)(3s)} = \frac{s^2+8s+15}{s^3+2s^2+4s+3}$$

The poles of the feedback system are the values of s that satisfy $s^3 + 2s^2 + 4s + 3 = 0$—that is, $s_1 = -1$, $s_2 = -0.5 + j1.66$ and $s_3 = -0.5 - j1.66$. The real part of each of these three poles is negative. Since all of the poles of the feedback system have negative real parts, the feedback system is stable. To see that this stable system is well behaved, consider its step response. This step response can be calculated by taking the inverse Laplace transform of $T(s)/s$. In this example, the step response of the feedback system is

$$\text{step response} = 5 - 11.09e^{-t}\cos(\sqrt{2t} + 63^\circ)$$

In contrast to the previous case, as time increases e^{-t} becomes zero so the second term of the step response dies out. This stable system does not exhibit the undesirable behavior typical of unstable systems.

STABILITY CRITERIA

Frequently, the information about a feedback system that is most readily available is the transfer functions of the forward and feedback blocks, $A(s)$ and $B(s)$. Stability criteria are tools for determining if a feedback system is stable by examining $A(s)$ and $B(s)$ directly, without first calculating $T(s)$ and then calculating its poles—that is, the roots of the denominator of $T(s)$. Two stability criteria will be discussed here: the Nyquist stability criteria and the use of Bode diagrams to determine the gain and phase margin.

The Nyquist stability criterion is based on a theorem in the theory of functions of a complex variable (1,3,4). This stability criterion requires a contour mapping of a closed curve in the s-plane using the function $A(s)B(s)$. The closed contour in the s-plane must enclose the right half of the s-plane and must not pass through any poles or zeros of $A(s)B(s)$. The result of this mapping is a closed contour in the $A(s)B(s)$-plane. Fortunately, the computer program MATLAB (5,6) can be used to generate an appropriate curve in the s-plane and do this mapping.

Rewriting the characteristic equation, Eq. (4), as

$$A(s)B(s) = -1 \tag{6}$$

suggests that the relationship of the closed contour in the $A(s)B(s)$-plane to the point $-1 + j0$ is important. Indeed, this is the case. The Nyquist stability criterion involves the number of encirclements of the point $-1 + j0$ by the curve in the $A(s)B(s)$-plane. Let

N = the number of encirclements, in the clockwise direction, of $-1 + j0$ by the closed curve in the $A(s)B(s)$-plane
Z = The number of poles of $T(s)$ in the right half of the s-plane
P = The number of poles of $A(s)B(s)$ in the right half of the s-plane

The Nyquist stability criterion states that N, Z, and P are related by

$$Z = P + N$$

A stable feedback system will not have any poles in the right half of the s-plane so $Z = 0$ indicates a stable system.

For example, suppose the forward and feedback blocks of the feedback system shown in Fig. 1 have the transfer functions described by Eq. (5). Then

$$A(s)B(s) = \frac{3s^2+15s}{s^3-s^2-11s+3} = \frac{3s^2+15s}{(s-3.73)(s-0.26)(s+3)} \tag{7}$$

Figure 3 shows the Nyquist plot for this feedback system. This plot was obtained using the MATLAB commands

```
num=[0  3  15 0]; %Coefficients of the
                   numerator of A(s)B(s)
den=[1 -1 -11 3]; %Coefficients of the
                   denominator of A(s)B(s)
nyquist (num,den)
```

Since $A(s)B(s)$ has two poles in the right half of the s-plane, $P = 2$. The Nyquist plot shows two counterclockwise encirclements of $-1 + j0$ so $N = -2$. Then $Z = P + N = 0$, indicating that the feedback system is stable.

Feedback systems need to be stable in spite of variations in the transfer functions of the forward and feedback blocks. The gain and phase margins of a feedback system give an indication of how much $A(s)$ and $B(s)$ can change without causing the system to become unstable. The gain and phase

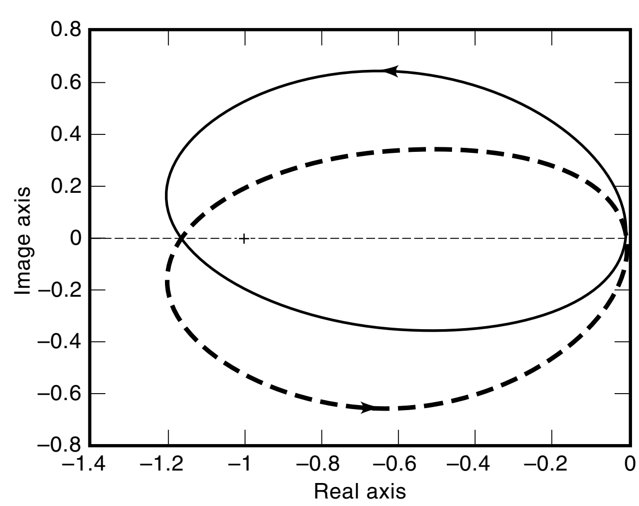

Figure 3. A Nyquist plot produced using MATLAB.

margins can be determined using Bode diagrams. To obtain the Bode diagrams, first let $s = j\omega$ so that Eq. (6) becomes

$$A(j\omega)B(j\omega) = -1$$

The value of $A(j\omega)B(j\omega)$ will, in general, be complex. Two Bode diagrams are used to determine the gain and phase margins. The magnitude Bode diagram is a plot of 20 $\log[|A(j\omega)B(j\omega)|]$ versus ω. The units of 20 $\log[|A(j\omega)B(j\omega)|]$ are decibels. The abbreviation for decibel is dB. The magnitude Bode diagram is sometimes referred to as a plot of the magnitude of $A(j\omega)B(j\omega)$, in dB, versus ω. The phase Bode diagram is a plot of the angle of $A(j\omega)B(j\omega)$ versus ω.

It is necessary to identify two frequencies: ω_g, the gain crossover frequency, and ω_p, the phase crossover frequency. To do so, first take the magnitude of both sides of Eq. (7) to obtain

$$|A(j\omega)B(j\omega)| = 1 \tag{8}$$

Converting to decibels gives

$$20\log[|A(j\omega)B(j\omega)|] = 0 \tag{9}$$

Equation (8) or (9) is used to identify a frequency, ω_g, the gain crossover frequency. That is, ω_g is the frequency at which

$$|A(j\omega_g)||B(j\omega_g)| = 1$$

Next, take the angle of both sides of Eq. (4) to

$$\angle(A(j\omega)B(j\omega)) = 180^\circ \tag{10}$$

Equation (10) is used to identify a frequency, ω_p, the gain crossover frequency. That is, ω_p is the frequency at which

$$\angle A(j\omega_p) + \angle B(j\omega_p) = 180^\circ \tag{11}$$

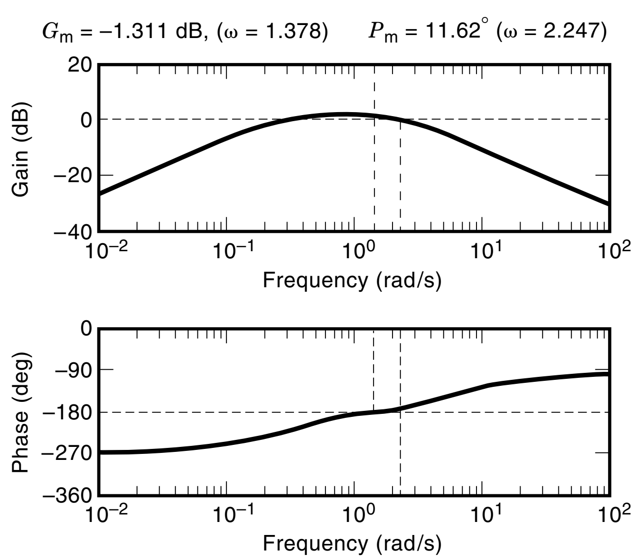

Figure 4. Bode plot used to determine the phase and gain margins. The plots were produced using MATLAB.

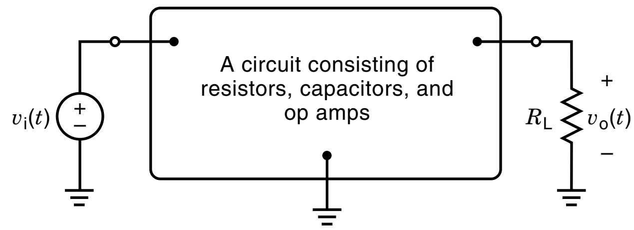

Figure 5. A circuit that is to be represented as a feedback system.

The gain margin of the feedback system is

$$\text{gain margin} = \frac{1}{|A(j\omega_p)|\,|B(j\omega_p)|} \tag{12}$$

The phase margin is

$$\text{phase margin} = 180^\circ - (\angle A(j\omega_g) + \angle B(j\omega_g)) \tag{13}$$

The gain and phase margins can be easily calculated using MATLAB. For example, suppose the forward and feedback blocks of the feedback system shown in Fig. 1 have the transfer functions described by Eq. (3). Figure 4 shows the Bode diagrams for this feedback system. These plots were obtained using the MATLAB commands

```
num=[0  3  15 0]; %Coefficients of the
                    numerator of A(s)B(s)
den=[1 -1 -11 3]; %Coefficients of the
                    denominator of A(s)B(s)
margin (num,den)
```

MATLAB has labeled the Bode diagrams in Fig. 4 to show the gain and phase margins. The gain margin of −1.331 dB indicates that a decrease in $|A(s)B(s)|$ of 1.331 dB or, equivalently, a decrease in gain by a factor of 0.858, at the frequency ω_p = 1.378 rad/s, would bring the system the boundary of instability. Similarly, the phase margin of 11.6° indicates that an increase in the angle of $A(s)B(s)$ of 11.6°, at the frequency ω_g = 2.247 rad/s, would bring the system the boundary of instability.

When the transfer functions $A(s)$ and $B(s)$ have no poles or zeros in the right half of the s-plane, then the gain and phase margins must both be positive in order for the system to be

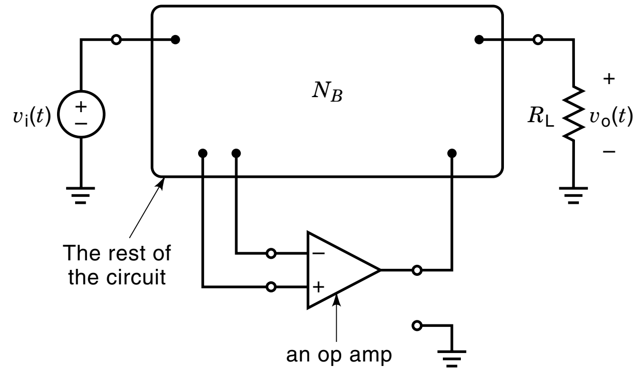

Figure 6. Identifying the subcircuit N_B by separating an op amp from the rest of the circuit.

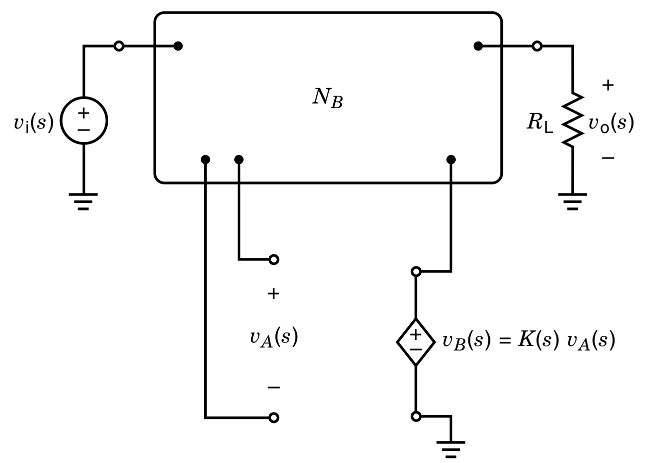

Figure 7. Replacing the op amp with a model of the op amp.

stable. As a rule of thumb (7), the gain margin should be greater than 6 dB and the phase margin should be between 30 and 60°. These gain and phase margins provide some protection against changes in $A(s)$ or $B(s)$.

STABILITY OF LINEAR CIRCUITS

The Nyquist criterion and the gain and phase margin can be used to investigate the stability of linear circuits. To do so requires that the parts of the circuit corresponding to the forward block and to the feedback block be identified. After this identification is made, the transfer functions $A(s)$ and $B(s)$ can be calculated.

Figures 5–8 illustrate a procedure for finding $A(s)$ and $B(s)$ (8). For concreteness, consider a circuit consisting of resistors, capacitors, and op amps. Suppose further that the input and outputs of this circuit are voltages. Such a circuit is shown in Fig. 5. In Fig. 6 one of the op amps has been separated from the rest of the circuit. This is done to identify the subcircuit N_B. The op amp will correspond to the forward block of the feedback system while N_B will contain the feedback block. N_B will be used to calculate $B(s)$. In Fig. 7, the op amp has been replaced by a model of the op amp (2). This model of the op amp indicates that the op amp input and output voltages are related by

$$V_B(s) = K(s)V_A(s) \tag{14}$$

The network N_B can be represented by the equation

$$\begin{pmatrix} V_o(s) \\ V_A(s) \end{pmatrix} = \begin{pmatrix} T_{11}(s) & T_{12}(s) \\ T_{21}(s) & T_{22}(s) \end{pmatrix} \begin{pmatrix} V_i(s) \\ V_B(s) \end{pmatrix} \tag{15}$$

Combining Eqs. (14) and (15) yields the transfer function of the circuit

$$T(s) = \frac{V_o(s)}{V_i(s)} = T_{11}(s) + \frac{T_{12}(s)K(s)T_{21}(s)}{1 - K(s)T_{22}(s)} \tag{16}$$

or

$$T(s) = \frac{V_o(s)}{V_i(s)} = \frac{T_{11}(s)(1 + K(s)T_{22}(s)) + T_{12}(s)K(s)T_{21}(s)}{1 + K(s)T_{22}(s)}$$

Equation (15) suggests a procedure that can be used to measure or calculate the transfer functions $T_{11}(s)$, $T_{12}(s)$, $T_{21}(s$, and $T_{22}(s)$. For example, Eq. (15) says that when $V_i(s) = 1$ and $V_B(s) = 0$, then $V_o(s) = T_{11}(s)$ and $V_A(s) = T_{21}(s)$. Figure 8 illustrates this procedure for determining $T_{11}(s)$ and $T_{21}(s)$. A short circuit is used to make $V_B(s) = 0$ and the voltage source voltage is set to 1 so that $V_i(s) = 1$. Under these conditions the voltages $V_o(s)$ and $V_A(s)$ will be equal to the transfer functions $T_{11}(s)$ and $T_{21}(s)$. Similarly, when $V_i(s) = 0$ and $V_B(s) = 1$, then $V_o(s) = T_{12}(s)$ and $V_A(s) = T_{22}(s)$. Figure 9 illustrates the procedure for determining $T_{12}(s)$ and $T_{22}(s)$. A short circuit is used to make $V_i(s) = 0$ and the voltage source voltage is set to 1 so that $V_{B1}(s) = 1$. Under these conditions the voltages $V_o(s)$ and $V_A(s)$ will be equal to the transfer functions $T_{11}(s)$ and $T_{21}(s)$.

Next, consider the feedback system shown in Fig. 10. (The feedback system shown in Fig. 1 is part, but not all, of the feedback system shown in Fig. 10. When $D(s) = 0$, $C_1(s) = 1$ and $C_2(s) = 1$; then Fig. 10 reduces to Fig. 1. Considering the system shown in Fig. 10, rather than the system shown in

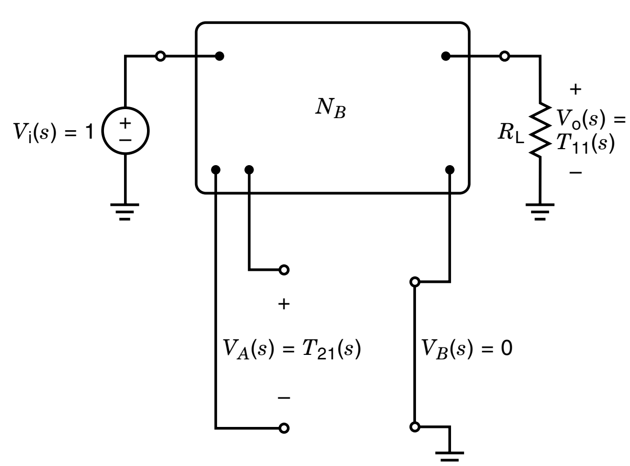

Figure 8. The subcircuit N_B is used to calculate $T_{12}(s)$ and $T_{22}(s)$.

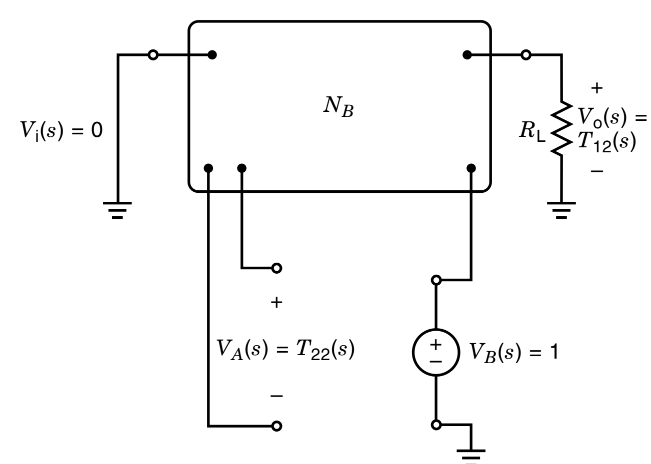

Figure 9. The subcircuit N_B is used to calculate $T_{11}(s)$ and $T_{21}(s)$.

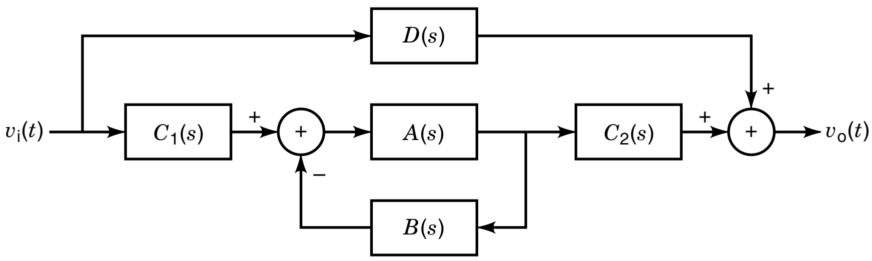

Figure 10. A feedback system that corresponds to a linear system.

Fig. 1, avoids excluding circuits for which $D(s) \neq 0$, $C_1(s) \neq 1$, or $C_2(s) \neq 1$.) The transfer function of this feedback system is

$$T(s) = \frac{V_o(s)}{V_i(s)} = D(s) + \frac{C_1(s)A(s)C_2(s)}{1 + A(s)B(s)} \tag{17}$$

or

$$T(s) = \frac{V_o(s)}{V_i(s)} = \frac{D(s)(1 + A(s)B(s)) + C_1(s)A(s)C_2(s)}{1 + A(s)B(s)}$$

Comparing Eqs. (16) and (17) shows that

$$A(s) = -K(s) \tag{18a}$$
$$B(s) = T_{22}(s) \tag{18b}$$
$$C_1(s) = T_{12}(s)$$
$$C_2(s) = T_{21}(s)$$
$$D(s) = T_{11}(s)$$

Finally, with Eqs. (18a) and (18b), the identification of $A(s)$ and $B(s)$ is complete. In summary,

1. The circuit is separated into two parts: an op amp and N_B, the rest of the circuit.
2. $A(s)$ is open-loop gain of the op amp, as shown in Fig. 7.
3. $B(s)$ is determined from the subcircuit N_B, as shown in Fig. 9.

As an example, consider the Sallen–Key bandpass filter (9) shown in Fig. 11. The transfer function of this filter is

$$T(s) = \frac{V_o(s)}{V_i(s)} = \frac{5460s}{s^2 + 199s + 4 \times 10^6} \tag{19}$$

The first step toward identifying $A(s)$ and $B(s)$ is to separate the op amp from the rest of the circuit, as shown in Fig. 12. Separating the op amp from the rest of the circuit identifies the subcircuit N_B. Next, N_B is used to calculate the transfer functions $T_{11}(s)$, $T_{12}(s)$, $T_{21}(s)$, and $T_{22}(s)$. Figure 13 corresponds to Fig. 8 and shows how $T_{12}(s)$ and $T_{22}(s)$ are calculated. Analysis of the circuit shown in Fig. 13 gives

$$T_{12}(s) = 1 \quad \text{and} \quad T_{22}(s) = \frac{0.259s^2 + 51.6s + 1.04 \times 10^6}{s^2 + 5660s + 4 \times 10^6} \tag{20}$$

(The computer program ELab, Ref. 10, provides an alternative to doing this analysis by hand. ELab will calculate the transfer function of a network in the form shown in Eq. (16)—that is, as a symbolic function of s. ELab is free and can be downloaded from http://sunspot.ece.clarkson.edu:1050/~svoboda/software.html on the World Wide Web.)

Figure 14 corresponds to Fig. 9 and shows how $T_{11}(s)$ and $T_{21}(s)$ are calculated. Analysis of the circuit shown in Fig. 14 gives

$$T_{11}(s) = 0 \quad \text{and} \quad T_{21}(s) = \frac{-1410s}{s^2 + 5660s + 4 \times 10^6} \tag{21}$$

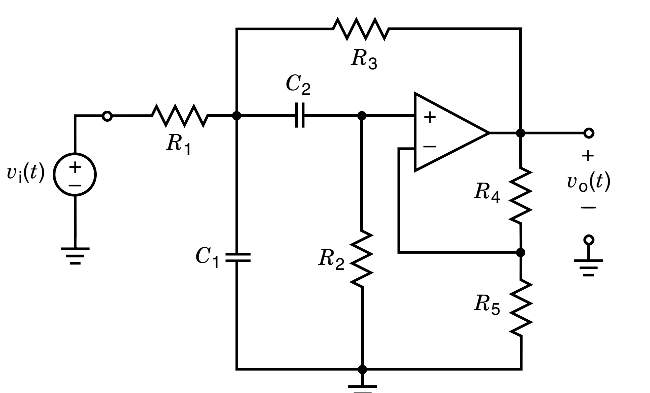

Figure 11. A Sallen-Key bandpass filter. $R_1 = R_2 = R_3 = R_5 = 7.07$ kΩ, $R_4 = 20.22$ kΩ, and $C_1 = C_2 = 0.1$ μF.

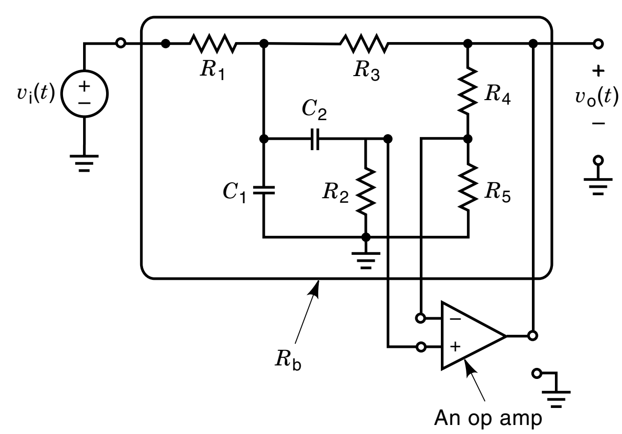

Figure 12. Identifying the subcircuit N_B by separating an op amp from the rest of the circuit.

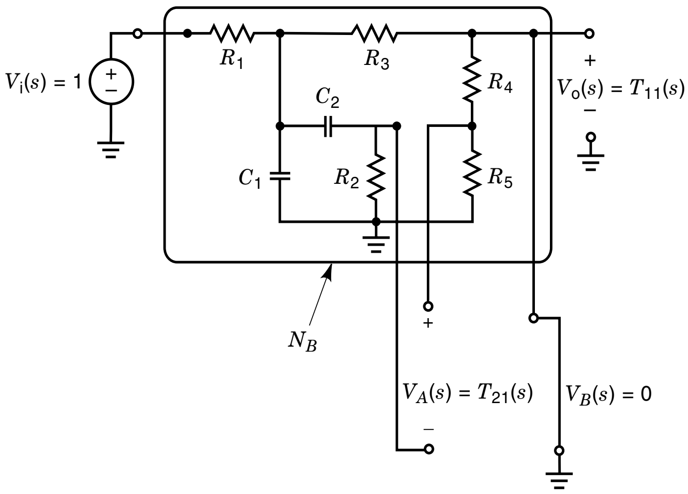

Figure 13. The subcircuit N_{B1} is used to calculate $T_{11}(s)$ and $T_{21}(s)$.

Substituting Eqs. (20) and (21) into Eq. (16) gives

$$T(s) = \frac{K(s)\left(\dfrac{-1410s}{s^2 + 5660s + 4 \times 10^6}\right)}{1 - K(s)\left(\dfrac{0.259s^2 + 51.6s + 1.04 \times 10^6}{s^2 + 5660s + 4 \times 10^6}\right)} \tag{22}$$

When the op amp is modeled as an ideal op amp, $K(s) \to \infty$ and Eq. (22) reduces to Eq. (19). This is reassuring but only confirms what was already known. Suppose that a more accurate model of the op amp is used. A frequently used op amp model (2) represents the gain of the op amp as

$$K(s) = -\frac{A_o}{s + \dfrac{B}{A_o}} \tag{23}$$

where A_o is the dc gain of the op amp and B is the gain-bandwidth product of the op amp (2). Both A_o and B are readily available from manufacturers specifications of op amps. For example, when the op amp is a μA741 op amp, then A_o = 200,000 and $B = 2\pi * 10^6$ rad/s, so

$$K(s) = -\frac{200,000}{s + 31.4}$$

Equation (18) indicates that $A(s) = -K(s)$ and $B(s) = T_{22}(s)$, so in this example

$$A(s) = \frac{200,000}{s + 31.4} \quad \text{and} \quad B(s) = 0.259\left(\frac{s^2 + 51.6s + 1.04 \times 10^6}{s^2 + 5600s + 4 \times 10^6}\right)$$

To calculate the phase and gain margins of this filter, first calculate

$$A(s)B(s) = \frac{51,800(s^2 + 51.6s + 1.04 \times 10^6)}{s^3 + 5974s^2 + 5777240s + 1246 \times 10^6}$$

Next, the MATLAB commands

```
num=20000*[0  0.259  51.6  1040000];
     %Numerator Coefficients
den=[1 5974 5777240 1256*10^6];
     %Denominator Coefficients
margin(num,den)
```

are used to produce the Bode diagram shown in Fig. 15. Figure 15 shows that the Sallen–Key filter will have an infinite gain margin and a phase margin of 76.5° when a μA741 op amp is used.

OSCILLATORS

Oscillators are circuits that are used to generate a sinusoidal output voltage or current. Typically, oscillators have no input. The sinusoidal output is generated by the circuit itself. This section presents the requirements that a circuit must satisfy if it is to function as an oscillator and shows how these requirements can be used to design the oscillator.

To begin, recall that the characteristic equation of a circuit is

$$1 + A(s)B(s) = 0$$

Suppose this equation is satisfied by a value of s of the form $s = 0 + j\omega_o$. Then

$$A(j\omega_o)B(j\omega_o) = -1 = 1e^{j180^\circ} \tag{24}$$

In this case, the steady-state response of the circuit will contain a sustained sinusoid at the frequency ω_o (11). In other words, Eq. (24) indicates that the circuit will function as an oscillator with frequency ω_o when $A(j\omega_o)B(j\omega_o)$ has a magnitude equal to 1 and a phase angle of 180°.

As an example, consider using Eq. (24) to design the Wien-bridge oscillator, shown in Fig. 16, to oscillate at ω_o = 1000 rad/s. The first step is to identify $A(s)$ and $B(s)$ using the procedure described in the previous section. In Fig. 17 the amplifier is separated from the rest of the network to identify the subcircuit N_B. Also, from Eqs. (14) and (18),

$$A(s) = -K$$

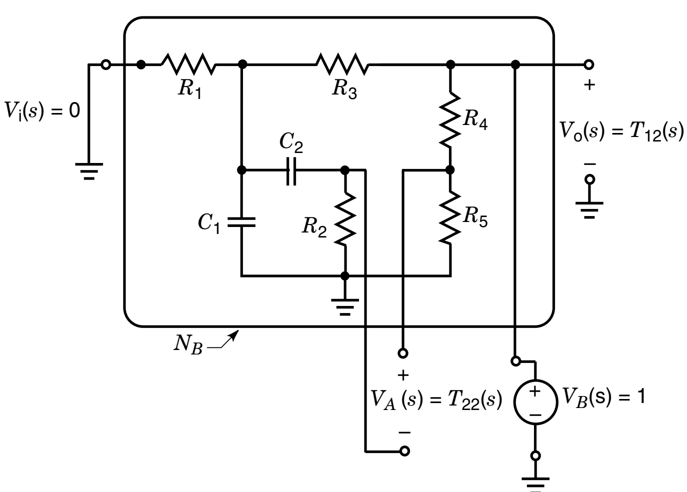

Figure 14. The subcircuit N_B is used to calculate $T_{12}(s)$ and $T_{22}(s)$.

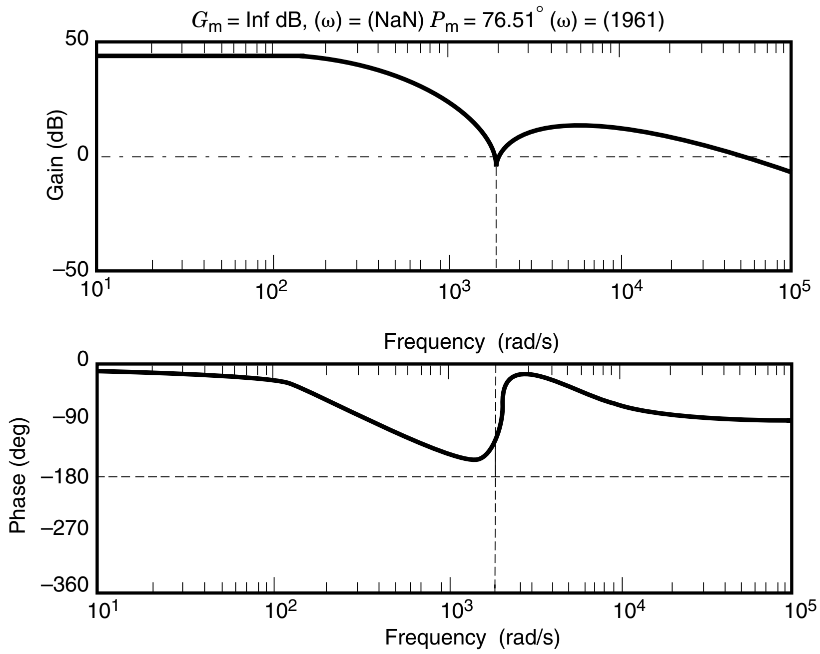

Figure 15. The Bode diagrams used to determine the phase and gain margins of the Sallen–Key bandpass filter.

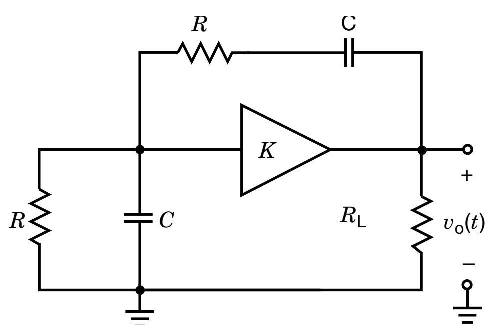

Figure 16. A Wien-bridge oscillator.

Next, the subcircuit N_B is used to determine $B(s) = T_{22}(s)$, as shown in Fig. 18. From Fig. 18 it is seen that

$$T_{22}(s) = \frac{\dfrac{\frac{1}{Cs} * R}{\frac{1}{Cs} + R}}{\dfrac{\frac{1}{Cs} * R}{\frac{1}{Cs} + R} + \left(R + \dfrac{1}{Cs}\right)} = \frac{1}{1 + \left(R + \dfrac{1}{Cs}\right)\dfrac{\left(R + \frac{1}{Cs}\right)}{\left(R * \frac{1}{Cs}\right)}}$$

$$= \frac{1}{1 + \left(R + \dfrac{1}{Cs}\right)\left(Cs + \dfrac{1}{R}\right)} = \frac{1}{3 + RCs + \dfrac{1}{RCs}}$$

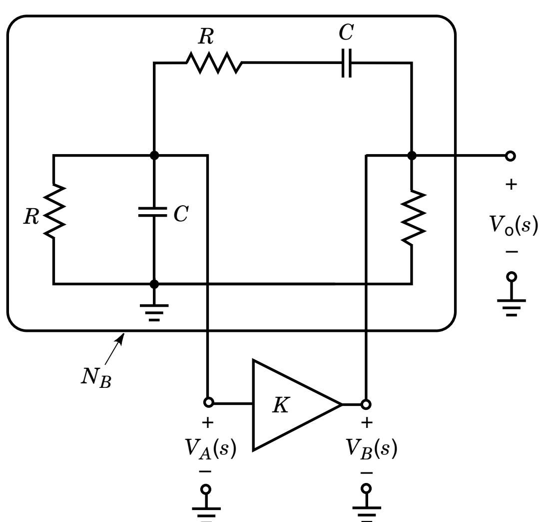

Figure 17. The amplifier is separated from the rest of the Wien-bridge oscillator to identify the subcircuit N_B.

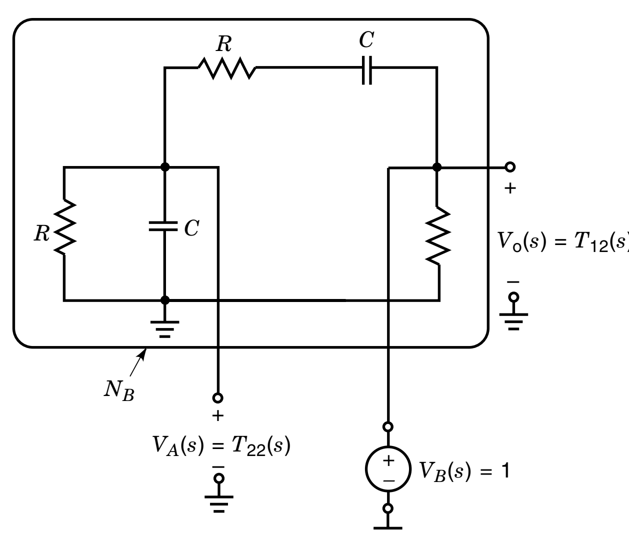

Figure 18. The subcircuit N_B is used to calculate $B(s) = T_{22}(s)$ for the Wien-bridge oscillator.

so

$$A(s)B(s) = \frac{-K}{3 + RCs + \dfrac{1}{RCs}}$$

Now let $s = 0 + j\omega_o$ to get

$$A(j\omega_o)B(j\omega_o) = \frac{-K}{3 + j\omega_o RC - j\dfrac{1}{\omega_o RC}} \quad (25)$$

The phase angle of $A(j\omega_o)B(j\omega_o)$ must be 180° if the circuit is to function as an oscillator. That requires

$$j\omega_o RC - j\frac{1}{\omega_o RC} = 0 \Rightarrow \omega_o = \frac{1}{RC} \quad (26)$$

Oscillation also requires that the magnitude of $A(j\omega_o)B(j\omega_o)$ be equal to 1. After substituting Eq. (26) into Eq. (25), this requirement reduces to

$$K = 3$$

That is, the amplifier gain must be set to 3. Design of the oscillator is completed by picking values of R and C to make ω_o = 1000 rad/s (e.g., R = 10 kΩ and C = 0.1 μF).

THE ROOT LOCUS

Frequently the performance of a feedback system is adjusted by changing the value of a gain. For example, consider the feedback system shown in Fig. 1 when

$$A(s) = \frac{N_A(s)}{D_A(s)} \quad \text{and} \quad B(s) = K \quad (27)$$

In this case, $A(s)$ is the ratio of two polynomials in s and $B(s)$ is the gain that is used to adjust the system. The transfer function of the feedback system is

$$T(s) = \frac{N_A(s)}{D_A(s) + KN_A(s)} = \frac{N(s)}{D(s)} \quad (28)$$

The poles of feedback system are the roots of the polynomial

$$D(s) = D_A(s) + KN_A(s) \quad (29)$$

Suppose that the gain K can be adjusted to any value between 0 and ∞. Consider the extreme values of K. When K= 0, $D(s) = D_A(s)$ so the roots of $D(s)$ are the same as the roots of $D_A(s)$. When $K = \infty$, $D_A(s)$ is negligible compared to $KN_A(s)$. Therefore $D(s) = KN_A(s)$ and the roots of $D(s)$ are the same as the roots of $N_A(s)$. Notice that the roots of $D_A(s)$ are the poles of $A(s)$ and the roots of $N_A(s)$ are the zeros of $A(s)$. As K varies from 0 and ∞, the poles of $T(s)$ start at the poles of $A(s)$ and migrate to the zeros of $A(s)$. The root locus is a plot of the paths that the poles of $T(s)$ take as they move across the s-plane from the poles of $A(s)$ to the zeros of $A(s)$.

A set of rules for constructing root locus plots by hand are available (1,4,7,13). Fortunately, computer software for constructing root locus plots is also available. For example, suppose that the forward and feedback blocks in Fig. 1 are described by

$$A(s) = \frac{s(s-2)}{(s+1)(s+2)(s+3)} = \frac{s^2 - 2s}{s^3 + 6s^2 + 11s + 6} \quad \text{and} \quad B(s) = K$$

The root locus plot for this system is obtained using the MATLAB (5,6) commands

```
num=([0 1 -2 0]);
den=([1 6 11 6]);
rlocus(num, den)
```

This root locus plot is shown in Fig. 19. After the root locus has been plotted, the MATLAB command

```
rlocfind(num, den)
```

can be used to find the value of the gain K corresponding to any point on the root locus. For example, when this command is given and the cursor is placed on the point where the locus crosses the positive imaginary axis, MATLAB indicates that

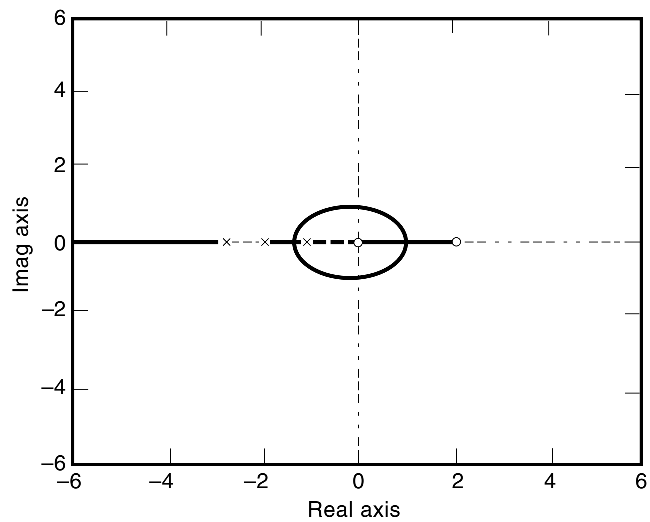

Figure 19. A root locus plot produced using MATLAB. The poles of $A(s)$ are marked by x's and the zeros of $A(s)$ are marked by o's. As K increases from zero to infinity, the poles of $T(s)$ migrate from the poles of $A(s)$ to the zeros of $A(s)$ along the paths indicated by solid lines.

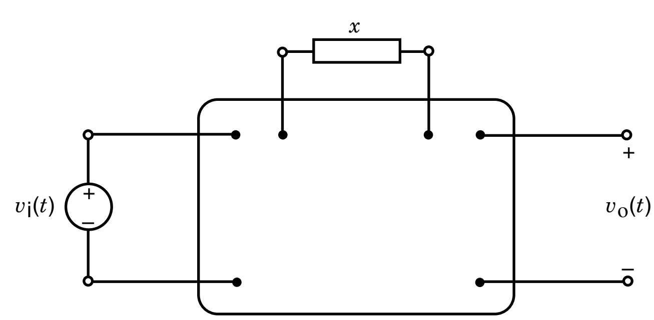

Figure 20. A single device is separated from the rest of the network. The parameter associated with this device is called x. The transfer function of the network will be a bilinear function of x.

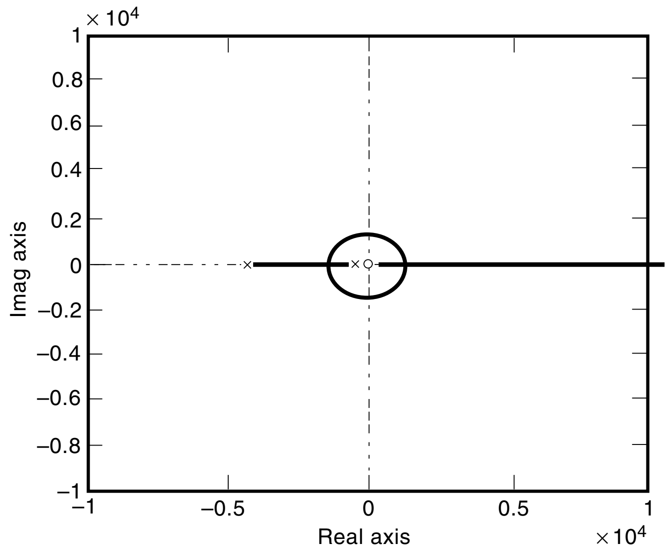

Figure 21. This root locus plot shows that the poles of the Sallen–Key bandpass filter move into the right of the s-plane as the gain increases.

gain corresponding to the point $0.0046 + j0.7214$ is $K = 5.2678$. For gains larger than 5.2678, two poles of $T(s)$ are in the right half of the s-plane so the feedback system is unstable.

The bilinear theorem (12) can be used to make a connection between electric circuits and root locus plots. Consider Fig. 20, where one device has been separated from the rest of a linear circuit. The separated device could be a resistor, a capacitor, an amplifier, or any two-terminal device (12). The separated device has been labeled as x. For example, x could be the resistance of a resistor, the capacitance of a capacitor, or the gain of an amplifier. The bilinear theorem states that the transfer function of the circuit will be of the form

$$T(s) = \frac{V_o(s)}{V_i(s)} = \frac{E(s) + xF(s)}{G(s) + xH(s)} = \frac{N(s)}{D(s)} \tag{30}$$

where $E(s)$, $F(s)$, $G(s)$, and $H(s)$ are all polynomials in s. A transfer function of this form is said to be a bilinear function of the parameter x since both the numerator and denominator polynomials are linear functions of the parameter x. The poles of $T(s)$ are the roots of the denominator polynomial

$$D(s) = G(s) + xH(s) \tag{31}$$

As x varies from 0 to ∞, the poles of $T(s)$ begin at the roots of $G(s)$ and migrate to the roots of $H(s)$. The root locus can be used to display the paths that the poles take as they move from the roots of $G(s)$ to the roots of $H(s)$. Similarly, the root locus can be used to display the paths that the zeros of $T(s)$ take as they migrate from the roots of $E(s)$ to the roots of $F(s)$.

For example, consider the Sallen–Key bandpass filter shown in Fig. 11. When

$$R_1 = R_2 = R_3 = 7.07\ \text{k}\Omega, C_1 = C_2 = 0.1\ \mu\text{F}, \quad \text{and} \quad K = 1 + \frac{R_4}{R_5}$$

then the transfer function of this Sallen–Key filter is

$$\begin{aligned} T(s) &= \frac{K(1414s)}{s^2 + (4-K)(1414s) + 4 \times 10^6} \\ &= \frac{K(1414s)}{(s^2 + 5656s + 4 \times 10^6) + K(-1414s)} \end{aligned} \tag{32}$$

As expected, this transfer function is a bilinear function the gain K. Comparing Eqs. (30) and (32) shows that $E(s) = 0$, $F(s) = 1414s$, $G(s) = s^2 + 5656s + 4 \times 10^5$, and $H(s) = -1414s$. The root locus describing the poles of the filter is obtained using the MATLAB commands

```
G=([1   5656  4*10^6]);
H=([0  -1414    0]);
rlocus(H,G)
```

Figure 21 shows the resulting root locus plot. The poles move into the right half of the s-plane, and the filter becomes unstable when $K > 4$.

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