ELLIPTIC FILTERS

An electrical filter will be defined as an electrical system that can process electrical signals on the basis of the frequencies composing that signal. This signal processing can affect both the magnitude and phase of each individual frequency component of the signal. For example, the output signal of an antenna may represent an electrical signal that requires magnitude processing. The output signal of the antenna has a fairly wide spectrum of frequencies, and yet we would like only a small range of these frequencies, such as those centered around our favorite radio station, for example, to be processed for our listening pleasure. One solution is to use a band-pass filter in one of the stages following the antenna. The circuit would process that signal in such a way that the band of frequencies containing the information from the station would be passed, and the signals outside of that band would be rejected or would not pass through. Although this example is very much simplified in comparison to what actually happens, it nonetheless illustrates the general idea of filtering.

Because of a need to filter signals in a variety of ways, several "standard" types of filters or signal processing schemes have evolved. These are low-pass, high-pass, band-pass, band-reject, and all-pass filters. Low-pass filters strive to allow frequencies below some predetermined cutoff frequency to pass, while rejecting those frequencies above the cutoff frequency. High-pass filters strive to allow frequencies above some predetermined cutoff frequency to pass, while rejecting those frequencies below the cutoff frequency. Band-pass filters allow a band of frequencies to pass, while rejecting frequencies outside of that band. Band-reject filters reject a band of frequencies, allowing frequencies outside that band to pass. The main objective of these four types of filters is to process the signal's magnitude as a function of frequency. The all-pass filter lets all signals pass through, but selects a band of frequencies for phase angle processing. The choice of filter depends on the application.

Filter design can be broken down into two broad phases. The first phase is the selection of a transfer function possessing the mathematical properties of filtering. A transfer function describes the relationship between the input signal and the output signal. We will use it in the sense that for a given input signal, we will have a specific output signal. Since filters process electrical signals according to the frequency content, the transfer function for a filter is a function of $s = j\omega = j2\pi f$, where ω is the frequency in radians/s and f is the frequency in hertz.

The second phase of filter synthesis is realization of a circuit that possesses the same transfer function as the mathematical function selected to do the filtering. The circuit may be an analog, digital, or a mixed analog–digital circuit depending on the application.

THE APPROXIMATION PROBLEM

When filtering, engineers tend to think in terms of ideal filters. For example, when deciding to use a low-pass filter, the engineer typically desires that all frequencies above a defined cutoff frequency should be eliminated. An ideal low-pass transfer function magnitude response with a cutoff frequency of 1 rad/s is shown in Fig. 1, and the ideal low-pass transfer function phase characteristics are shown in Fig. 2. For the magnitude plot, all frequencies below 1 rad/s are passed, with a gain of one, and all frequencies above 1 rad/s are rejected. It is a "brick wall" function. It is intuitively obvious that this is an ideal magnitude characteristic. The ideal phase characteristic is not so intuitive. The important feature of the ideal phase characteristics are not the values of the phase angle, but that the phase response is linear. A transfer function that has linear phase characteristics means that separate signals composed of two different frequencies applied at the same instant of time at the input of the filter will arrive, after processing, at the output of the filter at the same time. If the two input signals add together to create a distinct response in the time domain at the input, it may be important that they reconstruct together at the output to maintain the "shape" of the input signal. Sometimes this is important, and sometimes it is not. A deviation from the linear phase response results in phase distortion. The functions shown in Figs. 1 and 2 are normalized filters. That is, they have cutoff frequencies of 1 rad/s and maximum gains of 0 dB, or 1 V/V in the passband. It is conventional to begin a filter design with a normalized filter. This allows for a common starting point for all low-pass filters, for example, and is also a convenient way of comparing the characteristics of other different types of low-pass filter

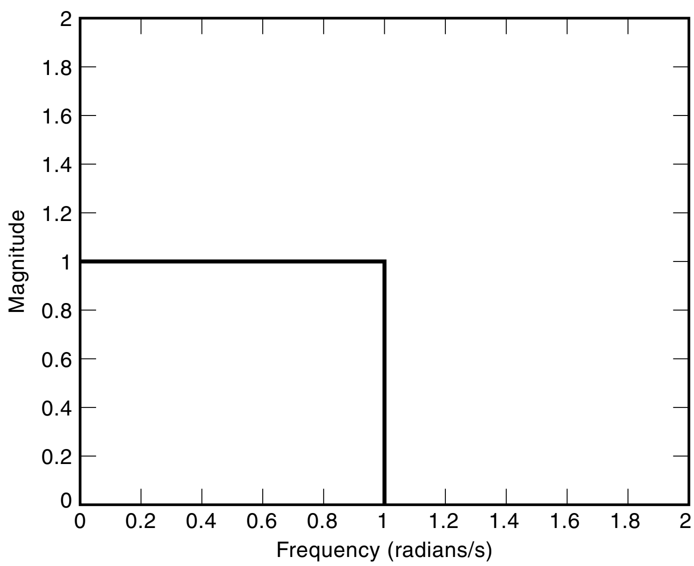

Figure 1. The magnitude versus frequency plot of an ideal low-pass filter transfer function shows that all frequencies of a signal below 1 rad/s are passed while those above 1 rad/s are rejected.

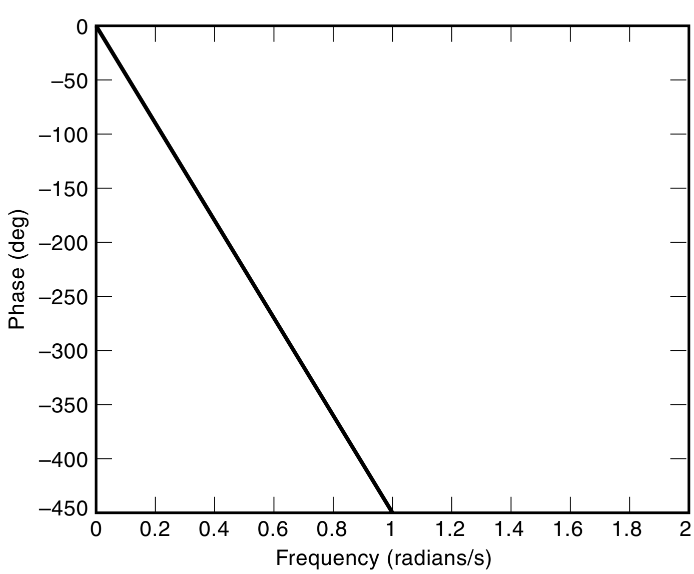

Figure 2. The ideal low-pass filter function phase characteristics may be summarized as a linear phase response.

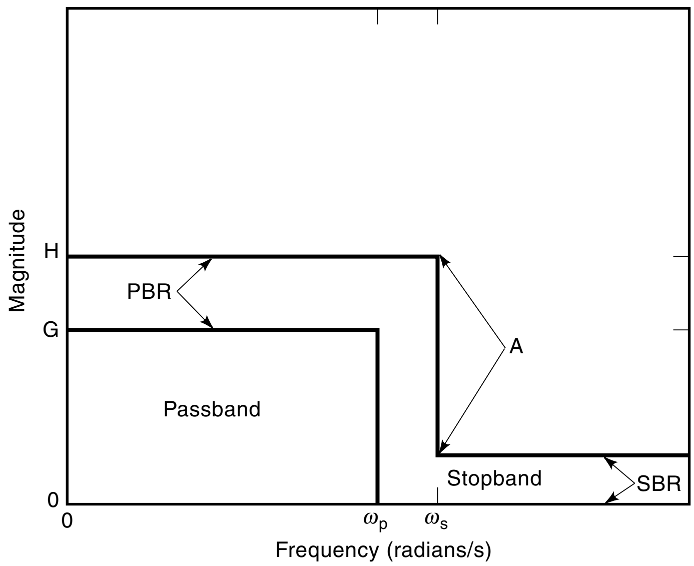

Figure 3. PBR, SBR, A, passband, and stopband are ways of characterizing an actual filter function's magnitude versus frequency response to that of an ideal response. For an ideal filter function, PBR = SBR = 0 V/V and ω_s and ω_p are equal.

functions with each other. Moreover, numerous tables exist that provide the coefficients or poles and zeros of filter transfer functions. These tables provide information for normalized filters. Since there is an infinite number of possibilities for cutoff frequencies, it would be impractical, if not impossible, to generate tables for all possible cases. Thus, the tables deal only with the normalized case. It is trivial to scale a filter for a desired cutoff frequency from the normalized frequency.

The first step in low-pass filter design is to find a transfer function of s having the same characteristics as the transfer function depicted in Fig. 1 and being realizable with a circuit. Without having transfer function with an infinite number of terms, it is impossible to devise a transfer function with those characteristics. Hence, from this simple example arises the *approximation problem.* That is, may we find a transfer function magnitude response that approaches that shown in Fig. 1? In general, the higher the order of the filter, the closer the transfer function magnitude response will approach the ideal case. However, the higher the order of a filter, the more complex the design and the more components that are needed to realize the transfer function. Thus, the concept of trade-offs and compromises arise. In general, a set of filter specifications must be determined before selecting the transfer function. The specifications may viewed as how far the actual filter response may deviate from the ideal response.

Since it is impossible to come up with an ideal transfer function that is practically realizable, several terms have been defined and have been accepted as convention that allows the description of the deviation of a practical filter function from the ideal filter function. These terms are depicted in Fig. 3 and may be referred to as filter specifications. The specifications are: the passband, the stopband, the passband ripple (PBR), the stopband ripple (SBR), and the stopband attenuation, A. PBR, SBR, and A are usually specified in dB. There are three distinct regions. The passband is located from 0 rad/s to ω_p rad/s. The second region is the stopband region located from ω_s to infinity. Lastly, the transition region is composed of the range between ω_p and ω_s. Figure 3 should be interpreted as follows: Signals at or below ω_p will have a gain of at least of G dB and at the most H dB, and signals above ω_s will be attenuated by at least A dB or will have a maximum gain of SBR dB. Note that $(G–H)$ dB = PBR in dB. Filter types other than low-pass filters have similar specifications, and the reader is encouraged to investigate these (1,2).

Past research in network theory has resulted in several classic mathematical approximations to the ideal filter magnitude function. Each of these were designed to optimize a property of the filter function. The low-pass filter approximations are usually of the form

$$|H(j\omega)|^2 = \frac{1}{1+\epsilon^2 T^2(\omega)} \qquad (1)$$

By replacing $T(\omega)$ with different functions, different approximations arise. The standard approximations to the ideal magnitude response are the Butterworth Approximation, the Chebychev Approximation, the Inverse-Chebychev Approximation, and the Elliptic Approximation. Each has strong points and weak points. A fifth classic approximation worth mentioning is the Bessel function. This approximation will not be discussed, because it strives to approximate the ideal phase response. This article will focus on the Elliptic Approximation.

THE ELLIPTIC APPROXIMATION

Before a mathematical discussion on the Elliptic Approximation is begun, it is useful to examine a plot of the magnitude of an elliptic filter function. A fifth order elliptic filter low-pass transfer function magnitude response is depicted in Fig. 4. The passband exists for $\omega \le \omega_p$. The stopband exists for $\omega \ge \omega_s$. The passband and stopband may be characterized as *equiripple.* That is, the amplitude oscillates between a maxi-

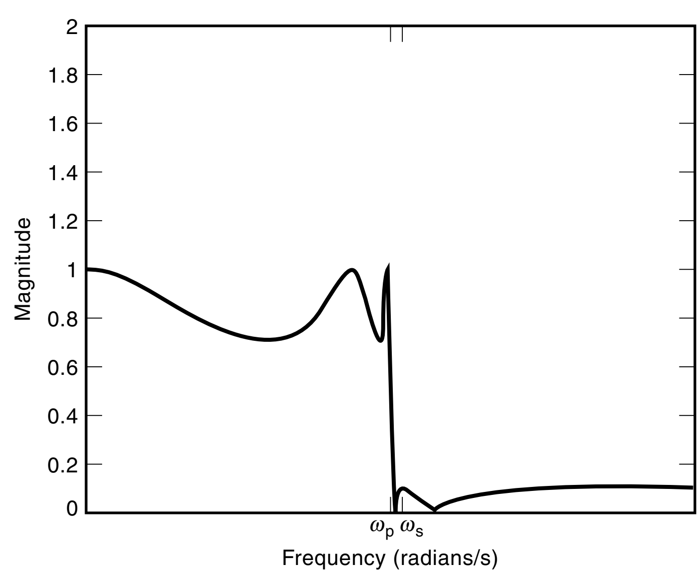

Figure 4. The magnitude characteristics of a fifth-order elliptic filter show an equiripple passband and stopband, and at zero at $\omega = \infty$.

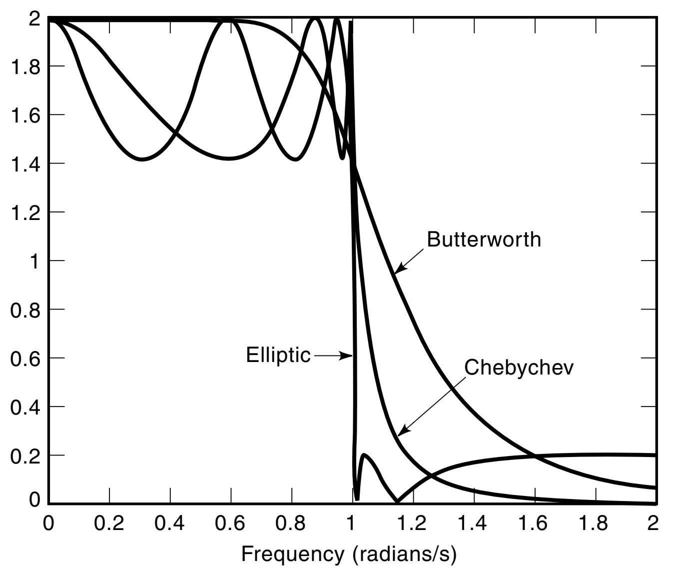

Figure 5. The fifth-order elliptic function magnitude characteristics show a much sharper transition from the passband to the stopband than the Butterworth and Chebychev function characteristics.

mum and minimum throughout a fraction of each band. If the order of the filter is even, there are $n/2$ peaks in the passband, and $n/2$ minimums or zeros in the stopband. If the order of the filter is odd, there are $(n - 1)/2$ peaks, plus one at $\omega = 0$ in the passband. Also for the odd order case, there are $(n - 1)/2$ minimums or zeros, plus one at $\omega = \infty$, in the stopband.

In discussing the properties of the elliptic filter, it is important to compare its characteristics with the other classic filter types. A comparison of fifth-order, low-pass, normalized, Butterworth, Chebychev, and elliptic filter magnitude functions is given in Fig. 5, and comparison of the phases is shown in Fig. 6. From these comparisons, Table 1 may be generated.

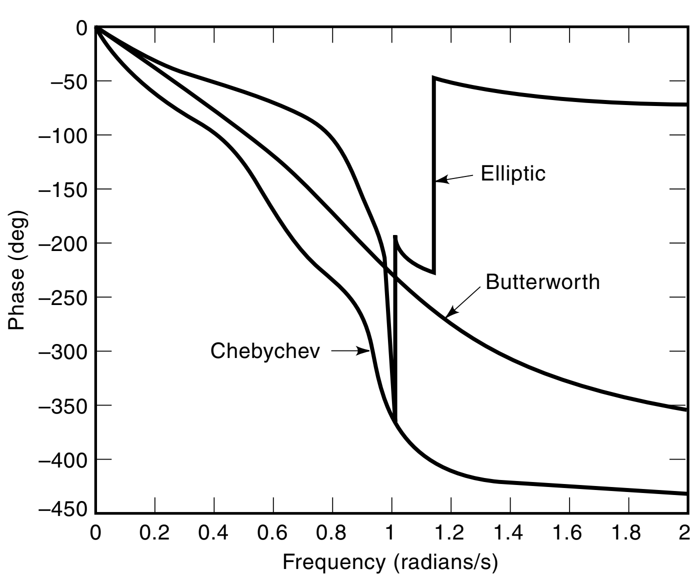

Figure 6. The fifth-order elliptic function phase characteristics deviate much farther from the desired ideal linear phase characteristics than the Butterworth and Chebychev function characteristics.

Table 1. Filter Comparisons

Filter Type	Transition Region	Linear Phase Properties
Butterworth	Poor	Good
Chebychev	Good	Poor
Elliptic	Best	Very poor

From this discussion, the main attribute of the elliptic filter may be stated. That is, for a given filter order, it provides the sharpest cutoff characteristics; and thus out of all three filters, it best approximates the ideal low-pass magnitude function in terms of a sharp transition region. This is very important if filtering is required for frequencies near each other, if we would like to pass one of these signals, and reject another. The compromise in using the elliptic filter is its very poor phase characteristics.

The theory behind the mathematics of the elliptic filter is complicated and is not suitable for this publication. Interested readers may consult Refs. 2 and 3. A summary of the mathematics is discussed in this article.

The general form of the elliptic filter magnitude squared transfer function is given by Eq. (1). For the low-pass elliptic filter function, $T(j\omega)$ is replaced with $R_n(j\omega)$. $R_n(j\omega)$ has two different forms: one for an even-order function and one for an odd-order function. $R_n(j\omega)$ will be described for a normalized low-pass filter. For the even-order case we have

$$R_n(\omega) = M \prod_{i=1}^{n/2} \frac{\omega^2 - (\omega_s/\omega_i)^2}{\omega^2 - \omega_i^2} \tag{2}$$

For the odd-order case we have

$$R_n(\omega) = N\omega \prod_{i=1}^{(n-1)/2} \frac{\omega^2 - (\omega_s/\omega_i)^2}{\omega^2 - \omega_i^2} \tag{3}$$

M and N are normalization constants and are chosen so that $R_n(1) = 1$. The ω_i are calculated for the even or odd case. For the even case we have

$$\omega_i = \frac{\omega_s}{sn\left[\dfrac{(2i-1)K\left(\dfrac{1}{\omega_s}\right)}{n}\right]} \tag{4}$$

and for the odd case we obtain

$$\omega_i = \frac{\omega_s}{sn\left[\dfrac{2iK\left(\dfrac{1}{\omega_s}\right)}{n}\right]} \tag{5}$$

$K(k)$ is the complete elliptic integral of the first kind and is defined as

$$K(x) = \int_0^{\pi/2} (1 - k^2 \sin^2 x)^{-1/2}\, dx \tag{6}$$

and sn is the Jacobian elliptic sine function.

The order of the filter function may be determined by rounding up n to the nearest integer in the expression

$$n = \frac{K\left(\dfrac{1}{\omega_s}\right)K'\left(\dfrac{1}{L}\right)}{K'\left(\dfrac{1}{\omega_s}\right)K\left(\dfrac{1}{L}\right)} \tag{7}$$

where L is defined as

$$L = \sqrt{\frac{10^{0.1\,\mathrm{PBR}} - 1}{10^{0.1\,A} - 1}} \tag{8}$$

and PBR and A are in decibels. Lastly,

$$K'(k) = K(\sqrt{(1-k^2)}) \tag{9}$$

When $R_n(\omega)$ is found, the substitution $s = \omega/j$ is made, and $R_n(\omega/j)$ may be inserted into Eq. (1). The poles of $H(s)H(s^*)$ can be found. This is a standard synthesis technique (4). The left half-plane poles and half of the zeros are selected and combined to give the final form of the elliptic filter transfer function. For n even, we have

$$H(s) = H\frac{\prod_{i=1}^{n/2}(s^2 + \omega_i^2)}{a_0 + a_1 s + \cdots + a_{n-1}s^{n-1} + a_n s^n} \tag{10}$$

For the case of n odd, we obtain

$$H(s) = H\frac{\prod_{i=1}^{n-1/2}(s^2 + \omega_i^2)}{a_0 + a_1 s + \cdots + a_{n-1}s^{n-1} + a_n s^n} \tag{11}$$

Note that the even-order transfer function given by Eq. (10) has no zeros at infinity while the odd-order transfer function of Eq. (11) has a single zero at infinity. It may be convenient to have the denominator in the form of coefficients or in terms of products of poles and zeros, depending on the type of realization.

Because of the complexity of the calculations required to find the transfer function, the usual method of finding $H(s)$ is usually either with a computer program or using one of the numerous tables that have been generated and published (2,5).

REALIZATIONS OF ELLIPTIC FILTERS

There are an infinite number of possible synthesis algorithms that may be used. In this section we describe one.

The first step in elliptic filter design is to find a transfer function that will meet a set of specifications that have been determined from the application. The design usually begins with specifying the passband frequency ω_p, PBR, ω_s, and A. If filter tables are to be used, the frequencies of the filter specifications must first be normalized. This is achieved by dividing ω_s and ω_p by ω_p. Other normalizations are possible. This results in a normalized passband frequency of 1, and a normalized stopband frequency of ω_s/ω_p. If a highpass filter is desired, the specifications must be transformed further, by inverting ω_p to $1/\omega_p$. Once the desired transfer function is determined, a method of realization is selected. The realizations may be analog or digital. The analog realizations may be passive or active. The choice depends on many practical issues (1).

Passive Realizations

Passive realizations utilize capacitors, inductors, and resistors. The source and load resistances are considered part of the realization. Systematic procedures exist for the synthesis of passive filters to realize elliptic transfer functions. Moreover, normalized passive filters are available as part of table look-up approaches (5). Examples of passive elliptic filters are depicted in Fig. 7.

Even-order passive realizations synthesized from Eq. (10) will have negative elements because they do not have at least a zero at infinity. This problem can be solved by shifting the

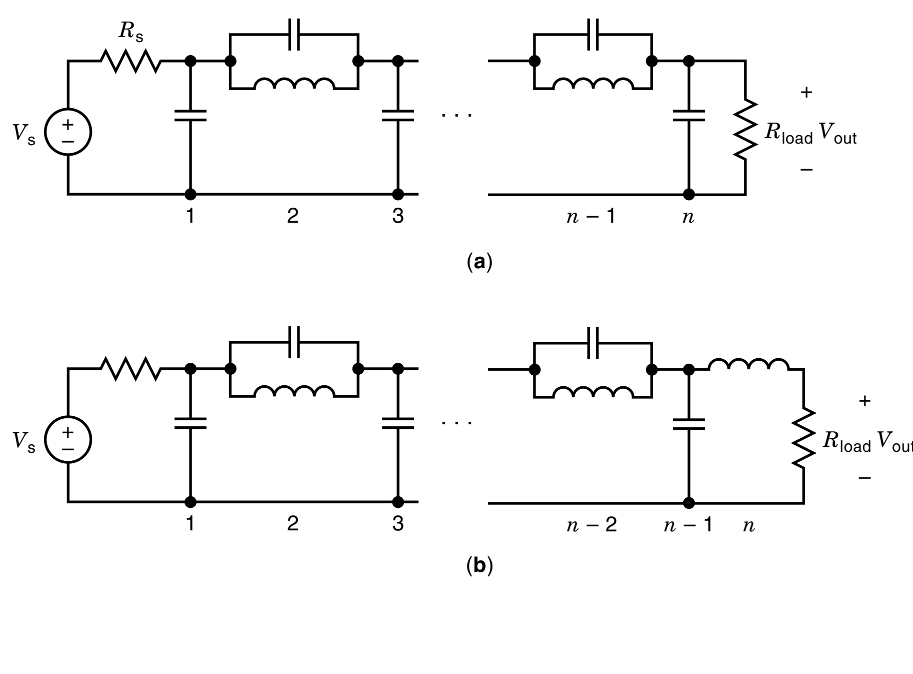

Figure 7. Typical nth-order low-pass passive elliptic filters are realized with inductors and capacitors, and include the source and load resistances. The circuit in (a) is for the odd-order case and has n capacitors and $(n–1)/2$ inductors. The circuit in (b) is for the even-order case and has $(n–1)$ capacitors and $n/2$ inductors.

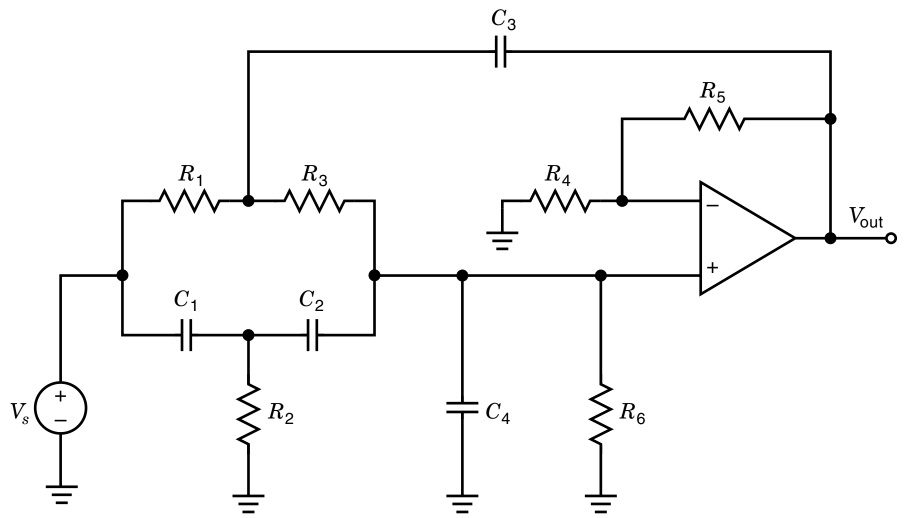

Figure 8. A typical second-order stage used as part of an active *RC* realization consists of an operational amplifier, resistors, and capacitors and is basically an active *RC* notch filter.

highest frequency zero to infinity. The resulting elliptic function filter function will have a double zero at infinity and the passive filter realization will now have positive elements but unequal terminating resistances. This even-order elliptic transfer function is known as case B, while the original even-order transfer function given by Eq. (10) is called case A. Equal terminating resistances can be obtained by shifting the first maximum to the origin. The resulting even-order elliptic transfer function is known as case C (2,5). The new filter functions will be of the forms given by Eq. (12) for case B and Eq. (13) for case C.

$$H_B(s) = H\frac{\prod_{i=2}^{n/2}(s^2+\omega_{Bi}^2)}{b_0 + b_1 s + \cdots + b_{n-1}s^{n-1} + b_n s^n} \tag{12}$$

$$H_C(s) = H\frac{\prod_{i=2}^{n/2}(s^2+\omega_{Ci}^2)}{c_0 + c_1 s + \cdots + c_{n-1}s^{n-1} + c_n s^n} \tag{13}$$

The magnitude response in the case B and case C filter functions are now slightly modified from the original type A.

One may ask if the passive realizations may utilize only discrete elements. At the time of this writing, there is considerable interest in fabricating on-chip or on-package inductors in integrated circuit (IC) design. If progress on these inductors continues at today's present rate, it is not inconceivable that passive synthesis could become commonplace in integrated filter design.

Active Realizations

Elliptic filters may be synthesized with active *RC* filters. A typical design procedure starts with dividing the elliptic filter function into second-order sections. If the order of the filter is odd, there will be one first-order section as well. All second-order sections consist of complex conjugate poles and a pair of zeros. An active filter stage may be used to create each second-order stage. The active filter stages composing the second-order stages are notch filters that allow for ω_z of the notch to be different from ω_0 of the complex pole pair (not all notch filters do). Once the filter is chosen, coefficients of the filter are equated with coefficients of the active filter second-order section. An example of such a filter section is shown in Fig. 8.

Once second-order sections have been synthesized, they may be cascaded together to form the entire circuit. If a first-order stage is used for an odd-order filter, a simple *RC* filter may be added on. The first-order stage may also include a voltage buffer. Active realizations may also be constructed using switched-capacitor circuits.

Another popular method of elliptic filter synthesis is to synthesize an active filter based on a passive realization. Generally, these types of filters replace inductors in the passive circuit with simulated inductors. One type of simulated inductor is composed of an active *RC* circuit configured so that the impedance of the circuit takes the form of the input impedance of an inductor. Active inductors may be also be realized by transconductors and capacitors. Filters using this type of active inductor are called g_m–C filters. They represent the current state of the art in high-frequency active integrated filter design using CMOS technology.

Another active filter based on a passive realization scales the *R*'s, *C*'s, and *L*'s of a passive configuration by $1/s$. The resulting circuit contains capacitors, *D* elements, and resistors. The *D* element is a two-terminal device with an impedance of K/s^2. Although the device doesn't exist as a passive element, active circuits may be synthesized that achieve an input impedance having this form.

Digital Realizations

Many of the techniques of digital filter synthesis are analogous to those used in analog filters. In particular, one of the most systematic approaches to recursive digital filter design is to first find a transfer function that meets specifications in the analog domain, and then port it over to the digital domain. The transformation takes the transfer function from the s domain into the z domain. The variable z plays the same role in digital design that s plays in the analog domain. Often, pre-distortion is applied, to account for errors the frequency response that can occur in the transformation. The reader is encouraged to consult Ref. 6 for more information on digital filtering.

FREQUENCY AND MAGNITUDE SCALING

Frequently, a normalized design is the first step in filter realization. A frequency-normalized filter is designed for a passband frequency of 1 rad/s. A typical normalized realization has component values on the order of ohms, farads, and henries. The design is then frequency scaled so the frequency normalized response is *shifted* into place. That is, the passband and stopband frequencies are transformed from normalized values to the design values. The procedure is performed by finding the scaling constant, ω_p, and replacing s with s/ω_p in the circuit. This results in new values for the capacitances and inductances, while the values for resistances remain unchanged. In circuit circumstances it may be desirable to frequency scale the normalized transfer function first and then do the circuit synthesis.

Frequency scaling usually results in values for capacitors and inductors that are close to practical, but still not practical. Moreover, the impedances of the resistors remain unchanged. The next step in denormalizing a normalized realization is to impedance scale. Impedance scaling amounts to multiplying each impedance by a specified constant. The constant is picked so after scaling, the components have reasonable values. If all impedances are scaled by the same factor, voltage transfer function remains the same. With good selection of the constant, practical values may be achieved.

HIGH-PASS ELLIPTIC FILTERS

The discussions in the preceding sections treat low-pass elliptic filters. There is little difference when discussing the properties of the high-pass elliptic filter.

The first step in high-pass filter design is to normalize the high-pass parameters to the parameters that describe the normalized low-pass filter. The parameters that describe the high-pass are identical to the low-pass filter. One difference is that $\omega_s < \omega_p$. In general, a low-pass filter transfer function may be transformed into a high-pass transfer function by a s to $1/s$ transformation. This simply means that everywhere s appears in the transfer function, $1/s$ is substituted.

Once the normalized low-pass elliptic transfer function has been determined and a normalized circuit has been synthesized, a low-pass to high-pass transformation is applied. This means that everywhere in the circuit, s is replaced with $1/s$. This results in capacitors becoming inductors, and inductors becoming capacitors. If, in an active RC circuit for example, inductors are not desired, the circuit may be magnitude scaled by $1/s$. This results in the inductors becoming resistors and the resistors becoming capacitors.

Alternatively, if a normalized low-pass elliptic function has been determined, it is possible to apply the s to $1/s$ transform on the transfer function, resulting in a normalized high-pass elliptic transfer function. It is now possible to synthesize a circuit directly from this transfer function.

BANDPASS ELLIPTIC FILTERS

Bandpass filters may be classified as wideband or narrowband. A wideband bandpass filter seeks to allow a wide range of frequencies to pass with equal magnitude scaling, ideally. A narrowband filter seeks to allow only one frequency, or a very small band of frequencies, to pass. One definition of narrowband versus wideband filters is given by Ref. 1. This particular definition states that if the ratio of the upper cutoff frequency to the lower cutoff frequency is greater than one octave, the filter is considered a wideband filter.

Synthesis of wideband bandpass filters may be performed by a cascade of a high-pass filter and a low-pass filter. The lower bound of the definition of wideband results in the separation of the high-pass and low-pass filters being such that there is minimal interaction between the filters. If the ratio is smaller than one octave, the cutoff frequencies are too close together and the filters interact and must be treated as one filter. Narrowband filters require different synthesis techniques.

Like the high-pass filter functions, bandpass filter functions may be synthesized from low-pass filter functions. This is done by performing the transformation

$$s = \frac{1}{\mathrm{BW}}\left(\frac{s^2+\omega_0^2}{s}\right) \tag{14}$$

on a normalized low-pass filter function, where ω_0 is the center frequency and BW is the bandwidth of the filter. This transform may also be used in the design of wideband bandpass filters.

BANDREJECT ELLIPTIC FILTERS

Like bandpass filters, bandreject filters may also be classified as wideband or narrowband. A wideband bandreject filter seeks to block a wide range of frequencies while allowing frequencies outside that band to pass with ideally equal magnitude scaling. A narrowband bandreject filter seeks to block only one frequency or a very small band of frequencies. Like the definition of narrowband versus wideband bandpass filter definition, Ref. 1 gives a definition for narrowband versus wideband bandreject filters. The definition is identical to that of the bandpass filter.

Synthesis of wideband bandreject filters may be performed by a cascade of a high-pass filter and a low-pass filter. The lower bound of the definition of wideband results in the separation of the high-pass and low-pass filters being such that there is minimal interaction between the filters. If the ratio is smaller than one octave, the cutoff frequencies are too close together and the filters interact and must be treated as one filter. Narrowband filters require different synthesis techniques.

Bandreject filters may also be synthesized from normalized low-pass filter functions by performing a transform of

$$s = \mathrm{BW}\left(\frac{s}{s^2+\omega_0^2}\right) \tag{15}$$

on a normalized low-pass filter, where ω_0 is the center frequency and BW is the bandwidth of the filter. This transform may also be used in the design of wideband bandpass filters.

SUMMARY

Elliptic filters are a class of filters used to shape the magnitude of an electric signal. They may be used in applications for any of the standard magnitude processing filters. In comparisons to other available filters, the elliptic filter provides the sharpest transition from the passband to the stopband for a given order. The magnitude response is equiripple in the passband and the stopband. The drawback of the elliptic filters is very poor phase characteristics in comparison to other filter types. Furthermore, evaluation of elliptic filter parameters is considerably more difficult than other filter approximation functions due to the use of elliptic sine functions and elliptic integrals.

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EMBEDDING METHODS. See HIGH DEFINITION TELEVISION.

EMC, TELEPHONE. See TELEPHONE INTERFERENCE.