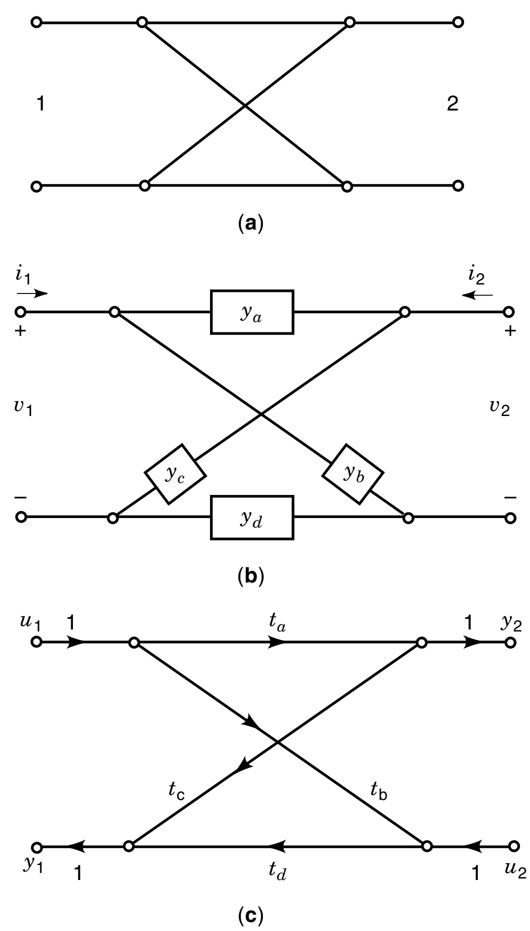

Figure 1. (a) Generic lattice structure. (b) Analog lattice two-port with v_1 and i_1 being the input port variables and v_2 and i_2 the output port variables. Here y_a, y_b, y_c, and y_d represent the one-port admittances of the four branches of the lattice. (c) Digital lattice signal flow graph. Here the branches are transmittances and the terminal variables are signal inputs (u_1 and u_2) and outputs (y_1 and y_2).

LATTICE FILTERS

This article discusses filters of a special topology called lattice filters which can be very useful for system phase correction. Here the focus is on the analog lattice described in terms of admittance, scattering, and transfer scattering matrices. A synthesis technique based on the constant-resistance method that yields a cascade realization in terms of degree-one or degree-two real lattices is included. Also included is an example to illustrate the technique.

DEFINITION

A lattice structure is one of the form of Fig. 1(a). In the case of analog circuits, it is taken to be the two-port of Fig. 1(b) with the port variables being voltages and currents, in which case the branches are typically represented by their one-port impedances or admittances. When the lattice is a digital lattice, the structure represents a signal flow graph where the branches are transmittances and the terminal variables are signal inputs and outputs, as shown in Fig. 1(c). Here we treat the analog lattice only; a treatment of the digital lattice can be found in Refs. 1–3.

ANALOG LATTICE

The analog lattices are most useful for the design of filters based upon the principle of constant R structures. These are especially useful for phase correction via all-pass structures as we now show through the use of symmetrical constant R lattices (4, Chap. 12; 5, p. 223; 6, Chap. 5).

We assume that the lattice branches are described by the respective admittances, y_a, y_b, y_c, y_d in which case the two-port admittance matrix Y has symmetry around the main and the skew diagonals

$$Y = \begin{bmatrix} y_{11} & y_{12} \\ y_{12} & y_{11} \end{bmatrix} \tag{1}$$

$$y_{11} = \frac{(y_a + y_b)(y_c + y_d)}{y_a + y_b + y_c + y_d} \tag{2}$$

$$y_{12} = \frac{y_b y_c - y_a y_d}{y_a + y_b + y_c + y_d} \tag{3}$$

In the case where the lattice is symmetrical as well about an horizontal line drawn through its middle, called a symmetrical lattice,

$$y_d = y_a \qquad \text{and} \qquad y_c = y_b \tag{4}$$

we see by inspection

$$Y = \frac{1}{2}\begin{bmatrix} y_a + y_b & y_b - y_a \\ y_b - y_a & y_a + y_b \end{bmatrix} \tag{5}$$

From Eq. (5) we note that the mutual (off-diagonal) terms can have zeros in the right half s-plane even when y_a and y_b may not. Consequently, the lattice can give nonminimum phase responses in which case it can be very useful for realizing a desired phase shift, possibly for phase correction.

SYNTHESIS BY THE CONSTANT-*R* LATTICE METHOD

Admittance Matrix

The constant-R lattice is defined by using dual arms. Specifically, writing $G = 1/R$, we obtain

$$Ry_b = \frac{1}{Ry_a} \Longrightarrow y_a y_b = G^2 \tag{6}$$

which results in

$$Y_R = \frac{1}{2y_a}\begin{bmatrix} G^2 + y_a^2 & G^2 - y_a^2 \\ G^2 - y_a^2 & G^2 + y_a^2 \end{bmatrix} \tag{7}$$

The name of this structure results from its beautiful property that if it is terminated at port 2 on an R-ohm resistor, the input impedance is an R-ohm resistor, as calculated from the input admittance

$$Y_{in} = \frac{\det Y + Gy_{11}}{G + y_{22}} = \frac{G\left(y_a + y_b + 2\frac{y_a y_b}{G}\right)}{y_a + y_b + 2G} = G \tag{8}$$

and as illustrated in Fig. 2. The transfer voltage ratio is given by

$$\frac{V_2}{V_1} = \frac{-y_{21}}{G + y_{22}} = \frac{y_b - y_a}{y_b + y_a + 2G} = \frac{y_a - G}{y_a + G} \tag{9}$$

Also we have

$$V_1 = \frac{V_{in}}{2} \tag{10}$$

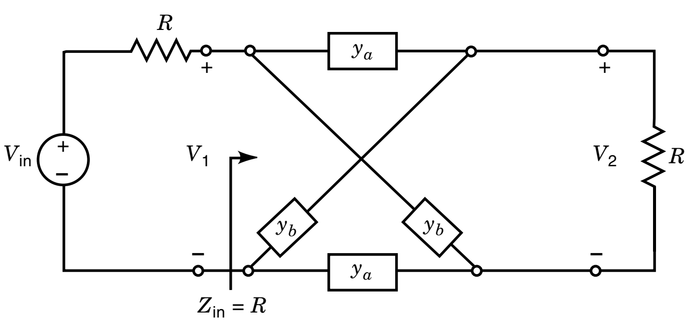

Figure 2. Symmetric analog lattice terminated on an R-ohm resistance at the output port.

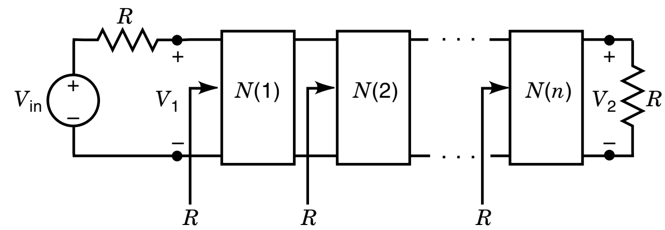

Figure 3. Cascade of n constant-R two-port lattices of the type shown in Fig. (1b) terminated on an R-ohm resistance at the output port.

From Eq. (6) we can obtain the lattice arm impedances y_a and $y_b = G^2/y_a$ since Eq. (9) gives

$$y_a = G\frac{1 + \frac{V_2}{V_1}}{1 - \frac{V_2}{V_1}} \tag{11}$$

In order for a passive synthesis to proceed, y_a must be positive real, the requirement for which is that $y_a(s)$ be analytic in the right half s-plane, $\mathrm{Re}(s) > 0$, and

$$\mathrm{Re}\{y_a(s)\} \ge 0 \quad \text{in} \quad \mathrm{Re}(s) > 0 \tag{12}$$

Translated into the voltage transfer function, after some algebra on Eq. (11), this is seen to be equivalent to

$$\left|\frac{V_2}{V_1}\right| \le 1 \quad \text{in} \quad \mathrm{Re}(s) > 0 \tag{13}$$

In other words, if the voltage transfer function is rational in s and bounded in magnitude by 1 in the right-half plane, it is guaranteed to be synthesized by a passive symmetrical constant-R lattice with an R-ohm termination.

However, this synthesis in one whole piece of V_2/V_1 may require rather complex lattice arms, in which case we can take advantage of the constant-R property to obtain a cascade of lattices. Toward this consider Fig. 3, which shows a cascade of constant-R two-ports loaded in R. As is clear from Fig. 3 we obtain a factorization of the voltage transfer function into the product of n voltage transfer functions, one for each section:

$$\frac{V_2}{V_{in}} = \frac{1}{2}\left[\frac{V_2}{V_1}\right]_{N(1)}\left[\frac{V_2}{V_1}\right]_{N(2)}\cdots\left[\frac{V_2}{V_1}\right]_{N(n)} \tag{14}$$

In order to synthesize a given realizable voltage transfer function, we can perform a factorization of V_2/V_1 into desirably simple factors and realize each factor by a corresponding constant-R lattice. The factorization can be done by factoring the given transfer function into its poles and zeros and associating appropriate pole–zero pairs with the V_2/V_1 terms of Eq. (14). Usually the most desirable factors are obtained by associating the poles and zeros into degree-one or degree-two real factors.

Lossless Synthesis

A particularly interesting case is when the lattice is lossless, which is expressed by

$$y_a(-s) = -y_a(s) \qquad \text{for a lossless lattice} \tag{15}$$

from which we see by Eq. (9) that

$$\frac{V_2(s)}{V_1(s)}\frac{V_2(-s)}{V_1(-s)} = 1 \qquad \text{for a lossless lattice} \tag{16}$$

In this lossless case we see that for $s = j\omega$ the magnitude of the voltage transfer function, from port 1 to 2, is unity; the circuit is all-pass and serves to only introduce phase shift for phase correction and for the design of constant time-delay networks (7, pp. 144–152). If V_2/V_1 is written as the ratio of a numerator polynomial, $N(s)$, over a denominator polynomial, $D(s)$, then in the all-pass case we have $N(s) = \pm D(-s)$, in which case the phase shift becomes twice that of the numerator, which is then

$$\angle\left(\frac{V_2(j\omega)}{V_1(j\omega)}\right) = 2\ \arctan\left[\frac{\mathrm{Im}(N(j\omega))}{\mathrm{Re}(N(j\omega))}\right] \tag{17}$$

By placing the zeros of $N(s)$ one can usually obtain a desirable phase shift. In particular, maximally flat delay can be obtained by choosing $N(s)$ to be a Bessel polynomial (7, p. 151).

Example. For $R = 5$, design a cascade of two lattices and compare with an equivalent single lattice for the all-pass function

$$\begin{aligned}\frac{V_{\text{out}}(s)}{V_{\text{in}}(s)} &= \frac{1}{2}\frac{s^2 - 3s + 2}{s^2 + 3s + 2} \\ &= \frac{1}{2}\frac{(s-2)(s-1)}{(s+2)(s+1)}\end{aligned} \tag{18}$$

For the first lattice of a cascade of two, using Eqs. (11) and (6) with $V_2/V_1 = (s - 2)/(s + 2)$, this gives

$$y_a = \frac{sG}{2} = \frac{s}{10} \quad \text{and} \quad y_b = \frac{2G}{s} = \frac{1}{2.5s} \tag{19}$$

and for the second lattice, with $V_2/V_1 = (s - 1)/(s + 1)$, we obtain

$$y_a = Gs = \frac{s}{5} \quad \text{and} \quad y_b = \frac{G}{s} = \frac{1}{5s} \tag{20}$$

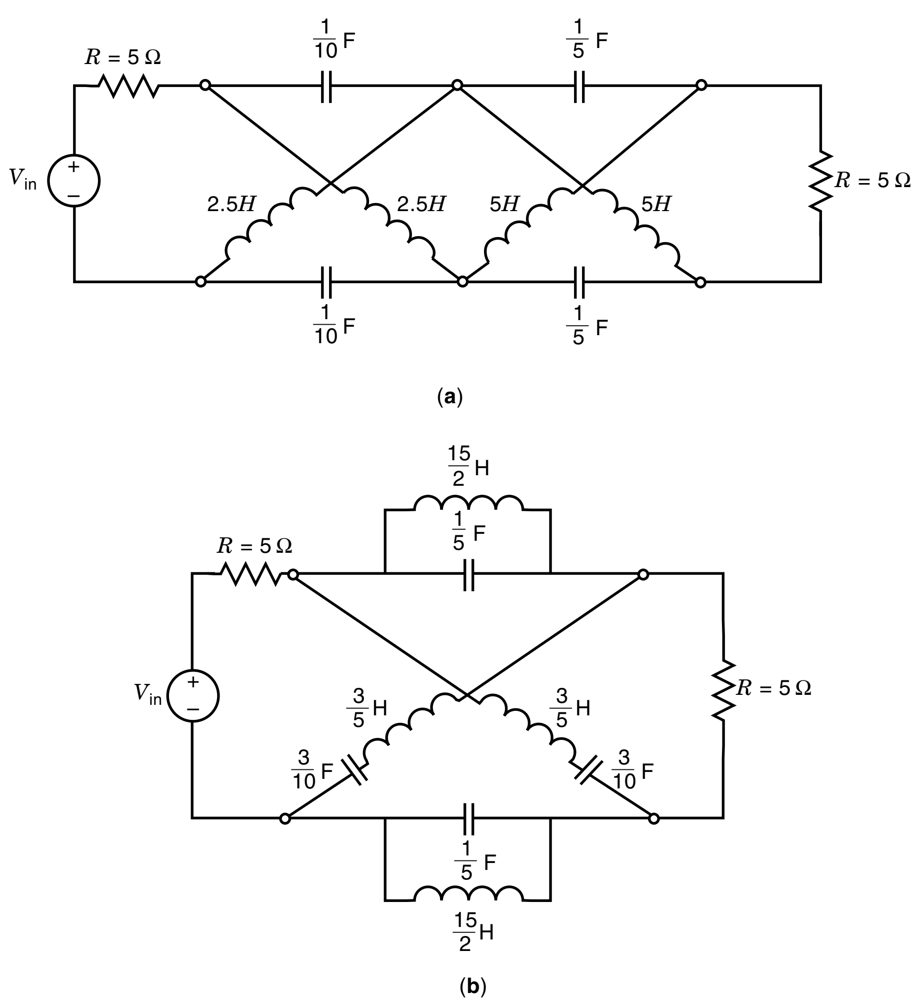

Figure 4. Lossless lattice synthesis of an all-pass transfer function of degree two. (a) Synthesis using a cascade of two lattices of degree 1 Arms. (b) Equivalent realization using a single lattice of degree 2 Arms.

In the case of a single lattice, for V_2/V_1 twice the first expression of Eq. (18), we have

$$y_a = \frac{G(s^2+2)}{3s} = G\left(\frac{s}{3} + \frac{1}{\frac{3}{2}s}\right) \quad \text{and} \quad y_b = \frac{3Gs}{s^2+2} = \frac{G}{\frac{s}{3} + \frac{1}{\frac{3}{2}s}} \tag{21}$$

The final cascade of lattices and equivalent lattice are given in Fig. 4(a) and Fig. 4(b), respectively.

Scattering Matrix

It is also of interest to look at the scattering matrix referenced to R, S, for the constant-R lattice which can be found from the augmented admittance matrix, Y_{aug}, of the lattice filter as illustrated in Fig. 5(a):

$$S = I_2 - 2RY_{\text{aug}} \tag{22}$$

where I_2 is the 2×2 identity matrix. By symmetry, we have from Fig. 5(b)

$$y_{\text{aug}11} = y_{\text{aug}22} = y_{\text{in}} = \frac{1}{2R} \tag{23}$$

and thus

$$s_{11} = s_{22} = 1 - 2R\frac{1}{2R} = 0 \tag{24}$$

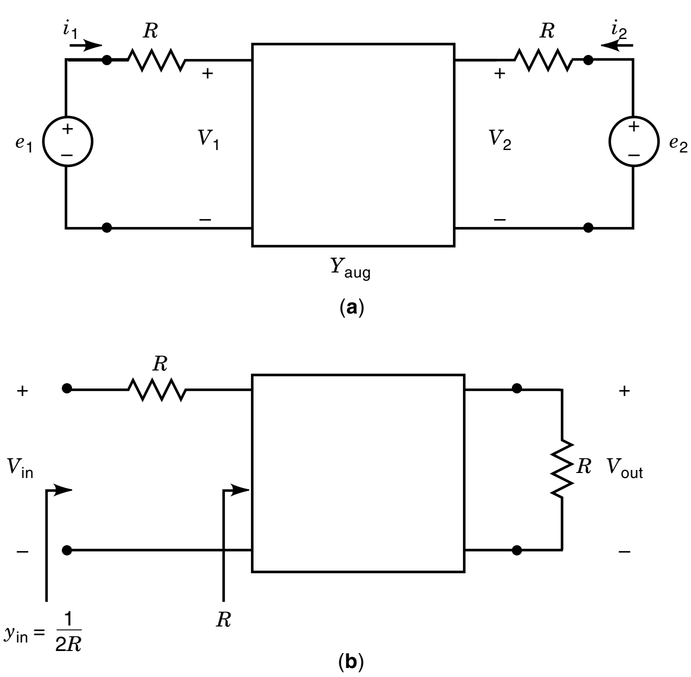

Figure 5. (a) Network pertinent to the interpretation of the scattering parameters. (b) The R-terminated two-port used to evaluate the input admittance y_{in}. This two-port configuration is obtained from Fig. (5a) by setting $e_2 = 0$ and applying an input voltage V_{in}.

The entries s_{12} and s_{21} are calculated in terms of y_a and G using Eq. (11):

$$s_{12} = s_{21} = 2\left[\frac{V_2}{e_1}\right]_{e_2=0} = 2\frac{V_{\text{out}}}{V_{\text{in}}} = \frac{V_2}{V_1} = \frac{y_a - G}{y_a + G} \tag{25}$$

The above results give the following scattering matrix:

$$S = \begin{pmatrix} 0 & \dfrac{y_a - G}{y_a + G} \\ \dfrac{y_a - G}{y_a + G} & 0 \end{pmatrix} \tag{26}$$

The zeros on the diagonal of S indicate that the constant-R lattice is matched to its terminations. Since cascade synthesis can proceed via factorization of the transfer scattering matrix (3), it is of interest to note that the transfer scattering matrix, $T(s)$, is given by

$$T(s) = \frac{1}{s_{12}}\begin{pmatrix} 1 & -s_{22} \\ s_{11} & \det S \end{pmatrix} = \begin{pmatrix} \dfrac{y_a + G}{y_a - G} & 0 \\ 0 & \dfrac{Y_a - G}{y_a + G} \end{pmatrix} \tag{27}$$

When working with the digital lattices of Fig. 1(c), the transfer scattering matrix is particularly convenient since its factorization is readily carried out using Richard's functions extractions of degree-one and degree-two sections [see (3) for details].

TRADE-OFFS AND SENSITIVITY

Despite its versatility, the lattice structure presents several disadvantages of a practical nature. As seen in Fig. 4, there is no possibility of a common ground between the input and the output terminals of a lattice circuit. Although generally it is difficult to obtain a transformation of the lattice to a circuit with common input–output ground, a Darlington synthesis can be undertaken with the desired result (8, Chap. 6). The lattice also uses at least twice the minimum number of components required since the upper arms repeat the lower arms. Furthermore, since the transmission zeros are a function of the difference of component values as seen by Eq. (5), small changes in these may distort the frequency response, the phase in particular, considerably (6, p. 148). However, if corresponding arm components simultaneously change in a lossless lattice, so that the constant-R property is preserved, then the sensitivity of $|V_2(j\omega)/V_1(j\omega)|$ is zero since it is identically 1.

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LAW. See Contracts; Law administration; Software management via law-governed regularities.