NETWORK PARAMETERS

The Laplace transform is commonly used to solve network equations. However, the mere computation of the solution of network equations is one of the many important applications for this elegant tool of network analysis. Our purpose here is to use this transform to define network function and to study the different ways of its representation, the superposition theorem, the characterizations and representations of one-port and two-port networks (1–3).

NETWORK PARAMETERS

We begin by considering a system of differential equations associated with an electrical network, most conveniently written in matrix notation as

$$\mathbf{W}(p)\mathbf{x}(t) = \mathbf{f}(t) \tag{1}$$

where $\mathbf{W}(p)$, $\mathbf{x}(t)$, and $\mathbf{f}(t)$ are used to represent the coefficient matrix of the differential operator p, the unknown vector $\mathbf{x}(t)$ and the known *forcing* or *excitation vector* $\mathbf{f}(t)$. On taking the Laplace transform on both sides, we obtain a system of linear algebraic equations

$$\mathbf{W}(s)\mathbf{X}(s) = \mathbf{F}(s) + \mathbf{h}(s) \tag{2}$$

where $\mathbf{X}(s)$ and $\mathbf{F}(s)$ denote the Laplace transforms of $\mathbf{x}(t)$ and $\mathbf{f}(t)$, respectively, and $\mathbf{h}(s)$ is a vector that includes the contributions due to initial conditions. The coefficient matrix $\mathbf{W}(s)$ in the complex frequency variable s is obtained from $\mathbf{W}(p)$, with s replacing p. An analysis of Eq. (2) is often referred to as analysis in the *frequency domain*, in contrast to the analysis of Eq. (1), which is called analysis in the *time domain*.

Network Functions

The unknown transform vector $\mathbf{X}(s)$ can be obtained immediately by inverting the matrix $\mathbf{W}(s)$:

$$\mathbf{X}(s) = \mathbf{W}^{-1}(s)[\mathbf{F}(s) + \mathbf{h}(s)] \tag{3}$$

provided that det $\mathbf{W}(s)$ is not identically zero.

Consider a linear time-invariant network that contains a single independent voltage or current source as the input with arbitrary waveform. Assume that all initial conditions in the network have been set to zero. Let the response be either a voltage across any two nodes of the network or a current in any branch of the network. Such a response is known as the *zero-state response.* Then, the network function $H(s)$ is defined by

$$H(s) = \frac{\text{the Laplace transform of the zero-state response}}{\text{the Laplace transform of the input or another zero-state response}} \tag{4}$$

Network functions generally fall into two classes depending on whether the terminals to which the response relates are the same or different from the input terminals. For the same pair of terminals, it is referred to as the driving-point or input function; and for different pairs of terminals, the transfer function. Since the input and the response may either be a current or a voltage, the network function may be a driving-point impedance, a driving-point admittance, a transfer impedance, a transfer admittance, a transfer voltage ratio, or a transfer current ratio. Our objective here is to obtain some general and broad properties of network functions, recognizing that each of the network functions mentioned has its own distinct characteristics.

Example. We write the nodal equations for the network of Fig. 1 for $t \geq 0$ after the switch S is closed and compute the input impedance Z_{in} facing the current source I and the transfer current ratio relating the transform current I_6 to the transform current source I.

The nodal equations are found to be

$$\begin{bmatrix} \frac{1}{s} + 2.6178 & -\frac{1}{s} & 0 \\ -\frac{1}{s} & \frac{1}{s} + s & -s \\ 0 & 1.618 - s & 2s + 1 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix} = \begin{bmatrix} I \\ 0 \\ 0 \end{bmatrix} \tag{5}$$

By using Cramer's rule, the nodal voltage V_1 can be expressed in terms of the source current I as

$$Z_{\text{in}}(s) = \frac{V_1}{I} = \frac{0.382(s^3 + 2.618s^2 + 2s + 1)}{s^3 + 3s^2 + 3s + 1} \tag{6}$$

$$\frac{I_6}{I} = \frac{0.382(s - 1.618)}{s^3 + 3s^2 + 3s + 1} \tag{7}$$

Principle of Superposition

The principle of superposition is intimately tied up with the concept of linearity, and is applicable to any linear network, whether it is time invariant or time varying. It is fundamental in characterizing network behavior and is very useful in solving linear network problems. For our purposes, we shall restrict ourselves to the class of linear time invariant networks.

Consider an arbitrary linear time-invariant network with many input excitations describable by a system of linear algebraic equations:

$$\mathbf{W}(s)\mathbf{X}(s) = \mathbf{F}(s) + \mathbf{h}(s) \tag{8}$$

where

$$\mathbf{F}(s) = [F_1 \quad F_2 \quad \dots \quad F_n]'$$
$$\mathbf{h}(s) = [h_1 \quad h_2 \quad \dots \quad h_n]'$$

and the prime denotes matrix transpose. Suppose that the kth row variable X_k of $\mathbf{X}(s)$ is the desired response. By appealing to Cramer's rule, we obtain from Eq. (8)

$$X_k(s) = \frac{\sum_{i=1}^{n} \Delta_{ik}(s)F_i(s)}{\Delta(s)} + \frac{\sum_{j=1}^{n} \Delta_{jk}(s)h_j(s)}{\Delta(s)} \tag{9}$$

Observe that $F_i(s)$ ($i = 1, 2, \dots, n$) are due to the contributions of independent sources. Therefore, to compute the complete response transform X_k, we may consider each of the transform sources F_i one at a time and then add the partial responses so determined to obtain X_k. If F_i represents a

J. Webster (ed.), *Wiley Encyclopedia of Electrical and Electronics Engineering*.

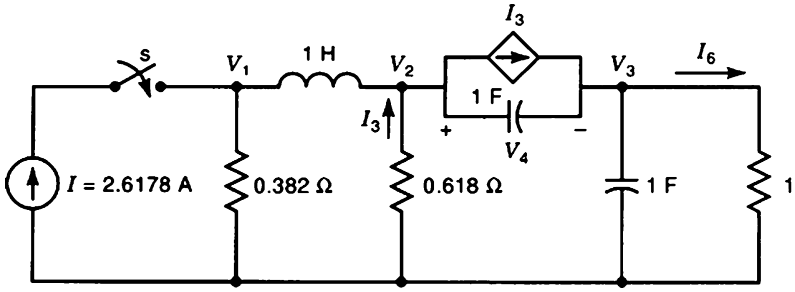

Figure 1. A network used to illustrate network functions.

linear combination of many sources, each source can again be considered separately, one at a time, and then add these partial responses to obtain the complete response. This is in essence the superposition principle.

Superposition Theorem. For a linear system, the zero-state response due to all the independent sources acting simultaneously is equal to the sum of the zero-state responses due to each independent source acting one at a time. If, in addition, the system is time invariant, the same holds in the frequency domain.

Two aspects of superposition are important to emphasize. The first is the additivity property. The other is the homogeneity property, which states that if all sources are multiplied by a constant, the response is also multiplied by the same constant.

Different versions of the superposition principle can be advanced. It states that in a linear time-invariant system the zero-input response is a linear function of the initial state, the zero-state response is a linear function of the input, and the complete response is the sum of the zero-input response and the zero-state response. Thus, the complete response of a linear network to a number of excitations applied simultaneously is the sum of the responses of the network when each of the excitations is applied individually. This statement remains valid even if we consider the initial capacitor voltages and inductor currents themselves to be separate excitations. Of course, the controlled sources cannot be considered as separate excitations. In the case of linear time-invariant networks, the same holds in the frequency domain or in the transform network.

We apply the principle of superposition to compute the inductor current i_2 in the network of Fig. 2. When the voltage source is short-circuited, the inductor current $i_2'(t)$ is found to be

$$i_2'(t) = -\frac{6}{13}e^{-3t} + \frac{2}{5}e^{-4t} + \frac{4}{65}\cos 2t + \frac{7}{65}\cos 2t \tag{10}$$

When the current source is removed, the inductor current $i_2''(t)$ is obtained as

$$i_2''(t) = \frac{10\sqrt{2}}{13}e^{-3t} - \frac{6\sqrt{2}}{5}e^{-4t} + \frac{28\sqrt{2}}{65}\cos 2t - \frac{16\sqrt{2}}{65}\cos 2t \tag{11}$$

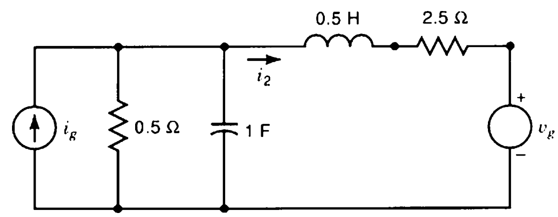

Figure 2. A network used to illustrate the principle of superposition.

The inductor current $i_2(t)$ is the algebraic sum of these two currents:

$$\begin{aligned} i_2(t) &= i_2'(t) + i_2''(t) \\ &= -1.548e^{-3t} + 2.096e^{-4t} + 0.713\cos(2t - 2.448) \end{aligned} \tag{12}$$

Two-Port Networks

A network is a structure comprised of a finite number of interconnected elements with a set of accessible terminal pairs called ports at which voltages and currents can be measured and the transfer of electromagnetic energy into or out of the structure can be made. The situation is similar to ships leaving or entering the ports. Fundamental to the concept of a port is the assumption that the instantaneous current entering one terminal of the port is always equal to the instantaneous current leaving the other terminal of the port. This assumption is crucial in subsequent derivations and resulting conclusions. If it is violated, the terminal pair does not constitute a port. A network with one such accessible port is called a *one-port network* or simply a *one-port*, as represented in Fig. 3(a). If a network is accessible through two such ports as shown in Fig. 3(b), the network is called a *two-port network* or simply a *two-port*. The nomenclature can be extended to networks having n accessible ports called the *n-port networks* or *n-ports.*

Figure 4 is a general representation of a one-port that is electrically and magnetically isolated except at the port with sign convention for the references of port voltage and current as indicated. Likewise, Fig. 5 is a general representation of a two-port that is electrically and magnetically isolated except at the two ports with sign convention for the references of port voltages and currents as indicated. By focusing attention on the ports, we are interested in the behavior of the network only at the ports. Our discussion will be entirely in terms of the transform network, under the assumption that the one-port or two-port is devoid of

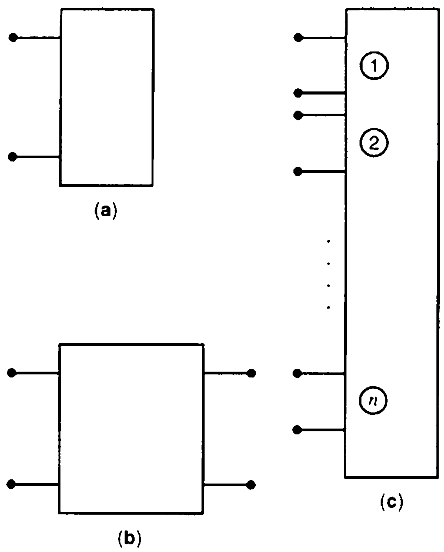

Figure 3. Symbolic representations of a one-port network (a), a two-port network (b), and an n-port network (c).

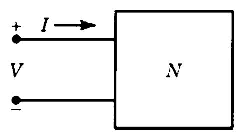

Figure 4. A general representation of a one-port with port voltage and current shown explicitly.

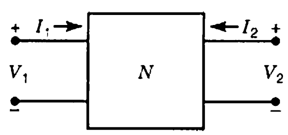

Figure 5. A general representation of a two-port with port voltages and currents shown explicitly.

independent sources inside and has zero initial conditions.

Short-Circuit Admittance Parameters. Refer to Fig. 5. There are four variables associated with the two ports: V_1, V_2, I_1, and I_2. Suppose that we choose port voltages V_1 and V_2 as the independent variables. Then, the port currents I_1 and I_2 are related to the port voltages V_1 and V_2 by the equation

$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \tag{13}$$

or, in matrix form,

$$\mathbf{I}(s) = \mathbf{Y}(s)\mathbf{V}(s) \tag{14}$$

where $\mathbf{I}(s) = [I_1 \quad I_2]'$ is the port-current vector and $\mathbf{V}(s) = [V_1 \quad V_2]'$ is the port-voltage vector. Equation (13) can be represented equivalently by the network of Fig. 6. The four admittance parameters $y_{ij}(i, j = 1, 2)$ are called the short-circuit admittance parameters or simply the y-parameters. The coefficient matrix $\mathbf{Y}(s)$ is referred to as the short-circuit admittance matrix or simply the admittance matrix. To calculate these parameters, we set either V_1 or V_2 to zero and obtain

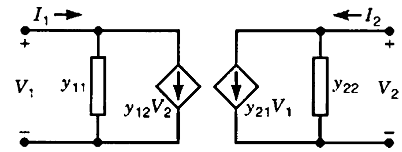

Figure 6. Representation of a two-port in terms of its short-circuit admittance parameters y_{ij}.

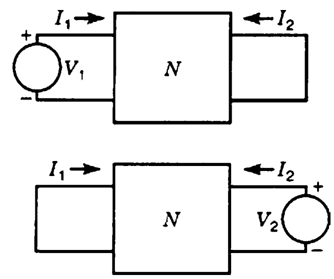

Figure 7. Networks used to compute the short-circuit admittance parameters y_{ij} of a two-port.

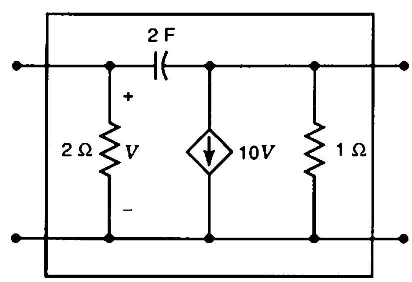

Figure 8. A small-signal network model of a transistor.

$$\begin{aligned} y_{11} &= \left.\frac{I_1}{V_1}\right|_{V_2=0}, \quad y_{12} = \left.\frac{I_1}{V_2}\right|_{V_1=0} \\ y_{21} &= \left.\frac{I_2}{V_1}\right|_{V_2=0}, \quad y_{22} = \left.\frac{I_2}{V_2}\right|_{V_1=0} \end{aligned} \tag{15}$$

The choice of the name short circuit becomes obvious. In computing y_{11} and y_{21}, the port V_2 is short-circuited, whereas for y_{12} and y_{22}, the port V_1 is short-circuited, as depicted in Fig. 7.

Example. Consider the equivalent network of a transistor amplifier shown in Fig. 8. Applying Eq. (16) yields

$$\begin{aligned} y_{11} &= \left.\frac{I_1}{V_1}\right|_{V_2=0} = 0.5 + 2s, \quad y_{12} = \left.\frac{I_1}{V_2}\right|_{V_1=0} = -2s \\ y_{21} &= \left.\frac{I_2}{V_1}\right|_{V_2=0} = 10 - 2s, \quad y_{22} = \left.\frac{I_2}{V_2}\right|_{V_1=0} = 1 + 2s \end{aligned} \tag{16}$$

giving the short-circuit admittance matrix as

$$\mathbf{Y}(s) = \begin{bmatrix} 0.5 + 2s & -2s \\ 10 - 2s & 1 + 2s \end{bmatrix} \tag{17}$$

Open-Circuit Impedance Matrix. Instead of choosing the port voltages V_1 and V_2 as the independent variables, sup-

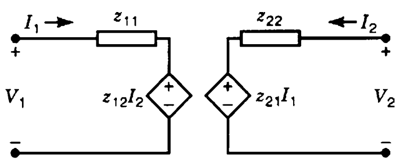

Figure 9. Representation of a two-port in terms of its open-circuit impedance parameters z_{ij}.

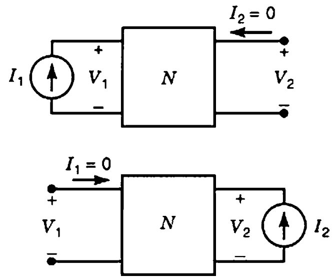

Figure 10. Networks used to compute the open-circuit impedance parameters z_{ij} of a two-port.

pose that we choose port currents I_1 and I_2 as the independent variables. Then, V_1 and V_2 are related to I_1 and I_2 by the equation

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \equiv \begin{bmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} \tag{18}$$

or, in matrix form,

$$\mathbf{V}(s) = \mathbf{Z}(s)\mathbf{I}(s) \tag{19}$$

Equation (18) can be represented equivalently by the network of Fig. 9. The four impedance parameters z_{ij} $(i,j = 1,2)$ are called the open-circuit impedance parameters or simply the z parameters. The coefficient matrix $\mathbf{Z}(s)$ is referred to as the open-circuit impedance matrix or simply the impedance matrix. Obviously, if $\mathbf{Z}(s)$ is not identically singular, its inverse is the short-circuit admittance matrix or

$$\mathbf{Y}(s) = \mathbf{Z}^{-1}(s) \tag{20}$$

and vice versa. To calculate these parameters, we set either I_1 or I_2 to zero and obtain

$$z_{11} = \left.\frac{V_1}{I_1}\right|_{I_2=0}, \quad z_{12} = \left.\frac{V_1}{I_2}\right|_{I_1=0}$$
$$z_{21} = \left.\frac{V_2}{I_1}\right|_{I_2=0}, \quad z_{22} = \left.\frac{V_2}{I_2}\right|_{I_1=0} \tag{21}$$

The choice of the name open circuit becomes obvious. In computing z_{11} and z_{21}, the port I_2 is open-circuited, whereas for z_{12} and z_{22}, the port I_1 is open-circuited, as depicted in Fig. 10.

Example. Consider the equivalent network of a transistor amplifier shown in Fig. 8. Applying Eq. (21) yields

$$z_{11} = \left.\frac{V_1}{I_1}\right|_{I_2=0} = \frac{4s+2}{46s+1}, \quad z_{12} = \left.\frac{V_1}{I_2}\right|_{I_1=0} = \frac{4s}{46s+1}$$
$$z_{21} = \left.\frac{V_2}{I_1}\right|_{I_2=0} = \frac{4s-20}{46s+1}, \quad z_{22} = \left.\frac{V_2}{I_2}\right|_{I_1=0} = \frac{4s+1}{46s+1} \tag{22}$$

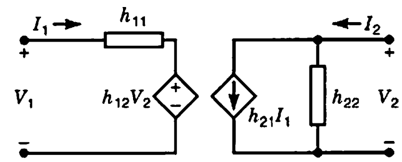

Figure 11. Representation of a two-port in terms of its hybrid parameters h_{ij}.

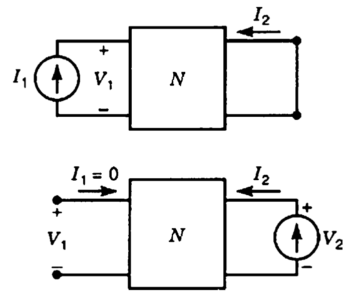

Figure 12. Networks used to compute the hybrid parameters h_{ij} of a two-port.

giving the open-circuit impedance matrix as

$$\mathbf{Z}(s) = \frac{1}{46s+1} \begin{bmatrix} 4s+2 & 4s \\ 4s-20 & 4s+1 \end{bmatrix} \tag{23}$$

The Hybrid Parameters. Suppose that we choose port variables I_1 and V_2 as the independent variables. Then, the remaining port variables V_1 and I_2 are related to I_1 and V_2 by the equation

$$\begin{bmatrix} V_1 \\ I_2 \end{bmatrix} \equiv \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix} \begin{bmatrix} I_1 \\ V_2 \end{bmatrix} \tag{24}$$

or, in matrix form,

$$\mathbf{y}(s) = \mathbf{H}(s)\mathbf{u}(s) \tag{25}$$

where $\mathbf{y}(s) = [V_1 \quad I_2]'$ and $\mathbf{u}(s) = [I_1 \quad V_2]'$. Equation (24) can be represented equivalently by the network of Fig. 11. The four immittance parameters h_{ij} $(i,j = 1,2)$ are called the hybrid parameters or simply the h parameters. The coefficient matrix $\mathbf{H}(s)$ is referred to as the hybrid matrix. To calculate these parameters, we set either I_1 or V_2 to zero and obtain

$$h_{11} = \left.\frac{V_1}{I_1}\right|_{V_2=0}, \quad h_{12} = \left.\frac{V_1}{V_2}\right|_{I_1=0}$$
$$h_{21} = \left.\frac{I_2}{I_1}\right|_{V_2=0}, \quad h_{22} = \left.\frac{I_2}{V_2}\right|_{I_1=0} \tag{26}$$

In computing h_{11} and h_{21}, the port V_2 is short-circuited, whereas for h_{12} and h_{22}, the port I_1 is open-circuited, as depicted in Fig. 12. Thus, h_{11} is the short-circuit input impedance, h_{21} is the short-circuit forward current ratio, h_{12} is the open-circuit reverse voltage ratio, and h_{22} is the open-circuit output admittance. These parameters are not only dimensionally mixed but also under a mixed set of terminal conditions. For this reason they are called hybrid parameters.

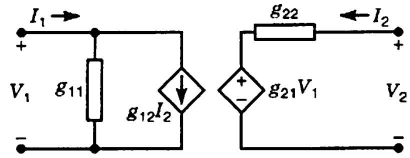

Figure 13. Representation of a two-port in terms of its inverse hybrid parameters g_{ij}.

Example. Consider the equivalent network of a transistor amplifier shown in Fig. 8. Applying Eq. (27) yields

$$h_{11} = \frac{V_1}{I_1}\bigg|_{V_2=0} = \frac{2}{4s+1}, \quad h_{12} = \frac{V_1}{V_2}\bigg|_{I_1=0} = \frac{4s}{4s+1}$$
$$h_{21} = \frac{I_2}{I_1}\bigg|_{V_2=0} = \frac{20-4s}{4s+1}, \quad h_{22} = \frac{I_2}{V_2}\bigg|_{I_1=0} = \frac{46s+1}{4s+1} \tag{27}$$

giving the hybrid matrix as

$$\mathbf{H}(s) = \frac{1}{4s+1}\begin{bmatrix} 2 & 4s \\ 20-4s & 46s+1 \end{bmatrix} \tag{28}$$

Inverse Hybrid Parameters. Suppose now that we choose V_1 and I_2 as the independent variables. Then I_1 and V_2 are related to V_1 and I_2 by the equation

$$\begin{bmatrix} I_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{bmatrix}\begin{bmatrix} V_1 \\ I_2 \end{bmatrix} \tag{29}$$

or, in matrix form,

$$\mathbf{u}(s) = \mathbf{G}(s)\mathbf{y}(s) \tag{30}$$

Equation (30) can be represented equivalently by the network of Fig. 13. The four immittance parameters g_{ij} (i,j = 1,2) are called the inverse hybrid parameters or simply the g parameters. The coefficient matrix $\mathbf{G}(s)$ is referred to as the inverse hybrid matrix. To calculate these parameters, we set either V_1 or I_2 to zero and obtain

$$g_{11} = \frac{I_1}{V_1}\bigg|_{I_2=0}, \quad g_{12} = \frac{I_1}{I_2}\bigg|_{V_1=0}$$
$$g_{21} = \frac{V_2}{V_1}\bigg|_{I_2=0}, \quad g_{22} = \frac{V_2}{I_2}\bigg|_{V_1=0} \tag{31}$$

If $\mathbf{G}(s)$ is not identically singular, its inverse is the hybrid matrix or

$$\mathbf{H}(s) = \mathbf{G}^{-1}(s) \tag{32}$$

Transmission Parameters. Another useful set of parameters is formed by choosing V_2 and $-I_2$ as the independent variables. Then V_1 and I_1 are related to V_2 and $-I_2$ by the equation

$$\begin{bmatrix} V_1 \\ I_1 \end{bmatrix} = \begin{bmatrix} A & B \\ C & D \end{bmatrix}\begin{bmatrix} V_2 \\ -I_2 \end{bmatrix} \tag{33}$$

The four immittance parameters A, B, C, and D are called the transmission parameters, which are also known as the chain parameters or the $ABCD$ parameters. The coefficient matrix is referred to as the transmission matrix. The first two names come from the fact that they are the natural ones to use in a cascade, tandem, or chain connection of two-ports. We remark that there is a negative sign associated with I_2, being a consequence of our choice of reference for I_2 in Fig. 5. To calculate these parameters, we set either V_2 or I_2 to zero and obtain

$$A = \frac{V_1}{V_2}\bigg|_{I_2=0}, \quad -B = \frac{V_1}{I_2}\bigg|_{V_2=0}$$
$$C = \frac{I_1}{V_2}\bigg|_{I_2=0}, \quad -D = \frac{I_1}{I_2}\bigg|_{V_2=0} \tag{34}$$

Example. Consider again the equivalent network of a transistor amplifier shown in Fig. 8. Applying Eq. (34) yields

$$A = \frac{V_1}{V_2}\bigg|_{I_2=0} = \frac{2s+1}{2s-10}, \quad -B = \frac{V_1}{I_2}\bigg|_{V_2=0} = \frac{-1}{2s-10}$$
$$C = \frac{I_1}{V_2}\bigg|_{I_2=0} = \frac{23s+0.5}{2s-10}, \quad -D = \frac{I_1}{I_2}\bigg|_{V_2=0} = -\frac{2s+0.5}{2s-10} \tag{35}$$

giving the transmission matrix as

$$\mathbf{T}(s) = \frac{1}{2s-10}\begin{bmatrix} 2s+1 & 1 \\ 23s+0.5 & 2s+0.5 \end{bmatrix} \tag{36}$$

By interchanging the roles of the excitation and the response in Eq. (33), we obtain yet another set of parameters called the inverse transmission or inverse chain parameters, and their corresponding matrix the inverse transmission or inverse chain matrix, the details of which are omitted.

Interrelations Among the Parameters Sets

The various ways of representing the external behaviors of a two-port are presented in the foregoing. Each finds useful applications, depending on the problem on hand. Table 1 gives the interrelationships among the different sets of parameters.

Interconnection of Two-Ports

Simple two-ports are interconnected to yield more complicated and practical two-ports. Two two-ports are said to be connected in cascade or tandem if the output terminals of one two-port are connected to the input terminals of the other, as depicted in Fig. 14. This type of connection is most conveniently described by the transmission parameters. From Fig. 14 we have for the two-port N_b

$$\begin{bmatrix} V_{2a} \\ -I_{2a} \end{bmatrix} = \begin{bmatrix} V_{1b} \\ I_{1b} \end{bmatrix} = \begin{bmatrix} A_b & B_b \\ C_b & D_b \end{bmatrix}\begin{bmatrix} V_{2b} \\ -I_{2b} \end{bmatrix} \tag{37}$$

and for two-port N_a

$$\begin{bmatrix} V_{1a} \\ I_{1a} \end{bmatrix} = \begin{bmatrix} A_a & B_a \\ C_a & D_a \end{bmatrix}\begin{bmatrix} V_{2a} \\ -I_{2a} \end{bmatrix} \tag{38}$$

where the subscripts a and b are used to distinguish the transmission parameters of N_a and N_b. Combining Eqs.

Table 1. Conversion Chart for Two-Port Parameters ($\Delta_x = x_{11}x_{22} - x_{12}x_{21}$)

To	From: Z	Y	T	T^{-1}	H	G
z Parameters	$\begin{matrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{matrix}$	$\begin{matrix} \frac{y_{22}}{\Delta_y} & \frac{y_{12}}{\Delta_y} \\ \frac{y_{21}}{\Delta_y} & \frac{y_{11}}{\Delta_y} \end{matrix}$	$\begin{matrix} \frac{A}{C} & \frac{\Delta_T}{C} \\ \frac{1}{C} & \frac{D}{C} \end{matrix}$	$\begin{matrix} \frac{D'}{C'} & \frac{1}{C'} \\ \frac{\Delta_T}{C'} & \frac{A'}{C'} \end{matrix}$	$\begin{matrix} \frac{\Delta_h}{h_{22}} & \frac{h_{12}}{h_{22}} \\ \frac{h_{21}}{h_{22}} & \frac{1}{h_{22}} \end{matrix}$	$\begin{matrix} \frac{1}{g_{11}} & \frac{g_{12}}{g_{11}} \\ \frac{g_{21}}{g_{11}} & \frac{\Delta_g}{g_{11}} \end{matrix}$
y Parameters	$\begin{matrix} \frac{z_{22}}{\Delta_z} & \frac{z_{12}}{\Delta_z} \\ \frac{z_{21}}{\Delta_z} & \frac{z_{11}}{\Delta_z} \end{matrix}$	$\begin{matrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{matrix}$	$\begin{matrix} \frac{D}{B} & \frac{\Delta_T}{B} \\ \frac{1}{B} & \frac{A}{B} \end{matrix}$	$\begin{matrix} \frac{A'}{B'} & \frac{1}{B'} \\ \frac{\Delta_T}{B'} & \frac{D'}{B'} \end{matrix}$	$\begin{matrix} \frac{1}{h_{11}} & \frac{h_{12}}{h_{11}} \\ \frac{h_{21}}{h_{11}} & \frac{\Delta_h}{h_{11}} \end{matrix}$	$\begin{matrix} \frac{\Delta_g}{g_{22}} & \frac{g_{12}}{g_{22}} \\ \frac{g_{21}}{g_{22}} & \frac{1}{g_{22}} \end{matrix}$
Transmission parameters	$\begin{matrix} \frac{z_{11}}{z_{21}} & \frac{\Delta_z}{z_{21}} \\ \frac{1}{z_{21}} & \frac{z_{22}}{z_{21}} \end{matrix}$	$\begin{matrix} \frac{y_{22}}{y_{21}} & \frac{1}{y_{21}} \\ \frac{\Delta_y}{y_{21}} & \frac{y_{11}}{y_{21}} \end{matrix}$	$\begin{matrix} A & B \\ C & D \end{matrix}$	$\begin{matrix} \frac{D'}{\Delta_T} & \frac{B'}{\Delta_T} \\ \frac{C'}{\Delta_T} & \frac{A'}{\Delta_T} \end{matrix}$	$\begin{matrix} \frac{\Delta_h}{h_{21}} & \frac{h_{11}}{h_{21}} \\ \frac{h_{22}}{h_{21}} & \frac{1}{h_{21}} \end{matrix}$	$\begin{matrix} \frac{1}{g_{21}} & \frac{g_{22}}{g_{21}} \\ \frac{g_{11}}{g_{21}} & \frac{\Delta_g}{g_{21}} \end{matrix}$
Inverse transmission parameters	$\begin{matrix} \frac{z_{22}}{z_{12}} & \frac{\Delta_z}{z_{12}} \\ \frac{1}{z_{12}} & \frac{z_{11}}{z_{12}} \end{matrix}$	$\begin{matrix} \frac{y_{11}}{y_{12}} & \frac{1}{y_{12}} \\ \frac{\Delta_y}{y_{12}} & \frac{y_{22}}{y_{12}} \end{matrix}$	$\begin{matrix} \frac{D}{\Delta_T} & \frac{B}{\Delta_T} \\ \frac{C}{\Delta_T} & \frac{A}{\Delta_T} \end{matrix}$	$\begin{matrix} A' & B' \\ C' & D' \end{matrix}$	$\begin{matrix} \frac{1}{h_{12}} & \frac{h_{11}}{h_{12}} \\ \frac{h_{22}}{h_{12}} & \frac{\Delta_h}{h_{12}} \end{matrix}$	$\begin{matrix} \frac{\Delta_g}{g_{12}} & \frac{g_{22}}{g_{12}} \\ \frac{g_{11}}{g_{12}} & \frac{1}{g_{12}} \end{matrix}$
h Parameters	$\begin{matrix} \frac{\Delta_z}{z_{22}} & \frac{z_{12}}{z_{22}} \\ \frac{z_{21}}{z_{22}} & \frac{1}{z_{22}} \end{matrix}$	$\begin{matrix} \frac{1}{y_{11}} & \frac{y_{12}}{y_{11}} \\ \frac{y_{21}}{y_{11}} & \frac{\Delta_y}{y_{11}} \end{matrix}$	$\begin{matrix} \frac{B}{D} & \frac{\Delta_T}{D} \\ \frac{1}{D} & \frac{C}{D} \end{matrix}$	$\begin{matrix} \frac{B'}{A'} & \frac{1}{A'} \\ \frac{\Delta_T}{A'} & \frac{C'}{A'} \end{matrix}$	$\begin{matrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{matrix}$	$\begin{matrix} \frac{g_{22}}{\Delta_g} & \frac{g_{12}}{\Delta_g} \\ \frac{g_{21}}{\Delta_g} & \frac{g_{11}}{\Delta_g} \end{matrix}$
g Parameters	$\begin{matrix} \frac{1}{z_{11}} & \frac{z_{12}}{z_{11}} \\ \frac{z_{21}}{z_{11}} & \frac{\Delta_z}{z_{11}} \end{matrix}$	$\begin{matrix} \frac{\Delta_y}{y_{22}} & \frac{y_{12}}{y_{22}} \\ \frac{y_{21}}{y_{22}} & \frac{1}{y_{22}} \end{matrix}$	$\begin{matrix} \frac{C}{A} & \frac{\Delta_T}{A} \\ \frac{1}{A} & \frac{B}{A} \end{matrix}$	$\begin{matrix} \frac{C'}{D'} & \frac{1}{D'} \\ \frac{\Delta_T}{D'} & \frac{B'}{D'} \end{matrix}$	$\begin{matrix} \frac{h_{22}}{\Delta_h} & \frac{h_{12}}{\Delta_h} \\ \frac{h_{21}}{\Delta_h} & \frac{h_{11}}{\Delta_h} \end{matrix}$	$\begin{matrix} g_{11} & g_{12} \\ g_{21} & g_{22} \end{matrix}$

(37) and (38) gives

$$\begin{bmatrix} V_{1a} \\ I_{1a} \end{bmatrix} = \begin{bmatrix} A_a & B_a \\ C_a & D_a \end{bmatrix} \begin{bmatrix} A_b & B_b \\ C_b & D_b \end{bmatrix} \begin{bmatrix} V_{2b} \\ -I_{2b} \end{bmatrix} \tag{39}$$

showing that the coefficient matrix, being the product of two matrices, is the transmission matrix of the composite two-port N. Thus, the transmission matrix of two two-ports connected in cascade is equal to the product of the transmission matrices of the individual two-ports:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A_a & B_a \\ C_a & D_a \end{bmatrix} \begin{bmatrix} A_b & B_b \\ C_b & D_b \end{bmatrix} \tag{40}$$

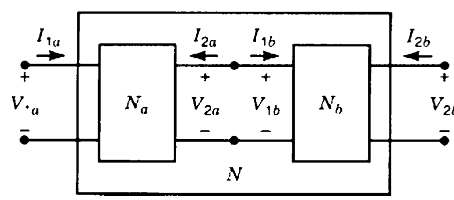

Figure 14. Symbolic representation of two two-ports connected in cascade.

Another useful connection is depicted in Fig. 15 where the input terminals and output terminals of the individual two-ports are connected in parallel, and is called a parallel connection. This connection forces the equality of the terminal voltages of the two-ports, and is most conveniently described by the short-circuit admittance parameters. From Fig. 15 we have

$$\begin{bmatrix} I_1 \\ I_2 \end{bmatrix} = \begin{bmatrix} I_{1a} \\ I_{2a} \end{bmatrix} + \begin{bmatrix} I_{1b} \\ I_{2b} \end{bmatrix} = \begin{bmatrix} y_{11a} & y_{12a} \\ y_{21a} & y_{22a} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} + \begin{bmatrix} y_{11b} & y_{12b} \\ y_{21b} & y_{22b} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$$
$$= \begin{bmatrix} y_{11a}+y_{11b} & y_{12a}+y_{12b} \\ y_{21a}+y_{21b} & y_{22a}+y_{22b} \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix} \tag{41}$$

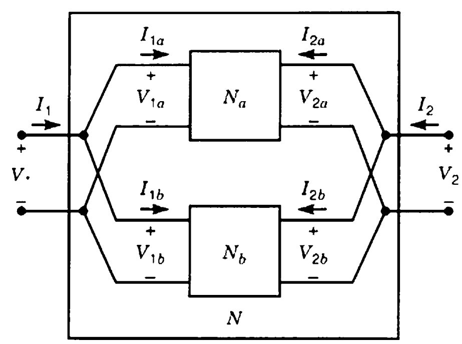

Figure 15. Symbolic representation of two two-ports connected in parallel.

showing that the short-circuit admittance matrix of the composite two-port N is the sum of those of the component two-ports N_a and N_b.

We remark that the validity of Eq. (41) is based on the assumption that the instantaneous current entering one terminal of a two-port is equal to the instantaneous current leaving the other terminal of the two-port after the interconnection. If this condition is violated, the statement that when two two-ports are connected in parallel, their admittance matrices add is no longer valid. To ensure that the nature of the ports are not altered after the connection, we employ the Brune's test as shown in Fig. 16: the voltage marked V is zero. If Brune's test is not satisfied, an ideal transformer with turns ratio 1:1 is required, and this transformer needs to be inserted either at the output or input port of one of the two-ports.

Example. Figure 17 is a simple RC twin-Tee used in the design of equalizers. This two-port N can be considered as a parallel connection of two two-ports N_a and N_b of Fig. 18. It is easy to verify that the Brune's test is satisfied and the short-circuit admittance matrix $\mathbf{Y}(s)$ of the twin-Tee is simply the sum of those $\mathbf{Y}_a(s)$ and $\mathbf{Y}_b(s)$ of the component

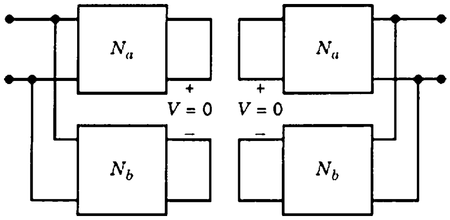

Figure 16. Brune's test for parallel connection of two two-ports.

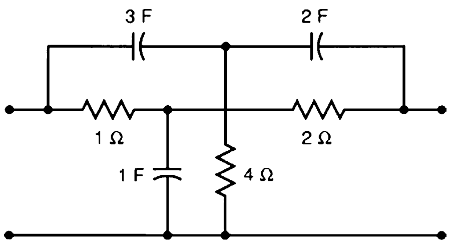

Figure 17. A twin-Tee used in the design of equalizers.

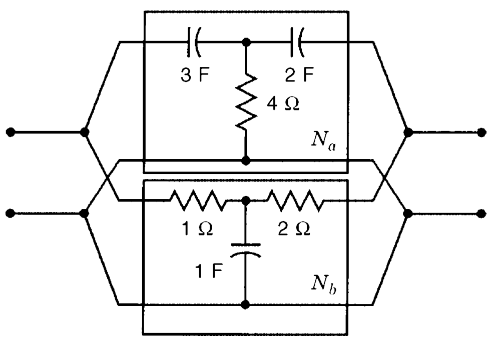

Figure 18. The parallel connection of two two-ports to form the twin-Tee of Fig. 17.

two-ports N_a and N_b:

$$\mathbf{Y}(s) = \mathbf{Y}_a(s) + \mathbf{Y}_b(s) = \frac{1}{40s^2 + 62s + 3} \begin{bmatrix} 48s^3 + 118s^2 + 31s + 1 & -48s^3 - 72s^2 - 20s - 1 \\ -48s^3 - 72s^2 - 20s - 1 & 48s^3 + 96s^2 + 27s + 1 \end{bmatrix} \tag{42}$$

Two two-ports N_a and N_b are said to be connected in series if they are connected as shown in Fig. 19. This connection forces the equality of the terminal currents of the two-ports, and is most conveniently described by the open-circuit impedance parameters. From Fig. 19 we have

$$\begin{bmatrix} V_1 \\ V_2 \end{bmatrix} = \begin{bmatrix} V_{1a} \\ V_{2a} \end{bmatrix} + \begin{bmatrix} V_{1b} \\ V_{2b} \end{bmatrix} = \begin{bmatrix} z_{11a} & z_{12a} \\ z_{21a} & z_{22a} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} + \begin{bmatrix} z_{11b} & z_{12b} \\ z_{21b} & z_{22b} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix}$$
$$= \begin{bmatrix} z_{11a} + z_{11b} & z_{12a} + z_{12b} \\ z_{21a} + z_{21b} & z_{22a} + z_{22b} \end{bmatrix} \begin{bmatrix} I_1 \\ I_2 \end{bmatrix} \tag{43}$$

showing that the open-circuit impedance matrix of the composite two-port N is the sum of those of the component two-ports N_a and N_b.

Note again that the validity of Eq. (43) is based on the assumption that the instantaneous current entering one terminal of a two-port is equal to the instantaneous current leaving the other terminal of the two-port after the interconnection. If this condition is violated, the previous statement is no longer valid. To test to see if this condition is satisfied, we employ the Brune's test as shown in Fig. 20: the voltage marked V is zero. If Brune's test is not satisfied, an ideal transformer with turns ratio 1:1 is required, and this transformer needs to be inserted either at the output or input port of one of the two-ports.

Combinations of the parallel and series connections are possible such as the series-parallel and parallel-series connections shown in Figs. 21 and 22.

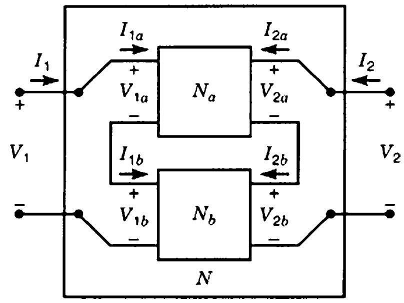

Figure 19. Symbolic representation of two two-ports connected in series.

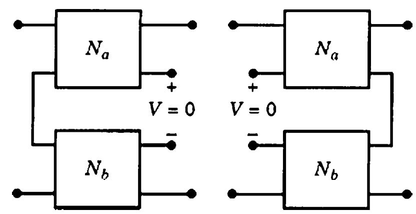

Figure 20. Brune's test for series connection of two two-ports.

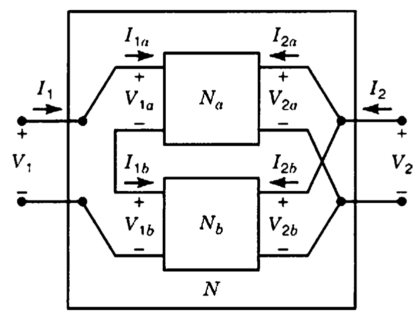

Figure 21. Symbolic representation of two two-ports connected in series-parallel.

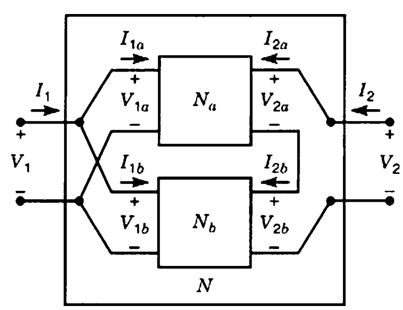

Figure 22. Symbolic representation of two two-ports connected in parallel-series.

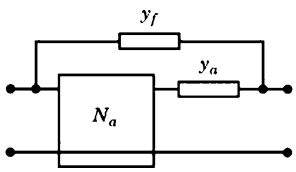

Figure 23. A feedback network N.

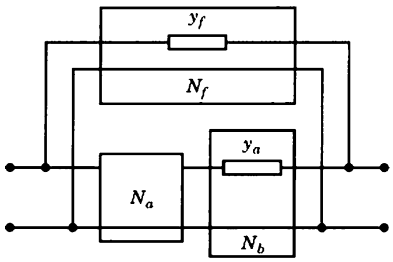

Figure 24. A decomposition of the feedback network N into three two-ports N_a, N_b, and N_f.

Example. Consider the feedback network N of Fig. 23. To compute its short-circuit admittance matrix $\mathbf{Y}(s)$, it is advantageous to consider N as being composed of two two-ports N_a and N_b connected in cascade and then in parallel with another N_f as depicted in Fig. 24. The transmission matrix of the two two-ports N_a and N_b connected in cascade, being the product of their transmission matrices, is given by

$$\frac{1}{y_a y_{21a}} \begin{bmatrix} y_{22a} & 1 \\ \Delta_y & y_{11a} \end{bmatrix} \begin{bmatrix} -y_a & -1 \\ 0 & -y_a \end{bmatrix} \tag{44}$$

where y_{ija} are the y-parameters of N_a and $\Delta_y = y_{11a}y_{22a} - y_{12a}y_{21a}$. The corresponding admittance matrix of Eq. (44) is found from Table 1 as

$$\frac{y_a}{y_a + y_{22a}} \begin{bmatrix} y_{11a} + \dfrac{\Delta_y}{y_a} & y_{12a} \\ y_{21a} & y_{22a} \end{bmatrix} \tag{45}$$

The short-circuit admittance matrix $\mathbf{Y}(s)$ of the overall two-port N of Fig. 23 is obtained as

$$\mathbf{Y}(s) = \frac{y_a}{y_a + y_{22a}} \begin{bmatrix} y_{11a} + \dfrac{\Delta_y}{y_a} & y_{12a} \\ y_{21a} & y_{22a} \end{bmatrix} + \begin{bmatrix} y_f & -y_f \\ -y_f & y_f \end{bmatrix} \tag{46}$$

Power Gains

Refer to the two-port network of Fig. 5. The simplest measure of power flow in N is the *power gain* G_p defined as the ratio of the average power delivered to the load P_2 to the average power entering the input port P_1:

$$G_p = \frac{P_2}{P_1} \tag{47}$$

which is a function of the two-port parameters and the load impedance Z_2, being independent of the source impedance Z_1. For a passive and lossless two-port network, $G_p = 1$.

The second measure of power flow is called the *available power gain* G_a defined as the ratio of the maximum available average power P_{2a} at the load to the maximum available average power P_{1a} at the source:

$$G_p = \frac{P_{2a}}{P_{1a}} \tag{48}$$

Therefore, it is a function of the two-port parameters and the source impedance Z_1, being independent of the load impedance Z_2.

Finally, the third and most useful measure of power flow is known as the *transducer power gain* G defined as the ratio of average power P_2 delivered to the load to the maximum available average power P_{1a} at the source:

$$G = \frac{P_2}{P_{1a}} \tag{49}$$

Clearly, it is a function of the two-port parameters and the source and load impedances Z_1 and Z_2. It is important because it compares the average power delivered to the load with the average power that the source is capable of supplying under the optimum terminations, thereby making this the most meaningful description of the power transfer capabilities of a two-port network. Notice that the three power gains can only be meaningfully defined on the real-frequency axis $s = j\omega$. In other words, we have substituted $s = j\omega$ in all the equations, even though they are not explicitly shown.

To show how these power gains can be expressed in terms of the two-port parameters of Fig. 5 and Z_1 and Z_2, we substitute $V_2 = -I_2 Z_2$ in Eq. (18) and solve for I_1 and I_2, yielding

$$\frac{I_2}{I_1} = -\frac{z_{21}}{z_{22} + Z_2} \tag{50}$$

The average power P_1 entering the input port and the average power P_2 delivered to the load Z_2 are given by

$$P_1 = |I_1|^2 \operatorname{Re} Z_{11} \tag{51}$$

$$P_2 = |I_2|^2 \operatorname{Re} Z_2 \tag{52}$$

where Z_{11} is the impedance looking into the input port with the output port terminating in Z_2.

The maximum available average power P_{1a} at the input port is attained, when the source impedance Z_1 and the input impedance Z_{11} are conjugately matched, or $Z_{11} = \mathring{Z}_1$, the complex conjugate of Z_1, giving

$$P_{1a} = \frac{|V_s|^2}{4 \operatorname{Re} Z_1} \tag{53}$$

where V_s is the voltage source at the input port.

To express Z_{11} in terms of the two-port parameters z_{ij} and Z_2, we substitute $V_2 = -I_2 Z_2$ in Eq. (18) and solve for I_1, yielding

$$Z_{11} = \frac{V_1}{I_1} = z_{11} - \frac{z_{12}z_{21}}{z_{22} + Z_2} \tag{54}$$

Combining Eqs. (51)–(55) obtains

$$G_p = \frac{P_2}{P_1} = \frac{|z_{21}|^2 \operatorname{Re} Z_2}{|z_{22} + Z_2|^2 \operatorname{Re} Z_{11}} = \frac{|z_{21}|^2 \operatorname{Re} Z_2}{|z_{22} + Z_2|^2 \operatorname{Re}(z_{11} - \frac{z_{12}z_{21}}{z_{22}+Z_2})} \tag{55}$$

$$G = \frac{P_2}{P_{1a}} = \frac{4|z_{21}|^2 \operatorname{Re} Z_1 \operatorname{Re} Z_2}{|(z_{11} + Z_1)(z_{22} + Z_2) - z_{12}z_{21}|^2} \tag{56}$$

For the available power gain, we first compute Thévenin equivalent voltage $V_{\rm eq}$ and impedance $Z_{\rm eq}$ looking into the output port of Fig. 5, when the input port is terminated in a series combination of a voltage source $V_{\rm s}$ and source impedance Z_1:

$$Z_{eq} = z_{22} - \frac{z_{12}z_{21}}{z_{11} + Z_1} \tag{57}$$

$$V_{eq} = \frac{z_{21}V_s}{z_{11} + Z_1} \tag{58}$$

Using this Thévenin equivalent network, the maximum available average power at the output port is attained when $Z_2 = \overset{\circ}{Z}_{eq}$, the complex conjugate of $Z_{\rm eq}$, obtaining

$$P_{2a} = \frac{|z_{21}|^2|V_s|^2}{4|z_{11} + Z_1|^2 \operatorname{Re} Z_{eq}} \tag{59}$$

The available power gain is found to be

$$G_a = \frac{P_{2a}}{P_{1a}} = \frac{|z_{21}|^2 \operatorname{Re} Z_1}{|z_{11} + Z_1|^2 \operatorname{Re} Z_{eq}} \tag{60}$$

which in conjunction with Eq. (57) gives

$$P_a = \frac{P_{2a}}{P_{1a}} = \frac{|z_{21}|^2 \operatorname{Re} Z_1}{|z_{11} + Z_1|^2 \operatorname{Re}(z_{22} - \frac{z_{12}z_{21}}{z_{11}+Z_1})} \tag{61}$$

Likewise, we can evaluate the three power gains in terms of other two-port parameters as follows:

$$\begin{aligned} G_p = \frac{P_2}{P_1} &= \frac{|z_{21}|^2 \operatorname{Re} Z_2}{|z_{22} + Z_2|^2 \operatorname{Re}(z_{11} - \dfrac{z_{12}z_{21}}{z_{22} + Z_2})} = \frac{|y_{21}|^2 \operatorname{Re} Y_2}{|y_{22} + Y_2|^2 \operatorname{Re}(y_{11} - \dfrac{y_{12}y_{21}}{y_{22} + Y_2})} \\ &= \frac{|h_{21}|^2 \operatorname{Re} Y_2}{|h_{22} + Y_2|^2 \operatorname{Re}(h_{11} - \dfrac{h_{12}h_{21}}{h_{22} + Y_2})} \end{aligned} \tag{62}$$

$$\begin{aligned} G = \frac{P_2}{P_{1a}} &= \frac{4|z_{21}|^2 \operatorname{Re} Z_1 \operatorname{Re} Z_2}{|(z_{11} + Z_1)(z_{22} + Z_2) - z_{12}z_{21}|^2} = \frac{4|y_{21}|^2 \operatorname{Re} Y_1 \operatorname{Re} Y_2}{|(y_{11} + Y_1)(y_{22} + Y_2) - y_{12}y_{21}|^2} \\ &= \frac{4|h_{21}|^2 \operatorname{Re} Z_1 \operatorname{Re} Y_2}{|(h_{11} + Z_1)(h_{22} + Y_2) - h_{12}h_{21}|^2} \end{aligned} \tag{63}$$

$$\begin{aligned} P_a = \frac{P_{2a}}{P_{1a}} &= \frac{|z_{21}|^2 \operatorname{Re} Z_1}{|z_{11} + Z_1|^2 \operatorname{Re}(z_{22} - \dfrac{z_{12}z_{21}}{z_{11} + Z_1})} = \frac{|y_{21}|^2 \operatorname{Re} Y_1}{|y_{11} + Y_1|^2 \operatorname{Re}(y_{22} - \dfrac{y_{12}y_{21}}{y_{11} + Y_1})} \\ &= \frac{|h_{21}|^2 \operatorname{Re} Z_1}{|h_{11} + Z_1|^2 \operatorname{Re}(h_{22} - \dfrac{h_{12}h_{21}}{h_{11} + Z_1})} \end{aligned} \tag{64}$$

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