WIENER FILTERS

In dealing with the operational aspects of dynamic and static (or memoryless) physical systems one often has to process many measured (observed) signals for extracting meaningful and useful information regarding the system under investigation. Such a sequence of signal processing steps (whether analog or digital, and whether implemented in hardware or software) forms the thrust of the field of estimation. Strictly speaking, filtering is a special form of estimation. *Filters* (or more generally estimators) are devices (hardware and/or software) that process noisy measurements to extract useful information. A *Wiener filter,* in particular, is a specialized linear (nonadaptive) filter, and it is optimal under certain idealized conditions. It was named after its developer, the famous mathematician and originator of the field of cybernetics, Norbert Wiener, who derived and helped implement the first filter during World War II.

Estimation is the process of inferring the value(s) of a variable of interest, using some relevant measurements. Almost everything around us can be considered a dynamic system. Nearly all physical systems have some dynamic nature, and precise estimation of any quantity that relates to them must take into consideration their dynamics. For example, the flow dynamics of a river system can be used to estimate future flood levels in surrounding communities, and the accurate position of a spacecraft can be estimated using radar tracking information. Furthermore, ship navigation can be accomplished by estimation methods using gyroscopic measurements.

Estimation of a quantity or a variable can take numerous forms depending on the problems being studied. In particular, when dealing with dynamic systems, estimation problems can be classified into three categories (1):

1. *State estimation,* the process of inferring the state(s) (or outputs related to the state) of a system using measurements collected from the system itself and a prespecified mathematical model of the system,
2. *System identification,* the process of inferring a mathematical model of a system using measurements collected from the system itself, and,
3. *Adaptive estimation,* the process of simultaneous state estimation and system identification.

The function performed by Wiener filters is that of a specialized state estimation. In an effort to put these filters in the proper framework, an attempt must be made to further categorize state estimation. For the remainder of this article state estimation will refer to the estimation of system states or outputs, where the latter are functions of the states.

In state estimation problems, the estimate of the variable of interest is usually denoted by $\hat{\mathbf{x}}(t|t)$, indicating the estimated value at time t given measurements up to and including the time t. The actual, and quite often unknown, value of the variable of interest is denoted by $\mathbf{x}(t)$, and the measurements are usually denoted by $\mathbf{y}(t)$ in the case of a system output and $\mathbf{u}(t)$ in the case of a system input. The estimate of the measured output $\mathbf{y}(t)$ is usually denoted $\hat{\mathbf{y}}(t/t)$. In this article and in most recent presentations of this subject, all developments are presented in the discrete-time domain (1a). The wide use of digital computers and the increased use of digital signal processors makes this presentation the preferred approach. The concepts presented are equally applicable in the continuous-time domain. In fact, the original concepts about optimal predictors and optimal filters were first derived in the continuous-time domain. The interested reader is referred to Grewal and Andrews (2) and Wiener (3). The following three types of state estimation problems can now be defined (1,4):

1. *Smoothing:* given the measurements, $\mathbf{y}(t + \lambda)$ for λ positive integer, up to and including the time instant (t +

λ), the state estimate $\hat{\mathbf{x}}(t|t+\lambda)$ at a past time t is determined.

2. *Filtering:* given the measurements, $\mathbf{y}(t)$, up to and including the time instant t, the state estimate $\hat{\mathbf{x}}(t|t)$, at the present time t is determined.
3. *Prediction:* given the measurements $\mathbf{y}(t-\lambda)$, up to and including the time instant $(t-\lambda)$, the state estimate $\hat{\mathbf{x}}(t|t-\lambda)$, at the future time t is determined. If $\lambda=1$, this is referred to as single-step-ahead prediction, and if $\lambda = p$, where $p > 1$, this is referred to as p-step-ahead or multistep-ahead prediction.

In defining state estimation, it is assumed that there exists a relation (dynamic and/or static) between the measurements $\mathbf{y}(t)$ and the state $\mathbf{x}(t)$ to be estimated. In many systems encountered in engineering, however, one does not always have an accurate knowledge of this relation. Therefore, assumptions must be made regarding the dynamics of the system, and these assumptions usually take the form of a mathematical model of the system investigated. The mathematical model (or relation) between the measurements and the estimated variables can be of varying complexity, ranging from extracting a signal corrupted by additive white Gaussian sensor noise, to estimating a time-varying parameter in a complex process control problem. The nature of this relationship dictates the complexity of the state estimation problem to be solved.

Even though the earliest sign of an "estimation theory" can be traced back to the mid-17th century work of Galileo Galilei, credit for the origin of linear estimation theory is given to Karl Friedrich Gauss, for his late-18th century invention called the method of least squares, to study the motion of heavenly bodies. In one way or another the least squares method by Gauss forms the basis for a number of estimation theories developed in the ensuing 200 years, including the Kalman filter (5). Following the work of Gauss, the next major breakthrough in estimation theory came from the work of A. N. Kolmogorov in 1941 and N. Wiener in 1942. In the early years of World War II, Wiener was involved in a military project at the Massachusetts Institute of Technology (MIT) regarding the design of automatic controllers for directing antiaircraft fire using radar information. As the speeds of the airplane and the bullet were comparable, it was necessary to account for the motion of the airplane by shooting the bullet "into the future." Therefore, the controller needed to predict the future position of the airplane using noisy radar tracking information. This work led first to the development of a linear optimum predictor, followed by a linear optimum filter, the so-called Wiener filter. Both the predictor and the filter were optimal in the mean-squared sense and they were derived in the continuous-time domain. The filter design equations were solved in the frequency domain (3). Later in 1947, Levinson formulated the Wiener filter in the discrete-time domain (6). An analogous, but by no means identical, derivation of the optimal linear predictor was developed by Kolmogorov in the discrete-time domain (7), prior to the widespread publication of the work by Wiener in 1949.

WIENER FILTERS—LINEAR OPTIMAL FILTERING

A more detailed treatment of Wiener filters is now presented by first defining the precise estimation problem they address. This is followed by a brief mathematical description of the filter structure and the method used to design it. The section ends with a brief treatment of Wiener filter performance. For the interested reader, additional details of the Wiener filter derivation can be found in the excellent textbook by Haykin (8). A mathematically rigorous treatment of the continuous-time filter derivation for scalar and vector signals can be found in the original text by Wiener (3).

Filtering Problem Statement

As mentioned at the beginning of this article, the Wiener filter belongs to the class of linear (nonadaptive) optimal filters. Considering the discrete-time treatment of a single-input (and a single-output) filter, let us assume that at time t the signal to be filtered (the input to the filter) consists of an infinite number of measured input samples $u(t)$, $u(t-1)$, ... The input samples are assumed to be random variables, and, therefore, the input signal represents a random (or stochastic) process characterized by some known statistics. The filter is described by an infinite number of constant coefficients, b_0, b_1, . . ., known as the filter impulse response. At any discrete-time instant t, the filter generates an output (the filtered signal) denoted by $\hat{y}(t/t)$. This output is an estimate of the desired filter response denoted by $y(t)$. Both the filter input and the desired filter response must be known in order to design a Wiener filter. However, only the former is needed for filter operation. The deviation of the filter output from the desired filter response gives rise to the estimation error, which becomes the key measure of filter performance. A block diagram representation of a Wiener filter is shown in Fig. 1.

Thus far the only assumption that has been made regarding Wiener filters is of their linear structure. The fact that the filter operates in the discrete-time domain is not a restrictive assumption, rather a necessity of the digital world. The continuous-time version of the filter can be developed at the expense of some mathematical complications. Additionally, and

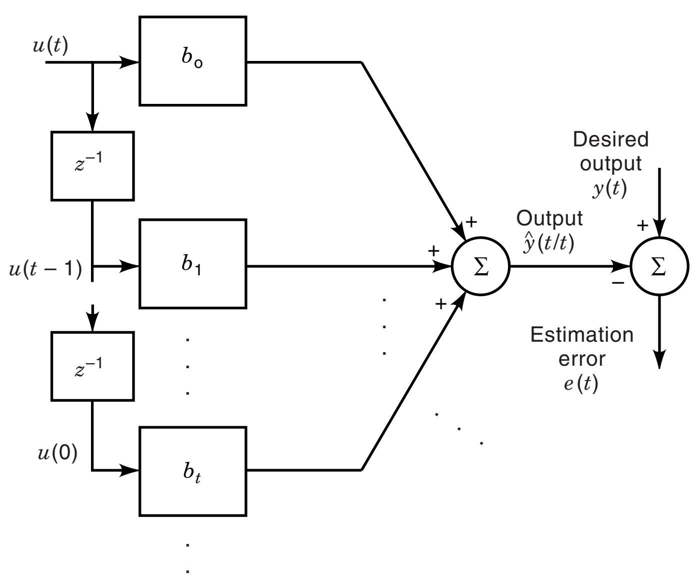

Figure 1. Block diagram of infinite impulse response (IIR) implementation of a Wiener filter.

without loss of generality, an implicit assumption is made regarding real-valued observations. The Wiener filter can be derived for complex-valued observations, commonly encountered in communications applications. The interested reader is referred to the Wiener filter presentation by Haykin (8), which assumes complex-valued observations.

The filtering problem addressed by Wiener filters can now be defined as follows:

> Design a linear (discrete-time) filter by completely defining all of its unknown parameters, such that for any given set of inputs $u(t)$, $u(t-1)$, . . ., the estimation error, defined as the difference between the desired filter response, $y(t)$, and the filter output, $\hat{y}(t/t)$, is minimized in some statistical sense.

In order to solve the forementioned problem for filter design, two important issues must be dealt with as follows:

1. How to select (or restrict) the structure of the filter impulse response?
2. What statistical criterion to use for optimizing the filter design?

The first issue must deal with whether the filter should have a finite (FIR) or an infinite impulse response (IIR). That is, whether the filter should have only "feedforward," or both "feedforward" and "feedback" signal flow paths. This is of great practical implication because design and implementation of IIR filters introduces many complications. Therefore, even though the general theory of Wiener filters was developed based on IIR filters, practical applications are usually treated employing some form of a FIR filter. The reason for this choice is the inherent stability of FIR filters, compared to the potential instabilities that can result from IIR filters. Nevertheless, properly designed IIR filters are guaranteed to be stable. Adaptation of such filters, however, raises serious complications.

The second issue is of mathematical importance. The choice of a complex statistical criterion to be optimized results in increased complexity in filter design equations. Generally, in designing a filter a cost (or objective) function is selected that is then minimized by choosing the appropriate filter parameters. The choice of the cost function varies, although the following are possible options:

1. Mean-square estimation error;
2. Expectation of the absolute value of the estimation error; and
3. Expectation of higher powers of the absolute value of the estimation error.

Additionally, combinations of the above objective functions are often used in attempts to minimize the effects of bad data. This is the subject of robust estimation, and the interested reader is referred to Söderström and Stoica (9) and Ljung (10). The mean-square estimation error is a popular choice, because it results in a convex optimization problem and a relatively simple set of filter design equations.

Filter Theory and Design

To develop the equations used in the design of Wiener filters, an expression for the filter output must first be developed. Let us now return to the depiction of Fig. 1. Considering an infinite observation horizon for the filter inputs $u(t)$, $u(t-1)$, . . ., and an IIR filter structure described by the coefficients b_0, b_1, . . ., we can express the filter response $\hat{y}(t/t)$ as

$$\hat{y}(t/t) = \sum_{k=0}^{\infty} b_k u(t-k), t = 0, 1, 2, \ldots \tag{1}$$

Filter Theory Development. The objective of a Wiener filter is to provide an optimal estimate of the desired filter response $y(t)$, where the estimate is optimal in some mean-squared sense. To obtain an optimal estimate, the estimation error is defined as

$$e(t) \equiv y(t) - \hat{y}(t/t) \tag{2}$$

The estimation error is utilized in the following filter objective function to be minimized:

$$J = E\{e^2(t)\} \tag{3}$$

Furthermore, in order to proceed with the Wiener filter development, it is assumed that the filter input and desired filter response, $u(t)$ and $y(t)$ respectively, are zero-mean jointly (wide-sense) stationary stochastic processes. For definitions and other clarifications regarding stationary stochastic processes, the reader is referred to Papoulis (11).

The filter objective function J can now be minimized by computing the IIR filter coefficients b_k such that the gradient of J with respect to each one coefficient becomes simultaneously zero. As a result of simultaneously setting all of the gradients to zero, the optimality of the Wiener filter is in the mean-squared-error sense. Taking the derivative of the objective function given by Eq. (3) with respect to the IIR filter coefficients b_0, b_1, ... and setting them to zero we obtain

$$\nabla_k J = \frac{\partial J}{\partial b_k} = E\left\{2e(t)\frac{\partial e(t)}{\partial b_k}\right\} = 0, k = 0, 1, 2, \ldots \tag{4}$$

Using the IIR filter response expressed by Eq. (1) and the estimation error of Eq. (2), the error gradient in Eq. (4) can further be expressed as

$$\frac{\partial e(t)}{\partial b_k} = -\frac{\partial \hat{y}(t/t)}{\partial b_k} = -u(t-k) \tag{5}$$

Utilizing this expression of the gradient, the following condition is obtained for minimizing the filter objective J:

$$\nabla_k J = -2E\{e(t)u(t-k)\} = 0 \tag{6}$$

In light of the objective function convexity, the filter becomes optimal whenever Eq. (6) is satisfied. Let us denote with a zero subscript the characteristics of the optimal filter. That is, b_{ok} represents the optimal IIR filter coefficients, $\hat{y}_o(t/t)$ represents the optimal filter response (or filter output), and $e_o(t)$ represents the optimal filter estimation error. The value of the objective function for the optimal filter is denoted by J_{min}. The optimality conditions for the Wiener filter can now be expressed as

$$E\{e_o(t)u(t-k)\} = 0, k = 0, 1, 2, \ldots \tag{7}$$

and the minimum mean-squared estimation error is given by

$$J_{min} = E\{e_o^2(t)\} \tag{8}$$

Equation (7) brings-up some important points regarding the operation of optimal filters. Specifically, this equation implies that if the filter operates in its optimal condition, then at each time t the (optimal) estimation error is orthogonal with the filter input. In other words, at each time t the optimal estimation error is uncorrelated with the filter input. The implication of this observation is consistent with filter optimality. It indicates that if a filter operates in optimal conditions, then all useful information carried by the filter inputs must have been extracted by the filter, and must appear in the filter response. The (optimal) estimation error must contain no information that is correlated with the filter input; rather it must contain only information that could not have been extracted by the filter.

Filter Design Equations. The previous section presented the development of the Wiener filter theory, but it did not address issues related to the design of such filters. The filter optimality condition, Eq. (7), becomes the starting point for Wiener filter design. The IIR filter structure can still be used prior to the selection of a more appropriate structure.

In view of filter response Eq. (1), and the definition of Eq. (2), substitution of the optimal estimation error in Eq. (7) results in the following expression:

$$E\{u(t-k)(y(t) - \sum_{k=0}^{\infty} b_{ok} u(t-k))\} = 0, k = 0, 1, 2, \ldots \tag{9}$$

Further expanding and manipulating the expectations in the above equation results in

$$\sum_{j=0}^{\infty} b_{oj} E\{u(t-k)u(t-j)\} = E\{u(t-k)y(t)\}, k = 0, 1, 2, \ldots \tag{10}$$

Notice that the preceding equation includes the unknown filter coefficients and observed quantities. The expectations present in Eq. (10) are the autocorrelation of the filter input and the cross correlation between the filter input and the desired filter response. Defining the forementioned autocorrelation and cross correlation by $R(j-k)$ and $P(-k)$, respectively, Eq. (10) can be expressed as

$$\sum_{j=0}^{\infty} b_{oj} R(j-k) = P(-k), k = 0, 1, 2, \ldots \tag{11}$$

The set of (infinite) equations given by Eq. (11) is called the *Wiener–Hopf* equations.

The structure of the assumed filter must be further simplified before attempting to solve the design equations in Eq. (11). The solution to these equations can be greatly simplified by further assumptions regarding the optimal filter structure. It should be noted that in the original formulation by Wiener, Eq. (11) was derived in the continuous-time domain at the expense of significant mathematical complications. Assuming an M-th order FIR filter, Eq. (11) is simplified as

$$\sum_{j=0}^{M-1} b_{oj} R(j-k) = P(-k), k = 0, 1, 2, \ldots (M-1) \tag{12}$$

where b_{o0}, b_{o1}, . . ., $b_{o(M-1)}$ are the optimal FIR filter coefficients. The assumed FIR filter structure is depicted in Fig. 2.

The Wiener–Hopf equations, Eq. (12), can be solved using one of the many numerical analysis methods for linear algebraic equations. These equations can be reformulated in matrix form

$$\mathbf{R}\mathbf{b}_o = \mathbf{P} \tag{13}$$

where the autocorrelation matrix is defined as

$$\mathbf{R} = \begin{bmatrix} R(0) & R(1) & \cdots & R(M-1) \\ R(1) & R(0) & \cdots & R(M-2) \\ \vdots & \vdots & \vdots & \vdots \\ R(M-1) & R(M-2) & \cdots & R(0) \end{bmatrix} \tag{14}$$

the cross-correlation matrix is defined as

$$\mathbf{P} = [P(0), P(-1), \ldots, P(1-M)]^T \tag{15}$$

and where the vector containing the filter coefficients is defined as

$$\mathbf{b}_o = [b_{o0}, b_{o1}, \ldots, b_{o(M-1)}]^T \tag{16}$$

Assuming the correlation matrix $\mathbf{R}$ is nonsingular, Eq. (13) can now be solved for $\mathbf{b}_o$

$$\mathbf{b}_o = \mathbf{R}^{-1}\mathbf{P} \tag{17}$$

representing the coefficients of the optimal filter. Design of an optimal Wiener filter requires computation of the right-hand side of Eq. (17). This computation requires knowledge of the autocorrelation and cross-correlation matrices $\mathbf{R}$ and $\mathbf{P}$. Both of these matrices depend on observations of the filter input and the desired filter response.

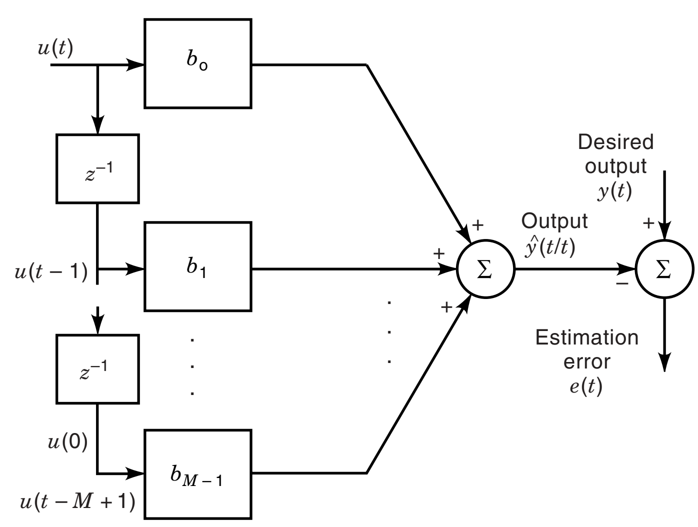

Figure 2. Finite impulse response (FIR) implementation of a Wiener filter.

Filter Performance

The performance of Wiener filters can be explored by expressing the filter objective function in terms of the filter parameters, that is, in terms of the impulse response coefficients b_0, b_1, . . ., b_{M-1}. Then, the objective function can be investigated as a function of these coefficients.

For the M-th order FIR Wiener filter shown in Fig. 2, let us rewrite the objective function of Eq. (3) in terms of the filter inputs and the desired filter response as follows:

$$J = E\{(y(t) - \sum_{k=0}^{M-1} b_k u(t-k))^2\} \tag{18}$$

This equation can be expanded as

$$\begin{aligned} J =& E\{y^2(t)\} - 2\sum_{k=0}^{M-1} b_k E\{u(t-k)y(t)\} \\ &+ \sum_{k=0}^{M-1}\sum_{j=0}^{M-1} b_k^2 E\{u(t-k)u(t-j)\} \end{aligned} \tag{19}$$

Now, using the definition for the variance

$$\sigma_y^2 = E\{y^2(t)\} \tag{20}$$

along with the autocorrelation and cross correlation, $R(j-k)$ and $P(-k)$, the objective function can be rewritten as

$$J = \sigma_y^2 - 2\sum_{k=0}^{M-1} b_k P(-k) + \sum_{k=0}^{M-1}\sum_{j=0}^{M-1} b_k^2 R(j-k) \tag{21}$$

Equation (21) can now be written in vector form as

$$J(\mathbf{b}) = \sigma_y^2 - 2\mathbf{b}^T\mathbf{P} + \mathbf{b}^T\mathbf{R}\mathbf{b} \tag{22}$$

where the objective function dependence on the filter parameters $\mathbf{b}$ is explicitly shown, and where the other parameters are as previously defined.

In view of the joint stationarity assumptions placed upon the filter input and the desired filter response signals, the objective function equation (21) or (22) is a quadratic function of the filter impulse response parameters b_k. Therefore, performance optimization of the Wiener filter is a quadratic optimization problem with a unique minimum. This unique minimum J_{min} occurs when the filter parameter values correspond to the optimal filter $\mathbf{b}_o$ such that

$$J_{\min} = J(\mathbf{b}_o) \tag{23}$$

The shape of the quadratic performance index depicting the optimal and suboptimal filters is shown in Fig. 3. The optimal filter parameters are computed by solving Eq. (17) for b_ks. For other values of the parameters b_k, the resulting filter is suboptimal. In fact, a truly optimal Wiener filter is realized if M is allowed to approach infinity, that is, $M \to \infty$. Therefore, all FIR implementations of a Wiener filter result in suboptimal performance. This practical limitation imposed by the need to select a finite M can be overcome by allowing "feedback" paths in the filter structure. Feedback allows implementation of a Wiener filter via finite-order IIR filters. (IIR filters are not discussed in this article.) In practical applications and irrespective of the selected filter structure, use of suboptimal filters is quite often inevitable because of violations in the assumptions underlying the optimal Wiener filter, such as the assumptions of filter input and desired response stationarity and other practical implementation considerations.

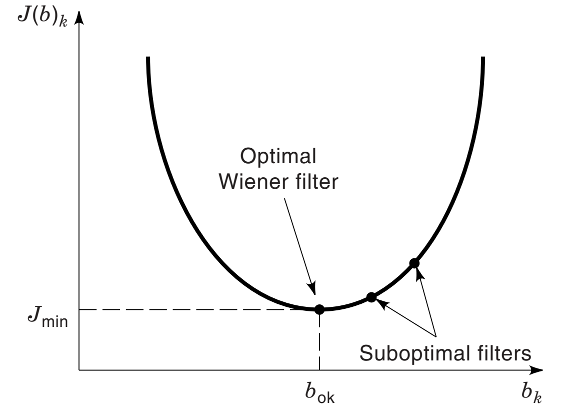

Figure 3. Wiener filter objective function depicting the impact of filter parameters on filter optimality.

The filter performance analysis can be taken a step further to determine the optimal filter performance in terms of the statistics of the filter input and the desired filter response. The optimal filter response can be expressed as

$$\hat{y}_o(t/t) = \sum_{k=0}^{M-1} b_{ok} u(t-k) = \mathbf{b}_o^T\mathbf{u}(t) \tag{24}$$

The filter response, a function of the filter input, is a stochastic process itself. The variance of the filter response can be expressed as

$$\sigma_{\hat{y}}^2 = E\{\mathbf{b}_o^T\mathbf{u}(t)\mathbf{u}^T(t)\mathbf{b}_o\} = \mathbf{b}_o^T E\{\mathbf{u}(t)\mathbf{u}^T(t)\}\mathbf{b}_o = \mathbf{b}_o^T\mathbf{R}\mathbf{b}_o \tag{25}$$

or, using Eqs. (13) and (17),

$$\sigma_{\hat{y}}^2 = \mathbf{b}_o^T\mathbf{P} = \mathbf{P}^T\mathbf{b}_o = \mathbf{P}^T\mathbf{R}^{-1}\mathbf{P} \tag{26}$$

Applying the definition given in Ref. 2 for the optimal filter, the desired filter response can be expressed as

$$y(t) = \hat{y}_o(t/t) + e_o(t) \tag{27}$$

Taking the expectation of the square of both sides of Eq. (27) results in the following relation between the variance of the filter response and the desired response:

$$\sigma_y^2 = \sigma_{\hat{y}}^2 + J_{\min} \tag{28}$$

Using Eqs. (26) and (28), the minimum mean-squared error of the objective function can be expressed as

$$J_{\min} = \sigma_y^2 - \mathbf{P}^T\mathbf{R}^{-1}\mathbf{P} = \sigma_y^2 - \mathbf{P}^T\mathbf{b_o} \tag{29}$$

Defining the normalized mean-squared error of the optimal filter as

$$\epsilon_0 = \frac{J_{\min}}{\sigma_y^2} \tag{30}$$

Eq. (28) can be expressed as

$$\epsilon_0 = 1 - \frac{\sigma_{\hat{y}}^2}{\sigma_y^2} \tag{31}$$

Equation (31) is a performance index for the optimal Wiener filter, expressed as a function of the variance of the filter response and the desired filter response. In view of Eq. (28), this performance index takes values in the range

$$0 \leq \epsilon_0 \leq 1 \tag{32}$$

If the optimal filter results in zero mean-squared estimation error, then the performance index ϵ_0 becomes zero. As the mean-squared estimation error of the optimal filter increases, the performance index ϵ_0 approaches one. The variation in the performance index for a family of Wiener filters is depicted in Fig. 4.

It should be noted that the range given by Eq. (32) is valid only for the optimal Wiener filter. For such a filter, the cross-correlation between the filter error and filter output is zero, resulting in the expression of Eq. (28). For suboptimal filters, the term $E\{\hat{y}(t/t) \cdot e(t)\}$ must be included on the right-hand-side of Eq. (28). Then, the performance index given by Eq. (31) does not have an upper-bound of 1.

LIMITATIONS OF WIENER FILTERS

Following the successful application of Wiener filters during World War II, it became clear that the functionality offered by such a device (a filter) would be immense in many technological applications. Careful consideration of the filter derivation, however, immediately points out a number of key limitations. The three main assumptions of Wiener filters for optimal operation are as follows:

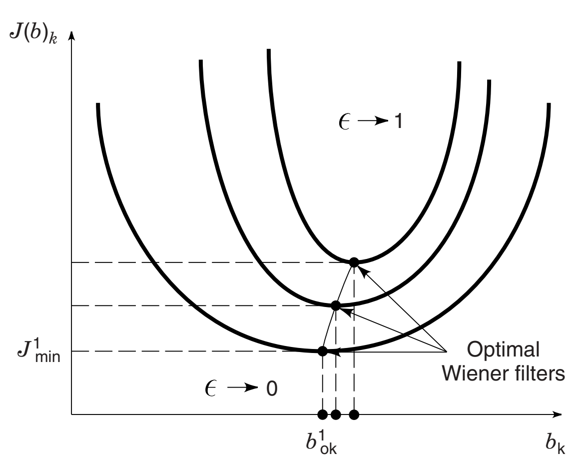

Figure 4. Wiener filter performance index depicting famlies of filter performance curves and location of optimal filters.

1. The filter input must consist of an infinite number of observations;
2. The filter input must be stationary; and
3. The filter output must be related to the filter input by a linear relation.

Furthermore, for designing an optimal Wiener filter the statistics of the desired filter response must be available, and it also must be stationary. Finally, there are additional limitations imposed by the underlying theory of Wiener filters, but these limitations relate to filter implementation.

Nonstationary Finite Duration Signals

A close look at the key assumptions made in deriving the Wiener filter reveals that stationarity of the filter input is required. The desired filter response statistics must also be available, and further the filter input and the desired filter response must be zero mean, jointly *wide sense stationary*. Stationarity of a stochastic signal implies that its statistical properties are invariant to a shift in time. Furthermore, these signals are required to be observed for an infinite observation interval (or window).

In engineering practice many signals that serve as filter inputs are not stationary and they are subject to finite observation intervals. This is especially true in control engineering applications of filters. In such applications the essence of the filter function is performed during periods in which the system generating the signals to be filtered is undergoing a transient. This results in signals with varying means, rendering them nonstationary. Additionally, in many engineering applications the desired filter response statistics may not be available. This is further complicated by the requirement of a zero mean, stationarity desired filter response. Finally, it is worth mentioning that practical implementation forbids excessively long observation intervals for all signals involved in the filter operation.

The limitation of Wiener filters in processing nonstationary, finite observation interval signals was well known to Wiener himself, who during the years following the development of the original linear optimal filter rigorously investigated its possible extensions (12). During the 1950s, many researchers attempted to extend the applicability of Wiener filters to nonstationary, finite observation interval signals with little success. Although some theoretical results were obtained that eliminated these assumptions, the rather complicated results did not find much use in practical filter design. The two main difficulties were associated with filter update as the number of observations increased, and the treatment of the multiple (vector) signal case. Both of these limitations of the Wiener filters were eliminated by the development of the Kalman filter. This development made the assumption of a stationary, infinite observation horizon filter input unnecessary.

Nonlinear Systems

Another limitation of Wiener filters results from the assumed linear relation between the filter input and the desired filter response. This implies a linear relation between the filter input and output also. Having a linear structure, Wiener filters can not effectively address filtering problems in which the filter inputs and outputs must be related by some nonlinear functional form. That is, if the filter inputs and the desired filter response is inherently nonlinear, then use of Wiener filters results in suboptimal filtering.

During the 1950s, Wiener conducted extensive research on the use of a special class of nonlinear functional form to relate the filter inputs and the desired filter response. He used the so-called Volterra series, which was first studied in 1880 as a generalization of the Taylor series expansion of a function. Wiener used Volterra series to model the input-output relationship of a system (10). The integral approach of Wiener's formulation of the nonlinear extensions to the Wiener filters, and the lack of computational capabilities during that period, limited the application and use of his developments until the mid-1980s (13).

In engineering practice, and especially in the many industrial applications of filtering and of estimation in general, nonlinearities are widely encountered facts of life. These nonlinearities are currently handled in an ad hoc manner, because an effective nonlinear filtering method has yet to be developed. The development of the Extended Kalman Filter (EKF), as a means to account for nonlinear process and/or noise dynamics, has not eliminated the problems associated with practical nonlinear filtering problems at all. The main reason for this inadequacy is the inherent modeling uncertainties in many nonlinear filtering problems. The modeling uncertainties render the EKF quite often ineffective. More recently during the 1990s, a different type of functional relation, based on the so-called artificial neural networks, has shown promise in nonlinear input-output modeling (14). In principle, the application of these mathematical tools has followed the initial attempts by Wiener on the use of Volterra series to extend the capabilities of linear optimal filters by the use of black-box nonlinear models.

Relation to Kalman Filters

The Kalman filter, probably the most significant and technologically influential development in estimation theory during this century, first appeared in the literature in 1959. It is not a secret that the Kalman filter was initially developed as a means of circumventing some of the limitations of the Wiener filters. The relation between these two filters can be best understood by considering the model-based nature of the "filtering operation" performed on a signal generated by a system. In effect, the Wiener filter attempts to model the filter inputs and outputs by an "input–output" (or transfer function) model. On the contrary, the Kalman filter poses the following question: Why not apply the concept of state-space to Wiener filters? The answer to this question was the mathematical development of linear optimal filtering theory based on "state-space" models. In doing so, many of the limitations imposed by the Wiener filter are eliminated. Some of the similarities and relative advantages of the Wiener and Kalman filters can be summarized as follows (2):

1. Both the Wiener and Kalman filters have linear structure.
2. The Wiener filter assumes an infinite observation horizon for the filter inputs, compared to the Kalman filter assumption of a finite observation horizon.
3. The Wiener filter assumes stationary stochastic processes as filter inputs, whereas the inputs to the Kalman filter may be nonstationary.
4. The Wiener filter assumes the availability of a desired filter response, whereas the Kalman filter assumes the availability of a model of the system to be filtered.
5. The Wiener filter inputs and outputs are related using an input–output (or transfer function) model represented by the impulse response coefficients. The Kalman filter inputs and outputs are related by a state-space model.
6. Derivation of the linear optimum filter based on the principles of Kalman filtering requires less mathematical sophistication than the equivalent derivation using the principles of Wiener filtering.
7. The Kalman filter provides a better framework than the Wiener filter for the detection and rejection of outliers (or bad data).
8. The Wiener filter implementation in analog electronics can operate at much higher effective throughput than the (digital) Kalman filter.
9. The Kalman filter is best suited for digital implementation. Although this implementation might be slower, it offers greater accuracy than that which is achievable with analog filters.

Variations of the Wiener Filter

The first true variation of the Wiener filter came from Levinson in 1947, who reformulated the original derivation in the discrete-time domain (6). During the 1950s several attempts were made to relax the infinite observation horizon and the stationarity requirements of the original Wiener filter formulation. These attempts resulted in mathematically very complex variations of the Wiener filter. Furthermore, handling of the vector case was excessively difficult. These complications resulted in Swerling's early attempts at recursive algorithms (8). They were followed by Kalman's derivation of the Wiener filter in the time-domain.

One of the most widely used variants of the Wiener filter is the so-called linearly constrained minimum variance (LCMV) filter. The derivation of this filter was motivated by the need to relax the presence of the desired filter response in Wiener filter design. In some filtering applications the desired filter response is not always known or available. Furthermore, in some applications it is desired to minimize a mean-squared error criterion subject to some constraints. In such circumstances, the LCMV is utilized as an alternative to Wiener filters. Design of an LCMV filter requires the solution of a constrained optimization problem. This is accomplished by using one of the many constrained optimization methods, such as the method of Lagrange multipliers. Another variation of Wiener filters is the linear optimal predictor, which preceded the development of the Wiener filter and which laid the foundations of the linear optimal filtering theory.

WIENER FILTER APPLICATIONS

Since its introduction in the 1940s, the Wiener filter has found many practical applications in science and technology. As with all other filters that were developed following World

War II, the Wiener filter extracts information from noisy signals. Nevertheless, the inherent assumptions made in deriving the Wiener filter place certain limitations on its applicability to many practical problems. In fact, these limitations have been among the primary motivations for the development of the Kalman filter.

Despite the apparent superiority of the Kalman filter, it is still advantageous to implement a Wiener filter when proper conditions arise. Such conditions include either filtering stationary or quasi-stationary signals, or equivalently filtering signals from systems operating under steady-state or quasi-steady-state conditions.

General Uses

In general, Wiener filters are applicable to problems in which all signals of interest can be assumed stationary, and the desired filter response can be either expressed analytically and/or measured. These preconditions limit, to a large extent, the use of Wiener filters. For example, many filtering problems encountered in control applications are characterized by nonstationary signals. Furthermore, very often the desired filter response and its statistics are not explicitly known and/or measured. An exception to this class of problems is the area of target-tracking and navigation, the very first application of Wiener filters. Additionally, Wiener filters have found many applications in communication systems, for example, in channel equalization and beamforming problems. Wiener filters have also found wide use in two-dimensional image processing applications.

In tracking applications, Wiener filters are used to estimate the position, velocity, and acceleration of a maneuvering target from noisy measurements. The target being tracked may be an aircraft, a missile, or a ship. Radar and other instruments measure the range, azimuth, and elevation angles of the target. If a Doppler radar is available, then range-rate information is also included. If the target moves at constant velocity, then the Wiener filter position estimates might be quite accurate. Because of the limitations inherent in the Wiener filter, however, evasive maneuvers of the target cannot be accounted for with accuracy. Estimation of target position is usually part of an overall system to improve the accuracy of a fire control system.

A Simple Example

In this section we present a very simplified example of the Wiener filter in the discrete-time domain. As the example progresses, comments regarding realistic applications will be made to inform the reader of some of the difficulties involved in real-world filtering applications.

The three key issues involved in Wiener filter design are:

1. What are the statistics of the desired response?
2. How is the filter input related to the desired response?
3. How is the order of the Wiener filter selected?

In many real-world filtering applications the statistics of the desired response are not easily quantified. Furthermore, in many instances the desired response may not be a well-behaved stochastic process. Similarly, the relation between the desired response and the inputs to the Wiener filter may not be simple. As a result, the third forementioned issue, the order of the Wiener filter, is not easily determined.

For the sake of this example, let us assume that the desired filter response is generated as the output of a zero-mean, white-noise driven linear time-invariant system with a transfer function $H(z)$. The white-noise is denoted by $w(t)$. This assumption greatly simplifies the analysis of the statistical properties of the desired filter response. Furthermore, let us assume that the desired response and the filter inputs are also related by a linear time-invariant system with a transfer function $G(z)$, corrupted by additive zero-mean, white noise. It is desired to design a Wiener filter such that the difference between the filter response $\hat{y}(t/t)$ and the desired response $y(t)$ is minimized in the mean-squared sense. The block diagram for this example is shown in Fig. 5.

Let us now assume, for simplicity, that the desired filter response is generated by the following first-order transfer function:

$$H(z) = \frac{1}{1 + h_1 z^{-1}} = \frac{1}{1 + 0.5z^{-1}} \tag{33}$$

where the zero-mean, white noise input $w(t)$ driving $H(z)$ has a variance $\sigma_w^2 = 0.35$. It is further assumed that the desired response is related to the filter input by the following, also first-order, transfer function:

$$G(z) = \frac{1}{1 + g_1 z^{-1}} = \frac{1}{1 - 0.75z^{-1}} \tag{34}$$

The output of the transfer function $G(z)$ is further corrupted by the additive zero-mean, white noise $n(t)$, with a variance $\sigma_n^2 = 0.15$.

To design a Wiener filter, we need to characterize two correlation functions related to the desired filter response and the filter input. Specifically, we need to compute the autocorrelation $\mathbf{R}$ of the filter input, $u(t)$, and the cross-correlation $\mathbf{P}$ between the filter input and the desired response $y(t)$. Additionally, we need to compute the variance of the desired response. This is accomplished by observing that the variance of the output of a linear filter driven by white noise is related to the variance of its input [for details of the appropriate equations, the reader is referred to Haykin (8) and Papoulis (9)]. This calculation results in

$$\sigma_y^2 = \frac{\sigma_w^2}{1 - h_1^2} = 0.47 \tag{35}$$

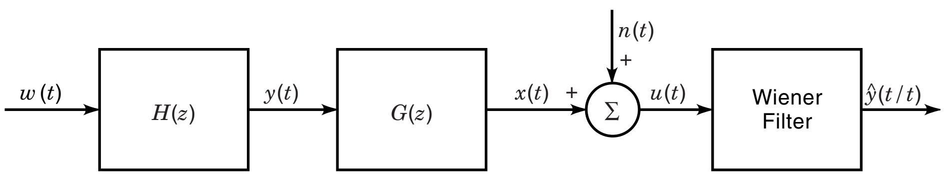

Figure 5. Wiener filter example block diagram.

In calculating the correlation matrices, it helps to observe that the two transfer functions specified in the preceding completely define the structural information needed for the design of the Wiener filter. The Wiener filter input can be expressed as the response of a white-noise driven second-order filter $G(z)H(z)$, corrupted by additive noise. Therefore, the autocorrelation matrix of the filter input is a two-dimensional matrix and the Wiener filter can be chosen as a second-order FIR filter. For more realistic problems, the precise characterization of the transfer functions $G(z)$ and $H(z)$ makes filter design a challenging task.

Let us now return to the calculation of the correlation matrices. The autocorrelation matrix $\mathbf{R}$ can be calculated as the sum of the autocorrelations of the uncorrupted response of $G(z)$ and additive noise $n(t)$. Furthermore, the autocorrelation of the uncorrupted response of $G(z)$ can be calculated in terms of the statistical properties of the desired filter response, $y(t)$, and the coefficients of $G(z)$. The cross-correlation matrix $\mathbf{P}$ can be calculated using similar arguments, in terms of the statistical properties of the desired response and the filter input. For this example these calculations result in the following numerical results:

$$\mathbf{R} = \begin{bmatrix} 0.6348 & 0.4 \\ 0.4 & 0.6348 \end{bmatrix} \tag{36}$$

$$\mathbf{P} = [0.1848 \quad 0.0364]^T \tag{37}$$

The Wiener filter coefficients can now be computed using Eq. (17),

$$\mathbf{b}_o = \mathbf{R}^{-1}\mathbf{P} = [0.4320 - 0.2092]^T \tag{38}$$

In view of Eq. (22), the filter objective function $J(b_0, b_1)$, can now be expressed in terms of the filter coefficients

$$J(b_0, b_1) = 0.47 - 0.1848b_0 - 0.0364b_1 + 0.8b_0b_1 + 0.6348(b_0^2 + b_1^2) \tag{39}$$

The error performance surface expressed by Eq. (39) is depicted in Fig. 6.

The optimal Wiener filter, corresponding to the filter coefficients given by Eq. (38), has minimum mean-squared error given by Eq. (29). The numerical value of this mean-squared error is

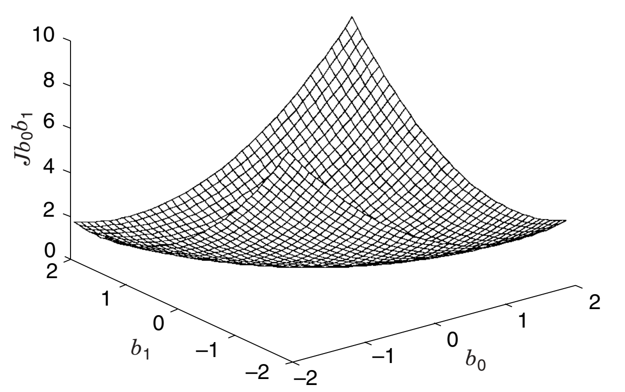

Figure 6. Error performance surface for Wiener filter example.

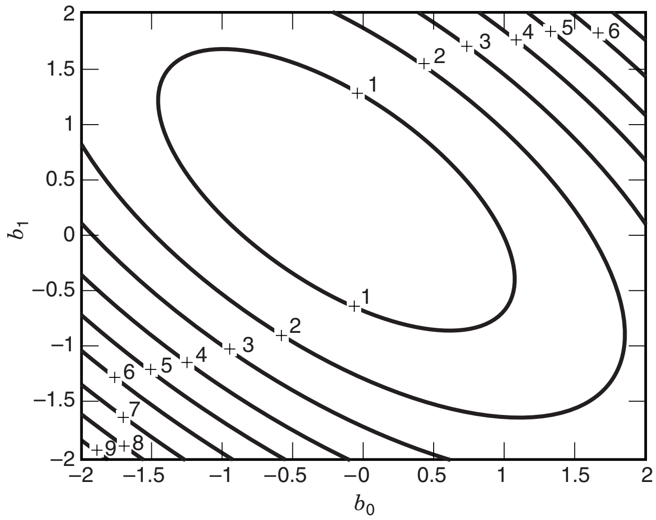

Figure 7. Error performance contours for Wiener filter example.

$$J_{\min} = 0.47 - [0.1848 \quad 0.0364][0.4230 - 0.2092]^T = 0.3961 \tag{40}$$

The variations in the filter error performance, from optimal to suboptimal, are best visualized by the contour plot shown in Fig. 7. The objective function value corresponding to the optimal filter is at the center of the ellipse depicted by contour value 1.

A PRACTICAL EXAMPLE: ELECTRIC MOTOR RESPONSE FILTER

In the final section of this article, we present a more practical—though still simplified—application of a Wiener filter. In particular, we filter the electrical response of an induction motor assumed to be operating under constant load conditions and without the presence of a variable speed drive. The power supply voltage applied to an induction motor is considered to be a motor input, whereas the electric current drawn by the motor is considered to be a motor output. In this application of Wiener filters, the motor current is estimated (or filtered) using voltage measurements, and the filter response is compared to the actual motor current measurements. If properly designed, such a filter could be utilized in practice to detect changes in the motor electrical response that might be due to power supply variations, load variations, incipient motor faults, or a combination of these conditions.

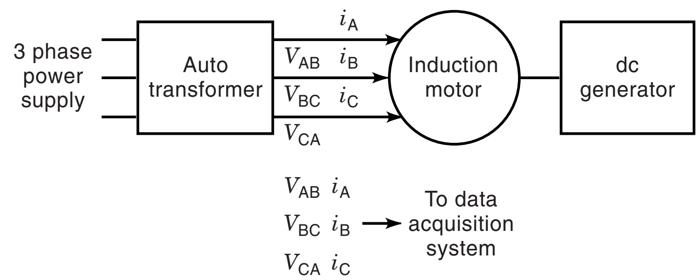

Figure 8. Depiction of experimental set-up for induction motor filter application.

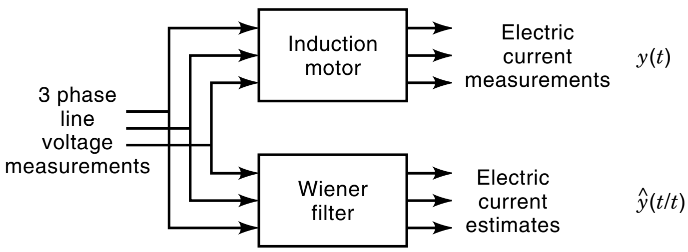

Figure 9. Input–output depiction of motor filter application.

In this example, several key assumptions are made regarding motor operation that simplify this application problem, as follows:

1. The motor is assumed to be connected to a balanced power supply.
2. The motor is assumed to consist of three balanced stator windings.
3. The motor is assumed to be operating under constant load conditions.

Under these simplifying assumptions, the three motor phases can be decoupled, and filtering of a single motor current phase can be pursued based on a single line voltage measurement. In this example, voltage and current measurements were obtained from the experimental set-up depicted in Fig. 8.

The three key issues involved in Wiener filter design are as follows:

1. What are the statistics of the desired response?
2. How is the filter input related to the desired response?
3. What is the best way to select the order of the Wiener filter?

In this application, the desired filter response is the measured motor current. This is a nonstationary signal with mean 60 Hz fundamental sinusoid. We could attempt to detrend the fundamental signal and then proceed with the filtering process. In this application, however, detrending was not pursued. Furthermore, the filter input (i.e., the measured motor line voltage) is related to the desired filter response via the generally nonlinear induction motor dynamics. Therefore, in this example two of the key assumptions of Wiener filters are violated, and an optimal filter cannot be designed. The violated assumptions are the nonstationarity of the filter response and the nonlinear relation between the filter input and the desired filter response. A block diagram depicting the filter input and filter response is given in Fig. 9.

To design a suboptimal Wiener filter, we need to deal with the issue of filter order. In this application, the exact filter order is not easily determined and an iterative approach must be followed. In order to determine a satisfactory filter order, one must compare the performance of various filters against a predetermined criterion. In this study, we have used two error criteria for this comparison—the normalized mean-squared error (NMSE) and the relative error (RE), defined as follows:

$$\text{NMSE} \equiv \frac{\sum[\hat{y}(t/t) - y(t)]^2}{\sum y^2(t)} \tag{41}$$

$$\text{RE} \equiv \frac{|\hat{y}(t/t) - y(t)|}{y_{\text{rms}}} \tag{42}$$

where y_{rms} is the root-mean-square value of the measurements $y(t)$ over a specific time interval and where all other

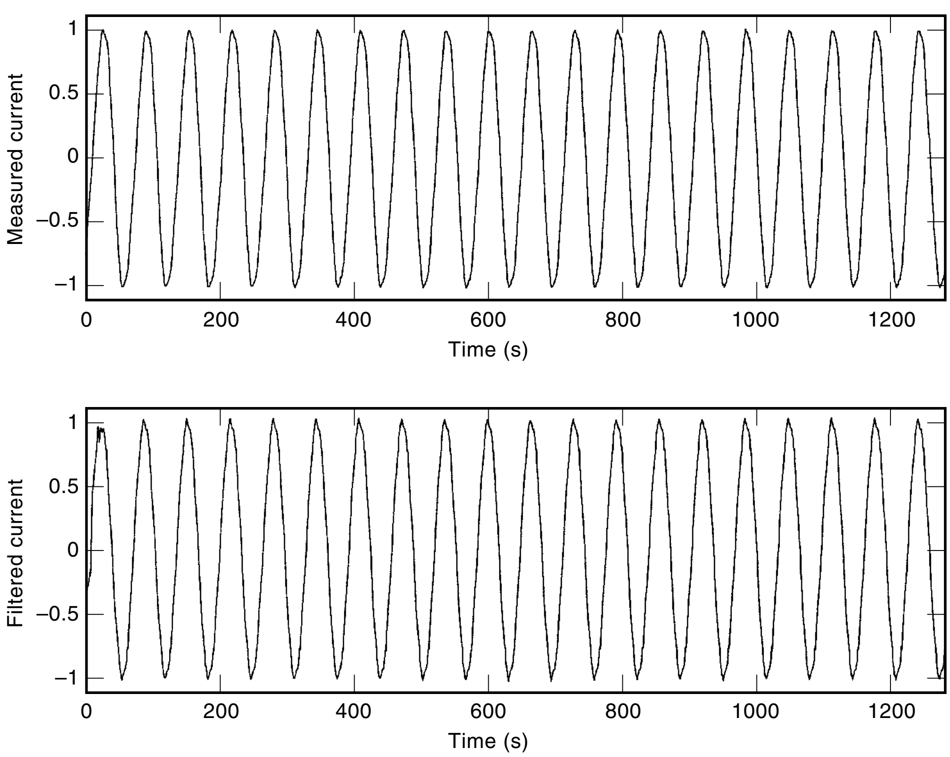

Figure 10. Normalized desired filter response and filter output.

variables are as previously defined. The two sums in the above error equations are carried over a sufficiently long interval to enable meaningful results. A first-order Wiener filter results in 0.5% NMSE. As the order of the filter increases, the NMSE decreases from approximately 0.1% (for a 5th order filter) to 0.06% (for a 10th order filter), 0.05% (for a 20th order filter), and 0.04% (for a 30th order filter). Further increase in the filter order does not produce any significant decrease in the NMSE.

A 10th order Wiener filter has been designed and the results are now presented. The normalized desired filter response and filter output are both shown in Fig. 10. The peak RE for the steady-state filter response shown in Fig. 10 is 11.2%, and NMSE for the interval shown in Fig. 10 is 0.06%. With the exception of the initial few cycles, during which the peak RE reaches 46%, the accuracy of the filter is acceptable considering that several of the key Wiener filter theory assumptions have been violated. Additionally, the good accuracy of these results is the direct consequence of the simplifying assumptions made in this application example. Relaxing some of these key assumptions, such as allowing for nonconstant motor load conditions, makes filter design a much more difficult task.

BIBLIOGRAPHY

1. F. C. Schweppe, *Uncertain Dynamic Systems,* Englewood Cliffs, NJ: Prentice-Hall, 1973.

1a. G. C. Goodwing and K. S. Sin, *Adaptive Filtering, Prediction and Control,* Englewood Cliffs, NJ: Prentice-Hall, 1984.

2. M. S. Grewal and A. P. Andrews, *Kalman Filtering,* Englewood Cliffs, NJ: Prentice-Hall, 1993.

3. N. Wiener, *Extrapolation, Interpolation, and Smoothing of Stationary Time Series,* Cambridge, MA: MIT Press, 1949. (The original work was published in a classified National Defense Research Report in February 1942.)

4. A. Gelb, *Applied Optimal Estimation,* Cambridge, MA: MIT Press, 1974.

5. H. W. Sorenson, Least-squares estimation: from Gauss to Kalman, *IEEE Spectrum,* **7**: 63–68, 1970.

6. N. Levinson, The Wiener RMS (root mean square) error criterion in filter design and prediction, *J. Math. Phys.,* **XXV** (4): 261–278, 1947.

7. A. N. Kolmogorov, Interpolation and extrapolation of stationary random sequences, translated by W. Doyle and J. Selin, Rept. RM-3090-PR, RAND Corp., Santa Monica, Calif., 1962. (Originally published in *Bulletin de l'academie des sciences USSR,* pp. 3–14, 1941.)

8. S. Haykin, *Adaptive Filter Theory,* 3rd ed., Upper Saddle River, NJ: Prentice-Hall, 1996.

9. T. Söderström and P. Stoica, *System Identification,* Englewood Cliffs, NJ: Prentice-Hall, 1989.

10. L. Ljung, *System Identification: Theory for the User,* Englewood Cliffs, NJ: Prentice-Hall, 1987.

11. A. Papoulis, *Probability, Random Variables, and Stochastic Processes,* New York: McGraw-Hill, 1991.

12. N. Wiener, *Nonlinear Problems in Random Theory,* Cambridge, MA: MIT Press, 1958.

13. I. J. Leontaritis and S. A. Billings, Model selection and validation methods for nonlinear systems, *Int. J. Control,* **45**: 311–341, 1987.

14. S. Haykin, *Neural Networks: A Comprehensive Foundation,* 2nd ed., Upper Saddle River, NJ: Prentice-Hall, 1999.

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WIMP (WINDOWS, ICONS, MENUS, AND POINTING DEVICES) INTERFACES. See GRAPHICAL USER INTERFACES.

WINDOWS, FIR FILTERS. See FIR FILTERS, WINDOWS.

WINDOWS, SPECTRAL ANALYSIS. See SPECTRAL ANALYSIS WINDOWING.