BESSEL FUNCTIONS

Friedrich Wilhelm Bessel led a fascinating and productive life (1–4). Bessel was born on July 22, 1784, in Minden, Westphalia, and died of cancer on March 17, 1846, in Königsberg, Prussia. His father was a civil servant, and his mother was the daughter of a minister. Bessel had two brothers and six sisters.

He began his education at the Gymnasium in Minden but left at the age of 14 after having difficulty learning Latin. His brothers went on to receive University degrees, finding careers as judges in high courts. Bessel became an apprentice in an import–export business. He independently studied textbooks, educating himself in Latin, Spanish, English, geography, navigation, astronomy, and mathematics.

In 1804 Bessel wrote a paper calculating the orbit of Halley's comet. His paper so impressed the comet expert Heinrich Olbers, that Olbers encouraged him to continue the work and become a professional astronomer. In 1806 Bessel obtained a position in the Lilienthal observatory near Bremen. In 1809 he was appointed the Director and Professor of Astronomy at the observatory in Königsberg. Commensurate with the position, he was awarded an honorary degree by Karl F. Gauss at the University of Göttingen.

In 1811 Bessel was awarded the Lalande Prize from the Institute of France for his refraction tables. In 1815 he was awarded a prize by the Berlin Academy of Sciences for his work in determining precession from proper star motions. Also, in 1825, he was elected as a Fellow of the Royal Society of London.

Perhaps his most famous accomplishment was solving a three-century dream of astronomers—the determination of the parallax of a star. However, of special interest to engineers, physicists, and mathematicians was the development of the Bessel function. His functions were derived in the study of the movement and perturbation of bodies under mutual gravitation. In 1824 his functions were used in a treatise on elliptic planetary motion.

The Bessel functions are frequently found in problems involving circular cylindrical boundaries. They arise in such fields as electromagnetics, elasticity, fluid flow, acoustics, and communications.

MATHEMATICAL FOUNDATION AND BACKGROUND OF BESSEL FUNCTIONS

Bessel's differential equation has roots in an elementary transformation of Riccati's equation (5). Three earlier mathematicians studied special cases of Bessel's equation (6). In 1732 Daniel Bernoulli studied the problem of a suspended heavy flexible chain. He obtained a differential equation that can be transformed into the same form as that used by Bessel. In 1764 Leonhard Euler studied the vibration of a stretched circular membrane and derived a differential equation essentially the same as Bessel's equation. In 1770 Joseph-Louis Lagrange derived an infinite series solution to the problem of the elliptic motion of a planet. His series coefficients are related to Bessel's later solution to the same problem. Various particular cases were solved by Bernoulli, Euler, and Lagrange, but it was Bessel who arrived at a systematic solution and the subsequent Bessel functions.

Bessel functions arise as a solution to the following differential equation

$$x\frac{d}{dx}\left(x\frac{dy}{dx}\right) + (x^2 - \nu^2)y = 0 \tag{1}$$

Equation (1), by applying the chain rule, may also be expressed as

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - \nu^2)y = 0 \tag{2}$$

Solutions to the preceding differential equation can be in one of three forms: Bessel functions of the first, second, or third kind and order ν. Bessel functions (7) of the first kind are denoted $J_{\pm\nu}(x)$. Bessel functions of the second kind are called Weber (Heinrich Weber) or Neumann (Carl Neumann) functions and are denoted $Y_\nu(x)$. They are sometimes also labeled $N_\nu(x)$. Bessel functions of the third kind are denoted $H_\nu^{(1)}(x)$, $H_\nu^{(2)}(x)$ and are called Hankel functions (Hermann Hankel). $H_\nu^{(1)}(x)$ is called the Hankel function of the first kind and order ν where $H_\nu^{(2)}(x)$ is the Hankel function of the second kind. Bessel functions of the second and third kind are linear functions of the Bessel function of the first kind. A variation of Eq. (2) will yield what is called the modified Bessel functions of the first [$I_\nu(x)$] and second [$K_\nu(x)$]kind. The modified Bessel functions will be discussed later.

The solution of Eq. (2) for noninteger orders can be expressed as a linear combination of Bessel functions of the first kind with positive and negative orders. It is given as

$$y(x) = AJ_\nu(x) + BJ_{-\nu}(x) \tag{3}$$

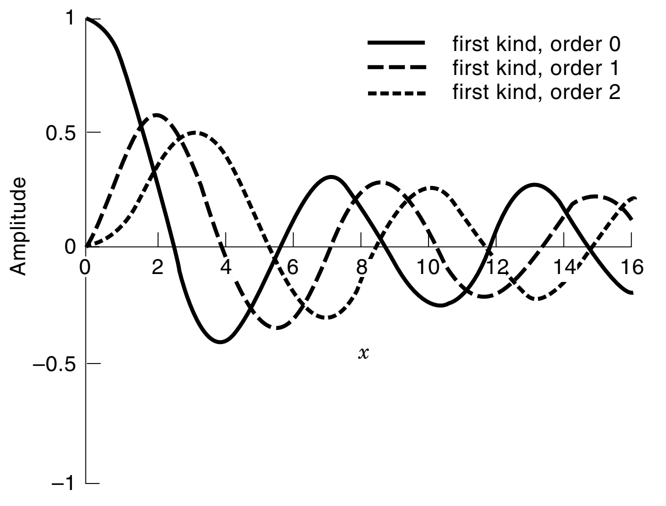

Figure 1. Plot of three typical Bessel functions of the first kind, orders 0, 1, and 2.

where A and B are arbitrary constants to be found by applying boundary conditions. Figure 1 shows a plot of Bessel functions of the first kind of orders 0, 1, and 2.

By allowing $A = \cot\text{an}(\nu\pi)$ and $B = -\csc(\nu\pi)$ and assuming ν is a noninteger, we arrive at the Weber function solution to Bessel's differential equation

$$Y_\nu(x) = \frac{J_\nu(x)\cos(\nu\pi) - J_{-\nu}(x)}{\sin(\nu\pi)} \tag{4}$$

Figure 2 shows a plot of the Weber function of orders 0, 1, and 2.

We can also find a linear combination of the Bessel functions of the first and second kind to derive the complex Hankel functions. The Hankel function of the first kind is given as

$$\begin{aligned} H_\nu^{(1)}(x) &= J_\nu(x) + jY_\nu(x) \\ &= j\csc(\nu\pi)[e^{-j\nu\pi}J_\nu(x) - J_{-\nu}(x)] \end{aligned} \tag{5}$$

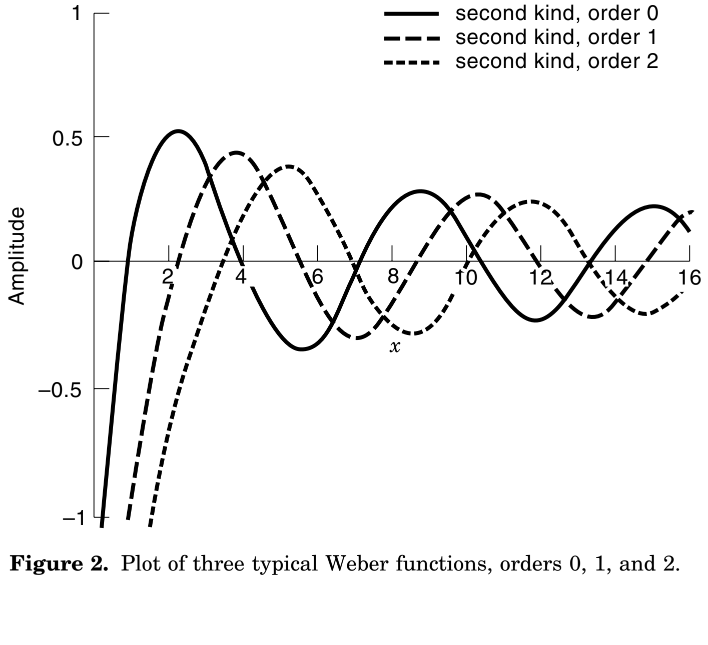

Figure 2. Plot of three typical Weber functions, orders 0, 1, and 2.

The Hankel function of the second kind is given as

$$\begin{aligned} H_\nu^{(2)}(x) &= J_\nu(x) + jY_\nu(x) \\ &= j\csc(\nu\pi)[J_{-\nu}(x) - e^{j\nu\pi}J_\nu(x)] \end{aligned} \tag{6}$$

It can be seen that $H_\nu^{(2)}(x)$ is the conjugate of $H_\nu^{(1)}(x)$.

Ascending Series Solution

Bessel's equation, Eq. (2), can be solved by using the method of Frobenius (Georg Frobenius). The method of Frobenius is the attempt to find nontrivial solutions to Eq. (2), which take the form of an infinite power series in x multiplied by x to some power ν. As a consequence, the Bessel functions of the first kind and order ν can then be expressed as

$$\begin{aligned} J_\nu(x) &= \sum_{m=0}^{\infty}(-1)^m \frac{1}{m!(m+\nu)!}\left(\frac{x}{2}\right)^{2m+\nu} \\ &= \sum_{m=0}^{\infty}(-1)^m \frac{1}{m!\Gamma(m+\nu+1)}\left(\frac{x}{2}\right)^{2m+\nu} \end{aligned} \tag{7}$$

where the Gamma function $\Gamma(m + \nu + 1)$ has been used to replace the factorial $(m + \nu)!$.

The series terms in Eq. (7) have alternating signs. Looking at the first few terms in the series we have

$$\begin{aligned} J_\nu(x) = \frac{1}{\nu!}\left(\frac{x}{2}\right)^\nu \Bigg[1 - \left(\frac{x}{2}\right)^2 \frac{1}{(1+\nu)} \\ + \left(\frac{x}{2}\right)^4 \frac{1}{2(2+\nu)(1+\nu)} - \cdots\Bigg] \end{aligned} \tag{8}$$

In the case of $\nu = 0$, we have the ascending series for the Bessel function of the first kind and order 0 given as

$$J_0(x) = 1 - \left(\frac{x}{2}\right)^2 + \frac{1}{(2!)^2}\left(\frac{x}{2}\right)^4 - \frac{1}{(3!)^2}\left(\frac{x}{2}\right)^6 + \cdots \tag{9}$$

To obtain the series solution for Bessel functions of negative order, merely substitute $-\nu$ for ν to get

$$J_{-\nu}(x) = \sum_{m=0}^{\infty}(-1)^m \frac{1}{m!\Gamma(m-\nu+1)}\left(\frac{x}{2}\right)^{2m-\nu} \tag{10}$$

If ν is an integer n, then the pair of Bessel functions for positive and negative orders is

$$J_n(x) = \sum_{m=0}^{\infty}(-1)^m \frac{1}{m!\Gamma(m+n+1)}\left(\frac{x}{2}\right)^{2m+n} \tag{11}$$

$$J_{-n}(x) = \sum_{m=0}^{\infty}(-1)^m \frac{1}{m!\Gamma(m-n+1)}\left(\frac{x}{2}\right)^{2m-n} \tag{12}$$

However, in the negative order series [Eq. (12)] the Gamma function $\Gamma(m - n + 1)$ is ∞ when $m < n$. In this case, all the terms in the series are zero for $m < n$. The series can then be rewritten as

$$J_{-n}(x) = \sum_{m=n}^{\infty}(-1)^m \frac{1}{m!\Gamma(m-n+1)}\left(\frac{x}{2}\right)^{2m-n} \tag{13}$$

By letting $m' = m - n$ the series can be reexpressed as

$$J_{-n}(x) = \sum_{m'=0}^{\infty} (-1)^{m'+n} \frac{1}{m'!\Gamma(m'+n+1)} \left(\frac{x}{2}\right)^{2m'+n} \quad (14)$$

In the integer order case by comparing Eqs. (11) and (14), it can be shown that

$$J_{-n}(x) = (-1)^n J_n(x) \quad (15)$$

The same procedure can be performed for the Weber function to show that

$$Y_{-n}(x) = (-1)^n Y_n(x) \quad (16)$$

Integral Solutions

The Bessel function solution can not only be defined in terms of the ascending power series above, but it also can be expressed in several integral forms. An extensive list of integral forms can be found in Refs. 7 and 8. When the order ν is not necessarily an integer, then the Bessel function can be expressed as Poisson's integral

$$J_\nu(x) = \left(\frac{x}{2}\right)^{\nu} \frac{1}{\sqrt{\pi}\Gamma(\nu+1/2)} \int_0^{\pi} \cos(z\cos\beta)\sin^{2\nu}(\beta)\,d\beta \quad (17)$$

Equation (17) is valid when letting $\cos\beta = z$, Eq. (17) can be written as

$$J_\nu(x) = 2\left(\frac{x}{2}\right)^{\nu} \frac{1}{\sqrt{\pi}\Gamma(\nu+1/2)} \int_0^{1} (1-z^2)^{\nu-1/2}\cos(zx)\,dz \quad (18)$$

In the case where ν is an integer, several more integral representations can used. For $\nu = 2m$ = even

$$J_{2m}(x) = \frac{2}{\pi}\int_0^{\pi} \cos(x\sin\beta)\cos(2m\beta)\,d\beta \qquad m > 0 \quad (19)$$

and for $\nu = 2m + 1$ = odd

$$J_{2m+1}(x) = \frac{2}{\pi}\int_0^{\pi} \sin(x\sin\beta)\sin[(2m+1)\beta]\,d\beta \qquad m > 0 \quad (20)$$

For an arbitrary integer n

$$J_n(x) = \frac{1}{\pi}\int_0^{\pi} \cos(x\sin\beta - n\beta)\,d\beta \quad (21)$$

Integrals Involving Bessel Functions

Many useful integrals involving Bessel functions may be found in Refs. 8 and 9. Several indefinite integrals follow. (A constant of integration may be necessary under certain conditions.)

$$\int J_\nu(x)\,dx = 2\sum_k J_{\nu+2k+1}(x) \quad (22)$$

$$\int x^{\nu+1} J_\nu(\alpha x)\,dx = \frac{1}{\alpha} x^{\nu+1} J_{\nu+1}(\alpha x) \quad (23)$$

$$\int x^{1-\nu} J_\nu(\alpha x)\,dx = -\frac{1}{\alpha} x^{1-\nu} J_{\nu-1}(\alpha x) \quad (24)$$

$$\int x^m J_n(x)\,dx = x^m J_{n+1}(x) - (m-n-1)\int x^{m-1} J_{n+1}(x)\,dx \quad (25)$$

$$\int x^m J_n(x)\,dx = -x^m J_{n-1}(x) + (m+n-1)\int x^{m-1} J_{n-1}(x)\,dx \quad (26)$$

Several definite integrals involving Bessel functions are given as

$$\int_0^{\infty} J_\nu(\alpha x)\,dx = \frac{1}{\alpha} \qquad [\mathrm{Re}\,\nu > -1,\ \alpha > 0] \quad (27)$$

$$\int_0^{a} J_0(x)\,dx = aJ_0(a) + \frac{\pi a}{2}[J_1(a)\mathbf{H}_0(a) - J_0(a)\mathbf{H}_1(a)] \qquad [a > 0] \quad (28)$$

$$\int_a^{\infty} J_0(x)\,dx = 1 - aJ_0(a) + \frac{\pi a}{2}[J_0(a)\mathbf{H}_1(a) - J_1(a)\mathbf{H}_0(a)][a > 0] \quad (29)$$

where $\mathbf{H}_0(a)$ and $\mathbf{H}_1(a)$ are the Struve functions defined by

$$\mathbf{H}_\nu(a) = \sum_{m=0}^{\infty} (-1)^m \frac{\left(\frac{a}{2}\right)^{2m+\nu+1}}{\Gamma(m+3/2)\Gamma(\nu+m+3/2)} \quad (30)$$

$$\int_0^{a} J_1(x)\,dx = 1 - J_0(a) \qquad [a > 0] \quad (31)$$

$$\int_a^{\infty} J_1(x)\,dx = J_0(a) \qquad [a > 0] \quad (32)$$

$$\left.\begin{aligned} \int_0^{\infty} J_\nu(\alpha x)J_{\nu-1}(\beta x)\,dx &= \frac{\beta^{\nu-1}}{\alpha^{\nu}} && [\beta < \alpha] \\ &= \frac{1}{2\beta} && [\beta = \alpha] \\ &= 0 && [\beta > \alpha] \end{aligned}\right\} \quad [\mathrm{Re}\,\nu > 0] \quad (33)$$

$$\int_0^{a} J_\nu(x)J_{\nu+1}(x)\,dx = \sum_{n=0}^{\infty} [J_{\nu+n+1}(a)]^2 \quad [\mathrm{Re}(\nu) > -1] \quad (34)$$

Recursion Relationships for $J_n(x)$ and $Y_n(x)$

By taking a derivative with respect to x (see Ref. 6 or 7) of Eq. (11), it can be shown that

$$xJ_n'(x) = nJ_n(x) + xJ_{n+1}(x) \quad (35)$$

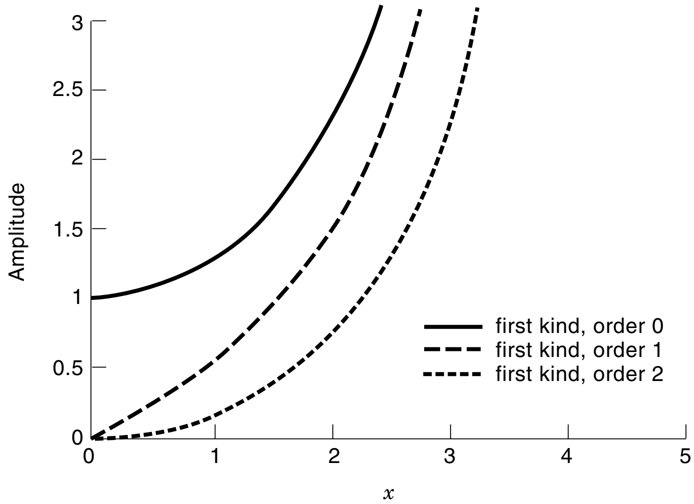

Figure 3. Modified Bessel functions of the first kind, orders 0, 1, and 2.

In a similar manner, we can also find that

$$xJ_n'(x) = nJ_n(x) - xJ_{n-1}(x) \tag{36}$$

By adding Eqs. (35) and (36) and normalizing by x, we can find that

$$J_n'(x) = \frac{1}{2}[J_{n-1}(x) - J_{n+1}(x)] \tag{37}$$

By subtracting Eq. (36) from Eq. (35), we get a recursion relationship for the Bessel function of the first kind and order $n + 1$ based upon orders n and $n - 1$.

$$J_{n+1}(x) = \frac{2n}{x} J_n(x) - J_{n-1}(x) \tag{38}$$

Modified Bessel Functions

A similar differential equation to that given in Eq. (2) can be derived by replacing x with the imaginary variable ix. We arrive at a variation on Bessel's differential equation given as

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - (x^2 + \nu^2)y = 0 \tag{39}$$

Equation (39) differs from Eq. (2) only in the sign of x^2 in parenthesis. The solution of Eq. (39) is defined as the modified Bessel function of the first kind and is given as

$$I_\nu(x) = \sum_{m=0}^{\infty} \frac{(x/2)^{2m+\nu}}{m!(m+\nu)!} = \sum_{m=0}^{\infty} \frac{(x/2)^{2m+\nu}}{m!\Gamma(m+\nu+1)} = i^{-\nu}J_\nu(ix) \tag{40}$$

The symbol $I_\nu(x)$ was chosen because $I_\nu(x)$ in Eq. (40) is related to the Bessel function $J_\nu(ix)$, which has an "imaginary" argument. One may note that the terms in the series of Eq. (40) are all positive, whereas the terms in Eq. (7) have alternating signs. The solution $I_{-\nu}(x)$ for $-\nu$ is linearly independent of $I_\nu(x)$ except when ν is an integer. When ν is equal to the integer n then $I_{-n}(x) = I_n(x)$. Figure 3 shows a graph of the modified Bessel functions of the first kind and orders 0, 1, and 2. We can define a second valid solution to Eq. (39) as a linear combination of the modified Bessel function of the first kind. The second solution is referred to as the modified Bessel function of the second kind $K_\nu(x)$ and is given as

$$K_\nu(x) = \frac{\pi}{2}\frac{I_{-\nu}(x) - I_\nu(x)}{\sin(\nu\pi)} \tag{41}$$

In the case where ν equals an integer n, the modified Bessel function $K_n(x)$ is found by

$$K_n(x) = \lim_{\nu\to n} K_\nu(x) \tag{42}$$

Figure 4 shows a graph of the modified Bessel functions of the second kind and orders 0, 1, and 2.

Integral Form of the Modified Bessel Function of the First Kind

In addition to the ascending series expression of Eq. (40), the modified Bessel function may be expressed in integral form. Several integral forms follow and are valid for $\mathrm{Re}(\nu) > 1/2$.

$$\begin{aligned} I_\nu(x) &= \frac{\left(\frac{x}{2}\right)^\nu}{\Gamma(\nu+1/2)\Gamma(1/2)} \int_0^\pi \cosh(x\cos\beta)\sin^{2\nu}(\beta)\,d\beta \\ &= \frac{\left(\frac{x}{2}\right)^\nu}{\Gamma(\nu+1/2)\Gamma(1/2)} \int_0^\pi e^{\pm x\cos\beta}\sin^{2\nu}(\beta)\,d\beta \\ &= \frac{\left(\frac{x}{2}\right)^\nu}{\Gamma(\nu+1/2)\Gamma(1/2)} \int_{-1}^1 (1-y^2)^{\nu-1/2}\cosh(xy)\,dy \end{aligned} \tag{43}$$

Approximations to Bessel Functions

The small argument approximation for the Bessel function is (6)

$$J_n(x) \approx \frac{1}{\Gamma(n+1)}\left(\frac{x}{2}\right)^n = \frac{1}{n!}\left(\frac{x}{2}\right)^n; \quad x \to 0 \tag{44}$$

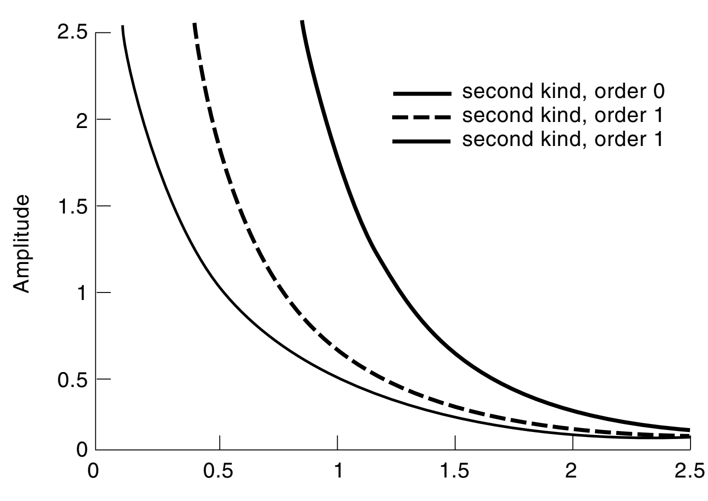

Figure 4. Modified Bessel functions of the second kind, orders 0, 1, and 2.

In the case of the $J_0(x)$ and the $J_1(x)$ Bessel functions,

$$J_0(x) = 1; \qquad J_1(x) = \frac{x}{2} \tag{45}$$

The large argument approximation is given as

$$J_\alpha(x) \approx \left(\frac{2}{\pi x}\right)^{1/2} \cos(x - \alpha\pi/2 - \pi/4); \quad x \to \infty \tag{46}$$

In the case of the $J_0(x)$ and the $J_1(x)$ Bessel functions we have

$$\begin{aligned} J_0(x) &= \left(\frac{2}{\pi x}\right)^{1/2} \cos(x - \pi/4); \\ J_1(x) &= \left(\frac{2}{\pi x}\right)^{1/2} \cos(x - 3\pi/4) \end{aligned} \tag{47}$$

The small argument approximation is only reasonably accurate, for orders 0 and 1, when $x < 0.5$. The large argument approximation, for orders 0 and 1, is only reasonably accurate for $x > 2.5$.

A 12th-order polynomial approximation is available in Abramowitz and Stegun (7), which is valid for $|x| \leq 3$. In the case of the 0th-order Bessel function, the approximation is given as

$$\begin{aligned} J_0(x) = 1 &- 2.2499997\left(\frac{x}{3}\right)^2 + 1.2656208\left(\frac{x}{3}\right)^4 \\ &- 0.3163866\left(\frac{x}{3}\right)^6 + 0.0444479\left(\frac{x}{3}\right)^8 \\ &- 0.0039444\left(\frac{x}{3}\right)^{10} + 0.0002100\left(\frac{x}{3}\right)^{12} + \epsilon \\ & |\epsilon| < 5 \times 10^{-8} \end{aligned} \tag{48}$$

In the case of the first-order Bessel function the approximation is given as

$$\begin{aligned} \frac{1}{x} J_1(x) = 0.5 &- 0.56249985\left(\frac{x}{3}\right)^2 + 0.21093573\left(\frac{x}{3}\right)^4 \\ &- 0.03954289\left(\frac{x}{3}\right)^6 + 0.00443319\left(\frac{x}{3}\right)^8 \\ &- 0.00031761\left(\frac{x}{3}\right)^{10} + 0.00001109\left(\frac{x}{3}\right)^{12} + \epsilon \\ & |\epsilon| < 1.3 \times 10^{-8} \end{aligned} \tag{49}$$

The polynomial requires an eight decimal place accuracy in the coefficients. Several other polynomial and rational approximations can be found in Luke (10–12).

A new approximating function can be developed that is simpler than Eqs. (48) and (49) and useful over the range $|x| \leq 5$. This function is adequate to replace the small argument approximation and bridges the gap to the large argument approximation of Eq. (46).

DERIVATION OF A NEW BESSEL FUNCTION APPROXIMATION

In studying the general problem of TM scattering from conducting strip gratings by using conformal mapping methods an integral was discovered with no previously known solution (13). The new integral is

$$f_{2n}(k) = \frac{2}{\pi} \int_0^\delta \frac{\cos(x)}{\sqrt{k^2 - \sin^2(x)}} \cos(2nx)\, dx \tag{50}$$

where

$$k = \sin\delta; \quad 0 \leq \delta \leq \pi/2$$

[Note the similarities between Eqs. (50) and (17).] Equation (50) can be reduced to a form identical to Eq. (17) by allowing δ to be vanishingly small. This application will be made after the function $f_{2n}(k)$ has been evaluated.

Several steps are undertaken in finding the solution for Eq. (50). By letting $\sin x = k \sin \alpha$, Eq. (50) can be reduced to the form

$$f_{2n}(k) = \frac{2}{\pi} \int_0^{\pi/2} \cos\{2n \arcsin[k \sin(\alpha)]\}\, d\alpha \tag{51}$$

Since $\arcsin(k \sin \alpha) = \pi/2 - \arccos(k \sin \alpha)$, $\cos(n\pi + \phi) = (-1)^n \cos(\phi)$, and using the definition of a Chebyshev polynomial (Eq. 22.3.15 in Ref. 7) we get

$$f_{2n}(k) = \frac{2}{\pi} \int_0^{\pi/2} (-1)^n T_{2n}(k \sin\alpha)\, d\alpha \tag{52}$$

with

$$T_{2n}(k\sin\alpha) = \text{Chebyshev polynomial of order } 2n$$

The Chebyshev polynomial of order $2n$ can be expressed as a finite sum (8) and is alternatively defined as

$$T_{2n}(x) = n \sum_{m=0}^{n} (-1)^m \frac{(2n-m-1)!}{m!(2n-2m)!} (2x)^{2n-2m} \tag{53}$$

by substituting Eq. (53) into Eq. (52), we get

$$\begin{aligned} f_{2n}(k) = \frac{2}{\pi}(-1)^n \Bigg[& n \sum_{m=0}^{n} (-1)^m \frac{(2n-m-1)!}{m!(2n-2m)!} (2k)^{2n-2m} \\ & \times \int_0^{\pi/2} (\sin\alpha)^{2n-2m}\, d\alpha \Bigg] \end{aligned} \tag{54}$$

However, the integral imbedded in Eq. (54) is $\pi/2$ when $2n = 2m$ and in general is given by

$$\int_0^{\pi/2} (\sin\alpha)^{2n-2m}\, d\alpha = \frac{\pi}{2} \frac{(2n-2m-1)!!}{(2n-2m)!!}, \qquad 2n > 2m \tag{55}$$

where

$$(2n-2m-1)!! = 1 \cdot 3 \cdot 5 \cdots (2n-2m-1)$$

and

$$(2n-2m)!! = 2 \cdot 4 \cdot 6 \cdots (2n-2m)$$

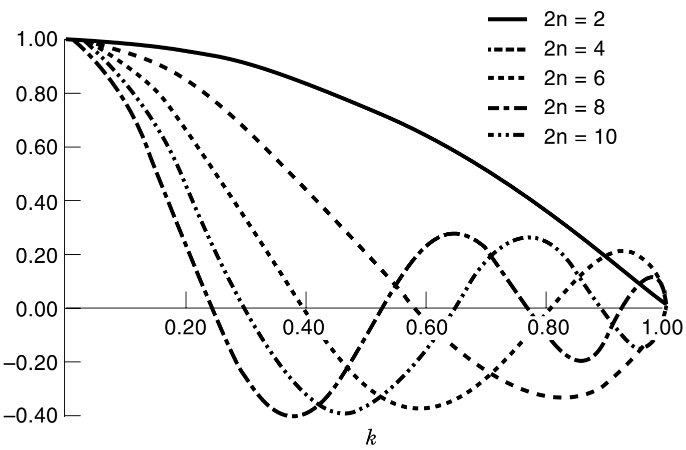

Figure 5. Plot of Bessel approximating function $f_{2n}(k)$.

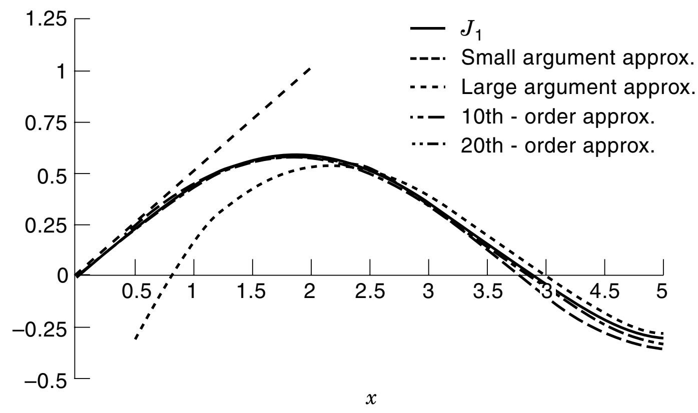

Figure 7. Comparison among $J_1(x)$, classic approximations, and the new approximations.

Upon substitution of Eq. (55) into Eq. (54) the final solution is given as

$$f_{2n}(k) = \sum_{m=0}^{n} b_m k^{2n-2m} \tag{56}$$

where the coefficients to the series are given as

$$b_m = n(-1)^{m+n} \frac{(2n-m-1)!2^{2n-2m}}{m!((2n-2m)!!)^2}$$

The solution in Eq. (56) is a closed form expression yielding an even-order polynomial of degree $2n$. The solutions $f_{2n}(k)$ for $2n = 0, \ldots, 10$ follow and are plotted in Fig. 5.

$$\begin{aligned}
f_0(k) &= 1\\
f_2(k) &= 1 - k^2\\
f_4(k) &= 1 - 4k^2 + 3k^4\\
f_6(k) &= 1 - 9k^2 + 18k^4 - 10k^6\\
f_8(k) &= 1 - 16k^2 + 60k^4 - 80k^6 + 35k^8\\
f_{10}(k) &= 1 - 25k^2 + 150k^4 - 350k^6 + 350k^8 - 126k^{10}
\end{aligned} \tag{57}$$

The function $f_{2n}(k)$ is only defined over the range $0 \leq k \leq 1$, and the coefficients are integers that always sum to 0 (except when $n = 0$). The new approximation has n maxima and minima over its domain.

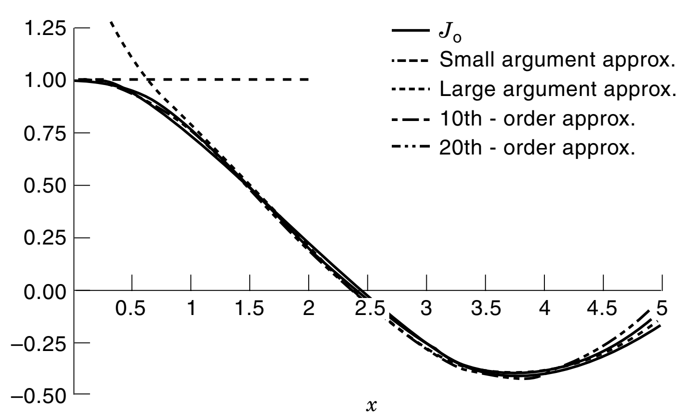

Figure 6. Comparison among $J_0(x)$, classic approximations, and the new approximations.

Bessel Function Approximation

By allowing δ to approach 0 in Eq. (50) and by manipulating the variables, it can easily be shown that

$$J_0(x) \approx f_{2n}\left(\frac{x}{2n}\right); \qquad n \geq 1 \tag{58}$$

Using the identity $J_1(x) = -J_0'(x)$ given in Eq. (36), we have

$$J_1(x) \approx -\frac{d}{dx} f_{2n}\left(\frac{x}{2n}\right) \tag{59}$$

The approximations in Eqs. (58) and (59) are appropriate for any x as long as $x/2n \leq 1$. The accuracy increases as $x/2n$ approaches 0. Therefore for small values of x, small-order polynomials are sufficient to approximate $J_0(x)$ and $J_1(x)$. Figure 6 compares the exact solution for $J_0(x)$, classic asymptotic solutions, and the polynomial approximation of Eq. (58) when $2n = 10$ and 20. Figure 7 compares the exact solution for $J_1(x)$, classic asymptotic solutions, and the polynomial approximation of Eq. (59) when $2n = 10$ and 20. It can be seen that the higher-order approximation is understandably better. However, the tenth-order polynomial is quite accurate for $x < 3.5$ in the $J_0(x)$ case and is reasonably accurate for $x < 3$ in the $J_1(x)$ case. If smaller arguments are anticipated, then lower-order polynomials seen in Eq. (57) are sufficient.

The polynomial approximations of Eqs. (58) and (59) are much simpler to express than the polynomials of Eqs. (48) and (49). They are also accurate over a greater range of x for the tenth- and higher-order polynomials.

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BETA TUNGSTEN SUPERCONDUCTORS, METALLURGY.

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