CALCULUS

Calculus has its foundation in taking a limit. For example, one can obtain the area of a circle as the limit of the areas of regular inscribed polygons as the number of sides increases without bound. This example can be extended to determining the perimeter of a circle or the volume of a sphere. Similarly in algebra, this limiting approach is used to seek the value of a repeating decimal. In plane analytic geometry, this concept is used to explain tangents to curves.

The two fundamental operations in calculus are differentiation and integration. Both of these fundamental tools have played an important role in the development of many scientific theories. The *Fundamental Theorem of Calculus* provides the connection between differentiation and integration and was discovered independently by Sir Isaac Newton and Baron Gottfried Wilhelm Leibniz.

In the remainder of this article, a brief history of development of calculus is presented. This introduction is followed by a discussion of the principle of differentiation. A similar discussion on integrals is presented next. Other relevant topics important to electrical engineers are also presented. Each section is augmented with examples using classic problems in engineering to illustrate the practical use of calculus.

HISTORY

The methods used by the Greeks for determining the area of a circle and a segment of a parabola, as well as the volumes of the cylinder, cone, and sphere, were in principle akin to the method of integration. During the first half of the 17th century, methods of more or less limited scope began to appear among mathematicians for constructing tangents, determining maxima and minima, and finding areas and volumes. In particular, Fermat, Pascal, Roberval, Descartes, and Huygens discussed methods of drawing tangents to particular curves and finding areas bounded by certain special curves. Each problem was considered by itself, and few general rules were developed. The essential ideas of the derivative and definite integral were, however, beginning to be formulated. With this mathematical heritage, Newton and Leibniz, working independently of each other during the latter half of the 17th century, defined the concepts of derivatives and integrals. Leibniz used the notation dy/dx for the derivative and introduced the integration symbol $\int$. The portion of mathematics that includes only topics that depend on calculus is called *analysis.* Included in this category are differential and integral equations, theory of functions of real and complex variables, and algebraic and elliptic functions.

Calculus has helped the development of other fields of science and engineering. Geometry and number theory make use of this powerful tool. In the development of modern physics and engineering, the concepts developed in calculus and its extensions are continually utilized. For example, in dealing with electricity, the current, I, through a circuit due to the flow of charge, Q, is expressed as $I \equiv dQ/dt$; the voltage, v, across an inductor, L, is defined as $v \equiv L\, dI/dt$; and the voltage through a capacitor, C, is defined as $v \equiv (1/C) \int I\, dt$.

NOTATION AND DEFINITIONS

Within this article, the parameters u, v, and w represent functions of independent variable x, while other alphabetic letters represent fixed real numbers. A variable in boldface type denotes a vector quantity.

Limits

Of fundamental importance to the field of calculus is the concept of the *limit*, which represents the value of an entity under a given extreme condition. For instance, a limit can be used to define the natural exponential function, e:

$$\lim_{h\to 0}(1+h)^{1/h} = e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n \tag{1}$$

Given here are rules for computing limits. The limit of a constant is the constant:

$$\lim_{x\to a} c = c \tag{2}$$

The limit of a function scaled by a constant is the constant times the limit of the function:

$$\lim_{x\to a} c\,f(x) = c\lim_{x\to a} f(x) \tag{3}$$

The limit of a sum (or difference) is the sum (or difference) of the limits:

$$\lim_{x\to a}[f(x)\pm g(x)] = \lim_{x\to a} f(x) \pm \lim_{x\to a} g(x) \tag{4}$$

The limit of a product is the product of the limits:

$$\lim_{x\to a} f(x)g(x) = [\lim_{x\to a} f(x)][\lim_{x\to a} g(x)] \tag{5}$$

The limit of a quotient is the quotient of the limits, if the denominator does not equal zero:

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} \tag{6}$$

The limit of a function raised to a positive integer power, n, is

$$\lim_{x\to a}[f(x)]^n = [\lim_{x\to a} f(x)]^n \tag{7}$$

The limit of a polynomial function $f(x) = b_n x^n + b_{n-1}x^{n-1} + \cdots + b_1 x + b_0$ is

$$\lim_{x\to a} f(x) = b_n a^n + b_{n-1}a^{n-1} + \cdots + b_1 a + b_0 \tag{8}$$

The limits of a function are sometimes broken into left-hand and right-hand limits. A function $f(t)$ has a limit at a if and only if the right-hand and left-hand limits at a exist and are equal.

L'Hôpital's Rule. If $f(x)/g(x)$ has the indeterminate form 0/0 or ∞/∞ at $x = a$, then

$$\lim_{x\to a}\frac{f(x)}{g(x)} = \lim_{x\to a}\frac{f'(x)}{g'(x)} \tag{9}$$

provided that the limit exists or becomes infinite.

Limits Example. A common application using limits is the initial and final value theorems. Consider a time function, $f(t) = 5\,e^{-2t}$, whose transformation to the Laplacian domain is

$$F(s) = \frac{5}{s+2}$$

The *final value* may be obtained from

$$\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s) = \lim_{s\to 0}\frac{5s}{s+2} = \frac{0}{2} = 0$$

Likewise, the *initial value* is determined from

$$\lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s) = \lim_{s\to\infty}\frac{5s}{s+2} = \frac{\infty}{\infty}$$

Hence, L'Hôpital's rule must be used to find its initial value:

$$\lim_{t\to 0} f(t) = \lim_{s\to\infty}\frac{\frac{d}{ds}(5s)}{\frac{d}{ds}(s+2)} = \lim_{s\to\infty}\frac{5}{1} = 5$$

Continuity

A function $y = f(x)$ is continuous at $x = a$ if and only if all three of the following conditions are satisfied:

1. $f(a)$ exists where a is in the domain of $f(x)$;
2. $\lim_{x\to a} f(x)$ exists; and
3. $\lim_{x\to a} f(x) = f(a)$.

If any of these three conditions fails to hold, then $f(x)$ is *discontinuous* at a. If $f(x)$ is continuous at every point of its domain, $f(x)$ is said to be a *continuous function.* The sine and cosine are examples of continuous functions.

DIFFERENTIAL CALCULUS

Derivative

If y is a single-valued function of x, $y = f(x)$, the *derivative* of y with respect to x is defined to be

$$\frac{dy}{dx} = \lim_{\Delta x\to 0}\left[\frac{f(x+\Delta x) - f(x)}{\Delta x}\right] \qquad (10)$$

This quantity is often written as $dy/dx = \lim_{\Delta x\to 0}(\Delta y/\Delta x)$, where Δx is an arbitrary increment of x and $\Delta y = f(x+\Delta x) - f(x)$. The derivative of a function, $y = f(x)$, may be represented in several different ways:

$$\frac{dy}{dx} = D_x y = y' = f'(x) \qquad (11)$$

Likewise, a second derivative can be denoted by

$$\frac{d^2y}{dx^2} = D_x^2 y = y'' = f''(x) \qquad (12)$$

The symbol D_x is referred to as the *differential operator.* Inverse functions are denoted as $f^{-1}(x)$. Therefore,

$$f^{-1}(y) = f^{-1}[f(x)] = x \qquad (13)$$

This operation should not be confused with the reciprocal of a function; that is,

$$[f(x)]^{-1} = 1/f(x) \qquad (14)$$

It is important to note that dy/dx is not a quotient. It is a number that is approached by the quotient $\Delta y/\Delta x$ in the limit. The symbols dy and dx, as they appear in dy/dx, have no meaning by themselves. The term dy/dx represents the limit of $\Delta y/\Delta x$.

The *differential* of y for a given value of x is defined as

$$dy = d[f(x)] = \frac{dy}{dx}dx = \frac{d[f(x)]}{dx}dx = f'(x)\,dx \qquad (15)$$

Each derivative expression has a differential formula associated with it. For example, the *chain rule*

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx} \qquad (16)$$

has an equivalent differential formula:

$$d(uv) = u\,dv + v\,du \qquad (17)$$

Application. A basic application of the first derivative is the calculation of speed $v(t)$ and acceleration $a(t)$ from a position function $s(t)$:

$$\begin{aligned} v(t) &= \frac{d}{dt}s(t) = s'(t) \\ a(t) &= \frac{d}{dt}v(t) = v'(t) \\ &= \frac{d}{dt}\left(\frac{d}{dt}s(t)\right) = \frac{d^2}{dt^2}s(t) = s''(t) \end{aligned}$$

This latter expression illustrates the concept of higher-order derivatives—in this case, the second derivative. The derivative is important in many applications—for example, determining the tangents to curves and finding the maxima and minima of a given function.

Tangents

The concept of derivative is best illustrated by considering the construction of a tangent to a curve. Consider a parabola that is represented by the equation $y = x^2$ as shown in Fig. 1. Let Q be any point on the parabola, distinct from another point P. The line that joins Q and P is a secant to the parabola. As Q approaches P, the secant rotates about P. In the limit, as Q is infinitesimally near P, without attaining it, the secant approaches the line that touches the parabola at P without cutting across it. This line is *tangent* to the parabola at point P.

The angle between the secant and the x-axis is the *inclination angle, θ*. The *slope* of a line is defined as the trigonometric tangent of the inclination of the line. To determine the slope of the tangent, it must be noted that in the limit, as Q approaches P, the inclination angle of the secant approaches that of the slope at P. If lines PM and QM are perpendicular to each other, then the slope of PQ is QM/PM.

Let P have coordinates (x, y). As noted earlier, Q is any point on the parabola with coordinates $(x+\Delta x, y+\Delta y)$,

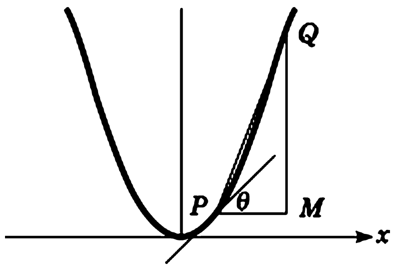

Figure 1. The tangent to the curve at P is the secant PQ in the limit as Q approaches P. The slope of the tangent at this point is the trigonometric tangent of θ as defined by the ratio of QM and PM.

where Δx and Δy equal PM and QM, respectively. Therefore, the slope of PQ is $\Delta y/\Delta x$. The slope of the tangent at P is then the value of this ratio as Q approaches P—that is, as Δx and Δy approach zero. Using the equation of the parabola, $y = x^2$ we obtain:

$$\begin{aligned} y + \Delta y &= (x + \Delta x)^2 \\ &= x^2 + 2x\,\Delta x + (\Delta x)^2. \end{aligned}$$

By definition, $y = x^2$, therefore:

$$\Delta y = 2x\,\Delta x + (\Delta x)^2, \quad \text{or} \quad \Delta y/\Delta x = 2x + \Delta x.$$

For the parabola, as Δx approaches zero, $\Delta y/\Delta x$, the slope of the tangent at P approaches $2x$.

To generalize, consider any function of x, say $y = f(x)$. For the points P and Q on y, the limit of $\Delta y/\Delta x$, as Q approaches P is the *derivative* of $f(x)$, evaluated at point P. The expression dy/dx represents the derivative of the function $y = f(x)$ for any value of x. As was demonstrated in the previous paragraphs, when $y = x^2$ we have $dy/dx = 2x$.

Since each value of x corresponds to a definite value of dy/dx, the derivative of y is also a function of x. The process of finding the derivative of a function is called *differentiation,* as was demonstrated for $y = x^2$. It is important to point out that there are classes of functions for which derivatives do not exist. For example, in the limit as Δx approaches zero, the function's value may either become infinite or oscillate without reaching a limit. In particular, the function $f(x) = |x|$ is not differentiable at $x = 0$ since the right-hand limit (which is 1) does not equal the left-hand limit (which is -1).

Partial Derivatives

Extension of differentiation to multivariable functions is the important field of *partial differential equations.* Applications of this type involve surfaces and finding the maxima and minima of these functions. Selected operations specific to partial differential equations are listed below. The reader is referred to calculus texts for a more extensive discussion on this topic.

Consider a function of the form $z = f(x, y)$. The first partial derivatives of f with respect to x and y, where y and x are held constant, respectively, can be defined as

$$\begin{aligned} \frac{\partial f}{\partial x} &= \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x} \\ \frac{\partial f}{\partial y} &= \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y} \end{aligned} \tag{18}$$

The function has three different second partial derivatives:

$$\begin{aligned} \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) &= \frac{\partial^2 f}{\partial x^2} \\ \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) &= \frac{\partial^2 f}{\partial y^2} \\ \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) &= \frac{\partial^2 f}{\partial x\,\partial y} \end{aligned} \tag{19}$$

If the function and its partial derivatives are continuous, then the order of differentiation is immaterial for the mixed derivatives and they satisfy the following relationship:

$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) \tag{20}$$

Mean Value Theorem

The *Mean Value Theorem* states that if $f(x)$ is defined and continuous on the closed interval $[a,b]$ and differentiable on the open interval (a,b), then there is at least one number c in (a,b) (that is, $a < c < b$) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a} \tag{21}$$

For a continuous function $f(x,y,z)$ with continuous partial derivatives, the mean value theorem is

$$\begin{aligned} &f(x_0 + h, y_0 + k, z_0 + \ell) - f(x_0, y_0, z_0) \\ &= h\frac{\partial f}{\partial x} + k\frac{\partial f}{\partial y} + \ell\frac{\partial f}{\partial z} \end{aligned} \tag{22}$$

Maxima and Minima

Consider a function $y = f(x)$ that has a derivative for every x in a given range. At a point where y reaches a maximum or a minimum, the slope of the tangent to the function is zero. Because the first derivative of a function represents the slope of the function at any point, the second derivative represents the rate of change of the slope. Hence, a positive second derivative indicates an increasing slope, whereas a negative second derivative denotes a decreasing slope. The *concavity* of a function is determined using the second derivative of the function:

If $f''(x) > 0$, then the function is concave upward.
If $f''(x) < 0$, the function is concave downward (convex).

A *point of inflection* denotes the location where curvature of the function changes from convex to concave, and the second derivative of the function is zero. Maxima, minima, and points of inflection are also known as *critical points* of a function. The derivative tests for critical points are listed in Table 1.

For all continuous functions, a maximum or minimum is located where the first derivative equals zero, and a point

Table 1. Conditions for Existence of Critical Points of a Function

First Derivative	Second Derivative	Critical Point
Zero	Negative	Maximum (local/global)
Zero	Positive	Minimum (local/global)
Any value	Zero	Probably an inflection point

of inflection is located where the second derivative equals zero. The converse of these statements is not true. For example, a straight horizontal line has a zero slope at all points but this does not indicate a critical point. Also, any linear function ($y = mx + b$) has a zero-valued second derivative, but this does not indicate points of inflection. Table 2 shows the necessary and sufficient conditions for existence of the maximum and minimum points of the function $z = f(x, y)$ using partial derivatives.

Critical Points Example. Consider the use of calculus to find the critical points of an alternating-current (*ac*) voltage source. Without specific knowledge of the cosine function, the critical points are found where the first derivative is zero; the second derivative is then used to classify the nature of these points. The voltage and its first and second derivatives are

$$v(t) = V_M \cos(\omega t + \theta)$$
$$v'(t) = \frac{d}{dt}v(t) = -V_M\omega \sin(\omega t + \theta)$$
$$v''(t) = \frac{d^2}{dt^2}v(t) = -V_M\omega^2 \cos(\omega t + \theta)$$

The critical points—that is, where $v'(t) = 0$—are located at $t = (n\pi - \theta)/\omega$, where n is an integer. Substitution of these values of t into the second derivative finds two results:

$$v''[t = (n\pi - \theta)/\omega] = -V_M\omega^2 \cos(n\pi) = \begin{cases} V_M\omega^2 & n \text{ odd} \\ -V_M\omega^2 & n \text{ even} \end{cases}$$

Hence, maxima exist at even values of n and minima at odd n values.

The points of inflection occur where the second derivative is zero, $v''(t) = 0$, specifically here for $t = [(2n + 1)\pi/2 - \theta]/\omega$. These points of inflection identify concavity changes. Regions of specific concavity behavior can be ascertained using $v''(t)$, namely,

$$v''\left(\frac{(2n+1)\pi/2 - \theta}{\omega} < t < \frac{(2n+3)\pi/2 - \theta}{\omega}\right) \begin{cases} < 0 & n \text{ odd} \Rightarrow \text{concave down} \\ > 0 & n \text{ even} \Rightarrow \text{concave up} \end{cases}$$

These results are shown in Fig. 2.

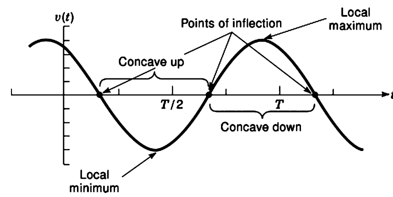

Figure 2. Critical points for an ac voltage source, $v(t) = V_M \cos(\omega t + \theta)$. The maxima and minima are located where $v'(t) = 0$. The points of inflection occur at $v''(t) = 0$ where the concavity of the curve changes.

Differentiation Rules

The following formulas represent the fundamental rules of differentiation. The derivatives of elaborate functions can be systematically evaluated using these rules. All arguments in trigonometric functions are measured in radians, and all inverse trigonometric and hyperbolic functions represent principal values.

Constants. The derivative of a constant is zero:

$$\frac{d}{dx}(a) = 0 \tag{23}$$

Scaling. If u is multiplied by a constant b, so is its derivative:

$$\frac{d}{dx}(bu) = b\frac{du}{dx} \tag{24}$$

Linearity. The derivative of the sum or difference of two or more functions is the sum or difference of the derivatives of the functions:

$$\frac{d}{dx}(u \pm v \pm \cdots) = \frac{du}{dx} \pm \frac{dv}{dx} \pm \cdots \tag{25}$$

Product Rule. The derivative of the product of two functions is

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx} \tag{26}$$

For three functions the product rule is

$$\frac{d}{dx}(uvw) = uv\frac{dw}{dx} + uw\frac{dv}{dx} + vw\frac{du}{dx} \tag{27}$$

which can be generalized to the product of more functions.

Quotient Rule. The derivative of the ratio of two functions can be expressed as

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{1}{v^2}\left(v\frac{du}{dx} - u\frac{dv}{dx}\right) \tag{28}$$

Chain Rule. Let y be a function of u, which in turn depends on x; then

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} \tag{29}$$

Table 2. The Necessary and Sufficient Conditions for Existence of the Maximum and Minimum Points of the Function $z = f(x, y)$ Based on Its Partial Derivatives

Partial Derivatives	Conditions for Maximum	Conditions for Minimum
$\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$	Both zero	Both zero
$\frac{\partial^2 z}{\partial x^2}\frac{\partial^2 z}{\partial y^2} - \left(\frac{\partial^2 z}{\partial x\, \partial y}\right)^2$	Positive	Positive
$\frac{\partial^2 z}{\partial x^2}$ and $\frac{\partial^2 z}{\partial y^2}$	Both negative	Both positive

Given $w = f(u, v)$, $u = g(x, y)$, and $v = h(x, y)$, the chain rule for partial derivatives may be applied as

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial x}$$
$$\frac{\partial w}{\partial y} = \frac{\partial w}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial w}{\partial v}\frac{\partial v}{\partial y} \quad (30)$$

Derivative of Integrals. Given t as an independent variable, we obtain

$$\frac{d}{dt}\int_a^t y(x)\,dx = y(t) \quad (31)$$

Power Rule.

$$\frac{d}{du}u^n = nu^{n-1}\frac{du}{dx} \quad (32)$$

The derivatives of a few selected functions appear in Table 3.

Differentiation Example. A classic network problem requiring differential calculus is the determination of an analytical expression for the load resistance that results in the maximum power transfer in a direct-current (dc) circuit. Consider a reduced circuit consisting of a voltage source, v, in series with a Thévenin equivalent resistance, R_{Th}, and the load resistance, R_{L}. The power delivered to the load is

$$P = i^2 R_{\mathrm{L}} = \left(\frac{v}{R_{\mathrm{Th}} + R_{\mathrm{L}}}\right)^2 R_{\mathrm{L}}$$

A maximum/minimum for P will occur where its derivative with respect to R_{L} is zero; that is, $dP/dR_L = 0$. To determine the derivative, the quotient (or product), power, and chain rules along with the scaling property are utilized:

$$\begin{aligned}
\frac{dP}{dR_L} &= \frac{d}{dR_L}\left[v^2\frac{R_L}{(R_{Th}+R_L)^2}\right]\\
&= \frac{v^2}{(R_{Th}+R_L)^4}\left[(R_{Th}+R_L)^2\frac{dR_L}{dR_L} - R_L\frac{d(R_{Th}+R_L)^2}{dR_L}\right]\\
&= \frac{v^2}{(R_{Th}+R_L)^4}\left[(R_{Th}+R_L)^2(1) - R_L\,2(R_{Th}+R_L)\frac{d(R_{Th}+R_L)}{dR_L}\right]\\
&= \frac{v^2}{(R_{Th}+R_L)^3}[(R_{Th}+R_L) - R_L\,2] = \frac{v^2(R_{Th}-R_L)}{(R_{Th}+R_L)^3}\,.
\end{aligned}$$

Setting this last expression equal to zero yields the classic solution of $R_{\mathrm{L}} = R_{\mathrm{Th}}$.

Mathematically speaking at this point, it is indeterminate as to whether this value of R_L provides the minimum or maximum power transfer. To verify that this solution is indeed the maximum, the second derivative of the power with respect to the load resistance at the point $R_{\mathrm{L}} = R_{\mathrm{Th}}$ is calculated:

$$\frac{d^2P}{dR_{\mathrm{L}}^2} = \frac{d}{dR_{\mathrm{L}}}\left(\frac{dP}{dR_{\mathrm{L}}}\right) = \frac{d}{dR_{\mathrm{L}}}\left[\frac{v^2(R_{\mathrm{Th}} - R_{\mathrm{L}})}{(R_{\mathrm{Th}} + R_{\mathrm{L}})^3}\right]$$

The product (versus quotient) rule is used here to broaden the scope of this example:

$$\begin{aligned}
\frac{d^2P}{dR_{\mathrm{L}}^2} &= v^2\frac{d}{dR_{\mathrm{L}}}\left[(R_{\mathrm{Th}} - R_{\mathrm{L}})(R_{\mathrm{Th}} + R_{\mathrm{L}})^{-3}\right]\\
&= v^2\Big[(R_{\mathrm{Th}} - R_{\mathrm{L}})\frac{d}{dR_{\mathrm{L}}}(R_{\mathrm{Th}} + R_{\mathrm{L}})^{-3}\\
&\quad + (R_{\mathrm{Th}} + R_{\mathrm{L}})^{-3}\frac{d}{dR_{\mathrm{L}}}(R_{\mathrm{Th}} - R_{\mathrm{L}})\Big]\\
&= v^2\left[(R_{\mathrm{Th}} - R_{\mathrm{L}})(-3)(R_{\mathrm{Th}} + R_{\mathrm{L}})^{-4} + (R_{\mathrm{Th}} + R_{\mathrm{L}})^{-3}(-1)\right]\\
&= \frac{v^2}{(R_{\mathrm{Th}} + R_{\mathrm{L}})^4}\left[-3(R_{\mathrm{Th}} - R_{\mathrm{L}}) - (R_{\mathrm{Th}} + R_{\mathrm{L}})\right]\\
&= \frac{2v^2(R_{\mathrm{L}} - 2R_{\mathrm{Th}})}{(R_{\mathrm{Th}} + R_{\mathrm{L}})^4}
\end{aligned}$$

Finally, the second derivative at the point of interest is

$$\left.\frac{d^2P}{dR_{\mathrm{L}}^2}\right|_{R_{\mathrm{L}}=R_{\mathrm{Th}}} = v^2\left.\frac{2R_{\mathrm{L}} - 4R_{\mathrm{Th}}}{(R_{\mathrm{Th}} + R_{\mathrm{L}})^4}\right|_{R_{\mathrm{L}}=R_{\mathrm{Th}}} = v^2\frac{-2R_{\mathrm{Th}}}{(2R_{\mathrm{Th}})^4}$$

Since the second derivative is negative for all R_{Th}, it may be concluded that the maximum power transfer does occur at $R_{\mathrm{L}} = R_{Th}$.

Power Series

A *power series* is an infinite series of the form

$$\sum_{m=0}^{\infty} a_m(x - x_0)^m = a_0 + a_1(x - x_0) + a_2(x - x_0)^2 + \cdots \quad (33)$$

where x_0 is the *center*. Variables x, x_0, and $a_0, a_1, a_2, \ldots$ are real.

Table 3. Derivatives of a Few Commonly Used Functions

$\frac{d}{dx}u^n - nu^{n\ 1}\frac{du}{dx}$	$\frac{d}{dx}a^u - a^u\ln(a)\frac{du}{dx}, a > 0$
$\frac{d}{dx}\ln(u) - \frac{1}{u}\frac{du}{dx}$	$\frac{d}{dx}e^{au} - ae^{au}\frac{du}{dx}$
$\frac{d}{dx}\sin(au) - a\cos(au)\frac{du}{dx}$	$\frac{d}{dx}\cos(au) - \quad a\sin(au)\frac{du}{dx}$

Maclaurin Series. The Maclaurin series uses the origin, $x_0 = 0$, as its reference point to expand a function:

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \cdots + \frac{f^{(n)}(0)}{n!}x^n + \cdots \tag{34}$$

Use of the Maclaurin series leads quickly to series expansions for the exponential, (co)sine, and hyperbolic (co)sine functions as given below:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots + \frac{x^n}{n!} + \cdots \tag{35}$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + (-1)^n\frac{x^{2n+1}}{(2n+1)!} + \cdots \tag{36}$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + (-1)^n\frac{x^{2n}}{(2n)!} + \cdots \tag{37}$$

$$\sinh(x) = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \cdots + \frac{x^{2n+1}}{(2n+1)!} + (18) \tag{38}$$

$$\cosh(x) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \cdots + \frac{x^{2n}}{(2n)!} + \cdots \tag{39}$$

Maclaurin Series Example. The Maclaurin series may be used to expand e^x to find Euler's identities. Begin with

$$\begin{aligned} e^{\pm j\theta} &= 1 + (\pm j\theta) + \frac{(\pm j\theta)^2}{2!} + \frac{(\pm j\theta)^3}{3!} + \frac{(\pm j\theta)^4}{4!} + \frac{(\pm j\theta)^5}{5!} + \cdots \\ &= 1 \pm j\theta - \frac{\theta^2}{2!} \mp \frac{j\theta^3}{3!} + \frac{\theta^4}{4!} \pm \frac{j\theta^5}{5!} \cdots \\ &= 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} \cdots \pm j\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} \cdots\right) \\ &= \cos(\theta) \pm j\sin(\theta) \end{aligned}$$

Adding and subtracting these two sinusoidal expressions

$$e^{j\theta} = \cos(\theta) + j\sin(\theta)$$
$$e^{-j\theta} = \cos(\theta) - j\sin(\theta)$$

along with a division by 2 and $2j$, respectively, form *Euler's identities:*

$$\cos(\theta) = \frac{e^{j\theta} + e^{-j\theta}}{2} \quad \text{and} \tag{40}$$

$$\sin(\theta) = \frac{e^{j\theta} - e^{-j\theta}}{2j} \tag{41}$$

In the special case of $\theta = \pi$, the identity becomes Euler's formula of

$$e^{j\pi} + 1 = 0$$

This formula connects both the fundamental values (of 0, 1, j, e and π) and the basic mathematical operators (addition, multiplication, raised power and equals).

Taylor Series. The Taylor series is more general than the Maclaurin series because it uses an arbitrary reference point, x_0:

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \frac{f'''(x_0)}{3!}(x - x_0)^3 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n + \cdots \tag{42}$$

Binomial Series. Related is the binomial series expansion, which converges for $x^2 < a^2$:

$$\begin{aligned} (a + x)^n &= a^n + na^{n-1}x + \frac{n(n-1)}{2!}a^{n-2}x^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}x^3 + \cdots + x^n \\ &= a^n + \binom{n}{1}a^{n-1}x + \binom{n}{2}a^{n-2}x^2 + \cdots + \binom{n}{k}a^{n-k}x^k + \cdots + x^n \end{aligned} \tag{43}$$

where the binomial coefficients are given by

$$\binom{n}{k} = \frac{n!}{k!(n-k!)} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k!} \tag{44}$$

Binomial Series Example. The binomial series expansion may be used to derive the classic expression for kinetic energy from the relativistic expression below:

$$E_k = m_0c^2\left[\frac{1}{\sqrt{1-\beta^2}} - 1\right]$$

where $\beta = v/c$, the fraction of light speed an object is traveling. The reciprocated square root term is expanded using the binomial formula above where $n = -1/2$, $a = 1$, and $x = -\beta^2$, which meets the convergence restriction. The expansion then is

$$(1-\beta^2)^{-1/2} = (1)^{-1/2} + \frac{-1}{2}(1)^{-3/2}(-\beta^2) + \frac{\frac{-1}{2}\left(\frac{-1}{2}-1\right)}{2!}(1)^{-5/2}(-\beta^2)^2 + \cdots = 1 + \frac{1}{2}\beta^2 + \frac{3}{8}\beta^4 + \cdots$$

If $v < c$, the β^4 and higher terms become insignificant. Substituting the expansion into the relativistic expression for kinetic energy yields

$$E_k \approx m_0c^2\left(1 + \frac{1}{2}\frac{v^2}{c^2} - 1\right) = \frac{1}{2}m_0v^2$$

Numerical Differentiation

Numerical differentiation, although perhaps less common than numerical integration (presented later), is, to a first order, a straightforward extension of Equation (14). For small values of Δx, the first derivative at x_x is

$$f\prime(x_i) = \frac{f(x_i + \Delta x) - f(x_i)}{\Delta x} \quad (45)$$

If Δx is positive, the above expression is referred to as a forward-difference formula, whereas if Δx is negative, it is termed a backward-difference formula. Greater accuracy can be obtained using formulas that employ data points on both sides of x_i. For instance, although $f(x_i)$ does not explicitly appear in the following equations, they are known as three-point and five point formulas respectively

$$f\prime(x_i) = \frac{f(x_i + \Delta x) - f(x_i - \Delta x)}{2\,\Delta x} \quad (46)$$

$$f\prime(x_i) = \frac{f(x_i - 2\,\Delta x) - 8\,f(x_i - \Delta x) + 8\,f(x_i + \Delta x) - f(x_i + 2\,\Delta x)}{12\,\Delta x} \quad (47)$$

INTEGRAL CALCULUS

Indefinite Integrals

Differentiation and integration are inverse operations. There are two fundamental issues associated with integral calculus. The first is to find integrals or antiderivatives of a function, that is, given an expression, find another function that has the first function as its derivative. The second problem is to evaluate a definite integral as a limit of a sum. As an example, consider $y = x^2$, which is an integral of $2x$. It is important to point out that the integral is not unique and that x^2 represents a family of functions with the same derivative. Therefore, the solution should be augmented with an *integration constant, c*, added to each expression to represent the *indefinite integral*. This is so because the derivative of a constant is zero. If $F(x)$ is an integral of $f(x)$, then

$$F(x) = \int f(x)\,dx + c \quad (48)$$

The addition of the integration constant represents all integrals of a function. The symbol $\int$, a medieval S, stands for *summa* (sum).

The process of finding the integral of a function is called *integration.* While the determination of the derivative of a function is rather straightforward since definite rules exist, there is no general method for finding the integral of a mathematical expression. Calculus gives rules for integrating large classes of functions. When these rules fail, approximate or numerical methods permit the evaluation of the integral for a given value of x.

Selected indefinite integrals are given in Table 4. Although extensive integral tables exist, there are expressions whose integrals are not listed. Therefore, it is important to be cognizant of rules such as integration by parts or some form of transformation to arrive at the integral of the desired mathematical expression.

Integration Rules

Properties that hold for the definite integral include scaling and linearity:

$$\int_a^b [cf(x) \pm kg(x)]dx = c\int_a^b f(x)\,dx \pm k\int_a^b g(x)\,dx \quad (49)$$

They also include particular properties due to the limits of integration:

$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx \quad (50)$$

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx \quad (51)$$

Transformations

Transformation is one method to facilitate evaluating integrals. Perhaps the simplest form of transformation is substitution. Other complex types of transformation are also possible, and some integral tables suggest appropriate substitutions for integrals, which are similar to the integrals in the table. Experience as well as intuition are the two most important factors in finding the right transformation. In performing the substitution with the definite integrals, it is important to change the limits. Particularly, the *change of limits rule* states that if the integral $\int f(g(x))g'(x)\,dx$ is subjected to the substitution $u = g(x)$, so that the integral becomes $\int f(u)\,du$, then

$$\int_a^b f(g(x))g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du \quad (52)$$

Substitution Example. To determine the area of the ellipse $x^2/a^2 + y^2/b^2 = 1$, as shown in Fig. 3, the function may be rearranged to

$$y = b\sqrt{1-(x/a)^2}$$

Table 4. Definite Integrals Without Constant of Integration

General Forms

$$\int u\,dv = uv - \int v\,du$$

$$\int a\,dx = ax$$

$$\int x^n dx = \frac{x^{n+1}}{n+1}, \qquad n \neq -1$$

$$\int \frac{1}{x}dx = \ln(x)$$

$$\int b^{ax}dx = \frac{b^{ax}}{a\ln(b)}, \qquad b > 0$$

$$\int a^x \ln(a)\,dx = a^x, \qquad a > 0$$

Forms with $a + bx$

$$\int (a+bx)^n dx = \frac{(a+bx)^{n+1}}{(n+1)b}, \qquad n \neq -1$$

$$\int \frac{1}{a+bx}dx = \frac{1}{b}\ln(a+bx)$$

$$\int \frac{x}{a+bx}dx = \frac{x}{b} - \frac{a}{b^2}\ln(a+bx)$$

$$\int \frac{1}{x(a+bx)}dx = -\frac{1}{a}\ln\left(\frac{a+bx}{x}\right)$$

$$\int \frac{1}{a+bx+cx^2}dx = \begin{cases} \dfrac{2}{\sqrt{4ac-b^2}}\tan^{-1}\left(\dfrac{2cx+b}{\sqrt{4ac-b^2}}\right), & 4ac-b^2>0 \\ \dfrac{2}{\sqrt{b^2-4ac}}\tanh^{-1}\left(\dfrac{2cx+b}{\sqrt{b^2-4ac}}\right), & 4ac-b^2<0 \end{cases}$$

Forms with $\sqrt{a+bx}$

$$\int \sqrt{a+bx}\,dx = \frac{2}{3b}\sqrt{(a+bx)^3}$$

$$\int x\sqrt{a+bx}\,dx = \frac{-2(2a-3bx)}{15b^2}\sqrt{(a+bx)^3}$$

$$\int \frac{1}{\sqrt{a+bx}}dx = \frac{2}{b}\sqrt{a+bx}$$

$$\int \frac{x}{\sqrt{a+bx}}dx = \frac{-2(2a-bx)}{3b^2}\sqrt{a+bx}$$

$$\int \frac{1}{x\sqrt{a+bx}}dx = \begin{cases} \dfrac{1}{\sqrt{a}}\ln\left(\dfrac{\sqrt{a+bx}-\sqrt{a}}{\sqrt{a+bx}+\sqrt{a}}\right), & a>0 \\ \dfrac{2}{\sqrt{-a}}\tan^{-1}\left(\sqrt{\dfrac{a+bx}{-a}}\right), & a<0 \end{cases}$$

Forms with $a^2 \pm x^2$ or $x^2 \pm a^2$

$$\int \frac{1}{a^2+x^2}dx = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$$

$$\int \frac{1}{a^2-x^2}dx = \frac{1}{a}\tanh^{-1}\left(\frac{x}{a}\right)$$

$$\int \frac{x}{a^2 \pm x^2}dx = \pm\frac{1}{2}\ln(a^2 \pm x^2)$$

$$\int \frac{1}{x^2-a^2}dx = \frac{1}{2a}\ln\left[\frac{x-a}{x+a}\right], \qquad x^2 > a^2$$

Forms with $\sqrt{a^2 \pm x^2}$ or $\sqrt{x^2 \pm a^2}$

$$\int \frac{1}{\sqrt{a^2-x^2}}dx = \sin^{-1}\left(\frac{x}{|a|}\right)$$

$$\int \sqrt{x^2 \pm a^2}\,dx = \frac{1}{2}\left[x\sqrt{x^2 \pm a^2} \pm a^2\ln\left(x+\sqrt{x^2 \pm a^2}\right)\right]$$

$$\int \frac{1}{\sqrt{x^2 \pm a^2}}dx = \ln\left(x+\sqrt{x^2 \pm a^2}\right)$$

$$\int \frac{x}{\sqrt{x^2 \pm a^2}}dx = \sqrt{x^2 \pm a^2}$$

$$\int \frac{1}{x\sqrt{x^2-a^2}}dx = \frac{1}{|a|}\sec^{-1}\left(\frac{x}{a}\right)$$

$$\int \frac{1}{x\sqrt{a^2 \pm x^2}}dx = \frac{-1}{a}\ln\left(\frac{a+\sqrt{a^2 \pm x^2}}{x}\right)$$

Forms with $a \pm bx^2$

$$\int \frac{1}{a+bx^2}dx = \begin{cases} \dfrac{1}{\sqrt{ab}}\tan^{-1}\left(\dfrac{x\sqrt{ab}}{a}\right), & ab>0 \\ \dfrac{1}{\sqrt{-ab}}\tanh^{-1}\left(\dfrac{x\sqrt{-ab}}{a}\right), & ab<0 \end{cases}$$

$$\int \frac{x}{a+bx^2}dx = \frac{1}{2b}\ln(a+bx^2)$$

$$\int \frac{1}{x(a+bx^2)}dx = \frac{1}{2a}\ln\left(\frac{x^2}{a+bx^2}\right)$$

Logarithmic Forms

$$\int \ln(x)\,dx = x\ln(x) - x$$

$$\int x\ln(x)\,dx = \frac{x^2}{2}\ln(x) - \frac{x^2}{4}$$

$$\int x^n\ln(ax)\,dx = \frac{x^{n+1}}{n+1}\ln(ax) - \frac{x^{n+1}}{(n+1)^2}, \qquad n \neq -1$$

$$\int \frac{\ln(x)}{x}dx = \frac{1}{2}[\ln(x)]^2$$

$$\int \frac{\ln(x)}{x^2}dx = \frac{-\ln(x)}{x} - \frac{1}{x}$$

$$\int \frac{1}{x\ln(x)}dx = \ln[\ln(x)]$$

$$\int [\ln(x)]^2 dx = x[\ln(x)]^2 - 2x\ln(x) + 2x$$

Exponential Forms

$$\int e^{ax}dx = \frac{e^{ax}}{a}$$

$$\int xe^{ax}dx = \frac{e^{ax}}{a^2}(ax-1)$$

$$\int x^2e^{ax}dx = \frac{e^{ax}}{a^3}(a^2x^2 - 2ax + 2)$$

$$\int xe^{-x^2}dx = \frac{-1}{2}e^{-x^2}$$

$$\int \frac{1}{1+e^x}dx = x - \ln(1+e^x) = \ln\left(\frac{e^x}{1+e^x}\right)$$

$$\int \frac{1}{a+be^{px}}dx = \frac{x}{a} - \frac{1}{ap}\ln(a+be^{px})$$

$$\int e^{ax}\sin(bx)\,dx = \frac{e^{ax}}{a^2+b^2}[a\sin(bx) - b\cos(bx)]$$

$$\int e^{ax}\cos(bx)\,dx = \frac{e^{ax}}{a^2+b^2}[a\cos(bx) + b\sin(bx)]$$

$$\int e^{ax}\sin^2(bx)\,dx = \frac{e^{ax}}{a^2+4b^2}\left\{[a\sin(bx) - 2b\cos(bx)]\sin(bx) + \frac{2b^2}{a}\right\}$$

$$\int e^{ax}\cos^2(bx)\,dx = \frac{e^{ax}}{a^2+4b^2}\left\{[a\cos(bx) + 2b\sin(bx)]\cos(bx) + \frac{2b^2}{a}\right\}$$

$$\int xe^{ax}\sin(bx)\,dx = \frac{xe^{ax}}{a^2+b^2}[a\sin(bx) - b\cos(bx)] - \frac{e^{ax}}{(a^2+b^2)^2}[(a^2-b^2)\sin(bx) - 2ab\cos(bx)]$$

$$\int xe^{ax}\cos(bx)\,dx = \frac{xe^{ax}}{a^2+b^2}[a\cos(bx) + b\sin(bx)] - \frac{e^{ax}}{(a^2+b^2)^2}[(a^2-b^2)\cos(bx) + 2ab\sin(bx)]$$

Table 4. *(Continued)*

Trigonometric Forms with Sine

$$\int \sin(ax)\,dx = -\frac{1}{a}\cos(ax)$$

$$\int \sin(a+bx)\,dx = \frac{-1}{b}\cos(a+bx)$$

$$\int \sin^2(ax)\,dx = \frac{x}{2} - \frac{\sin(2ax)}{4a} = \frac{x}{2} - \frac{1}{2a}\sin(ax)\cos(ax)$$

$$\int \sin(ax)\sin(bx)\,dx = \frac{\sin[(a-b)x]}{2(a-b)} - \frac{\sin[(a+b)x]}{2(a+b)}, \qquad a^2 \neq b^2$$

$$\int x\sin(ax)\,dx = \frac{1}{a^2}\sin(ax) - \frac{x}{a}\cos(ax)$$

$$\int x^2\sin(ax)\,dx = \frac{2x}{a^2}\sin(ax) - \frac{a^2x^2-2}{a^3}\cos(ax)$$

Trigonometric Forms with Cosine

$$\int \cos(ax)\,dx = \frac{1}{a}\sin(ax)$$

$$\int \cos(a+bx)\,dx = \frac{1}{b}\sin(a+bx)$$

$$\int \frac{1}{1+\cos(ax)}dx = \frac{1}{a}\tan\left(\frac{ax}{2}\right)$$

$$\int \frac{\cos(ax)}{1+\cos(ax)}dx = x - \frac{1}{a}\tan\left(\frac{ax}{2}\right)$$

$$\int \cos^2(ax)\,dx = \frac{x}{2} + \frac{\sin(2ax)}{4a} = \frac{x}{2} + \frac{1}{2a}\sin(ax)\cos(ax)$$

$$\int \cos(ax)\cos(bx)\,dx = \frac{\sin[(a-b)x]}{2(a-b)} + \frac{\sin[(a+b)x]}{2(a+b)}, \qquad a^2 \neq b^2$$

$$\int x\cos(ax)\,dx = \frac{1}{a^2}\cos(ax) + \frac{x}{a}\sin(ax)$$

$$\int x^2\cos(ax)\,dx = \frac{2x}{a^2}\cos(ax) + \frac{a^2x^2-2}{a^3}\sin(ax)$$

Trigonometric Forms with Sine and Cosine

$$\int \sin(ax)\cos(bx)\,dx = -\frac{\cos[(a-b)x]}{2(a-b)} - \frac{\cos[(a+b)x]}{2(a+b)}, \qquad a^2 \neq b^2$$

$$\int \sin(ax)\cos(ax)\,dx = \frac{1}{2a}\sin^2(ax)$$

$$\int \sin^2(ax)\cos^2(ax)\,dx = \frac{x}{8} - \frac{1}{32a}\sin(4ax)$$

$$\int \sin(ax)\cos^m(ax)\,dx = \frac{-\cos^{m+1}(ax)}{(m+1)a}$$

$$\int \sin^m(ax)\cos(ax)\,dx = \frac{\sin^{m+1}(ax)}{(m+1)a}$$

$$\int \frac{\sin(ax)}{\cos^2(ax)}dx = \frac{1}{a\cos(ax)} = \frac{\sec(ax)}{a}$$

$$\int \frac{\cos(ax)}{\sin^2(ax)}dx = \frac{-1}{a\sin(ax)} = \frac{-\csc(ax)}{a}$$

$$\int \frac{x+\sin(x)}{1+\cos(x)}dx = x\tan\left(\frac{x}{2}\right)$$

Trigonometric Forms Without Sine and Cosine

$$\int \tan(ax)\,dx = -\frac{1}{a}\ln[\cos(ax)] = \frac{1}{a}\ln[\sec(ax)]$$

$$\int \cot(ax)\,dx = \frac{1}{a}\ln[\sin(ax)] = \frac{-1}{a}\ln[\csc(ax)]$$

$$\int \sec(ax)\,dx = \frac{1}{a}\ln[\sec(ax) + \tan(ax)] = \frac{1}{a}\ln\left[\tan\left(\frac{ax}{2} + \frac{\pi}{4}\right)\right]$$

$$\int \csc(ax)\,dx = \frac{1}{a}\ln[\csc(ax) - \cot(ax)] = \frac{1}{a}\ln\left[\tan\left(\frac{ax}{2}\right)\right]$$

$$\int \tan^2(ax)\,dx = \frac{1}{a}\tan(ax) - x$$

$$\int \sec^2(ax)\,dx = \int \frac{1}{\cos^2(ax)}dx = \frac{1}{a}\tan(ax)$$

$$\int \csc^2(ax)\,dx = \int \frac{1}{\sin^2(ax)}dx = \frac{-1}{a}\cot(ax)$$

Inverse Trigonometric Forms

$$\int \sin^{-1}(ax)\,dx = x\sin^{-1}(ax) + \frac{\sqrt{1-a^2x^2}}{a}$$

$$\int \cos^{-1}(ax)\,dx = x\cos^{-1}(ax) - \frac{\sqrt{1-a^2x^2}}{a}$$

$$\int \tan^{-1}(ax)\,dx = x\tan^{-1}(ax) - \frac{1}{2a}\ln(1+a^2x^2)$$

Hyperbolic Forms

$$\int \sinh(ax)\,dx = \frac{1}{a}\cosh(ax)$$

$$\int \cosh(ax)\,dx = \frac{1}{a}\sinh(ax)$$

$$\int \tanh(ax)\,dx = \frac{1}{a}\ln[\cosh(ax)]$$

$$\int \sinh^{-1}\left(\frac{x}{a}\right)dx = x\sinh^{-1}\left(\frac{x}{a}\right) - \sqrt{x^2+a^2}$$

$$\int x\sinh(ax)\,dx = \frac{x\cosh(ax)}{a} - \frac{\sinh(ax)}{a^2}$$

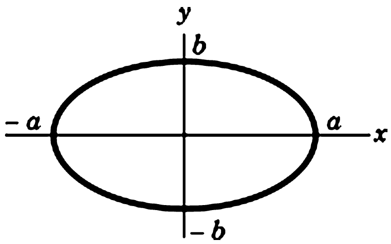

Figure 3. The area of the region encompassed by the ellipse $x^2/a^2 + y^2/b^2 = 1$ may be obtained by taking advantage of the symmetric structure of the function. To this end, the total area is twice the area of the region above the x-axis, which is equal to $\int^{+a}{}_{-a}$ $b1 - (x/a)^2\ dx$.

Taking advantage of the symmetric nature of the function, the area of the ellipse is twice the area of its upper half:

$$A = 2\int_{-a}^{a} b\sqrt{1-(x/a)^2}\,dx$$

Let $u = x/a$, which results in $du = dx/a$. When $x = -a$ we obtain $u = -1$; similarly, $u = 1$ for $x = a$. Thus

$$\begin{aligned} A &= 2\int_{-a}^{a} b\sqrt{1-(x/a)^2}dx = 2b\,a\int_{-a}^{a}(1/a)\sqrt{1-(x/a)^2}dx \\ &= 2b\,a\int_{-1}^{1}\sqrt{1-(u)^2}du \end{aligned}$$

We know that in general

$$\int\sqrt{c^2-u^2}dx = \frac{u}{2}\sqrt{c^2-u^2} + \frac{c^2}{2}\sin^{-1}\left(\frac{u}{c}\right)$$

Since here c = 1, the area is

$$\begin{aligned} A &= 2\,b\,a\int_{-1}^{1}\sqrt{1-(u)^2}du \\ &= 2\,b\,a\left[\frac{u}{2}\sqrt{1-u^2}+\frac{1}{2}\sin^{-1}(u)\right]\Big|_{-1}^{1} \\ &= 2\,b\,a\left\{\left[\frac{1}{2}\sqrt{1-(1)^2}+\frac{1}{2}\sin^{-1}(1)\right]\right. \\ &\quad \left.-\left[\frac{-1}{2}\sqrt{1-(-1)^2}+\frac{1}{2}\sin^{-1}(-1)\right]\right\} \\ &= 2\,b\,a\left\{\left[0+\left(\frac{1}{2}\right)\left(\frac{\pi}{2}\right)\right]-\left[0+\left(\frac{1}{2}\right)\left(\frac{-\pi}{2}\right)\right]\right\} \\ &= \pi\,a\,b. \end{aligned}$$

Integration Example. Calculation of the *root-mean-square* (*rms*) value of a function is a classic use of the integral. The rms value is found by first squaring the waveform, followed by computing its average, and finally by taking its square root. Consider the determination of the rms value of a sinusoidal current, $i(t) = I_M \cos(\omega t + \theta)$, of constant frequency, ω, and constant phase shift, θ. The rms current is found over a representative period, $T = 2\pi/\omega$:

$$I_{rms} = \sqrt{\frac{1}{T}\int_0^T i^2(t)\,dt} = \sqrt{\frac{1}{T}\int_0^T I_M^2\cos^2(\omega t+\theta)\,dt}$$

The solution to the integral may be found using a change of variables and the table of integrals. First, let $u = \omega t + \theta$, such that $du = \omega\,dt$. The variable change modifies the upper and lower limits of integration to $\omega T + \theta$ and θ, respectively. The expression for integral now appears as

$$\begin{aligned} \int_0^T I_M^2\cos^2(\omega t+\theta)\,dt &= \int_\theta^{\omega T+\theta} I_M^2\cos^2(u)\frac{du}{\omega} \\ &= \frac{I_M^2}{\omega}\int_\theta^{\omega T+\theta}\cos^2(u)\,du \end{aligned}$$

Using the table of integrals (Table 4), we obtain

$$\begin{aligned} \int_\theta^{\omega T+\theta}\cos^2(u)\,du &= \frac{u}{2}+\frac{\sin(2u)}{4}\Big|_\theta^{\omega T+\theta} \\ &= \left\{\frac{\omega T+\theta}{2}+\frac{\sin[2(\omega T+\theta)]}{4}\right\} \\ &\quad -\left\{\frac{\theta}{2}+\frac{\sin(2\theta)}{4}\right\} \\ &= \frac{\omega T}{2}+\frac{\sin[2(\omega T+\theta)]-\sin(2\theta)}{4} \\ &= \frac{2\pi}{2}+\frac{\sin[2(2\pi+\theta)]-\sin(2\theta)}{4} = \pi \end{aligned}$$

Thus, the rms value is

$$I_{rms} = \sqrt{\frac{1}{T}\frac{I_M^2}{\omega}\pi} = \sqrt{\frac{I_M^2\pi}{2\pi}} = \frac{I_M}{\sqrt{2}}$$

Integration by Parts

One of the most important techniques of integration is the principle of integration by parts. Let $f(x)$ and $g(x)$ be any two functions and let $G(x)$ be an antiderivative of $g(x)$. Using the product rule for derivatives, the integral of the product of the two functions can be derived as

$$\int f(x)g(x)\,dx = f(x)G(x) - \int f'(x)G(x)\,dx \tag{53}$$

For definite integrals we obtain

$$\int_a^b f(x)g(x)\,dx = f(x)G(x)\Big|_a^b - \int_a^b f'(x)G(x)\,dx \tag{54}$$

Integration by Parts Example. This example illustrates integration by parts in evaluating the *Laplace transform,* which is defined by

$$v = \int e^{-st}\,dt = -e^{-st}/s \tag{55}$$

Here we transform a ramp function, $f(t) = at$. Let $u = at$ and $dv = e^{-st}\ dt$. Hence, $du = a\ dt$ and $v = \int e^{-st}\,dt = -e^{-st}/s$.

The Laplace transform of a ramp is

$$\begin{aligned} F(s) &= \int_0^\infty at\, e^{-st}\, dt = at \frac{-e^{-st}}{s} |_0^\infty - \int_0^\infty \frac{-e^{-st}}{s} a\, dt \\ &= \left[a \cdot \infty \frac{-e^{-s\infty}}{s} - a \cdot 0 \frac{-e^{-s0}}{s} \right] + \frac{a}{s} \left(\frac{e^{-st}}{-s} \right) |_0^\infty \\ &= 0 + \frac{-a}{s^2} (e^{-s\infty} - e^{-s0}) = \frac{a}{s^2}. \end{aligned}$$

Definite Integrals

A definite integral is the limit of a sum. Common applications of the definite integral include determination of area, arc length, volume, and function average. These quantities can be approximated by sums obtained by dividing the given quantity into small parts and approximating each part. The definite integral allows one to arrive at the exact values of these quantities instead of their approximate values.

The symbol $\int^b{}_a f(x)\, dx$ is the *definite integral* of $f(x)\, dx$ on interval $[a, b]$. Let $f(x)$ be a single-valued function of x, defined at each point on $[a, b]$. Choose points x_i on the interval such that

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b$$

Let $\Delta x_i = x_i - x_{i-1}$. Choose in each interval Δx_i a point t_i. Form the sum

$$S_n = \sum_{i=1}^{n} f(t_i)\Delta x_i = f(t_1)\Delta x_1 + f(t_2)\Delta x_2 + \cdots + f(t_n)\,\Delta x_n$$

The limit of this sum, as the largest interval approaches zero, is defined as the *definite integral* $\int^b{}_a f(x)\, dx$, if it can exist. The existence of f is guaranteed if it is a continuous function on $[a, b]$. If $F(x)$ is a function whose derivative is $f(x)$, then it can be shown that

$$\int_a^b f(x)\, dx = F(b) - F(a) \tag{56}$$

This is essentially the fundamental theorem of calculus. If F does not exist, numerical methods may be used to obtain the value of the integral. Several definite integrals important in engineering are listed in Table 5. For a more comprehensive list of integrals, the reader is referred to a number of calculus texts.

Applications. One use of definite integrals is to find the areas bounded by certain curves. For example, the area bounded by the polar function $f(\theta)$ and the lines θ_a and θ_b is

$$A = \frac{1}{2} \int_{\theta_a}^{\theta_b} [f(\theta)]^2\, d\theta \tag{57}$$

Another application of integration is to find an average (mean) value:

$$\bar{x} = \frac{1}{b-a} \int_a^b x(t)\, dt \tag{58}$$

Integration is used to find arc length from point a to point b:

$$L = \int_a^b \sqrt{1 + [f'(x)]^2}\, dx \tag{59}$$

Or it is used to find arc length in polar coordinates:

$$L = \int_{\theta_a}^{\theta_b} \sqrt{[f(\theta)]^2 + [f'(\theta)]^2}\, d\theta \tag{60}$$

Multiple Integration

The double integral of $f(x, y)$ over some region R is the generalization of the definite integral and is denoted as

$$\iint_R f(x, y)\, dx\, dy \tag{61}$$

It is typically applied to find the volume encompassed by a surface, the center of gravity of a given structure, and moments of inertia.

Let $f(x, y)$ be a function of two variables, and let $g(x)$ and $h(x)$ be two functions of x alone. Furthermore, let a and b be real numbers. Then, an *iterated integral* is an expression of the form

$$\int_a^b \left(\int_{g(x)}^{h(x)} f(x, y)\, dy \right) dx \tag{62}$$

where $f(x, y)$ is first treated as a function of y alone. The inner integral is evaluated between the limits $y = g(x)$ and $y = h(x)$, which results in an expression that is a function of x alone. The resultant integrand is then evaluated between the limits of $x = a$ and $x = b$.

A similar principle applies to function of three or more independent variables. A change of variables in multiple integrals is generally accomplished with the aid of the Jacobian.

For a transformation of the form

$$x = f(u, v, w) \quad y = g(u, v, w) \quad z = h(u, v, w)$$

the Jacobian of the transformation is defined as

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} \tag{63}$$

Special Functions

Various other special functions exist. The gamma function is defined by the integral

$$\Gamma(n) = \int_0^\infty t^{n-1} e^{-t}\, dt \quad, n > 0 \tag{64}$$

The error function is given by

$$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt \tag{65}$$

Table 5. Definite Integrals

$$\int_0^\infty \frac{a}{a^2+x^2}\,dx = \begin{cases} \pi/2, & a>0 \\ 0, & a=0 \\ -\pi/2, & a<0 \end{cases}$$

$$\int_1^\infty \frac{1}{x^m}\,dx = \frac{1}{m-1}, \quad m>1$$

$$\int_0^\infty \frac{1}{(1+x)\sqrt{x}}\,dx = \pi$$

$$\int_0^\infty x^n e^{-ax}\,dx = \frac{n!}{a^{n+1}}, \quad a>0, n \text{ a positive integer}$$

$$\int_0^\infty e^{-a^2x^2}\,dx = \frac{\sqrt{\pi}}{2a}, \quad a>0$$

$$\int_0^\infty e^{-ax}\cos(bx)\,dx = \frac{a}{a^2+b^2}, \quad a>0$$

$$\int_0^\infty e^{-ax}\sin(bx)\,dx = \frac{b}{a^2+b^2}, \quad a>0$$

$$\int_0^\infty \frac{\sin(mx)}{x}\,dx = \begin{cases} \pi/2, & m>0 \\ 0, & m=0 \\ -\pi/2, & m<0 \end{cases}$$

$$\int_0^\infty \frac{\cos(x)}{x}\,dx = \infty$$

$$\int_0^\infty \frac{\tan(x)}{x}\,dx = \frac{\pi}{2}$$

$$\int_0^\infty \cos(ax^2)\,dx = \int_0^\infty \sin(ax^2)\,dx = \frac{1}{2}\sqrt{\frac{\pi}{2a}}$$

$$\int_0^\pi \sin(mx)\sin(nx)\,dx = \begin{cases} 0, & m,n \text{ integers and } m \neq n \\ \pi/2, & m,n \text{ integers and } m = n \end{cases}$$

$$\int_0^\pi \cos(mx)\cos(nx)\,dx = \begin{cases} 0, & m,n \text{ integers and } m \neq n \\ \pi/2, & m,n \text{ integers and } m = n \end{cases}$$

$$\int_0^\pi \sin(mx)\cos(nx)\,dx = \begin{cases} 0, & m+n \text{ even} \\ \dfrac{2m}{m^2-n^2}, & m+n \text{ odd} \end{cases}$$

$$\int_0^\infty \frac{\sin^2(px)}{x^2}\,dx = \frac{\pi p}{2}$$

$$\int_0^\infty \frac{1-\cos(px)}{x^2}\,dx = \frac{\pi p}{2}$$

$$\int_0^\infty \operatorname{sinc}(ax)\,dx = \int_0^\infty \frac{\sin(\pi ax)}{\pi ax} = \frac{1}{2a}, \quad a>0$$

$$\int_0^\infty \operatorname{sinc}^2(ax)\,dx = \int_0^\infty \frac{\sin^2(\pi ax)}{(\pi ax)^2} = \frac{1}{2a}, \quad a>0$$

$$\int_0^\infty \frac{\sin(x)}{\sqrt{x}}\,dx = \int_0^\infty \frac{\cos(x)}{\sqrt{x}}\,dx = \sqrt{\frac{\pi}{2}}$$

The complementary error function is simply: $\operatorname{erfc}(x) = 1 - \operatorname{erf}(x)$.

Numerical Integration

Numerical methods may be used to approximate the definite integral in cases where either an analytical solution is unavailable or the function is unknown (as in the case of sampled data). The simplest numerical integration uses the Riemann sum in which the integral symbol becomes a summation, and the dx term becomes a partition, $\Delta x_i = [x_i - x_{i-1}]$, in $[a, b]$:

$$\int_a^b f(x)\,dx \approx \sum_{i=1}^{n} f(w_i)\,\Delta x_i \tag{66}$$

where w_i is any number, usually the midpoint, in partition Δx_i. The partition is typically a constant proportional to the number of partitions, $\Delta x = (b-a)/n$ (rectangle rule). As the magnitude of Δx decreases, the accuracy of the numerical estimates increases. A traditional approach of testing the solution convergence is to repeatedly halve the partition width until an acceptable error is reached.

Trapezoidal Rule. The trapezoidal rule improves the numerical estimate of the integral (as compared with the rectangle rule above) by fitting a piecewise linear approximation to each subinterval using its endpoints (see Fig. 4):

$$\int_a^b f(x)\,dx \approx \Delta x \left[\frac{f(a)+f(b)}{2} + \sum_{i=1}^{n-1} f(a+i\Delta x)\right] \tag{67}$$

Simpson's Rule. Simpson's rule is a further improvement employing a piecewise quadratic approximation. In this method, the number of subintervals must be even (i.e.,

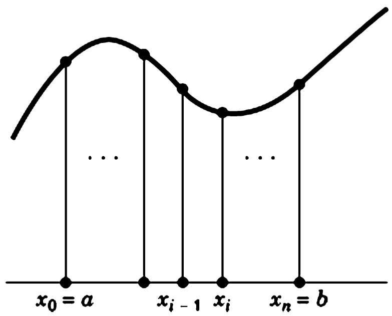

Figure 4. For trapezoidal numerical integration the curve is subdivided into equal increments between the left-hand limit at $x_0 = a$ and the right-hand limit at $x_n = b$. The area within each subinterval is approximated as $(x_i - x_{i-1})f(x_i) + f(x_{i-1})/2$. The integral is then numerically approximated by the summation of the subinterval areas.

$m = 2n$) and $\Delta x = (b - a)/m$. The numerical area is

$$\int_a^b f(x)\,dx \approx \frac{\Delta x}{3}\left\{f(a) - f(b) + \sum_{i=1}^{n}[4f(a + (2i-1)\Delta x) + 2f(a + 2i\Delta x)]\right\} \quad (68)$$

Calculus Software

With the advent of powerful personal computers, software has been developed for solving calculus problems and providing graphical visualization of their solutions. Many of these programs rely on symbolic processing that was pioneered in artificial intelligence. Caution should, however, be heeded in the use of these programs as they can result in nonsensical solutions. Some of the more advanced and commercial programs are Maple®, Mathematica®, Matlab®, and MathCad®. A discussion on these programs and their use in solving calculus problems is omitted here due to the evolving nature of such software, but the reader is referred to the Internet for Web-based calculus software resources.

ADDITIONAL TOPICS IN CALCULUS

Although differentiation and integration form the pillars of the use of calculus in engineering there are other mathematical tools, such as vectors and the convergence theorem, which transcend the boundaries of calculus. These topics are presented here.

Transformation of Coordinates

In some engineering applications, it is necessary to transform a given mathematical expression from one coordinate system to another. Examples of this transformation are those for the Laplacian operator, which appear later in this section. For the coordinate systems that appear in Fig. 5, the transformations appear in Table 6.

Vector Calculus

Consider a vector function

$$\boldsymbol{v}(x,y,z) = v_x(x,y,z)\boldsymbol{i} + v_y(x,y,z)\boldsymbol{j} + v_z(x,y,z)\boldsymbol{k} = \langle v_x, v_y, v_z\rangle \quad (69)$$

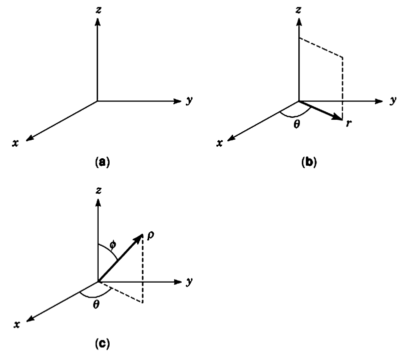

Figure 5. Cartesian (a), cylindrical (b), and spherical (c) coordinate systems are used in many engineering analyses. To facilitate an analysis, the coordinates of a given point may be transformed from one coordinate system to another. The transformation rules appear in Table 6.

where $\boldsymbol{i}$, $\boldsymbol{j}$, and $\boldsymbol{k}$ are unit vectors in the positive x, y, and z directions, respectively. The magnitude of the vector is $\sqrt{v_x^2 + v_y^2 + v_z^2}$. The *dot product* (also referred to as the scalar or inner product) of $\boldsymbol{v}$ and $\boldsymbol{w}$ is defined as

$$\boldsymbol{v}\cdot\boldsymbol{w} \equiv v_x w_x + v_y w_y + v_z w_z = |\boldsymbol{v}||\boldsymbol{w}|\cos(\theta) \quad (70)$$

where θ is the angle between $\boldsymbol{v}$ and $\boldsymbol{w}$. Two vectors are orthogonal if and only if $\boldsymbol{v}\cdot\boldsymbol{w} = \boldsymbol{0}$.

The *cross product* or vector product of $\boldsymbol{v}$ and $\boldsymbol{w}$ is defined as

$$\boldsymbol{v}\times\boldsymbol{w} \equiv \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix} \quad \text{with} \quad |\boldsymbol{v}\times\boldsymbol{w}| = |\boldsymbol{v}||\boldsymbol{w}|\sin(\theta) \quad (71)$$

Two vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ are parallel if and only if $\boldsymbol{v}\times\boldsymbol{w} = \boldsymbol{0}$.

The *vector differential operator* ∇ ("del") is defined in three dimensions as

$$\nabla \equiv \frac{\partial}{\partial x}\boldsymbol{i} + \frac{\partial}{\partial y}\boldsymbol{j} + \frac{\partial}{\partial z}\boldsymbol{k} \quad (72)$$

The *gradient* of a scalar field, $f(x,y,z)$, is defined as

$$\text{grad}\, f = \nabla f = \frac{\partial f}{\partial x}\boldsymbol{i} + \frac{\partial f}{\partial y}\boldsymbol{j} + \frac{\partial f}{\partial z}\boldsymbol{k} \quad (73)$$

The *divergence* of a vector field is the dot product of the gradient operator and the vector field:

$$\text{div}\,\boldsymbol{v} = \nabla\cdot\boldsymbol{v} = \frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \quad (74)$$

The *curl* of a vector field is the cross product of the gradient and the vector function:

$$\text{curl}\,\boldsymbol{v} = \nabla\times\boldsymbol{v} = \left(\frac{\partial v_z}{\partial y} - \frac{\partial v_y}{\partial z}\right)\boldsymbol{i} + \left(\frac{\partial v_x}{\partial z} - \frac{\partial v_z}{\partial x}\right)\boldsymbol{j} + \left(\frac{\partial v_y}{\partial x} - \frac{\partial v_x}{\partial y}\right)\boldsymbol{k} \quad (75)$$

The curl of any gradient is the zero vector, $\nabla\times(\nabla f) = \boldsymbol{0}$. The divergence of any curl is zero, $\nabla\cdot(\nabla\times\boldsymbol{F}) = 0$. The *divergence* of a gradient of f is its Laplacian, denoted as $\nabla^2 f$ or Δf. For the Cartesian coordinate system the Laplacian is repre-

Table 6. Coordinate System Transformations

Cartesian	Cylindrical	Spherical
x	$r\cos\theta$	$\rho\sin\phi\cos\theta$
y	$r\sin\theta$	$\rho\sin\phi\sin\theta$
z	z	$\rho\cos\phi$
–	$r^2 = x^2 + y^2$	$\rho^2 = x^2 + y^2 + z^2$
$dV = dx\,dy\,dz$	$r\,dr\,d\theta\,dz$	$\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$

sented as

$$\Delta = \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \tag{76}$$

for the cylindrical coordinate system it is represented as

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{\partial^2}{\partial z^2} \tag{77}$$

and for the spherical coordinate system it is represented as

$$\nabla^2 = \frac{1}{\rho^2}\frac{\partial}{\partial\rho}\left(\rho^2\frac{\partial}{\partial\rho}\right) + \frac{1}{\rho^2\sin\phi}\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial}{\partial\phi}\right) + \frac{1}{\rho^2\sin^2\phi}\frac{\partial^2}{\partial\theta^2} \tag{78}$$

Functions that satisfy Laplace's equation, $\nabla^2 f = 0$, are said to be *harmonic.*

Vector Calculus Example. Let $\boldsymbol{v} = x^2y\boldsymbol{i} + z\boldsymbol{j} + xyz\boldsymbol{i}$. The divergence of the vector is

$$\begin{aligned}\operatorname{div}\boldsymbol{v} &= \frac{\partial}{\partial x}(x^2y) + \frac{\partial}{\partial y}(z) + \frac{\partial}{\partial z}(xyz)\\ &= 2xy + 0 + xy = 3xy\end{aligned}$$

The curl of $\boldsymbol{v}$ is

$$\boldsymbol{\nabla}\times\boldsymbol{v} = \begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ \partial/\partial x & \partial/\partial y & \partial/\partial z\\ x^2y & z & xyz\end{vmatrix} = (xz-1)\boldsymbol{i} - yz\boldsymbol{j} - x^2\boldsymbol{k}$$

Gauss's and Stokes's Theorems. Maxwell's equations for electromagnetic fields are derived using the concepts of vector calculus applied to Faraday's law, Ampere's law, and Gauss's laws for electric and magnetic fields. The derivation is accomplished using Stokes's theorem and Gauss's divergence theorem.

The divergence theorem of Gauss provides a transformation of volume integrals into surface intervals, and conversely. Given a vector function $\boldsymbol{F}$ with continuous first partial derivatives in a region R bounded by a closed surface S

$$\iiint_R \nabla\cdot\boldsymbol{F}\,dV = \iint_S \boldsymbol{F}\cdot\boldsymbol{n}\,dS \tag{79}$$

where $\boldsymbol{n}$ is the outer unit normal to S. Physically, the flux of $\boldsymbol{F}$ across a closed surface is the integral of the divergence of $\boldsymbol{F}$ over the region.

Stokes's theorem provides a transformation of surface integrals into line integrals, and vice versa. The surface integral of the normal component of curl $\boldsymbol{F}$ over S equals the line integral of the tangential component of $\boldsymbol{F}$ taken along the simple (nonintersecting) closed curve C, which forms the boundary of the open surface S

$$\iint_S (\nabla\times\boldsymbol{F})\cdot\boldsymbol{n}\,dS = \oint_C \boldsymbol{F}\cdot d\boldsymbol{r} \tag{80}$$

where $\boldsymbol{r}$ is the position vector of the point on C. Stokes's theorem is a generalization of Green's theorem to three dimensions.

Singularity Functions in Engineering

Although strictly speaking they are not part of calculus, there are several singularity functions used in engineering problems worth examining while the subjects of differentiation and integration are explored. Two of the most common singularity functions are the unit step, $u(t)$, and the unit impulse or delta function, $\delta(t)$. The unit step is defined as

$$u(t-\tau) = \begin{cases}0, & t < \tau\\ 1, & t > \tau\end{cases} \tag{81}$$

The unit step function is discontinuous at $t = \tau$, where it abruptly jumps from zero to unity. Two unit step functions are oftentimes combined into a gate function as $u(t-\tau) - u[t-(\tau+T)]$, which is a pulse of period T. The delta function is a pulse of infinitesimal width and area (strength) of one, and it is defined as

$$\int_{-\infty}^{\infty} f(t)\delta(t-\tau)\,dt = f(\tau) \tag{82}$$

Hence, the unit step function is the integral of the unit impulse:

$$u(t) = \int_{-\infty}^{t} \delta(x)\,dx \tag{83}$$

The integration of the step function results in a ramp function.

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