EIGENVALUES AND EIGENFUNCTIONS

DEFINITION OF EIGENVALUE AND EIGENFUNCTION

Many physical system models deal with a square matrix $A = [a_{i,j}]_{n\times n}$ and its eigenvalues and eigenvectors. The eigenvalue problem aims to find a nonzero vector $x = [x_1]_{1\times n}$ and scalar λ such that they satisfy the following equation:

$$Ax = \lambda x \tag{1}$$

where λ is the *eigenvalue* (or characteristic value or proper value) of matrix A, and x is the corresponding right *eigenvector* (or characteristic vector or proper vector) of A.

The necessary and sufficient condition for Eq. (1) to have a nontrivial solution for vector x is that the matrix $(\lambda I - A)$ is singular. Equivalently, the last requirement can be rewritten as a *characteristic equation* of A:

$$\det(\lambda I - A) = 0 \tag{2}$$

where I is the identity matrix. All n roots of the characteristic equation are all n eigenvalues $[\lambda_1, \lambda_2, \ldots, \lambda_n]$. Expansion of $\det(\lambda I - A)$ as a scalar function of λ gives the *characteristic polynomial* of A:

$$L(\lambda) = a_n\lambda^n + a_{n=1}\lambda^{n-1} + \cdots + a_1\lambda + a_0 \tag{3}$$

where λ^k, $k = 1, \ldots, n$, are the corresponding kth powers of λ, and a_k, $k = 0, \ldots, n$, are the coefficients determined via the elements a_{ij} of A.

Each eigenvalue also corresponds to a left eigenvector l, which is the right eigenvector of matrix A^T where the superscript T denotes the transpose of A. The left eigenvector satisfies the equation

$$(\lambda I - A^T)l = 0 \tag{4}$$

The set of all eigenvalues is called the *spectrum* of A.

Eigenfunction is defined for an operator in the functional space. For example, oscillations of an elastic object can be described by

$$\varphi'' = L\varphi \tag{5}$$

where $L\varphi$ is some differential expression. If a solution of Eq. (5) has the form $\varphi = T(t)u(x)$, then with respect to function $u(x)$, the following equation holds:

$$L(u) + \lambda u = 0 \tag{6}$$

In a restricted region and under some homogenous conditions on its boundary, parameter λ is called eigenvalue, and nonzero solutions of Eq. (6) are called eigenfunctions. More descriptions of this eigenfunction are given in the sequel (1–7).

Along with the eigenvalues, *singular values* are often used. If a matrix ($m \times n$) can be transformed in the following form:

$$U^*AV = \begin{bmatrix} S & 0 \\ 0 & 0 \end{bmatrix}, \text{ where } S = \text{diag}[\sigma_1, \sigma_2, \ldots, \sigma_r] \tag{7}$$

where U and V are ($m \times m$) and ($n \times n$) orthogonal matrix respectively, and all $\sigma_k \geq 0$, then expression (7) is called a singular value decomposition. The values $\sigma_1, \sigma_2, \ldots, \sigma_r$ are called singular values of A, and r is the rank of A. If A is a symmetric matrix, then matrices U and V coincide, and σ_k are equal to the absolute values of eigenvalues of A. The singular decomposition (7) is often used in the least square method, especially when A is ill conditioned (1), where condition number of a square matrix is defined as $k(A) = \|A^{-1}\| \cdot \|A\|$; a large $k(A)$ or ill-conditioned A is unwanted when solving linear equations, since a small variation in the system during computation causes a large displacement in the solution.

SOME PROPERTIES OF EIGENVALUES AND EIGENVECTORS

Eigenvectors corresponding to distinct eigenvalues are linearly independent.

Eigenvalues of a real matrix appear as real numbers or complex conjugate pairs.

A symmetric real matrix has all real eigenvalues.

The product of all eigenvalues of A is equal to the determinant of A; in other words,

$$\lambda_1\lambda_2, \ldots, \lambda_n = \det A \tag{8}$$

Eigenvalues of a triangle or diagonal matrix are the diagonal components of the matrix.

The sum of all eigenvalues of a matrix is equal to its *trace;* that is,

$$\lambda_1 + \lambda_2 + \cdots + \lambda_n = \text{tr}A = a_{11} + a_{22} + \cdots + a_{nn} \tag{9}$$

Eigenvalues for A^k are $\lambda_1^k, \lambda_2^k, \ldots, \lambda_n^k$, e.g., eigenvalues for A^{-1} are $\lambda_1^{-1}, \lambda_2^{-1}, \ldots, \lambda_n^{-1}$.

A symmetric matrix A can be put in a diagonal form with eigenvalues as the elements along the diagonal as shown below:

$$A = T\Lambda T^* = T\,\text{diag}[\lambda_1, \lambda_2, \ldots, \lambda_n]T^* \tag{10}$$

where $T = [t_{ij}]_{n\times n}$ is the *transformation matrix* and T^* is its complex conjugate transpose matrix, $T^* = [t_{ji}^*]_{n\times n}$.

However non-semi-simple matrices cannot be put into diagonal form, though, they can be put into the so-called *Jordan form.* For a non-semi-simple *multiple eigenvalue* λ, the eigenvector u^1 is dependent on ($m - 1$) generalized eigenvectors $u^2, \ldots, u^m$:

$$\begin{aligned} Au^1 &= \lambda u^1 \\ Au^2 &= \lambda u^2 + u^1 \\ &\ldots \\ Au^m &= \lambda u^m + u^{m-1} \\ Au^{m+1} &= \lambda_{m+1} u^{m+1} \\ &\ldots \\ Au^n &= \lambda_n u^n \end{aligned} \tag{11}$$

From these equations, the matrix form can be obtained as follows:

$$T^{-1}AT = J \tag{12}$$

where J is a matrix containing a *Jordan block,* and T is the *modal matrix* containing the generalized eigenvectors, $T = [u^1u^2, \ldots, u^m, \ldots, u^n]$. For example, when the number of multiple eigenvalues is 3, matrix J takes the form:

$$J = \begin{bmatrix} \lambda & \delta & & & & \\ & \lambda & \delta & & 0 & \\ & & \lambda & & & \\ & & & \lambda_{m+1} & & \\ & 0 & & & \lambda_{m+2} & \\ & & & & & \lambda_n \end{bmatrix} \tag{13}$$

where $\delta = 0$ or 1 (1).

EIGENVALUE ANALYSIS FOR ORDINARY DIFFERENTIAL EQUATIONS

The eigenvalue approach is applied to *solving the ordinary differential equations* (ODE) given in the following linear form:

$$\frac{dx}{dt} = Ax + Bu \tag{14}$$

where A is the *state matrix* and u is the vector of controls. When A is a matrix with all different eigenvalues λ_i and Eq. (14) is homogeneous (that is $u = 0$), then a solution of Eq. (14) can be found in the following general form:

$$x(t) = \sum_{l=1}^{n} c_i e^{\lambda_i t} \tag{15}$$

where c_i are coefficients that are determined by the initial conditions $x(0)$. For the case of $m < n$ different eigenvalues, the general solution of Eq. (14) for $u = 0$ is

$$x(t) = \sum_{i=1}^{m} \sum_{k=0}^{K_m-1} c_{ik} t^k e^{\lambda_i t} \tag{16}$$

where K_m is the multiplicity of λ_1. If the system is inhomogeneous (that is u is nonzero), a solution of Eq. (14) can be found as a sum of a general solution for the homogeneous system (15) and (16) and a particular solution of the inhomogeneous system.

The elements of Eqs. (15) and (16) corresponding to each real eigenvalue $\lambda_i = \alpha_i$ or to each pair of complex conjugate eigenvalues $\lambda_i = \alpha_i \pm j\omega_i$ are called *aperiodic and oscillatory modes* of the system motion, respectively. The eigenvalue real part α_i is called damping of the mode i, and the imaginary part ω_i determines the frequency of oscillations.

When A is a matrix with all different eigenvalues, by substituting

$$x = Tx', u = Tu' \tag{17}$$

the original ODE can be transformed into

$$T dx'/dt = ATx' + Tu' \tag{18}$$

If T is a nonsingular matrix chosen so that

$$T^{-1}AT = \Lambda = \text{diag}[\lambda_1, \lambda_2, \ldots, \lambda_n] \tag{19}$$

we get a modal form of ODE:

$$dx'/dt = T^{-1}ATx' + u' = \Lambda x' + u' \tag{20}$$

In the modal form, state variables x' and equations are independent, and T is the eigenvector matrix. Diagonal elements of the matrix Λ are eigenvalues of A, which can be used to solve ODE (1).

For a general nth order differential equation,

$$A_n \frac{d^n x}{dt^n} + A_{n-1}\frac{d^{n-1}x}{dt^{n-1}} + \cdots + A_1 \frac{dx}{dt} + A_0 x = 0 \tag{21}$$

Besides solving it through transferring it into a set of first order differential equations (1,5), it can also be solved using the original coordinate. The matrix polynomial of system (21) follows:

$$L(\lambda) = A_n\lambda^n + A_{n-1} + \cdots + A_1\lambda + A_0 \tag{22}$$

The solutions and eigenvalues as well as eigenvectors of the system (21) can be obtained by solving the eigenvalue equation:

$$L(\lambda)u = 0 \tag{23}$$

where $L(\lambda)$ is the matrix (22) containing an eigenvalue λ having the corresponding eigenvector u. If vectors u^1, u^2, . . ., u^m, where $m < n$, satisfies the equation:

$$\begin{aligned} &L(\lambda)u^1 = 0 \\ &L(\lambda)u^2 + \frac{1}{1!}\frac{dL(\lambda)}{d\lambda}u^1 = 0 \\ &\cdots \\ &L(\lambda)u^m + \frac{1}{1!}\frac{dL(\lambda)}{d\lambda}u^{m-1} + \cdots + \frac{1}{(m-1)!}\frac{d^{m-1}L(\lambda)}{d\lambda^{m-1}}u^1 = 0 \end{aligned} \tag{24}$$

then

$$x(t) = [t^{m-1}u^1/(m-1)! + \cdots + tu^{m-1}/1! + u^m]e^{\lambda_1} \tag{25}$$

is a solution to the ODE system (1). The set of equations (24) defines the *Jordan Chain* of the multiple eigenvalue λ and the eigenvector u^1.

EIGENVALUES AND EIGENFUNCTIONS FOR INTEGRAL EQUATIONS

An integral equation takes the following general form (1):

$$x(t) = f(t) + \lambda \int_0^t k(\zeta, t)x(\zeta)\, d\zeta \tag{26}$$

In eigenanalysis, we concentrate on the integral equation, which can be rewritten as

$$x(t) = f(t) + \lambda \int_a^b \sum_{i=1}^n r_i(\zeta)q_i(\zeta)x(\zeta)\, d\zeta \tag{27}$$

where

$$\int_a^b r_i(\zeta)q_l(\zeta)x(\zeta)\, d\zeta = \text{const.} \tag{28}$$

So Eq. (27) can be reduced to the following problem:

$$x(t) = f(t) + \lambda \sum_{j=1}^n x_j q_j(t) \tag{29}$$

By substitution, we have:

$$x_i = \int_a^b r_i(\zeta)[f(t) + \lambda \sum_{j=1}^n x_j q_j(t)]\, d\zeta, i = 1, 2, \ldots, n \tag{30}$$

The equation of the system can be obtained as

$$(I - \lambda A)x = b \tag{31}$$

where

$$\begin{aligned} x &= [x_1, x_2, \ldots, x_n]^T \\ A &= [a_{i,j}] = \left[\int_a^b r_l(\zeta)q_j(\zeta)\, d\zeta\right] \\ b &= [b_i] = \left[\int_a^b r_l(\zeta)q_j(\zeta)\, d\zeta\right] \end{aligned}$$

The values of λ which satisfy

$$\text{der}[I - \lambda A] = 0 \tag{32}$$

are the eigenvalues of the integral equation.

To find $x(t)$ by solving an integral equation similar to (26) except for the interval, which is $[a, b]$ instead of $[0, t]$, the eigenfunction approach can also be used. First, $x(t)$ is rewritten as

$$x(t) = f(t) + \sum_{n=1}^{\infty} a_n\phi_n(t) \tag{33}$$

where $\phi_1(t)$, $\phi_2(t)$, . . . are eigenfunctions of the system, and satisfy

$$\phi(t) = \lambda_n \int_a^b k(\zeta, t)\phi_n(\zeta)\, d\zeta \tag{34}$$

where λ_1, λ_2, . . . are eigenvalues of the integral equation. After substituting the eigenfunction into the integral equation and further simplification, the solution $x(t)$ is obtained as

$$x(t) = f(t) + \sum_{n=1}^{\infty} \frac{\lambda f_n}{\lambda_n - \lambda}\phi_n(t) \tag{35}$$

where $f_n = \int_a^b f(\zeta)\phi(\zeta)d\zeta$.

LINEAR DYNAMIC MODELS AND EIGENVALUES

State Space Modeling

In control systems, where the purpose of control is to make a variable adhere to a particular value, the system can be modeled by using the *state space equation* and *transfer functions.* The state space equation is

$$\begin{aligned} \dot{x} &= Ax + Bu \\ y &= Cx + Du \end{aligned} \tag{36}$$

where x is the $(n \times 1)$ vector of *state variables,* $\dot{x}$ is its first-order derivative vector, u is the $(p \times 1)$ *control vector,* and y is the $(q \times 1)$ *output vector.* Accordingly, A is the $(n \times n)$ *state matrix,* B is a $(n \times p)$ matrix, C is a $(q \times n)$ matrix, and D is of $(q \times p)$ dimension.

Model Analysis on the Base of Eigenvalues and Eigenvectors

Model analysis is based on the *state space representation* (36). It also explores *eigenvalues, eigenvectors,* and *transfer functions* (8–10).

Consider a case where matrix D is a zero matrix. Then the state space model can be transformed using *Laplace transformation* in a *transfer function* that maps input into output:

$$G(s) = C(sI - A)^{-1}B \tag{37}$$

where s is the Laplace complex variable, and $G(s)$ is composed of denominator $a(s)$ and a numerator $b(s)$:

$$\begin{aligned} G(s) &= b(s)/a(s) \\ &= (b_0 s^n + b_1 s^{n-1} + \cdots + b_n)/(s^n + a_1 s^{n-1} + \cdots + a_n) \end{aligned} \tag{38}$$

The *closed-loop transfer function* for a *feedback system* is

$$G_c(s) = [I + G(s)H(s)]^{-1}G(s) \tag{39}$$

where $H(s)$ is a feedback transfer function.

The system model (36) can be analyzed using the observability and controllability concepts. *Observability* indicates whether all the system's modes can be observed by monitoring only the sensed outputs. *Controllability* decides whether the system state can be moved from an initial point to any other point in the state space within infinite time, and if every mode is connected to the controlled input. The concepts can be described more precisely as follows (8,9,11):

1. For a linear system, if within an infinite time interval, $t_0 < t < t_1$, there exists a piecewise continuous control signal $u(t)$, so that the system states can be moved from any initial mode $x(t_0)$ to any final mode $x(t_1)$, then the system is said to be controllable at the time t_0. If every system mode is controllable, then the system is state controllable. If at least one of the states is not controllable, then the system is not controllable.
2. For a linear system, if within an infinite time interval, $t_0 < t < t_1$, every initial mode $x(t_0)$ can be observed exclusively by the sensed value $y(t)$, then the system is said to be fully observable.

Matrix transformations are required to assess observability and controllability. To study controllability, it is necessary to introduce the control canonical form as follows:

$$A_c = \begin{bmatrix} -a_1 & -a_2 & \dots & -a_n \\ 1 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 1 & 0 \end{bmatrix}, B_c = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{40}$$
$$C_c = [b_1 \quad b_2 \quad \dots \quad b_n], \quad D_c = 0$$

where the subscript c denotes that the associated matrix is in control canonical form.

For a linear time-invariant system, the necessary and sufficient condition for the system state controllability is the full rank of controllability matrix Q_c. The controllability matrix is

$$Q_c = [B \vdots AB \vdots \cdots \vdots A_{n-1}B] \tag{41}$$

and the system is controllable if and only if Rank $Q_c = n$.

When the linear time-invariant system has distinct eigenvalues, then after the modal transformation, the new system becomes

$$\dot{z} = T^{-1}ATz + T^{-1}Bu \tag{42}$$

where T^{-1}AT is diagonal matrix. Under such condition, the sufficient and necessary conditions for state controllability is that there are no rows in the matrix $T^{-1}B$ containing all zero elements.

When matrix A has multiple eigenvalues, and every multiple eigenvalue corresponds to the same eigenvector, then the system can be transformed into the new state space form, which is called the *Jordan canonical form:*

$$\dot{z} = Jz + T^{-1}Bu \tag{43}$$

where the matrix J is Jordan canonical matrix. Then the sufficient and necessary condition for state controllability is that not all the elements in the matrix $T^{-1}B$, corresponding to the last row of every Jordan sub-matrix in matrix J, are zero.

The *output controllability sufficient and necessary condition* for linear time-invariant system is that the matrix $[CB \vdots CAB \vdots \cdots \vdots CA^{n-1}B \vdots D]$ is full rank; that is,

$$\text{rank}[CB \vdots CAB \vdots \cdots \vdots CA^{n-1}B \vdots D] = n \tag{44}$$

Similarly, the sufficient and necessary observability condition for linear time-invariant system is that the observability matrix is full rank; that is,

$$Q_D = [C \vdots CA \vdots \cdots \vdots CA^{n-1}]^T \tag{45}$$

and *rank* $Q_0 = n$. When the system has distinct eigenvalues, then after a linear nonsingular transformation, the system takes the form (when control vector u is zero):

$$\begin{aligned} \dot{z} &= T^{-1}ATz \\ y &= CTz \end{aligned} \tag{46}$$

then the condition for observability is that there are no rows in the matrix CT which have only zero elements. Even though the system has multiple eigenvalues, and every multiple eigenvalue corresponds to the same eigenvector, the system after transformation looks like

$$\begin{aligned}\dot{z} &= Jz \\ y &= CTz\end{aligned} \tag{47}$$

where J is the Jordan matrix. The observability condition is that there are no columns corresponding to the first row of each Jordan submatrix having only zero elements.

EIGENVALUES AND STABILITY

Since the time-dependent characteristic of a mode corresponding to an eigenvalue λ_i is given by $e^{\lambda_i t}$, the stability of the system matrix can be determined by the eigenvalues of the system state matrix, as in the following (see Ref. 37).

A real eigenvalue corresponds to a nonoscillatory mode. A negative real eigenvalue represents a decaying mode. The larger its magnitude, the faster the decay. A positive real eigenvelue represents aperiodic instability.

Complex eigenvalues occur in conjugate pairs, and each pair corresponds to an oscillatory mode. The real component of the eigenvalues gives the damping, and the imaginary component defines the frequency of oscillation. A negative real part indicates a damped oscillation and a positive one represents oscillation of increasing amplitude. For a complex pair of eigenvalues, $\lambda = -\sigma \pm j\omega$, the frequency of oscillation in hertz can be calculated by

$$f = \omega/2\pi \tag{48}$$

which represents the actual or damped frequency. The damping ratio is given by

$$\zeta = \frac{\sigma}{\sqrt{\sigma^2 + \overline{\omega}^2}} \tag{49}$$

From the point of view of a system modeled by a transfer function, the concept of natural frequency is given based on complex poles which correspond to the complex eigenvalues of the state matrix A, as in Eq. (36). Let the complex poles be $s = -\sigma \pm j\omega$, and the denominator corresponding to them be $d(s) = (s + \sigma)^2 + \omega^2$. Then its transfer function is represented in polynomial form as $H(s) = \omega_n^2/(s^2 + 2\zeta\omega_n s + \omega_n^2)$, where $\sigma = \zeta\omega_n$ and $\omega = \omega_n\sqrt{(1 - \zeta^2)}$. This introduces the definition of the undamped natural frequency, ω_n, and again the damping ratio, ζ.

More fundamentally, the Lyapunov stability theory forms a basis for stability analysis. There are two approaches to evaluate system stability (4,8,9,11,12):

1. the first Lyapunov method and
2. the second Lyapunov method.

The *first Lyapunov method* is based on eigenvalue and eigenvector analysis for linearized systems and small disturbances. It finds its application in many areas, for example, in the area of power systems engineering.

To study small-signal stability, it is necessary to clarify some basic concepts regarding the following *Differential-Algebraic Equation* (DAE):

$$\begin{aligned}\dot{x} &= f(x, y, p) \quad f: R^{n+m+q} \to R^n \\ 0 &= g(x, y, p) \quad g: R^{m+n+q} \to R^m\end{aligned} \tag{50}$$

where $x \subset R^n$, $y \subset R^m$, $p \subset R^q$; x is the vector of dynamic state variables, y is the vector of static or instantaneous state variables, and p is a selected system parameter affecting the studied system behavior. Variable y usually represents a state variable whose dynamics is instantaneously completed as compared to that of the dynamic state variable x. Parameter p belongs to the system parameters which have no dynamics at all at least if modeled by Eq. (50) (13). For example, in power system engineering, typical dynamic state variables are chosen from the time-dependent variables such as machine angle and machine speed. The static variables are the load flow variables including bus voltages and angles. Parameter p can be selected from static load powers, or control system parameters.

A system is said to be in its *equilibrium condition* when the derivatives of its state variables are equal to zero, which means there is no variation of the state variables. For the system modeled in Eq. (50), this condition is given as follows:

$$\begin{aligned}0 &= f(x, y, p) \\ 0 &= g(x, y, p)\end{aligned} \tag{51}$$

Solutions (x_0, y_0, p_0), of the preceding system are the system *equilibrium points.* Small-signal stability analysis uses the system represented in linearized form, which is done by differentiating the original system respect to the system variables and parameters around its equilibrium point (x_0, y_0, p_0). This linearization is necessary for the Lyapunov method and via computing system eigenvalues and eigenvectors.

For the original system (50), its linearized form is given in the following:

$$\begin{aligned}\Delta\dot{x} &= \frac{\partial f}{\partial x}\Delta x + \frac{\partial f}{\partial y}\Delta y \\ 0 &= \frac{\partial g}{\partial x}\Delta x + \frac{\partial g}{\partial y}\Delta y\end{aligned} \tag{52}$$

For simplicity, system (52) is rewritten as

$$\begin{aligned}\Delta\dot{x} &= A\Delta x + B\Delta y \\ 0 &= C\Delta x + D\Delta y\end{aligned} \tag{53}$$

where matrices A, B, C, and D are the partial derivatives' matrices. If the algebraic matrix D is not singular (i.e., det $D \neq 0$), the state matrix A_s is given as

$$A_s = A - BD^{-1}C \tag{54}$$

which is studied in stability analysis using the eigenvalue and eigenvector approach.

The use of the first Lyapunov method involves the following steps (12,14,15):

1. linearization of the original system (50) as in (52);
2. elimination of the algebraic variables to form the reduced dynamic state matrix A_s;

3. computation of the eigenvalues and eigenvectors of the state matrix A_s;
4. stability study of the system (16):
   a. If eigenvalues of the state matrix are located in the left-hand side of the complex plane, then the system is said to be small-signal stable at the studied equilibrium point;
   b. If the rightmost eigenvalue is zero, the system is on the edge of small-signal aperiodic instability;
   c. If the rightmost complex conjugate pair of eigenvalues has a zero real part and a nonzero imaginary part, the system is on the edge of oscillatory instability depending on the transversality condition (17);
   d. If the system has eigenvalue with positive real parts, the system is not stable;
   e. For the stable case, analyze several characteristics including damping and frequencies for all modes, eigenvalue sensitivities to the system parameters, excitability, observability, and controllability of the modes.

More precise definitions for the first Lyapunov method have been addressed in the literature (16,18). The general time-varying or nonautonomous form is as follows:

$$\dot{x} = f(t, x, u) \tag{55}$$

where t represents time, x is the vector of state variables, and u is the vector of system input. In a special case of the system (55), f is not explicitly dependent on time t; that is,

$$\dot{x} = f(x) \tag{56}$$

and the system is said to be autonomous or time-invariant. Such a system does not change its behavior at different times (16).

An equilibrium point x_0 of the autonomous system (56) is

1. stable if, for each $\epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$ such that $\|x(0)\| < \delta \Rightarrow \|x(t)\| < \epsilon$, $\forall t \geq 0$;
2. unstable otherwise;
3. asymptotically stable, if it is stable, and δ can be chosen such that: $\|x(0)\| < \delta \Rightarrow \lim_{t\to\infty} x(0) = x_0$.

The definition can be represented in the form of eigenvalue approach as given in the Lyapunov first-method theorem.

Let x_0 be an equilibrium point for the autonomous system (54), where f: $D \to R^n$ is continuously differentiable and D is a neighborhood of the origin. Let the system Jacobian be $A = (\partial f/\partial x)(x)|_{x=x_0}$, and $\lambda = [\lambda_1, \lambda_2, \ldots, \lambda_n]$ be the eigenvalues of A, then the origin is asymptotically stable if Re $\lambda_i < 0$ for all eigenvalues of A, or the origin is unstable if Re $\lambda_i > 0$ for one or more eigenvalues of A.

Let us take one step further. The stability definition for a time-varying system, where the system behavior depends on the origin at the initial time t_0, is as follows. The equilibrium point x_0 for the system (55) is (16)

1. stable, if for each $\epsilon > 0$, there exists $\delta = \delta(\epsilon, t_0) > 0$ such that $\|x(t_0)\| < \delta \Rightarrow \|x(t)\| < \epsilon$, $\forall t \geq t_0 \geq 0$;
2. uniformly stable, if for each $\epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$, such that $\|x(t_0)\|\ \delta \Rightarrow \|x(t)\| < \epsilon$, $\forall t \geq t_0 \geq 0$;
3. unstable otherwise;
4. asymptotically stable, if it is stable and there is $c = c(t_0) > 0$ such that, for all $\|x(t_0)\| < c$, $\lim_{t\to\infty} x(t) = x_0$;
5. uniformly asymptotically stable if it is uniformly stable and there is a time invariant $c > 0$, such that for all $\|x(t_0)\| < c$, $\lim_{t\to\infty} x(t) = x_0$. This holds for each $\epsilon > 0$, if there is $T = T(\epsilon) > 0$, such that $\|x(t)\| < \epsilon$, $\forall t \geq t_0 + T(\epsilon)$, $\forall \|x(t_0)\| < c$;
6. globally uniformly asymptotically stable if it is uniformly stable and for each pair of positive numbers ϵ and c, there is a $T = T(\epsilon, c) > 0$, such that $\|x(t)\| < \epsilon$, $\forall t \geq t_0 + T(\epsilon, c)$, $\forall \|x(t_0)\| < c$.

The corresponding stability theorem follows.

Let $f(t, x, u)|_{(t^*, x^*, u^*)} = 0$ be an equilibrium point for the nonlinear time-varying system (55), where f: $[0, \infty) \times D \to R^n$ is continuously differentiable, $D = \{x \in R^n \| \|x\|_2 < r\}$, the Jacobian matrix is bounded and Lipschitz on D, uniformly in t. $A(t) = (\partial f/\partial x)(t, x)|_{x=x_0}$ is the Jacobian; then the origin is exponentially stable for the nonlinear system if it is an exponentially stable equilibrium point for the linear system $\dot{x} = A(t)x$.

The second Lyapunov method isa potentially most reliable and powerful method for the original nonlinear and nonautonomous (or time-varying) systems. But it relys on the Lyapunov function, which is hard to find for many physical systems.

EIGENVALUES AND BIFURCATIONS

The bifurcation theory has a rich mathematical description and literature for various areas of applications. Many physical systems can be modeled by the general form

$$x' = f(x, p) \tag{57}$$

where x is vector of the system state variables, and p is the system's parameter, which may vary during system operation in normal as well as contingency conditions.

Bifurcations occur where, by slowly varying certain system parameters in some direction, the system properties change qualitatively or quantitatively at a certain point (14,19). Local bifurcations can be detected by monitoring the behavior of eigenvalues of the systems operation point. In some direction of parameter variation, the system may become unstable because of the singularity of the system dynamic state matrix associated with zero eigenvalue or because of a pair of complex conjugate eigenvalues crossing the imaginary axes of the complex plane. These two phenomena are saddle node and Hopf bifurcations, respectively. Other conditions that may drive the system state into instability may also occur. These include singularity-induced bifurcations, cyclic fold, period doubling, and blue sky bifurcations or even chaos (15,19,20).

For the general system (57), a point (x_0, p_0) is said to be a *saddle node bifurcation* point if it is an equilibrium point of the system; in other words, $f(x_0, p_0) = 0$, the system Jacobian matrix, $f_y(x_0, p_0)$ has a simple zero eigenvalue $\lambda(p_0) = 0$, and the transversal conditions hold (17,21). More generally (24), the saddle bifurcation satisfy the following conditions:

1. The point is the system's equilibrium point [i.e., $f(x_0, p_0) = 0$].
2. The Jacobian matrix, $f_y(x_0, p_0)$ has a simple and unique eigenvalue $\lambda(p_0) = 0$ with the corresponding right and left eigenvectors l and r, respectively.
3. Transversality condition of the first-order derivative: $l^T f_y(x_0, p_0) \neq 0$
4. Transversality condition of the second-order derivative: $l^T[f_{yy}(x_0, p_0)r]r \neq 0$

A Hopf bifurcation occurs when the following conditions are satisfied:

1. The point is a system operation equilibrium point [i.e., $f(x_0, p_0) = 0$];
2. The Jacobian matrix $f_y(x_0, p_0)$ has a simple pair of pure imaginary eigenvalues $\lambda(p_0) = 0 \pm j\omega$ and no other eigenvalues with zero real part;
3. Transversality condition: $d[\mathrm{Re}\ \lambda(p_0)]/dp \neq 0$.

The last condition guarantees the transversal crossing of the imaginary axis. The sign of $d[\mathrm{Re}\ \lambda(p_0)]/dp$ determines whether there is a birth or death of a *limit cycle* at (x_0, p_0). Depending on the direction of transversal crossing the imaginary axis, Hopf bifurcation can be further categorized into supercritical and subcritical ones. The *supercritical Hopf bifurcation* happens when the critical eigenvalue moves from the left half plane to the right half plane. The *subcritical Hopf bifurcation* occurs when the eigenvalue moves from the left half plane to the right half plane and is unstable. The system transients are diverged into an oscillatory style at the vicinity of the subcritical Hopf bifurcation points.

Singularity-induced bifurcations occur when the system's equilibrium approaches singularity, and some of the system eigenvalues become unbounded along the real axis (i.e., $\lambda_i \to \infty$). In case of the DAE model (50), the singularity of the algebraic Jacobian $D = g_y$ causes the singularity-induced bifurcations. In that case, singular perturbations or noise techniques must be used to analyze the system dynamics (22). When singularity-induced bifurcation occurs, the system behavior becomes hardly predictable and may cause fast claps type instability (22).

A graphical illustration of these three major bifurcations is given in Fig. 1.

Methods of computing bifurcations can be categorized into direct and indirect approaches. The direct method has been practiced by many researchers in this area (13–15,17,19, 20,22–32). For example, the direct method computes the Hopf bifurcation condition by solving directly the set of equations (15,17,20,26):

$$f(x, p_0 + \tau\Delta p) = 0 \tag{58}$$

$$A'_s(x, p_0 + \tau\Delta p)l' + \omega l'' = 0 \tag{59}$$

$$A'_s(x, p_0 + \tau\Delta p)l'' - \omega l' = 0 \tag{60}$$

$$\|l\| = 1 \tag{61}$$

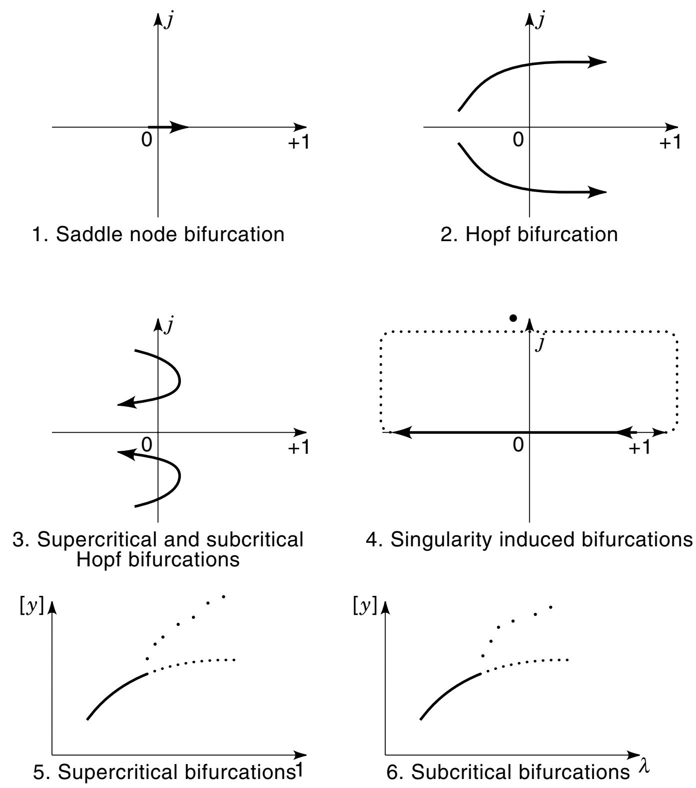

Figure 1. Bifurcation diagrams for different bifurcations. +1, real axis; j, imaginary axis; 1–4: eigenvalue trajectories as a result of system parameter variation; 5, 6: system state variable branch diagrams. The branching properties of the system state variable movement determine the type of bifurcations.

where $A_s = A - BD^{-1}C = f_x - f_y g_y^{-1} g_x$ is the state matrix, $0 + j\omega$ is its eigenvalue, $l = l' + jl''$ is the corresponding left eigenvector, $p = p_0 + \tau\Delta p$ is the system parameter vector varying from the point p_0 in direction Δp. By taking zero ω and l'', saddle node bifurcation can be computed as well.

Indirect methods are mainly Newton–Raphson type method using predictor and corrector to trace the bifurcation diagram. A detailed description of the continuation methods can be found in Refs. 17, 23, 25–27, and 33.

As an example of applied bifurcation analysis, let us consider a task from the area of power system analysis (20,26). The power system model is composed of two generators and one load bus. The system is shown in Fig. 2.

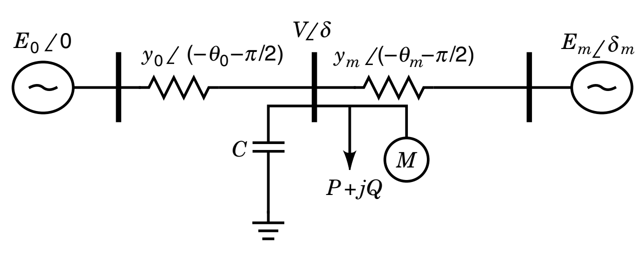

Figure 2. A simple power system model. The system dynamics are introduced mainly by the induction motor and generators.

Static and an induction motor load are connected with the load bus in the middle of the network. A capacitor device is also connected with the same bus to provide reactive power supply and control the voltage magnitude; E and δ_m are generator terminal voltage and angle, respectively; V is load bus voltage; δ is load bus voltage angle; Y is line conductance; and M stands for induction motor load. The system is modeled by the following equations:

$$
\begin{aligned}
\delta'_{\mathrm{m}} &= \omega \\
M\dot{\omega} &= -\delta_{\mathrm{m}}\omega + P_{\mathrm{m}} \\
&\quad + E_{\mathrm{m}} y_{\mathrm{m}} V \sin(\delta - \delta_{\mathrm{m}} - \theta_{\mathrm{m}}) \\
&\quad + E_{\mathrm{m}}^2 y_{\mathrm{m}} \sin\theta_{\mathrm{m}} \\
K_{\mathrm{qw}}\dot{\delta} &= -K_{\mathrm{qv2}}V^2 - K_{\mathrm{qv}}V + E'_0 y'_0 V \cos(\delta + \theta''_0) \\
&\quad + E_{\mathrm{m}} y_{\mathrm{m}} V \cos(\delta - \delta_{\mathrm{m}} + \theta_{\mathrm{m}}) \\
&\quad - (y'_0 \cos\theta'_0 + y_{\mathrm{m}} \cos\theta'_{\mathrm{m}} V^2 - Q_0 - Q_1 \\
TK_{\mathrm{qw}}K_{\mathrm{pv}}\dot{V} &= K_{\mathrm{qw}}K_{\mathrm{qv}}^2 V^2 + (K_{\mathrm{pw}}K_{\mathrm{qv}} - K_{\mathrm{qw}}K_{\mathrm{pv}})V \\
&\quad + \sqrt{(K_{\mathrm{qw}}^2 + K_{\mathrm{pw}}^2)}[-E'_0 y'_0 V \cos(\delta + \theta'_0 - h) \\
&\quad - E_{\mathrm{m}} y_{\mathrm{m}} V \cos(\delta - \delta_{\mathrm{m}} + \theta_{\mathrm{m}} - \eta) + (y'_0 \cos(\theta'_0 - \eta) \\
&\quad + y_{\mathrm{m}} \cos(\theta_{\mathrm{m}} - \eta))V^2] - K_{\mathrm{qw}}(P_0 + P_1) \\
&\quad + K_{\mathrm{pw}}(Q_0 + Q_1)
\end{aligned}
$$

where $\eta = \tan^{-1}(K_{\mathrm{qw}}/K_{\mathrm{pw}})$. The active and reactive loads are featured by the following equations:

$$
\begin{aligned}
P_{\mathrm{d}} &= P_0 + P_1 + K_{\mathrm{pw}}\delta + K_{\mathrm{pv}}(V + TV') \\
Q_{\mathrm{d}} &= Q_0 + Q_1 + K_{\mathrm{qw}}\delta + K_{\mathrm{qv}}V + K_{\mathrm{qv2}}V^2
\end{aligned}
$$

The system parameter Q_1 is selected as the bifurcation parameter to be increased slowly. Voltage V is taken as a dependent parameter for illustration. Figure 3 shows the dynamics for the system in the form of a Q–V curve (19).

The eigenvalue trajectory around a Hopf bifurcation point is given in Fig. 4 where both supercritical and subcritical Hopf bifurcations can be seen.

NUMERICAL METHODS FOR THE EIGENVALUE PROBLEM

Computing Eigenvalues and Eigenvectors

Although roots of the characteristic polynomial $L(\lambda) = a_n\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0$ are eigenvalues of the matrix A, a direct calculation of these roots is not recommended because of the rounded errors and high sensitivity of the roots to coefficients a_i (1).

We start by introducing the *power method,* which locates the largest eigenvalue. Suppose a matrix A has eigenvalues $\Lambda = [\lambda_1, \lambda_2, \ldots, \lambda_n]^{\mathrm{T}}$ and the corresponding right eigenvectors

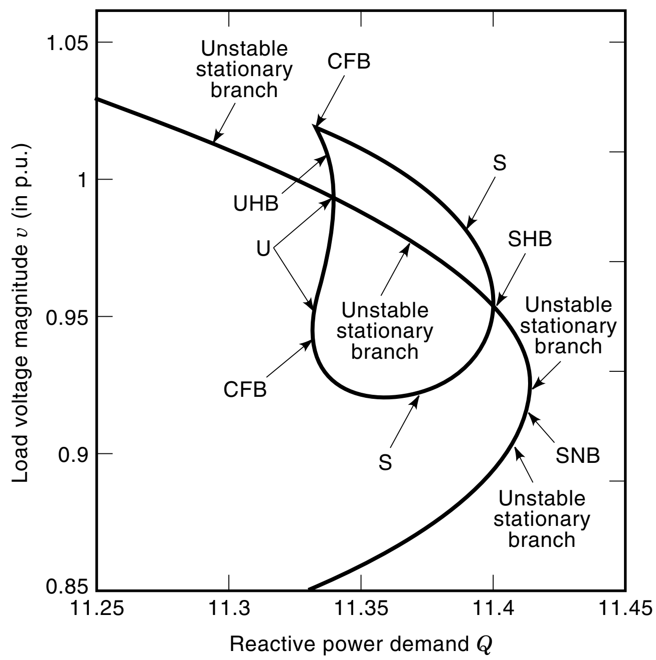

Figure 3. The Q–V curve branch diagrams. S—stable periodic branch; U—unstable periodic branch; SNB—saddle node bifurcation; SHB—stable (supercritical) Hopf bifurcation; UHB—unstable (subcritical) Hopf bifurcation; CFB—cyclic fold bifurcation. These bifurcations are associated with system eigenvalue behavior while the reactive load power Q_1 is consistently increased. This shows that for a simple dynamic system, as given in Fig. 3, stability-related phenomena are very rich.

$R = [r_1, r_2, \ldots, r_n]^{\mathrm{T}}$, and $|\lambda 1| \geq |\lambda_2| \geq \cdots \geq |\lambda_n|$. For any vector $x \neq 0$, we have

$$x = \sum_{i=1}^{n} c_i r_i \tag{62}$$

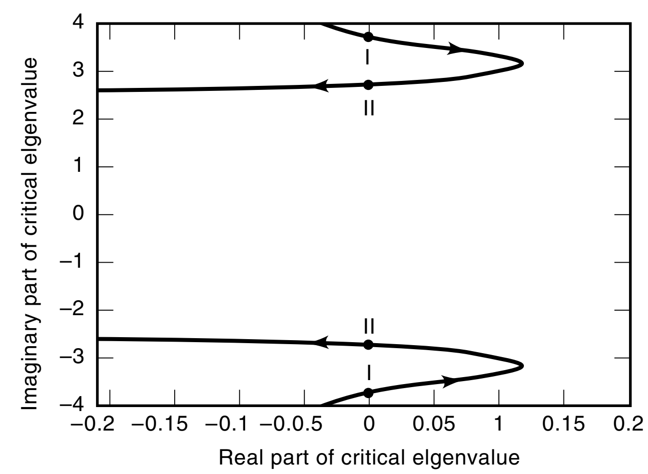

Figure 4. An illustration of subcritical (I) and supercritical (II) Hopf bifurcations. (I) corresponds to movement of the eigenvalue real part from the left to the right side of the s-plane; (II) indicates a reverse movement.

By multiplying (62) by A, A^2, . . ., it can be obtained that

$$
\begin{aligned}
x^{(1)} &= Ax = \sum_{i=1}^{n} c_i \lambda_1 r_i \\
x^{(2)} &= Ax^{(1)} = \sum_{i=1}^{n} c_i \lambda_i^2 r_i \\
&\cdots \\
x^{(m)} &= Ax^{(m-1)} = \sum_{i=1}^{n} c_i \lambda_i^m r_i
\end{aligned}
\tag{63}
$$

After a number of iterations, $x^{(m)} \to \lambda_1^m c_1 r_1$. Therefore, λ_1 can be obtained by dividing the corresponding elements of $x^{(m)}$ and $x^{(m-1)}$ after a sufficient number of iterations, and the eigenvector can be obtained by scaling $x^{(m)}$ directly. Other eigenvalues and eigenvectors can be computed by applying the same method to the new matrix:

$$A_1 = A - \lambda_1 r_1 v^{1*} \tag{64}$$

where v^1 is the reciprocal vector of the first eigenvector r_1. It can be observed that the matrix A_1 has the same eigenvalues as A except the first eigenvalue, which is set to zero by the transformation. By applying the method successively, all eigenvalues and corresponding right eigenvectors of matrix A can be located. The applicability of this method is restricted by computational errors. Convergence of the method depends on separation of eigenvalues determined by the ratios $\|\lambda_i/\lambda_1\|$, $\|\lambda_i/\lambda_2\|$, etc. As evident, the method can compute only one eigenvalue and eigenvector at a time.

The *Schur algorithm* can also be used to locate eigenvalues from λ_2 while knowing λ_1 by applying the power method to A_1 after the following transformation (1):

$$\begin{bmatrix} \lambda_1 & B_1 \\ 0 & A_1 \end{bmatrix} \begin{bmatrix} 1 \\ y^{(2)} \end{bmatrix} = \lambda_2 \begin{bmatrix} 1 \\ y^{(2)} \end{bmatrix} \tag{65}$$

A general idea of the *inverse power method* is to use the power method determining the minimum eigenvalues. By shifts, any eigenvalue can be made the minimum one. The inverse method can compute eigenvectors accurately even when the eigenvalues are not well separated. The method implies the following. Let λ_i^* be an approximation of one of the eigenvalues λ_i of A. The steps involved follow:

- Obtain a tridiagonal matrix T by reduction of matrix A;
- Find zeros of $(T - \lambda_i^* I)^{-1} y_0$, i.e. $\{z_1|(T - \lambda_i^* I)z_1 = y_0)$;
- Set $y_1 = z_1/\|z_1\|$;
- Solve $(T - \lambda_i^* I)^{-1} z_2 = y_1$, etc.

The eigenvector l_i of λ_i is approximated by $y_k = z_k/\|z_k\|$ provided y_0 contains a nonzero term in l_i. If $|\lambda_i^* - \lambda_i|$ is sufficiently small, the inverse iteration method obtains the eigenvector associated with λ_i within only several iterations.

The *QR method* is one of the most popular algorithms for computing eigenvalues and eigenvectors. By using a factorization of the product of a unitary matrix Q and an upper-triangular matrix R, this method involves the following iteration process:

$$
\begin{aligned}
A_1 &= RQ = Q_1 R_1 \\
&\cdots \\
A_i &= R_{i-1} Q_{i-1} = Q_i R_i = Q_{i-1}^{-1} A_{l-1} Q_{l-1}
\end{aligned}
\tag{66}
$$

where the tth unitary matrix Q_t is obtained by solving

$$Q_t^{\mathrm{T}} A = R_t$$

and Q_t^{T} is determined in a factorized form, such as the product of plane rotations or of elementary Hermitians. Then, the matrix $R_t Q_t$ is obtained by successive post-multiplication of R_t with the transposition of the factors of Q_t^{T} (5).

After a number of iterations, diagonal elements of R_m approximate eigenvalues for A (1). To reduce the number of iterations and speed up computations because less computational effort is required at each iteration, the studied matrix A is initially reduced to the Hessenberg form, which is preserved during iterations (1,5).

A more *general form of the eigenvalue problem* can be modeled as

$$Ax = \lambda Bx \tag{67}$$

If A and B are nonsingular, the problem can be transformed into the standard form of eigenvalue problems by expressing

$$B^{-1}Ax = \lambda x \quad \text{or} \quad A^{-1}Bx = \lambda^{-1} x \tag{68}$$

Then methods discussed earlier can be applied to solve the problem.

There are cases when the computation can be simplified (1). When both A and B are symmetric and B is positive definite, matrix B can be decomposed as $B = C^T\mathrm{C}$ where C is a nonsingular triangular matrix. Then the problem can be expressed as

$$Ax = \lambda C^T C x \tag{69}$$

If vector y is chosen so that $y = Cx$, the final transformation is obtained as

$$(C^{\gamma})^{-1} A C^1 y = Gy = \lambda y \tag{70}$$

and the problem is simplified into the eigenvalue problem with matrix G. Techniques dealing with other situations of the generalized eigenvalue problem can be found in Ref. 34.

In many cases, matrix A is a sparse matrix with many zero elements. Different techniques solving the *sparse matrix eigenvalue problem* are derived. The approaches can be categorized into two major branches: (1) problems where the *LU factorization* is possible, or (2) where it is impossible.

In the first case, after transformation of the generalized eigenvalue problem as $B^{-1}Ax = \lambda x$, or $y = L^T x$, so that $L^{-1}AL^{-T}y = \lambda y$, the resulting matrices may not necessarily be sparse. There are several aspects of the problem. First, the matrix should be represented in such a way that it dispenses with zero elements and allows new elements to be inserted as they are generated by the elimination process during the decomposition; second, pivoting must be performed during the elimination process to preserve sparsity and ensure numerical stability (31,35). The power method is sometimes used for large sparse matrix problems to compute eigenvalues.

When matrices A and B become very large, performing the LU factorization for the general eigenvalue problem becomes more and more difficult. In this case, a function should be constructed so that it reaches its minimum at one or more of the eigenvectors, and the problem is to minimize this function with an appropriate numerical method (31). For example, the successive search method can be used to minimize this function. Also, other gradient methods can be employed.

Among all these computation methods for eigenvalue problems, many factors influence the efficiency of a particular method. For a large matrix, whether it is dense or sparse, the power method is suitable when only a few large eigenvalues and corresponding eigenvectors are required. The inverse iteration method is the most robust and accurate in calculating eigenvectors. Nevertheless, the most popular general method for eigenvalue and eigenvector computations is the QR method. However, in many cases, especially when the matrix is Hermitian or real symmetric, many methods can provide satisfactory results.

Localization of Eigenvalues

Along with the direct method based on computation of eigenvalues, there are several indirect methods to determine a domain in the complex plane where the eigenvalues are located. A particular interest for the stability studies is to decide whether all eigenvalues have negative real parts. Some methods can also count the number of stable and unstable modes without solving the general eigenvalue problem. Also, there are methods that determine a bounded region where the eigenvalues are located.

The following algebraic results can help to identify the stability of a matrix (4):

- If the matrix $A \in R_{n\times n}$ is stable and $W \in R_{n\times n}$ is positive (nonnegative) definite, then there exists a real positive (positive or nonnegative) definite matrix V such that $AV' + VA = -W$.
- Let $V \in R_{n\times n}$ be positive definite, define the real symmetric matrix W by $A'V + VA = -W$. Then A is stable if for the right eigenvector r associated with every distinct eigenvalue of A, there holds the relation $r^*Wr > 0$ where r^* means conjugate transpose of eigenvector r.
- If W is positive definite, then A is stable if $A'V + VA = -W$ has a positive definite solution matrix V.

Also, the stability problem can be studied by locating the eigenvalues using coefficients of the characteristic polynomial $\det(\lambda I - A) = 0$ rather then the matrix itself. The *Routh–Hurwitz criterion* is one of these approaches. For the monic polynomial with real coefficients,

$$f(z) = z^n + a_1 z^{n-1} + \cdots + a_n \tag{71}$$

and the *Hurwitz matrices* are defined as

$$\begin{aligned} H_1 &= a_1 \\ H_2 &= \begin{bmatrix} a_1 & 1 \\ a_3 & a_2 \end{bmatrix} \\ &\cdots \\ H_n &= \begin{bmatrix} a_1 & 1 & 0 & 0 & & 0 \\ a_3 & a_2 & a_1 & 1 & & 0 \\ a_3 & a_4 & a_3 & a_2 & & \cdot \\ \cdot & \cdot & \cdot & & \cdot & \cdot \\ a_{2n-1} & \cdot & \cdot & & a_{n+1} & a_n \end{bmatrix} \end{aligned} \tag{72}$$

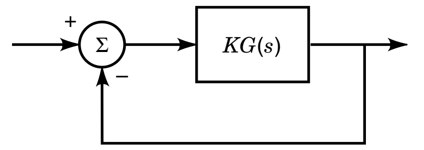

Figure 5. Block diagram for the feedback system: $Y(s)/R(s) = H(s) = KG(s)/[1 + KG(s)]$.

The criteria say that all the zeros of the polynomial $f(z)$ have negative real parts iff $\det H_i > 0$, for $i = 1, 2, \ldots, n$. This also indicates that the eigenvalues λ of the matrix associated with the characteristic polynomial $f(\lambda)$ have all negative real parts, so the matrix is stable.

The *Nyquist stability criterion* is another indirect approach to evaluating stability conditions. For the feedback system given in Fig. 5, it relates the system open loop frequency response to the number of closed-loop poles in the right half of the complex plane (8).

Stability of the system is analyzed by studying the Nyquist plot (polar plot) of the open loop transfer function $KG(j\omega)$. Because it is based on the poles of the closed-loop system, which is decided by $1 + KG(s) = 0$, the point -1 is the critical point for study of the curve $KG(j\omega)$ in the polar plot. The following steps are involved. First, draw the magnitude and angle of $KG(j\omega)$. Second, count the number of clockwise encirclements of -1 as N. Third, find the number of unstable poles of $G(s)$, which is P. The system is stable if the number of unstable closed-loop roots $Z = N - P$ is zero, which means that there are no closed-loop poles in the right half plane. There are other methods exploiting godographs of the system transfer function as functions of ω.

The *Gershgorin's theorem* is also used in eigenvalue localization. It states that any of the eigenvalues of a matrix $A = [a_{i,j}]_{n\times n}$ lies in at least one of the circular discs with centers $a_{i,i}$ and radii as sum of $\|a_{i,j}\|$ for all $i \neq j$. If there are s such circular discs forming a connected domain isolated from other discs, then A has exactly s eigenvalues within this domain (5,36). This theorem finds its application in perturbation analysis of eigenvalues.

Mode Identification

Identification of a mode of a system finds its application in many engineering tasks. Based on nonlinear simulations or measurements, system identification techniques can be used for this purpose. The least-squares method is among those widely used. The major approaches in system modeling and identification include system identification based on an FIR (MA) system model, system identification based on all All-Pole (AR) system model, and system identification based on a Pole-Zero (ARMA) system model. As one of the typically used methods in identfiying modes of a dynamic system, Prony's method is a procedure for fitting a signal $y(t)$ to a weighted sum of exponential terms of the form:

$$\hat{y}(t) = \sum_{i=1}^{n} R_i e^{\lambda_i t} \tag{73a}$$

or in a discrete form:

$$\hat{y}(k) = \sum_{i=1}^{n} R_i z_i^k \tag{73b}$$

where $\hat{y}(t)$, $\hat{y}(k)$ are the Prony approximation to $y(t)$, R_i is signal residue, λ_i is the s-plane mode, z_i is the z-plane mode, and n is the Prony fit order. Supposing that the signal y is a linear function of past values, the modes and signal residues can be calculated by the following equation:

$$y(k) = a_1 y(k-1) + a_2 y(k-1) + \cdots + a_n y(k-n) \tag{74}$$

which can be applied repeatedly to form the linear set of equations as shown in Eq. (75), where N is the number of sample points:

$$\begin{bmatrix} y(n+0) & y(n-1) & \cdots & y(1) \\ y(n+1) & y(n+0) & \cdots & y(2) \\ & & \cdots & \\ y(N-1) & y(N-2) & \cdots & y(N-n) \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \cdots \\ a_n \end{bmatrix} = \begin{bmatrix} y(n+1) \\ y(n+2) \\ \cdots \\ y(N) \end{bmatrix} \tag{75}$$

From Eq. (75), the coefficients a_i can be calculated. The modes z_i are the roots of the polynomial: $z^n - a_1 z^{n-1} - \cdots - a_n = 0$. The signal residues R_i can be calculated by solving the linear equations:

$$\begin{bmatrix} z_1^1 & z_2^1 & \cdots & z_n^1 \\ z_1^2 & z_2^2 & \cdots & z_n^2 \\ & & \cdots & \\ z_1^N & z_2^N & \cdots & z_n^N \end{bmatrix} \begin{bmatrix} R_1 \\ R_2 \\ \\ R_n \end{bmatrix} = \begin{bmatrix} y(1) \\ y(2) \\ \\ y(N) \end{bmatrix} \tag{76}$$

from which the s-plane modes λ_i can be computed by $\lambda_i = \log_e(z_i)/\Delta t$, where Δt is the sampling time interval (36a). A similar estimation method is the Shanks' method, which employs a least-squares criterion (36b).

SOME PRACTICAL APPLICATIONS OF EIGENVALUES AND EIGENVECTORS

Some Useful Comments

In the area of stability and control, eigenvalues give such important information as *damping, phase,* and *magnitude* of oscillations (15,20,37). For example, for the system dynamic state matrix A_s critical eigenvalue $\lambda_i = \alpha_i \pm j\omega_i$, which is the eigenvalue with the largest real part α_i, the damping constant is $\sigma = \alpha_i$, and frequency of oscillation is ω_i in radius per second unit, or $\omega_i/2\pi$ in hertz.

The *eigenvalue sensitivity analysis* is often needed to assess the influence of certain system parameters p on damping and enhance system stability (2,15):

$$\frac{\partial \alpha_j}{\partial p_i} = \mathrm{Re}\left\{ \frac{l_j^T \dfrac{\partial A_s}{\partial p_i} r_j}{l_j^T r_j} \right\} \tag{77}$$

l_j and r_j are the corresponding left and right eigenvectors for the jth eigenvalue α_j, and $\partial A_s/\partial p_i$ is the sensitivity of the dynamic state matrix to the ith parameter p_i.

The left and right vectors are also associated with important features of the system dynamics.

The left eigenvector is a normal vector to the equal damping surfaces, and the right eigenvector shows the initial dynamics of the system at a disturbance (15). They also provide an efficient mathematical approach to locating these equal damping surfaces in the parameter spaces (15,20).

The elements of the right and left eigenvectors are dependent on units and scaling associated with the state variables. This may cause difficulties when these eigenvectors are applied individually for identification of the relationship between the states and the modes. The participation matrix P is needed to solve the problem. The participation matrix combines the right and left eigenvectors and can serve as a measure of the association between the state variables and the modes. It is defined as

$$P = [P_1 P_2, \ldots, P_n] \qquad \text{with} \qquad P_i = \begin{bmatrix} P_{1i} \\ P_{2i} \\ \vdots \\ P_{ni} \end{bmatrix} = \begin{bmatrix} \rho_{1i}\vartheta_{i1} \\ \rho_{2i}\vartheta_{i2} \\ \vdots \\ \rho_{ni}\vartheta_{in} \end{bmatrix} \tag{78}$$

where ρ_{ki} is the kth entry of the right eivengector r_i, and θ_{ik} is the kth entry of the left eigenvector l_i. The element is the participation factor, which measures the relative participation of the kth state variable in the ith mode, and vice versa. Regarding the eigenvector normalization, the sum of the participation factors associated with any mode ($\sum_{i=1}^{n} P_{ki}$) or with any state variable ($\sum_{k=1}^{n} P_{ki}$) is equal to 1 (37).

A Power System Example

Let us take a power system example in DAE form, using a comprehensive numerical method (15) to calculate the following important small-signal stability characteristic points:

- load flow feasibility points, beyond where there exists no solution for the system load flow equations;
- aperiodic and oscillatory stability points;
- min/max damping points.

The method employs the following constrained optimization problem:

$$a^2 \Rightarrow \text{min/max} \tag{79}$$

subject to

$$f(x, p_0 + \tau\Delta p) = 0 \tag{80}$$

$$As(x, p_0 + \tau\Delta p)l' - al' + \omega l'' = 0 \tag{81}$$

$$As(x, p_0 + \tau\Delta p)l'' - al'' - \omega l' = 0 \tag{82}$$

$$l_l' - 1 = 0 \tag{83}$$

$$l_i'' = 0 \tag{84}$$

where a is the real part of system eigenvalue of interest, ω is the imaginary part; l' and l'' are real and imaginary parts of the corresponding left eigenvector l; $l_i' + jl_i''$ is the ith element of the left eigenvector l; $p_0 + \tau\Delta p$ specifies a ray in the space of p; and A_s stands for the state matrix. In the preceding set,

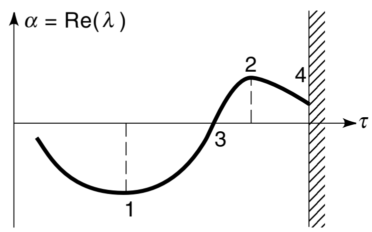

Figure 6. Different solutions of the problem: 1, 2—minimum and maximum damping; 3—saddle ($\omega = 0$) or Hopf ($\omega \neq 0$) bifurcations; 4—load flow feasibility boundary. $\alpha = \text{Re}(\lambda)$: real part of system eigenvalue; τ = system parameter variation factor. These characteristic points can be located in one approach using a general method, as described in text.

(80) is the load flow equation and conditions (81)–(84) provide an eigenvalue with the real part of a and the corresponding left eigenvector.

The problem may have a number of solutions, and each one of them presents a different aspect of the small-signal stability problem as shown in Fig. 6.

The minimum and maximum damping points correspond to zero derivative $da/d\tau$. The constraint set (80)–(84) gives all unknown variables at these points. The minimum and maximum damping, determined for all oscillatory modes of interest, provides essential information about damping variations caused by a directed change of power system parameters.

The saddle node or Hopf bifurcations correspond to $a = 0$. They indicate the small-signal stability limits along the specified loading trajectory $p_0 + \tau\Delta p$. Besides revealing the type of instability (aperiodic for $\omega = 0$ or oscillatory for $\omega \neq 0$), the constraint set (80)–(84) gives the frequency of the critical oscillatory mode. The left eigenvector $l = l' + jl''$ (together with the right eigenvector $r = r' + jr''$ which can be easily computed in turn) determines such essential factors as sensitivity of a with respect to p, the mode, shape, participation factors, observability, and excitability of the critical oscillatory mode (29,37,38).

The load flow feasibility boundary points (80) reflect the maximal power transfer capabilities of the power system. Those conditions play a decisive role when the system is stable everywhere on the ray $p_0 + \tau\Delta p$ up to the load flow feasibility boundary. The optimization procedure stops at these points because the constraint (80) cannot be satisfied anymore.

The problem (79)–(84) takes into account only one eigenvalue each time. The procedure must be repeated for all eigenvalues of interest. The choice of eigenvalues depends upon the concrete task to be solved. The eigenvalue sensitivity, observability, excitability, and controllability factors (29,37,39) can help to determine the eigenvalues of interest and to trace them during optimization.

The result of optimization depends on the initial guesses for all variables in (79)–(84). To get all characteristic points for a selected eigenvalue, different initial points may be computed for different values of τ.

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