FOURIER ANALYSIS

Fourier analysis is a collection of related techniques for representing general functions as linear combinations of simple functions or functions with certain special properties. In the classical theory, these simple functions (called *basis functions*) are sinusoids (sine or cosine functions). The modern theory uses many other functions as the basis functions. Every basis function carries certain characteristics that can be used to describe the functions of interest; it plays the role of a building block for the complicated structures of the functions we want to study. The choice of a particular set of basis functions reflects how much we know and what we want to find out about the functions we want to analyze. In this article, we will restrict our discussion (except for the last section about wavelets) to the classical theory of Fourier analysis. For a broader point of view, see FOURIER TRANSFORM.

In applications, Fourier analysis is used either simply as an efficient computational algorithm or as a tool for analyzing the properties of the signals, functions of time, or space variables at hand. (In this article, we will use the terms signal and function interchangeably.)

A very important component of modern technology is the processing of signals of various forms in order to extract the most significant characteristics carried in the signals. In practice, most of the signals in their raw format, are given as functions of the time or space variables, so we also call the domain of the signal the *time* (or *space*) *domain*. This time- or space-domain representation of a signal is not always the best for most applications. In many cases, the most distinguished information is hidden in the *frequency content* or *frequency spectrum* of a signal. Fourier analysis is used to accomplish the representation of signals in the frequency domain.

Fourier analysis allows us to calculate the "weights" (amplitudes) of the different frequency sinusoids which make up the signal. Given a signal, we can view the process of analyzing the signal by Fourier analysis as one of transforming the original signal into another form that reveals its properties (in the frequency domain) that cannot be directly seen in the original form of the signal.

The most useful tools in Fourier analysis are the following three types of transforms: Fourier series, discrete Fourier transform, and (continuous) Fourier transform. With each transform there is associated an inverse transform that recovers (in a sense to be discussed later) the original signal from the transformed one. The process of calculating a transform is also referred to as Fourier spectral analysis; the process of recovering the original function from its transform by using the inverse transform is called Fourier synthesis.

The wide use of Fourier analysis in engineering must be credited to the existence of the fast Fourier transform (FFT), a fast computer implementation of the discrete Fourier transform. Areas where Fourier analysis (via FFT) has been successfully applied, include applied mechanics, biomedical engineering, computer vision,numerical methods, signal and image processing, and sonics and accoustics.

Fourier analysis is closely related to the sampling of signals. In order to analyze signals using a computer, a continuous time signal must be sampled (at either equally or unequally spaced time intervals). Instead, they are given by a set of sample values. The resulting discrete-time signal is called the *sampled version* of the original continuous-time signal. There are two types of sampling: uniform sampling and nonuniform sampling. We only discuss the uniform sampling in this article. How often must a signal be sampled in order that all the frequencies present should be detected? This is discussed with sampling theorems.

Recently, a new set of tools under the generic name *wavelets analysis* has found various applications. Wavelets analysis can be view as an enhancement of the classical Fourier analysis. In wavelets analysis, the basis functions are not sinusoids but functions with zero average and other additional properties. These basis functions are localized in both time and frequency domains.

FOURIER SERIES

History

The history of Fourier analysis can be dated back at least to the year 1747 when Jean Le Rond d'Alembert (1717–1783) derived the "wave equation" which governs the vibration of a string. Other mathematicians involved in the study of Fourier analysis include Leonard Euler (1707–1783), Daniel Bernoulli (1700–1782), and Joseph Louis Lagrange (1736–1813). An enormous and important step was made by Jean Baptiste Joseph Fourier (1768–1830) when he took up the study of heat conduction. He used sines and cosines in his study of the flow of heat. He submitted a basic paper on heat conduction to the Academy of Sciences of Paris in 1807 in which he announced his belief in the possibility of representing every function $f(x)$ on the interval (a, b) by a trigonometric series of the form (with $P = b - a$)

$$\frac{1}{2}A_0 + \sum_{n=1}^{\infty}\left(A_n \cos\frac{2\pi nx}{P} + B_n \sin\frac{2\pi nx}{P}\right) \tag{1}$$

where

$$A_n = \frac{2}{P}\int_a^b f(x)\cos\frac{2\pi nx}{P}\,dx \qquad (n = 0,\ 1,\ 2,\ \ldots) \tag{2}$$

$$B_n = \frac{2}{P}\int_a^b f(x)\sin\frac{2\pi nx}{P}\,dx \qquad (n = 1,\ 2,\ 3,\ \ldots) \tag{3}$$

Because of its lack of rigor, the paper was rejected by a committee consisting of Lagrange, Laplace, and Legendre. Fourier then revised the paper and resubmitted it in 1811. The paper was judged again by the three aforementioned mathematicians as well as others. Showing great insight, Academy awarded Fourier the Grand Prize of the Academy despite the defects in his reasoning. This 1811 paper was not published in its original form in the *Mémoires* of the Academy until 1824 when Fourier became the secretary of the Academy. (It is worthwhile to point out that there were good reasons that Fourier's theorem was criticized by his contemporaries: At that time, the modern concepts of function and limit were not available.)

As a result of Fourier's work, the sequences $\{A_n\}_{n=0}^{\infty}$ and $\{B_n\}_{n=0}^{\infty}$ defined by Eqs. (2) and (3) are now universally known as the (real) Fourier coefficients of $f(x)$ (though these formulae were known to Euler and Lagrange before Fourier). The term, $A_1 \cos(2\pi \times /P) + \beta_1 \sin(2\pi \times /P)$, is called the principal (spectral) component of the expansion; and the number $\omega_0 = 1/P$ is called the principal (or fundamental) frequency.

Since Fourier coefficients are defined by integrals, the function f must be integrable. In searching for a more general concept of integration (so as to include more functions in Fourier analysis), Bernhardt Riemann (1826–1866) introduced the definition of integral now associated with his name, the Riemann integral. Later, Henri Lebesgue (1875–1941) constructed an even more general integral, the Lebesgue integral.

Because changing the values of a function at finitely many points will not change the value of its integral, we will not distinguish two functions if they are the same except at finitely many points.

The Complex Form of Fourier Series

Given a function $f(x)$ on (a, b), to calculate its Fourier series of the form shown in Eq. (1) we have to use two equations [Eqs. (2) and (3)] to obtain the coefficients A_n and B_n. This is why we sometimes want to use an alternative form of Fourier series, the complex form. To rewrite Eq. (1), we use Euler's identity (as usual, with $j = \sqrt{-1}$)

$$e^{j\phi} = \cos\phi + j\sin\phi$$

Then the trigonometric series in Eq. (1) can be put in a formally equivalent form,

$$\sum_{n=-\infty}^{\infty} c_n e^{j2\pi nx/P} \tag{4}$$

in which, on writing $B_0 = 0$, we have

$$c_n = \frac{1}{2}(A_n + B_n), \quad c_{-n} = \frac{1}{2}(A_n - B_n), \qquad n = 0, 1, 2, \ldots$$

From Eqs. (2) and (3), we can derive

$$c_n = \frac{1}{P}\int_a^b f(x)e^{-j2\pi nx/P}\,dx, \qquad n = 0, \pm1, \pm2, \ldots \tag{5}$$

The numbers $\{c_n\}_{n=-\infty}^{\infty}$ are called the "complex" Fourier coefficients of $f(x)$. The two series in Eqs. (1) and (4) are referred to as the real and complex Fourier series of $f(x)$, respectively.

The Orthogonality Relations

Before we explore Fourier series further, it is important to point out the facts that provided the heuristic basis for the formulae in Eqs. (2), (3), and (5) for the Fourier coefficients. These facts, which can be proved by simple and straightforward calculations, are expressed in the following orthogonality relations. In the real form, we have

$$\frac{1}{P}\int_a^b \cos\frac{2\pi mx}{P}\cos\frac{2\pi nx}{P}\,dx = \begin{cases} 0 & \text{for } m \neq n \\ \frac{1}{2} & \text{for } m = n \neq 0 \\ 1 & \text{for } m = n = 0 \end{cases} \tag{6}$$

$$\frac{1}{P}\int_a^b \sin\frac{2\pi mx}{P}\sin\frac{2\pi nx}{P}\,dx = \begin{cases} 0 & \text{for } m \neq n \\ \frac{1}{2} & \text{for } m = n \neq 0 \\ 1 & \text{for } m = n = 0 \end{cases} \tag{7}$$

$$\frac{1}{P}\int_a^b \sin\frac{2\pi mx}{P}\cos\frac{2\pi nx}{P}\,dx = 0 \tag{8}$$

and in the complex form, we have

$$\frac{1}{P}\int_a^b e^{j2\pi mx/P}\,e^{-j2\pi nx/P}\,dx = \begin{cases} 0 & \text{for } m \neq n \\ 1 & \text{for } m = n \end{cases} \tag{9}$$

where m and n are integers, and the interval of integration $[a, b]$ can be replaced by any other interval of length P.

Note that to express the orthogonality among trigonometric functions, we need three identities, namely, Eqs. (6), (7), and (8); but to do the same among exponential functions, we need only one identity, Eq. (9). In general, it is more convenient to compute the complex Fourier series first and then change it to the "real" form in sine and cosine functions. From the definition, we can easily verify that if $f(x)$ is real-valued, then its complex Fourier series can always be put into a real-valued trigonometric series. We will illustrate this in our examples.

Examples of Fourier Series

Example 1. Find the Fourier series of $f(x) = \pi - x$ on interval $(0, 2\pi)$.

SOLUTION. We use Eq. (5) to find the complex Fourier coefficients first. For $n = 0$, we have $c_0 = (1/2\pi)\int_0^{2\pi}(\pi - x)\,dx = 0$. For $n \neq 0$, using integration by parts, we have

$$\begin{aligned} c_n &= \frac{1}{2\pi}\int_0^{2\pi}(\pi - x)e^{-jnx}\,dx \\ &= \frac{1}{2\pi}\left\{-\frac{1}{jn}e^{-jnx}(\pi - x)\Big|_0^{2\pi} - \frac{1}{jn}\int_0^{2\pi} e^{-jnx}\,dx\right\} \\ &= \frac{1}{2\pi}\left\{\frac{2\pi}{jn} + \frac{1}{(jn)^2}e^{-jnx}\Big|_0^{2\pi}\right\} = -\frac{j}{n} \end{aligned}$$

Hence, the complex Fourier series of $f(x)$ on $(0, 2\pi)$ is given by

$$\sum_{n=-\infty}^{\infty}{}' -\frac{j}{n}e^{jnx} \tag{10}$$

where the prime on the sum is used to indicate that the $n = 0$ term is omitted.

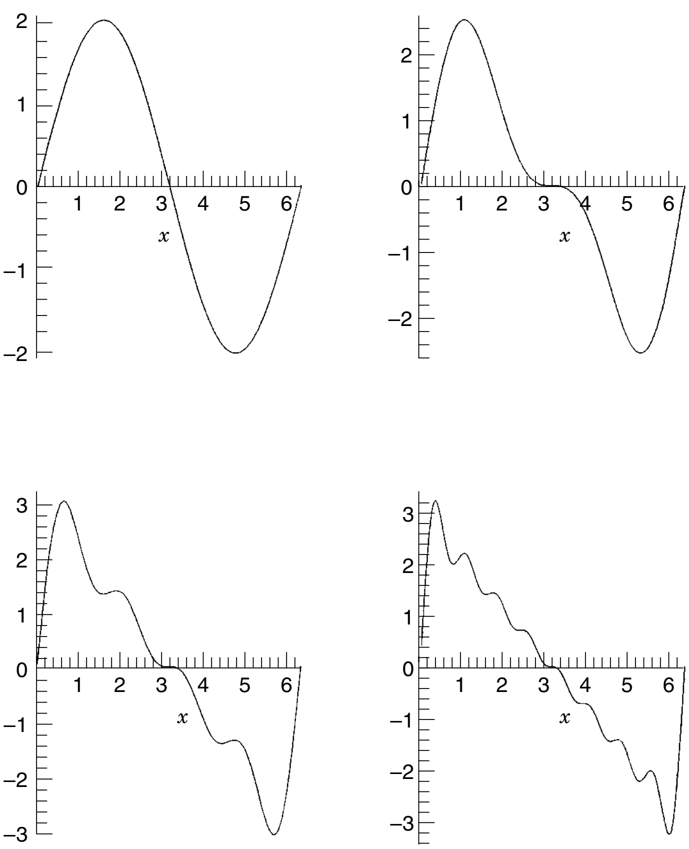

Figure 1. $S_1(x)$, $S_2(x)$, $S_4(x)$, and $S_8(x)$.

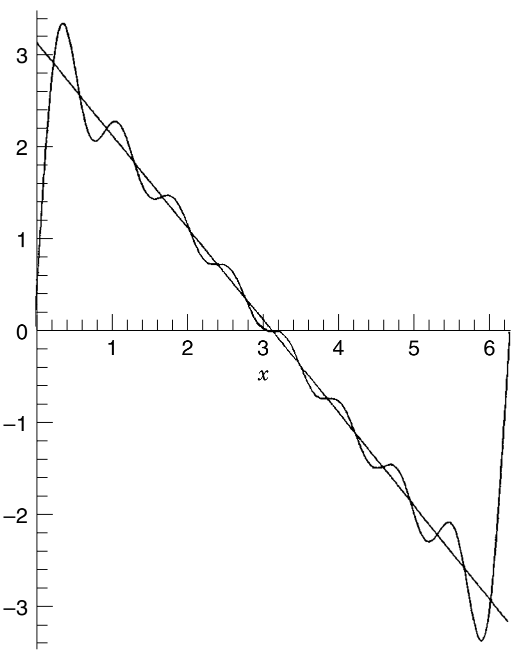

Figure 2. $f(x) = \pi - x$ and $S_8(x)$ on $(0, 2\pi)$.

Grouping $-(j/n)e^{jnx}$ and $-(j/-n)e^{-jnx}$, we obtain $(2/n)$ sin nx, so we can write Eq. (10) in the real form:

$$\sum_{n=1}^{\infty} \frac{2}{n} \sin nx$$

In Fig. 1, we show the graphs of the partial sums:

$$S_m(x) = \sum_{n=1}^{m} \frac{2}{n} \sin nx$$

for $m = 1, 2, 4$, and 8. In Fig. 2, we show both $f(x) = \pi - x$ and $S_8(x)$ on the interval $(0, 2\pi)$.

Notice that the graph of $S_8(x)$ is a wavy approximation to the original function $f(x) = \pi - x$ on $(0, 2\pi)$. Outside of the interval, the graph of $S_8(x)$ is approximating the *periodic extension* (*with period* 2π) $f_p(x)$ of $f(x)$ (see Fig. 3).

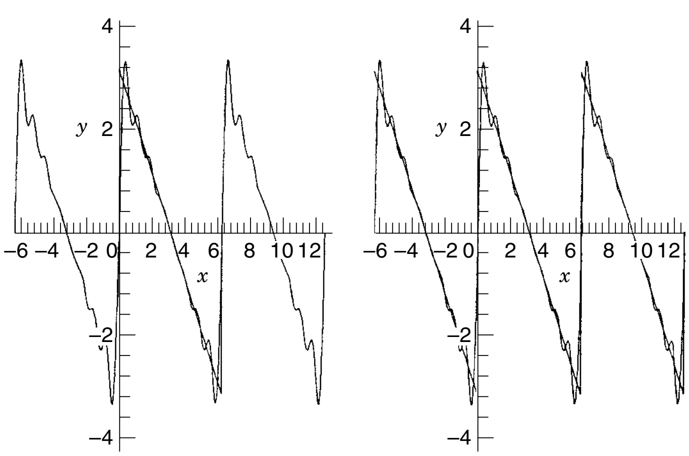

Figure 3. $f(x)$ and $S_8(x)$ on $(-2\pi, 4\pi)$, $f_p(x)$ and $S_8(x)$ on $(-2\pi, 4\pi)$.

Example 2. Find the Fourier series of $f(x)$ defined by

$$f(x) = \begin{cases} 0, & -1 < x < 0 \\ \dfrac{1}{2}, & x = 0 \\ 1, & 0 < x < 1 \end{cases}$$

on the interval $(-1, 1)$.

SOLUTION. The complex Fourier series of $f(x)$ is given by

$$\frac{1}{2} + \sum_{n=-\infty}^{\infty}{}' \frac{j}{2\pi n}[(-1)^n - 1]e^{j\pi nx} \tag{11}$$

We can write Eq. (11) in the following real form:

$$\frac{1}{2} + \sum_{n=1}^{\infty} \frac{1}{\pi n}[1 - (-1)^n] \sin \pi nx = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{\pi(2n-1)} \sin \pi(2n-1)x$$

Let

$$S_m(x) = \frac{1}{2} + \sum_{n=1}^{m} \frac{2}{\pi(2n-1)} \sin \pi(2n-1)x$$

denote the partial sum. In Fig. 4, we show the graphs of $S_m(x)$ for $m = 12$, 24, and 36, along with the graph of $f(x)$ and $S_{36}(x)$.

Convergence

Does the Fourier series of a function $f(x)$ converge to $f(x)$? Fourier's assertion that the answer is yes was initially greeted with a great amount of disbelief as we mentioned earlier. In fact, the answer depends on what sense of convergence is understood. Fourier was right, and the answer is always yes provided that things are interpreted suitably. Pointwise convergence is one of the many choices; it is also the first one considered in the study of Fourier series. Because of this, there is a lot of pointwise convergence theorems, although most of them are sufficient conditions. Dirichlet was the first mathematician who carefully studied the validity of pointwise convergence of the Fourier series. We will state two such results that more or less cover most application problems from physics and engineering.

We say that the function $f(x)$ is piecewise continuous on (a, b) if (1) f is continuous on (a, b) except perhaps at finitely many *exceptional* points and (2) at each x_* of the exceptional points, both one-sided limits

$$\lim_{x \to x_*^-} f(x) =: f(x_*-) \text{ and } \lim_{x \to x_*^+} f(x) =: f(x_*+)$$

exist. [At the endpoints, a and b, we assume both $\lim_{x\to a^+} f(x) =: f(a+)$ and $\lim_{x \to b^-} f(x) =: f(b-)$ exist.]

Next, we say that function $f(x)$ is piecewise smooth on (a, b) if both $f(x)$ and its first derivative $f'(x)$ are piecewise continuous on (a, b).

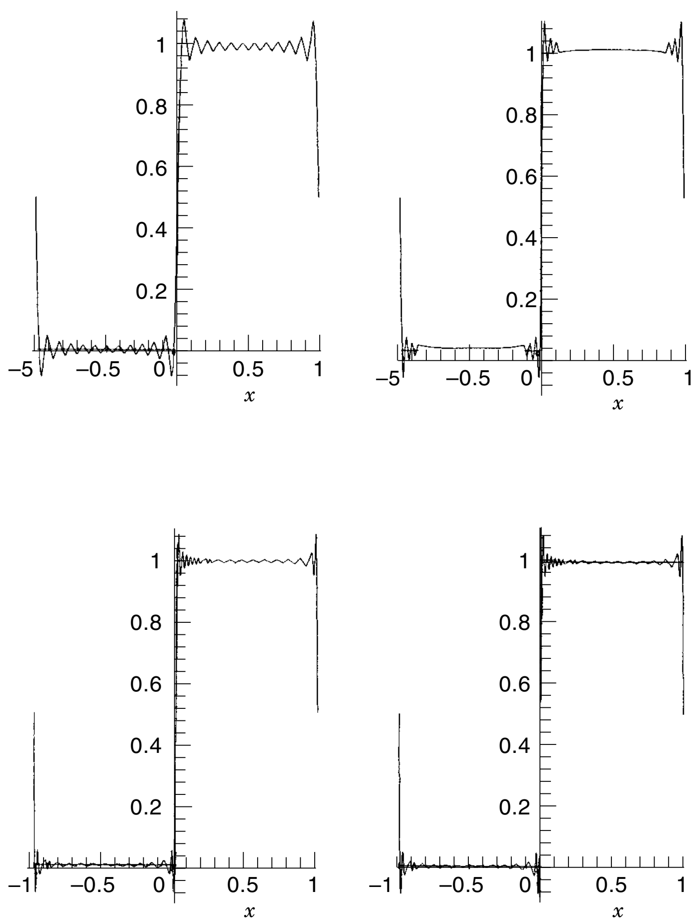

Figure 4. $S_{12}(x)$, $S_{24}(x)$, $S_{36}(x)$, and $f(x)$ and $S_{36}(x)$.

Dirichlet Theorem. If $f(x)$ is piecewise smooth on (a, b), and $S_m(x)$ denotes the mth partial sums of the Fourier series of $f(x)$, then

$$\lim_{m\to\infty} S_m(x) = \begin{cases} \frac{1}{2}[f(x-) + f(x+)], & \text{for } a < x < b \\ \frac{1}{2}[f(a+) + f(b-)], & \text{for } x = a \text{ or } b \end{cases}$$

In particular, $\lim_{m\to\infty} S_m(x) = f(x)$ for every $x \in (a, b)$ at which f is continuous.

The functions in Examples 1 and 2 are both piecewise smooth functions. For example, in Example 1 the Fourier series converges to $f(x)$ at every point in $(0, 2\pi)$ and to 0 at the endpoints 0 and 2π.

Dirichlet's Theorem can be extended to more functions. We say that the function $f(x)$ is of bounded variation on (a, b) if the sums of the form

$$|f(x_1) - f(a)| + |f(x_2) - f(x_1)| + \cdots + |f(b) - f(x_k)|$$

are bounded for all k and for all choices of $(a<)x_1 < x_2 < \cdots < x_k(<b)$.

It is known that a function is of bounded variation if and only if it can be written as the difference of two monotonic (either increasing or decreasing) functions (see Ref. 1).

Dirichlet–Jordan Theorem. If $f(x)$ is of bounded variation on interval (a, b), and if $S_m(x)$ denotes the mth partial sum of its Fourier series, then

$$\lim_{m\to\infty} S_m(x) = \begin{cases} \frac{1}{2}[f(x-) + f(x+)], & \text{for } a < x < b \\ \frac{1}{2}[f(a+) + f(b-)], & \text{for } x = a \text{ or } b \end{cases}$$

It is not hard to show that a piecewise smooth function can be represented as the difference of two increasing functions; so a piecewise smooth function is of bounded variation. Therefore, the Dirichlet–Jordan Theorem is more general than the Dirichlet Theorem.

Limitations of Pointwise Convergence

Although it is undeniably of great intrinsic interest to know that a certain function admits a pointwise representation by its Fourier series, it must be pointed out without delay that, in many situations, simple pointwise convergence is not the appropriate thing to look at.

It has been known since 1876 that the Fourier series of a continuous function may diverge at infinitely many points, and the Fourier series of a integrable function may diverge at all points. For almost a century, whether the Fourier series of a general continuous function is guaranteed to converge at least at some points remained in doubt. An affirmative answer was obtained by L. Carleson in 1966 with a deep theorem asserting that the Fourier series of every square-integrable function must converge to the function at "almost every" point. See the article by Hunt in Ref. 2. Therefore, we have to restrict functions to certain special types in order to achieve the pointwise convergence. Although the theorems presented above are sufficient for many purposes, they do not give the whole picture. See Zygmund's book (1) for more details.

Other Types of Convergence. Here we briefly mention some other types of convergence that may be used when studying the Fourier series. First, there is *uniform convergence,* which is stronger than pointwise convergence. Next, there is *pth power mean convergence,* according to which the series $\sum_{n=-\infty}^{\infty} c_n e^{j2\pi nx/P}$ converges to $f(x)$ on the interval (a, b) (with $P = b - a$) if

$$\lim_{m\to\infty} \int_a^b \left| f(x) - \sum_{n=-m}^{m} c_n e^{j2\pi nx/P} \right|^p dx = 0$$

The case when $p = 2$ is especially simple and useful. Finally, there is *distributional convergence* which is defined as follows: The series $\sum_{n=-\infty}^{\infty} c_n e^{j2\pi nx/P}$ is said to distributionally converge to $f(x)$ on the interval (a, b) $(P = b - a)$ if

$$\lim_{m\to\infty} \int_a^b u(x) \sum_{n=-m}^{m} c_n e^{j2\pi nx/P}\, dx = \int_z^b u(x) f(x)\, dx$$

for every infinitely differentiable periodic function $u(x)$ with period P.

Fourier Sine Series and Fourier Cosine Series

A function $f(x)$ on $(-L, L)$ is said to be an even function if $f(-x) = f(x)$ for $x \in (-L, L)$; it is said to be an odd function if $f(-x) = -f(x)$ for $x \in (-L, L)$. The graph of an even function is symmetric with respect to the y axis in the xy plane, while the graph of an odd function is symmetric with respect to the origin. Note that $\cos(\pi nx/L)$ is an even function in x ($n = 0, 1, 2, \ldots$), and $\sin(\pi nx/L)$ is an odd function in x ($n = 1, 2, 3, \ldots$). By exploring the symmetry, we can show that if $f(x)$ is an even function on $(-L, L)$, then its real Fourier series on $(-L, L)$ contains no sine terms; and if $f(x)$ is an odd function, then its real Fourier series is a series of sines only. Of course, a trigonometric series containing no sines must be even, if the series ever converges. Similarly, a trigonometric series containing only sines must be odd.

When a function $f(x)$ is defined on $(0, L)$, we can extend it to a larger interval $(-L, L)$. Among the infinitely many possible extensions, we consider the following two. The first one is to extend the function so that it is an even function on $(-L, L)$ (we still use $f(x)$ to denote the extension):

$$f(x) = \begin{cases} f(x), & 0 < x < L \\ f(-x), & -L < x < 0 \end{cases}$$

The other way is to extend it as an odd function on $(-L, L)$:

$$f(x) = \begin{cases} f(x), & 0 < x < L \\ -f(-x), & -L < x < 0 \end{cases}$$

Note that in the above definitions of the extensions of $f(x)$ we did not give any definition for the extension at $x = 0$. Because of an earlier remark, changing one value of a function will not change its Fourier series; so at $x = 0$ we can define $f(x)$ in any way we want. For example, in view of the convergence theorems, we may define $f(0) = f(0+)$ in the even extension, and $f(0) = 0$ in the odd extension.

Now we can form the Fourier series of $f(x)$ on the larger interval $(-L, L)$ after the extension. When we use the even extension, the Fourier series of $f(x)$ on $(-L, L)$ is given by

$$\frac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n \cos\frac{\pi nx}{L} \tag{12}$$

with

$$A_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{\pi nx}{L}\,dx = \frac{1}{L}\left[\int_{-L}^{0} f(x)\cos\frac{\pi nx}{L}\,dx + \int_{0}^{L} f(x)\cos\frac{\pi nx}{L}\,dx\right]$$

The last two integrals are equal since $f(x)\cos(\pi nx/L)$ is even. So the formula for the real Fourier coefficient A_n can be simplified to

$$A_n = \frac{2}{L}\int_{0}^{L} f(x)\cos\frac{\pi nx}{L}\,dx \tag{13}$$

Notice that this formula uses only the function $f(x)$ on its original domain $(0, L)$. This motivates the following definition of Fourier cosine series.

Definition. Let $f(x)$ be a function defined on $(0, L)$. Then its Fourier cosine series on $(0, L)$ is given by Eq. (12), where A_n are defined by Eq. (13) for $n = 0, 1, 2, \ldots$.

Similarly, we can define the Fourier sine series by using the odd extension of a function defined on $(0, L)$.

Definition. Let $f(x)$ be a function defined on $(0, L)$. Then its Fourier sine series on $(0, L)$ is given by

$$\sum_{n=1}^{\infty} B_n \sin\frac{\pi nx}{L}\,dx$$

where B_n's are defined by

$$B_n = \frac{2}{L}\int_{0}^{L} f(x)\sin\frac{\pi nx}{L}\,dx$$

for $n = 0, 1, 2, \ldots$.

In Fig. 5, we show the graph of function $f(x) = \pi - x$ on $(0, \pi)$, the even extension of $f(x)$ on $(-\pi, \pi)$, and the odd extension of $f(x)$ on $(-\pi, \pi)$; in Fig. 6, we sketch some partial sums of the Fourier cosine and Fourier sines series of the even and the odd extensions, respectively. It is clear from graphs that on the interval $(0, \pi)$, the Fourier cosine series provides much better approximation to $f(x)$ than the Fourier sine series. This is because the even extension is continuous while the odd extension has a jump at $x = 0$.

Gibb's Phenomenon

In Fig. 4, notice the sharp peaks near 0, the point of discontinuity of $f(x)$. That this is not an isolated case was first explained in 1899 (in a letter to *Nature*) by Josiah Gibbs in re-

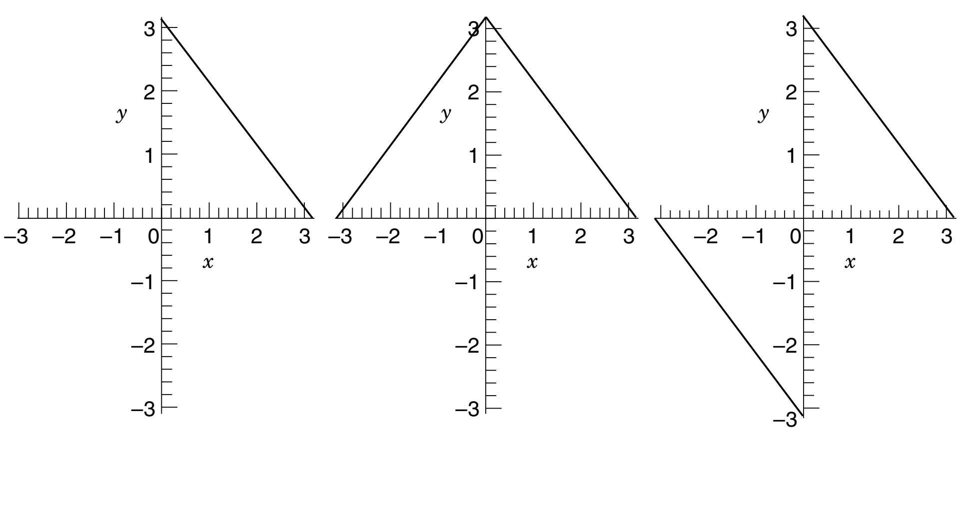

Figure 5. Graphs of $f(x)$ on $(0, \pi)$, and its even and odd extensions on $(-\pi, \pi)$.

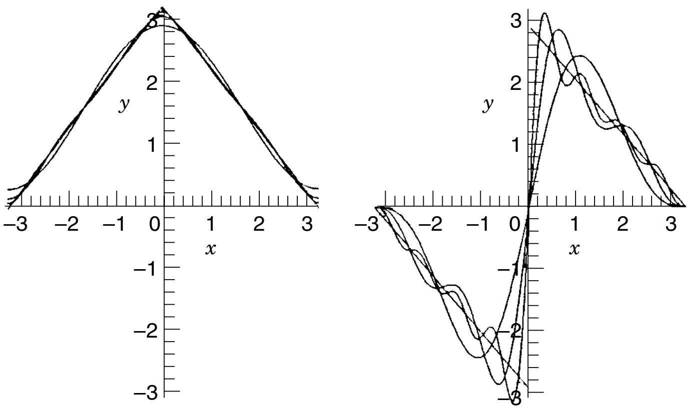

Figure 6. The even extension of $f(x)$ and the Fourier cosine series on $(-\pi, \pi)$; the odd extension and the Fourier sine series on $(-\pi, \pi)$.

sponse to a question of the American physicist Albert Michelson, who observed such a phenomenon in his experiments on superposition of harmonics. It can be proved that Gibbs's phenomenon occurs at x_* whenever it is a discontinuous point of a piecewise smooth function $f(x)$. In Fig. 7, we show a close-up of what is happening near the right side of 0 for the function $f(x)$ in Example 2, together with the partial sums of its Fourier series S_{12}, S_{24}, and S_{36}. Note that the amount of overshot is almost unchanged.

The Discrete Fourier Transform

As a motivation for the definition of the discrete Fourier transform, let us consider the computation of the complex Fourier coefficients,

$$c_n = \frac{1}{P}\int_a^b f(x)e^{-j2\pi nx/P}\,dx \tag{14}$$

when we only know the values of $f(x)$ at evenly spaced points in (a, b), say $x_k = a + k\delta$ for $k = 0, 1, \ldots, N-1$ with $\delta =$

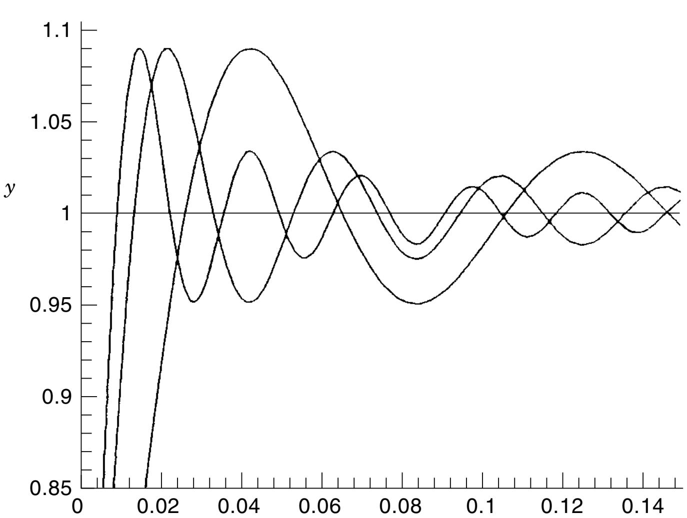

Figure 7. Gibbs' phenomenon.

P/N. We have to approximate the integral in Eq. (14). To do this, we use a left-endpoint Riemann sum:

$$\begin{aligned} c_n &\approx \frac{1}{P}\sum_{k=0}^{N-1} f(x_k)e^{-j2\pi na/P}\,e^{-j2\pi nk\delta/P}\,\frac{P}{N} \\ &= \frac{1}{Ne^{j2\pi na/P}}\sum_{k=0}^{N-1} f(x_k)e^{-j2\pi nk/N} \end{aligned} \tag{15}$$

The final summation in Eq. (15) motivates the following definition.

Definition. Let $\{h_k\}_{k=0}^{N-1}$ be a set of complex numbers. The discrete Fourier transform of $\{h_k\}_{k=0}^{N-1}$ is denoted by $\{H_n\}$ and is given by

$$H_n = \sum_{k=0}^{N-1} h_k e^{-j2\pi nk/N}$$

for $n = 0, \pm 1, \pm 2, \ldots$.

Although H_n is defined for all integers n, there are only at most N distinct values since $H_{n+N} = H_n$. So, we can just use $\{H_n\}_{n=0}^{N-1}$. Therefore, the discrete Fourier transform maps a set of N numbers ($\{h_k\}_{k=0}^{N-1}$) into a set of N numbers ($\{H_n\}_{n=0}^{N-1}$).

Using the terminology of the discrete Fourier transform, we can say that Eq. (15) gives c_k, the kth complex Fourier coefficient of the function $f(x)$, as approximately $(Ne^{j2\pi na/P})^{-1}F_k$, with $\{F_k\}$ being the discrete Fourier transform of $\{f(x_k)\}_{k=0}^{N-1}$. Of course, in order to get satisfactory approximation of c_k, we have to choose a large value for N. Based on numerical observations (see, for example, Ref. 3), it seems that we need to make $N \geq 8|k|$ to ensure some degree of good approximation of c_k.

As in the case of Fourier series, the orthogonality property plays a very important role in the discrete Fourier transform. We now have

$$\sum_{k=0}^{N-1} e^{j2\pi mk/N}\,e^{-j2\pi nk/N} = \begin{cases} 0, & \text{if } m \neq n \\ N, & \text{if } m = n \end{cases}$$

With the orthogonality property, we derive the following inversion formula for the discrete Fourier transform: If $\{H_n\}_{n=0}^{N-1}$ is the discrete Fourier transform of $\{h_k\}_{k=0}^{N-1}$, then

$$h_k = \frac{1}{N}\sum_{n=0}^{N-1} H_n e^{j2\pi nk/N}, \qquad k = 0, 1, \ldots, N-1 \tag{16}$$

Equation (16) defines the discrete *inverse* Fourier transform of $\{H_n\}_{n=0}^{N-1}$. Note that there are only two differences in the definitions of the discrete Fourier transform and the discrete inverse Fourier transform: (1) opposite signs in the exponential and (2) presence or absence of a factor $1/N$. This means that an algorithm for calculating the discrete Fourier transforms can also calculate the discrete inverse Fourier transforms with minor changes.

In the literature of Fourier analysis, there are alternative ways to define the discrete Fourier transforms. One variation is in the factor $1/N$; some authors use it in the definition of the discrete Fourier transforms instead of putting it in the discrete inverse Fourier transforms like we do here. Another difference is the ranges of the indices; there are good reasons in favor of (or against) the use of indices running either between 0 and $N - 1$ or from $-N/2$ to $N/2$. With little modification, anything that can be done for one version of discrete Fourier transform can also be done for other versions. For this reason, we now state an alternate definition.

Discrete Fourier Transform (Alternate Version). (1) Let N be an even positive integer and let $\{h_k\}_{k=-N/2+1}^{N/2}$ be a set of complex numbers. Then its discrete Fourier transform is given by

$$H_n = \sum_{k=-N/2+1}^{N/2} h_k e^{-j2\pi nk/N}, \qquad k = -\frac{N}{2}+1, \ldots, \frac{N}{2}$$

(2) If N is an odd positive integer and $\{h_k\}_{k=-N/2+1}^{N/2}$ is a set of N complex numbers, then its discrete Fourier transform is given by

$$H_n = \sum_{k=-(N-1)/2}^{(N-1)/2} h_k e^{-j2\pi nk/N}, \qquad k = -\frac{N-1}{2}, \ldots, \frac{N-1}{2}$$

Continuous Fourier Transform

Now, we briefly discuss the last of the three types of transforms in Fourier analysis: the continuous Fourier transform. It is also referred to simply as the Fourier transform. It applies to functions of a continuous variable that runs on the whole real line $(-\infty, \infty)$. Given a signal $f(x)$, the *Fourier transform* $F(\xi)$ of $f(x)$ is defined by

$$F(\xi) = \int_{-\infty}^{\infty} f(x) e^{-j2\pi\xi x}\, dx, \qquad -\infty < \xi < \infty$$

where ξ is usually called the *frequency variable.* Due to the presence of the complex exponential $e^{-j2\pi\xi x}$ in the integrand of the above integral, the values of $F(\xi)$ may be complex. So, to specify $F(\xi)$, it is necessary to display both the magnitude and the angle of $F(\xi)$. From the definition, like the Fourier coefficients, Fourier transform of a function $f(x)$ is defined only if the above integral makes sense. A function $f(x)$ is said to be absolutely integrable if

$$\int_{-\infty}^{\infty} |f(x)|\, dx < \infty \tag{17}$$

Let L^1 denote the set of all absolutely integrable functions. [That is, L^1 denotes the set of all Lebesgue integrable functions defined on $(-\infty, \infty)$.] It is a well-known fact in the theory of Lebesgue integration that if $|f(x)|$ is Riemann integrable on $(-\infty, \infty)$, then $f(x)$ is in L^1. For the readers not familiar with the Lebesgue integration, it is safe to interpret all integrals as Riemann integrals since most signals in practice are Riemann integrable.

Equation (17) guarantees that the Fourier transform $F(\xi)$ of $f(x)$ is well-defined. Actually, in this case, $F(\xi)$ is a (uniformly) continuous function of ξ in $(-\infty, \infty)$. But, a continuous function on $(-\infty, \infty)$ is not necessarily in L^1. For example, the constant function $f(x) = 1$, $-\infty < x < \infty$, has infinite area under its graph over $(-\infty, \infty)$. Thus, even if $f(x)$ is in L^1 so that $F(\xi)$ is uniformly continuous, still we cannot assert that $F(\xi)$ is in L^1. The following result shows the magic when we know that $F(x)$ is in L^1.

The Inversion Theorem. Let $f(x)$ be in L^1 and let $F(\xi)$ denote its Fourier transform. Assume that $F(\xi)$ is in L^1. Then

$$f(x) = \int_{-\infty}^{\infty} F(\xi) e^{j2\pi\xi x}\, d\xi, \tag{18}$$

for almost every x in $(-\infty, \infty)$.

This theorem tells us that under the suitable conditions, we can recover a function from its Fourier transform. This is why the integral in Eq. (18) is called the *inverse Fourier transform* of $F(\xi)$. Notice the positive sign in the exponential.

When we treat Fourier transforms as operations on functions, it is more convenient to use the following notation to indicate that $F(\xi)$ is the Fourier transform of $f(x)$:

$$\mathscr{F}(f) = F \text{ or } f(x) \xrightarrow{\mathscr{F}} F(\xi)$$

Similarly, the Fourier inverse transform is denoted by $\mathscr{F}^{-1}$.

Examples

Given a function, it is not always easy to find its Fourier transform explicitly. For simple functions, tables of Fourier transform formulas are available in most books on Fourier transforms. Symbolic mathematical packages all have Fourier transform routines. Here we look at two important cases where it is possible to find the Fourier transform explicitly.

Example 1 We verify that $F(\xi) = \sin \pi\xi/(\pi\xi)$ when

$$f(x) = \begin{cases} 1, & \text{for } |x| \le 1/2 \\ 0, & \text{for } |x| > 1/2 \end{cases}$$

Indeed, for $\xi \ne 0$, we have

$$\begin{aligned} F(\xi) &= \int_{-\infty}^{\infty} f(x) e^{-jz\pi\xi x}\, dx \\ &= \int_{-1/2}^{1/2} e^{-jz\pi\xi x}\, dx \\ &= -\frac{e^{-jz\pi\xi x}}{jz\pi\xi}\Bigg|_{x=-1/2}^{x=1/2} \\ &= \frac{\sin \pi\xi}{\pi\xi} \end{aligned}$$

Example 2. Let $f(x) = e^{-\pi x^2}$. Then

$$F(\xi) = \int_{-\infty}^{\infty} e^{-\pi x^2} e^{-j2\pi\xi x}\, dx = \int_{-\infty}^{\infty} e^{-(\sqrt{\pi}x)^2 - 2(\sqrt{\pi}x)(j\sqrt{\pi}\xi)}\, dx$$

The integrand is equal to

$$e^{-(\sqrt{\pi}x + j\sqrt{\pi}\xi)^2 - \pi\xi^2}$$

By using Cauchy's Theorem to shift the path of integration from the real axis to the horizontal line $\mathrm{Im}(z) = \sqrt{\pi}\xi$ [with fixed ξ in $(-\infty, \infty)$], one can derive that

$$F(\xi) = e^{-\pi\xi^2}$$

Notice that in this example both the function and its Fourier transform are given by the same formula.

Some Important Results

We now discuss some most important results in Fourier transforms. Among them, the most significant one is related to a very useful operation in Fourier analysis, the convolution of two functions. If $f(x)$ and $g(x)$ are two functions on $(-\infty, \infty)$, their *convolution* is the function $f * g(x)$ defined by

$$f * g(x) = \int_{-\infty}^{\infty} f(y)g(x-y)\,dy$$

provided that the integral exists. Note that if $f * g$ is well-defined, then so is $g * f$, and $f * g(x) = g * f(x)$. There are various assumptions on $f(x)$ and $g(x)$ to ensure that the convolution $f * g(x)$ is defined for all x in $(-\infty, \infty)$. For example:

1. Assume that $f(x)$ is in L^1 and $g(x)$ is bounded (say $|g(x)| < C$ for all x). Then $f * g(x)$ is defined for all x since

$$\int_{-\infty}^{\infty} |f(y)g(x-y)|\,dy \le C\int_{-\infty}^{\infty} |f(x)| < \infty$$

2. Assume that $|f(x)|^2$ and $|g(x)|^2$ are in L^1. Then, using the Cauchy–Schwarz inequality, we have

$$\int_{-\infty}^{\infty} |f(y)g(x-y)|\,dy \le \sqrt{\int_{-\infty}^{\infty} |f(y)|^2\,dy}\sqrt{\int_{-\infty}^{\infty} |g(x-y)|^2\,dy} < \infty$$

   We will use L^2 to denote the set of all functions $f(x)$ such that $|f(x)|2$ is in L^1.
3. Assume that both $f(x)$ and $g(x)$ are in L^1. Then it can be shown that $f * g(x)$ exists for "almost every" x, and $f * g(x)$ is itself in L^1 (see Ref. 4, Sec. 8.1).

The most important property of the Fourier transform is the following result.

Convolution Theorem. Suppose that $f(x)$ and $g(x)$ are in L^1, and $F(\xi)$ and $G(\xi)$ are their Fourier transforms, respectively. Then the Fourier transform of $f * g(x)$ is given by $F(\xi)G(\xi)$; that is,

$$\mathscr{F}(f * g)(\xi) = F(\xi)G(\xi) \quad \text{or} \quad f * g(x) \xrightarrow{\mathscr{F}} F(\xi)G(\xi)$$

The next result is closely related to sampling theory.

Poisson Summation. Let $f(x)$ be a continuous function in L^1 and $F = \mathscr{F}(f)$. If $\sum_{n=-\infty}^{\infty} f(x - 2nL)$ defines a continuous function on $(-L, L)$, and if $\sum_{-\infty}^{\infty} |F(n/2L))|$ converges, then

$$\sum_{-\infty}^{\infty} f(x-2nL) = \sum_{-\infty}^{\infty} \frac{1}{2L} F\left(\frac{n}{2L}\right) e^{jn\pi x/L}$$

Since the integral of the absolute-value squared of a function can be interpreted as its energy, the following identity, Parseval's identity, expresses the fact that a signal's energy is equal to its frequency energy.

Parseval's Identity. Let $f(x)$ be a function in $L^1 \cap L^2$, and let $F(\xi)$ be its Fourier transform. Then

$$\int_{-\infty}^{\infty} |f(x)|^2\,dx = \int_{-\infty}^{\infty} |F(x)|^2\,dx$$

SIGNAL SAMPLING

When we analyze signals using a computer, we are no longer working with continuous time signals, but rather with discrete time functions. This requires the sampling of the continuous signals. Let $f(x)$ be a signal. Let us assume that we sample at equally spaced time intervals, Δ, so that the sequence of sampled signal of $f(x)$ is

$$f_n = f(n\Delta), \qquad n = 0, \pm 1, \pm 2, \ldots$$

We call the number $1/\Delta$ the *sampling rate;* it is the number of samples recorded per second, if time is measured in seconds. Half of the sampling rate is a critical value called the *Nyquist critical frequency,* denoted by f_c; that is, $f_c = 1/(2\Delta)$. The importance of the Nyquist critical frequency can be seen in the following result.

Sampling Theorem. Suppose $f(x)$ is a continuous function in L^1 and $F(\xi) = 0$ for all $|\xi| > f_c$. Then $f(x)$ is completely determined by its values f_n at $n\Delta$, $n = 0, \pm 1, \pm 2, \ldots$. In fact,

$$f(x) = \Delta \sum_{n=-\infty}^{\infty} f_n \frac{\sin[2\pi f_c(x - n\Delta)]}{\pi(x - n\Delta)}$$

A function $f(x)$ whose Fourier transform $F(\xi)$ vanishes outside of a finite interval is said to be *bandwidth-limited.* Therefore, a bandwidth-limited signal whose frequencies are bounded in $[-f_c, f_c]$ can be fully recovered from its sampled values at $n\Delta$, $n = 0, \pm 1, \pm 2, \ldots$ if the sampling rate is twice its Nyquist critical frequency f_c, that is, $1/\Delta = 2f_c$.

NUMERICAL COMPUTATION

Fast Fourier Transform

The fast Fourier transform (FFT) is a family of methods for computing the discrete Fourier transform of a function with minimum computational effort. The FFT became well known after the publication of the article by Cooley and Tukey in 1965, although it had been used in various forms by others before this. Various forms of FFT are available as subroutines in almost every mathematical software package, such as Matlab, Mathematica, and Maple, to name a few. In this section, we provide some basics of the FFT so that the reader will be able to make the best use of it.

Recall that the discrete Fourier transform of N numbers $[h_k]_{k=0}^{N-1}$ is given by

$$H_n = \sum_{k=0}^{N-1} h_k e^{-j2\pi nk/N}, \qquad n = 0, 1, \ldots, N-1$$

Let $W_N = e^{-j2\pi/N}$. Then W_N is an Nth root of the unity; that is,

$$W_N^N = 1$$

Notice that if we compute the transformed points $[H_n]_{n=0}^{N-1}$ directly by their definitions, we need to use N multiplications for each H_n. So, the N numbers H_n, $n = 0, 1, \ldots, N - 1$, would require N^2 multiplications. This can result in a great deal of computation when N is large. It turns out that the discrete Fourier transform of a data set of length N can be computed by using the FFT algorithm, which requires only $(N \log_2 N)/2$ multiplications. This is a significant decrease in the N^2 multiplications required in the direct evaluation of the transform. For example, if $N = 1024$, the direct evaluation requires $N^2 = 1{,}048{,}576$ multiplications. In contrast, the FFT algorithm requires $(1024 \log_2 1024)/2 = 5120$ multiplications.

Suppose $N = 2^M$, where M is a positive integer. Let us split the sum for each n into even and odd parts:

$$\begin{aligned} H_n &= \sum_{k=0}^{N-1} h_k W_N^{nk} \\ &= \sum_{k=0}^{N/2-1} h_{2k} W_N^{n(2k)} + \sum_{k=0}^{N/2-1} h_{2k+1} W_N^{n(2k+1)} \\ &= \sum_{k=0}^{N/2-1} h_{2k} W_{N/2}^{nk} + W_N^n \sum_{k=0}^{N/2-1} h_{2k+1} W_{N/2}^{nk} \end{aligned}$$

Let us write

$$H_n = H_n^0 + W_N^n H_n^1 \tag{19}$$

with

$$H_n^0 = \sum_{k=0}^{N/2-1} h_{2k} W_{N/2}^{nk}$$

and

$$H_n^1 = \sum_{k=0}^{N/2-1} h_{2k+1} W_{N/2}^{nk}$$

Note that $[H_n^0]_{n=0}^{N/2-1}$ and $[H_n^1]_{n=0}^{N/2-1}$ are, respectively, the discrete Fourier transforms of the even components $[h_{2k}]_{k=0}^{N/2-1}$ and the odd components $[h_{2k+1}]_{k=0}^{N/2-1}$. Note also that $[H_n^0]_{n=0}^{N/2-1}$ and $[H_n^1]_{n=0}^{N/2-1}$ are of length $N/2$. So, we have

$$H_{n+N/2}^0 = H_n^0 \quad \text{and} \quad H_{n+N/2}^1 = H_n^1, \qquad n = 0, 1, \ldots, \frac{N}{2} - 1$$

This, together with the fact that $W_N^{n+N/2} = -W_N^n$ ($n = 1, 2, \ldots, N/2 - 1$), allows us to write the equation in Eq. (19) as

$$H_n = H_n^0 + W_N^n H_n^1 \quad \text{and} \quad H_{n+N/2} = H_n^0 - W_N^n H_n^1 \tag{20}$$

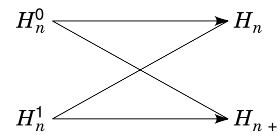

Figure 8. The butterfly diagram.

for $n = 1, 2, \ldots, N/2 - 1$. This pair of calculations is called *combined formulae* or *butterfly relations* since it can be visualized in the so-called *butterfly diagram* (see Fig. 8).

The splitting of $\{H_n\}$ into two half length (i.e., $N/2$) discrete Fourier transforms $\{H_n^0\}$ and $\{H_n^1\}$ can be applied now on a smaller scale. In other words, we define H_n^{00} and H_n^{01} to be the discrete Fourier transforms of the even and odd components of $\{h_{2k}\}_{k=0}^{N/2-1}$ and define H_n^{10} and H_n^{11} to be the discrete Fourier transforms of the even and odd components of $\{h_{2k+1}\}_{k=0}^{N/2-1}$, respectively. We then get

$$H_n^0 = H_n^{00} + W_{N/2}^n H_n^{01} \; H_{n+N/4}^0 = H_n^{00} - W_{N/2}^n H_n^{01}$$

and

$$H_n^1 = H_n^{10} + W_{N/2}^n H_n^{11} \; H_{n+N/4}^1 = H_n^{10} - W_{N/2}^n H_n^{11}$$

for $n = 0, 1, \ldots, N/4 - 1$.

If we continue with this process of halving the length of the discrete Fourier transforms, then after $M = \log_2 N$ steps we reach the point where we are performing the transforms on data of length 1, which is trivial since the discrete Fourier transform of a data set of length 1 is the identity transform. For each of the M steps, there are $N/2$ multiplications, and so there are about $MN = (N \log_2 N)/2$ multiplications needed for the FFT as we claimed at the beginning of this section.

Another important step in FFT is that the repeated processes of splitting the data set into the even and odd subsets of half length can be realized by bits reversal. That is, we order $\{h_k\}$ so that h_k is put at the k'th place, where $k' = a_p a_{p-1} \ldots a_1$ (base 2) if $k = a_1 a_2 \ldots a_p$ (base 2). After the reordering of $\{h_k\}$, we start to use the butterfly relations to combine the adjacent pairs to get 2-point transforms, then combine the 2-point transforms to get the 4-point transforms, and so on, until the final transform $\{H_n\}$ is formed from two $N/2$-point transforms. So, there are two major steps in the FFT algorithm: The first step sorts the data into bit-reversed order. This can be done without additional storage and involves at most $N/2$ swaps of the elements of a data set of length N. The second step calculates, in turn, transforms of length $2, 4, \ldots, N$. Figure 9 shows steps in the FFT algorithm for $N = 8$.

We have only discussed the case when N is a power of 2, the so-called radix 2 case. The algorithm first reorders the input data in bit-reversed order, then builds up the transform in $\log_2 N$ steps. This is referred to as the *decimation-in-time* FFT. It is also possible to go through the $\log_2 N$ steps of transforms and then rearrange the output into the bit-reversed order. This is the *decimation-in-time* FFT.

There are higher-dimensional generalizations of the FFT for transforming complex functions defined over a two- or higher-dimensional grid (see Ref. 5).

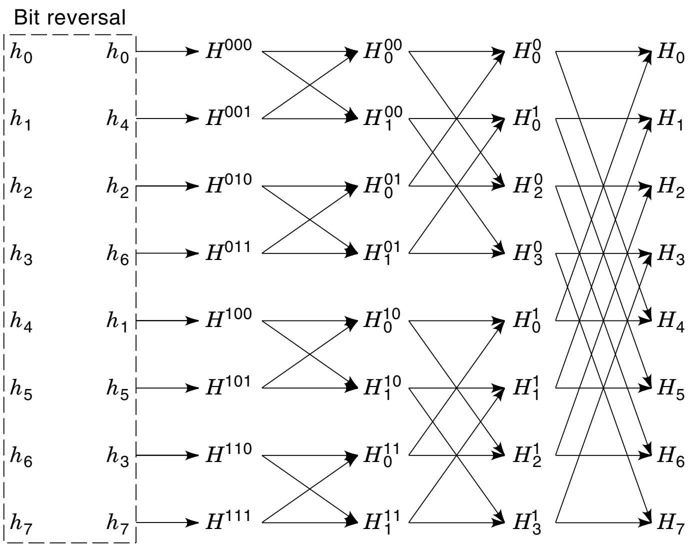

Figure 9. The decimation-in-time (radix 2) FFT for a data set of length $N = 8$.

Fast Sine and Cosine Transform

Discrete sine and cosine transforms can be derived from the Fourier sine and cosine series discussed earlier.

Discrete sine and cosine transforms. For a real sequence $\{h_k\}_{k=1}^{N-1}$ the discrete sine transform (DST), $\{H_n^{S}\}_{n=1}^{N-1}$, is given by

$$H_n^{S} = \sum_{k=1}^{N-1} h_k \sin \frac{\pi n k}{N}, \qquad n = 1, 2, \ldots, N-1$$

For a real sequence $\{h_k\}_{k=0}^{N-1}$ the discrete cosine transform (DCT), $\{H_n^{C}\}_{n=0}^{N-1}$, is given by

$$H_n^{C} = \sum_{k=0}^{N-1} h_k \cos \frac{\pi n k}{N}, \qquad n = 1, 2, \ldots, N-1$$

The form of DCT given above is only one of several commonly used DCTs. DSTs and DCTs are related to odd/even symmetry of DFTs, as with Fourier sine and cosine series. See Ref. 6. To compute the DSTs and DCTs, there exist the fast sine transform and fast cosine transform. These algorithms are usually implemented by using the FFT routine (see Refs. 3 and 5).

WAVELETS APPROACH

In Fourier analysis, every function is expanded into a series or an integral of sines and cosines which are themselves analytic functions. When approximating a function with a point of discontinuity, the partial sums of its Fourier series do not converge uniformly in any neighborhood of the point of discontinuity; they do a very poor job in approximating sharp spikes. (Recall that near an isolated point of discontinuity of a function of bounded variation, the Gibbs' phenomenon occurs.) For many years, scientists have searched for more appropriate functions than sines and cosines to represent functions with discontinuities. With the construction of smooth, compactly supported, orthogonal wavelet basis functions (now referred to as the Daubechies wavelets) by Ingrid Daubechies in 1988, wavelet analysis emerged as a powerful toolbox leading to new and varied applications in, for example, data compression, signal and image processing, nuclear engineering, geology, and such pure mathematics as solving differential equations.

A (mother) wavelet $\Phi(x)$ is a function in L^1 that has zero average:

$$\int_{-\infty}^{\infty} \Phi(x)\, dx = 0$$

Useful wavelets satisfy further conditions that we will not specify in this article since a more detailed discussion on wavelets is given in WAVELET TRANSFORMS. Here, we just try to give a very brief introduction of the wavelet theory and compare it with the classical Fourier analysis discussed in this article. In Fig. 10, we show two typical mother wavelets: Haar wavelet and Daubechies wavelet (DAUB6).

The Discrete Wavelet Transform

Unlike Fourier analysis, in which we represent functions by series of sines and cosines, in wavelet analysis we represent

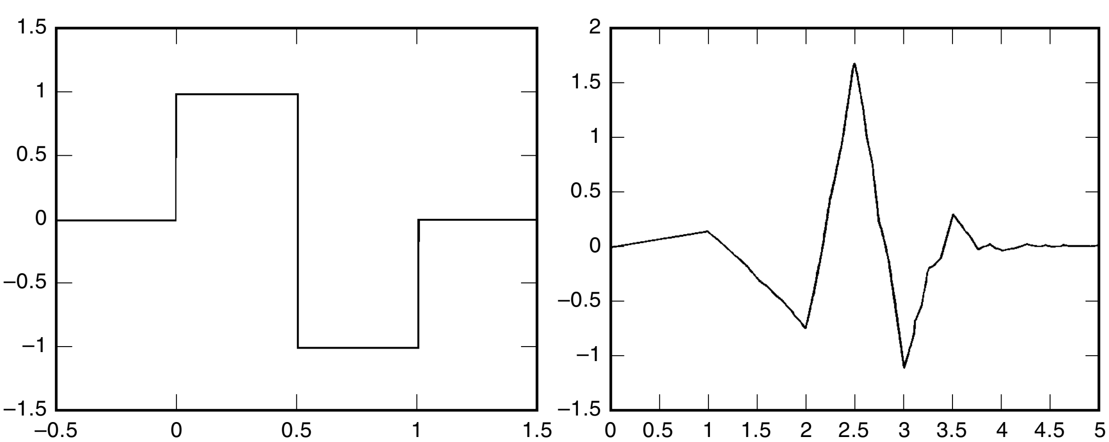

Figure 10. Haar wavelet and Daubechies wavelet (DAUB6).

functions by series of dilations and translations of a single function called the *mother wavelet* $\Phi(x)$:

$$\Phi_{s,l}(x) = 2^{-s/2}\Phi(2^{-s}x - l), \qquad s, l = 0, \pm 1, \pm 2, \ldots$$

So, we would like to write

$$f(x) = \sum_{s,l=-\infty}^{\infty} f_{s,l}\, \Phi_{s,l}(x)$$

See WAVELET TRANSFORMS or Ref. 7. The set of coefficients $\{f_{s,l}\}$ is called the discrete wavelet transform of $f(x)$ (with respect to mother wavelet $\Phi(x)$). Of course, as in the case of Fourier series, the above "equality" needs suitable interpretations. More important to applications is the fact that mother wavelets Φ can be chosen best adapted to the problems at hand. This is possible because there are many different mother wavelets with different properties available. There is even a fairly general scheme for generating various mother wavelets, a procedure called *multiresolution analysis.*

To compute the wavelet transform, we face the same complexity issue that we previously faced for the computation of the discrete Fourier transform. Fortunately, there exists a "fast" wavelet transform algorithm that requires only order n operations to transform an n-sample vector.

Wavelet Analysis Versus Fourier Analysis

We start with the similarities between these two. The discrete Fourier transform and discrete wavelet transform are both linear operations that can be carried out in "almost linear" time; that is, about $n \log_2 n$ or n operations (addition, multiplication) are needed to transform a sample vector of size n. Another similarity is that both basis functions, sines and cosines in Fourier analysis, and dilations and translations of a mother wavelet in wavelet analysis are localized in frequency, making spectral analysis possible. When used to represent smooth or stationary signals, both Fourier and wavelet methods perform almost equally well.

The most striking difference between these two kinds of transforms is that wavelet functions are localized in time (or space) domain, while Fourier sines and cosines are not. When analyzing nonstationary signals, it is often desirable to be able to acquire a correlation between the time and frequency domains of a signal. The Fourier analysis provides information about the frequency domain, but time localized information is essentially lost in the process. In contrast to the Fourier analysis, the wavelet transforms allow exceptional localization in time domain via translations of the mother wavelet, as well as in frequency (scale) domain via dilations.

CONCLUSIONS

Given a signal, Fourier analysis decomposes it into its frequency components. This decomposition can be used to represent the original signal if it is smooth (or piecewise smooth) and time-invariant (stationary). When a signal has lots of jumps (points of discontinuity), at each jump Gibbs' phenomenon may occur. In the case of analyzing transient or nonstationary signals, wavelet transforms provide much desired tools. The mathematical theory of wavelets has been more or less established in the last decade. The future of wavelets lies in the discovery of its applications in every discipline of engineering. See FREQUENCY DOMAIN CIRCUIT ANALYSIS, WAVELET TRANSFORMS.

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FOURIER SERIES. See POWER SYSTEM HARMONICS.