GEOMETRY

Geometry is a comprehensive branch of mathematics that discusses the properties of space and objects in space. Geometry deals with points and with the measurement and the relationship of lines, angles, two-dimensional surfaces, and three-dimensional solids. The word *geometry* is derived from the Greek words for "earth" and "measure," and the Greeks should be credited for developing geometry as a systematic science. Geometric figures have been found in cave paintings and pottery that date back as far as 12,500 B.C. The Egyptians had excellent capabilities with regard to applying geometry (e.g., to build pyramids), but they did not lay out the laws, rules, and theorems systematically by modern standards. Greek philosophers, mathematicians, and astronomers provided a strong mathematical foundation for the empirical art and transformed it to a systematic science. Among the Greek pioneers, Pythagoras (ca. 540 B.C.), arranged sets of objects in geometrical shapes (Fig. 1), Plato (ca. 400 B.C.) tried to explain the nature of the universe using geometrical principles, and his pupil Aristotle developed laws for logical reasoning.

Euclid, who lived in Alexandria (ca. 330 to 275 B.C.), finally systematized all the work of Pythagoras, Plato, Aristotle and other learned scientists and philosophers in his book *Elements.* Euclid's mathematical accomplishments were taught in Plato's academy, and *Elements* was considered to be the one true geometry for almost two thousand years. Euclid is also supposed to have written four books on conic sections, but that work has been completely lost.

Euclid is considered to be one of the most influential mathematicians of all time. His greatness lies in the fact that he developed geometry logically as an exact science and that his work has been studied for more than 24 centuries all over the world. A complete text of all 13 books of Euclid can be read and studied using a Java applet called *Geometry Applet* (1). Another Greek mathematician, Menaechmus, first studied conic sections in fourth century B.C. Archimedes (ca. 287 to 212 B.C.) used Euclid's principles and found the area of an ellipse, the area of a sector of a parabola, the volume of a sphere, and the like. Apollonius (ca. 225 B.C.) wrote eight books on conic sections (only seven have survived). Apollonius was the first to develop the concept of coordinates, which was later perfected by the French mathematician René Descartes (1596–1650). The modern study of geometry makes extensive use of the Cartesian system of coordinates, named after him.

Euclidean geometry can be viewed as a study of *congruent figures.* In his 13 books, Euclid set forth the elementary part of the deductive system of geometry. Many principles were axioms and regarded as obviously true. (Example: All right angles are equal.) Some were "common notions." (Example: The whole is greater than a part.) Some were derived from experience and eventually developed into powerful theorems that have established strong foundations for modern-day mathematicians.

For almost 2000 years Euclid's 13 books defined geometry, and all schools and colleges used Euclid's books in one form or the other. The *Elements* was translated from Greek to Arabic (ca. 800 A.D.), then to Latin (ca. 1120 A.D.), and was first translated into English at the end of the sixteenth century.

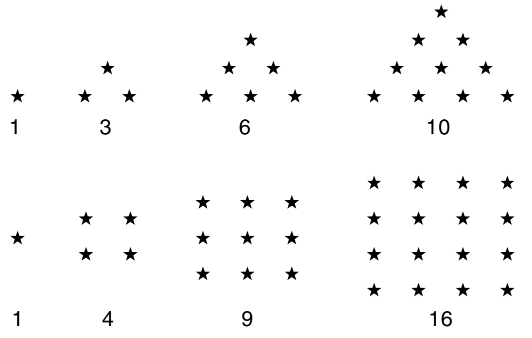

Figure 1. Objects arranged geometrically: 1, 3, 6, 10 are triangular numbers [$n(n + 1)/2$]; 1, 4, 9, 16 are square numbers (n^2).

The five chief axioms (or postulates) of Euclid are as follows:

1. A straight line can be drawn from any given point to another point: Given two points, there is an interval that joins them.
2. A finite straight line can be drawn continuously: An interval can be prolonged indefinitely.
3. A circle is determined by its center and its radius. A circle can be constructed when its center and a point on it are given.
4. All right angles are equal.
5. If a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, the two straight lines, if produced indefinitely, meet on that side on which the angles are less than the two right angles.

The fifth postulate is the most famous of Euclid's axioms, and is called the *parallel axiom.* It is also the most controversial. Many questions have been raised concerning it. Euclid himself must have been aware of the fact that the fifth postulate is lengthy (compared with the first four) in making its statement. Euclid proved his first 28 propositions without making use of the fifth axiom. Mathematicians are intrigued by the fifth postulate and have tried in vain to prove it or disprove it. An equivalent statement, *Playfair's axiom,* states: "Through a point not on a given line, there passes not more than one parallel to the line."

The fifth axiom is considered inapplicable or invalid in *hyperbolic geometry,* because rays may converge at first, but later diverge after attaining a minimum distance. The fifth axiom is satisfied only marginally in the *elliptic geometry.* Various mathematicians have contributed to such non-Euclidean geometries. Some were attempting to verify the validity of the fifth postulate. The Italian logician Girolamo Saccheri; the German mathematicians Georg Simon Klügel, Carl Friedrich Gauss, and Friedrich Ludwig Wachter; the Russian Nikolay Ivanovich Lobachevsky; and the Hungarian father and son Farkas Bolyai and János Bolyai are among the most prominent contributors. Non-Euclidean geometry and the theory of relativity have dramatically changed the thinking of nineteenth- and twentieth-century scientists and mathematicians.

The collection of propositions based only on the first four axioms of Euclid is known as *absolute geometry.* It should be noted that in fact we might well be living in a non-Euclidean universe.

In addition to the above-mentioned five axioms, the five important "common notions" are listed below:

1. Things equal to the same thing are equal.
2. If equals are added to equals, the wholes are equal.
3. If equals are subtracted from equals, the remainders are equal.
4. Things that coincide with one another are equal.
5. The whole is greater than a part.

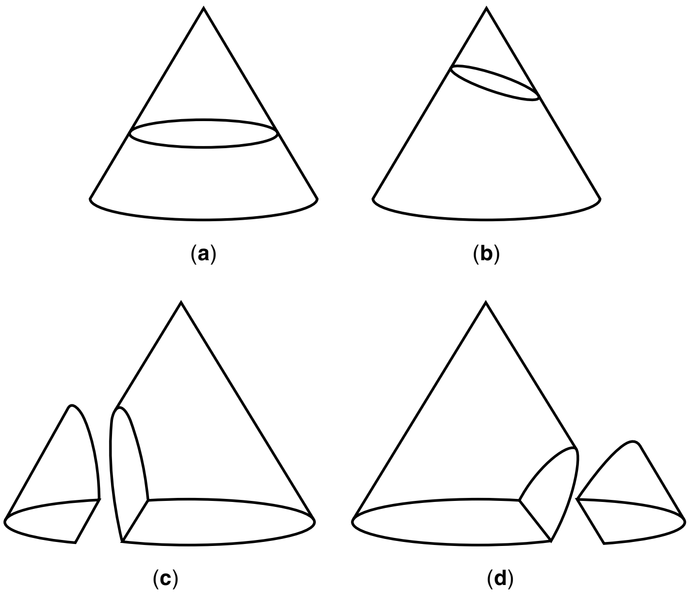

Figure 2. Conic sections: (a) circle, (b) ellipse, (c) hyperbola, and (d) parabola.

ANALYTIC GEOMETRY

In 1637, the French mathematician René Descartes applied algebra to geometry to calculate the dimensions of geometrical figures. The idea of negative distances developed by Sir Isaac Newton in the seventeenth century helped to perfect this new branch of geometry, also called *coordinate geometry,* wherein lines and curves could be represented by sets of equations. In two dimensions, an extensive study of mathematical expressions for conic sections was made possible. Figure 2 shows four important conic sections: the circle, ellipse, parabola, and hyperbola. Since Euclid, it is the German mathematician David Hilbert who has made the greatest impact on the mathematical world of geometry. His publication of *Grundlagen der Geometrie* in 1899 formalized the notions of hyperbolic geometry and elliptic geometry. A simplified view of the two non-Euclidean geometries can be seen in Fig. 3(a) and (b).

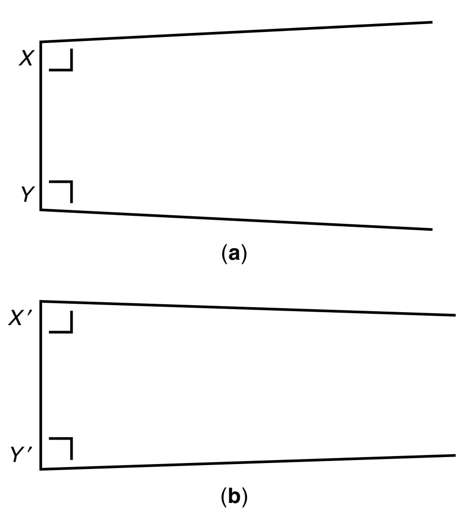

Figure 3. (a) Hyperbolic geometry: rays share perpendicular line, then diverge; (b) elliptic geometry: rays share perpendicular line then converge.

Consider two perpendicular rays emerging from the same side of a line that connects points X and Y. If two rays diverge, as they extend farther and farther from the line XY, then the geometry is said to be hyperbolic. If the two rays converge, then it is said to be elliptic. In considering this type of non-Euclidean universe, one has to develop intuition without the help of observation.

Hilbert, in his 21 axioms, generalized the foundations of geometry and replaced the classical concepts that had been derived from intuition. His generalization of classical geometry has found application in science and industry.

The German mathematicians Leonhard Euler and Gaspard Monge are credited with the development of *differential geometry* in the eighteenth century. They studied the geometry of curves and surfaces in space as an application of calculus. Another German mathematician, Bernhard Riemann, who had a thorough understanding of the limitations of Euclidean geometry, developed what is known as *double elliptic geometry.* This new geometry played a vital role in defining a four-dimensional Riemannian space and in developing the theory of relativity.

PROJECTIVE AND ANALYTIC GEOMETRY

The German astronomer Johannes Kepler extended the Euclidean plane to three dimensions and helped in the development of solid analytic geometry. Analytic geometry along with calculus enabled the development of a wide variety of special curves that are extremely useful in space-age applications. For example, a special case of the cubical parabola ($y = ax^3$) was studied by the German mathematician Gottfried Wilhelm Leibniz in 1675 and is shown in Fig. 4.

Some of the famous geometrical patterns and curves and shapes that are derived from the equation $r^x = a^x\theta$ are also shown in Fig. 5. When $x = 1$, we have $r = a\theta$ and the curve is known as the *spiral of Archimedes.* When $x = 2$ it is called Fermat's spiral and follows the equation $r^2 = a^2\theta$. Johann Bernoulli studied the case $x = -1$, or $a = r\theta$.

Another important curve follows an algebraic equation that is of the general form

$$y = e^{-0.5x}$$

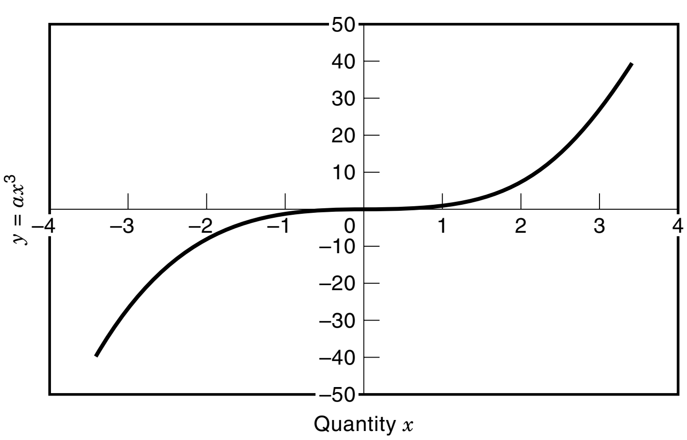

Figure 4. Cubical parabola.

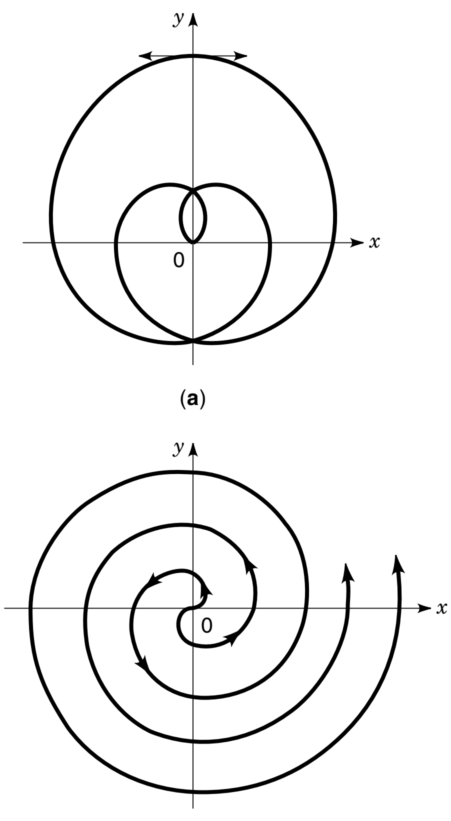

Figure 5. Geometrical patterns generated from equation $r^x = \alpha^x\theta$: (a) spiral of Archimedes ($r = \alpha\theta$); (b) Fermat's spiral ($r^2 = \alpha^2\theta$).

A version of this equation results in the well-known *bell-shaped curve* (Fig. 6). Although Abraham de Moivre, a French mathematician, was the first to study it, it is commonly known as the Gaussian distribution curve. It is associated with the French mathematician Pierre Simon Laplace as well.

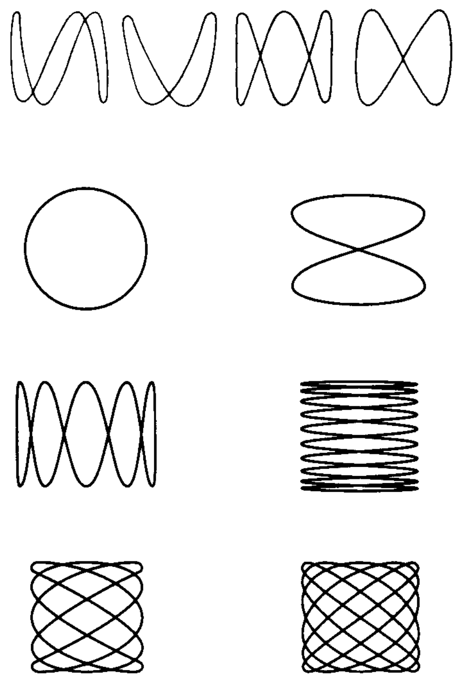

Figure 7. Bowditch curves or Lissajous figures.

Electrical engineers have greatly benefited from Lissajous figures, shown in Fig. 7. These are also known as Bowditch curves after Nathaniel Bowditch. By their use an oscilloscope can be used to compare frequencies. An electron beam can be deflected either in the x direction or in the y direction using suitable voltages at appropriate frequencies. If $x = A \sin \omega t$ and $y = B \sin(k\omega t + \phi)$, then the oscilloscope displays patterns as shown in Fig. 7 for different frequencies.

One of the most important of all special curves in the cycloid, which is obviously of great interest to engineers who study the motion of wheels. It follows the equations

$$x = a(\phi - \sin\phi) \quad \text{and} \quad y = a(1 - \cos\phi)$$

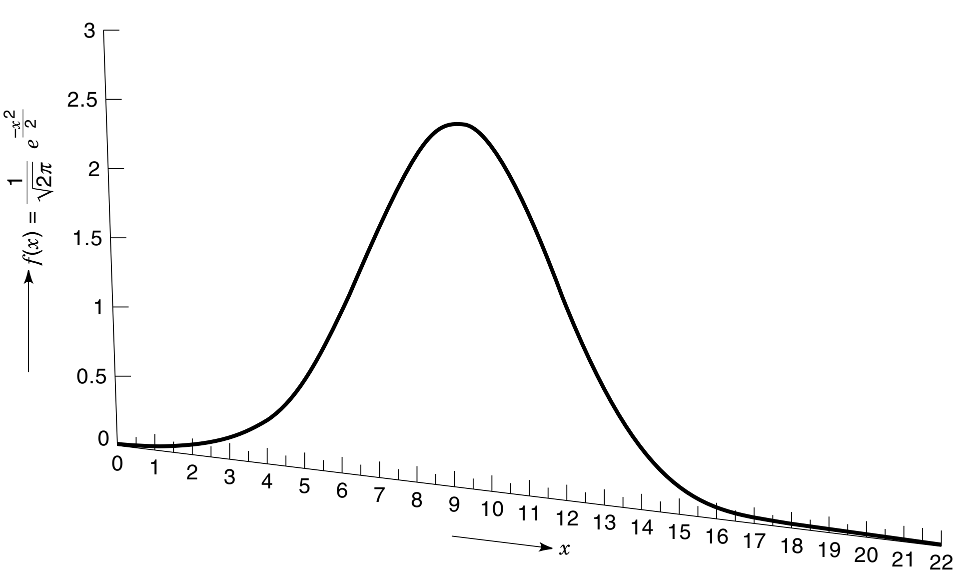

Figure 6. Bell-shaped curve or Gaussian distribution.

The curve is the path of a point on the circumference of a circle of radius a that rolls along a straight line without slipping or sliding. Galileo (1564–1642) was among the first to study the cycloid.

DIFFERENTIAL GEOMETRY

One can easily observe that the concept of the derivative of a function is similar to drawing a tangent to a curve and finding its slope at the given point. The area under a curve can almost always be determined by evaluating the integral of the equation defining the curve. Thus, differential calculus and integral calculus can be used to study the geometry of two-dimensional curves and three-dimensional surfaces. Leonhard Euler and Gaspard Monge are to be credited for the development of differential geometry. A clear understanding of the concepts of topology, elliptic operators, the Gauss–Bonnet formula, manifolds, and tensor bundles is essential for studying the principles of differential geometry in depth.

BIBLIOGRAPHY

D. E. Joyce, *Geometry Applet,* http://aleph0.clarku.edu/~djoyce/java/Geometry/Geometry.html.

Reading List

J. Stillwell, *Numbers and Geometry,* New York: Springer, 1998.

H. T. Croft, K. J. Falconer, and R. K. Guy, *Unsolved Problems in Geometry,* New York: Springer-Verlag, 1991.

G. Stephenson, *Worked Examples in Mathematics for Scientists and Engineers,* London: Longman, 1985.

G. Temple, *100 Years of Mathematics,* New York: Springer-Verlag, 1981.

J. Fang, *Mathematicians from Antiquity to Today,* Hauppauge, NY: Paideia Press, 1972.

D. E. Smith and J. Ginsburg, *A History of Mathematics in America Before 1900,* New York: Arno Press, 1980.

F. Cajori, *A History of Mathematics,* New York: Chelsea Publishing Co., 1980.

E. T. Bell, *The Development of Mathematics,* New York: McGraw-Hill, 1940.

R. Billstein and J. W. Lott, *Mathematics For Liberal Arts,* Reading, MA: Benjamin/Cummings, 1986.

R. A. Barnett and M. R. Ziegler, *Applied Mathematics,* 3rd Ed., San Francisco, CA: Dellen Publishing Co., 1989.

G. B. Arfken and H. J. Weber, *Mathematical Methods for Physicists,* 4th Ed., San Diego, CA: Academic Press, 1995.

C. E. Pearson (ed.), *Handbook of Applied Mathematics,* 2nd Ed., New York: Van Nostrand, 1983.

MYSORE NARAYANAN
Miami University