GREEN'S FUNCTION METHODS

The Green's function method is a powerful technique for solving boundary-value problems. Green's function was named after George Green (1793–1841), who developed a general method to obtain solutions of Poisson's equation in potential theory. This method was described in an essay by Green entitled "On the application of mathematical analysis to the theories of electricity and magnetism," published in 1828.

To illustrate the Green's function method, consider the electric potential produced by a point electric charge q_1 placed at $\mathbf{r}_1$ in an unbounded homogeneous free space. It is well known from the elementary theory of electricity (1) that this potential at $\mathbf{r}$ is given by

$$\phi_1(\mathbf{r}) = \frac{q_1}{4\pi\epsilon|\mathbf{r}-\mathbf{r}_1|} \tag{1}$$

where $|\mathbf{r} - \mathbf{r}_1|$ denotes the distance between the points $\mathbf{r}$ and $\mathbf{r}_1$ and ϵ is a constant called the permittivity. If there is another point charge q_2 placed at $\mathbf{r}_2$, the potential produced by this charge is

$$\phi_2(\mathbf{r}) = \frac{q_2}{4\pi\epsilon|\mathbf{r}-\mathbf{r}_2|} \tag{2}$$

The total potential produced by q_1 and q_2 is then the linear superposition of ϕ_1 and ϕ_2:

$$\phi(\mathbf{r}) = \phi_1(\mathbf{r}) + \phi_2(\mathbf{r}) = \frac{q_1}{4\pi\epsilon|\mathbf{r}-\mathbf{r}_1|} + \frac{q_2}{4\pi\epsilon|\mathbf{r}-\mathbf{r}_2|} \tag{3}$$

If there are N point charges in the space, the total potential is given by

$$\phi(\mathbf{r}) = \sum_{i=1}^{N} \phi_i(\mathbf{r}) = \sum_{i=1}^{N} \frac{q_i}{4\pi\epsilon|\mathbf{r}-\mathbf{r}_i|} \tag{4}$$

where Σ denotes the summation over all point charges and ϕ_i denotes the potential due to the ith point charge placed at $\mathbf{r}_i$. The procedure described above is known as the principle of linear superposition.

Next, consider the electric potential produced by a volume electric charge whose charge density is denoted by $\rho(\mathbf{r})$. To find the potential, we divide the volume of the charge into many small cubes. The charge within each small cube is then given by

$$q_i \approx \rho(\mathbf{r}_i)\Delta V_i \tag{5}$$

where $\mathbf{r}_i$ denotes the center of the ith cube and ΔV_i denotes the volume of the cube. Since each cube is very small, it can be approximated as a point charge, whose potential is given by

$$\phi_i(\mathbf{r}) \approx \frac{q_i}{4\pi\epsilon|\mathbf{r}-\mathbf{r}_i|} \approx \frac{\rho(\mathbf{r}_i)\Delta V_i}{4\pi\epsilon|\mathbf{r}-\mathbf{r}_i|} \tag{6}$$

According to the principle of linear superposition, the total potential is then given by

$$\phi(\mathbf{r}) = \sum_{i=1}^{N} \phi_i(\mathbf{r}) \approx \sum_{i=1}^{N} \frac{\rho(\mathbf{r}_i)\Delta V_i}{4\pi\epsilon|\mathbf{r}-\mathbf{r}_i|} \tag{7}$$

Clearly, the approximation improves as the volume is divided into smaller cubes. In the limit when $\Delta V_i \to 0$, Eq. (7) becomes exact. Hence, one obtains

$$\phi(\mathbf{r}) = \lim_{\Delta V_i \to 0} \sum_{i=1}^{\infty} \frac{\rho(\mathbf{r}_i)\Delta V_i}{4\pi\epsilon|\mathbf{r}-\mathbf{r}_i|} \tag{8}$$

which can be written in the integral form as

$$\phi(\mathbf{r}) = \int_V \frac{\rho(\mathbf{r}')\, dV'}{4\pi\epsilon|\mathbf{r}-\mathbf{r}'|} \tag{9}$$

where V denotes the volume of the electric charge.

The potential produced by a point source of unit strength is called the Green's function. In the example above, the Green's function is

$$G(\mathbf{r},\mathbf{r}') = \frac{1}{4\pi\epsilon|\mathbf{r}-\mathbf{r}'|} \tag{10}$$

and the total potential can then be written as

$$\phi(\mathbf{r}) = \int_V \rho(\mathbf{r}')G(\mathbf{r},\mathbf{r}')\, dV' \tag{11}$$

It is clear that the Green's function method treats an arbitrary source for the potential as a linear superposition of weighted point sources. It then finds the potential as the corresponding linear superposition of the potentials produced by the point sources.

Obviously, once the Green's function corresponding to the potential due to a point source is found, the potential produced by an arbitrary distribution of sources can be obtained easily. Therefore, for a specific boundary-value problem, instead of finding the potential for each new source encountered by solving Poisson's equation repeatedly, one can find the Green's function for that problem only once and obtain solutions to any sources by the principle of linear superposition. The procedure of finding the Green's function is usually much simpler than finding the solution to an arbitrary source. To a large extent, a Green's function plays the same role as an impulse response of a linear circuit system. The system response to any input function can be determined by convolving the input function with the impulse response of the system. The Green's function method has since been expanded to deal with a large number of different partial differential equations.

In electrodynamics, both the source (electric current density) and the response (electric or magnetic field) are vectors, each of which has three components. Since each component of an electric current density can produce all three components of the electric or magnetic field, one has nine Green's functions that relate the response to the source. This unwieldiness can be alleviated by introducing the concept of the dyadic Green's function. A dyadic Green's function, which can be expressed as a 3×3 matrix, can be considered as a compact representation of the nine scalar Green's functions. The first use of dyadic Green's function was made by Julian Schwinger. The subject was also covered by Morse and Feshbach (2) in their well-known treatise on the methods of theoretical physics. A more comprehensive treatment of the dyadic Green's functions in electromagnetic theory was presented by Tai (3), who has done much original work on this topic. In his well-known book, Tai derived dyadic Green's functions for a variety of electromagnetic problems of practical importance. Discussions on dyadic Green's functions can also be found in Collin (4), Kong (5), and Chew (6).

As shall be shown later, the Green's function method not only provides a solution to many boundary-value problems involving canonical geometries, but it also leads to integral equations for problems involving more complex geometries. These integral equations form the basis for a numerical solution of complex boundary-value problems.

SCALAR GREEN'S FUNCTIONS

When both the source and response are scalar functions, the corresponding Green's function is also scalar and, hence, the name scalar Green's function.

The Delta Function

Since the Green's function method is based on the representation of an arbitrary source by the superposition of point sources, the mathematical representation of a point source will first be described. Consider an electric charge of unit strength located at point $\mathbf{r}'$. When the volume of the charge approaches zero, the charge density can be described by a function

$$\delta(\mathbf{r}-\mathbf{r}') = \begin{cases} \infty & \text{for } \mathbf{r} = \mathbf{r}' \\ 0 & \text{for } \mathbf{r} \neq \mathbf{r}' \end{cases} \tag{12}$$

Since the total charge remains at unity,

$$\int_V \delta(\mathbf{r}-\mathbf{r}')\, dV = \begin{cases} 1 & \text{for } \mathbf{r}' \text{ in } V \\ 0 & \text{for } \mathbf{r}' \text{ not in } V \end{cases} \tag{13}$$

The function defined in Eqs. (12) and (13) is known as the Dirac delta function, named after P. A. M. Dirac. Clearly, given an arbitrary function $f(\mathbf{r})$, which is continuous at $\mathbf{r} = \mathbf{r}'$,

$$\int_V f(\mathbf{r})\delta(\mathbf{r}-\mathbf{r}')\, dV = \begin{cases} f(\mathbf{r}') & \text{for } \mathbf{r}' \text{ in } V \\ 0 & \text{for } \mathbf{r}' \text{ not in } V \end{cases} \tag{14}$$

This expression represents a volume source $f(\mathbf{r}')$ as a linear superposition of an infinite number of point sources $\delta(\mathbf{r} - \mathbf{r}')$.

In one dimension, the delta function can be considered as the limit of a function,

$$\delta(x-x') = \lim_{\epsilon \to 0} u_\epsilon(x-x') \tag{15}$$

where $u_\epsilon(x - x')$ is called a delta family. It can be a rectangular function of width ϵ and height $1/\epsilon$, or a triangular function of width 2ϵ and height $1/\epsilon$, or a Gaussian function $e^{-(x-x')^2/2\epsilon^2}/\epsilon\sqrt{2\pi}$, all centered at $x = x'$. The important feature of the delta function is not its shape, but the fact that its effective width approaches zero, while its area remains at unity, that is,

$$\int_a^b \delta(x-x')\,dx = \begin{cases} 1 & \text{for } x' \text{ in } (a,b) \\ 0 & \text{for } x' \text{ not in } (a,b) \end{cases} \quad (16)$$

such that

$$\int_a^b f(x)\delta(x-x')\,dx = \begin{cases} f(x') & \text{for } x' \text{ in } (a,b) \\ 0 & \text{for } x' \text{ not in } (a,b) \end{cases} \quad (17)$$

The delta function so defined is not a function in the classical sense. For this reason, it is called a symbolic or generalized function (7).

Clearly, the delta function is a symmetric function

$$\delta(x-x') = \delta(x'-x) \quad (18)$$

The three-dimensional delta function in the rectangular, cylindrical, and spherical coordinate systems is related to the one-dimensional delta function by

$$\delta(\mathbf{r}-\mathbf{r}') = \delta(x-x')\delta(y-y')\delta(z-z') \quad (19)$$

$$\delta(\mathbf{r}-\mathbf{r}') = \frac{\delta(\rho-\rho')\delta(\phi-\phi')\delta(z-z')}{\rho} \quad (20)$$

$$\delta(\mathbf{r}-\mathbf{r}') = \frac{\delta(r-r')\delta(\theta-\theta')\delta(\phi-\phi')}{r^2\sin\theta} \quad (21)$$

All of the above satisfy Eq. (13).

One-Dimensional Green's Function

To introduce the concept of Green's function in one dimension, consider an infinitely long transmission line with a distributed current source $K(x)$ (3), as illustrated in Fig. 1. Using Kirchhoff's voltage and current laws, one finds the relations between the voltage and current as

$$\frac{dV(x)}{dx} + (j\omega L + R)I(x) = 0 \quad (22)$$

$$\frac{dI(x)}{dx} + (j\omega C + G)V(x) = K(x) \quad (23)$$

where ω denotes the angular frequency and L, C, R, and G are the inductance, capacitance, resistance, and conductance of the transmission line per unit length. Eliminating $I(x)$ in Eqs. (22) and (23), one obtains the differential equation for the voltage as

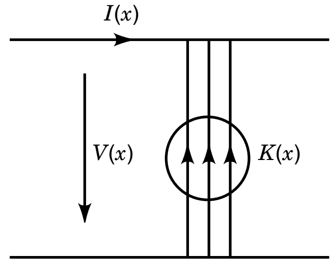

Figure 1. An infinitely long transmission line excited by a distributed current source.

$$\frac{d^2V(x)}{dx^2} - \gamma^2 V(x) = -(j\omega L + R)K(x) \quad (24)$$

where $\gamma^2 = (j\omega L + R)(j\omega C + G)$. Since the line is infinitely long, there is no reflected wave; hence, $V(x)$ satisfies the boundary conditions

$$\frac{dV(x)}{dx} + \gamma V(x) = 0 \qquad \text{for } x \to \infty \quad (25)$$

$$\frac{dV(x)}{dx} - \gamma V(x) = 0 \qquad \text{for } x \to -\infty \quad (26)$$

Since these boundary conditions are imposed when $|x| \to \infty$, they are also called radiation conditions.

Instead of solving for $V(x)$ directly from Eqs. (24)–(26), one can consider the solution of the following differential equation

$$\frac{d^2g_0(x,x')}{dx^2} - \gamma^2 g_0(x,x') = -\delta(x-x') \quad (27)$$

where $g_0(x)$ satisfies the same radiation conditions as $V(x)$. Since $g_0(x, x')$ is a point source response and $V(x)$ in Eq. (24) is due to the source $(j\omega L + R)K(x)$, according to the principle of linear superposition, $V(x)$ can be expressed as a convolution of $g_0(x, x')$ with $(j\omega L + R)K(x)$:

$$V(x) = \int_{-\infty}^{\infty} (j\omega L + R)K(x')g_0(x,x')\,dx' \quad (28)$$

It is evident that once we obtain $g_0(x, x')$, the voltage on the transmission line can be evaluated via a simple integration using Eq. (28).

To find $g_0(x, x')$, note that since

$$\frac{d^2g_0(x,x')}{dx^2} - \gamma^2 g_0(x,x') = 0 \qquad \text{for } x > x' \text{ or } x < x' \quad (29)$$

one has

$$g_0(x,x') = Ae^{-\gamma x} \qquad \text{for } x > x' \quad (30)$$

$$g_0(x,x') = Be^{\gamma x} \qquad \text{for } x < x' \quad (31)$$

where the radiation conditions in Eqs. (25) and (26) were used to determine the sign in front of γ. To determine the unknown coefficients A and B, consider Eq. (27). First, note that $g_0(x, x')$ must be continuous at $x = x'$, that is,

$$g_0(x,x')|_{x=x'+0} = g_0(x,x')|_{x=x'-0} \quad (32)$$

where $x = x' + 0$ stands for the right-hand side of x' and $x = x' - 0$ stands for the left-hand side of x' since a discontinuity in $g_0(x, x')$ at $x = x'$ would result in a derivative on $\delta(x - x')$ on the left-hand side of Eq. (27). Next, integrate Eq. (27) over the region from $x' - \epsilon$ to $x' + \epsilon$ and in the limit when $\epsilon \to 0$,

$$\left.\frac{dg_0(x,x')}{dx}\right|_{x=x'+0} - \left.\frac{dg_0(x,x')}{dx}\right|_{x=x'-0} = -1 \quad (33)$$

Applying these two conditions to Eqs. (30) and (31), one finds

$$g_0(x, x') = \frac{1}{2\gamma} e^{-\gamma(x-x')} \qquad \text{for } x > x' \tag{34}$$

$$g_0(x, x') = \frac{1}{2\gamma} e^{\gamma(x-x')} \qquad \text{for } x < x' \tag{35}$$

or, more compactly,

$$g_0(x, x') = \frac{1}{2\gamma} e^{-\gamma|x-x'|} \tag{36}$$

This is the Green's function for the infinitely long transmission line.

Two- and Three-Dimensional Green's Functions

Consider the electric and magnetic fields produced by a time harmonic electric source whose current density is denoted by $\mathbf{J}(\mathbf{r})$ and charge density is denoted by $\rho(\mathbf{r})$. These fields satisfy Maxwell's equations given by (1)

$$\nabla \times \mathbf{E}(\mathbf{r}) = -j\omega\mathbf{B}(\mathbf{r}) \tag{37}$$

$$\nabla \times \mathbf{H}(\mathbf{r}) = j\omega\mathbf{D}(\mathbf{r}) + \mathbf{J}(\mathbf{r}) \tag{38}$$

$$\nabla \cdot \mathbf{D}(\mathbf{r}) = \rho(\mathbf{r}) \tag{39}$$

$$\nabla \cdot \mathbf{B}(\mathbf{r}) = 0 \tag{40}$$

and the constitutive relations given by $\mathbf{B} = \mu\mathbf{H}$ and $\mathbf{D} = \epsilon\mathbf{E}$, where μ is the magnetic permeability and ϵ is the electric permittivity. Again, assume that the space is homogeneous. Taking the curl of Eq. (37), one has

$$\nabla \times \nabla \times \mathbf{E}(\mathbf{r}) = -j\omega\mu\nabla \times \mathbf{H}(\mathbf{r}) \tag{41}$$

Using Eq. (38) in Eq. (41), one obtains

$$\nabla \times \nabla \times \mathbf{E}(\mathbf{r}) - k^2\mathbf{E}(\mathbf{r}) = -j\omega\mu\mathbf{J}(\mathbf{r}) \tag{42}$$

where $k^2 = \omega^2\mu\epsilon$. Since $\nabla \times \nabla \times \mathbf{E} = \nabla(\nabla \cdot \mathbf{E}) - \nabla^2\mathbf{E}$, Eq. (42) can be written as

$$\nabla^2\mathbf{E}(\mathbf{r}) + k^2\mathbf{E}(\mathbf{r}) = j\omega\mu\mathbf{J}(\mathbf{r}) + \frac{1}{\epsilon}\nabla\rho(\mathbf{r}) \tag{43}$$

where Eq. (39) has been applied. Similarly, one obtains the equation for $\mathbf{H}$ as

$$\nabla^2\mathbf{H}(\mathbf{r}) + k^2\mathbf{H}(\mathbf{r}) = -\nabla \times \mathbf{J}(\mathbf{r}) \tag{44}$$

Equations (43) and (44) are called inhomogeneous Helmholtz wave equations.

If one uses ϕ to represent each component of $\mathbf{E}$ or $\mathbf{H}$ in a Cartesian coordinate system, then ϕ satisfies the inhomogeneous Helmholtz equation

$$\nabla^2\phi(\mathbf{r}) + k^2\phi(\mathbf{r}) = -f(\mathbf{r}) \tag{45}$$

When $\phi(\mathbf{r})$ propagates in an infinite unbounded space, there is no reflected wave. Hence, $\phi(\mathbf{r})$ satisfies the radiation condition

$$r\left(\frac{\partial\phi}{\partial r} + jk\phi\right) = 0 \qquad \text{for } r \to \infty \tag{46}$$

where r represents the radial variable in spherical coordinates. Instead of solving for $\phi(\mathbf{r})$ directly from Eqs. (45) and (46) for each $f(\mathbf{r})$, one first finds its Green's function, which is the solution of the following partial differential equation

$$\nabla^2 G_0(\mathbf{r}, \mathbf{r}') + k^2 G_0(\mathbf{r}, \mathbf{r}') = -\delta(\mathbf{r} - \mathbf{r}') \tag{47}$$

subject to the radiation condition in Eq. (46). If G_0 can be found, using the principle of linear superposition, one obtains

$$\phi(\mathbf{r}) = \int_V G_0(\mathbf{r}, \mathbf{r}') f(\mathbf{r}')\, dV' \tag{48}$$

where V is the support of $f(\mathbf{r})$, which is the volume having nonzero $f(\mathbf{r})$.

To find G_0, we introduce a new coordinate system with its origin located at $\mathbf{r}'$. Thus, the problem has a spherical symmetry with respect to this point. Equation (47) then becomes

$$\frac{1}{r_1^2}\frac{d}{dr_1}\left[r_1^2\frac{dG_0(\mathbf{r}_1, 0)}{dr_1}\right] + k^2 G_0(\mathbf{r}_1, 0) = -\delta(\mathbf{r}_1) \tag{49}$$

where $\mathbf{r}_1 = \mathbf{r} - \mathbf{r}'$. When $r_1 \neq 0$, Eq. (49) can be written as

$$\frac{d^2[r_1 G_0(\mathbf{r}_1, 0)]}{dr_1^2} + k^2 r_1 G_0(\mathbf{r}_1, 0) = 0 \tag{50}$$

which has a well-known solution

$$r_1 G_0(\mathbf{r}_1, 0) = Ae^{-jkr_1} \qquad \text{or} \qquad G_0(\mathbf{r}_1, 0) = A\frac{e^{-jkr_1}}{r_1} \tag{51}$$

The sign in the exponent is chosen such that Eq. (51) satisfies the radiation condition in Eq. (46). To determine the unknown coefficient A, substitute Eq. (51) into Eq. (49) and integrate over a small sphere centered at $\mathbf{r}_1 = 0$ with its radius $\epsilon \to 0$. The result is $A = (4\pi)^{-1}$. Therefore,

$$G_0(\mathbf{r}_1, 0) = \frac{e^{-jkr_1}}{4\pi r_1} \tag{52}$$

and in the original coordinates, it becomes

$$G_0(\mathbf{r}, \mathbf{r}') = \frac{e^{-jk|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r} - \mathbf{r}'|} \tag{53}$$

Following the same procedure, one can obtain the two-dimensional Green's function for the Helmholtz equation as

$$G_0(\boldsymbol{\rho}, \boldsymbol{\rho}') = \frac{1}{4j} H_0^{(2)}(k|\boldsymbol{\rho} - \boldsymbol{\rho}'|) \tag{54}$$

where $\boldsymbol{\rho} = x\hat{x} + y\hat{y}$ and $H_0^{(2)}(k|\boldsymbol{\rho} - \boldsymbol{\rho}'|)$ is the zeroth-order Hankel function of the second kind.

When one deals with the static electric field, Maxwell's equations for $\mathbf{E}(\mathbf{r})$ reduce to

$$\nabla \times \mathbf{E}(\mathbf{r}) = 0 \qquad \text{and} \qquad \nabla \cdot \mathbf{E}(\mathbf{r}) = \frac{\rho(\mathbf{r})}{\epsilon} \tag{55}$$

These two equations can be solved conveniently by introducing the electric potential $\phi(\mathbf{r})$, which is defined as

$$\mathbf{E}(\mathbf{r}) = -\nabla\phi(\mathbf{r}) \tag{56}$$

The first equation in Eq. (55) is automatically satisfied because of the identity $\nabla \times \nabla\phi(\mathbf{r}) \equiv 0$. Substituting Eq. (56) into the second equation in Eq. (55), one obtains

$$\nabla^2\phi(\mathbf{r}) = -\frac{\rho(\mathbf{r})}{\epsilon} \tag{57}$$

This equation is known as Poisson's equation, which can be considered as a special case of Eq. (45) with $k = 0$.

Using the procedure described in this section, one obtains the three-dimensional Green's function for Poisson's equation as

$$G_0(\mathbf{r}, \mathbf{r}') = \frac{1}{4\pi|\mathbf{r} - \mathbf{r}'|} \tag{58}$$

and the two-dimensional Green's function as

$$G_0(\boldsymbol{\rho}, \boldsymbol{\rho}') = -\frac{1}{2\pi}\ln|\boldsymbol{\rho} - \boldsymbol{\rho}'| \tag{59}$$

Classification of Green's Functions

The Green's functions derived above are for the infinite unbounded space where no other objects are present. They are called the free-space Green's functions and are denoted by the subscript "0." When the region of interest is bounded, one then has to consider boundary conditions for the Green's function. Different boundary conditions lead to different Green's functions. For this reason, Green's functions are classified into three categories: Green's function of the first, second, and third kind (3).

The Green's function of the first kind, denoted by G_1, satisfies the Dirichlet boundary condition, that is,

$$G_1(\mathbf{r}, \mathbf{r}') = 0 \qquad \text{for } \mathbf{r} \text{ on } S \tag{60}$$

where S denotes the boundary of the problem. For a half space with an infinite ground plane coincident with the $z = 0$ plane, the Green's function of the first kind for Poisson's equation is given by

$$G_1(\mathbf{r}, \mathbf{r}') = G_0(\mathbf{r}, \mathbf{r}') - G_0(\mathbf{r}, \mathbf{r}_i') = \frac{1}{4\pi|\mathbf{r} - \mathbf{r}'|} - \frac{1}{4\pi|\mathbf{r} - \mathbf{r}_i'|} \tag{61}$$

where $\mathbf{r}_i' = \mathrm{r}' - 2z'\hat{z} = x'\hat{x} + y'\hat{y} - z'\hat{z}$. This result can be derived conveniently using the method of images. It is easy to see that the Dirichlet boundary condition is satisfied by $G_1(\mathbf{r}, \mathbf{r}')$ in the $z = 0$ plane.

The Green's function of the second kind, denoted by G_2, satisfies the Neumann boundary condition, that is,

$$\frac{\partial G_2(\mathbf{r}, \mathbf{r}')}{\partial n} = 0 \qquad \text{for } \mathbf{r} \text{ on } S \tag{62}$$

where S denotes the boundary of the problem and $\partial/\partial n$ denotes the normal derivative. For a half space with an infinite magnetic (symmetry) plane coincident with the $z = 0$ plane, the Green's function of the second kind for Poisson's equation is given by

$$G_2(\mathbf{r}, \mathbf{r}') = G_0(\mathbf{r}, \mathbf{r}') + G_0(\mathbf{r}, \mathbf{r}_i') = \frac{1}{4\pi|\mathbf{r} - \mathbf{r}'|} + \frac{1}{4\pi|\mathbf{r} - \mathbf{r}_i'|} \tag{63}$$

where $\mathbf{r}_i'$ is the same as the one in Eq. (61). It satisfies the Neumann boundary condition in the $z = 0$ plane.

The Green's function of the third kind is defined for problems involving two or more media. It can be denoted as $G^{(ij)}(\mathbf{r}, \mathbf{r}')$, where i indicates the medium where the field point $\mathbf{r}$ is located and j indicates the medium where the source point $\mathbf{r}'$ is located. Consider, for example, a potential problem involving two half spaces. The upper half space (medium 1) above $z = 0$ has a permittivity of ϵ_1, and the lower half space (medium 2) has a permittivityof ϵ_2. The Green's function for Poisson's equation is given by (8)

$$\begin{aligned} G^{(11)}(\mathbf{r}, \mathbf{r}') &= G_0(\mathbf{r}, \mathbf{r}') - \frac{\epsilon_2 - \epsilon_1}{\epsilon_2 + \epsilon_1}G_0(\mathbf{r}, \mathbf{r}_i') \\ &= \frac{1}{4\pi|\mathbf{r} - \mathbf{r}'|} - \frac{\epsilon_2 - \epsilon_1}{\epsilon_2 + \epsilon_1}\frac{1}{4\pi|\mathbf{r} - \mathbf{r}_i'|} \end{aligned} \tag{64}$$

and

$$G^{(21)}(\mathbf{r}, \mathbf{r}') = \frac{2\epsilon_2}{\epsilon_2 + \epsilon_1}G_0(\mathbf{r}, \mathbf{r}') = \frac{2\epsilon_2}{\epsilon_2 + \epsilon_1}\frac{1}{4\pi|\mathbf{r} - \mathbf{r}'|} \tag{65}$$

Exchanging ϵ_1 and ϵ_2 in $G^{(11)}$ and $G^{(21)}$, one obtains the expressions for $G^{(22)}$ and $G^{(12)}$, respectively. This method of obtaining the Green's functions of the third kind works only for Poisson's equation, but not for the Helmholtz equation because the standard image method is not applicable to the Helmholtz equation in this case.

Eigenfunction Expansion

In addition to the conventional method described earlier, another general method for deriving Green's functions is called the method of Ohm-Rayleigh or the method of eigenfunction expansion (3). In this section, one rederives the Green's functions in Eq. (36) and Eq. (53) to illustrate the process of the Ohm-Rayleigh method.

Consider first the solution of Eq. (27). Expand $g_0(x, x')$ in terms of a Fourier integral

$$g_0(x, x') = \int_{-\infty}^{\infty} A(h)e^{jhx}\,dh \tag{66}$$

The e^{jhx}, which is the solution of the homogeneous differential equation $d^2\psi(x)/dx^2 + h^2\psi(x) = 0$, is called the eigenfunction and h^2 is the corresponding eigenvalue. Therefore, Eq. (66) can be considered as the eigenfunction expansion of $g_0(x, x')$. To determine $A(h)$, substitute Eqs. (66) into Eq. (27) and note that

$$\delta(x - x') = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{jh(x-x')}\,dh \tag{67}$$

This yields

$$A(h) = \frac{e^{-jhx'}}{2\pi(h^2 + \gamma^2)} \tag{68}$$

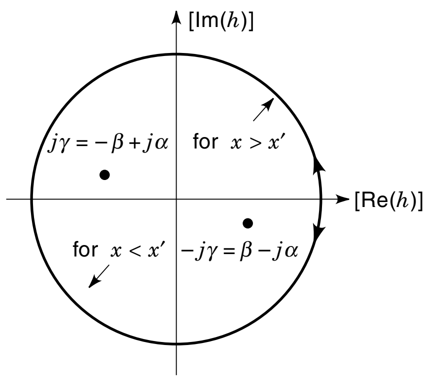

Figure 2. Locations of the two poles in the complex plane and the closed contours for integration.

Hence,

$$g_0(x, x') = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{jh(x-x')}}{h^2 + \gamma^2} \, dh \tag{69}$$

This is known as the spectral representation of $g_0(x, x')$. The integral in this equation can be evaluated using Cauchy's residue theorem (9). For this, one needs to form a closed contour for the integral in Eq. (69). In order to satisfy the boundary conditions in Eqs. (25) and (26), for $x - x' > 0$ the infinite integration path must be closed in the upper half-plane and for $x - x' < 0$ the infinite path must be closed in the lower half-plane, as shown in Fig. 2. The application of Cauchy's residue theorem yields

$$g_0(x, x') = \frac{1}{2\gamma} \begin{cases} e^{-\gamma(x-x')} & \text{for } x > x' \\ e^{\gamma(x-x')} & \text{for } x < x' \end{cases} \tag{70}$$

which is the same as Eqs. (34) and (35).

Next, consider the solution of Eq. (47). First expand $G_0(\mathbf{r}, \mathbf{r}')$ in terms of Fourier integrals

$$G_0(\mathbf{r}, \mathbf{r}') = \int_{-\infty}^{\infty} A(\mathbf{h}) e^{j\mathbf{h}\cdot\mathbf{r}} \, d\mathbf{h} \tag{71}$$

where $\mathbf{h} = h_x\hat{x} + h_y\hat{y} + h_z\hat{z}$. The $e^{j\mathbf{h}\cdot\mathbf{r}}$, which is the solution of the homogeneous partial differential equation $\nabla^2\psi(\mathbf{r}) + h^2\psi(\mathbf{r}) = 0$, is called the eigenfunction and $h^2 = |\mathbf{h}|^2$ is the corresponding eigenvalue. Again, Eq. (71) can be considered as the eigenfunction expansion of $G_0(\mathbf{r}, \mathbf{r}')$. Substituting Eq. (71) into Eq. (47), and noting that

$$\delta(\mathbf{r} - \mathbf{r}') = \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} e^{j\mathbf{h}\cdot(\mathbf{r}-\mathbf{r}')} \, d\mathbf{h} \tag{72}$$

one finds

$$A(\mathbf{h}) = \frac{e^{-j\mathbf{h}\cdot\mathbf{r}'}}{(2\pi)^3(h^2 - k^2)} \tag{73}$$

Therefore,

$$G_0(\mathbf{r}, \mathbf{r}') = \frac{1}{(2\pi)^3} \int_{-\infty}^{\infty} \frac{e^{j\mathbf{h}\cdot(\mathbf{r}-\mathbf{r}')}}{h^2 - k^2} \, d\mathbf{h} \tag{74}$$

This is the spectral representation of the three-dimensional Green's function. To evaluate the spectral integral, let

$$h_x = h\sin\theta\cos\varphi, \qquad h_y = h\sin\theta\sin\varphi, \qquad h_z = h\cos\theta \tag{75}$$

so that

$$d\mathbf{h} = h^2 \sin\theta \, dh \, d\theta \, d\varphi \tag{76}$$

Furthermore, because of the spherical symmetry of G_0 with respect to the point $\mathbf{r}'$, the value of G_0 is independent of the direction of $\mathbf{r} - \mathbf{r}'$. Therefore, one can choose an arbitrary $\mathbf{r} - \mathbf{r}'$ for the evaluation of G_0. If one chooses the direction of $\mathbf{r} - \mathbf{r}'$ to coincide with the z-direction, Eq. (74) may be written as

$$\begin{aligned} G_0(\mathbf{r}, \mathbf{r}') &= \frac{1}{(2\pi)^3} \int_0^{\infty} \int_0^{\pi} \int_0^{2\pi} \frac{e^{jh\cos\theta|\mathbf{r}-\mathbf{r}'|}}{h^2 - k^2} h^2 \sin\theta \, dh \, d\theta \, d\varphi \\ &= \frac{j}{(2\pi)^2|\mathbf{r}-\mathbf{r}'|} \int_0^{\infty} [e^{-jh|\mathbf{r}-\mathbf{r}'|} - e^{jh|\mathbf{r}-\mathbf{r}'|}] \frac{h\,dh}{h^2 - k^2} \\ &= \frac{j}{(2\pi)^2|\mathbf{r}-\mathbf{r}'|} \int_{-\infty}^{\infty} \frac{he^{-jh|\mathbf{r}-\mathbf{r}'|}}{h^2 - k^2} \, dh \end{aligned} \tag{77}$$

This integral can now be evaluated using Cauchy's residue theorem. The integrand has two poles: one at $h = k$ and the other at $h = -k$. Although the problem considered here is lossless, treat it as a limiting case of a lossy problem for which k has a small negative imaginary part. Consequently, the pole at $h = k$ is on the lower side of the real axis and the pole at $h = -k$ is on the upper side of the real axis. In order to satisfy the radiation condition in Eq. (46), the infinite integration path must be closed in the lower half-plane, as shown in Fig. 3. Applying Cauchy's residue theorem, one obtains

$$G_0(\mathbf{r}, \mathbf{r}') = \frac{e^{-jk|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|} \tag{78}$$

which is the same as Eq. (53).

Finally, note that, although the process of the Ohm-Rayleigh method is more involved than the conventional method, it is more general and can be used to find Green's functions in many problems.

Green's Functions in a Bounded Region

As can be seen in the preceding section, the spectrum (eigenvalue) for infinite-space problems is continuous and, as a re-

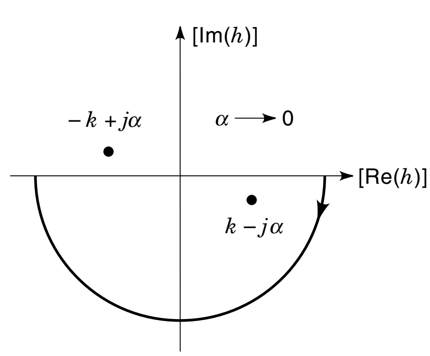

Figure 3. Locations of the two poles in the complex plane and the closed contour for integration.

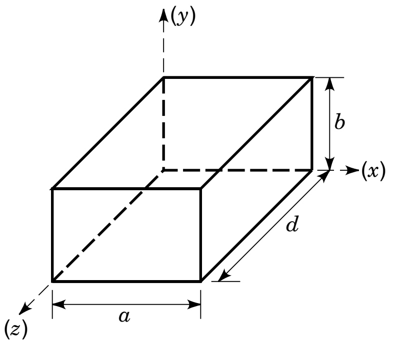

Figure 4. A grounded rectangular cavity.

sult, the spectral representation of the Green's function involves spectral integrals. When the region of interest is finite, the spectrum will be discrete. To demonstrate this fact, consider a grounded rectangular cavity of dimension $a \times b \times d$, depicted in Fig. 4. The Green's function for Poisson's equation satisfies the partial differential equation

$$\nabla^2 G_1(\mathbf{r}, \mathbf{r}') = -\delta(\mathbf{r} - \mathbf{r}') \tag{79}$$

and the Dirichlet boundary condition

$$G_1(\mathbf{r}, \mathbf{r}') = 0 \qquad \text{for } \mathbf{r} \text{ on cavity's walls} \tag{80}$$

This Green's function can be derived in a number of different ways, such as the conventional method, the method of images, and the Ohm-Rayleigh method. Here, the Ohm-Rayleigh method is employed. First, consider the solution of

$$\nabla^2 \psi + h^2 \psi = 0 \tag{81}$$

subject to the condition in Eq. (80). Using the method of separation of variables, one finds

$$\psi_{mnp} = \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \sin\frac{p\pi z}{d} \tag{82}$$

which is the eigenfunction of Eq. (81) with eigenvalue $h^2 = (m\pi/a)^2 + (n\pi/b)^2 + (p\pi/d)^2$. This can be used to expand G_1:

$$G_1(\mathbf{r}, \mathbf{r}') = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{p=1}^{\infty} A_{mnp} \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \sin\frac{p\pi z}{d} \tag{83}$$

Substituting this expression into Eq. (79), one has

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{p=1}^{\infty} A_{mnp} h^2 \sin\frac{m\pi x}{a} \sin\frac{n\pi y}{b} \sin\frac{p\pi z}{d} = \delta(x - x')\delta(y - y')\delta(z - z') \tag{84}$$

The coefficient A_{mnp} can be determined by multiplying both sides by $\sin(m'\pi x/a)$ $\sin(n'\pi y/b)$, $\sin(p'\pi z/d)$ and integrating over x, y, and z. The result is

$$G_1(\mathbf{r}, \mathbf{r}') = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\sum_{p=1}^{\infty} \frac{8}{abdh^2} \sin\frac{m\pi x}{a} \sin\frac{m\pi x'}{a} \sin\frac{n\pi y}{b} \sin\frac{n\pi y'}{b} \sin\frac{p\pi z}{d} \sin\frac{p\pi z'}{d} \tag{85}$$

The triple summation can be reduced to a double summation using the formula (4)

$$\sum_{p=1}^{\infty} \frac{1}{h^2} \sin\frac{p\pi z}{d} \sin\frac{p\pi z'}{d} = \frac{d}{2k_c \sinh k_c d} \sinh k_c z_< \sinh k_c (d - z_>) \tag{86}$$

where $kc = \sqrt{(m\pi/a)^2 + (n\pi/b)^2}$, $z_< = z$ when $z < z'$, and $z_< = z'$ when $z' < z$, and $z_> = z$ when $z > z'$, and $z_> = z'$ when $z' > z$. As a result, Eq. (85) becomes

$$G_1(\mathbf{r}, \mathbf{r}') = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{4}{ab} \sin\frac{m\pi x}{a} \sin\frac{m\pi x'}{a} \sin\frac{n\pi y}{b} \sin\frac{\sinh k_c z_< \sinh k_c (d - z_>)}{k_c \sinh k_c d} \tag{87}$$

Next, consider the problem of a parallel-plate waveguide, which is finite in the y direction and infinite in the x and z directions, as shown in Fig. 5. Assuming that the source is uniform in the z direction, the Green's function of the first kind for the Helmholtz equation satisfies the partial differential equation

$$\nabla^2 G_1(\boldsymbol{\rho}, \boldsymbol{\rho}') + k^2 G_1(\boldsymbol{\rho}, \boldsymbol{\rho}') = -\delta(\boldsymbol{\rho} - \boldsymbol{\rho}') \tag{88}$$

and the boundary conditions

$$G_1(\boldsymbol{\rho}, \boldsymbol{\rho}') = 0 \qquad \text{for } y = 0, b \tag{89}$$

and the radiation conditions

$$\frac{\partial G_1(\boldsymbol{\rho}, \boldsymbol{\rho}')}{\partial x} + jkG_1(\boldsymbol{\rho}, \boldsymbol{\rho}') = 0 \qquad \text{for } x \to \infty \tag{90}$$

$$\frac{\partial G_1(\boldsymbol{\rho}, \boldsymbol{\rho}')}{\partial x} - jkG_1(\boldsymbol{\rho}, \boldsymbol{\rho}') = 0 \qquad \text{for } x \to -\infty \tag{91}$$

The eigenfunction for this problem is found as

$$\psi_n(h_x) = e^{jh_x x} \sin\frac{n\pi y}{b} \tag{92}$$

from which G_1 can be expanded as

$$G_1(\boldsymbol{\rho}, \boldsymbol{\rho}') = \int_{-\infty}^{\infty} \sum_{n=1}^{\infty} A_n(h_x) e^{jh_x x} \sin\frac{n\pi y}{b} \, dh_x \tag{93}$$

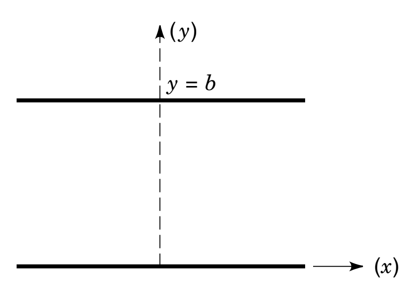

Figure 5. A parallel-plate waveguide.

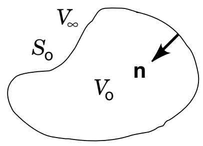

Figure 6. An object occupying volume V_o.

Substituting this expression into Eq. (88), one obtains

$$\int_{-\infty}^{\infty}\sum_{n=1}^{\infty}A_n(h_x)\left[k^2-h_x^2-\left(\frac{n\pi}{b}\right)^2\right]e^{jh_x x}\sin\frac{n\pi y}{b}\,dh_x = -\delta(x-x')\delta(y-y') \quad (94)$$

The coefficient $A_n(h_x)$ can be determined by multiplying both sides by $e^{jh_x' x}\sin(n'\pi y/b)$ and integrating over x and y. The result is

$$G_1(\boldsymbol{\rho},\boldsymbol{\rho}') = \frac{1}{\pi b}\int_{-\infty}^{\infty}\sum_{n=1}^{\infty}\left[h_x^2+\left(\frac{n\pi}{b}\right)^2-k^2\right]^{-1}e^{jh_x(x-x')}\sin\frac{n\pi y}{b}\sin\frac{n\pi y'}{b}\,dh_x \quad (95)$$

Using Cauchy's residue theorem, one can evaluate the spectral integral in a similar manner to that for the transmission line case, yielding

$$G_1(\boldsymbol{\rho},\boldsymbol{\rho}') = \frac{1}{b}\sum_{n=1}^{\infty}\frac{1}{\gamma_x}e^{-\gamma_x|x-x'|}\sin\frac{n\pi y}{b}\sin\frac{n\pi y'}{b} \quad (96)$$

where $\gamma_x = \sqrt{(n\pi/b)^2-k^2}$.

Scalar Integral Equations

Originally, Green's function methods were developed for finding the general solution of a boundary-value problem whose Green's function can be derived. For many practical problems, the Green's function cannot be derived. As a result, one must resort to a numerical method for the solution of the problem. One such numerical method is based on an integral equation derived using the Green's function method.

To demonstrate the formulation of integral equations, consider the problem of a scalar wave produced by a source $f(\mathbf{r})$ in the presence of an arbitrarily shaped object immersed in an infinite medium, as illustrated in Fig. 6. Exterior to the object, the wave function $\phi(\mathbf{r})$ satisfies the inhomogeneous Helmholtz equation in Eq. (45) and the radiation boundary condition in Eq. (46). Since the object has an arbitrary shape, no closed-form Green's function can be found for this problem. However, one can establish an integral equation for this problem using the free-space Green's function given in Eq. (53), which is the solution of Eq. (47) under the condition in Eq. (46).

First, multiply Eq. (45) with G_0, Eq. (47) with ϕ, and integrate the difference of the resultant equations over the entire exterior volume, yielding

$$\int_{V_\infty}[G_0(\mathbf{r},\mathbf{r}')\nabla^2\phi(\mathbf{r})-\phi(\mathbf{r})\nabla^2 G_0(\mathbf{r},\mathbf{r}')]\,dV = -\int_{V_s}G_0(\mathbf{r},\mathbf{r}')f(\mathbf{r})\,dV+\int_{V_\infty}\phi(\mathbf{r})\delta(\mathbf{r}-\mathbf{r}')\,dV \quad (97)$$

where V_∞ denotes the infinite space exterior to the object and V_s denotes the support of $f(\mathbf{r})$. Applying the second scalar Green's theorem (1)

$$\int_V(a\nabla^2 b-b\nabla^2 a)\,dV = \int_S\left(a\frac{\partial b}{\partial n}-b\frac{\partial a}{\partial n}\right)dS \quad (98)$$

where S denotes the surface enclosing V, one obtains

$$\int_{S_o+S_\infty}\left[G_0(\mathbf{r},\mathbf{r}')\frac{\partial\phi(\mathbf{r})}{\partial n}-\phi(\mathbf{r})\frac{\partial G_0(\mathbf{r},\mathbf{r}')}{\partial n}\right]dS - \int_{V_\infty}\phi(\mathbf{r})\delta(\mathbf{r}-\mathbf{r}')\,dV = -\int_{V_s}G_0(\mathbf{r},\mathbf{r}')f(\mathbf{r})\,dV \quad (99)$$

where S_o denotes the surface of the object and S_∞ denotes a spherical surface with a radius approaching infinity. Since both G_0 and ϕ satisfy Eq. (46), the surface integral over S_∞ vanishes. Consequently, one has

$$\int_{S_o}\left[G_0(\mathbf{r},\mathbf{r}')\frac{\partial\phi(\mathbf{r})}{\partial n}-\phi(\mathbf{r})\frac{\partial G_0(\mathbf{r},\mathbf{r}')}{\partial n}\right]dS-\int_{V_\infty}\phi(\mathbf{r})\delta(\mathbf{r}-\mathbf{r}')\,dV = -\int_{V_s}G_0(\mathbf{r},\mathbf{r}')f(\mathbf{r})\,dV \quad (100)$$

where the normal unit vector on S_o points toward the interior of the object. Using Eq. (14), one obtains

$$\int_{S_o}\left[G_0(\mathbf{r},\mathbf{r}')\frac{\partial\phi(\mathbf{r})}{\partial n}-\phi(\mathbf{r})\frac{\partial G_0(\mathbf{r},\mathbf{r}')}{\partial n}\right]dS+\int_{V_s}G_0(\mathbf{r},\mathbf{r}')f(\mathbf{r})\,dV = \begin{cases}\phi(\mathbf{r}') & \text{for } \mathbf{r}' \text{ in } V_\infty\\ 0 & \text{for } \mathbf{r}' \text{ in } V_o\end{cases} \quad (101)$$

where V_o denotes the volume of the object. Exchanging $\mathbf{r}$ and $\mathbf{r}'$ and using the symmetry property of G_0 [i.e., $G_0(\mathbf{r}', \mathbf{r}) = G_0(\mathbf{r}, \mathbf{r}')$], one has

$$\int_{S_o}\left[G_0(\mathbf{r},\mathbf{r}')\frac{\partial\phi(\mathbf{r}')}{\partial n'}-\phi(\mathbf{r})\frac{\partial G_0(\mathbf{r},\mathbf{r}')}{\partial n'}\right]dS' + \int_{V_s}G_0(\mathbf{r},\mathbf{r}')f(\mathbf{r}')\,dV' = \begin{cases}\phi(\mathbf{r}) & \text{for } \mathbf{r} \text{ in } V_\infty\\ 0 & \text{for } \mathbf{r} \text{ in } V_o\end{cases} \quad (102)$$

Equation (102) is an important result, which has several implications. First, notice that when the object is absent, the surface integral vanishes. Hence,

$$\phi(\mathbf{r}) = \int_{V_s}G_0(\mathbf{r},\mathbf{r}')f(\mathbf{r}')\,dV' \quad (103)$$

which is the same as Eq. (48). This may be called the incident field impinging on the object and be denoted as $\phi^{\text{inc}}(\mathbf{r})$. Second, when there is no source in V_∞, Eq. (102) becomes

$$\phi(\mathbf{r}) = \int_{S_o}\left[G_0(\mathbf{r},\mathbf{r}')\frac{\partial\phi(\mathbf{r}')}{\partial n'}-\phi(\mathbf{r}')\frac{\partial G_0(\mathbf{r},\mathbf{r}')}{\partial n'}\right]dS' \quad (104)$$

for $\mathbf{r}$ in V_∞. Since there is no source in V_∞, the field on S_o must be produced by the source inside S_o. This equation indicates that the field in a source-free region can be calculated based on the knowledge of the potential and its normal derivative on the surface enclosing the region. This is the mathematical representation of the well-known Huygens' principle for a scalar wave.

Equation (102) also provides the foundation to establish an integral equation for ϕ and $\partial\phi/\partial n$ on the surface of the object. If the object is impenetrable with a hard surface where ϕ satisfies the boundary condition

$$\phi(\mathbf{r}) = 0 \qquad \text{for } \mathbf{r} \text{ on } S_o \tag{105}$$

Eq. (102) becomes

$$\phi^{\text{inc}}(\mathbf{r}) + \int_{S_o} G_0(\mathbf{r},\mathbf{r}')\frac{\partial\phi(\mathbf{r}')}{\partial n'}\,dS' = \begin{cases}\phi(\mathbf{r}) & \text{for } \mathbf{r} \text{ in } V_\infty \\ 0 & \text{for } \mathbf{r} \text{ in } V_o\end{cases} \tag{106}$$

Applying this equation on S_o, one obtains

$$\int_{S_o} G_0(\mathbf{r},\mathbf{r}')\frac{\partial\phi(\mathbf{r}')}{\partial n'}\,dS' = -\phi^{\text{inc}}(\mathbf{r}) \qquad \text{for } \mathbf{r} \text{ on } S_o \tag{107}$$

which is the integral equation for $\partial\phi/\partial n$ on S_o.

If the object is impenetrable with a soft surface where ϕ satisfies the boundary condition

$$\frac{\partial\phi(\mathbf{r})}{\partial n} = 0 \qquad \text{for } \mathbf{r} \text{ on } S_o \tag{108}$$

Eq. (102) becomes

$$\phi^{\text{inc}}(\mathbf{r}) - \int_{S_o} \phi(\mathbf{r}')\frac{\partial G_0(\mathbf{r},\mathbf{r}')}{\partial n'}\,dS' = \begin{cases}\phi(\mathbf{r}) & \text{for } \mathbf{r} \text{ in } V_\infty \\ 0 & \text{for } \mathbf{r} \text{ in } V_o\end{cases} \tag{109}$$

Applying this equation on S_o, one obtains

$$\frac{1}{2}\phi(\mathbf{r}) + ⨍_{S_o} \phi(\mathbf{r}')\frac{\partial G_0(\mathbf{r},\mathbf{r}')}{\partial n'}\,dS' = \phi^{\text{inc}}(\mathbf{r}) \qquad \text{for } \mathbf{r} \text{ on } S_o \tag{110}$$

where $⨍$ denotes the integral excluding the contribution from the singular point which is known as the principal value integral. This result is obtained as follows: The integral over S_o in Eq. (109) is divided into an integral over a small circular disk with center at $\mathbf{r}$ plus the remaining integral which is represented as a principal value integral $⨍$ in the limit as the area of the isolated disk approaches zero. If $\mathbf{r}$ approaches S_o from the outside, the integral over the vanishingly small disk can be evaluated to give $-\phi(\mathbf{r})/2$. If $\mathbf{r}$ approaches S_o from the inside, the integral gives $\phi(\mathbf{r})/2$. In either case, one obtains Eq. (110), which is the integral equation for ϕ on S_o.

If the object is penetrable and homogeneous, apply Eq. (102) on S_o to obtain

$$\frac{1}{2}\phi(\mathbf{r}) - ⨍_{S_o}\left[G_0(\mathbf{r},\mathbf{r}')\frac{\partial\phi(\mathbf{r}')}{\partial n'} - \phi(\mathbf{r}')\frac{\partial G_0(\mathbf{r},\mathbf{r}')}{\partial n'}\right]dS' = \phi^{\text{inc}}(\mathbf{r}) \qquad \text{for } \mathbf{r} \text{ on } S_o \tag{111}$$

To solve for ϕ and $\partial\phi/\partial n$ on S_o, another equation is needed, which can be derived by considering the interior of the object. The wave function inside the object satisfies the Helmholtz equation

$$\nabla^2\phi(\mathbf{r}) + \tilde{k}^2\phi(\mathbf{r}) = 0 \tag{112}$$

where $\tilde{k}$ characterizes the property of the object. Multiplying this equation by the Green's function for unbounded space filled with material characterized by $\tilde{k}$:

$$\tilde{G}_0(\mathbf{r},\mathbf{r}') = \frac{e^{-j\tilde{k}|\mathbf{r}-\mathbf{r}'|}}{4\pi|\mathbf{r}-\mathbf{r}'|} \tag{113}$$

and applying a similar derivation as before, one has

$$-\int_{S_o}\left[\tilde{G}_0(\mathbf{r},\mathbf{r}')\frac{\partial\phi(\mathbf{r}')}{\partial n'} - \phi(\mathbf{r}')\frac{\partial\tilde{G}_0(\mathbf{r},\mathbf{r}')}{\partial n'}\right]dS' = \begin{cases}0 & \text{for } \mathbf{r} \text{ in } V_\infty \\ \phi(\mathbf{r}) & \text{for } \mathbf{r} \text{ in } V_o\end{cases} \tag{114}$$

When this is applied on S_o, one obtains the second integral equation

$$\frac{1}{2}\phi(\mathbf{r}) + ⨍_{S_o}\left[\tilde{G}_0(\mathbf{r},\mathbf{r}')\frac{\partial\phi(\mathbf{r}')}{\partial n'} - \phi(\mathbf{r}')\frac{\partial\tilde{G}_0(\mathbf{r},\mathbf{r}')}{\partial n'}\right]dS' = 0 \qquad \text{for } \mathbf{r} \text{ on } S_o \tag{115}$$

which can be used together with Eq. (111) for a numerical solution of ϕ and $\partial\phi/\partial n$ on S_o.

If the object is penetrable and inhomogeneous, the wave function still satisfies Eq. (112); however, $\tilde{k}$ now is a function of $\mathbf{r}$. In this case, one can write Eqs. (45) and (112) in one equation

$$\nabla^2\phi(\mathbf{r}) + k^2\phi(\mathbf{r}) = -f(\mathbf{r}) - [\tilde{k}^2(\mathbf{r}) - k^2]\phi(\mathbf{r}) \tag{116}$$

Multiplying this equation by G_0 and integrating over the infinite volume, one obtains

$$\phi(\mathbf{r}) - \int_{V_o}[\tilde{k}^2(\mathbf{r}') - k^2]G_0(\mathbf{r},\mathbf{r}')\phi(\mathbf{r}')\,dV' = \int_{V_s} G_0(\mathbf{r},\mathbf{r}')f(\mathbf{r}')\,dV' \tag{117}$$

This is the integral equation that can be used to solve for ϕ in V_o. Unlike the previous integral equations, this equation involves the volume integral. For this reason, it is often referred to as the volume integral equation, whereas the previous ones are often referred to as the surface integral equations. Integral equations for more complicated objects may involve both volume and surface integrals (10).

DYADIC GREEN'S FUNCTIONS

When both the source and response are vector functions, the corresponding Green's function is a dyad, and hence, the name dyadic Green's function.

Definition of Dyad

A dyad, denoted by $\overline{\mathbf{D}}$, is formed by two vectors

$$\overline{\mathbf{D}} = \mathbf{AB} \tag{118}$$

This entity by itself does not have any physical interpretation as a vector. However, when it acts upon another vector, the result becomes meaningful. The major role of a dyad is that its scalar product with a vector produces another vector of different magnitude and direction. For example, its anterior scalar product with vector **C** yields

$$\mathbf{C}\cdot\overline{\mathbf{D}} = (\mathbf{C}\cdot\mathbf{A})\mathbf{B} \tag{119}$$

which is a vector. Its posterior scalar product with vector **C** yields

$$\overline{\mathbf{D}}\cdot\mathbf{C} = \mathbf{A}(\mathbf{B}\cdot\mathbf{C}) \tag{120}$$

which is also a vector. Apparently, the resulting vectors in Eqs. (119) and (120) are different. In addition to the two scalar products, there are two vector products. The anterior vector product is defined as

$$\mathbf{C}\times\overline{\mathbf{D}} = (\mathbf{C}\times\mathbf{A})\mathbf{B} \tag{121}$$

and the posterior vector product is defined as

$$\overline{\mathbf{D}}\times\mathbf{C} = \mathbf{A}(\mathbf{B}\times\mathbf{C}) \tag{122}$$

Clearly, these products are dyads.

The dyad defined in Eq. (118) is a special entity, since it contains only six independent components, three in each of the two vectors. A more general dyad, also called a tensor, is defined as

$$\overline{\mathbf{D}} = \mathbf{D}_x\hat{x} + \mathbf{D}_y\hat{y} + \mathbf{D}_z\hat{z} \tag{123}$$

where $\mathbf{D}_x$, $\mathbf{D}_y$, and $\mathbf{D}_z$ are vectors. Therefore, Eq. (123) can be expressed as

$$\begin{aligned}\overline{\mathbf{D}} = {} & D_{xx}\hat{x}\hat{x} + D_{yx}\hat{y}\hat{x} + D_{zx}\hat{z}\hat{x} + D_{xy}\hat{x}\hat{y} + D_{yy}\hat{y}\hat{y} + D_{zy}\hat{z}\hat{y} \\ & + D_{xz}\hat{x}\hat{z} + D_{yz}\hat{y}\hat{z} + D_{zz}\hat{z}\hat{z}\end{aligned} \tag{124}$$

which contains nine independent components.

A special dyad is called the unit dyad or identity dyad, defined as

$$\overline{\mathbf{I}} = \hat{x}\hat{x} + \hat{y}\hat{y} + \hat{z}\hat{z} \tag{125}$$

It is evident that

$$\mathbf{C}\cdot\overline{\mathbf{I}} = \overline{\mathbf{I}}\cdot\mathbf{C} = \mathbf{C} \tag{126}$$

Free-Space Dyadic Green's Functions

Consider the electric and magnetic fields produced by an electric current source $\mathbf{J}(\mathbf{r})$ in an unbounded space. Maxwell's equations for this problem are given in Eqs. (37)–(40), which lead to Eq. (42), reproduced here as

$$\nabla\times\nabla\times\mathbf{E}(\mathbf{r}) - k^2\mathbf{E}(\mathbf{r}) = -j\omega\mu\mathbf{J}(\mathbf{r}) \tag{127}$$

The above is the vector wave equation, which is the analog of the scalar Helmholtz wave equation. It describes electromagnetic wave phenomena that are very pervasive in modern technologies, such as in communications, microwave, computer chips, and so forth.

Just as in the scalar case, one can derive a dyadic Green's function $\overline{\mathbf{G}}_{e0}$, whose end result is to relate $\mathbf{E}(\mathbf{r})$ and $\mathbf{J}(\mathbf{r})$ by

$$\mathbf{E}(\mathbf{r}) = -j\omega\mu\int_V \overline{\mathbf{G}}_{e0}(\mathbf{r},\mathbf{r}')\cdot\mathbf{J}(\mathbf{r}')\,dV' \tag{128}$$

where V is the support of the current $\mathbf{J}(\mathbf{r})$. Using Eq. (128) in Eq. (127), one obtains

$$\begin{aligned}& -j\omega\mu\int_V \nabla\times\nabla\times\overline{\mathbf{G}}_{e0}(\mathbf{r},\mathbf{r}')\cdot\mathbf{J}(\mathbf{r}')\,dV' \\ & \qquad + j\omega\mu k^2\int_V \overline{\mathbf{G}}_{e0}(\mathbf{r},\mathbf{r}')\cdot\mathbf{J}(\mathbf{r}')\,dV' \\ & \qquad = -j\omega\mu\mathbf{J}(\mathbf{r}) = -j\omega\mu\int_V \overline{\mathbf{I}}\delta(\mathbf{r}-\mathbf{r}')\cdot\mathbf{J}(\mathbf{r}')\,dV'\end{aligned} \tag{129}$$

For arbitrary $\mathbf{J}(\mathbf{r})$, the above could be satisfied only if

$$\nabla\times\nabla\times\overline{\mathbf{G}}_{e0}(\mathbf{r},\mathbf{r}') - k^2\overline{\mathbf{G}}_{e0}(\mathbf{r},\mathbf{r}') = \overline{\mathbf{I}}\delta(\mathbf{r}-\mathbf{r}') \tag{130}$$

The $\overline{\mathbf{G}}_{e0}(\mathbf{r},\mathbf{r}')$ is called the dyadic Green's function of the electric type that relates vector field **E** to vector current **J**.

Taking the curl of Eq. (128) and using Maxwell's equations, one obtains

$$\mathbf{H}(\mathbf{r}) = \int_V \nabla\times\overline{\mathbf{G}}_{e0}(\mathbf{r},\mathbf{r}')\cdot\mathbf{J}(\mathbf{r}')\,dV' = \int_V \overline{\mathbf{G}}_{m0}(\mathbf{r},\mathbf{r}')\cdot\mathbf{J}(\mathbf{r}')\,dV' \tag{131}$$

where $\overline{\mathbf{G}}_{m0}(\mathbf{r},\mathbf{r}') = \nabla\times\overline{\mathbf{G}}_{e0}(\mathbf{r},\mathbf{r}')$ is called the dyadic Green's function of the magnetic type. It satisfies the equation

$$\nabla\times\nabla\times\overline{\mathbf{G}}_{m0}(\mathbf{r},\mathbf{r}') - k^2\overline{\mathbf{G}}_{m0}(\mathbf{r},\mathbf{r}') = \nabla\times[\overline{\mathbf{I}}\delta(\mathbf{r}-\mathbf{r}')] \tag{132}$$

Therefore, the task of finding the dyadic Green's function of the electric type is reduced to the task of solving Eq. (130). Equation (130) can be made less difficult by taking the posterior scalar product with an arbitrary vector **a**, yielding

$$\nabla\times\nabla\times\overline{\mathbf{G}}_{e0}(\mathbf{r},\mathbf{r}')\cdot\mathbf{a} - k^2\overline{\mathbf{G}}_{e0}(\mathbf{r},\mathbf{r}')\cdot\mathbf{a} = \mathbf{a}\delta(\mathbf{r}-\mathbf{r}') \tag{133}$$

Recognizing that $\overline{\mathbf{G}}_{e0}(\mathbf{r},\mathbf{r}')\cdot\mathbf{a}$ represents a vector, one may use the vector identity $\nabla\times\nabla\times\mathbf{A} = \nabla(\nabla\cdot\mathbf{A}) - \nabla^2\mathbf{A}$ to find

$$\begin{aligned}& -\nabla^2\overline{\mathbf{G}}_{e0}(\mathbf{r},\mathbf{r}')\cdot\mathbf{a} - k^2\overline{\mathbf{G}}_{e0}(\mathbf{r},\mathbf{r}')\cdot\mathbf{a} \\ & \qquad = \mathbf{a}\delta(\mathbf{r}-\mathbf{r}') - \nabla[\nabla\cdot\overline{\mathbf{G}}_{e0}(\mathbf{r},\mathbf{r}')\cdot\mathbf{a}]\end{aligned} \tag{134}$$

Taking the divergence of Eq. (133) and making use of the fact that $\nabla\cdot(\nabla\times\mathbf{A}) \equiv 0$, it can be seen that

$$\nabla\cdot\overline{\mathbf{G}}_{e0}(\mathbf{r},\mathbf{r}')\cdot\mathbf{a} = -\frac{1}{k^2}\nabla\cdot[\mathbf{a}\delta(\mathbf{r}-\mathbf{r}')] \tag{135}$$

Using this in Eq. (134), one obtains

$$\nabla^2\overline{\mathbf{G}}_{e0}(\mathbf{r},\mathbf{r}')\cdot\mathbf{a} + k^2\overline{\mathbf{G}}_{e0}(\mathbf{r},\mathbf{r}')\cdot\mathbf{a} = -\left(1+\frac{\nabla\nabla\cdot}{k^2}\right)[\mathbf{a}\delta(\mathbf{r}-\mathbf{r}')] \tag{136}$$

By making use of the fact that

$$\nabla^2 G_0(\mathbf{r},\mathbf{r}') + k^2 G_0(\mathbf{r},\mathbf{r}') = -\delta(\mathbf{r}-\mathbf{r}') \tag{137}$$

and the fact that $1 + \nabla\nabla\cdot/k^2$ is a linear operator that commutes with ∇^2, it can be deduced that

$$\overline{\mathbf{G}}_{e0}(\mathbf{r}, \mathbf{r}') \cdot \mathbf{a} = \left(1 + \frac{\nabla\nabla\cdot}{k^2}\right)[\mathbf{a}G_0(\mathbf{r}, \mathbf{r}')] \tag{138}$$

Writing the above as

$$\overline{\mathbf{G}}_{e0}(\mathbf{r}, \mathbf{r}') \cdot \mathbf{a} = \left(\overline{\mathbf{I}} + \frac{\nabla\nabla}{k^2}\right) G_0(\mathbf{r}, \mathbf{r}') \cdot \mathbf{a} \tag{139}$$

and since **a** is an arbitrary vector, one deduces that

$$\overline{\mathbf{G}}_{e0}(\mathbf{r}, \mathbf{r}') = \left(\overline{\mathbf{I}} + \frac{\nabla\nabla}{k^2}\right) G_0(\mathbf{r}, \mathbf{r}') \tag{140}$$

The free-space dyadic Green's function of the magnetic type can be derived as

$$\overline{\mathbf{G}}_{m0}(\mathbf{r}, \mathbf{r}') = \nabla \times \overline{\mathbf{G}}_{e0}(\mathbf{r}, \mathbf{r}') = \nabla \times [\overline{\mathbf{I}}G_0(\mathbf{r}, \mathbf{r}')] = \nabla G_0(\mathbf{r}, \mathbf{r}') \times \overline{\mathbf{I}} \tag{141}$$

The above is the explicit representation of the dyadic Green's function in terms of the scalar Green's function $G_0(\mathbf{r}, \mathbf{r}')$. It is to be noted that the aforementioned relationship between the dyadic Green's function and the scalar Green's function $G_0(\mathbf{r}, \mathbf{r}')$ is only valid for a homogeneous unbounded space such as a free space. Such a relation does not hold true in a cavity, waveguide, or half-space. For example, the dyadic Green's functions for a half-space (above $z = 0$) are given by (3)

$$\overline{\mathbf{G}}_{e1}(\mathbf{r}, \mathbf{r}') = \left(\overline{\mathbf{I}} - \frac{\nabla\nabla'}{k^2}\right)[G_0(\mathbf{r}, \mathbf{r}') - G_0(\mathbf{r}, \mathbf{r}_i')] + 2\hat{z}\hat{z}G_0(\mathbf{r}, \mathbf{r}_i') \tag{142}$$

and

$$\overline{\mathbf{G}}_{m2}(\mathbf{r}, \mathbf{r}') = \nabla G_0(\mathbf{r}, \mathbf{r}') \times \overline{\mathbf{I}} + \nabla G_0(\mathbf{r}, \mathbf{r}_i') \times \overline{\mathbf{I}}_i \tag{143}$$

where $\mathbf{r}_i' = x'\hat{x} + y'\hat{y} - z'\hat{z}$ and $\overline{\mathbf{I}}_i = -\overline{\mathbf{I}} + 2\hat{z}\hat{z}$. It can be verified that $\overline{\mathbf{G}}_{e1}(\mathbf{r}, \mathbf{r}')$ and $\overline{\mathbf{G}}_{m2}(\mathbf{r}, \mathbf{r}')$ satisfy the boundary conditions

$$\hat{z} \times \overline{\mathbf{G}}_{e1}(\mathbf{r}, \mathbf{r}') = 0 \qquad \text{for } z = 0 \tag{144}$$

and

$$\hat{z} \times \nabla \times \overline{\mathbf{G}}_{m2}(\mathbf{r}, \mathbf{r}') = 0 \qquad \text{for } z = 0 \tag{145}$$

respectively. For this reason, $\overline{\mathbf{G}}_{e1}(\mathbf{r}, \mathbf{r}')$ is called the electric-type dyadic Green's function of the first kind and $\overline{\mathbf{G}}_{m2}(\mathbf{r}, \mathbf{r}')$ is called the magnetic-type dyadic Green's function of the second kind. The classification of dyadic Green's functions is similar to that of scalar Green's functions.

Eigenfunction Expansion

As in the scalar case, the Ohm–Rayleigh method or the method of eigenfunction expansion is a general method to derive dyadic Green's functions (3). For vector problems, the eigenfunctions are vector functions, known as vector wave functions. There are three kinds of vector wave functions (1), defined by

$$\mathbf{L}(\mathbf{r}) = \nabla\psi(\mathbf{r}) \tag{146}$$

$$\mathbf{M}(\mathbf{r}) = \nabla \times [c\psi(\mathbf{r})] \tag{147}$$

$$\mathbf{N}(\mathbf{r}) = \frac{1}{\kappa}\nabla \times \mathbf{M}(\mathbf{r}) \tag{148}$$

where **c** is a vector called the pilot vector and ψ satisfies the homogeneous Helmholtz wave equation

$$\nabla^2\psi(\mathbf{r}) + \kappa^2\psi(\mathbf{r}) = 0 \tag{149}$$

It can be shown that **L**, **M**, and **N** satisfy the vector equations

$$\nabla^2\mathbf{L}(\mathbf{r}) + \kappa^2\mathbf{L}(\mathbf{r}) = 0 \tag{150}$$

$$\nabla \times \nabla \times \mathbf{M}(\mathbf{r}) - \kappa^2\mathbf{M}(\mathbf{r}) = 0 \tag{151}$$

$$\nabla \times \nabla \times \mathbf{N}(\mathbf{r}) - \kappa^2\mathbf{N}(\mathbf{r}) = 0 \tag{152}$$

and **M** can be expressed in terms of **N** as

$$\mathbf{M}(\mathbf{r}) = \frac{1}{\kappa}\nabla \times \mathbf{N}(\mathbf{r}) \tag{153}$$

Since $\nabla \times \mathbf{L}(\mathbf{r}) = \nabla \times \nabla\psi(\mathbf{r}) = 0$, **L** is known as the irrotational vector wave function. Since $\nabla \cdot \mathbf{M}(\mathbf{r}) = 0$ and $\nabla \cdot \mathbf{N}(\mathbf{r}) = 0$, **M** and **N** are known as the solenoidal vector wave functions.

For a rectangular waveguide illustrated in Fig. 7, ψ is given by

$$\psi_{\substack{e\\o}mn}(h, \mathbf{r}) = \begin{Bmatrix} \cos k_x x \cos k_y y \\ \sin k_x x \sin k_y y \end{Bmatrix} e^{-jhz} \tag{154}$$

where $k_x = m\pi/a$ and $k_y = n\pi/b$. The vector wave functions **L**, **M**, and **N** are given by

$$\mathbf{L}_{\substack{e\\o}mn}(h, \mathbf{r}) = \nabla\psi_{\substack{e\\o}mn}(h, \mathbf{r}) \tag{155}$$

$$\mathbf{M}_{\substack{e\\o}mn}(h, \mathbf{r}) = \nabla \times [\hat{z}\psi_{\substack{e\\o}mn}(h, \mathbf{r})] \tag{156}$$

$$\mathbf{N}_{\substack{e\\o}mn}(h, \mathbf{r}) = \frac{1}{\kappa}\nabla \times \nabla \times [\hat{z}\psi_{\substack{e\\o}mn}(h, \mathbf{r})] \tag{157}$$

where the pilot vector is $\mathbf{c} = \hat{z}$. This causes **M** to be transverse to $\hat{z}$.

The vector wave functions are always orthogonal to each other. For those in a rectangular waveguide, it can be shown

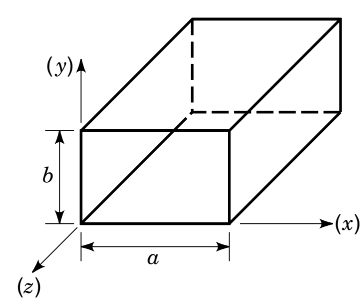

Figure 7. A rectangular waveguide.

that

$$\int_V \mathbf{U}_{\substack{e\\o}mn}(h,\mathbf{r})\cdot\mathbf{V}_{\substack{e\\o}m'n'}(-h',\mathbf{r})\,dV=0 \tag{158}$$

where $\mathbf{U}$, $\mathbf{V}$ = $\mathbf{L}$, $\mathbf{M}$, $\mathbf{N}$, except when $\mathbf{U}_{\substack{e\\o}mn}(h, \mathbf{r}) = \mathbf{V}_{\substack{e\\o}mn}(h, \mathbf{r})$. They form a complete set and, therefore, can be employed to expand any vector functions.

The electric-type dyadic Green's function of the first kind satisfies the equation

$$\nabla\times\nabla\times\overline{\mathbf{G}}_{e1}(\mathbf{r},\mathbf{r}')-k^2\overline{\mathbf{G}}_{e1}(\mathbf{r},\mathbf{r}')=\overline{\mathbf{I}}\delta(\mathbf{r}-\mathbf{r}') \tag{159}$$

and the boundary condition

$$\hat{n}\times\overline{\mathbf{G}}_{e1}(\mathbf{r},\mathbf{r}')=0 \qquad \text{on the waveguide walls} \tag{160}$$

It is clear that only $\mathbf{L}_{omn}$, $\mathbf{M}_{emn}$, and $\mathbf{N}_{omn}$ satisfy Eq. (160) and, therefore, can be used to expand $\overline{\mathbf{G}}_{e1}$:

$$\overline{\mathbf{G}}_{e1}(\mathbf{r},\mathbf{r}')=\int_{-\infty}^{\infty}\sum_{m,n}[\mathbf{L}_{omn}(h,\mathbf{r})\mathbf{A}_{omn}(h)+\mathbf{M}_{emn}(h,\mathbf{r})\mathbf{B}_{emn}(h) + \mathbf{N}_{omn}(h,\mathbf{r})\mathbf{C}_{omn}(h)]\,dh \tag{161}$$

Substituting this expansion into Eq. (159), one obtains

$$\int_{-\infty}^{\infty}\sum_{m,n}\{-k^2\mathbf{L}_{omn}(h,\mathbf{r})\mathbf{A}_{omn}(h)+(\kappa^2-\kappa^2)[\mathbf{M}_{emn}(h,\mathbf{r})\mathbf{B}_{emn}(h) + \mathbf{N}_{omn}(h,\mathbf{r})\mathbf{C}_{omn}(h)]\}\,dh=\mathbf{I}\delta(\mathbf{r}-\mathbf{r}') \tag{162}$$

Taking the anterior scalar product of Eq. (162) with $\mathbf{L}_{om'n'}(-h', \mathbf{r})$, $\mathbf{M}_{em'n'}(-h', \mathbf{r})$, and $\mathbf{N}_{om'n'}(-h', \mathbf{r})$, respectively, integrating over the entire volume of the waveguide, and applying the orthogonal relation in Eq. (158), one can find

$$\mathbf{A}_{omn}(h)=-\frac{k_{cmn}^2}{k^2\kappa^2}C_{mn}\mathbf{L}_{omn}(-h,\mathbf{r}') \tag{163}$$

$$\mathbf{B}_{emn}(h)=\frac{1}{\kappa^2-k^2}C_{mn}\mathbf{M}_{emn}(-h,\mathbf{r}') \tag{164}$$

$$\mathbf{C}_{emn}(h)=\frac{1}{\kappa^2-k^2}C_{mn}\mathbf{N}_{omn}(-h,\mathbf{r}') \tag{165}$$

where $k_{cmn}^2 = k_x^2 + k_y^2$ and $C_{mn} = (2 - \delta_0)/(\pi ab k_{cmn}^2)$ with $\delta_0 = 1$ when $m = 0$ or $n = 0$ and $\delta_0 = 0$, where both m and n are nonzero. Therefore,

$$\overline{\mathbf{G}}_{e1}(\mathbf{r},\mathbf{r}')=\int_{-\infty}^{\infty}\sum_{m,n}C_{mn}\left\{-\frac{k_{cmn}^2}{k^2\kappa^2}\mathbf{L}_{omn}(h,\mathbf{r})\mathbf{L}_{omn}(-h,\mathbf{r}') + \frac{1}{\kappa^2-k^2}[\mathbf{M}_{emn}(h,\mathbf{r})\mathbf{M}_{emn}(-h,\mathbf{r}') + \mathbf{N}_{omn}(h,\mathbf{r})\mathbf{N}_{omn}(-h,\mathbf{r}')]\right\}dh \tag{166}$$

Through some mathematical manipulations and the application of Cauchy's residue theorem (3), one can simplify Eq. (166) as

$$\overline{\mathbf{G}}_{e1}(\mathbf{r},\mathbf{r}')=-\frac{1}{k^2}\hat{z}\hat{z}\delta(\mathbf{r}-\mathbf{r}')-\frac{j}{ab}\sum_{m,n}\frac{2-\delta_0}{k_{cmn}^2k_{gmn}} [\mathbf{M}_{emn}(\pm k_{gmn},\mathbf{r})\mathbf{M}_{emn}(\pm k_{gmn},\mathbf{r})\mathbf{M}_{emn}(\mp k_{gmn},\mathbf{r}') + \mathbf{N}_{omn}(\pm k_{gmn},\mathbf{r})\mathbf{N}_{omn}(\mp k_{gmn},\mathbf{r}')] \quad z \gtrless z' \tag{167}$$

where $k_{gmn} = \sqrt{k^2 - k_{cmn}^2}$.

In addition to the method described above, Tai (3) proposed the method of $\overline{\mathbf{G}}_m$, in which $\overline{\mathbf{G}}_m$ is derived first and $\overline{\mathbf{G}}_e$ is then derived from $\nabla \times \overline{\mathbf{G}}_m = \overline{\mathbf{I}}\delta(\mathbf{r} - \mathbf{r}') + k^2\overline{\mathbf{G}}_e$. Since $\overline{\mathbf{G}}_m$ is completely solenoidal, its expansion requires only $\mathbf{M}$ and $\mathbf{N}$ and, therefore, the derivation becomes simpler.

The Ohm–Rayleigh method can be used to derive a variety of dyadic Green's functions. Table 1 lists the problems for which the dyadic Green's functions have been derived.

Table 1. Problems with Available Dyadic Green's Functions

Geometry of Problem	References
Parallel-plate waveguide	(3)
Rectangular waveguide	(3), (11)
Rectangular waveguide with two dielectrics	(3), (12)
Cylindrical waveguide	(3), (13), (14)
Coaxial waveguide	(3), (15)
Rectangular cavity	(3), (16)
Cylindrical cavity	(3), (13), (17)
Spherical cavity	(3), (18)
Circular conducting cylinder	(3)
Circular dielectric cylinder	(3)
Circular coated cylinder	(3)
Elliptical conducting cylinder	(3)
Conducting wedge and half-sheet	(3)
Conducting sphere and cone	(3)
Homogeneous and inhomogeneous spheres	(3)
Planar layered medium	(3), (5), (6)
Planar anisotropic layered medium	(19)
Conductor-backed layered medium	(20), (21), (22)
Cylindrically layered medium	(6), (23)
Spherically layered medium	(3), (6), (24)
Moving medium	(3), (25), (26)

Vector Integral Equations

Consider the problem of the electric and magnetic fields produced by an electric current source $\mathbf{J}(\mathbf{r})$ in the presence of an arbitrarily shaped object immersed in an infinite homogeneous medium (see Fig. 6). Exterior to the object, the electric field satisfies the vector wave equation in Eq. (127) and the radiation condition at infinity is given by

$$r[\nabla\times\mathbf{E}(\mathbf{r})+jk\hat{r}\times\mathbf{E}(\mathbf{r})]=0 \qquad \text{for } r\to\infty \tag{168}$$

Multiplying Eq. (127) by $\overline{\mathbf{G}}_{e0}(\mathbf{r}, \mathbf{r}')$, Eq. (130) by $\mathbf{E}(\mathbf{r})$, and integrating the difference of the resultant equations over the exterior region, one obtains

$$\int_{V_\infty}\{[\nabla\times\nabla\times\mathbf{E}(\mathbf{r})]\cdot\overline{\mathbf{G}}_{e0}(\mathbf{r},\mathbf{r}')-\mathbf{E}(\mathbf{r})\cdot[\nabla\times\nabla\times\overline{\mathbf{G}}_{e0}(\mathbf{r},\mathbf{r}')]\}\,dV = -j\omega\mu\int_{V_s}\mathbf{J}(\mathbf{r})\cdot\overline{\mathbf{G}}_{e0}(\mathbf{r},\mathbf{r}')\,dV-\int_{V_\infty}\mathbf{E}(\mathbf{r})\cdot\overline{\mathbf{I}}\delta(\mathbf{r}-\mathbf{r}')\,dV \tag{169}$$

where V_∞ denotes the infinite space exterior to the object and V_s denotes the support of $\mathbf{J}(\mathbf{r})$. Applying the vector-dyadic Green's second identity (27)

$$\int_V \left[(\nabla\times\nabla\times\mathbf{A})\cdot\overline{\mathbf{D}} - \mathbf{A}\cdot(\nabla\times\nabla\times\overline{\mathbf{D}})\right]dV = \int_S \left[(\hat{n}\times\mathbf{A})\cdot(\nabla\times\overline{\mathbf{D}}) + (\hat{n}\times\nabla\times\mathbf{A})\cdot\overline{\mathbf{D}}\right]dS \quad (170)$$

where V is a volume enclosed by S, one has

$$\int_{S_o+S_\infty} \left\{[\hat{n}\times\mathbf{E}(\mathbf{r})]\cdot[\nabla\times\overline{\mathbf{G}}_{e0}(\mathbf{r},\mathbf{r}')] + \hat{n}\times\nabla\times\mathbf{E}(\mathbf{r})]\cdot\overline{\mathbf{G}}_{e0}(\mathbf{r},\mathbf{r}')\right\}dS = -j\omega\mu\int_{V_s}\mathbf{J}(\mathbf{r})\cdot\overline{\mathbf{G}}_{e0}(\mathbf{r},\mathbf{r}')\,dV - \int_{V_\infty}\mathbf{E}(\mathbf{r})\delta(\mathbf{r}-\mathbf{r}')\,dV \quad (171)$$

where S_o denotes the surface of the object, S_∞ denotes a large spherical surface whose radius approaches infinity, and $\hat{n}$ is the normal unit vector pointing away from V_∞. Since both $\mathbf{E}(\mathbf{r})$ and $\overline{\mathbf{G}}_{e0}(\mathbf{r}, \mathbf{r}')$ satisfy the radiation condition, the surface integral over S_∞ vanishes. As a result,

$$-\int_{S_o} \left\{[\hat{n}\times\mathbf{E}(\mathbf{r})]\cdot[\nabla\times\overline{\mathbf{G}}_{e0}(\mathbf{r},\mathbf{r}')] + [\hat{n}\times\nabla\times\mathbf{E}(\mathbf{r})]\cdot\overline{\mathbf{G}}_{e0}(\mathbf{r},\mathbf{r}')\right\}dS - j\omega\mu\int_{V_s}\mathbf{J}(\mathbf{r})\cdot\overline{\mathbf{G}}_{e0}(\mathbf{r},\mathbf{r}')\,dV = \begin{cases}\mathbf{E}(\mathbf{r}') & \text{for } \mathbf{r}' \text{ in } V_\infty \\ 0 & \text{for } \mathbf{r}' \text{ in } V_o\end{cases} \quad (172)$$

which can also be written as

$$-\int_{S_o} \left\{[\nabla\times\overline{\mathbf{G}}_{e0}(\mathbf{r},\mathbf{r}')]\cdot[\hat{n}'\times\mathbf{E}(\mathbf{r}')] - j\omega\mu\overline{\mathbf{G}}_{e0}(\mathbf{r},\mathbf{r}')\cdot[\hat{n}'\times\mathbf{H}(\mathbf{r}')]\right\}dS' - j\omega\mu\int_{V_s}\overline{\mathbf{G}}_{e0}(\mathbf{r},\mathbf{r}')\cdot\mathbf{J}(\mathbf{r}')\,dV' = \begin{cases}\mathbf{E}(\mathbf{r}) & \text{for } \mathbf{r} \text{ in } V_\infty \\ 0 & \text{for } \mathbf{r} \text{ in } V_o\end{cases} \quad (173)$$

where V_o denotes the volume of the object.

Similar to Eq. (102) in the scalar case, Eq. (173) is an important result, which has several implications. First, notice that when the object is absent, the surface integral vanishes. Hence,

$$\mathbf{E}(\mathbf{r}) = -j\omega\mu\int_{V_s}\overline{\mathbf{G}}_{e0}(\mathbf{r},\mathbf{r}')\cdot\mathbf{J}(\mathbf{r}')\,dV' \quad (174)$$

which is the same as Eq. (128). The above can be regarded as the incident field and denoted as $\mathbf{E}^{\text{inc}}(\mathbf{r})$. Second, when there is no source in V_∞, Eq. (173) becomes

$$\mathbf{E}(\mathbf{r}) = -\int_{S_o}\left\{[\nabla\times\overline{\mathbf{G}}_{e0}(\mathbf{r},\mathbf{r}')]\cdot[\hat{n}'\times\mathbf{E}(\mathbf{r}')] - j\omega\mu\overline{\mathbf{G}}_{e0}(\mathbf{r},\mathbf{r}')\cdot[\hat{n}'\times\mathbf{H}(\mathbf{r}')]\right\}dS' \quad (175)$$

for $\mathbf{r}$ in V_∞. Since there is no source in V_∞, the field on S_o must be produced by the source inside S_o. This equation indicates that the field in a source-free region can be calculated based on the knowledge of the tangential electric and magnetic fields on the surface enclosing the region. This is the mathematical representation of the well-known Huygens' principle for a vector field.

Equation (173) also provides the foundation to establish an integral equation for $\hat{n}\times\mathbf{E}$ and $\hat{n}\times\mathbf{H}$ on the surface of the object. If the object is a perfect conductor, $\hat{n}\times\mathbf{E}(\mathbf{r}) = 0$ for $\mathbf{r}$ on S_o. Consequently, Eq. (173) becomes

$$\mathbf{E}(\mathbf{r}) = \mathbf{E}^{\text{inc}}(\mathbf{r}) + j\omega\mu\int_{S_o}\overline{\mathbf{G}}_{e0}(\mathbf{r},\mathbf{r}')\cdot[\hat{n}'\times\mathbf{H}(\mathbf{r}')]\,dS' \quad (176)$$

for $\mathbf{r}$ in V_∞. Substituting this into $\hat{n}\times\mathbf{E}(\mathbf{r}) = 0$ for $\mathbf{r}$ on S_o, we obtain an integral equation, which can be solved for $\hat{n}\times\mathbf{H}(\mathbf{r})$.

If the object is a homogeneous body, one can derive another integral representation for the field inside S_o using the unbounded-space dyadic Green's function for the interior medium. When this and Eq. (173) are applied at S_o, one obtains two integral equations, which can be solved for $\hat{n}\times\mathbf{E}(\mathbf{r})$ and $\hat{n}\times\mathbf{H}(\mathbf{r})$.

If the object is an inhomogeneous dielectric body, the electric field satisfies the vector wave equation

$$\nabla\times\nabla\times\mathbf{E}(\mathbf{r}) - \tilde{k}^2(\mathbf{r})\mathbf{E}(\mathbf{r}) = -j\omega\mu\mathbf{J}(\mathbf{r}) \quad (177)$$

This can be written as

$$\nabla\times\nabla\times\mathbf{E}(\mathbf{r}) - k^2\mathbf{E}(\mathbf{r}) = -j\omega\mu\mathbf{J}(\mathbf{r}) + [\tilde{k}^2(\mathbf{r}) - k^2]\mathbf{E}(\mathbf{r}) \quad (178)$$

Multiplying Eq. (177) by $\overline{\mathbf{G}}_{e0}(\mathbf{r}, \mathbf{r}')$ and integrating the resultant equation over the entire space, one obtains

$$\mathbf{E}(\mathbf{r}) = \mathbf{E}^{\text{inc}}(\mathbf{r}) + \int_{V_o}\overline{\mathbf{G}}_{e0}(\mathbf{r},\mathbf{r}')\cdot[\tilde{k}^2(\mathbf{r}') - k^2]\mathbf{E}(\mathbf{r}')\,dV' \quad (179)$$

This is the mathematical representation of the volume equivalence principle. It provides a volume integral equation which can be solved for $\mathbf{E}(\mathbf{r})$.

We note that the formulation described in this section can be repeated for the magnetic field in a similar manner. As a result, different integral equations exist for the same problem, which provide different approaches to the solution of the problem.

Singularity of the Dyadic Green's Function

As shown in Eq. (128), the electric field produced by the current $\mathbf{J}$ in an unbounded space can be written as

$$\mathbf{E}(\mathbf{r}) = -j\omega\mu\int_V\overline{\mathbf{G}}_{e0}(\mathbf{r},\mathbf{r}')\cdot\mathbf{J}(\mathbf{r}')\,dV' \quad (180)$$

where $\overline{\mathbf{G}}_{e0}(\mathbf{r}, \mathbf{r}')$ is defined by Eq. (140). Many electromagneticists have tried to fathom the meaning of Eq. (180). Strictly speaking, the integral does not converge because of the $1/|\mathbf{r}-\mathbf{r}'|$ singularity in $G_0(\mathbf{r}, \mathbf{r}')$. After being operated upon by the double ∇ operator in Eq. (140), $\overline{\mathbf{G}}_{e0}(\mathbf{r}, \mathbf{r}')$ contains terms of the form $1/|\mathbf{r}-\mathbf{r}'|^3$, rendering the integral in Eq. (180) ill-defined. A remedy to this is to rewrite Eq. (180) as

$$\mathbf{E}(\mathbf{r}) = -j\omega\mu\left(\overline{\mathbf{I}} + \frac{\nabla\nabla}{k^2}\right)\cdot\int_V G_0(\mathbf{r},\mathbf{r}')\mathbf{J}(\mathbf{r}')\,dV' \quad (181)$$

This equation is well-defined for all $\mathbf{r}$ and $\mathbf{r}'$, but lacks the compactness of Eq. (180).

Equation (180) can be made meaningful in a generalized function sense. To this end, one defines $\overline{\mathbf{G}}_{e0}(\mathbf{r}, \mathbf{r}')$ as a generalized function

$$\overline{\mathbf{G}}_{e0}(\mathbf{r}, \mathbf{r}') = PV\overline{\mathbf{G}}_{e0}(\mathbf{r}, \mathbf{r}') - \frac{\overline{\mathbf{L}}\delta(\mathbf{r} - \mathbf{r}')}{k^2} \tag{182}$$

where PV implies the invokement of a principal volume integral whose value depends on the shape of the principal volume chosen. For the sake of uniqueness, $\overline{\mathbf{L}}$ also depends on the shape of the principal volume. A principal volume integral is defined as

$$\int_V PV\overline{\mathbf{G}}_{e0}(\mathbf{r}, \mathbf{r}') \cdot \mathbf{J}(\mathbf{r}')\, dV' = PV \int_V \overline{\mathbf{G}}_{e0}(\mathbf{r}, \mathbf{r}') \cdot \mathbf{J}(\mathbf{r}')\, dV' = \lim_{V_\delta \to 0} \int_{V - V_\delta} \overline{\mathbf{G}}_{e0}(\mathbf{r}, \mathbf{r}') \cdot \mathbf{J}(\mathbf{r}')\, dV' \tag{183}$$

where V_δ is called the exclusion volume. Unfortunately, even though the integral above converges, its value is nonunique in the sense that it depends on the shape of V_δ. The nonuniqueness in the first term of Eq. (182) is rectified by the choice of a shape-dependent $\overline{\mathbf{L}}$. The value of $\overline{\mathbf{L}}$ in Eq. (182) for various exclusion volumes is given by (28–33)

$$\overline{\mathbf{L}} = \frac{\overline{\mathbf{I}}}{3} \quad \text{for spheres and cubes} \tag{184}$$

$$\overline{\mathbf{L}} = \hat{z}\hat{z} \quad \text{for disks perpendicular to the } z\text{-axis} \tag{185}$$

$$\overline{\mathbf{L}} = \frac{\hat{x}\hat{x} + \hat{y}\hat{y}}{2} \quad \text{for needles parallel to the } z\text{-axis} \tag{186}$$

To understand how the above $\overline{\mathbf{L}}$s are derived, one can start with the classically legitimate Eq. (181) and split the integral into two terms with the definition of a principal volume integral:

$$\mathbf{E}(\mathbf{r}) = -j\omega\mu \lim_{V_\delta \to 0} \left(\overline{\mathbf{I}} + \frac{\nabla\nabla}{k^2}\right) \cdot \int_{V - V_\delta} G_0(\mathbf{r}, \mathbf{r}')\mathbf{J}(\mathbf{r}')\, dV' - j\omega\mu \lim_{V_\delta \to 0} \left(\overline{\mathbf{I}} + \frac{\nabla\nabla}{k^2}\right) \cdot \int_{V_\delta} G_0(\mathbf{r}, \mathbf{r}')\mathbf{J}(\mathbf{r}')\, dV' \tag{187}$$

In the first term, $V_\delta > 0$ and hence, it is legitimate to exchange the order of differentiation and integration so that it becomes the first term of Eq. (182). In the second term in Eq. (187), the term that does not allow the exchange of the order of integration and differentiation is the term involving the $\nabla\nabla$ operator. Focusing on it more carefully, one has

$$\lim_{V_\delta \to 0} \nabla\nabla \cdot \int_{V_\delta} G_0(\mathbf{r}, \mathbf{r}')\mathbf{J}(\mathbf{r}')\, dV' = \lim_{V_\delta \to 0} \nabla \left[\int_{V_\delta} G_0(\mathbf{r}, \mathbf{r}')\nabla' \cdot \mathbf{J}(\mathbf{r}')\, dV' - \int_{S_\delta} \hat{n}' \cdot \mathbf{J}(\mathbf{r}')G_0(\mathbf{r}, \mathbf{r}')\, dS' \right] \tag{188}$$

To arrive at the above, one has made use of the fact that $\nabla G_0(\mathbf{r}, \mathbf{r}') = -\nabla' G_0(\mathbf{r}, \mathbf{r}')$, and $[\nabla' G_0(\mathbf{r}, \mathbf{r}')] \cdot \mathbf{J}(\mathbf{r}') = \nabla' \cdot [G_0(\mathbf{r}, \mathbf{r}')\mathbf{J}(\mathbf{r}')] - G_0(\mathbf{r}, \mathbf{r}')\nabla' \cdot \mathbf{J}(\mathbf{r}')$. Using Gauss's theorem on the term involving $\nabla' \cdot [G_0(\mathbf{r}, \mathbf{r}')\mathbf{J}(\mathbf{r}')]$ finally gives rise to Eq. (188). The first integral on the right-hand side of Eq. (188) vanishes since if $\mathbf{J}(\mathbf{r}')$ is regular, $\nabla' \cdot \mathbf{J}(\mathbf{r}') = \rho(\mathbf{r}')/j\omega$ is also regular and the integral is finally proportional to V_δ. In the second integral, S_δ is the surface bounding V_δ. Hence, $\hat{n}' \cdot \mathbf{J}(\mathbf{r}')$ is the surface charge on S_δ due to the sudden truncation of $\mathbf{J}(\mathbf{r}')$ within the volume V_δ. This integral gives the potential observed within V_δ due to this surface charge, and it is nonzero even when $V_\delta \to 0$. The gradient (outside the brackets) in turn yields the field generated by this surface charge. In other words, surface charges of opposite polarities on the wall of an infinitesimally small volume always generate a finite field within the small volume. This fact is also intimately related to the scale invariant nature of the Laplace equation which is Maxwell's equations at low frequency.

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GROUND EFFECTS, RADIOWAVE PROPAGATION.

See RADIOWAVE PROPAGATION GROUND EFFECTS.